LOGARITHMIC PARAFERMION VERTEX OPERATOR ALGEBRAS

by

Jean Auger

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Department of Mathematical and Statistical Sciences
University of Alberta

Abstract

This thesis is about the theory of vertex operator algebras and their representations. Its main results provide new examples of logarithmic C_2-cofinite vertex operator algebras. These include closure of the characters under modular transformations with explicit determination of the modular coefficients for an infinite family of parafermionic vertex operator algebras and proofs of C_2-cofiniteness for a few specific levels including for three new cases. The rest of the work presented in this thesis provides a new comprehension of the notoriously difficult proof of the Kac-Wakimoto character formula, but also categorical results and tools to study certain vertex operator algebras' categories involving infinite direct sums.

Dedicated to *i fagiani del Chianti*

e qualcuna di speciale.

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Introduction

The main topic of my PhD is the study of certain classes of vertex operator algebras. The notion of vertex operator algebra originated from physical theories developed in the late 20$^{\text{th}}$ century. A few decades earlier, physicists had advanced two theories that are now used to understand the physical world: quantum physics and general relativity. However, both theories fail to explain some observed phenomena such as black holes and dark matter. Such a shortcoming might be due to differences in the nature of the two theories: quantum physics is probabilistic, while general relativity is not. In order to fill these gaps, many have tried to find a new theoretical framework for physics that would account for both quantum physics and general relativity at the same time. Provided it exists, such a unified framework would certainly bring new insights into understanding phenomena that remain unexplained. A promising candidate has been thought to be the so-called String Theory in which 2-dimensional Conformal Field Theories play an important role [GG 2000]. Conformal Field Theories satisfy certain *conformal invariance* properties Conformal invariance has to do with invariance properties of the physical theory under the effect of Lorentz transformations [DFMS 1997] and as a consequence, a Virasoro Lie algebra action appears naturally in 2-dimensional Conformal Field Theory. Introductions to Conformal Field Theory can be found in [DFMS 1997] and [Gab 2000], while an axiomatic treatment is outlined in [GG 2000]. The highly developed language in which all these physical theories are written requires more than just bare Lie Theory to be understood. Along with other mathematical motivations, the study of vertex operator algebras came to be a field on its own, which can be thought of as the study of some natural invariants of the physical theories, see [BLL^{+} 2015], [BR 2018] for instance. Specifically, the space of states of a physical system in a Conformal Field Theory is often seen as a representation for a certain type of vertex operator algebra. A mathematician would then say that the chiral symmetry of the Conformal Field Theory is expressed by the action of that same vertex operator algebra [Gab 2000], [DFMS 1997].

Regardless of any physical veracity, the mathematical study of vertex operator algebras has since unveiled a lot of unexpected connections between many areas in the very same mathematics. A vertex operator algebra is an object of multifaceted nature. For example, from an algebraic perspective the structure of a vertex operator algebra is strongly connected to Lie Theory, and its representations provide new insights into Number Theory (see [LW 1981], [KLRS 2017] for instance). On the other hand, the physical origin of vertex operator algebras makes them strongly linked with modular forms [DFMS 1997], complex analysis, geometry, topology and even probability theory [CR 2013a], [MR 2007]. Notable connections to topology involve the so-called Jones polynomial and Reshetikhin-Turaev invariants in relation to knots [RT 1991].

As an algebraic structure, a vertex operator algebra "lives trough its actions" in the sense that what really characterises it are its representations. Therefore, Category Theory is a very natural tool to study vertex operator algebras. It does provide a framework to make sense of the algebras and their modules as a whole. Remarkably, certain vertex operator module categories exhibit very rich structures involving a tensor product bifunctor, see [MS 1989], [Hua 2005], [Hua 2010] and also [HL 2013], [HLZ 2007], [Miy 2003], [Miy 2010], [CG 2017]. The study of tensor products for vertex operator algebras is not an easy task in general, but it is certainly rewarding for both the general understanding of these structures and for the potential applications and insights in Physics and Geometry. Typically, physicists use tensor product fusion rules to compute correlation functions and other statistical features of quantum systems. It is difficult to have precise descriptions of tensor products for general vertex operator algebras. Even to properly define tensor products is a challenge in general. Under suitable assumptions, the Verlinde formula relates the tensor product to modular transformation coefficients of characters of modules [Ver 1988], [Hua 2005]. The Verlinde formula is quite useful in practice because it lets one understand tensor products through modular forms. In the appropriate settings, the Verlinde formula can also take categorical data as input: such data are derived from traces of monodromy homomorphisms of modules called the Hopf links. Historically, key landmark results include that somple, rational and C_2-cofinite vertex operator algebras of CFT-type that are isomorphic to their graded restricted dual have a category of module with a modular tensor category structure [Hua 2005], [Lep 2005], [Hua 2008]. Those are special types of ribbon braided monoidal categories with duality morphisms. In this way, the worlds of braided monoidal categories and modular forms meet with that of algebra and physics. Such connections are fascinating and it is often not known how general they are.

Currently, well understood vertex operator algebras are often too specific for the most relevant applications in Physics and other mathematical fields. The theory of vertex operator algebras is still quite recent and its technical challenges make interesting examples rather hard to approach. This is also valid for the study of modules of vertex operator algebras about which so little is known in general. For example, it is hard to prove the existence of projective covers of simple modules, even for a simple vertex operator algebra. As mentioned above, natural tensor product bifunctors are often hard to define and to make use of [HLZ 2007], [CHY 2018], see also [Miy 2003] for instance. Over time, mathematicians and physicists have developed many ways of constructing new vertex operator algebras building upon existing ones, but what people actually know about the corresponding module categories is often very limited. Currently, the well understood Representation Theory settings for vertex operator algebras orbit around the rational vertex operator algebras while only little is known about non-rational vertex operator algebras: even less about more generic structures [Mil 2014], [Fuc 2007], [CG 2017].

We are going to focus primarily on irrational vertex operator algebras that are C_2-cofinite and related matters. We will call such vertex operator algebras *logarithmic* and C_2-cofinite. The use of the adjective logarithmic here comes from the fact that the non-semisimplicity of modules for such vertex operator algebras often leads to the appearance of logarithms of variable in formal variable expansions of *intertwing operators* used to define well-behaved tensor products. The property of C_2-cofiniteness was introduced by Zhu [Zhu 1996] and is seen as a rather technical finiteness condition although other interpretations have since been given [GG 2009] including a notable geometrical one [Ara 2012]. The importance of C_2-cofinite vertex operator algebras is that they have finitely many simple modules and they share a key property for physical applications: a natural modularity behaviour of its ring of characters [Miy 2004]. Logarithmic C_2-cofinite vertex operator algebras have non-simple indecomposable modules, a feature that makes them complicated to study. Despite all obstacles, it is believed that they share key features of their rational C_2-cofinite cousins: a category of modules with a log-modular tensor category, modularity behaviour of characters and the Verlinde formula [CG], [CG 2017], [GR 2017]. Until now, examples of logarithmic C_2-cofinite vertex operator algebras are quite rare and not much is known about them in general [AM 2013], [CG]. Most of the better known logarithmic C_2-cofinite structures that are known are related to the so-called triplet $\mathcal{W}(p)$-algebras whose categorical and modularity features are well detailed in [TW 2013], see also [AM 2008a].

The aim of the thesis is to provide new details about certain logarithmic vertex operator algebras that are thought to be C_2-cofinite and prove C_2-cofiniteness whenever possible. In Chapter 1, we present a detailed proof of the Kac-Wakimoto character formula [KW 1988], which helped mathematicians to discover modular behaviours in infinite dimensional Lie theory and beyond. This formula has direct applications to logarithmic affine vertex operator algebras. In Chapter 2, we present a direct sum completion of a monoidal braided category and apply this framework to the problem of constructing even lattice vertex operator algebras, a basic vertex operator algebra extension setting. The sum completion fills a gap in the literature and combined with the techniques of [CKM 2017] and [CKL 2015], it will help understand much more complex settings on solid grounds. Note that the results of Chapter 2 have been accepted for publication in a journal [AR 2018]. In Chapter 3, we study logarithmic parafermion vertex operator algebras associated with the affine simple vertex operator algebra $L_k(\mathfrak{sl}_2)$ at admissible rational level k. We show that a family of these parafermion vertex operator algebras have a category of module whose characters have a modular behaviour comparable to that described in [Miy 2004]. We then conjecture that our family of parafermion vertex algebras is C_2-cofinite. The results of Chapter 3 have also been accepted for publication in a journal [ACR 2018]. In Chapter 4, we prove C_2-cofiniteness of a number of logarithmic parafermion vertex operator algebras of Chapter 3 using computational methods.

Chapter 1

The Kac-Wakimoto Character Formula

This chapter presents an accessible proof of the famous Kac-Wakimoto character formula [KW 1988]. Understanding the proof of this formula has remained a notoriously challenging task for many mathematicians, including representation theorists. Consider a Kac-Moody Lie algebra with symmetrisable Cartan matrix, then a slightly modified Weyl-Kac character formula holds for certain irreducible Verma quotients $L(\lambda)$'s with specific allowed weights λ's. The Kac-Wakimoto formula is:

$$\mathrm{ch}\,[L(\lambda)] = \sum_{w \in W^{\lambda+\rho}} \varepsilon(w)\,\mathrm{ch}\,[M(w \bullet \lambda)] \ .$$

For the rest of the thesis, the weights for which the corresponding irreducible Verma quotient will satisfy this formula will be called *admissible weights*.

The Kac-Wakimoto character formula, just as other character formulas known in Representation Theory, relates some algebraic objects to certain power series. Interpreting the resulting power series in various ways proved to be a very prolific point of view as notable objects emerged from this analysis, e.g. meromorphic functions, vector-valued modular forms, etc. These ideas allowed to borrow combinatorial and analysis concepts and techniques to study Representation Theory. In such ways, character formulas have established many connections between Algebra and other mathematical topics such as Number Theory and even Analysis and Topology. This is especially the case when the algebraic objects originate from infinite dimensional Representation Theory. These bridges have actually worked both ways: for example characters have sometimes helped to understand and motivate complex combinatorial identities, see for instance [KLRS 2017] and [LW 1981].

The character formula from V. Kac and M. Wakimoto is especially notable because it revealed the presence of so-called *modular-phenomena* in the context of affine Lie algebras and integrable modules [KP 1984], [Fre 1984]. It turns out that the admissible weight modules are precisely those on which one can define a natural vertex algebra module structure for an action of the associated simple affine vertex operator algebra. A fascinating aspect of characters is that they do reflect properties of the associated algebraic structure or its module categories. Y. Zhu proved in [Zhu 1996] that the linear span of characters of the simple modules of a rational C_2-cofinite vertex operator algebra gives rise to a finite dimensional vector-valued modular form, a result which was later generalised by M. Miyamoto to more general C_2-cofinite vertex operator algebras [Miy 2004]. As it will be seen at the begining of Chapter 3, the Kac-Wakimoto character formula gives a starting point to study module categories of affine vertex operator algebras.

The proof of the Kac-Wakimoto formula presented in this section follows the original paper of V. Kac and M. Wakimoto. In this Chapter, the proof is presented in four logical steps presented in four different sections:

Section 1.1: a few results dealing with roots and weights;

Section 1.2: a few categorical results;

Section 1.3: a proof that a translation functor $\underline{\mathbf{T}}$ maps $M(w \bullet \lambda)$ to $M(w \bullet \mu)$;

Section 1.4: the proof of the Kac-Wakimoto formula itself.

Precious references for understanding [KW 1988] have been [DGK 1982] and also [Ioh 1997]. Note also that certain ideas and tools developed for the proof of Kac-Wakimoto's formula have analogs in the category $\mathcal{O}$ for a finite dimensional semisimple Lie algebra, see [Hum 2008] and also the comments from Section 13.6 of the same reference.

Notation

Throughout the chapter, the following notation will be employed:

- $a \geq 0 \ \ (a \in \mathbb{C}) \quad \Leftrightarrow \quad$ Re $a > 0$ or $a \in \mathbb{R}_{\geq 0} i$;
- $L = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$ is the triangular decomposition of a Kac-Moody Lie algebra with symmetrisable Cartan matrix;
- $\mathfrak{h} \subseteq L$ is its Cartan subalgebra i.e. a maximal toral Lie subalgebra of L;
- $P_+ \subseteq \mathfrak{h}^*$ is the set of dominant integral weights of L;
- Q_+ is the set of positive roots of L;
- $R^\vee = \{$ Real coroots $\} \subseteq \mathfrak{h}$;
- $\rho \in \mathfrak{h}^*$ is an element such that $\rho(s) = 1$ for all simple coroots s. It is usually called *the Weyl element*;

- W is the Weyl group of L;

- $w \bullet \lambda = w(\lambda + \rho) - \rho$ where for $w \in W$ and $\lambda \in \mathfrak{h}$;

- $(-|-)$ = the symmetric, invariant and non-degenerate bilinear form on L normalised so that the longest roots of L have squared length 2;

- $(-,-)$ = the induced bilinear form on $\mathfrak{h}^*$;

- $K = \left\{ \lambda \in \mathfrak{h}^* \;\middle|\; \begin{array}{l} \#\{r \in R_+^\vee \,|\, \lambda(r) < 0)\} < \infty \\ \beta \text{ positive isotropic root } \Rightarrow (\lambda, \beta) \neq 0 \end{array} \right\} \subseteq \mathfrak{h}^*$;

- $K^L = -\rho + K$;

- $\mathcal{O}$ = the category of L-modules such that:

  ($\mathcal{O}1$) M is $\mathfrak{h}$-semisimple with finite dimensional weight spaces;

  ($\mathcal{O}2$) any weight of M is contained in a finite union $\displaystyle\bigcup_{r=1}^{N}(\mu_r - Q_+)$,

  where the μ_r are weights of L.

  For more details, see [Kac 1974];

- $\mathcal{O}^L$ = the subcategory of the category $\mathcal{O}$ formed of objects whose irreducible constituents are all $L(\lambda)$'s for $\lambda \in -\rho + K$;

- $C = \left\{ \lambda \in \mathfrak{h}^* \;\middle|\; \begin{array}{l} r \in R_+^\vee \;\Rightarrow\; \lambda(r) \geq 0 \\ \beta \text{ positive isotropic root } \Rightarrow (\lambda, \beta) \neq 0 \end{array} \right\} \subseteq K \subseteq \mathfrak{h}^*$;

- $R_\lambda^\vee = \{r \in R^\vee \,|\, \lambda(r) \in \mathbb{Z}\}$ for a given $\lambda \in \mathfrak{h}^*$;

- $\Pi_\lambda^\vee = \{$ the simple elements of $R_\lambda^\vee\}$ for a given $\lambda \in \mathfrak{h}^*$;

- $W^\lambda = \langle\, \sigma_r \,|\, r \in R^\vee_\lambda \,\rangle = \langle\, \sigma_s \,|\, s \in \Pi^\vee_\lambda \,\rangle \leq W$ where W is the Weyl group of the root system of L.

Here are a few remarks:

Remark 1.1. Both the sets $\{a \in \mathbb{C} \,|\, a \geq 0\}$ (see the above notation) and its complement are closed under addition.

Remark 1.2. K is W-invariant under the standard Weyl action on $\mathfrak{h}^*$.

Remark 1.3. K^L is W-invariant under the dot action on $\mathfrak{h}^*$.

Remark 1.4. $C \subseteq K$ and in each W-orbit in K, there exists a unique element of C in that orbit.

1.1 Preliminary Results

In this first section, I present some relevant results for later use in the proof of the Kac-Wakimoto character formula. All these results are related to weights, roots and Weyl group properties or to some basic properties of the category $\mathcal{O}$.

Result 1.5. *$L(\mu)$ is a constituent (subquotient) of $M(\lambda)$ if and only if there exists positive roots $\beta_1, \dots, \beta_N$ and $n_1, \dots, n_N \in \mathbb{N}$ such that*

(1) $\mu = \lambda - \sum_{\ell=1}^{N} n_\ell \beta_\ell$;

(2) $2\Big(\big(\lambda - \sum_{\ell=1}^{j-1} n_\ell \beta_\ell\big) + \rho \,,\, \beta_j\Big) = n_j\Big(\beta_j \,,\, \beta_j\Big)$ *for all* $j \in \{0, \dots, N\}$.

Proof: Can be found in [KK 1979], see Theorem 2.

Result 1.6. *Let $\lambda \in K^L$. Then:*

$$L(\mu) \text{ is a constituent of } M(\lambda) \qquad \Longrightarrow \qquad \mu \in K^L \ .$$

Proof: By Result 1.5, we just need to prove that $\mu' = \lambda - n_1\beta_1 \in K^L = -\rho + K$. We know that $2(\lambda + \rho, \beta_1) = n_1(\beta_1, \beta_1)$ where $n_1 \in \mathbb{N}\backslash\{0\}$ to avoid tackling a triviality. Then, since $\lambda + \rho \in K^L$, the root β_1 cannot be isotropic i.e. cannot be such that $(\beta, \beta) \neq 0$. Next, we can write

$$\begin{aligned}
\mu' &= \lambda - n_1\beta_1 \\
&= \lambda - 2\frac{(\lambda + \rho, \beta_1)}{(\beta_1, \beta_1)}\beta_1 \\
&= (\lambda + \rho) - 2\frac{(\lambda + \rho, \beta_1)}{(\beta_1, \beta_1)}\beta_1 - \rho \ .
\end{aligned}$$

It means that if β_1 were a real root, we could write $\mu' = \sigma_{\beta_1} \bullet \lambda$ and conclude that $\mu' \in -\rho + K$ by the W-invariance of the set K^L under the dot action.

Suppose then that β_1 is an imaginary root i.e. that $(\beta_1, \beta_1) < 0$. In this case $k\beta_1$ is also a positive root for each $k \in \mathbb{N}\backslash\{0\}$ and this leads to

$$(\lambda + \rho, k\beta_1) = k(\lambda + \rho, \beta_1) = kn_1(\beta_1, \beta_1) < 0 \quad \text{for each } k \in \mathbb{N}\backslash\{0\} \ .$$

However, since $\lambda + \rho \in K$, this situation cannot occur. Therefore β_1 is a real root, $\sigma_{\beta_1} \in W$ and so $\mu' = \sigma_{\beta_1} \bullet \lambda \in K^L$ since K^L is W-invariant under the dot action.

Q.E.D.

Definition 1.7. Define an equivalence relation $\sim$ on K^L as follows:

$\lambda \sim \mu \quad \Longleftrightarrow \quad$ there exist $\lambda = \lambda_0, \lambda_1, \ldots, \lambda_{n-1}, \lambda_n = \mu$ such that for all $i \in \{0, \ldots, n-1\}$, we have either:

a) $L(\lambda_i)$ is a constituent of $M(\lambda_{i+1})$, or

b) $L(\lambda_{i+1})$ is a constituent of $M(\lambda_i)$.

For any $\lambda \in K^L$, we will denote by $[\![\lambda]\!]$ its equivalence class under $\sim$. Note that a) and b) just above can be understood in terms of weights and roots by Result 1.5.

Remark 1.8. Note that we cannot conclude a priori that $\sim$ is the relation defining the *extension blocks* of the category $\mathcal{O}^L$, although it probably can be shown with further analysis. See Section 1.13 of [Hum 2008] for the definition of the *linkage* equivalence relation in the Representation Theory setting of finite-dimensional semisimple Lie algebras.

Proposition 1.9. *The category $\mathcal{O}^L$ formed of objects whose constituents are all $L(\mu)$'s with $\mu \in K^L$, is a* full subcategory *of the category $\mathcal{O}$.*

Definition 1.10. For a given $\lambda \in K^L$, we define a category $\mathcal{O}^L_{[\![\lambda]\!]}$ formed of the objects whose constituents are all $L(\mu)$'s for μ's that all lie in the equivalence class $[\![\lambda]\!] \subseteq K^L$. These are full subcategories of the category $\mathcal{O}^L$.

Result 1.11. *The category $\mathcal{O}^L$ decomposes as:*

$$\mathcal{O}^L = \bigoplus_{\lambda \in K^L} \mathcal{O}^L_{[\![\lambda]\!]} \ .$$

Proof: Can be found in [DGK 1982], see Theorem 5.7.

Remark 1.12. Corollary 5.4 of [DGK 1982] shows that $\mathcal{O}^L$ also contains all Verma modules $M(\lambda)$'s where $\lambda \in K^L$.

Remark 1.13. In fact, given $\lambda, \mu \in K^L$ such that $[\![\lambda]\!] \neq [\![\mu]\!]$ (i.e. such that $\lambda \not\sim \mu$), Theorem 4.5 of [DGK 1982] states that

$$\left.\begin{array}{r} V \in \mathrm{Ob}\,\mathcal{O}^L_{[\![\lambda]\!]} \\ W \in \mathrm{Ob}\,\mathcal{O}^L_{[\![\mu]\!]} \end{array}\right\} \qquad \Longrightarrow \qquad \mathrm{Ext}^1_L(V, W) = 0 \ .$$

This is a property of the decomposition of $\mathcal{O}^L$ given in Result 1.11 that an extension block decomposition of $\mathcal{O}^L$ would also share.

Focusing on the category $\mathcal{O}^L$ rather than $\mathcal{O}$ allows one to get a precise description of the equivalence relation $\sim$ in terms of the Weyl group and some of its subgroups W^η's for some $\eta \in \mathfrak{h}^*$. Here is a useful thing we can say about these subgroups:

Result 1.14. *Let $\eta \in \mathfrak{h}^*$ and let $w \in W$. The following statements are true:*

(1) $w\big(R^\eta\big) = R^{w(\eta)}$;

(2) $wW^\eta w^{-1} = W^{w(\eta)}$. *In particular, if $g \in W^\eta$, then $W^\eta = W^{g(\eta)}$.*

<u>Proof:</u> To prove (1), let $r \in R^\eta$. Then

$$2\frac{(\eta, r)}{(r, r)} = 2\frac{\big(w(\eta), w(r)\big)}{\big(w(r), w(r)\big)} \in \mathbb{Z}\ .$$

So we obtain $w(r) \in R^{w(\eta)}$. Finally, since the previous line is an equality and since W is a group, the proof of (1) is complete.

To prove (2), let $g \in W^\eta$. Then write $g = \prod_i \sigma_{r_i}$ where $r_i \in R^\eta$ for all indices i. Next, we can write

$$wgw^{-1} = \prod_i \left(w\sigma_{r_i}w^{-1}\right) = \prod_i \sigma_{w(r_i)}\ .$$

By part (1), $w(r_i) \in R^{w(\eta)}$ for all indices i, so by definition $wgw^{-1} \in W^{w(\eta)}$. Again, since W is a group, the proof is complete.

Q.E.D.

Result 1.15. *The following statements are true:*

(1) *for $\lambda \in K^L$, we have:* $[\![\lambda]\!] = W^{\lambda+\rho} \bullet \lambda$;

(2) *for* $\lambda \in K^L$ *and* $w \in W$*, we have:* $[\![w \bullet \lambda]\!] = \big(wW^{\lambda+\rho}\big) \bullet \lambda$.

Proof: Property (2) is easy to derive from (1). Let's start by proving (1).

If $\mu \in K^L$ is such that $[\![\mu]\!] = [\![\lambda]\!]$, then by the definition of $\sim$ we can assume (without loss of generality), that $\mu = \lambda - n\alpha$ where $n \in \mathbb{N}$, α is a positive root and $2(\lambda+\rho, \alpha) = n(\alpha, \alpha)$.

By assumption, α is a positive root, but it is also a real one for the same reason as in the proof of Result 1.6. With this key information at hand, we can write:

$$\mu = \lambda - n\alpha = (\lambda+\rho) - 2\frac{(\lambda+\rho,\alpha)}{(\alpha,\alpha)}\alpha - \rho = \sigma_\alpha \bullet \lambda \ .$$

As $\alpha^\vee \in R^\vee_{\lambda+\rho}$, we know that $\sigma_\alpha \in W^{\lambda+\rho}$ and this concludes this part of the proof.

We now need to prove that any element of $W^{\lambda+\rho} \bullet \lambda$ is equivalent to λ under $\sim$. As $W^{\lambda+\rho}$ is generated by the reflections σ_r where $r \in \Pi^\vee_{\lambda+\rho} \subseteq R^\vee_{\lambda+\rho,+}$, it will be sufficient to prove that $[\![\sigma_r \bullet \lambda]\!] = [\![\lambda]\!]$ for a fixed $r \in \Pi^\vee_{\lambda+\rho}$.

Let us now denote by β the positive real root corresponding to the positive real coroot $r \in \Pi^\vee_{\lambda+\rho}$ and let $n \in \mathbb{Z}$ be such that $\big(\lambda+\rho\big)(r) = n$. Then we can write:

$$\sigma_r \bullet \lambda = \lambda - 2\big(\lambda+\rho\big)(r)\beta = \lambda - n\beta \ ,$$

where β is a root with $2(\lambda+\rho, \beta) = n(\beta, \beta)$ for a $n \in \mathbb{Z}$. Note here that we have

$$\big(\lambda+\rho\big)(r) = n \ , \quad \text{and} \quad \big(\lambda - n\beta + \rho\big)(r) = -n \ .$$

If $n \in \mathbb{N}$, then setting $\lambda' = \lambda$ and $\mu' = \lambda - ns$, we see by Result 1.5, that $L(\mu')$ is a constituent of $M(\lambda')$. Else, if $-n \in \mathbb{N}$, we see that $L(\lambda')$ is a constituent of $M(\mu')$. In both cases, the definition of $\sim$ allows to conclude that indeed, $[\![\sigma_r \bullet \lambda]\!] = [\![\lambda]\!]$.

This finishes the proof of (1).

To prove (2), we use part (1) that tells us we have $[\![\lambda]\!] = W^{\lambda+\rho} \bullet \lambda$ for any $\lambda \in K^L$. Let $w \in W$, then we can write

$$\begin{aligned}
[\![w \bullet \lambda]\!] &= \left(W^{w\bullet(\lambda)+\rho}\right) \bullet (w \bullet \lambda) \\
&= \left(W^{w\bullet(\lambda)+\rho} w\right) \bullet \lambda \\
&= \left(W^{w(\lambda+\rho)-\rho+\rho} w\right) \bullet \lambda \\
&= \left(W^{w(\lambda+\rho)} w\right) \bullet \lambda \\
&= \left(w W^{\lambda+\rho} w^{-1} w\right) \bullet \lambda \qquad \text{by Result 1.14} \\
&= \left(w W^{\lambda+\rho}\right) \bullet \lambda \,.
\end{aligned}$$

Q.E.D.

Corollary 1.16. *If $\lambda, \mu \in K^L$ are such that $[\![\mu]\!] = [\![\lambda]\!]$, then we have $W^{\mu+\rho} = W^{\lambda+\rho}$.*

Proof: The hypothesis $[\![\mu]\!] = [\![\lambda]\!]$ gives $\mu = g \bullet \lambda$ for a certain $g \in W^{\lambda+\rho}$ by part (1) of the Result 1.15. Thus, we know that $W^{\mu+\rho} = W^{g\bullet\lambda+\rho} = W^{g(\lambda+\rho)}$.

Then, using part (2) of Corollary 1.14, we get $W^{\lambda+\rho} = W^{g(\lambda+\rho)}$ because g belongs to $W^{\lambda+\rho}$. This gives $W^{\lambda+\rho} = W^{\mu+\rho}$ and the proof is complete.

Q.E.D.

Remark 1.17. Corollary 1.16 explains how a thing such as part (1) of Result 1.15 is possible at all, knowing that $\mu \sim \lambda \;\Leftrightarrow\; \lambda \sim \mu$. To see it, one has to keep in mind that $\sim$ is an equivalence relation.

To conclude Section 1.1, we report here a relevant result about a hypothesis that will be often used in the rest of the chapter:

Result 1.18. *Let $\lambda, \mu \in \mathfrak{h}^*$ be such that $\big(W(\mu-\lambda)\big) \cap P_+ \neq \emptyset$, then $W^\lambda = W^\mu$.*

Proof: First note that $\big(W(\mu-\lambda)\big) \cap P_+ \neq \emptyset$ implies that it contains precisely one element. This is because there is at most one dominant integral element in any given W-orbit inside of $\mathfrak{h}^*$.

Let θ be the dominant integral element in $\big(W(\mu-\lambda)\big) \cap P_+ \neq \emptyset$ and let $\omega \in W$ be such that $\omega(\mu-\lambda) = \theta$. Then

$$\mu = \lambda + \omega^{-1}(\theta) \ , \quad \text{and} \quad \mu + \rho = \lambda + \rho + \omega^{-1}(\theta) \ .$$

We are almost ready to get started. Recall that $W^{\mu+\rho} = \langle\, \sigma_r \,|\, r \in R^\vee_{\mu+\rho} \rangle$. Fix any $r \in R^\vee_{\mu+\rho}$. In the following, it will be proved that this $r \in R^\vee_{\lambda+\rho}$.

Let s be the real root corresponding to the real coroot r. As s is a real root, it is in the W-orbit of a simple root, say α. So let's write $s = w(\alpha)$ for some $w \in W$.

Next, we know that $w^{-1}\omega^{-1}(\theta)$ is a weight of the module $L(\theta)$, so $w^{-1}\omega^{-1}(\theta) \in \theta - Q_+$. Therefore, we can write:

$$w^{-1}\omega^{-1}(\theta) = \theta - \sum_{\text{finite}} c_i \alpha_i \ .$$

where $c_i \in \mathbb{N}$ and α_i is a simple root for every i's. Then, we can write:

$$\begin{aligned}
\big(\mu+\rho\big)(r) &= \big(\lambda+\rho\big)(r) + \big(\omega^{-1}(\theta)\big)(r) \\
&= \big(\lambda+\rho\big)(r) + 2\frac{\Big(\big(\omega^{-1}(\theta)\big), s\Big)}{(s,s)}\omega^{-1}(\theta) \\
&= \big(\lambda+\rho\big)(r) + 2\frac{\Big(\big(w^{-1}\omega^{-1}(\theta)\big), \alpha\Big)}{(\alpha,\alpha)}w^{-1}\omega^{-1}(\theta)\big) \\
&= \big(\lambda+\rho\big)(r) + 2\frac{(\theta,\alpha)}{(\alpha,\alpha)}\theta - \sum_{\text{finite}} c_i \cdot 2\frac{(\alpha_i,\alpha)}{(\alpha,\alpha)}\alpha_i \ . \qquad (1.19)
\end{aligned}$$

Now, since $r \in R^\vee_{\lambda+\rho}$, the first term in the previous line is an integer. Since $\theta \in P_+$ and since α is a simple root, the second term is also an integer. Also, each term of the sum is an integer c_i multiplying a Cartan integer since the α_i's are simple roots, just as α is. Putting the above information together, this tells us that $\big(\mu + \rho\big)(r) \in \mathbb{Z}$. As we started with an arbitrary $r \in R^\vee_{\lambda+\rho}$, this means that $R^\vee_{\lambda+\rho} \subseteq R^\vee_{\mu+\rho}$.

Finally, as we can reproduce another big formula just like (1.19) starting from any other $r \in R^\vee_{\mu+\rho}$ instead, we obtain the other inclusion so that

$$R^\vee_{\lambda+\rho} = R^\vee_{\mu+\rho} \ .$$

As $W^{\lambda+\rho}$ and $W^{\mu+\rho}$ are generated by the same set, they are equal and the proof is complete.

Q.E.D.

1.2 Translation Functors and Composition Series

In this second step, I present the key elements to prove the Kac-Wakimoto character formula. These key elements are some translation functors and a specific weak composition series. The results of this section are only related to properties of the categories $\mathcal{O}$ and $\mathcal{O}^L$.

The mathematician J.C. Jantzen defined certain translation functors that will be key for our purpose. Here is an account of the elements that are needed to prove the Kac-Wakimoto character formula.

Let $\mathcal{C}$ be a category that has an “extension-block”-like decomposition:

$$\mathcal{C} = \bigoplus_{b \in \text{ ext-block}} \mathcal{C}_b \ . \tag{1.20}$$

Remark 1.21. An extension block decomposition in a suitable category $\mathcal{C}$ is a partition of its objects according to an equivalence relation comparable to that given in Definition 1.7 up to a certain modifications. For two objects $M, N \in \mathrm{Ob}(\mathcal{C})$, we would write that M is in the same extension block as N if and only if either they are isomorphic or there exist a finite sequence of objects $\{C_i\}_{i=0}^{n} \subseteq \mathrm{Ob}(\mathcal{C})$ with $C_0 = M$, $C_n = N$ and for which $\mathrm{Ext}_{\mathcal{C}}(C_i, C_{i+1})$ or $\mathrm{Ext}_{\mathcal{C}}(C_{i+1}, C_i) \neq 0$ for $i \in \{0, \dots, n-1\}$. The equivalence class of an object under this relation is then called its extension block or its *linkage class*. One should now compare (1.20) to Proposition 1.11. In a block decomposition, objects from different extension-blocks must have no non-trivial morphisms and extensions between them (the latter corresponds to Remark 1.13 in our setting). An illustrative reference for understanding the parallel between an extension block decomposition and the decomposition of $\mathcal{O}^L$ of Proposition 1.11 is [Hum 2008]: in particular Section 1.13 is most relevant. However, [Hum 2008] treats of finite-dimensional semisimple Lie algebra settings and not of Kac-Moody Lie algebras.

Fix a $\theta \in P_+$ for which one has $M \otimes L(\theta) \in \mathrm{Ob}\,\mathcal{C}$ for all $M \in \mathrm{Ob}\,\mathcal{C}$. Then you can define a *translation functor* from the "ext-block" b_1 to another "ext-block" b_2 by setting:

$$\begin{array}{rccc} \underline{\mathbf{T}}(\theta)_{b_1}^{b_2} : & \mathcal{C}_{b_1} & \longrightarrow & \mathcal{C}_{b_2} \\ & M & \longmapsto & [M \otimes L(\theta)]_{b_2\text{-comp}} \end{array} . \tag{1.22}$$

where $[C]_{b_2\text{-comp}}$ means the b_2-component of a module C, in its decomposition as a direct sum of submodules with respect to the "ext-block" decomposition (1.20).

Remark 1.23. For the original setting in which the translation functors first appeared, see the book [Jan 1979] (in German) by J.C. Jantzen.

Remark 1.24. In [DGK 1982], the authors mention that J.C. Jantzen makes use of such functors in finite dimensional Lie theory settings.

Remark 1.25. We will use some translation functors with the category $\mathcal{O}^L$ and its decomposition given by Result 1.11.

Some very important results for later use follow:

Result 1.26. *The translation functors* (1.22) *defined with respect to the decomposition of category* $\mathcal{O}^L$ *given of Result 1.11 are additive and exact.*

Proof: It follows from Section 5 of [DGK 1982].

Remark 1.27. For more details on how to prove Result 1.26, see also Chapter 7 of [Hum 2008] for details in the case of category $\mathcal{O}$ for finite-dimensional semisimple Lie algebras. For instance, a module M in category $\mathcal{O}$ for an affine Lie algebra associated to a finite-dimensional semisimple Lie algebra $\mathfrak{g}$ will also be in category $\mathcal{O}$ for $\mathfrak{g}$ (pulling back its action to a $\mathfrak{g}$-action). In such a case, properties of the translation functors can be deduced from their validity in category $\mathcal{O}$ for $\mathfrak{g}$.

Result 1.28. *Let* $\theta \in P_+$ *and let* $M \in \operatorname{Ob} \mathcal{O}^L$*. Then* $M \otimes L(\theta) \in \operatorname{Ob} \mathcal{O}^L$.

Proof: Can be found in [DGK 1982], see Proposition 5.9.

Definition 1.29. For a module in $C \in \operatorname{Ob} \mathcal{O}$, a *weak composition series* is an increasing filtration of some of its submodules

$$\{0\} = P_0 \subseteq P_1 \subseteq P_2 \subseteq \cdots \subseteq C \ ,$$

such that

(1) $\bigcup_i P_i = C$;

(2) P_{i+1}/P_i is a highest weight module for any given i;

(3) if the highest weight of P_{i+1}/P_i is greater than that of P_{j+1}/P_j, then $i < j$ (i.e. highest weights of the quotients decrease along with the indices);

(4) for any weight η of C, there exist an index i_η such that $\left(C/P_{i_\eta}\right)_\eta = 0$.

Remark 1.30. Filtrations from Definition 1.29 are called a *weak composition series* partly because the successive quotients are not required to be irreducible modules like in a usual composition series. Also, note that the filtration is not required to be finite.

Result 1.31. *Let $\theta \in P_+$ and let V be a highest weight module of highest weight $\lambda \in \mathfrak{h}^*$, then $V \otimes L(\theta)$ has a weak composition series*

$$\{0\} = P_0 \subseteq P_1 \subseteq P_2 \subseteq \cdots \subseteq C \ ,$$

such that for any i, P_{i+1}/P_i is a highest weight module of highest weight $\lambda + \nu_i$ where ν_i is a weight of $L(\theta)$. As part of coming from a weak composition series, the set of weights $\{\nu_i\}_i$ satisfy

$$\nu_i > \nu_j \quad \Longrightarrow \quad i < j \ .$$

If we take $V = M(\lambda)$, then $P_{i+1}/P_i \cong M(\lambda + \nu_i)$ for all i.

Proof: Can be found in [DGK 1982], see Lemma 5.8.

Remark 1.32. The (only) properties of Result 1.31 that we will really be making use of are the following:

- $\bigcup_i P_i = M \otimes L(\theta)$;
- $P_{i+1}/P_i \cong M(\lambda + \nu_i)$ for all i's when $V = M(\lambda)$ to start with.

Remark 1.33. Result 1.31 is in fact Lemma 5.8 of [DGK 1982]. While their Lemma 5.8 may seem more precise than Result 1.31, it is not the case. In the current exposition of the proof of the Kac-Wakimoto formula, I will not fix a particular notation to denote weights of $L(\theta)$ since it would not be used later on.

Coming back to the category $\mathcal{O}^L = \bigoplus_{\lambda \in K^L} \mathcal{O}^L_{[\![\lambda]\!]}$, here is a corollary to the above remarks:

Corollary 1.34. *Let $\theta \in P_+$, let V be a highest weight module of highest weight $\lambda \in K^L$ (so that $V \in \mathrm{Ob}\,\mathcal{O}^L$) and fix $\eta \in K^L$. Then the module $\big[V \otimes L(\theta)\big]_{[\![\eta]\!]\text{-comp}}$ has a weak composition series*

$$\{0\} = \tilde{P}_0 \subseteq \tilde{P}_1 \subseteq \tilde{P}_2 \subseteq \cdots \subseteq \big[V \otimes L(\theta)\big]_{[\![\eta]\!]\text{-comp}} ,$$

such that $\tilde{P}_{i+1}/\tilde{P}_i$ is a highest weight module with highest weight $\lambda + \tilde{\nu}_i \in [\![\eta]\!]$ where $\tilde{\nu}_i$ is a weight of $L(\theta)$.

As part of coming from a weak composition series, the set of weight $\{\tilde{\nu}_i\}_i$ satisfy

$$\tilde{\nu}_i > \tilde{\nu}_j \quad \Longrightarrow \quad i < j \, .$$

If we take $V = M(\lambda)$, then $\tilde{P}_{i+1}/\tilde{P}_i \cong M(\lambda + \tilde{\nu}_i)$ for all i.

Proof: Take the weak composition series for $V \otimes L(\theta)$ given by Result 1.31 and keep only the P_i's such that after relabeling, the highest weights of any successive quotient is in the correct equivalence class of $\sim$, namely $[\![\eta]\!]$.

Q.E.D.

Remark 1.35. For more details on Corollary 1.34 and/or on its justification, see Remark 5.11 in [DGK 1982].

To conclude Section 2, here is a word on translation functors in the case of $\mathcal{O}^L$. We define them just as of line (1.22). However, there is a more relevant one to care about. Let $\lambda, \mu \in K^L$ be such that $\big(W(\mu - \lambda)\big) \cap P_+ = \{\theta\}$. Then consider the specific translation functor

$$\underline{\mathbf{T}}(\theta)_{[\![\lambda]\!]}^{[\![\mu]\!]} : \begin{array}{rcl} \mathcal{O}^L_{[\![\lambda]\!]} & \longrightarrow & \mathcal{O}^L_{[\![\mu]\!]} \\ M & \longmapsto & [M \otimes L(\theta)]_{[\![\mu]\!]\text{-comp}} \end{array} , \tag{1.36}$$

which will be the most interesting translation functor. It is also this specific functor that will allow us to prove the Kac-Wakimoto character formula.

Remark 1.37. A reason why the specific functor (1.36) is relevant for proving the Kac-Wakimoto formula follows from the fact that the assumption

$$\#\big(W(\mu - \lambda)\big) \cap P_+ = \#\{\theta\} = 1 ,$$

combined with properties of $\underline{\mathbf{T}}$, will be sufficient to prove an existence and unicity result.

1.3 Translation of Certain Verma Modules

In this third step, I present a detailed proof of Lemma 1 from [KW 1988]. This is the last preliminary step to proving the character formula. The purpose of this lemma is to prove that the specific translation functor (1.36) is the right object to focus upon.

Lemma 1.38. **(Lemma 1 of [KW 1988])** *Let* $\lambda, \mu \in K^L$ *be weights such that* $\big(W(\mu - \lambda)\big) \cap P_+ \neq \emptyset$. *Suppose that* $\big(\lambda + \rho\big)(r) \neq 0$ *for all* $r \in R^\vee_{\lambda+\rho}$. *Further, assume that*

$$\begin{aligned} \big(\lambda + \rho\big)(r) > 0 \qquad &\Longrightarrow \qquad \big(\mu + \rho\big)(r) \geq 0 \,, \\ \big(\lambda + \rho\big)(r) < 0 \qquad &\Longrightarrow \qquad \big(\mu + \rho\big)(r) \leq 0 \,. \end{aligned}$$

Then for any $g \in W^{\lambda+\rho} = W^{\mu+\rho}$*, we have*

$$\underline{\mathbf{T}}(\theta)^{[\![\mu]\!]}_{[\![\lambda]\!]}\big(M(g \bullet \lambda)\big) = M(g \bullet \mu) \,. \tag{1.39}$$

<u>**Proof:**</u> Corollary 1.34 from Section 1.2 describes a weak composition series for the translated Verma module of the left handside of equation (1.39).

In order to figure anything out about such a specific weak composition series, we need to invesigate the possibility of having $g \bullet \lambda + \nu \in [\![\mu]\!]$ where ν is a weight of $L(\theta)$. Firstly, observe that Result 1.15 (1) gives $[\![\mu]\!] = W^{\mu+\rho} \bullet \mu = W^{\lambda+\rho} \bullet \mu$ so we will need to solve the equation

$$g \bullet \lambda + \nu = \tilde{g} \bullet \mu \tag{1.40}$$

where ν is a weight of $L(\theta)$ and $\tilde{g} \in W^{\mu+\rho} = W^{\lambda+\rho}$.

Let's first derive an equivalent formula to solve. We can write:

$$\begin{aligned}
(1.40) \quad &\iff \quad g^{-1} \bullet (g \bullet \lambda + \nu) = g^{-1} \bullet \tilde{g} \bullet \mu \\
&\iff \quad g^{-1}\big(g(\lambda+\rho) - \rho + \nu + \rho\big) - \rho = (g^{-1}\tilde{g}) \bullet \mu \\
&\iff \quad (\lambda+\rho) + g^{-1}(\nu) - \rho = (g^{-1}\tilde{g}) \bullet \mu \\
&\iff \quad \lambda + g^{-1}(\nu) = (g^{-1}\tilde{g}) \bullet \mu \\
&\iff \quad \lambda + \psi = h \bullet \mu \qquad (1.41)
\end{aligned}$$

where ψ is a weight of $L(\theta)$ and $h \in W^{\mu+\rho} = W^{\lambda+\rho}$.

Next, let $w \in W$ be the unique element such that $w \bullet \lambda \in \rho + C$. Then it will be applied to both sides of (1.41):

$$\begin{aligned}
(1.41) \quad &\iff \quad w \bullet (\lambda + \psi) = w \bullet h \bullet \mu \\
&\iff \quad w(\lambda + \psi + \rho) - \rho = (wh) \bullet \mu \\
&\iff \quad w(\lambda+\rho) + w(\psi) - \rho = (wh) \bullet \mu \\
&\iff \quad w(\lambda+\rho) + w(\psi) - \rho = \big(wh\big)(\mu+\rho) - \rho \\
&\iff \quad w(\lambda+\rho) + w(\psi) = \big(wh\big)(\mu+\rho) \\
&\iff \quad w(\lambda+\rho) + w(\psi) = \Big(whw^{-1}\Big)\big(w(\mu+\rho)\big) \\
&\iff \quad w(\lambda+\rho) + \bar{\psi} = \bar{g}\big(w(\mu+\rho)\big) \qquad (1.42)
\end{aligned}$$

where $\bar{\psi}$ is a weight of $L(\theta)$ and $\bar{g} \in W^{w(\mu+\rho)} = W^{w(\lambda+\rho)}$ using Result 1.14. Also, note that $w(\lambda+\rho) \in C$ in (1.42).

The next step to prove this lemma is to show that the problems (1.40) $\Leftrightarrow$ (1.41) $\Leftrightarrow$ (1.42) admit precisely one solution. Recall that w, λ and μ are fixed.

Existence: By assumption, $\big(W(\mu-\lambda)\big)\cap P_+ = \{\theta\}$. Let's call $\omega \in W$ an element so that $\omega(\mu-\lambda)=\theta$. This leads to the equation

$$\lambda + \omega^{-1}(\theta) = \mu \ . \tag{1.43}$$

Since $\omega^{-1}(\theta)$ is a weight of $L(\theta)$ and since $\mu \in [\![\mu]\!] = W^{\mu+\rho} \bullet \mu$, line (1.43) represents a solution for equation (1.41) $\Leftrightarrow$ (1.40) $\Leftrightarrow$ (1.42).

Unicity: Assume that $\bar{\psi} \in P\big(L(\theta)\big)$ and $\bar{g} \in W^{w(\mu+\rho)} = W^{w(\lambda+\rho)}$ provides a solution of (1.42), i.e. that

$$w(\lambda+\rho) + \bar{\psi} = \bar{g}\big(w(\mu+\rho)\big) \ .$$

Then as $w(\lambda+\rho) \in C$, we have

$$\big(w(\lambda+\rho)\big)(r) > 0 \qquad \text{for all } r \in R_+^{w(\lambda+\rho)} = R_+^{w(\mu+\rho)} \ . \tag{1.44}$$

The two assumptions of the lemma then give:

$$\big(w(\mu+\rho)\big)(r) \geq 0 \qquad \text{for all } r \in R_+^{w(\lambda+\rho)} = R_+^{w(\mu+\rho)} \ . \tag{1.45}$$

Let's set $\bar{\lambda} = w(\lambda+\rho)$ and $\bar{\mu} = w(\mu+\rho)$ so that we can write

$$\bar{\lambda} + \bar{\psi} = \bar{w}(\bar{\mu}) \tag{1.46}$$

where $\bar{\psi} \in P\big(L(\theta)\big)$ and $\bar{w} \in W^{\bar{\lambda}} = W^{\bar{\mu}}$. We'll end up showing that the equation (1.46) represents the same solution to (1.41) as the one given in the unicity part.

We can make use of the form $(-,-)$ on $\mathfrak{h}^* \times \mathfrak{h}^*$ and write

$$\begin{aligned}(\bar{\psi},\bar{\psi}) &= \big(\bar{w}(\bar{\mu})-\bar{\lambda}, \bar{w}(\bar{\mu})-\bar{\lambda}\big) \\ &= \big(\bar{w}(\bar{\mu}),\bar{w}(\bar{\mu})\big)+(\bar{\lambda},\bar{\lambda})-2\big(\bar{w}(\bar{\mu}),\bar{\lambda}\big) \\ &= \big(\bar{\mu},\bar{\mu}\big)+(\bar{\lambda},\bar{\lambda})-2\big(\bar{w}(\bar{\mu}),\bar{\lambda}\big)\ . \end{aligned} \tag{1.47}$$

By Proposition 3 (i) of [MP 1995], $R^{\bar{\lambda}} = R^{\bar{\mu}}$ is a subroot system of the whole root system and its Weyl group is $W^{\bar{\lambda}} = W^{\bar{\mu}}$. This last fact is easy to check. We then use Exercise 3.12 of [Kac 1990] to obtain

$$\bar{w}(\bar{\mu}) = \bar{\mu} - \sum_{\text{finite}} \bar{\mu}(s_i)\beta_i \tag{1.48}$$

where for any i, $s_i \in \Pi^{\bar{\mu}}$ and β_i is a positive real root corresponding to some real coroot r_i of $R^{\bar{\mu}}_+$. Note that the coefficients in the sum of (1.48) are all in $\mathbb{N}$ by Equation (1.45).

Let's then use (1.48) to rewrite $\big(\bar{w}(\bar{\mu}),\bar{\lambda}\big)$ differently:

$$\begin{aligned}\big(\bar{w}(\bar{\mu}),\bar{\lambda}\big) &= (\bar{\mu},\bar{\lambda}) - \sum_{\text{finite}} \bar{\mu}(s_i)\,(\beta_i,\bar{\lambda}) \\ &= (\bar{\mu},\bar{\lambda}) - \sum_{\text{finite}} \bar{\mu}(s_i)\,\bar{\lambda}(r_i) \\ &\in (\bar{\mu},\bar{\lambda}) - \mathbb{N} \qquad \text{by the paragraph just above and (1.44)}\ . \end{aligned}$$

Therefore, we get $\big(\bar{w}(\bar{\mu}),\bar{\lambda}\big) \leq (\bar{\mu},\bar{\lambda})$ with equality $\Leftrightarrow$ $\bar{w}(\bar{\mu}) = \bar{\mu}$. Combined to the line (1.47), this gives

$$(\bar{\psi},\bar{\psi}) \geq \big(\bar{\mu}-\bar{\lambda},\bar{\mu}-\bar{\lambda}\big) = (\theta,\theta) \qquad \text{with equality} \Leftrightarrow \quad \bar{w}(\bar{\mu}) = \bar{\mu}\ . \tag{1.49}$$

On the other hand, Proposition 11.4 a) of [Kac 1990] shows that we do always have

$$(\bar{\psi}, \bar{\psi}) \leq (\theta, \theta) \qquad \text{with equality } \Leftrightarrow \quad \bar{\psi} \in W(\theta)\ . \tag{1.50}$$

Both the inequalities (1.49) and (1.50) being true, we are forced to admit that there is equality and so we have

$$\bar{w}(\bar{\mu}) = \bar{\mu} \quad \text{and} \quad \bar{\psi} = f(\theta) \quad \text{for some } f \in W\ .$$

It follows that the equation (1.46) can be rewritten as

$$\bar{\lambda} + f(\theta) = \bar{\mu}\ . \tag{1.51}$$

From the previous equality, we can deduce that

$$\begin{aligned} f(\theta) &= \bar{\mu} - \bar{\lambda} \\ &= w(\mu + \rho) - w(\lambda + \rho) \\ &= w(\mu - \lambda) \\ &= w\omega^{-1}(\theta)\ . \end{aligned} \tag{1.52}$$

The equation (1.51) can then be rewritten as

$$\begin{aligned} \bar{\lambda} + f(\theta) &= \bar{\mu} \\ w(\lambda + \rho) + w\omega^{-1}(\theta) &= w(\mu + \rho) \\ \lambda + \rho + \omega^{-1}(\theta) &= \mu + \rho \\ \lambda + \omega^{-1}(\theta) &= \mu\ . \end{aligned} \tag{1.53}$$

As the equations (1.53) and (1.43) are the same, we conclude that the existing solution of the problems (1.40) $\Leftrightarrow$ (1.41) $\Leftrightarrow$ (1.42) is unique.

Finally, the problem (1.40) of solving $g \bullet \lambda + \nu \in [\![\mu]\!]$ where ν is a weight of $L(\theta)$ admits precisely one solution in the current setting. This solution is given by applying the g dot action on both sides of (1.43):

$$\begin{aligned} g \bullet \big(\lambda + \omega^{-1}(\theta)\big) &= g \bullet \mu \\ g\big(\lambda + \rho + \omega^{-1}(\theta)\big) - \rho &= g \bullet \mu \\ g(\lambda + \rho) + g\omega^{-1}(\theta) - \rho &= g \bullet \mu \\ g(\lambda + \rho) - \rho + g\omega^{-1}(\theta) &= g \bullet \mu \\ g \bullet \lambda + g\omega^{-1}(\theta) &= g \bullet \mu\ . \end{aligned}$$

We then conclude from the Corollary 1.34 that the module

$$\underline{\mathbf{T}}(\theta)^{[\![\mu]\!]}_{[\![\lambda]\!]}\big(M(g \bullet \lambda)\big) = \big[M(g \bullet \lambda) \otimes L(\theta)\big]_{[\![\mu]\!]\text{-comp}}$$

has a weak composition series

$$\{0\} = \tilde{P}_0 \subseteq \tilde{P}_1 \tag{1.54}$$

where $\tilde{P}_1/\tilde{P}_0 \cong \tilde{P}_1 \cong M\big(g \bullet \lambda + g\omega^{-1}(\theta)\big) = M(g \bullet \mu)$.

By the definition of a weak composition series, we must have

$$\underline{\mathbf{T}}(\theta)^{[\![\mu]\!]}_{[\![\lambda]\!]}\big(M(g \bullet \lambda)\big) = \textstyle\bigcup_i \tilde{P}_i = \tilde{P}_1 \cong M(g \bullet \mu)\ ,$$

where we used (1.54).

Q.E.D.

1.4 The Kac-Wakimoto Formula

In this fourth and last step, I present a detailed proof of Theorem 1 from [KW 1988]. This is the proof of the Kac-Wakimoto formula. We basically just use general properties about the category $\mathcal{O}$, a topological lemma about Weyl chambers from [MP 1995] and some translation functors from Section 1.3.

Lemma 1.55. *Let $\lambda \in K^L$ for which $R^\vee_{\lambda+\rho} \neq \emptyset$. Assume that $\lambda(t) > 0$ for all $t \in \Pi^\vee_{\lambda+\rho,+}$. Then for any fixed $s \in \Pi^\vee_{\lambda+\rho}$, there exists a $\mu \in \mathfrak{h}^*$ such that*

(i) $\big(W(\mu - \lambda)\big) \cap P_+ \neq \emptyset$;

(ii) $\big(\mu + \rho\big)(s) = 0$;

(iii) $\big(\lambda + \rho\big)(t) > 0$ *for all* $t \in \Pi^\vee_{\mu+\rho} \backslash \{s\} = \Pi^\vee_{\lambda+\rho} \backslash \{s\}$.

Proof: Can be found in [MP 1995], see Lemma 6.8.6.

Theorem 1.56. (Theorem 1 of [KW 1988]) *Let $\lambda \in K^L$ for which $\lambda(t) > 0$ for all $t \in \Pi^\vee_{\lambda+\rho}$. Then*

$$\mathrm{ch}\,[L(\lambda)] = \sum_{w \in W^{\lambda+\rho}} \varepsilon(w)\ \mathrm{ch}\,[M(w \bullet \lambda)] \tag{1.57}$$

where $\varepsilon(w)$ is the sign of the Weyl group element w.

