University of Alberta

Mathematical Models of Sustainable Agriculture and Environment

by

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Dedication

I dedicate this thesis to my father, **Eassah Gyasi Agyemang** for his love, care, inspiration, understanding and support for me and all my brothers and sisters. Paapa, it is your deep sense of understanding of your children future conditions and the strength of your belief in the institutions of family, religion and education that has made me who I am today. To you Dad, I owe this achievement of my life. May Allah Almighty continue to shower his blessings and grace upon you. May Allah with his infinite grace endow you with contented and anxiety free life. May He grant you good health and long life and fulfil your sincere desires and enable you to gain His nearness through spiritual enhancement - Ameen.

Abstract

The concept of agricultural and environmental sustainability has been a subject of interest since the late 1980's. In this thesis, we study agricultural and environmental sustainability by studying the dynamics of the interactions between agriculture, industry and the environment. Three different models involving systems of nonlinear differential equations are presented.

The first is a competition model for two industries competing for market from agriculture, while agriculture depends on the ecosphere for survival. Our second model deals with the interaction between normal agriculture, renewable agriculture, industry and the ecosphere. As an approximation to competition for land between renewable agriculture and normal agriculture, we consider a model involving competition for land by two farming groups within a locality as our last model.

All the three models are analyzed for boundary equilibria, their local and global stabilities. Sufficient conditions for the persistence of each system in each model and the existence of an interior four dimensional equilibrium are established. Numerical examples are used to illustrate some of the analytical results obtained.

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Contents

List of some of the symbols used in chapter 2, their respective meanings and dimensions.

Symbol	Meaning	Dimension
x	Agricultural assets	Asset
y_1	Industrial assets one	Asset
y_2	Industrial assets two	Asset
z	Ecospheric assets	dimensionless
$\dot{x}$	Instantaneous time rate of change of x	$\frac{Asset}{Time}$
$\dot{y_1}$	Instantaneous time rate of change of y_1	$\frac{Asset}{Time}$
$\dot{y_2}$	Instantaneous time rate of change of y_2	$\frac{Asset}{Time}$
$\dot{z}$	Instantaneous time rate of change of z	$\frac{1}{Time}$
α	Growth rate coefficient of x due to agricultural activities for fixed z	$\frac{1}{Time}$
β	Per asset diminishing return rate coefficient for x in the absence of industry	$\frac{1}{[Asset].[Time]}$
γ_1	Per asset terms of trade rate coefficient between agriculture and industry one	$\frac{1}{[Asset].[Time]}$
γ_2	Per asset terms of trade rate coefficient between agriculture and industry two	$\frac{1}{[Asset].[Time]}$
θ	Cost rate coefficient of agriculture to restore the ecosphere	$\frac{1}{Time}$
ξ_1	Constant depreciation rate coefficient of industrial assets one	$\frac{1}{Time}$
ξ_2	Constant depreciation rate coefficient of industrial assets two	$\frac{1}{Time}$

Symbol	Meaning	Dimension
η_1	Per asset (linear) depreciation rate coefficient of industrial assets one	$\frac{1}{[Asset].[Time]}$
η_2	Per asset (linear) depreciation rate coefficient of Industrial assets two	$\frac{1}{[Asset].[Time]}$
ρ_1	Per asset competitive rate coefficient of y_2 acting on y_1	$\frac{1}{[Asset].[Time]}$
ρ_2	Per asset competitive rate coefficient of y_1 acting on y_2	$\frac{1}{[Asset].[Time]}$
δ_1	Per asset growth rate of industry one in dealing with agriculture	$\frac{1}{[Asset].[Time]}$
δ_2	Per asset growth rate of industry two in dealing with agriculture	$\frac{1}{[Asset].[Time]}$
κ	Per asset degradation rate coefficient of the ecosphere due to agricultural activities	$\frac{1}{[Asset].[Time]}$
ϑ	Natural restoration rate coefficient of the ecosphere	$\frac{1}{Time}$
ϕ	Per asset effort input rate of agriculture to restore the ecosphere	$\frac{1}{[Asset].[Time]}$
$F_{(.)}(.)$	Equilibria of the four dimensional system	
$E_{(.)}(.)$	Equilibria of a subsystem of the four dimensional system	
V_F	Variation matrix of the four dimensional system evaluated at the equilibrium F	
$V(.)$	Liapunov function	

List of some of the symbols used in chapter 3, their respective meanings and dimensions.

Symbol	Meaning	Dimension
x_1	Normal agricultural assets	Asset
x_2	Renewable agricultural assets	Asset
y	Industrial assets	Asset
z	Ecospheric assets	dimensionless
$\dot{x_1}$	Instantaneous time rate of change of x_1	$\frac{Asset}{Time}$
$\dot{x_2}$	Instantaneous time rate of change of x_2	$\frac{Asset}{Time}$
$\dot{y}$	Instantaneous time rate of change of y	$\frac{Asset}{Time}$
$\dot{z}$	Instantaneous time rate of change of z	$\frac{1}{Time}$
α_1	Growth rate coefficient of x_1 due to normal agricultural activities for fixed z	$\frac{1}{Time}$
α_2	Growth rate coefficient of x_2 due to renewable agricultural activities for fixed z	$\frac{1}{Time}$
β_1	Per asset diminishing return rate coefficient for x_1 in the absence of y and x_2	$\frac{1}{[Asset].[Time]}$
β_2	Per asset diminishing return rate coefficient for x_2 in the absence of y and x_1	$\frac{1}{[Asset].[Time]}$
γ_1	Per asset terms of trade rate coefficient between normal agriculture and industry	$\frac{1}{[Asset].[Time]}$
γ_2	Per asset terms of trade rate coefficient between renewable agriculture and industry	$\frac{1}{[Asset].[Time]}$
θ	Cost rate coefficient of normal agriculture to restore the ecosphere	$\frac{1}{Time}$

Symbol	Meaning	Dimension
ρ_1	Per asset competitive rate coefficient of x_2 acting on x_1	$\frac{1}{[Asset].[Time]}$
ρ_2	Per asset competitive rate coefficient of x_1 acting on x_2	$\frac{1}{[Asset].[Time]}$
ξ	Constant depreciation rate coefficient of industrial assets	$\frac{1}{Time}$
η	Per asset (linear) depreciation rate coefficient of industrial assets	$\frac{1}{[Asset].[Time]}$
δ	Per asset growth rate of industry in dealing with normal agriculture	$\frac{1}{[Asset].[Time]}$
κ	Per asset degradation rate coefficient of the ecosphere due to normal agricultural activities	$\frac{1}{[Asset].[Time]}$
ϑ	Natural restoration rate coefficient of the ecosphere	$\frac{1}{Time}$
ϕ	Per asset effort input rate of normal agriculture to restore the ecosphere	$\frac{1}{[Asset].[Time]}$
$F_{(.)}(.)$	Equilibria of the four dimensional system	
$E_{(.)}(.)$	Equilibria of a subsystem of the four dimensional system	
V_F	Variation matrix of the four dimensional system evaluated at the equilibrium F	
$V(.)$	Liapunov function	
D_n	Determinant of $n \times n$ matrix	

List of some of the symbols used in chapter 4, their respective meanings and dimensions.

Symbol	Meaning	Dimension
x_1	Agricultural assets of farming group A	Asset
x_2	Agricultural assets of farming group B	Asset
y	Industrial assets	Asset
z	Percentage of land owned by farming group A in a given area	dimensionless
$1-z$	Percentage of land owned by farming group B in the given area	dimensionless
$\dot{x_1}$	Instantaneous time rate of change of x_1	$\frac{Asset}{Time}$
$\dot{x_2}$	Instantaneous time rate of change of x_2	$\frac{Asset}{Time}$
$\dot{y}$	Instantaneous time rate of change of y	$\frac{Asset}{Time}$
$\dot{z}$	Instantaneous time rate of change of z	$\frac{1}{Time}$
$f(z)$	Growth rate function of x_1 due to group A farmers agricultural activities on z	$\frac{1}{Time}$
$g(1-z)$	Growth rate function of x_2 due to Group B agricultural activities on $(1-z)$	$\frac{1}{Time}$
β_1	Per asset diminishing return rate coefficient for x_1 in the absence of y	$\frac{1}{[Asset].[Time]}$
β_2	Per asset diminishing return rate coefficient for x_2 in the absence of y	$\frac{1}{[Asset].[Time]}$
$p\gamma$	Per asset terms of trade rate coefficient between x_1 and industry	$\frac{1}{[Asset].[Time]}$

Symbol	Meaning	Dimension
γ	Per asset terms of trade rate coefficient between x_2 and industry	$\frac{1}{[Asset].[Time]}$
ξ	Constant depreciation rate coefficient of industrial assets	$\frac{1}{Time}$
η	Per asset (linear) depreciation rate coefficient of industrial assets	$\frac{1}{[Asset].[Time]}$
$q\delta$	Per asset growth rate of industry in dealing with x_1	$\frac{1}{[Asset].[Time]}$
δ	Per asset growth rate of industry in dealing with x_2	$\frac{1}{[Asset].[Time]}$
α	Per asset growth rate coefficient of z due to group A farming activities	$\frac{1}{[Asset].[Time]}$
p	A constant such that $0 \leq p \leq 1$	
q	A constant such that $0 \leq q \leq 1$	
K_z	Carrying capacity of z	
$f(1) = \beta_1 X_{1K}$	Maximum growth rate of x_1 in the absence of y and x_2	$\frac{1}{Time}$
$g(1) = \beta_2 X_{2K}$	Maximum growth rate of x_2 in the absence of y and x_1	$\frac{1}{Time}$
$F_{(.)}(.)$	Equilibria of the four dimensional system	
V_F	Variation matrix of the four dimensional system evaluated at the equilibrium F	
$V(.)$	Liapunov function	
D_n or B_n	Determinant of a given $n \times n$ matrix	

Chapter 1

Introduction

1.1 Overview of the problem

Even though the concept of sustainability has become common since the late 1980's, and has provoked a lot of interest and discussion in various academic disciplines (especially in economics) and non-academic circles, there is still no precise and concise well-established agreed upon definition [35,38,64]. There are even some who wonder whether there is a need to define sustainability at all [35] and others who argue that defining sustainability is pointless unless such a definition can be made operational [61].

As a mathematical work and not a treatise in economics, this dissertation will not discuss the large number of various views on sustainability (for this see Gold [35], Heal [38], Klaassen et al [50], Ozkaynak et al [61] and Pezzey [64]). Instead we adopt certain definitions and interpretations of sustainability from the literature which address some of the concerns common to most concepts of sustainability, and use these as a basis for our study. By adopting some specific definitions and interpretations from the mass of literature, we are by no means claiming that these are better than the others, nor are we disagreeing with those who think it is not important to define sustainability. It is only to serve as a basis for our study and give us some sense of the direction in which this study must go and what results to expect.

We define sustainability throughout this thesis as doing things that we can safely continue indefinitely: doing things that can be continued over long periods without unacceptable consequences, or without high risks of unacceptable consequences [35,38,39,40,41,45,64]. Thus a sustainable system of interacting assets is a system in which the assets are capable of meaningful and purposeful interaction while maintaining their usefulness indefinitely, without high risk of unacceptable consequences. This simple statement can in fact be interpreted in many ways (see [30,50,62,64]). One of these interpretations is that no asset within such a system can go extinct (Pearce quoted in Klassen [50]) and this is the interpretation we adopt. We adopt this interpretation because it is closely related to the concept of persistence in dynamical systems (which is basically that solutions with positive initial conditions remain positive). Based on this definition, we can define sustainable agriculture as that agriculture that can safely continue indefinitely without high risks of unacceptable consequences and similarly for sustainable industry and sustainable environment.

This definition of sustainable agriculture prompts the following question. What type of agriculture is sustainable? Is it the primitive (traditional) system of agriculture which makes little or no use of modern agricultural methods and as a result is characterized by low yields, or is it the modern (industrial) agriculture which is characterized by higher yields? To answer this question, we consider the following three examples.

(i) Agriculture has been an important part of the United States economy since its earliest days when about ninety percent of the work force was employed on farms. Agricultural productivity (the amount of agricultural income an investment in a unit area of agricultural land generates over time) of the United States has increased rapidly since then, producing more with less labor. Just before World War II, about twenty percent of the United States population was employed on farms, while today less than three percent of the work force in the United States produces enough food and fiber to meet all domestic needs as well as supply food and fiber to about ten percent of the total external population. This growth in productivity can be attributed to rapid improvements in general agricultural methods, such as the use of electric- and gasoline-powered farm machinery, chemical and organic fertilizers, pesticides, the development and use of hybrid strains, and government policies and subsidies that favor maximization of agricultural productivity. However, associated with these improvements in general agricultural methods are soil erosion, which leads to depletion and degradation of topsoil and soil nutrients, and pollution of underground water crucial for both good human health and productivity [72]. In other words, agronomists and governments in the United States were mostly interested in ways of increasing productivity to meet domestic and some overseas consumption, thereby focusing much attention on the short-term profitability and social responsibility components of agricultural sustainability, leaving out the other aspect such as long-term (economic) profitability and intergenerational equity.

(ii) The scenario in Australia is quite similar to that of the United States in the sense that farm policies in Australia favor more of the short-term profitability and social responsibility. In order to promote growth in the agricultural, industrial and economic sectors of the Australian economy, and also to encourage immigrants into Australia to occupy its huge empty spaces, various government policies were enacted to govern land settlement. Some of these policies required landowners to farm on a prescribed percentage of their land within a given period of time in order to show their stewardship [8,13,44]. As a result, farmers usually cleared more land than they could actually use to grow crops and fiber, resulting in an unprecedented exposure of both farmed and unfarmed lands to soil erosion and depletion of soil nutrients [44]. Other policies, such as the establishment of small subsistence farms for families which were not large enough to operate resourcefully, most of the time turned out to be disastrous and unsuccessful, leading to even higher environmental degradation. Huge government subsidies on fertilizers in the 1960's and 70's also led to a massive increase in fertilizer use in Australia over that period, which in turn contributed to a high level of soil acidification and underground water contamination [23,44,81].

(iii) The problems described above can also be found in a worse form in developing countries. Because of poverty and high population growth (i.e. about three percent per year) in most of the developing regions of the world such as Sub-Saharan Africa, most of the farmers there cannot afford to practice modern (industrialized) agriculture. Few of the farmers in Sub-Saharan Africa can afford modern farm machinery, fertilizers, pesticides and technology to aid them in their practice because there are few if any government subsidies for such farm supplies. In addition most of these farmers do not qualify for significant loans from the banks. As a result, farmers do intensify (increase) land use to meet increasing food and fiber needs of the increasing population. This has led to, among other things, a sharp decline in the carrying capacity of agricultural lands in Sub-Saharan Africa because expansion of farming activities has led to increasing scarcity of and competition for land. This is due to the fact that farmers have to cultivate a large area of land just to produce the little necessary for the survival of their family and community. Intensification and expansion on poor quality lands keeps growing, and farmers do this without proper farm management practices and with little outside help. This has led to an acceleration of soil degradation in the already poor soil, resulting in lower yields of crops in many of these countries [7,22]. Also, farmers and their laborers (who are usually family members) endure poor economic and social conditions in rural communities.

From the above discussion, it is evident that both traditional and modern agriculture can lead to environmental degradation, pollution, loss of pollination and vegetation, and loss of biodiversity in the long run. Thus both modern and traditional systems of agriculture are not presently economically feasible, socially dependable, just and humane and above all not ecologically sound and friendly for both present and future generations, and hence not sustainable [35,45,47]. If both agricultures we know are not sustainable then which form of agriculture is sustainable? One possible answer to this question is posted on the web site of Union of the Concerned Scientists [73] and reported by Gold [35] in the following way "sustainable agriculture does not mean a return to either the low yields or poor farmers that characterized the 19th century. Rather sustainability builds on current agricultural achievements, adopting a sophisticated approach that can maintain high yields and farm profits without undermining the resources on which agriculture depends." To be able to build on the current(modern) agricultural achievements efficiently, we have to understand how modern agriculture, which is founded on a base of industrial and technological improvements, interacts with the environment.

The purpose of this dissertation is to study sustainable agriculture and environment by studying the dynamics of the interactions between agriculture, industry and the environment. We will carefully discuss what we mean by "industry" further on. Our study will relate sustainability to concepts from dynamical systems: persistence, uniform persistence and existence of a positive interior equilibrium, rather than use the popular concepts of utility functions from economics. Economists define utility as wellbeing or welfare and hence utility functions are customarily used to describe the wellbeing of an individual. Usually this function is maximized because it is believed that normal people making rational choices about their wellbeing or welfare will act so as to maximize it [62,64]. For more on sustainability as related to the economic concept of utility functions see [15,16,39,50,64]. In our dynamical systems approach we are interested in sustainable agriculture and environment and hence a positive interior global equilibrium for a system composed of agriculture, industry and ecosphere. We do not maximize such an equilibrium. What we do is model all the assets in the interaction and study all the possible outcomes of such interactions. Such outcomes may consist of extinctions (which are not desirable) and persistence. The reason for doing this is that if we know all the cases under which extinctions occur, then we can avoid such conditions and hence get persistence. We do not consider whether extinction or persistence is "good" or "bad" for the farmer or a farming group, but simply derive conditions for them to occur. We only consider persistence as "good" in terms of "component representation" of the entire system in the sense that we don't want any component to go extinct.

How is persistence, uniform persistence and global stability of an interior equilibrium of a system related to sustainability of that system? Theoretically speaking, a component of a system is persistent if it remains positive for all times (indefinitely) and is uniformly persistent if there exists a positive value (call it ϵ) which is independent of the initial value of the component such the values of this component at all time (indefinitely) remain greater than or equal to ϵ. A system is (uniformly) persistent if all its components are (uniformly) persistent. Thus if a system is uniformly persistent then it is sustainable in the sense that none of the components will go extinct or approach extinction. Finally, equilibria of a system are the time independent solutions of the system. Thus if a system has a global interior equilibrium, and if we can show that the system will eventually approach such an equilibrium, then no extinctions will occur. This is the goal of our analysis.

1.2 Mathematical modelling of agricultural-industrial-ecospheric systems

Modern bioeconomics (i.e. the integration of thoughts and concepts from economics and biology for the purpose of enhancing and developing both economics and biology by broadening and extending their theoretical and empirical bases) began in the 1970's with the pioneering work of G. Tullock, G.Becker, J. Hirshleifer and M. Ghiselin [53,80]. Since then there has been a lot of studies and publications in this area and this eventually led to the launching of the Journal of Bioeconomics in 1998 [53]. For more on publications in bioeconomics, see "A Bibliography for Bioeconomics" by Ghiselin [34].

An area of bioeconomics which is somewhat related to the study in this dissertation is the mathematical modelling of optimal management of renewable resources pioneered by Colin Clark. In [15], Clark writes " it should be stated at the outset that our subject matter is the *conservation* of *productive* resources, rather than the *preservation* of natural environment ". He also writes in [16] " this book has a similar emphasis, being concerned with the *economics* of the *sustainable* use of *biological* resources and with the understanding why such resource have often been used in a *nonsustainable* manner". Thus, from the start Clark specifically excludes interactions between the overall environment and the harvesting of resources. He is only concerned with the portion of the environment related to productive renewable resources (eg. fish, timber, wildlife) being harvested.

Clark's models rely on the standard Schaefer model which models the supply of *open-access* renewable resources. In the simplest case, the Schaefer model takes the form

$$\frac{dX}{dt} = F(X) - h(t),$$

where $F(X) = rX[1 - \frac{X}{K}]$ and $h(t) = qEX$ are the stock growth and harvest functions respectively. X is the stock size, r is the intrinsic growth rate, K is the carrying capacity for X, q is the catchability coefficient, and E is the harvesting effort. Clark examined the economic parameters relative to a sustained-yield harvesting with maximum returns for the harvester, using the parameters:

$$R = ph - cE,$$

where R is profits or sustained economic rent, p is the output price, c is the effort price, and

$$Y = \frac{rc}{pq}[1 - \frac{c}{pqK}]$$

is the sustained yield. The sustained yield Y represents the stationary solution for the above Schaefer model with $pqX = c$. With this model, under some modifications and some assumptions, Clark was able to discuss and explain the common phenomena of biological and economic over-exploitation. With his models he was able to predict and identify exploitation patterns and the features of socially optimal productive resource exploitation policies, and suggested that government management and regulation of open-access productive resource stock is essential if misuse is to be prevented. This was and is only the beginning of a very sophisticated study of optimal management of a componentwise renewable resource (eg fish, timber,wildlife) model. Since then many mathematical bioeconomics models have been formulated for componentwise productive renewable resources. These include the work by Amir et al [2], Amstrong et al [4], Boman et al [6], Conrad [18,19] and Ussif et al [74,75].

In the past 12 years, a few mathematical models have been formulated to study the interaction between agricultural, industrial, and ecospheric wealth or assets based on the predator-prey paradigm from the dynamical systems points of view. Most economists and environmental scientists consider this approach as very unorthodox because such models among other things do not include the economic rent and thus do not have production and utility functions. Despite the aforementioned, these models have been used successfully to determine conditions under which a system of interacting agricultural, industrial and ecospheric wealth or assets exhibit persistence or otherwise. These include the work of Apedaille et al [5], Solomonovich et al [68,69,70], Freedman et al [30] and Agyemang [1]. In Apedaille et al [5] and Solomonovich et al [68,69,70] agricultural and industrial wealth was measured in dollars and a dollar value of the environmental quality was assigned. As a result of measuring wealth in dollars, it was assumed that the growth of industrial and agricultural wealth must of necessity saturate. In 1994, Apedaille et al [5], held the environment constant and the direct interaction between agriculture and industry was considered. They determined the stability and bifurcation properties of the two-dimensional system and predicted that agricultural economy ages by slowing its learning, seeking and obtaining greater stability and eventually leading to trapping. In 1997, Solomonovich et al [68] introduced the environment into their model with a minimum safe standard policy or threshold. In 1998, Solomonovich et al [70] allowed the ecosphere to degenerate or recover either through farming, adding of nutrients or through natural causes. They showed that it was possible to eliminate chaotic uncertainty and replace it with predictability by simply modifying some of the model parameters. Later in 2001, Solomonovich et al [69] modified the ecospheric equation in their 1998 model to include a term they called "agricultural investment in the ecosphere" and the cost of the investment into the agricultural equation and demonstrated how timely investment in the ecosphere could promote a high level of stable agricultural income.

In 2001, Agyemang [1] modelled the interaction between agricultural, industrial and ecospheric assets. Here assets were used instead of wealth (dollar value) because it was believed that assets are a better measure of industrial and agricultural wealth in the sense that there are some agricultural assets which can hardly be priced. Choosing appropriate units for bioeconomic models has always been a problem, which is why Landa et al [53] wrote " bioeconomists are struggling with the equivalence of certain kinds of units that may or may not play an important role in biological and economic theory. ... any more than to say that a professor is the unit of teaching." It should be noted that wealth is part of, and not opposed to assets. Also for these models it does not matter as to whether agricultural and industrial assets are measured in the same units or not (but the models are much easier to understand, if one assumes the assets are measured in the same units). As to the ecosphere, a better measure is a value between zero and one given by (asset level)/(maximum possible asset level) that is, the normalized asset units for the ecosphere. It was shown that if the ecosphere was assumed to be in a state of equilibrium, then depending on the choice of the model parameters, industrial assets may either go extinct or both industrial and agricultural assets will persist uniformly. In the case of the three dimensional system corresponding to an ecosphere not in equilibrium, only local stability conditions of the equilibria were given. It was shown locally that ecospheric assets always survived the interactions but the survival of industrial and agricultural assets depended critically on the choice of the model parameters.

Freedman et al [30], in 2003 also considered the interaction of agriculture assets with industry assets and the environment quality. They analyzed the agricultural-industrial and agricultural-environmental sub-models and left the analysis of the full model to a future work.

Our purpose in this thesis is to determine conditions under which a system of interacting agriculture, industry and the ecosphere can exist indefinitely with all three components healthy and viable. This is already a challenging problem. Beginning from our work in this thesis, we can continue in the future to the (much more difficult) study of price/yield/demand interaction.

1.3 Definitions

In this section we define the key words and phrases (some of which we have encountered already) used in the forthcoming chapters.

1.3.1 Agricultural definitions

Normal Agriculture [cf. 56]:

Normal agriculture is the occupation, business or science of cultivating the land, producing crops and raising animals.

Renewable Agriculture:

A naturally occurring agricultural resource on earth that is capable of being replenished by natural ecological cycles or sound management practices itself (eg. forest ,wildlife, birds, fish, etc).

Agriculture:

Agriculture is the union of renewable and normal agricultures. That is the occupation, business or science of cultivating the land, producing crops and raising animals and all naturally occurring agricultural resource on earth that can replenish itself and all harvesting agents of an industrial economy in particular (cf. [1,5,70]).

Sustainability [cf. 35,38,39,40,45]:

Doing things that we can safely continue indefinitely: doing things that can be continued over long periods without unacceptable consequences, or without high risks of unacceptable consequences.

Sustainable Management[cf. 20]:

Managing a renewable resource such as a forest to meet the needs of the present without compromising the ability of future generations to meet their needs.

Industry :

Industry is an organized economic activity connected with production, manufacturing, or construction of fertilizer products, agricultural pest control chemicals, mechanized agricultural equipment, agricultural hand-tools, farm machinery and equipment, farm conveyors, etc. and all industrial accessories used for agricultural purposes.

Cost or Expense [57]:

Cost or expense is what is given, done or undergone to obtain something or to survive. This cost should not be confused with economic cost or opportunity cost which is the value of the best alternative that was forgone in order to pursue the current endeavour, that is cost is part of opportunity cost.

Utility [cf. 57,62,64]:

Utility is a measure of the wellbeing, welfare, happiness or satisfaction of something or some individual.

Preservation [57] :

The action of restoring, protecting or safeguarding a portion of a resource from unnatural disturbance. This does not necessary imply preserving a portion of the resource in its present state, for natural events and natural ecological processes are expected to continue.

Conservation [57] :

Management of the use of a resource so that it may yield the greatest sustainable benefit for the present generations while maintaining its potential to meet the needs and aspirations of future generations. It includes the preservation, maintenance, sustainable utilization, restoration and enhancement of the resource.

Sustained yield [57]:

A method of resource management that calls for an approximate balance between net growth and amount harvested.

Open-access resource [16]:

A resource in which exploitation is completely uncontrolled.

Competition [57]:

Competition is the act of striving or vying against another force or forces for the same limited resources for the purpose of achieving dominance or survival.

Effort Input:

Effort input is the amount of resource devoted to or put into a particular activity in its operations to achieve a result or output.

Ecosphere :

This refers to the quality of the land and the environment. It also refers to the natural world, within which people, animals and plants live, and includes the whole complex of climatic, edaphic and biotic factors, such as light or food supply, that influence the life of an organism or an ecological community. (cf. [1,5]).

Environment [cf. 56]:

This is the union of the ecosphere and renewable agriculture.

Asset [cf. 56]:

An asset is something that is useful and contributes to the success of an activity. This include cash value, the property of something to which a cash value can be assigned, or the property of something to which cash value cannot be assigned etc.

1.3.2 Mathematical definitions

Steady State (cf. [59,63,70,78]):

Let

$$\frac{dx}{dt} = f(x), \quad x(t_0) = x_0 \tag{1.1}$$

be a (non)linear system in $\Re^n$. Let G be an open subset of $\Re^n$ containing x^*. Then x^* is said to be a steady state or an equilibrium of Equation (1.1) if $f(x^*) = 0$.

Jacobian Matrix J(x) (cf. [27,63]):

The Jacobian matrix J(x) of Equation (1.1) is defined as

$$J(x) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(x) & . \; . \; . & \frac{\partial f_1}{\partial x_n}(x) \\ . & & \\ . & & \\ . & & \\ \frac{\partial f_n}{\partial x_1}(x) & . \; . \; . & \frac{\partial f_n}{\partial x_n}(x) \end{bmatrix}. \tag{1.2}$$

Hyperbolicity (cf. [10,11,59]):

Let x^* be an equilibrium of Equation (1.1). Let $J(x^*)$ be the Jacobian of Equation (1.1) about x^* and the eigenvalues of $J(x^*)$ be defined by the set

$$\sigma(J(x^*)) = \{\lambda_i | det(J(x^*) - \lambda I_n) = 0, \quad i = 1, 2, ...n\}. \tag{1.3}$$

The equilibrium x^* is called hyperbolic if the eigenvalues corresponding to $J(x^*)$ are such that they have non-zero real parts. Furthermore if $\sigma(J(x^*))$ contain n eigenvalues, then

- x^* is a hyperbolic saddle point if there exist some $\lambda_i \in \sigma(J(x^*))$ with Re $\lambda_i > 0$ and also some $\lambda_j \in \sigma(J(x^*))$ with Re $\lambda_j < 0$, for $i, j \in \{1, 2, ..., n\}$, but there exit no $\lambda_k \in \sigma(J(x^*))$ with Re $\lambda_k = 0$;
- x^* is a hyperbolic sink (stable) point if for all $\lambda_i \in \sigma(J(x^*))$ we have Re $\lambda_i < 0$, $i \in \{1, 2, ..., n\}$;
- x^* is a hyperbolic source point if for all $\lambda_i \in \sigma(J(x^*))$ we have Re $\lambda_i > 0$, $i \in \{1, 2, ..., n\}$.

Differentiability (cf. [49,63]):

The function $f : \Re^n \to \Re^n$ is said to be differentiable at $x_0 \in \Re^n$ if $J(x_0)$ satisfies $\lim_{\|h\| \to 0} \frac{\|f(x_0+h) - f(x_0) - J(x_0)h\|}{\|h\|} = 0$, where $\|.\|$ denotes the norm on $\Re^n$.

Continuity (cf. [49,63]):

Let $\|.\|$ be the norm on $\Re^n$. Then $f : \Re^n \to \Re^n$ is said to be continuous at $x_0 \in \Re^n$ if for all $\epsilon > 0$ there exists a $\delta' > 0$ such that $\|x - x_0\| < \delta'$ implies that $\|f(x) - f(x_0)\| < \epsilon$.

If f is continuous at each point $x \in G$, where G is an open subset of $\Re^n$, then we say f is continuous on $G \subset \Re^n$ and we write $f \in C(G)$.

Continuous Differentiability (cf. [49,63]):

If $f : G \rightarrow \Re^n$ is differentiable on G and the Jacobian J(x) is continuous on G, then we say f is a continuously differentiable function on G and write $f \in C'(G)$.

Flow of a Differential Equation (cf. [49,63]):

Let G be an open subset of $\Re^n$ and $f \in C'(G)$. For $X_0 \in G$, let $\Phi(t, X_0) = \Phi_t(X)$ be the solution of Equation (1.1) defined on its maximal interval of existence. Then Φ_t is called the flow of Equation (1.1).

Stability of Steady States (cf. [49,63,78]):

Suppose x^* is a steady state of Equation (1.1).

- Then x^* is said to be a stable point if for all $\epsilon > 0$ there exists a $\delta' > 0$ such that for all x in the δ'-neighborhood of x^* and $t \geq 0$, we have $\Phi_t(x)$ in the ϵ-neighborhood of x^*.

- The steady state x^* is said to be unstable if it is not stable.

- The steady state x^* is said to be asymptotically stable if it is stable and there exists a $\delta' > 0$ such that for all x in the the δ'-neighborhood of x^*, we have $\lim_{t\to\infty} \Phi_t(x) = x^*$.

Acyclicity (cf.[12,59]):

Consider Equation (1.1) and let $x_0 \in \Re^n_+ = \{x \in \Re^n | x_i > 0, i = 1, 2, ..., n\}$. Let $\partial\Re^n_+$ denote the boundary of $\Re^n_+$. Let x_1^* and x_2^* be equilibria or invariants sets on $\partial\Re^n_+$. Let $O(x)$ denote the orbit of a point $x \in \partial\Re^n_+$ such that $\alpha(x) = x_1^*$ and $\omega(x) = x_2^*$, respectively the alpha and omega limit sets of x. Then x_1^* is said to be connected or chained to x_2^* and is denoted by

$$x_1^* \rightarrow x_2^*.$$

Let $\mathcal{S}$ denote the set of all equilibria or invariants sets of Equation (1.1) as defined by

$$\mathcal{S} = \{x_1^*, x_2^*, ..., x_k^*\}.$$

If for all cyclic permutations of i and all $x_i^* \in \mathcal{S}$ with $i = (1, 2, ..., k)$ we have

$$x_i^* \rightarrow x_{i+1}^* \rightarrow x_{i+2}^* \rightarrow ... \rightarrow x_k^* \nrightarrow x_i^*$$

then the system (1.1) is said to be acyclic.

Dissipativity (cf. [12,33,59]):

Consider the system given by Equation (1.1). Then the system describing the evolution of $x(t)$ is said to be dissipative if

$$\limsup_{t\rightarrow\infty} \parallel x(t) \parallel \leq L,$$

where L is a positive constant. That is the trajectories of the system are asymptotically bounded.

For dissipative systems, the existence of an interior equilibrium is a consequence of uniform persistence [11,12].

Persistence and uniform persistence(cf. [11,12,28,31]):

Consider Equation (1.1) and let $x(t) = \{x_i(t)\}_{i=1}^n$. Then x_i is said to persist if

$$x_i(t) > 0, \quad t \geq 0 \quad and \quad \liminf_{t\rightarrow\infty} x_i(t) \geq 0.$$

x_i is said to persist uniformly if there exist a $\delta' > 0$ such that

$$x_i(t) > 0, \quad t \geq 0 \quad and \quad \liminf_{t\rightarrow\infty} x_i(t) \geq \delta'.$$

independent of $x_i(0)$.

System (1.1) is said to persist(uniformly) if each component $x_i, \quad i = \{1, 2, ..., n\}$ persists (uniformly).

Local stable and unstable manifolds (cf. [63]):

Let ϕ_t be the flow of the nonlinear system (1.1). Let $x^\star$ be a hyperbolic equilibrium of (1.1) and N be a neighborhood of $x^\star$. The local stable and unstable manifolds of (1.1) at $x^\star$ are defined by

$$S(x^\star) = \{x \in N \mid \phi_t(x) \to x^\star \quad \text{as} \quad t \to \infty \quad \text{and} \quad \phi_t(x) \in N \quad \text{for} \quad t \geq 0\}$$

and

$$U(x^\star) = \{x \in N \mid \phi_t(x) \to x^\star \quad \text{as} \quad t \to -\infty \quad \text{and} \quad \phi_t(x) \in N \quad \text{for} \quad t \leq 0\}$$

respectively. $S(x^\star)$ and $U(x^\star)$ are also known as the local stable and unstable manifolds of $x^\star$ respectively.

Global stable and unstable manifolds(cf. [63,78]):

Let ϕ_t be the flow of the nonlinear system (1.1) and $x^\star$ be an equilibrium point. The global stable and unstable manifolds of $x^\star$ are defined by

$$W^s(x^\star) = \bigcup_{t \leq 0} \phi_t(S(x^\star))$$

and

$$W^u(x^\star) = \bigcup_{t \geq 0} \phi_t(U(x^\star))$$

respectively, where $S(x^\star)$ and $U(x^\star)$ are the local stable and unstable manifolds of $x^\star$ respectively.

Leading principal minor of a matrix (cf. [36,52,77]):

Let A be an $n \times n$ matrix over the reals, given by

$$A = \begin{bmatrix} a_{11} & a_{12} & . & . & . & a_{1n-1} & a_{1n} \\ . & & & & & & \\ . & & & & & & \\ . & & & & & & \\ a_{n1} & a_{n2} & . & . & . & a_{nn-1} & a_{nn} \end{bmatrix}.$$

Let $R = \{1, 2, 3, ..., k\}$ and $S = \{1, 2, 3, ..., k\}$ be subsets of $\{1, 2, ..., n\}$. Then

1. the k-th order leading principal minor of A defined by R and S is the determinant

$$D_{RS} = D_k = det \begin{bmatrix} a_{11} & a_{12} & . & . & . & a_{1k-1} & a_{1k} \\ . & & & & & & \\ . & & & & & & \\ . & & & & & & \\ a_{k1} & a_{k2} & . & . & . & a_{kk-1} & a_{kk}, \end{bmatrix},$$

2. $D_k = det(A)$, the determinant of A if $k = n$.

1.4 Basic and standard theorems

In this section, we state without proofs some of the standard and basic theorems that will be used in the forthcoming chapters.

Theorem 1.1 (Liapunov). *Consider the system given by Equation (1.1). Let G be an open subset of* $\Re^n$ *containing* x^*. *Suppose that* $f \in C'(G)$ *and that* $f(x^*) = 0$. *Suppose further that there exists a function* $V \in C'(G)$ *satisfying* $V(x^*) = 0$ *and* $V(x) > 0$ *if* $x \neq x^*$. *Then*

- *if* $\frac{dV(x)}{dt} \leq 0$ *for all* $x \in G$*, then* x^* *is stable;*

- *if* $\frac{dV(x)}{dt} < 0$ *for all* $x \neq x^* \in G$*, then* x^* *is asymptotically stable;*

- *if* $\frac{dV(x)}{dt} > 0$ *for all* $x \neq x^* \in G$*, then* x^* *is unstable, [cf. 54,59,63,78].*

Theorem 1.2 (Isolatedness). *Consider system (1.1) and let* $f(x)$ *be analytic and let* $x_0^*, x_1^*, x_2^*, ..., x_n^*$ *denotes the equilibria or steady states of (1.1). Then a sufficient condition for* x_0^* *to be isolated, is that the Jacobian matrix due to the linearization of (1.1) in the neighborhood of* x_0^**, denoted by* $J(x_0^*)$*, is such that* $J(x_0^*)$ *is non-singular, [59].*

Theorem 1.3 (Routh-Hurwitz). *The characteristic equation of a matrix*

$$B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}, \tag{1.4}$$

is given by

$$\lambda^3 + \tau_2\lambda^2 + \tau_1\lambda + \tau_0 = 0, \tag{1.5}$$

where

- $\tau_2 = -(b_{11} + b_{22} + b_{33})$;

- $\tau_1 = b_{11}b_{22} + b_{11}b_{33} + b_{22}b_{33} - b_{12}b_{21} - b_{13}b_{31} - b_{23}b_{32}$;

- $\tau_0 = b_{12}b_{21}b_{33} + b_{13}b_{31}b_{22} + b_{11}b_{32}b_{23} - b_{11}b_{22}b_{33} - b_{12}b_{31}b_{23} - b_{13}b_{21}b_{32}$.

Necessary and sufficient conditions for all the eigenvalues of the matrix B *(i.e. all the roots of Equation (1.5)) to have negative real parts is that*

- $\tau_2 > 0$,

- $\tau_0 > 0$,

- $\tau_2\tau_1 - \tau_0 > 0$,

[3,24,27,63].

Theorem 1.4 (Negative definiteness of a quadratic form). *Let A be a symmetric $n \times n$ matrix and X be an $n \times 1$ vector and $V = X^t AX$, where $(.)^t$ denotes the transpose. Then*

- *V is negative if $X^t AX$ is negative definite,*

- *$X^t AX$ is negative definite if A is negative definite,*

- *A is negative definite if the eigenvalues of A all have negative real parts,*

[54,59,63].

Theorem 1.5 (Frobenius 1879). *Let A be a symmetric $n \times n$ matrix over the reals, given by*

$$A = \begin{bmatrix} a_{11} & a_{12} & . & . & . & a_{1n-1} & a_{1n} \\ . & & & & & & \\ . & & & & & & \\ . & & & & & & \\ a_{n1} & a_{n2} & . & . & . & a_{nn-1} & a_{nn} \end{bmatrix}.$$

Let D_k denote the sequence of (leading) principal minors of the matrix A. In particular

$$D_1 = a_{11}, \quad D_2 = det \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix},$$

$$D_3 = det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad ..., \quad D_n = det(A),$$

where $a_{ij} = a_{ji}$ *for all i,j.*

Then necessary and sufficient condition for the real, symmetric matrix A *to be negative definite is that the principal minors of a, starting with that of the first order be alternating negative and positive, i.e.*

$$D_1 < 0, \quad D_2 > 0, \quad D_3 < 0, \quad ..., \quad (-1)^n D_n > 0,$$

[25,59].

Theorem 1.6 (Hartman-Grobman). *Consider the system given by Equation (1.1). Let* G *be an open subset of* $\Re^n$ *containing the origin. Suppose that* $f \in C'(G)$ *and let* ϕ_t *be the flow of (1). Suppose that* $f(0) = 0$ *and the matrix* $A = J(0)$ *has no eigenvalue with zero real part. Then there exists a homeomorphism* H *of an open set* U *containing the origin onto open sets* V *containing the origin such that for each* $x_0 \in U$*, there is an open interval* $I_0 \subset \Re$ *containing zero such that for all* $x_0 \in U$ *and* $t \in I_0$

$$H \circ \phi_t(x_0) = e^{At} H(x_0);$$

i.e H *maps trajectories of (1.1) near the origin onto trajectories of* $\dot{x} = Ax$ *near the origin and preserves the parametrization by time, [63].*

Theorem 1.7 (Bendixon's negative criterion). *Consider the system*

$$\begin{aligned} \dot{x_1} &= F_1(x_1, x_2) \\ \dot{x_2} &= F_2(x_1, x_2) \end{aligned} \tag{1.6}$$

on a simply connected domain $G \subset \Re^2$*. Suppose*

- $F_1, F_2 \in C'(G, \Re)$ *such that* F_1, F_2 *have continuous first partial derivatives* $\frac{\partial F_1}{\partial x_1}, \frac{\partial F_2}{\partial x_2}$, *on* G;

- $\frac{\partial F_1}{\partial x_1} + \frac{\partial F_2}{\partial x_2}$ *does not change sign on* G, *and does not vanish identically in an open subset of* G.

Then there are no non-trivial closed paths (periodic solutions or limit cycles) in G, *[10,59].*

Theorem 1.8 (Bendixson-Dulac). *Consider system (1.6). Suppose*

- $B(x_1, x_2) \in C'(G, \Re)$;

- $\frac{\partial [F_1(x_1,x_2)B(x_1,x_2)]}{\partial x_1} + \frac{\partial [F_2(x_1,x_2)B(x_1,x_2)]}{\partial x_2}$ *does not change sign or vanish identically on any open subset of a simply connected domain* G.

Then there are no closed orbits lying entirely in G, *[10,59].*

Theorem 1.9 (Butler- McGehee Lemma). *Let* E *be an isolated hyperbolic equilibrium in the omega limit set* $\Omega(X)$ *of an orbit* $O(X)$. *Then either* $\Omega(X) = E$ *or there exist points* E^s *in* $W^s(E) \cap \Omega(X)$ *and* E^u *in* $W^u(E) \cap \Omega(X)$, *where* $W^s(E)$ *and* $W^u(E)$ *are the stable and unstable manifolds respectively of* E, *such that* $E^s \neq E$ *and* $E^u \neq E$, *[29,31,32].*

Theorem 1.10 (Uniform persistence). *Let* $\mathcal{F}$ *be a continuous flow on a locally compact metric space* E *with a metric* d *and boundary* ∂E. *Let* $\partial \mathcal{F}$ *be the restriction of* $\mathcal{F}$ *on the boundary* ∂E *(assumed invariant under* $\mathcal{F}$*). Then* $\mathcal{F}$ *is uniformly persistent if*

- $\mathcal{F}$ *is dissipative*

- *$\mathcal{F}$ is persistent*
- *$\partial\mathcal{F}$ is isolated and*
- *$\partial\mathcal{F}$ is acyclic,*

[11,29].

Theorem 1.11 (Local center manifold,[14,63]). *Consider the system (1.1). Let G be an open subset of $\Re^n$ containing the origin, and let $f \in C^1(G)$; suppose that $f(0) = 0$ and that the $n \times n$ matrix $A = J(0) =$diag$[C, P, Q]$, where the square matrix C has c eigenvalues with zero real parts, the square matrix P has s eigenvalues with negative real parts, and the square matrix Q has r eigenvalues with positive real parts. Then there exists C^1 functions $h_1(u)$ and $h_2(u)$ satisfying*

$$\begin{aligned} J(h_1(u))[Cu + F(u, h_1(u), h_2(u))] - Ph_1(u) - G(u, h_1(u), h_2(u)) = 0 \\ J(h_2(u))[Cu + F(u, h_1(u), h_2(u))] - Qh_1(u) - H(u, h_1(u), h_2(u)) = 0, \end{aligned} \tag{1.7}$$

in the neighborhood of the origin such that the nonlinear system (1.1), which can be written in the form

$$\begin{bmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{bmatrix} = \begin{bmatrix} C & 0 & 0 \\ 0 & P & 0 \\ 0 & 0 & Q \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} + \begin{bmatrix} F(u, v, w) \\ G(u, v, w) \\ H(u, v, w) \end{bmatrix}, \tag{1.8}$$

with $F(0) = G(0) = H(0) = 0$ and $J(F(0)) = J(G(0)) = J(H(0))$ is topologically conjugate to the C^1 system

$$\begin{bmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{bmatrix} = \begin{bmatrix} C & 0 & 0 \\ 0 & P & 0 \\ 0 & 0 & Q \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} + \begin{bmatrix} F(u, h_1(u), h_2(u)) \\ 0 \\ 0 \end{bmatrix}, \tag{1.9}$$

for $(u, v, w) \in \Re^c \times \Re^s \times \Re^r$ in the neighborhood of the origin [10,41,49].

We observe that in the above theorem, the local center manifold is given by

$$W^c(0) = \{(u, v, w) \in \Re^c \times \Re^s \times \Re^r | v = h_(u) and w = h_2(u) for |u| < \epsilon\} \qquad (1.10)$$

for some $\epsilon > 0$, where $h_1 \in C^n(N_\epsilon(0))$, $h_2 \in C^n(N_\epsilon(0))$ and

$$h_1(0) = h_2(0) = J(h_1(0)) = J(h_2(0)) = 0. \qquad (1.11)$$

1.5 Epilogue to the introduction

The idea to formulate a mathematical model using the predator-prey paradigm for the study and analysis of the interaction between agriculture, industry and the environment is relatively new. Apedaile et al [5] and Solomonovich et al [68,69,70] did some work in this direction by considering the interaction of agriculture, industry and environment in terms of their wealth. The other known works are the works of Agyemang [1] and Freedman et al [30] who studied the interactions in terms of their assets rather than wealth.

In the forthcoming chapters, the predator-pray paradigm will be used to model the long term trajectories for the shares of agricultural assets, industrial assets and the environmental assets. In our models, we will be interested in long term shares of assets between all the components of the system. That is we are not interested in the quantitative outcome of our models but only interested in the qualitative outcome. The models in the forthcoming chapters are general local models and not global models. All our models contain constant parameters, which imply that technology remains constant. In future works, we can allow the parameters to vary, incorporating technological change.

The chapters are structured such that Chapter 2 deals with competition between two industries. Here, we consider two industries competing with each other for the same market from agriculture. We formulate a mathematical model for this interaction. Equilibria for the system are obtained and their stability determined. Sufficient criteria are also obtained for the extinction of both industries and persistence of all the components of the system consisting of agriculture, two industries and the ecosphere. In chapter 3, we model the interaction between normal agriculture, renewable agriculture, industry and the ecosphere. Equilibria for the system and their stabilities are determined. Global stability conditions for some of the equilibria are given. Conditions for persistence of the system and extinctions of some of the components of the system are also given. In Chapter 4, we introduce a competition for land model involving two farming groups within a given area. Sufficient criteria are obtained for the persistence of the system. We use Liapunov functions to establish sufficient criteria for the extinction of some of the components of the system. Numerical examples are used throughout to illustrate results in this thesis.

Chapter 2

Competition Model for Two Industries

2.1 Introduction

In our previous work [1], we studied the interactions between agriculture, industry and the ecosphere. The interactive dynamics of the above system was modelled by a system of three ordinary differential equations using a predator-prey paradigm. Criteria for local stability of the steady states for the systems were given.

In modelling the interactions between agricultural, industrial and ecospheric assets, we make use of the following assumptions.

- Industry generates its assets from agriculture.
- Agriculture generates its assets from the ecosphere.
- The process of agricultural assets generation can be enhanced by industry (eg. machinery, pesticides) at some cost. Thus the interaction between industry and

agriculture can be considered to be parasitism, mutualism or commensalism depending on the assets creation by this process and the cost associated with it.

- In general we expect agriculture and industry to replenish the ecosphere. But for simplicity we will assume that the direct effect of industry on the ecosphere and vise versa is minimal compared to that of agriculture on the ecosphere.

- Agricultural assets generation depends on the quality of the environment and hence allow a possibility of diminishing returns for agriculture.

- Industrial assets creation faces both fixed and variable expenses independent of agriculture and the ecosphere.

In this chapter however, we consider two industries competing with each other for the same resources (market) from agriculture, whereas agriculture generates its assets from the ecosphere. Here agriculture is the union of normal agriculture and renewable agriculture such as wildlife, forest, trees, etc. The ecosphere mainly refers to the quality of the land and the industry refers to industry associated with agriculture. Thus, we will be considering a system of four ordinary differential equations.

In §2.2, we develop the model of the interaction between the two industries, agriculture and the ecosphere using the predator-prey paradigm. We determine the equilibria in the case when the interaction between each industry and agriculture favors the industry but not agriculture in §2.4, and perform the local stability analysis of each equilibrium in §2.5. Global stability analysis of each equilibrium is done in §2.6 and persistence theory is used to establish the existence of a 4-dimensional equilibrium in §2.7. Criteria for the extinction of both industries are given in §2.8. Numerical examples are used to illustrate some of the results we obtain in §2.9. In §2.10-§2.12, we study the case where the interaction between one industry and agriculture favors both the industry and agriculture whereas in the other it favors only industry. §2.13 contains a discussion and conclusions.

2.2 The model

Let $x(t)$, $y_1(t)$,$y_2(t)$ and $z(t)$ denote respectively, agricultural assets, industrial assets one, industrial assets two and ecospheric assets. The model for the above system is as follows

$$\dot{x} = (\alpha z - \beta x + \gamma_1 y_1 + \gamma_2 y_2 - \theta(1-z))x \tag{2.1}$$

$$\dot{y_1} = (-\xi_1 - \eta_1 y_1 - \rho_1 y_2 + \delta_1 x)y_1 \tag{2.2}$$

$$\dot{y_2} = (-\xi_2 - \eta_2 y_2 - \rho_2 y_1 + \delta_2 x)y_2 \tag{2.3}$$

$$\dot{z} = -\kappa xz + \vartheta(1-z)z + \phi(1-z)x \tag{2.4}$$

with initial conditions $x_0 = x(0) \geq 0$, $y_{10} = y_1(0) \geq 0$, $y_{20} = y_2(0) \geq 0$ and $z_0 = z(0), 0 \leq z_0 \leq 1$, where all parameters are assumed to be positive constants except γ_1 and γ_2 which can be any real numbers. α is the growth rate coefficient of agricultural assets due to agricultural activities for fixed z , β is the per asset diminishing returns rate coefficient for agriculture in the absence of industry, $\gamma_1(\gamma_2)$ is the per asset terms of trade rate coefficient between agriculture and industry one (two), $\xi_1(\xi_2)$ is the constant depreciation rate coefficient of industry one (two), $\eta_1(\eta_2)$ is the per asset (linear) depreciation rate of industry one (two), $\delta_1(\delta_2)$is the per asset growth rate of industry one (two), $\rho_1(\rho_2)$ is the per asset competitive rate coefficient of industry two (one) acting on industry one (two), κ is the per asset degradation rate coefficient of the ecosphere due to agricultural activities, ϑ is the natural restoration rate coefficient of the ecosphere, $\phi(1-z)x$ is the effort input rate of agriculture to restore the ecosphere and $\theta(1-z)x$ is the net cost rate to agriculture to restore the ecosphere.

Here $0 \leq z(t) \leq 1$ and $z(t)$ has no units. We also require that the constant depreciation rate of each industry must be less than its maximum growth rate (else such industry will go extinct). Thus mathematically we require

$$\xi_i < (\frac{\alpha}{\beta})\delta_i, \quad i = 1, 2 \tag{2.5}$$

for the case when γ_1 and γ_2 are both non-positive.

We state here that we have a monopsonistic market. That is we have two providers (sellers) of services which are industry one and industry two, and only one buyer which is agriculture. If γ_1, the per asset terms of trade coefficient between agriculture and industry one is positive then industry one's influence causes a net gain in agricultural assets. That is the cost to agriculture in dealing with industry one is less than the gain in assets obtained by agriculture in the process. If the cost to agriculture in dealing with industry one equals all the benefits obtained by agriculture in the process, then $\gamma_1 = 0$. Moreover if industry one's influence causes a net loss in agricultural assets then $\gamma_1 < 0$. Industry two interacts similarly.

The analysis of the above model will be done in two parts. Part I, when both γ_1 and γ_2 are non-positive and Part II, when at least one of γ_1 or γ_2 is positive.

2.3 Part I: Both γ_1 and γ_2 are non-positive

Theorem 2.1. *If γ_1 and γ_2 are non-positive, then the system given by Equations (1.1)-(1.4) is dissipative, with attraction region contained in* $\mathbf{A} = \{(x, y_1, y_2, z) : 0 \leq x \leq \alpha/\beta, 0 \leq y_1 \leq (\alpha\delta_1 - \beta\xi_1)/(\beta\eta_1), 0 \leq y_2 \leq (\alpha\delta_2 - \beta\xi_2)/(\beta\eta_2), 0 \leq z \leq 1\}$.

Proof:

From Equation (2.4) we have $\dot{z}\mid_{z=0}= \phi x \geq 0$. Also $\dot{z}\mid_{z=1}= -\kappa x \leq 0$. Hence for $0 \leq z_0 \leq 1$, we have $0 \leq z(t) \leq 1$ for all $t \geq 0$.