Proof: Since we work within in the category $\mathcal{O}$, any module V has a character given in terms of a sum of characters $\mathrm{ch}\,[L(\eta)]$'s for certain η's in $\mathfrak{h}^*$.

Because $M(\lambda)$ is in $\mathcal{O}^L$, we have (by Result 1.6) that $\mathrm{ch}\,[M(\lambda)]$ is given in terms of a sum of $\mathrm{ch}\,[L(\eta)]$'s for certain η's in K^L.

In the case of $M(\lambda)$, we have that $P\big(M(\lambda)\big) \subseteq \lambda - \mathbb{N}Q_+$. Consider the set

$$P = \{\lambda = p_0, p_1, p_2, ...\} = [\![\lambda]\!] \cap \Big(\lambda - \mathbb{N}Q_+\Big)\ , \tag{1.58}$$

for which $p_i \in p_j - \mathbb{N}Q_+$ for all $i > j$. Also, recall that $[\![\lambda]\!] = W^{\lambda+\rho} \bullet \lambda$. Next, since $\lambda \in K^L$, Result 1.5 lets us write $\left[M(\lambda) : L(\eta)\right] \neq 0 \;\Rightarrow\; [\![\eta]\!] = [\![\lambda]\!]$ and so we have

$$\left[M(\lambda) : L(\eta)\right] \neq 0 \qquad \Longrightarrow \qquad \eta = p_j \text{ for some } j \; .$$

For any fixed $i \in \mathbb{N}$, we then obtain

$$\text{ch}\left[M(p_i)\right] = \sum_{j \geq i} \left[M(p_i) : L(p_j)\right] \; \text{ch}\left[L(p_j)\right] \; . \tag{1.59}$$

Note that $\left[M(p_j) : L(p_j)\right] = 1$ for any given $j \in \mathbb{N}$ because of the basic properties following from the definition of a Verma module. Next, we can view the set of Equations (1.59) as an infinite triangular "linear system":

$$\begin{array}{ccccc} (\text{ Multiplicities }) & \cdot & (\text{ ch}\left[L(p)\right]\text{'s }) & = & \text{ch}\left[M(p)\right]\text{'s} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ \star & 1 & 0 & 0 & 0 \\ \star & \star & 1 & 0 & 0 \\ \vdots & \vdots & \ddots & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} & \cdot & \begin{pmatrix} L(\lambda) \\ L(p_1) \\ L(p_2) \\ \vdots \\ \vdots \end{pmatrix} & = & \begin{pmatrix} M(\lambda) \\ M(p_1) \\ M(p_2) \\ \vdots \\ \vdots \end{pmatrix} . \end{array} \tag{1.60}$$

Formally inverting the triangular "linear system" (1.60) gives us the formula:

$$\text{ch}\left[L(\lambda)\right] = \sum_{w \in W^{\lambda+\rho}} m(w, \lambda) \; \text{ch}\left[M(w \bullet \lambda)\right] \tag{1.61}$$

for some integers $m(w, \lambda)$. The only differences between formulas (1.61) and (1.57) are the coefficients of the corresponding sums. Understandably, the last objective will be to justify that $m(w, \lambda) = \varepsilon(w)$ for any given $w \in W^{\lambda+\rho}$. In order to achieve that, let's first fix an $s \in \Pi^\vee_{\lambda+\rho} \neq \emptyset$. With $s \in \Pi^\vee_{\lambda+\rho}$ fixed, Lemma 1.55 ensures the existence of a $\mu \in K^L$ such that

$$\begin{aligned} &\big(W(\mu-\lambda)\big) \cap P_+ = \{\theta\} \quad \text{(fixing the notation);} \\ &\big(\mu+\rho\big)(s) = 0; \\ &\big(\mu+\rho\big)(t) > 0 \quad \text{for all} \quad t \in \Pi^\vee_{\mu+\rho}\backslash\{s\} = \Pi^\vee_{\lambda+\rho}\backslash\{s\} \end{aligned} \tag{1.62}$$

The pair (λ, μ) does satisfy the conditions of Lemma 1.38 and so the corresponding translation functor $\underline{\mathbf{T}}(\theta)^{[\![\mu]\!]}_{[\![\lambda]\!]}$ does map $M(w.\lambda)$ to $M(w.\mu)$. Note that this functor was also both exact and additive.

Let β_s be the positive root corresponding to the positive coroot s. The assumption from the theorem together with the choice of $s \in \Pi^\vee_{\lambda+\rho}$ give $\big(\lambda+\rho\big)(s) \in \mathbb{N}\backslash\{0\}$. In fact, Result 1.5 then gives that $\big[M(\lambda) : L(\sigma_s \bullet \lambda)\big] \neq 0$.

Next, we have

$$M(\sigma_s \bullet \lambda) \subseteq N(\lambda) \tag{1.63}$$

where $N(\lambda)$ is the maximal submodule of $M(\lambda)$ for which the quotient is the irreducible $L(\lambda)$. It follows from line (1.63) that there is a surjection

$$\frac{M(\lambda)}{M(\sigma_s \bullet \lambda)} \twoheadrightarrow \frac{M(\lambda)}{N(\lambda)} \cong L(\lambda)\ .$$

Equivalently, there is an exact sequence

$$\frac{M(\lambda)}{M(\sigma_s \bullet \lambda)} \to L(\lambda) \to 0\ . \tag{1.64}$$

Let's apply the exact functor $\underline{\mathbf{T}}(\theta)^{[\![\mu]\!]}_{[\![\lambda]\!]}$ to the sequence (1.64). Using the description of the effect of the functor on Verma modules given at the line (1.39) from

Section 1.3, we obtain the exact sequence

$$\frac{M(\mu)}{M(\sigma_s \bullet \mu)} \to \underline{\mathbf{T}}(\theta)_{[\![\lambda]\!]}^{[\![\mu]\!]}\big(L(\lambda)\big) \to 0\ . \tag{1.65}$$

As presented at line (1.62), the choice of μ gives

$$\sigma_s \bullet \mu = \mu + \rho - \big(\mu+\rho\big)(s)\beta_s - \rho = \mu - \big(\mu+\rho\big)(s)\beta_s = \mu\ .$$

Therefore, $M(\sigma_s.\mu) = M(\mu)$ and the exact sequence (1.65) really is

$$0 \to \underline{\mathbf{T}}(\theta)_{[\![\lambda]\!]}^{[\![\mu]\!]}\big(L(\lambda)\big) \to 0\ . \tag{1.66}$$

This means that $\underline{\mathbf{T}}(\theta)_{[\![\lambda]\!]}^{[\![\mu]\!]}\big(L(\lambda)\big) \cong \{0\}$ is itself the trivial module.

Now, since the translation functors are additive, their application on modules commutes with taking characters. From this relevant fact, we rewrite the formula (1.61) as we apply the functor $\underline{\mathbf{T}}(\theta)_{[\![\lambda]\!]}^{[\![\mu]\!]}$. The result is

$$0 = \mathrm{ch}\left[\underline{\mathbf{T}}(\theta)_{[\![\lambda]\!]}^{[\![\mu]\!]}\big(L(\lambda)\big)\right] = \sum_{w\in W^{\lambda+\rho}} m(w,\lambda)\ \mathrm{ch}\left[M(w\bullet\mu)\right]\ . \tag{1.67}$$

Thus, the coefficient of any Verma module of the right handside of (1.67) will be zero. Let's focus on the coefficient of $M(w\bullet\mu)$ in the right handside of (1.67) where $w \in W^{\lambda+\rho} = W^{\mu+\rho}$ is any fixed group element.

To obtain the coefficient of $M(w.\mu)$ in (1.67), we must find all the elements g of $W^{\lambda+\rho} = W^{\mu+\rho}$ such that

$$g\bullet\mu = w\bullet\mu \qquad \Longleftrightarrow \qquad g^{-1}w \in \mathrm{Stab}_{W^{\lambda+\rho}\bullet}\{\mu\}\ . \tag{1.68}$$

Let's then proceed to find $\mathrm{Stab}_{W^{\lambda+\rho}\bullet}\{\mu\}$. First, recall that

$$W^{\lambda+\rho} = \langle\, \sigma_t \mid t \in \Pi^\vee_{\lambda+\rho} \,\rangle \ . \tag{1.69}$$

With this in mind, the properties of μ at line (1.62) lead to

$$\mathrm{Stab}_{W^{\lambda+\rho}\bullet}\{\mu\} = \{\mathrm{Id}, \sigma_s\} \ .$$

We deduce that the elements g from the line (1.68) we are looking for are w and $w\sigma_s$. Finally, we can write that the coefficient of $M(w.\mu)$ in the right handside of the equation (1.67) is $m(w,\lambda) + m(w\sigma_s,\lambda)$. From this same equation we have

$$\begin{aligned} 0 &= m(w,\lambda) + m(w\sigma_s,\lambda) \\ -m(w\sigma_s,\lambda) &= m(w,\lambda) \ . \end{aligned} \tag{1.70}$$

In conclusion, since line (1.70) is independent of μ that was chosen to suit our arbitrarily fixed $s \in \Pi^\vee_{\lambda+\rho}$ and since $w \in W^{\lambda+\rho}$ is also arbitrary, equation (1.70) holds independently of the choices of pairs (w, s).

Using the fact that $W^{\lambda+\rho}$ is generated by the Weyl reflections σ_t for $t \in \Pi^\vee_{\lambda+\rho}$, we conclude that $m(w,\lambda) = \varepsilon(w)$. It is then possible to rewrite the equation (1.61) as:

$$\mathrm{ch}\,[L(\lambda)] = \sum_{w\in W^{\lambda+\rho}} \varepsilon(w)\ \mathrm{ch}\,[M(w\bullet\lambda)] \ ,$$

which is the Kac-Wakimoto character formula.

Q.E.D.

Chapter 2

Direct Sum Completion of a Braided Monoidal Category

This chapter is devoted to establishing a categorical framework that allows for considering infinite direct sums within braided monoidal categories settings. This plays an important role in the study of some vertex operator algebras constructions such as simple currents extensions of infinite order and for linking some of the related module categories. Most of the content of this chapter is summarised in [AR 2018].

Since the appearance of vertex operator algebras in relation to Physics' Conformal Field Theories, notable examples of vertex operator algebras have displayed rich module category structures that involve tensor products as well as rich additional structures [MS 1989], [Hua 2005], [Lep 2005], [Hua 2010], see also [CG 2017] for a recent related work. For instance, the class of rational and C_2-cofinite vertex operator algebras have been shown to have so-called *modular tensor category*, see [Hua 2005]. For most physical applications in Conformal Field Theory, the module category of a vertex operator algebra should possess certain properties and structural elements. Perhaps the most obvious example is that of a *fusion product* of modules: on the mathematical side of the picture, a well defined fusion product means an exact tensor product bifunctor for which we can express a product of two indecomposable modules as a finite linear combinations of other modules with coefficients in $\mathbb{N}$. For related comments, see [FHST 2004]. Huang has also proved that a *Rational* Conformal Field Theory led to having a rational vertex operator algebra, that is a vertex operator algebra with a (finite) modular tensor category of modules, see [Hua 2005].

The non-rational Conformal Field Theories [CR 2013a] are both much more challenging and rewarding to study than the rational ones. The associated logarithmic vertex operator features non-semisimple modules and allows for non-trivial extensions, but such theories can relate to more important physical phenomena. For a long time, not much has been known about logarithmic vertex operator algebras' Representation Theory, but their categories of modules are certainly rich mathematical objects [Hua 2005], [Hua 2010], [Fuc 2007]. Serious work has been done in the last decade to define appropriate tensor products for logarithmic vertex operator algebras, see [HL 2013] and [HLZ 2007]. Even though the applicability of this logarithmic tensor theory is still very difficult to decide (see [Hua 2017] or even [CHY 2018] for instance), ideas about the structure of the corresponding module categories are being developed for logarithmic and C_2-cofinite vertex operator algebras [Miy 2003], [Miy 2010], [CG 2017].

A general obstacle to developments on logarithmic vertex operator algebra is the lack of examples. Even in the case of logarithmic C_2-cofinite algebras, only the *triplet vertex operator algebras* [TW 2013], [AM 2008a], some superalgebra analogues [AM 2008b], [AM 2009], and the even part of the symplectic fermions superalgebra [Abe 2007] have been known. To remedy this lack of examples, one usually attempts to construct new examples from algebras that already have some of the required properties. As the following chapters of this thesis will show, a successful application of this procedure has been to consider simple current extensions of vertex operator algebras, see [CKL 2015], [CKM 2017] for instance. In the next chapters, we consider infinite order simple current extensions of parafermionic vertex operator algebras. Since these extensions are of infinite order, they involve objects made from infinite direct sums. Therefore, we have addressed this problem by developing a suitable notion of a direct sum completion. My colleague M. Rupert and I wrote a paper [AR 2018] on this chapter's topic. Currently with T. Creutzig, S. Kanade and M. Rupert, we are preparing a paper [ACKR] in which applying the direct sum completion and the extension theory for vertex operator algebras [CKM 2017] to study a module category for the logarithmic $\mathcal{B}_p$ vertex operator algebra of [CRW 2014] that relate to some Argyres-Douglas Theories [Cre 2017] (in Physics). In our paper, we describe a category of local $\mathcal{B}_p$-modules whose characters satisfy a Verlinde-type formula. For p odd, the modular and Hopf links S-matrices coincide up to normalisation and the Grothendieck ring of the semisimplification of our category is a $\mathbb{Z}_+$-ring. We also show that the character of $\mathcal{B}_p$ matches that of a certain subregular quantum Hamiltonian reduction of $\mathfrak{sl}_{p-1}$.

In the first section of the chapter, we develop a proper background on categorical limits in order to formulate a suitable definition of a direct sum completion. In the second section, we propose a definition of the direct sum completion $\mathcal{C}_\oplus$ of a base category $\mathcal{C}$ that is either $\mathbb{K}$-linear additive category, $\mathbb{K}$-linear additive and monoidal category or $\mathbb{K}$-linear, additive, monoidal and braided with possibly twist isomorphisms. We conclude the chapter by illustrating the use of the direct sum completion and of the theory of vertex operator algebra extensions [HKJL 2015], [CKL 2015], [CKM 2017] in a basic vertex operator algebra setting: constructing the even lattice vertex operator algebra from the Heisenberg vertex operator algebra. This last part is also treated in [AR 2018].

Notation

Throughout the chapter, the following notation will be employed:

- $\mathbb{K}$ is a field;
- $\mathcal{C}$ is a category with some specified additional structure;
- $\oplus$ is a direct sum or coproduct on a given category;
- f.s.$(S) = \{$finite subsets of $S\}$ for any set S;
- $\otimes$ is a tensor product bifunctor on a given category;
- $a_{-,-,-}$ are the natural associativity isomorphisms in a given monoidal category;
- $\mathbb{1}$ is a tensor product identity in a monoidal category;
- l_{-} and r_{-} are left and right unit constraints on a monoidal category;
- $c_{-,-}$ is a braiding on a monoidal category;
- θ_{-} is a twist on a braided monoidal category;
- $L = \sqrt{2N}\ \mathbb{Z}$ is an even lattice with standard product as its bilinear form $(\ell_1 \cdot \ell_2) = \ell_1\ell_2$;
- $L^* = \frac{1}{\sqrt{2N}}\mathbb{Z} = \{x \in L \otimes_{\mathbb{Z}} \mathbb{C} \mid x\ell \in \mathbb{Z}$ for all $\ell \in L\}$ is the lattice dual to L.

- $\underline{\mathcal{F}} : \mathcal{C}^{loc} \to \mathrm{Rep}^0 V_L$ is the induction functor from local modules of $\mathcal{C}$ to the category of untwisted V_L-modules where V_L is seen as an algebra object, see [CKM 2017] and Appendix B for details.

For an overview of background concepts on Category Theory topics, see Appendix B. For more details on braided monoidal categories, tensor categories and modular categories, a useful reference is the book [EGNO 2015].

2.1 Categorical Limits

In this section, we review notions of categorical limits that are needed to understand the abstract Ind-category of [AGV 1971]. The latter is often pointed to as a framework that allows for a rigorous treatment of infinite direct sums. However, no one seems to have explained how this could be done since the Ind-category is too abstract to be manipulated with ease. In the following, we go through the key notion of categorical limit in order to explain what should be an Ind-category. This section serves as an inspiration for the rest of the chapter.

2.1.1 Limits Version 1

In order to approach the notion of infinite direct sums in categories, they have to be viewed as coproducts, which are themselves special cases of categorical limits. References on these topics include [Rot 2009] and [PP 1979].

Fix a category $\mathcal{C}$. Here is a first notion of a categorical limit in $\mathcal{C}$.

Definition 2.1. A *directed set* is a pair $(I, \leq)$ where I is a set and where $\leq$ is a relation on I that satisfies:

- $\leq$ is reflexive and transitive;

- for any given two elements $i, j \in I$, there exists a $k \in I$ such that $i \leq k$ and $j \leq k$.

For limits to make sense, notions of a progression of objects must be established. In this way a limit can be interpreted as the “final stage” of the progressions. The correct categorical notions of relevant “progressions of objects” is that of a system of objects in $\mathcal{C}$. In a system of objects in $\mathcal{C}$, a direction is to be given via a directed set.

Definition 2.2. A *direct system* in $\mathcal{C}$ is a directed set $(I, \leq)$ together with a pair

$$\big(\{X_i\}_{i\in I}, \{t_{ij} : X_i \to X_j\}_{i\leq j \text{ in } I}\big)$$

composed of a family of objects of $\mathcal{C}$ and a family of transition morphisms between them. These transition maps represent the relation $\leq$ on I and “preserve its orientation”. This means that they satisfy

- $t_{ii} = \mathrm{Id}_{X_i}$ for any $i \in I$, and;
- $t_{jk} \circ t_{ij} = t_{ik} : X_i \to X_j \to X_k$ whenever $i \leq j \leq k$ in I.

Definition 2.3. An *inverse system* in $\mathcal{C}$ is a directed set $(I, \leq)$ together with a pair

$$\big(\{X_i\}_{i\in I}, \{t_{ij} : X_j \to X_i\}_{i\leq j \text{ in } I}\big)$$

composed of a family of objects of $\mathcal{C}$ and a family of transition morphisms between them.

These transition maps represent the relation $\leq$ on I and “invert its orientation”. This means that they satisfy

- $t_{ii} = \mathrm{Id}_{X_i}$ for any $i \in I$, and

- $t_{ij} \circ t_{jk} = t_{ik} : X_k \to X_j \to X_i$ whenever $i \leq j \leq k$ in I.

The categorical limits for a system of either type is defined as follows:

Definition 2.4. The *limit of a direct system* $\left(\{X_i\}_{i\in I}, \{t_{ij} : X_i \to X_j\}_{i\leq j\in I}\right)$ in $\mathcal{C}$ is (if it exists) a pair

$$\left(L, \{\lambda_i : L \to X_i\}_{i\in I}\right) ,$$

made of an object L of $\mathcal{C}$ together with morphisms λ_i compatible with the transition morphisms of the direct system. Furthermore, L must be universal with respect to this property.

This means that for any given object Y in $\mathcal{C}$ such that a family of morphisms $y_i : Y \to X_i$ is compatible with the transition morphisms of the direct system, there exists a unique map $u : Y \to L$ such that $y_i = \lambda_i \circ u$ for any given $i \in I$.

Definition 2.5. The *limit of an inverse system* $\left(\{X_i\}_{i\in I}, \{t_{ij} : X_j \to X_i\}_{i\leq j\in I}\right)$ in $\mathcal{C}$ is (if it exists) a pair

$$\left(L, \{\lambda_i : X_i \to L\}_{i\in I}\right)$$

made of an object L of $\mathcal{C}$ together with morphisms λ_i compatible with the transition morphisms of the inverse system. Furthermore, L must be universal with respect to this property.

This means that for any given object Y in $\mathcal{C}$ such that a family of morphisms $y_i : Y \to X_i$ is compatible with the transition morphisms of the direct system, there exists a unique map $u : Y \to L$ such that $y_i = \lambda_i \circ u$ for any given $i \in I$.

Table 2.1 reports equivalent terminologies that are commonly employed for the above notions of limits in $\mathcal{C}$.

Remark 2.6 ($\bigoplus$ and $\prod$ as limits). In a given category $\mathcal{C}$, both finite and infinite direct sums, if they exist, can be seen as colimits (of direct systems). Similarly, finite

Limit of a direct system	Limit of an inverse system
colimit	limit
inductive limit	projective limit
direct limit	inverse limit

Table 2.1: Equivalent common terminologies for categorical limits.

and infinite products, if they exist, can be seen as limits (of inverse systems). Given an index set S over which to consider a direct sum or a product. The construction of the corresponding systems in $\mathcal{C}$ goes as follow:

- the relevant directed set $(I, \leq)$ is given by $I = \text{f.s.}(S)$ and the relation $\leq$ is the inclusion $\subseteq$ of subsets;
- the objects composing the relevant systems are the partial direct sums or the partial direct products over elements of I;
- the transition morphisms of the relevant systems are the natural injections for direct sums and the natural projections for products.

If it exists, the limit of such a system will be a direct sum or a product, in accordance with the choice of the system.

2.1.2 Presheaves, Yoneda's Lemma and Representability

References for the content of this section include [Sch 1972], [PP 1979], [ML 1998].

Let us fix a $\mathbb{K}$-linear category $\mathcal{C}$. This means that $\mathcal{C}$ is an additive category for which the Hom-spaces are vector spaces, compositions of morphisms are $\mathbb{K}$-linear (in both slots) and where $\oplus : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is $\mathbb{K}$-bilinear.

Definition 2.7. Define $\widehat{\mathcal{C}}$ to be the category of $\mathbb{K}$-linear contravariant functors

$$\mathcal{C} \to \mathrm{Vect} \ .$$

Morphisms of functors are simply natural transformations and the category $\widehat{\mathcal{C}}$ is called the category of *vector space presheaves over* $\mathcal{C}$.

Definition 2.8. Let A be an object of $\mathcal{C}$. Define the following presheaf:

$$h_A(-) = \mathrm{Hom}_{\mathcal{C}}(-, A) \in \mathrm{Ob}\, \widehat{\mathcal{C}} \ .$$

Note that since $\mathrm{Hom}_{\mathcal{C}}(-,-)$ is a bifunctor $\mathcal{C} \times \mathcal{C}^{op} \to \mathrm{Vect}$ the mapping $A \mapsto h_A(-)$ is natural in A. In particular, the following mappings form a functor:

$$\begin{aligned} h:\quad \mathcal{C} &\longrightarrow \widehat{\mathcal{C}} \\ X &\longmapsto h_X(-) = \mathrm{Hom}_{\mathcal{C}}(-, X) \\ [f : A \to B] &\longmapsto [(? \circ f) : h_B(-) \to h_A(-)] \end{aligned} \quad . \tag{2.9}$$

Definition 2.10. An object $R(-)$ of $\widehat{\mathcal{C}}$ is said to be *representable* if there exists an object A of $\mathcal{C}$ such that $R(-) \cong h_A(-)$.

Lemma 2.11. (Yoneda's Lemma) *Let A be an object of $\mathcal{C}$ and $R(-)$ be an object of $\widehat{\mathcal{C}}$. Then there is an isomorpism*

$$\begin{aligned} \mathbf{Yoneda}_{A,R} :\ \mathrm{Hom}_{\widehat{\mathcal{C}}}(h_A, R) &\longrightarrow R(A) \\ \eta &\longmapsto \eta_A(\mathrm{Id}_A) \end{aligned} \ ,$$

that is natural in both A and $R(-)$.

Proof: Let $\eta \in \mathrm{Hom}_{\widehat{\mathcal{C}}}(h_A, R)$. Then for any $X \in \mathrm{Ob}\,\mathcal{C}$, one has a map η_X

$$\eta_X : \; h_A(X) = \mathrm{Hom}_{\mathcal{C}}(X, A) \longrightarrow R(X)$$

that is natural in X. In particular, for any $f \in \mathrm{Hom}_{\mathcal{C}}(X, A)$, one has to fix a value $\eta_X(f) \in R(X)$. As $\mathrm{Id}_A \in \mathrm{Hom}_{\mathcal{C}}(A, A)$ is such that $f = \mathrm{Id}_A \circ f = \big(h_A(\mathrm{Id}_A)\big)(f)$, write the following commutative diagram

$$\begin{tikzcd}
h_A(A) = \mathrm{Hom}_{\mathcal{C}}(A, A) \arrow[r, "\eta_A"] \arrow[d, "- \circ f"'] & R(A) \arrow[d, "R(f)"] \\
h_X(A) = \mathrm{Hom}_{\mathcal{C}}(X, A) \arrow[r, "\eta_X"'] & R(X)
\end{tikzcd}$$

By commutativity, $\eta_X(f) = \Big(R(f)\Big)\big(\eta_A(\mathrm{Id}_A)\big)$. This means that $\eta_A(\mathrm{Id}_A) \in R(A)$ actually determines the behaviour of the whole natural transformation, $\eta \in \mathrm{Hom}_{\widehat{\mathcal{C}}}(h_A, R)$.

Next we define $\mathbf{Yoneda}_{A,R}(\eta) = \eta_A(\mathrm{Id}_A)$. This map is bijective because we can define an inverse map

$$\begin{aligned}
\Theta_{A,R} : \quad R(A) &\longrightarrow \mathrm{Hom}_{\widehat{\mathcal{C}}}(h_A, R) \\
r &\longmapsto \eta^r = (\eta^r_X)_{X \in \mathrm{Ob}\,\mathcal{C}}
\end{aligned}$$

where $\eta^r_X(X \xrightarrow{f} A) = \big(R(f)\big)(r)$. This η^r is a well defined natural transformation and indeed, the following equalities hold:

$$\mathbf{Yoneda}_{A,R} \circ \Theta_{A,R} = \mathrm{Id}_{\mathrm{Hom}_{\widehat{\mathcal{C}}}(h_A,R)} \quad \text{and} \quad \Theta_{A,R} \circ \mathbf{Yoneda}_{A,R} = \mathrm{Id}_{R(A)} \;.$$

It will now be shown the naturality of $\mathbf{Yoneda}_{A,R}$ in A and R. First, we fix $g \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ to obtain the natural transformation $(g \circ -) : h_A \to h_B$. Then for any $\theta \in \mathrm{Hom}_{\widehat{\mathcal{C}}}(h_B, R)$ one has

$$\begin{aligned}
\mathbf{Yoneda}_{A,R}\left(\theta(g \circ -)\right) &= \theta_A(g \circ \mathrm{Id}_A) \in R(A) \\
&= \theta_A(g) \\
&= \Big(R(g)\Big)\big(\theta_B(\mathrm{Id}_B)\big) \\
&= \Big(R(g)\Big)\big(\,\mathbf{Yoneda}_{B,R}(\theta)\big)\ .
\end{aligned}$$

It means that $\mathbf{Yoneda}_{A,R}$ is natural in A. Finally, we will show its naturality in R. If $\tau \in \mathrm{Hom}_{\widehat{\mathcal{C}}}(R, G)$, we get that $(\tau \circ -) : \mathrm{Hom}_{\widehat{\mathcal{C}}}(h_A, R) \to \mathrm{Hom}_{\widehat{\mathcal{C}}}(h_A, G)$. Then for any $\eta \in \mathrm{Hom}_{\widehat{\mathcal{C}}}(h_A, R)$, one has

$$\begin{aligned}
\mathbf{Yoneda}_{A,G}(\tau \circ \eta) &= \big(\tau \circ \eta\big)_A(\mathrm{Id}_A) \in G(A) \\
&= \tau_A\big(\eta_A(\mathrm{Id}_A)\big) \\
&= \tau_A\big(\,\mathbf{Yoneda}_{A,R}(\eta)\big)\ .
\end{aligned}$$

Q.E.D.

Corollary 2.12. *The covariant functor h from line (2.9) is fully faithful (i.e. it is bijective on morphisms). Moreover, one has $h_A \cong h_B \;\Leftrightarrow\; A \cong B$.*

Proof: If $\eta : h_A \to h_B$ is an isomorphism of functors, then there is an $\eta^{-1} : h_B \to h_A$ is an isomorphism of functors and we have $\eta \circ \eta^{-1} = \mathrm{Id}_{h_A}$ and $\eta^{-1} \circ \eta = \mathrm{Id}_{h_B}$.

This means that

$$
\begin{aligned}
\mathrm{Id}_A &= (\mathrm{Id}_{h_A})_A(\mathrm{Id}_A) \\
&= \mathbf{Yoneda}_{A,h_A}(\mathrm{Id}_{h_A}) \\
&= \mathbf{Yoneda}_{A,h_A}(\eta \circ \eta^{-1}) \\
&= \eta_B\big(\mathbf{Yoneda}_{B,h_A}(\eta^{-1})\big) \in \mathrm{Hom}_{\mathcal{C}}(B,B) \\
&= \Big(h_B(\mathbf{Yoneda}_{B,h_A}(\eta^{-1})\Big)\big(\eta_A(\mathrm{Id}_A)\big) \\
&= \big(\mathbf{Yoneda}_{B,h_A}(\eta^{-1})\big) \circ \big(\eta_A(\mathrm{Id}_A)\big) \\
&= \big((\eta^{-1})_B(\mathrm{Id}_B)\big) \circ \big(\eta_A(\mathrm{Id}_A)\big).
\end{aligned}
$$

Similarly, $\mathrm{Id}_B = (\eta^{-1})_A\big(\mathbf{Yoneda}_{A,h_B}(\eta)\big) \in \mathrm{Hom}_{\mathcal{C}}(A,A)$. This will imply that

$$\mathbf{Yoneda}_{A,h_B}(\eta^{-1}) \in \mathrm{Hom}_{\mathcal{C}}(A,B) \quad \text{and} \quad \mathbf{Yoneda}_{B,h_A}(\eta) \in \mathrm{Hom}_{\mathcal{C}}(B,A)\ ,$$

are inverses of each other. Proving this statement is straightforward.

Q.E.D.

2.1.3 Limits Version 2

References for the contents of this section include [AGV 1971] and [PP 1979].

We will revisit the notion of limit in a more structured way. The first thing to mention is that directed sets can be interpreted as categories and then, systems in a fixed category $\mathcal{C}$ appear as functors from a directed set to $\mathcal{C}$.

Definition 2.13. A directed set $(I, \leq)$, as in Definition 2.1, is naturally interpreted as constituting a category by setting:

$$\mathrm{Ob}(I,\leq)=I, \qquad \mathrm{Hom}_{(I,\leq)(i,j)}=\begin{cases}\{i\to j\} & \text{if } i\leq j\\ \emptyset & \text{otherwise}\end{cases},$$

where the composition is given by concatenation of arrows. This is indeed a category since the associativity of the composition is ensured by the transitivity of $\leq$ and the existence of identity morphisms $i\to i$ is ensured by the reflexivity of $\leq$.

The "upper bound property" of $\leq$ translates into the statement that for any pair of objects i and j, there exists a k such that $i\to k$ and $j\to k$ are morphisms in this categorical interpretation.

Definition 2.14. A *direct system* in $\mathcal{C}$ is a covariant functor $V:\ (I,\leq)\to\mathcal{C}$ where $(I,\leq)$ is a directed set viewed as a category via Definition 2.13.

Definition 2.15. An *inverse system* in $\mathcal{C}$ is a contravariant functor $R:\ (I,\leq)\to\mathcal{C}$ where $(I,\leq)$ is a directed set viewed as a category via Definition 2.13.

Remark 2.16. The equivalence between notions of direct systems from Definition 2.14 and Definition 2.2 can be given by $V(i)=X_i$ and $V(i\to j)=t_{i,j}$. A similar equivalence holds for Definitions 2.15 and 2.13 of inverse systems.

A great advantage of these new notions of systems in $\mathcal{C}$ is that one can then naturally speak of a category of systems of a given type in $\mathcal{C}$. For instance, if $V_I : I\to\mathcal{C}$ and $V_J : J\to\mathcal{C}$ are two direct systems in $\mathcal{C}$, a natural notion of a morphism between V_I and V_J could be that of a pair

$$(u,\psi) \tag{2.17}$$

where $u: I\to J$ is a functor and where $\psi\in\mathrm{Cov}(I,\mathcal{C})\in\mathrm{Hom}_{\mathcal{C}}(V_I,V_J\circ u)$.

In order to define new notions of limits in a category $\mathcal{C}$, we will need the notion of a constant functor.

Definition 2.18. Let $X \in \operatorname{Ob}\mathcal{C}$. For any categoty I, the *constant functor* is defined by:

$$\begin{aligned} k_X :\ I &\longrightarrow \widehat{\mathcal{C}} \\ i &\longmapsto X \\ [f : i \to j] &\mapsto [\mathrm{Id}_X : X \to X] \end{aligned} \quad . \tag{2.19}$$

Note that k_X is always both a covariant and a contravariant functor $I \to \mathcal{C}$.

Definition 2.20. Let $\mathcal{C}$ be a fixed $\mathbb{K}$-linear category. The *limit of a direct system* $V : I \to \mathcal{C}$ with values in $\mathcal{C}$ is the following functor:

$$\begin{aligned} (\lim V) :\ \mathcal{C} &\longrightarrow \mathrm{Vect} \\ X &\longmapsto \mathrm{Hom}_{\mathrm{Cov}(I,\mathcal{C})}(k_X, V) \end{aligned} \quad .$$

Note that $(\lim V) \in \widehat{\mathcal{C}}$ for a direct system V.

Definition 2.21. Let $\mathcal{C}$ be a fixed $\mathbb{K}$-linear category. The *limit of an inverse system* $R : I \to \mathcal{C}$ with values in $\mathcal{C}$ is the functor

$$\begin{aligned} (\lim R) :\ \mathcal{C} &\longrightarrow \mathrm{Vect} \\ X &\longmapsto \mathrm{Hom}_{\mathrm{Cont}(I,\mathcal{C})}(R, k_X) \end{aligned} \quad .$$

Note that $(\lim R) : \mathcal{C} \to \mathrm{Vect}$ is a covariant functor for an inverse system R.

The limits presented in Definitions 2.20 and 2.21 are such that the representability of their limit functors is equivalent to the existence of a limit in terms of Definitions 2.4 and 2.5. Concretely, if a limit functor, as of Definitions 2.20 or 2.21, is represented by an object $L \in \operatorname{Ob}\mathcal{C}$, then L is precisely the limit of the corresponding system in the sense of Definitions 2.4 and 2.5. As the converse also holds, we conclude that Definitions 2.20 and 2.21 are simply more general notions of limits.

Another important element to note is that since limits of functors $I \to \mathcal{C}$ are themselves functors $\mathcal{C} \to \mathrm{Vect}$, an intuitive notion of a morphism in a category of systems can be put forth: that a morphism between two systems in $\mathcal{C}$ should be given by a morphism of their respective limit functors. This provides a weaker notion of morphism than that of (2.17) which often turns out to be more practical.

2.1.4 The Ind-Category

References for this section include [AGV 1971] and [PP 1979].

Given a $\mathbb{K}$-linear category $\mathcal{C}$, the Ind-category $\mathrm{Ind}(\mathcal{C})$ is composed of direct systems. The notion of morphism between two systems is more subtle and has to do with morphisms in $\widehat{\mathcal{C}}$ between the limits.

Consider the fully faithful functor $h : \mathcal{C} \to \widehat{\mathcal{C}}$. By Corollary 2.12, working in $\mathcal{C}$ up to isomorphism is the same as working in $\widehat{\mathcal{C}}$ up to isomorphism. Taking limits of functors whose index sets are directed sets commute with evaluations $\mathrm{ev}_X : \widehat{\mathcal{C}} \to \mathrm{Vect}$ where $X \in \mathrm{Ob}\,\mathcal{C}$. Additionally, limits of such direct systems in $\widehat{\mathcal{C}}$ are alway representable in $\widehat{\mathcal{C}}$. Concretely, given a direct system $V : I \to \mathcal{C}$ one can make it a direct system in $\widehat{\mathcal{C}}$ via composition by the covariant functor h as follows:

$$(h \circ V) : \; I \to \mathcal{C} \xrightarrow{h} \widehat{\mathcal{C}}\,.$$

Since $\lim(h \circ V) : I \to \widehat{\mathcal{C}}$ is always representable, it has to be is represented by an object of $L\big(\lim(h \circ V)\big) \in \mathrm{Ob}\,\widehat{\mathcal{C}}$. The Ind-category associated with $\mathcal{C}$ has the following definition:

Definition 2.22. The category $\mathrm{Ind}(\mathcal{C})$ is defined by the following:

$$\begin{aligned}\mathrm{Ob}\,\mathrm{Ind}(\mathcal{C}) &= \{V \in \mathrm{Cov}(I, \mathcal{C}) \mid I \text{ is a directed set}\} \\ &= \{\text{direct systems in } \mathcal{C} \text{ whose index category is a directed set}\} \ .\end{aligned}$$

Also, for any two covariant functors $V_I : I \to \mathcal{C} \xrightarrow{h} \widehat{\mathcal{C}}$ and $V_J : J \to \mathcal{C} \xrightarrow{h} \widehat{\mathcal{C}}$, we define

$$\mathrm{Hom}_{\mathrm{Ind}(\mathcal{C})}(V_I, V_J) = \mathrm{Hom}_{\widehat{\mathcal{C}}}\big(L(\lim V_I), L(\lim V_J)\big) \ .$$

where $L(\lim V)$ is the object of $\widehat{\mathcal{C}}$ that represents the limit of V.

Remark 2.23. This notion of an Ind-category is a little less general than in [AGV 1971], but it is necessary for treating the upcoming topics. To have the most general notion of Ind-category, one can further weaken the properties of the index category I in a way such that limits are still defined with decent properties.

The notion of $\mathrm{Ind}(\mathcal{C})$ displayed in Definition 2.22 actually points how to naturally define morphisms between formal direct sum of objects.

2.2 Direct Sum Completion

A notion of a direct sum completion of an additive category $\mathcal{C}$ is detailed below. In few words, it will be a concrete presentation of the subcategory of $\mathrm{Ind}(\mathcal{C})$ whose direct systems correspond to coproducts or direct sums. See Remark 2.6 for a description of such systems.

At term, we will consider a category $\mathcal{C}$ that will also be $\mathbb{K}$-linear, monoidal, braided and with twist isomorphisms. We will then want to show that $\mathcal{C}_\oplus$ naturally retains such characteristics. We expect that a natural inclusion functor $\mathcal{C} \to \mathcal{C}_\oplus$ would then be fully faithful and preserve the pertinent supplementary properties.

2.2.1 The Direct Sum Completion of an Additive Category $\mathcal{C}$

Given an additive category $\mathcal{C}$, we will define the category $\mathcal{C}_\oplus$ as follows:

Definition 2.24. Define $\mathcal{C}_\oplus$ by setting:

$$\operatorname{Ob}\mathcal{C}_\oplus = \left\{ \bigoplus_{s\in S} X_s \;\middle|\; \begin{array}{l} S \text{ is a set} \\ X_s \in \operatorname{Ob}\mathcal{C} \text{ for every } s \in S \end{array} \right\},$$

$$\operatorname{Hom}_{\mathcal{C}_\oplus}\left(\bigoplus_{s\in S} X_s, \bigoplus_{t\in T} Y_t\right) = \left\{\left(\alpha, \{f_{s,t}\}_{s\in S}^{t\in\alpha(s)}\right)\right\} \big/ \sim$$

where

- $\alpha : \{\text{finite subsets of } S\} \to \{\text{finite subsets of } T\}$ is a function that commute with unions. For any singleton $\{s\} \subseteq S$, we employ the notation $\alpha(s) = \alpha\big(\{s\}\big)$;

- $f_{s,t} \in \operatorname{Hom}_{\mathcal{C}}(X_s, Y_t)$ for any $t \in \alpha(s)$;

- $\sim$ is an equivalence relation defined by:

$$\left(\alpha, \{f_{s,t}\}_{s\in S}^{t\in\alpha(s)}\right) \sim \left(A, \{F_{s,t}\}_{s\in S}^{t\in A(s)}\right) \quad \Leftrightarrow \quad \begin{cases} f_{s,t} = 0_{s,t} & \text{if } t \in \alpha(s)\backslash A(s) \\ f_{s,t} = F_{s,t} & \text{if } t \in \alpha(s) \cap A(s) \\ F_{s,t} = 0_{s,t} & \text{if } t \in A(s)\backslash\alpha(s) \end{cases};$$

- the composition of a pair of morphisms $\left(\beta, \{g_{t,r}\}_{t\in T}^{r\in\beta(t)}\right) : \bigoplus_{t\in T} Y_t \to \bigoplus_{r\in R} Z_r$ and $\left(\alpha, \{f_{s,t}\}_{s\in S}^{t\in\alpha(s)}\right) : \bigoplus_{s\in S} X_s \to \bigoplus_{t\in T} Y_t$ is defined to be the equivalence class of

$$\left(\beta\circ\alpha,\ \left\{\sum_{\substack{t\in\alpha(s)\\ \text{s.t. } r\in\beta(t)}} g_{t,r}\circ f_{s,t}\right\}_{s\in S}^{r\in\left(\beta\circ\alpha\right)(s)}\right);$$

- the identity morphism of $\bigoplus_{s\in S} X_s$ is the equivalence class of $\left(\mathrm{Id}_{\text{f.s.}(S)}, \{\mathrm{Id}_{X_s}\}_{s\in S}\right)$.

Proposition 2.25. *The elements of Definition 2.24 really define a category $\mathcal{C}_\oplus$. In particular, it means that:*

- *$\sim$ is indeed an equivalence relation;*
- *the composition is compatible with $\sim$ in both arguments;*
- *the composition is associative;*
- *the identity morphisms of an object indeed preserve any morphism via composition on both sides.*

Proof: See proof of Proposition 3.2 of [AR 2018].

Remark 2.26. The equivalence relation $\sim$ appearing in Definition 2.24 is relevant in the definition of morphisms. It allows to restrict their study to pairs $\left(\alpha, \{f_{s,t}\}_{s\in S}^{t\in\alpha(s)}\right)$ such that $f_{s,t}\neq 0$ whenever defined. This equivalence relation will also find use in defining an additive structure on $\mathcal{C}_\oplus$.

Remark 2.27. The notion of morphism can be thought of in a more concrete manner. Fix objects $\bigoplus_{s\in S} X_s$ and $\bigoplus_{t\in T} Y_t$, then the following notions are equivalent :

- an element of $\mathrm{Hom}_{\mathcal{C}_\oplus}\left(\bigoplus_{s\in S} X_s, \bigoplus_{t\in T} Y_t\right) = \left\{\left(\alpha, \{f_{s,t}\}_{s\in S}^{t\in\alpha(s)}\right)\right\}\Big/_{\sim}$, and;

- an element of $\prod_{s\in S}\Big(\bigoplus_{t\in T}\operatorname{Hom}_{\mathcal{C}}(X_s,Y_t)\Big)$,

the latter being less easy to deal with for theoretical purposes. To pass from the first notion to the second, we identify $\Big(\alpha,\{f_{s,t}\}_{s\in S}^{t\in\alpha(s)}\Big)\to\big(f_{s,t}\big)_{s\in S,\,t\in\alpha(s)}$.

Remark 2.28. If our base category $\mathcal{C}$ were the skeleton of the Heisenberg vertex operator algebra module category (i.e. a semisimple category with simple objects F_λ for $\lambda\in\mathbb{C}$, the *Fock spaces*), even the definition of morphisms of its completion $\mathcal{C}_\oplus$ appears simplified. Indeed, Schur's Lemma guarantees that

$$s\neq t\quad\Longrightarrow\quad\operatorname{Hom}_{\mathcal{C}}(F_s,F_t)=\{0\}\ .$$

Thus, a general morphism of $\operatorname{Hom}_{\mathcal{C}_\oplus}\big(\bigoplus_{s\in S}F_s,\bigoplus_{t\in T}F_t\big)$ is of the shape

$$(\alpha,\{f_{s,t}\}_{s\in S}^{t\in\alpha(s)})\sim(\tilde{\alpha}:\{s_i\}_{i=1}^n\mapsto\{s_i\,|\,s_i\in T\}_{i=1}^n,\{f_{s,s}\}_{s\in S\cap T})\ .$$

So, elements of $\operatorname{Hom}_{\mathcal{C}_\oplus}\big(\bigoplus_{s\in S}F_s,\bigoplus_{t\in T}F_t\big)$ naturally correspond to n-tuples of scalars

$$(\varphi_s)_{s\in S\cap T}\in\prod_{S\cap T}k\ .$$

However, note that categories $\mathcal{C}_\oplus$ and $\operatorname{Rep}^0 A$ for an algebra object $A\in\operatorname{Ob}\mathcal{C}_\oplus$ are quite different. In $\operatorname{Rep}^0 A$, we will have fewer morphisms.

Definition 2.29. *Addition* on $\operatorname{Hom}_{\mathcal{C}_\oplus}\big(\bigoplus_{s\in S}X_s,\bigoplus_{t\in T}Y_t\big)$ is defined as follows:

$$\Big(\alpha_1,\{f^1_{s,t}\}_{s\in S}^{t\in\alpha_1(s)}\Big)+\Big(\alpha_2,\{f^2_{s,t}\}_{s\in S}^{t\in\alpha_2(s)}\Big)=\left(\alpha_1\cup\alpha_2,\{\sigma_{s,t}\}_{s\in S}^{t\in\big(\alpha_1\cup\alpha_2\big)(s)}\right)\tag{2.30}$$

where

- for any finite subset $A\subseteq S$, define $\big(\alpha_1\cup\alpha_2\big)(A)=\alpha_1(A)\cup\alpha_2(A)$;

- for any $s \in S$ and $t \in \big(\alpha_1 \cup \alpha_2\big)(s)$, define

$$\sigma_{s,t} = \begin{cases} f^1_{s,t} & \text{if } t \in \alpha_1(s)\backslash\alpha_2(s) \\ f^1_{s,t} + f^2_{s,t} & \text{if } t \in \alpha_1(s) \cap \alpha_2(s) \\ f^2_{s,t} & \text{if } t \in \alpha_2(s)\backslash\alpha_1(s) \end{cases} .$$

Given any $\lambda \in \mathbb{F}$, a *scalar multiplication* by λ on $\mathrm{Hom}_{\mathcal{C}_\oplus}\big(\bigoplus_{s\in S} X_s, \bigoplus_{t\in T} Y_t\big)$ can be defined as follows:

$$\lambda \cdot \Big(\alpha, \{f_{s,t}\}_{s\in S}^{t\in\alpha(s)}\Big) = \Big(\alpha, \{\lambda f_{s,t}\}_{s\in S}^{t\in\alpha(s)}\Big) . \tag{2.31}$$

Definition 2.32. A *zero object* is defined as $0_{\mathcal{C}_\oplus} = \bigoplus_{0\in\{0\}} 0_0$ where $0_0 = 0 \in \mathrm{Ob}(\mathcal{C})$. Also, the *zero morphism* in $\mathrm{Hom}_{\mathcal{C}_\oplus}\big(\bigoplus_{s\in S} X_s, \bigoplus_{t\in T} Y_t\big)$ is defined as $\big(\Omega, \emptyset\big)$ where $\Omega(A) = \emptyset \subseteq T$ for every finite subset $A \subseteq S$. In particular, this gives

$$\mathrm{Hom}_{\mathcal{C}_\oplus}(0_{\mathcal{C}_\oplus}, 0_{\mathcal{C}_\oplus}) = \left\{ \begin{array}{c} \text{The equivalence} \\ \text{class of } \big(\Omega, \emptyset\big) \end{array} \right\} . \tag{2.33}$$

Definition 2.34. We define *direct sums* in $\mathcal{C}_\oplus$ as follows. Given a pair of objects $\bigoplus_{s\in S} X_s$ and $\bigoplus_{t\in T} Y_t$ of $\mathcal{C}_\oplus$, we have an object $\bigoplus_{a\in S\sqcup T} A_a$ where

$$A_a = \begin{cases} X_a & \text{if } a \in S \\ Y_t & \text{if } a \in T \end{cases} ,$$

with projection and inclusion morphisms p_S, p_T, i_S, i_T satisfying

$$p_S \circ i_S = \mathrm{Id}_{\bigoplus_{s\in S} X_s}\ , \qquad\qquad p_T \circ i_T = \mathrm{Id}_{\bigoplus_{t\in T} Y_t}\ , \tag{2.35}$$

$$i_S \circ p_S \ + \ i_T \circ p_T = \mathrm{Id}_{\bigoplus_{a\in S\sqcup T} A_a}\ , \tag{2.36}$$

and defined as:

$$\begin{aligned}
p_S &= \Big(\pi_S, \{\mathrm{Id}_{X_a}\}_{a\in S\sqcup T}\Big) \in \mathrm{Hom}_{\mathcal{C}_\oplus}\left(\bigoplus_{a\in S\sqcup T} A_a, \bigoplus_{s\in S} X_s\right), \\
p_T &= \Big(\pi_T, \{\mathrm{Id}_{Y_a}\}_{a\in S\sqcup T}\Big) \in \mathrm{Hom}_{\mathcal{C}_\oplus}\left(\bigoplus_{a\in S\sqcup T} A_a, \bigoplus_{t\in T} Y_t\right), \\
i_S &= \Big(\iota_S, \{\mathrm{Id}_{X_s}\}_{s\in S}\Big) \in \mathrm{Hom}_{\mathcal{C}_\oplus}\left(\bigoplus_{s\in S} X_s, \bigoplus_{a\in S\sqcup T} A_a\right), \\
i_T &= \Big(\iota_T, \{\mathrm{Id}_{Y_t}\}_{t\in T}\Big) \in \mathrm{Hom}_{\mathcal{C}_\oplus}\left(\bigoplus_{s\in S} X_s, \bigoplus_{a\in S\sqcup T} A_a\right)
\end{aligned}$$

where

- $\pi_S(A) = \{a \in A \mid a \in S\} \subseteq S$ for any finite subset $A \in S \sqcup T$;
- $\pi_T(A) = \{a \in A \mid a \in T\} \subseteq T$ for any finite subset $A \in S \sqcup T$;
- $\iota_S(B) = B \subseteq S \sqcup T$ for any finite subset $B \subseteq S$;
- $\iota_T(C) = C \subseteq S \sqcup T$ for any finite subset $C \subseteq T$.

Proposition 2.37. *The elements of Definitions 2.29, 2.32 and 2.34 make $\mathcal{C}_\oplus$ an additive $\mathbb{K}$-linear category. In particular:*

- *the addition (2.30) and scalar multiplication (2.31) are compatible with the equivalence relation $\sim$ in both arguments;*
- *the scalar multiplication is distributive with respect to the addition;*

- *the* Hom*-spaces of* $\mathcal{C}_\oplus$ *form* $\mathbb{K}$*-vector spaces with the zero morphisms* $(\Omega, \emptyset)$ *and where the inverse of a morphism* $(\alpha, \{f_{s,t}\})$ *is just* $(\alpha, \{-f_{s,t}\})$*;*
- *the composition of morphisms in* $\mathcal{C}_\oplus$ *is* $\mathbb{K}$*-bilinear;*
- *the equalities* (2.33), (2.35)*and* (2.36) *indeed hold.*

Proof: See proof of Proposition 3.6 of [AR 2018].

Proposition 2.38. *Let* $\mathcal{C}$ *be an additive category. In the additive category* $\mathcal{C}_\oplus$*, arbitrary infinite coproducts exist. Let* $\left(\bigoplus_{s\in S_i} X_s^i\right)_{i\in I}$ *be a family of objects in* $\mathcal{C}_\oplus$*; define their coproduct as the object* $\bigoplus_{a\in\bigsqcup_{i\in I} S_i} A_a$ *where*

$$A_a = X_a^{i_0} \qquad \textit{for } a \in S_{i_0} \subseteq \bigsqcup_{i\in I} S_i \ .$$

The structural injections are given by

$$\mathrm{inj}_{S_{i_0}} = \left(\iota_{S_{i_0}}, \{\mathrm{Id}_{X_s^{i_0}}\}_{s\in S_{i_0}}\right) \in \mathrm{Hom}_{\mathcal{C}_\oplus}\left(\bigoplus_{s\in S_{i_0}} X_s^{i_0}, \bigoplus_{a\in\bigsqcup_{i\in I} S_i} A_a\right)$$

where $\iota_{S_{i_0}}(B) = B \subseteq \bigsqcup_{i\in I} S_i$ *for any finite subset* $B \subseteq S_{i_0}$*.*

Proof: This is Proposition 3.8 of [AR 2018].

Fix $\mathcal{C}$ to be a $\mathbb{K}$-linear category. The above results show that $\mathcal{C}_\oplus$ also has a natural $\mathbb{K}$-linear and additive structure. This section's last results are meant to justify the choice of "direct sum completion" to qualify the construction $\mathcal{C}_\oplus$ with respect to $\mathcal{C}$.

Definition 2.39. We define an *inclusion functor* $\mathcal{I} : \mathcal{C} \to \mathcal{C}_\oplus$ as follows:

$$X \longmapsto \bigoplus_{0\in\{0\}} X_0 \quad \text{where } X_0 = X \ ,$$
$$[X \xrightarrow{f} Y] \longmapsto \left(\mathrm{Id}_{\{0\}}, \{f_{0,0} = f\}_{0\in\{0\}}^{0\in\{0\}}\right)$$

Proposition 2.40. *The inclusion functor $\mathcal{I}$ is fully faithful and $\mathbb{K}$-linear. In other words, there are natural $\mathbb{K}$-linear bijections*

$$\mathrm{Hom}_{\mathcal{C}_\oplus}\big(\mathcal{I}(X),\mathcal{I}(Y)\big) = \mathrm{Hom}_{\mathcal{C}}(X,Y)\ .$$

Moreover, every $\bigoplus_{s\in S} X_s \in \mathrm{Ob}(\mathcal{C}_\oplus)$ is a coproduct of its terms $\mathcal{I}(X_s) \in \mathrm{Ob}(\mathcal{C}_\oplus)$.

Proof: See proof of Proposition 3.10 of [AR 2018].

2.2.2 Monoidal Structure on $\mathcal{C}_\oplus$

In this section, let $\mathcal{C}$ denote a $\mathbb{K}$-linear monoidal category with tensor product $\otimes$, associativity isomorphisms $\{a_{X,Y,Z}\}_{X,Y,Z\in\mathrm{Ob}(\mathcal{C})}$ and unit $(\mathbb{1},l,r)$. The goal of this section is to define a natural monoidal structure on $\mathcal{C}_\oplus$.