From Equation (2.1), we have $\dot{x} \leq (\alpha - \beta x)x$. Comparing with $\dot{u} = (\alpha - \beta u)u$, then separating variables and integrating, we obtain

$$\frac{1}{\alpha} \ln \left(\frac{u}{\alpha - \beta u}\right) = t + constant.$$

$$u(t) = \frac{\alpha \exp \alpha(t + constant)}{1 + \beta \exp \alpha(t + constant)}.$$

$$i.e, \quad \lim_{t \to \infty} u(t) = \alpha/\beta.$$

Hence if $x(0) = u(0)$, then $x(t) \leq u(t)$, for all $t \geq 0$. Thus

$$0 \leq \limsup_{t \to \infty} x(t) \leq \alpha/\beta.$$

From Equation (2.2) we have $\dot{y}_1 \leq (-\xi_1 - \eta_1 y_1 + \alpha\delta_1/\beta)y_1$. Compare this with $\dot{u} = (-\xi_1 - \eta_1 u + \alpha\delta_1/\beta)u$. Separating variables and integrating, we obtain

$$u(t) = \frac{(-\xi_1 + \alpha\delta_1/\beta) \exp\left(-\xi_1 + \alpha\delta_1/\beta\right)(t + constant)}{1 + \eta_1 \exp\left(-\xi_1 + \alpha\delta_1/\beta\right)(t + constant)},$$

$$i.e, \quad \lim_{t \to \infty} u(t) = (-\xi_1 + \alpha\delta_1/\beta)/\eta_1 = \frac{\alpha\delta_1 - \beta\xi_1}{\beta\eta_1}.$$

Hence if $y_1(0) = u(0)$, then $y_1(t) \leq u(t)$, for all $t \geq 0$. Then

$$0 \leq \limsup_{t \to \infty} y_1(t) \leq \frac{\alpha\delta_1 - \beta\xi_1}{\beta\eta_1}.$$

Similarly, one can show that

$$0 \leq \limsup_{t \to \infty} y_2(t) \leq \frac{\alpha\delta_2 - \beta\xi_2}{\beta\eta_2}. \quad \square$$

2.4 Equilibria

In this section we determine all the possible equilibria for the system (2.1)-(2.4). Equilibrium conditions for the system given by Equations (2.1) - (2.4) are found as solutions of the algebraic system

$$(\alpha z - \beta x + \gamma_1 y_1 + \gamma_2 y_2 - \theta(1 - z))x = 0 \tag{2.6}$$

$$(-\xi_1 - \eta_1 y_1 - \rho_1 y_2 + \delta_1 x)y_1 = 0 \tag{2.7}$$

$$(-\xi_2 - \eta_2 y_2 - \rho_2 y_1 + \delta_2 x)y_2 = 0 \tag{2.8}$$

$$-\kappa xz + \vartheta(1 - z)z + \phi(1 - z)x = 0. \tag{2.9}$$

We are only interested in nonnegative equilibria. We shall denote by $F_a(x, y_1, y_2, z)$ the equilibria lying on the a-axis, by $F_{ab}(x, y_1, y_2, z)$ the ones in the positive (a,b)-plane, by $F_{abc}(x, y_1, y_2, z)$ the ones in the positive (a,b,c)-octant and by $F^*(x, y_1, y_2, z)$ the positive interior equilibrium of the whole system whenever they exist.

It is easy and trivial to see that $F_0(0, 0, 0, 0)$ and $F_z(0, 0, 0, 1)$ are equilibria. Also the only axial equilibrium for the system is $F_z(0, 0, 0, 1)$, since there cannot be a positive equilibrium lying completely on the x-axis, y_1-axis or y_2-axis.

2.4.1 Positive interior planar equilibria

In this subsection, we determine all the two-dimensional equilibria for the system (2.1)-(2.4).

In the positive interior of the (x, y_1)-plane, we have $y_2 = z = 0$ and $x > 0$, $y_1 > 0$. Hence the algebraic system (2.6)-(2.9) reduces to

$$-\beta x + \gamma_1 y_1 - \theta = 0 \tag{2.10}$$

$$(-\xi_1 - \eta_1 y_1 + \delta_1 x) = 0 \tag{2.11}$$

$$\phi x = 0. \tag{2.12}$$

From the last equation, since $\phi \neq 0$, we have that $x = 0$. Thus there is no equilibrium lying entirely in the positive interior of the (x, y_1)-plane. Similarly there is no equilibrium lying in the positive interior of the (x, y_2)-plane.

In the positive interior of the (x, z)-plane, we have $x > 0$, $z > 0$ and $y_1 = y_2 = 0$. Here the algebraic system (2.6)-(2.9) reduces to

$$\alpha z - \beta x - \theta(1 - z) = 0 \tag{2.13}$$

$$-\kappa x z + \vartheta(1 - z)z + \phi(1 - z)x = 0. \tag{2.14}$$

We solve Equations (2.13)and (2.14) to get $x = \bar{x} = \frac{(\alpha+\theta)\bar{z}-\theta}{\beta}$ and $z = \bar{z}$, where $\bar{z}$ is the solution of the equation

$$(\kappa\alpha + \kappa\theta + \vartheta\beta)\bar{z}^2 - (\kappa\theta + \vartheta\beta - \phi\beta)\bar{z} - \phi\beta = 0. \tag{2.15}$$

It is not difficult to see that Equation (2.15) has only one positive non-zero root given by

$$\bar{z} = \frac{\kappa\theta + \vartheta\beta - \phi\beta + \sqrt{(\kappa\theta + \vartheta\beta + \phi\beta)^2 + 4\beta\phi\kappa\alpha}}{2(\kappa\alpha + \kappa\theta + \vartheta\beta)}. \tag{2.16}$$

From Equation (2.16), we have the following

$$\begin{aligned}
\bar{z} &= \frac{\kappa\theta + \vartheta\beta - \phi\beta + \sqrt{(\kappa\theta + \vartheta\beta + \phi\beta)^2 + 4\beta\phi\kappa\alpha}}{2(\kappa\alpha + \kappa\theta + \vartheta\beta)} \\
&> \frac{\kappa\theta + \vartheta\beta - \phi\beta + \kappa\theta + \vartheta\beta + \phi\beta}{2(\kappa\alpha + \kappa\theta + \vartheta\beta)} \\
&= \frac{\kappa\theta + \vartheta\beta}{\kappa(\theta + \alpha) + \vartheta\beta} \\
&> \frac{\theta}{\theta + \alpha}.
\end{aligned}$$

Hence this equilibrium always exists and we denote it by $F_{xz}(\bar{x}, 0, 0, \bar{z})$. It is the only interior planar equilibrium for the system since no equilibria lie in the interior of the (y_1, y_2)-plane, the (y_1, z)-plane or the (y_2, z)-plane. One can verify the previous statement directly from the equations of our model or by virtue of the fact that industrial assets can only depreciate in the absence of agricultural assets and hence must go extinct.

2.4.2 Positive three-dimensional equilibria

Here, we determine all the possible three-dimensional equilibria for the system (2.1)-(2.4).

In the positive interior of the (x, y_1, y_2)-octant, we have $z = 0$, $x > 0$, $y_1 > 0$, and $y_2 > 0$. The algebraic system (2.6)-(2.9) becomes

$$-\beta x + \gamma_1 y_1 + \gamma_2 y_2 - \theta = 0 \tag{2.17}$$

$$-\xi_1 - \eta_1 y_1 - \rho_1 y_2 + \delta_1 x = 0 \tag{2.18}$$

$$-\xi_2 - \eta_2 y_2 - \rho_2 y_1 + \delta_2 x = 0 \tag{2.19}$$

$$\phi x = 0. \tag{2.20}$$

From Equation (2.20) we observe that $x = 0$ since $\phi \neq 0$. Hence there is no equilibrium lying completely in the positive interior of the (x, y_1, y_2)-octant. Similarly, it can be shown that there is no equilibrium lying completely in the positive interior of the (y_1, y_2, z)-octant.

In the positive interior of the (x, y_1, z)-octant, we have $x > 0$, $y_1 > 0$, $y_2 = 0$, and $z > 0$. So the algebraic system (2.6)-(2.9) reduces to

$$\alpha z - \beta x + \gamma_1 y_1 - \theta(1 - z) = 0 \tag{2.21}$$

$$-\xi_1 - \eta_1 y_1 + \delta_1 x = 0 \tag{2.22}$$

$$-\kappa x z + \vartheta(1-z)z + \phi(1-z)x = 0. \tag{2.23}$$

Solving for y_1 from Equation (2.22), we get

$$y_1 = \breve{y}_1 = \frac{-\xi_1 + \delta_1 \breve{x}}{\eta_1}, \tag{2.24}$$

which exists if

$$\breve{x} > \frac{\xi_1}{\delta_1}. \tag{2.25}$$

Substituting Equation (2.24) into (2.21) we get

$$(\alpha\eta_1 + \theta\eta_1)\breve{z} + (\gamma_1\delta_1 - \beta\eta_1)\breve{x} - \gamma_1\xi_1 - \theta\eta_1 = 0, \tag{2.26}$$

and solving for $\breve{x}$ we get

$$x = \breve{x} = \frac{(\alpha\eta_1 + \theta\eta_1)\breve{z} - \gamma_1\xi_1 - \theta\eta_1}{(\beta\eta_1 - \gamma_1\delta_1)}. \tag{2.27}$$

However, from Equation (2.25) we require $\breve{x} > \frac{\xi_1}{\delta_1}$. Hence from Equation (2.27), we must have

$$\breve{x} = \frac{(\alpha\eta_1 + \theta\eta_1)\breve{z} - \gamma_1\xi_1 - \theta\eta_1}{(\beta\eta_1 - \gamma_1\delta_1)} > \frac{\xi_1}{\delta_1}.$$

Thus for the existence of this equilibrium we require

$$\breve{z} > \frac{\beta\xi_1 + \theta\delta_1}{\alpha\delta_1 + \theta\delta_1}, \tag{2.28}$$

but, since $\breve{z} \leq 1$, from Equation (2.28) we require

$$\beta\xi_1 < \delta_1\alpha,$$

which is clearly satisfied from Equation (2.5).

Substituting Equation (2.27) into (2.23) and simplifying, we get

$$a\breve{z}^2 - b\breve{z} + c = 0, \tag{2.29}$$

where

$$a = (\alpha\eta_1 + \theta\eta_1)(\kappa + \phi) + \vartheta(\beta\eta_1 - \gamma_1\delta_1)$$

$$b = \vartheta(\beta\eta_1 - \gamma_1\delta_1) + (\kappa + \phi)(\theta\eta_1 + \gamma_1\xi_1) + \phi(\alpha\eta_1 + \theta\eta_1)$$

$$c = \phi(\theta\eta_1 + \gamma_1\xi_1).$$

We solve (2.29) for $\breve{z}$ to get

$$z = \breve{z} = \frac{b \pm \sqrt{b^2 - 4ac}}{2a}. \tag{2.30}$$

Theorem 2.2. *There exists at most one equilibrium lying completely in the positive interior of the (x, y_1, z)-octant.*

Proof:

$$\begin{aligned}
b^2 - 4ac &= \{\vartheta(\beta\eta_1 - \gamma_1\delta_1) + (\kappa + \phi)(\theta\eta_1 + \gamma_1\xi_1) + \phi(\alpha\eta_1 + \theta\eta_1)\}^2 \\
&- 4\phi(\theta\eta_1 + \gamma_1\xi_1)\{(\alpha\eta_1 + \theta\eta_1)(\kappa + \phi) + \vartheta(\beta\eta_1 - \gamma_1\delta_1)\} \\
&= \{\vartheta(\beta\eta_1 - \gamma_1\delta_1) + \kappa(\theta\eta_1 + \gamma_1\xi_1) + \phi(\alpha\eta_1 + \theta\eta_1)\}^2 + \{\phi(\theta\eta_1 + \gamma_1\xi_1)\}^2 \\
&- 2\phi(\theta\eta_1 + \gamma_1\xi_1)\{\vartheta(\beta\eta_1 - \gamma_1\delta_1) + \kappa(\theta\eta_1 + \gamma_1\xi_1) + \phi(\alpha\eta_1 + \theta\eta_1)\} \\
&+ 4\phi\kappa(\theta\eta_1 + \gamma_1\xi_1)(\gamma_1\xi_1 - \alpha\eta_1) \\
&= \{\vartheta(\beta\eta_1 - \gamma_1\delta_1) + (\kappa - \phi)(\theta\eta_1 + \gamma_1\xi_1) + \phi(\alpha\eta_1 + \theta\eta_1)\}^2 \\
&+ 4\phi\kappa(\theta\eta_1 + \gamma_1\xi_1)(\gamma_1\xi_1 - \alpha\eta_1).
\end{aligned}$$

We observe from the above that if $\theta\eta_1 + \gamma_1\xi_1 \leq 0$, then $b^2 - 4ac \geq b^2$. Hence there can be at most one positive root for $\check{z}$. Hence the result of the theorem.

On the other hand if $\theta\eta_1 + \gamma_1\xi_1 > 0$, then we have

$$\begin{aligned}
b^2 - 4ac &= \{\vartheta(\beta\eta_1 - \gamma_1\delta_1) + (\kappa - \phi)(\theta\eta_1 + \gamma_1\xi_1) + \phi(\alpha\eta_1 + \theta\eta_1)\}^2 \\
&+ 4\phi\kappa(\theta\eta_1 + \gamma_1\xi_1)(\gamma_1\xi_1 - \alpha\eta_1) \\
&= \{\vartheta(\beta\eta_1 - \gamma_1\delta_1) + \kappa(\theta\eta_1 + \gamma_1\xi_1) + \phi(\alpha\eta_1 - \gamma_1\xi_1)\}^2 \\
&+ 4\phi\kappa(\theta\eta_1 + \gamma_1\xi_1)(\gamma_1\xi_1 - \alpha\eta_1) \\
&= \{\vartheta(\beta\eta_1 - \gamma_1\delta_1)\}^2 + \{\kappa(\theta\eta_1 + \gamma_1\xi_1) - \phi(\alpha\eta_1 - \gamma_1\xi_1)\}^2 \\
&+ 2\vartheta(\beta\eta_1 - \gamma_1\delta_1)\{\kappa(\theta\eta_1 + \gamma_1\xi_1) + \phi(\alpha\eta_1 - \gamma_1\xi_1)\} \\
&= \{\vartheta(\beta\eta_1 - \gamma_1\delta_1) - \kappa(\theta\eta_1 + \gamma_1\xi_1) + \phi(\alpha\eta_1 - \gamma_1\xi_1)\}^2 \\
&+ 4\kappa\vartheta(\beta\eta_1 - \gamma_1\delta_1)(\theta\eta_1 + \gamma_1\xi_1) \\
&> 0.
\end{aligned}$$

Since $b^2 - 4ac > 0$ and $b^2 - 4ac < b^2$, $\check{z}$ has two positive roots with the smallest root given by

$$\begin{aligned}
\check{z}_- &= \frac{b-\sqrt{b^2-4ac}}{2a} \\
&= \frac{\vartheta(\beta\eta_1-\gamma_1\delta_1)+(\kappa+\phi)(\theta\eta_1+\gamma_1\xi_1)+\phi(\alpha\eta_1+\theta\eta_1)}{2\{(\alpha\eta_1+\theta\eta_1)(\kappa+\phi)+\vartheta(\beta\eta_1-\gamma_1\delta_1)\}} \\
&- \frac{\sqrt{\{\vartheta(\beta\eta_1-\gamma_1\delta_1)-\kappa(\theta\eta_1+\gamma_1\xi_1)+\phi(\alpha\eta_1-\gamma_1\xi_1)\}^2+4\kappa\vartheta(\beta\eta_1-\gamma_1\delta_1)(\theta\eta_1+\gamma_1\xi_1)}}{2\{(\alpha\eta_1+\theta\eta_1)(\kappa+\phi)+\vartheta(\beta\eta_1-\gamma_1\delta_1)\}} \\
&< \frac{\vartheta(\beta\eta_1-\gamma_1\delta_1)+(\kappa+\phi)(\theta\eta_1+\gamma_1\xi_1)+\phi(\alpha\eta_1+\theta\eta_1)}{2\{(\alpha\eta_1+\theta\eta_1)(\kappa+\phi)+\vartheta(\beta\eta_1-\gamma_1\delta_1)\}} \\
&- \frac{\sqrt{\{\vartheta(\beta\eta_1-\gamma_1\delta_1)-\kappa(\theta\eta_1+\gamma_1\xi_1)+\phi(\alpha\eta_1-\gamma_1\xi_1)\}^2}}{2\{(\alpha\eta_1+\theta\eta_1)(\kappa+\phi)+\vartheta(\beta\eta_1-\gamma_1\delta_1)\}} \\
&= \frac{(\kappa+\phi)(\theta\eta_1+\gamma_1\xi_1)}{(\alpha\eta_1+\theta\eta_1)(\kappa+\phi)+\vartheta(\beta\eta_1-\gamma_1\delta_1)} \\
&< \frac{(\kappa+\phi)\theta\eta_1}{(\alpha\eta_1+\theta\eta_1)(\kappa+\phi)+\vartheta(\beta\eta_1-\gamma_1\delta_1)}.
\end{aligned}$$

Now suppose $\check{z}_- > \frac{\beta\xi_1+\theta\delta_1}{\alpha\delta_1+\theta\delta_1}$. Then at least we must have

$$\frac{\beta\xi_1+\theta\delta_1}{\alpha\delta_1+\theta\delta_1} < \frac{(\kappa+\phi)\theta\eta_1}{(\alpha\eta_1+\theta\eta_1)(\kappa+\phi)+\vartheta(\beta\eta_1-\gamma_1\delta_1)},$$

$$\Longrightarrow (\kappa+\phi)\theta\eta_1(\alpha\delta_1+\theta\delta_1) > (\beta\xi_1+\theta\delta_1)\{(\alpha\eta_1+\theta\eta_1)(\kappa+\phi)+\vartheta(\beta\eta_1-\gamma_1\delta_1)\},$$

$$\Longrightarrow (\kappa+\phi)(\alpha+\theta)\eta_1\{\theta\delta_1-(\beta\xi_1+\theta\delta_1)\} > (\beta\xi_1+\theta\delta_1)\vartheta(\beta\eta_1-\gamma_1\delta_1),$$

$$\Longrightarrow -(\kappa+\phi)(\alpha+\theta)\eta_1\beta\xi_1 > (\beta\xi_1+\theta\delta_1)\vartheta(\beta\eta_1-\gamma_1\delta_1).$$

Which is a contradiction. Hence $\check{z}_- < \frac{\beta\xi_1+\theta\delta_1}{\alpha\delta_1+\theta\delta_1}$ is not an admissible equilibrium. □

We now proceed to show by persistence theory that there exists a unique equilibrium in the positive interior of the (x, y_1, z)-octant if $\bar{x} > \frac{\xi_1}{\delta_1}$. Consider system (2.1)-(2.4) restricted to $\mathbf{R}^+_{xy_1z} = \{(x, y_1, z) : 0 \le x \le \infty, 0 \le y_1 \le \infty, 0 \le z \le 1\}$ as represented by

$$\dot{x} = (\alpha z - \beta x + \gamma_1 y_1 - \theta(1-z))x \tag{2.31}$$

$$\dot{y_1} = (-\xi_1 - \eta_1 y_1 + \delta_1 x)y_1 \tag{2.32}$$

$$\dot{z} = -\kappa xz + \vartheta(1-z)z + \phi(1-z)x \tag{2.33}$$

with initial conditions $x_0 = x(0) \geq 0$, $y_{10} = y_1(0) \geq 0$ and $z_0 = z(0), 0 \leq z_0 \leq 1$. The possible equilibria in $\mathbf{R}^+_{xy_1z}$ are given by $E_0(0,0,0)$, $E_z(0,0,1)$, $E_{xz}(\bar{x}, 0, \bar{z})$ and possibly $E_{xy_1z}(\breve{x}, \breve{y}_1, \breve{z})$. We note that the existence of $E_{xy_1z}(\breve{x}, \breve{y}_1, \breve{z})$ in $\mathbf{R}^+_{xy_1z}$ implies the existence of an equilibrium in the (x, y_1, z)-octant of the original system. The existence of $E_0(0,0,0)$, $E_z(0,0,1)$ and $E_{xz}(\bar{x}, 0, \bar{z})$ follows similarly as in sections (2.4) and (2.4.1).

The variational matrix V for system (2.31)-(2.33) in $\mathbf{R}^+_{xy_1z}$ is given by

$$V = \begin{bmatrix} m_{11} & \gamma_1 x & (\alpha+\theta)x \\ \delta_1 y_1 & m_{22} & 0 \\ -(\kappa+\phi)z+\phi & 0 & m_{33} \end{bmatrix}, \tag{2.34}$$

where

$$m_{11} = \alpha z - 2\beta x + \gamma_1 y_1 - \theta(1-z)$$

$$m_{22} = -\xi_1 - 2\eta_1 y_1 + \delta_1 x$$

$$m_{33} = -\kappa x + \vartheta(1-z) - \vartheta z - \phi x.$$

The variational matrix V about the equilibrium $E_0(0,0,0)$ is given by

$$V_{E_0} = \begin{bmatrix} -\theta & 0 & 0 \\ 0 & -\xi_1 & 0 \\ \phi & 0 & \vartheta \end{bmatrix}. \tag{2.35}$$

The eigenvalues of V_{E_0} are given by $-\theta, -\xi_1$ and ϑ. Thus we have the following lemma.

Lemma 2.1. *The equilibrium $E_0(0,0,0)$ in $\mathbf{R}^+_{xy_1z}$ is a hyperbolic saddle point, locally unstable in the z-direction and stable in all other directions. In particular the dimension of the stable manifold is two and of the unstable manifold is one.*

The variational matrix V about the equilibrium $E_z(0,0,1)$ is given by

$$V_{E_z} = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & -\xi_1 & 0 \\ -\kappa & 0 & -\vartheta \end{bmatrix}. \tag{2.36}$$

The eigenvalues of V_{E_z} are given by $\alpha, -\xi_1$ and $-\vartheta$. Hence we have the following lemma.

Lemma 2.2. *The equilibrium $E_z(0,0,1)$ in $\mathbf{R}^+_{xy_1z}$ is a hyperbolic saddle point, locally unstable in the x-direction and stable in all other directions. In particular the dimensions of the stable manifold is two and of the unstable manifold is one.*

The variational matrix V about the equilibrium $E_{xz}(\bar{x},0,\bar{z})$ is given by

$$V_{E_{xz}} = \begin{bmatrix} -\beta\bar{x} & \gamma_1\bar{x} & (\alpha+\theta)\bar{x} \\ 0 & -\xi_1+\delta_1\bar{x} & 0 \\ -(\kappa+\phi)\bar{z}+\phi & 0 & -(\frac{\vartheta\bar{z}^2+\phi\bar{x}}{\bar{z}}) \end{bmatrix}. \tag{2.37}$$

The eigenvalues of $V_{E_{xz}}$ are given by $\lambda_2 = -\xi_1 + \delta_1\bar{x}$ and λ_1 and λ_3 which are the eigenvalues of J_{22}. The eigenvalues of J_{22} are given by the eigenvalues of

$$J_{22} = \begin{bmatrix} -\beta\bar{x} & (\alpha+\theta)\bar{x} \\ -(\kappa+\phi)\bar{z}+\phi & -(\frac{\vartheta\bar{z}^2+\phi\bar{x}}{\bar{z}}) \end{bmatrix}. \tag{2.38}$$

$$Trace(J_{22}) = -(\beta\bar{x} + \frac{\vartheta\bar{z}^2+\phi\bar{x}}{\bar{z}}) < 0$$

and

$$det(J_{22}) = (\frac{\vartheta \bar{z}^2 + \phi \bar{x}}{\bar{z}})(\beta \bar{x}) + (\alpha + \theta)((\kappa + \phi)\bar{z} - \phi)\bar{x} > 0.$$

(Same proof as that of section (2.5.3)). Thus both λ_1 and λ_3 have negative real parts. Hence we have the following two lemmas.

Lemma 2.3. *The equilibrium $E_{xz}(\bar{x}, 0, \bar{z})$ in $\mathbf{R}^+_{xy_1z}$ is locally asymptotically stable if $\lambda_2 = -\xi_1 + \delta_1 \bar{x} < 0$.*

Lemma 2.4. *The equilibrium $E_{xz}(\bar{x}, 0, \bar{z})$ in $\mathbf{R}^+_{xy_1z}$ is a hyperbolic saddle point if $\lambda_2 = -\xi_1 + \delta_1 \bar{x} > 0$ and it is locally stable (attracting) in the $x-$direction and the $z-$direction and unstable in the y_1- direction.*

In order to state and prove the main result of this subsection, we state and prove the following lemmas which we will require in the proof of our main result.

Lemma 2.5. *The subsystem given by Equations (2.31) - (2.33) in $\mathbf{R}^+_{xy_1z}$ is dissipative with attraction region contained in $\mathbf{B} = \{(x, y_1, z) : 0 \leq x \leq \alpha/\beta, 0 \leq y_1 \leq (\alpha\delta_1 - \beta\xi_1)/(\beta\eta_1), 0 \leq z \leq 1\}$.*

Proof:

The proof is similar to the proof of Theorem 2.1. One can also use the fact that a subsystem of a dissipative system is dissipative to arrive at a proof of this lemma.

Lemma 2.6. *There are no closed curves, orbits or paths in the (x,z)-plane (in fact in all the planes) of $\mathbf{R}^+_{xy_1z}$.*

Proof:

Consider

$$\begin{aligned}
\mathbf{D}(x,z) &= \frac{\partial}{\partial x}(\frac{\dot{x}}{xz}) + \frac{\partial}{\partial z}(\frac{\dot{z}}{xz}) \\
&= \frac{\partial}{\partial x}(\frac{\alpha z - \beta x + \gamma_1 y_1 - \theta(1-z)}{z}) + \frac{\partial}{\partial z}(-\kappa + \frac{\vartheta}{x} - \frac{\vartheta z}{x} + \frac{\phi}{z} - \phi) \\
&= -(\frac{\beta}{z} + \frac{\vartheta}{x} + \frac{\phi}{z^2}) \\
&< 0.
\end{aligned}$$

The results follows directly from Dulac's theorem. □

Equipped with the above lemmas, we now state and prove the main result of this subsection for subsystem (2.31)-(2.33).

Theorem 2.3. *If $\lambda_2 = -\xi_1 + \delta_1 \bar{x} > 0$, the subsystem in $\mathbf{R}^+_{xy_1z}$ exhibits uniform persistence and hence there exists an interior equilibrium $E_{xy_1z} = (\breve{x}, \breve{y}_1, \breve{z})$ for the subsystem.*

Proof:

Let $\breve{E} = (\breve{x}, \breve{y}_1, \breve{z})$ with $\breve{x} > 0, \breve{y}_1 > 0$ and $\breve{z} > 0$ be a point in the interior of $\mathbf{R}^+_{xy_1z}$, $O(\breve{E})$ be the orbit through the point $\breve{E}$ and $\Omega(\breve{E})$, the omega limit set of $\breve{E}$. The proof of the above theorem is completed by showing the following.

1. The system given by Equations (2.31)-(2.33) is dissipative and solutions initiating in the interior $\mathbf{R}^+_{xy_1z}$ are eventually uniformly bounded.

2. The boundary flow is isolated and acyclic with respect to **B**.

3. $E_0(0,0,0) \notin \Omega(\breve{E})$.

4. $E_z(0,0,1) \notin \Omega(\breve{E})$.

5. $E_{xz}(\bar{x}, 0, \bar{z}) \notin \Omega(\breve{E})$.

We have already shown in Lemma 2.5 that the system given by Equations (2.31)-(2.33) is dissipative with attraction region contained in $\mathbf{B}$. Thus all solutions initiating in the interior of $\mathbf{R}^+_{xy_1z}$ will eventually enter $\mathbf{B}$ and hence are eventually bounded.

Also from Equations (2.35) - (2.37), it is easy to see that the variational matrices V_{E_0}, V_{E_Z} and $V_{E_{xz}}$ about the equilibria $E_0(0,0,0), E_z(0,0,1)$ and $E_{xz}(\bar{x}, 0, \bar{z})$ respectively are non-singular, and hence the equilibria are isolated.

It is also trivial to show that $E_z(0,0,1)$ is globally asymptotically stable with respect to solutions initiating in interior of $\mathbf{R}^+_z$. Also the subsystem given by Equations (2.31)and (2.33) is dissipative since it is a subsystem of a dissipative system. The equilibrium $E_{xz}(\bar{x}, 0, \bar{z})$ is the only positive interior equilibrium in $\mathbf{R}^+_{xz}$, and also the only locally asymptotically stable equilibrium for the subsystem given by Equations (2.31) and (2.33) in $\mathbf{R}^+_{xz}$, and hence $E_{xz}(\bar{x}, 0, \bar{z})$ is globally asymptotically stable with respect to solutions initiating in the interior of $\mathbf{R}^+_{xz}$.

The global asymptotic stability of $E_z(0,0,1)$and $E_{xz}(\bar{x}, 0, \bar{z})$ with respect to $\mathbf{R}^+_z$ and $\mathbf{R}^+_{xz}$ implies the boundary flow is acyclic.

Clearly from Lemma 2.5, all solutions initiating in the interior of $\mathbf{R}^+_{xy_1z}$ are (eventually) bounded in forward time, and hence $\Omega(\breve{E})$ is bounded.

Now suppose $E_0 = E_0(0,0,0) \in \Omega(\breve{E})$. Then since E_0 is a hyperbolic saddle point (Lemma 2.1), $E_0 \neq \Omega(\breve{E})$. Hence by Theorem 1.9, there exists at least one point $E_0^s \in \Omega(\breve{E}) \cap W^s(E_0) \setminus \{E_0\}$, where $W^s(E_0)$ is the strong stable manifold of E_0. But $W^s(E_0)$ is the (x, y_1)-plane. If E_0^s is on either of the positive x-axis or the positive y_1-axis, then since the entire orbit $O(E_0^s)$ through E_0^s is contained in $\Omega(\breve{E})$, the whole of the positive x-axis or y_1-axis respectively is contained in $\Omega(\breve{E})$, which is a contradiction to the fact that $\Omega(\breve{E})$ is bounded (Lemma 2.5). On the other hand if E_0^s is in the interior of the (x, y_1)-plane, then since there is no equilibrium in the interior of the (x, y_1)-plane, $\Omega(\breve{E})$ must be unbounded, giving a contradiction. Hence $E_0 = E_0(0,0,0) \notin \Omega(\breve{E})$.

We now show $E_z = E_z(0,0,1) \notin \Omega(\breve{E})$. Suppose $E_z(0,0,1) \in \Omega(\breve{E})$. Since E_z is a hyperbolic saddle point by Lemma 2.2, there exists $E_z^s \in \Omega(\breve{E}) \cap W^s(E_z) \setminus \{E_z\}$. But $W^s(E_z)$ is the (y_1, z)-plane. If E_z^s is in the interior of the (y_1, z)-plane , then $O(E_z^s)$ is unbounded since there is no equilibrium in the interior of the (y_1, z)-plane and also there is no equilibrium lying completely on the y_1-axis, which in turn implies $\Omega(\breve{E})$ is unbounded, which is a contraction. If E_z^s is on the y-axis, then either $O(E_z^s) \subset \Omega(\breve{E})$ is unbounded or the alpha limit set of E_z^s, $\alpha(E_z^s) = E_0 \in \Omega(\breve{E})$, both contradictions. On the other hand if E_z^s is on the z-axis then $\alpha(E_z^s) = E_0 \in \Omega(\breve{E})$, a contradiction. Hence $E_z = E_z(0,0,1) \notin \Omega(\breve{E})$.

Suppose $E_{xz} = E_{xz}(\bar{x}, 0, \bar{z}) \in \Omega(\breve{E})$. Then by Lemma 2.4 and our hypothesis, E_{xz} is a hyperbolic saddle point and hence there exists $E_{xz}^s \in \Omega(\breve{E}) \cap W^s(E_{xz}) \setminus \{E_{xz}\}$. But $W^s(E_{xz})$ is the (x, z)-plane. Then either $O(E_{xz}^s)$ is unbounded implying $\Omega(\breve{E})$ is unbounded or either $\alpha(E_{xz}^s) = E_0 \in \Omega(\breve{E})$ or $\alpha(E_{xz}^s) = E_z \in \Omega(\breve{E})$, all of which are contradictions. Hence $E_{xz} = E_{xz}(\bar{x}, 0, \bar{z}) \notin \Omega(\breve{E})$.

Finally, if there exists any point $E \in \Omega(\breve{E})$ such that E is on the boundary of $\mathbf{R}^+_{xy_1z}$, then the closure of the orbit through E must either contain E_0, E_z, E_{xz} or is unbounded. This gives a contradiction. Thus the subsystem in $\mathbf{R}^+_{xy_1z}$ exhibits persistence.

The system is uniformly persistent because it is persistent, dissipative and the boundary flow is isolated and acyclic by Theorem 1.10. $\square$

Theorem 2.4. *There exists a unique equilibrium $F_{xy_1z}(\check{x}, \check{y}_1, 0, \check{z})$ in the positive interior of the (x, y_1, z)-octant if $-\xi_1 + \delta_1\bar{x} > 0$.*

Proof:

The existence of $F_{xy_1z}(\breve{x}, \breve{y}_1, 0, \breve{z})$ is a consequence of Theorem 2.3. The uniqueness part follows from Theorem 2.2. $\square$

Similarly one can prove the following theorem.

Theorem 2.5. *There exists a unique equilibrium $F_{xy_2z}(\breve{x}, 0, \breve{y}_2, \breve{z})$ in the positive interior of the (x, y_2, z)-octant if $-\xi_2 + \delta_2\bar{x} > 0$.*

Later in our work, we will try to establish sufficient conditions for the existence of an interior equilibrium for the whole system, $F^\star(x^\star, {y_1}^\star, {y_2}^\star, z^\star)$. We proceed in the next section to determine the local stabilities of the equilibria above.

2.5 Local stability analysis of equilibria

By applying the Hartman-Grobman theorem (i.e. Theorem 1.6), we know that the local behavior of a nonlinear system near a hyperbolic fixed point can qualitatively be determined by the behavior of the corresponding linearized system near the origin. Thus the local stability of a hyperbolic fixed point of a nonlinear system can be determined from the variational matrix V about the equilibrium. The real parts of the eigenvalues of the variational matrix V determines the local stability of an equilibrium. The variational matrix V for the system (2.1)-(2.4) is given by

$$V = \begin{bmatrix} v_{11} & \gamma_1 x & \gamma_2 x & (\alpha+\theta)x \\ \delta_1 y_1 & v_{22} & -\rho_1 y_1 & 0 \\ \delta_2 y_2 & -\rho_2 y_2 & v_{33} & 0 \\ -(\kappa+\phi)z+\phi & 0 & 0 & v_{44} \end{bmatrix}, \tag{2.39}$$

where

$$v_{11} = \alpha z - 2\beta x + \gamma_1 y_1 + \gamma_2 y_2 - \theta(1-z)$$

$$v_{22} = -\xi_1 - 2\eta_1 y_1 - \rho_1 y_2 + \delta_1 x$$

$$v_{33} = -\xi_2 - 2\eta_2 y_2 - \rho_2 y_1 + \delta_2 x$$

$$v_{44} = -\kappa x + \vartheta(1-z) - \vartheta z - \phi x.$$

2.5.1 Local stability analysis of $F_0(0,0,0,0)$

The variational matrix V about the equilibrium $F_0(0,0,0,0)$ is given by

$$V_{F_0} = \begin{bmatrix} -\theta & 0 & 0 & 0 \\ 0 & -\xi_1 & 0 & 0 \\ 0 & 0 & -\xi_2 & 0 \\ \phi & 0 & 0 & \vartheta \end{bmatrix}. \tag{2.40}$$

The eigenvalues of V_{F_0} are given by $-\theta, -\xi_1, -\xi_2$ and ϑ.

Theorem 2.6. *The equilibrium $F_0(0,0,0,0)$ is a hyperbolic saddle point, locally unstable in the z-direction (repelling) and stable in all other directions. In particular the dimension of the stable manifold is three and of the unstable manifold is one.*

2.5.2 Local stability analysis of $F_z(0,0,0,1)$

The variational matrix V about the equilibrium $F_z(0,0,0,1)$ is given by

$$V_{F_z} = \begin{bmatrix} \alpha & 0 & 0 & 0 \\ 0 & -\xi_1 & 0 & 0 \\ 0 & 0 & -\xi_2 & 0 \\ -\kappa & 0 & 0 & -\vartheta \end{bmatrix}. \tag{2.41}$$

The eigenvalues of V_{F_z} are given by $\alpha, -\xi_1, -\xi_2$ and $-\vartheta$.

Theorem 2.7. *The equilibrium $F_z(0,0,0,1)$ is a hyperbolic saddle point, locally unstable in the x-direction and stable in all other directions. In particular the dimensions of the stable manifold is three and of the unstable manifold is one.*

2.5.3 Local stability analysis of $F_{xz}(\bar{x}, 0, 0, \bar{z})$

The variational matrix V about the equilibrium $F_{xz}(\bar{x}, 0, 0, \bar{z})$ is given by

$$V_{F_{xz}} = \begin{bmatrix} -\beta\bar{x} & \gamma_1\bar{x} & \gamma_2\bar{x} & (\alpha+\theta)\bar{x} \\ 0 & -\xi_1+\delta_1\bar{x} & 0 & 0 \\ 0 & 0 & -\xi_2+\delta_2\bar{x} & 0 \\ -(\kappa+\phi)\bar{z}+\phi & 0 & 0 & -(\frac{\vartheta\bar{z}^2+\phi\bar{x}}{\bar{z}}) \end{bmatrix}. \tag{2.42}$$

The eigenvalues of $V_{F_{xz}}$ are given by $\lambda_2 = -\xi_1 + \delta_1\bar{x}$ and $\lambda_3 = -\xi_2 + \delta_2\bar{x}$ and λ_1 and λ_4 which are the eigenvalues of J_{22} given by

$$J_{22} = \begin{bmatrix} -\beta\bar{x} & (\alpha+\theta)\bar{x} \\ -(\kappa+\phi)\bar{z}+\phi & -(\frac{\vartheta\bar{z}^2+\phi\bar{x}}{\bar{z}}) \end{bmatrix}. \tag{2.43}$$

$$Trace(J_{22}) = -(\beta\bar{x} + \frac{\vartheta\bar{z}^2+\phi\bar{x}}{\bar{z}}) < 0$$

and

$$\begin{aligned} det(J_{22}) &= (\frac{\vartheta\bar{z}^2+\phi\bar{x}}{\bar{z}})(\beta\bar{x}) + (\alpha+\theta)((\kappa+\phi)\bar{z}-\phi)\bar{x} \\ &= (\frac{\vartheta\bar{z}^2+\phi\bar{x}}{\bar{z}})(\beta\bar{x}) + (\alpha+\theta)(\kappa\bar{z}\bar{x}+\phi(\bar{z}-1)\bar{x}) \\ &= (\frac{\vartheta\bar{z}^2+\phi\bar{x}}{\bar{z}})(\beta\bar{x}) + (\alpha+\theta)\vartheta(1-\bar{z})\bar{z} \\ &> 0. \end{aligned}$$

Thus both λ_1 and λ_4 have negative real parts.

Theorem 2.8. *The equilibrium $F_{xz}(\bar{x}, 0, 0, \bar{z})$ is locally asymptotically stable if $\lambda_2 = -\xi_1 + \delta_1\bar{x} < 0$ and $\lambda_3 = -\xi_2 + \delta_2\bar{x} < 0$.*

Theorem 2.9. *The equilibrium $F_{xz}(\bar{x}, 0, 0, \bar{z})$ is a hyperbolic saddle point if either $\lambda_2 = -\xi_1 + \delta_1\bar{x} > 0$ or $\lambda_3 = -\xi_2 + \delta_2\bar{x} > 0$ or both. It is always locally stable (attracting) in the x and $z-$directions.*

2.5.4 Local stability analysis of $F_{xy_1z}(\breve{x}, \breve{y}_1, 0, \breve{z})$

The variational matrix V about the equilibrium $F_{xy_1z}(\breve{x}, \breve{y}_1, 0, \breve{z})$ is given by

$$V_{F_{xy_1z}} = \begin{bmatrix} -\beta\breve{x} & \gamma_1\breve{x} & \gamma_2\breve{x} & (\alpha+\theta)\breve{x} \\ \delta_1\breve{y}_1 & -\eta_1\breve{y}_1 & -\rho_1\breve{y}_1 & 0 \\ 0 & 0 & -\xi_2 - \rho_2\breve{y}_1 + \delta_2\breve{x} & 0 \\ -(\kappa+\phi)\breve{z} + \phi & 0 & 0 & -(\frac{\vartheta\breve{z}^2+\phi\breve{x}}{\breve{z}}) \end{bmatrix}. \quad (2.44)$$

The eigenvalues of $V_{F_{xy_1z}}$ are given by $\lambda_3 = -\xi_2 - \rho_2\breve{y}_1 + \delta_2\breve{x}$ and λ_1, λ_2 and λ_4 which are the eigenvalues of J_{33} given by

$$J_{33} = \begin{bmatrix} -\beta\breve{x} & \gamma_1\breve{x} & (\alpha+\theta)\breve{x} \\ \delta_1\breve{y}_1 & -\eta_1\breve{y}_1 & 0 \\ -(\kappa+\phi)\breve{z} + \phi & 0 & -(\frac{\vartheta\breve{z}^2+\phi\breve{x}}{\breve{z}}) \end{bmatrix}. \quad (2.45)$$

Let

$$b_1 = \beta\breve{x} + \eta_1\breve{y}_1 + (\frac{\vartheta\breve{z}^2 + \phi\breve{x}}{\breve{z}}) > 0$$

$$\begin{aligned} b_2 =& (\beta\breve{x})(\frac{\vartheta\breve{z}^2 + \phi\breve{x}}{\breve{z}}) + ((\kappa+\phi)\breve{z} - \phi)(\alpha+\theta)\breve{x} \\ &+ \eta_1\breve{y}_1(\frac{\vartheta\breve{z}^2 + \phi\breve{x}}{\breve{z}}) + \beta\eta_1\breve{y}_1\breve{x} - \delta_1\gamma_1\breve{x}\breve{y}_1 \\ =& (\frac{\vartheta\breve{z}^2 + \phi\breve{x}}{\breve{z}})(\beta\breve{x}) + (\alpha+\theta)(1-\breve{z})\vartheta\breve{z} \\ &+ \eta_1\breve{y}_1(\frac{\vartheta\breve{z}^2 + \phi\breve{x}}{\breve{z}}) + (\beta\eta_1 - \gamma_1\delta_1)\breve{x}\breve{y}_1 > 0 \end{aligned}$$

$$\begin{aligned} b_3 =& \eta_1\breve{y}_1((\frac{\vartheta\breve{z}^2+\phi\breve{x}}{\breve{z}})\beta\breve{x} + ((\kappa+\phi)\breve{z}-\phi)(\alpha+\theta)\breve{x}) \\ & -(\frac{\vartheta\breve{z}^2+\phi\breve{x}}{\breve{z}})\gamma_1\breve{x}\delta_1\breve{y}_1 \\ =& (\frac{\vartheta\breve{z}^2+\phi\breve{x}}{\breve{z}})(\beta\eta_1-\gamma_1\delta_1)\breve{x}\breve{y}_1 + (\alpha+\theta)(1-\breve{z})\eta_1\vartheta\breve{z}\breve{y}_1 > 0. \end{aligned}$$

The eigenvalues of J_{33} are given by

$$\sigma(J_{33}) = \{\lambda_j | \lambda_j^3 + b_1\lambda_j^2 + b_2\lambda_j + b_3 = 0, j = 1,2,4\}.$$

Also we note that

$$\begin{aligned} b_1b_2 - b_3 =& (\beta\breve{x} + \eta_1\breve{y}_1 + (\frac{\vartheta\breve{z}^2+\phi\breve{x}}{\breve{z}}))(\frac{\vartheta\breve{z}^2+\phi\breve{x}}{\breve{z}})(\beta\breve{x}+\eta_1\breve{y}_1) \\ & + (\beta\eta_1-\gamma_1\delta_1)(\beta\breve{x}+\eta_1\breve{y}_1)\breve{x}\breve{y}_1 \\ & + (\beta\breve{x} + (\frac{\vartheta\breve{z}^2+\phi\breve{x}}{\breve{z}}))(\alpha+\theta)(1-\breve{z})\vartheta\breve{z} \\ & > 0. \end{aligned}$$

Hence all the eigenvalues of J_{33} have negative real parts.

Theorem 2.10. *If $\lambda_3 = -\xi_2 - \rho_2\breve{y}_1 + \delta_2\breve{x} > 0$, then $F_{xy_1z}(\breve{x}, \breve{y}_1, 0, \breve{z})$ is a hyperbolic saddle point locally unstable in the y_2-direction and stable in all other three directions. In particular, the (x, y_1, z)-space forms the stable manifold and the unstable manifold is the y_2-axis.*

Proof:

The proof of this theorem follows directly from the above and the application of Routh-Hurwitz criteria. □

Theorem 2.11. *If $\lambda_3 = -\xi_2 - \rho_2\breve{y}_1 + \delta_2\breve{x} < 0$, then $F_{xy_1z}(\breve{x}, \breve{y}_1, 0, \breve{z})$ is locally asymptotically stable.*

Proof:

The proof of this theorem follows directly from the application of Routh-Hurwitz criteria. □

Similarly, one can prove the following two theorems below.

Theorem 2.12. *If* $\lambda_2 = -\xi_1 - \rho_1 \breve{y}_2 + \delta_1 \breve{x} > 0$, *then* $F_{xy_2z}(\breve{x}, 0, \breve{y}_2, \breve{z})$ *is a hyperbolic saddle point locally unstable in the* y_1*-direction and stable in all other three directions. In particular, the* (x, y_2, z)*-space forms the stable manifold and the unstable manifold is the* y_1*-axis.*

Theorem 2.13. *If* $\lambda_2 = -\xi_1 - \rho_1 \breve{y}_2 + \delta_1 \breve{x} < 0$, *then* $F_{xy_2z}(\breve{x}, 0, \breve{y}_2, \breve{z})$ *is locally asymptotically stable.*

2.6 Global stability analysis of equilibria

In this section, criteria for the global asymptotic stability of the boundary equilibria, $F_z(0,0,0,1)$, $F_{xz}(\bar{x}, 0, 0, \bar{z})$, $F_{xy_1z}(\breve{x}, \breve{y}_1, 0, \breve{z})$ and $F_{xy_1z}(\breve{x}, 0, \breve{y}_2, \breve{z})$ with respect to solutions initiating from the interior of $\mathbf{R}_z^+$, $\mathbf{R}_{xz}^+$, $\mathbf{R}_{xy_1z}^+$ and $\mathbf{R}_{xy_2z}^+$ respectively will be established.

2.6.1 Global asymptotic stability of $E_z(1)$

Consider system (2.1)-(2.4) restricted to $\mathbf{R}_z^+ = \{0 \leq z \leq 1\}$ as depicted by

$$\dot{z} = \vartheta(1 - z)z.$$

It is easy to see that the one-dimensional equilibrium $E_z(1)$ exists and consequently $F_z(0,0,0,1)$ exists. In this subsection we show that $E_z(1)$ is globally asymptotically stable and hence $F_z(0,0,0,1)$ is globally asymptotically stable with respect to solutions initiating from the interior of $\mathbf{R}_z^+$.

Theorem 2.14. *The equilibrium $E_z(1)$ and consequently $F_z(0,0,0,1)$ is globally asymptotically stable with respect to solution trajectories initiating from the interior of $\mathbf{R}_z^+$.*

Proof:

In $\mathbf{R}_z^+$, we choose a Liapunov function V(z) defined by

$$V(z) = \frac{(z-1)^2}{2}.$$

The derivative of V(z) along the solution curves in $\mathbf{R}_z^+$ is given by

$$\begin{aligned} \dot{V} &= \dot{z}(z-1) \\ &= -\vartheta(1-z)^2 z < 0. \quad \square \end{aligned}$$

2.6.2 Global asymptotic stability of $E_{xz}(\bar{x}, \bar{z})$

In this subsection, criteria for the global asymptotic stability of $E_{xz}(\bar{x}, \bar{z})$ with respect to solutions initiating in the interior of the $\mathbf{R}_{xz}^+ = \{0 \le x \le \infty, 0 \le z \le 1\}$ will be given. In $\mathbf{R}_{xz}^+$, system (2.1)- (2.4) reduces to

$$\begin{aligned} \dot{x} &= (\alpha z - \beta x - \theta(1-z))x \\ \dot{z} &= -\kappa x z + \vartheta(1-z)z + \phi(1-z)x. \end{aligned} \tag{2.46}$$

We observe that the system (2.46) is dissipative since it is a subsystem of a dissipative system. Also there are no closed curves, orbits or paths in $\mathbf{R}_{xz}^+$ (cf. Lemma 2.6). In $\mathbf{R}_{xz}^+$, there are only three equilibria, namely $E_0(0,0)$, $E_z(0,1)$ and $E_{xz}(\bar{x}, \bar{z})$. Both $E_0(0,0)$ and $E_z(0,1)$ are locally unstable. $E_{xz}(\bar{x}, \bar{z})$ is the only locally asymptotically stable equilibrium and hence it has to be globally asymptotically stable. This leads to the next theorem.

Theorem 2.15. *The equilibrium $E_{xz}(\bar{x}, \bar{z})$ and consequently $F_{xz}(\bar{x}, 0, 0, \bar{z})$ is globally asymptotically stable with respect to solution trajectories emanating from the positive interior of $\mathbf{R}^+_{xz}$.*

Proof:

Let m and n be positive constants defined by $m = \frac{\vartheta(1-\bar{z})}{\bar{x}}$ and $n = \theta + \alpha$.

In $\mathbf{R}^+_{xz}$, we choose a Liapunov function V(x,z) defined by

$$V(x,z) = m\{x - \bar{x} - \bar{x}\ln(\frac{x}{\bar{x}})\} + n\{z - \bar{z} - \bar{z}\ln(\frac{x}{\bar{z}})\}. \tag{2.47}$$

The derivative of (2.47) along the solution curves of (2.46) in $\mathbf{R}^+_{xz}$ is given by

$$\begin{aligned}
\dot{V} &= \frac{m\dot{x}(x-\bar{x})}{x} + \frac{n\dot{z}(z-\bar{x})}{z} \\
&= m(\alpha z - \beta x - \theta(1-z))(x-\bar{x}) + n(-\kappa x + \vartheta(1-z) - \phi x)(z-\bar{z}) + n\phi x(z-\bar{z})/z \\
&= m(\alpha(z-\bar{z}) - \beta(x-\bar{x}) + \theta(z-\bar{z}))(x-\bar{x}) + n(-\kappa(x-\bar{x}) - \vartheta(z-\bar{z}) - \phi(x-\bar{x}))(z-\bar{z}) \\
&+ m(\alpha\bar{z} - \beta\bar{x} - \theta(1-\bar{z}))(x-\bar{x}) + n(-\kappa\bar{x} + \vartheta(1-\bar{z}) - \phi\bar{x})(z-\bar{z}) + n\phi x(z-\bar{z})/z \\
&= m(\alpha(z-\bar{z}) - \beta(x-\bar{x}) + \theta(z-\bar{z}))(x-\bar{x}) + n(-\kappa(x-\bar{x}) - \vartheta(z-\bar{z}) - \phi(x-\bar{x}))(z-\bar{z}) \\
&+ n\phi(\frac{x}{z} - \frac{\bar{x}}{\bar{z}})(z-\bar{z}) \\
&= -m\beta(x-\bar{x})^2 - n(\vartheta + \phi\frac{x}{z\bar{z}})(z-\bar{z})^2 \\
&+ (m(\alpha+\theta) - n(\kappa - \phi + \frac{\phi}{\bar{z}}))(x-\bar{x})(z-\bar{z}) \\
&= -m\beta(x-\bar{x})^2 - n(\vartheta + \phi\frac{x}{z\bar{z}})(z-\bar{z})^2 \\
&+ (m(\alpha+\theta) - \frac{n\vartheta(1-\bar{z})}{\bar{x}})(x-\bar{x})(z-\bar{z}) \\
&= -m\beta(x-\bar{x})^2 - n(\vartheta + \phi\frac{x}{z\bar{z}})(z-\bar{z})^2 \\
&< 0 \quad for \quad all \quad (x,z) \neq (\bar{x}, \bar{z}). \quad \square
\end{aligned}$$

2.6.3 Global asymptotic stability of $E_{xy_1z}(\breve{x}, \breve{y}_1, \breve{z})$

In this subsection, criteria for the global asymptotic stability of $F_{xy_1z}(\breve{x}, \breve{y}_1, 0, \breve{z})$ with respect to solutions initiating in the interior of the $\mathbf{R}^+_{xy_1z} = \{0 \le x \le \infty, 0 \le y_1 \le \infty, 0 \le z \le 1\}$ will be given. In fact, we will show that if the equilibrium $E_{xy_1z}(\breve{x}, \breve{y}_1, \breve{z})$ exists, then it is globally asymptotically stable. Thus we have the following theorem.

Theorem 2.16. *The equilibrium $F_{xy_1z}(\breve{x}, \breve{y}_1, 0, \breve{z})$ is globally asymptotically stable with respect to solution trajectories emanating from the positive interior of $\mathbf{R}^+_{xy_1z}$ if it exists.*

Proof:

In $\mathbf{R}^+_{xy_1z}$, the system of Equations (2.1)- (2.4) reduces to

$$\begin{aligned} \dot{x} &= (\alpha z - \beta x + \gamma_1 y_1 - \theta(1-z))x \\ \dot{y_1} &= (-\xi_1 - \eta_1 y_1 + \delta_1 x)y_1 \\ \dot{z} &= -\kappa x z + \vartheta(1-z)z + \phi(1-z)x. \end{aligned} \tag{2.48}$$

Let m, n and p be positive constants defined by $m = \kappa + \phi - \frac{\phi}{\breve{z}} = \frac{\vartheta(1-\breve{z})}{\breve{x}}$, $n = \frac{-\gamma_1 m}{\delta_1}$ and $p = \theta + \alpha$.