Definition 2.41. A tensor product on $\otimes_{\mathcal{C}_\oplus}:\ \mathcal{C}_\oplus \times \mathcal{C}_\oplus \to \mathcal{C}_\oplus$ is defined as follows:

- it sends a pair of objects $\big(\bigoplus_{s\in S} X_s \bigoplus_{t\in T} Y_t\big)$ to the object $\bigoplus_{(s,t)\in S\times T}(X_s \otimes_{\mathcal{C}} Y_t)$;
- it sends a pair of morphisms $\Big(\alpha, \{f_{s,\tilde{s}}\}_{s\in S}^{\tilde{s}\in\alpha(s)}\Big), \Big(\beta, \{g_{t,\tilde{t}}\}_{t\in T}^{\tilde{t}\in\beta(t)}\Big)$ to the morphism described by

$$\alpha\otimes\beta\ :\ \big\{(s_i,t_i)\big\}_{i=1}^{n} \longmapsto \bigcup_{i=1}^{n}\big(\alpha(s_i)\times\beta(t_i)\big)\ , \tag{2.42}$$

$$\Big\{f_{s,\tilde{s}}\otimes g_{t,\tilde{t}}\Big\}_{(s,t)\in S\times T}^{(\tilde{s},\tilde{t})\in\alpha(s)\times\beta(t)=\big(\alpha\otimes\beta\big)(s,t)}\ . \tag{2.43}$$

Note that the rule $\emptyset\times A=\emptyset$ for any set A is assumed in the above.

Definition 2.44. Let $\bigoplus_{s\in S} X_s, \bigoplus_{t\in T} Y_t, \bigoplus_{r\in R} Z_r \in \mathrm{Ob}(\mathcal{C}_\oplus)$. *Associativity morphisms* for the tensor product $\otimes_{\mathcal{C}_\oplus}$ are defined as follows:

$$a^{\mathcal{C}_\oplus}_{(\oplus_S X_s, \oplus_T Y_t, \oplus_R Z_r)} = \\ \left(\alpha : \left\{\Big((s_i,t_i),r_i)\Big)\right\}_{i=1}^{n} \mapsto \left\{\Big(s_i,(t_i,r_i)\Big)\right\}_{i=1}^{n}, \left\{a_{X_s,Y_t,Z_r}\right\}_{\big((s,t),r\big)\in(S\times T)\times R}\right).$$

Definition 2.45. We define a *unit object* $\mathbb{1}_{\mathcal{C}_\oplus} = \mathcal{I}(\mathbb{1}) = \bigoplus_{0\in\{0\}} \mathbb{1}_0$ where $\mathbb{1}_0$ is simply $\mathbb{1} \in \mathrm{Ob}(\mathcal{C})$ and we define a *left unit* $l^{\mathcal{C}_\oplus}_{-}$ by

$$l^{\mathcal{C}_\oplus}_{\bigoplus_{s\in S} X_s} = \left(\alpha : \{(0,s_i)\}_{i=1}^{n} \mapsto \{s_i\}_{i=1}^{n}, \left\{l_{X_s}\right\}^{s\in\alpha(0,s)}_{(0,s)\in\{0\}\times S}\right),$$

in $\mathrm{Hom}_{\mathcal{C}_\oplus}\left(\mathbb{1}_{\mathcal{C}_\oplus} \otimes \bigoplus_{s\in S} X_s, \bigoplus_{s\in S} X_s\right)$. *Right units* are defined similarly.

Proposition 2.46. *The elements of Definitions 2.41, 2.44, 2.45 define a monoidal structure on $\mathcal{C}_\oplus$. In particular:*

- *$\otimes_{\mathcal{C}_\oplus}$ is a bifunctor $\mathcal{C}_\oplus \times \mathcal{C}_\oplus \to \mathcal{C}_\oplus$;*
- *$(\mathbb{1}_{\mathcal{C}_\oplus}, l^{\mathcal{C}_\oplus}_{-}, r^{\mathcal{C}_\oplus}_{-})$ is a unit for this tensor product;*
- *the isomorphisms $a^{\mathcal{C}_\oplus}_{-,-,-}$ are well defined and trinatural;*
- *the pentagon and triangle axioms are satisfied.*

Proof: See proof of Proposition 3.14 of [AR 2018].

The proof of the following natural proposition is then straightforward.

Proposition 2.47. *The inclusion functor $\mathcal{I} : \mathcal{C} \to \mathcal{C}_\oplus$ from Definition 2.39 is monoidal.*

Proof: Long but straightforward at this point.

2.2.3 Braiding and Twist on $\mathcal{C}_\oplus$

In this subsection, let $\mathcal{C}$ denote a braided, $\mathbb{K}$-linear and monoidal category with braidings $\{c_{X,Y}\}_{X,Y\in\mathrm{Ob}(\mathcal{C})}$ and possibly with twist isomorphisms $\{\theta_X\}_{X\in\mathrm{Ob}(\mathcal{C})}$ satisfying the balancing axiom $\theta_{X\otimes Y} = (\theta_X \otimes \theta_Y) \circ c_{Y,X} \circ c_{X,Y}$ for every $X, Y \in \mathrm{Ob}\,\mathcal{C}$.

Definition 2.48. Let $\bigoplus_{s\in S} X_s, \bigoplus_{t\in T} Y_t \in \mathrm{Ob}(\mathcal{C}_\oplus)$. *Braiding isomorphisms* in $\mathcal{C}_\oplus$ are:

$$c^{\mathcal{C}_\oplus}_{(\oplus_S X_s, \oplus_T Y_t)} = \left(\alpha : \left\{(s_i, t_i)\right\}_{i=1}^{n} \mapsto \left\{(t_i, s_i)\right\}_{i=1}^{n}, \left\{c_{X_s,Y_t}\right\}_{(s,t)\in S\times T}\right).$$

Definition 2.49. If $\mathcal{C}$ has twist isomorphisms, let $\bigoplus_{s\in S} X_s \in \mathrm{Ob}(\mathcal{C}_\oplus)$. *Twist isomorphisms* in $\mathcal{C}_\oplus$ are:

$$\theta^{\mathcal{C}_\oplus}_{(\bigoplus_{s\in S} X_s)} = \left(\mathrm{Id}_{\mathrm{f.s.}(S)}, \left\{\theta_{X_s}\right\}_{s\in S}\right).$$

Proposition 2.50. *The braiding and twist of Definitions 2.48 and 2.49 makes $\mathcal{C}_\oplus$ a braided $\mathbb{K}$-linear monoidal category (with twists if $\mathcal{C}$ has twists). In particular:*

- *the braiding $c^{\mathcal{C}_\oplus}_{-,-}$ and twist $\theta^{\mathcal{C}_\oplus}_{-}$ are natural isomorphisms in $\mathcal{C}_\oplus$ (in every argument);*
- *the hexagon and balancing axioms are satisfied.*

Proof: See proof of Proposition 3.18 of [AR 2018].

Finally, the following (straightforward) proposition motivates the choice of the word completion for $\mathcal{C}_\oplus$ in the context of braided monoidal categories:

Proposition 2.51. *The inclusion functor $\mathcal{I} : \mathcal{C} \to \mathcal{C}_\oplus$ of Definition 2.39 is braided monoidal and preserves twist.*

With this setup, a braided $\mathbb{K}$-linear category $\mathcal{C}$ with twists can be replaced by $\mathcal{C}_\oplus$ where infinite direct sums are needed. We will make use of the framework $\mathcal{C}_\oplus$ in the following section to illustrate how it matches with what one would expect.

2.3 Application: The Even Lattice Vertex Operator Algebra From Simple Heisenberg Currents

In this section, we make use of the direct sum completion framework in order to realise the usual module category of an even lattice vertex operator algebra as a module category for an algebra object in $\mathcal{C}_\oplus$. This serves as an illustration of the works presented in [CKM 2017] and [CKL 2015] in the context of well understood basic algebras where the need for dealing with infinite direct sums is also needed.

Let H be the rank 1 Heisenberg vertex operator algebra and let V_L be a rank 1 lattice vertex operator algebras (see Appendix A for a review). The goal of this section is to explicitly realise the category of untwisted weak V_L-modules as the category $\operatorname{Rep}^0 V_L$ where V_L interpreted as an algebra object in a direct sum completion of an appropriate category of H-modules (see Appendix B for a review). By the methods of [CKM 2017], we will determine the braided monoidal structure on the skeleton of category of untwisted weak V_L-modules.

In this section, there will be three important categories to distinguish:

- $\mathcal{C}$: the skeleton the category of H-modules described in Appendix A;
- $\mathcal{C}_\oplus$: the direct sum completion of $\mathcal{C}$ as described in the previous section;
- $\operatorname{Rep}^0 V_L$: the category of untwisted weak modules for the algebra object $V_L \in \operatorname{Ob} \mathcal{C}_\oplus$. Objects in this category are those for which the double braiding

with V_L is the identity.

The tensor products of the above three categories will be noted by $\otimes$, $\otimes_{\mathcal{C}_\oplus}$ and $\otimes_A$, respectively.

2.3.1 The Commutative Algebra Object V_L in $\mathcal{C}_\oplus$

Let $L = \sqrt{2N}\,\mathbb{Z}$ be an even lattice and consider the even lattice vertex operator algebra V_L. It is well known that the simple vertex operator algebra V_L decomposes as $\bigoplus_{\ell \in L} F_\ell$ when seen as an H-module. Henceforth, we will interpret V_L as an algebra object in $\mathcal{C}_\oplus$ as follows:

The object $V_L = \bigoplus_{\ell \in L} F_\ell \in \operatorname{Ob} \mathcal{C}_\oplus$ can be seen as a simple commutative algebra object by fixing multiplication an unit morphisms in $\mathcal{C}_\oplus$[1]. Since Fock spaces F_λ with $\lambda \in \mathbb{R}$ are the simple H-modules of $\mathcal{C}$, they are also the simple modules in $\mathcal{C}_\oplus$. By Schur's Lemma, we can interpret component morphisms of $\mathcal{C}_\oplus$ between Fock spaces as complex numbers and so the multiplication and unit maps of the algebra object V_L must be of the following form:

$$\begin{aligned}
\mu \quad &\sim \quad \Big(\big\{(\ell_1, \ell_2)\big\} \mapsto \{\ell_1 + \ell_2\}, \{\mu_{\ell_1,\ell_2}\}_{(\ell_1,\ell_2)\in L^2}\Big) \in \operatorname{Hom}_{\mathcal{C}_\oplus}(V_L \otimes_{\mathcal{C}_\oplus} V_L, V_L)\ , \\
u \quad &\sim \quad \Big(\{0\} \mapsto \{0\}, \{u_0\}_{0\in\{0\}}\Big) \in \operatorname{Hom}_{\mathcal{C}_\oplus}(F_0, V_L)
\end{aligned}$$

where $\mu_{\ell_1,\ell_2} \in \mathbb{C}^\times$ for all $\ell_1, \ell_2 \in L$ and where we can set $u_0 = 1$.

Note that having all the scalars μ_{ℓ_1,ℓ_2} non-zero is important to ensure V_L is indeed a simple algebra. Setting the scalar $u_0 = 1$ simply means that the map u is taken to be the natural inclusion $H = F_0 \subset \bigoplus_{\ell \in L} F_\ell = V_L$.

The associativity, commutativity and unit requirements for V_L to be an algebra

[1]Note that since $\#L = \infty$, we need to use the direct sum completion $\mathcal{C}_\oplus$ here.

object forces the following relations:

$$\text{(ASSOCIATIVITY)} \qquad \mu_{\ell_1,\ell_2}\mu_{\ell_1+\ell_2,\ell_3} = \mu_{\ell_1,\ell_2+\ell_3}\mu_{\ell_2,\ell_3} \;, \tag{2.52}$$

$$\text{(COMMUTATIVITY)} \qquad \mu_{\ell_1,\ell_2} = e^{\pi i(\ell_1\cdot\ell_2)}\mu_{\ell_2,\ell_1} \;, \tag{2.53}$$

$$\text{(UNIT)} \qquad \mu_{\ell,0} = 1 \;, \tag{2.54}$$

$$\text{(UNIT)} \qquad \mu_{0,\ell} = 1 \tag{2.55}$$

for all ℓ, ℓ_1, ℓ_2, $\ell_3 \in L$. In other words, the multiplication map μ gives rise to a 2-cocycle

$$\begin{aligned} K(\mu): \quad L \times L &\longrightarrow \mathbb{C}^\times \\ (\ell_1,\ell_2) &\longmapsto \mu_{\ell_1,\ell_2} \end{aligned} \;. \tag{2.56}$$

We can make sense of $K(\mu)$ as a cocycle of the usual group cohomology set[2] $\mathrm{H}^2(L^2;\mathbb{C}^\times)$. However, in [DF 2012], the authors prefer to see $K(\mu)$ in another type of cohomology set defined for abelian groups. Theorem 4.5 of [DF 2012] shows that with an even lattice $L = \sqrt{2N}\mathbb{Z}$, the requirements (2.52)–(2.55) fix the cohomology class of $\mu \in \mathrm{H}^2(L^2;\mathbb{C}^\times)$.

Remark 2.57. One can show that cohomologous cocycles that satisfy the requirements (2.52)–(2.55) lead to isomorphic algebra objects (see [CGR 2017] for a justification for the even lattice case). In [AR 2018], we deliberately chose to fix $\mu_{\ell_1,\ell_2} = 1$ for all $\ell_1, \ell_2 \in L$ in order to facilitate all forthcoming computations. However, the work presented in this thesis maintains μ a general cocycle.

Remark 2.58. Note that in $\mathcal{C}$, the twist θ_{F_ℓ} for any $\ell \in L = \sqrt{2N}\,\mathbb{Z}$ is trivial because

$$\theta_{F_{\sqrt{2N}r}} = e^{\pi i(\sqrt{2N}r)^2}\,\mathrm{Id}_{F_{\sqrt{2N}r}} = e^{2\pi i N^2 r^2}\,\mathrm{Id}_{F_{\sqrt{2N}r}} \tag{2.59}$$

[2] $\mathrm{H}^2(L^2;\mathbb{C}^\times)$ is the second group cohomology of the abelian group L with values in $\mathbb{C}^\times$, which is a trivial L-module.

where $r \in \mathbb{Z}$. It follows that the twist of V_L can be seen as an object of $\mathcal{C}_\oplus$ is $\theta_{V_L}^{\mathcal{C}_\oplus} = \mathrm{Id}_{V_L}$ according to Definition 2.49. This fact will allow to make sense of a natural twist on $\theta_{V_L}^{\mathrm{Rep}^0 V_L}$ on the category $\mathrm{Rep}^0 V_L$.

2.3.2 Objects and Tensor Products in $\mathrm{Rep}^0 V_L$

Consider the algebra object $V_L = \bigoplus_{\ell \in L} F_\ell$ as of Section 2.3.1. Recall that $\mathcal{C}$ is a semisimple, skeletal and strict category (see Appendix B).

By applying Theorem 4.5 of [CKM 2017] to our setting, we obtain that all simple objects of $\mathrm{Rep}^0 V_L$ are induced by simple objects of $\mathcal{C}^{loc}$. Recall that $\mathcal{C}^{loc}$ is the full subcategory of $\mathcal{C}$ consisting of objects that induce to $\mathrm{Rep}^0 V_L$. Thus, we arrive at the following result

Lemma 2.60. *The simple modules of* $\mathrm{Rep}^0 V_L$ *are can be realised by the induction of Fock spaces with highest weight contained in* L^*.

Proof: Since $\mathcal{C}$ is semisimple with simple modules being Fock spaces with real highest weight, it is sufficient to check which Fock spaces F_λ with $\lambda \in \mathbb{R}$ are induced in $\mathrm{Rep}^0 V_L$. By Theorem 3.15 of [CKL 2015], $\underline{\mathcal{F}}(F_\lambda) \in \mathrm{Ob}\,\mathrm{Rep}^0 V_L$ if and only if the double braiding (or monodromy) $c_{F_\lambda, F_{\sqrt{2N}}} \circ c_{F_{\sqrt{2N}}, F_\lambda} = \mathrm{Id}_{F_{\sqrt{2N}+\lambda}}$ where $F_{\sqrt{2N}}$ is a simple current sufficient to build V_L. Since the two braidings are $c_{F_\lambda, F_{\sqrt{2N}}} = c_{F_{\sqrt{2N}}, F_\lambda} = e^{\pi i \lambda \sqrt{2N}} \mathrm{Id}_{F_{\sqrt{2N}+\lambda}}$, we conclude that F_λ induces to $\mathrm{Rep}^0 V_L$ if and only if $e^{2\pi i \lambda \sqrt{2N}} = 1$, which means that λ should be in L^*, the dual of L.

Q.E.D.

Remark 2.61. Fix $\lambda \in L^*$ so that $\underline{\mathcal{F}}(F_\lambda) = \bigoplus_{\ell \in L} F_\ell \otimes F_\lambda$ is in $\mathrm{Ob}\,\mathrm{Rep}^0 V_L$. Just as for the multiplication map of V_L (see Section 2.3.1), Schur's Lemma and the notion of morphisms in $\mathcal{C}_\oplus$ allow to explicit the V_L-action map corresponding to $\underline{\mathcal{F}}(F_\lambda)$ as:

$$\left(\left\{(\ell_1, \ell_2)\right\} \mapsto \{\ell_1 + \ell_2\}, \{\mu_{\ell_1,\ell_2}\}_{(\ell_1,\ell_2)\in L^2}\right) \in \mathrm{Hom}_{\mathcal{C}_\oplus}\left(V_L \otimes_{\mathcal{C}_\oplus} \underline{\mathcal{F}}(F_\lambda), \underline{\mathcal{F}}(F_\lambda)\right) .$$

We now proceed to study the tensor product of induced modules and to classify the simple modules that the simple modules of $\mathrm{Rep}^0 V_L$. It was proven in [CKM 2017] (see also Theorem B.16) that $\underline{\mathcal{F}}$ is a tensor functor so we know there should be natural V_L-isomorphisms

$$\underline{\mathcal{F}}\left(F_\alpha\right) \otimes_A \underline{\mathcal{F}}\left(F_\beta\right) \cong \underline{\mathcal{F}}\left(F_{\alpha+\beta}\right) \tag{2.62}$$

for all $\alpha, \beta \in L^*$.

As we expect to retrieve the well known fact that the isomorphism classes of simple V_L-modules are in bijection with the set

$$\mathcal{S} = \left\{\underline{\mathcal{F}}(F_{\frac{a}{\sqrt{2N}}}) \mid a \in \{0, \dots, 2N-1\}\right\} , \tag{2.63}$$

we will prove the following important Lemma:

Lemma 2.64. *Let $\lambda \in L^*$ and $\ell \in L$. Then $\underline{\mathcal{F}}\left(F_{\lambda+\ell}\right) \cong \underline{\mathcal{F}}\left(F_\lambda\right)$ as a V_L-module.*

<u>Proof:</u> In the skeletal category $\mathcal{C}$, we have $F_{\lambda+\ell} = F_\ell \otimes F_\lambda$. Thus, for any $\ell_1 \in L$, we can write:

$$F_{\ell_1} \otimes F_{\lambda+\ell} \cong F_{\ell_1} \otimes F_\ell \otimes F_\lambda \cong F_{\ell_1+\ell} \otimes F_\lambda .$$

Trying to reproduce the above line in the definition of a V_L-module isomorphism that is hopefully compatible with the V_L-actions, one defines:

$$\mathrm{Shift}^\ell = \left(\{\ell_1\} \mapsto \{\ell_1 + \ell\}, \{\mu_{\ell_1,\ell}\}_{\ell_1\in L}\right) \in \mathrm{Hom}_{\mathcal{C}_\oplus}\left(\underline{\mathcal{F}}(F_{\lambda+\ell}), \underline{\mathcal{F}}(F_\lambda)\right) . \tag{2.65}$$

Obviously, the morphism of $\mathcal{C}_\oplus$ that is Shift^ℓ is invertible with inverse $\mathrm{Shift}^{-\ell}$. At this point, it is sufficient to show that Shift^ℓ intertwines the V_L-actions on these two induced modules (see Remark 2.61). For this, recall that $\mathcal{C}_\oplus$ is a strict monoidal category because $\mathcal{C}$ is in the first place. It follows that in

$$\mathrm{Hom}_{\mathcal{C}_\oplus}\left(V_L \otimes_{\mathcal{C}_\oplus} \underline{\mathcal{F}}(F_{\lambda+\ell}), \underline{\mathcal{F}}(F_\lambda)\right) ,$$

we have the following relations:

$$\begin{aligned}\mathrm{Shift}^\ell \circ \mu &= \left(\left\{(\ell_A, \ell_1)\right\} \mapsto \{\ell_A + \ell_1 + \ell\}, \{\mu_{\ell_A,\ell_1} \cdot \mu_{\ell_A+\ell_1,\ell}\}_{(\ell_A,\ell_1)\in L^2}\right)\\ &= \left(\left\{(\ell_A, \ell_1)\right\} \mapsto \{\ell_A + \ell_1 + \ell\}, \{\mu_{\ell_1,\ell} \cdot \mu_{\ell_A,\ell_1+\ell}\}_{(\ell_A,\ell_1)\in L^2}\right)\\ &= \mu \circ \mathrm{Shift}^\ell\end{aligned}$$

where we make use of the cocycle property (2.52) of $K(\mu)$. We conclude that $\underline{\mathcal{F}}(F_{\lambda+\ell}) \cong \underline{\mathcal{F}}(F_\lambda)$ in $\mathrm{Rep}^0 V_L$.

Q.E.D.

Corollary 2.66. *The set $\mathcal{S}$ of line (2.63) is a complete set of representatives of isomorphism classes of simple V_L-modules without redundancies.*

Proof: It is sufficient to show that for all $\alpha, \beta \in L^*$, we have $\underline{\mathcal{F}}(F_\alpha) \cong \underline{\mathcal{F}}(F_\beta)$ if and only if $\alpha \equiv \beta \in L^*/L$.

By Lemma 2.64, $\alpha \equiv \beta \in L^*/L$ implies that the two modules are isomorphic. However when $\alpha \not\equiv \beta$, Schur's Lemma and the definition of morphisms in $\mathcal{C}_\oplus$ implies that all component maps between the two induced modules at the level of Fock spaces are 0, and so is any such morphism in $\mathrm{Rep}^0 A$.

Q.E.D.

Now that the simple modules of $\mathrm{Rep}^0 V_L$ are classified in Corollary 2.66, we turn to constructing explicit V_L-isomorphisms that realise (2.62). This is crucial to explicitly compute the associativity isomorphisms for the skeleton of $\operatorname{Rep}^0 V_L$ with simple objects $\mathcal{S}$ that will be considered in the subsequent subsection. To achieve a meaningful understanding of (2.62), we will need the following analysis.

For $\alpha, \beta \in L^*$, we would like to clarify the definition of the tensor product $\underline{\mathcal{F}}\,(F_\alpha) \otimes_{V_L} \underline{\mathcal{F}}\,(F_\beta)$ and compare it to $\underline{\mathcal{F}}(F_{\alpha+\beta})$. Since the resulting object of a tensor product $\otimes_{V_L}$ is really a quotient of the corresponding object resulting of $\otimes_{\mathcal{C}_\oplus}$, let's start by writing out their tensor product in $\mathcal{C}_\oplus$:

$$\underline{\mathcal{F}}\,(F_\alpha) \otimes_{\mathcal{C}_\oplus} \underline{\mathcal{F}}\,(F_\beta) = \bigoplus_{\ell_1, \ell_2 \in L} \left(F_{\ell_1} \otimes F_\alpha \otimes F_{\ell_2} \otimes F_\beta\right) \ , \tag{2.67}$$

$$\underline{\mathcal{F}}(F_{\alpha+\beta}) = \bigoplus_{\ell_3 \in L} \left(F_{\ell_3} \otimes F_{\alpha+\beta}\right) \ . \tag{2.68}$$

The above two decompositions strongly suggest that the passage from line (2.67) to (2.68) can be achieved by somehow adding labels ℓ_1 and ℓ_2 to produce a corresponding label ℓ_3. However we need to write out the action maps m^{left} and m^{right} that are identified with each other in Definition B.8. In the appropriate morphism spaces, one can write:

$$m^{\text{left}} = \left(\left\{(\ell_1, \ell_A, \ell_2)\right\} \mapsto \left\{(\ell_A + \ell_1, \ell_2)\right\}, \left\{\mu_{\ell_A, \ell_1} \cdot e^{\pi i \left((\ell_1 + \alpha)\cdot \ell_A\right)}\right\}_{(\ell_1, \ell_A, \ell_2) \in L^3}\right) \ , \tag{2.69}$$

$$m^{\text{right}} = \left(\left\{(\ell_1, \ell_A, \ell_2)\right\} \mapsto \left\{(\ell_1, \ell_A + \ell_2)\right\}, \left\{\mu_{\ell_A, \ell_2}\right\}_{(\ell_1, \ell_A, \ell_2) \in L^3}\right) \ . \tag{2.70}$$

We then fix $\ell_1, \ell_2, \tilde{\ell}_1, \tilde{\ell}_2 \in L$ such that $\ell_1 + \ell_2 = \tilde{\ell}_1 + \tilde{\ell}_2$. According to Definition B.9, the above left and right actions maps have indicates us that in the quotient space $\underline{\mathcal{F}}\,(F_\alpha) \otimes_{V_L} \underline{\mathcal{F}}\,(F_\beta)$, one must view as equivalent the following operations on components of $\underline{\mathfrak{F}}(F_\alpha) \otimes_{\mathcal{C}_\oplus} \underline{\mathfrak{F}}(F_\beta)$:

$$\begin{aligned}&\bullet \text{ multiplying } F_{\ell_1} \otimes F_\alpha \otimes F_{\ell_2} \otimes F_\beta \text{ by } \mu_{\ell_1-\tilde{\ell}_1,\tilde{\ell}_1} e^{\pi i\left((\tilde{\ell}_1+\alpha)\cdot(\ell_1-\tilde{\ell}_1)\right)} \quad ;\\ &\bullet \text{ multiplying } F_{\tilde{\ell}_1} \otimes F_\alpha \otimes F_{\tilde{\ell}_2} \otimes F_\beta \text{ by } \mu_{\tilde{\ell}_2-\ell_2,\ell_2} \quad .\end{aligned} \tag{2.71}$$

This means that the components $F_{\ell_1} \otimes F_\alpha \otimes F_{\ell_2} \otimes F_\beta$ and $F_{\tilde{\ell}_1} \otimes F_\alpha \otimes F_{\tilde{\ell}_2} \otimes F_\beta$ of $\underline{\mathfrak{F}}(F_\alpha) \otimes_{\mathcal{C}_\oplus} \underline{\mathfrak{F}}(F_\beta)$ are redundant in the quotient $\underline{\mathfrak{F}}(F_\alpha) \otimes_{V_L} \underline{\mathfrak{F}}(F_\beta)$. Consequently, we can write:

$$\begin{aligned}\underline{\mathfrak{F}}(F_\alpha) \otimes_{V_L} \underline{\mathfrak{F}}(F_\beta) &= \bigoplus_{(\ell_1,\ell_2)\in\frac{L^2}{\ker(+)}} (F_{\ell_1} \otimes F_\alpha \otimes F_{\ell_2} \otimes F_\beta)\\ &\cong \bigoplus_{t\in L} (F_t \otimes F_\alpha \otimes F_\beta) \\ &= \bigoplus_{t\in L} (F_t \otimes F_{\alpha+\beta}) \\ &= \underline{\mathfrak{F}}(F_{\alpha+\beta}) \quad .\end{aligned} \tag{2.72}$$

The map at line (2.72) should constitute an V_L-module isomorphism. The following proposition formalises this result:

Lemma 2.73. *The the V_L-module isomorphism needed at line (2.72) can be defined as follows:*

$$f^{\alpha,\beta} = \Big(\big\{(\ell_1,\ell_2)\big\} \mapsto \{\ell_1+\ell_2\}, \{\mu_{\ell_1,\ell_2} e^{\pi i\alpha\ell_2}\}_{(\ell_1,\ell_2)\in\frac{L^2}{\ker(+)}=L}\Big) \ .$$

Note that this V_L-isomorphism corresponds to that of Theorem 2.59 (2) of [CKM 2017].

Proof: This is also Lemma 4.6 of [AR 2018] when the cocycle $K(\mu)$ is taken to be trivial. See also Proposition 5.7 of [DF 2012]. To show that $f^{\alpha,\beta}$ is well defined, we must compare its effects on two equivalent components of $\underline{\mathfrak{F}}(F_\alpha) \otimes_{V_L} \underline{\mathfrak{F}}(F_\beta)$.

With the same notation as of line (2.71), we see that $f^{\alpha,\beta}$ is well defined if and only if

$$\mu_{\ell_1,\ell_2}e^{\pi i(\alpha\cdot\ell_2)} \;=\; \mu_{\tilde\ell_1,\tilde\ell_2}e^{\pi i(\alpha\cdot\tilde\ell_2)}\cdot\frac{\mu_{\tilde\ell_2-\ell_2,\ell_2}}{\mu_{\ell_1-\tilde\ell_1,\tilde\ell_1}}e^{\pi i\left((\tilde\ell_1+\alpha)\cdot(\ell_1-\tilde\ell_1)\right)}\;. \tag{2.74}$$

Recall that $\alpha,\beta\in L^*$ and that $\ell_1,\ell_2,\tilde\ell_1,\tilde\ell_2\in L$ are such that $\ell_1+\ell_2=\tilde\ell_1+\tilde\ell_2$. We proceed to check (2.74) by writing:

$$\begin{aligned}
&& 1 &= \frac{\mu_{\tilde\ell_1,\tilde\ell_2}\mu_{\tilde\ell_2-\ell_2,\ell_2}}{\mu_{\ell_1,\ell_2}\mu_{\ell_1-\tilde\ell_1,\tilde\ell_1}}e^{\pi i\left((\tilde\ell_1+\alpha)\cdot(\ell_1-\tilde\ell_1)+(\alpha\cdot\tilde\ell_2)-(\alpha\cdot\ell_2)\right)}\\
&\Leftrightarrow & 1 &= \frac{\mu_{\tilde\ell_1,\tilde\ell_2}\mu_{\tilde\ell_2-\ell_2,\ell_2}}{\mu_{\ell_1,\ell_2}\mu_{\ell_1-\tilde\ell_1,\tilde\ell_1}}e^{\pi i(\tilde\ell_1\cdot\ell_1)}\\
&\Leftrightarrow & 1 &= \frac{\mu_{\tilde\ell_1,\tilde\ell_2}\mu_{\tilde\ell_2-\ell_2,\ell_2}}{\mu_{\ell_1,\ell_2}\mu_{\ell_1-\tilde\ell_1,\tilde\ell_1}}e^{\pi i\left(\tilde\ell_1\cdot(\ell_1-\tilde\ell_1)\right)}\\
&\Leftrightarrow & 1 &= \frac{\mu_{\tilde\ell_1,\tilde\ell_2}\mu_{\tilde\ell_2-\ell_2,\ell_2}\mu_{\ell_1-\tilde\ell_1,\ell_1}}{\mu_{\ell_1,\ell_2}\mu_{\ell_1-\tilde\ell_1,\tilde\ell_1}\mu_{\ell_1,\ell_1-\tilde\ell_1}}\\
&\Leftrightarrow & 1 &= \frac{\mu_{\tilde\ell_1,\tilde\ell_2}\mu_{\tilde\ell_2-\ell_2,\ell_2}}{\mu_{\ell_1,\ell_2}\mu_{\ell_1,\ell_1-\tilde\ell_1}}
\end{aligned}$$

where at the last step, one sets $A=\tilde\ell_1$, $B=\ell_1-\tilde\ell_1=\tilde\ell_2-\ell_2$ and $C=\ell_2$ and writes down the cocycle property $\mu_{A,B}\mu_{A+B,C}=\mu_{A,B+C}\mu_{B,C}$ to conclude equality. This check means that $f^{\alpha,\beta}$ is a well defined morphism in $\mathcal{C}_\oplus$.

It remains to be shown that $f^{\alpha,\beta}$ intertwines the V_L-actions on and that it is both injective and surjective. The morphism $f^{\alpha,\beta}$ will be one also in $\operatorname{Rep}^0 V_L$ if the following diagram of V_L-actions commutes:

$$\begin{tikzcd}
F_{\ell_A}\otimes F_{\ell_1}\otimes F_\alpha\otimes F_{\ell_2}\otimes F_\beta \arrow[r, "\mu_{\ell_A,\ell_1}"] \arrow[d, "\mu_{\ell_1,\ell_2}e^{\pi i(\alpha\cdot\ell_2)}"'] & F_{\ell_A+\ell_1}\otimes F_\alpha\otimes F_{\ell_2}\otimes F_\beta \arrow[d, "\mu_{\ell_A+\ell_1,\ell_2}e^{\pi i(\alpha\cdot\ell_2)}"] \\
F_{\ell_A}\otimes F_{\ell_1+\ell_2}\otimes F_{\alpha+\beta} \arrow[r, "\mu_{\ell_A,\ell_1+\ell_2}"'] & F_{\ell_A+\ell_1+\ell_2}\otimes F_{\alpha+\beta}
\end{tikzcd}\;.$$

The above diagram does indeed commute thanks to the cocycle property of $K(\mu)$. Finally, let's address the injectivity and surjectivity of $f^{\alpha,\beta}$. They both simply follow from the facts that: 1) at the level of Fock spaces, the component maps of $f^{\alpha,\beta}$ are isomorphisms (scalar multiples of the identity), and 2) that there is a natural bijection

$$+: \quad L^2/\ker(+) \quad \overset{1:1}{\longleftrightarrow} \quad L \ ,$$

between the index sets of the domain and codomain direct sums of f.

Q.E.D.

2.3.3 Associativity, Braidings and Twists in $\mathrm{Rep}^0 A$

In this last section, we recover the structure of the skeleton of the monoidal category $\mathrm{Rep}^0 V_L$: we will compute braiding, twist, duals and associativity morphisms associated to triples simple modules of the set $\mathcal{S}$ defined in (2.63). In order to do so, we have to work in $\mathrm{Rep}^0 V_L$ while making sure to keep manipulating objects from $\mathcal{S}$ at every step.

Here are a few technical concepts that will be useful to stay within $\mathcal{S}$. Consider the following short exact sequences of abelian groups:

$$0 \to L \to L^* \xrightarrow{g} L^*/L \to 0$$

where g is the quotient map. This choice of complete set of representatives $\mathcal{S}$ as in line (2.63) effectively corresponds to the following choice of a section of the above short exact sequence:

$$s: \quad \begin{aligned} L^*/L &\longrightarrow L^* \\ \overline{\lambda} &\longmapsto \tfrac{a}{\sqrt{2N}} \end{aligned} \qquad \text{where } a \in \{0, \dots, 2N-1\} \text{ and } g\left(\frac{a}{\sqrt{2N}}\right) = \overline{\lambda} \ .$$

From the same short exact sequence, one has an isomorphism $L^* \cong L \oplus (L^*/L)$

as abelian groups. The section s gives us a nice way to precise this isomorphism. Since s is injective, one has

$$L \oplus s(L^*/L) \cong L^* \;, \tag{2.75}$$

and we can make sense of the addition in L^* on the left-hand side of (2.75) by defining a cocycle $t^{\mathcal{S}}$ as follows:

$$(\ell_1, \overline{\alpha}) + (\ell_2, \overline{\beta}) = \left(\ell_1 + \ell_2 + \overbrace{s(\overline{\alpha}) + s(\overline{\beta}) - s(\overline{\alpha + \beta})}^{t^{\mathcal{S}}(\overline{\alpha},\overline{\beta})}, \overline{\alpha} + \overline{\beta}\right) \in L \oplus s(L^*/L) \;.$$

Note that $t^{\mathcal{S}} : L^*/L \times L^*/L \to L$ is a well defined group 2-cocycle. Explicitly, the cocycle $t^{\mathcal{S}}$ takes values in the set $\left\{0, \sqrt{2N}\right\}$:

$$t^{\mathcal{S}} : \; (\overline{\alpha}, \overline{\beta}) \quad \longmapsto \quad \begin{cases} \sqrt{2N} & \text{if } s(\overline{\alpha}) + s(\overline{\beta}) > \sqrt{2N} \\ 0 & \text{if } s(\overline{\alpha}) + s(\overline{\beta}) \leq \sqrt{2N} \end{cases} \;. \tag{2.76}$$

Let's now make sense of the braiding morphisms on the skeleton of $\operatorname{Rep}^0 V_L$ given by $\mathcal{S}$. By Lemma 2.73, we know how to handle the tensor isomorphisms (2.62). Plus, we know that $\underline{\mathcal{F}}$ is a braided tensor functor, so the braiding on the objects of $\mathcal{C}^{loc}$ should induce to a braiding on $\operatorname{Rep}^0 V_L$. First, we check that the morphism $c^{\mathcal{C}_\oplus}_{\underline{\mathcal{F}}(F_\alpha), \underline{\mathcal{F}}(F_\beta)}$ of $\mathcal{C}_\oplus$ is well defined on the tensor product of $\otimes_{V_L}$. Second, we consider the following diagram:

$$\begin{tikzcd} \underline{\mathcal{F}}(F_\alpha) \otimes \underline{\mathcal{F}}(F_\beta) \arrow[d, "c^{\mathcal{C}_\oplus}_{\underline{\mathcal{F}}(F_\alpha), \underline{\mathcal{F}}(F_\beta)}"'] & \underline{\mathcal{F}}(F_{\alpha+\beta}) \arrow[l, "(f^{\alpha,\beta})^{-1}"'] & \underline{\mathcal{F}}\left(F_{\alpha+\beta-t^{\mathcal{S}}(\overline{\alpha},\overline{\beta})}\right) \arrow[l, "\mathrm{Shift}^{-t^{\mathcal{S}}(\overline{\alpha},\overline{\beta})}"'] \\ \underline{\mathcal{F}}(F_\beta) \otimes \underline{\mathcal{F}}(F_\alpha) \arrow[r, "f^{\beta,\alpha}"'] & \underline{\mathcal{F}}(F_{\alpha+\beta}) \arrow[r, "\left(\mathrm{Shift}^{-t^{\mathcal{S}}(\overline{\alpha},\overline{\beta})}\right)^{-1}"] & \underline{\mathcal{F}}\left(F_{\alpha+\beta-t^{\mathcal{S}}(\overline{\alpha},\overline{\beta})}\right) \end{tikzcd} \;.$$

Let's compute the scalar multiple of the identity corresponding to a fixed single component from the above diagram. To do so, fix $\ell_{start} = \ell_1 + \ell_2$ with $\ell_1, \ell_2 \in L$:

$$\begin{aligned}
&\mu_{\ell_{start}, -t^{\mathcal{S}}(\overline{\alpha},\overline{\beta})} \cdot \frac{1}{\mu_{\ell_1 + t^{\mathcal{S}}(\overline{\alpha},\overline{\beta}), \ell_2} e^{\pi i \alpha \ell_2}} \cdot e^{\pi i \left(\ell_1 + t^{\mathcal{S}}(\overline{\alpha},\overline{\beta}) + \alpha\right)(\ell_2 + \beta)} \\
&\quad \cdot \mu_{\ell_2, \ell_1 + t^{\mathcal{S}}(\overline{\alpha},\overline{\beta})} e^{\pi i \beta \left(\ell_1 + t^{\mathcal{S}}(\overline{\alpha},\overline{\beta})\right)} \cdot \frac{1}{\mu_{\ell_{start}, -t^{\mathcal{S}}(\overline{\alpha},\overline{\beta})}} \\
&\quad = e^{\pi i(-\alpha \ell_2)} e^{\pi i \left(\ell_1 + t^{\mathcal{S}}(\overline{\alpha},\overline{\beta})\right)\ell_2} e^{\pi i(-\alpha \ell_2)} \\
&\qquad \cdot e^{\pi i \left(\ell_1 + t^{\mathcal{S}}(\overline{\alpha},\overline{\beta}) + \alpha\right)(\ell_2 + \beta)} e^{\pi i \beta \left(\ell_1 + t^{\mathcal{S}}(\overline{\alpha},\overline{\beta})\right)} \quad \text{by (2.53)} \\
&\quad = e^{\pi i \alpha \beta} \; .
\end{aligned}$$

This scalar then corresponds to that of the induced V_L-morphism $\underline{\mathcal{F}}(c^{\mathcal{C}_\oplus}_{F_\alpha, F_\beta})$. Note that the previous computations would have given the same result should we chose to group $t^{\mathcal{S}}(\overline{\alpha}, \overline{\beta})$ with ℓ_2 instead of ℓ_1. We conclude that one can define a braiding on the skeleton of $\mathrm{Rep}^0 V_L$ by setting

$$c^{\mathrm{Rep}^0 V_L}_{\underline{\mathcal{F}}(F_\alpha), \underline{\mathcal{F}}(F_\beta)} = \underline{\mathcal{F}}(c^{\mathcal{C}_\oplus}_{F_\alpha, F_\beta}) \; . \tag{2.77}$$

Next, let's make sense of a compatible twist isomorphisms for the skeleton of $\mathcal{S}$. Recall Remark 2.58 in which we show that $\theta^{\mathcal{C}_\oplus}_{V_L} = \mathrm{Id}_{V_L}$. Fix $a \in \{0, \ldots, 2N-1\}$ so that $\underline{\mathcal{F}}\left(F_{\frac{a}{\sqrt{2N}}}\right) \in \mathcal{S}$. The natural candidate is just the twist of the underlying object of $\underline{\mathcal{F}}\left(F_{\frac{a}{\sqrt{2N}}}\right)$ in $\mathcal{C}_\oplus$. By the balancing axiom in $\mathcal{C}_\oplus$, we can write

$$\begin{aligned}
\theta^{\mathcal{C}_\oplus}_{V_L \otimes_{\mathcal{C}_\oplus} \underline{\mathcal{F}}\left(F_{\frac{a}{\sqrt{2N}}}\right)} &= \left(c^{\mathcal{C}_\oplus}_{V_L, \underline{\mathcal{F}}\left(F_{\frac{a}{\sqrt{2N}}}\right)} \circ c^{\mathcal{C}_\oplus}_{\underline{\mathcal{F}}\left(F_{\frac{a}{\sqrt{2N}}}\right), V_L} \right) \circ \left(\theta^{\mathcal{C}_\oplus}_{V_L} \otimes \theta^{\mathcal{C}_\oplus}_{\underline{\mathcal{F}}\left(F_{\frac{a}{\sqrt{2N}}}\right)} \right) \\
&= \left(\theta^{\mathcal{C}_\oplus}_{V_L} \otimes \theta^{\mathcal{C}_\oplus}_{\underline{\mathcal{F}}\left(F_{\frac{a}{\sqrt{2N}}}\right)} \right) \qquad \text{since } \underline{\mathcal{F}}\left(F_{\frac{a}{\sqrt{2N}}}\right) \in \operatorname{Rep}^0 V_L \\
&= 1 \otimes \theta^{\mathcal{C}_\oplus}_{\underline{\mathcal{F}}\left(F_{\frac{a}{\sqrt{2N}}}\right)} \qquad \text{since } \theta^{\mathcal{C}_\oplus}_{V_L} = \mathrm{Id}_{V_L} \\
&= \underline{\mathcal{F}}\left(\theta^{\mathcal{C}}_{F_{\frac{a}{\sqrt{2N}}}}\right) .
\end{aligned}$$

Therefore, the twist isomorphisms $\theta^{\mathcal{C}_\oplus}_{-}$ automatically commute with the V_L-actions on the induced modules. By the discussion leading up to Lemma 2.60, the induced modules constitute all the modules of $\operatorname{Rep}^0 V_L$. We conclude that one can simply define a twist $\theta^{\operatorname{Rep}^0 V_L}_{-}$ on $\operatorname{Rep}^0 V_L$ as follows:

$$\theta^{\operatorname{Rep}^0 V_L}_{\underline{\mathcal{F}}\left(F_{\frac{a}{\sqrt{2N}}}\right)} = \underline{\mathcal{F}}\left(\theta^{\mathcal{C}}_{F_{\frac{a}{\sqrt{2N}}}}\right) . \tag{2.78}$$

This twist then satisfies the banalcing axiom in $\operatorname{Rep}^0 V_L$ since it does in $\mathcal{C}_\oplus$ and the braiding on $\operatorname{Rep}^0 V_L$ is also induced.

We are now ready to compute the associativity morphisim of the skeleton of $\operatorname{Rep}^0 V_L$ with simple objects labelled by the set $\mathcal{S}$ of line (2.63).

Proposition 2.79. *Let $\lambda_1, \lambda_2, \lambda_3 \in \{\frac{a}{\sqrt{2N}}\}_{a=0}^{2N-1}$ so that $\underline{\mathcal{F}}(F_{\lambda_1})$, $\underline{\mathcal{F}}(F_{\lambda_2})$ $\underline{\mathcal{F}}(F_{\lambda_3})$ are all three in the set $\mathcal{S}$. Then the associativity map is as follows:*

$$a^{\operatorname{Rep}^0 A}_{\underline{\mathcal{F}}(F_{\lambda_1}), \underline{\mathcal{F}}(F_{\lambda_2}), \underline{\mathcal{F}}(F_{\lambda_3})} = (-1)^{\lambda_1 \cdot t^{\mathcal{S}}(\overline{\lambda}_2, \overline{\lambda}_3)} \ \mathrm{Id}_{\underline{\mathcal{F}}\left(F_{s(\overline{\lambda}_1 + \overline{\lambda}_2 + \overline{\lambda}_3)}\right)} .$$

<u>**Proof:**</u> In our computation similar to that of Proposition 4.5 of [AR 2018], we must

now take into account the scalars produced by the cocycle $K(\mu)$ of line (2.56).

I a) Applying f^{λ_1,λ_2};

b) then $\mu_{\ell_1+\ell_2,t^{\mathrm{s}}(\overline{\lambda}_1,\overline{\lambda}_2)}$;

c) then $f^{s(\overline{\lambda}_1+\overline{\lambda}_2),\lambda_3}$;

d) and finally $\mu_{\ell_1+\ell_2+\ell_3+t^{\mathrm{s}}(\overline{\lambda}_1,\overline{\lambda}_2),t^{\mathrm{s}}(\overline{\lambda}_1+\overline{\lambda}_2,\overline{\lambda}_3)}$:

$$
\begin{aligned}
&(F_{\ell_1}\otimes F_{\lambda_1})\otimes(F_{\ell_2}\otimes F_{\lambda_2})\otimes(F_{\ell_3}\otimes F_{\lambda_3})\\
&\quad\cong(F_{\ell_1+\ell_2}\otimes F_{\lambda_1+\lambda_2})\otimes(F_{\ell_3}\otimes F_{\lambda_3})\\
&\quad\cong\Big(F_{\ell_1+\ell_2+c(\overline{\lambda}_1,\overline{\lambda}_2)}\otimes F_{s(\overline{\lambda}_1+\overline{\lambda}_2)}\Big)\otimes(F_{\ell_3}\otimes F_{\lambda_3})\\
&\quad\cong F_{\ell_1+\ell_2+c(\overline{\lambda}_1,\overline{\lambda}_2)+\ell_3}\otimes F_{s(\overline{\lambda}_1+\overline{\lambda}_2)+\lambda_3}\\
&\quad\cong F_{\ell_1+\ell_2+\ell_3+t^{\mathrm{s}}(\overline{\lambda}_1,\overline{\lambda}_2)+t^{\mathrm{s}}(\overline{\lambda}_1+\overline{\lambda}_2,\overline{\lambda}_3)}\otimes F_{s(\overline{\lambda}_1+\overline{\lambda}_2+\overline{\lambda}_3)}\ .
\end{aligned}
$$

The operations a), b), c), d) above correspond to multiplying by the following scalar:

$$
\begin{aligned}
&\mu_{\ell_1,\ell_2}e^{\pi i(\lambda_1\cdot\ell_2)}\mu_{\ell_1+\ell_2,t^{\mathrm{s}}(\overline{\lambda}_1,\overline{\lambda}_2)}\mu_{\ell_1+\ell_2+t^{\mathrm{s}}(\overline{\lambda}_1,\overline{\lambda}_2),\ell_3}\\
&\cdot e^{\pi i\left(s(\overline{\lambda}_1+\overline{\lambda}_2)\cdot\ell_3\right)}\mu_{\ell_1+\ell_2+\ell_3+t^{\mathrm{s}}(\overline{\lambda}_1,\overline{\lambda}_2),t^{\mathrm{s}}(\overline{\lambda}_1+\overline{\lambda}_2,\overline{\lambda}_3)}\ .
\end{aligned}
$$

II a) Applying f^{λ_2,λ_3};

b) then $\mu_{\ell_2+\ell_3,t^{\mathrm{s}}(\overline{\lambda}_2,\overline{\lambda}_3)}$;

c) then $f^{\lambda_1,s(\overline{\lambda}_2+\overline{\lambda}_3)}$;

d) and finally $\mu_{\ell_1+\ell_2+\ell_3+t^{\mathrm{S}}(\overline{\lambda}_2,\overline{\lambda}_3),t^{\mathrm{S}}(\overline{\lambda}_1,\overline{\lambda}_2+\overline{\lambda}_3)}$;

$$\begin{aligned}
&(F_{\ell_1}\otimes F_{\lambda_1})\otimes(F_{\ell_2}\otimes F_{\lambda_2})\otimes(F_{\ell_3}\otimes F_{\lambda_3})\\
&\quad\cong(F_{\ell_1}\otimes F_{\lambda_1})\otimes(F_{\ell_2+\ell_3}\otimes F_{\lambda_2+\lambda_3})\\
&\quad\cong(F_{\ell_1}\otimes F_{\lambda_1})\otimes\left(F_{\ell_2+\ell_3+t^{\mathrm{S}}(\overline{\lambda}_2,\overline{\lambda}_3)}\otimes F_{s(\overline{\lambda}_2+\overline{\lambda}_3)}\right)\\
&\quad\cong F_{\ell_1+\ell_2+\ell_3+t^{\mathrm{S}}(\overline{\lambda}_2,\overline{\lambda}_3)}\otimes F_{\lambda_1+s(\overline{\lambda}_2+\overline{\lambda}_3)}\\
&\quad\cong F_{\ell_1+\ell_2+\ell_3+t^{\mathrm{S}}(\overline{\lambda}_2,\overline{\lambda}_3)+t^{\mathrm{S}}(\overline{\lambda}_1,\overline{\lambda}_2+\overline{\lambda}_3)}\otimes F_{s(\overline{\lambda}_1+\overline{\lambda}_2+\overline{\lambda}_3)}\ .
\end{aligned}$$

The operations a), b), c), d) above correspond to multiplying by the following scalar:

$$\begin{aligned}
&\mu_{\ell_2,\ell_3}e^{\pi i(\lambda_2\cdot\ell_3)}\mu_{\ell_2+\ell_3,t^{\mathrm{S}}(\overline{\lambda}_2,\overline{\lambda}_3)}\cdot\\
&\qquad\mu_{\ell_1,\ell_2+\ell_3+t^{\mathrm{S}}(\overline{\lambda}_2,\overline{\lambda}_3)}e^{\pi i\left(\lambda_1\cdot\left(\ell_2+\ell_3+t^{\mathrm{S}}(\overline{\lambda}_2,\overline{\lambda}_3)\right)\right)}\mu_{\ell_1+\ell_2+\ell_3+t^{\mathrm{S}}(\overline{\lambda}_2,\overline{\lambda}_3),t^{\mathrm{S}}(\overline{\lambda}_1,\overline{\lambda}_2+\overline{\lambda}_3)}\ .
\end{aligned}$$

III The associativity scalar is then the quotient the scalar of **I** by that of **II**:

$$\begin{aligned}
&\frac{\mu_{\ell_1,\ell_2}\mu_{\ell_1+\ell_2,t^{\mathrm{S}}(\overline{\lambda}_1,\overline{\lambda}_2)}\mu_{\ell_1+\ell_2+t^{\mathrm{S}}(\overline{\lambda}_1,\overline{\lambda}_2),\ell_3}\mu_{\ell_1+\ell_2+\ell_3+t^{\mathrm{S}}(\overline{\lambda}_1,\overline{\lambda}_2),t^{\mathrm{S}}(\overline{\lambda}_1+\overline{\lambda}_2,\overline{\lambda}_3)}}{\mu_{\ell_2,\ell_3}\mu_{\ell_2+\ell_3,t^{\mathrm{S}}(\overline{\lambda}_2,\overline{\lambda}_3)}\mu_{\ell_1,\ell_2+\ell_3+t^{\mathrm{S}}(\overline{\lambda}_2,\overline{\lambda}_3)}\mu_{\ell_1+\ell_2+\ell_3+t^{\mathrm{S}}(\overline{\lambda}_2,\overline{\lambda}_3)t^{\mathrm{S}}(\overline{\lambda}_1,\overline{\lambda}_2+\overline{\lambda}_3)}}\cdot\\
&\qquad e^{\pi i\left(\lambda_1\cdot\ell_2+s(\overline{\lambda}_1+\overline{\lambda}_2)\cdot\ell_3\right)-\left(\lambda_2\cdot\ell_3+\lambda_1\cdot\left(\ell_2+\ell_3+t^{\mathrm{S}}(\overline{\lambda}_2,\overline{\lambda}_3)\right)\right)}\ .
\end{aligned}\tag{2.80}$$

where a number of simplifications (omitted here) can be made thanks to the cocycle property (2.52) of $K(\mu)$. After performing the much needed simplifications at line (2.80), we obtain the following scalar:

$$\frac{\mu_{c(\overline{\lambda}_1,\overline{\lambda}_2),c(\overline{\lambda}_1+\overline{\lambda}_2,\overline{\lambda}_3)}}{\mu_{c(\overline{\lambda}_2,\overline{\lambda}_3),c(\overline{\lambda_1},\overline{\lambda}_2+\overline{\lambda}_3)}}(-1)^{\lambda_1\cdot c(\overline{\lambda}_2,\overline{\lambda}_3)}\ .\tag{2.81}$$

As a check that the simplifications of (2.80) really lead to (2.81), observe that (2.81) matches perfectly the associativity scalar computed in Proposition A.2 of [DF 2012].

IV Given the description of the cocycle $t^{\mathcal{S}}$ from line (2.76), we can check explicitly that the quotient of cocycle values in line (2.81) is always equal to one. It follows that the associativity scalar is precisely $(-1)^{\lambda_1 \cdot c(\overline{\lambda}_2, \overline{\lambda}_3)}$.

Q.E.D.

Remark 2.82. The exponent of -1 in the associativity map of Proposition 2.79 makes sense. It is indeed an integer since $\lambda_3 \in L^*$ and $t^{\mathcal{S}}(\overline{\lambda}_2, \overline{\lambda}_3) \in L$.