In $\mathbf{R}^+_{xy_1z}$, we choose a Liapunov function $V(x, y_1, z)$ defined by

$$\begin{aligned} V(x, y_1, z) = {} & m\{x - \breve{x} - \breve{x}\ln(\frac{x}{\breve{x}})\} + n\{y_1 - \breve{y}_1 - \breve{y}_1\ln(\frac{y_1}{\breve{y}_1})\} \\ & + p\{z - \breve{z} - \breve{z}\ln(\frac{z}{\breve{z}})\}. \end{aligned} \tag{2.49}$$

The derivative of (2.49) along the solution curves of (2.48) in $\mathbf{R}^+_{xy_1z}$ is given by

$$\begin{aligned}
\dot{V} &= \frac{m\dot{x}(x-\breve{x})}{x} + \frac{n\dot{y}_1(y_1-\breve{y}_1)}{y_1} + \frac{p\dot{z}(z-\breve{z})}{z} \\
&= m(\alpha z - \beta x + \gamma_1 y_1 - \theta(1-z))(x-\breve{x}) + n(-\xi_1 - \eta_1 y_1 + \delta_1 x)(y_1-\breve{y}_1) \\
&+ p(-\kappa x + \vartheta(1-z) - \phi x)(z-\breve{z}) + p\phi x(z-\breve{z})/z \\
&= m(\alpha(z-\breve{z}) - \beta(x-\breve{x}) + \gamma_1(y_1-\breve{y}_1) + \theta(z-\breve{z}))(x-\breve{x}) \\
&+ n(-\eta_1(y_1-\breve{y}_1) + \delta_1(x-\breve{x}))(y_1-\breve{y}_1) + p(-\kappa(x-\breve{x}) - \vartheta(z-\breve{z}) - \phi(x-\breve{x}))(z-\breve{z}) \\
&+ m(\alpha\breve{z} - \beta\breve{x} + \gamma_1\breve{y}_1 - \theta(1-\breve{z}))(x-\breve{x}) + n(-\xi_1 - \eta_1\breve{y}_1 + \delta_1\breve{x})(y_1-\breve{y}_1) \\
&+ p(-\kappa\breve{x} + \vartheta(1-\breve{z}) - \phi\breve{x})(z-\breve{z}) + p\phi x(z-\breve{z})/z \\
&= m(\alpha(z-\breve{z}) - \beta(x-\breve{x}) + \gamma_1(y_1-\breve{y}_1) + \theta(z-\breve{z}))(x-\breve{x}) \\
&+ n(-\eta_1(y_1-\breve{y}_1) + \delta_1(x-\breve{x}))(y_1-\breve{y}_1) + p(-\kappa(x-\breve{x}) - \vartheta(z-\breve{z}) - \phi(x-\breve{x}))(z-\breve{z}) \\
&+ p\phi(\frac{x}{z} - \frac{\breve{x}}{\breve{z}})(z-\breve{z}) \\
&= -m\beta(x-\breve{x})^2 - n\eta_1(y_1-\breve{y}_1)^2 - p(\vartheta + \phi\frac{x}{z\breve{z}})(z-\breve{z})^2 \\
&+ (m(\alpha+\theta) - p(\kappa + \phi - \frac{\phi}{\breve{z}}))(x-\breve{x})(z-\breve{z}) + (m\gamma_1 + n\delta_1)(x-\breve{x})(y_1-\breve{y}_1) \\
&= -m\beta(x-\breve{x})^2 - n\eta_1(y_1-\breve{y}_1)^2 - p(\vartheta + \phi\frac{x}{z\breve{z}})(z-\breve{z})^2 \\
&< 0 \quad for \quad all \quad (x, y_1, z) \neq (\breve{x}, \breve{y}_1, \breve{z}). \quad \square
\end{aligned}$$

Similarly one can prove the following theorem.

Theorem 2.17. *The equilibrium $F_{xy_2z}(\breve{x}, 0, \breve{y}_2, \breve{z})$ is globally asymptotically stable with respect to solution trajectories emanating from the positive interior of $\mathbf{R}^+_{xy_2z}$ if it exists.*

2.7 Existence of $F^{\star}(x^{\star}, {y_1}^{\star}, {y_2}^{\star}, z^{\star})$

In the previous sections, we have given criteria for the existences of the boundary equilibria and also given criteria for such equilibria to be globally asymptotically stable with respect to solutions emanating on the axis, planes or octants for which such equilibria exist.

In this section however, we shall present results on uniform persistence and also give sufficient conditions for the existence of a positive interior equilibrium $F^{\star}(x^{\star}, {y_1}^{\star}, {y_z}^{\star}, z^{\star})$ for the four dimensional system.

Theorem 2.18. *Assume that the system given by Equations (2.1)-(2.4) is such that*

1. $F_{xz}(\bar{x}, 0, 0, \bar{z})$ *is a hyperbolic saddle point and repelling in the* y_1 *and* y_2*-directions locally (cf. Theorem 2.9)*

2. $F_{xy_1z}(\breve{x}, \breve{y}_1, 0, \breve{z})$ *is a hyperbolic saddle point and repelling in the* y_2*-direction (cf. Theorem 2.10)*

3. $F_{xy_2z}(\breve{x}, 0, \breve{y}_2, \breve{z})$ *is a hyperbolic saddle point repelling in the* y_1*-direction (cf. Theorem 2.12).*

Then the system given by Equations (2.1)-(2.4) exhibits uniform persistence.

Proof:

The proof of this will be done using Theorem 1.9 (Butler and McGehee) as we did in the proof of Theorem 2.3. Let

$\mathbf{A} = \{(x, y_1, y_2, z) : 0 \leq x \leq \alpha/\beta, 0 \leq y_1 \leq (\alpha\delta_1 - \beta\xi_1)/(\beta\eta_1), 0 \leq y_2 \leq (\alpha\delta_2 - \beta\xi_2)/(\beta\eta_2), 0 \leq z \leq 1\} \subset \mathbf{R}^{+}_{xy_1y_2z}$.

We have already shown in Theorem 2.1 that $\mathbf{A}$ is positively invariant and that all solutions initiating in the interior of $\mathbf{R}^+_{xy_1y_2z}$ will eventually enter $\mathbf{A}$ proving that the system given by Equations (2.1)-(2.4) is dissipative.

We have also shown that the only compact invariant sets on the boundary of $\mathbf{R}^+_{xy_1y_2z}$ are $F_0(0,0,0,0)$, $F_z(0,0,0,1)$, $F_{xz}(\bar{x},0,0,\bar{z})$, $F_{xy_2z}(\breve{x},\breve{y}_1,0,\breve{z})$, and $F_{xy_2z}(\breve{x},0,\breve{y}_2,\breve{z})$. We proved in Theorems 2.14 and 2.15 that the equilibria $F_z(0,0,0,1)$ and $F_{xz}(\bar{x},0,0,\bar{z})$ are globally asymptotically stable with respect to solutions initiating in the interior of $\mathbf{R}^+_z$ and $\mathbf{R}^+_{xz}$ respectively and they always exist. We also proved in Theorems 2.16 and 2.17 that the equilibria $F_{xy_2z}(\breve{x},\breve{y}_1,0,\breve{z})$ and $F_{xy_2z}(\breve{x},0,\breve{y}_2,\breve{z})$ are globally asymptotically stable with respect to solutions initiating in the interior of $\mathbf{R}^+_{xy_1z}$ and $\mathbf{R}^+_{xy_2z}$ if only they exist.

The global asymptotic stability of $F_z(0,0,0,1)$ and $F_{xz}(\bar{x},0,0,\bar{z})$ with respect to solutions initiating in the interior of $\mathbf{R}^+_z$ and $\mathbf{R}^+_{xz}$ respectively, together with the fact that $F_{xy_2z}(\breve{x},\breve{y}_1,0,\breve{z})$ and $F_{xy_2z}(\breve{x},0,\breve{y}_2,\breve{z})$ are globally asymptotically stable with respect to solutions initiating in the interior of $\mathbf{R}^+_{xy_1z}$ and $\mathbf{R}^+_{xy_2z}$ respectively, implies that the boundary flow of the system is acyclic and isolated.

Let $F^\star = (x^\star, {y_1}^\star, {y_2}^\star, z^\star)$ with $x^\star > 0, {y_1}^\star > 0, {y_2}^\star > 0$ and $z^\star > 0$ be a point in the interior of $\mathbf{R}^+_{xy_1y_2z}$ and $O(F^\star)$ be the orbit through the point $F^\star$. The proof of the above theorem is completed by showing the following:

1. $F_0(0,0,0,0) \notin \Omega(F^\star)$.

2. $F_z(0,0,0,1) \notin \Omega(F^\star)$.

3. $F_{xz}(\bar{x},0,0,\bar{z}) \notin \Omega(F^\star)$.

4. $F_{xy_1z}(\breve{x},\breve{y}_1,0,\breve{z}) \notin \Omega(F^\star)$.

5. $F_{xy_2z}(\breve{x}, 0, \breve{y}_2, \breve{z}) \notin \Omega(F^\star)$.

We note that, since the system given by Equations (2.1)-(2.4) is dissipative, $\Omega(F^\star)$ is bounded.

Suppose $F_0 = F_0(0,0,0,0) \in \Omega(F^\star)$. Since F_0 is a hyperbolic saddle point (Theorem 2.6), $F_0(0,0,0,0) \neq \Omega(F^\star)$. Hence by Theorem 1.9, there exists at least one point $F_0^s \in \Omega(F^\star) \cap W^s(F_0) \setminus \{F_0\}$. But $W^s(F_0)$ is the positive $(x, y_1, y_2)-$ octant, and all orbits in the positive $(x, y_1, y_2) \setminus \{F_0\}$ - octant are unbounded, giving a contradiction, since all orbits in $\Omega(F^\star)$ are bounded. Thus $F_0 \notin \Omega(F^\star)$.

Now suppose $F_z = F_z(0,0,0,1) \in \Omega(F^\star)$. Since F_z is a hyperbolic saddle point (Theorem 2.7), there exists $F_z^s \in \Omega(F^\star) \cap W^s(F_z) \setminus \{F_z\}$. But $W^s(F_z)$ is the positive $(y_1, y_2, z)-$ octant, and F_z is globally asymptotically stable with respect to solutions initiating in the interior of $\mathbf{R}_z^+$. This implies that the closure of the orbit through F_z^s either contains F_0 or is unbounded (since the closure of all orbits in the positive $(y_1, y_2, z) \setminus \{F_z\}-$ octant either contain F_0 or are unbounded). This is a contradiction. Hence $F_z \notin \Omega(F^\star)$.

Now assume $F_{xz} = F_{xz}(\bar{x}, 0, 0, \bar{z}) \in \Omega(F^\star)$. Then there exists $F_{xz}^s \in \Omega(F^\star) \cap W^s(F_{xz}) \setminus \{F_{xz}\}$. By assumption, F_{xz} is a hyperbolic saddle point and repelling in the y_1 and y_2-directions locally. Hence $W^s(F_{xz})$ is the (x, z)-plane. But F_{xz} is globally asymptotically stable with respect to solutions initiating in the interior of $\mathbf{R}_{xz}^+$. Thus the closure of all orbits in the positive $(x, z) \setminus \{F_{xz}\}$-plane either contains either F_0 or F_z or are unbounded. This implies either $\Omega(F^\star)$ is unbounded or either $F_z \in \Omega(F^\star)$ or $F_0 \in \Omega(F^\star)$, all of which are contradictions.

Similarly, if $F_{xy_1z} = F_{xy_1z}(\breve{x}, \breve{y}_1, 0, \breve{z}) \in \Omega(F^\star)$, then there exists a point $F_{xy_1z}^s \in \Omega(F^\star) \cap W^s(F_{xy_1z}) \setminus \{F_{xy_1z}\}$ by Theorem 1.9. But F_{xy_1z} is globally asymptotically stable with respect to solutions initiating in the interior of $\mathbf{R}_{xy_1z}^+$. Thus the closure of every orbit in the positive $(x, y_1, z) \setminus \{F_{xy_1z}\}$-octant either contains one of F_0, F_z and F_{xz} or is unbounded. This implies either $\Omega(F^\star)$ is unbounded or contains one of F_0, F_z and F_{xz}, all of which are contradictions.

Similarly, one can show that $F_{xy_2z} = F_{xy_2z}(\breve{x}, 0, \breve{y}_2, \breve{z}) \notin \Omega(F^\star)$.

Finally, if there exists a point $F \in \Omega(F^\star)$ such that F is on the boundary of $\mathbf{R}^+_{xy_1y_2z}$, then the closure of the orbit through F must either contain F_0, F_z, F_{xz}, F_{xy_1z}, F_{xy_2z} or be unbounded. This gives a contradiction.

This shows that the system is uniformly persistent. □

Theorem 2.19. *If the conditions of Theorem 2.18 are satisfied, then the system given by Equations (2.1)-(2.4) exhibits uniform persistence and contains an equilibrium of the form* $F^\star = (x^\star, {y_1}^\star, {y_2}^\star, z^\star)$ *with* $x^\star > 0, {y_1}^\star > 0, {y_2}^\star > 0$ *and* $z^\star > 0$.

Proof:

The existence of an equilibrium of the form $F^\star = (x^\star, {y_1}^\star, {y_2}^\star, z^\star)$ with $x^\star > 0$, ${y_1}^\star > 0, {y_2}^\star > 0$ and $z^\star > 0$, is a direct consequence of Theorem 2.18. □

2.8 Criteria for extinction of both industrial assets

In this section, necessary and sufficient criteria for total extinction or elimination of both industries will be given. Even though this work is aimed at determining conditions for sustainability of the entire system consisting of agriculture, industry one, industry two and the ecosphere, knowing all the necessary and sufficient conditions for total extinction of both industries can help avoid such extinction.

In order to obtain the conditions for the total extinction of both industries, we use a Liapunov function to establish criteria for the global asymptotic stability for the equilibrium $F_{xz}(\bar{x}, 0, 0, \bar{z})$ with respect to $\mathbf{R}^+_{xy_1y_2z}$.

Let m and n be positive constants defined by $m = \frac{\vartheta(1-\bar{z})}{\bar{x}}$ and $n = \theta + \alpha$. Let $p = \frac{-m\gamma_1}{\delta_1}$ and $q = \frac{-m\gamma_2}{\delta_2}$. In $\mathbf{R}^+_{xy_1y_2z}$, we choose a Liapunov function $V(x, y_1, y_2, z)$ defined by

$$V(x, y_1, y_2, z) = m\{x - \bar{x} - \bar{x}\ln(\frac{x}{\bar{x}})\} + n\{z - \bar{z} - \bar{z}\ln(\frac{z}{\bar{z}})\} + py_1 + qy_2. \tag{2.50}$$

The derivative of (2.50) along the solution curves of system (2.1)-(2.4) in $\mathbf{R}^+_{xy_1y_2z}$ is given by

$$\begin{aligned}
\dot{V} &= \frac{m\dot{x}(x-\bar{x})}{x} + \frac{n\dot{z}(z-\bar{x})}{z} + p\dot{y_1} + q\dot{y_2} \\
&= m(\alpha z - \beta x + \gamma_1 y_1 + \gamma_2 y_2 - \theta(1-z))(x-\bar{x}) + p(-\xi_1 - \eta_1 y_1 - \rho_1 y_2 + \delta_1 x)y_1 \\
&+ n(-\kappa x + \vartheta(1-z) - \phi x)(z-\bar{z}) + n\phi x(z-\bar{z})/z + q(-\xi_2 - \eta_2 y_2 - \rho_2 y_1 + \delta_2 x)y_2 \\
&= m(\alpha(z-\bar{z}) - \beta(x-\bar{x}) + \theta(z-\bar{z}))(x-\bar{x}) + n(-\kappa(x-\bar{x}) - \vartheta(z-\bar{z}) \\
&- \phi(x-\bar{x}))(z-\bar{z}) + n\phi(\frac{x}{z} - \frac{\bar{x}}{\bar{z}})(z-\bar{z}) + (x-\bar{x})(m\gamma_1 y_1 + m\gamma_2 y_2 + p\delta_1 y_1 + q\delta_2 y_2) \\
&+ p(-\xi_1 - \eta_1 y_1 - \rho_1 y_2 + \delta_1\bar{x})y_1 + q(-\xi_2 - \eta_2 y_2 - \rho_2 y_1 + \delta_2\bar{x})y_2 \\
&= -m\beta(x-\bar{x})^2 - n(\vartheta + \phi\frac{x}{z\bar{z}})(z-\bar{z})^2 + p(-\xi_1 - \eta_1 y_1 - \rho_1 y_2 + \delta_1\bar{x})y_1 \\
&+ (m(\alpha+\theta) - n(\kappa - \phi + \frac{\phi}{\bar{z}}))(x-\bar{x})(z-\bar{z}) + q(-\xi_2 - \eta_2 y_2 - \rho_2 y_1 + \delta_2\bar{x})y_2 \\
&= -m\beta(x-\bar{x})^2 - n(\vartheta + \phi\frac{x}{z\bar{z}})(z-\bar{z})^2 + p(-\xi_1 - \eta_1 y_1 - \rho_1 y_2 + \delta_1\bar{x})y_1 \\
&+ (m(\alpha+\theta) - \frac{n\vartheta(1-\bar{z})}{\bar{x}})(x-\bar{x})(z-\bar{z}) + q(-\xi_2 - \eta_2 y_2 - \rho_2 y_1 + \delta_2\bar{x})y_2 \\
&= -m\beta(x-\bar{x})^2 - n(\vartheta + \phi\frac{x}{z\bar{z}})(z-\bar{z})^2 + p(-\xi_1 - \eta_1 y_1 - \rho_1 y_2 + \delta_1\bar{x})y_1 \\
&+ q(-\xi_2 - \eta_2 y_2 - \rho_2 y_1 + \delta_2\bar{x})y_2.
\end{aligned}$$

We observe from the above that if $-\xi_1 + \delta_1\bar{x} < 0$, $-\xi_2 + \delta_2\bar{x} < 0$ and $(x, y_1, y_2, z) \neq (\bar{x}, 0, 0, \bar{z})$, then $\dot{V} < 0$. Hence we have the following theorem:

Theorem 2.20. *The equilibrium $F_{xz}(\bar{x}, 0, 0, \bar{z})$ is globally asymptotically stable with respect to $\mathbf{R}^+_{xy_1y_2z}$ if and only if $F_{xz}(\bar{x}, 0, 0, \bar{z})$ is locally asymptotically stable with respect to $\mathbf{R}^+_{xy_1y_2z}$.*

2.9 Numerical examples

In this section we illustrate some of our results with numerical examples. All the values of parameters in our examples are chosen for numerical convenience and do not represent any actual agricultural-industrial-ecospheric system.

2.9.1 Example 1: Both industries go extinct (Theorem 2.20)

In example 1, we set

$$\alpha = \vartheta = 2, \quad \beta = 1/10, \quad \theta = 6/5, \quad \phi = 1,$$

$$\kappa = 1/10, \quad \gamma_1 = \gamma_2 = -1/10, \quad \xi_1 = \xi_2 = 2 \quad \rho_1 = \rho_2 = 1/50,$$

$$\eta_1 = \eta_2 = 1/100, \quad \delta_1 = \delta_2 = 1/10.$$

The initial conditions are

$$x(0) = 3, \quad y_1(0) = 10, \quad y_2(0) = 5, \quad z(0) = 0.3.$$

This example represents the case where the constant depreciation rate of each industrial asset is greater than its maximum growth rate. In other words, condition (2.5) is violated and hence each industry goes extinct (see Figure 2.1).

2.9.2 Example 2: One industrial asset goes extinct

In this example the parameters are

$$\alpha = \vartheta = 2, \quad \beta = 1/10, \quad \theta = 6/5, \quad \phi = 1, \quad \rho_1 = \rho_2 = 1/50, \quad \eta_1 = 1/98$$

$$\eta_2 = 1/100, \quad \kappa = 1/10, \quad \gamma_1 = \gamma_2 = -1/10, \quad \xi_1 = \xi_2 = 2 \quad \delta_1 = \delta_2 = 1/4.$$

The initial conditions are $x(0) = 3, \quad y_1(0) = 10, \quad y_2(0) = 10, \quad z(0) = 0.2$. In this example we have chosen the parameters such that conditions for Theorem 2.4 and Theorem 2.5 are satisfied. Hence each subsystem in $\mathbf{R}^+_{xy_1z}$ and $\mathbf{R}^+_{xy_1z}$ exhibits uniform persistence. However conditions 2 and 3 of Theorem 2.18 are violated and the system is not persistent in $\mathbf{R}^+_{xy_1y_2z}$. In $\mathbf{R}^+_{xy_1y_2z}$ the industry with the slightest initial (competitive) advantage outcompetes the other thereby driving it to extinction. This is illustrated in Figures 2.2.

In Figure 2.2, industry two has the same properties as industry one except that it does not depreciate as fast as industry one. As a result of this, industry two drives industry one to extinction in the competition. This example shows that although the subsystems in $\mathbf{R}^+_{xy_1z}$ and $\mathbf{R}^+_{xy_2z}$ exhibit uniform persistence, the system itself is not persistent in $\mathbf{R}^+_{xy_1y_2z}$ because of the competition between the two industries.

2.9.3 Example 3: Persistence

In this example, we choose our parameters such that the system exhibits uniform persistence (i.e. we choose the parameters such that all the conditions of Theorem 2.18 are satisfied). In this example we set

$$\alpha = 2, \quad \vartheta = 2, \quad \beta = 1/10, \quad \theta = 6/5, \quad \phi = 1,$$

$$\kappa = 1/10, \quad \gamma_1 = \gamma_2 = -1/10, \quad \xi_1 = 1, \quad \xi_2 = 5/2,$$

$$\delta_1 = 1/4, \quad \delta_2 = 1/2, \quad \eta_1 = \eta_2 = 1/25 \quad \rho_1 = 1/50, \quad \rho_2 = 1/100.$$

Here the possible equilibria for the system are $F_0(0,0,0,0)$, $F_{xz}(17.3459, 0, 0, 0.9571)$, $F_z(0,0,0,1)$, $F_{xy_1z}(5.8950, 11.8440, 0, 0.9294)$, $F_{xy_2z}(5.9432, 0, 11.7915, 0.9293)$ and $F_{xy_1y_1z}(5.5490, 7.1429, 5.0786, 0.9304)$. All the conditions of Theorem 1.18 are satisfied and hence the system exhibits uniform persistence (see Figure 2.3).

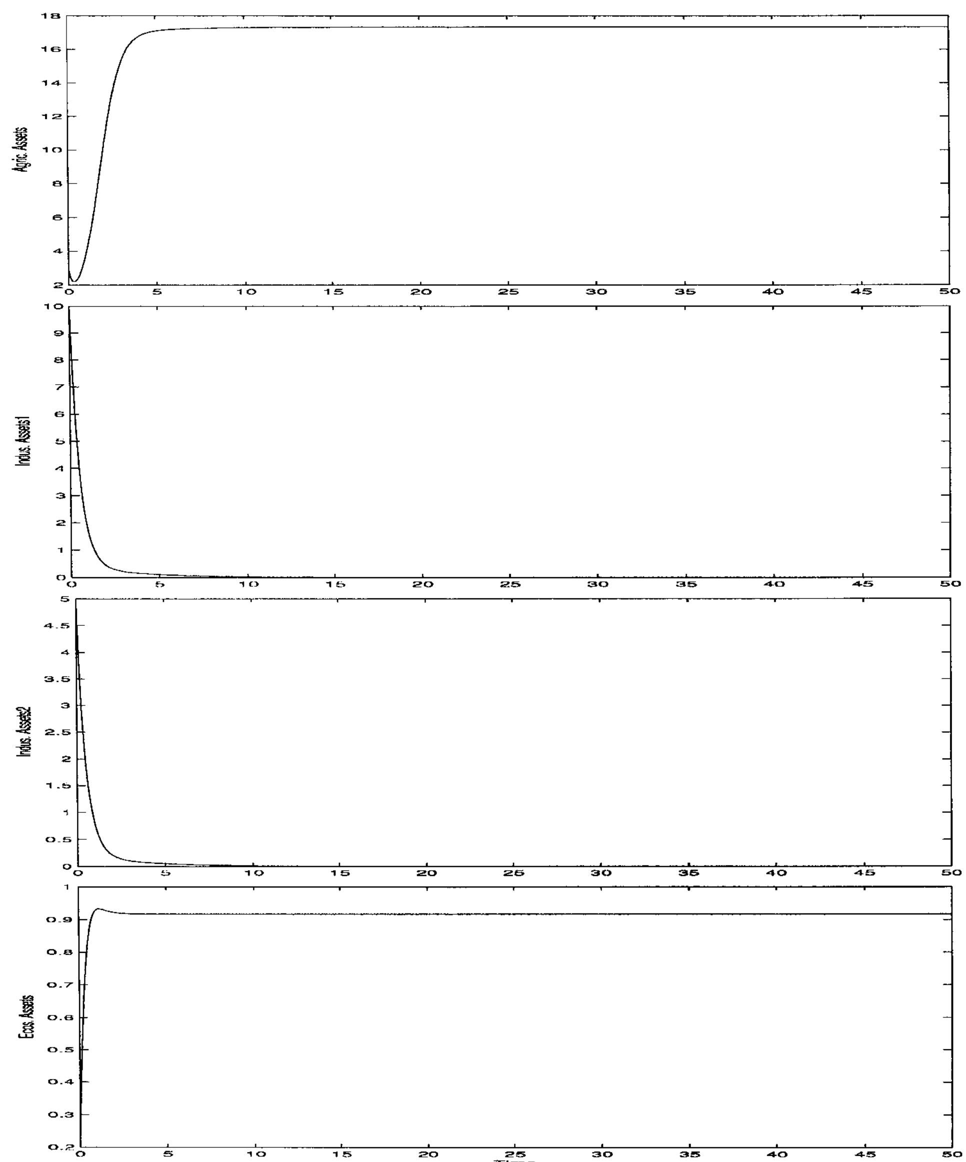

Figure 2.1: Constant depreciation rate of each industry is greater than its maximum growth rate. Both industries go extinct.

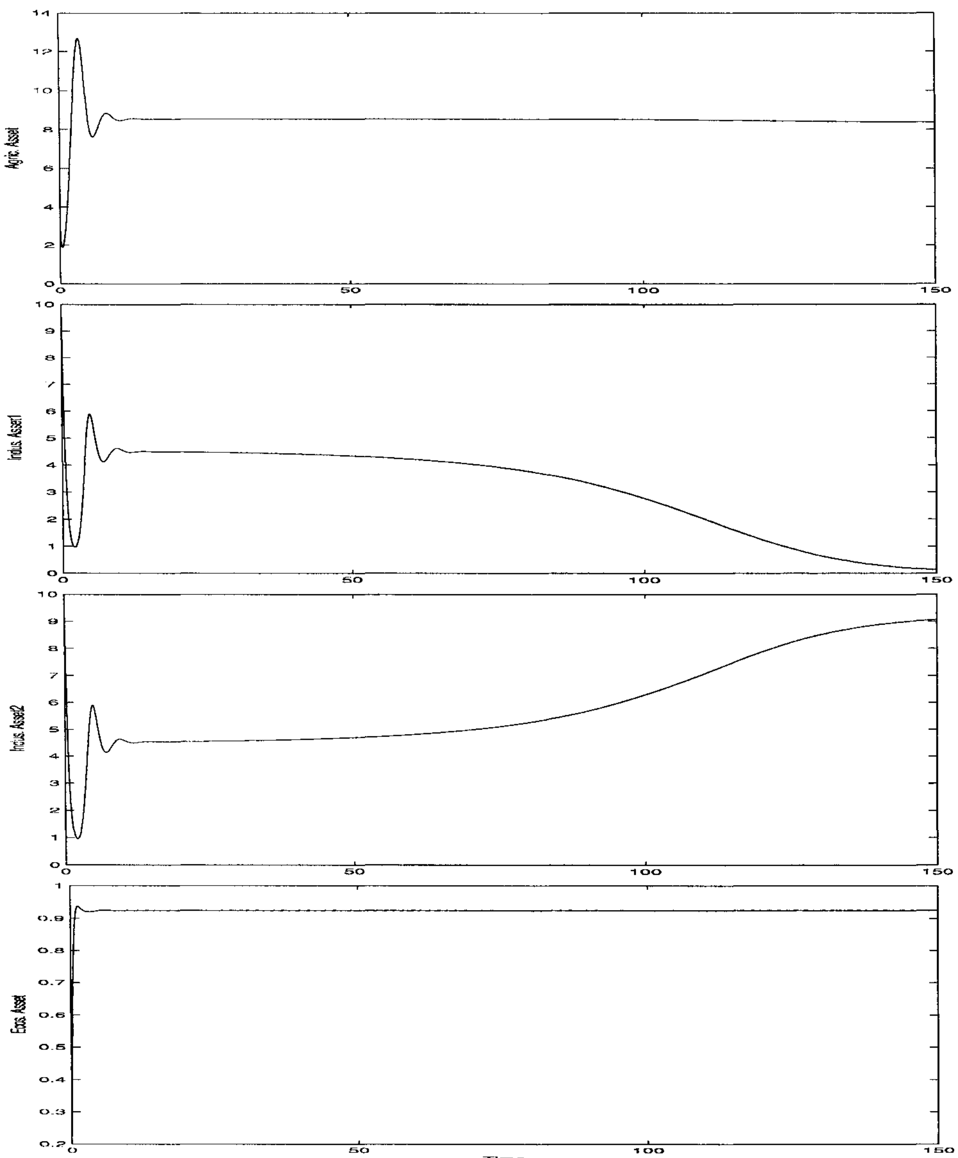

Figure 2.2: Each subsystem in $\mathbf{R}^+_{xy_1z}$ and $\mathbf{R}^+_{xy_2z}$ exhibits uniform persistence. Linear depreciation rate of Industry one is slightly greater than that of Industry two. Industry one is driven extinct.

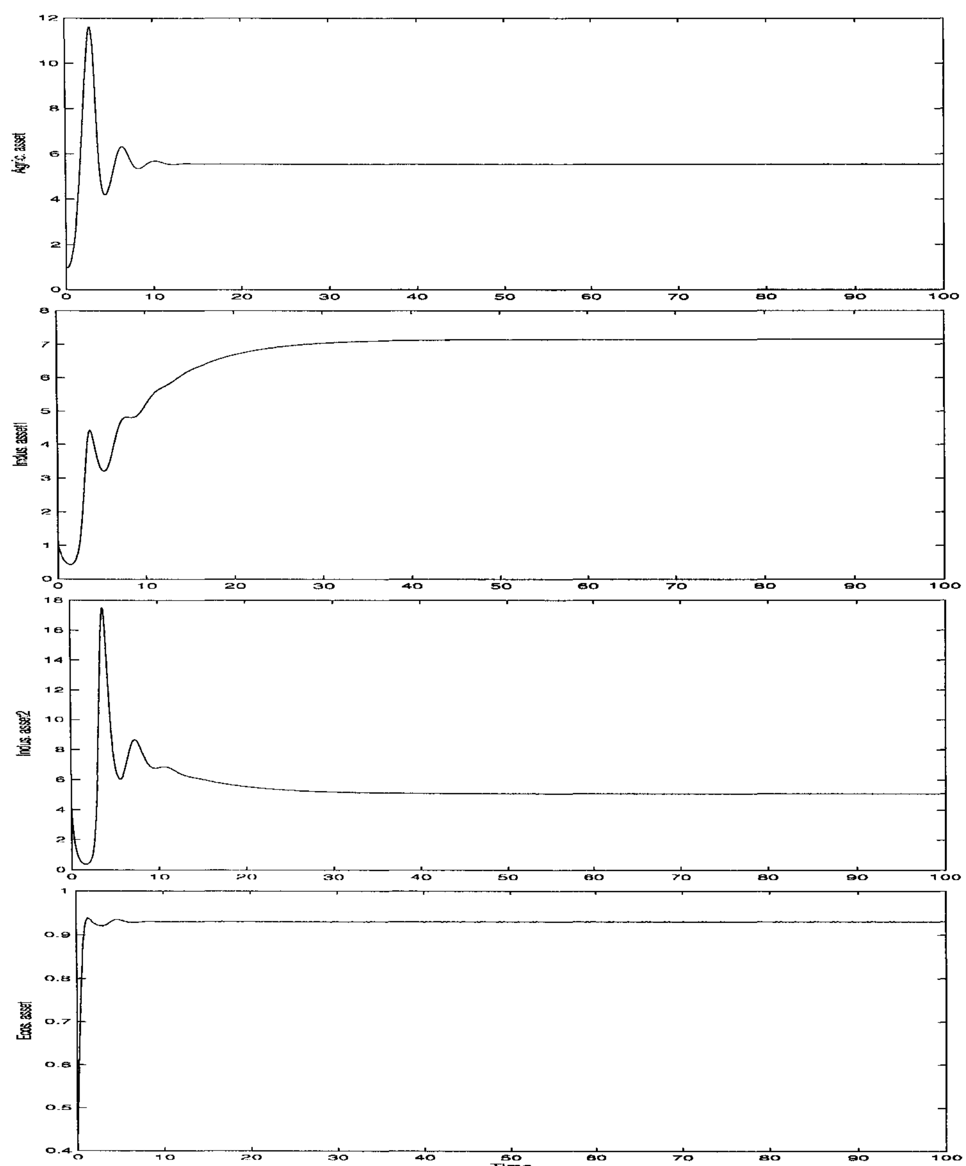

Figure 2.3: The system exhibits uniform persistence.

2.10 $\gamma_2 \leq 0$ and $\gamma_1 > 0$ with $\beta\eta_1 - \gamma_1\delta_1 > 0$

In this section, we consider system (2.1)- (2.4) with $\gamma_1 > 0$, $\gamma_2 \leq 0$, and $\beta\eta_1 - \gamma_1\delta_1 > 0$. We will determine all the equilibria and discuss their stabilities and illustrate some of our results with numerical examples.

Theorem 2.21. *If $\gamma_1 > 0$ and $\gamma_2 \leq 0$ with $\beta\eta_1 - \gamma_1\delta_1 > 0$, then the system given by Equations (2.1)-(2.4) is dissipative with attraction region contained in* $\mathbf{A} = \{(x, y_1, y_2, z) : 0 \leq x \leq M, 0 \leq y_1 \leq (\alpha\delta_1 - \beta\xi_1)/(\beta\eta_1 - \gamma_1\delta_1), 0 \leq y_2 \leq (M\delta_2 - \xi_2)/(\eta_2), 0 \leq z \leq 1\}$, *where* $M = \frac{\alpha\eta_1 - \gamma_1\xi_1}{\beta\eta_1 - \gamma_1\delta_1}$, *provided* $\alpha\eta_1 - \gamma_1\xi_1 > 0$, $\alpha\delta_1 - \beta\xi_1 > 0$ *and* $M > \frac{\xi_2}{\delta_2}$.

Proof:

We proved in Theorem 2.1 that $0 \leq z \leq 1$.

From Equations (2.1) and (2.2), we have

$$\begin{aligned} \dot{x} &\leq (\alpha - \beta x + \gamma_1 y_1)x \\ \dot{y_1} &\leq (-\xi_1 - \eta_1 y_1 + \delta_1 x)y_1. \end{aligned} \tag{2.51}$$

Compare this with

$$\begin{aligned} \dot{u} &= (\alpha - \beta u + \gamma_1 v)u \\ \dot{v} &= (-\xi_1 - \eta_1 v + \delta_1 u)v. \end{aligned} \tag{2.52}$$

We note that under the conditions of this theorem, system (2.52) has only three equilibria given by $E_0(0,0)$, $E_u(\alpha/\beta, 0)$ and $E_{uv}(u^\star, v^\star) = E_{uv}(\frac{\alpha\eta_1 - \gamma_1\xi_1}{\beta\eta_1 - \gamma_1\delta_1}, \frac{\alpha\delta_1 - \beta\xi_1}{\beta\eta_1 - \gamma_1\delta_1})$.

The variational matrix V for system (2.52) about the equilibrium $E_0(0,0)$ is given by

$$V_{E_0} = \begin{bmatrix} \alpha & 0 \\ 0 & -\xi_1 \end{bmatrix}. \tag{2.53}$$

The eigenvalues of V_{E_0} are given by $\alpha > 0$ and $-\xi_1 < 0$. Thus the equilibrium $E_0(0,0)$ is a hyperbolic saddle point, locally unstable in the u-direction and locally stable in the v-direction.

Similarly the variational matrix V for system (2.52) about the equilibrium $E_u(\alpha/\beta, 0)$ is given by

$$V_{E_u} = \begin{bmatrix} -\alpha & \frac{\alpha\gamma_1}{\beta} \\ 0 & \frac{\alpha\delta_1 - \beta\xi_1}{\beta} \end{bmatrix}. \tag{2.54}$$

The eigenvalues of V_{E_u} are given by $-\alpha < 0$ and $\frac{\alpha\delta_1 - \beta\xi_1}{\beta} > 0$. Thus the equilibrium $E_u(\alpha/\beta, 0)$ is a hyperbolic saddle point, locally stable in the u-direction and locally unstable in the v-direction.

Also the variational matrix V for system (2.52) about the equilibrium $E_{uv}(u^\star, v^\star)$ is given by

$$V_{E_{uv}} = \begin{bmatrix} -\beta u^\star & \gamma_1 u^\star \\ \delta_1 v^\star & -\eta_1 v^\star \end{bmatrix}. \tag{2.55}$$

We observe that

$$trace(V_{Euv}) = -\beta u^\star - \eta_1 v^\star < 0$$

and

$$det(V_{Euv}) = (\beta\eta_1 - \gamma_1\delta_1)u^\star v^\star > 0.$$

Thus both eigenvalues of V_{Euv} have negative real parts. Hence the equilibrium $E_{uv}(u^\star, v^\star)$ is locally asymptotically stable.

Now consider

$$\begin{aligned}
\mathbf{D}(u,v) &= \frac{\partial}{\partial u}(\frac{\dot{u}}{uv}) + \frac{\partial}{\partial v}(\frac{\dot{v}}{uv}) \\
&= \frac{\partial}{\partial u}(\frac{\alpha - \beta u + \gamma_1 v}{v}) + \frac{\partial}{\partial v}(\frac{-\xi_1 - \eta_1 v + \delta_1 u}{u}) \\
&= -(\frac{\beta}{v} + \frac{\eta_1}{u}) \\
&< 0.
\end{aligned}$$

This shows that there are no closed curves, orbits or paths in the (u, v)-plane. Therefore the only locally asymptotically stable equilibrium in the $(u, v)-$plane has to be a globally stable equilibrium. Thus all solutions initiating in the positive interior of the $(u, v)-$plane will eventually approach $(u^\star, v^\star)$. That is

$$\lim_{t\to\infty}(u(t), v(t)) = (u^\star, v^\star).$$

Hence system (2.52) is dissipative.

But if $x(0) = u(0)$ and $y_1(0) = v(0)$, then for all $t \geq 0$, we have $x(t) \leq u(t)$ and $y_1(t) \leq v(t)$. Thus

$$0 \leq \limsup_{t\to\infty} x(t) \leq u^\star = \frac{\alpha\eta_1 - \gamma_1\xi_1}{\beta\eta_1 - \gamma_1\delta_1}$$

and

$$0 \leq \limsup_{t\to\infty} y_1(t) \leq v^\star = \frac{\alpha\delta_1 - \beta\xi_1}{\beta\eta_1 - \gamma_1\delta_1}.$$

The proof of

$$0 \leq \limsup_{t\to\infty} y_2(t) \leq (M\delta_2 - \xi_2)/(\eta_2)$$

is similar to the proof in Theorem 2.1. □

Alternatively, one could also prove the global stability of equilibrium $E_{uv}(u^\star, v^\star)$ by considering the following Liapunov function V(u,v) defined by

$$V(u,v) = m\{u - u^\star - u^\star(\ln(\frac{u}{u^\star})\} + n\{v - v^\star - v^\star(\ln(\frac{v}{v^\star})\}, \tag{2.56}$$

where $m = \frac{\delta^2}{\beta}$, $n = \frac{b^2}{\eta_1}$ and $b = \frac{\beta\eta_1 + \sqrt{\beta\eta_1(\beta\eta_1 - \gamma_1\delta_1)}}{\beta}$. The derivative of (2.56) along the solution curves of (2.52) is given by

$$\begin{aligned}
\dot{V} &= \frac{m\dot{u}(u-u^\star)}{u} + \frac{n\dot{v}(v-v^\star)}{v} \\
&= -m\beta(u-u^\star)^2 - n\eta_1(v-v^\star)^2 + (m\gamma_1 + n\delta_1)(u-u^\star)(v-v^\star) \\
&= -\delta^2(u-u^\star)^2 - b^2(v-v^\star)^2 + (m\gamma_1 + n\delta_1)(u-u^\star)(v-v^\star) \\
&= -(\delta(u-u^\star) - b(v-v^\star))^2 \\
&\leq 0.
\end{aligned}$$

2.10.1 Equilibria for the system with $\gamma_2 \leq 0$, $\gamma_1 > 0$, $\beta\eta_1 - \gamma_1\delta_1 > 0$

Clearly $F_0(0,0,0,0)$ and $F_z(0,0,0,1)$ are equilibria. $F_z(0,0,0,1)$ is the only (axial) one-dimensional equilibrium (see section 2.4).

Also $F_{xz}(\bar{x},0,0,\bar{z})$ is the only (planar) two-dimensional equilibrium for the system (see Section 2.4.1) and it always exists.

Also there exists a unique three-dimensional equilibrium $F_{xy_2z}(\check{x},0,\check{y}_2,\check{z})$ in the positive $(x,y_2,z)-$ octant if $-\xi_2 + \delta_2\bar{x} > 0$ (see Theorem 2.5).

However, in the positive interior of the (x,y_1,z)-octant, we have $x > 0$, $y_1 > 0$, $y_2 = 0$, and $z > 0$. So the algebraic system (2.6)-(2.9) reduces to

$$\alpha z - \beta x + \gamma_1 y_1 - \theta(1-z) = 0 \tag{2.57}$$

$$-\xi_1 - \eta_1 y_1 + \delta_1 x = 0 \tag{2.58}$$

$$-\kappa xz + \vartheta(1-z)z + \phi(1-z)x = 0. \tag{2.59}$$

Solving for y_1 from Equation (2.58), we get

$$y_1 = \check{y}_1 = \frac{-\xi_1 + \delta_1\check{x}}{\eta_1}, \tag{2.60}$$

which exists if

$$\breve{x} > \frac{\xi_1}{\delta_1}. \tag{2.61}$$

Substituting Equation (2.60) into (2.57) we get

$$(\alpha\eta_1 + \theta\eta_1)\breve{z} + (\gamma_1\delta_1 - \beta\eta_1)\breve{x} - \gamma_1\xi_1 - \theta\eta_1 = 0, \tag{2.62}$$

Using Equation (2.62) and solving for $\breve{x}$ we get

$$x = \breve{x} = \frac{(\alpha\eta_1 + \theta\eta_1)\breve{z} - \gamma_1\xi_1 - \theta\eta_1}{(\beta\eta_1 - \gamma_1\delta_1)}. \tag{2.63}$$

However for existence (Equation (2.61)), we require

$$\breve{x} = \frac{(\alpha\eta_1 + \theta\eta_1)\breve{z} - \gamma_1\xi_1 - \theta\eta_1}{(\beta\eta_1 - \gamma_1\delta_1)} > \frac{\xi_1}{\delta_1},$$

which is trivially satisfied if

$$\breve{z} > \frac{\beta\xi_1 + \theta\delta_1}{\alpha\delta_1 + \theta\delta_1}. \tag{2.64}$$

But, since $\breve{z} \leq 1$, from Equation (2.64) we require

$$\beta\xi_1 < \delta_1\alpha. \tag{2.65}$$

Substituting Equation (2.63) into (2.59) and simplifying, we get

$$a\breve{z}^2 - b\breve{z} + c = 0, \tag{2.66}$$

where

$$a = (\alpha\eta_1 + \theta\eta_1)(\kappa + \phi) + \vartheta(\beta\eta_1 - \gamma_1\delta_1) > 0$$

$$b = \vartheta(\beta_1\eta_1 - \gamma_1\delta_1) + (\kappa + \phi)(\theta\eta_1 + \gamma_1\xi_1) + \phi(\alpha\eta_1 + \theta\eta_1) > 0$$

$$c = \phi(\theta\eta_1 + \gamma_1\xi_1) > 0.$$

We solve (2.66) for $\breve{z}$ to get

$$z = \breve{z} = \frac{b \pm \sqrt{b^2 - 4ac}}{2a}. \tag{2.67}$$

Theorem 2.22. *There exists at most one equilibrium lying completely in the positive interior of the (x, y_1, z)-octant if $\beta_1\eta_1 - \gamma_1\delta_1 > 0$.*

Proof:

Clearly, since a, b and c are all positive, $b^2 - 4ac < b^2$. We also have

$$\begin{aligned}
b^2 - 4ac &= \{\vartheta(\beta_1\eta_1 - \gamma_1\delta_1) + (\kappa - \phi)(\theta\eta_1 + \gamma_1\xi_1) + \phi(\alpha\eta_1 + \theta\eta_1)\}^2 \\
&+ 4\phi\kappa(\theta\eta_1 + \gamma_1\xi_1)(\gamma_1\xi_1 - \alpha\eta_1) \\
&= \{\vartheta(\beta_1\eta_1 - \gamma_1\delta_1) + \kappa(\theta\eta_1 + \gamma_1\xi_1) + \phi(\alpha\eta_1 - \gamma_1\xi_1)\}^2 \\
&+ 4\phi\kappa(\theta\eta_1 + \gamma_1\xi_1)(\gamma_1\xi_1 - \alpha\eta_1) \\
&= \{\vartheta(\beta_1\eta_1 - \gamma_1\delta_1)\}^2 + \{\kappa(\theta\eta_1 + \gamma_1\xi_1) - \phi(\alpha\eta_1 - \gamma_1\xi_1)\}^2 \\
&+ 2\vartheta(\beta_1\eta_1 - \gamma_1\delta_1)\{\kappa(\theta\eta_1 + \gamma_1\xi_1) + \phi(\alpha\eta_1 - \gamma_1\xi_1)\} \\
&= \{\vartheta(\beta_1\eta_1 - \gamma_1\delta_1) - \kappa(\theta\eta_1 + \gamma_1\xi_1) + \phi(\alpha\eta_1 - \gamma_1\xi_1)\}^2 \\
&+ 4\kappa\vartheta(\beta_1\eta_1 - \gamma_1\delta_1)(\theta\eta_1 + \gamma_1\xi_1) \\
&> 0.
\end{aligned}$$

Since $b^2 - 4ac > 0$ and $b^2 - 4ac < b^2$, $\breve{z}$ has two positive roots with the smallest root given by

$$
\begin{aligned}
\check{z}_- &= \frac{b-\sqrt{b^2-4ac}}{2a} \\
&= \frac{\vartheta(\beta_1\eta_1-\gamma_1\delta_1)+(\kappa+\phi)(\theta\eta_1+\gamma_1\xi_1)+\phi(\alpha\eta_1+\theta\eta_1)}{2\{(\alpha\eta_1+\theta\eta_1)(\kappa+\phi)+\vartheta(\beta\eta_1-\gamma_1\delta_1)\}} \\
&- \frac{\sqrt{\{\vartheta(\beta_1\eta_1-\gamma_1\delta_1)-\kappa(\theta\eta_1+\gamma_1\xi_1)+\phi(\alpha\eta_1-\gamma_1\xi_1)\}^2+4\kappa\vartheta(\beta_1\eta_1-\gamma_1\delta_1)(\theta\eta_1+\gamma_1\xi_1)}}{2\{(\alpha\eta_1+\theta\eta_1)(\kappa+\phi)+\vartheta(\beta\eta_1-\gamma_1\delta_1)\}} \\
&< \frac{\vartheta(\beta_1\eta_1-\gamma_1\delta_1)+(\kappa+\phi)(\theta\eta_1+\gamma_1\xi_1)+\phi(\alpha\eta_1+\theta\eta_1)}{2\{(\alpha\eta_1+\theta\eta_1)(\kappa+\phi)+\vartheta(\beta\eta_1-\gamma_1\delta_1)\}} \\
&- \frac{\sqrt{\{\vartheta(\beta_1\eta_1-\gamma_1\delta_1)-\kappa(\theta\eta_1+\gamma_1\xi_1)+\phi(\alpha\eta_1-\gamma_1\xi_1)\}^2}}{2\{(\alpha\eta_1+\theta\eta_1)(\kappa+\phi)+\vartheta(\beta\eta_1-\gamma_1\delta_1)\}} \\
&= \frac{(\kappa+\phi)(\theta\eta_1+\gamma_1\xi_1)}{(\alpha\eta_1+\theta\eta_1)(\kappa+\phi)+\vartheta(\beta\eta_1-\gamma_1\delta_1)} \\
&< \frac{(\kappa+\phi)\theta\eta_1}{(\alpha\eta_1+\theta\eta_1)(\kappa+\phi)+\vartheta(\beta\eta_1-\gamma_1\delta_1)}.
\end{aligned}
$$

Now suppose $\check{z}_- > \frac{\beta\xi_1+\theta\delta_1}{\alpha\delta_1+\theta\delta_1}$, then at least we must have

$$\frac{\beta\xi_1+\theta\delta_1}{\alpha\delta_1+\theta\delta_1} < \frac{(\kappa+\phi)\theta\eta_1}{(\alpha\eta_1+\theta\eta_1)(\kappa+\phi)+\vartheta(\beta\eta_1-\gamma_1\delta_1)},$$

$$\Longrightarrow (\kappa+\phi)\theta\eta_1(\alpha\delta_1+\theta\delta_1) > (\beta\xi_1+\theta\delta_1)\{(\alpha\eta_1+\theta\eta_1)(\kappa+\phi)+\vartheta(\beta\eta_1-\gamma_1\delta_1)\},$$

$$\Longrightarrow (\kappa+\phi)(\alpha+\theta)\eta_1\{\theta\delta_1-(\beta\xi_1+\theta\delta_1)\} > (\beta\xi_1+\theta\delta_1)\vartheta(\beta\eta_1-\gamma_1\delta_1),$$

$$\Longrightarrow -(\kappa+\phi)(\alpha+\theta)\eta_1\beta\xi_1 > (\beta\xi_1+\theta\delta_1)\vartheta(\beta\eta_1-\gamma_1\delta_1).$$

Which is a contradiction. Hence $\check{z}_- < \frac{\beta\xi_1+\theta\delta_1}{\alpha\delta_1+\theta\delta_1}$ is not an admissible equilibrium. □

In Theorem 2.22, we proved that there may or may not be a three-dimensional equilibrium in the positive interior of the $(x, y_1, z)-$ octant. In the next theorem, we give a sufficient condition for the existence of a unique equilibrium in the positive interior of the $(x, y_1, z)-$ octant if $\beta\eta_1-\gamma_1\delta_1>0$.

Theorem 2.23. *There exists a unique three-dimensional equilibrium $F_{xy_1z}(\check{x}, \check{y}_1, 0, \check{z})$ in the positive interior of $(x, y_1, z)-$ octant if*

1. $\beta\eta_1 - \gamma_1\delta_1 > 0$

2. $\beta\xi_1 < \delta_1\alpha$

3. $(\delta_1\alpha - \beta\xi_1)\{\vartheta(\beta\eta_1 - \gamma_1\delta_1) + \phi(\alpha\eta_1 + \theta\eta_1)\} \geq \kappa\eta_1(\theta + \alpha)(\beta\xi_1 + \theta\delta_1)$.

Proof:

From Equation (2.67), the largest of the two roots for $\breve{z}$ is given by

$$\begin{aligned}
\breve{z}_+ &= \frac{b + \sqrt{b^2 - 4ac}}{2a} \\
&= \frac{\vartheta(\beta\eta_1 - \gamma_1\delta_1) + (\kappa + \phi)(\theta\eta_1 + \gamma_1\xi_1) + \phi(\alpha\eta_1 + \theta\eta_1)}{2\{(\alpha\eta_1 + \theta\eta_1)(\kappa + \phi) + \vartheta(\beta\eta_1 - \gamma_1\delta_1)\}} \\
&+ \frac{\sqrt{\{\vartheta(\beta\eta_1 - \gamma_1\delta_1) - \kappa(\theta\eta_1 + \gamma_1\xi_1) + \phi(\alpha\eta_1 - \gamma_1\xi_1)\}^2 + 4\kappa\vartheta(\beta\eta_1 - \gamma_1\delta_1)(\theta\eta_1 + \gamma_1\xi_1)}}{2\{(\alpha\eta_1 + \theta\eta_1)(\kappa + \phi) + \vartheta(\beta\eta_1 - \gamma_1\delta_1)\}} \\
&> \frac{\vartheta(\beta\eta_1 - \gamma_1\delta_1) + (\kappa + \phi)(\theta\eta_1 + \gamma_1\xi_1) + \phi(\alpha\eta_1 + \theta\eta_1)}{2\{(\alpha\eta_1 + \theta\eta_1)(\kappa + \phi) + \vartheta(\beta\eta_1 - \gamma_1\delta_1)\}} \\
&+ \frac{\sqrt{\{\vartheta(\beta\eta_1 - \gamma_1\delta_1) - \kappa(\theta\eta_1 + \gamma_1\xi_1) + \phi(\alpha\eta_1 - \gamma_1\xi_1)\}^2}}{2\{(\alpha\eta_1 + \theta\eta_1)(\kappa + \phi) + \vartheta(\beta\eta_1 - \gamma_1\delta_1)\}} \\
&= \frac{\vartheta(\beta\eta_1 - \gamma_1\delta_1) + \phi\eta_1(\theta + \alpha)}{(\alpha\eta_1 + \theta\eta_1)(\kappa + \phi) + \vartheta(\beta\eta_1 - \gamma_1\delta_1)} \\
&\geq \frac{\beta\xi_1 + \theta\delta_1}{\alpha\delta_1 + \theta\delta_1} \quad \text{if condition 3 is satisfied.} \qquad \square
\end{aligned}$$

2.10.2 Local and global stability analysis

In this subsection, criteria for local and global (asymptotic) stability of the equilibria $F_0(0,0,0,0)$, $F_z(0,0,0,1)$, $F_{xz}(\bar{x},0,0,\bar{z})$, $F_{xy_1z}(\check{x},\check{y}_1,0,\check{z})$ and $F_{xy_2z}(\breve{x},0,\breve{y}_2,\breve{z})$ for the case $\beta\eta_1 - \gamma_1\delta_1 > 0$ are given. Under the condition $\beta\eta_1 - \gamma_1\delta_1 > 0$, the conditions for local asymptotic stability of these equilibria are the same as those given in Theorems 2.6-2.13. One can also easily verify that the global stability conditions given in Theorems 2.14-2.15 for $F_z(0,0,0,1)$ and $F_{xz}(\bar{x},0,0,\bar{z})$ and Theorem 2.17 for

$F_{xy_2z}(\breve{x}, 0, \breve{y}_2, \breve{z})$ also holds if $\beta\eta_1 - \gamma_1\delta_1 > 0$. We now proceed to give the version of Theorem 2.16 in this case for $F_{xy_1z}(\breve{x}, \breve{y}_1, 0, \breve{z})$.

Global stability of $E_{xy_1z}(\breve{x}, \breve{y}_1, \breve{z})$

Criteria for the global stability of $F_{xy_1z}(\breve{x}, \breve{y}_1, 0, \breve{z})$ with respect to solutions initiating in the interior of the $\mathbf{R}^+_{xy_1z} = \{0 \le x \le \infty, 0 \le y_1 \le \infty, 0 \le z \le 1\}$ will be given. In fact, we will show that if the equilibrium $E_{xy_1z}(\breve{x}, \breve{y}_1, \breve{z})$ exists, then it is globally stable. Thus we have the following theorem.

Theorem 2.24. *The equilibrium $F_{xy_1z}(\breve{x}, \breve{y}_1, 0, \breve{z})$ is globally stable with respect to solutions trajectories emanating from the positive interior of $\mathbf{R}^+_{xy_1z}$ if it exists.*

Proof:

In $\mathbf{R}^+_{xy_1z}$, the system of Equations (2.1)- (2.4) reduces to

$$\begin{aligned} \dot{x} &= (\alpha z - \beta x + \gamma_1 y_1 - \theta(1-z))x \\ \dot{y}_1 &= (-\xi_1 - \eta_1 y_1 + \delta_1 x)y_1 \\ \dot{z} &= -\kappa xz + \vartheta(1-z)z + \phi(1-z)x. \end{aligned} \tag{2.68}$$

Let m, n and p be positive constants defined by $m = \kappa + \phi - \frac{\phi}{\breve{z}} = \frac{\vartheta(1-\breve{z})}{\breve{x}}$, $n = \frac{m}{\delta_1^2}\{\beta\eta_1 + (\beta\eta_1 - \gamma_1\delta_1) + 2\sqrt{\beta\eta_1(\beta\eta_1 - \gamma_1\delta_1)}\}$ and $p = \theta + \alpha$.

In $\mathbf{R}^+_{xy_1z}$, we choose a Liapunov function $V(x, y_1, z)$ defined by

$$\begin{aligned} V(x,y_1,z) = m\{x - \breve{x} - \breve{x}(\ln(\frac{x}{\breve{x}})\} + n\{y_1 - \breve{y}_1 - \breve{y}_1(\ln(\frac{y_1}{\breve{y}_1})\} \\ + p\{z - \breve{z} - \breve{z}(\ln(\frac{z}{\breve{z}})\}. \end{aligned} \tag{2.69}$$

The derivative of (2.69) along the solution curves of (2.68) in $\mathbf{R}^+_{xy_1z}$ is given by

$$
\begin{aligned}
\dot{V} &= \frac{m\dot{x}(x-\breve{x})}{x} + \frac{n\dot{y}_1(y_1-\breve{y}_1)}{y_1} + \frac{p\dot{z}(z-\breve{z})}{z} \\
&= m(\alpha z - \beta x + \gamma_1 y_1 - \theta(1-z))(x-\breve{x}) + n(-\xi_1 - \eta_1 y_1 + \delta_1 x)(y_1 - \breve{y}_1) \\
&+ p(-\kappa x + \vartheta(1-z) - \phi x)(z-\breve{z}) + p\phi x(z-\breve{z})/z \\
&= m(\alpha(z-\breve{z}) - \beta(x-\breve{x}) + \gamma_1(y_1-\breve{y}_1) + \theta(z-\breve{z}))(x-\breve{x}) \\
&+ n(-\eta_1(y_1-\breve{y}_1) + \delta_1(x-\breve{x}))(y_1-\breve{y}_1) + p(-\kappa(x-\breve{x}) - \vartheta(z-\breve{z}) - \phi(x-\breve{x}))(z-\breve{z}) \\
&+ m(\alpha\breve{z} - \beta\breve{x} + \gamma_1\breve{y}_1 - \theta(1-\breve{z}))(x-\breve{x}) + n(-\xi_1 - \eta_1\breve{y}_1 + \delta_1\breve{x})(y_1-\breve{y}_1) \\
&+ p(-\kappa\breve{x} + \vartheta(1-\breve{z}) - \phi\breve{x})(z-\breve{z}) + p\phi x(z-\breve{z})/z \\
&= m(\alpha(z-\breve{z}) - \beta(x-\breve{x}) + \gamma_1(y_1-\breve{y}_1) + \theta(z-\breve{z}))(x-\breve{x}) \\
&+ n(-\eta_1(y_1-\breve{y}_1) + \delta_1(x-\breve{x}))(y_1-\breve{y}_1) + p(-\kappa(x-\breve{x}) - \vartheta(z-\breve{z}) - \phi(x-\breve{x}))(z-\breve{z}) \\
&+ p\phi(\frac{x}{z} - \frac{\breve{x}}{\breve{z}})(z-\breve{z}) \\
&= -m\beta(x-\breve{x})^2 - n\eta_1(y_1-\breve{y}_1)^2 - p(\vartheta + \phi\frac{x}{z\breve{z}})(z-\breve{z})^2 \\
&+ (m(\alpha+\theta) - p(\kappa + \phi - \frac{\phi}{\breve{z}}))(x-\breve{x})(z-\breve{z}) + (m\gamma_1 + n\delta_1)(x-\breve{x})(y_1-\breve{y}_1) \\
&= -\{\sqrt{m\beta}(x-\breve{x}) - \sqrt{n\eta_1}(y_1-\breve{y}_1)\}^2 - p(\vartheta + \phi\frac{x}{z\breve{z}})(z-\breve{z})^2 \\
&\leq 0 \quad for \quad all \quad (x, y_1, z). \qquad \square
\end{aligned}
$$

2.10.3 Existence of $F^\star(x^\star, y_1^\star, y_2^\star, z^\star)$

In this subsection, we present uniform persistence results and give a sufficient criterion for the existence of a positive interior equilibrium $F^\star(x^\star, y_1^\star, y_2^\star, z^\star)$ for the system with $\gamma_1 > 0$, $\gamma_2 \leq 0$ and $\beta\eta_1 - \gamma_1\delta_1 > 0$. We recall that the system given by Equation (2.1)-(2.4) is dissipative under the conditions $\alpha\eta_1 - \gamma_1\xi_1 > 0$, $\alpha\delta_1 - \beta\xi_1 > 0$, $M > \frac{\xi_2}{\delta_2}$ and $\beta\eta_1 - \gamma_1\delta_1 > 0$.

Theorem 2.25. *Assume that the system given by Equations (2.1)-(2.4) is such that:*

1. $\alpha\eta_1 - \gamma_1\xi_1 > 0$, $\alpha\delta_1 - \beta\xi_1 > 0$, $M > \frac{\xi_2}{\delta_2}$ *and* $\beta\eta_1 - \gamma_1\delta_1 > 0$.

2. $F_{xz}(\bar{x}, 0, 0, \bar{z})$ *is a hyperbolic saddle point and repelling in the* y_1 *and* y_2 *directions locally (cf. Theorem 2.9).*

3. $F_{xy_1z}(\breve{x}, \breve{y}_1, 0, \breve{z})$ *is a hyperbolic saddle point and repelling in the* y_2 *direction (cf. Theorem 2.10).*

4. $F_{xy_2z}(\breve{x}, 0, \breve{y}_2, \breve{z})$ *is a hyperbolic saddle point repelling in the* y_1 *direction (cf. Theorem 2.12).*

Then the system given by Equations (2.1)-(2.4) exhibits uniform persistence.

Proof:

Similar to Theorem 2.18.

2.10.4 Numerical examples

In this subsection we illustrate some of our results in this section with numerical examples. In examples 4-6, we set

$$\alpha = 2, \quad \vartheta = 2, \quad \beta = 1/10, \quad \theta = 6/5, \quad \phi = 1,$$

$$\gamma_1 = 1/125, \quad \gamma_2 = -1/10, \quad \xi_1 = 2, \quad \xi_2 = 5/8,$$

$$\delta_1 = 1/4, \delta_2 = 1/8, \quad \eta_1 = \eta_2 = 2/25, \quad \rho_1 = 1/50, \quad \rho_2 = 1/100$$

The remaining parameter κ is varied to give three examples. The above parameters are chosen such that the entire system is dissipative.

Example 4:

We set $\kappa = 1$. Here the possible set of equilibria for the system is $F_0(0,0,0,0)$, $F_z(0,0,0,1)$, $F_{xz}(5.45,0,0,0.54)$ and $F_{xy_2z}(5.20,0,0.31,0.54)$. In this case, condition 3 of Theorem 2.23 is not satisfied and hence there is no equilibrium in the positive (xy_1z)-octant. Hence the locally asymptotic equilibrium $F_{xy_2z}(5.20,0,0.31,0.54)$ becomes a globally asymptotic equilibrium for the entire system.(see Figure 2.4).

Example 5:

We set $\kappa = 1/2$. Here the possible set of equilibria for the system is $F_0(0,0,0,0)$, $F_z(0,0,0,1)$, $F_{xz}(10.21,0,0,0.69)$, $F_{xy_1z}(10.88,9.03,0,0.69)$ and $F_{xy_2z}(7.17,0,3.39,0.70)$. In this case, all the conditions of Theorem 2.23 are satisfied and hence there exists an equilibrium in the positive (xy_1z)-octant which is locally unstable. Hence the locally asymptotic equilibrium $F_{xy_2z}(7.17,0,3.39,0.70)$ becomes a globally asymptotically stable equilibrium for the entire system (see Figure 2.5). Note that the system does not persist because condition 4 of Theorem 2.25 is not satisfied.

Example 6:

We set $\kappa = 1/10$. Here the possible set of equilibria for the system is $F_0(0,0,0,0)$, $F_z(0,0,0,1)$, $F_{xz}(17.34,0,0,0.91)$, $F_{xy_1z}(20.41,38.79,0,0.91)$ $F_{xy_2z}(9.88,0,7.62,0.91)$ and $F_{xy_1y_2z}(10.29,5.28,7.62,0.92)$. In this case, all the conditions of Theorem 2.23 are satisfied and hence there exists an equilibrium in the positive (xy_1z)-octant which is locally unstable. Here all the conditions of Theorem 2.25 are satisfied and hence the system persists uniformly (see Figure 2.6).

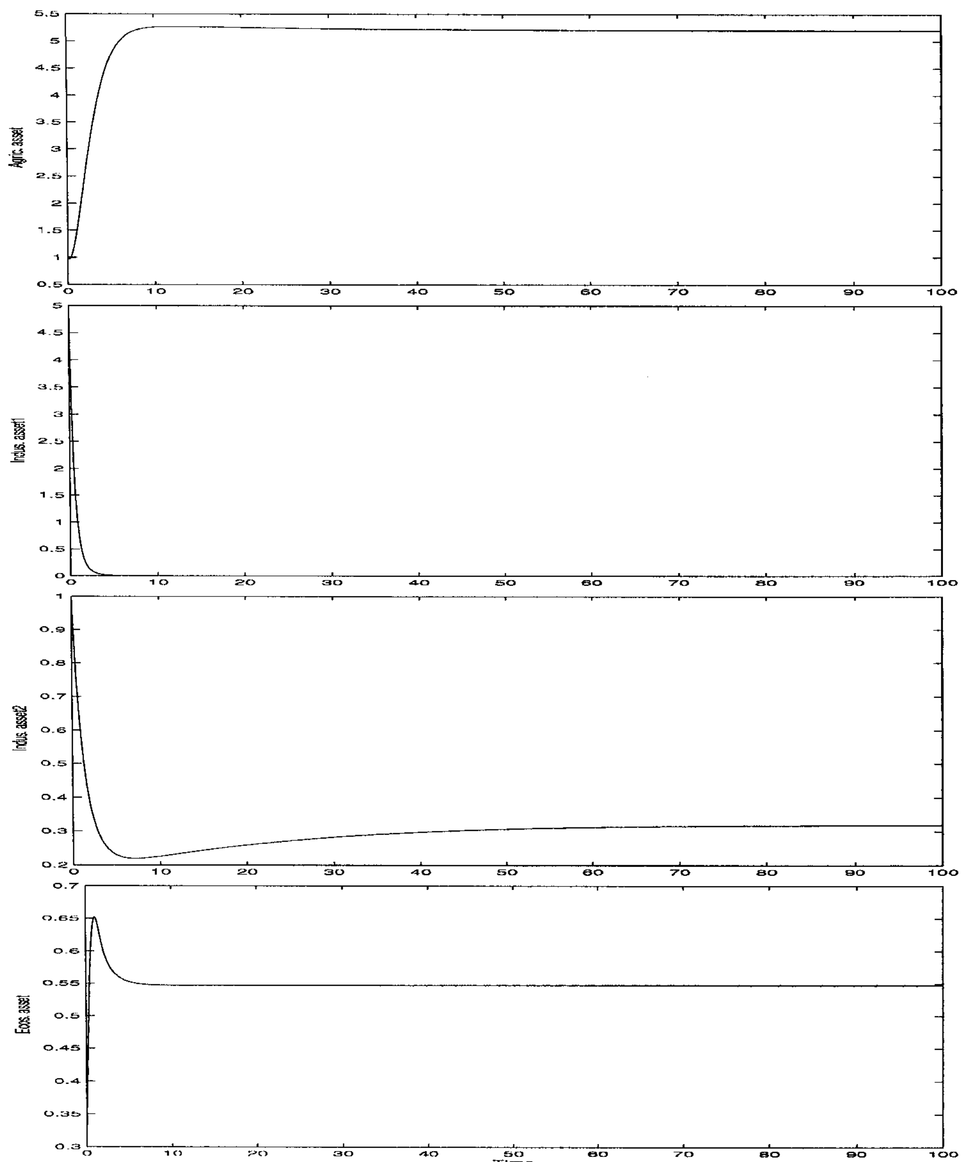

Figure 2.4: The equilibria for the system are $F_0(0,0,0,0)$, $F_z(0,0,0,1)$, $F_{xz}(5.45,0,0,0.54)$ and $F_{xy_2z}(5.20,0,0.31,0.54)$. $F_{xy_2z}(5.20,0,0.31,0.54)$ is globally asymptotic stable equilibrium for the entire system.

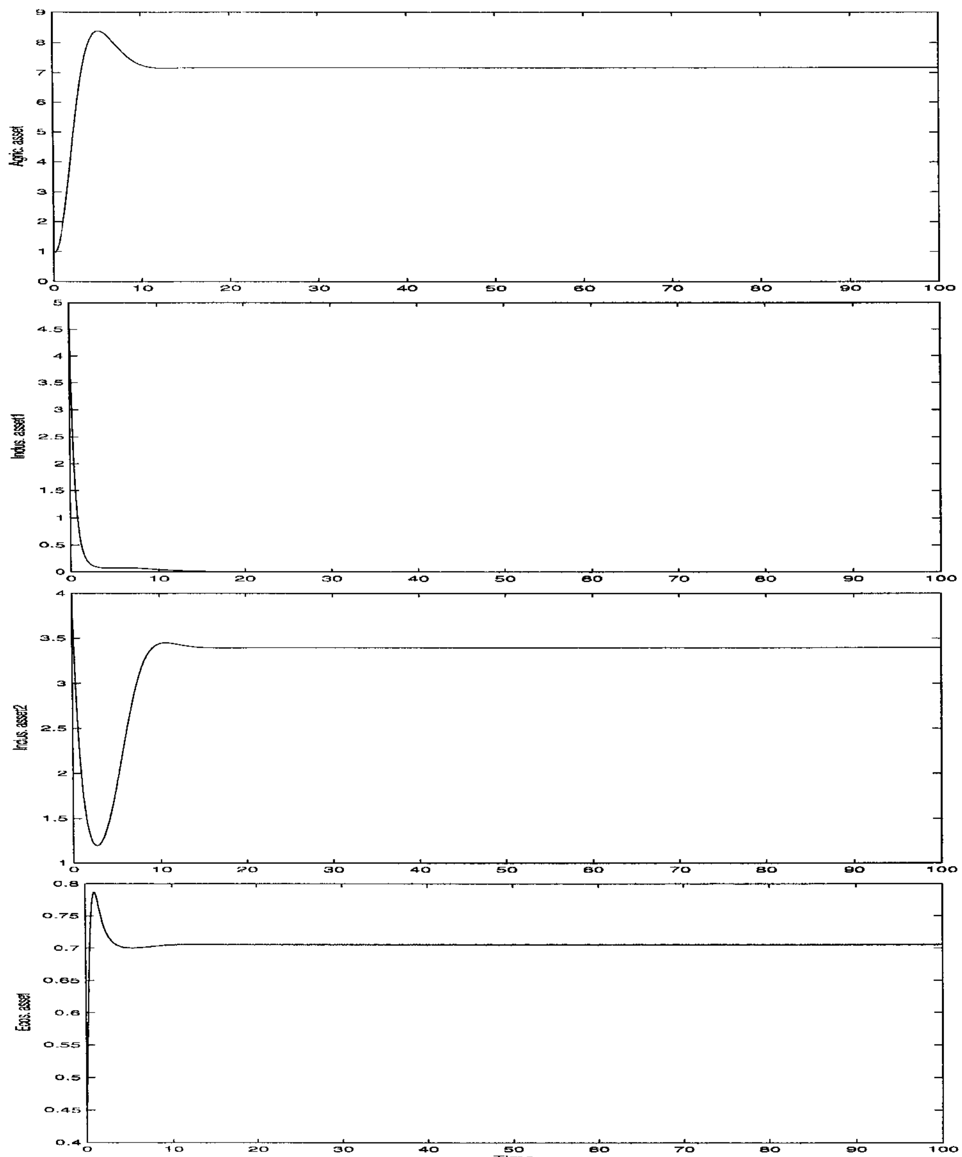

Figure 2.5: $F_0(0,0,0,0)$, $F_z(0,0,0,1)$, $F_{xz}(10.21,0,0,0.69)$, $F_{xy_1z}(10.88,9.03,0,0.69)$ and $F_{xy_2z}(7.17,0,3.39,0.70)$ are equilibria. Industry one is driven extinct.

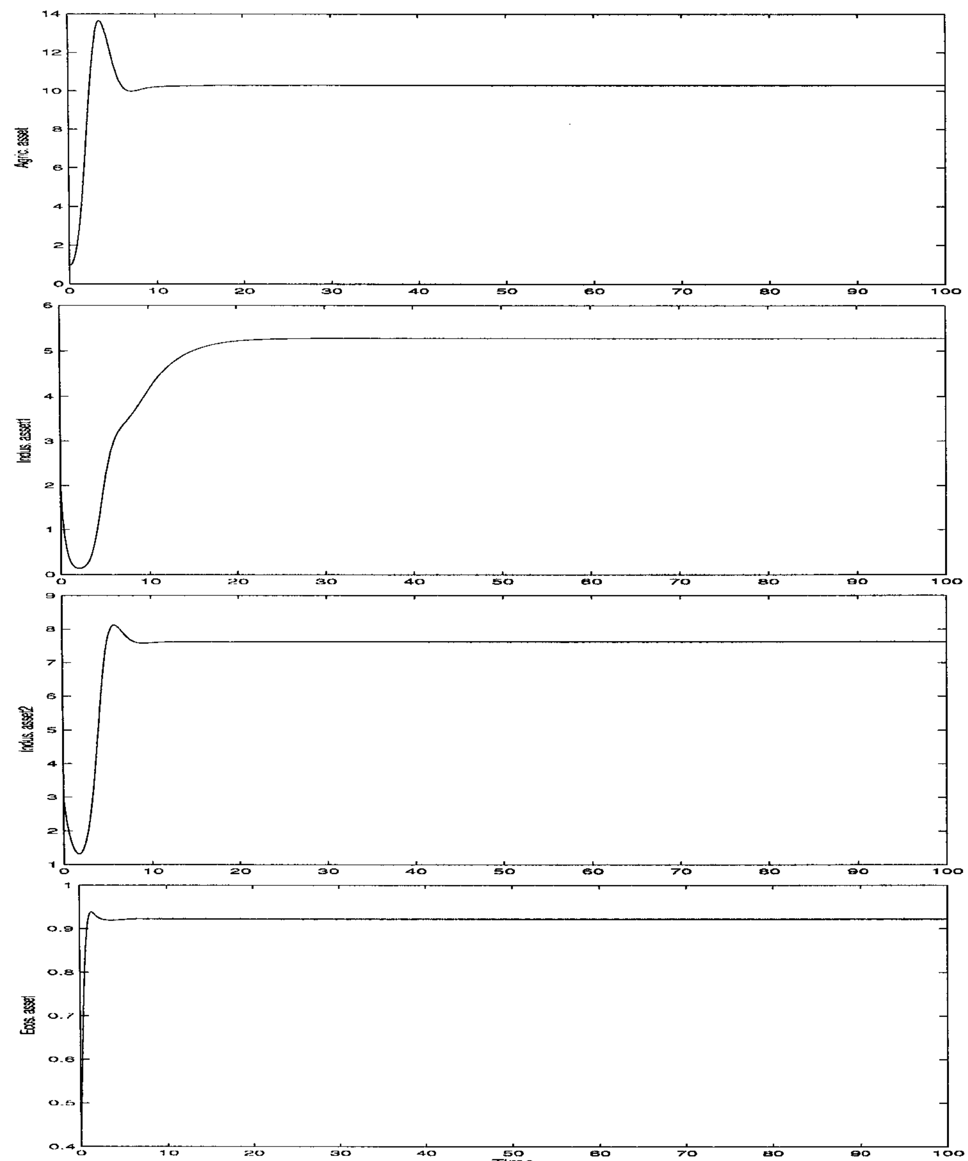

Figure 2.6: $F_0(0,0,0,0)$, $F_z(0,0,0,1)$, $F_{xz}(17.34,0,0,0.91)$, $F_{xy_1z}(20.41,38.79,0,0.91)$ $F_{xy_2z}(9.88,0,7.62,0.91)$ and $F_{xy_1y_2z}(10.29,5,28,7.62,0.92)$ are equilibria. Persistence of solutions.

2.11 $\gamma_2 \leq 0$ and $\gamma_1 > 0$ with $\beta\eta_1 - \gamma_1\delta_1 = 0$

In this section, we consider system (2.1)- (2.4) with $\gamma_1 > 0$, $\gamma_2 \leq 0$, and $\beta\eta_1 - \gamma_1\delta_1 = 0$. We will determine all the equilibria and discuss their stabilities and illustrate some of our results with numerical examples.

2.11.1 Equilibria for the system with $\gamma_2 \leq 0$, $\gamma_1 > 0$, $\beta\eta_1 - \gamma_1\delta_1 = 0$

Clearly $F_0(0,0,0,0)$ and $F_z(0,0,0,1)$ are equilibria. $F_z(0,0,0,1)$ is the only (axial) one-dimensional equilibrium (see Section 2.4) and $F_{xz}(\bar{x},0,0,\bar{z})$ is the only (planar) two-dimensional equilibrium for the system (see Section 2.4.1) and they always exist. Also there exists a unique three-dimensional equilibrium $F_{xy_2z}(\breve{x},0,\breve{y}_2,\breve{z})$ in the positive $(x,y_2,z)-$ octant if $-\xi_2 + \delta_2\bar{x} > 0$ (see Theorem 2.5).

However, in the positive interior of the (x,y_1,z)-octant, we have $x > 0$, $y_1 > 0$, $y_2 = 0$, and $z > 0$. So the algebraic system (2.6)-(2.9) reduces to

$$\alpha z - \beta x + \gamma_1 y_1 - \theta(1-z) = 0 \tag{2.70}$$

$$-\xi_1 - \eta_1 y_1 + \delta_1 x = 0 \tag{2.71}$$

$$-\kappa x z + \vartheta(1-z)z + \phi(1-z)x = 0. \tag{2.72}$$

Solving for y_1 from Equation (2.71), we get

$$y_1 = \breve{y}_1 = \frac{-\xi_1 + \delta_1\breve{x}}{\eta_1}, \tag{2.73}$$

which exists if

$$\breve{x} > \frac{\xi_1}{\delta_1}. \tag{2.74}$$

Substituting Equation (2.73) into (2.70) we get

$$(\alpha\eta_1 + \theta\eta_1)\breve{z} + -\gamma_1\xi_1 - \theta\eta_1 = 0. \tag{2.75}$$

Solving (2.75) for $\breve{z}$, we get

$$\breve{z} = \frac{\theta\eta_1 + \gamma_1\xi_1}{\theta\eta_1 + \alpha\eta_1}, \tag{2.76}$$

which exists if $\gamma_1\xi_1 \leq \alpha\eta_1$.

From Equation (2.72), we have

$$\breve{x} = \frac{\vartheta(1-\breve{z})\breve{z}}{\kappa\breve{z} - \phi(1-\breve{z})}.$$

For existence, Equation (2.74) must be satisfied and thus we must have $\kappa\breve{z} - \phi(1-\breve{z}) > 0$ and

$$\frac{\vartheta(1-\breve{z})\breve{z}}{\kappa\breve{z} - \phi(1-\breve{z})} > \frac{\xi_1}{\delta_1}.$$

We have the following Theorem:

Theorem 2.26. *There exists a unique three-dimensional equilibrium in the positive interior of the $(x, y_1, z)-$ octant if:*

1. $\beta\eta_1 - \gamma_1\delta_1 = 0$
2. $\gamma_1\xi_1 \leq \alpha\eta_1$,
3. $\kappa\breve{z} - \phi(1-\breve{z}) > 0$
4. $\frac{\vartheta(1-\breve{z})\breve{z}}{\kappa\breve{z}-\phi(1-\breve{z})} > \frac{\xi_1}{\delta_1}$, *where* $\breve{z} = \frac{\theta\eta_1+\gamma_1\xi_1}{\theta\eta_1+\alpha\eta_1}$.

2.11.2 Local and global stability analysis

In this subsection, criteria for local and global (asymptotic) stability of the equilibria $F_0(0,0,0,0)$, $F_z(0,0,0,1)$, $F_{xz}(\bar{x},0,0,\bar{z})$, $F_{xy_1z}(\breve{x},\breve{y}_1,0,\breve{z})$ and $F_{xy_2z}(\breve{x},0,\breve{y}_2,\breve{z})$ for the case $\beta\eta_1 - \gamma_1\delta_1 > 0$ are given. Under the condition $\beta\eta_1 - \gamma_1\delta_1 = 0$, the conditions for local asymptotic stability of these equilibria are the same as those given in Theorems 2.6-2.13. One can also easily verify that the global stability conditions given in Theorems 2.14-2.15 for $F_z(0,0,0,1)$ and $F_{xz}(\bar{x},0,0,\bar{z})$ and Theorem 2.17 for $F_{xy_2z}(\breve{x},0,\breve{y}_2,\breve{z})$ also holds if $\beta\eta_1 - \gamma_1\delta_1 = 0$. We now proceed to give the version of Theorem 2.24 in this case for $F_{xy_1z}(\breve{x},\breve{y}_1,0,\breve{z})$.

Global stability of $E_{xy_1z}(\breve{x},\breve{y}_1,\breve{z})$

Criteria for the global stability of $F_{xy_1z}(\breve{x},\breve{y}_1,0,\breve{z})$ with respect to solutions initiating in the interior of the $\mathbf{R}^+_{xy_1z} = \{0 \le x \le \infty, 0 \le y_1 \le \infty, 0 \le z \le 1\}$ is given. In fact, we will show that if the equilibrium $E_{xy_1z}(\breve{x},\breve{y}_1,\breve{z})$ exists, then it is globally stable. Thus we have the following theorem.

Theorem 2.27. *The equilibrium $F_{xy_1z}(\breve{x},\breve{y}_1,0,\breve{z})$ is globally stable with respect to solutions trajectories emanating from the positive interior of $\mathbf{R}^+_{xy_1z}$ if it exists.*

Proof:

In $\mathbf{R}^+_{xy_1z}$, the system of Equations (2.1)- (2.4) reduces to

$$\begin{aligned}
\dot{x} &= (\alpha z - \beta x + \gamma_1 y_1 - \theta(1-z))x \\
\dot{y}_1 &= (-\xi_1 - \eta_1 y_1 + \delta_1 x)y_1 \\
\dot{z} &= -\kappa x z + \vartheta(1-z)z + \phi(1-z)x.
\end{aligned} \tag{2.77}$$

Let m, n and p be positive constants defined by $m = \kappa + \phi - \frac{\phi}{\breve{z}} = \frac{\vartheta(1-\breve{z})}{\breve{x}}$, $n = \frac{m\beta\eta_1}{\delta_1^2}$ and $p = \theta + \alpha$.

In $\mathbf{R}^+_{xy_1z}$, we choose a Liapunov function $V(x, y_1, z)$ defined by

$$\begin{aligned}
V(x,y_1,z) &= m\{x - \breve{x} - \breve{x}(\ln(\frac{x}{\breve{x}})\} + n\{y_1 - \breve{y}_1 - \breve{y}_1(\ln(\frac{y_1}{\breve{y}_1})\} \\
&\quad + p\{z - \breve{z} - \breve{z}(\ln(\frac{z}{\breve{z}})\}.
\end{aligned} \tag{2.78}$$

The derivative of (2.78) along the solution curves of (2.77) in $\mathbf{R}^+_{xy_1z}$ is given by

$$
\begin{aligned}
\dot{V} &= \frac{m\dot{x}(x-\breve{x})}{x} + \frac{n\dot{y}_1(y_1-\breve{y}_1)}{y_1} + \frac{p\dot{z}(z-\breve{z})}{z} \\
&= m(\alpha z - \beta x + \gamma_1 y_1 - \theta(1-z))(x-\breve{x}) + n(-\xi_1 - \eta_1 y_1 + \delta_1 x)(y_1 - \breve{y}_1) \\
&+ p(-\kappa x + \vartheta(1-z) - \phi x)(z-\breve{z}) + p\phi x(z-\breve{z})/z \\
&= m(\alpha(z-\breve{z}) - \beta(x-\breve{x}) + \gamma_1(y_1-\breve{y}_1) + \theta(z-\breve{z}))(x-\breve{x}) \\
&+ n(-\eta_1(y_1-\breve{y}_1) + \delta_1(x-\breve{x}))(y_1-\breve{y}_1) + p(-\kappa(x-\breve{x}) - \vartheta(z-\breve{z}) - \phi(x-\breve{x}))(z-\breve{z}) \\
&+ m(\alpha\breve{z} - \beta\breve{x} + \gamma_1\breve{y}_1 - \theta(1-\breve{z}))(x-\breve{x}) + n(-\xi_1 - \eta_1\breve{y}_1 + \delta_1\breve{x})(y_1-\breve{y}_1) \\
&+ p(-\kappa\breve{x} + \vartheta(1-\breve{z}) - \phi\breve{x})(z-\breve{z}) + p\phi x(z-\breve{z})/z \\
&= m(\alpha(z-\breve{z}) - \beta(x-\breve{x}) + \gamma_1(y_1-\breve{y}_1) + \theta(z-\breve{z}))(x-\breve{x}) \\
&+ n(-\eta_1(y_1-\breve{y}_1) + \delta_1(x-\breve{x}))(y_1-\breve{y}_1) + p(-\kappa(x-\breve{x}) - \vartheta(z-\breve{z}) - \phi(x-\breve{x}))(z-\breve{z}) \\
&+ p\phi(\frac{x}{z} - \frac{\breve{x}}{\breve{z}})(z-\breve{z}) \\
&= -m\beta(x-\breve{x})^2 - n\eta_1(y_1-\breve{y}_1)^2 - p(\vartheta + \phi\frac{x}{z\breve{z}})(z-\breve{z})^2 \\
&+ (m(\alpha+\theta) - p(\kappa + \phi - \frac{\phi}{\breve{z}}))(x-\breve{x})(z-\breve{z}) + (m\gamma_1 + n\delta_1)(x-\breve{x})(y_1-\breve{y}_1) \\
&= -\{\sqrt{m\beta}(x-\breve{x}) - \sqrt{n\eta_1}(y_1-\breve{y}_1)\}^2 - p(\vartheta + \phi\frac{x}{z\breve{z}})(z-\breve{z})^2 \\
&\leq 0 \quad for \quad all \quad (x, y_1, z). \qquad \square
\end{aligned}
$$

2.11.3 Numerical examples

In this subsection we illustrate some of our results in this section with numerical examples. We will also use a numerical example to illustrate that there may exist a 4-dimensional equilibrium depending on the choice of the model parameters. The existence of a 4-dimensional equilibrium for the system in this case can be calculated algebraically from the system, but due to its complexity, we omit it.

In examples 7-10, we assign the model coefficients with the following values

$$\alpha = 2, \quad \vartheta = 2, \quad \beta = 1/10, \quad \theta = 6/5, \quad \phi = 1,$$

$$\gamma_1 = 1/50, \quad \gamma_2 = -1/10, \eta_2 = 1/50, \quad \kappa = 2,$$

$$\delta_1 = 1/4, \delta_2 = 1/2, \quad \eta_1 = 1/10, \quad \rho_1 = 1/50, \quad \rho_2 = 1/50.$$

The remaining parameters ξ_1 and ξ_2 are varied to give four examples.

Example 7:

We set $\xi_1 = 1$ and $\xi_2 = 5/2$. The only possible set of equilibria for the system is $F_0(0,0,0,0)$, $F_z(0,0,0,1)$, $F_{xz}(1.7142, 0, 0, 0.4286)$. In this case, condition 4 of Theorem 2.26 is not satisfied and hence there is no equilibrium in the positive (xy_1z)-octant. Also there is no equilibrium in the positive (xy_2z)-octant because $-\xi_2 + \delta_2\bar{x} < 0$ (see Theorem 2.4). Hence the locally asymptotic equilibrium $F_{xz}(1.7142, 0, 0, 0.4286)$ becomes a globally asymptotic equilibrium for the entire system (see Figure 2.7).

Example 8:

We set $\xi_1 = 1/4$ and $\xi_2 = 5/2$. Here the possible set of equilibria for the system is $F_0(0,0,0,0)$, $F_z(0,0,0,1)$, $F_{xz}(1.7142, 0, 0, 0.4286)$, $F_{xy_1z}(1.8752, 2.1883, 0, 0.4200)$. In this case, all the conditions of Theorem 2.26 are satisfied and hence there exists an equilibrium in the positive (xy_1z)-octant which is locally stable (see Theorem 2.10). Under this condition $F_{xz}(1.7142, 0, 0, 0.4286)$ becomes locally unstable (see Theorem 2.9), hence the locally asymptotic equilibrium $F_{xy_1z}(1.8752, 2.1883, 0, 0.4200)$ becomes a globally asymptotic equilibrium for the entire system (see Figure 2.8). Note that the system does not persist.

Example 9:

We set $\xi_1 = 1/4$ and $\xi_2 = 1/8$. Here the possible set of equilibria for the system is $F_0(0,0,0,0)$, $F_z(0,0,0,1)$, $F_{xz}(1.7142, 0, 0, 0.4286)$, $F_{xy_1z}(2.2052, 6.0268, 0, 0.4064)$

and $F_{xy_2z}(0.5439, 0, 7.3481, 0.6217)$. The equilibrium $F_{xy_2z}(0.5439, 0, 7.3481, 0.6217)$ is a globally asymptotically stable equilibrium for the system (see Figure 2.9). Note that the system does not persist.

Example 10:

We set $\xi_1 = 1/4$ and $\xi_2 = 2/3$. Here the possible set of equilibria for the system is $F_0(0, 0, 0, 0)$, $F_z(0, 0, 0, 1)$, $F_{xz}(1.7142, 0, 0, 0.4286)$, $F_{xy_1z}(1.8751, 2.1883, 0, 0.4200)$, $F_{xy_2z}(1.3780, 0, 1.1240, 0.4543)$, and $F_{xy_1y_2z}(1.4115, 0.7979, 1.1559, 0.4503)$. In this case the system persists (see Figure 2.10).

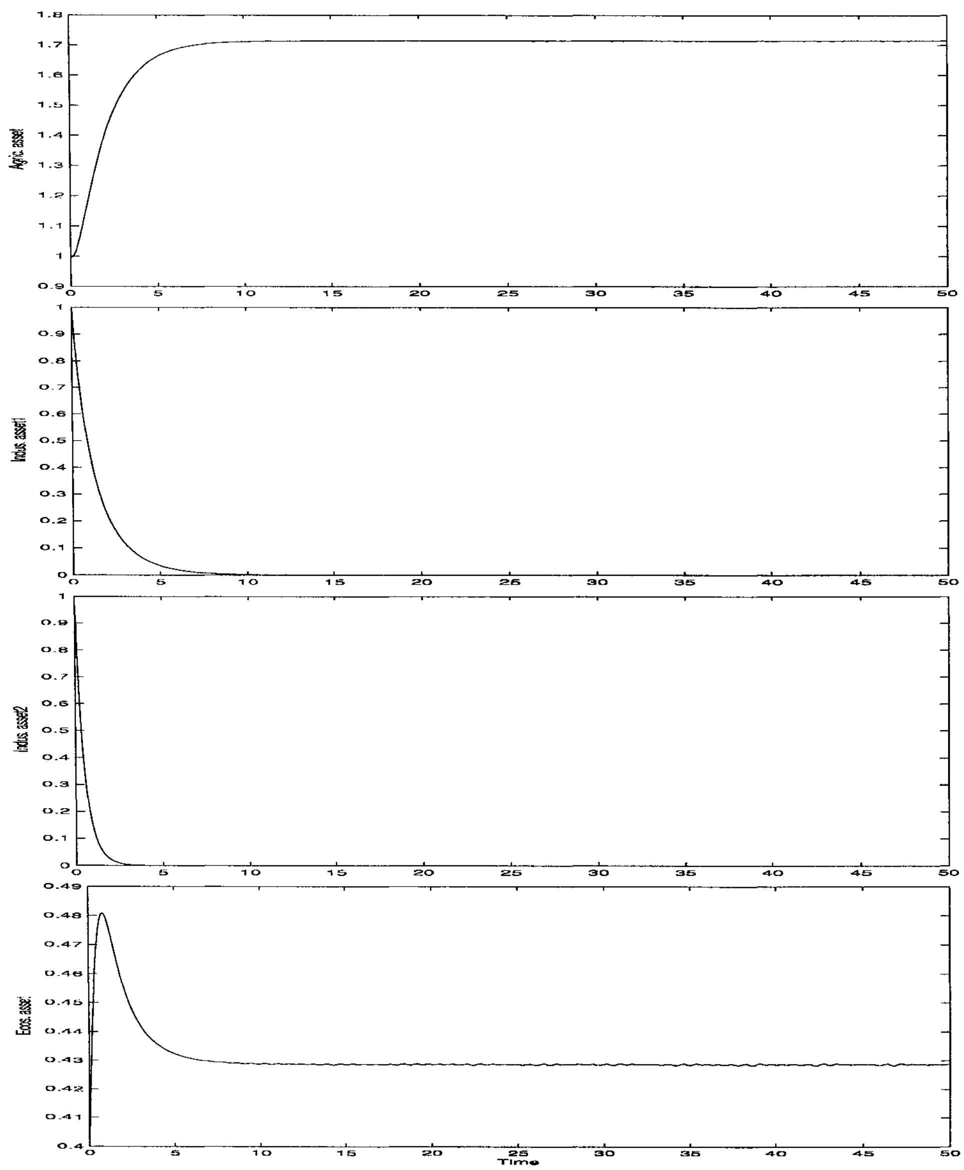

Figure 2.7: $F_0(0,0,0,0)$, $F_z(0,0,0,1)$, $F_{xz}(1.7142,0,0,0.4286)$ are equilibria. Both industries are driven extinct.

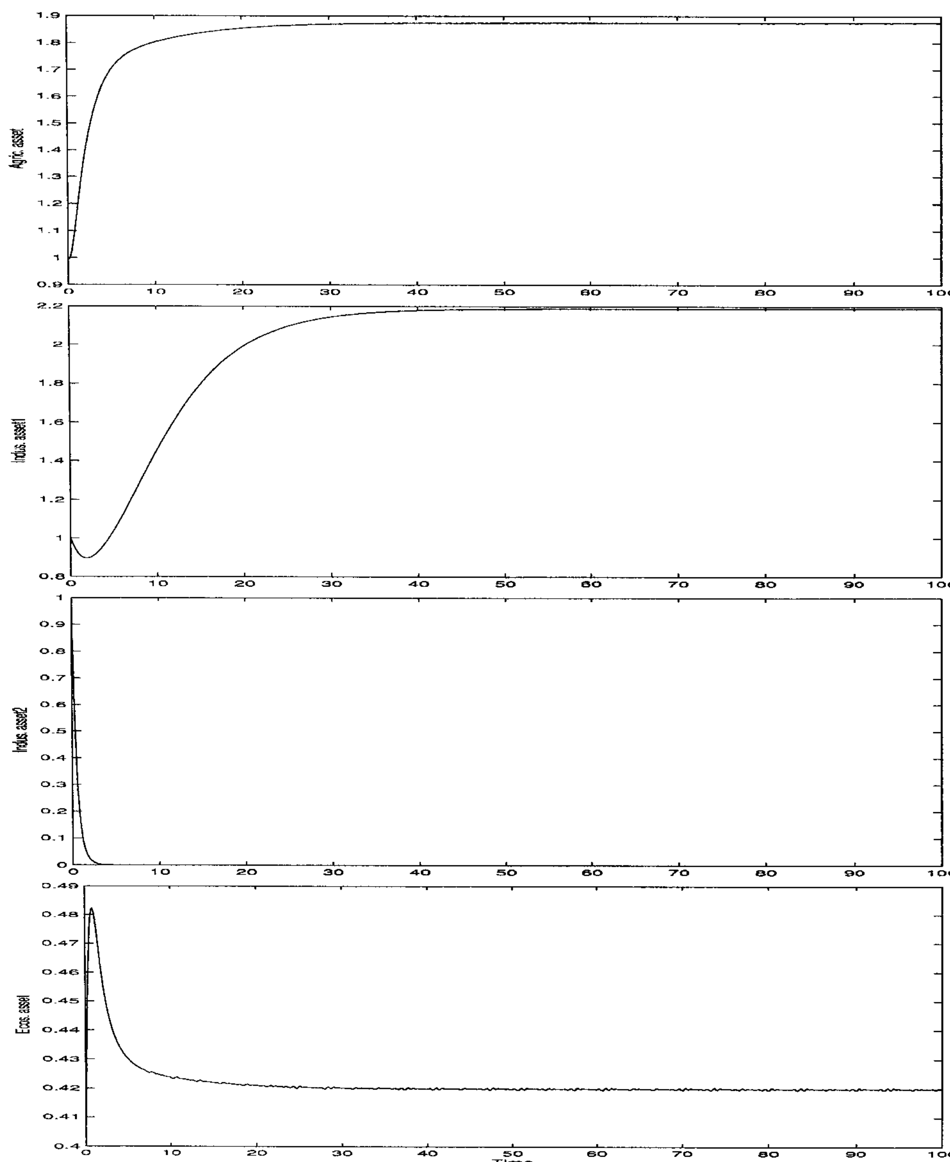

Figure 2.8: $F_{xy_1z}(1.8752, 2.1883, 0, 0.4286)$, $F_{xz}(1.7142, 0, 0, 0.4286)$, $F_0(0, 0, 0, 0)$ and $F_z(0, 0, 0, 1)$ are equilibria. Industry two is driven extinct.

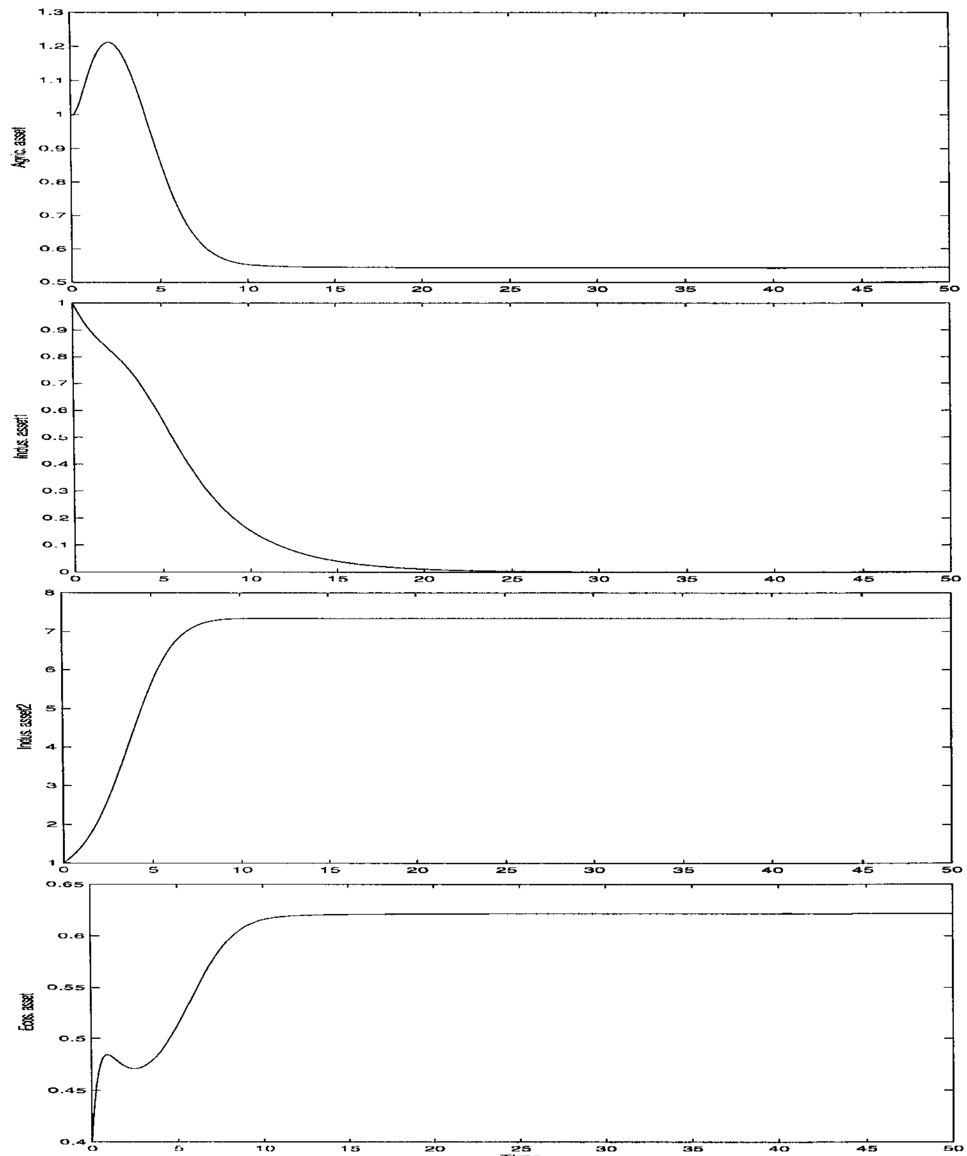

Figure 2.9: $F_0(0,0,0,0)$, $F_{xz}(1.7142,0,0,0.4286)$, $F_{xy_1z}(2.2052,6.0268,0,0.4064)$, $F_{xy_2z}(0.5439,0,7.3481,0.6217)$ and $F_z(0,0,0,1)$ are equilibria. Industry one is driven extinct.

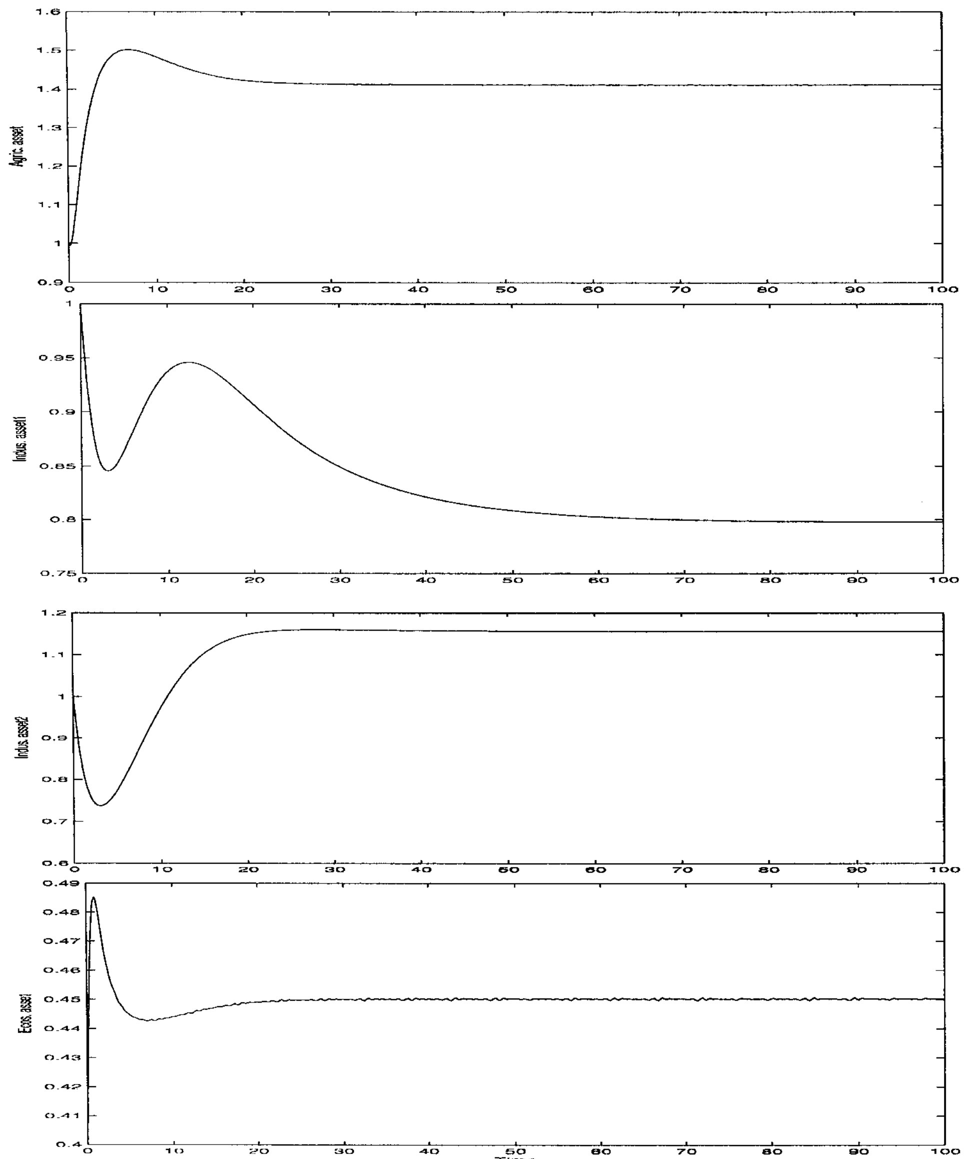

Figure 2.10: $F_{xy_1y_2z}(1.4115, 0.7979, 1.1559, 0.4503)$, $F_{xy_1z}(2.2052, 6.0268, 0, 0.4064)$, $F_{xz}(1.7142, 0, 0, 0.4286)$, $F_{xy_2z}(1.3780, 0, 1.1240, 0.4543)$, $F_z(0, 0, 0, 1)$ and $F_0(0, 0, 0, 0)$ are equilibria. The system persists.