Finally, we can gather the scalars associated with associativity, braiding and twists from Proposition 2.79 and Equations (2.77), (2.78) to state a result equivalent to Theorem 4.7 of [AR 2018]:

Theorem 2.83. *The skeleton of the semisimple category* $\operatorname{Rep}^0 V_L$ *with simple objects given by the set* $\mathcal{S}$ *of line* (2.63) *is a braided monoidal category with a twist. Its tensor product is*

$$\underline{\mathcal{F}}(F_x) \otimes_{V_L} \underline{\mathcal{F}}(F_y) = \underline{\mathcal{F}}(F_{s(\overline{x}+\overline{y})})$$

where $\underline{\mathcal{F}}(F_x)$, $\underline{\mathcal{F}}(F_y) \in \mathcal{S}$ *and* $t^{\mathcal{S}}$ *is as of line* (2.76). *This skeletal category has associativity, braidings and twists given by*

$$\begin{aligned} a^{V_L}_{\underline{\mathcal{F}}(F_x),\underline{\mathcal{F}}(F_y),\underline{\mathcal{F}}(F_z)} &= (-1)^{x \cdot t^{\mathcal{S}}(y,z)} \operatorname{Id}_{\underline{\mathcal{F}}(F_{s(\overline{x}+\overline{y}+\overline{z})})} , \\ c^{V_L}_{\underline{\mathcal{F}}(F_x),\underline{\mathcal{F}}(F_y)} &= e^{\pi i x y} \operatorname{Id}_{\underline{\mathcal{F}}(F_{s(\overline{x}+\overline{y})})} , \\ \theta^{V_L}_{\underline{\mathcal{F}}(F_x)} &= e^{\pi i x^2} \operatorname{Id}_{\underline{\mathcal{F}}(F_x)} \end{aligned}$$

where $\underline{\mathcal{F}}(F_x), \underline{\mathcal{F}}(F_y), \underline{\mathcal{F}}(F_z) \in \mathcal{S}$. *Notice that the above associativity, braiding and twist scalars match their respective analogues for* V_L *seen as a vertex operator* *algebra.*

Remark 2.84. Duals in $\mathrm{Rep}^0 V_L$ can also be induced from $\mathcal{C}^{loc}$. We omit the details here, but the interested reader may want to read Exercise 2.10.6 and Section 2.10 of [EGNO 2015]. Recall that the dual of a Fock space F_λ in $\mathcal{C}^{loc}$ is given by $F_{-\lambda}$ and that evaluation and coevaluation morphisms can be fixed in terms of scalar multiples of the identity (the category $\mathcal{C}^{loc}$ is skeletal as well here).

Considering the duals of $\mathcal{S}$ in the skeleton of $\mathrm{Rep}^0 V_L$ effectively allows to recover its ribbon category structure. One could even go further and deduce from this picture the whole modular tensor structure of it. This includes categorical traces, Hopf-links, the S-matrix and the famous Verlinde formula.

Chapter 3

Modularity of Parafermion Vertex Algebras

In this chapter, we establish the modular behaviour of the characters of the modules of a certain category of representations for some *parafermion* vertex operator algebras. Note that modularity properties are of notable significance in Conformal Field Theory, see Chapter 10 of [DFMS 1997] for instance. The type of parafermion vertex operator algebra studied here is *logarithmic*, which means that it has reducible indecomposable modules, thus non-trivial extensions. This chapter's main result is especially important because within this logarithmic vector operator algebra setting, it is precisely the type of modular behaviour expected of a C_2-cofinite vertex operator algebras. This gives very good reasons to conjecture that the family of extended parafermions vertex operator algebras we study is a new family of examples of logarithmic C_2-cofinite vertex operator algebras. Note that this chapter includes the main results of the article [ACR 2018] that has now been accepted for publication in the journal *Letters in Mathematical Physics*.

We consider the category $\mathcal{A}_k$ of relaxed highest weight $L_k(\mathfrak{sl}_2)$-modules and their spectral flow twists as in [CR 2012] and [CR 2013b]. This category allowed the authors to establish modular properties and an adapted Verlinde formula that predicts non-negative integral fusion rules for what would be the Grothendieck ring of this category (see Appendix B). With the objective of constructing new examples of C_2-cofinite vertex operator algebras, we consider the Heisenberg commutant

$$\mathsf{C}_k(\mathfrak{sl}_2) = \mathrm{Com}\left(H, L_k(\mathfrak{sl}_2)\right) ,$$

known as the parafermions and construct a big extension $\mathsf{B}_k(\mathfrak{sl}_2)$ of $\mathsf{C}_k(\mathfrak{sl}_2)$:

$$L_k(\mathfrak{sl}_2) \xrightarrow{\text{commutant}} \mathsf{C}_k(\mathfrak{sl}_2) \xrightarrow{\infty\text{-extension}} \mathsf{B}_k(\mathfrak{sl}_2) . \tag{3.1}$$

Recall that C_2-cofinite vertex operator algebras have a finite number of simple modules whose characters have a certain modular behaviour. Tools were developed in [CKLR 2016] to analyse the commutants of type $\mathsf{C}_k(\mathfrak{sl}_2)$ including a Schur-Weyl type duality result that establishes a correspondance of simple modules. From $\mathcal{A}_k$, one could then produce a family of $\mathsf{C}_k(\mathfrak{sl}_2)$-modules whose characters are terms of known functions with good modular properties. Studying a big extension $\mathsf{B}_k(\mathfrak{sl}_2)$ effectively lowered the number of simple modules to a finite number. Under two natural assumptions, we were able to analyse the finitely many simple characters of the extended parafermions $\mathsf{B}_k(\mathfrak{sl}_2)$ for $k < 0$ admissible. We show that the simple $\mathsf{B}_k(\mathfrak{sl}_2)$-characters display the modular behaviour of a C_2-cofinite vertex operator algebra. Our conclusion is to conjecture that for all $k < 0$ admissible, $\mathsf{B}_k(\mathfrak{sl}_2)$ are C_2-cofinite. In Chapter 4 we will prove that our conjecture holds for certain values of k.

In the first section, we introduce a certain category $\mathcal{A}_k$ of $L_k(\mathfrak{sl}_2)$-modules and its simple objects. We then introduce the parafermions algebras $\mathsf{C}_k(\mathfrak{sl}_2)$ and $\mathsf{B}_k(\mathfrak{sl}_2)$, a simple current extension of $\mathsf{C}_k(\mathfrak{sl}_2)$ of infinite order. In the second section, we compute characters of simple modules for $\mathsf{C}_k(\mathfrak{sl}_2)$ that we obtain from the simple $L_k(\mathfrak{sl}_2)$-modules of $\mathcal{A}_k$. Inducing the appropriate subset of the simple $\mathsf{C}_k(\mathfrak{sl}_2)$-modules to untwisted $\mathsf{B}_k(\mathfrak{sl}_2)$-modules leaves us with finitely many simple $\mathsf{B}_k(\mathfrak{sl}_2)$-module of which we determine the character. In the last section, we decompose the finitely many simple characters of $\mathsf{B}_k(\mathfrak{sl}_2)$ in terms of Jacobi-theta functions and derivatives. By an explicit computation, we then check the closure of the $\mathbb{C}[\tau]$-span of characters under modular transformation. We end up with a finite dimensional vector-valued modular form of which we also determine an upper bound the dimension.

Notation

Throughout the chapter, the following notation will be employed:

- $\mathbb{H} \subset \mathbb{C}$ is the upper-half plane of the complex numbers;
- k such that $k+2 \in \mathbb{Q}_{>0} \backslash \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\right\}$ is called an *admissible* level of the affine vertex operator algebras associated to $\mathfrak{sl}_2$;
- $t = k + 2 = \frac{u}{v}$ with $gcd(u, v) = 1$ where $k \in \mathbb{Q}$ is an admissible $\mathfrak{sl}_2$ level;
- $w = -kv = 2v - u$ is the numerator of k up to a sign; in the case where $k < 0$ and is admissible that we will consider in the current chapter, this will ensure $w \in \mathbb{N}$;
- $V_k(\mathfrak{sl}_2)$ is the universal affine vertex operator algebra of level k associated to the finite dimensional Lie algebra $\mathfrak{sl}_2(\mathbb{C})$;

- $L_k(\mathfrak{sl}_2)$ is the simple quotient of $V_k(\mathfrak{sl}_2)$ with the inherited vertex operator algebra structure; for k admissible, the kernel of the quotient map is generated by a single singular vector while for non-admissible k, then $L_k(\mathfrak{sl}_2) = V_k(\mathfrak{sl}_2)$;

- H is the Heisenberg vertex operator algebra generated by the field associated to the Cartan subalgebra of $\mathfrak{sl}_2$; note that $H \subset V_k(\mathfrak{sl}_2)$ also makes sense as a subalgebra of $L_k(\mathfrak{sl}_2)$;

- $\mathfrak{h} \subset \mathfrak{sl}_2$ is the Cartan subalgebra of $\mathfrak{sl}_2$;

- $\mathsf{Q} = 2\mathbb{Z}$ is the root lattice of $\mathfrak{sl}_2$ that has been identified with $2\mathbb{Z}$;

- $\langle \mathsf{w} \rangle \ltimes \langle \sigma \rangle = \mathbb{Z}_2 \ltimes \mathbb{Z}$ is the affine Weyl group of $\mathfrak{sl}_2$; the generator w is the finite dimensional Weyl group generator and σ is the spectral flow; the two generators are subject to the relation $\mathsf{w}\sigma\mathsf{w} = \sigma^{-1}$;

- $\eta(q) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n)$ is Dedekind's η function; it converges to a modular function on $\mathbb{H}$.

3.1 The Extended Parafermions $\mathsf{B}_k(\mathfrak{sl}_2)$

In this section, we review relaxed highest weight modules for $L_k(\mathfrak{sl}_2)$ and their spectral flow twists. We then define the main objects of this study: the parafermion vertex operator algebras $\mathsf{C}_k(\mathfrak{sl}_2)$ and $\mathsf{B}_k(\mathfrak{sl}_2)$.

3.1.1 Relaxed Highest Weight Modules and Spectral Flow

We recall the basics of the vertex operator algebra $L_k(\mathfrak{sl}_2)$ from Appendix A and let k be an admissible level. We will first consider the class of *relaxed highest weight* $L_k(\mathfrak{sl}_2)$-modules. Those will be induced relaxed highest weight modules for $\mathfrak{sl}_2$

Consider the following triangular decomposition of the Lie algebra $\widehat{\mathfrak{sl}_2}$:

$$\underbrace{\{x_n \mid x \in \mathfrak{sl}_2 \text{ and } n < 0\}}_{\mathfrak{N}_-} \oplus \underbrace{\{x_0, \kappa \mid x \in \mathfrak{sl}_2\}}_{\mathfrak{T} \cong \mathfrak{sl}_2 \oplus \mathbb{C}.\kappa} \oplus \underbrace{\{x_n \mid x \in \mathfrak{sl}_2 \text{ and } n > 0\}}_{\mathfrak{N}_+} \tag{3.2}$$

where κ is the central element.

Definition 3.3. Consider the triangular decomposition (3.2) of $\widehat{\mathfrak{sl}_2}$. A *relaxed highest weight* vector v_λ of relaxed highest weight λ in an $\widehat{\mathfrak{sl}_2}$-module is a vector such that $\mathfrak{N}_+.v_\lambda = 0$ and $h_0 \in \mathfrak{h} \subset \mathfrak{T}$ acts as $h_0.v_\lambda = \lambda(h_0)v_\lambda$. For the rest of the chapter, we will think of λ as the complex number $\lambda(h_0) \in \mathbb{C}$ since the Cartan subalgebra of $\mathfrak{sl}_2$ is $\mathbb{C}.h_0$.

Remark 3.4. A usual highest weight vector for $\widehat{\mathfrak{sl}_2}$ is then a relaxed highest weight vector with the additional property that $e_0.v_\lambda = 0$.

Fix an admissible level k so that $\kappa \in \mathfrak{T}$ always act as multiplication by k. Simple $\widehat{\mathfrak{sl}_2}$-modules of level k include the unique simple quotients $L(\lambda)$ of the Verma modules

$$V(\lambda) = U(\mathfrak{N}_-) \otimes_{U(\mathfrak{T}\oplus\mathfrak{N}_+)} \mathbb{C}.v_\lambda \ ,$$

generated by a highest weight vector v_λ of highest weight λ. However only few among them turn out to make sense as modules for the simple affine vertex operator algebra[1] $L_k(\mathfrak{sl}_2)$. As shown in [AM 1995] and [DLM 1997], the allowed highest weights λ for $L(\lambda)$ compatible with the natural structure of $L_k(\mathfrak{sl}_2)$-module are those of the form

$$\lambda_{r,s} = (r-1) - st \tag{3.5}$$

where $r \in \{1, \ldots, u-1\}$ and $s \in \{0, \ldots, v-1\}$. Throughout the chapter, we will

[1]This is due to the fact that the singular vector that generates the ideal $I_{\text{sing}} \subset V_k(\mathfrak{sl}_2)$ has to act as zero on $L(\lambda)$.

employ the following notation:

- $\mathcal{L}_r$ = the simple quotient $L(\lambda_{r,0})$ of the Verma module $V(\lambda_{r,0})$ for $r \in \{1, \dots, u-1\}$;

- $\mathcal{D}^+_{r,s}$ = the simple quotient $L(\lambda_{r,s})$ of the Verma module $V(\lambda_{r,s})$ for $r \in \{1, \dots, u-1\}$ and $s \in \{0, \dots, v-1\}$.

One of the main reasons for the notational distinction for whether $s = 0$ or $s \neq 0$ is that the $\mathfrak{T}$-module $\{x \in V(\lambda_{r,s}) \mid \mathfrak{N}_+.x = 0\}$ is an infinite dimensional $\mathfrak{T}$-module for $s \neq 0$ while it is a finite dimensional $\mathfrak{T}$-modules for $s = 0$.

Remark 3.6. Note that as an $L_k(\mathfrak{sl}_2)$-module, we have $L_k(\mathfrak{sl}_2) \cong \mathcal{L}_1$.

Given the Sugawara conformal vector $\omega \in L_k(\mathfrak{sl}_2)$ as in Appendix A, one can show that any highest weight vector v_λ has a conformal weight (a L_0-eigenvalue) of

$$\frac{\lambda(\lambda+2)}{4t} = \frac{\lambda(\lambda+2)}{4(k+2)} \ .$$

Applying this to the allowed highest weights $\lambda = \lambda_{r,s}$ yields the conformal dimensions of the simple highest weight $L_k(\mathfrak{sl}_2)$-modules $\mathcal{L}_r$ and $\mathcal{D}^+_{r,s}$:

$$\Delta_{r,s} = \frac{\lambda_{r,s}(\lambda_{r,s}+2)}{4t} = \frac{(r-st)^2-1}{4t} \ . \tag{3.7}$$

Before moving on, note that for any of $r \in \{1, \dots, u-1\}$ and $s \in \{0, \dots, v-1\}$ the following hold:

$$\lambda_{u-r,v-s} = -\lambda_{r,s} - 2 \quad \text{and} \quad \Delta_{u-r,v-s} = \Delta_{r,s} \ . \tag{3.8}$$

When it comes to the relaxed highest weight simple $L_k(\mathfrak{sl}_2)$-modules, we have the following fundamental result:

Theorem 3.9. **(from [AM 1995])** *Let $k = -2+\frac{u}{v}$ be an admissible level for $L_k(\mathfrak{sl}_2)$. Then the simple relaxed highest weight modules are exhausted up to isomorphisms in the following list:*

- *the $\mathcal{L}_r$, for $r \in \{1, \ldots, u-1\}$;*
- *if $v > 1$, the $\mathcal{D}^+_{r,s}$, for $r \in \{1, \ldots, u-1\}$ and $s \in \{1, \ldots, v-1\}$;*
- *if $v > 1$, the conjugates $\mathcal{D}^-_{r,s} = \mathsf{w}(\mathcal{D}^+_{r,s})$[2], for $r \in \{1, \ldots, u-1\}$ and $s \in \{1, \ldots, v-1\}$;*
- *if $v > 1$, the $\mathcal{E}_{\lambda;\Delta_{r,s}}$, for $r \in \{1, \ldots, u-1\}$, $s \in \{1, \ldots, v-1\}$ and $\lambda \in \mathfrak{h}^*$ with $\lambda \neq \lambda_{r,s}, \lambda_{u-r,v-s} \bmod \mathsf{Q}$.*

Remark 3.10. The simple modules $\mathcal{E}_{\lambda;\Delta_{r,s}}$ are often referred to as *typical* $L_k(\mathfrak{sl}_2)$-modules as they are classified by a continuous set of parameters as opposed to $\mathcal{L}_r$ and $\mathcal{D}^\pm_{r,s}$ which are referred to as atypical modules. In practice, the $L_k(\mathfrak{sl}_2)$-modules $\mathcal{E}_{\lambda;\Delta_{r,s}}$ are generated by a single relaxed highest weight vector that is not a highest weight vector. Interestingly, these modules also have an interpretation in terms of Whittaker modules [Ada 2017].

Modules of type $\mathcal{E}_{\lambda;\Delta_{r,s}}$ but with $\lambda = \lambda_{r,s}$ also make sense. However these are indecomposable non-simple modules (see Remark 2.4 of [ACR 2018] for a bit more details). Notably, these typical indecomposable modules are non-split extensions of typical modules (see also [KR 2018]). As this is quite useful for the character analysis to come, we fix the following notation:

$$\mathcal{E}^+_{r,s} = \mathcal{E}_{\lambda_{r,s};\Delta_{r,s}}\ , \qquad\qquad \mathcal{E}^-_{r,s} = \mathsf{w}(\mathcal{E}^+_{r,s}) \tag{3.11}$$

[2]This corresponds to the module $\mathcal{D}^+_{r,s}$ whose $L_k(\mathfrak{sl}_2)$ action map is twisted by the automorphism w coming from the Weyl group of $\mathfrak{sl}_2$. More details on this can be found in Appendix A.

for $r \in \{1, \ldots, u-1\}$ and $s \in \{1, \ldots, v-1\}$.

As explained in [CR 2013b], the typical indecomposables (3.11) are extensions of atypical simple modules [RW 2015a] as follows:

$$0 \longrightarrow \mathcal{D}^{+}_{r,s} \longrightarrow \mathcal{E}^{+}_{r,s} \longrightarrow \mathcal{D}^{-}_{u-r,v-s} \longrightarrow 0 \,, \tag{3.12}$$

$$0 \longrightarrow \mathcal{D}^{-}_{r,s} \longrightarrow \mathcal{E}^{-}_{r,s} \longrightarrow \mathcal{D}^{+}_{u-r,v-s} \longrightarrow 0 \,. \tag{3.13}$$

We will then consider new simple $L_k(\mathfrak{sl}_2)$-modules that are obtained by twisting the $L_k(\mathfrak{sl}_2)$-action by automorphisms of $\widehat{\mathfrak{sl}_2}$ coming from the affine Weyl group[3] associated to the affinisation of $\mathfrak{sl}_2$. From Appendix A, recall that the affine Weyl group associated with $\mathfrak{sl}_2$ is generated by the conjugation w and the spectral flow σ [CR 2013b] under the relation $\mathsf{w}\sigma\mathsf{w} = \sigma^{-1}$. As automorphisms are invertible, applying twists by a fixed automorphism constitutes an exact functor on the module categories. Twisting the action by an automorphism thus preserves short exact sequences. An important consequence for us is that twisting a simple module by an automorphism results in another simple module. Additionally, for each $\ell \in \mathbb{Z}$ one can directly apply spectral flow twists σ^{ℓ} to the short exact sequences (3.12) and (3.13) to produce new short exact sequences.

For the rest of the chapter, we will be interested in the relaxed highest weight $L_k(\mathfrak{sl}_2)$-modules of Theorem 3.9 and their twists by the automorphisms w and σ^{ℓ} ($\ell \in \mathbb{Z}$). However we must consider a few identifications among the twisted relaxed highest weights that we have introduced up to now:

Proposition 3.14. (see [CR 2013b]. Also reported in [ACR 2018] as Remark 2.5 and Proposition 2.6) *Given $k = -2 + \frac{u}{v}$ admissible, we have the following*

[3] Recall that if M is an $L_k(\mathfrak{sl}_2)$-module, the action is given through vertex operators with coefficients in $\mathrm{End}(M)$; such coefficients must then satisfy the defining relations of $\widehat{\mathfrak{sl}_2}$ as explained in Remark A.9

isomorphisms:

$$\mathsf{w}(\mathcal{L}_r) \cong \mathcal{L}_r \ , \qquad \mathsf{w}(\mathcal{E}_{\lambda;\Delta_{r,s}}) \cong \mathcal{E}_{-\lambda;\Delta_{r,s}} \ , \qquad \mathsf{w}(\mathcal{E}^{\pm}_{r,s}) \cong \mathcal{E}^{\mp}_{r,s} \ .$$

Also, if $v > 1$ *we have the following additional isomorphism:*

$$\sigma(\mathcal{L}_r) \cong \mathcal{D}^{+}_{u-r,v-1}, \quad \sigma^{-1}(\mathcal{L}_r) \cong \mathcal{D}^{-}_{u-r,v-1} \qquad \textit{for } r \in \{1,\dots,u-1\} \ ,$$
$$\sigma^{-1}(\mathcal{D}^{+}_{r,s}) \cong \mathcal{D}^{-}_{u-r,v-1-s} \qquad \textit{for } r \in \{1,\dots,u-1\} \textit{ and } s \in \{1,\dots,v-2\} \ .$$

From the isomorphisms of 3.14 and the twisted short exact sequences, we deduce the following resolutions of atypical modules $\mathcal{L}_r$ and $\mathcal{D}_{r,s}$, respectively, in terms of typical indecomposable modules $\mathcal{E}^{+}_{r,s}$ of line (3.11):

$$\begin{aligned} &\cdots \longrightarrow \sigma^{3v-1}(\mathcal{E}^{+}_{r,v-1}) \longrightarrow \cdots \longrightarrow \sigma^{2v+2}(\mathcal{E}^{+}_{r,2}) \longrightarrow \sigma^{2v+1}(\mathcal{E}^{+}_{r,1}) \\ &\quad \longrightarrow \sigma^{2v-1}(\mathcal{E}^{+}_{u-r,v-1}) \longrightarrow \cdots \longrightarrow \sigma^{v+2}(\mathcal{E}^{+}_{u-r,2}) \longrightarrow \sigma^{v+1}(\mathcal{E}^{+}_{u-r,1}) \\ &\quad \longrightarrow \sigma^{v-1}(\mathcal{E}^{+}_{r,v-1}) \longrightarrow \cdots \longrightarrow \sigma^{2}(\mathcal{E}^{+}_{r,2}) \longrightarrow \sigma(\mathcal{E}^{+}_{r,1}) \longrightarrow \mathcal{L}_r \longrightarrow 0 \ , \end{aligned} \tag{3.15}$$

$$\begin{aligned} 0 \longrightarrow &\sigma^{v-s}(\mathcal{L}_{u-r}) \longrightarrow \sigma^{v-1-s}(\mathcal{E}^{+}_{r,v-1}) \longrightarrow \cdots \\ &\longrightarrow \sigma^{2}(\mathcal{E}^{+}_{r,s+2}) \longrightarrow \sigma(\mathcal{E}^{+}_{r,s+1}) \longrightarrow \mathcal{D}^{+}_{r,s} \longrightarrow 0 \ . \end{aligned} \tag{3.16}$$

These resolutions are originally presented in [CR 2013b], but are reported as Remark 2.14 in [ACR 2018]. Among other useful things, they will help us to express characters of the simple $L_k(\mathfrak{sl}_2)$-modules of Theorem 3.9.

For the rest of this chapter, we will be considering the following families of simple L_k-modules:

- the $\sigma^{\ell}(\mathcal{L}_r)$, for $\ell \in \mathbb{Z}$ and $r \in \{1,\dots,u-1\}$;

- if $v > 1$, the $\sigma^\ell(\mathcal{D}^+_{r,s})$, for $\ell \in \mathbb{Z}$, $r \in \{1, \dots, u-1\}$ and $s \in \{1, \dots, v-1\}$;
- if $v > 1$, the $\sigma^\ell(\mathcal{E}_{\lambda;\Delta_{r,s}})$, for $\ell \in \mathbb{Z}$, $r \in \{1, \dots, u-1\}$, $s \in \{1, \dots, v-1\}$ and $\lambda \in (\mathbb{R}/2\mathbb{Z})\backslash\{\lambda_{r,s}, \lambda_{u-r,v-s}\}$.

The fact that we are interested in the $\sigma^\ell(\mathcal{E}_{\lambda;\Delta_{r,s}})$ with real λ's has some physical sense and follows logically from the studies of [CR 2012] and [CR 2013b].

Definition 3.17. For the rest of the chapter, the category $\mathcal{A}_k$ of $L_k(\mathfrak{sl}_2)$-modules we consider is the full subcategory of $L_k(\mathfrak{sl}_2)$-modules in which the simple objects are listed above and in which the objects are subquotients of iterated tensor (fusion) products of a finite number of simple objects. This presupposes that the tensor product behaves well enough between such modules. Although such checks are very difficult for logarithmic vertex operator algebras, there is evidence to justify our interest in the category $\mathcal{A}_k$ (see [ACR 2018] for details).

Next are some important assumptions that for the rest of this chapter:

Assumption 3.18. The spectral flow twists behave well with respect to tensor products of $L_k(\mathfrak{sl}_2)$-modules in the sense that there are the following natural isomorphisms

$$\sigma^{\ell_1}(M) \otimes_{L_k(\mathfrak{sl}_2)} \sigma^{\ell_2}(N) \cong \sigma^{\ell_1+\ell_1}(M \otimes_{L_k(\mathfrak{sl}_2)} N) \ .$$

Assumption 3.19. For k admissible, the Grothendieck fusion rules of $\mathcal{A}_k$ are well defined and the fusion coefficients are computed by the Verlinde formula of [RW 2015b] and [CR 2013a].

No proof of Assumption 3.18 is known. However, Assumptions 3.18 and 3.19 are supported by a number of results. For instance, the authors of [CR 2013b] compute fusion rules for all modules of $\mathcal{A}_k$ (see Definition 3.17) in perfect agreement with what was already known for the specific cases $k = -\frac{1}{2}$ [Rid 2011] or $k = -\frac{4}{3}$ [Gab 2001].

Remark 3.20. It is worth noting recent progress on fusion for affine vertex operator algebras. The affine case is of particular interest to eventually explain Assumptions 3.18 and 3.19 introduced also in [ACR 2018]. On the Category Theory front, [CHY 2018] and [Cre 2018] establish that the category of ordinary modules for an affine vertex operator algebra has a braided tensor structure in the sense of Huang-Lepowsky-Zhang and in certain cases, it is also a fusion category. The authors in [CHY 2018] can even verify fusion rules for the category of ordinary modules. For instance, this means that the category of $L_k(\mathfrak{sl}_2)$-modules generated by the relaxed highest weight modules of Theorem 3.9 have their fusion rules as expected and stated in Chapter 3. Of course, much more has to be done in order to prove that the category Definition 3.17 also fits in this vertex tensor theory mold, but these recent advances let us believe that this will be a possibility in the future. Additionnally, D. Adamović realises the relaxed highest weight $L_k(\mathfrak{sl}_2)$-modules as modules for a rational Virasoro vertex operator algebra tensored with a lattice vertex operator algebra [Ada 2018]: such methods can possibly be extended to prove certain tensor product fusion rules among more modules. Also, let's mention Proposition 2.4 of [Li 1997] that can help to construct intertwining operators for modules whose action have been twisted by an endomorphisms of a so-called *universal enveloping algebra* of a vertex operator algebra, see [FZ 1992] for details.

A third assumption will be introduced in the next section.

3.1.2 The Parafermions Algebras $\mathsf{C}_k(\mathfrak{sl}_2)$ and $\mathsf{B}_k(\mathfrak{sl}_2)$

Let k be an admissible level for $\mathfrak{sl}_2$.

Definition 3.21. Let $H \subset L_k(\mathfrak{sl}_2)$ be the Heisenberg vertex operator algebra generated by the vertex operator fields of vectors from the Cartan Lie algebra $\mathfrak{h} = \mathbb{C}.h_0 \subset \widehat{\mathfrak{sl}_2}$. We define the vertex operator algebra $\mathsf{C}_k(\mathfrak{sl}_2)$ as the commutant of the fields of H in $L_k(\mathfrak{sl}_2)$:

$$\mathsf{C}_k(\mathfrak{sl}_2) = \mathrm{Com}\left(H, L_k(\mathfrak{sl}_2)\right) .$$

Now consider the vertex operator algebra $H \otimes \mathsf{C}_k(\mathfrak{sl}_2) \subset L_k(\mathfrak{sl}_2)$. By [CKM 2017], this is a vertex algebra extension.

The duality results of Schur-Weyl flavour developed in [CKLR 2016] combined with the tools of [CKL 2015], allow us to formulate key results reported in [ACR 2018] as follows:

Result 3.22. (Results 1.1–1.3 of [ACR 2018]) *As an $\left(H \otimes \mathsf{C}_k(\mathfrak{sl}_2)\right)$-module, the vertex operator algebra $L_k(\mathfrak{sl}_2)$ decomposes as follows:*

$$\mathrm{Res}^{L_k(\mathfrak{sl}_2)}_{H \otimes \mathsf{C}_k(\mathfrak{sl}_2)} L_k(\mathfrak{sl}_2) \cong \bigoplus_{\mu \in \mathsf{Q}} F_\mu \otimes \mathcal{C}_\mu \tag{3.23}$$

where the F_μ's are usual Fock spaces for H. The $\mathcal{C}_\mu$'s are simple currents of $\mathsf{C}_k(\mathfrak{sl}_2)$ with $C_0 \cong \mathsf{C}_k(\mathfrak{sl}_2)$ and for which

$$\mathcal{C}_\lambda \otimes_{\mathsf{C}_k(\mathfrak{sl}_2)} \mathcal{C}_\mu \cong \mathcal{C}_{\lambda+\mu} . \tag{3.24}$$

Additionnally, for $k < 0$ the decomposition (3.23) *is* multiplicity-free *in the sense that $\lambda \neq \mu$ implies $\mathcal{C}_\lambda \ncong \mathcal{C}_\mu$.*

So the restriction of $L_k(\mathfrak{sl}_2)$ to a $\left(H \otimes \mathsf{C}_k(\mathfrak{sl}_2)\right)$-module produces simple currents $\mathcal{C}_\lambda$'s of $\mathsf{C}_k(\mathfrak{sl}_2)$ with nice tensor product properties. The simple currents involved in (3.24) are of infinite order and so a simple current extension of $\mathsf{C}_k(\mathfrak{sl}_2)$ by some C_λ would have a much reduced number of simple modules. Together with the modularity, such simple current extensions would make good candidates for C_2-cofiniteness.

Let's now see how we can make sure that an extension of $\mathsf{C}_k(\mathfrak{sl}_2)$ by some simple current $\mathcal{C}_\lambda$ exists. The following argument is reported from Section 1.1 of [ACR 2018]. Given a lattice $L \in \mathbb{C} \otimes_{\mathbb{Z}} \mathsf{Q}$, we can give a natural vertex operator algebra structure to $V_L \cong \bigoplus_{\lambda \in L} F_\lambda$ if and only if V_L is $\mathbb{Z}$-graded by conformal weight [DL 1993]. Likewise, a result of [Li 2001] implies that $\bigoplus_{\lambda \in L} \mathcal{C}_\lambda$ has a natural vertex operator algebra structure if an only if it is $\mathbb{Z}$-graded by conformal weight.

For all values of admissible level k, the largest lattice L ensuring that the space $\bigoplus_{\lambda \in L} \mathcal{C}_\lambda$ has natural structure of vertex operator algebra is

$$L = -2vk\mathbb{Z} = 2w\mathbb{Z} = w\mathsf{Q} \ ,$$

as explained in Section 4.1 of [ACR 2018]. We can now define the extended parafermion algebra accordingly:

Definition 3.25. Let k be admissible. From the simple currents (3.24), the *extended parafermions* vertex operator algebra at level k is defined as

$$\mathsf{B}_k(\mathfrak{sl}_2) = \bigoplus_{\lambda \in L} \mathcal{C}_\lambda = \bigoplus_{\ell \in \mathbb{Z}} \mathcal{C}_{2w\ell} \ .$$

Note that $\mathsf{B}_k(\mathfrak{sl}_2)$ is a simple current extension of $\mathsf{C}_k(\mathfrak{sl}_2)$ of infinite order and can be understood with the framework of Chapter 2 and [CKM 2017].

Remark 3.26. For $k < 0$, the $\mathsf{B}_k(\mathfrak{sl}_2)$-modules coming from (3.1) and the $L_k(\mathfrak{sl}_2)$-module category will have weights that are bounded from below. This key property effectively allows the $\mathsf{B}_k(\mathfrak{sl}_2)$-characters to have meromorphic continuations on the upper half plane $\mathbb{H}$: otherwise, the series would not even possibly converge. In the following modularity analysis, we will thus consider k to be negative.

Remark 3.27. The same vertex operator algebra $\mathsf{B}_k(\mathfrak{sl}_2)$ also has an interpretation in terms of a commutant inside an infinite order simple current extension of $L_k(\mathfrak{sl}_2)$ built using spectral flow alone. For more details, see Section 4.1 of [ACR 2018].

Let's focus once again on the unextended parafermions $\mathsf{C}_k(\mathfrak{sl}_2)$. The following Schur-Weyl type duality result of [CKLR 2016] also has a meaning for indecomposable $L_k(\mathfrak{sl}_2)$-modules on which H acts semisimply:

Result 3.28. *If M is an indecomposable $L_k(\mathfrak{sl}_2)$-module on which H acts semisimply, then it decomposes as an $\big(H \otimes \mathsf{C}_k(\mathfrak{sl}_2)\big)$-module as follows:*

$$\operatorname{Res}^{L_k(\mathfrak{sl}_2)}_{H\otimes \mathsf{C}_k(\mathfrak{sl}_2)} M \cong \bigoplus_{\mu\in\alpha+\mathsf{Q}} F_\mu \otimes T_\mu \tag{3.29}$$

for some $\alpha \in \mathbb{C}\otimes_{\mathbb{Z}} \mathsf{Q}$ and some indecomposable $\mathsf{C}_k(\mathfrak{sl}_2)$-modules T_μ. Moreover, this decomposition is structure-preserving: if M has socle series $0 \subset N^1 \subset \cdots \subset N^{r-1} \subset M$ and we define $\mathsf{C}_k(\mathfrak{sl}_2)$-modules T^i_μ by

$$\operatorname{Res}^{L_k(\mathfrak{sl}_2)}_{H\otimes \mathsf{C}_k(\mathfrak{sl}_2)} N^i \cong \bigoplus_{\mu\in\alpha+\mathsf{Q}} F_\mu \otimes T^i_\mu, \quad \text{for } i \in \{1,\ldots,r-1\},$$

then $0 \subset T^1_\mu \subset \cdots \subset T^{r-1}_\mu \subset T_\mu$ is the socle series of T_μ for all $\mu \in \alpha+\mathsf{Q}$.

Recall that the H-actions on relaxed highest weight $L_k(\mathfrak{sl}_2)$-modules of $\mathcal{A}_k$ are semisimple as they are weight modules. Therefore, all the simple and indecomposable modules we consider in the category $\mathcal{A}_k$ of $L_k(\mathfrak{sl}_2)$-modules will produce, respectively, certain simple and indecomposable modules which have the same structures. We will then want to see which of them are induced in the category of untwisted modules for the extended parafermions algebra $\mathsf{B}_k(\mathfrak{sl}_2)$. To accomplish this, we can relate to the theory of extensions for logarithmic vertex operator algebras [CKM 2017] if we make the following assumption:

Assumption 3.30. The vertex tensor category theory of Huang-Lepowsky-Zhang [HLZ 2007] may be applied to the smallest $\mathsf{C}_k(\mathfrak{sl}_2)$-module category containing all simple and indecomposable objects produced by Result 3.28 from the simple and indecomposable $L_k(\mathfrak{sl}_2)$-modules of $\mathcal{A}_k$ (see Definition 3.17).

We will be interested in untwisted $\mathsf{B}_k(\mathfrak{sl}_2)$-modules where $\mathsf{C}_k(\mathfrak{sl}_2) \subset \mathsf{B}_k(\mathfrak{sl}_2)$ is a vertex operator algebra extension. From this point of view (see [CKM 2017] and [KJO 2002]). We recall that the untwisted $\mathsf{B}_k(\mathfrak{sl}_2)$-modules are just the ones in $\operatorname{Rep}^0 \mathsf{B}_k(\mathfrak{sl}_2)$. From the same references, we know that the induction functor $\underline{\mathcal{F}}$ from $\mathsf{C}_k(\mathfrak{sl}_2)$ to $\mathsf{B}_k(\mathfrak{sl}_2)$-modules is given by $\mathsf{B}_k(\mathfrak{sl}_2) \otimes_{\mathsf{C}_k(\mathfrak{sl}_2)} -$ and that it should constitute a tensor functor.

Remark 3.31. We now investigate when our $\mathsf{C}_k(\mathfrak{sl}_2)$-modules is induced to an untwisted $\mathsf{B}_k(\mathfrak{sl}_2)$-module. From [CKL 2015], we have that a $\mathsf{C}_k(\mathfrak{sl}_2)$-module with a one dimensional endomorphism space induces to an untwisted module if and only if it is $\mathbb{Z}$-graded. Moreover, simple $\mathsf{C}_k(\mathfrak{sl}_2)$-modules that induce to untwisted $\mathsf{B}_k(\mathfrak{sl}_2)$-modules give simple $\mathsf{B}_k(\mathfrak{sl}_2)$-modules (see Result 1.6 of [ACR 2018]).

3.2 Obtaining $\mathsf{B}_k(\mathfrak{sl}_2)$-Characters

We now fix k admissible such that $k < 0$.

With the parafermions vertex operator algebras $\mathsf{C}_k(\mathfrak{sl}_2)$ and $\mathsf{B}_k(\mathfrak{sl}_2)$ defined above and Assumptions 3.18, 3.19 and 3.30 in place, we are now set to produce a number of $\mathsf{B}_k(\mathfrak{sl}_2)$-characters from the simple $L_k(\mathfrak{sl}_2)$-modules in $\mathcal{A}_k$. Modular transformations will then be applied to these $\mathsf{B}_k(\mathfrak{sl}_2)$-characters in the subsequent section.

3.2.1 Relevant Characters of $L_k(\mathfrak{sl}_2)$ and $\mathsf{C}_k(\mathfrak{sl}_2)$

We now gather the characters of the indecomposable objects from the category $\mathcal{A}_k$ of $L_k(\mathfrak{sl}_2)$-modules. Recall from [CR 2013b] that the spectral flow twist of a $L_k(\mathfrak{sl}_2)$-action changes the effects of all the operators $\kappa, h_0, L_0 \in \widehat{\mathfrak{sl}_2}$. We will then introduce variables y and z in the characters to keep track of such effects. Thus, the $L_k(\mathfrak{sl}_2)$-character of a weight module M of level k is given by:

$$\mathrm{ch}\left[M\right](y;z;q) = \mathrm{tr}_M(y^{\kappa} z^{h_0} q^{L_0 - \frac{c}{24}}) = y^k \sum_{n\in\mathbb{Z}} (\dim M_{\lambda,n})\; z^{\lambda} q^{(n+h_M)-\frac{c}{24}}$$

where $c = \frac{3k}{2t}$ is the central charge of $L_k(\mathfrak{sl}_2)$ with the Sugawara conformal vector. The effect of twisting the $L_k(\mathfrak{sl}_2)$-action on characters is as follows:

$$\begin{aligned} \mathrm{ch}\left[\sigma^{\ell}(M)\right](y;z;q) &= \mathrm{ch}\left[M\right]\left(yz^{\ell}q^{\frac{\ell^2}{4}}; zq^{\frac{\ell}{2}}; q\right) \quad \text{for all } \ell \in \mathbb{Z} \\ \mathrm{ch}\left[\mathsf{w}(M)\right](y;z;q) &= \mathrm{ch}\left[M\right]\left(y; z^{-1}; q\right)\ . \end{aligned}$$

Let's start by characters of the typical relaxed highest weight modules of the form $\mathcal{E}_{\lambda;\Delta_{r,s}}$ for any $\lambda \in \mathbb{R}/2\mathbb{Z}$. From [CR 2013b], [Ada 2017] and [KR 2018], we have the following formula:

$$\mathrm{ch}\left[\mathcal{E}_{\lambda;\Delta_{r,s}}\right](y;z;q) = y^k \frac{z^{\lambda}\chi_{r,s}^{M(u,v)}(q)}{\eta(q)^2} \sum_{n\in\mathbb{Z}} z^{2n} \tag{3.32}$$

where $\chi_{r,s}^{M(u,v)}(q)$ is the character of a simple module for the simple Virasoro vertex operator algebra $M(u,v)$ and $\eta(q)$ is Dedekind's η function. The Virasoro vertex operator algebra $M(u,v)$ is often referred to as the *Virasoro minimal model* of central charge $1-\frac{6(v-u)^2}{uv}$. The characters $\chi_{r,s}^{M(u,v)}(q)$ are given at line (2.12) of [ACR 2018] along with the corresponding simple module's fusion rules in Section 2.3 of the same reference (see also section 8.1.2 of [DFMS 1997]).

While we could use the Kac-Wakimoto character formula of Chapter 1 to obtain the characters of the $L_k(\mathfrak{sl}_2)$-modules of the form $\mathcal{L}_r$ and $\mathcal{D}^+_{r,s}$ (in case $v>1$) for appropriate $r\in\{1,\dots,u-1\}$ and $s\in\{1,\dots,v-1\}$, we prefer using Proposition 3.14 and the resolutions (3.15) and (3.16) to do so. The reason for this preference is that both the simple characters $\chi_{r,s}^{M(u,v)}(q)$ and $\eta(q)$ appearing in (3.32) have well known modular behaviours [CR 2013b]. In this way, we can obtain all the simple characters of $\mathcal{A}_k$.

Proposition 3.33. (Proposition 2.10 of [ACR 2018]) *Let $k=-2+\frac{u}{v}$ be an admissible level and assume that $v>1$. Then, we have the following character formulae:*

$$\operatorname{ch}\left[\sigma^\ell(\mathcal{E}_{\lambda;\Delta_{r,s}})\right]=\frac{y^k z^{\ell k} q^{\ell^2 k/4}\chi_{(r,s)}^{\mathsf{M}(u,v)}(q)}{\eta(q)^2}\sum_{\mu\in\lambda+\mathsf{Q}} z^\mu q^{\ell\mu/2}\,,$$

$$\operatorname{ch}\left[\sigma^\ell(\mathcal{E}^+_{r,s})\right]=\frac{y^k z^{\ell k} q^{\ell^2 k/4}\chi_{(r,s)}^{\mathsf{M}(u,v)}(q)}{\eta(q)^2}\sum_{\mu\in\lambda_{r,s}+\mathsf{Q}} z^\mu q^{\ell\mu/2}\,,$$

$$\operatorname{ch}\left[\sigma^\ell(\mathcal{L}_r)\right]=\sum_{s'=1}^{v-1}(-1)^{s'-1}\sum_{m=0}^{\infty}\Big(\operatorname{ch}\left[\sigma^{2mv+s'+\ell}(\mathcal{E}^+_{r,s'})\right]-\operatorname{ch}\left[\sigma^{2(m+1)v-s'+\ell}(\mathcal{E}^+_{u-r,v-s'})\right]\Big)\,,$$

$$\operatorname{ch}\left[\sigma^\ell(\mathcal{D}^+_{r,s})\right]=\sum_{s'=s+1}^{v-1}(-1)^{s'-s-1}\operatorname{ch}\left[\sigma^{s'-s+\ell}(\mathcal{E}^+_{r,s'})\right]+(-1)^{v-1-s}\operatorname{ch}\left[\sigma^{v-s+\ell}(\mathcal{L}_{u-r})\right]\,.$$

If $k<0$, then the infinite sum in $\operatorname{ch}\left[\sigma^\ell(\mathcal{L}_r)\right]$ converges in the sense of formal power series in z, meaning that the coefficient of each power of z converges to a *meromorphic function of q for $|q| < 1$.*

We can now proceed to using the Schur-Weyl duality stated above as Result 3.28 to obtain the corresponding simple $\mathsf{C}_k(\mathfrak{sl}_2)$-characters. Here is a procedure that allows us to compute pertinent $\mathsf{C}_k(\mathfrak{sl}_2)$-characters from corresponding $L_k(\mathfrak{sl}_2)$-characters:

1. the characters of a module M that is both a module for $H \otimes \mathsf{C}_k(\mathfrak{sl}_2)$ and $L_k(\mathfrak{sl}_2)$ coincide since the operators κ, h_0 and L_0 of these two vertex operator algebras coincide; in fact we have

$$\mathrm{tr}_M \left(y^{\kappa} z^{h_0} q^{L_0^{L_k(\mathfrak{sl}_2)} - \frac{c}{24}} \right) = \mathrm{tr}_M \left(y^{\kappa} z^{h_0} q^{L_0^H - \frac{c^H}{24}} \cdot q^{L_0^{\mathsf{C}_k(\mathfrak{sl}_2)} - \frac{c^{\mathsf{C}_k(\mathfrak{sl}_2)}}{24}} \right) \ ;$$

2. since h_0, L_0^H and κ commute with $L_0^{\mathsf{C}_k(\mathfrak{sl}_2)}$, we have

$$\mathrm{ch}\left[F_\lambda \otimes N\right](y;z;q) = \mathrm{ch}\left[F_\lambda\right](y;z;q) \cdot \mathrm{ch}\left[N\right](q) \ ,$$

for any $\mathsf{C}_k(\mathfrak{sl}_2)$-module N with finite dimensional weight spaces[4];

3. the character of a simple H-module (a Fock space) is given by $\mathrm{ch}\left[F_\lambda\right](y;z;q) = y^k \frac{z^\lambda q^{-\frac{\lambda^2}{4k}}}{\eta(q)}$.

Since Proposition 3.33 gives the simple characters of $\mathcal{A}_k$ in terms of the typical modules of the form $\mathrm{ch}\left[\mathcal{E}_{\lambda;\Delta_{r,s}}\right]$, Proposition 3.1 of [ACR 2018] establishes the existence of corresponding $\mathsf{C}_k(\mathfrak{sl}_2)$-modules denoted by $\mathcal{C}^{\mathcal{E}}_{\mu;r,s}$ with $\mu \in \lambda + \mathsf{Q}$ and whose characters are given by

$$\mathrm{ch}\left[\mathcal{C}^{\mathcal{E}}_{\lambda;r,s}\right](q) = \mathrm{ch}\left[\mathcal{C}^{\pm}_{\mu;r,s}\right](q) = \frac{\chi^{\mathsf{M}(u,v)}_{(r,s)}(q)}{\eta(q)} q^{-\mu^2/4k} \ . \tag{3.34}$$

[4]Observe that the character of N does not depend on z anymore.

To be exact, these atypical $\mathsf{C}_k(\mathfrak{sl}_2)$-modules appear as follows:

$$\mathrm{Res}^{L_k\mathfrak{sl}_2}_{H\otimes\mathsf{C}_k(\mathfrak{sl}_2)}\,\sigma^\ell(\mathcal{E}_{\lambda;\Delta_{r,s}}) \cong \bigoplus_{\mu\in\lambda+\mathsf{Q}} F_{\mu+\ell k}\otimes\mathcal{C}^{\mathcal{E}}_{\mu;r,s}$$

for any choice of $\lambda\in\mathbb{R}/2\mathbb{Z}$. When $\lambda\notin\{\lambda_{r,s},\lambda_{u-r,v-s}\}$, the modules $\mathcal{C}^{\mathcal{E}}_{\mu;r,s}$ are simple. Else, $\lambda\in\{\lambda_{r,s},\lambda_{u-r,v-s}\}$ and the $\mathcal{C}^{\pm}_{\mu;r,s}$ are indecomposables of Loewy length 2 just as the $\mathcal{E}^{\pm}_{r,s}$ are of Loewy length 2 for $L_k(\mathfrak{sl}_2)$.

Combined with Proposition 3.1 of [ACR 2018], Proposition 3.33 above allows us to obtain the simple $\mathsf{C}_k(\mathfrak{sl}_2)$-characters of all corresponding types:

Proposition 3.35. (Proposition 3.3 of [ACR 2018]) *The atypical irreducible $L_k(\mathfrak{sl}_2)$-modules decompose into $\big(H\otimes\mathsf{C}_k(\mathfrak{sl}_2)\big)$-modules as*

$$\mathrm{Res}^{L_k\mathfrak{sl}_2}_{H\otimes\mathsf{C}_k(\mathfrak{sl}_2)}\,\sigma^\ell(\mathcal{L}_r) \cong \bigoplus_{\mu\in\lambda_{r,0}+\mathsf{Q}} F_{\mu+\ell k}\otimes\mathcal{C}^{\mathcal{L}}_{\mu;r}\ ,$$
$$\mathrm{Res}^{L_k\mathfrak{sl}_2}_{H\otimes\mathsf{C}_k(\mathfrak{sl}_2)}\,\sigma^\ell(\mathcal{D}^+_{r,s}) \cong \bigoplus_{\mu\in\lambda_{r,s}+\mathsf{Q}} F_{\mu+\ell k}\otimes\mathcal{C}^{\mathcal{D}}_{\mu;r,s}$$

where the $\mathcal{C}^{\mathcal{L}}_{\mu;r}$ and $\mathcal{C}^{\mathcal{D}}_{\mu;r,s}$ are irreducible highest weight C_k-modules characterised by the following resolutions:

$$\begin{aligned}\cdots &\longrightarrow \mathcal{C}^+_{\mu-(3v-1)k;r,v-1}\longrightarrow\cdots\longrightarrow\mathcal{C}^+_{\mu-(2v+2)k;r,2}\longrightarrow\mathcal{C}^+_{\mu-(2v+1)k;r,1}\\ &\longrightarrow\mathcal{C}^+_{\mu-(2v-1)k;u-r,v-1}\longrightarrow\cdots\longrightarrow\mathcal{C}^+_{\mu-(v+2)k;u-r,2}\longrightarrow\mathcal{C}^+_{\mu-(v+1)k;u-r,1}\\ &\longrightarrow\mathcal{C}^+_{\mu-(v-1)k;r,v-1}\longrightarrow\cdots\longrightarrow\mathcal{C}^+_{\mu-2k;r,2}\longrightarrow\mathcal{C}^+_{\mu-k;r,1}\longrightarrow\mathcal{C}^{\mathcal{L}}_{\mu;r}\longrightarrow 0\ ,\end{aligned}$$

$$\begin{aligned}0&\longrightarrow\mathcal{C}^{\mathcal{L}}_{\mu-(v-s)k;u-r}\longrightarrow\mathcal{C}^+_{\mu-(v-1-s)k;r,v-1}\longrightarrow\cdots\\ &\longrightarrow\mathcal{C}^+_{\mu-2k;r,s+2}\longrightarrow\mathcal{C}^+_{\mu-k;r,s+1}\longrightarrow\mathcal{C}^{\mathcal{D}}_{\mu;r,s}\longrightarrow 0\ .\end{aligned}$$

Their characters are given by:

$$\mathrm{ch}\left[\mathcal{C}^{\mathcal{L}}_{\mu;r}\right](q) = \sum_{s=1}^{v-1}(-1)^{s-1}\frac{\chi^{\mathsf{M}(u,v)}_{(r,s)}(q)}{\eta(q)}\sum_{m=0}^{\infty}\Big(q^{-(\mu-sk+2wm)^2/4k} - q^{-(\mu+sk+2w(m+1))^2/4k}\Big)\,,$$
$$\mathrm{ch}\left[\mathcal{C}^{\mathcal{D}}_{\mu;r,s}\right](q) = \sum_{s'=s+1}^{v-1}(-1)^{s'-s-1}\frac{\chi^{\mathsf{M}(u,v)}_{(r,s')}(q)}{\eta(q)}q^{-(\mu-(s'-s)k)^2/4k} + (-1)^{v-1-s}\,\mathrm{ch}\left[\mathcal{C}^{\mathcal{L}}_{\mu-(v-s)k;u-r}\right]\,.$$

We now have the characters of infinitely many simple and indecomposable $\mathsf{C}_k(\mathfrak{sl}_2)$-modules of the form $\mathrm{ch}\left[\mathcal{C}^{\mathcal{L}}_{\mu;r}\right]$, $\mathrm{ch}\left[\mathcal{C}^{\mathcal{D}}_{\mu;r,s}\right]$ and $\mathrm{ch}\left[\mathcal{C}^{\mathcal{E}}_{\lambda;r,s}\right]$. This is what we were looking for. Many identifications and symmetries among those characters are noted in several remarks in [ACR 2018]. The conformal dimensions of these $\mathsf{C}_k(\mathfrak{sl}_2)$-modules are given by Proposition 3.8 of the same reference.

Fusion rules among the $\mathsf{C}_k(\mathfrak{sl}_2)$-modules of the form $\mathrm{ch}\left[\mathcal{C}^{\mathcal{L}}_{\mu;r}\right]$, $\mathrm{ch}\left[\mathcal{C}^{\mathcal{D}}_{\mu;r,s}\right]$ and $\mathrm{ch}\left[\mathcal{C}^{\mathcal{E}}_{\lambda;r,s}\right]$ are specified in Section 3.2 of [ACR 2018].

Section 3.3 of [ACR 2018] then identifies the simple vertex operator algebra $\mathsf{C}_k(\mathfrak{sl}_2)$ at a few specific levels $k \in \mathbb{Q}$ to other simple logarithmic vertex operator algebras studied in the literature.

3.2.2 Characters of $\mathsf{B}_k(\mathfrak{sl}_2)$

We will now see what type of untwisted simple $\mathsf{B}_k(\mathfrak{sl}_2)$-characters ($k < 0$ admissible) we obtain by lifting the appropriate $\mathsf{C}_k(\mathfrak{sl}_2)$-modules of the form $\mathcal{C}^{\mathcal{L}}_{\mu;r}$, $\mathcal{C}^{\mathcal{D}}_{\mu;r,s}$ and $\mathcal{C}^{\mathcal{E}}_{\lambda;r,s}$ determined above. Let's start by relating the simple $\mathsf{C}_k(\mathfrak{sl}_2)$-modules of Section 3.2.1 to those needed in the construction of $\mathsf{B}_k(\mathfrak{sl}_2)$. Consider the simple $\mathsf{C}_k(\mathfrak{sl}_2)$-modules $\mathcal{C}_\mu$ of (3.23) and of Definition 3.25. Then we have that $\mathcal{C}_\mu \cong \mathcal{C}^{\mathcal{L}}_{\mu;1}$ and $\mathcal{C}_0 \cong \mathsf{C}_k(\mathfrak{sl}_2) \cong \mathcal{C}^{\mathcal{L}}_{0;1}$ for all $\mu \in \mathsf{Q} = 2\mathbb{Z}$. In particular, we can identify the extended parafermion vertex operator algebra:

$$\mathsf{B}_k(\mathfrak{sl}_2) \cong \bigoplus_{\lambda \in L} \mathcal{C}^{\mathcal{L}}_{\lambda;1} = \bigoplus_{\ell \in \mathbb{Z}} \mathcal{C}^{\mathcal{L}}_{2w\ell;1} \tag{3.36}$$

where $w = -kv > 0$ (note that w is the absolute value of the numerator of k).

Recall that we are interested in untwisted $\mathsf{B}_k(\mathfrak{sl}_2)$-modules. By the theory of vertex operator algebra extensions of [CKM 2017], the induction from $\mathsf{C}_k(\mathfrak{sl}_2)$-modules to $\mathsf{B}_k(\mathfrak{sl}_2)$-modules is given by the tensor product with $\mathsf{B}_k(\mathfrak{sl}_2) \otimes_{\mathsf{C}_k(\mathfrak{sl}_2)} -$. Fusion rules for $\mathsf{C}_k(\mathfrak{sl}_2)$ could be determined in Section 3.2 of [ACR 2018] and we know that the modules $\mathcal{C}^{\mathcal{L}}_{2w\ell;1}$ are simple currents of infinite order (Remark 3.10 of [ACR 2018]).