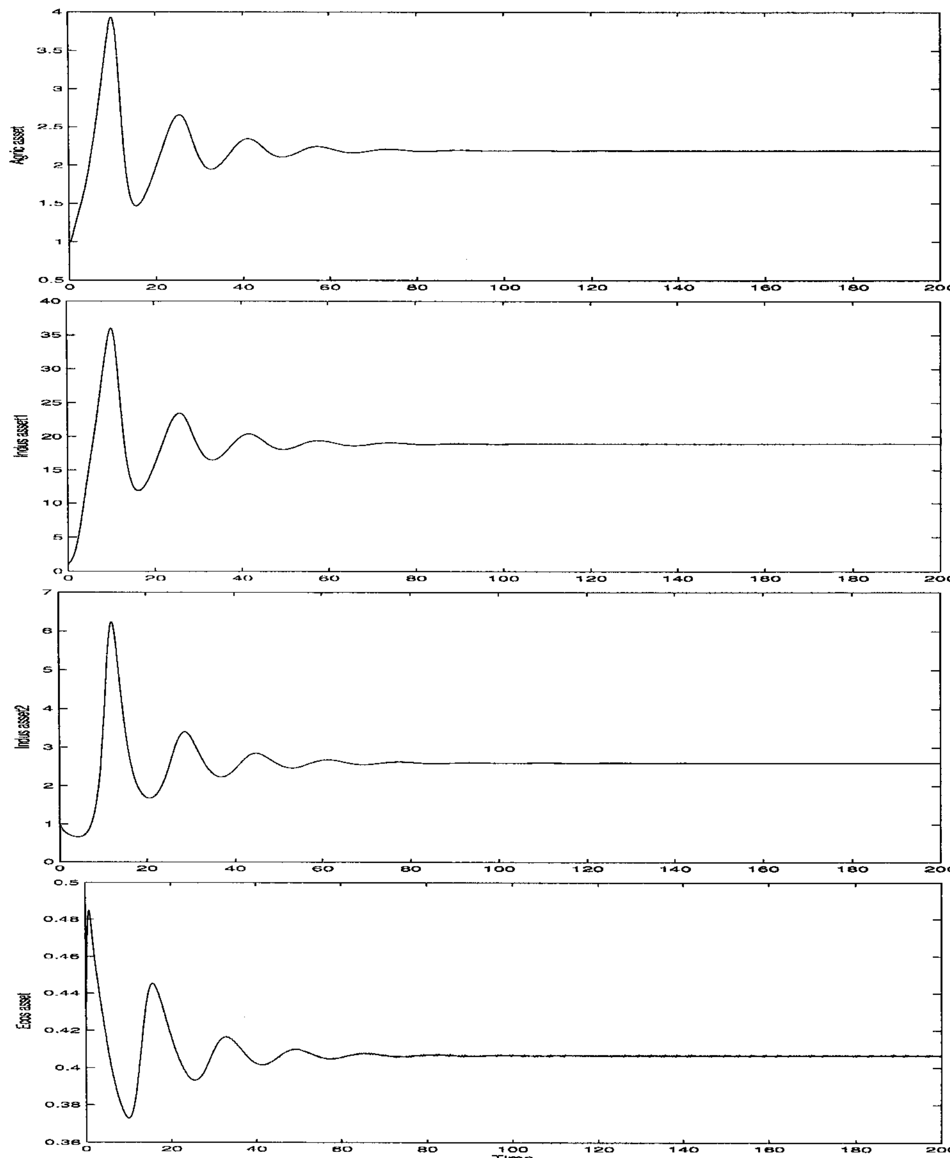

Figure 2.11: $F_{xy_1y_2z}(2.1919, 18.8959, 2.5986, 0.4068)$, $F_{xy_2z}(1.3780, 0, 1.1240, 0.4534)$, $F_{xz}(1.7142, 0, 0, 0.4286)$, $F_z(0, 0, 0, 1)$and $F_0(0, 0, 0, 0)$ are equilibria. The system persists.

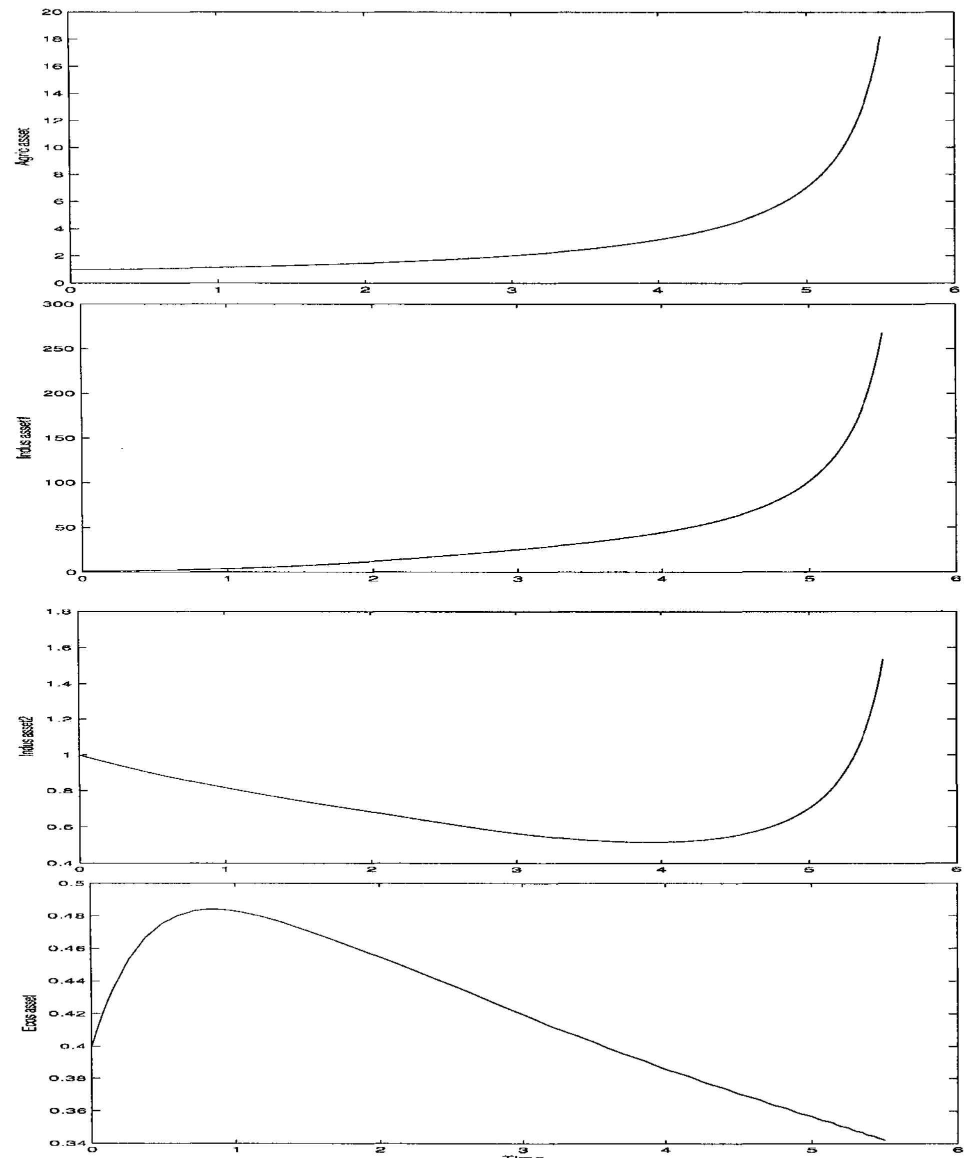

Figure 2.12a: $F_{xy_2z}(1.3780, 0, 1.1240, 0.4534)$, $F_{xz}(1.7142, 0, 0, 0.4286)$, $F_0(0, 0, 0, 0)$ and $F_z(0, 0, 0, 1)$, are equilibria.

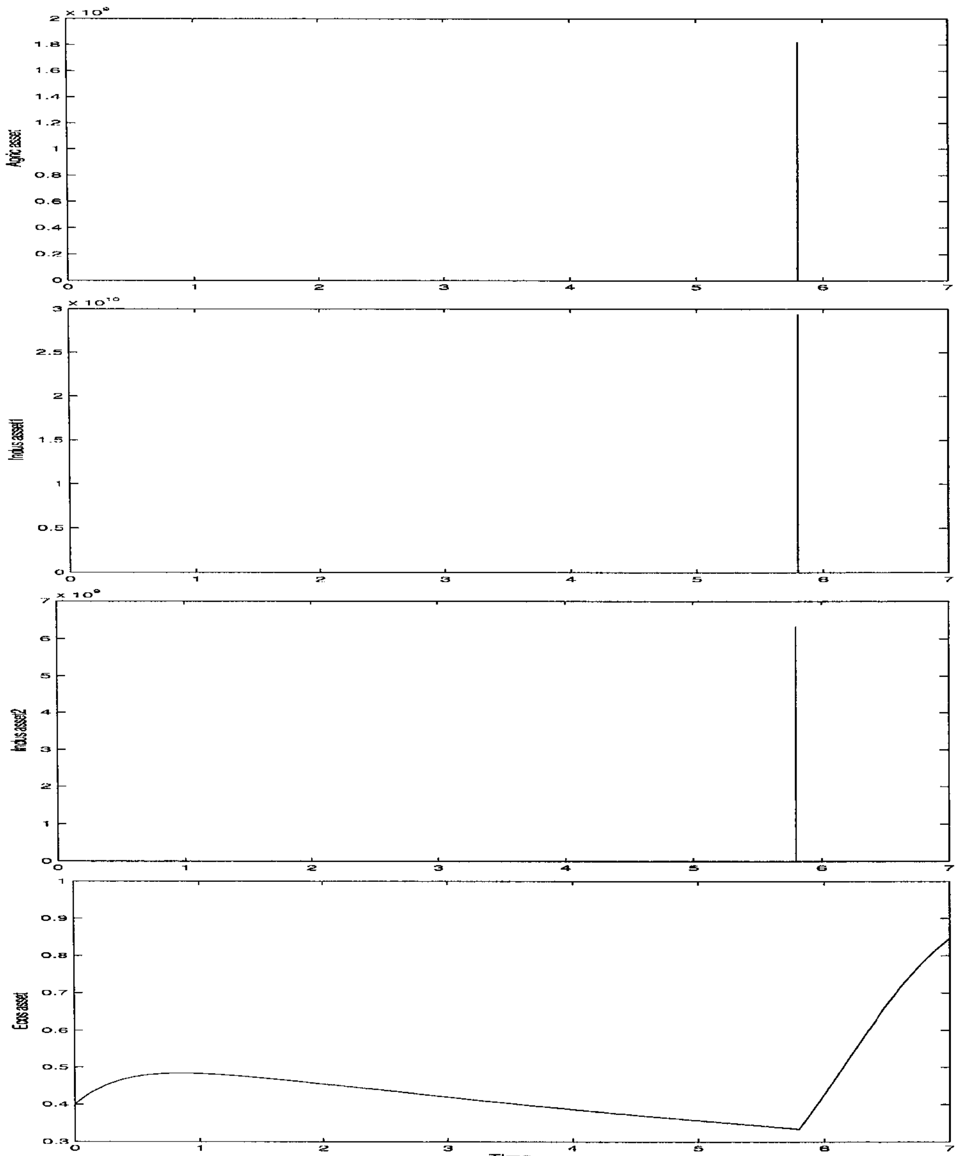

Figure 2.12b: $F_{xy_2z}(1.3780, 0, 1.1240, 0.4534)$, $F_{xz}(1.7142, 0, 0, 0.4286)$, $F_0(0, 0, 0, 0)$ and $F_z(0, 0, 0, 1)$, are equilibria.

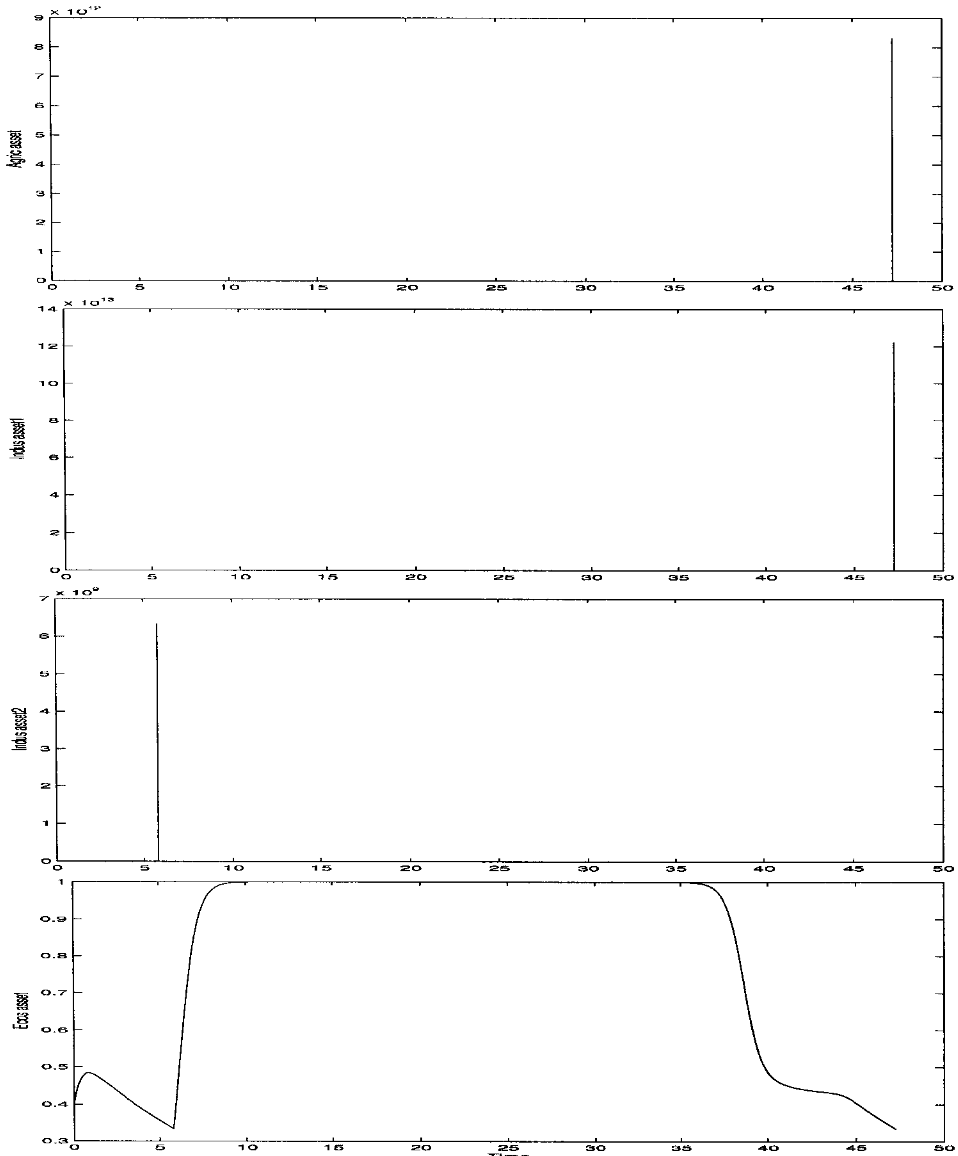

Figure 2.12c: $F_{xy_2z}(1.3780, 0, 1.1240, 0.4534)$, $F_{xz}(1.7142, 0, 0, 0.4286)$, $F_0(0, 0, 0, 0)$ and $F_z(0, 0, 0, 1)$, are equilibria.

2.12 $\gamma_2 \leq 0$ and $\gamma_1 > 0$ with $\beta\eta_1 - \gamma_1\delta_1 < 0$

We illustrate this case by considering two numerical examples.

Example 11:

Here all the parameters of the model are chosen to be the same as those in example 5.4 except $\delta_1 = 1$ instead of $\delta_1 = 1/4$. Here the possible set of equilibria for the system is $F_0(0,0,0,0)$, $F_z(0,0,0,1)$, $F_{xz}(1.7142, 0, 0, 0.4286)$, $F_{xy_2z}(1.3780, 0, 1.1240, 0.4543)$ and $F_{xy_1y_2z}(2.1919, 18.8959, 2.5986, 0.4068)$. In this example,the globally asymptotically stable equilibrium is $F_{xy_1y_2z}(2.1919, 18.8959, 2.5986, 0.4068)$ (see Figure 2.11). That is, if $\beta\eta_1 - \gamma_1\delta_1 < 0$ and $\mid \beta\eta_1 - \gamma_1\delta_1 \mid$ is "small" then the qualitative behaviour of the four dimensional system in this case is similar to the case when $\beta\eta_1 - \gamma_1\delta_1 = 0$ even though the behaviour of the subsystem in $\mathbf{R}^+_{xy_1z}$ in the two cases are entirely different.

Example 12:

Here all the parameters of the model are chosen to be the same as those in example 2:10 except $\delta_1 = 5/3$ instead of $\delta_1 = 1/4$. Here the possible set of equilibria for the system is $F_z(0,0,0,1)$, $F_{xz}(1.7142, 0, 0, 0.4286)$, $F_{xy_2z}(1.3780, 0, 1.1240, 0.4534)$ and $F_0(0,0,0,0)$. All the above equilibria are unstable. The output of the numerical solutions are illustrated in Figures 2.12a, 2.12b and 2.12c for different time intervals. Here $\beta\eta_1 - \gamma_1\delta_1 < 0$ and $\mid \beta\eta_1 - \gamma_1\delta_1 \mid$ is "large". The behaviour of the system in $\mathbf{R}^+_{xy_1z}$ in this case and that in Example 2.11 are qualitatively similar but the behaviour of the four dimensional system in the case is completely different form that in Example 2.11 in the sense that there is persistence in Example 2.11 while we have non-persistence in Example 2.12.

In $\mathbf{R}^+_{xy_1y_2z}$, we observe from Figures 2.12a and 2.12b that industry one, industry two and agriculture are concave upwards on $(0, 5.81)$ while the ecosphere is concave downwards on the same interval. In particular, on $(4, 5.81)$ industry one, industry two and agriculture grow very fast while the ecosphere declines within the same period. At $t = 5.81$ the ecosphere can no longer support the huge growth in agriculture, industry one and industry two and so there is a sudden collapse of agriculture, industry one and industry two. This collapse leads to the eventual growth in the ecosphere assets (see Figures 2.12b and 2.12c). The ecosphere grows to its maximum possible levels, which eventually leads to a growth in agriculture and industry one. Industry two could not recover from the collapse because the per asset terms of trade rate coefficient between agriculture and industry two is negative. Also the competition between the two industries did not help in the recovery of industry two. On $(46, 47.2)$, there is a huge growth in industry one and agriculture and a decline in the ecosphere within the same period.

2.13 Summary and conclusions

In the preceding sections of this chapter, we used mathematical model to discuss the interaction between agriculture, ecosphere and two industries competing with each other. We used mathematical tools such as differential equation analysis, persistence theory and linear systems theory to analyze our model.

In the case where agriculture looses to each industry as a result of their interactions, complete local and global analysis of the system's equilibria are done. We proved that the equilibria $F_0(0,0,0,0)$, $F_z(0,0,0,1)$, and $F_{xz}(\bar{x},0,0,\bar{z})$ always exist and obtained sufficient criteria for the existence of $F_{xy_1z}(\breve{x},\breve{y}_1,0,\breve{z})$, $F_{xy_2z}(\breve{x},0,\breve{y}_2,\breve{z})$ and $F_{xy_1y_2z}(x^\star,y_1^\star,y_2^\star,z^\star)$. We obtained a sufficient condition for $F_{xy_1z}(\breve{x},\breve{y}_1,0,\breve{z})$ to be globally asymptotically stable with respect to solutions initiating in the interior of the positive (xy_1z)-octant. This condition can easily be modified to give persistence conditions and a global asymptotic stability condition for the system in the case where we have only one industry interacting with agriculture and the ecosphere, that analysis which was omitted by Agyemang in [1].

We established criteria for the extinction of both industrial assets, that is the global asymptotic stability of the equilibrium $F_{xz}(\bar{x}, 0, 0, \bar{z})$. We proved that if $\bar{x} < min\{\frac{\xi_1}{\delta_1}, \frac{\xi_2}{\delta_2}\}$ then $F_{xz}(\bar{x}, 0, 0, \bar{z})$ is globally asymptotically stable. This condition can be interpreted as saying if the ratio of the constant depreciation rates of each industry to its per asset growth rate is "high" then both industries will go extinct. This condition can happen only in two ways, either agriculture is not buying much from the industries because farmers cannot afford their products, can get a cheaper product from somewhere else or don't need the industrial products, or just because the products from industries are so "bad" they depreciate very fast. In most cases, it is the former rather than the latter which is often the case. This condition explains why industries in some farming communities in developed countries collapse and as such farmers have to travel long distances to get supplies such as fertilizers, insecticides, machinery, etc. It also explains why some industries in developing countries collapse and as such governments in such countries have to import their industrial needs from other countries at a higher cost. As a result of higher cost of imported industrial products, many of the local farmers cannot afford such products and as such continue with their primitive and traditional bad farming practices which is currently found in many developing countries. Thus for sustainable agriculture, this is a condition which has to be avoided.

In section 2.7, we established the existence of a positive interior equilibrium $F_{xy_1y_2z}(x^\star, y_1^\star, y_2^\star, z^\star)$. Since under conditions for the existence of this equilibrium, all other equilibria have to be unstable. This equilibrium must be globally asymptotically stable because the system is dissipative. That is we will have a sustainable system, sustainable in the sense that all the components of the system, such as industry one, industry two, agriculture and the ecosphere will exist positively with each component viable and healthy. Also since this equilibrium exists, one will be able to predict the outcome of future levels of agricultural, industrial one and industrial two assets and the quality of the ecosphere.

In the case where the interaction between one industry (in this case industry one) and agriculture is that of a mutualist rather than parasitism, we showed that solutions are eventually uniformly bounded (dissipative) if the product of the per asset growth rate of industry one and the per asset terms of trade rate coefficient between industry one and agriculture is less than the product of the per asset diminishing returns rate coefficient of agriculture and the linear depreciation rate coefficient of industry one. This condition can only be satisfied if industry one does not grow too "much" or the terms of trade is not too "much". For most practical purposes we will favor the latter rather than the former because if industrial growth is less, it is more likely that it will go extinct, which will violate the condition of persistence.

In section 2.12, we observed from numerical simulations that if $\beta\eta_1 - \gamma_1\delta_1 < 0$, and the difference is really "small", then the system behaves almost as if $\beta\eta_1 - \gamma_1\delta_1 = 0$. On the other hand, if $\beta\eta_1 - \gamma_1\delta_1 < 0$ and $\mid \beta\eta_1 - \gamma_1\delta_1 \mid$ is "large" both industries and agriculture grow relatively huge over a period of time and the ecosphere declines over the same period of time, and eventually agriculture, industry one and industry two collapse. The collapse of both industries and agriculture leads to a growth in the ecosphere which subsequently leads to growth in industry one and agriculture but industry two does not recover.

Chapter 3

Competition between Normal and Renewable Agriculture

3.1 Introduction

In chapter 2, we studied the interactions between agriculture, industry and the ecosphere, where agriculture was considered as a combination of renewable and normal agriculture. In this chapter, we consider renewable and normal agriculture separately. Normal Agriculture is the occupation, business or science of cultivating the land, producing crops and raising animals. Renewable Agriculture is the naturally occurring agricultural resource on earth that can replenish itself or grow back again such as forest, wildlife, etc. We once again consider industry as that industry associated with agriculture and the ecosphere as the quality of land and the environment. Sometimes the term natural ecosystems or environment will be used for the combination of the ecosphere and renewable agriculture. The term agriculture will be reserved for the union of renewable and normal agriculture, i.e. the occupation, business or science of cultivating the land, producing crops and raising animals and all naturally occurring agricultural resource from the earth that can replenish itself.

Since the time of creation, renewable agriculture and the ecosphere has been providing the infrastructure on which every generation lives. This infrastructure can best be described as a home for humanity in the sense that it provides humanity with much of what it needs to be comfortable, secure and prosper [40]. According to Gretchen Daily [20] and Geoffrey Heal [40], it is because "the environment performs critical life-support services, upon which the well-being of all societies depend. These include:

- purification of air and water
- mitigation of droughts and floods
- generation and preservation of soils and renewal of their fertility
- detoxification and decomposition of wastes
- pollination of crops and natural vegetation
- dispersal of seeds
- cycling and movement of nutrients
- control of the vast majority of potential normal agricultural pests
- maintenance of biodiversity, from which humanity has derived key elements of its normal agricultural, medicinal, and industrial enterprise
- protection of coastal shores from erosion by waves
- protection from the sun's harmful ultraviolet rays

- stabilization of the climate
- moderation of weather extremes and their impacts
- provision of aesthetic beauty and intellectual stimulation that lift the human spirit".

It is this home or basic infrastructure that many environmentalists and agriculturists think that our normal agricultural activities are affecting in quite unprecedented ways. It is also believed that the more pressure we exert on it due to our normal agricultural activities the more wear and tear we subject this home to and this can force our home to exceed its capacity. It is the belief that this may happen and if it does then the future generations will have to spend a lot in repairing and maintaining such a basic system (home) so as to function well. However the cost of maintaining and repairing this home is not the only worrying aspect, but the fact that some of this infrastructure if damaged or depleted cannot be built back by humans no matter how advanced we are now and will be in the future [40].

As the world population keeps on growing and the demand for food for survival is high, much pressure is put on normal agriculture to produce more. But the more the production from normal agriculture, the greater the pressure we exert on renewable agriculture and the ecosphere (i.e. natural environment), and the greater this pressure the more the renewable agriculture and the ecosphere get damaged and depleted. One of the frustrating but important questions we have to deal with is; how are we going to produce more (enough) to meet our needs now and over a long period of time without leaving a damaged and depleted natural environment for future generations [46,47]? It is this question that we try to address from a mathematical point of view in this chapter.

We model the interactive dynamics of the share of assets between normal agriculture, renewable agriculture, industry and the ecosphere using a system of four ordinary non-linear differential equations. We study the long term effects of each of these assets on each other.

In some developed countries where wilderness is gradually disappearing, there is a current trend towards the blurring of normal and renewable agriculture. For example, wild fisheries are being replaced by aquaculture, mariculture and fish farming; native forests by silviculture and plantations; pristine ecosystems by recreated nature reserves, national parks, marine reserves, etc. In such a situation separating agriculture into the two components of normal and renewable agriculture has to be done with care. Whereas one can easily classify all recreated nature reserves, national parks and marine reserves as renewable agriculture, there is no way forward as to how to classify aquaculture, mariculture, fish farming, silviculture and plantations. For example, we can classify aquaculture, mariculture or fish farming involving a few number of fish species as normal agriculture, but it will be very unrealistic to classify the world's largest aquarium, Georgia aquarium in Atlanta, Georgia, USA which houses over 100,000 animals representing 500 species from around the globe as normal agriculture. While our model in this chapter may not be able to capture "all the complex dynamics" of such current trends towards blurring normal and renewable agriculture, it will be able to do so in countries where wilderness, wild fisheries, native forests, pristine ecosystems, etc. exist.

This chapter consists of twelve sections. In §3.2 we develop the model. Equilibria for the system are determined in §3.3 and we perform a local stability analysis of these equilibria in §3.4. Global stability analysis of the equilibria is done in §3.5. Sufficient conditions for the existences of an interior equilibrium (persistence) are given in §3.6.

Criteria for extinction of normal agriculture and industry is given in §3.7. Criteria for extinction of renewable agriculture and industry is given in §3.8. Criteria for extinction of industry is given in §3.9 followed with criteria for the extinction of renewable agriculture in §3.10. This is then followed with some numerical examples in §3.11, and a brief discussion in §3.12.

3.2 The model

In modelling the interactions between normal agriculture, renewable agriculture, industry and the ecosphere we make use of the following assumptions and facts:

- Industry generates its assets from normal agriculture by selling to it.
- Industrial assets generation faces both fixed and variable expenses independent of agriculture and the ecosphere.
- Both renewable and normal agricultural asset creation depends on the ecosphere and hence we allow for a possibility of diminishing returns for both assets.
- Normal agricultural asset generation has a negative impact on the ecosphere and hence it cost normal agriculture to replenish the ecosphere.
- In the absence of normal agricultural activity the ecosphere can replenish itself.
- There is competition for assets between renewable agriculture and normal agriculture.
- Normal agricultural asset creation may be enhanced by industry at some cost.

- Industry has negative effects on renewable agriculture. For example, it is known that the quantity of nitrogen added to the soil through fertilizers now exceed the total fixed through natural processes [21,43,76]. As a result less than half of the nitrogen added to soil is taken up by plants and the rest runs off into underground water and ends up in lakes, the seas or seeps through the ground into acquifers. This has led to changes in marine vegetation, a decrease in coral reefs systems, fish populations, an outbreak of algae which has killed millions of fish, damaged fisheries and has rendered some beaches unusable (i.e. decrease in renewable agriculture assets) [40,43].

Let $x_1(t), x_2(t), y(t)$ and $z(t)$ represent normal agricultural assets, renewable agricultural assets, industrial assets and ecospheric assets respectively. Then the above reasoning motivates the model given by the following system of four ordinary differential equations

$$\dot{x_1} = (\alpha_1 z - \beta_1 x_1 + \gamma_1 y - \rho_1 x_2 - \theta(1-z))x_1 \tag{3.1}$$

$$\dot{x_2} = (\alpha_2 z - \beta_2 x_2 - \gamma_2 y - \rho_2 x_1)x_2 \tag{3.2}$$

$$\dot{y} = (-\xi - \eta y + \delta x_1)y \tag{3.3}$$

$$\dot{z} = -\kappa x_1 z + \vartheta(1-z)z + \phi(1-z)x_1 \tag{3.4}$$

with initial conditions $x_{10} = x_1(0) \geq 0$, $x_{20} = x_2(0) \geq 0$, $y_0 = y(0) \geq 0$, and $z_0 = z(0), 0 \leq z_0 \leq 1$, where all parameters are assumed to be positive constants except γ_1 which can be any real constant. $\alpha_1(\alpha_2)$ is the growth rate coefficient of normal(renewable) agriculture due to normal(renewable) agricultural activity for fixed z, $\beta_1(\beta_2)$ is the per asset diminishing returns rate coefficient for normal(renewable) agriculture in the absence of industry and renewable(normal) agriculture, $\gamma_1(\gamma_2)$ is the per asset terms of trade coefficient between normal(renewable) agriculture and industry, ξ is the constant depreciation rate coefficient of industry, η is the per asset (linear) depreciation rate coefficient of industry, δ is the per asset growth rate for industry in dealing with normal agriculture, $\rho_1(\rho_2)$ is the per asset competitive rate coefficient of renewable(normal) agriculture acting on normal(renewable) agriculture, κ is the per asset degradation rate coefficient of the ecosphere due to normal agricultural activities, ϑ is the natural restoration rate coefficient for the ecosphere, $\phi(1-z)x$ is the rate of effort input to restore the ecosphere by normal agriculture and $\theta(1-z)x$ is the net cost rate to normal agriculture to restore the ecosphere.

We state here that we have a local and natural monopolistic and monopsonistic market. That is we have only one provider (seller) of services which is industry and only one buyer which is normal agriculture. If γ_1, the per asset terms of trade coefficient between normal agriculture and industry is positive then industrial influence causes a net gain in agricultural assets. That is the cost to normal agriculture in dealing with industry is less than the benefits obtained by normal agriculture in the process. If the cost to normal agriculture in dealing with industry equals all the benefits obtained by agriculture in the process then $\gamma_1 = 0$. Moreover if industrial influence causes a net loss in agricultural assets then $\gamma_1 < 0$. It has been argued in [5,30] that, it is too often the case that industrial influence causes a net loss in agricultural assets.

Theorem 3.1. *If $\beta_1\eta - \gamma_1\delta > 0$, $\alpha_1\eta - \gamma_1\xi > 0$ and $\alpha_1\delta - \beta_1\xi > 0$, then the system given by equations (3.1)-(3.4) is dissipative, with attraction region contained in $\mathbf{A} = \{(x_1, x_2, y, z) : 0 \leq x_1 \leq M, 0 \leq x_2 \leq \frac{\alpha_2}{\beta_2}, 0 \leq y \leq N, 0 \leq z \leq 1\}$, where $M = max(\frac{\alpha_1}{\beta_1}, \;\; \frac{\alpha_1\eta-\gamma_1\xi}{\beta_1\eta-\gamma_1\delta})$ and $N = max(\frac{\alpha_1\delta-\beta_1\xi}{\beta_1\eta}, \;\; \frac{\alpha_1\delta-\beta_1\xi}{\beta_1\eta-\gamma_1\delta})$.*

Proof:

The part of the proof concerning the x_2 and z-variables is the same as the proof for the x and z-variables respectively, in Theorem 2.1 with the appropriate change of model parameters.

As for the x_1 and y variables, if $\gamma_1 < 0$ then we have

$$0 \leq \limsup_{t\to\infty} x_1(t) \leq \frac{\alpha_1}{\beta_1}$$

and

$$0 \leq \limsup_{t\to\infty} y(t) \leq \frac{\alpha_1\delta - \beta_1\xi}{\beta_1\eta}$$

(cf. Theorem 2.1).

On the other hand if $\gamma_1 > 0$, $\beta_1\eta - \gamma_1\delta > 0$, $\alpha_1\eta - \gamma_1\xi > 0$ and $\alpha_1\delta - \beta_1\xi > 0$. Then we have

$$0 \leq \limsup_{t\to\infty} x_1(t) \leq \frac{\alpha_1\eta - \gamma_1\xi}{\beta_1\eta - \gamma_1\delta}$$

and

$$0 \leq \limsup_{t\to\infty} y(t) \leq= \frac{\alpha_1\delta - \beta_1\xi}{\beta_1\eta - \gamma_1\delta}$$

(cf. Theorem 2.21). □

3.3 Equilibria

In this section, we attempt to describe all the possible configurations of equilibria for the system given by Equations (3.1)-(3.4). We are only interested in nonnegative equilibria. We shall denote by $F_a(x_1, x_2, y, z)$ the equilibria lying on the a-axis, $F_{ab}(x_1, x_2, y, z)$ the ones in the positive (a,b)-plane, by $F_{abc}(x_1, x_2, y, z)$ the ones in the positive (a,b,c)-octant and by $F^*(x_1, x_2, y, z)$ the positive interior equilibrium of the whole system.

Equilibrium conditions for the system given by (3.1)-(3.4) are found as solutions of the algebraic system

$$\dot{x_1} = \dot{x_2} = \dot{y} = \dot{z} = 0.$$

Thus, the equilibria are determined by the system

$$(\alpha_1 z - \beta_1 x_1 + \gamma_1 y - \rho_1 x_2 - \theta(1-z))x_1 = 0 \tag{3.5}$$

$$(\alpha_2 z - \beta_2 x_2 - \gamma_2 y - \rho_2 x_1)x_2 = 0 \tag{3.6}$$

$$(-\xi - \eta y + \delta x_1)y = 0 \tag{3.7}$$

$$-\kappa x_1 z + \vartheta(1-z)z + \phi(1-z)x_1 = 0. \tag{3.8}$$

It is trivial to see that $F_0(0,0,0,0)$ is an equilibrium.

3.3.1 Axial (one-dimensional) equilibria

(i) $z-$axis: $x_1 = x_2 = y = 0$ and $z \neq 0$.

Algebraic system (3.5)-(3.8) reduces to

$$\vartheta(1-z) = 0 \quad \Longrightarrow z = 1. \tag{3.9}$$

Hence there exist a nonnegative equilibrium $F_z(0,0,0,1)$.

(ii) x_1-axis: $z = x_2 = y = 0$ and $x_1 \neq 0$.

Algebraic system (3.5)-(3.8) reduces to

$$-\beta_1 x_1 - \theta = 0, \quad \phi x_1 = 0.$$

This system does not have nonnegative solutions and hence there are no equilibria on the x_1-axis.

(iii) x_2-axis: $z = x_1 = y = 0$ and $x_2 \neq 0$.

Algebraic system (3.5)-(3.8) reduces to

$$-\beta_2 x_2 = 0.$$

This system does not have non-zero solutions and hence there are no equilibria on the x_2-axis.

(iv) $y-$axis: $z = x_1 = x_2 = 0$ and $y \neq 0$.

Algebraic system (3.5)-(3.8) reduces to

$$-\xi - \eta y = 0.$$

This system does not have nonnegative solutions and hence there are no equilibria on the $y-$axis.

3.3.2 Planar (two-dimensional) equilibria

(i) $(x_1, x_2)-$plane: $z = y = 0$, $x_1 \neq 0$ and $x_2 \neq 0$.

Algebraic system (3.5)-(3.8) reduces to the system

$$-\beta_1 x_1 - \rho_1 x_2 - \theta = 0 \tag{3.10}$$

$$-\beta_2 x_2 - \rho_2 x_1 = 0 \tag{3.11}$$

$$\phi x_1 = 0 \tag{3.12}$$

which has no solutions for positive x_1 and x_2. Hence there are no equilibria in the $(x_1, x_2)-$plane.

(ii) $(x_1, y)-$plane: $z = x_2 = 0$, $x_1 \neq 0$ and $y \neq 0$.

Algebraic system (3.5)-(3.8) reduces to the system

$$-\beta_1 x_1 + \gamma_1 y - \theta = 0 \tag{3.13}$$

$$-\xi - \eta y + \delta x_1 = 0 \tag{3.14}$$

$$\phi x_1 = 0, \tag{3.15}$$

which has no solutions for positive x_1 and y. Hence there are no equilibria in the $(x_1, y)-$plane.

(iii) $(x_1, z)-$plane: $y = x_2 = 0$, $x_1 \neq 0$ and $z \neq 0$.

Algebraic system (3.5)-(3.8) reduces to the system

$$\alpha_1 z - \beta_1 x_1 - \theta(1 - z) = 0 \tag{3.16}$$

$$-\kappa x_1 z + \vartheta(1 - z)z + \phi(1 - z)x_1 = 0. \tag{3.17}$$

This has one and only one positive solution (see §2.4.1) given by

$$x_1 = \bar{x_1} = \frac{(\alpha_1 + \theta)\bar{z} - \theta}{\beta_1}, \tag{3.18}$$

$$\bar{z} = \frac{\kappa\theta + \vartheta\beta_1 - \phi\beta_1 + \sqrt{(\kappa\theta + \vartheta\beta_1 + \phi\beta_1)^2 + 4\beta_1\phi\kappa\alpha_1}}{2(\kappa\alpha_1 + \kappa\theta + \vartheta\beta_1)}. \tag{3.19}$$

Hence, this equilibrium always exists and we denote it by $F_{x_1 z}(\bar{x_1}, 0, 0, \bar{z})$.

(iv) $(x_2, y)-$plane: $z = x_1 = 0$, $x_2 \neq 0$ and $y \neq 0$.

Algebraic system (3.5)-(3.8) reduces to the system

$$-\beta_2 x_2 - \gamma_2 y = 0 \tag{3.20}$$

$$-\xi - \eta y = 0 \tag{3.21}$$

which has no solutions for positive x_2 and y. Hence, there are no equilibria in the $(x_2, y)-$plane.

(v) $(x_2, z)-$plane: $y = x_1 = 0$, $x_2 \neq 0$ and $z \neq 0$.

Algebraic system (3.5)-(3.8) reduces to the system

$$\alpha_2 z - \beta_2 x_2 = 0 \tag{3.22}$$

$$\vartheta(1-z)=0 \tag{3.23}$$

which has one and only one positive solution given by

$$\bar{x}_2=\frac{\alpha_2}{\beta_2}, \tag{3.24}$$

$$\bar{z}=1. \tag{3.25}$$

Hence, this equilibrium always exists and we denote it by $F_{x_2z}(0,\frac{\alpha_2}{\beta_2},0,1)$.

(vi) $(y,z)-$plane: $x_2=x_1=0$, $z\neq 0$ and $y\neq 0$.

Algebraic system (3.5)-(3.8) reduces to the system

$$-\xi-\eta y=0 \tag{3.26}$$

$$\vartheta(1-z)=0 \tag{3.27}$$

which has no solutions for positive y and z. Hence, there are no equilibria in the $(y,z)-$plane.

3.3.3 Positive octant (three-dimensional) equilibria

(i) $(x_1,x_2,y)-$octant: $z=0$, $x_1\neq 0, x_2\neq 0$ and $y\neq 0$.

Algebraic system (3.5)-(3.8) reduces to the system

$$-\beta_1x_1+\gamma_1y-\rho_1x_2-\theta(1-z)=0 \tag{3.28}$$

$$-\beta_2x_2-\gamma_2y-\rho_2x_1=0 \tag{3.29}$$

$$-\xi-\eta y+\delta x_1=0 \tag{3.30}$$

$$\phi x_1=0 \tag{3.31}$$

which has no positive solutions for x_1,x_2 and y, since $\phi\neq 0$. Thus, there are no equilibria in the positive $(x_1,x_2,y)-$octant.

(ii) $(x_2, y, z)-$octant: $x_1 = 0$, $x_2 \neq 0, y \neq 0$ and $z \neq 0$.

Algebraic system (3.5)-(3.8) reduces to the system

$$\alpha_2 z - \beta_2 x_2 - \gamma_2 y = 0 \tag{3.32}$$

$$-\xi - \eta y = 0 \tag{3.33}$$

$$+\vartheta(1 - z) = 0 \tag{3.34}$$

which has no positive solutions for x_2, y and z. Thus, there are no equilibria in the positive $(x_2, y, z)-$octant.

(iii) $(x_1, y, z)-$octant: $x_2 = 0$, $x_1 \neq 0, y \neq 0$ and $z \neq 0$.

Algebraic system (3.5)-(3.8) reduces to the system

$$\alpha_1 z - \beta_1 x_1 + \gamma_1 y - \theta(1 - z) = 0 \tag{3.35}$$

$$-\xi - \eta y + \delta x_1 = 0 \tag{3.36}$$

$$-\kappa x_1 z + \vartheta(1 - z)z + \phi(1 - z)x_1 = 0. \tag{3.37}$$

If $\gamma_1 \leq 0$, then the above system has one and only one positive solution for x_1, y and z if $-\xi + \delta \bar{x_1} > 0$ (see Theorem 2.4 and §2.4.2). If on the other hand $\gamma_1 > 0$ and $\beta_1\eta - \gamma_1\delta > 0$, then the above system has one and only one positive solutions for x_1, y and z if

1. $\beta_1\xi < \delta\alpha_1$ and

2. $(\delta\alpha_1 - \beta_1\xi)\{\vartheta(\beta_1\eta - \gamma_1\delta) + \phi(\alpha_1\eta + \theta\eta)\} \geq \kappa\eta(\theta + \alpha_1)(\beta_1\xi + \theta\delta)$ (see Theorem 2.23).

However if $\gamma_1 > 0$ and $\beta_1\eta - \gamma_1\delta = 0$, then there exists a unique solution to the above system if

1. $\gamma_1\xi \leq \alpha_1\eta$,

2. $\kappa\breve{z} - \phi(1-\breve{z}) > 0$,

3. $\frac{\vartheta(1-\breve{z})\breve{z}}{\kappa\breve{z}-\phi(1-\breve{z})} > \frac{\xi}{\delta}$, where $\breve{z} = \frac{\theta\eta+\gamma_1\xi}{\theta\eta+\alpha_1\eta}$ (see Theorem 2.26).

If this equilibrium exist, denote it by $F_{x_1yz}(\breve{x}_1, 0, \breve{y}, \breve{z})$.

(iv) $(x_1, x_2, z)-$octant: $y = 0$, $x_1 \neq 0, x_2 \neq 0$ and $z \neq 0$.

Algebraic system (3.5)-(3.8) reduces to the system

$$\alpha_1 z - \beta_1 x_1 - \rho_1 x_2 - \theta(1-z) = 0, \tag{3.38}$$

$$\alpha_2 z - \beta_2 x_2 - \rho_2 x_1 = 0, \tag{3.39}$$

$$-\kappa x_1 z + \vartheta(1-z)z + \phi(1-z)x_1 = 0. \tag{3.40}$$

Theorem 3.2. *The system given by equations (3.38) -(3.40) has at most one positive solution for x_1, x_2 and z. That is, the system (3.1)-(3.4) possesses at most one equilibrium in the positive $(x_1, x_2, z)-$octant.*

Proof:

We rewrite Equation (3.40) as

$$x_1 = f_1(z) = \frac{\vartheta(1-z)z}{\kappa z - \phi(1-z)}. \tag{3.41}$$

Also solving (3.39) for x_2, we get

$$x_2 = \frac{\alpha_2 z - \rho_2 x_1}{\beta_2}. \tag{3.42}$$

Substituting Equation (3.42) into (3.38), we get

$$(\alpha_1\beta_2 - \rho_1\alpha_2 + \theta\beta_2)z + (\rho_1\rho_2 - \beta_1\beta_2)x_1 - \theta\beta_2 = 0. \tag{3.43}$$

If $\rho_1\rho_2 - \beta_1\beta_2 = 0$ then z has at most one positive solution and hence, x_1 has at most one positive solution (by Equation (3.41)) and x_2 has at most only one positive solution (by Equation (3.42)). On the other hand if $\rho_1\rho_2 - \beta_1\beta_2 \neq 0$, then we can rewrite Equation (3.43) as

$$x_1 = f_2(z) = \frac{(\alpha_1\beta_2 - \rho_1\alpha_2 + \theta\beta_2)z}{\beta_1\beta_2 - \rho_1\rho_2} - \frac{\theta\beta_2}{\beta_1\beta_2 - \rho_1\rho_2}. \tag{3.44}$$

So if an equilibrium exists, then at least the graphs of $f_1(z)$ and $f_2(z)$ must intersect in the positive (x_1, z)-plane. Let us analyze the behaviours of $f_1(z)$ and $f_2(z)$. The function $f_2(z)$ is a line which passes through the points $(0, x_c = \frac{-\theta\beta_2}{\beta_1\beta_2-\rho_1\rho_2})$ and $(1, \frac{\alpha_1\beta_2-\rho_1\alpha_2}{\beta_1\beta_2-\rho_1\rho_2})$.

The function $f_1(z)$ has a vertical asymptote at $z = z_a = \frac{\phi}{\kappa+\phi}$. This function, $f_1(z)$ also passes through the points $(0,0)$ and $(1,0)$. It is monotonically decreasing and concave down on the interval $0 \leq z < z_a = \frac{\phi}{\kappa+\phi}$. It is also monotonically decreasing and concave up on the interval $z_a = \frac{\phi}{\kappa+\phi} < z \leq 1$. Hence $f_1(z)$ and $f_2(z)$ can only intersect at most once in the positive (x_1, z)–plane (see Figures 3.1 and 3.2).
□

Theorem 3.3. *The system (3.1)-(3.4) possesses exactly one positive equilibrium $F_{x_1x_2z}(\breve{x}_1, \breve{x}_2, 0, \breve{z})$ in the positive (x_1, x_2, z)-octant if either*

$$\begin{aligned}
&\beta_1\beta_2 - \rho_1\rho_2 = 0\\
&\alpha_1\beta_2 - \rho_1\alpha_2 \geq 0\\
&max(\frac{\phi}{\phi+\kappa}, \frac{\rho_2\breve{x}_1}{\alpha_2}) < \breve{z} \leq 1,
\end{aligned} \tag{3.45}$$

or

$$\beta_1\beta_2 - \rho_1\rho_2 \neq 0$$
$$\frac{\alpha_1\beta_2 - \rho_1\alpha_2}{\beta_1\beta_2 - \rho_1\rho_2} > 0 \tag{3.46}$$
$$\frac{\rho_2\breve{x}_1}{\alpha_2} < \breve{z} \leq 1,$$

where $\breve{z}$ *and* $\breve{x}_1$ *is the solution of (3.41) and (3.43).*

Proof:

If $\beta_1\beta_2 - \rho_1\rho_2 = 0$ and $\alpha_1\beta_2 - \rho_1\alpha_2 \geq 0$, then from (3.43) we have $0 < z = \breve{z} \leq 1$. If max $(\frac{\phi}{\phi+\kappa}, \frac{\rho_2\breve{x}_1}{\alpha_2}) < \breve{z} \leq 1$, then from Equations (3.41) and (3.42) we have a positive solution for $x_1 = \breve{x}_1$ and $x_2 = \breve{x}_2$ respectively.

On the other hand if $\beta_1\beta_2 - \rho_1\rho_2 \neq 0$ and $\frac{\alpha_1\beta_2-\rho_1\alpha_2}{\beta_1\beta_2-\rho_1\rho_2} > 0$, then from Equations (3.41) and (3.44) we have that there exist a unique positive solution for $x_1 = \breve{x}_1$ and $z = \breve{z}$ (see Figures 3.1 and 3.2). If $\frac{\rho_2\breve{x}_1}{\alpha_2} < \breve{z} \leq 1$ is satisfied then Equation (3.42) has a positive solution for $x_2 = \breve{x}_2$. □

We shall establish sufficient conditions for the existence of a positive interior equilibrium $F^*(x_1, x_2, y, z)$ using persistence theory in §3.6.

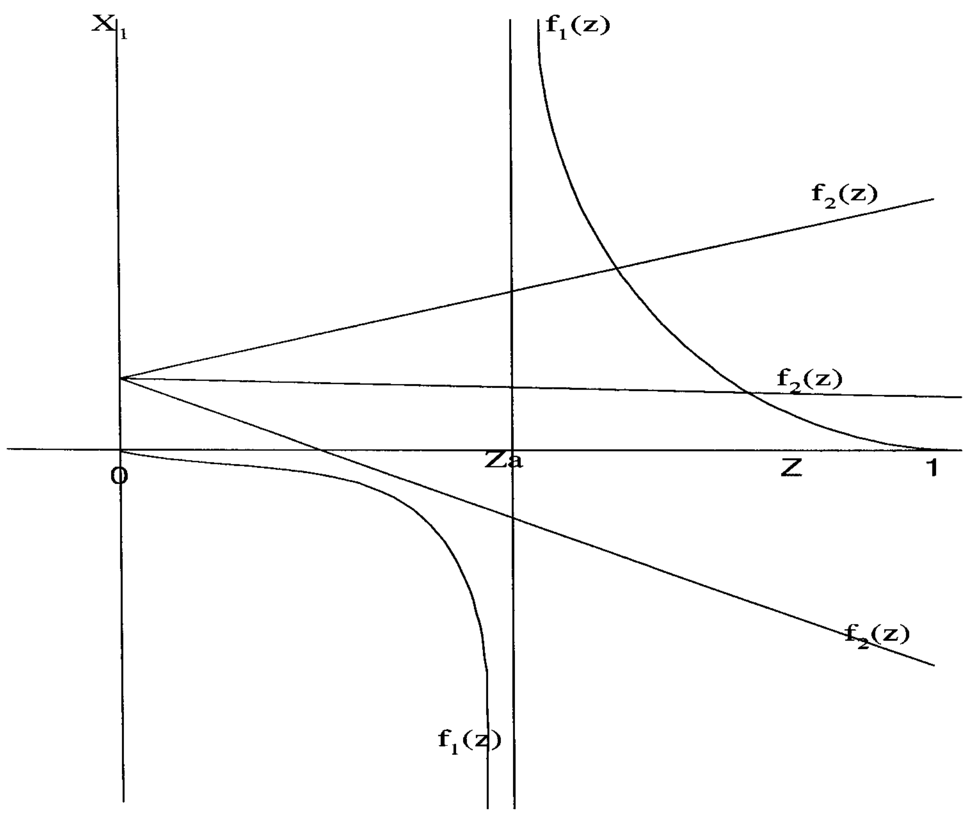

Figure 3.1: Different graphs of $f_1(z)$ and $f_2(z)$ when $\beta_1\beta_2 - \rho_1\rho_2 < 0$.

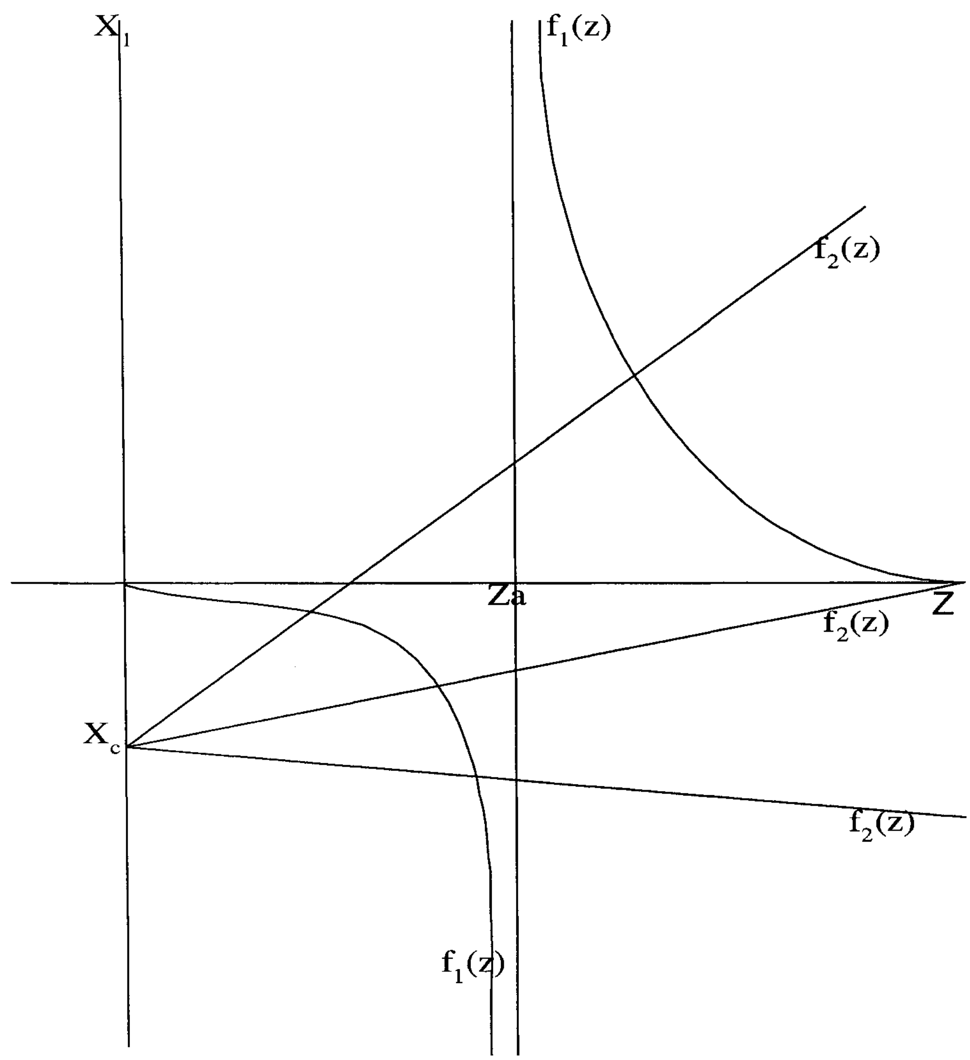

Figure 3.2: Different graphs of $f_1(z)$ and $f_2(z)$ when $\beta_1\beta_2 - \rho_1\rho_2 > 0$.

3.4 Local stability analysis of equilibria

In this section we determine the local stability properties of all the equilibria determined in §3.3. The local stability properties of these equilibria are determined by the signs of the the real parts of the eigenvalues of the variational matrix V of the system (3.1)-(3.4) evaluated at each equilibrium. The variational matrix V for the system (3.1)-(3.4) is given by

$$V = \begin{bmatrix} v_{11} & -\rho_1 x_1 & \gamma_1 x_1 & (\alpha_1 + \theta)x_1 \\ -\rho_2 x_2 & v_{22} & -\gamma_2 x_2 & \alpha_2 x_2 \\ \delta y & 0 & v_{33} & 0 \\ -(\kappa + \phi)z + \phi & 0 & 0 & v_{44} \end{bmatrix}, \tag{3.47}$$

where

$$v_{11} = \alpha_1 z - 2\beta_1 x_1 + \gamma_1 y - \theta(1 - z) - \rho_1 x_2$$

$$v_{22} = \alpha_2 z - 2\beta_2 x_2 - \gamma_2 y - \rho_2 x_1$$

$$v_{33} = -\xi - 2\eta y + \delta x_1$$

$$v_{44} = -\kappa x_1 + \vartheta(1 - 2z) - \phi x_1.$$

3.4.1 Local stability analysis of $F_0(0, 0, 0, 0)$

The variational matrix V about the equilibrium $F_0(0, 0, 0, 0)$ is given by

$$V_{F_0} = \begin{bmatrix} -\theta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\xi & 0 \\ \phi & 0 & 0 & \vartheta \end{bmatrix}. \tag{3.48}$$

The eigenvalues of V_{F_0} are given by $-\theta, 0, -\xi$ and ϑ. This equilibrium is non-hyperbolic and unstable since one of the eigenvalues is positive.

Theorem 3.4. *The equilibrium* $F_0(0,0,0,0)$ *is a non-hyperbolic and locally unstable. In fact* $F_0(0,0,0,0)$ *is a topological saddle point.*

Proof:

Consider the vector field given by the system (3.1)-(3.4). We also shown that the eigenvalues of (3.1)-(3.4) linearized about $(x_1, x_2, y, z) = (0,0,0,0)$ are $-\theta, 0, -\xi$, and ϑ. We will like to determine the flow on the center manifold, since this flow cannot be determined based on the above linearization.

The eigenvalues of V_{F_0} are given by $0, -\theta - \xi$ and ϑ with corresponding eigenvectors

$$\begin{bmatrix} -\frac{\vartheta+\theta}{\phi} \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, and \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}. \tag{3.49}$$

We first put the system (3.1)-(3.4) into standard form. Using the eigenbasis (3.49), we obtain the transformation

$$\begin{bmatrix} x_1 \\ x_2 \\ y \\ z \end{bmatrix} = \begin{bmatrix} -\frac{\vartheta+\theta}{\phi} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ v \\ w \end{bmatrix} \tag{3.50}$$

with inverse

$$\begin{bmatrix} u_1 \\ u_2 \\ v \\ w \end{bmatrix} = \begin{bmatrix} -\frac{\phi}{\vartheta+\theta} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{\phi}{\vartheta+\theta} & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ y \\ z \end{bmatrix}, \tag{3.51}$$

which transforms system (3.1)-(3.4) into

$$\begin{bmatrix} \dot{u_1} \\ \dot{u_2} \\ \dot{v} \\ \dot{w} \end{bmatrix} = \begin{bmatrix} -\theta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\xi & 0 \\ 0 & 0 & 0 & \vartheta \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ v \\ w \end{bmatrix}$$
$$+ \begin{bmatrix} -((\alpha_1\phi+\theta\phi+\beta_1\vartheta+\beta_1\theta)u_1-\rho_1\phi u_2+\gamma_1\phi v+\phi(\alpha_1+\theta)w)(\frac{\vartheta+\theta}{\phi^2})u_1 \\ ((\alpha_2\phi+\rho_2\vartheta+\rho_2\theta)u_1-\phi\beta_2 u_2-\phi\gamma_2 v+\alpha_2 w)\frac{u_2}{\phi} \\ (-(\frac{\delta_1\vartheta+\delta_1\theta}{\phi})u_1-\eta v)v \\ ((\kappa\vartheta+\kappa\theta+\theta\phi)u_1-\vartheta\phi w)(\frac{u_1+w}{\phi}) \end{bmatrix}. \tag{3.52}$$

Now using the center manifold theorem (Theorem 1.11), the local center manifold of (3.52) is given by

$$W^c(0) = \{(u_2,(u_1,v),w) \in \Re^c \times \Re^s \times \Re^r \quad |u_1 = h_1(u_2), v = h_2(u_2), w = h_3(u_2) \quad for \quad |u_2| < \epsilon\} \tag{3.53}$$

for some $\epsilon > 0$, where $h_1 \in C^n(N_\epsilon(0))$, $h_2 \in C^n(N_\epsilon(0))$, $h_3 \in C^n(N_\epsilon(0))$ and

$$h_1(0) = h_2(0) = h_3(0) = J(h_1(0)) = J(h_2(0)) = J(h_3(0)) = 0. \tag{3.54}$$

If we approximate $h_1(u_2), h_2(u_2)$ and $h_3(u_2)$ by their power series expansion, then from (3.54) we have that

$$u_1 = h_1(u_2) = au_2^2 + O(u_2^3)$$

$$v = h_2(u_2) = bu_2^2 + O(u_2^3)$$

$$w = h_3(u_2) = du_2^2 + O(u_2^3),$$

where a, b, c are constants to be determined. Hence we have that the flow on the center manifold $W^c(0)$ is defined by the differential equation

$$\dot{u}_2 = -\beta_2 u_2^2 + O(u_2^3). \tag{3.55}$$

Thus the flow on the center manifold is always away from the origin. □

3.4.2 Local stability analysis of $F_z(0, 0, 0, 1)$

The variational matrix V about the equilibrium $F_z(0, 0, 0, 1)$ is given by

$$V_{F_z} = \begin{bmatrix} \alpha_1 & 0 & 0 & 0 \\ 0 & \alpha_2 & 0 & 0 \\ 0 & 0 & -\xi & 0 \\ -\kappa & 0 & 0 & -\vartheta \end{bmatrix}. \tag{3.56}$$

The eigenvalues of V_{F_z} are given by $\alpha_1, \alpha_2, -\xi$ and $-\vartheta$.

Theorem 3.5. *The equilibrium $F_z(0, 0, 0, 1)$ is a hyperbolic saddle point, unstable in the x_1 and x_2-directions and locally stable in the y and z-directions. In particular the dimensions of the stable manifold is two and of the unstable manifold is two.*

3.4.3 Local stability analysis of $F_{x_1z}(\bar{x_1}, 0, 0, \bar{z})$

The variational matrix V about the equilibrium $F_{x_1z}(\bar{x_1}, 0, 0, \bar{z})$ is given by

$$V_{F_{x_1z}} = \begin{bmatrix} -\beta_1\bar{x_1} & -\rho_1\bar{x_1} & \gamma_1\bar{x_1} & (\alpha_1+\theta)\bar{x_1} \\ 0 & \alpha_2\bar{z}-\rho_2\bar{x_1} & 0 & 0 \\ 0 & 0 & -\xi+\delta\bar{x_1} & 0 \\ -(\kappa+\phi)\bar{z}+\phi & 0 & 0 & -(\frac{\vartheta\bar{z}^2+\phi\bar{x_1}}{\bar{z}}) \end{bmatrix}. \tag{3.57}$$

The eigenvalues of $V_{F_{x_1z}}$ are given by $\lambda_2 = \alpha_2\bar{z} - \rho_2\bar{x_1}$ and $\lambda_3 = -\xi + \delta\bar{x_1}$ and λ_1 and λ_4 which are the eigenvalues of J_{22} given by

$$J_{22} = \begin{bmatrix} -\beta_1\bar{x_1} & (\alpha+\theta)\bar{x_1} \\ -(\kappa+\phi)\bar{z}+\phi & -(\frac{\vartheta\bar{z}^2+\phi\bar{x_1}}{\bar{z}}) \end{bmatrix}. \tag{3.58}$$

$$Trace(J_{22}) = -(\beta_1\bar{x_1} + \frac{\vartheta\bar{z}^2+\phi\bar{x_1}}{\bar{z}}) < 0$$

and

$$\begin{aligned} det(J_{22}) &= (\frac{\vartheta\bar{z}^2+\phi\bar{x_1}}{\bar{z}})(\beta_1\bar{x_1}) + (\alpha_1+\theta)((\kappa+\phi)\bar{z}-\phi)\bar{x_1} \\ &= (\frac{\vartheta\bar{z}^2+\phi\bar{x_1}}{\bar{z}})(\beta_1\bar{x_1}) + (\alpha_1+\theta)(\kappa\bar{z}\bar{x_1}+\phi(\bar{z}-1)\bar{x_1}) \\ &= (\frac{\vartheta\bar{z}^2+\phi\bar{x_1}}{\bar{z}})(\beta_1\bar{x_1}) + (\alpha_1+\theta)\vartheta(1-\bar{z})\bar{z} \\ &> 0. \end{aligned}$$

Thus both λ_1 and λ_4 have negative real parts. Hence, we have the following two theorems:

Theorem 3.6. *The equilibrium $F_{x_1z}(\bar{x_1}, 0, 0, \bar{z})$ is locally asymptotically stable if $\lambda_2 = \alpha_2\bar{z} - \rho_2\bar{x_1} < 0$ and $\lambda_3 = -\xi + \delta\bar{x_1} < 0$.*

Theorem 3.7. *The equilibrium $F_{x_1z}(\bar{x}_1, 0, 0, \bar{z})$ is a hyperbolic saddle point if either $\lambda_2 = \alpha_2\bar{z} - \rho_2\bar{x}_1 > 0$ or $\lambda_3 = -\xi + \delta\bar{x}_1 > 0$ or both. It is always locally stable (attracting) in the x_1 and z-directions.*

3.4.4 Local stability analysis of $F_{x_2z}(0, \frac{\alpha_2}{\beta_2}, 0, 1)$

The variational matrix V about the equilibrium $F_{x_2z}(0, \frac{\alpha_2}{\beta_2}, 0, 1)$ is given by

$$V_{F_{x_2z}} = \begin{bmatrix} \frac{\alpha_1\beta_2-\rho_1\alpha_2}{\beta_2} & 0 & 0 & 0 \\ -\frac{\rho_2\alpha_2}{\beta_2} & -\alpha_2 & -\frac{\gamma_2\alpha_2}{\beta_2} & \frac{\alpha_2^2}{\beta_2} \\ 0 & 0 & -\xi & 0 \\ -\kappa & 0 & 0 & -\vartheta \end{bmatrix}. \tag{3.59}$$

The eigenvalues of $V_{F_{x_2z}}$ are given by $\frac{\alpha_1\beta_2-\rho_1\alpha_2}{\beta_2}$, $-\alpha_2$, $-\xi$ and $-\vartheta$. Hence, we have the following two theorems:

Theorem 3.8. *The equilibrium $F_{x_2z}(0, \frac{\alpha_2}{\beta_2}, 0, 1)$ is locally asymptotically stable if $\alpha_1\beta_2 - \rho_1\alpha_2 < 0$.*

Theorem 3.9. *The equilibrium $F_{x_2z}(0, \frac{\alpha_2}{\beta_2}, 0, 1)$ is a hyperbolic saddle point if $\alpha_1\beta_2 - \rho_1\alpha_2 > 0$ and it is a repeller in the x_1-direction.*

3.4.5 Local stability analysis of $F_{x_1yz}(\breve{x}_1, 0, \breve{y}, \breve{z})$

The variational matrix V about the equilibrium $F_{x_1yz}(\breve{x}_1, 0, \breve{y}, \breve{z})$ is given by

$$V_{F_{xy_1z}} = \begin{bmatrix} -\beta_1\breve{x}_1 & -\rho_1\breve{x}_1 & \gamma_1\breve{x}_1 & (\alpha_1+\theta)\breve{x}_1 \\ 0 & \alpha_2\breve{z} - \gamma_2\breve{y} - \rho_2\breve{x}_1 & 0 & 0 \\ \delta\breve{y} & 0 & -\eta\breve{y} & 0 \\ -(\kappa+\phi)\breve{z}+\phi & 0 & 0 & -(\frac{\vartheta\breve{z}^2+\phi\breve{x}_1}{\breve{z}}) \end{bmatrix}. \tag{3.60}$$

The eigenvalues of $V_{F_{x_1yz}}$ are given by $\lambda_2 = \alpha_2\breve{z} - \gamma_2\breve{y} - \rho_2\breve{x}_1$ and λ_1, λ_3 and λ_4 which are the eigenvalues of J_{33} given by

$$J_{33} = \begin{bmatrix} -\beta_1\breve{x}_1 & \gamma_1\breve{x}_1 & (\alpha_1+\theta)\breve{x}_1 \\ \delta\breve{y} & -\eta\breve{y} & 0 \\ -(\kappa+\phi)\breve{z}+\phi & 0 & -(\frac{\vartheta\breve{z}^2+\phi\breve{x}_1}{\breve{z}}) \end{bmatrix}. \tag{3.61}$$

Let

$$\begin{aligned} b_1 &= \beta_1\breve{x}_1 + \eta\breve{y} + (\frac{\vartheta\breve{z}^2+\phi\breve{x}_1}{\breve{z}}) > 0 \\ b_2 =&(\beta_1\breve{x}_1)(\frac{\vartheta\breve{z}^2+\phi\breve{x}_1}{\breve{z}}) + ((\kappa+\phi)\breve{z}-\phi)(\alpha_1+\theta)\breve{x}_1 \\ &+ \eta\breve{y}(\frac{\vartheta\breve{z}^2+\phi\breve{x}_1}{\breve{z}}) + \beta_1\eta\breve{y}\breve{x}_1 - \delta\gamma_1\breve{x}_1\breve{y} \\ b_3 =&\eta\breve{y}((\frac{\vartheta\breve{z}^2+\phi\breve{x}_1}{\breve{z}})\beta_1\breve{x}_1 + ((\kappa+\phi)\breve{z}-\phi)(\alpha_1+\theta)\breve{x}_1) \\ &- (\frac{\vartheta\breve{z}^2+\phi\breve{x}_1}{\breve{z}})\gamma_1\breve{x}_1\delta\breve{y}. \end{aligned}$$

Then the eigenvalues of J_{33} are given by

$$\sigma(J_{33}) = \{\lambda_j | \lambda_j^3 + b_1\lambda_j^2 + b_2\lambda_j + b_3 = 0, j = 1,3,4\}.$$

We note that

$$\begin{aligned} b_1b_2 - b_3 =&(\beta_1\breve{x}_1 + \eta\breve{y} + (\frac{\vartheta\breve{z}^2+\phi\breve{x}_1}{\breve{z}}))(\frac{\vartheta\breve{z}^2+\phi\breve{x}_1}{\breve{z}})(\beta_1\breve{x}_1 + \eta\breve{y}) \\ &+ (\beta_1\eta - \gamma_1\delta)(\beta_1\breve{x}_1 + \eta\breve{y})\breve{x}_1\breve{y} \\ &+ (\beta_1\breve{x}_1 + (\frac{\vartheta\breve{z}^2+\phi\breve{x}_1}{\breve{z}}))(\alpha_1+\theta)(1-\breve{z})\vartheta\breve{z}. \end{aligned}$$

Lemma 3.1. *If $\beta_1\eta - \gamma_1\delta \geq 0$, then $b_1 > 0$, $b_2 > 0$, $b_3 > 0$ and $b_1b_2 - b_3 > 0$ and hence, all the eigenvalues of J_{33} have negative real parts.*

Proof:

See §2.5.4.

The results of Lemma 3.1 lead us to the following Theorems:

Theorem 3.10. *If $\lambda_2 = \alpha_2\breve{z} - \gamma_2\breve{y} - \rho_2\breve{x}_1 > 0$, then $F_{x_1yz}(\breve{x}_1, 0, \breve{y}, \breve{z})$ is a hyperbolic saddle point unstable in the x_2-direction and stable in all other three directions provided $\beta_1\eta - \gamma_1\delta \geq 0$. In particular, the (x_1, y, z)-space forms the stable manifold and the unstable manifold is the x_2-axis.*

Theorem 3.11. *If $\lambda_2 = \alpha_2\breve{z} - \gamma_2\breve{y} - \rho_2\breve{x}_1 < 0$, then $F_{x_1yz}(\breve{x}_1, 0, \breve{y}, \breve{z})$ is locally asymptotically stable provided $\beta_1\eta - \gamma_1\delta \geq 0$.*

3.4.6 Local stability analysis of $F_{x_1x_2z}(\breve{x}_1, \breve{x}_2, 0, \breve{z})$

The variational matrix V about the equilibrium $F_{x_1x_2z}(\breve{x}_1, \breve{x}_2, 0, \breve{z})$ is given by

$$V_{F_{x_1x_2z}} = \begin{bmatrix} -\beta_1\breve{x}_1 & -\rho_1\breve{x}_1 & \gamma_1\breve{x}_1 & (\alpha_1+\theta)\breve{x}_1 \\ -\rho_2\breve{x}_2 & -\beta_2\breve{x}_2 & -\gamma_2\breve{x}_2 & \alpha_2\breve{x}_2 \\ 0 & 0 & -\xi+\delta_1\breve{x}_1 & 0 \\ -(\kappa+\phi)\breve{z}+\phi & 0 & 0 & -(\frac{\vartheta\breve{z}^2+\phi\breve{x}_1}{\breve{z}}) \end{bmatrix}. \tag{3.62}$$

The eigenvalues of $V_{F_{x_1x_2z}}$ are given by $\lambda_3 = -\xi + \delta_1\breve{x}_1$ and λ_1, λ_2 and λ_4 which are the eigenvalues of J_{33} given by

$$J_{33} = \begin{bmatrix} -\beta_1\breve{x}_1 & -\rho_1\breve{x}_1 & (\alpha_1+\theta)\breve{x}_1 \\ -\rho_2\breve{x}_2 & -\beta_2\breve{x}_2 & \alpha_2\breve{x}_2 \\ -(\kappa+\phi)\breve{z}+\phi & 0 & -(\frac{\vartheta\breve{z}^2+\phi\breve{x}_1}{\breve{z}}) \end{bmatrix}. \tag{3.63}$$

Let

$$b_1 = \beta_1\breve{x}_1 + \beta_2\breve{x}_2 + (\frac{\vartheta\breve{z}^2+\phi\breve{x}_1}{\breve{z}}) > 0,$$

$$
\begin{aligned}
b_2 =&(\beta_1\breve{x}_1)(\frac{\vartheta\breve{z}^2+\phi\breve{x}_1}{\breve{z}}) + ((\kappa+\phi)\breve{z}-\phi)(\alpha_1+\theta)\breve{x}_1 \\
&+ \beta_2\breve{x}_2(\frac{\vartheta\breve{z}^2+\phi\breve{x}_1}{\breve{z}}) + \beta_1\beta_2\breve{x}_1\breve{x}_2 - \rho_1\rho_2\breve{x}_1\breve{x}_2, \\
=&(\frac{\vartheta\breve{z}^2+\phi\breve{x}_1}{\breve{z}})(\beta_1\breve{x}_1+\beta_2\breve{x}_2) + (\alpha_1+\theta)(1-\breve{z})\vartheta\breve{z} \\
&+ (\beta_1\beta_2-\rho_1\rho_2)(\breve{x}_1\breve{x}_2), \\
b_3 =&\beta_2\breve{x}_2(\frac{\vartheta\breve{z}^2+\phi\breve{x}_1}{\breve{z}})\beta_1\breve{x}_1 + ((\kappa+\phi)\breve{z}-\phi)(\alpha_1+\theta)\breve{x}_1\beta_2\breve{x}_2 \\
&- (\frac{\vartheta\breve{z}^2+\phi\breve{x}_1}{\breve{z}})\rho_1\breve{x}_1\rho_2\breve{x}_2 - ((\kappa+\phi)\breve{z}-\phi)\rho_1\breve{x}_1\alpha_2\breve{x}_2, \\
=&(\frac{\vartheta\breve{z}^2+\phi\breve{x}_1}{\breve{z}})(\beta_1\beta_2-\rho_1\rho_2)\breve{x}_1\breve{x}_2 + \vartheta(\alpha_1\beta_2+\beta_2\theta-\alpha_2\rho_1)(1-\breve{z})\breve{z}\breve{x}_2.
\end{aligned}
$$

Then

$$
\begin{aligned}
b_1b_2 - b_3 =& \beta_1 b_2\breve{x}_1 + \vartheta\alpha_2\rho_1(1-\breve{z})\breve{z}\breve{x}_2 + (\frac{\vartheta\breve{z}^2+\phi\breve{x}_1}{\breve{z}})^2(\beta_1\breve{x}_1+\beta_2\breve{x}_2) \\
&+ (\frac{\vartheta\breve{z}^2+\phi\breve{x}_1}{\breve{z}})\{\vartheta(\alpha_1+\theta)(1-\breve{z})\breve{z} + (\beta_2\beta_1\breve{x}_1+\beta_2^2\breve{x}_2)\breve{x}_2{}^2\} \\
&+ \beta_2(\beta_1\beta_2-\rho_1\rho_2)\breve{x}_1\breve{x}_2{}^2.
\end{aligned}
$$

We observe that if the equilibrium $F_{x_1x_2z}(\breve{x}_1, \breve{x}_2, 0, \breve{z})$ exists and $\beta_1\beta_2 - \rho_1\rho_2 \geq 0$ then $b_1 > 0, b_2 > 0, b_3 > 0$ and $b_1b_2 - b_3 > 0$. Thus, we have the following lemma.

Lemma 3.2. *If the equilibrium $F_{x_1x_2z}(\breve{x}_1, \breve{x}_2, 0, \breve{z})$ exists and $\beta_1\beta_2 - \rho_1\rho_2 \geq 0$. Then $b_1 > 0$, $b_2 > 0$, $b_3 > 0$ and $b_1b_2 - b_3 > 0$ and hence, all the eigenvalues of J_{33} have negative real parts.*

Proof:

Follows from Routh-Hurwitz theorem (Theorem 1.3).

The results of Lemma 3.2 lead us to the following Theorems:

Theorem 3.12. *If $\lambda_3 = -\xi + \delta_1\breve{x}_1 > 0$, then $F_{x_1x_2z}(\breve{x}_1, \breve{x}_2, 0, \breve{z})$ is a hyperbolic saddle point unstable in the y-direction and stable in all other three directions provided $\beta_1\beta_2-$*

$\rho_1\rho_2 \geq 0$. *In particular, the (x_1, x_2, z)-space forms the stable manifold and the unstable manifold is the y-axis.*

Theorem 3.13. *If $\lambda_3 = -\xi + \delta_1\check{x}_1 < 0$, then $F_{x_1x_2z}(\check{x}_1, \check{x}_2, 0, \check{z})$ is locally asymptotically stable provided $\beta_1\beta_2 - \rho_1\rho_2 \geq 0$.*

3.5 Global stability analysis of equilibria

In this section, criteria for the global asymptotic stability of the boundary equilibria, $F_z(0,0,0,1)$, $F_{x_1z}(\bar{x_1}, 0, 0, \bar{z})$, $F_{x_2z}(0, \frac{\alpha_2}{\beta_2}, 0, 1)$, $F_{x_1yz}(\check{x}_1, 0, \check{y}, \check{z})$ and $F_{x_1,x_2,z}(\check{x}_1, \check{x}_2, 0, \check{z})$ with respect to solutions initiating from the interior of $\mathbf{R}^+_z$, $\mathbf{R}^+_{x_1z}$, $\mathbf{R}^+_{x_2z}$, $\mathbf{R}^+_{x_1yz}$ and $\mathbf{R}^+_{x_1x_2z}$ respectively will be established. These conditions are necessary for the boundary flow of system (3.1)-(3.4) to be acyclic and isolated which in turn is a necessary condition for persistence.

3.5.1 Global asymptotic stability of $E_z(1)$

We have already established in §2.6.1 that $E_z(1)$ and consequently $F_z(0,0,0,1)$ is globally asymptotically stable with respect to solution trajectories initiating from the interior of $\mathbf{R}^+_z$.

3.5.2 Global asymptotic stability of $E_{x_1z}(\bar{x_1}, \bar{z})$

We have shown in §3.3.2 that the equilibrium $F_{x_1z}(\bar{x_1}, 0, 0, \bar{z})$ always exists, hence the 2-dimensional equilibrium $E_{x_1z}(\bar{x_1}, \bar{z})$ for the system (3.1)-(3.4) restricted to $\mathbf{R}^+_{x_1z}$ also exists. In this section we shall establish criteria for the global asymptotic stability of the 2-dimensional equilibrium, $E_{x_1z}(\bar{x_1}, \bar{z})$ with respect to the system (3.1)-(3.4) restricted to $\mathbf{R}^+_{x_1z}$. This consequently will be the criteria for the global asymptotic stability of $F_{x_1z}(\bar{x}_1, 0, 0, \bar{z})$ with respect to solutions emanating from the interior of $\mathbf{R}^+_{x_1z}$.

Theorem 3.14. *The equilibrium $F_{x_1z}(\bar{x}_1, 0, 0, \bar{z})$ is globally asymptotically stable with respect to solution trajectories emanating from the interior of $\mathbf{R}^+_{x_1z}$.*

Proof:

The proof of this theorem is the same as the proof of Theorem 2.15 in §2.6.2 with the appropriate change of model parameters.

3.5.3 Global stability of $E_{x_2z}(\frac{\alpha_2}{\beta_2}, 1)$

The existence of the 2-dimensional equilibrium $E_{x_2z}(\frac{\alpha_2}{\beta_2}, 1)$ for the system (3.1)-(3.4) restricted to $\mathbf{R}^+_{x_2z}$ is a direct consequence of the existence of the equilibrium $F_{x_2z}(0, \frac{\alpha_2}{\beta_2}, 0, 1)$ for the system (3.1)-(3.4). In this subsection criteria for the global asymptotic stability of $E_{x_2z}(\frac{\alpha_2}{\beta_2}, 1)$ or equivalently $F_{x_2z}(0, \frac{\alpha_2}{\beta_2}, 0, 1)$ with respect to solution initiating from the interior of $\mathbf{R}^+_{x_2z}$ will be established. In $\mathbf{R}^+_{x_2z}$, the system (3.1)-(3.4) reduces to

$$\begin{aligned} \dot{x_2} &= (\alpha_2 z - \beta_2 x_2)x_2 \\ \dot{z} &= \vartheta(1-z)z. \end{aligned} \tag{3.64}$$

In $\mathbf{R}^+_{x_2z}$, we choose a Liapunov function $V(x_2, z)$ defined by

$$V(x_2, z) = V = \frac{\alpha_2 z - \alpha_2 - \alpha_2 ln(z)}{2\vartheta} + 2\{\frac{\beta_2 x_2}{\alpha_2} - 1 - ln(\frac{x_2\beta_2}{\alpha_2})\}. \tag{3.65}$$

The derivative of (3.58) along the solution curves of (3.57) in $\mathbf{R}^+_{x_2z}$ is given by

$$\begin{aligned}
\dot{V} &= \frac{(z-1)\alpha_2\dot{z}}{2\vartheta z} + \frac{2\dot{x}_2(\beta_2 x_2 - \alpha_2)}{\alpha_2 x_2} \\
&= -\frac{\alpha_2(1-z)^2}{2} - \frac{2(\alpha_2 - \beta_2 x_2)^2}{\alpha_2} + 2(1-z)(\alpha_2 - \beta_2 x_2) \\
&= -[\sqrt{\frac{\alpha_2}{2}}(1-z) - \sqrt{\frac{2}{\alpha_2}}(\alpha_2 - \beta_2 x_2)]^2 \\
&\leq 0.
\end{aligned}$$

Hence we have the following theorem:

Theorem 3.15. *The equilibrium $E_{x_2z}(\frac{\alpha_2}{\beta_2}, 1)$ and consequently $F_{x_2z}(0, \frac{\alpha_2}{\beta_2}, 0, 1)$ is globally stable with respect to solution trajectories initiating from the interior of $\mathbf{R}^+_{x_2z}$.*

3.5.4 Global stability of $E_{x_1yz}(\breve{x}_1, \breve{y}, \breve{z})$

The existence of the 3-dimensional equilibrium $E_{x_1yz}(\breve{x}_1, \breve{y}, \breve{z})$ for the system (3.1)-(3.4) restricted to $\mathbf{R}^+_{x_1yz}$ is a direct consequence of the existence of the equilibrium $F_{x_1yz}(\breve{x}_1, 0, \breve{y}, \breve{z})$ for the system (3.1)-(3.4).

Theorem 3.16. *The 3-dimensional equilibrium $E_{x_1yz}(\breve{x}_1, \breve{y}, \breve{z})$ for the system (3.1)-(3.4) restricted to $\mathbf{R}^+_{x_1yz}$ is globally (asymptotically) stable if it exists and $\gamma_1 < 0$ or $\beta_1\eta - \gamma_1\delta \geq 0$.*

Proof:

See Theorems 2.16, 2.24 and 2.27.

3.5.5 Global stability of $E_{x_1x_2z}(\breve{x}_1, \breve{x}_2, \breve{z})$

The existence of the 3-three dimensional equilibrium $E_{x_1x_2z}(\breve{x}_1, \breve{x}_2, \breve{z})$ for the system (3.1)-(3.4) restricted to $\mathbf{R}^+_{x_1x_2z}$ is a direct consequence of the existence of the equilibrium $F_{x_1yz}(\breve{x}_1, \breve{x}_2, 0, \breve{z})$ for system (3.1)-(3.4).

Consider system (3.1)-(3.4) restricted to $\mathbf{R}^+_{x_1x_2z}$ as represented by:

$$\begin{aligned}
\dot{x_1} &= (\alpha_1 z - \beta_1 x_1 - \rho_1 x_2 - \theta(1-z))x_1 \\
\dot{x_2} &= (\alpha_2 z - \beta_2 x_2 - \rho_2 x_1)x_2 \\
\dot{z} &= -\kappa x_1 z + \vartheta(1-z)z + \phi(1-z)x_1.
\end{aligned} \tag{3.66}$$

Let m, n and p be positive constants defined by $m\rho_1 = \kappa + \phi - \frac{\phi}{\breve{z}} = \frac{\vartheta(1-\breve{z})}{\breve{x}_1}$, $n\rho_2 = \frac{\vartheta(1-\breve{z})}{\breve{x}_1}$ and $p\rho_1 = \theta + \alpha_1$.

In $\mathbf{R}^+_{x_1x_2z}$, we choose a Liapunov function $V = V(x_1, x_2, z)$ defined by

$$\begin{aligned}
V(x_1, x_2, z) = m\{x_1 - \breve{x}_1 - \breve{x}_1(\ln(\frac{x_1}{\breve{x}_1})\} + n\{x_2 - \breve{x}_2 - \breve{x}_2(\ln(\frac{x_2}{\breve{x}_2})\} \\
+ p\{z - \breve{z} - \breve{z}(\ln(\frac{z}{\breve{z}})\}.
\end{aligned} \tag{3.67}$$

The derivative of (3.60) along the solution curves of (3.59) in $\mathbf{R}^+_{x_1x_2z}$ is given by

$$\begin{aligned}
\dot{V} &= \frac{m\dot{x}_1(x_1-\breve{x}_1)}{x_1} + \frac{n\dot{x}_2(x_2-\breve{x}_2)}{x_2} + \frac{p\dot{z}(z-\breve{z})}{z} \\
&= m(\alpha_1 z - \beta_1 x_1 - \rho_1 x_2 - \theta(1-z))(x_1-\breve{x}_1) + n(\alpha_2 z - \beta_2 x_2 - \rho_2 x_1)(x_2-\breve{x}_2) \\
&+ p(-\kappa x_1 + \vartheta(1-z) - \phi x_1)(z-\breve{z}) + p\phi x_1(z-\breve{z})/z \\
&= m(\alpha_1(z-\breve{z}) - \beta_1(x_1-\breve{x}_1) - \rho_1(x_2-\breve{x}_2) + \theta(z-\breve{z}))(x_1-\breve{x}_1) \\
&+ n(\alpha_2(z-\breve{z}) - \beta_2(x_2-\breve{x}_2) - \rho_2(x_1-\breve{x}_1))(x_2-\breve{x}_2) \\
&+ p(-\kappa(x_1-\breve{x}_1) - \vartheta(z-\breve{z}) - \phi(x_1-\breve{x}_1))(z-\breve{z}) + p\phi(\frac{x_1}{z} - \frac{\breve{x}_1}{\breve{z}})(z-\breve{z}) \\
&= -m\beta_1(x_1-\breve{x}_1)^2 - n\beta_2(x_2-\breve{x}_2)^2 - p(\vartheta + \phi\frac{x_1}{z\breve{z}})(z-\breve{z})^2 \\
&+ (m(\alpha_1+\theta) - p(\kappa + \phi - \frac{\phi}{\breve{z}}))(x_1-\breve{x}_1)(z-\breve{z}) - (m\rho_1 + n\rho_2)(x_1-\breve{x}_1)(x_2-\breve{x}_2) \\
&+ n\alpha_2(x_2-\breve{x}_2)(z-\breve{z}) \\
&= -m\beta_1(x_1-\breve{x}_1)^2 - n\beta_2(x_2-\breve{x}_2)^2 - p(\vartheta + \phi\frac{x_1}{z\breve{z}})(z-\breve{z})^2 \\
&- (m\rho_1 + n\rho_2)(x_1-\breve{x}_1)(x_2-\breve{x}_2) + n\alpha_2(x_2-\breve{x}_2)(z-\breve{z}) \\
&\leq X^t B X,
\end{aligned} \tag{3.68}$$

where

$$X = \begin{bmatrix} x_1-\breve{x}_1 \\ x_2-\breve{x}_2 \\ z-\breve{z} \end{bmatrix}, \tag{3.69}$$

$$B = \begin{bmatrix} -m\beta_1 & -\frac{m\rho_1+n\rho_2}{2} & 0 \\ -\frac{m\rho_1+n\rho_2}{2} & -n\beta_2 & \frac{n\alpha_2}{2} \\ 0 & \frac{n\alpha_2}{2} & -p\vartheta \end{bmatrix}, \tag{3.70}$$

and X^t denotes the transpose of X.

The characteristic equation of the matrix B in (3.63) is given by

$$\lambda^3 + \tau_2\lambda^2 + \tau_1\lambda + \tau_0 = 0,$$

where

$$\tau_2 = m\beta_1 + n\beta_2 + p\vartheta,$$
$$\tau_1 = mn\beta_1\beta_2 + mp\beta_1\vartheta + np\beta_2\vartheta - (\frac{m\rho_1 + n\rho_2}{2})^2 - (\frac{n\alpha_2}{2})^2,$$
$$\tau_0 = mnp\beta_1\beta_2\vartheta - m\beta_1(\frac{n\alpha_2}{2})^2 - p\vartheta(\frac{m\rho_1 + n\rho_2}{2})^2.$$

Theorem 3.17. *The 3-dimensional equilibrium $E_{x_1x_2z}(\breve{x}_1, \breve{x}_2, \breve{z})$ and consequently $F_{x_1x_2z}(\breve{x}_1, \breve{x}_2, \breve{z})$ is globally asymptotically stable with respect to solutions initiating in the interior of $\mathbf{R}^+_{x_1x_2z}$ if*

$$4\rho_2\breve{x}_1(\theta + \alpha_1)(\beta_1\beta_2 - \rho_1\rho_2) - \beta_1\rho_1\alpha_2^2(1 - \breve{z}) > 0. \tag{3.71}$$

Proof:

We observe that if (3.64) is satisfied then

$$\beta_1\beta_2 - \rho_1\rho_2 > 0. \tag{3.72}$$

We also know that the characteristic equation of the matrix B in (3.63) is given by

$$\lambda^3 + \tau_2\lambda^2 + \tau_1\lambda + \tau_0 = 0,$$

where

$$\tau_2 = m\beta_1 + n\beta_2 + p\vartheta,$$
$$\tau_1 = mn\beta_1\beta_2 + mp\beta_1\vartheta + np\beta_2\vartheta - (\frac{m\rho_1 + n\rho_2}{2})^2 - (\frac{n\alpha_2}{2})^2,$$
$$\tau_0 = mnp\beta_1\beta_2\vartheta - m\beta_1(\frac{n\alpha_2}{2})^2 - p\vartheta(\frac{m\rho_1 + n\rho_2}{2})^2.$$

Clearly

$$\tau_2 = m\beta_1 + n\beta_2 + p\vartheta > 0.$$

Also

$$\begin{aligned}
\tau_0 &= mnp\beta_1\beta_2\vartheta - m\beta_1(\frac{n\alpha_2}{2})^2 - p\vartheta(\frac{m\rho_1 + n\rho_2}{2})^2 \\
&= \frac{m^2 p\rho_1\beta_1\beta_2\vartheta}{\rho_2} - m\beta_1(\frac{m\rho_1\alpha_2}{2\rho_2})^2 - p\vartheta(m\rho_1)^2 \\
&= \frac{m^2}{4\rho_2^2}(4p\rho_1\rho_2\beta_1\beta_2\vartheta - m\beta_1\rho_1^2\alpha_2^2 - 4p\rho_1^2\rho_2^2\vartheta) \\
&= \frac{m^2}{4\rho_2^2}(4p\rho_1\rho_2\vartheta\{\beta_1\beta_2 - \rho_1\rho_2\} - m\beta_1\rho_1^2\alpha_2^2) \\
&= \frac{m^2}{4\rho_2^2}(4\rho_2\vartheta(\theta + \alpha_1)\{\beta_1\beta_2 - \rho_1\rho_2\} - \beta_1\rho_1\alpha_2^2\vartheta(\frac{1-\breve{z}}{\breve{x}_1})) \\
&= \frac{m^2\vartheta}{4\rho_2^2\breve{x}_1}(4\rho_2\breve{x}_1(\theta + \alpha_1)\{\beta_1\beta_2 - \rho_1\rho_2\} - \beta_1\rho_1\alpha_2^2(1-\breve{z})) \\
&\geq 0.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\tau_1 &= mn\beta_1\beta_2 + mp\beta_1\vartheta + np\beta_2\vartheta - (\frac{m\rho_1 + n\rho_2}{2})^2 - (\frac{n\alpha_2}{2})^2 \\
&= (mn\beta_1\beta_2 - (m\rho_1)^2) + (np\beta_2\vartheta - (\frac{n\alpha_2}{2})^2) + mp\beta_1\vartheta \\
&= \frac{m^2\rho_1(\beta_1\beta_2 - \rho_1\rho_2)}{\rho_2} + \frac{\tau_0 + p\vartheta(m\rho_1)^2}{m\beta_1} + mp\beta_1\vartheta \\
&> 0.
\end{aligned}$$

Lastly,

$$\begin{aligned}
\tau_2\tau_1 - \tau_0 &= \{m\beta_1 + n\beta_2 + p\vartheta\}\{\frac{m^2\rho_1(\beta_1\beta_2 - \rho_1\rho_2)}{\rho_2} + \frac{\tau_0 + p\vartheta(m\rho_1)^2}{m\beta_1} + mp\beta_1\vartheta\} - \tau_0 \\
&= (n\beta_2 + p\vartheta)\tau_1 + m\beta_1\{\frac{m^2\rho_1(\beta_1\beta_2 - \rho_1\rho_2)}{\rho_2} + mp\beta_1\vartheta\} + p\vartheta(m\rho_1)^2 \\
&> 0.
\end{aligned}$$

Hence by the Routh-Hurwitz criterion (Theorem 1.3), all the eigenvalues of B have negative real parts. That is X^tBX is negative definite by Theorem 1.4. □

3.6 Existence of interior equilibrium

In this section we shall present results on persistence, uniform persistence and finally give sufficient criteria for the existence of a positive interior equilibrium $F^\star(x_1^\star, x_2^\star, y^\star, z^\star)$.

Theorem 3.18. *Assume that the system given by Equations (3.1)-(3.4) is such that*

1. $\beta_1\eta - \gamma_1\delta > 0$, $\alpha_1\eta - \gamma_1\xi > 0$ *and* $\alpha_1\delta - \beta_1\xi > 0$

2. $F_{x_1z}(\bar{x}_1, 0, 0, \bar{z})$ *is a hyperbolic saddle point and repelling in the* x_2 *and* y*-directions locally (cf. Theorem 3.7)*

3. $F_{x_2z}(0, \frac{\alpha_2}{\beta_2}, 0, 1)$ *is a hyperbolic saddle point and repelling in the* x_1*-direction locally (cf. Theorem 3.9)*

4. $F_{x_1yz}(\breve{x}_1, 0, \breve{y}, \breve{z})$ *is a hyperbolic saddle point and repelling in the* x_2*-direction (cf. Theorem 3.10)*

5. $F_{x_1x_2z}(\breve{x}_1, \breve{x}_2, 0, \breve{z})$ *is a hyperbolic saddle point repelling in the* y*-direction (cf. Theorem 3.12)*

6. *The 3-dimensional equilibrium* $E_{x_1x_2z}(\breve{x}_1, \breve{x}_2, \breve{z})$ *and consequently* $F_{x_1x_2z}(\breve{x}_1, \breve{x}_2, \breve{z})$ *is globally asymptotically stable with respect to solutions initiating in the interior of* $\mathbf{R}^+_{x_1x_2z}$.

Then the system given by Equations (3.1)-(3.4) exhibits uniform persistence.

Proof:

The proof of this theorem will be done using Theorem 1.9 (Butler and McGehee) as we did in the proof of Theorem 2.18.

We have shown in Theorem 3.1 that if condition 1 is satisfied then $\mathbf{A} = \{(x_1, x_2, y, z) : 0 \leq x_1 \leq M, 0 \leq x_2 \leq \frac{\alpha_2}{\beta_2}, 0 \leq y \leq N, 0 \leq z \leq 1\}$, where $M = max(\frac{\alpha_1}{\beta_1}, \ \frac{\alpha_1\eta - \gamma_1\xi}{\beta_1\eta - \gamma_1\delta})$ and $N = max(\frac{\alpha_1\delta - \beta_1\xi}{\beta_1\eta}, \ \frac{\alpha_1\delta - \beta_1\xi}{\beta_1\eta - \gamma_1\delta})$ is positively invariant and any solution initiating at a point in $\mathbf{R}^+_{x_1x_2yz}$ will eventually enter $\mathbf{A}$ and hence, is eventually bounded (dissipative).

We have also shown that the only compact invariants sets on the boundary of $\mathbf{R}^+_{x_1x_2yz}$ are $F_0(0,0,0,0)$, $F_z(0,0,0,1)$, $F_{x_1z}(\bar{x}_1, 0, 0, \bar{z})$, $F_{x_2z}(0, \frac{\alpha_2}{\beta_2}, 0, 1)$, $F_{x_1yz}(\breve{x}_1, 0, \breve{y}, \breve{z})$, and $F_{x_1x_2z}(\breve{x}_1, \breve{x}_2, 0, \breve{z})$. Let $F^\star = (x_1^\star, x_2{}^\star, y^\star, z^\star)$ with $x_1^\star > 0, x_2{}^\star > 0, y^\star > 0$ and $z^\star > 0$ be a point in the interior of $\mathbf{R}^+_{x_1x_2yz}$ and $O(F^\star)$ be the orbit through the point $F^\star$. The proof of the above theorem is completed by showing the following:

1. $F_0(0,0,0,0) \notin \Omega(F^\star)$.

2. $F_z(0,0,0,1) \notin \Omega(F^\star)$.

3. $F_{x_1z}(\bar{x}_1, 0, 0, \bar{z}) \notin \Omega(F^\star)$.

4. $F_{x_2z}(0, \frac{\alpha_2}{\beta_2}, 0, 1) \notin \Omega(F^\star)$.

5. $F_{x_1yz}(\breve{x}_1, 0, \breve{y}, \breve{z}) \notin \Omega(F^\star)$.

6. $F_{x_1x_2z}(\breve{x}_1, \breve{x}_2, 0, \breve{z}) \notin \Omega(F^\star)$.

7. No other point on the boundary of $\mathbf{R}^+_{x_1x_2yz}$ belongs to $\Omega(F^\star)$.

We have shown in Theorem 3.4, that $F_0(0,0,0,0)$ is a topological saddle point, hence the proof of the above facts (1)-(7) follow similarly as the proof in Theorem 2.18. The fact that boundary flow of the system (3.1)-(3.4) is acyclic and isolated is also similar to the proof in Theorem 2.18. $\square$

Theorem 3.19. *If all the conditions Theorem 3.18 are satisfied, then the system given by Equations (3.1)-(3.4) exhibits uniform persistence and contains an equilibrium of the form* $F^\star = (x_1^\star, x_2{}^\star, y^\star, z^\star)$ *with* $x_1^\star > 0, x_2{}^\star > 0, y^\star > 0$ *and* $z^\star > 0$.

Proof:

This is a direct consequence of Theorem 3.18. □

3.7 Criteria for extinction of normal agriculture and industry

In this section, we shall derive sufficient conditions for the extinction of both normal agriculture and industry. This corresponds to a situation where there will be no cultivation of land, production of crops and raising of animals for human consumption.

In order to obtain the conditions for the total extinction of both normal agriculture and industry, we use a Liapunov function to establish criteria for the global asymptotic stability for the equilibrium $F_{x_2 z}(0, \frac{\alpha_2}{\beta_2}, 0, 1)$ with respect to $\mathbf{R}^+_{x_1 x_2 y z}$. We choose the Liapunov function $V = V(x_1, x_2, y, z)$ as given by

$$V = mx_1 + n\{x_2 - \frac{\alpha_2}{\beta_2} - \frac{\alpha_2}{\beta_2}\ln(\frac{\beta_2 x_2}{\alpha_2})\} + py + q(z - 1 - \ln(z)). \tag{3.73}$$

The derivative of (3.73) along the solution curves of system given by Equations (3.1)-(3.4) is given by

$$
\begin{aligned}
\dot{V} &= m\dot{x}_1 + \frac{n\dot{x}_2}{x_2}(x_2 - \frac{\alpha_2}{\beta_2}) + p\dot{y} + \frac{q\dot{z}}{z}(z-1) \\
&= m(\alpha_1 z - \beta_1 x_1 + \gamma_1 y - \rho_1 x_2 - \theta(1-z))x_1 + n(\alpha_2 z - \beta_2 x_2 - \gamma_2 y - \rho_2 x_1)(x_2 - \frac{\alpha_2}{\beta_2}) \\
&+ p(-\xi - \eta y + \delta x_1)y + q(-\kappa x_1 + \vartheta(1-z) - \phi x_1)(z-1) + \frac{q\phi(z-1)x_1}{z} \\
&= -m\beta_1 x_1^2 - n\beta_2(x_2 - \frac{\alpha_2}{\beta_2})^2 - p\eta y^2 - q(\vartheta + \frac{\phi x_1}{z})(z-1)^2 + x_1(x_2 - \frac{\alpha_2}{\beta_2})(-m\rho_1 - n\rho_2) \\
&+ x_1 y(m\gamma_1 + p\delta_1) + x_1(z-1)(m(\alpha_1+\theta) - q\kappa) - n\gamma_2 y(x - \frac{\alpha_2}{\beta_2}) + n\alpha_2(x_2 - \frac{\alpha_2}{\beta_2})(z-1) \\
&- p\xi y + m(\frac{\beta_2\alpha_1 - \rho_1\alpha_2}{\beta_2}).
\end{aligned} \tag{3.74}
$$

Now suppose $\gamma_1 < 0$, that is the normal agricultural growth can only be enhanced by the ecosphere. Suppose also that $\beta_2\alpha_1 - \rho_1\alpha_2 < 0$. Then from (3.74), we have

$$
\begin{aligned}
\dot{V} &< -m\beta_1 x_1^2 - n\beta_2(x_2 - \frac{\alpha_2}{\beta_2})^2 - p\eta y^2 - q(\vartheta + \frac{\phi x_1}{z})(z-1)^2 \\
&+ x_1(x_2 - \frac{\alpha_2}{\beta_2})(-m\rho_1 - n\rho_2) + x_1 y(m\gamma_1 + p\delta_1) + x_1(z-1)(m(\alpha_1+\theta) - q\kappa) \\
&- n\gamma_2 y(x - \frac{\alpha_2}{\beta_2}) + n\alpha_2(x_2 - \frac{\alpha_2}{\beta_2})(z-1).
\end{aligned} \tag{3.75}
$$

Now we choose m,n,p and q positive constants such that

$$
m = \frac{1}{\rho_1}, \quad n = \frac{1}{\rho_2}, \quad p = \frac{-\gamma_1}{\delta_1\rho_1}, \quad q = \frac{\alpha_1+\theta}{\kappa\rho_1}.
$$

Then from (3.68) we have

$$\dot{V} < -m\beta_1 x_1^2 - n\beta_2(x_2 - \frac{\alpha_2}{\beta_2})^2 - p\eta y^2 - q(\vartheta + \frac{\phi x_1}{z})(z-1)^2$$
$$- 2x_1(x_2 - \frac{\alpha_2}{\beta_2}) - n\gamma_2 y(x - \frac{\alpha_2}{\beta_2}) + n\alpha_2(x_2 - \frac{\alpha_2}{\beta_2})(z-1)$$
$$\leq -m\beta_1 x_1^2 - n\beta_2(x_2 - \frac{\alpha_2}{\beta_2})^2 - p\eta y^2 - q\vartheta(z-1)^2 - 2x_1(x_2 - \frac{\alpha_2}{\beta_2}) - n\gamma_2 y(x - \frac{\alpha_2}{\beta_2})$$
$$+ n\alpha_2(x_2 - \frac{\alpha_2}{\beta_2})(z-1)$$
$$= X^t B X, \tag{3.76}$$

where

$$X = \begin{bmatrix} x_1 \\ x_2 - \frac{\alpha_2}{\beta_2} \\ y \\ z-1 \end{bmatrix}, \tag{3.77}$$

$$B = \begin{bmatrix} -m\beta_1 & -1 & 0 & 0 \\ -1 & -n\beta_2 & -\frac{n\gamma_2}{2} & \frac{n\alpha_2}{2} \\ 0 & -\frac{n\gamma_2}{2} & -p\eta & 0 \\ 0 & \frac{n\alpha_2}{2} & 0 & -q\vartheta \end{bmatrix}, \tag{3.78}$$

and X^t denotes the transpose of X. Let D_k denote the sequence of principal minors of the matrix B. Then

$$D_1 = -m\beta_1 < 0, \tag{3.79}$$

$$D_2 = det \begin{bmatrix} -m\beta_1 & -1 \\ -1 & -n\beta_2 \end{bmatrix} = \frac{\beta_1\beta_2 - \rho_1\rho_2}{\rho_1\rho_2}, \tag{3.80}$$

$$D_3 = det \begin{bmatrix} -m\beta_1 & -1 & 0 \\ -1 & -n\beta_2 & -\frac{n\gamma_2}{2} \\ 0 & -\frac{n\gamma_2}{2} & -p\eta \end{bmatrix} = -p\eta D_2 + m\beta_1(\frac{n\gamma_2}{2})^2, \tag{3.81}$$

$$D_4 = det \begin{bmatrix} -m\beta_1 & -1 & 0 & 0 \\ -1 & -n\beta_2 & -\frac{n\gamma_2}{2} & \frac{n\alpha_2}{2} \\ 0 & -\frac{n\gamma_2}{2} & -p\eta & 0 \\ 0 & \frac{n\alpha_2}{2} & 0 & -q\vartheta \end{bmatrix} = -q\vartheta D_3 - mp\beta_1\eta(\frac{n\alpha_2}{2})^2 \qquad (3.82)$$

Theorem 3.20. *The real symmetric matrix B and consequently the quadratic form X^tBX in (3.82) is negative definite if $D_4 > 0$.*

Proof:

Suppose $D_4 > 0$. Then from Equation (3.82), we have

$$-q\vartheta D_3 > mp\beta_1\eta(\frac{n\alpha_2}{2})^2.$$

Therefore we have $D_3 < 0$. Also if $D_3 < 0$, then from (3.81) we have

$$p\eta D_2 > m\beta_1(\frac{n\gamma_2}{2})^2.$$

Hence we have $D_2 > 0$. Thus we have the following:

$$D_1 < 0, \quad D_2 > 0, \quad D_3 < 0, \quad D_4 > 0$$

for the symmetric matric B. Hence by Frobenius theorem (Theorem 1.5), B is negative definite. □

The above theorem leads to the following theorem which is the main result of this section.

Theorem 3.21. *Suppose*

1. $\gamma_1 < 0$,

2. $\beta_2\alpha_1 - \rho_1\alpha_2 < 0$,

3. $D_4 > 0$.

Then the equilibrium $F_{x_2z}(0, \frac{\alpha_2}{\beta_2}, 0, 1)$ *is globally asymptotically stable with respect to* $\mathbf{R}^+_{x_1x_2yz}$, *i.e. both normal agriculture and industry will become extinct.*

3.8 Criteria for extinction of renewable agriculture and industry

In this section, we shall derive sufficient conditions for the extinction of both renewable agriculture and industry. This corresponds to a situation where all wildlife, forest,birds, etc. have become extinct as a result of human normal farming activities, a situation that can be harmful to the the production or crops and raising of animals (i.e. normal agriculture).