Remark 3.31 points out that the simple $\mathsf{C}_k(\mathfrak{sl}_2)$-modules lift to untwisted $\mathsf{B}_{\mathbb{Z}}(\mathfrak{sl}_2)$-modules if and only if they are $\mathbb{Z}$-graded (see Result 1.6 of [ACR 2018]). For the indecomposables $\mathcal{C}^{\pm}_{\mu;r,s}$, an argument shows that they induce in the category of untwisted $\mathsf{B}_k(\mathfrak{sl}_2)$-modules whenever $\mu \in L'$, the dual lattice of L which is $\frac{1}{v}\mathbb{Z}$ here. We deduce the following definitive answer:

Proposition 3.37. (Proposition 4.3 of [ACR 2018]) *The typical $\mathsf{C}_k(\mathfrak{sl}_2)$-modules $\mathcal{C}^{\mathcal{E}}_{\mu;r,s}$ lift to irreducible highest weight $\mathsf{B}_k(\mathfrak{sl}_2)$-modules (denoted by $\mathcal{B}^{\mathcal{E}}_{\mu;r,s}$), only if $\mu \in L'$. The atypical irreducible $\mathsf{C}_k(\mathfrak{sl}_2)$-modules $\mathcal{C}^{\mathcal{L}}_{\mu;r}$ and $\mathcal{C}^{\mathcal{D}}_{\mu;r,s}$ always lift to irreducible highest weight $\mathsf{B}_k(\mathfrak{sl}_2)$-modules (denoted by $\mathcal{B}^{\mathcal{L}}_{\mu;r}$ and $\mathcal{B}^{\mathcal{D}}_{\mu;r,s}$, respectively). Likewise, the atypical standard $\mathsf{C}_k(\mathfrak{sl}_2)$-modules $\mathcal{C}^{\pm}_{\mu;r,s}$ always lift to length 2 indecomposable $\mathsf{B}_k(\mathfrak{sl}_2)$-modules, denoted by $\mathcal{B}^{\pm}_{\mu;r,s}$. The corresponding decompositions as $\mathsf{C}_k(\mathfrak{sl}_2)$-modules take the unified form*

$$\mathrm{Res}^{L_k\mathfrak{sl}_2}_{H \otimes \mathsf{C}_k(\mathfrak{sl}_2)} \mathcal{B}^{\bullet}_{\mu;\star} \cong \bigoplus_{\lambda \in \mu + \mathsf{L}} \mathcal{C}^{\bullet}_{\lambda;\star} \tag{3.38}$$

for appropriate $\bullet$ *and* $\star$*. We have* $\mathcal{B}^{\bullet}_{\lambda;\star} \cong \mathcal{B}^{\bullet}_{\mu;\star}$ *when* $\lambda = \mu \bmod L$*.*

Remark 3.39. The allowed indices deduced from Proposition 3.35 for the three types of $\mathsf{B}_k(\mathfrak{sl}_2)$-modules of Proposition 3.37 are summarised in Table 3.1

module type	$\mathcal{B}^{\mathcal{E}}_{\mu;r,s}$	$\mathcal{B}^{\mathcal{L}}_{\mu;r}$	$\mathcal{B}^{\mathcal{D}}_{\mu;r,s}$
allowed $\mu \bmod L$	$\frac{1}{v}\mathbb{Z} = L'$	$\lambda_{r,0} + \mathsf{Q}$	$\lambda_{r,s} + \mathsf{Q}$

Table 3.1: Allowed indices for three types of untwisted $\mathsf{B}_k(\mathfrak{sl}_2)$-modules. Note that $\mathsf{Q} = 2\mathbb{Z}$, $r \in \{1, \dots, u-1\}$ and $s \in \{1, \dots, v-1\}$ for $v > 1$. Also, recall from (3.5) and (3.8) that $\lambda_{r,s} = (r-1) - st$ with $\lambda_{u-r,v-s} \equiv -\lambda_{r,s} \bmod \mathsf{Q}$

From Remark 3.39, we see that the number of isomorphism classes of the $\mathsf{B}_k(\mathfrak{sl}_2)$-modules given in Proposition 3.37 is finite. While there might exist other untwisted $\mathsf{B}_k(\mathfrak{sl}_2)$-modules, we believe that we have them all when $\mathsf{B}_k(\mathfrak{sl}_2)$ is C_2-cofinite. This is the conjecture we will formulate at the end of this chapter.

Let's move on to computing the characters of the untwisted $\mathsf{B}_k(\mathfrak{sl}_2)$-modules from Proposition 3.37. According to (3.38), the characters of the $\mathcal{B}^{\mathcal{E}}_{\mu;r,s}$, $\mathcal{B}^{\mathcal{L}}_{\mu;r}$ and $\mathcal{B}^{\mathcal{D}}_{\mu;r,s}$ are given by infinite sums of $\mathsf{C}_k(\mathfrak{sl}_2)$-characters of type $\mathcal{E}, \mathcal{L}, \mathcal{D}$, respectively. Omitting computational technicalities reported in details in Section 4 of [ACR 2018], the resulting $\mathsf{B}_k(\mathfrak{sl}_2)$-characters can be expressed as follows:

$$\mathrm{ch}\left[\mathcal{B}^{\mathcal{E}}_{\mu;r,s}\right](q) = \mathrm{ch}\left[\mathcal{B}^{\pm}_{\mu;r,s}\right](q) = \frac{\chi^{\mathsf{M}(u,v)}_{(r,s)}(q)}{\eta(q)} \sum_{\lambda\in\mu+\mathsf{L}} q^{-\lambda^2/4k} , \tag{3.40}$$

$$\mathrm{ch}\left[\mathcal{B}^{\mathcal{L}}_{\mu;r}\right](q) = \sum_{s=1}^{v-1}(-1)^{s-1}\frac{\chi^{\mathsf{M}(u,v)}_{(r,s)}(q)}{\eta(q)} \cdot \sum_{\lambda\in\mu+\mathsf{L}}\sum_{m=0}^{\infty}\left(q^{-(\lambda+2mw-sk)^2/4k} - q^{-(\lambda+2(m+1)w+sk)^2/4k}\right) , \tag{3.41}$$

$$\mathrm{ch}\left[\mathcal{B}^{\mathcal{D}}_{\mu;r,s}\right](q) = \sum_{s'=s+1}^{v-1}(-1)^{s'-s-1}\,\mathrm{ch}\left[\mathcal{B}^{\mathcal{E}}_{\mu-(s'-s)k;r,s'}\right](q) + (-1)^{v-1-s}\,\mathrm{ch}\left[\mathcal{B}^{\mathcal{L}}_{\mu-(v-s)k;u-r}\right](q) \ . \tag{3.42}$$

where the indices are as of Table 3.1.

3.3 Modularity Behaviour of the $\mathsf{B}_k(\mathfrak{sl}_2)$-Characters

In this last section, we explicitly describe the effect of the modular transformations on the simple $\mathsf{B}_k(\mathfrak{sl}_2)$-characters (3.40), (3.41) and (3.42) where $k<0$ is admissible. We will show that their linear span defines a finite dimensional vector-valued modular form.

In order to do so, we should first recall that characters $\mathrm{ch}[M](q) = \mathrm{ch}[M](\tau)$ will be interpreted as complex-valued functions supported on a suitable open set of $\mathbb{H}$, where $q : \tau \longmapsto e^{2\pi i\tau}$ ($\tau \in \mathbb{H}$). The modular group $PSL_2(\mathbb{Z})$ acts on $\mathbb{H}$ via MÃűbius transformations

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : \tau \longmapsto \frac{a\tau+b}{c\tau+d} .$$

So we will let $PSL_2(\mathbb{Z})$ act on characters. As the two modular transformations $S : \tau \mapsto -\frac{1}{\tau}$ and $T : \tau \mapsto \tau + 1$ generate the action of $PSL_2(\mathbb{Z})$, it will be sufficient to consider their effects alone.

The modular transformation T is quite easy to describe given the definition of the character of a $\mathbb{N}$-graded module M with finite dimensional weight spaces:

$$\begin{aligned}
\operatorname{ch}[M](\tau+1) &= \sum_{n\in\mathbb{Z}} (\dim M_n) e^{2\pi i(\tau+1)(n+h_M-\frac{c}{24})} \\
&= \sum_{n\in\mathbb{Z}} (\dim M_n) e^{2\pi i(\tau)(n+h_M-\frac{c}{24})} \cdot e^{2\pi i(n+h_M-\frac{c}{24})} \\
&= e^{2\pi i(h_M-\frac{c}{24})} \sum_{n\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} (\dim M_n) e^{2\pi i(\tau)(n+h_M-\frac{c}{24})} \\
&= e^{2\pi i(h_M-\frac{c}{24})} \operatorname{ch}[M](\tau) \qquad (3.43)
\end{aligned}$$

where h_M is the conformal dimension of M and c is the central charge of the vertex operator algebra. Note that in our case the conformal vector of $\mathsf{B}_k(\mathfrak{sl}_2)$ coincides with that of $\mathsf{C}_k(\mathfrak{sl}_2)$ and so their central charges are equal. Their conformal dimensions are also equal modulo $\mathbb{Z}$ by Proposition 3.8 of [ACKR]. Therefore (3.43) already provides a description of the effect of the modular generator T on the $\mathsf{B}_k(\mathfrak{sl}_2)$-characters (3.40), (3.41) and (3.42).

It is much more difficult, however, to describe the effect of the modular transformation S on characters. In Section 3.3.1, we will decompose the $\mathsf{B}_k(\mathfrak{sl}_2)$-characters (3.40), (3.41) and (3.42) in terms of functions with known modular properties. In Section 3.3.2, we will then compute modularity coefficients and gather results and conclusions on the modularity behaviour of the $\mathsf{B}_k(\mathfrak{sl}_2)$-characters.

3.3.1 Decompositions of $\mathsf{B}_k(\mathfrak{sl}_2)$-Characters

Note that the simple $\mathsf{B}_k(\mathfrak{sl}_2)$-characters (3.42) are expressed in terms of those of (3.40) and (3.41). Therefore, it will suffice to analyse the effect of the modular generator S on the characters $\operatorname{ch}\left[\mathcal{B}^{\mathcal{E}}_{\mu;r,s}\right]$ with $\mu \in L' = \frac{1}{v}\mathbb{Z}$ and $\operatorname{ch}\left[\mathcal{B}^{\mathcal{L}}_{\lambda;r}\right]$ with $\lambda \in \lambda_r + \mathsf{Q}$.

Let's first rewrite characters of type $\mathcal{B}^{\mathcal{E}}_{\mu;r,s}$ ($\mu \in L'$) in a more effective way. We first fix any $\mu \in L'$. The sum supported on L that appears in the characters of $\mathcal{B}^{\mathcal{E}}_{\mu;r,s}$ given in (3.40) can be interpreted as Jacobi-theta functions[5] of the lattice L; such functions will be denoted by

$$\vartheta_{\mu+L}(z;q) = \sum_{\lambda \in \mu+\mathsf{L}} z^{\lambda} q^{-\lambda^2/4k} \,, \tag{3.44}$$

so that

$$\operatorname{ch}\left[\mathcal{B}^{\mathcal{E}}_{\mu;r,s}\right](q) = \frac{\chi^{\mathsf{M}(u,v)}_{(r,s)}(q)}{\eta(q)} \vartheta_{\mu+L}(q) \,. \tag{3.45}$$

Likewise, we will rewrite characters of type $\operatorname{ch}\left[\mathcal{B}^{\mathcal{L}}_{\lambda;r}\right]$ ($\lambda \in \lambda_r + \mathsf{Q}$) in a better way than (3.41). Recall that

$$\operatorname{ch}\left[\mathcal{B}^{\mathcal{L}}_{\mu;r}\right](q) = \sum_{s=1}^{v-1} (-1)^{s-1} \frac{\chi^{\mathsf{M}(u,v)}_{(r,s)}(q)}{\eta(q)} \sum_{\lambda \in \mu+\mathsf{L}} \sum_{m=0}^{\infty} \Big(q^{-(\lambda+2mw-sk)^2/4k} - q^{-(\lambda+2(m+1)w+sk)^2/4k} \Big) \,.$$

As we already know about the modular properties of $\eta(q)$ and $\chi^{\mathsf{M}(u,v)}_{(r,s)}(q)$ (see [DFMS 1997] for the latter), we should focus on the double sum of the previous

[5]The function $\vartheta_{\mu+L}(q) = \vartheta_{\mu+L}(1;q)$ is invariant under translations by L and $\mu \mapsto -\mu$. These properties have to do with affine Weyl group associated to $\mathfrak{sl}_2$.

expression. For this, we introduce the following notation[6]:

$$\vartheta'_{\mu+L}(z;q) = -\frac{z}{2w} \cdot \frac{\partial}{\partial z}\big(\vartheta_{\mu+L}(z;q)\big) = -\frac{1}{2w} \sum_{\lambda\in\mu+\mathsf{L}} \lambda z^{\lambda} q^{-\lambda^2/4k} \,, \tag{3.46}$$

$$A_{\lambda}(q) = \sum_{\ell\in\mathbb{Z}} \sum_{m=0}^{\infty} \Big(q^{-(\lambda-2w(\ell+m))^2/4k} - q^{-(\lambda+2w(\ell+m+1))^2/4k} \Big) \,. \tag{3.47}$$

Note that $A_{\lambda}(q)$ is not absolutely convergent. Manipulating it with care, we show that it can be rewritten in terms of the lattice Jacobi-theta functions of lines (3.44) and (3.46):

Proposition 3.48. (Lemma 4.10 of [ACR 2018]) *For any $\lambda \in L'$, we have*

$$A_{\lambda}(q) = 2\vartheta'_{\lambda+L}(q) + \left(1 + \frac{\lambda}{w}\right)\vartheta_{\lambda+L}(q) \,.$$

With this result at hand, we have what it takes to re-write $\mathrm{ch}\left[\mathcal{B}^{\mathcal{L}}_{\mu;r}\right]$ in terms of functions with known modular behaviour. We note that this character is invariant under the transformation $\mu \mapsto -\mu$ and so we prefer to use the more symmetric form

$$\mathrm{ch}\left[\mathcal{B}^{\mathcal{L}}_{\mu;r}\right](q) = \frac{1}{2}\left(\mathrm{ch}\left[\mathcal{B}^{\mathcal{L}}_{\mu;r}\right](q) + \mathrm{ch}\left[\mathcal{B}^{\mathcal{L}}_{-\mu;r}\right](q)\right) \,,$$

in the what follows. In this way, we obtain a useful expression:

Proposition 3.49. (Proposition 4.11 of [ACR 2018]) *We have*

$$\begin{aligned}\mathrm{ch}\left[\mathcal{B}^{\mathcal{L}}_{\mu;r}\right](q) = \sum_{s=1}^{v-1} (-1)^{s-1} \frac{\chi^{\mathsf{M}(u,v)}_{(r,s)}(q)}{\eta(q)} \Bigg(& \vartheta'_{\mu+sk+L}(q) - \vartheta'_{\mu-sk+L}(q) \\ & + \frac{\mu-(v-s)k}{2w}\vartheta_{\mu+sk+L}(q) - \frac{\mu+(v-s)k}{2w}\vartheta_{\mu-sk+L}(q) \Bigg)\end{aligned} \tag{3.50}$$

[6]The function $\vartheta'_{\mu+L}(q) = \vartheta'_{\mu+L}(1;q)$ is invariant under translations by L and antisymmetric with respect to the transformation $\mu \mapsto -\mu$.

for all $r \in \{1, \ldots, u-1\}$ *and* $\mu \equiv \lambda_{r,0} \bmod \mathsf{Q}$.

In the expression (3.50), we can recognise instances of simpler characters $\mathrm{ch}\left[\mathcal{B}^{\mathcal{E}}_{\mu;r,s}\right]$ of line (3.45). We can also rename the "leftover terms" of $\mathrm{ch}\left[\mathcal{B}^{\mathcal{L}}_{\mu;r}\right]$ as

$$\Gamma_{\mu;r}(q) = \sum_{s=1}^{v-1}(-1)^{s-1}\frac{\chi^{\mathsf{M}(u,v)}_{(r,s)}(q)}{\eta(q)}\Big(\vartheta'_{\mu+sk+\mathsf{L}}(q) - \vartheta'_{\mu-sk+\mathsf{L}}(q)\Big) \tag{3.51}$$

for $\mu \equiv \lambda_{r,0} \bmod \mathsf{Q}$.

Remark 3.52. The effect of the modular transformation S on $\mathrm{ch}\left[\mathcal{B}^{\mathcal{L}}_{\mu;r}\right]$ is completely determined by its effects on $\Gamma_{\mu;r}$ and $\mathrm{ch}\left[\mathcal{B}^{\mathcal{E}}_{\mu;r,s}\right]$.

Before closing the current section, we briefly discuss linear relations among the different expressions $\Gamma_{\mu;r}$. The Kac table (see Table) also gives symmetries of the Virasoro characters $\chi^{\mathsf{M}(u,v)}_{(r,s)}(q)$ In addition, we consider the Weyl symmetries of

$$\mathrm{Kac}(u,v) = \tfrac{\{1,\ldots,u-1\}\times\{1,\ldots,v-1\}}{(r,s)\sim(u-r,v-s)}$$

Table 3.2: Kac's table of symmetries for the Virasoro characters $\chi^{\mathsf{M}(u,v)}_{(r,s)}(q)$.

$\vartheta'_{\mu+L}$ under translations by L or reflections $\mu \mapsto -\mu$ to deduce the following result:

Lemma 3.53. (Lemma 4.13 of [ACR 2018]) *For* $r \in \{1, \ldots, u-1\}$ *and* $\mu \equiv \lambda_{r,0} \bmod \mathsf{Q}$*, we have the following identities:*

$$\Gamma_{\mu;r} = \Gamma_{\mu+2w;r}\ , \qquad \Gamma_{\mu;r} = \Gamma_{-\mu;r}\ ,$$
$$\Gamma_{\mu;r} = (-1)^{v-1}\Gamma_{w+\mu;u-r}\ , \qquad \Gamma_{\mu;r} = (-1)^{v-1}\Gamma_{w-\mu;u-r}\ .$$

Setting $V_k = \mathrm{Span}_{\mathbb{C}}\{\Gamma_{\mu;r} \mid \mu \equiv r-1 \bmod 2\mathbb{Z}\}$, the above identities allow us to find a nicely reduced generating set (perhaps even a base) for V_k:

Proposition 3.54. (Proposition 4.14 of [ACR 2018]) *The vector space V_k is generated by the set $\mathbf{Gen}_k$ given by the elements $\Gamma_{\mu;r}$ with indices from Table 3.3 subject to $\mu \equiv \lambda_{r,0} \equiv r - 1 \bmod \mathsf{Q}$.*

	μ	r
u odd	$\{0, 1, \ldots, w\}$	$\left\{1, 2, \ldots, \frac{1}{2}(u-1)\right\}$
u even	$\{0, 1, \ldots, w\}$	$\left\{1, 2, \ldots, \frac{1}{2}u - 1\right\}$
	$\left\{0, 1, \ldots, \frac{w}{2}\right\}$	$\left\{\frac{1}{2}u\right\}$

.

Table 3.3: Indices of the elements $\Gamma_{\mu;r}$ generating the set $\mathbf{Gen}_k$.

In particular, we have

$$\dim_{\mathbb{C}} V_k \leq \begin{cases} \frac{1}{4}(u-1)(w+1) & \textit{if } u \textit{ is odd,} \\ \frac{1}{4}uw - \frac{1}{2}(v-1-u) & \textit{if } u \textit{ is even.} \end{cases}$$

We are now ready to compute the effects of the modular transformation S on functions of the form $\mathrm{ch}\left[\mathcal{B}^{\mathcal{E}}_{\mu;r,s}\right](q)$ and $\Gamma_{\mu;r}(q)$ with appropriate indices.

3.3.2 Modularity Behaviour of Characters

Recall from (10.12) of [DFMS 1997] that Dedekind's η-function obeys the following S-transformation rule

$$\eta\left(-\frac{1}{\tau}\right) = \sqrt{-i\tau}\,\eta(\tau)\,. \tag{3.55}$$

We now proceed to describing the behaviour of the characters $\mathrm{ch}\left[\mathcal{B}^{\mathcal{E}}_{\mu;r,s}\right]$ for $\mu \in L'$ and of $\mathrm{ch}\left[\mathcal{B}^{\mathcal{L}}_{\lambda;r}\right]$ where $\lambda \in \lambda_r + \mathsf{Q}$ under the S-transformation.

Let's start with characters of type $\mathrm{ch}\left[\mathcal{B}^{\mathcal{E}}_{\frac{m}{v};r,s}\right]$ for some $m \in \{0,1,\ldots,p\}$. The Jacobi-theta functions $\vartheta_{\mu+L}(z;q)$ for $\mu \in L'$ of line (3.44) have well known behaviour under the modular S-transformation (see (4.23) and (4.24) of [ACKR] for instance). We then set $p = \frac{1}{2}|L'/L|$ and fix $\frac{m}{v}$ for some $m \in \{0,1,\ldots,p\}$. The following holds:

$$\vartheta_{\frac{m}{v}+L}\left(-\frac{1}{\tau}\right) = \sqrt{-i\tau}\sum_{\ell=0}^{p} \mathrm{S}^{\vartheta}_{m\ell}\,\vartheta_{\ell/v+L}(\tau) \tag{3.56}$$

where

$$\mathrm{S}^{\vartheta}_{m\ell} = \begin{cases} \sqrt{\frac{1}{2p}}\cos\left(\frac{\pi\ell m}{p}\right) & \text{if } \ell \in p\mathbb{Z}, \\ \sqrt{\frac{2}{p}}\cos\left(\frac{\pi\ell m}{p}\right) & \text{otherwise.} \end{cases} \tag{3.57}$$

The S-transformation rules of the minimal model characters $\chi^{\mathsf{M}(u,v)}_{(r,s)}(q)$ are

$$\chi^{\mathsf{M}(u,v)}_{(r,s)}\left(-\frac{1}{\tau}\right) = \sum_{(r',s')} \mathrm{S}^{\mathsf{M}(u,v)}_{(r,s)\,(r',s')}\,\chi^{\mathsf{M}(u,v)}_{(r',s')}(\tau) \tag{3.58}$$

where

$$\mathrm{S}^{\mathsf{M}(u,v)}_{(r,s)\,(r',s')} = -2\sqrt{\frac{2}{uv}}(-1)^{rs'+r's}\sin\frac{v\pi rr'}{u}\sin\frac{u\pi ss'}{v} \tag{3.59}$$

and the summation is taken from the Kac table. See Section 10.6 of [DFMS 1997] for a proof of the formulas (3.58) and (3.58).

Combining the S-transformation rules (3.55), (3.56) and (3.58), we can formulate an interesting result:

Proposition 3.60. (Proposition 4.9 of [ACR 2018]) *The S-transformation rules of the typical $\mathsf{B}_k(\mathfrak{sl}_2)$-characters are*

$$\mathrm{ch}\left[\mathcal{B}^{\mathcal{E}}_{m/v;r,s}\right]\left(-\frac{1}{\tau}\right) = \sum_{(r',s')}\sum_{\ell=0}^{p} \mathrm{S}^{\mathsf{M}(u,v)}_{(r,s)\,(r',s')}\mathrm{S}^{\vartheta}_{m\ell}\,\mathrm{ch}\left[\mathcal{B}^{\mathcal{E}}_{\ell/v;r',s'}\right](\tau) \tag{3.61}$$

where $\mathrm{S}^{\mathsf{M}(u,v)}_{(r,s)\,(r',s')}$ *and* $\mathrm{S}^{\vartheta}_{m\ell}$ *are defined in* (3.59) *and* (3.57)*, respectively. Note that the sum* $\sum_{(r',s')}$ *in* (3.61) *runs over Kac's table, see Table 3.2.*

Following indications from Remark 3.52, we now only need to apply the S transform to the expressions $\Gamma_{\mu;r}(q)$ with $\mu \equiv r-1 \bmod 2\mathbb{Z}$ as in (3.51). In order to do so, we record the effect of S on the differentiated Jacobi-theta function. Set $p = \frac{1}{2}|L'/L|$ as before. Then for $m \in \{1, \dots, p-1\}$[7]:

$$\vartheta'_{\frac{m}{v}+L}\left(-\frac{1}{\tau}\right) = (-i\tau)^{\frac{3}{2}} \sum_{\ell=1}^{p-1} \mathrm{S}'_{m\ell}\vartheta'_{\frac{\ell}{v}+L}(\tau) \tag{3.62}$$

where

$$\mathrm{S}'_{m\ell} = \sqrt{\frac{2}{p}}\sin\left(\frac{\pi\ell m}{p}\right) .$$

All is then set up for complicated computations of the effects of the modular transformation S on $\Gamma_{\mu;r}(q)$. Here is the outcome:

Theorem 3.63. (Theorem 4.17 of [ACR 2018]) *The elements of* V_k *constitute a finite dimensional vector-valued modular form of weight* 1 *with*

$$\Gamma_{\mu;r}\left(-\frac{1}{\tau}\right) = -i\tau \sum_{(\mu';r')\in\mathbf{Gen}_k} \mathrm{S}^{\Gamma}_{(\mu;r)(\mu';r')}\Gamma_{\mu';r'}(\tau) ,$$

$$\Gamma_{\mu;r}(\tau+1) = e^{2\pi i(\delta^{\mathcal{L}}_{\mu;r}-\frac{\tilde{c}}{24})}\Gamma_{\mu;r}(\tau) ,$$

where

$$\delta^{\mathcal{L}}_{\mu;r} = \begin{cases} \Delta_{r,0} - \frac{\mu^2}{4k} & \text{if } |\mu| \leq r-1 \\ \Delta_{r,0} - \frac{\mu^2}{4k} + \frac{|\mu|-r+1}{2} & \text{if } |\mu| > r-1 \end{cases} ,$$

[7]For $m=1$ or $m=p$, we can check that $\vartheta'_{\frac{m}{v}+L}(\tau)$ vanishes. However, we can also check that the corresponding S-matrices elements also always vanish. Therefore, we may as well include 1 and p in the domain m here to simplify notation.

$\tilde{c} = 2 - \frac{6}{t}$, $\mathbf{Gen}_k$ *is the generating set of* V_k *given in Proposition 3.54 and*

$$\mathrm{S}^{\Gamma}_{(\mu;r)(\mu';r')} = \frac{2A_{\mu';r'}}{\sqrt{uw}} \sin\left(\frac{\pi r r'}{t}\right) \cos\left(\frac{\pi \mu\mu'}{k}\right) ,$$

$$\textit{with } A_{\mu';r'} = \begin{cases} \frac{1}{2} & \textit{if } r' = \frac{u}{2} \textit{ and } \mu' \in w\mathbb{Z}, \\ 2 & \textit{if } r' \neq \frac{u}{2} \textit{ and } \mu' \notin w\mathbb{Z}, \\ 1 & \textit{otherwise.} \end{cases}$$

Theorem 3.63 ensures that the linear span of $\mathsf{B}_k(\mathfrak{sl}_2)$-characters

$$\mathrm{Span}_{\mathbb{C}} \left\{ \mathrm{ch}\left[\mathcal{B}^{\mathcal{E}}_{\mu;r,s}\right](q), \mathrm{ch}\left[\mathcal{B}^{\mathcal{L}}_{\mu;r}\right](q), \mathrm{ch}\left[\mathcal{B}^{\mathcal{D}}_{\mu;r,s}\right](q) \;\middle|\; \begin{array}{l} \text{indices are as} \\ \text{in Remark 3.39} \end{array} \right\},$$

is also a finite dimensional vector-valued modular form. Its dimension is given in Corollary 4.18 of [ACR 2018] as $p(u-1)(v-1) + 2\dim_{\mathbb{C}} V_k$. In conclusion, we believe that the modularity results of this chapter suggest the following conjecture:

Conjecture 3.64. *The logarithmic vertex operator algebras* $\mathsf{B}_k(\mathfrak{sl}_2)$ *is* C_2*-cofinite for admissible negative* k.

Assuming our conjecture holds, the results of [Miy 2004] on the modularity behaviour of characters of a C_2-cofinite vertex operator algebra lead us to believe that the elements of V_k (see Proposition 3.54) can be identified with the so-called *pseudo-trace functions*. The pseudo-trace functions of [Miy 2004] are usually difficultly identifiable, although conjectures are being formulated in [CG] and [CG 2017], see also [GR 2017] on pseudo-trace functions. We hope that our results in [ACR 2018] will furnish some inspiration in this direction.

Remark 3.65. We note that Theorem 3.63 and the other modularity results for the $\mathsf{B}_k(\mathfrak{sl}_2)$-characters of this chapter are not the most general modularity behaviours expected for a C_2-cofinite vertex operator algebra. The most general objects that are modular invariant associated to a C_2-cofinite vertex operator algebra are in fact the torus one-point functions: the general modular invariance result can be found as Theorem 5.2 of [Miy 2004].

Denote by $E_{2r}(q)$ ($r \in \mathbb{N}$) the corresponding Eisenstein q-series. We introduce the torus one point functions here:

Definition 3.66. Let V be a vertex operator algebra. Denote by $O_q(V)$ the submodule of $V[E_4(q), E_6(q)] \subset V[[q]]$ generated by elements of the type $v_{[0]}(u)$ where $u, v \in V$. Then the space $\mathcal{C}_1(V)$ of *torus one-point functions* is the $\mathbb{C}$-linear space of functions

$$S : \; V[E_4(q), E_6(q)] \otimes \mathbb{H} \to \mathbb{C} \; ,$$

satisfying the following conditions:

- for $u \in V\big(SL(\mathbb{Z})\big)$, $S(u,\tau)$ is holomorphic in $\tau \in \mathbb{H}$;
- S is linear in $\mathbb{C}[E_4(q), E_6(q)]$;
- $S(u,\tau) = 0$ for $u \in O_q(V)$;
- For $u \in V$, one has

$$S\left(L_{[-2]}(u), \tau\right) = \frac{1}{2\pi i}\frac{\partial}{\partial \tau} S(u,\tau) + \sum_{r=1}^{\infty} E_{2r}(\tau) S\left(L_{[2r-2](u)}, \tau\right) \; .$$

M. Miyamoto obtains the modular behaviour of the one-point functions in his Theorem 5.2 [Miy 2004]. In fact, given a vertex operator algebra V and a weak $\mathbb{N}$-graded module M, then the trace functions

$$\mathrm{tr}_M \left(o(v) q^{L_0 - \frac{c}{24}}\right) \; , \tag{3.67}$$

belong to $\mathcal{C}_1(V)$ where c is the central charge of V and $o(v)$ is the grade-preserving operator of v (for $v \in V$ homogeneous). Specialising $v = \mathbf{1} \in V$ in Formula (3.67), we recover Definition A.15 of the character of a module that we have been using in this chapter and [ACR 2018].

In his Theorem 5.5 [Miy 2004], Miyamoto argues that $\mathcal{C}_1(V)$ is spanned by trace and pseudo-trace functions associated to generalised Verma modules. What we have proved in this chapter and [ACR 2018] is modular invariance of one-point functions specialised at $v = \mathbf{1} \in V$. Specialisation of trace functions (3.67) to $v = \mathbf{1} \in V$ induces a $\mathbb{C}$-algebra map

$$\mathrm{ev}_{v=\mathbf{1}} : \mathrm{tr}_M \left(o(v) q^{L_0 - \frac{c}{24}} \right) \longmapsto \mathrm{tr}_M \left(q^{L_0 - \frac{c}{24}} \right) = \mathrm{ch}\left[M \right] (q) \ .$$

Proving modularity of one-point functions $\mathcal{C}_1\left(\mathsf{B}_k(\mathfrak{sl}_2)\right)$ thus requires more work than what we have done in this chapter and [ACR 2018]. However, our modularity result still gives us partial evidence to have identified a new family of C_2-cofinite vertex operator algebras.

Chapter 4

C_2-Cofiniteness of Certain Parafermion Vertex Algebras

In this chapter, we prove C_2-cofiniteness of eight examples of logarithmic vertex operator algebras of which five are totally new. These non-rational C_2-cofinite vertex operator algebras are the extended parafermion algebras $\mathsf{B}_k(\mathfrak{sl}_2)$ of Chapter 3 and [ACR 2018] at the following specific admissible negative levels[1]:

$$A = \left\{-\frac{1}{2}, -\frac{4}{3}, -\frac{8}{5}, -\frac{5}{4}, -\frac{7}{5}, -\frac{12}{7}, -\frac{16}{9}\right\} \cup \left\{-\frac{2}{3}\right\} . \tag{4.1}$$

The above levels correspond to those for which our computational approach has been successful so far. The central charges of $\mathsf{B}_k(\mathfrak{sl}_2)$ for $k \in A$ are reported in Table 4.1.

We use K. Thielemans' Mathematica package *OPEdefs.m* (see [Thi 1991]) as a basis to perform vertex algebra computations within the universal affine vertex operator algebra $V_k(\mathfrak{sl}_2)$. For obtaining this chapter's results, I have developed and

[1]As of now, the C_2-cofiniteness of $\mathsf{B}_{-\frac{2}{3}}(\mathfrak{sl}_2)$ is true assuming a conjecture. For more details, see Remark 4.20 of [ACR 2018].

level	k	$-\frac{1}{2}$	$-\frac{4}{3}$	$-\frac{8}{5}$	$-\frac{5}{4}$	$-\frac{7}{5}$	$-\frac{12}{7}$	$-\frac{16}{9}$	$-\frac{2}{3}$
u	$\mathrm{num}(k+2)$	3	2	2	3	3	2	2	4
v	$\mathrm{denom}(k+2)$	2	3	5	4	5	7	9	3
c	$\frac{2(k-1)}{k+2}$	-2	-7	-13	-6	-8	-19	-25	$-\frac{5}{2}$

Table 4.1: Parameters u, v and central charges c of $\mathsf{B}_k(\mathfrak{sl}_2)$ for specific levels k studied in Chapter 4.

commented additional Mathematica functions and packages.

Two of the admissible negative levels in A produce algebras $\mathsf{B}_k(\mathfrak{sl}_2)$ known to be C_2-cofinite, while C_2-cofiniteness for $k = -\frac{2}{3}$ has been proven assuming another conjecture. The extended parafermion $\mathsf{B}_{-\frac{1}{2}}(\mathfrak{sl}_2)$ is isomorphic to the triplet vertex operator algebra $W(1,2)$ and $\mathsf{B}_{-\frac{4}{3}}(\mathfrak{sl}_2)$ is isomorphic to a $\mathbb{Z}_2$-orbifold of the triplet $W(1,3)$ while $\mathsf{B}_{-\frac{2}{3}}(\mathfrak{sl}_2)$ is isomorphic to the even part of the supertriplet $sW(1,3)$ (see Section 4.3 and Remark 4.20 of [ACR 2018] for more details). Note that the triplet vertex operator algebras are some of the few known types of algebras that are both logarithmic and C_2-cofinite [TW 2013].

Computational approaches have already been used in [DLY 2009] and [ALY 2014] to establish rationality and C_2-cofiniteness of the parafermion algebra $\mathsf{C}_k(\mathfrak{sl}_2)$ for $k \in \mathbb{N}\backslash\{0\}$. Thanks to investigations on the structure the parafermion vertex operator algebra in [DLWY 2010], [DW 2010], we know that $\mathsf{C}_k(\mathfrak{sl}_2)$ has a finite number of strong generators[2]

$$\left\{L = L^{\text{para}}, W_3, W_4, W_5\right\}$$

where the latter three are Virasoro primary vectors of conformal dimension matching

[2]That vertex operator algebra V is strongly generated by a set of generators means that every vertex operator field $\mathbf{Y}(v,z)$ for $v \in V$ is a normally-ordered polynomial in the generators and their derivatives.

their subscript. By Corollary 2.6.2 of [Ara 2012], it can then be argued that we have a natural surjection of commutative algebras

$$\mathbb{C}[x_2, x_3, x_4, x_5] \twoheadrightarrow \frac{\mathsf{C}_k(\mathfrak{sl}_2)}{C_2\left(\mathsf{C}_k(\mathfrak{sl}_2)\right)}\ , \tag{4.2}$$

where the product on the C_2-quotient is given by $(a, b) \mapsto a_{(-1)}(b)$ is the vector whose vertex operator field is the normally ordered product of those of a and b:

$$\mathbf{Y}(a_{(-1)}(b), z) = :\mathbf{Y}(a, z)\mathbf{Y}(b, z):\ .$$

By computing enough relations amongst the generators $\{W_i\}_{i=2}^5$, one can aim to show that the surjective map (4.2) factors through a finite dimensional quotient $\mathbb{C}[x_2, x_3, x_4, x_5]/I$, which immediately implies C_2-cofiniteness of $\mathsf{C}_k(\mathfrak{sl}_2)$. In this chapter, we follow a similar approach for our extended parafermion algebra $\mathsf{B}_k(\mathfrak{sl}_2)$ where k is a negative and admissible level instead of being integral and positive.

Currently, the computations behind this chapter's results rely on our capacity to compute explicitly the non-trivial singular vector of $s_k \in V_k(\mathfrak{sl}_2)$ where k is admissible. Let $k = -2 + \frac{u}{v}$ be admissible. Thanks to the Kac-Kazhdan formula for the Shapovalov determinant [KK 1979], we know what h_0 and L_0 weight the singular vector s_k should have. However, the dimension for the appropriate homogeneous subspaces grow rapidly when varying k and determining s_k by direct computation become out of reach when the conformal dimension of s_s is above 11. Though we have considered making use of the expression of s_k in terms of Jack polynomials [RW 2015a], we have not yet fully implemented the required free field realisation of $V_k(\mathfrak{sl}_2)$.

In the first section, we discuss C_2-cofiniteness, generators for $\mathsf{C}_k(\mathfrak{sl}_2)$, $\mathsf{B}_k(\mathfrak{sl}_2)$

and describe a strategy for computing C_2-cofiniteness. In the second and last section, we give results of computations in the form of polynomial relations that must hold in the C_2-quotient of $\mathsf{B}_k(\mathfrak{sl}_2)$. A few of these relations hold for general k as seen in [DLY 2009] while most others depend on the specific value of $k \in A$ where A is the set (4.1). In the second section, we focus on each $k \in A$ separatedly by giving more complete lists of polynomial relations to hold in $\frac{\mathsf{B}_k(\mathfrak{sl}_2)}{C_2(\mathsf{B}_k(\mathfrak{sl}_2))}$ and we use Gröbner bases (see [CLO 1997] for instance) to establish C_2-cofiniteness.

Notation

Throughout the chapter, the following notation will be employed:

- for a vertex operator V with strong generators $\{G_i\}$, we will write $G_i(z) = \mathbf{Y}(G_i, z)$;
- $C_2(V) = \{a_{-2}(b) \in V \mid a, b \in V\}$ where V is a given vertex operator algebra;
- $\frac{V}{C_2(V)}$ is the C_2-quotient associated to a given vertex operator algebra V. It is known to have a natural associative and commutative Poisson algebra structure, see [Ara 2012] for instance;
- let k be admissible. Then s_k is the singular vector[3] of $V_k(\mathfrak{sl}_2)$. In particular, its coefficients must all act as zero on $L_k(\mathfrak{sl}_2) = V_k(\mathfrak{sl}_2)/\langle s_k \rangle$;
- let k be admissible. Then s_k^{para} is the singular vector of $\mathrm{Com}\left(H, V_k(\mathfrak{sl}_2)\right)$ whose unique simple quotient is $\mathsf{C}_k(\mathfrak{sl}_2) = \mathrm{Com}\left(H, L_k(\mathfrak{sl}_2)\right)$. In particular, the coefficients of its vertex operator must all act as zero on $\mathsf{C}_k(\mathfrak{sl}_2)$.

[3]Just like for $\widehat{\mathfrak{sl}_2}$, this means that $e_n.s_k = f_{n+1}.s_k = h_{n+1}.s_k = 0$ for all $n \in \mathbb{N}$ and that s_k is an h_0-eigenvector.

4.1 Principles, Approach and Methods

In this section, we summarise the approach and the computing methods employed to prove C_2-cofiniteness of the extended parafermions vertex operator algebra $\mathsf{B}_k(\mathfrak{sl}_2)$ with $k \in A$ as in (4.1).

The parafermion vertex operator algebra $\mathsf{C}_k(\mathfrak{sl}_2)$ with $k \in \mathbb{N}\backslash\{0\}$ has been studied in [DLY 2009], [DLWY 2010], [DW 2010]. The C_2-cofiniteness property has been estaablished for all positive integral level parafermion algebras [ALY 2014], see also [DLY 2009]. In our case, however, we want to consider k to be admissible and negative instead of integral and positive.

4.1.1 Our Approach

Fix k admissible and negative. A quick review of some key the properties and consequences of C_2-cofiniteness can be found in Appendix A. Recall that we define the C_2-space of a given vertex operator algebra V as

$$C_2\left(V\right) = \mathrm{Span}_{\mathbb{C}}\{a_{-2}(b) \mid a, b \in V\}$$

and that V is called C_2-cofinite if $\dim_{\mathbb{C}} \frac{V}{C_2(V)} < \infty$. In particular, the existence of the conformal vector $\omega \in V$ allow us to show that $a_{-m}(b) \in C_2\left(V\right)$ for all $m \geq 2$. Relevant properties of the C_2-quotient $\frac{V}{C_2(V)}$ of an arbitrary vertex operator algebra V can also be found in Section 3 of [ALY 2014] or Section 2.3 of [BR 2018]. We recall here certain basic facts about the C_2-quotient of a vertex operator algebra:

Result 4.3. *For a vertex operator algebra V, the C_2-quotient $\frac{V}{C_2(V)}$ has a natural* *structure of a commutative associative Poisson algebra with multiplication*

$$(\overline{a}, \overline{b}) \longmapsto \overline{a_{-1}(b)} \ ,$$

and Poisson bracket

$$(\overline{a}, \overline{b}) \longmapsto \overline{a_0(b)} \ .$$

In particular:

- *the multiplication is associative and commutative;*
- *the Poisson bracket satisfies a Leibniz-type rule with respect to the multiplication:*

$$\overline{a_0\big(b_{-1}(c)\big)} = \overline{\big(a_0(b)\big)_{-1}(c)} + \overline{b_{-1}\big(a_0(b)\big)} \ ;$$

- *the Poisson bracket satisfies a Jacobi identity similar to that of Lie algebras.*

Remark 4.4. Recall that the mapping $a \mapsto \mathbf{Y}(a, z)$ for a in a given vertex operator algebra is always injective. If one works with vertex operators instead of vectors, the multiplication on $\frac{V}{C_2(V)}$ is just the normally ordered product of the two given vertex operators:

$$\Big(\mathbf{Y}(a, b), \mathbf{Y}(b, z)\Big) \longmapsto \ : \mathbf{Y}(a, z)\mathbf{Y}(b, z) : \ .$$

The following result is also helpful:

Result 4.5. *Let V be a vertex operator algebra and a vertex ideal $I \lhd V$. Then one has $C_2(V/I) \cong (C_2(V) + I)\,/I$ and so*

$$\frac{V/I}{C_2\,(V/I)} \cong \frac{V/I}{(C_2(V) + I)\,/I} \cong V/\,(C_2(V) + I) \ .$$

This means that the C_2-quotient of V/I can be understood in terms of that of V subject to any additional relations coming from the ideal I.

Now recall that $V_k(\mathfrak{sl}_2)$ has a unique maximal ideal $I = \langle s_k \rangle$ generated by a singular vector and the corresponding quotient is $L_k(\mathfrak{sl}_2) = V_k(\mathfrak{sl}_2)/\langle s_k \rangle$. We will show in the subsection below that by applying certain powers of the operator f_0 to s_k, we obtain a unique singular vector $s_k^{\text{para}} \in \text{Com}\left(H, V_k(\mathfrak{sl}_2)\right)$. Then since the Heisenberg vertex algebra $H \subset V_k(\mathfrak{sl}_2)$ also injects naturally into the simple quotient $L_k(\mathfrak{sl}_2)$, one can argue similarly as in [DLWY 2010] and obtain that

$$\mathsf{C}_k(\mathfrak{sl}_2) \cong \text{Com}\left(H, V_k(\mathfrak{sl}_2)\right)/\langle s_k^{\text{para}} \rangle \ .$$

Another important result from [DLWY 2010] is that $\mathsf{C}_k(\mathfrak{sl}_2)$ is strongly generated by its conformal vector L along with three Virasoro primary vectors W_3, W_4 and W_5 of conformal dimensions $3, 4, 5$, respectively. By Remark 4.1 of [ACR 2018], we can strongly generate the extended parafermion vertex operator $\mathsf{B}_k(\mathfrak{sl}_2) = \bigoplus_{\ell \in \mathbb{Z}} \mathcal{C}^{\mathcal{L}}_{2w\ell;1}$ with the generators of $\mathsf{C}_k(\mathfrak{sl}_2)$ and two additional Virasoro primary vectors $W_\pm$ associated with the highest weight states of the simple $\mathsf{C}_k(\mathfrak{sl}_2)$-modules $\mathcal{C}^{\mathcal{L}}_{\pm 2w;1}$. Moreover, the simple current extension $\mathsf{C}_k(\mathfrak{sl}_2) \subset \mathsf{B}_k(\mathfrak{sl}_2)$ is by an abelian intertwining algebra governed by the additive group of the lattice $L = 2w\mathbb{Z}$ so the algebra $\frac{\mathsf{B}_k(\mathfrak{sl}_2)}{C_2(\mathsf{B}_k(\mathfrak{sl}_2))}$ will also be polynomial in its strong generators.

Strong generators are key here because the C_2-quotient is also generated by them as shown in Corollary 2.6.2 of [Ara 2012]. By the discussion above, the following key result holds:

Result 4.6. *For k admissible and negative, the set $\{L, W_3, W_4, W_5\}$ strongly gen-*

erates $\mathsf{C}_k(\mathfrak{sl}_2)$ *and the set*

$$\{L, W_3, W_4, W_5, W_+, W_-\} \ , \tag{4.7}$$

strongly generates $\mathsf{B}_k(\mathfrak{sl}_2)$. *Consequently, there is a surjective algebra homomorphism*

$$\mathbb{C}[x_2, x_3, x_4, x_5, x_+, x_-] \twoheadrightarrow \frac{\mathsf{B}_k(\mathfrak{sl}_2)}{C_2\left(\mathsf{B}_k(\mathfrak{sl}_2)\right)} \ , \tag{4.8}$$

sending x_2 *to* L*,* x_i *to* W_i *for* $i \in \{1, 2, 3\}$ *and* $x_\pm$ *to* $W_\pm$*.*

In practice, Results 4.5 and 4.6 allow us to compute relations amongst the generators and the singular vector s_k^{para} in order to show C_2-cofiniteness of $\mathsf{B}_k(\mathfrak{sl}_2)$. A sufficient number of such relations, would allow us to bound the dimension of its C_2-quotient space by a finite number. The principle is as follow: for every relation $r(L, W_3, W_4, W_5, W_\pm)$, form a the corresponding polynomial $r(x_2, x_3, x_4, x_5)$ and let $P \lhd \mathbb{C}[x_2, x_3, x_4, x_5, x_+, x_-]$ be the the ideal generated by all corresponding such polynomials. By construction, the surjection (4.8) factors through

$$\mathbb{C}[x_2, x_3, x_4, x_5, x_+, x_-]/P \ . \tag{4.9}$$

Therefore, showing that the dimension of the above quotient is finite will immediately imply that $\mathsf{B}_k(\mathfrak{sl}_2)$ is C_2-cofinite!

Let us describe how we can find relations in the kernel of (4.8). By Remark 4.1 of [ACR 2018], we deduce the following two relations in $\frac{\mathsf{B}_k(\mathfrak{sl}_2)}{C_2(\mathsf{B}_k(\mathfrak{sl}_2))}$:

$$x_+^2 = 0 \ , \qquad\qquad x_-^2 = 0 \ . \tag{4.10}$$

This is because $\big(: W_+W_+ : \big)(z) \ = 0$ and $\big(: W_-W_- : \big)(z) \ = 0$ for conformal weight reasons. For obtaining other necessary relations among the generators (4.7), we will look for certain types of relations in $\frac{\mathsf{B}_k(\mathfrak{sl}_2)}{C_2(\mathsf{B}_k(\mathfrak{sl}_2))}$, in particular:

- relations coming from a parafermion singular vector $s_k^{\text{para}} \in \operatorname{Com}\big(H, V_k(\mathfrak{sl}_2)\big)$. In practice, all the operator product expansion coefficients $\gamma_i^{G,s_k^{\text{para}}}(w)$ have to be set to zero where $i \in \mathbb{N}$ and where $G \in \{L, W_3, W_4, W_5\}$ is a strong generator of $\mathsf{C}_k(\mathfrak{sl}_2)$;

- genuine relations among the generators W_3, W_4, W_5 in $\mathsf{C}_k(\mathfrak{sl}_2)$. Relations of this type were also computed in [DLY 2009] by writing $\big(: W_4W_4 :\big)(z)$, $\big(: W_4W_5 :\big)(z)$ and $\big(: W_5W_5 :\big)(z)$ as linear combinations of the other normally ordered monomial of appropriate conformal weight in the strong generators $\{L, W_3, W_4, W_5\}$ of $\mathsf{C}_k(\mathfrak{sl}_2)$;

- a relation of the form $x_+x_- = p(x_2, x_3, x_4, x_5)$ where p is a polynomial. This is predictable because $\big(: W_+W_- :\big)(z)$ must lie in $\mathsf{C}_k(\mathfrak{sl}_2)$ by construction of our extended parafermion vertex operator algebra $\mathsf{B}_k(\mathfrak{sl}_2)$.

Remark 4.11. We expect that relations coming from the parafermion singular vector to be the most challenging to obtain since singular vectors are very hard to explicitly determine. However, they should be the most rewarding when it comes to producing relations that must hold in the C_2-quotient $\frac{\mathsf{B}_k(\mathfrak{sl}_2)}{C_2(\mathsf{B}_k(\mathfrak{sl}_2))}$.

Certain semi-explicit forms for the singular vector of $V_k(\mathfrak{sl}_2)$ are known, but none of them is workable. For instance, s_k is given in [MFF 1986] as a monomial in e_n, f_n for certain $n \in \mathbb{Z}$ with fractional powers. However to make sense of such an expression in $U\left(\widehat{\mathfrak{sl}_2}\right)$ is all but straightforward and cannot be done in a sensible way for a generic k. There is also an expression for s_k in terms of Jack polynomials [RW 2015a] if one realises $V_k(\mathfrak{sl}_2)$ in the Wakimoto free field realisation, but I have not yet been able to take full advantage of this.

There is a problem, however, with computing a polynomial relation for $\big(: W_+W_- : \big)(z)$ in terms of fields of $\mathsf{C}_k(\mathfrak{sl}_2)$: the generically high conformal dimension of the space in which such a computation would be performed:

$$2(v+1)w = 2(v+1)(2v-u) \ .$$

We will instead try to compute a polynomial relation in fields of $\mathsf{C}_k(\mathfrak{sl}_2)$ for certain generalised fields $G_{\pm 2}$ that we will be introduced in the next section. These fields $G_{\pm 2}$ will be those associated to a vertex operator algebra action of $\mathsf{C}_k(\mathfrak{sl}_2)$ on the highest weight vectors of $\mathcal{C}^{\mathcal{L}}_{\pm 2;1}$, respectively[4] .

We now want to develop a concrete strategy to determine C_2-cofiniteness of $\mathsf{B}_k(\mathfrak{sl}_2)$ using Mathematica and a personal computer. Recall that we can only manipulate vertex operator algebra fields of integral conformal dimensions with *OPEdefs.m*. To meet our objective, we will make use of auxiliary generalised vertex operator fields taken from a certain auxiliary positive definite lattice vertex operator algebra:

Result 4.12. *Suppose there exists a positive definite even lattice $\tilde{L}^k$ such that:*

- *$M/\tilde{L}^k \cong (\mathsf{Q}/w\mathsf{Q}) = (2\mathbb{Z})/(2w\mathbb{Z}) \cong \mathbb{Z}/w\mathbb{Z}$ for some bigger lattice M such that $\tilde{L}^k \subset M \subset \left(\tilde{L}^k\right)'$;*
- *there is $m \in M$ such that $\overline{m} = m + \tilde{L}^k$ generates $M/\tilde{L}^k$ and*

$$\frac{1}{2}(m \cdot m) = \left\lceil \frac{v}{w} \right\rceil - \frac{v}{w} \ \in \ \mathbb{Q} \cap [0,1] \ . \tag{4.13}$$

Then the following procedure allows to deduce that $\mathsf{B}_k(\mathfrak{sl}_2)$ is C_2-cofinite:

[4] In this sense, $G_{\pm 2}(z)$ can be thought of as being w^{th}-roots of the fields $W_\pm(z)$, respectively.

1. *we fix an isomorphism of abelian groups* $\tau : M/\tilde{L}^k \to \mathsf{Q}/w\mathsf{Q}$ *such that* $\tau(m) = 2 + w\mathsf{Q} \in \mathsf{Q}/w\mathsf{Q}$*;*

2. *just as in Chapter 3, a result of [Li 2001] shows that*

$$A_{M,k} = \bigoplus_{\overline{m}\in M/\tilde{L}^k} \left(\bigoplus_{\ell\in\mathsf{Q}} \mathcal{C}^{\mathcal{L}}_{\ell+\tau(m);1}\right) \otimes V_{\tilde{L}^k+m} \ , \tag{4.14}$$

   is a vertex operator algebra since it is $\mathbb{Z}$*-graded by conformal weight;*

3. *we prove* C_2*-cofiniteness of* $A_{k,M}$ *by computing the normally ordered product* $\big(: G_2G_{-2} : \big)(u) \otimes \big(: V_mV_{-m} : \big)(u) \propto \big(: G_2G_{-2} : \big)(u)$ *in a space of low conformal dimension and adding it to the other polynomial relations we already have;*

4. *we note that* $M/\tilde{L}^k$ *naturally acts on* $A_{k,M}$ *by automorphisms via*

$$\overline{m}.x = e^{\pi i(m\cdot n)}x \ ,$$

   for any $x \in \left(\bigoplus_{\ell\in\mathsf{Q}} \mathcal{C}^{\mathcal{L}}_{\ell+\tau(n);1}\right) \otimes V_{\tilde{L}^k+n}$ *where* $n \in \tilde{L}^k$*. By construction, the orbifold (or invariant) sub-vertex operator algebra*

$$A^{M/\tilde{L}^k}_{k,M} = \mathsf{B}_k(\mathfrak{sl}_2) \otimes V_{\tilde{L}^k} \ , \tag{4.15}$$

   of the C_2*-cofinite* $A_{k,M}$ *under the action of the finite solvable abelian group* $M/\tilde{L}^k \cong \mathbb{Z}/w\mathbb{Z}$ *has to be* C_2*-cofinite as well, see [Miy 2015], [CM 2016] for details.*

5. *that* $\mathsf{B}_k(\mathfrak{sl}_2)$ *is* C_2*-cofinite will follow form from the fact that the tensor product vertex operator algebra (4.15) is* C_2*-cofinite.*

Remark 4.16. We will not address the problem of the existence of the suitable even lattice $\tilde{L}^k$ in detail here. Note however that for all $k \in A$ as of (4.1), we could always find a lattice of the form $\bigoplus_{i=1}^{a} A_{n_i}$ where $n_i \in \mathbb{N}\backslash\{0\}$ and A_{n_i} is the root lattice of finite dimensional simple Lie algebra $\mathfrak{sl}_{n_i+1}$ whose invariant bilinear form is normalised so that the roots have length 2.

4.1.2 Methods of Computation

Fix $k = -2 + \frac{u}{v}$ admissible and negative. Recall that L^{aff} is given by the Sugawara conformal vector in $L_k(\mathfrak{sl}_2)$, that $L^H = \frac{1}{4k} h_{-1}^2 \mathbf{1}$ (where $\mathbf{1}$ is the highest weight vector that generates $L_k(\mathfrak{sl}_2)$) and that

$$L = L^{\text{para}} = L^{\text{aff}} - L^H \ . \tag{4.17}$$

Any computations can only be performed in a freely generated vertex operator algebra since computers cannot directly manipulate and operate in any coset defined by an equivalence relations. We will therefore work with fields of the universal vertex operator algebra $V_k(\mathfrak{sl}_2)$ and construct parafermion fields in the universal parafermion vertex operator algebra[5]

$$\mathrm{Com}\left(H, V_k(\mathfrak{sl}_2)\right) = \{a \in V_k(\mathfrak{sl}_2) \mid h_n.a = 0 \text{ for all } n \in \mathbb{N}\} \ .$$

It follows that $L_0^H = \frac{1}{2k} \sum_{n>0} n\, h_{-n} h_n$ acts as zero on elements of $\mathrm{Com}\left(H, V_k(\mathfrak{sl}_2)\right)$. In particular, we have the following useful fact:

Result 4.18. *Let* $v \in \mathrm{Com}\left(H, V_k(\mathfrak{sl}_2)\right) \subset V_k(\mathfrak{sl}_2)$. *Then the conformal weights of*

[5]Recall from last section that $\mathrm{Com}\left(H, V_k(\mathfrak{sl}_2)\right)$ has a unique maximal submodule generated by a parafermion singular vector. The corresponding simple quotient $\mathsf{C}_k(\mathfrak{sl}_2)$.

$v \in \mathrm{Com}\left(H, V_k(\mathfrak{sl}_2)\right)$, $v \in V_k(\mathfrak{sl}_2)$, $\overline{v} \in \mathsf{C}_k(\mathfrak{sl}_2)$ *and* $\overline{v} \in L_k(\mathfrak{sl}_2)$ *all coincide.*

Our computations will be performed using the software Mathematica using K. Thielemans' package *OPEdefs.m* [Thi 1991] for computing operator product expansions and normally ordered products of fields as well as a number of custom functions. The usual Poincaré-Birkhoff-Witt basis of $V_k(\mathfrak{sl}_2)$ combined with Result 4.18 allows to represent the homogeneous elements of $\mathsf{C}_k(\mathfrak{sl}_2)$ with linear combinations of monomials of the form

$$h_{-i_q}^{p_{h,q}} \cdots h_{i_1}^{p_{h,1}} \cdot e_{-i_r}^{p_{e,r}} \cdots e_{i_1}^{p_{e,1}} \cdot f_{-i_s}^{p_{h,s}} \cdots f_{i_1}^{p_{f,1}} \mathbf{1} + \langle s_k^{\text{para}} \rangle \tag{4.19}$$

where $q, r, s \in \mathbb{N}$, $i_\ell > \cdots > i_1$ for every $\ell \in \{q, r, s\}$, where $p_{x,\ell} \in \mathbb{N}$ for every $(x, \ell) \in \{(h, q), (e, r), (f, s)\}$ and

$$\sum_{a=1}^{r} p_{e,a} - \sum_{b=1}^{s} p_{f,b} = 0$$

whenever any of the two above sums make sense. The vector (4.19) has conformal weight $\sum_{\ell \in \{q,r,s\}} \sum_{r=1}^{\ell} i_r$.

To setup computations in $\mathsf{C}_k(\mathfrak{sl}_2)$ with Mathematica, we just define strong generator vertex operator fields $L(z) = L^{\text{para}}(z)$ by (4.17), and $W_3(z)$, $W_4(z)$ and $W_5(z)$ as linear combinations of terms of the form (4.19) with the additional restrictions for being a primary Virasoro vector for $L \in \mathsf{C}_k(\mathfrak{sl}_2)$. The general expressions we obtain in this way will be reported in the next section. Note that our expressions for $W_3(z)$, $W_4(z)$ and $W_5(z)$ are equivalent to what has been computed in [DLY 2009] and used in [ALY 2014].

Remark 4.20. In principle, Lemma 4.1 of [ALY 2014] could help us finding the Virasoro primary vectors by eliminating the possibility for certain terms of the form (4.19) to appear. While this is rather useless here, one could try to apply the same idea to the problem of computing the singular vector $s_k^{\text{para}} \in V_k(\mathfrak{sl}_2)$.

Singular vector relations. The first type of relations for the C_2-quotient that we aim for are those coming from singular vectors (also called null fields). Recall that for k admissible, there exists a unique single singular vector $s_k \in V_k(\mathfrak{sl}_2)$ that generates an ideal whose quotient is $L_k(\mathfrak{sl}_2)$. By the Kac-Kazhdan formula [KK 1979] for the Shapovalov determinant, we find that the singular vector $s_k \in V_k(\mathfrak{sl}_2)$ has to satisfy:

$$h_0.s_k = 2(u-1) \cdot s_k \ , \qquad L_0^{L_k(\mathfrak{sl}_2)}.s_k = (u-1)v \cdot s_k \ . \tag{4.21}$$

For more explanations, see for instance [CR 2013b]. This will allow us to explicitly compute s_k for $k \in A$ as of line (4.1).

The singular vector s_k of $V_k(\mathfrak{sl}_2)$ must act as zero on the simple affine $L_k(\mathfrak{sl}_2)$. By applying appropriate powers of the operator $f_0 \in U\left(\widehat{\mathfrak{sl}_2}\right)$ to the affine singular vector s_k produces a parafermion singular vector s_k^{para} in $\mathrm{Com}\left(H, V_k(\mathfrak{sl}_2)\right)$. Indeed, operating via $f_0 \in U(\mathfrak{sl}_2) \subset U(\widehat{\mathfrak{sl}_2})$ shifts the h_0-weight of any weight vector by -2 and this operation commutes with the quadratic Casimir element $L_0^{L_k(\mathfrak{sl}_2)} \in U(\mathfrak{sl}_2) \subset U(\widehat{\mathfrak{sl}_2})$ so that:

$$\begin{aligned} h_n.(f_0^{u-1}.s_k) &= 0 \quad \text{for all } n \geq 0 \ , \\ L_0^{\mathsf{C}_k(\mathfrak{sl}_2)}.(f_0^{u-1}.s_k) &= L_0^{L_k(\mathfrak{sl}_2)}.(f_0^{u-1}.s_k) - \left(\frac{1}{2k}\sum_{n>0} n\, h_{-n}h_n\right).(f_0^{u-1}.s_k) \\ &= L_0^{L_k(\mathfrak{sl}_2)}.(f_0^{u-1}.s_k) \\ &= (u-1)v \cdot (f_0^{u-1}.s_k) \ . \end{aligned}$$

We can thus define the parafermion singular vector as follows:

$$s_k^{\text{para}} = f_0^{u-1}.s_k \in \mathrm{Com}\left(H, V_k(\mathfrak{sl}_2)\right) . \tag{4.22}$$

In terms of vertex operator algebra operations, we obtain s_k^{para} from (4.22) as the $(u-1)^{\text{th}}$ operator product expansion coefficient of the vertex operator fields $\mathbf{Y}(f_{-1}^{u-1}, z)\mathbf{Y}(s, w)$. It follows that determining explicitely s_k^{para} is equivalent to determining $s_k \in V_k(\mathfrak{sl}_2)$. In practice, we have been able to determine s_k^{para} and s_k^{para} only for values of k such that $(u-1)v \leq 11$. Remark 4.20 outlines an idea on how to potentially ease the computation of s_k^{para}, which may allow for computing singular vectors of conformal weight higher than 11.

As s_k^{para} is set to zero in $\mathsf{C}_k(\mathfrak{sl}_2)$ (and in $\mathsf{B}_k(\mathfrak{sl}_2)$ too), its corresponding vertex operator field $\mathbf{Y}(s_k^{\text{para}}, z)$ should also be set to zero. Therefore, for every strong generator $X \in \{L, W_3, W_4, W_5\}$, we can produce null relations by taking OPE coefficients with the parafermion generators:

$$\mathbf{Y}(s_k^{\text{para}}, z)\mathbf{Y}(X, w) = \sum_{r=0}^{N(X)} \frac{\gamma_r^{s_k^{\text{para}}, X}(w)}{(z-w)^r} . \tag{4.23}$$

Then all fields $\gamma_r^{s_k^{\text{para}}, X}(w)$ in (4.23) have to be set to zero in $\mathsf{C}_k(\mathfrak{sl}_2)$. We can then express each $\gamma_r^{s_k^{\text{para}}, X}(w)$ with $X \in \{L, W_3, W_{,4}, W_5\}$ as normally ordered polynomials in the generating vertex operator fields $L(w), W_3(w), W_4(w)$ and $W_5(w)$ and set all derivatives to zero to obtain valid null relations in the C_2-quotients $\frac{\mathsf{C}_k(\mathfrak{sl}_2)}{C_2(\mathsf{C}_k(\mathfrak{sl}_2))}$ and $\frac{\mathsf{B}_k(\mathfrak{sl}_2)}{C_2(\mathsf{B}_k(\mathfrak{sl}_2))}$.

Relations for unextended generators. The second type of relation for the C_2-quotient to aim for comes from genuine relations amongst the parafermion generators

$\{L, W_3, W_4, W_5\}$. As noted in [DLY 2009], the parafermion algebra $\mathsf{C}_k(\mathfrak{sl}_2)$ is not freely generated and so there exist non-trivial linear relations among them. The existence of such relations can also be checked theoretically by comparing the low q-powers of the character of $\mathrm{ch}\left[\mathsf{C}_k(\mathfrak{sl}_2)\right](q) = \mathrm{ch}\left[\mathcal{C}^{\mathcal{L}}_{0;1}\right](q)$ given in Proposition 3.35 with the character of a freely generated algebra by generators of conformal dimension $2, 3, 4$ and 5:

$$\prod_{r=1}^{\infty} \frac{1}{(1-q^{n+2})(1-q^{n+3})(1-q^{n+4})(1-q^{n+5})} \,. \tag{4.24}$$

For every pair $X, Y \in \{L, W_3, W_4, W_5\}$, we can express $\big(: XY : \big)(z)$ as a normally ordered polynomial in the generating fields $L(z), W_3(z), W_4(z)$ and $W_5(z)$ and their derivatives. For every non-trivial normally ordered polynomial found in this way, we obtain potential relations for the C_2-quotient by removing all monomials containing derivatives and conserve any non-trivial equalities. Following [DLY 2009], we expect non-trivial relations of this type for the C_2-quotient to be arising in spaces of conformal dimensions $8, 9$ and 10.

Suppose that two strong generators X and Y can be expressed in a non trivial normally ordered polynomial $r^{X,Y}$ in the strong generators and their derivatives as follows:

$$\big(: XY : \big)(z) = r^{X,Y}(L, W_3, W_4, W_5 \text{ and their derivatives}) \,.$$

Then the field

$$N^{X,Y}(z) = \big(: XY : \big)(z) - r^{X,Y}(L, W_3, W_4, W_5 \text{ and their derivatives}) \,,$$

is a null field. This means that we can also find set all the operator product expansion coefficients of $N^{X,Y}(z)$ and $G(w)$ to zero. This process however does not seem to produce new relations for the C_2-quotient in general.

Relation for extended generators. A priori, the third type of relations in the C_2-quotient of $\mathsf{B}_k(\mathfrak{sl}_2)$ that we would aim for is of the type

$$\big(: W_+(z) W_- : \big)(z) \quad \equiv p\big(L(z), W_3(z), W_4(z), W_5(z)\big) + C_2\left(\mathsf{B}_k(\mathfrak{sl}_2)\right) \tag{4.25}$$

where p is a normally ordered polynomial. However, recall that the fields $W_\pm(z) = \mathbf{Y}(W_\pm, z)$ are those of the generating highest weight vectors $W_\pm$ of the $\mathsf{C}_k(\mathfrak{sl}_2)$-modules $\mathcal{C}^{\mathcal{L}}_{\pm 2w;1}$ used to construct the extension $\mathsf{B}_k(\mathfrak{sl}_2)$. Their corresponding h_0-weight is $\pm 2w$ and their conformal dimensions are given in Remark 4.1 and Proposition 3.8 of [ACR 2018] by $(v+1)(2v-u) = (v+1)w$. As argued at the end of the preceding section, this number is generically too high to perform computations. Indeed, as conformal dimension increases, bases for the correponding homogeneous spaces for $\mathsf{C}_k(\mathfrak{sl}_2)$ grow very fast: to have an idea, expand the product (4.24) as a sum and consider the growth of the coefficients of q^n as n grows. Even if one had explicit expressions for both $W_\pm(z)$, computing a relation of type (4.25) would probably be impossible for generic k.

At the end of last section, we outlined a procedure to circumvent this difficulty. We must now introduce the generalised fields $G_{\pm 2}(z)$. Recall from Chapter 3 that

$$\mathrm{Res}^{L_k \mathfrak{sl}_2}_{H \otimes \mathsf{C}_k(\mathfrak{sl}_2)} L_k(\mathfrak{sl}_2) = \bigoplus_{2n \in 2\mathbb{Z}} F_{2n} \otimes \mathcal{C}^{\mathcal{L}}_{2n;1} \tag{4.26}$$

where $2\mathbb{Z}$ corresponds to the root lattice Q of the finite dimensional Lie algebra $\mathfrak{sl}_2$. In order to build the simple current extension $\mathsf{B}_k(\mathfrak{sl}_2)$ of $\mathsf{C}_k(\mathfrak{sl}_2) = \mathcal{C}^{\mathcal{L}}_{0;1}$, we only used the simple currents $\mathcal{C}^{\mathcal{L}}_{2wn;1}$ where $w = -kv$ is the absolute value of the numerator of k. This choice is minimal with the property of ensuring that $\mathsf{B}_k(\mathfrak{sl}_2)$ is $\mathbb{Z}$-graded by eigenvalues of $L^{\text{para}} \in \mathsf{C}_k(\mathfrak{sl}_2) \subset \mathsf{B}_k(\mathfrak{sl}_2)$ at any negative admissible level k. Note that the fields $W_\pm(z)$ are those corresponding to the $\mathsf{C}_k(\mathfrak{sl}_2)$-actions on the highest weight vectors that generate $\mathcal{C}^{\mathcal{L}}_{\pm 2w;1}$, respectively. In particular, we have

$$\big(: e^w : \big)(z) = V_{2w}(z) \otimes W_+(z) \ , \tag{4.27}$$

$$\big(: f^w : \big)(z) = V_{-2w}(z) \otimes W_-(z) \ . \tag{4.28}$$

However, we will rather focus on generalised fields $G_{\pm 2}(z)$ introduced below:

Definition 4.29. The generalised vertex operator fields $G_{\pm 2}(z)$ correspond to the highest weight generating vector of the $\mathsf{C}_k(\mathfrak{sl}_2)$-module $\mathcal{C}^{\mathcal{L}}_{\pm 2;1}$. In particular:

$$e(z) = V_2(z) \otimes G_2(z) \quad \text{from } F_2 \otimes \mathcal{C}^{\mathcal{L}}_{2;1} \subset L_k(\mathfrak{sl}_2) \ , \tag{4.30}$$

$$f(z) = V_{-2}(z) \otimes G_{-2}(z) \quad \text{from } F_{-2} \otimes \mathcal{C}^{\mathcal{L}}_{-2;1} \subset L_k(\mathfrak{sl}_2) \ . \tag{4.31}$$

Remark 4.32. We say that $G_{\pm 2}(z)$ are generalised vertex operator fields for $\mathsf{C}_k(\mathfrak{sl}_2)$ because their conformal weight is generically fractional since k is both negative and admissible. We have recorded conformal dimensions of the above pertinent vertex fields appearing in the current discussion in Table 4.2. Note that these values are deduced from Proposition 3.8 of [ACR 2018].

Here is a first technical result on the generalised fields $G_{\pm 2}(z)$:

Lemma 4.33. *The generalised fields $G_{\pm 2}(z)$ from Definition 4.29 appear in the*

field	$e(z), f(z)$	$G_{\pm 2}(z)$	$V_{\pm 2}(z)$
conf. dim.	1	$1+\frac{v}{w}$	$-\frac{v}{w}$

Table 4.2: Conformal dimensions of certain fields. Note that all fields in the above table make sense in terms of action maps for the vertex operator algebra $L_k(\mathfrak{sl}_2)$ or $H \otimes \mathsf{C}_k(\mathfrak{sl}_2)$ or both. In particular, the fields $G_{\pm 2}(z)$, $V_{\pm 2}(z)$ mutually commute.

following expressions:

$$G_2(z)G_{-2}(u) = \sum_{\ell=-2}^{\infty} X_\ell(u) \cdot (z-u)^{\ell - \frac{2v}{w}}$$

where

- $X_{-2}(u) = -k\,\mathrm{Id}$*;*
- $X_{-1}(u) = 0$*;*
- $X_\ell(u)$ *is a polynomial in fields of* $\mathsf{C}_k(\mathfrak{sl}_2)$ *of homogeneous conformal dimension* ℓ *for all* $\ell \geq 0$*.*

Moreover, we have

$$\frac{1}{n!}(: e^{(n)} f :)(u) = \sum_{t=0}^{n+2} \tilde{Y}_t(u) \otimes X_{n-t}(u) \qquad \textit{for all } n \in \mathbb{N}\,, \tag{4.34}$$

where

- $\tilde{Y}_0(u)$ *is a scalar multiple of the identity;*
- $\tilde{Y}_t(u)$ *is a polynomial in* $h(u)$ *and its derivatives of homogeneous conformal dimension* t *for all* $t > 0$*. In particular, the constant term of these polynomials is* 0*.*

Proof: Recall from $L_k(\mathfrak{sl}_2)$ the following OPE relations:

$$\begin{aligned} e(z)f(u) =& \frac{-k\cdot \mathrm{Id}}{(z-u)^2} + \frac{h(u)}{z-u} + \ :e(z)f(u): \\ =& \frac{-k\cdot \mathrm{Id}}{(z-u)^2} + \frac{h(u)}{z-u} \ + \sum_{n=0}^{\infty} \frac{1}{n!}\big(:e^{(n)}f:\big)(u)\cdot (z-u)^n \end{aligned}$$

We note that

$$V_2(z)V_{-2}(u) = (z-u)^{\frac{2v}{w}} \cdot \sum_{t=0} \tilde{Y}_t(u)\cdot (z-u)^t \ ,$$

has all the claimed properties. Using (4.30) and (4.31), we write

$$e(z)f(u) = G_2(z)G_{-2}(u)\cdot (z-u)^{\frac{2v}{w}} \cdot \sum_{t=0} \tilde{Y}_t(u)\cdot (z-u)^t \ , \tag{4.35}$$

so we deduce that $G_2(z)G_{-2}(u)$ has an expansion of the form

$$G_2(z)G_{-2}(u)\cdot (z-u)^{\frac{2v}{w}} = \sum_{\ell=-2}^{\infty} X_\ell(u)\cdot (z-u)^\ell$$

where $X_\ell(z)$ is a field of $\mathsf{C}_k(\mathfrak{sl}_2)$ for all $\ell \geq -2$ with all the claimed properties. In particular, we find that $X_{-2}(u)$ is a scalar multiple of the identity, $X_{-1}=0$ since its conformal dimension has to be 1. Also by inspection of (4.35), we obtain (4.34).

Q.E.D.

In practice, we compute the required polynomial relation to complete the proof that $A_{k,M}$ (4.13) is C_2-cofinite as follows:

1. by assumption, the auxiliary lattice vertex operator algebra $\tilde{V}_{\tilde{L}^k}$ contains a

primary vector m as in (4.13) vertex operator fields $V_{\pm m}(z)$ which satisfy

$$\begin{aligned} V_m(z)V_{-m}(u) &= \sum_{b=0}^{\infty} B_b(u) \cdot (z-u)^{b-(m\cdot m)} \\ &= \sum_{b=0}^{\infty} B_b(u) \cdot (z-u)^{b-2\lceil \frac{v}{w} \rceil + \frac{2v}{w}} \end{aligned}$$

where $B_0(u)$ is a scalar multiple of the identity and $B_b(u)$ is a polynomial in the bosonic generators of the auxiliary lattice $\tilde{L}^k$ and their derivatives of homogeneous conformal dimension b for all $b > 0$;

2. Consider the fields $G_2(z) \otimes V_m(z)$ and $G_{-2}(z) \otimes V_{-m}(z)$ that complete L, W_3, W_4 and W_5 to form a strong generating set of $A_{k,M}$ from (4.14). By construction, the normally ordered product of $G_2(z) \otimes V_m(z)$ and $G_{-2}(z) \otimes V_{-m}(z)$ lies in

$$\mathsf{C}_k(\mathfrak{sl}_2) \otimes \mathrm{Span}_{\mathbb{C}}\{\text{`` bosons associated to } \tilde{L}^k \text{''}\} .$$

On the other hand, we know that we can write

$$G_2(z)G_{-2}(u) \otimes V_m(z)V_{-m}(u) = \sum_{\substack{\ell=-2 \\ b=0}}^{\infty} X_\ell(u) \otimes B_b(u) \cdot (z-u)^{\ell+b-2\lceil \frac{v}{w} \rceil} .$$

The normally ordered product is thus the coefficient of $(z-u)^0$, which is

$$\sum_{b\geq 0}^{2\lceil \frac{v}{w} \rceil+2} X_{2\lceil \frac{v}{w} \rceil - b}(u) \otimes B_b(u)$$

Next, we note $B_b(u) \equiv 0$ modulo $C_2(A_{k,M})$ so that the only part of this coefficient that could possibly have a non-trivial contribution for a relation in the C_2-quotient of $A_{k,M}$ is the parafermion field $X_{2\lceil \frac{v}{w} \rceil}(u)$;

3. now that we know that we need to determine $X_{2\lceil \frac{v}{w} \rceil}(u)$, here is what we do to compute it using the affine fields defined in Mathematica. Write down (4.34) specialised at $n = 2\left\lceil \frac{v}{w} \right\rceil$ so that it reads

$$\frac{1}{\left(2\left\lceil \frac{v}{w} \right\rceil\right)!}\left(: e^{\left(2\left\lceil \frac{v}{w} \right\rceil\right)} f : \right)(u) = \sum_{t=0}^{2\left\lceil \frac{v}{w} \right\rceil + 2} \tilde{Y}_t(u) \otimes X_{\left(2\left\lceil \frac{v}{w} \right\rceil\right) - t}(u) \; ; \qquad (4.36)$$

4. effectively, we just have to compute the left-hand side of (4.36) as a linear combinations of fields $h(z)$, $L(z)$,$W_3(z)$,$W_4(z)$, $W_5(z)$ and their derivatives and then set all the terms with $h(z)$ (and its derivatives) to zero. This way, we obtain $X_{2\lceil \frac{v}{w} \rceil}(z)$ (modulo the C_2-space of $A_{k,M}$) as per the right-hand side of (4.36).

Remark 4.37. Surprisingly, we do not need to define any auxiliary fields from $\tilde{L}^k$ to obtain a $\pm$ relation for C_2-cofiniteness. Although it may seem surprising on first look, one may observe that the existence of an $\tilde{L}^k$ with the properties stated in the the previous section is only required to ensure that one can define the vertex operator algebra $A_{k,M}$ as of (4.14). It is then $A_{k,M}$ whose very existence in turn implies that we may use the procedure described at the end of the preceding section.

The non-necessity of using auxiliary fields from $\tilde{L}^k$ in the computations with Mathematica has in fact showed up in my early attempts to preform the computation. In these attempts, coefficients associated with the auxiliary fields often turned out to be zero. It has since been satisfactory to observe how the theory caught up with the experiment!

Remark 4.38. In practice, to find a lattice $\tilde{L}^k$ for a given k, we look at the denominator in Equation (4.13) multiplied by 2. This denominator will generically be w if it is an odd number, or $\frac{w}{2}$ if w was even to start with. In Remark 4.16, we have mentioned to have found lattices $\tilde{L}^k$ to be a direct sum of lattices of type A_n for certain $n \in \mathbb{N}\backslash\{0\}$: the reason is that V_{A_n} should have a fundamental weight $\omega_1 \in (A_1)'$ should have a rational norm where the denominator a number closely related to n. We intend to develop more on this in the near future. In particular, a suitable choice of $\tilde{L}^k$ might even be made explicit for certain specific families of levels k such as boundary admissible.

4.2 Results

In this section, we record the polynomial relations found to bound the dimension of the space $\frac{A_{k,M}}{C_2\left(A_{k,M}\right)}$ for each of the $k \in A$ as of line (4.1). We then deduce C_2-cofiniteness of $A_{k,M}$ and of the extended parafermion vertex operator algebra $\mathsf{B}_k(\mathfrak{sl}_2)$.

4.2.1 Relations for Generic Negative Admissible Level k

Just as in [DLY 2009] and [ALY 2014], we have determined the strong generating fields $L(z)$, $W_3(z)$, $W_4(z)$ and $W_5(z)$ in terms of the fields $e(z)$, $f(z)$ and $h(z)$ from $L_k(\mathfrak{sl}_2)$. The explicit expressions can be found in Appendix C. In the C_2-quotient, non-trivial polynomial relations among the strong generating fields $L(z)$, $W_3(z)$, $W_4(z)$ and $W_5(z)$ have been found in spaces of conformal dimension 8, 9 and 10.

Recall from Result 4.6 that the map (4.8) factors through a quotient

$$\mathbb{C}[x_2, x_3, x_4, x_5, x_+, x_-]/P$$

where P is an ideal, see (4.9). The relations in the spaces of conformal dimensions

8, 9 and 10 already give us three relations in the C_2-quotient of $\mathsf{C}_k(\mathfrak{sl}_2) \subset \mathsf{B}_k(\mathfrak{sl}_2) \subset A_{k,M}$. After changing the normally ordered polynomials in fields $L(z)$, $W_3(z)$, $W_4(z)$, $W_5(z)$ by the corresponding polynomial in x_2, x_3, x_4 and x_5, we obtain the following three polynomial relations:

$$\begin{aligned}
x_4^2 = &\frac{18(72k^7+518k^6+1391k^5+1652k^4+732k^3)}{16k+17}x_2^2x_4 \\
&-\frac{162(104k^7+852k^6+2518k^5+3207k^4+1494k^3)}{64k+107}x_2x_3^2 \\
&+\frac{4536}{(16k+17)^2}(72k^{14}+828k^{13}+3934k^{12}+9713k^{11}+12324k^{10}+ \\
&4856k^9-6368k^8-8304k^7-2880k^6)x_2^4+\frac{3(2k+3)}{2(3k+4)}x_3x_5
\end{aligned}$$

$$\begin{aligned}
x_4x_5 = &\,972(12k^6+52k^5+75k^4+36k^3)x_3^3 \\
&+\frac{252(6k^7+41k^6+104k^5+116k^4+48k^3)}{16k+17}x_2^2x_5 \\
&-\frac{108(2148k^7+13736k^6+32679k^5+34306k^4+13416k^3)}{64k+107}x_2x_3x_4 \\
&-\frac{3888}{(16k+17)(64k+107)}(8088k^{14}+91220k^{13}+423682k^{12}+1014691k^{11} \\
&+1216690k^{10}+344852k^9-833704k^8-951744k^7-316800k^6)x_2^3x_3
\end{aligned}$$

$$
\begin{aligned}
x_5^2 &= 648(18k^6 + 75k^5 + 104k^4 + 48k^3)x_3^2x_4 \\
&\quad - \frac{432(498k^7 + 3121k^6 + 7269k^5 + 7466k^4 + 2856k^3)}{64k + 107}x_2x_3x_5 \\
&\quad - 20736(18k^{12} + 183k^{11} + 770k^{10} + 1716k^9 + 2136k^8 + 1408k^7 + 384k^6)x_2^3x_4 \\
&\quad + \frac{23328}{(16k + 17)(64k + 107)^2}(3123072k^{15} + 46891896k^{14} \\
&\quad + 309206212k^{13} + 1175980402k^{12} + 2844159575k^{11} + 4537865348k^{10} \\
&\quad + 4777381292k^9 + 3200532544k^8 + 1238077632k^7 + 210673152k^6)x_2^2x_3^2 \\
&\quad - \frac{373248}{16k + 17}(216k^{19} + 3204k^{18} + 20658k^{17} + 75103k^{16} + 165574k^{15} \\
&\quad + 215512k^{14} + 128048k^{13} - 49744k^{12} - 142624k^{11} - 95232k^{10} - 23040k^9)x_2^5
\end{aligned}
$$

Note that scaling $W_3(z)$, $W_4(z)$ and $W_5(z)$ slightly differently, one can get rid of all denominators in the above three relations. As a consequence, we should have three non-trivial relations of this type for generic values of k.

4.2.2 C_2-Cofiniteness of $\mathsf{B}_k(\mathfrak{sl}_2)$ at Specific Levels

In this section we establish C_2-cofiniteness of $\mathsf{B}_k(\mathfrak{sl}_2)$ for all $k \in A$ by the means of a combination an adaptation of Result 4.6 for $A_{k,M}$ instead of $\mathsf{B}_k(\mathfrak{sl}_2)$ and Result 4.12. With the computer, we must compute polynomial relations.

Following the ideas developed in this chapter, we explicit lists of polynomial relations composed of relations of the following types

- one for $s_k^{\text{para}} \in \mathsf{C}_k(\mathfrak{sl}_2)$, the parafermion singular vector;
- several others corresponding to the null fields that s_k^{para} produces by taking the operator product expansion coefficients with the strong generators of $\mathsf{C}_k(\mathfrak{sl}_2)$;

- three for the relations in spaces of conformal dimension 8, 9 and 10, respectively. We can just substitute the value of k in the relations obtained at the previous section;

- the two relations $x_\pm^2 = 0$ where $x_\pm$ corresponds to $G_\pm(z) \otimes V_{\pm m}(z)$;

- one for $X_{2\lceil \frac{v}{w} \rceil}(z)$ computed from expressing

$$\left(: e^{\left(2\lceil \frac{v}{w} \rceil\right)} f : \right)(z) \ ,$$

in terms of $h(z)$, $L(z)$, $W_3(z)$, $W_4(z)$,$W_5(z)$ and their derivatives. We then set $h(z)$ and all derivatives to zero. The result should allow to show that $A_{k,M}$ is C_2-cofinite, which then implies C_2-cofiniteness of $\mathsf{B}_k(\mathfrak{sl}_2)$.

After listing all relevant polynomials, we consider the ideal

$$P_k \in \mathbb{C}[x_2, x_3, x_4, x_5, x_+, x_-] \ ,$$

generated by all of them. We then compute a Gröbner basis of P_k and argue that P_k has finite codimension by computing the associated affine variety. By an adaptation of Result 4.6 for $A_{k,M}$ and Result 4.12, we will would establish C_2-cofiniteness of the extended logarithmic parafermion vertex operator algebra $\mathsf{B}_k(\mathfrak{sl}_2)$.

Let's now treat the cases $k \in A$ from line (4.1) one by one. Before doing so, we record a few special levels where the singular vector is proportional to a strong generator of $\mathsf{C}_k(\mathfrak{sl}_2)$ in Table 4.3.

Case $k = -\frac{1}{2}$. In this case, $w = 1$ and the parafermion singular vector can be taken to be $s^{\text{para}}_{-\frac{1}{2}} = W_4$, so we record the relation $\boxed{x_4 = 0}$.

$k \in A$	$-\frac{1}{2}$	$-\frac{4}{3}$	$-\frac{8}{5}$
$s_k^{\text{para}} \propto$	W_4	W_3	W_5

Table 4.3: Special coincidences of parafermion singular vector.

Taking $x_4 = 0$ into account, the non-trivial null relations we obtain from the operator product expansion of $s^{\text{para}}_{-\frac{1}{2}}$ with the strong generating fields $L(z)$, $W_2(z)$,$W_3(z)$, $W_4(z)$ and W_5 are:

$$\boxed{0 = 2025x_2^3 - 648x_3^2}\,, \quad \boxed{0 = \frac{171}{2}x_2x_5}\,, \quad \boxed{0 = x_2^4 - \frac{8}{25}x_2x_3^2 - \frac{16}{4725}x_3x_5}\,.$$

The three relations in conformal dimension 8, 9 and 10 together with $x_4 = 0$ found above lead to the following two non-redundant relations:

$$\boxed{0 = \frac{30375}{8}x_2^3x_3 - \frac{315}{8}x_2^2x_5 - 1215x_3^3}\,, \quad \boxed{x_5^2 = \frac{27}{14}x_2x_3x_5 - x_5^2}\,.$$

Next, we find relations for the variables $x_\pm$. In this specific case, $w = 1$ and $v = 2$ so $W_\pm(z) = G_{\pm 2}(z) \in \mathcal{C}^{\mathcal{L}}_{2;1}$. Moreover (4.34) allows us to directly compute $\big(: W_+W_- : \big)(u)$ by setting $2v = 4$ and let all $h(u)$ and their derivatives to be zero in the resulting expression. This is why in this very specific case, we do not even need the existence of any lattice with special properties. Using also $x_4 = 0$ from the singular vector, we obtain the relation:

$$\boxed{x_+x_- = 144x_2^3 - \frac{128}{3}x_3^2}$$

where we note that $\frac{128}{3*144} \neq \frac{8}{25}$ so this relation is not a priori redundant. Next, as $\big(:$

$W_\pm W_\pm :\big)(z) \ = 0$ modulo $C_2\left(\mathsf{B}_k(\mathfrak{sl}_2)\right)$, so we also record: $\boxed{0 = \left(144x_2^3 - \frac{128}{3}x_3^2\right)^2}$.

Eliminating the variables x_4 found to be zero above, we consider the ideal $P_{-\frac{1}{2}}$ generated by the above boxed polynomials:

$$P_{-\frac{1}{2}} \lhd \mathbb{C}[x_2, x_3, x_5, x_+, x_-] \subset \mathbb{C}[x_2, x_3, x_4, x_5, x_+, x_-] \ .$$

To determine it has finite codimension, we compute a Gröbner basis with lexicographic variable ordering $x_2 < x_3 < x_5 < x_+ < x_-$:

$$\left\{x_-^2, x_+^2, x_-x_+x_5, x_5^2, x_3x_5, 256x_3^2 - 75x_-x_+, x_2x_5, 32x_2^3 - 3x_-x_+\right\} \ .$$

Let $B_{-\frac{1}{2}}$ denote the ideal generated by the polynomials of the above Gröbner basis. Recall that then the affine variety $V\left(P_{-\frac{1}{2}}\right)$ coincides with the affine variety $V\left(B_{-\frac{1}{2}}\right)$. By Elimination Theory and the properties of Gröbner bases [CLO 1997], we can easily compute $V\left(B_{-\frac{1}{2}}\right)$ as follows:

1. $0 \in \mathbb{C}$ is the only point of $V\left(\langle x_-^2\rangle\right) = V\left(\langle x_-\rangle\right)$ where $\langle x_-^2\rangle = B_{-\frac{1}{2}} \cap \mathbb{C}[x_-]$ is the fourth elimination ideal;

2. $0 \in \mathbb{C}$ is the only point a such that $(a, 0) \in \mathbb{C}^2$ is actually in

$$V\left(\langle x_-^2, x_+^2\rangle\right) = V\left(\langle x_-, x_+\rangle\right)$$

   where $\langle x_-^2, x_+^2\rangle = B_{-\frac{1}{2}} \cap \mathbb{C}[x_+, x_-]$ is the third elimination ideal;

3. $0 \in \mathbb{C}$ is the only point $a \in \mathbb{C}$ such that $(a, 0, 0) \in \mathbb{C}^3$ is actually in

$$V\left(\langle x_-^2, x_+^2, x_-x_+x_5, x_5^2\rangle\right) = V\left(\langle x_-, x_+, x_-x_+x_5, x_5\rangle\right)$$

where $\langle x_-^2, x_+^2, x_-x_+x_5, x_5^2\rangle = B_{-\frac{1}{2}} \cap \mathbb{C}[x_5, x_+, x_-]$ is the second elimination ideal;

4. $0 \in \mathbb{C}$ is the only point $a \in \mathbb{C}$ such that $(a, 0, 0, 0) \in \mathbb{C}^4$ is actually in

$$V\left(\langle x_-^2, x_+^2, x_-x_+x_5, x_5^2, x_3x_5, 256x_3^2 - 75x_-x_+\rangle\right) \ ,$$

which is the varitety of the first elimination ideal;

5. $0 \in \mathbb{C}$ is the only point $a \in \mathbb{C}$ such that $(a, 0, 0, 0, 0) \in \mathbb{C}^5$ is actually in the full affine variety $V\left(B_{-\frac{1}{2}}\right)$.

It follows that $V\left(B_{-\frac{1}{2}}\right) = V\left(P_{-\frac{1}{2}}\right) = \left\{(0,0,0,0)\right\} \subset \mathbb{C}^4$, and so this affine variety is zero dimensional. By Theorem 5.3.6 of [CLO 1997], this statement directly implies that

$$\dim_{\mathbb{C}}\left(\frac{\mathbb{C}[x_2, x_3, x_4, x_5, x_+, x_-]}{P_{-\frac{1}{2}}}\right) < \infty \ .$$

Finally, since the map (4.8) factors through $\mathbb{C}[x_2, x_3, x_5, x_+, x_-]/P_{-\frac{1}{2}}$ (by construction), we conclude that $\mathsf{B}_{-\frac{1}{2}}(\mathfrak{sl}_2)$ is C_2-cofinite. In particular, it is a logarithmic C_2-cofinite vertex operator algebra.

Case $k = -\frac{4}{3}$. In this case, the parafermion singular vector can be taken to be $s^{\text{para}}_{-\frac{4}{3}} = W_3$, so we record the relation $\boxed{x_3 = 0}$.

Note that since we can find $W_4(z)$ and $W_5(z)$ in the operator product expansion coefficients of combinations of $W_3(z) = 0$ and $L(z)$, we expect to eventually find $x_4 = 0 = x_5$. Taking the $x_3 = 0$ into account, the only non-trivial null relation we obtain from the operator product expansion coefficients of $s^{\text{para}}_{-\frac{4}{3}}$ with the strong generators of $\mathsf{C}_{-\frac{4}{3}}(\mathfrak{sl}_2)$ is $\boxed{\frac{1024}{27}x_2x_4 = 0}$.

There is no genuine relation in conformal dimension 8 in this specific case. However the genuine relations in conformal dimension 9 and 10 together with the relation $x_3 = 0$ found above lead to: $\boxed{x_4x_5 = 0}$ and $\boxed{x_5^2 = 0}$.

Next, the variables $x_\pm \leftrightarrow G_\pm(z) \otimes V_{\pm m}(z) \bmod C_2\left(A_{-\frac{4}{3},A_3}\right)$. Recall that we always have the two relations $\boxed{x_\pm^2 = 0}$. The polynomial relation we find (having implemented $x_3 = 0$ found above) is

$$\boxed{x_+x_- = x_2^2 - \frac{27}{512}x_4} \ .$$

As $x_\pm^2 = 0$, we also record: $\boxed{0 = \left(x_2^2 - \frac{27}{512}x_4\right)^2}$.

Eliminating the variable x_3 found to be zero above, we consider the ideal $P_{-\frac{4}{3}}$ generated by the above boxed polynomials:

$$P_{-\frac{4}{3}} \lhd \mathbb{C}[x_2, x_4, x_5, x_+, x_-] \subset \mathbb{C}[x_2, x_3x_4, x_5, x_+, x_-] \ .$$

To determine if it has finite codimension, we compute a Gröbner basis with lexicographic variable ordering $x_2 < x_4 < x_5 < x_+ < x_-$:

$$\left\{x_-^2, x_+^2, x_5^2, x_4x_5, 27x_4^2 + 512x_-x_+x_4, x_2x_4, 512x_2^2 - 512x_-x_+ - 27x_4\right\} \ .$$

Let $B_{-\frac{4}{3}}$ denote the ideal generated by the polynomials of the above Gröbner basis. We compute the variety $V\left(B_{-\frac{4}{3}}\right)$ as follows:

1. $0 \in \mathbb{C}$ is the only point of $V\left(\langle x_-^2\rangle\right)$ where $\langle x_-^2\rangle = B_{-\frac{4}{3}} \cap \mathbb{C}[x_-]$;

2. $0 \in \mathbb{C}$ is the only point a such that $(a, 0) \in \mathbb{C}^2$ is actually in $V\left(\langle x_-^2, x_+^2\rangle\right)$;

3. $0 \in \mathbb{C}$ is the only point $a \in \mathbb{C}$ such that $(a,0,0) \in \mathbb{C}^3$ is actually in $V\left(\langle x_-^2, x_+^2, x_5^2\rangle\right)$;

4. $0 \in \mathbb{C}$ is the only point $a \in \mathbb{C}$ such that $(a,0,0,0) \in \mathbb{C}^4$ is actually in $V\left(\langle x_-^2, x_+^2, x_5^2, x_4x_5, 27x_4^2 + 512x_-x_+x_4\rangle\right)$;

5. $0 \in \mathbb{C}$ is the only point $a \in \mathbb{C}$ such that $(a,0,0,0) \in \mathbb{C}^5$ is the such that $(a,0,0,0,0) \in \mathbb{C}^4$ is actually in the full $V\left(B_{-\frac{4}{3}}\right)$.

So $V\left(P_{-\frac{4}{3}}\right) = V\left(B_{-\frac{4}{3}}\right) = \{(0,0,0,0,0)\}$ is a point. Then this variety is zero dimensional and we conclude by Theorem 5.3.6 of [CLO 1997] that $P_{-\frac{4}{3}}$ has finite codimension in $\mathbb{C}[x_2, x_4, x_5, x_+, x_-]$. As for the other case, this imply that $A_{-\frac{4}{3}, A_3}$ and $\mathsf{B}_{-\frac{4}{3}}(\mathfrak{sl}_2)$ are C_2-cofinite vertex operator algebras!

Case $k = -\frac{8}{5}$. In this case, the parafermion singular vector can be taken to be $s^{\text{para}}_{-\frac{8}{5}} = W_5$, so we record the relation $\boxed{x_5 = 0}$.

Taking the $x_5 = 0$ into account, the non-trivial null relation obtained from the operator product expansion coefficients of $s^{\text{para}}_{-\frac{8}{5}}$ with $L(z)$, $W_3(z)$, $W_4(z)$ and $W_5(z)$ are

$$\boxed{0 = -\frac{2629632}{3359375}x_2^3 + \frac{1}{20}x_4x_2 - \frac{3}{40}x_3^2},$$

$$\boxed{0 = \frac{10857140453376}{10498046875}x_2^3 - \frac{1032192}{15625}x_4x_2 + \frac{1548288}{15625}x_3^2},$$

$$\boxed{0 = \frac{3514368}{1596875}x_4x_2^2 - \frac{2261495808}{1596875}x_3^2x_2 + \frac{23005}{1533}x_4^2}.$$

The genuine relations in conformal dimensions 8, 9 and 10 together with $x_5 = 0$ found above lead to:

$$\boxed{x_4^2 = \frac{5555405979648}{451416015625}x_2^4 - \frac{626688}{671875}x_4x_2^2 + \frac{1492992}{15625}x_3^2x_2},$$

$$\boxed{0 = \frac{13959180582912}{10498046875}x_3x_2^3 - \frac{1327104}{15625}x_3x_4x_2 + \frac{1990656}{15625}x_3^3}\,,$$

$$\boxed{0 = \frac{187240672945137254 4}{164031982421875}x_2^5 + \frac{695784701952}{244140625}x_4x_2^3 - \frac{212910118797312}{10498046875}x_3^2x_2^2 + \frac{5308416}{15625}x_3^2x_4}\,.$$

Next, the variables $x_\pm \leftrightarrow G_\pm(z) \otimes V_{\pm m}(z) \bmod C_2\left(A_{-\frac{8}{5},A_7\oplus A_3}\right)$. Recall that we always have $\boxed{x_\pm^2 = 0}$. Using $x_4 = 0$ found above, we obtain

$$\boxed{x_+x_- = -\frac{73}{430}x_2^2 - \frac{3125}{73728}x_4}\,.$$

We also record that the right side of the above equation squares to zero.

Eliminating the variables x_5 previously found to be zero, we consider the ideal $P_{-\frac{8}{5}}$ and compute its associated affine variety by obtaining a Gröbner basis with lexicographic variable ordering $x_2 < x_3 < x_4 < x_+ < x_-$ using Mathematica. Proceeding with the routine use of the elimination ideals, it is then straightforward to check that $V\left(P_{-\frac{8}{5}}\right) = \big\{(0,0,0,0,0)\big\} \subset \mathbb{C}^5$.

As in the other cases treated above, this implies that the vertex operator algebras $A_{-\frac{8}{5},A_7\oplus A_3}$ and $\mathsf{B}_{-\frac{8}{5}}(\mathfrak{sl}_2)$ are C_2-cofinite!

Case $k = -\frac{5}{4}$. In this case, the parafermion singular vector $s^{\text{para}}_{-\frac{5}{4}}$ is in the space of conformal dimension 8. It has a complicated shape, but its reduction to the C_2-quotient of $\mathsf{B}_{-\frac{5}{4}}(\mathfrak{sl}_2)$ gives the polynomial relations[6]

$$0 = x_2^4 - \frac{37670912x_4x_2^2}{362112375} + \frac{4194304x_4^2}{2930765625} - \frac{67108864x_3x_5}{9052809375}\,,$$

$$0 = \frac{28659x_4x_2^2}{965633} + x_3^2x_2 - \frac{22528x_4^2}{7815375} - \frac{1707008x_3x_5}{362112375}\,,$$

[6]More than one relation shows up because the singular vector $s^{\text{para}}_{-\frac{5}{4}}$ lies in the space of conformal dimension 8, which is greater than 7. One of these relations is thus redundant with a genuine relation among generators in conformal dimension 8.

which is in fact the singular vector up to some genuine null relation modulo $C_2\left(\mathsf{B}_{-\frac{5}{4}}(\mathfrak{sl}_2)\right)$.The non-redundant null relations obtained from the operator product expansion coefficients of $s^{\text{para}}_{-\frac{5}{4}}$ with $L(z)$, $W_3(z)$, $W_4(z)$ and $W_5(z)$ are the following:

$$\begin{aligned} 0 = & -\frac{167030078104 x_4 x_2^2}{570326990625} - \frac{17484568 x_3^2 x_2}{1771875} \\ & + \frac{393892347904 x_4^2}{13847867578125} + \frac{29846297452544 x_3 x_5}{641617864453125} \ , \end{aligned}$$

$$\begin{aligned} 0 = & -\frac{221407084584 x_3^3}{42849964375} + \frac{453890642996 x_2 x_4 x_3}{1156949038125} \\ & - \frac{2526520076 x_2^2 x_5}{42849964375} + \frac{1763563466752 x_4 x_5}{433855889296875} \ , \end{aligned}$$

$$0 = -\frac{4321903270941 x_4 x_2^2}{43260358400} - \frac{150804399 x_3^2 x_2}{44800} + \frac{4423595704 x_4^2}{455896875} + \frac{335187910844 x_3 x_5}{21123221875} \ ,$$

$$\begin{aligned} 0 = & \frac{30231524208134 39 x_4 x_2^3}{18942969337216} - \frac{587080074087856 x_4^2 x_2}{277484902400625} + \frac{1820781698047432 x_3 x_5 x_2}{92494967466875} \\ & - \frac{1695724461924352 x_5^2}{34685612800078125} - \frac{393219331912544 x_3^2 x_4}{92494967466875} \ , \end{aligned}$$

$$\begin{aligned} 0 = & \frac{9050014582371 x_3^3}{8775672704} - \frac{12368520021641 x_2 x_4 x_3}{157962108672} \\ & + \frac{206543016213 x_2^2 x_5}{17551345408} - \frac{187723064332 x_4 x_5}{231389807625} \ . \end{aligned}$$

The genuine relations in dimension 8, 9 and 10 are:

$$x_4^2 = -\frac{1993359375}{4194304} x_2^4 + \frac{3375}{64} x_4 x_2^2 + \frac{113625}{1024} x_3^2 x_2 + 3 x_3 x_5$$

$$x_4 x_5 = -\frac{8536640625}{2097152} x_3 x_2^3 + \frac{23625}{2048} x_5 x_2^2 + \frac{83625}{1024} x_3 x_4 x_2 - \frac{30375}{256} x_3^3 \ ,$$

$$
\begin{aligned}
x_5^2 =& \frac{2883251953125}{67108864} x_2^5 - \frac{34171875}{32768} x_4 x_2^3 + \frac{4875890625}{1048576} x_3^2 x_2^2 \\
&+ \frac{22875}{512} x_3 x_5 x_2 - \frac{10125}{256} x_3^2 x_4 \ .
\end{aligned}
$$

Next, the variables $x_\pm \leftrightarrow G_{\pm 2}(z) \otimes V_{\pm m}(z) \bmod C_2\left(A_{-\frac{5}{4}, A_4}\right)$. Recall that we always have $\boxed{x_\pm^2 = 0}$. In this case, we find that: $\boxed{x_+ x_- = \frac{256 x_4}{5625} - \frac{15 x_2^2}{8}}$. We also record that the right side of the above equation squares to zero.

Let $P_{-\frac{5}{4}}$ denote the ideal generated by all the above polynomials. We obtain a Gröbner basis with the computer and follow the elimination process as above to determine its associated affine variety. In this case, the elimination process is straightforward and we deduce that $V\left(P_{-\frac{5}{4}}\right) = \{(0,0,0,0,0,0)\}$.

Therefore $\mathsf{B}_{-\frac{5}{4}}(\mathfrak{sl}_2)$ is another C_2-cofinite logarithmic vertex operator algebra!

Case $k = -\frac{7}{5}$. In this case, the parafermion singular vector $s^{\mathrm{para}}_{-\frac{7}{5}}$ is in the space of conformal dimension 10. It has a complicated shape, but its reduction to the C_2-quotient of $\mathsf{B}_{-\frac{7}{5}}(\mathfrak{sl}_2)$ gives the polynomial relations

$$
\begin{aligned}
0 = x_2^5 &- \frac{1189133300781 25 x_4^2 x_2}{110818158196028724} - \frac{17005358886718 75 x_3 x_5 x_2}{110818158196028724} \\
&- \frac{2843620452880859375 x_5^2}{8210295704427376103712} - \frac{8627197265625 x_3^2 x_4}{12313128688447636}
\end{aligned}
$$

$$
\begin{aligned}
0 = x_2^2 x_3^2 &- \frac{776755484375 x_4 x_3^2}{107695003105956} - \frac{16058659375 x_2 x_5 x_3}{1648392904683} \\
&- \frac{189225000000 x_2 x_4^2}{8974583592163} + \frac{604660931640625 x_5^2}{239367221703422043 84}
\end{aligned}
$$

$$0 = x_4x_2^3 - \frac{2932006609375x_4^2x_2}{53847501552978} - \frac{14994586121875x_3x_5x_2}{107695003105956} + \frac{5616852935546875x_5^2}{7978907390114068128} + \frac{1118394421875x_3^2x_4}{35898334368652} ,$$

which is in fact the singular vector up to some genuine null relation modulo $C_2\left(\mathsf{C}_{-\frac{5}{4}}(\mathfrak{sl}_2)\right)$.The non-redundant null relations obtained from the operator product expansion coefficients of $s^{\text{para}}_{-\frac{7}{5}}$ with $L(z)$, $W_3(z)$, $W_4(z)$ and $W_5(z)$ are the two following:

$$0 = -\frac{939048374945}{111560627598}x_4x_2^3 + \frac{2753296041861593152109375}{6007261067834505325886844}x_4^2x_2 + \frac{1408064173071956846642187 5}{1201452213566901065177368 8}x_3x_5x_2 - \frac{527449662143034579362304687 5x_5^2}{890131915987445661168608996544} - \frac{3500754881364238366406 25x_3^2x_4}{1334946903963223405752632} ,$$

$$0 = -\frac{915008736546408x_4x_2^3}{1185805990625} + \frac{20440469814780467561260x_4^2x_2}{486496685117792786353} + \frac{52267342104431038147358x_3x_5x_2}{486496685117792786353} - \frac{97894657293747217929643 75x_5^2}{180217832035035159776605 32} - \frac{11695321907661647534490x_3^2x_4}{486496685117792786353} ,$$

The genuine relations in dimension 8, 9 and 10 are:

$$x_4^2 = \frac{3972771432}{244140625}x_2^4 + \frac{109074}{15625}x_4x_2^2 + \frac{12946878}{453125}x_3^2x_2 - \frac{3}{2}x_3x_5 ,$$

$$x_4x_5 = \frac{2556821480976}{7080078125}x_3x_2^3 - \frac{28812}{15625}x_5x_2^2 - \frac{10705716}{453125}x_3x_4x_2 + \frac{333396}{15625}x_3^3 ,$$

$$
\begin{aligned}
x_5^2 = \frac{9082327572638208}{3814697265625} x_2^5 - \frac{65868380928}{244140625} x_4 x_2^3 \\
+ \frac{118177938609696}{205322265625} x_3^2 x_2^2 - \frac{2074464}{453125} x_3 x_5 x_2 - \frac{222264}{15625} x_3^2 x_4 \ .
\end{aligned}
$$

Next, we find relations for the variables $x_\pm \leftrightarrow G_{\pm 2}(z) \otimes V_{\pm m}(z) \bmod C_2\left(A_{-\frac{7}{5},A_6}\right)$. Recall that we always have $\boxed{x_\pm^2 = 0}$. In this case, $W_\pm(z) \in \mathcal{C}^{\mathcal{L}}_{14;1}$ since $w = 2v - u = 7$. Then using two bosonic auxiliary fields we find that:

$$
\boxed{x_+ x_- = \left(\frac{3125}{43218} x_4 - \frac{67}{105} x_2^2\right)^7} \ .
$$

We also record that the right side of the above equation squares to zero.

Let $P_{-\frac{7}{5}}$ denote the ideal generated by all the above polynomials. We obtain a Gröbner basis with the computer and follow the elimination process as above to determine its associated affine variety. In this case, the elimination process is straightforward and we deduce $V\left(P_{-\frac{7}{5}}\right) = \big\{(0,0,0,0,0,0)\big\}$. Therefore $A_{-\frac{7}{5},A_6}$ and $\mathsf{B}_{-\frac{7}{5}}(\mathfrak{sl}_2)$ are C_2-cofinite vertex operator algebras.