In order to obtain conditions for the total extinction of both renewable agriculture and industry, we use a Liapunov function to establish criteria for the global asymptotic stability for the equilibrium $F_{x_1z}(\bar{x_1}, 0, 0, \bar{z})$ with respect to $\mathbf{R}^+_{x_1x_2yz}$. We choose the Liapunov function $V = V(x_1, x_2, y, z)$ as given by

$$V = m\{x_1 - \bar{x_1} - \bar{x_1}\ln(\frac{x_1}{\bar{x_1}})\} + nx_2 + py + q\{z - \bar{z} - \bar{z}\ln(\frac{z}{\bar{z}})\}. \tag{3.83}$$

The derivative of (3.83) along the solution curves of the system given by Equations (3.1)-(3.4) is

$$
\begin{aligned}
\dot{V} &= \frac{m\dot{x}_1}{x_1}(x_1 - \bar{x_1}) + n\dot{x}_2 + p\dot{y} + \frac{q\dot{z}}{z}(z - \bar{z}) \\
&= m(\alpha_1 z - \beta_1 x_1 + \gamma_1 y - \rho_1 x_2 - \theta(1 - z))(x_1 - \bar{x_1}) \\
&+ n(\alpha_2 z - \beta_2 x_2 - \gamma_2 y - \rho_2 x_1)x_2 + p(-\xi - \eta y + \delta x_1)y \\
&+ q(-\kappa x_1 + \vartheta(1 - z) - \phi x_1)(z - \bar{z}) + \frac{q\phi(z - \bar{z})x_1}{z} \\
&= -m\beta_1(x_1 - \bar{x_1})^2 - n\beta_2 x_2^2 - p\eta y^2 - q(\vartheta + \phi_1\frac{x_1}{z\bar{z}})(z - \bar{z})^2 + (x_1 - \bar{x_1})x_2(-m\rho_1 - n\rho_2) \\
&+ (x_1 - \bar{x_1})y(m\gamma_1 + p\delta_1) + (x_1 - \bar{x_1})(z - \bar{z})\{m(\alpha_1 + \theta) - q(\kappa + \phi - \frac{\phi}{\bar{z}})\} \\
&+ x_2 y(-n\gamma_2) + x_2(z - \bar{z})(n\alpha_2) + n(\alpha_2\bar{z} - \rho_2\bar{x_1})x_2 + p(-\xi + \delta_1\bar{x_1})y.
\end{aligned}
\tag{3.84}
$$

Now suppose

1. $\alpha_2\bar{z} - \rho_2\bar{x_1} < 0$,

2. $-\xi + \delta_1\bar{x_1} < 0$.

We observe that the above two conditions are necessary and sufficient conditions for the equilibrium $F_{x_1 z}(\bar{x_1}, 0, 0, \bar{z})$ to be locally asymptotically stable (Theorem 3.6). Then from Equation (3.84) we have

$$
\begin{aligned}
\dot{V} &< -m\beta_1(x_1 - \bar{x_1})^2 - n\beta_2 x_2^2 - p\eta y^2 - q(\vartheta + \phi_1\frac{x_1}{z\bar{z}})(z - \bar{z})^2 + (x_1 - \bar{x_1})x_2(-m\rho_1 - n\rho_2) \\
&+ (x_1 - \bar{x_1})y(m\gamma_1 + p\delta_1) + (x_1 - \bar{x_1})(z - \bar{z})\{m(\alpha_1 + \theta) - q(\kappa + \phi - \frac{\phi}{\bar{z}})\} \\
&+ x_2 y(-n\gamma_2) + x_2(z - \bar{z})(n\alpha_2) \\
&\le -m\beta_1(x_1 - \bar{x_1})^2 - n\beta_2 x_2^2 - p\eta y^2 - q\vartheta(z - \bar{z})^2 + (x_1 - \bar{x_1})x_2(-m\rho_1 - n\rho_2) \\
&+ (x_1 - \bar{x_1})y(m\gamma_1 + p\delta_1) + (x_1 - \bar{x_1})(z - \bar{z})\{m(\alpha_1 + \theta) - q(\kappa + \phi - \frac{\phi}{\bar{z}})\} \\
&+ x_2 y(-n\gamma_2) + x_2(z - \bar{z})(n\alpha_2).
\end{aligned}
\tag{3.85}
$$

Now we assume $\gamma_1 < 0$ and choose m,n,p and q positive constants such that

$$m = \frac{1}{\rho_1}, \quad n = \frac{1}{\rho_2}, \quad p = \frac{-\gamma_1}{\delta_1 \rho_1}, \quad q = \frac{(\alpha_1 + \theta)\bar{z}}{((\kappa + \phi)\bar{z} - \phi)\rho_1}.$$

Then from (3.85) we have

$$\begin{aligned}\dot{V} &< -m\beta_1(x_1 - \bar{x_1})^2 - n\beta_2 x_2^2 - p\eta y^2 - q\vartheta(z - \bar{z})^2 - 2(x_1 - \bar{x_1})x_2 + x_2 y(-n\gamma_2) \\ &+ x_2(z - \bar{z})(n\alpha_2) \\ &= X^t B X,\end{aligned} \tag{3.86}$$

where

$$X = \begin{bmatrix} x_1 - \bar{x_1} \\ x_2 \\ y \\ z - \bar{z} \end{bmatrix}, \tag{3.87}$$

$$B = \begin{bmatrix} -m\beta_1 & -1 & 0 & 0 \\ -1 & -n\beta_2 & -\frac{n\gamma_2}{2} & \frac{n\alpha_2}{2} \\ 0 & -\frac{n\gamma_2}{2} & -p\eta & 0 \\ 0 & \frac{n\alpha_2}{2} & 0 & -q\vartheta \end{bmatrix}, \tag{3.88}$$

and X^t denotes the transpose of X.

Theorem 3.22. *Suppose*

1. $\gamma_1 < 0$

2. $\alpha_2 \bar{z} - \rho_2 \bar{x_1} < 0$

3. $-\xi + \delta_1 \bar{x_1} < 0$

4. $det(B) > 0$.

Then the equilibrium $F_{x_1 z}(\bar{x}_1, 0, 0, \bar{z})$ is globally asymptotically stable with respect to $\mathbf{R}^+_{x_1 x_2 y z}$, i.e. both renewable agriculture and industry will become extinct.

Proof:

See the proof of Theorem 3.20 in §3.7.

3.9 Criteria for extinction of industry

In this section, we shall establish sufficient conditions for the total extinction of industry. We do this as before, by using a Liapunov function to establish criteria for the global asymptotic stability of the equilibrium $F_{x_1 x_2 z}(\breve{x}_1, \breve{x}_2, 0, \breve{z})$ with respect to $\mathbf{R}^+_{x_1 x_2 y z}$. We choose the Liapunov function $V = V(x_1, x_2, y, z)$ as given by

$$V = m\{x_1 - \breve{x}_1 - \breve{x}_1 \ln(\frac{x_1}{\breve{x}_1})\} + n\{x_2 - \breve{x}_2 - \breve{x}_2 \ln(\frac{x_2}{\breve{x}_2})\} + py + q\{z - \breve{z} - \breve{z}\ln(\frac{z}{\breve{z}})\}. \quad (3.89)$$

The derivative of (3.89) along the solution curves of the system given by Equations (3.1)-(3.4) is

$$\begin{aligned}
\dot{V} &= \frac{m\dot{x}_1}{x_1}(x_1 - \breve{x}_1) + \frac{n\dot{x}_2}{x_2}(x_1 - \breve{x}_2) + p\dot{y} + \frac{q\dot{z}}{z}(z - \breve{z}) \\
&= m(\alpha_1 z - \beta_1 x_1 + \gamma_1 y - \rho_1 x_2 - \theta(1-z))(x_1 - \breve{x}_1) \\
&+ n(\alpha_2 z - \beta_2 x_2 - \gamma_2 y - \rho_2 x_1)(x_2 - \breve{x}_2) + p(-\xi - \eta y + \delta x_1)y \\
&+ q(-\kappa x_1 + \vartheta(1-z) - \phi x_1)(z - \breve{z}) + \frac{q\phi(z-\breve{z})x_1}{z} \\
&= -m\beta_1(x_1 - \breve{x}_1)^2 - n\beta_2(x_2 - \breve{x}_2)^2 - p\eta y^2 - q(\vartheta + \phi_1\frac{x_1}{z\breve{z}})(z - \breve{z})^2 \\
&+ (x_1 - \breve{x}_1)(x_2 - \breve{x}_2)(-m\rho_1 - n\rho_2) + (x_1 - \breve{x}_1)y(m\gamma_1 + p\delta_1) \\
&+ (x_1 - \breve{x}_1)(z - \breve{z})\{m(\alpha_1 + \theta) - q(\kappa + \phi - \frac{\phi}{\breve{z}})\} + (x_2 - \breve{x}_2)y(-n\gamma_2) + p(-\xi + \delta_1\breve{x}_1)y \\
&+ (x_2 - \breve{x}_2)(z - \breve{z})(n\alpha_2).
\end{aligned} \quad (3.90)$$

If we assume that $F_{x_1x_2z}(\breve{x}_1, \breve{x}_2, 0, \breve{z})$ is locally asymptotically stable (i.e. $-\xi+\delta_1\breve{x}_1 < 0$ by Theorem 3.13) and $\gamma_1 < 0$ and then choose m,n,p and q positive constants such that

$$m = \frac{1}{\rho_1}, \quad n = \frac{1}{\rho_2}, \quad p = \frac{-\gamma_1}{\delta_1\rho_1}, \quad q = \frac{(\alpha_1+\theta)\breve{z}}{((\kappa+\phi)\breve{z}-\phi)\rho_1},$$

then from (3.90) we have

$$\begin{aligned}\dot{V} &< -m\beta_1(x_1-\bar{x}_1)^2 - n\beta_2(x_2^2-\breve{x}_2) - p\eta y^2 - q\vartheta(z-\bar{z})^2 - 2(x_1-\breve{x}_1)(x_2-\breve{x}_2) \\ &+ (x_2-\breve{x}_2)y(-n\gamma_2) + (x_2-\breve{x}_2)(z-\bar{z})(n\alpha_2) \\ &= X^tBX,\end{aligned} \tag{3.91}$$

where

$$X = \begin{bmatrix} x_1-\breve{x}_1 \\ x_2-\breve{x}_2 \\ y \\ z-\breve{z} \end{bmatrix}, \tag{3.92}$$

$$B = \begin{bmatrix} -m\beta_1 & -1 & 0 & 0 \\ -1 & -n\beta_2 & -\frac{n\gamma_2}{2} & \frac{n\alpha_2}{2} \\ 0 & -\frac{n\gamma_2}{2} & -p\eta & 0 \\ 0 & \frac{n\alpha_2}{2} & 0 & -q\vartheta \end{bmatrix}, \tag{3.93}$$

and X^t denotes the transpose of X.

Theorem 3.23. *If*

1. $\gamma_1 < 0$

2. $det(B) > 0$

3. $F_{x_1x_2z}(\breve{x}_1, \breve{x}_2, 0, \breve{z})$ *is locally asymptotically stable with respect to* $\mathbf{R}^+_{x_1x_2yz}$,

then $F_{x_1x_2z}(\breve{x}_1, \breve{x}_2, 0, \breve{z})$ *is globally asymptotically with respect to* $\mathbf{R}^+_{x_1x_2yz}$.

Proof:

See the proof of Theorem (3.20) in §3.7.

3.10 Criteria for extinction of renewable agriculture

In this section, we shall establish sufficient conditions for the total extinction of renewable agriculture. We do this as before, by using a Liapunov function to establish criteria for the global asymptotic stability for the equilibrium $F_{x_1yz}(\breve{x}_1, 0, \breve{y}, \breve{z})$ with respect to $\mathbf{R}^+_{x_1x_2yz}$. We choose the Liapunov function $V = V(x_1, x_2, y, z)$ as given by

$$V = m\{x_1 - \breve{x}_1 - \breve{x}_1 \ln(\frac{x_1}{\breve{x}_1})\} + nx_2 + p\{y - \breve{y} - \breve{y}\ln(\frac{y}{\breve{y}})\} + q\{z - \breve{z} - \breve{z}\ln(\frac{z}{\breve{z}})\}. \quad (3.94)$$

The derivative of (3.94) along the solution curves of the system given by Equations (3.1)-(3.4) is

$$\dot{V} = \frac{m\dot{x}_1}{x_1}(x_1 - \breve{x}_1) + n\dot{x}_2 + \frac{p\dot{y}}{y}(y - \breve{y}) + \frac{q\dot{z}}{z}(z - \breve{z})$$
$$= m(\alpha_1 z - \beta_1 x_1 + \gamma_1 y - \rho_1 x_2 - \theta(1-z))(x_1 - \breve{x}_1)$$
$$+ n(\alpha_2 z - \beta_2 x_2 - \gamma_2 y - \rho_2 x_1)x_2 + p(-\xi - \eta y + \delta x_1)(y - \breve{y})$$
$$+ q(-\kappa x_1 + \vartheta(1-z) - \phi x_1)(z - \breve{z}) + \frac{q\phi(z - \breve{z})x_1}{z}$$
$$= -m\beta_1(x_1 - \breve{x}_1)^2 - n\beta_2 x_2^2 - p\eta(y - \breve{y})^2 - q(\vartheta + \phi_1\frac{x_1}{z\breve{z}})(z - \breve{z})^2$$
$$+ (x_1 - \breve{x}_1)x_2(-m\rho_1 - n\rho_2) + (x_1 - \breve{x}_1)(y - \breve{y})(m\gamma_1 + p\delta_1)$$
$$+ (x_1 - \breve{x}_1)(z - \breve{z})\{m(\alpha_1 + \theta) - q(\kappa + \phi - \frac{\phi}{\breve{z}})\} + x_2(y - \breve{y})(-n\gamma_2) + x_2(z - \breve{z})(n\alpha_2)$$
$$+ n(\alpha_2\breve{z} - \gamma_2\breve{y} - \rho_2\breve{x}_1)x_2. \tag{3.95}$$

If we assume that $\alpha_2\breve{z} - \gamma_2\breve{y} - \rho_2\breve{x}_1 < 0$ (a condition required for this equilibrium to be locally asymptotically stable, see Theorem 3.11) and $\gamma_1 < 0$ and then choose m,n,p and q positive constants such that

$$m = \frac{1}{\rho_1}, \quad n = \frac{1}{\rho_2}, \quad p = \frac{-\gamma_1}{\delta_1\rho_1}, \quad q = \frac{(\alpha_1 + \theta)\breve{z}}{((\kappa + \phi)\breve{z} - \phi)\rho_1},$$

Then from (3.95) we have

$$\dot{V} < -m\beta_1(x_1 - \bar{x}_1)^2 - n\beta_2 x_2^2 - p\eta(y - \breve{y})^2 - q\vartheta(z - \bar{z})^2 - 2(x_1 - \breve{x}_1)x_2$$
$$+ x_2(y - \breve{y})(-n\gamma_2) + x_2(z - \bar{z})(n\alpha_2) \tag{3.96}$$
$$= X^t B X,$$

where

$$X = \begin{bmatrix} x_1 - \breve{x}_1 \\ x_2 \\ y - \breve{y} \\ z - \breve{z} \end{bmatrix}, \tag{3.97}$$

$$B = \begin{bmatrix} -m\beta_1 & -1 & 0 & 0 \\ -1 & -n\beta_2 & -\frac{n\gamma_2}{2} & \frac{n\alpha_2}{2} \\ 0 & -\frac{n\gamma_2}{2} & -p\eta & 0 \\ 0 & \frac{n\alpha_2}{2} & 0 & -q\vartheta \end{bmatrix}, \tag{3.98}$$

and X^t denotes the transpose of X.

Theorem 3.24. *If*

1. $\gamma_1 < 0$
2. $det(B) > 0$
3. $\alpha_2 \breve{z} - \gamma_2 \breve{y} - \rho_2 \breve{x}_1 < 0,$

then $F_{x_1yz}(\breve{x}_1, 0, \breve{y}, \breve{z})$ *is globally asymptotically with respect to* $\mathbf{R}^+_{x_1x_2yz}$.

Proof:

See the proof of Theorem (3.20) in §3.7.

3.11 Numerical examples

In this section we describe some examples to illustrate some of our results. In all of these examples, parameter values are chosen for numerical convenience to illustrate mathematical results and may not represent any actual agricultural-industrial-ecospheric systems. As a result the numerical values of the numerical output of our computations are not significant (i.e. we are just interested in the qualitative behaviour of the numerical output).

3.11.1 Example 3.1

In this example, we set

$$\alpha_1 = 3, \quad \beta_1 = 1/10, \quad \gamma_1 = -1/49, \quad \rho_1 = 1/10, \quad \theta = 6/5$$

$$\alpha_2 = 1, \quad \beta_2 = 1/10, \quad \gamma_2 = 1/10, \quad \rho_2 = 1/5,$$

$$\xi = 1, \quad \eta = 1/20, \quad \delta = 1/4, \quad \kappa = 2, \quad \vartheta = 2, \quad \phi = 1.$$

In this example the possible set of equilibria for the system is $F_0(0,0,0,0)$, $F_z(0,0,0,1)$, $F_{x_1z}(3.7499,0,0,0.3751)$, and $F_{x_2z}(0,10,0,1)$. Here $F_{x_1z}(3.7499,0,0,0.3751)$ is globally asymptotically stable with respect to solutions initiating in the interior of $\mathbf{R}^+_{x_1x_2yz}$ (see Figure 3.3). Also all the conditions of Theorem 3.21 are satisfied and hence renewable agriculture and industry become extinct.

3.11.2 Example 3.2

We set

$$\alpha_1 = 2, \quad \beta_1 = 1/10, \quad \gamma_1 = -1/49, \quad \rho_1 = 1/10, \quad \theta = 6/5$$

$$\alpha_2 = 3, \quad \beta_2 = 1/10, \quad \gamma_2 = 1/10, \quad \rho_2 = 1/25,$$

$$\xi = 1/4, \quad \eta = 1/20, \quad \delta = 1/4, \quad \kappa = 2, \quad \vartheta = 2, \quad \phi = 1.$$

In this example the possible set of equilibria for the system is $F_0(0,0,0,0)$, $F_z(0,0,0,1)$, $F_{x_1z}(1.7142,0,0,0.4286)$, $F_{x_2z}(0,30,0,1)$ and $F_{x_1yz}(1.5374,0,2.6874,0.4402)$. Moreover, $F_{x_2z}(0,30,0,1)$ is globally asymptotically stable with respect to solutions initiating in the interior of $\mathbf{R}^+_{x_1x_2yz}$ (see Figure 3.4). Further more all the conditions of Theorem 3.20 are satisfied and hence both normal agriculture and industry become extinct.

3.11.3 Example 3.3

In this example, the coefficients are given by the following:

$$\alpha_1 = 2, \quad \beta_1 = 1/10, \quad \gamma_1 = -1/49, \quad \rho_1 = 1/50, \quad \theta = 6/5$$

$$\alpha_2 = 2, \quad \beta_2 = 1/10, \quad \gamma_2 = 1/10, \quad \rho_2 = 1/50,$$

$$\xi = 1, \quad \eta = 1/5, \quad \delta = 1/4, \quad \kappa = 1, \quad \vartheta = 2, \quad \phi = 1.$$

The equilibria for the system are $F_0(0,0,0,0)$, $F_z(0,0,0,1)$, $F_{x_1z}(5.4545, 0, 0, 0.5455)$, $F_{x_1yz}(5.2118, 0, 1.5149, 0.5476)$, $F_{x_2z}(0, 20, 0, 1)$, and $F_{x_1x_2z}(3.9153, 10.4722, 0, 0.5632)$. $F_{x_1x_2z}(3.9153, 10.4722, 0, 0.5632)$ is globally asymptotically stable with respect to solutions initiating in the interior of $\mathbf{R}^+_{x_1x_2yz}$ (see Figure 3.5). We note observe that all the conditions of Theorem 3.22 are satisfied and hence industry goes extinct.

3.11.4 Example 3.4

Here, the coefficients of the model are chosen as:

$$\alpha_1 = 2, \quad \beta_1 = 1/10, \quad \gamma_1 = -1/49, \quad \rho_1 = 1/50, \quad \theta = 6/5$$

$$\alpha_2 = 2, \quad \beta_2 = 1/10, \quad \gamma_2 = 1/10, \quad \rho_2 = 1/50,$$

$$\xi = 1/4, \quad \eta = 1/20, \quad \delta = 1/4, \quad \kappa = 1, \quad \vartheta = 2, \quad \phi = 1.$$

The equilibria for the system are $F_0(0,0,0,0), F_z(0,0,0,1), F_{x_1z}(5.4545, 0, 0, 0.5455)$, $F_{x_2z}(0, 20, 0, 1), F_{x_1yz}(3.5724, 0, 12.8617, 0.5686)$ and $F_{x_1x_2z}(3.9159, 10.4733, 0, 0.5629)$, with $F_{x_1yz}(3.5724, 0, 12.8617, 0.5686)$ being globally asymptotically stable with respect to solutions initiating in the interior of $\mathbf{R}^+_{x_1x_2yz}$ (see Figure 3.6). Also all the conditions of Theorem 3.23 are satisfied and hence renewable agriculture goes extinct.

3.11.5 Examples 3.5: Persistence

In this subsection, we choose the parameter of our modelled system such that the system persists. In these examples, the coefficients are given by the following:

$$\alpha_1 = 2, \quad \beta_1 = 1/10, \quad \rho_1 = 1/50, \quad \theta = 6/5, \quad \gamma_1 = -1/49$$

$$\alpha_2 = 2, \quad \beta_2 = 1/10, \quad \gamma_2 = 1/10, \quad \rho_2 = 1/50,$$

$$\xi = 1/4, \quad \eta = 1/20, \quad \delta = 1/4, \quad \kappa = 2, \quad \vartheta = 2, \quad \phi = 1.$$

We obtain the following equilibria: $F_0(0, 0, 0, 0)$, $F_{x_2z}(0, 20, 0, 1)$, $F_{x_1z}(1.7142, 0, 0, 0.4286)$, $F_{x_1yz}(1.5374, 0, 2.6874, 0.4402)$, $F_z(0, 0, 0, 1)$, $F_{x_1x_2z}(1.2133, 9.1589, 0, 0.4704)$ and $F_{x_1x_2yz}(1.2126, 8.0933, 1.0696, 0.4703)$. In this case, $F_{x_1x_2yz}(1.2126, 8.0933, 1.0696, 0.4703)$ is globally asymptotically stable with respect to solutions initiating in the interior of $\mathbf{R}^+_{x_1x_2yz}$ (see Figure 3.7).

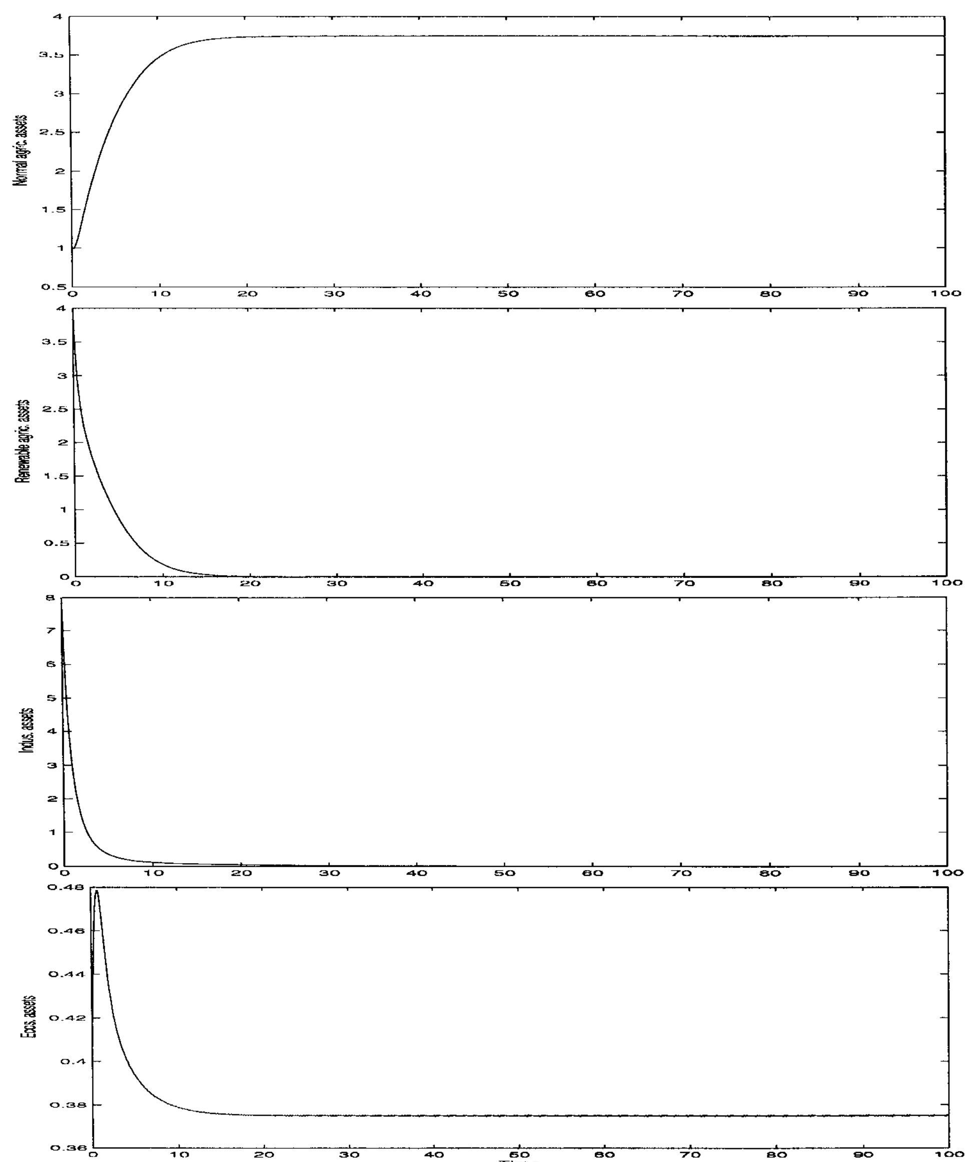

Figure 3.3: $F_{x_1 z}(3.7499, 0, 0, 0.3751)$ is globally asymptotically stable, that is both renewable agriculture and industry are driven extinct.

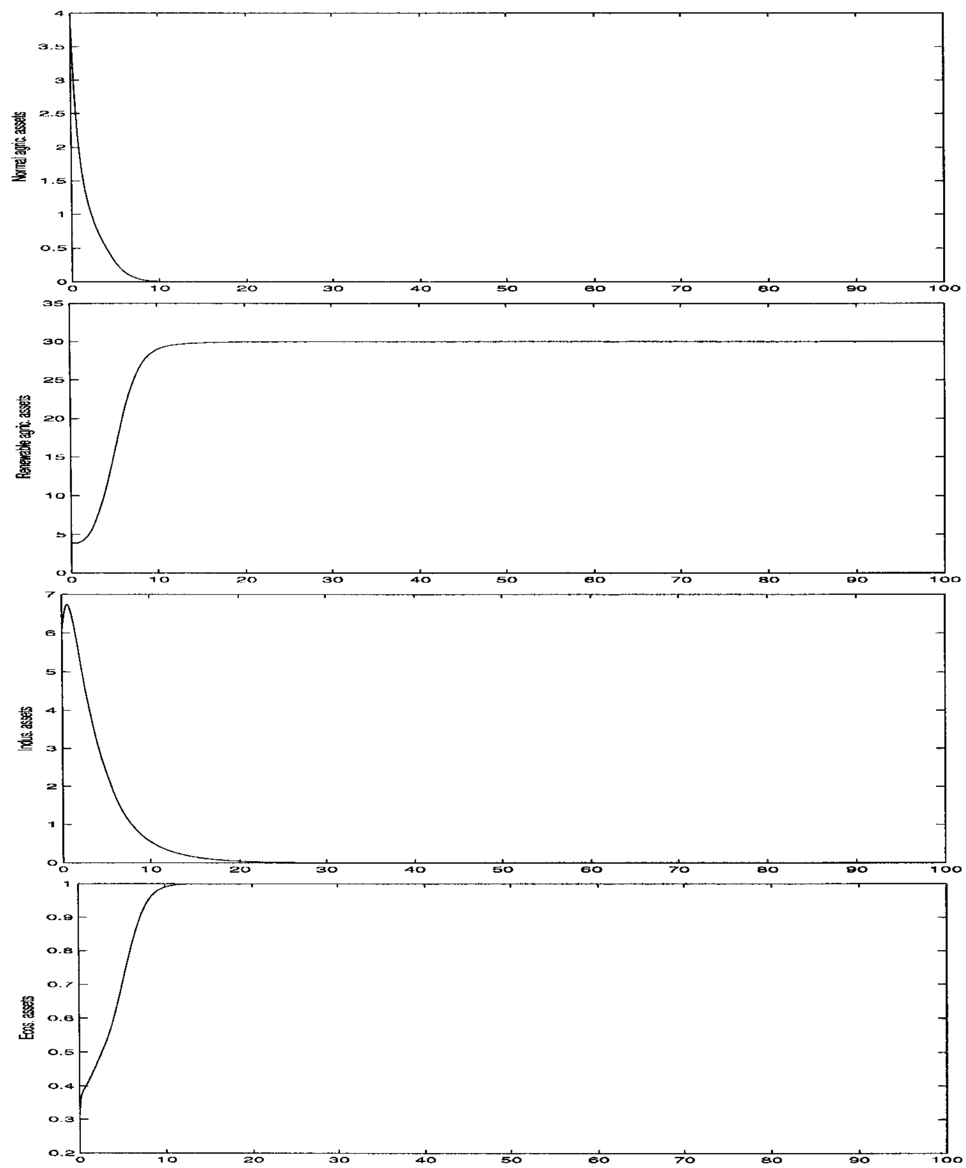

Figure 3.4: $F_{x_2z}(0, 30, 0, 1)$ is globally asymptotically stable, that is both normal agriculture and Industry are driven extinct.

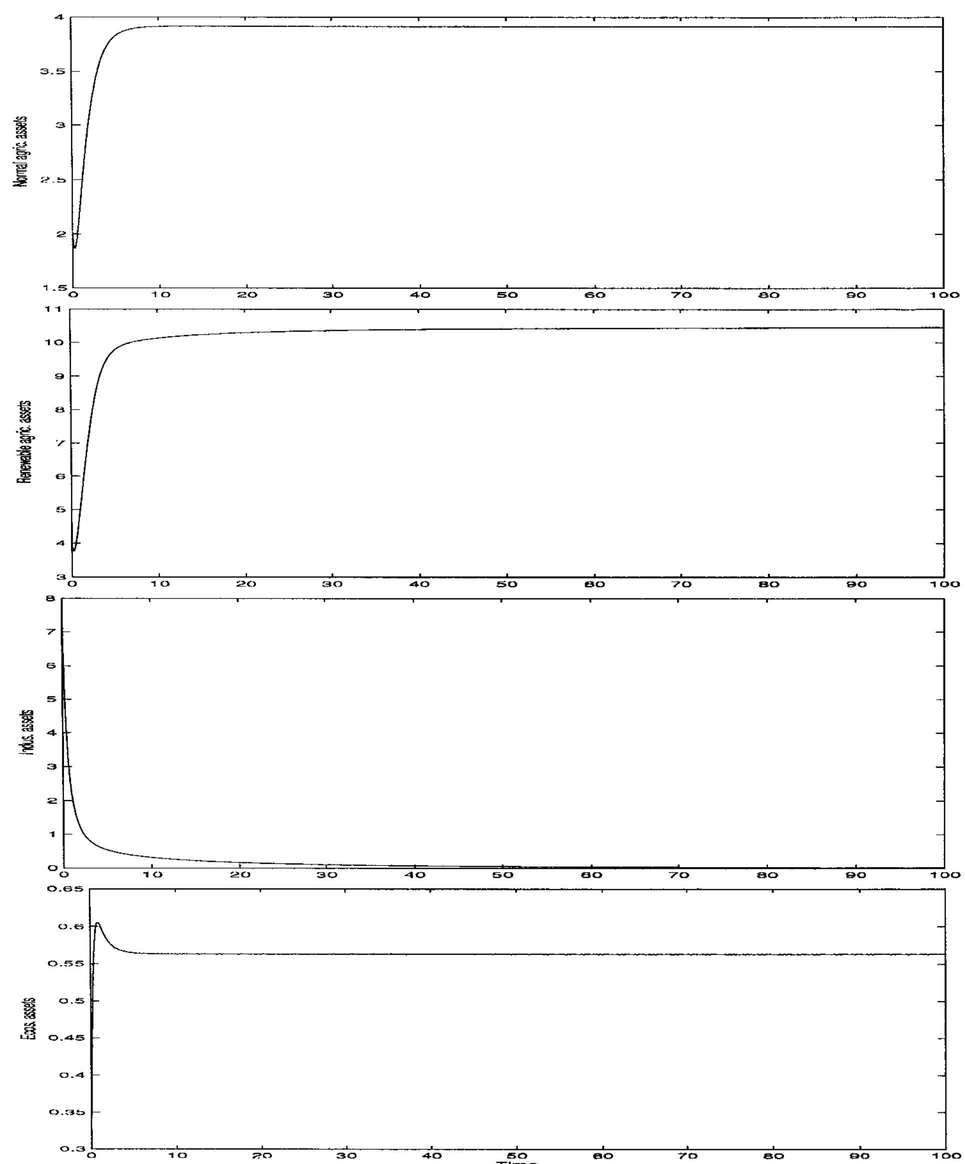

Figure 3.5: $F_{x_1 x_2 z}(3.9153, 10.4722, 0, 0.5632)$ is globally asymptotically stable. Industry is driven extinct.

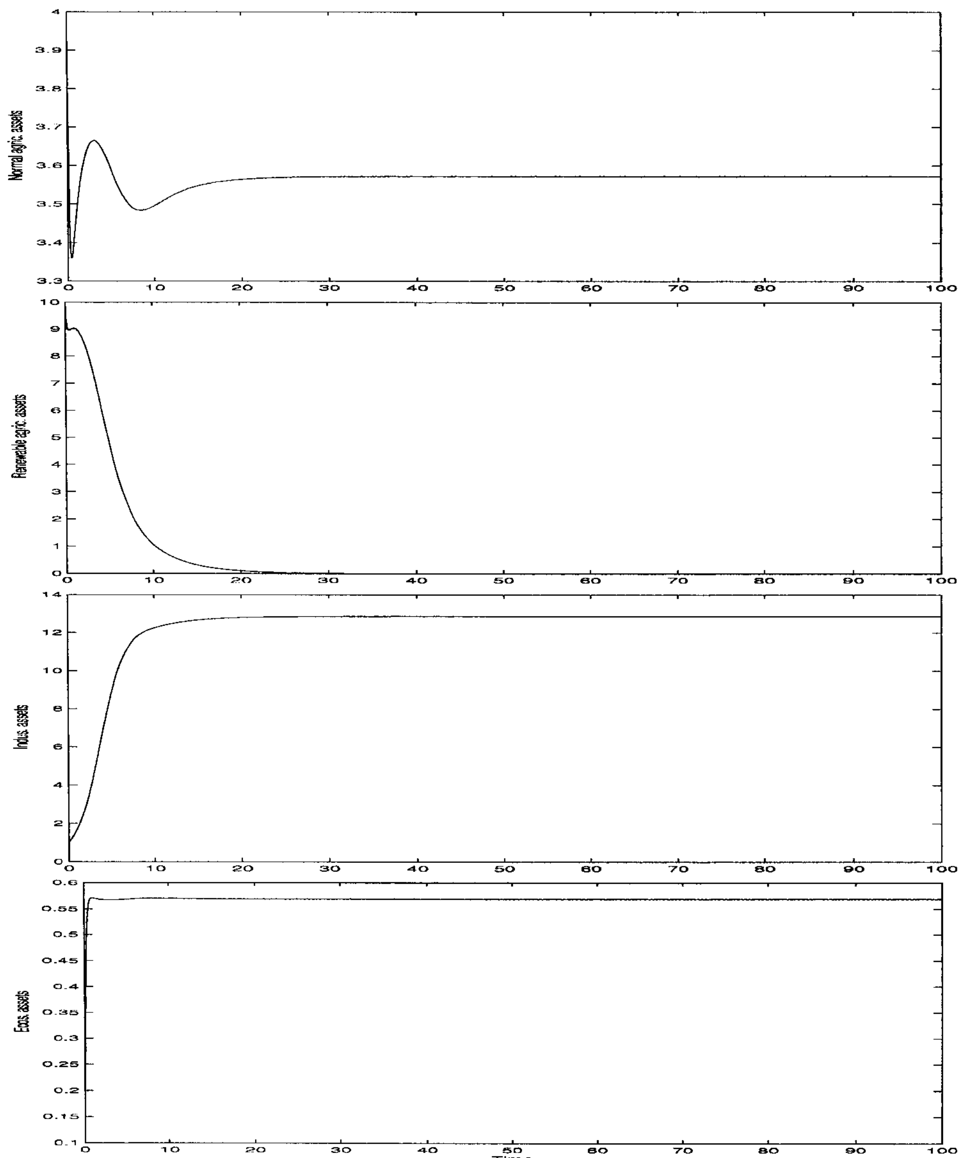

Figure 3.6: $F_{x_1yz}(3.5724, 0, 12.8617, 0.5686)$ is globally asymptotically stable. Renewable agriculture is driven extinct

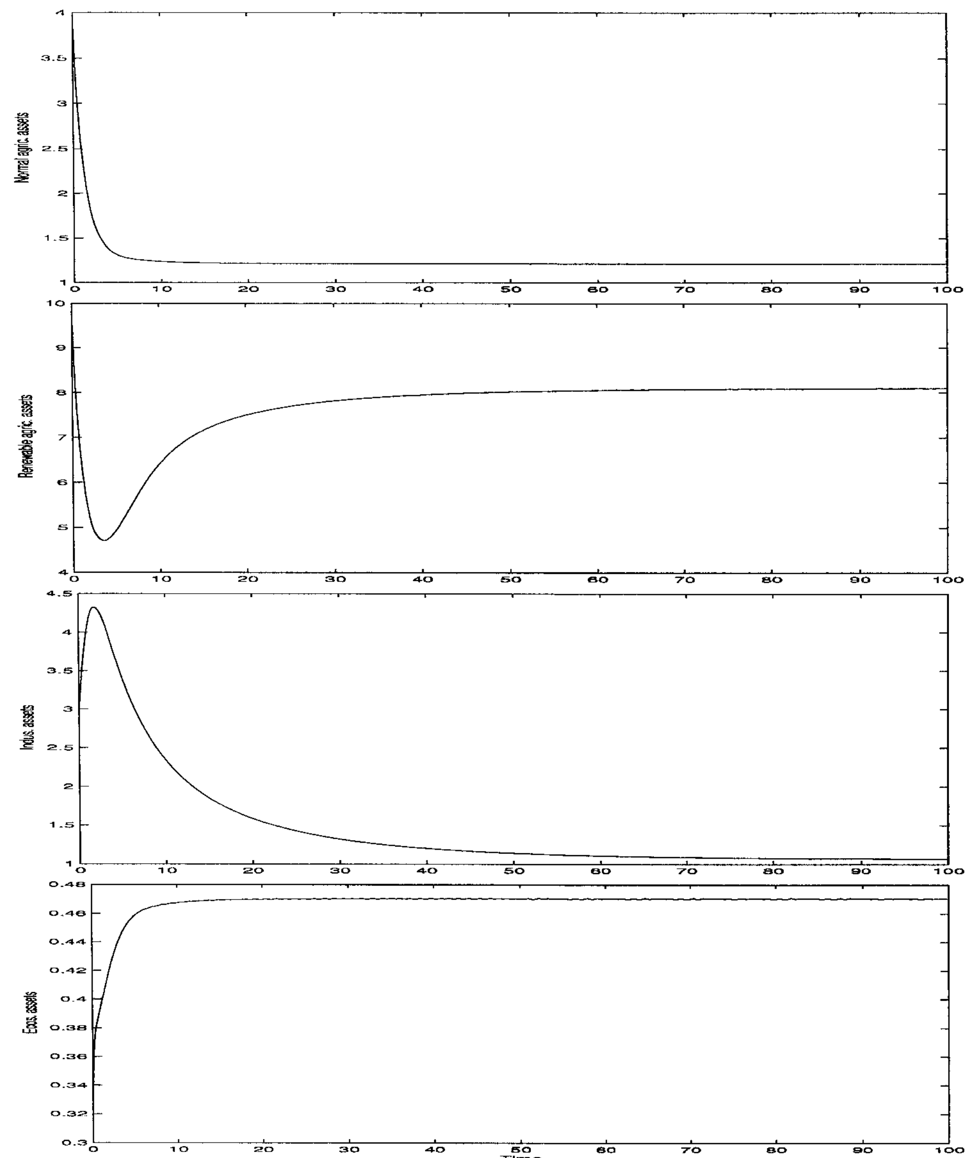

Figure 3.7: $F_{x_1x_2yz}(1.2126, 8.0933, 1.0696, 0.4703)$ is globally asymptotically stable. The system persists.

3.12 Summary and conclusions

In the preceding sections of this chapter, we used mathematical model to discuss the interaction between a natural environment (i.e the ecosphere and renewable agriculture), our normal farming activities (i.e normal agriculture) and industry as related to agriculture. We used mathematical tools such as differential equation analysis, persistence theory, Liapunov functions and linear systems theory to analyze our model.

We explicitly recognize agriculture as being of two parts: (a) normal agriculture which is cultivation of land, raising of animals and cultivation of crops and (b)renewable agriculture which comprises of naturally occurring agricultural products on earth that can replenish itself such as the natural populations of fish, wild animals, trees and forest. By separating agriculture in to these two parts, we are able to throw more light into the question of sustainabilty. Unlike the model in previous chapter, where we were able to determine under what conditions agriculture (that is both normal and renewable) go extinct, in this chapter we were more precise in determining which part of the agriculture will go extinct and conditions under which it occurs. In this chapter also, we were able to determine the conditions under which natural environment (i.e. the ecosphere which is the quality of the environment and renewable agriculture) will be degraded or depleted. This aggregate is one of the key components of sustainabilty which most people are interested in studying.

We established sufficient conditions under which the equilibrium $F_{x_2z} = (0, \frac{\alpha_2}{\beta_2}, 0, 1)$ is globally asymptotically stable with respect to $\mathbf{R}^+_{x_1x_2yz}$ in Theorem 3.21 of §3.7. Under the conditions of Theorem 3.21, both normal agriculture and industry go extinct, the ecosphere grows to the maximum possible level it could ever attain (its carrying capacity) and renewable agriculture such as populations of fish, wildlife, trees and forest grows to its carrying capacity. The first condition of Theorem 3.21 which is

$\gamma_1 < 0$ (i.e. the per asset terms of trade rate coefficient between normal agriculture and industry is negative) can be interpreted economically as a net loss by normal agriculture during the transfer of assets between industry and normal agriculture. The second condition, $\frac{\alpha_1}{\rho_1} < \frac{\alpha_2}{\beta_2}$ simply says, the net growth in assets of normal agriculture is less that that renewable agriculture. The third condition is complex to interpret. Thus in order to preserve the natural environment at its highest possible quality level in the long term, at least the above two conditions has to be met.

Criteria for the extinction of both renewable agriculture and industry (i.e. the global asymptotic stability of $F_{x_1 z} = (\bar{x_1}, 0, 0, \bar{z}))$ was established in Theorem 3.22 of §3.8. We stated some of the negative effects an economy may experience if its industrial assets go extinct, and these effects can easily be seen in developing countries (see section 2.13). Such effects on normal agriculture and the ecosphere can not be over emphasized. We note that the natural environment which is normally referred to as natural home or infrastructure of human societies by Geoffrey Heal [40] is made up of the ecosphere and renewable agriculture. Thus driving renewable agriculture to extinction implies destroying our vegetation, forest, animals, birds and fisheries. In fact a total loss of biodiveristy will occur. This is a fact which was easily established using our system of differential equations, but this is not just a mathematical fact but a real life fact and this has been one of the major concerns of environmentalists.

Criteria for the extinction of industrial assets only is established in §3.9 Theorem 3.22 and the extinction of renewable agriculture only is given in §3.10 Theorem 3.23.

In §3.6 Theorem 3.18 and 3.19, we established sufficient conditions for the persistence of the system and the existence of an interior equilibrium $F_{x_1 x_2 y z}(x_1^\star, x_2^\star, y^\star, z^\star)$. Theorem 3.18 and 3.19 give us the hope that if we understand the dynamics of the interaction between agriculture, industry and the ecosphere very well, then we can at least preserve the system to some extent over a long period of time.

The main purpose of the model in this chapter was to study and discuss sustainable normal agriculture which in turn leads to sustainable natural environment and industry. Our model suggests that our present farming activities can lead to extinction of renewable agriculture and industry, extinction of normal agriculture and industry, extinction of industry only, extinction of renewable agriculture only and can also lead to persistence of the entire system depending on the way we perform our normal farming activities and our understanding of the way agriculture, industry and the ecosphere interacts. Thus our model suggests that the problem of agricultural sustainability can best be handled if we actually learn to understand the interaction of agriculture, industry and the ecosphere and the effects of each of these assets on each other.

Chapter 4

A Model Incorporating Competition for Land

4.1 Introduction

There is a worldwide increase in awareness of the importance of "foresting" growth, and in maintaining the integrity of existing game ranches, national reserves and parks and recreation (i.e. renewable agriculture). As a result many environmentalists are fighting worldwide to either increase the land for renewable agriculture or to maintain the current existing ones.

Similarly, as the world's population grows (rapidly in some regions), there is an increase in demand for meat and useful plant biomass (i.e. normal agriculture) for sustenance. But an increase in demand for useful plant biomass and animals amounts to either increasing land for farming or to increasing yields of useful plant biomass and animals per acre of farm land. While the latter has been the common practice in most developed countries, the former can be found in both developed and developing countries [43]. Thus one can say that increasing the demand for normal agriculture to produce more leads to an increase in the demand for more land for farming.

It is evident from the aforementioned that one of the key components underlying the study of sustainable agriculture, industry and environment is competition for land. There is an increase in competition for land between preserving, retaining, restoring and sustaining the natural ecosystem in which all living organisms live, and agriculture and industry. This is a fact that one hardly comes across explicitly in most of the studies in agricultural and environmental sustainability.

A model that will describe the interactions between agriculture, industry and the environment competing for land is complex and involves a system of five or more differential equations. As a result of this complexity, in this thesis we will consider as an approximation, a competition for land model involving two farming groups within a locality. A case in point is say, the two farming groups in Wheatland County of Southern Alberta, Hutterites and non-Hutterites. We leave more complex models to future research.

Hutterites have a different culture from non-Hutterites. They live in a separate community with their own schools going only to grade 8. There is no private possession of properties among Hutterites. They are the largest family-type communal group in Canada. They usually live on large acreages of communally owned land and maintain a high degree of geographic isolation so as to maintain their religious and cultural structure. Their religious and cultural structure is characterized by the economy of human effort, communal sharing of wealth and property, a high degree of security for the individual, large-scale farming, larger families, proper and effective upbringing of their youth, German dialect and modest traditional dress [42]. Hutterites in general only acquire land and do not sell land for religious and cultural reasons. This is certainly the case of the Hutterites in Wheatland County.

Hutterites can also be found in other parts of North America such as Saskatchewan, Manitoba, British Columbia, North and South Dakota, Minnesota, Montana and Washington. There are many other groups which interact similarly. Some of these groups are the Mennonites of British Columbia and Manitoba; Amish of Pennsylvania and Illinois; Amana colonies in Iowa; and Doukabours of British Columbia. Thus the case of Hutterites in Wheatland County, Alberta, Canada is representative of a more general concern. We consider the case of Hutterites in Wheatland County because we have more data on them as compared to the other groups.

Wheatland County is bounded on the north by Kneehill County, on the north east by the Red Deer River, on the south east by the county of Newell No.4, on the south by the Bow River and on the west by the Municipal District of Rockyview No. 44. In this county, there is competition for land mainly between Hutterite colonies and very large commercial private operations. There is however, a third group which is individual farmers, but most of them have given up their lands to either Hutterites or large commercial private operations because of the unusually high cost of land. Land is costly due to pressure from Hutterite colonies to expand, i.e. form new colonies and therefore bid higher than the otherwise market value of land. Also Hutterites usually want to live close to each other and as such do not often want to move to other places and so bid higher prices for the land near to their existing colonies. In cases where individual farms exists, they will be counted among the large commercial private operations.

There are about 7030 parcels of land in Wheatland County, each measuring about a quarter of a square mile (i.e $1/4mile^2$ or 160 acres) excluding the Siksika Nation Reserve on the north of Bow River. In 1998, Hutterites owned about 700 parcels of the land in Wheatland county. This number increased to about 775 parcels of the land in 2001 and remained almost unchanged in 2005.

The main purpose of this chapter is to present and analyze a mathematical model for the competition for farm land by two farming groups, A and B in a given area. In the next section, we discuss the model, and in the section thereafter dissipativity and boundedness criteria are given. In §4.4, we determine all the equilibria for our model. We discuss the local stability properties of all the equilibria in §4.5 and global stability of the interior equilibrium is studied in §4.6. Criteria for the extinction of industrial assets and one of the agricultural assets is given in §4.7 and §4.8 respectively. Numerical examples are given in §4.9 followed by discussion and conclusions.

4.2 The model

In this section, we present a mathematical model for the competition for farm land by two farming groups, A and B in a given area. The model consists of a system of four ordinary nonlinear differential equations. Let $x_1(t)$ represent the agricultural assets of farming group A, $x_2(t)$ the agricultural assets of farming group B, $y(t)$ the industrial assets, and $z(t)$ the percentage of land in the given area owned by farming group A at time $t \geq 0$. Thus, the percentage of land in the given area owned by group B at time $t \geq 0$ is $1 - z(t)$. We consider x_1 and x_2 as competing for farm land and consider y as a predator capable of destroying or enhancing both x_1 and x_2, but one which has more effect on x_2 than on x_1.

We suppose that the generation of industrial assets encounters both fixed and variable expenses independent of the land and both agricultural assets. We further suppose that the industrial assets are enhanced by both agricultural assets, but the enhancement by x_2 is higher than that of x_1. We assume that the percentage of land owned by group A can only grow because group A farmers do not ever sell land for cultural and religious reasons and that this growth is bounded above by its limiting factor or carrying capacity. Since the land for any given area is bounded and the growth in agricultural assets depends on land, we assume that in the absence of industry, the growth in both x_1 and x_2 will be bounded, that is the law of diminishing returns applies.

The above reasoning motivates the model given by the following system of equations

$$\dot{x_1} = x_1[f(z) - \beta_1 x_1 + p\gamma y] \tag{4.1}$$

$$\dot{x_2} = x_2[g(1-z) - \beta_2 x_2 + \gamma y] \tag{4.2}$$

$$\dot{y} = y[-\xi - \eta y + \delta(x_2 + q x_1)] \tag{4.3}$$

$$\dot{z} = \alpha z x_1 (1 - \frac{z}{K_z}) \tag{4.4}$$

with initial conditions $x_{10} = x_1(0) \geq 0$, $x_{20} = x_2(0) \geq 0$, $y_0 = y(0) \geq 0$, $0 < z_0 = z(0) \leq K_z \leq 1$, where $0 \leq p, q \leq 1$. All parameters are assumed to be positive except γ which can be any real number. $f(z)$ is the growth rate function of x_1 due to group A agricultural activities on z, β_1 is the per asset diminishing returns rate coefficient for x_1 in the absence of y, $p\gamma$ is the per asset terms of trade rate coefficient between x_1 and y, γ is the per asset terms of trade rate coefficient between x_2 and y, $g(1-z)$ is the growth rate function x_2 due to group B agricultural activities on $(1-z)$, β_2 is the per asset diminishing returns rate coefficient for x_2 in the absence of y. ξ and η are the constant and linear depreciation rate coefficients of industry, δ is the per asset growth rate coefficient of y in dealing with x_2 and K_z is the carrying capacity for z. In interpretation of Hutterites and non-Hutterites, x_1 would represent the agricultural assets of Hutterites.

The following additional hypotheses are assumed to hold:

- In the absence of both land and industry, both x_1 and x_2 will go extinct, that is $f(0) = g(0) = 0$.
- The growth in agricultural assets, x_1 of farming group A is an increasing function of the percentage of land owned by group A, that is $f'(z) > 0$.
- The growth in agricultural assets, x_2 of farming group B is an increasing function of the percentage of land owned by group B, that is $g'(1-z) > 0$.
- There exists a constant X_{1K} such that $f(1) = \beta_1 X_{1K}$.
- There exists a constant X_{2K} such that $g(1) = \beta_2 X_{2K}$.
- f(z) and g(1-z) are continuously differentiable on $0 \leq z \leq 1$.

4.3 Dissipativity and boundedness

In this section, we shall show that the model equations are bounded and dissipative with respect to a region in $\mathbf{R}^{+}_{xy_1y_2z}$ for some given parametric configurations.