Case $k = -\frac{12}{7}$. In this case, the parafermion singular vector $s^{\text{para}}_{-\frac{12}{7}}$ is in the space of conformal dimension 7. It has a complicated shape, but its reduction to the C_2-quotient of $\mathsf{B}_{-\frac{12}{7}}(\mathfrak{sl}_2)$ gives the polynomial relation

$$
\boxed{0 = -\frac{163200}{3330187} x_3 x_2^2 + \frac{49}{1679616} x_5 x_2 - \frac{49}{1119744} x_3 x_4} \ .
$$

The non-redundant null relations obtained from the OPEs of $s^{\text{para}}_{-\frac{12}{7}}$ with $L(z)$, $W_3(z)$, $W_4(z)$ and $W_5(z)$ are the following:

$$
0 = \frac{163200}{475741} x_3 x_2^2 - \frac{343}{1679616} x_5 x_2 + \frac{343}{1119744} x_3 x_4 \ ,
$$

$$0 = -\frac{28576}{6936489}x_4x_2^2 + \frac{13675752}{4881233}x_3^2x_2 + \frac{1634689}{359437824}x_4^2 - \frac{154399}{59906304}x_3x_5 \ ,$$

$$0 = -\frac{52264275148800}{4085843062357}x_3x_2^2 + \frac{572}{75117}x_5x_2 - \frac{286}{25039}x_3x_4 \ ,$$

$$0 = -\frac{24786000}{16200233}x_3^3 - \frac{9652920}{4216499}x_2x_4x_3 - \frac{4789520}{1182617009}x_2^2x_5 + \frac{728875}{489691008}x_4x_5 \ ,$$

$$\begin{aligned}0 = \ & -\frac{1097780428800}{30224851643}x_4x_2^2 + \frac{269514805955788800}{10911171443123}x_3^2x_2 \\ & + \frac{587153600}{14643699}x_4^2 - \frac{110915200}{4881233}x_3x_5 \ ,\end{aligned}$$

$$\begin{aligned}0 = & \frac{1972560521134080}{78830730920893}x_4x_2^3 + \frac{1319315915480}{114578576847}x_4^2x_2 + \frac{64531360800}{12730952983}x_3x_5x_2 \\ & - \frac{1098641005}{243046531584}x_5^2 + \frac{109811372740}{12730952983}x_3^2x_4 \ .\end{aligned}$$

The genuine relations in dimension 8, 9 and 10 are:

$$x_4^2 = \frac{238505686990848}{10537174213447}x_2^4 + \frac{1866240}{8588377}x_4x_2^2 - \frac{1355450112}{2235331}x_3^2x_2 + \frac{9}{16}x_3x_5 \ ,$$

$$x_4x_5 = \ -\frac{69224603880259584}{19197865347787}x_3x_2^3 + \frac{5971968}{1226911}x_5x_2^2 + \frac{3430895616}{2235331}x_3x_4x_2 + \frac{120932352}{117649}x_3^3 \ ,$$

$$\begin{aligned}x_5^2 = & \frac{37982622214739628130304}{118874192647462777}x_2^5 + \frac{95105071448064}{13841287201}x_4x_2^3 \\ & - \frac{600419976231650328576}{364759441607953}x_3^2x_2^2 + \frac{5840584704}{2235331}x_3x_5x_2 + \frac{214990848}{117649}x_3^2x_4 \ .\end{aligned}$$

Next, we find relations for the variables $x_\pm \leftrightarrow G_{\pm 2}(z) \otimes V_{\pm m}(z) \bmod C_2\left(A_{-\frac{12}{7}, A_3 \oplus A_5}\right)$.

In this case, we obtain:

$$\boxed{x_+x_- = -\frac{107x_2^2}{1533} - \frac{16807x_4}{1119744}}\,.$$

We also record that the right side of the above equation squares to zero.

Let $P_{-\frac{12}{7}}$ denote the ideal generated by all the above polynomials. We obtain a Gröbner basis with the computer and follow the elimination process as above to determine its associated affine variety. In this case, the elimination process is straightforward and we deduce $V\left(P_{-\frac{12}{7}}\right) = \{(0,0,0,0,0,0)\}$. Therefore $A_{-\frac{12}{7},A_3\oplus A_5}$ and $\mathsf{B}_{-\frac{12}{7}}(\mathfrak{sl}_2)$ are other C_2-cofinite vertex operator algebra!

Case $k = -\frac{16}{9}$. In this case, the parafermion singular vector $s^{\text{para}}_{-\frac{16}{9}}$ is in the space of conformal dimension 9. It has a complicated shape, but its reduction to the C_2-quotient of $\mathsf{B}_{-\frac{16}{9}}(\mathfrak{sl}_2)$ gives the polynomial relations

$$0 = x_3^3 + \frac{6729402}{13223275}x_2x_4x_3 + \frac{78751}{44655650}x_2^2x_5 - \frac{126502641}{284131328000}x_4x_5\,,$$

$$0 = x_3x_2^3 - \frac{3322234521}{4318796185600}x_5x_2^2 + \frac{1782039771}{539849523200}x_3x_4x_2 - \frac{2434162932387}{2830366268194816000}x_4x_5\,,$$

which is in fact the singular vector up to some genuine null relation modulo $C_2\left(\mathsf{B}_{-\frac{16}{9}}(\mathfrak{sl}_2)\right)$. Null relations obtained from the operator product expansion coefficients of $s^{\text{para}}_{-\frac{16}{9}}$ with $L(z)$, $W_3(z)$, $W_4(z)$ and $W_5(z)$ include the following:

$$\begin{aligned}0 = &-\frac{87420035596x_4x_2^3}{5991911555535} - \frac{90842849951x_4^2x_2}{19950511390720} - \frac{58231219427x_3x_5x_2}{39901022781440}\\ &+\frac{66932590636017x_5^2}{26149534290044518400} - \frac{367439665479x_3^2x_4}{39901022781440}\end{aligned}$$

$$0 = \frac{316925050x_3^3}{73660347} + \frac{31839436x_2x_4x_3}{14541363} + \frac{57566981x_2^2x_5}{7587015741} - \frac{14094411x_4x_5}{7357726720}$$

$$0 = \frac{829728807356 x_5 x_2^3}{270502934862429} + \frac{761050557088 x_3 x_4 x_2^2}{875414028681} - \frac{16560666313 x_4 x_5 x_2}{21860666736640} - \frac{26653523 x_3 x_4^2}{5305987072} + \frac{133267615 x_3^2 x_5}{42447896576}$$

$$0 = -\frac{12810400536504303616 0 x_4 x_2^3}{1438853300800663941} - \frac{2031246124904360 x_4^2 x_2}{73101320977527} - \frac{651025033193860 x_3 x_5 x_2}{73101320977527} + \frac{31162214681 x_5^2}{1995051139072} - \frac{1369325153351740 x_3^2 x_4}{24367106992509}$$

The genuine relations in dimension 8, 9 and 10 are

$$x_4^2 = \frac{70652212019200 x_2^4}{4110143967801} + \frac{163840 x_4 x_2^2}{225261} - \frac{135577600 x_3^2 x_2}{400221} + \frac{5 x_3 x_5}{8}$$

$$x_4 x_5 = -\frac{1778975454003200 x_3 x_2^3}{2434162932387} + \frac{9175040 x_5 x_2^2}{2027349} + \frac{456785920 x_3 x_4 x_2}{400221} + \frac{1638400 x_3^3}{729}$$

$$x_5^2 = \frac{2116691824864133120 0 x_2^5}{785436540953661} + \frac{2748779069440 x_4 x_2^3}{387420489} - \frac{99807137219215360 00 x_3^2 x_2^2}{16498215430623} + \frac{673710080 x_3 x_5 x_2}{400221} + \frac{2621440}{729} x_3^2 x_4$$

Next, the variables $x_\pm \leftrightarrow G_{\pm 2}(z) \otimes V_{\pm m}(z) \bmod C_2\left(A_{-\frac{16}{9}, A_3 \oplus A_1 5}\right)$. In this case, we find that:

$$\boxed{x_+ x_- = -\frac{47 x_2^2}{1236} - \frac{6561 x_4}{655360}}\,.$$

We also record that the right side of the above equation squares to zero.

Let $P_{-\frac{16}{9}}$ denote the ideal generated by all the above polynomials. We compute a Gröbner basis with the computer and follow the elimination process as above. With a little more work at each elimination step, we show with the help of the computer that $V\left(P_{-\frac{16}{9}}\right) = \{(0,0,0,0,0,0)\}$. Therefore $A_{-\frac{16}{9}, A_3 \oplus A_{15}}$ and $\mathsf{B}_{-\frac{16}{9}}(\mathfrak{sl}_2)$ are other C_2-cofinite vertex operator algebras!

Case $k = -\frac{2}{3}$. The C_2-cofiniteness of $\mathsf{B}_{-\frac{2}{3}}(\mathfrak{sl}_2)$ is established in Remark 4.20 of [ACR 2018] assuming validity of a conjecture. This conjecture says that a certain vertex operator algebra result of [Miy 2015] should still hold for an appropriate vertex operator superalgebras. In the following, we record what can be inferred from the same procedure as above, although we cannot quite arrive to a definitive proof.

For this level, the parafermion singular vector $s^{\mathrm{para}}_{-\frac{2}{3}}$ is in the space of conformal dimension 9. It has a complicated shape, but its reduction to the C_2-quotient of $\mathsf{B}_{-\frac{2}{3}}(\mathfrak{sl}_2)$ gives the polynomial relation

$$\boxed{-\frac{496545}{176416}x_3^3 - \frac{108959319}{42560360}x_2x_4x_3 + \frac{752121}{1047470}x_2^2x_5 + \frac{5325183}{282265600}x_4x_5 = 0}\,.$$

The non-redundant null relations obtained from the operator product expansion coefficient of $s^{\mathrm{para}}_{-\frac{2}{3}}$ with $L(z)$, $W_3(z)$, $W_4(z)$ and $W_5(z)$ are the following:

$$0 = -\frac{6769089}{1047470}x_2^2x_5 + \frac{980633871}{42560360}x_2x_3x_4 + \frac{4468905}{176416}x_3^3 - \frac{47926647}{282265600}x_4x_5$$

$$\begin{aligned}0 =& \frac{7094834624}{2046195}x_2^3x_4 + \frac{3280059}{303140}x_2x_3x_5 + \frac{9312603}{60628}x_2x_4^2\\ &- \frac{2141834337}{4850240}x_3^2x_4 - \frac{2464860321}{6208307200}x_5^2\end{aligned}$$

$$\begin{aligned}0 =& -\frac{34917938085478840}{2132544429}x_2^3x_4 - \frac{44842051040}{8775903}x_2x_3x_5\\ &- \frac{1909704455200}{26327709}x_2x_4^2 + \frac{1830078450170}{8775903}x_3^2x_4 + \frac{363745459}{1940096}x_5^2\end{aligned}$$

The genuine relations in dimension 8, 9 and 10 are

$$x_4^2 = -\frac{45875200}{29241}x_2^4 - \frac{47360}{513}x_2^2x_4 + \frac{315200}{1737}x_2x_3^2 + \frac{5}{4}x_3x_5$$

$$x_4x_5 = \frac{36129996800}{2673243}x_2^3x_3 - \frac{35840}{513}x_2^2x_5 + \frac{1230080}{1737}x_2x_3x_4 - 1600x_3^3$$

$$x_5^2 = \frac{5242880}{567}x_2^3x_4 - \frac{794663321600}{1203850431}x_2^2x_3^2 + \frac{1761280}{36477}x_2x_3x_5 + \frac{77824}{189}x_2x_4^2 - 1280x_3^2x_4$$

Next, we find relations for $x_\pm \leftrightarrow G_{\pm 2}(z) \otimes V_{\pm m}(z) \bmod C_2\left(A_{-\frac{2}{3},A_1}\right)$. In this case, we obtain:

$$\boxed{x_+x_- = -\frac{1728}{19}x_2^3 - \frac{81}{20}x_2x_4 + \frac{81}{8}x_3^2}\,.$$

We also record that the right side of the above equation squares to zero.

Let $P_{-\frac{2}{3}}$ denote the ideal generated by all the above polynomials. We compute a Gröbner basis with the computer and follow the elimination process as above. With a little more work at each elimination step, we show with the help of the computer that

$$V\left(P_{-\frac{2}{3}}\right) = \big\{(0,0,0,0,0,0)\big\} \cup \{\text{an infinite number of points}\}$$

where the "infinite number of points" arise from each non-trivial solution of

$$0 = 224888313791044771 84x_4^5 + 265954664335840695x_5^4 \quad \text{where } x_5 \neq 0\ .$$

So we cannot conclude just as in the other cases. We could probably obtain the C_2-cofiniteness of $\mathsf{B}_{-\frac{2}{3}}(\mathfrak{sl}_2)$ with a few new polynomial relations. Results of [ALY 2014] suggest that applying the operator $(W_1)_1$ successively to $s_k^{\text{para}} \in \mathsf{C}_k(\mathfrak{sl}_2)$

up to three times might produce sufficient relations, however computations in the space of homogeneous dimensions 12 seem too complicated for the computing power of a personal computer of the second decade of the second millenium. Hopefully we can obtain a positive answer sometime in the near future.

Conclusion

4.3 Summary of the Results

4.3.1 Proof of Kac-Wakimoto Formula

In Chapter 1, we gave a detailed proof of the Kac-Wakimoto character formula for a Lie algebra with triangular decomposition. The key step of the proof is the application of certain exact translation functors to Verma highest weight modules whose highest weight λ satisfy certain integrality conditions. Results on weak composition series for Verma modules allow to analyse precisely their images under certain translation functors, which leads to a preliminary character formula. In the last technical step of the demonstration, we find coefficients of the preliminary formula by determining stabilisers of weights μ under the action of a certain subgroup W^{μ} of the full Weyl group W of the Lie algebra.

For certain general Lie algebras with triangular decomposition, one expects that character formulae can most likely be deduced from the determination of stabilisers for the action of certain Coxeter groups. Also, one could eventually determine if the Kac-Wakimoto character formula can also be generalised affine Lie superalgebras, and also to categories of relaxed highest weight modules for affine Lie algebras such as those appearing in [AM 1995], [CR 2012], [CR 2013b]: a positive result could help to generalise the analysis of Chapter 3 and [ACR 2018] to higher rank Lie algebras.

4.3.2 Direct Sum Completion of a Braided Monoidal Category

In Chapter 2, we defined a suitable direct sum completion $\mathcal{C}_\oplus$ of a category $\mathcal{C}$. If in addition the base category is a braided monoidal category with possibly a twist, then the completion naturally inherits these structural features. This work fills a gap in the literature and where many authors referred to the highly abstract notion of Ind-object of [AGV 1971], one can now point to the completion $\mathcal{C}_\oplus$.

Our main application of $\mathcal{C}_\oplus$ is to provide a working framework when dealing with a vertex operator algebra extension $V \subseteq E$, where E is seen as a V-module is an infinite direct sum of modules. In such a situation, the results of [HKJL 2015], [CKM 2017] show that $E \in \mathcal{C}_\oplus$ is an algebra object where $\mathcal{C}$ is a certain category of V-modules. The category of untwisted E-modules is braided equivalent to the category $\mathrm{Rep}^0 E$ of modules for E seen as an algebra object. In certain situations, we can find out much information about E-modules from the knowledge of a base category of V-modules and a natural induction functor. We displayed such an application to the simplest possible vertex operator algebra setting of $H \subset V_L$ where H is the rank 1 Heisenberg algebra and where V_L is the even lattice vertex operator algebra. Note that the content of this chapter has now been published [AR 2018].

Combined to [CKM 2017], our work in Chapter 2 and [AR 2018] provides a solid framework for studying lare vertex operator algebra extensions. For instance, when branching rules of a extension $V \subset E$ are infinite, our completion is of great help to make sense of and to manipulate objects in full rigour.

4.3.3 Modularity Behaviour of $\mathsf{B}_k(\mathfrak{sl}_2)$

In Chapter 3, we studied an infinite order extension $\mathsf{B}_k(\mathfrak{sl}_2)$ of the simple parafermion algebra $\mathsf{C}_k(\mathfrak{sl}_2)$ at negative admissible level k. In particular, $k \notin -2 + \mathbb{N}$ is rational and so $\mathsf{B}_k(\mathfrak{sl}_2)$ is a logarithmic vertex operator algebra. The outcome of our analysis is the modular behaviour of simple characters of $\mathsf{B}_k(\mathfrak{sl}_2)$ under two natural assumptions on our module categories. The modular behaviour found is of the type that is expected of a C_2-cofinite vertex operator algebras [Miy 2004]. Since also we have finitely many simple modules in the category of $\mathsf{B}_k(\mathfrak{sl}_2)$-modules considered, we conjecture that $\mathsf{B}_k(\mathfrak{sl}_2)$ are C_2-cofinite at all negative admissible levels. Again, the content of this chapter has recently been published [ACR 2018]. In this paper, we also deduce fusion rules of some indecomposable modules in the categories we have considered.

It is interesting to note that $\mathsf{B}_k(\mathfrak{sl}_2)$ as an algebra object in the direct sum completion of a certain category of $\mathsf{C}_k(\mathfrak{sl}_2)$-modules and we can use the point of view of [HKJL 2015], [CKM 2017] to support our study thanks to the direct sum completion from Chapter 2 and [AR 2018].

4.3.4 C_2-Cofiniteness of $\mathsf{B}_k(\mathfrak{sl}_2)$ for Certain k's

In Chapter 4, we have established the C_2-cofiniteness of certain logarithmic vertex operator algebras $\mathsf{B}_k(\mathfrak{sl}_2)$ at negative admissible level, namely those for $k \in A$ as of line (4.1). These furnish new examples of non-rational C_2-cofinite vertex operator algebras that may be useful in future developments of logarithmic vertex operator algebra theory. Since the choice of levels $k \in A$ has been mostly due to computational restrictions, these new examples add much weight to the conjecture resulting from Chapter 3 and [ACR 2018] stating that $\mathsf{B}_k(\mathfrak{sl}_2)$ would be C_2-cofinite for any negative admissible level k. We hope to improve our approach strategies in the future to deduce broader C_2-cofiniteness results for $\mathsf{B}_k(\mathfrak{sl}_2)$ and other parafermion vertex operator algebras.

In fact, the results obtained in this chapter suggest that the methods for establishing C_2-cofiniteness of $\mathsf{C}_k(\mathfrak{sl}_2)$ for $k \in \mathbb{N}\backslash\{0\}$ from [DLY 2009], [ALY 2014] could somehow be adapted to the non-integral case if one replaces $\mathsf{C}_k(\mathfrak{sl}_2)$ by a suitable infinite order extension $\mathsf{B}_k(\mathfrak{sl}_2)$.

4.4 Future Work

Results of Chapters 3 and 4 are promising and we hope that they will help us discovering more on C_2-cofiniteness for logarithmic vertex operator algebras. It is notable that most of the C_2-cofinite $\mathsf{B}_k(\mathfrak{sl}_2)$ with $k \in A$ as of line (4.1) are not related to the triplet vertex operator algebras or the symplectic fermions, which used to be about the only families of logarithmic vertex operator algebra known to also be C_2-cofinite.

Testing and comparing elements of theories for C_2-cofinite logarithmic vertex operator algebras with the new examples provided in this thesis might prove useful to further developements. Following [ADJR 2018], one could also compute the quantum dimensions of all indecomposable $\mathsf{B}_k(\mathfrak{sl}_2)$-modules considered in Chapter 3 and perform further analysis to check elements of the theory of [CG], [CG 2017], especially for $k \in A$ where C_2-cofiniteness is known. The key element of the theory of log-modular categories developed in [CG 2017] seems to be the open Hopf-links of projective indecomposable modules. As the two authors of the latter reference noted, even if serious work might be needed to identify such objects in our case it is probably worth investing such efforts. In [GR 2017], the authors offer a new approach to the important notion of pseudo-trace function [Miy 2004]. It might be instructive to compare pseudo-trace functions of $\mathsf{B}_k(\mathfrak{sl}_2)$ to the components of the finite dimensional vector-valued modular form of Chapter 3 and [ACR 2018]. Even just in the cases $k \in A$ as of line (4.1), such checks could help us make sense of the chosen $\mathsf{B}_k(\mathfrak{sl}_2)$-module category in a broader sense.

We will also aim to improve the scope of the C_2-cofiniteness results of Chapter 4. According to the structural results of [DW 2010], [DW 2011], the more general parafermion vertex operator algebras $\mathsf{C}_k(\mathfrak{g})$ associated with any finite dimensional simple Lie algebra $\mathfrak{g}$ can be sufficiently well understood through the $\mathfrak{sl}_2$-triples in $\mathfrak{g}$ and a good knowledge of $\mathsf{C}_{k'}(\mathfrak{sl}_2)$ where $k' \in \{k, 2k, 3k\}$. When $k \in \mathbb{N}\backslash\{0\}$, the simple and explicit form of the parafermion singular vector of $\mathsf{C}_k(\mathfrak{sl}_2)$ allowed to obtain C_2-cofinitness for all $\mathsf{C}_k(\mathfrak{g})$ [ALY 2014]. Interestingly, the authors of the latter reference proved that rationality of the C_2-cofinite parafermion $\mathfrak{sl}_2$-triples of $\mathsf{C}_k(\mathfrak{g})$ was not needed to conclude its C_2-cofiniteness. Therefore, we expect a similar approach to work for k negative and admissible where $\mathsf{C}_k(\mathfrak{g})$ would be replaced by a big extension $\mathsf{B}_k(\mathfrak{g})$ whose "$\mathfrak{sl}_2$-triples" would coincide with $\mathsf{B}_k(\mathfrak{sl}_2)$. When $\mathfrak{g}$ is simply laced, i.e. of type A-D-E, we have $k' = k$ and so a C_2-cofiniteness result for $\mathsf{B}_k(\mathfrak{g})$ might already be within reach for all $k \in A$ as in (4.1).

Even for $k \in A$, a challenging step for proving the C_2-cofiniteness of $\mathsf{B}_k(\mathfrak{sl}_2)$ at admissible negative level following the procedure of Chapter 4 is the explicit determination of the parafermion singular vector $s_k^{\text{para}} \in \mathsf{C}_k(\mathfrak{sl}_2)$. Determining singular vectors in a given symmetry algebra is a general problem of Representation Theory that is also important for physical applications [DFMS 1997], [AV 2014], [RW 2015a]. Despite the difficulties, a couple of semi-explicit forms for the singular vector s_k of $V_k(\mathfrak{sl}_2)$ have been known: there is the Malikov-Feign-Fuchs formula for s_k in terms of elements of the standard *negative* Borel subalgebra $U_k(\widehat{\mathfrak{sl}_2})$ with fractional exponents [MFF 1986] and more recently, there are expressions for s_k in terms of Jack polynomials in the Wakimoto free-field realisation of $V_k(\mathfrak{sl}_2)$ [RW 2015a].

Another possible work to pursue could be an analysis of the type [ALY 2014] in wich we consider the effects of the differential operator induced by $(W_1)_{(1)}$ on the C_2-quotient of $\mathsf{B}_k(\mathfrak{sl}_2)$. This might enable one to reduce the C_2-cofiniteness problem of the general $\mathsf{B}_k(\mathfrak{sl}_2)$ to a simpler one. Indeed, the conclusions of [ALY 2014] include that C_2-cofiniteness of $\mathsf{C}_k(\mathfrak{sl}_2)$ where $k \in \mathbb{N}\backslash\{0\}$ can be established by a number of polynomial relations that can be produced by few successive applications of the operator $(W_1)_{(1)}$ to a genuine null relation in conformal dimension 8 and to the singular vector s_k^{para} of $\mathsf{C}_k(\mathfrak{sl}_2)$. The fact that Gröbner bases of an ideal of a polynomial ring can be related to the ideal generated by the leading terms [CLO 1997] might be of use here.

Given that the parafermion vertex operator algebras $\mathsf{C}_k(\mathfrak{sl}_2)$ and $\mathsf{B}_k(\mathfrak{sl}_2)$ at specific levels $k \in \{-\frac{1}{2}, -\frac{2}{3}, -\frac{4}{3}\}$ are related to other well understood vertex operator algebras [ACR 2018], another interesting avenue is to explore and study these parafermion algebras at certain specific families of levels such as the boundary admissible ones $k \in \left\{-2 + \frac{2}{n} \mid 1 \neq n \text{ odd}\right\}$. Other interesting cases would be given by the non-dmissible levels of the form $k \in \left\{-2 + \frac{1}{n} \mid n \in \mathbb{N}\backslash\{0\}\right\}$: at such levels, one would not have the problem of determining a singular vector for it does not exist in the vertex operator algebra when k is not admissible, but we can consider a larger simple current extensions than $\mathsf{B}_k(\mathfrak{sl}_2)$. This means that such cases might be even more accessible.

In a work that is currently in preparation [ACKR], we study interesting features of a certain category of local modules of a logarithmic vertex operator algebra $\mathcal{B}_p$ of interest in current Physics. This vertex operator algebra can be related to the Argyres-

Douglas Theories whose Schur indices can be related to the character $\mathrm{ch}\left[\mathcal{B}_p\right](q)$ of the algebra [Cre 2017]. The tools for our analysis are simply [CKM 2017] and Chapter 2 or [AR 2018]. We plan to apply similar methods to understand more on the Representation Theory of large vertex operator algebra extensions in the future.

Obviously, much is left to do to understand logarithmic vertex operator algebras. However, I hope that the new examples of logarithmic C_2-cofinite vertex operator algebras introduced in this thesis will help construct a useful theory and to refine our understanding of these structures. This could eventually be an asset to understanding certain intriguing physical applications.

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Appendix A

Selected Basic Elements of Vertex Operator Algebras

In this appendix, you will find basic notions definitions about Vertex Operator Algebras and their Representation Theory. There is also a review of the Heisenberg and the even lattice vertex operator algebras which will be of use to understand the last section of Chapter 2. Introductive books on the notions of vertex operator algebras include [FBZ 2004] and [LL 2003]. A nice elementary account on vertex operator algebra axioms is presented in [Tui 2017].

Throughout Appendix A, all vector spaces are defined over the field $\mathbb{C}$ unless otherwise mentioned.

A.1 Vertex Operator Algebras

A.1.1 Basic Concepts and Definitions

Here are basic concepts about vertex operator algebras:

Definition A.1. A *vertex operator algebra* $V = (V, \mathbf{Y}, \mathbf{1}, \omega)$ consists of the following data:

(1) a $\mathbb{Z}$-graded vector space $V = \bigoplus_{n\in\mathbb{Z}} V_n$;

(2) a linear map $\mathbf{Y}(-, z) : V \to \big(\operatorname{End} V\big)((z))$ that associates a power series in z with coefficients in $\operatorname{End} V$ to an arbitrary vector $v \in V$. The image of v is often denoted $\mathbf{Y}(v, z) = \sum_{n\in\mathbb{Z}} v_{(n)} z^{-n-1}$ and is called a *vertex operator*. The meaning of $((z))$ in the symbol $\big(\operatorname{End} V\big)((z))$ can be summarized as follows:

$$\mathbf{Y}(u, z)v = \sum_{n\in\mathbb{Z}} u_{(n)}(v) z^{-n-1} \qquad \text{is a Laurent series in } z \text{ for all } u, v \in V,$$

so that all vertex operators are what is called a *field*;

(3) a distinguished vector $\mathbf{1} \in V$ subject to:

- $\mathbf{Y}(\mathbf{1}, z) = \operatorname{Id}_V z^0 = \operatorname{Id}_V$;
- $\mathbf{Y}(v, z)\mathbf{1} = \sum_n v_{(n)}(\mathbf{1}) z^{-n-1}$ has only non-negative powers of z and the coefficient of z^0 in the same power series is v;

(4) Another distinguished vector $\omega \in V$, called the *conformal vector*, subject to:

- writing $\mathbf{Y}(\omega, z) = \sum_n L_n z^{-n-2}$ gives

$$[L_n, L_m] = (n - m)L_{n+m} + \frac{n(n^2 - 1)}{12}\delta_{n+m,0}\, c \operatorname{Id}_V \tag{A.2}$$

where $c \in \mathbb{C}$ is a constant called the *central charge* of the vertex operator algebra;

- $V_n = \ker(L_0 - n \operatorname{Id}_V)$ for all n;
- $\mathbf{Y}\big(L_{-1}(v), z\big) = \frac{d}{dz}\mathbf{Y}(v, z)$ for all $v \in V$;

(5) for all $u, v \in V$, there exists an $N \in \mathbb{N}$ such that

$$(z-w)^N \cdot \left[\mathbf{Y}(u,z)\mathbf{Y}(v,w)\right] = 0$$

where $[-,-]$ is the usual commutator. This axiom is also called mutual locality of the fields and has other interpretation in terms of power series.

Remark A.3. For a topological and algebraical introduction to the notion of a vertex operator algebra, see [FBZ 2004]. For an elementary approach to the basic axioms (including that of locality) and results for vertex operator algebras, see [Tui 2017].

While Definition A.1 surely defines sensible objects, some vertex operator algebras we will often consider will have the following additional properties:

- the conformal grading on V is actually by $\mathbb{N}$: $V = \bigoplus_{n \in \mathbb{N}} V_n$;
- the space V_0 of lowest conformal dimension is one dimensional: $V_0 = \mathbb{C}.\mathbf{1}$.

Note that the two above assumptions are necessary for a simple vertex operator algebra V to have a unique nondegenerate symmetric invariant bilinear form [Li 1994]. Bilinear forms are relevant for physical applications and very relevant to important mathematical results.

From the axioms (3), it can be derived that the map $\mathbf{Y}$ is injective. This is known to physicists as the *state-field correspondance*. Therefore, one can vertex operator algebra V through its vertex operator fields rather than its vectors (the states).

The very useful Mathematica package *OPEdefs.m* [Thi 1991] written by K. Thielemans to achieve vertex operator algebra related computations actually only allows for manipulation of fields and vertex operators by the computer. The code developed here for obtaining C_2-cofiniteness results in Chapter 4 also manipulates fields.

An equivalent axiom to (5) is that for every pair of elements $u, v \in V$, a certain *Jacobi identity* is satisfied. This identity relates a composition $\mathbf{Y}\big(\mathbf{Y}(u,z)v, w\big)$ of vertex operators to "multiplications" $\mathbf{Y}(u,z)\mathbf{Y}(v,w)$ and $\mathbf{Y}(v,w)\mathbf{Y}(u,z)$ up to some formal power series factors. Just as for Lie algebras, this Jacobi identity is often thought of as a property that replaces associativity in the "algebra" V. For more details, see Section 3.2 of [FBZ 2004].

It is important to note that vertex operator algebras are not just like more basic algebraic structures like groups and algebras. If V is a vertex operator algebra, there is no straightforward binary multiplication $V \otimes V \to V$, but instead we can view the operation $\mathbf{Y}$ as a map $u \otimes v \longmapsto \mathbf{Y}(u,z)v$ resulting in a formal Laurent series with coefficients in V. This fundamental difference is also what makes vertex algebras special and it also suggests natural connexions with Analysis and Topology. For instance, the axiom of mutual locality in Definition A.1 has a precise interpretation in terms of power series, see Theorem 3.2.1 of [FBZ 2004] for instance.

Remark A.4. Although vertex operator algebras are quite different than classical types of algebras, they remain closely related with Lie algebras in particular. For example, it can be proved that the linear span of the collection of the coefficients $\textsc{Coeffs}(V)$ of the vertex operator series $\mathbf{Y}(u,z)$ for $u \in V$ actually form a Lie algebra. The Lie bracket of two *vertex coefficients* of $\textsc{Coeffs}(V)$ is then given by the formula

$$[u_n, v_m] = \sum_{r \geq 0} \binom{n}{r} \big(u_r(v)\big)_{n+m-r} \tag{A.5}$$

where $u, v \in V$. Note that even tho ugh n can be negative, the binomial coefficients above have a precise meaning. For more details on this formula, see (3.3.12) in [FBZ 2004].

Another key aspect of Definition A.1 is that equation (A.2) represents the defining relations of $\mathbf{Vir}_c$ the Virasoro Lie algebra of level c, which is also the central charge parameter of the vertex operator algebra. The presence of the conformal vector $\omega \in V$ effectively makes the vertex algebra V a $\mathbf{Vir}_c$-module. This is in fact what makes a vertex operator algebra potentially relevant for the Conformal Field Theories: the physical axiom of *conformal invariance* translates into a Virasoro algebra action on V.

Notice that the Virasoro element L_0 provides the grading of the vertex operator algebra $V = \bigoplus_{n\in\mathbb{Z}} V_n$. In fact, this grading on V bears the name of *conformal grading*: the subspace V_n is said to be the subspace of vectors of conformal weight or conformal dimension n.

As for any type of algebra, a very important concept to understand how pairs of elements multiply or interact with each other. For a vertex operator algebra V, two vertex operators $\mathbf{Y}(u,z)$ and $\mathbf{Y}(v,w)$ can be combined through a so-called *operator product expansion*. The naive combination $\mathbf{Y}(u,z)\mathbf{Y}(v,w)$ will always admit a unique expansion as follows:

$$\mathbf{Y}(u,z)\mathbf{Y}(v,w) = \sum_{r=0}^{N-1} \frac{\gamma_r(w)}{(z-w)^{r+1}} + \; :\mathbf{Y}(u,z)\mathbf{Y}(v,w): \tag{A.6}$$

where $\gamma_r(w)$ are also fields coming from V and where $:\mathbf{Y}(u,z)\mathbf{Y}(v,w):$ is a special type of operator called the *normally ordered product* of the fields $\mathbf{Y}(u,z)$ and $\mathbf{Y}(v,w)$. For a complete definition of the normally ordered product of two fields as of line (A.6), see Definition 2.2.2 of [FBZ 2004] and read the Section 3.3.4.

Remark A.7. The normally ordered product appearing in (A.6) has two formal variables. However, it also has a meaning in terms of a single variable, about $z = w$. The outcome is in fact a vertex operator that has a corresponding state vector in V. To see it, one can apply a formal Taylor expansion centred in $z = w$ to $\mathbf{Y}(u,z)$ (A.6) and compare with equation (3.3.1) of [FBZ 2004] to establish that:

$$:\mathbf{Y}(u,z)\mathbf{Y}(v,z): = \mathbf{Y}\big(u_{-1}(v),z\big) \ ,$$

which is in fact a special case of equation (3.3.8) of the same reference. The above equation means that the element $u_{-1}(v) \in V$ is a state vector whose vertex operator is the normally ordered product $:\mathbf{Y}(u,z), \mathbf{Y}(v,z):$.

Normally ordered product is that they are in general not associative nor commutative. The convention for writing them when more than two factors are involved is to write

$$: \ \mathbf{Y}(v_1,z_1)\dots\mathbf{Y}(v_r,z_r) \ := : \ \mathbf{Y}(v_1,z_1) : \cdots : \mathbf{Y}(v_{r-1},z_{r-1})\mathbf{Y}(v_r,z_r) \ : \cdots \ :$$

Now that the vertex operator algebras have been defined, we can define their modules, or representations.

Definition A.8. Let V be a vertex operator algebra. A *weak V-module* $(M, \mathbf{Y}_M)$ is defined as a vector space M with

(1) a linear map $\mathbf{Y}_M(-,z) : V \to \big(\operatorname{End} M\big)((z))$ that associates a field with coefficients in $\operatorname{End} M$ to an arbitrary $v \in V$. The image of v is often denoted $\mathbf{Y}(v,z) = \sum_{n\in\mathbb{Z}} v^M_{(n)} z^{-n-1}$;

(2) the distinguished vector $\mathbf{1} \in V$ is subject to $\mathbf{Y}_M(\mathbf{1},z) = \mathrm{Id}_M\, z^0 = \mathrm{Id}_M$;

(3) the coefficients of $\mathbf{Y}(\omega,z)$ span a Virasoro Lie algebra of level c corresponding to the central charge of V;

(4) $\mathbf{Y}_M\big(L_{-1}(v),z\big) = \frac{d}{dz}\mathbf{Y}_M(v,z)$ for all $v \in V$;

(5) the $\mathbf{Y}_M(u,z)$ and $\mathbf{Y}_M(v,w)$ are local with each other or, equivalently, $\mathbf{Y}_M$ should satisfy a Jacobi identity.

Remark A.9. Given any V-module M, one observes that for any $v \in V$ and any $n \in \mathbb{Z}$, we have $v_n \in \mathrm{End}(V)$ and $v_n^M \in \mathrm{End}(M)$. It can be proven that the natural association

$$\begin{aligned} \wedge^M : \quad \textsc{Coeffs}(V) &\longrightarrow \mathrm{End}(M) \\ v_n &\longmapsto v_n^M \end{aligned}$$

is a Lie algebra homomorphism. However, one truly needs more than just bare Lie theory to study vertex operator algebras.

While the Definition A.8 of weak modules surely exposes sensible mathematical objects, the vertex operator algebras we consider are $\mathbb{Z}$-graded by conformal weight (the eigenvalue of L_0) and so the modules we will consider in this thesis will also be graded objects:

Definition A.10. An $\mathbb{N}$-*graded weak module* M is a weak module as of Definition A.8 that satisfies the following additional properties:

(6) M is graded is by $\mathbb{N}$: $M = \bigoplus_{n\in\mathbb{N}} M_n$;

(7) for any $n, m \in \mathbb{Z}$, $r \in \mathbb{N}$ and $u \in V_n$, we have:

$$u_m|_{M_r} : M_r \to M_{r+(n-m-1)} \ ,$$

so that u_m is a homogeneous linear operator of degree $n-m-1$ on M.

In particular, L_0^M is a degree preserving linear operator and L_{-1}^M is a degree 1 linear operator on M.

Remark A.11. There exists also a notion of *ordinary V-module*. This is defined to be a $\mathbb{N}$-graded weak modules $M = \bigoplus_{n\in\mathbb{N}} M_n$ such that L_0^M is semisimple on M_{h_M+n} and $\dim_{\mathbb{C}} M_{h_M+n} < \infty$ for each $n \in \mathbb{N}$.

Remark A.12. $\mathbb{N}$-graded modules are more often considered rather than $\mathbb{Z}$-graded ones. This can be partly explained by the access it gives to convergence results for certain complex valued power series such as characters and trace functions that will later be discussed in this appendix. From another point of view, this can be motivated by the already vast and developed study of highest weight modules for Lie algebras with triangular decompositions. Highest weight modules of universal affine vertex operator algebras associated with a finite dimensional semisimple Lie algebra will be $\mathbb{N}$-graded and rather than just $\mathbb{Z}$-graded. Also, highest weight modules in general are of particular significance in some physical applications.

Let's now recall a few important result on simple modules here:

Result A.13. (see Lemma 2.7 of [Zhu 1996]) *Let V be a vertex operator algebra. If $M = \bigoplus_{n=0}^{\infty} M_n$ is an irreducible V-module such that M_0 has countable dimension, then there is a constant $h_M \in \mathbb{C}$ called the* conformal dimension of M *such that L_0^M acts on each M_n as the scalar $h_M + n$. In particular, every simple $\mathbb{N}$-graded weak V-module M with $\dim_{\mathbb{C}} M_0 < \infty$ has a conformal dimension h_M.*

The following result is also useful for many important examples of vertex operator algebras:

Result A.14. (see Lemma 1.2.2 of [Zhu 1996]) *If V has a countable basis, then any simple $\mathbb{N}$-graded weak V-module $M = \bigoplus_{n=0}^{\infty} M_n$ have a conformal dimension $h_M \in \mathbb{C}$ just as in Result A.13. In particular, L_0^M acts on each M_n as the scalar $h_M + n$.*

Unsurprisingly, modules for a given vertex operator algebra form categories, where the arrows are the maps between pairs of modules M, N that intertwine the corresponding vertex action maps $\mathbf{Y}_M$ and $\mathbf{Y}_N$.

Given an automorphism $g \in \mathrm{Aut}(V)$ of a vertex operator algebra V and given a V-module M with action map $\mathbf{Y}_M$, one can construct a new representation on the space M:

$$\tilde{M} = M \text{ as vector spaces} \qquad \text{and} \qquad \mathbf{Y}_{\tilde{M}}(v,z) = \mathbf{Y}_M\big(g^{-1}(v), z\big) \ .$$

If instead of $g \in \mathrm{Aut}(V)$ we are given a certain type of automorphism $\theta \in \mathrm{Aut}\left(\textsc{Coeffs}(V)\right)$ of the Lie algebra $\textsc{Coeffs}(V)$, we can also define a new vertex V-action map on the vector space M as follows:

$$\mathbf{Y}^{\theta}_M(v,z) = \sum_{n \in \mathbb{Z}} \theta^{-1}(v_n) z^{-n-1}$$

where $v \in V$.

When studying categories of modules and their characters like in Chapter 3, the above constructions of twisted modules are very useful. For instance, if M is a simple module, then $\tilde{M}$ will also be a simple module. Also invertibility of automorphisms ensure that twisting by a given one is functorial. Such considerations play a key role in [CR 2012], [CR 2013b] and [ACR 2018] for instance.

A.1.2 Characters

Let V be a vertex operator algebra. For any $\mathbb{N}$-graded weak V-module $M = \bigoplus_{n \in \mathbb{N}} M_n$ whose graded pieces are finite dimensional

$$\dim_{\mathbb{C}} M_n < \infty \quad \text{for all } n \in \mathbb{N} \ ,$$

one can define its *character* or *graded dimension* as follows:

Definition A.15. Let M be as above. Then its *character* is defined to be a formal power series

$$\mathrm{ch}\,[M]\,(q) = \sum_{n\in\mathbb{N}} \dim(M_n)\; q^{n-\frac{c}{24}}$$

where c is the central charge of V. The series $\mathrm{ch}\,[M]$ can be thought of as a vertex operator algebra module adaptation of the *Hilbert-PoincarÃľ series* of a $\mathbb{N}$-graded algebra: see Problem 2.8.11 of [EGH+ 2011] for instance. The character of M also corresponds to a formal graded trace of certain operators:

$$\mathrm{ch}\,[M]\,(q) = \mathrm{tr}_M(q^{L_0-\frac{c}{24}})\;. \tag{A.16}$$

A priori the variable q is formal, however the characters of certain vertex operator algebra modules can sometimes be re-interpreted as complex-valued functions by setting $q = e^{2\pi i\tau}$ where $\tau \in \mathbb{H} \subset \mathbb{C}$ is a variable from the the upper-half plane.

This observation has led to proofs that the complex functions that characters define are continuous and even holomorphic for certain classes of vertex operator algebras. In fact Y. Zhu proved that for rational and C_2-cofinite vertex operator algebras, the linear span of the simple module's characters is invariant under an action of $SL_2(\mathbb{Z})$ [Zhu 1996]. This means that the characters of the weak modules of a rational and C_2-cofinite seen as functions give rise to a finite dimensional *vector-valued modular form*. M. Miyamoto eventually generalised Y. Zhu's results on $SL_2(\mathbb{Z})$-invariance of character by proving that any C_2-cofinite vertex operator algebra gives rise to a finite dimensional vector-valued modular form [Miy 2004]. Known examples of non-rational C_2-cofinite vertex operator algebras are few, but include the triplet vertex operator algebras [AM 2008a], [TW 2013] and some of the extended parafermions algebras of [ACR 2018] and Chapter 4 at certain admissible negative levels as shown in Chapter 4. For the latter example, details about the vector-valued modular form can be found in Chapter 3.

Characters of modules also behave well with respect to short exact sequences that we sometimes study to understand categories. In fact, characters are additive with respect to them. This means that given a vertex operator algebra V, a category $\mathcal{C}$ of V-modules and $M, E, N \in \mathrm{Ob}(\mathcal{C})$ such that the sequence $0 \to M \to E \to N \to 0$ is exact, one has

$$\mathrm{ch}\,[E] = \mathrm{ch}\,[M] + \mathrm{ch}\,[N] \ . \tag{A.17}$$

Important consequences include that it is sufficient to consider characters of indecomposable modules in order to consider them all.

For *locally finite* abelian monoidal tensor categories with an exact tensor product bifunctor, it is possible to define its Grothendieck ring. Recall that a locally finite category is one where every object has finite length and where the dimension of every Hom space is finite dimensional for every pair of objects [EGNO 2015].

Definition A.18. Let $\mathcal{C}$ be a locally finite monoidal tensor category with exact tensor product. For an object $X \in \mathrm{Ob}(\mathcal{C})$, let $[X]$ denote its isomorphism class. We define the *Grothendieck ring* of $\mathbb{C}$ to be the ring

$$\mathcal{G}(\mathcal{C}) = \left(\sum_{X \in \mathrm{Ob}(\mathcal{C})} \mathbb{Z}.[X] \right) \Bigg/ \left\langle \begin{matrix} [E] = [A] + [B] \\ \text{whenever } 0 \to A \to E \to B \to 0 \text{ is exact} \end{matrix} \right\rangle \ .$$

where the addition and multiplication are given by

$$\begin{aligned} [X] + [Y] &= [X \oplus Y] \ , \\ [X] \cdot [Y] &= [X \otimes Y] \ , \end{aligned}$$

for every objects $X, Y \in \mathrm{Ob}(\mathcal{C})$. The multiplication on $\mathcal{G}(\mathcal{C})$ is usually referred to as the *Grothendieck fusion product*.

Thanks to the Krull-Schmidt theorem, every object of $\mathcal{C}$ is a direct sum of indecomposable modules. Now since all non-zero indecomposables have finite length, their images in $\mathcal{G}(\mathcal{C})$ are well defined non-zero $\mathbb{N}$-linear combinations of isomphism classes of simple objects of $\mathcal{C}$. Therefore, we recover the well known fact that $\mathcal{G}(\mathcal{C})$ is generated by the isomorphism classes of its simple objects as an abelian group. Moreover, one can argue that $\mathcal{G}(\mathcal{C})$ is a $\mathbb{Z}_+$-ring [EGNO 2015].

Remark A.19. Note that the exactness of $\otimes$ on $\mathcal{C}$ is required for the multiplication on $\mathcal{G}(\mathcal{C})$ to be distributive on the addition $\oplus$.The assumption for $\mathcal{C}$ to be abelian in Definition A.18 is important to make sense of exact sequences. Also, $\mathcal{C}$ should be locally finite to ensure that an element of $\mathcal{G}(\mathcal{C})$ is never an infinite sum of isomorphism classes of simple modules.

In principle, a well defined notion of $\mathcal{G}(\mathcal{C})$ might hold for slightly more general $\mathcal{C}$, which might be useful for certain non-abelian vertex operator algebra module categories. However, we will not develop further this topic here.

To better understand $\mathcal{C}$, it is evident that a knowledge of $\mathcal{G}(\mathcal{C})$ is key. The main difficulty for this is to grasp the multiplicative structure of $\mathcal{G}(\mathcal{C})$, that is, the tensor (fusion) product of arbitrary pairs of simple objects in $\mathcal{C}$.

For $X, Y, Z \in \mathrm{Irrep}(\mathcal{C})$, we define the Grothendieck fusion rules ${}^{\mathrm{Gr}}\mathbf{N}_{X,Y}^{Z} \in \mathbb{Z}$ to be the structure constants of $\mathcal{G}(\mathcal{C})$ appearing in

$$[X] \cdot [Y] = [X \otimes Y] = \sum_{Z \in \mathrm{Irrep}(\mathcal{C})} {}^{\mathrm{Gr}}\mathbf{N}_{X,Y}^{Z} [Z] \, . \tag{A.20}$$

Remark A.21. Since $\mathcal{G}(\mathcal{C})$ is a ring, it is a module over itself and the corresponding

$[X]$-action map for $X \in \operatorname{Irrep}(\mathcal{C})$ is

$$\begin{aligned} \operatorname{act}_{[X]} : \quad \mathcal{G}(\mathcal{C}) &\longrightarrow \mathcal{G}(\mathcal{C}) \\ [Y] &\longmapsto \textstyle\sum_{Z \in \operatorname{Irrep}(\mathcal{C})} \operatorname{Gr}\mathbf{N}_{X,Y}^{Z} [Z] \end{aligned}$$

for $Y \in \operatorname{Irrep}(\mathcal{C})$. In particular, when $\# \operatorname{Irrep}(\mathcal{C}) < \infty$ and we fix one representative per isomorphism classe if simple object in $\operatorname{Irrep}(\mathcal{C})$, we have square matrices

$$\operatorname{act}_{[X]} = \left(\operatorname{Gr}\mathbf{N}_{X,Y}^{Z}\right)_{Y,Z \in \operatorname{Irrep}(\mathcal{C})}$$

for every $X \in \operatorname{Irrep}(\mathcal{C})$. The analogy with matrices persists when $\# \operatorname{Irrep}(\mathcal{C}) = \infty$ often providing good notation and reference points for certain complex formulae one has to write.

Remark A.22. For $X, Y, Z \in \operatorname{Indecs}(\mathcal{C})$, we could define simply *fusion rules* $\mathbf{N}_{X,Y}^{Z}$ to be the structure constants

$$X \otimes Y = \sum_{Z \in \operatorname{Irrep}(\mathcal{C})} \mathbf{N}_{X,Y}^{Z} \, Z \ .$$

Notice the difference of the above equation with (A.20). Indeed there is an important distinction to make between fusion rules and Grothendieck fusion rules.

Known modularity results in Rational Conformal Field Theory settings have made possible the important discovery that the Grothendieck fusion rules of the simple modules could be obtained by their character modular transfomation data [MS 1989], [Hua 2005]. In addition, he observed that the S-transformation diagonalise the Grothendieck fusion rules. This connexion is quite important since in relates analytical and number theoretical data from characters to categorical data from tensor products. For physicists, Grothendieck fusion rules allow to compute invariant quantities and to perform certain geometrical constructions. This very important connexion is best summarised as the Verlinde formula which is best expressed for a simple rational C_2-cofinite vertex operator algebra R. We know that R has a semisimple module category $\mathcal{C}$ with a finite number of simple modules $\{M_i\}_{i=0}^N$ such that $M_0 = R$. The associated characters are $\mathrm{ch}\left[M_i\right](q) = \mathrm{ch}\left[M_i\right](\tau)$ where $q = e^{2\pi i \tau}$ for $\tau \in \mathbb{H}$ and behave as follows under modular transformation:

$$T.\,\mathrm{ch}\left[M_i\right](\tau) = \mathrm{ch}\left[M_i\right](\tau+1) = e^{2\pi i h_{M_i}}\,\mathrm{ch}\left[M_i\right](\tau)\ ,$$
$$S.\,\mathrm{ch}\left[M_i\right](\tau) = \mathrm{ch}\left[M_i\right]\left(-\frac{1}{\tau}\right) = \sum_{i=0}^{N} S_{i,j}\cdot \mathrm{ch}\left[M_i\right](\tau)$$

where $S_{i,j} \in \mathbb{C}$ for all $i, j \in \{0, \ldots, N\}$. In this setting, every module M_i has a dual M_{i*} where $i^* \in \{0, \ldots, N\}$. We can now write Verlinde's formula as follows:

$$\mathrm{Gr}\mathbf{N}_{M_i,M_j}^{M_k} = \sum_{r=0}^{N} \frac{S_{i,r}S_{j,r}S_{k*,r}}{S_{0,r}}\ . \tag{A.23}$$

Attempts to generalise such a formulas to logaritmic C_2-cofinite settings are the object much current research despite the many challenges [CR 2012], [CR 2013b], see also [CG], [GR 2017], [RW 2015b], [CM 2014], [CMR 2016] as well. Complications due to the presence of reducible indecomposable modules for logarithmic vertex operator algebras makes this situation challenging. Another challenge to overcome is the lack of examples of logarithmic C_2-cofinite vertex operator algebras. Hopefully, this thesis can improve the situation.

Modularity of rational C_2-cofinite vertex operator algebra was properly established in [Zhu 1996], while modularity of general C_2-cofinite vertex operator algebra was proven in [Miy 2004]. That a rational C_2-cofinite vertex operator algebra has a modular tensor category was finally proven in [Hua 2008]. Currently, some research is underway to describe categories of C_2-cofinite vertex operator algebras and some of their general features [CG 2017], [CG], see also [Miy 2003], [Miy 2010], [GR 2017].

A.1.3 C_2-Cofiniteness

The property of C_2-cofiniteness is unavoidable when it comes to vertex operator algebras. Originally introduced by Y. Zhu in [Zhu 1996], C_2-cofiniteness is a rather technical property that can be defined as follows:

Definition A.24. A vertex operator algebra V is said to be C_2-cofinite if the subspace

$$C_2\,(V) = \{a_{-2}(b) \in V \mid a, b \in V\} \subseteq V \ ,$$

has finite codimension in V. This just means that V is C_2-cofinite if $\dim_{\mathbb{C}} \frac{V}{C_2(V)} < \infty$.

Although quite technical, this property has a lot of theoretical consequences on the vertex operator algebra and its representation theory [Miy 2004]:

Result A.25. (see Theorem 2.7 of [Miy 2004]) *Let V be a vertex operator algebra. The following statements are equivalent:*

- *V is C_2-cofinite;*
- *every weak module is a direct sum of generalised eigenspaces of L_0.*

Therefore, the C_2-cofiniteness does have a simple conceptual meaning after all. However, it is not the only such conceptual meaning. It is now known that the C_2-quotient $\frac{V}{C_2(V)}$ of any vertex operator algebra V has a natural structure of Poisson algebra and gives rise to an algebraic variety [Ara 2012]. Using appropriate geometric methods, T. Arakawa could show that C_2-cofiniteness is also equivalent to the requirement that the C_2-quotient should have finitely many *symplectic leaves*[1] or *lisse*.

A C_2-cofinite vertex operator algebra V, always have a finite number of simple weak modules since the dimension of $\frac{V}{C_2(V)}$ bounds the dimension of Zhu's algebra $A(V)$ whose simple modules are in bijection with simple $\mathbb{N}$-graded weak V-modules [Zhu 1996]. Moreover, the conformal dimensions and conformal weights of a C_2-cofinite vertex operator algebras are always rational numbers [Miy 2004].

Perhaps the most important consequence of the C_2-cofiniteness property is the modularity behaviour of the finite dimensional span of *trace* and *pseudo-trace* functions [Miy 2004]. Moreover, it is proven that the torus 1-point functions of a C_2-cofinite vertex operator algebra are spanned by trace and pseudo-trace functions [Miy 2004]. While trace functions are closely related to characters of modules, pseudo-trace functions introduced in the latter reference are rather complicated to define. However a new approach to pseudo-trace functions developed in [GR 2017] seems to be a bit simpler to understand, see also [AN 2014] on the same topic. Note that modular invariance results are also expected in many physical situations [DFMS 1997], [GN 2003].

Another very important consequence of C_2-cofiniteness is that the characters of the $\mathbb{N}$-graded weak V-modules with finite dimensional graded pieces where V is a C_2-cofinite vertex operator algebra converge to holomorphic functions on the upper half plane. To see it, combine Theorem 4.4.1 of [Zhu 1996] with $a = \mathbf{1}$ and the general modularity result of [Miy 2004]. From this key result, we deduce that general C_2-cofinite vertex operator algebras give rise to finite dimensional vector-

[1]This notion has to do with properties of certain schemes defined on this type of algebraic varieties called *jet schemes*.

valued modular forms: the one spanned by its trace and pseudo-trace functions. For more on vector-valued modular forms, see [KM 2003], [Gan 2014].

Here are a few useful results about the C_2-quotient of an arbitrary vertex operator algebra V:

Result A.26. (Lemma 3.1 of [Ara 2012]) We have the following properties:

- $L_{-1}V \subset C_2\left(V\right)$;
- $a_n C_2\left(V\right) \subset C_2\left(V\right)$ for all $n \leq 0$ and $a \in V$;
- $a_n V \subset C_2\left(V\right)$ for all $n \leq 0$ and $a \in C_2\left(V\right)$;
- $a_1(b) = b_1(a) \bmod C_2\left(V\right)$ for all $a, b \in V$. Equivalently this property can be reformulated as $: \mathbf{Y}(a,z)\mathbf{Y}(b,z) : \equiv : \mathbf{Y}(b,z)\mathbf{Y}(a,z) : \bmod$ fields of $C_2\left(V\right)$.

It is mentioned in [BR 2018] that the associator with respect to the normally ordered product in V actually lies in $C_2\left(V\right)$. We deduce that $\frac{V}{C_2(V)}$ is a commutative associative algebra with the normally ordered product. It is also known that any set of generators for $\frac{V}{C_2(V)}$ furnishes a set of strong generators for V and vice-versa, see Corollary 2.6.2 in [Ara 2012]. In particular, C_2-cofinite vertex operator algebras should have a finite set of stong generators. Note that this was previously known [GN 2003].

Let's state a last useful result in this section:

Result A.27. (Lemma 3.2 of [Ara 2012]) Let $a \in V$ such that $a_0 V \subset C_2\left(V\right)$. Then

- $a_1 C_2\left(V\right) \subset C_2\left(V\right)$;
- $a_1(b_{-1}c) \equiv (a_1 b)_{-1}c + b_{-1}(a_1 c) \bmod C_2\left(V\right)$ for all $b, c \in V$.

In particular, a_1 induces a derivation on $\frac{V}{C_2(V)}$.

Note that $\omega \in V$ always satisfies the condition of Result A.27 and $\omega_1 = L_0$.

In fact, using filtrations, one can show that $\frac{V}{C_2(V)}$ is a $\mathbb{Z}$-graded commutative Poisson algebra where V is any vertex operator algebra: see Section 3 of [Ara 2012].

A.2 Three Basic Vertex Operator Algebras

A.2.1 Heisenberg Vertex Operator Algebras

The Heisenberg vertex operator algebras are among the most basic non-commutative vertex operator algebras, see [FBZ 2004] for instance.Their underlying vector space is that of a Verma module for a universal central extension of the loop algebra associated to a commutative finite dimensional Lie algebra. The text [Sch] treats of these Lie algebras in an effective and accessible manner.

Let $\mathfrak{h} = \bigoplus_{i=1}^{r} \mathbb{C}.b^i$ be a commutative Lie algebra and let

$$\widehat{\mathfrak{h}} = \left(\mathfrak{h} \otimes \mathbb{C}[t^{\pm 1}]\right) \oplus \mathbb{C}.\kappa$$

where $[\mathfrak{h}, \kappa] = 0$ and $[b^i \otimes t^n, b^j \otimes t^m] = n\delta_{n+m,0}\, \kappa$. For any $\star \in \{\mathbb{Z}_{>0}, \mathbb{Z}_{<0}\}$ set

$$\begin{aligned}
\widehat{\mathfrak{h}}_\star &= \mathrm{Span}_{\mathbb{C}} \left\{ b^i \otimes t^n \mid \text{for all } i \in \{1, \ldots, r\} \text{ and } n \in \star \right\} \\
&= \mathrm{Span}_{\mathbb{C}} \left\{ (b^i)_n \mid \text{for all } i \in \{1, \ldots, r\} \text{ and } n \in \star \right\} \ , \\
\widehat{\mathfrak{h}}_0 &= \mathrm{Span}_{\mathbb{C}} \left\{ b^i \otimes 1 \right\} \oplus \mathbb{C}.\kappa \\
&= \mathrm{Span}_{\mathbb{C}} \left\{ (b^i)_0 \right\} \oplus \mathbb{C}.\kappa \ ,
\end{aligned}$$

where the more convenient notation $(b^j)_m = b^j \otimes t^m$ is employed. Then the

Heisenberg vertex operator algebra H is a structure given to the underlying vector space of the Verma module

$$M(0,1) = U(\widehat{\mathfrak{h}}) \otimes_{U\left(\widehat{\mathfrak{h}}_{\geq 0}\right)} \mathbb{C}.\mathbf{1} \tag{A.28}$$

where $b^i \otimes t^n.\mathbf{1} = 0$ for all $n \in \mathbb{N}$ and where $\kappa.\mathbf{1} = \mathbf{1}$. More precisely, we set $H = M(0,1)$ as a vector space and consider its Poincaré-Birkhoff-Witt basis composed of monomials of the form

$$(b^{i_1})_{n_1} \cdots (b^{i_d})_{n_d}.\mathbf{1} \in H \tag{A.29}$$

where $d \in \mathbb{N}$, $i_j \in \{1, \ldots, r\}$ and $n_j \in \mathbb{Z}_{<0}$ for all $j \in \{1, \ldots, d\}$ satisfy $i_j \leq i_{j'}$ whenever $j < j'$ and $n_j < n_{j'}$ whenever $i_j = i_{j'}$ and $j < j'$. The vertex operator algebra structure H is then fixed by setting

- $\mathbf{1} \in H$ is the vacuum vector;
- we set $b^i(z) = \mathbf{Y}\Big((b^i)_{-1}.\mathbf{1}, z\Big) = \sum_{n \in \mathbb{Z}} (b^i)_n z^{-n-1}$;
- for v given by (A.29), we set

$$\mathbf{Y}(v, z) = \frac{:\partial_z^{-n_1-1} b^{i_1}(z) \cdots \partial_z^{-n_d-1} b^{i_d}(z):}{(-n_1-1)! \cdots (-n_d-1)!} \; ;$$

- the (standard) conformal vector is given by the Casimir element in $U(\mathfrak{h})$ as follow:

$$\omega_H = \frac{1}{2} \sum_{a=1}^{r} (b^a)_{-1} (b^a)_{-1}.\mathbf{1} \in H \; . \tag{A.30}$$

  It has central charge of $c_{\mathfrak{h}} = r = \dim_{\mathbb{C}} \mathfrak{h}$ and makes all generators b^i homogeneous elements of degree one.

The operator product expansion among pairs of generating fields is then

$$b^{a_1}(z)b^{a_2}(w) = \frac{2\cdot \mathbf{1}}{(z-w)^2} + \ :b^{a_1}(z)b^{a_2}(w): , \quad a_1, a_2 \in \{1, \ldots, r\}$$

Remark A.31. Recall that an H-module is automatically $\widehat{\mathfrak{h}}$-module where $\widehat{\mathfrak{h}}$ is a Lie algebra. For the vertex operator algebra H, a weight module M is a H-module that is semisimple as an $\mathfrak{h}$-module through the natural injection of Lie algebras $\mathfrak{h} \hookrightarrow \widehat{\mathfrak{h}}_0$. In particular, a simple $\mathbb{Z}$-graded weight module for H is parametrised by an element $\lambda \in \mathfrak{h}^* = \{x \in \mathfrak{h} \otimes_{\mathbb{Z}} \mathbb{C} \mid (x, h) \in \mathbb{Z} \text{ for all } h \in \mathfrak{h}\}$.

For simplicity, let H be the vertex operator algebra associated with the rank 1 Heisenberg Lie algebra $\mathfrak{h} = \mathrm{Span}_{\mathbb{C}}\{b_0.\}$. Recall that $\mathfrak{h}^* \cong \mathbb{C}$ via $(\lambda b_0, b_0) = 2\lambda \in \mathbb{C}$. Here are some key facts about the category $\mathcal{C}$ of finitely generated weight-modules for H:

- $\mathcal{C}$ is semisimple and its simple objects are the Fock spaces

$$\left\{M(\lambda, 1) = F_\lambda \mid \lambda \in \mathfrak{h}^* \leftrightarrow (\lambda b_0, b_0) \text{ where } \lambda \in \mathbb{C}\right\},$$

  whose tensor product rules are given by $F_\lambda \otimes F_\mu \cong F_{\lambda+\mu}$;

- $\mathcal{C}$ is a semisimple ribbon tensor category;

- under the identifications $F_\lambda \otimes F_\mu = F_{\lambda+\mu} = F_\mu \otimes F_\lambda$, the braiding of two Fock spaces is given by $c_{F_\lambda, F_\mu} = e^{\pi i \lambda\mu}\, \mathrm{Id}_{F_{\lambda+\mu}}$;

- associativity maps in $\mathcal{C}$ are trivial;

- the unit for the tensor product in $\mathcal{C}$ is $H = F_0$;

- duals are given by $\big(F_\lambda\big)^* = F_{-\lambda}$ and the associated evaluation and coevaluation morphisms are scalar multiples of the identity whose scalars can be fixed for every $\bar{\lambda} \in \mathbb{C}/\{\mu \sim -\mu\}$;

- twists isomorphism are given by $\theta_{F_\lambda} = e^{\pi i \lambda^2}\,\mathrm{Id}_{F_\lambda}$.

Note that these results for $\mathcal{C}$ hold more generally for the H based on a r-dimensional Heisenberg Lie algebra $\mathfrak{h}$. The main difference is that $\lambda \in \mathfrak{h}^*$ will be a r-tuple of complex numbers instead of a single one.

A.2.2 Even Lattice Vertex Operator Algebras

More details can be found in [Don 1993], [DL 1993], [LL 2003]. Certain explicit details can also be found in [FRS 2004]. For a compact overview of key elements of the representation theory of the lattice vertex operator algebras, see [Mil 2014].

Let $L = \sum_{i=1}^r \mathbb{Z}.g^i$ be a positive definite even lattice of rank r with bilinear form

$$(-,-) : L \times L \to \mathbb{C}\ .$$

In particular, this implies that $(\ell,\ell) \in 2\mathbb{N}\backslash\{0\}$ for all $\ell \in L$ and we deduce that (ℓ_1,ℓ_2) is an integer for every $\ell_1,\ell_2 \in L$. Set $g^i(z) = \sum_{n\in\mathbb{Z}} (g^i)_n\, z^{-n-1}$ for all $i \in \{1,\ldots,r\}$ and set

$$g^i(z)g^j(w) = \frac{(g^i,g^j)\cdot \mathbf{1}}{(z-w)^2} + \ :g^i(z)g^j(w):\ ,$$

for all $i,j \in \{1,\ldots,r\}$. This is equivalent to defining a Lie algebra structure on the vector space $\widehat{\mathfrak{h}^L} = L \otimes_{\mathbb{Z}} \mathbb{C}[t^{\pm 1}] \oplus \mathbb{C}.\kappa$ as follows:

$$\begin{aligned}\big[(g^i)_n, (g^j)_m\big] &= n(g^i, g^j)\, \delta_{n+m,0} \cdot \kappa \ , \\ \big[\kappa, (g^i)_n\big] &= 0 \ .\end{aligned}$$

We observe that, $\widehat{\mathfrak{h}^L}$ has a natural triangular decomposition

$$\underbrace{\mathrm{Span}_{\mathbb{C}}\{(g^i)_n \mid n \in \mathbb{Z}_{<0}\}}_{\widehat{\mathfrak{h}^L}_{<0}} \oplus \underbrace{(\mathrm{Span}_{\mathbb{C}}\{(g^i)_0\} \oplus \mathbb{C}.\kappa)}_{\widehat{\mathfrak{h}^L}_{0}} \oplus \underbrace{\mathrm{Span}_{\mathbb{C}}\{(g^i)_n \mid n \in \mathbb{Z}_{>0}\}}_{\widehat{\mathfrak{h}^L}_{>0}} \ .$$

As is usual for lattices, define $L^* = \{x \in L \otimes_{\mathbb{Z}} \mathbb{C} \mid (x, \ell) \in \mathbb{Z}\}$[2]. Eventually, L^* will be used to define $\mathbb{Z}$-gradation on certain spaces that will bear the structure of weak $\mathbb{N}$-graded V_L-modules.

Similarly to the case of the Heisenberg vertex operator algebra treated in the above subsection, define $H^L = M(0,1)$ by (A.28), with $\widehat{\mathfrak{h}^L}$ instead of $\widehat{\mathfrak{h}}$. The even lattice vertex operator algebra V_L is then the following extension of H^L:

$$V_L = \bigoplus_{a \in L} M(\lambda_a, 1)$$

where $\lambda_a(-) = (a, -) \in L^*$ is a weight function. By definition, any $v \in V_L$ can be written as a finite sum of Poincaré-Birkhoff-Witt monomials corresponding to some spaces $M(\lambda_a, 1)$. By the Reconstruction Theorem (see [FBZ 2004] for instance), the vertex operator fields for $v \in H^L = M(0,1)$ can be defined inductively by only the fields associated to the highest weight vectors $\mathbf{1}_{\lambda_a} \in M(\lambda_a, 1)$ for $a \in L$. The conformal vector ω_{V_L} is given by the Casimir element of $\mathfrak{h}^L$ just as (A.30) is for

[2]The notion of dual lattice L^* coincides with the notion of integral weight lattice of the toral Lie subalgebra $\widehat{\mathfrak{h}^L}_0 \subset \widehat{\mathfrak{h}^L}$.

H in the previous section. The fields $\mathbf{Y}(v,z)$ for $v \in H^L$ are defined just as for the Heisenberg vertexoperator algebra and the fields $\mathbf{Y}(\mathbf{1}_{\lambda_a}, z)$ are set so that they satisfy

$$\mathbf{Y}(L_0.\mathbf{1}_{\lambda_a}, z) = \frac{(\lambda_a, \lambda_a)}{2}\mathbf{Y}(\mathbf{1}_{\lambda_a}, z) \ ,$$
$$\mathbf{Y}((L_1.\mathbf{1}_{\lambda_a}, z) = \partial_z \mathbf{Y}(\mathbf{1}_{\lambda_a}, z) \ .$$

Using explicit formulas for the endomorphisms L_n, it can be shown that $L_n.\mathbf{1}_{\lambda_a} = 0$ for all $n \geq 2$. It follows that $\mathbf{Y}(\mathbf{1}_{\lambda_a}, z)$ is Virasoro primary vector for all $a \in L$.

The even lattice vertex operator algebra V_L is known to be a rational C_2-cofinite vertex operator algebra. In particular, the category of finitely generated V_L-module with a semisimple $\mathfrak{h}^L$-action has a finite set of simple modules. These correspond to the simple weak $\mathbb{N}$-graded V_L-modules:

$$V_{L+\gamma} = V_L \otimes M(\gamma, 1) \tag{A.32}$$

where $\bar{\gamma} \in L^*/L$. Fixing a complete set of representatives $S = \{L + \gamma_i\}_{i=1}^{\#L^*/L}$ of right cosets of L^*/L is the same as fixing a 2-cocycle

$$k^S : \quad \begin{aligned} L \times L &\longrightarrow L \\ (\gamma_i, \gamma_j) &\longmapsto \gamma_{i+j} \end{aligned} \qquad \text{where } \gamma_i + \gamma_j \equiv \gamma_{i+j} \in S \ .$$

Let $\mathcal{C}$ be the category of weak $\mathbb{N}$-graded V_L-module with a single object in each isomorphism class where S is the complete set of representatives of the simple objects. Then $\mathcal{C}$ is known to have the following semisimple ribbon tensor category structure:

- it has a finite set S of simple objects;

- the tensor product rules between any two such simple modules is given by

$$V_{L+\gamma_i} \otimes V_{L+\gamma_j} \cong V_{L+\gamma_{i+j}} \ ;$$

- the associativity isomorphisms are

$$a_{\gamma_a,\gamma_b,\gamma_c} = (-1)^{\left(\gamma_a, k^S(b,c)\right)} \operatorname{Id}_{V_{L+\gamma_{a+b+c}}} \ ;$$

- the braiding isomorphisms are

$$c_{\gamma_a,\gamma_b} = (-1)^{(\gamma_a,\gamma_b)} \operatorname{Id}_{V_{L+\gamma_{a+b}}} \ ;$$

- the twist isomophisms are

$$\theta_{\gamma_a} = (-1)^{(\gamma_a,\gamma_a)} \operatorname{Id}_{V_{L+\gamma_a}} \ ;$$

- the dual of the module $V_{L+\gamma_a}$ is $V_{L+\gamma-a}$. Corresponding evaluation and coevaluation maps are given as scalar multiple of the identity in the same way as for the Heisenberg algebra H^L.

Note that since the double braiding of $V_{L+\gamma}$ with V_L is $c_{0,\gamma} \circ c_{\gamma,0} = \operatorname{Id}_{V_\gamma}$ is always trivial, the modules of $\mathcal{C}$ are said to be *untwisted*.

A.2.3 Affine Vertex Operator Algebras

Let $\mathfrak{g}$ be a finite dimensional simple Lie algebra. Then one forms two important associated affine vertex operator algebras: the universal one and the simple one, which sometimes match.

Fix a basis $\{X^i\}_{i=1}^{\dim \mathfrak{g}}$ of $\mathfrak{g}$, a symmetric invariant bilinear form $(-,-)$ such that the length of the long roots of $\mathfrak{g}$ is 2. Consider $\widehat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t^{\pm 1}] \oplus \mathbb{C}.\kappa$ with Lie bracket

$$[x_n, y_m] = [x, y]_{n+m} + n(x, y)\delta_{n+m,0} \cdot \kappa \qquad \text{and} \qquad [\kappa, x_n] = 0 \ ,$$

for all $x, y \in \mathfrak{g}$ and $n, m \in \mathbb{Z}$. Fix $k \in \mathbb{C}$ and consider the following triangular decomposition of $\mathfrak{g}$:

$$\begin{aligned} \mathfrak{g}_{>0} &= \mathrm{Span}_{\mathbb{C}}\{x_n \mid x \in \mathfrak{g} \text{ and } n \in \mathbb{Z}_{<0}\} \ , \\ \mathfrak{g}_0 &= \mathrm{Span}_{\mathbb{C}}\{x_0 \mid x \in \mathfrak{g}\} \oplus \mathbb{C}.\kappa \ , \\ \mathfrak{g}_{<0} &= \mathrm{Span}_{\mathbb{C}}\{x_n \mid x \in \mathfrak{g} \text{ and } n \in \mathbb{Z}_{>0}\} \ . \end{aligned}$$

Similarly as in the previous two sections, we can define a vertex operator algebra structure on a Verma module. This time, the Verma module we consider has highest weight 0 (the h_0-eigenvalue) and level k (the κ-eigenvalue):

$$V(0, k) = U(\widehat{\mathfrak{g}}) \otimes_{U(\mathfrak{g}_{\geq 0})} \mathbb{C}.\mathbf{1}$$

where $\mathfrak{g}_{>0}.\mathbf{1} = e_0.\mathbf{1} = h_0.\mathbf{1} = 0$ and $\kappa.\mathbf{1} = k\mathbf{1}$. The vertex operator algebra structure is then given by:

- $\mathbf{1} \in V(0, k)$ is the vacuum vector;

- we set $x(z) = \mathbf{Y}(x_{-1}, z) = \sum_{n\in\mathbb{Z}} x_n z^{-n-1}$ for all $x \in \{h_i, e_i, f_i\}_{i=1}^r$;

- for a Poincaré-Birkhoff-Witt monomial $v = \left(\prod_{i=1}^{\dim \mathfrak{g}} \prod_{s=1}^{n_i} X^i_{a_i}\right).\mathbf{1}$ where

$n_i \in \mathbb{N}$ and $a_i \in \mathbb{Z}_{\leq 0}$ for all i, we define

$$\mathbf{Y}(v,z) = \frac{:\prod_{i=1}^{\dim \mathfrak{g}} \prod_{s=1}^{n_i} \partial_z^{-a_s-1} X^i(z):}{\prod_{i=1}^{\dim \mathfrak{g}} \prod_{s=1}^{n_i} (-a_s-1)!}$$

where the term ordering in the normally ordered product is the same than that of the Poincaré-Birkhoff-Witt base;

- the (Sugawara) conformal vector is given in a way that recalls the Casimir element as follows:

$$\omega_{\mathfrak{g}} = \left(\sum_{i=1}^{\dim \mathfrak{g}} (X^i)_{-1} (X^{i,*})_{-1} \right) .\mathbf{1} \in V(0,k)$$

where x^* is the dual of x with respect to the bilinear form $(-,-)$ for all $x \in \mathfrak{g}$. This conformal vector has central charge

$$c_{\mathfrak{g}} = \frac{(\dim_{\mathbb{C}} \mathfrak{g})k}{k+h^\vee}$$

where $h^\vee$ is the dual coxeter number[3] of $\mathfrak{g}$.

The operator product expansion among pairs of generating fields is then

$$X^i(z)X^j(w) = \frac{k\,(X^i,X^j)\cdot \mathbf{1}}{(z-w)^2} + \frac{[X^i,X^j](w)}{z-w} + \ :X^i(z)X^j(w):$$

where $i,j \in \{1,\ldots,\dim \mathfrak{g}\}$.

[3]This is the number $h^\vee$ such that the $2 = (r,r) = \frac{1}{h^\vee}(r,r)_{\text{Killing}}$ for all long roots $r \in \mathsf{Q}_{\mathfrak{g}}$ of $\mathfrak{g}$.

Appendix B

Notions of Category Theory

In this appendix, you will find basic notions definitions about Category Theory and extensions for vertex operator algebras in order to be able to better follow some parts of the thesis. A more compact version of this appendix can be found in Section 2 of [AR 2018].

For a brief introduction to basic notions of Homological Algebra in the specific context of Representation Theory, see [EGH$^+$ 2011]. For a more in depth coverage of Homological Algebra, one can find much in [Rot 2009] and [Wei 1994].

Historically, it was proven that rational and C_2-cofinite vertex operator algebras give rise to categories of modules with the rich structure of a modular tensor category [Hua 2008]. Motivated partly by physical considerations, it is expected that parts of this rich categorical structure is still shared in important non-rational settings [HLZ 2007], [CG 2017], [CG], see also the introductive parts of [CKL 2015], [CKM 2017]. In [HLZ 2007], considerable efforts have been put to define tensor products for a broad class of vertex operator algebras. The important family of logarithmic triplet vertex operator algebras are also known to have a rich category of modules [TW 2013]. Several studies of logarithmic settings [CR 2013b],[GR 2017], [Fuc 2007] lead to the expectation that monoidal braided tensor categories are key concepts for better understanding generic vertex operator algebras.

B.1 Background on Braided Monoidal Categories

Detailed references on monoidal categories and related topics include [EGNO 2015] and [BAK 2001]. In the latter reference, the very useful *graphical calculus* is introduced and detailed.

B.1.1 Categorical Background

A category $\mathcal{C}$ is a class of objects $\mathrm{Ob}\,\mathcal{C}$ and of morphisms $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ where $X, Y \in \mathrm{Ob}\,\mathcal{C}$. These are subjected to the assumption that there is an associative composition law

$$\circ : \ \mathrm{Hom}_{\mathcal{C}}(V, W) \times \mathrm{Hom}_{\mathcal{C}}(U, V) \to \mathrm{Hom}_{\mathcal{C}}(U, W) \ .$$

Additionnally, *identity morphisms* $\mathrm{Id}_X \in \mathrm{Hom}_{\mathcal{C}}(X, X)$ should also exist for each object $X \in \mathrm{Ob}\,\mathcal{C}$. An invertible morphism in $\mathcal{C}$ is usually called isomorphisms. Observe that $\mathrm{Ob}\,\mathcal{C}$ can then be partitioned in isomorphism classes by setting $X \sim Y$ whenever there exists an isomorphism between X and Y. We usually denote by $[X]$ the *isomorphism class* of $X \in \mathrm{Ob}\,\mathcal{C}$.

The second most fundamental notion is that of *functors* which are maps between categories. More precisely, functors are composition preserving maps between objects of two given categories. When the categories in question have additional structure e.g. additive, monoidal, braided, etc., one consider subsets of functors that preserve these additional structures.

One calls a category skeletal if it has only one object per isomorphism class. For instance, let R be a complete set of representative of isomorphism classes of a given category $\mathcal{C}$, we can then sometimes define a skeletal category whose objects are given by R and whose structure is essentially that of $\mathcal{C}$. In some situations, this can make some results appear more explicit such as in the analysis of the even lattice vertex operator algebra of the end of Chapter 2 and [AR 2018].

We move on to additive structures on categories:

Definition B.1. A category $\mathcal{C}$ is *additive* if

- $\mathrm{Hom}_{\mathcal{C}}(U, V)$ is an abelian group for every pair of objects $U, V \in \mathrm{Ob}(\mathcal{C})$ and composition of morphisms is bi-additive,
- $\mathcal{C}$ has a zero object 0 such that $\mathrm{Hom}_{\mathcal{C}}(0, 0) = 0$ is the trivial abelian group,
- $\mathcal{C}$ contains finite direct sums (finite coproducts). That is, for every pair of objects $V_1, V_2 \in \mathrm{Ob}(\mathcal{C})$, there exists $W = V_1 \oplus V_2 \in \mathrm{Ob}(\mathcal{C})$ and morphisms $p_1 : W \to V_1$, $p_2 : W \to V_2$, $i_1 : V_1 \to W$, $i_2 : V_2 \to W$ such that $p_1 \circ i_1 = \mathrm{Id}_{V_1}$, $p_2 \circ i_2 = \mathrm{Id}_{V_2}$, and $i_1 \circ p_1 + i_2 \circ p_2 = \mathrm{Id}_W$.

Let $\mathbb{F}$ be a field. An additive category $\mathcal{C}$ is called $\mathbb{F}$-*linear* if for each $U, V \in \mathrm{Ob}\,\mathcal{C}$, $\mathrm{Hom}_{\mathcal{C}}(U, V)$ is a vector space over $\mathbb{F}$ and the composition is $\mathbb{F}$-bilinear.

We then move on to the realm of tensor products and monoidal categories, central concepts for the study of vertex operator algebras:

A *tensor product* on a category $\mathcal{C}$ is a bifunctor

$$\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C} \ ,$$

that commutes with finite direct sums. Naturally, one expects such products to be associative and so we define *associativity constraints* to be trinatural isomorphisms

$$\left\{a_{U,V,W} : (U \otimes V) \otimes W \to U \otimes (V \otimes W)\right\}_{U,V,W \in \mathrm{Ob}(\mathcal{C})} .$$

An associativity constraint on $\mathcal{C}$ satisfies the *pentagon axiom* if the diagram

$$\begin{tikzcd}
(U \otimes (V \otimes W)) \otimes X \arrow[d, "a_{U,V\otimes W,X}"] & & ((U \otimes V) \otimes W) \otimes X \arrow[ll, "a_{U,V,W}\otimes \mathrm{Id}_X"'] & \\
& & & (U \otimes V) \otimes (W \otimes X) \arrow[ul, "a_{U\otimes V,W,X}"'] \arrow[dl, "a_{U,V,W\otimes X}"] \\
U \otimes ((V \otimes W) \otimes X) & & U \otimes (V \otimes (W \otimes X)) \arrow[ll, "\mathrm{Id}_U \otimes a_{V,W,X}"'] &
\end{tikzcd} ,$$

commutes for every choice of objects $U, V, W, X \in \mathrm{Ob}(\mathcal{C})$.

Another useful notion is that of a unit object $(\mathbb{1}, \iota)$ where $\mathbb{1} \in \mathrm{Ob}\,\mathcal{C}$ and $\iota : \mathbb{1} \otimes \mathbb{1} \to \mathbb{1}$. We say that left and right *unit constraints* are natural isomorphisms

$$\left\{l_V : \mathbb{1} \otimes V \to V\right\}_{V \in \mathrm{Ob}(\mathcal{C})} ,$$
$$\left\{r_V : V \otimes \mathbb{1} \to V\right\}_{V \in \mathrm{Ob}(\mathcal{C})} .$$

The unit constraints are said to satisfy the triangle axiom if the following diagram commutes:

$$\begin{tikzcd}
(U \otimes \mathbb{1}) \otimes V \arrow[rr, "a_{U,I,V}"] \arrow[dr, "r_U \otimes \mathrm{Id}_V"'] & & U \otimes (\mathbb{1} \otimes V) \arrow[dl, "\mathrm{Id}_U \otimes l_V"] \\
& U \otimes V &
\end{tikzcd} ,$$

for every pair of objects $U, V \in \mathrm{Ob}\,\mathcal{C}$. Finally, we can define a monoidal category:

Definition B.2. A monoidal category is a tuple $(\mathcal{C}, \otimes, a_{-,-,-}, \mathbb{1}, \iota, \ell_-, r_-)$ where $\mathcal{C}$ is a category with tensor product $\otimes$ with associativity constraint $a_{-,-,-}$, unit object $(\mathbb{1}, \iota)$ and unit constraints r_-, l_- which satisfy the pentagon and triangle axiom.

Then we call a functor $\underline{\mathcal{F}} : \mathcal{C} \to \mathcal{C}'$ monoidal if it comes with binatural isomorphisms

$$J_{X,Y} : \underline{\mathcal{F}}(X) \otimes_{\mathcal{C}'} \underline{\mathcal{F}}(Y) \xrightarrow{\cong} \underline{\mathcal{F}}(X \otimes_{\mathcal{C}} Y) \ ,$$

such that $\underline{\mathcal{F}}(\mathbb{1}_{\mathcal{C}}) \cong \mathbb{1}_{\mathcal{C}'}$ and successive uses of associatiivity and $\underline{\mathcal{F}}$ on the tensor product of three objects always commute.

Remark B.3. As noted in [EGNO 2015], a monoidal category can be shown to be equivalent to a srtict monoidal category or to a skeletal monoidal category, but not both at the same time.

Let $\mathcal{C}$ be a monoidal tensor category. A *commutativity constraint* on $\mathcal{C}$ is a family

$$\left\{c_{U,V} : U \otimes V \to V \otimes U\right\}_{U,V \in \mathrm{Ob}\,\mathcal{C}} \ ,$$

of natural isomorphisms. A *braiding* structure on $\mathcal{C}$ is a commutativity constraint which also satisfies the *hexagon axiom*, which is the commutativity of

$$\begin{tikzcd}
 & U \otimes (V \otimes W) \arrow[r, "c_{U,V\otimes W}"'] & (V \otimes W) \otimes U \arrow[dr, "a_{V,W,U}"] & \\
(U \otimes V) \otimes W \arrow[ur, "a_{U,V,W}"'] \arrow[dr, "c_{U,V} \otimes \mathrm{Id}_W"] & & & V \otimes (W \otimes U) \ , \\
 & (V \otimes U) \otimes W \arrow[r, "a_{V,U,W}"'] & V \otimes (U \otimes W) \arrow[ur, "\mathrm{Id}_V \otimes c_{U,W}"'] &
\end{tikzcd}$$

and of the analagous diagram for a^{-1}.

Definition B.4. A braided monoidal category is a monoidal category with a commutativity constraint c satisfying the hexagon axioms.

Braided functors are monoidal functors that preserve braiding isomorphisms.

A twist θ in a braided monoidal category $\mathcal{C}$ is a family of natural isomorphisms

$$\{\theta_V : V \to V\}_{V \in \mathrm{Ob}(\mathcal{C})} \ ,$$

such that the *balancing axiom* $\theta_{U\otimes V} = c_{V,U} \circ c_{U,V} \circ (\theta_U \otimes \theta_V)$ holds.

Finally, let's define duals in a monoidal category $\mathcal{C}$. A left dual for $X \in \operatorname{Ob}\mathcal{C}$ is an object X^* with *duality morphisms*

$$\overrightarrow{\mathrm{coev}}_X : \mathbb{1} \to X \otimes X^*, \qquad \overrightarrow{\mathrm{ev}}_X : X^* \otimes X \to \mathbb{1} \ ,$$

which satisfy the relations

$$r_X \circ (\mathrm{Id}_X \otimes \overrightarrow{\mathrm{ev}}_X) \circ a_{X,X^*,X} \circ (\overrightarrow{\mathrm{coev}}_X \otimes \mathrm{Id}_X) \circ l_X^{-1} = \mathrm{Id}_X \ ,$$
$$l_{X^*} \circ (\overrightarrow{\mathrm{ev}}_X \otimes \mathrm{Id}_{X^*}) \circ a^{-1}_{X^*,X,X^*} \circ (\mathrm{Id}_{X^*} \otimes \overrightarrow{\mathrm{coev}}_X) \circ r^{-1}_{X^*} = \mathrm{Id}_{X^*} \ .$$

If a left dual exists for every $V \in \operatorname{Ob}\mathcal{C}$, the category is called *left rigid*. Right duals are defined analogously.

When $\mathcal{C}$ is a braided monoidal, a left duality morphisms are said to be compatible with the braiding and twist if they satisfy the additional relation

$$(\theta_X \otimes \mathrm{Id}_{X^*}) \circ \overrightarrow{\mathrm{coev}}_X = (\mathrm{Id}_X \otimes \theta_{X^*}) \circ \overrightarrow{\mathrm{coev}}_X \ ,$$

for all $X \in \operatorname{Ob}\mathcal{C}$.

Definition B.5. A *ribbon category* is a rigid braided monoidal category where duality morphisms are compatible with braidings and where $(\theta_X)^* = \theta_{X^*}$.

B.1.2 Algebra Objects, $\operatorname{Rep} A$ and $\operatorname{Rep}^0 A$

This subsection is devoted to defining certain module category $\operatorname{Rep}^0 A$ for an algebra object A defined right below. For a more detailed overview of these concepts, see the book [EGNO 2015].

This category has a natural braided structure coming from the base category $\mathcal{C}$. The category $\operatorname{Rep}^0 A$ had first been introduced in [Par 1995]. This point of view has later been applied to semisimple Representation Theory settings in [KJO 2002] and more recently in non-semisimple settings for vertex operator algebras [HKJL 2015], [CKM 2017].

Definition B.6. Let $\mathcal{C}$ be a monoidal category with unit $\mathbb{1}$. An *algebra object* in $\mathcal{C}$ is a triple (A, μ, u) where μ and u are maps as follows:

$$\mu : \ A \otimes A \longrightarrow A \ , \qquad\qquad u : \ \mathbb{1} \longrightarrow A \ .$$

where μ plays the role of a multiplication map in A and u of a unit in A. Moreover, we require μ to be associative and u to be a unit for the multiplication μ in bot slots.

If $\mathcal{C}$ is also braided, then an algebra object $A \in \operatorname{Ob}\mathcal{C}$ is said to be commutative if $\mu \circ c_{A,A} = \mu$.

Definition B.7. Let (A, μ, u) be an associative unital and commutative algebra object in $\mathcal{C}$. Define $\operatorname{Rep} A$ to be the category whose objects are given by pairs (V, μ_V) where $V \in \operatorname{Ob}(\mathcal{C})$ and $\mu_V \in \operatorname{Hom}_{\mathcal{C}}(A \otimes V, V)$ are subject to the natural associativity and unit requirement of an action of A.

We define the morphisms between two objects as follows. Let (M, μ_M), (N, μ_N) be two objects of $\operatorname{Rep} A$ and set

$$\operatorname{Hom}_{\operatorname{Rep} A}(M, N) = \left\{ f \in \operatorname{Hom}_{\mathcal{C}}(M, N) \ \middle| \ f \circ \mu_M = \mu_N \circ (\operatorname{Id}_A \otimes f) \right\} \ .$$

Schematically, a morphism $f : M \to N$ in $\operatorname{Rep} A$ is a morphism of $\mathcal{C}$ such that the following diagram commutes:

$$
\begin{tikzcd}
A \otimes_{\mathcal{C}} M \arrow[r, "\mu_M"] \arrow[d, "\mathrm{Id}_A \otimes f"'] & M \arrow[d, "f"] \\
A \otimes_{\mathcal{C}} N \arrow[r, "\mu_N"'] & N
\end{tikzcd} .
$$

The next important element is the definition of a natural tensor product $\otimes_A$ for the category of categorical modules $\mathrm{Rep}\, A$.

Definition B.8. Let (X, μ_X), $(Y, \mu_Y) \in \mathrm{Ob}\,\mathrm{Rep}^0\, A$. Consider the tensor product

$$
X \otimes_{\mathcal{C}} (A \otimes_{\mathcal{C}} Y) \ ,
$$

and the following morphisms of $\mathcal{C}$:

$$
\begin{aligned}
m^{\text{left}} = (\mu_X \otimes \mathrm{Id}_Y) \circ c_{X,A} \circ a_{X,A,Y} : \quad & X \otimes_{\mathcal{C}} (A \otimes_{\mathcal{C}} Y) \longrightarrow X \otimes_{\mathcal{C}} Y \ , \\
m^{\text{right}} = \mathrm{Id}_X \otimes \mu_Y : \quad & X \otimes_{\mathcal{C}} (A \otimes_{\mathcal{C}} Y) \longrightarrow X \otimes_{\mathcal{C}} Y \ .
\end{aligned}
$$

Define the object of the tensor product $\otimes_A$ by:

$$
X \otimes_A Y = \frac{X \otimes_{\mathcal{C}} Y}{\mathrm{im}(m^{\text{left}} - m^{\text{right}})} \ . \tag{B.9}
$$

Additionnally, we can define the A-action map on $X \otimes_A Y$ as

$$
\mu_{V \otimes_A W} = \mu_V \otimes \mathrm{Id}_W \circ a^{-1}_{A,V,W} : A \otimes (V \otimes_A W) \to V \otimes_A W \ ,
$$

since the A-action via μ_V is identified with the action via μ_W on the quotient (B.9).

Remark B.10. The tensor product defined as of (B.9) is a *coequaliser* in terms of Category Theory vocabular. However it is quite close to the usual notion of tensor product for two modules M and N of an associative ring R. In the latter more familiar case, m^{left} is just $r.(m \otimes n) = (r.m) \otimes n$ while m^{right} is $r.(m \otimes n) = m \otimes (r.n)$. The product $M \otimes_R N$ is then defined to be analog of (B.9) where the rule $(r.m) \otimes n = m \otimes (r.n)$ for every $r \in R$, $m \in M$ and $n \in N$ is imposed.

Remark B.11. As mentioned in [EGNO 2015], the above definition of $\otimes_A$ makes it a right exact bifunctor $\operatorname{Rep} A \times \operatorname{Rep} A \to \operatorname{Rep} A$.

One can show that $\operatorname{Rep} A$ has natural associativity morphisms and that it is a monoidal category with unit $A \in \operatorname{Ob} \operatorname{Rep} A$ [EGNO 2015]. As noted in [Par 1995], the category $\operatorname{Rep} A$ might not have a natural braided structure, but a certain subcategory $\operatorname{Rep}^0 A$ always has. We define this category as follows:

Definition B.12. Let (A, μ, ι_A) be an associative unital and commutative algebra object in $\mathcal{C}$. Define $\operatorname{Rep}^0 A$ to be the full subcategory of $\operatorname{Rep} A$ whose objects (V, μ_V) satisfy

$$\mu_V \circ (c_{V,A} \circ c_{A,V}) = \mu_V \ .$$

The category $\operatorname{Rep}^0 A$ is often referred to as the category of *local* or *untwisted* A-modules.

Remark B.13. See [Par 1995] for the original setting in which $\operatorname{Rep}^0 A$ first appeared. This full subcategory of $\operatorname{Rep} A$ corresponds to those for which the use braidings in $\mathcal{C}$ to define a right module structure gives an equivalent object. For this reason, $\operatorname{Rep}^0 A$ was sometimes said to be the category of dyslexic A-modules.

One of the key element studied in [CKM 2017] is the following natural induction functor in relation to $\operatorname{Rep}^0 A$:

Definition B.14. Let A be a commutative unital algebra object in a braided monoidal category $\mathcal{C}$. We have an induction functor

$$\begin{aligned} \mathcal{F}: \quad \mathcal{C} &\longrightarrow \operatorname{Rep} A \\ M &\longmapsto \left(A \otimes_{\mathcal{C}} M \,,\, (\mu \otimes \mathrm{Id}_M) \circ \mathbf{A}^{\mathcal{C}}_{A,A,M}\right) \end{aligned} . \tag{B.15}$$

where $(\mu \otimes \mathrm{Id}_M) \circ \mathbf{A}^{\mathcal{C}}_{A,A,M} : A \otimes_{\mathcal{C}} (A \otimes_{\mathcal{C}} M) \to A \otimes_{\mathcal{C}} M$ is the corresponding A-action map.

It was proven in [KJO 2002] that $\mathcal{F}$ is a monoidal functor in the sense that it preserves tensor products up to natural isomorphisms [EGH$^+$ 2011]. Moreover, Theorem 1.6 of [KJO 2002] establishes that $\mathcal{F}$ is an exact and adjoint functor.

Further properties of $\mathcal{F}$ were studied in Section 2 of [CKM 2017]. Integrating the observations of [Par 1995], the following notable result was obtained:

Theorem B.16. (Theorem 2.67 of [CKM 2017]) *Let $\mathcal{C}^0$ denote the full subcategory of $\mathcal{C}$ consisting of objects that induce to $\operatorname{Rep}^0 A$. Then restricting the induction functor (B.15) to $\mathcal{C}^0$ gives a braided tensor functor*

$$\mathcal{C}^0 \to \operatorname{Rep}^0 A \,.$$

Concretely, this means that restricted to $\mathcal{C}^0$, (B.15) respects the tensor products and braidings of $\mathcal{C}$. If in addition one has $\theta_A = \mathrm{Id}_A$, then the balancing axiom for the twists in $\mathcal{C}$ show that the same morphisms define a twist isomorphisms on $\operatorname{Rep}^0 A$ and that the fucntor (B.15) also respects twists.

B.2 Simple Current Extensions

B.2.1 Extensions and Algebra Objects

A detailed framework to study general types of vertex operator algebra extensions has been developed in [CKM 2017] following important ideas from [KJO 2002], [HKJL 2015] and also [Par 1995].

Recall that a vertex operator algebra extension is an inclusion of vertex operator algebra $V \subseteq E$ where the conformal vectors coincide $\omega_V = \omega_E$. The main result of [CKM 2017] states that extensions of vertex operator algebra can be studied within the purely categorical framework of algebra objects reviewed in the preceding section.

Let $V \subset E$ be a vertex algebra extension. When the vertex tensor theory of Huang-Lepowsky-Zhang [HLZ 2007] applies to a given base category $\mathcal{C}$ of V-modules that contains $E \in \operatorname{Ob}\mathcal{C}$, the larger vertex operator algebra E can be seen as an algebra object in $\mathcal{C}$ category [KJO 2002], [HKJL 2015], [CKM 2017]. In fact, there is a one to one correspondance between algebra objects in $\mathcal{C}$ and vertex operator algebra extensions of V [CKM 2017]. Under this correspondance, the category $\operatorname{Rep}^0 E$ is braided equivalent to the category of *local* or *untwisted* E-modules as vertex operator algebras [HKJL 2015].

The availability of the categorical methods of [CKM 2017] to study some features of the Representation Theory of E is especially valuable when E is a logarithmic vertex operator algebra. In such a case, the vertex theoretic framework provided by [HLZ 2007] is very challenging to apply while this category theoretic framework is much more easy. Even though checking that a category $\mathcal{C}$ of vertex algebra modules is a difficult task, recent illustrative applications include:

- the even lattice vertex operator algebra [AR 2018] or Chapter 2, an infinite order simple current extension;
- $L_k\big(\mathfrak{osp}(1|2)\big)$ [CFK 2017], a finite extension that is not by simple currents;
- in principle, the $\mathcal{B}_p$ logarithmic vertex operator algebra [CR 2013a] [ACKR], see also [CRW 2014], [Cre 2017] and [BR 2018];
- in principle, the extended parafermions vertex operator algebras of [ACR 2018] and Chapters 3 and 4.

B.2.2 Extensions from Simple Currents

We summarise the setup for constructing a special type of extension for a simple vertex operator algebra V that is widely used in the mathematics and physics literature: a simple current extension. This can be useful to follow parts of Chapters 2 and 3.

Suppose that V is a vertex operator algebra with a ribbon module category $\mathcal{C}$ containing V and equipped with the vertex tensor structure of Huang-Lepowsky-Zhang [HLZ 2007]. Following [CKL 2015], we define a simple current as follows:

Definition B.17. A simple current in $\mathcal{C}$ is a simple object J which is invertible with respect to the tensor product. That is, there exists an object $J^{-1} \in \mathcal{C}$ satisfying $J \otimes J^{-1} \cong \mathbb{1}$.

Remark B.18. In particular, since the vertex operator algebra V is the tensor unit of $\mathcal{C}$, one can show that the existence of a simple current forces V to be simple. For a few more details on simple currents, see [CKL 2015].

We will now sketch how to construct a vertex operator algebra extension of V from a simple current J. In Theorem 1.3 of [CKL 2015], the author consider a simple current $J \in \mathrm{Ob}(\mathcal{C})$ and its powers such that the twists isomophisms satisfy $\theta_{J^{\otimes r}} = \pm \mathrm{Id}_{J^{\otimes r}}$ and $\theta_{J^{\otimes r}} = \theta_{J^{\otimes (r+2)}}$ for all $r \in \mathbb{Z}$ where $J^0 = V$ and $J^{\otimes -1} = J^*$. Set $n =$ order(J) $\in \mathbb{N} \cup \{\infty\}$ and also

$$G(J) = \begin{cases} \{0, .., n-1\} \leftrightarrow \mathbb{Z}/n\mathbb{Z} & \text{if } n \in \mathbb{N}, \\ \mathbb{Z} & \text{if } n = \infty. \end{cases}$$

Now consider the following:

$$E_S = \bigoplus_{r \in S} J^{\otimes r} , \tag{B.19}$$

where $S \leq G(J)$ is a subgroup of $G(J)$. By a result of [Li 2001], E_S has a natural structure of a vertex operator algebra if and only if it is integer-graded. See [CKL 2015] for more explanations.

Remark B.20. If $\#S = \infty$, we must see E_S as an object of the direct sum completion $\mathcal{C}_{\oplus}$ of $\mathcal{C}$. See Chapter 2 for more details.

Integer-graded objects of type (B.19) are examples of what is called a simple current extensions of V. In both Chapters 2 and 3, we consider simple currents extensions as the even lattice vertex operator algebra and the extended parafermion vertex operator algebras, respectively.

Appendix C

Explicit Fields and Relations Related to $\mathsf{C}_k(\mathfrak{sl}_2)$

In this appendix, we explicit the fields $\{L(z), W_3(z), W_4(z), W_5(z)\}$ that strongly generate the parafermion operator algebra $\mathsf{C}_k(\mathfrak{sl}_2)$ in terms of $L_k(\mathfrak{sl}_2)$ fields and we give the action of the differential operator $(W_3)_1$ on the strong generators modulo $\mathsf{B}_k(\mathfrak{sl}_2)$. The latter is motivated by the study of C_2-cofiniteness of [ALY 2014] at positive integral level.

Remark C.1. The primary fields $W_3(z)$,$W_4(z)$ and $W_5(z)$ below can all be scaled with a scalar so that we can eliminate any denominators.We chose to leave the denominators here since all our computations have been performed with $W_3(z)$,$W_4(z)$ and $W_5(z)$ as they are shown below. Indeed, the denominators in the expression below never vanish for $k \in A$ as of line (4.1).

C.1 Strong Generators of $\mathsf{C}_k(\mathfrak{sl}_2)$ From $L_k(\mathfrak{sl}_2)$ Fields

In the following, the products of fields should be interpreted as normally ordered products. The parafermion generating fields are as follows:

$$L(z) = -\frac{2ke(z)f(z) + h(z)^2 + kh(z)'}{2k^2 + 4k}$$

$$\begin{aligned} W_3(z) = \frac{1}{4}\big(&- 6f(z)(k-4)ke(z)' + 6e(z)(k-4)kf(z)' \\ &+ 12e(z)f(z)h(z)k + 4h(z)^3 + k^2h(z)'' + 6h(z)kh(z)'\big) \end{aligned}$$

$$\begin{aligned} W_4(z) = -\frac{1}{16k+17}\Big(&3k(2k+3)(36e(z)^2f(z)^2k^3 - 30e(z)^2f(z)^2k^2 \\ &+ 12f(z)k^4e(z)'' - 48f(z)k^3e(z)'' + 144f(z)k^2e(z)'' + 144f(z)ke(z)'' \\ &- 12ke(z)'((3k^3 - 23k^2 + 52k + 48)f(z)' + f(z)h(z)(5k^2 - 33k - 24)) \\ &+ 12e(z)k^4f(z)'' - 120e(z)k^3f(z)'' + 204e(z)k^2f(z)'' + 144e(z)kf(z)'' \\ &+ 12e(z)h(z)k(5k^2 - 33k - 24)f(z)' + 132e(z)f(z)h(z)^2k^2 \\ &+ 72e(z)f(z)h(z)^2k + 72e(z)f(z)k^3h(z)' - 60e(z)f(z)k^2h(z)' \\ &+ 33h(z)^4k + 18h(z)^4 + h(z)^{(3)}k^4 + h(z)^{(3)}k^3 + h(z)^{(3)}k^2 \\ &+ 12h(z)k^3h(z)'' + 12h(z)k^2h(z)'' + 12h(z)kh(z)'' + 15k^3(h(z)')^2 \\ &- 18k(h(z)')^2 + 66h(z)^2k^2h(z)' + 36h(z)^2kh(z)') \end{aligned}$$

$$
\begin{aligned}
W_5(z) = \frac{1}{128k+214}\Big(&9k^2(6k^2+17k+12)(-20e(z)^{(3)}f(z)k^5+200e(z)^{(3)}f(z)k^4\\
&-160e(z)^{(3)}f(z)k^3+880e(z)^{(3)}f(z)k^2+1920e(z)^{(3)}f(z)k\\
&+1200e(z)^2f(z)^2h(z)k^3-840e(z)^2f(z)^2h(z)k^2+60kf(z)'((2k^4\\
&-19k^3+64k^2-140k-288)e(z)''+e(z)(e(z)f(z)k(10k^2-87k+56)\\
&+h(z)^2(17k^2-164k-144))+e(z)k(9k^2-80k+44)h(z)')\\
&+180f(z)h(z)k^4e(z)''-900f(z)h(z)k^3e(z)''+6480f(z)h(z)k^2e(z)''\\
&+8640f(z)h(z)ke(z)''-60ke(z)'(10e(z)f(z)^2k^3-87e(z)f(z)^2k^2\\
&+56e(z)f(z)^2k+2k^4f(z)''-29k^3f(z)''+151k^2f(z)''-196kf(z)''\\
&-288f(z)''+8h(z)(k^3-14k^2+55k+72)f(z)'+17f(z)h(z)^2k^2\\
&-164f(z)h(z)^2k-144f(z)h(z)^2+f(z)(11k^2-94k+68)kh(z)')\\
&+20e(z)f(z)^{(3)}k^5-400e(z)f(z)^{(3)}k^4+1900e(z)f(z)^{(3)}k^3\\
&-2000e(z)f(z)^{(3)}k^2-1920e(z)f(z)^{(3)}k+180e(z)h(z)k^4f(z)''\\
&-3300e(z)h(z)k^3f(z)''+8160e(z)h(z)k^2f(z)''+8640e(z)h(z)kf(z)''\\
&+2280e(z)f(z)h(z)^3k^2+1440e(z)f(z)h(z)^3k+240e(z)f(z)k^4h(z)''\\
&-420e(z)f(z)k^3h(z)''+480e(z)f(z)k^2h(z)''+2400e(z)f(z)h(z)k^3h(z)'\\
&-1680e(z)f(z)h(z)k^2h(z)'+456h(z)^5k+288h(z)^5+h(z)^{(4)}k^5\\
&+3h(z)^{(4)}k^4+5h(z)^{(4)}k^3+20h(z)^{(3)}h(z)k^4+60h(z)^{(3)}h(z)k^3\\
&+100h(z)^{(3)}h(z)k^2+210h(z)^2k^3h(z)''+360h(z)^2k^2h(z)''\\
&+480h(z)^2kh(z)''+540h(z)k^3(h(z)')^2-720h(z)k(h(z)')^2\\
&+1140h(z)^3k^2h(z)'+720h(z)^3kh(z)'+90k^4h(z)'h(z)''\\
&-120k^2h(z)'h(z)'')\Big)
\end{aligned}
$$

C.2 Action of $(W_3)_1$ on Strong Generators of $\mathsf{C}_k(\mathfrak{sl}_2)$ Mod $C_2\left(\mathsf{C}_k(\mathfrak{sl}_2)\right)$

Up to re-sclaing, we essentially recover Lemma 4.4 of [ALY 2014][1].

Modulo $C_2\left(\mathsf{C}_k(\mathfrak{sl}_2)\right)$, the action of the homogeneous linear operator of degree 1 $(W_3)_1 \in \mathrm{End}_{\mathbb{C}}(\mathsf{C}_k(\mathfrak{sl}_2))$ on the strong generating fields $\{L(z), W_3(z), W_4(z), W_5(z)\}$ of $\mathsf{C}_k(\mathfrak{sl}_2)$ is given as follows:

$$
\begin{aligned}
(W_3)_1.x_2 &= 3x_3\\
(W_3)_1.x_3 &= \frac{72\left(3k^7+10k^6-4k^5-40k^4-32k^3\right)}{16k+17}x_2^2+x_4\\
(W_3)_1.x_4 &= \frac{11232\left(8k^8+20k^7-46k^6-189k^5-189k^4-54k^3\right)}{1024k^2+2800k+1819}x_2x_3+x_5\\
(W_3)_1.x_5 &= -\frac{180\left(1212k^7+7550k^6+17437k^5+17710k^4+6680k^3\right)}{64k+107}x_4x_2\\
&\quad+\frac{1620\left(492k^7+2864k^6+6247k^5+6051k^4+2196k^3\right)}{64k+107}x_3^2\\
&\quad-\frac{6480}{1024k^2+2800k+1819}(7272k^{14}+78756k^{13}+350998k^{12}+806843k^{11}\\
&\quad+932442k^{10}+271268k^9-550472k^8-610944k^7-194688k^6)x_2^3
\end{aligned}
$$

In particular, the same conclusion as in [ALY 2014] holds:

Result C.2. *As an associative and commutative (Poisson) algebra, the C_2-quotient $\frac{\mathsf{C}_k(\mathfrak{sl}_2)}{C_2(\mathsf{C}_k(\mathfrak{sl}_2))}$ is generated by*

$$\Big((W_3)_1\Big)^i(x_2) \quad \textit{for } i \in \{0,1,2,3\}\,.$$

Conjecturally, $(W_3)_1$ can be used as in [ALY 2014] to better analyse C_2-cofiniteness of $\mathsf{B}_k(\mathfrak{sl}_2)$ for negative admissible k as well. I would like to thank

[1] Note that both our $\mathfrak{sl}_2$ basis and our scalings of W_3,W_4 and W_5 differ from those of [ALY 2014].

T. Arakawa for bringing [ALY 2014] to my attention. In the near future, we hope to explore this avenue for possibly improving the methods of Chapter 4.