Theorem 4.1. *If γ is non-positive and $-\xi + \delta(X_{2K} + qX_{1K}) > 0$, then the system given by Equations (4.1)-(4.4) is dissipative, with attraction region contained in*

$$\mathbf{A} = \{(x_1, x_2, y, z) : 0 \leq x \leq X_{1K}, 0 \leq x_2 \leq X_{2K}, 0 \leq y \leq \frac{-\xi + \delta(X_{2K} + qX_{1K})}{\eta}, 0 \leq z \leq K_z\}.$$

Proof:

If $\gamma \leq 0$, then from (4.1)-(4.2) we have

$$\dot{x_1} < \beta_1 x_1 (X_{1K} - x_1) \tag{4.5}$$

$$\dot{x_2} < \beta_2 x_2(X_{2K} - x_2). \tag{4.6}$$

Thus from (4.5), we have

$$x_1(t) \leq max(x_{10}, X_{1K}).$$

We observe from (4.5) also that $\dot{x_1} < 0$ for $x_1 > X_{1K}$, and hence we have

$$\limsup_{t \to \infty} x_1(t) \leq X_{1K}. \tag{4.7}$$

Similarly, from (4.6) we obtain

$$\limsup_{t \to \infty} x_2(t) \leq X_{2K}. \tag{4.8}$$

Also from (4.4), we have

$$\limsup_{t \to \infty} z(t) \leq K_z, \tag{4.9}$$

since $z_0 \leq K_z$.

Similarly from Equation (4.3), we have

$$\begin{aligned} \dot{y} &\leq -\xi y - \eta y^2 + \delta(X_{2K} + qX_{1K})y \\ &= -\xi + \delta(X_{2K} + qX_{1K})\{1 - \frac{\eta y}{-\xi + \delta(X_{2K} + qX_{1K})}\}y. \end{aligned} \tag{4.10}$$

Thus if $-\xi + \delta(X_{2K} + qX_{1K}) > 0$, then we have

$$\limsup_{t \to \infty} y(t) \leq \frac{-\xi + \delta(X_{2K} + qX_{1K})}{\eta}. \qquad \square \tag{4.11}$$

4.4 Equilibria

In this section, we attempt to describe all the possible configurations of equilibria for the system given by Equations (4.1)-(4.4). We are only interested in nonnegative equilibria. We shall denote by $F_a(x_1, x_2, y, z)$ the equilibria lying on the a-axis,

$F_{ab}(x_1, x_2, y, z)$ the ones in the positive (a,b)-plane, by $F_{abc}(x_1, x_2, y, z)$ the ones in the positive (a,b,c)-octant and by $F^*(x_1, x_2, y, z)$ the positive interior equilibrium of the whole system whenever they exist.

The equilibria of the system given by Equations (4.1)-(4.4) are obtained by solving the system of isocline equations

$$x_1[f(z) - \beta_1 x_1 + p\gamma y] = 0 \tag{4.12}$$

$$x_2[g(1-z) - \beta_2 x_2 + \gamma y] = 0 \tag{4.13}$$

$$y[-\xi - \eta y + \delta(x_2 + qx_1)] = 0 \tag{4.14}$$

$$\alpha z x_1 (1 - \frac{z}{K_z}) = 0. \tag{4.15}$$

We observe that $F_0(0,0,0,0)$ is an equilibrium.

4.4.1 Axial (one-dimensional) equilibria

(i) $z-$axis: $x_1 = x_2 = y = 0$ and $z \neq 0$.
Since $x_1 = 0$, from (4.4) we have $\dot{z} = 0$. Thus we have $z = z_c$, a constant. Hence, there exists a nonnegative equilibrium on the $z-$axis, which we denote by $F_z(0,0,0,z_c)$.
(ii) x_1-axis: $z = x_2 = y = 0$ and $x_1 \neq 0$.

Algebraic system (4.12)-(4.15) reduces to the equation

$$f(0) - \beta_1 x_1 = 0 \quad \Longrightarrow x_1 = 0. \tag{4.16}$$

This does not have a nonzero solution for x_1. Hence there is no equilibrium on the x_1- axis.
(iii) x_2-axis: $x_1 = y = z = 0$ and $x_2 \neq 0$.

Algebraic system (4.12)-(4.15) reduces to

$$g(1) - \beta_2 x_2 = 0, \quad \Longrightarrow x_2 = X_{2K}. \tag{4.17}$$

Thus, there exist a nonnegative equilibrium on the x_2-axis. Denote this equilibrium by $F_{x_2}(0, X_{2K}, 0, 0)$.

(iv) $y-$axis: $x_1 = x_2 = z = 0$ and $y \neq 0$.

Algebraic system (4.12)-(4.15) reduces to

$$-\xi - \eta y = 0. \tag{4.18}$$

This does not have a nonnegative solution and hence there is no equilibrium on the $y-$axis.

4.4.2 Planar (two-dimensional) equilibria

(i) $(x_1, x_2)-$plane: $z = y = 0$, $x_1 \neq 0$ and $x_2 \neq 0$.

Algebraic system (4.12)-(4.15) reduces to the system

$$\begin{aligned} f(0) - \beta_1 x_1 &= 0 \\ g(1) - \beta_2 x_2 &= 0, \end{aligned} \tag{4.19}$$

which has no positive solution for x_1. Hence there is no equilibrium in the $(x_1, x_2)-$plane.

(ii) $(x_1, y)-$plane: $z = x_2 = 0$, $x_1 \neq 0$ and $y \neq 0$.

Algebraic system (4.12)-(4.15) reduces to the system

$$\begin{aligned} f(0) - \beta_1 x_1 + p\gamma y &= 0 \\ -\xi - \eta y + \delta q x_1 &= 0, \end{aligned} \tag{4.20}$$

which possesses one and only one solution

$$y = \bar{y} = \frac{\beta_1 \xi}{pq\gamma\delta - \beta_1 \eta},$$

$$x_1 = \bar{x_1} = \frac{p\xi\gamma}{pq\gamma\delta - \beta_1 \eta}.$$

Clearly, $\bar{y}$ can only be positive if

$$pq\gamma\delta - \beta_1 \eta > 0. \tag{4.21}$$

It follows from (4.21) that the system (4.1)-(4.4) has no equilibria in the $(x_1, y)-$plane when $\gamma \leq 0$. But if $\gamma > 0$ then there exists an equilibrium, $F_{x_1 y}(\frac{p\xi\gamma}{pq\gamma\delta - \beta_1\eta}, 0, \frac{\beta_1\xi}{pq\gamma\delta - \beta_1\eta}, 0)$ provided (4.21) is satisfied.

(iii) $(x_1, z)-$plane: $y = x_2 = 0$, $x_1 \neq 0$ and $z \neq 0$.

Algebraic system (4.12)-(4.15) reduces to the system

$$\begin{aligned} f(z) - \beta_1 x_1 = 0 \\ 1 - \frac{z}{K_z} = 0. \end{aligned} \tag{4.22}$$

This has one positive solution given by $z = K_z$ and $x_1 = \frac{f(K_z)}{\beta_1}$. This equilibrium always exists and we represent it by $F_{x_1 z}(\frac{f(K_z)}{\beta_1}, 0, 0, K_z)$.

(iv) $(x_2, y)-$plane: $z = x_1 = 0$, $x_2 \neq 0$ and $y \neq 0$.

Algebraic system (4.12)-(4.15) becomes

$$\begin{aligned} g(1) - \beta_2 x_2 + \gamma y = 0 \\ -\xi - \eta y + \delta x_2 = 0. \end{aligned} \tag{4.23}$$

The system (4.23) possesses a unique solution given by

$$y = \bar{y} = \frac{\beta_2(\delta X_{2K} - \xi)}{\beta_2 \eta - \gamma\delta} \tag{4.24}$$

and

$$x_2 = \bar{x_2} = \frac{\xi + \eta \bar{y}}{\delta} = \frac{\beta_2 \eta X_{2K} - \gamma \xi}{\beta_2 \eta - \gamma \delta}. \tag{4.25}$$

We note that if $\bar{y} > 0$ then so is $\bar{x_2}$. Hence we have the following theorem:

Theorem 4.2. *Suppose that*

$$(\delta X_{2K} - \xi)(\beta_2 \eta - \gamma \delta) > 0.$$

Then there exists a unique positive equilibrium in the $(x_2, y)-$plane.

If this equilibrium exists then denote it by $F_{x_2 y}(0, \frac{\beta_2 \eta X_{2K} - \gamma \xi}{\beta_2 \eta - \gamma \delta}, \frac{\beta_2(\delta X_{2K} - \xi)}{\beta_2 \eta - \gamma \delta}, 0)$.

(v) $(x_2, z)-$plane: $y = x_1 = 0$, $x_2 \neq 0$ and $z \neq 0$.

The system (4.12)-(4.15) reduces to the system

$$g(1-z) - \beta_2 x_2 = 0. \tag{4.26}$$

But since $x_1 = 0$, we have from (4.4) that $z = z_c$. Hence, there is a nonnegative equilibrium $F_{x_2 z}(0, \frac{g(1-z_c)}{\beta_2}, 0, z_c)$.

(vi) $(y, z)-$plane: $x_2 = x_1 = 0$, $z \neq 0$ and $y \neq 0$.

The system (4.12)-(4.15) reduces to

$$\begin{aligned} -\xi - \eta y &= 0 \\ x_1(1 - \frac{z}{K_z}) &= 0. \end{aligned} \tag{4.27}$$

This system does not have nonnegative solutions so there are no equilibria in the $(y, z)-$plane.

4.4.3 Positive octant (three-dimensional) equilibria

(i) $(x_1, x_2, y)-$octant: $z = 0$, $x_1 \neq 0, x_2 \neq 0$ and $y \neq 0$.

Algebraic system (4.12)-(4.15) reduces to the system

$$\begin{aligned} &f(0) - \beta_1 x_1 + p\gamma y = 0 \\ &g(1) - \beta_2 x_2 + \gamma y = 0 \\ &-\xi - \eta y + \delta(x_2 + q x_1) = 0, \end{aligned} \tag{4.28}$$

which has one and only one solution given by

$$y = \breve{y} = \frac{\beta_1\beta_2(\delta X_{2K} - \xi)}{\beta_1\beta_2\eta - \delta\gamma(\beta_1 + pq\beta_2)}, \tag{4.29}$$

$$x_1 = \breve{x}_1 = \frac{p\gamma\breve{y}}{\beta_1} = \frac{p\beta_2\gamma(\delta X_{2K} - \xi)}{\beta_1\beta_2\eta - \delta\gamma(\beta_1 + pq\beta_2)}, \tag{4.30}$$

$$x_2 = \breve{x}_2 = \frac{\beta_2 X_{2K} + \gamma\breve{y}}{\beta_2} = \frac{\beta_2 X_{2K}(\beta_1\eta - pq\gamma\delta) - \beta_1\gamma\xi}{\beta_1\beta_2\eta - \delta\gamma(\beta_1 + pq\beta_2)}. \tag{4.31}$$

Thus, we have the following theorem:

Theorem 4.3. *There exists a unique equilibrium, $F_{x_1x_2y}(\breve{x}_1, \breve{x}_2, \breve{y}, 0)$ in the positive $(x_1, x_2, y)-$octant if $\gamma > 0$ and $[\beta_1\beta_2\eta - \delta\gamma(\beta_1 + pq\beta_2)][\delta X_{2K} - \xi] > 0$.*

(ii) $(x_2, y, z)-$octant: $x_1 = 0$, $x_2 \neq 0, y \neq 0$ and $z \neq 0$.

The algebraic system (4.12)-(4.15) reduces to

$$\begin{aligned} &g(1 - z) - \beta_2 x_2 + \gamma y = 0 \\ &-\xi - \eta y + \delta x_2 = 0. \end{aligned} \tag{4.32}$$

Also since $x_1 = 0$, we have $z = z_c$. Thus, we have

$$z = z_c, \tag{4.33}$$

$$y = \breve{y} = \frac{\delta g(1 - z_c) - \beta_2 \xi}{\beta_2 \eta - \gamma \delta}, \tag{4.34}$$

$$x_2 = \breve{x}_2 = \frac{\eta g(1 - z_c) - \gamma \xi}{\beta_2 \eta - \gamma \delta}. \tag{4.35}$$

The above result leads to the following theorem:

Theorem 4.4. *If* $[\delta g(1-z_c)-\beta_2\xi][\beta_2\eta-\gamma\delta] > 0$, *then the system (4.1)-(4.4) possesses a unique equilibrium,* $F_{x_2,y,z}(0, \breve{x}_2, \breve{y}, z_c)$ *in the positive interior of the* $(x_2, y, z)-$*octant.*

(iii) $(x_1, y, z)-$octant: $x_2 = 0$, $x_1 \neq 0, y \neq 0$ and $z \neq 0$.

Algebraic system (4.12)-(4.15) reduces to the system

$$\begin{aligned} &f(z) - \beta_1 x_1 + p\gamma y = 0 \\ &-\xi - \eta y + \delta q x_1 = 0 \\ &1 - \frac{z}{K_z} = 0. \end{aligned} \tag{4.36}$$

Solving (4.36) for x_1, y and z, we get

$$z = K_z \tag{4.37}$$

$$y = \breve{y} = \frac{\delta q f(K_z) - \beta_1 \xi}{\beta_1 \eta - pq\delta\gamma} \tag{4.38}$$

$$x_1 = \breve{x}_1 = \frac{\eta f(K_z) - p\gamma\xi}{\beta_1 \eta - pq\delta\gamma}. \tag{4.39}$$

We observe that if $\breve{y} > 0$ then so is $\breve{x}_1$. Hence, we have the following theorem:

Theorem 4.5. *Suppose that* $[\delta q f(K_z) - \beta_1\xi][\beta_1\eta - pq\delta\gamma] > 0$. *Then there exists a unique positive equilibrium in the positive* $(x_1, y, z)-$*octant.*

Denote this equilibrium by $F_{x_1yz}(\breve{x}_1, 0, \breve{y}, K_z)$ if it exists.

(iv) $(x_1, x_2, z)-$octant: $y = 0,\ x_1 \neq 0, x_2 \neq 0$ and $z \neq 0$.

Algebraic system (4.12)-(4.15) reduces to the system

$$\begin{aligned} f(z) - \beta_1 x_1 &= 0 \\ g(1-z) - \beta_2 x_2 &= 0 \\ 1 - \frac{z}{K_z} &= 0. \end{aligned} \tag{4.40}$$

Solving (4.40) for x_1, x_2 and z, we get $z = K_z$, $x_1 = \frac{f(K_z)}{\beta_1}$ and $x_2 = \frac{g(1-K_z)}{\beta_2}$. This equilibrium always exists and we represent it by $F_{x_1x_2z}(\frac{f(K_z)}{\beta_1}, \frac{g(1-K_z)}{\beta_2}, 0, K_z)$.

4.4.4 Existence of interior equilibrium, $F^\star(x_1^\star, x_2{}^\star, y^\star, z^\star)$

The interior equilibria of the system given by Equations (4.1)-(4.4) are obtained by solving the system of isocline equations

$$f(z) - \beta_1 x_1 + p\gamma y = 0 \tag{4.41}$$

$$g(1-z) - \beta_2 x_2 + \gamma y = 0 \tag{4.42}$$

$$-\xi - \eta y + \delta(x_2 + q x_1) = 0 \tag{4.43}$$

$$1 - \frac{z}{K_z} = 0. \tag{4.44}$$

Solving for z from (4.44), we get

$$z = K_z. \tag{4.45}$$

Substituting Equation (4.45) into (4.41)-(4.42) we obtain

$$f(K_z) - \beta_1 x_1 + p\gamma y = 0 \tag{4.46}$$

$$g(1-K_z) - \beta_2 x_2 + \gamma y = 0. \tag{4.47}$$

Solving for x_1 from (4.46) and x_2 from (4.47) we get respectively

$$x_1 = \frac{f(K_z) + p\gamma y}{\beta_1} \tag{4.48}$$

and

$$x_2 = \frac{g(1 - K_z) + \gamma y}{\beta_2}. \tag{4.49}$$

We observe from (4.48) and (4.49) that

1. if $\gamma \geq 0$ and $y > 0$ then both x_1 and x_2 are positive,
2. if $\gamma < 0$ and $0 < y < min(-\frac{f(K_z)}{p\gamma}, -\frac{g(1-K_z)}{\gamma})$, then both x_1 and x_2 are positive.

Now substituting (4.48) and (4.49) into (4.43) and solving for y, we obtain

$$y = \frac{\beta_1\beta_2\xi - \delta[\beta_2 q f(K_z) + \beta_1 g(1 - K_z)]}{\gamma\delta[\beta_1 + pq\beta_2] - \beta_1\beta_2\eta}. \tag{4.50}$$

Thus, if $\{\beta_1\beta_2\xi - \delta[\beta_2 q f(K_z) + \beta_1 g(1-K_z)]\}\{\gamma\delta[\beta_1 + pq\beta_2] - \beta_1\beta_2\eta\} > 0$, then $y > 0$. The above results lead to the following theorems:

Theorem 4.6. *If $\gamma \geq 0$ and*

$$\{\beta_1\beta_2\xi - \delta[\beta_2 q f(K_z) + \beta_1 g(1 - K_z)]\}\{\gamma\delta[\beta_1 + pq\beta_2] - \beta_1\beta_2\eta\} > 0,$$

then the system given by Equations (4.1)-(4.4) contains a unique interior equilibrium of the form $F^\star = (x_1^\star, x_2{}^\star, y^\star, z^\star)$ with $x_1^\star > 0, x_2{}^\star > 0, y^\star > 0$ and $z^\star > 0$.

Proof:

The proof follows immediately from above calculations.

Theorem 4.7. *If*

1. $\gamma < 0$,

2. $\beta_1\beta_2\xi - \delta[\beta_2 q f(K_z) + \beta_1 g(1-K_z)] < 0$ *and*

3. $0 < y = y\star < min(-\frac{f(K_z)}{p\gamma}, -\frac{g(1-K_z)}{\gamma})$

then the system given by Equations (4.1)-(4.4) possesses a unique interior equilibrium of the form $F^\star(x_1^\star, x_2{}^\star, y^\star, z^\star)$ *with* $x_1^\star > 0, x_2{}^\star > 0, y^\star > 0$ *and* $z^\star > 0$.

Proof:

The proof follows directly from the above calculations.

4.5 Local stability analysis of equilibria

In this section, we discuss the local stability properties of all the equilibria determined in §4.4. We shall assume all the conditions for a given equilibrium are satisfied. The local stability properties of these equilibria are determined by the signs of the the real parts of the eigenvalues of the variational matrix V of system (4.1)-(4.4) evaluated at each equilibrium. The variational matrix V for system (4.1)-(4.4) is given by

$$V = \begin{bmatrix} f(z) - 2\beta_1 x_1 + p\gamma y & 0 & p\gamma x_1 & x_1 f'(z) \\ 0 & g(1-z) - 2\beta_2 x_2 + \gamma y & \gamma x_2 & -x_2 g'(1-z) \\ \delta q y & \delta y & -\xi - 2\eta y + \delta(x_2 + q x_1) & 0 \\ \alpha z(1 - \frac{z}{K_z}) & 0 & 0 & \alpha x_1(1 - \frac{2z}{K_z}) \end{bmatrix}. \qquad (4.51)$$

4.5.1 Local stability analysis of $F_0(0,0,0,0)$

The variational matrix V about the equilibrium $F_0(0,0,0,0)$ is given by

$$V_{F_0} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & g(1) & 0 & 0 \\ 0 & 0 & -\xi & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{4.52}$$

The eigenvalues of V_{F_0} are given by $0, 0, g(1)$ and $-\xi$. Thus, this equilibrium is non-hyperbolic and unstable since $g(1) > 0$.

4.5.2 Local stability analysis of $F_z(0,0,0,z_c)$

In this case,

$$V_{F_z} = \begin{bmatrix} f(z_c) & 0 & 0 & 0 \\ 0 & g(1-z_c) & 0 & 0 \\ 0 & 0 & -\xi & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{4.53}$$

The eigenvalues are equal to the corresponding diagonal elements of V_{F_z}, so the equilibrium is non-hyperbolic and unstable, since $f(z_c) > 0$ and $g(1 - z_c) > 0$.

4.5.3 Stability analysis of $F_{x_2}(0, X_{2K}, 0, 0)$

In this subsection, we consider the local and global stability properties of $F_{x_2}(0, X_{2K}, 0, 0)$. The local stability properties of $F_{x_2}(0, X_{2K}, 0, 0)$ are determined by the matrix

$$V_{F_z} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -\beta_2 X_{2K} & \gamma X_{2k} & -X_{2K} g'(1) \\ 0 & 0 & -\xi + \delta X_{2K} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{4.54}$$

The corresponding eigenvalues of V_{F_z} are $0, 0, -\xi + \delta X_{2K}$ and $-\beta_2 X_{2K}$. The equilibrium $F_{x_2}(0, X_{2K}, 0, 0)$ is non-hyperbolic and hence further analysis needs to be done to determine its local stability properties if $-\xi + \delta X_{2K} < 0$. If on the other hand if $-\xi + \delta X_{2K} > 0$, then the equilibrium $F_{x_2}(0, X_{2K}, 0, 0)$ is non hyperbolic and unstable.

Now, suppose $-\xi + \delta X_{2K} < 0$. To determine the stability of $F_{x_2}(0, X_{2K}, 0, 0)$ in this case, we choose a Liapunov function $V = V(x_1, x_2, y, z)$ defined by

$$V = ax_1 + b\{x_2 - X_{2K} - X_{2K} \ln(\frac{x_2}{X_{2K}})\} + cy + dz, \tag{4.55}$$

where a,b,c and d are positive constant to be determined.

The derivative of (4.55)along the solution curves of system (4.1)-(4.4) is given by

$$
\begin{aligned}
\dot{V} &= a\dot{x}_1 + b\frac{\dot{x}_2}{x_2}(x_2 - X_{2K}) + c\dot{y} + d\dot{z} \\
&= a(f(z) - \beta_1 x_1 + p\gamma y)x_1 + b(g(1-z) - \beta_2 x_2 + \gamma y)(x_2 - X_{2K}) \\
&+ c(-\xi - \eta y + \delta(x_2 + qx_1))y + d\alpha x_1(1 - \frac{z}{K_z})z \\
&= -a\beta_1 x_1^2 - b\beta_2(x_2 - X_{2K})^2 - c\eta y^2 - \frac{d\alpha x_1}{K_z}z^2 + c(-\xi + \delta X_{2K})y \\
&+ (ap\gamma + cq\delta)x_1 y + (b\gamma + c\delta)(x_2 - X_{2K})y + (a\tilde{f}(z) + d\alpha)x_1 z + b\tilde{g}(z)(x_2 - X_{2K})z \\
&\leq -a\beta_1 x_1^2 - b\beta_2(x_2 - X_{2K})^2 - c\eta y^2 - \frac{d\alpha x_1}{K_z}z^2 + (ap\gamma + cq\delta)x_1 y \\
&+ (b\gamma + c\delta)(x_2 - X_{2K})y + (a\tilde{f}(z) + d\alpha)x_1 z + b\tilde{g}(z)(x_2 - X_{2K})z,
\end{aligned} \tag{4.56}
$$

where $\tilde{f}(z)$ and $\tilde{g}(z)$ are defined respectively by

$$
\tilde{f}(z) = \begin{cases} \frac{f(z)}{z}, & z \neq 0 \\ f'(0), & z = 0 \end{cases} \tag{4.57}
$$

and

$$
\tilde{g}(z) = \begin{cases} \frac{g(1-z) - \beta_2 X_{2K}}{z}, & z \neq 0 \\ g'(1), & z = 0. \end{cases} \tag{4.58}
$$

Now, we consider two distinct cases:

1. $\gamma < 0$ and

2. $\gamma \geq 0$.

Case 1: $\gamma < 0$

In this case, we choose a, b, c and d as follows:

$$
a = q, \quad b = p, \quad d = \frac{K_z}{\alpha} \quad \textit{and} \quad c = \frac{-p\gamma}{\delta}.
$$

Equation (4.56) then becomes

$$\begin{aligned}\dot{V} &= -q\beta_1 x_1^2 - p\beta_2(x_2 - X_{2K})^2 + \frac{p\eta\gamma}{\delta}y^2 - x_1 z^2 \\ &+ (q\tilde{f}(z) + K_z)x_1 z + p\tilde{g}(z)(x_2 - X_{2K})z, \\ &= X^t B X,\end{aligned} \tag{4.59}$$

where

$$X = \begin{bmatrix} x_1 \\ x_2 - X_{2K} \\ y \\ z \end{bmatrix}, \tag{4.60}$$

$$B = \begin{bmatrix} -q\beta_1 & 0 & 0 & \frac{q\tilde{f}(z)+K_z}{2} \\ 0 & -p\beta_2 & 0 & \frac{p\tilde{g(z)}}{2} \\ 0 & 0 & \frac{p\eta\gamma}{\delta} & 0 \\ \frac{q\tilde{f}(z)+K_z}{2} & \frac{p\tilde{g(z)}}{2} & 0 & -x_1 \end{bmatrix}, \tag{4.61}$$

and X^t denotes the transpose of X.

Let B_k denote the sequence of principal minors of the matrix B. Then

$$B_1 = -q\beta_1 < 0, \tag{4.62}$$

$$B_2 = det \begin{bmatrix} -q\beta_1 & 0 \\ 0 & -p\beta_2 \end{bmatrix} = pq\beta_1\beta_2 > 0, \tag{4.63}$$

$$B_3 = det \begin{bmatrix} -q\beta_1 & 0 & 0 \\ 0 & -p\beta_2 & 0 \\ 0 & 0 & \frac{p\eta\gamma}{\delta} \end{bmatrix} = \frac{p^2 q\beta_1\beta_2\eta\gamma}{\delta} < 0, \tag{4.64}$$

$$B_4 = det \begin{bmatrix} -q\beta_1 & 0 & 0 & \frac{q\tilde{f}(z)+K_z}{2} \\ 0 & -p\beta_2 & 0 & \frac{p\tilde{g(z)}}{2} \\ 0 & 0 & \frac{p\eta\gamma}{\delta} & 0 \\ \frac{q\tilde{f}(z)+K_z}{2} & \frac{p\tilde{g(z)}}{2} & 0 & -x_1 \end{bmatrix} \tag{4.65}$$

$$= -x_1 B_3 + \frac{p\eta\gamma}{4\delta}(p^2 q\beta_1 \tilde{g(z)}^2 + p\beta_2(q\tilde{f}(z) + K_z)^2).$$

Theorem 4.8. *Suppose*

1. $\gamma < 0$,

2. $B_4 = -x_1 B_3 + \frac{p\eta\gamma}{4\delta}(p^2 q\beta_1 \tilde{g(z)}^2 + p\beta_2(q\tilde{f}(z) + K_z)^2) > 0.$

Then $F_{x_2}(0, X_{2K}, 0, 0)$ is globally asymptotically stable.

Proof:

Suppose $B_4 > 0$, then we have $B_1 < 0$, $B_2 > 0$, $B_3 < 0$ and $B_4 > 0$ for the real symmetric matrix B. Hence by Frobenius theorem (Theorem 1.5), B is negative definite. □

Case 2: $\gamma > 0$

In this case we choose a,b,c and d as follow:

$$a = \frac{q}{p\gamma},\ b = \frac{1}{\gamma},\ c = \frac{1}{\delta},\ d = \frac{K_z}{\alpha}.$$

Thus, Equation (4.57) becomes

$$\begin{aligned} \dot{V} &= -\frac{q\beta_1}{p\gamma}x_1^2 - \frac{\beta_2}{\gamma}(x_2 - X_{2K})^2 - \frac{\eta}{\delta}y^2 - x_1 z^2 + 2qx_1 y \\ &+ 2(x_2 - X_{2K})y + (\frac{q}{p\gamma}\tilde{f}(z) + K_z)x_1 z + \frac{1}{\gamma}\tilde{g}(z)(x_2 - X_{2K})z \\ &= X^t D X, \end{aligned} \tag{4.66}$$

where

$$X = \begin{bmatrix} x_1 \\ x_2 - X_{2K} \\ y \\ z \end{bmatrix}, \tag{4.67}$$

$$D = \begin{bmatrix} -\frac{q\beta_1}{p\gamma} & 0 & q & \frac{q\tilde{f}(z)}{2p\gamma} + \frac{K_z}{2} \\ 0 & -\frac{\beta_2}{\gamma} & 1 & \frac{\tilde{g(z)}}{2\gamma} \\ q & 1 & -\frac{\eta}{\delta} & 0 \\ \frac{q\tilde{f}(z)}{2p\gamma} + \frac{K_z}{2} & \frac{\tilde{g(z)}}{2\gamma} & 0 & -x_1 \end{bmatrix}, \tag{4.68}$$

and X^t denotes the transpose of X.

Let D_k denote the sequence of principal minors of the matrix D. Then

$$D_1 = -\frac{q\beta_1}{p\gamma} < 0, \tag{4.69}$$

$$D_2 = det \begin{bmatrix} -\frac{q\beta_1}{p\gamma} & 0 \\ 0 & -\frac{\beta_2}{\gamma} \end{bmatrix} = \frac{q\beta_1\beta_2}{p\gamma^2} > 0, \tag{4.70}$$

$$\begin{aligned} D_3 = det &\begin{bmatrix} -\frac{q\beta_1}{p\gamma} & 0 & q \\ 0 & -\frac{\beta_2}{\gamma} & 1 \\ q & 1 & -\frac{\eta}{\delta} \end{bmatrix} \\ &= \frac{-q}{p\gamma^2\delta}[\beta_1\beta_2\eta - \gamma\delta(\beta_1 + pq\beta_2)], \end{aligned} \tag{4.71}$$

$$D_4 = det \begin{bmatrix} -\frac{q\beta_1}{p\gamma} & 0 & q & \frac{q\tilde{f}(z)}{2p\gamma} + \frac{K_z}{2} \\ 0 & -\frac{\beta_2}{\gamma} & 1 & \frac{\tilde{g(z)}}{2\gamma} \\ q & 1 & -\frac{\eta}{\delta} & 0 \\ \frac{q\tilde{f}(z)}{2p\gamma} + \frac{K_z}{2} & \frac{\tilde{g(z)}}{2\gamma} & 0 & -x_1 \end{bmatrix} \tag{4.72}$$

$$= -D_3 x_1 - \frac{q\tilde{g(z)}}{\gamma}(\frac{q\tilde{f}(z)}{2p\gamma} + \frac{K_z}{2}) - \frac{q(\beta_1\eta - pq\gamma\delta)}{4p\delta\gamma^3}\tilde{g}^2(z)$$

$$- \frac{(\beta_2\eta - \gamma\delta)}{\delta\gamma}(\frac{q\tilde{f}(z)}{2p\gamma} + \frac{K_z}{2})^2.$$

Theorem 4.9. *Suppose*

1. $\gamma > 0$,
2. $D_3 < 0$,
3. $D_4 > 0$.

Then $F_{x_2}(0, X_{2K}, 0, 0)$ *is a globally asymptotically stable equilibrium.*

Proof:

This proof is similar to the proof of Theorem 4.8.

4.5.4 Local stability analysis of $F_{x_1y}(\bar{x_1}, 0, \bar{y}, 0)$

The variational matrix V about the equilibrium $F_{x_1y}(\bar{x_1}, 0, \bar{y}, 0)$ is given by

$$V_{F_{x_1y}} = \begin{bmatrix} -\beta_1\bar{x_1} & 0 & p\gamma\bar{x_1} & \bar{x_1}f'(0) \\ 0 & g(1) + \gamma\bar{y} & 0 & 0 \\ q\delta\bar{y} & \delta\bar{y} & -\eta\bar{y} & 0 \\ 0 & 0 & 0 & \alpha\bar{x_1} \end{bmatrix}. \tag{4.73}$$

The eigenvalues of $V_{F_{x_1y}}$ are given by $\lambda_2 = g(1) + \gamma\bar{y} > 0$, $\lambda_4 = \alpha\bar{x_1} > 0$ and λ_1 and λ_3 which are the eigenvalues of J_{22}, where J_{22} is given by

$$J_{22} = \begin{bmatrix} -\beta_1\bar{x_1} & p\gamma\bar{x_1} \\ \\ q\delta\bar{y} & -\eta\bar{y} \end{bmatrix}. \tag{4.74}$$

$$Trace(J_{22}) = -(\beta_1\bar{x_1} + \eta\bar{y}) < 0$$

and

$$det(J_{22}) = \bar{x_1}\bar{y}(\beta_1\eta - pq\gamma\delta) < 0,$$

that is one of the eigenvalues of J_{22} has a negative real part while the other has a positive real part.

Theorem 4.10. *The equilibrium $F_{x_1y}(\bar{x_1}, 0, \bar{y}, 0)$ is a hyperbolic saddle point always repelling in the x_2 and $z-$directions.*

4.5.5 Local stability analysis of $F_{x_1z}(\frac{f(K_z)}{\beta_1}, 0, 0, K_z)$

In this case,

$$V_{F_{x_1z}} = \begin{bmatrix} -f(K_z) & 0 & p\gamma\frac{f(K_z)}{\beta_1} & \frac{f(K_z)}{\beta_1}f'(K_z) \\ 0 & g(1-K_z) & 0 & 0 \\ 0 & 0 & -\xi + q\delta\frac{f(K_z)}{\beta_1} & 0 \\ 0 & 0 & 0 & -\alpha\frac{f(K_z)}{\beta_1} \end{bmatrix}. \tag{4.75}$$

The eigenvalues of $V_{F_{x_1z}}$ are equal to the corresponding diagonal elements of $V_{F_{x_1z}}$. That is, the equilibrium $F_{x_1z}(\frac{f(K_z)}{\beta_1}, 0, 0, K_z)$ is locally asymptotically stable in the x_1 and $z-$directions, and unstable in the x_2-direction. The stability in the $y-$direction depends on the sign of $-\xi + q\delta\frac{f(K_z)}{\beta_1}$.

Theorem 4.11. *The equilibrium* $F_{x_1 z}(\frac{f(K_z)}{\beta_1}, 0, 0, K_z)$ *is a hyperbolic saddle point always repelling in the* x_2-*direction.*

4.5.6 Stability analysis of $F_{x_2 y}(0, \frac{\beta_2 \eta X_{2K} - \gamma\xi}{\beta_2\eta - \gamma\delta}, \frac{\beta_2(\delta X_{2K} - \xi)}{\beta_2\eta - \gamma\delta}, 0)$.

For this equilibrium, the variational matrix (4.51) reduces to

$$V_{F_{x_2 y}} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -\beta_2\bar{x}_2 & \gamma\bar{x}_2 & -\bar{x}_2 g'(1) \\ q\delta\bar{y} & \delta\bar{y} & -\eta\bar{y} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{4.76}$$

We observe that if $\beta_2\eta - \gamma\delta < 0$, then one of the eigenvalues of $V_{F_{x_2 y}}$ has positive real part, hence the equilibrium $F_{x_2 y}(0, \frac{\beta_2 \eta X_{2K} - \gamma\xi}{\beta_2\eta - \gamma\delta}, \frac{\beta_2(\delta X_{2K} - \xi)}{\beta_2\eta - \gamma\delta}, 0)$ is non-hyperbolic and unstable. If on the other hand $\beta_2\eta - \gamma\delta > 0$, then further analysis needs to be done to determine its local stability properties.

Now, we assume $\beta_2\eta - \gamma\delta > 0$. Let $\bar{x}_2 = \frac{\beta_2 \eta X_{2K} - \gamma\xi}{\beta_2\eta - \gamma\delta}$ and $\bar{y} = \frac{\beta_2(\delta X_{2K} - \xi)}{\beta_2\eta - \gamma\delta}$. Since $\bar{x}_2 > 0$ and $\bar{y} > 0$, we have $\beta_2\eta X_{2K} - \gamma\xi > 0$ and $\delta X_{2K} - \xi > 0$.

To determine the stability of $F_{x_2 y}(0, \bar{x}_2, \bar{y}, 0)$ in this case, we choose a Liapunov function $V = V(x_1, x_2, y, z)$ defined by

$$V = ax_1 + b\{x_2 - \bar{x}_2 - \bar{x}_2 \ln(\frac{x_2}{\bar{x}_2})\} + c\{y - \bar{y} - \bar{y}\ln(\frac{x_2}{\bar{x}_2})\} + dz, \tag{4.77}$$

where a,b,c and d are positive constant to be determined.

The derivative of (4.77)along the solution curves the system (4.1)-(4.4) is given by

$$
\begin{aligned}
\dot{V} &= a\dot{x}_1 + b\frac{\dot{x}_2}{x_2}(x_2 - \bar{x}_2) + c\frac{\dot{y}}{y}(y - \bar{y}) + d\dot{z} \\
&= a(f(z) - \beta_1 x_1 + p\gamma y)x_1 + b(g(1-z) - \beta_2 x_2 + \gamma y)(x_2 - \bar{x}_2) \\
&+ c(-\xi - \eta y + \delta(x_2 + q x_1))(y - \bar{y}) + d\alpha x_1(1 - \frac{z}{K_z})z \qquad (4.78)\\
&= -a\beta_1 x_1^2 - b\beta_2(x_2 - \bar{x}_2)^2 - c\eta(y - \bar{y})^2 - \frac{d\alpha x_1}{K_z}z^2 + (ap\gamma + cq\delta)x_1(y - \bar{y}) \\
&+ (b\gamma + c\delta)(x_2 - \bar{x}_2)(y - \bar{y}) + (a\tilde{f}(z) + d\alpha)x_1 z + b\tilde{g}(1-z)(x_2 - \bar{x}_2)z,
\end{aligned}
$$

where $\tilde{f}(z)$ and $\tilde{g}(1-z)$ are defined respectively by

$$
\tilde{f}(z) = \frac{f(z) + p\gamma\bar{y}}{z}, \qquad (4.79)
$$

and

$$
\tilde{g}(z) = \begin{cases} \frac{g(1-z) - g(1)}{z}, & z \neq 0 \\ g'(1), & z = 0. \end{cases} \qquad (4.80)
$$

Now, we consider two distinct cases:

1. $\gamma < 0$ and

2. $\gamma \geq 0$.

Case 1: $\gamma < 0$

In this case, we choose a, b, c and d as follows:

$$
a = q, \quad b = p, \quad d = \frac{K_z}{\alpha} \quad and \quad c = \frac{-p\gamma}{\delta}.
$$

Equation (4.78) then becomes

$$
\begin{aligned}
\dot{V} &= -q\beta_1 x_1^2 - p\beta_2(x_2 - \bar{x}_2)^2 + \frac{p\eta\gamma}{\delta}(y - \bar{y})^2 - x_1 z^2 \\
&+ (q\tilde{f}(z) + K_z)x_1 z + p\tilde{g}(1-z)(x_2 - \bar{y})z, \qquad (4.81)\\
&= X^t B X,
\end{aligned}
$$

where

$$X = \begin{bmatrix} x_1 \\ x_2 - \bar{x}_2 \\ y - \bar{y} \\ z \end{bmatrix}, \tag{4.82}$$

$$B = \begin{bmatrix} -q\beta_1 & 0 & 0 & \frac{q\tilde{f}(z)+K_z}{2} \\ 0 & -p\beta_2 & 0 & \frac{p\tilde{g}(1-z)}{2} \\ 0 & 0 & \frac{p\eta\gamma}{\delta} & 0 \\ \frac{q\tilde{f}(z)+K_z}{2} & \frac{p\tilde{g}(1-z)}{2} & 0 & -x_1 \end{bmatrix}, \tag{4.83}$$

and X^t denotes the transpose of X.

The above result leads to the following theorem:

Theorem 4.12. *Suppose*

1. $\gamma < 0$,

2. $B_4 = -x_1 B_3 + \frac{p\eta\gamma}{4\delta}(p^2 q\beta_1 \tilde{g}^2(1-z) + p\beta_2(q\tilde{f}(z) + K_z)^2) > 0$

where

$$B_3 = det \begin{bmatrix} -q\beta_1 & 0 & 0 \\ 0 & -p\beta_2 & 0 \\ 0 & 0 & \frac{p\eta\gamma}{\delta} \end{bmatrix} = \frac{p^2 q\beta_1\beta_2\eta\gamma}{\delta} < 0.$$

Then $F_{x_2y}(0, \bar{x}_2, \bar{y}, 0)$ *is globally asymptotically stable.*

Proof:

The proof of this Theorem is similar to the proof of Theorem 4.8.

Case 2: $\gamma > 0$

In this case we choose a,b,c and d as follow:

$$a = \frac{q}{p\gamma},\ b = \frac{1}{\gamma},\ c = \frac{1}{\delta},\ d = \frac{K_z}{\alpha}.$$

Thus, Equation (4.78) becomes

$$\begin{aligned}\dot{V} &= -\frac{q\beta_1}{p\gamma}x_1^2 - \frac{\beta_2}{\gamma}(x_2-\bar{y})^2 - \frac{\eta}{\delta}(y-\bar{y})^2 - x_1z^2 + 2qx_1(y-\bar{y}) \\ &+ 2(x_2-\bar{x_2})(y-\bar{y}) + (\frac{q}{p\gamma}\tilde{f}(z) + K_z)x_1z + \frac{1}{\gamma}\tilde{g}(z)(x_2-\bar{x_2})z \\ &= X^tDX,\end{aligned} \tag{4.84}$$

where

$$X = \begin{bmatrix} x_1 \\ x_2-\bar{x_2} \\ y-\bar{y} \\ z \end{bmatrix}, \tag{4.85}$$

$$D = \begin{bmatrix} -\frac{q\beta_1}{p\gamma} & 0 & q & \frac{q\tilde{f}(z)}{2p\gamma}+\frac{K_z}{2} \\ 0 & -\frac{\beta_2}{\gamma} & 1 & \frac{\tilde{g}(1-z)}{2\gamma} \\ q & 1 & -\frac{\eta}{\delta} & 0 \\ \frac{q\tilde{f}(z)}{2p\gamma}+\frac{K_z}{2} & \frac{\tilde{g}(1-z)}{2\gamma} & 0 & -x_1 \end{bmatrix}, \tag{4.86}$$

and X^t denotes the transpose of X.

Let

$$\begin{aligned}D_3 &= det\begin{bmatrix} -\frac{q\beta_1}{p\gamma} & 0 & q \\ 0 & -\frac{\beta_2}{\gamma} & 1 \\ q & 1 & -\frac{\eta}{\delta} \end{bmatrix} \\ &= \frac{-q}{p\gamma^2\delta}[\beta_1\beta_2\eta - \gamma\delta(\beta_1 + pq\beta_2)],\end{aligned} \tag{4.87}$$

and

$$D_4 = det \begin{bmatrix} -\frac{q\beta_1}{p\gamma} & 0 & q & \frac{q\tilde{f}(z)}{2p\gamma} + \frac{K_z}{2} \\ 0 & -\frac{\beta_2}{\gamma} & 1 & \frac{\tilde{g}(1-z)}{2\gamma} \\ q & 1 & -\frac{\eta}{\delta} & 0 \\ \frac{q\tilde{f}(z)}{2p\gamma} + \frac{K_z}{2} & \frac{\tilde{g}(1-z)}{2\gamma} & 0 & -x_1 \end{bmatrix}$$

$$= -D_3 x_1 - \frac{q\tilde{g}(1-z)}{\gamma}(\frac{q\tilde{f}(z)}{2p\gamma} + \frac{K_z}{2}) - \frac{q(\beta_1\eta - pq\gamma\delta)}{4p\delta\gamma^3}\tilde{g}^2(1-z)$$
$$- \frac{(\beta_2\eta - \gamma\delta)}{\delta\gamma}(\frac{q\tilde{f}(z)}{2p\gamma} + \frac{K_z}{2})^2. \tag{4.88}$$

Theorem 4.13. *Suppose*

1. $\gamma > 0$,
2. $D_3 < 0$,
3. $D_4 > 0$.

Then $F_{x_2 y}(0, \bar{x_2}, \bar{y}, 0)$ *is a globally asymptotically stable equilibrium.*

Proof:

This proof is similar to the proof of Theorem 4.8.

4.5.7 Local stability analysis of $F_{x_2 z}(0, \frac{g(1-z_c)}{\beta_2}, 0, z_c)$

In this case, the variational matrix (4.52) reduces to

$$V_{F_{x_2 z}} = \begin{bmatrix} f(z_c) & 0 & 0 & 0 \\ 0 & -g(1-z_c) & \frac{\gamma g(1-z_c)}{\beta_2} & -\frac{g(1-z_c)}{\beta_2} g'(1-z_c) \\ 0 & 0 & -\xi + \delta\frac{g(1-z_c)}{\beta_2} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{4.89}$$

The eigenvalues of $V_{F_{x_2z}}$ are given by $0, f(z_c), -g(1-z_c)$ and $-\xi + \delta\frac{g(1-z_c)}{\beta_2}$. Hence, this equilibrium is non-hyperbolic and unstable since $f(z_c) > 0$.

4.5.8 Local stability analysis of $F_{x_1x_2y}(\breve{x}_1, \breve{x}_2, \breve{y}, 0)$

The variational matrix of system (4.1)-(4.4) in the positive $(x_1, x_2, y)-$ octant reduces to

$$V_{F_{x_1x_2y}} = \begin{bmatrix} -\beta_1\breve{x}_1 & 0 & p\gamma\breve{x}_1 & \breve{x}_1 f'(0) \\ 0 & \beta_2\breve{x}_2 & \gamma\breve{x}_2 & -\breve{x}_2 g'(1) \\ q\delta\breve{y} & \delta\breve{y} & -\eta y & 0 \\ 0 & 0 & 0 & \alpha\breve{x}_1 \end{bmatrix}. \tag{4.90}$$

We observe that one of the eigenvalues of $V_{F_{x_1x_2y}}$ is $\alpha\breve{x}_1 > 0$. Hence the equilibrium $F_{x_1x_2y}(\breve{x}_1, \breve{x}_2, \breve{y}, 0)$ is always unstable in the $z-$direction.

4.5.9 Local stability analysis of $F_{x_2,y,z}(0, \breve{x}_2, \breve{y}, z_c)$

The variational matrix V about the equilibrium $F_{x_2,y,z}(0, \breve{x}_2, \breve{y}, z_c)$ is given by

$$V_{F_{x_2yz}} = \begin{bmatrix} f(z_c) + p\gamma\breve{y} & 0 & 0 & 0 \\ 0 & -\beta_2\breve{x}_2 & \gamma\breve{x}_2 & -\breve{x}_2 g'(1-z_c) \\ q\delta\breve{y} & \delta\breve{y} & -\eta\breve{y} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{4.91}$$

The corresponding eigenvalues of $V_{F_{x_2yz}}$ are $0, f(z_c) + p\gamma\breve{y}$ and the eigenvalues corresponding to

$$J_{22} = \begin{bmatrix} -\beta_2\breve{x}_2 & \gamma\breve{x}_2 \\ \delta\breve{y} & -\eta\breve{y} \end{bmatrix}. \tag{4.92}$$

We observe that

$$Trace(J_{22}) = -(\beta_2\breve{x}_2 + \eta\breve{y}) < 0$$

and

$$det(J_{22}) = (\beta_2\eta - \gamma\delta)\breve{x}_2\breve{y}.$$

Hence we have the following results:

1. If $\gamma > 0$ and $\beta_2\eta - \gamma\delta > 0$, then one of the eigenvalues of $V_{F_{x_2yz}}$ has positive real part and hence this equilibrium is non-hyperbolic and unstable.

2. If $\gamma > 0$ and $\beta_2\eta - \gamma\delta < 0$ then two of the eigenvalues of $V_{F_{x_2yz}}$ have positive real parts, so this equilibrium is non-hyperbolic and unstable.

3. If $\gamma < 0$ and $f(z_c) + p\gamma\breve{y} > 0$ then one of the eigenvalues of $V_{F_{x_2yz}}$ is a positive real number, and hence this equilibrium is non-hyperbolic and unstable.

4. If $f(z_c) + p\gamma\breve{y} < 0$, then further analysis needs to be done to study the local stability properties of this equilibrium. See §4.8 for the discussion for this case.

4.5.10 Local stability analysis of $F_{x_1yz}(\breve{x}_1, 0, \breve{y}, K_z)$

In this case the variation matrix (4.51) reduces to

$$V_{F_{x_1yz}} = \begin{bmatrix} -\beta_1\breve{x}_1 & 0 & p\gamma\breve{x}_1 & \breve{x}_1 f'(K_z) \\ 0 & g(1-K_z)+\gamma\breve{y} & 0 & 0 \\ q\delta\breve{y} & \delta\breve{y} & -\eta\breve{y} & 0 \\ 0 & 0 & 0 & -\alpha\breve{x}_1 \end{bmatrix}, \tag{4.93}$$

and the eigenvalues of $V_{F_{x_1yz}}$ are $-\alpha\breve{x}_1$ and $g(1-K_z)+\gamma\breve{y}$ and the eigenvalues corresponding to

$$J_{22} = \begin{bmatrix} -\beta_1\breve{x}_1 & p\gamma\breve{x}_1 \\ q\delta\breve{y} & -\eta\breve{y} \end{bmatrix}. \tag{4.94}$$

We note that

$$Trace(J_{22}) = -(\beta_1 \breve{x}_1 + \eta \breve{y}) < 0$$

and

$$det(J_{22}) = (\beta_1 \eta - pq\gamma\delta)\breve{x}_2 \breve{y}.$$

We observe at this point that if $det(J_{22}) < 0$, then $\gamma > 0$. Hence we have the following theorem:

Theorem 4.14. *The equilibrium $F_{x_1 yz}(\breve{x}_1, 0, \breve{y}, K_z)$ is a locally asymptotically stable point if and only if $g(1-K_z)+\gamma\breve{y} < 0$. It is a local saddle point if $g(1-K_z)+\gamma\breve{y} > 0$.*

4.5.11 Local stability analysis of $F_{x_1 x_2 z}(\frac{f(K_z)}{\beta_1}, \frac{g(1-K_z)}{\beta_2}, 0, K_z)$

The variational matrix V about the equilibrium $F_{x_1 x_2 z}(\frac{f(K_z)}{\beta_1}, \frac{g(1-K_z)}{\beta_2}, 0, K_z)$ is given by

$$V_{F_{x_1 x_2 z}} = \begin{bmatrix} -f(K_z) & 0 & p\gamma\frac{f(K_z)}{\beta_1} & \frac{f(K_z)}{\beta_1} f'(K_z) \\ 0 & -g(1-K_z) & \gamma\frac{g(1-K_z)}{\beta_2} & -\frac{g(1-K_z)}{\beta_2} g'(1-K_z) \\ 0 & 0 & -\xi + \delta(\frac{g(1-K_z)}{\beta_2} + q\frac{f(K_z)}{\beta_1}) & 0 \\ 0 & 0 & 0 & -\alpha\frac{f(K_z)}{\beta_1} \end{bmatrix}. \tag{4.95}$$

The eigenvalue corresponding to the $y-$direction equals

$$\lambda_3 = -\xi + \delta(\frac{g(1-K_z)}{\beta_2} + q\frac{f(K_z)}{\beta_1}). \tag{4.96}$$

The other eigenvalues are $-f(K_z), -g(1-K_z)$ and $-\alpha\frac{f(K_z)}{\beta_1}$.

This results leads to the following theorem:

Theorem 4.15. *The equilibrium $F_{x_1 x_2 z}(\frac{f(K_z)}{\beta_1}, \frac{g(1-K_z)}{\beta_2}, 0, K_z)$ is a locally asymptotically stable point if $\lambda_3 = -\xi + \delta(\frac{g(1-K_z)}{\beta_2} + q\frac{f(K_z)}{\beta_1}) < 0$. If on the other hand if*

$\lambda_3 = -\xi + \delta(\frac{g(1-K_z)}{\beta_2} + q\frac{f(K_z)}{\beta_1}) > 0$ *then the equilibrium* $F_{x_1 x_2 z}(\frac{f(K_z)}{\beta_1}, \frac{g(1-K_z)}{\beta_2}, 0, K_z)$ *is a hyperbolic saddle point repelling in the* $y-$*direction.*

4.5.12 Local stability analysis of $F^\star(x_1^\star, x_2{}^\star, y^\star, z^\star)$

In this case the variational matrix (4.51) simplifies to

$$V_F^\star = \begin{bmatrix} -\beta_1 x_1^\star & 0 & p\gamma x_1^\star & x_1^\star f'(z^\star) \\ 0 & -\beta_2 x_2^\star & \gamma x_2^\star & -x_2^\star g'(z^\star) \\ q\delta y^\star & \delta y^\star & -\eta y^\star & 0 \\ 0 & 0 & 0 & -\alpha x_1^\star \end{bmatrix}. \tag{4.97}$$

The eigenvalues of $V_F^\star$ are $-\alpha x_1^\star$ and the eigenvalues of matrix

$$J_{33} = \begin{bmatrix} -\beta_1 x_1^\star & 0 & p\gamma x_1^\star \\ 0 & -\beta_2 x_2^\star & \gamma x_2^\star \\ q\delta y^\star & \delta y^\star & -\eta y^\star \end{bmatrix}. \tag{4.98}$$

Let

$$b_1 = -(-\beta_1 x_1^\star - \beta_2 x_2^\star - \eta y^\star) > 0$$

$$b_2 = \beta_1\beta_2 x_1^\star x_2^\star + (\beta_2\eta - \gamma\delta)x_2^\star y^\star + (\beta_1\eta - pq\delta\gamma)x_1^\star y^\star$$

$$b_3 = [\beta_1\beta_2\eta - \gamma\delta(\beta_1 + pq\beta_2)]x_1^\star x_2^\star y^\star.$$

Then the eigenvalues of J_{33} are given by the roots of

$$\lambda^3 + b_1\lambda^2 + b_2\lambda + b_3 = 0. \tag{4.99}$$

But

$$
\begin{aligned}
b_1b_2 - b_3 &= \{\beta_1 x_1^\star + \beta_2 x_2^\star + \eta y^\star\}\{\beta_1\beta_2 x_1^\star x_2^\star + (\beta_2\eta - \gamma\delta)x_2^\star y^\star + (\beta_1\eta - pq\delta\gamma)x_1^\star y^\star\} \\
&\quad - [\beta_1\beta_2\eta - \gamma\delta(\beta_1 + pq\beta_2)]x_1^\star x_2^\star y^\star \\
&= \beta_1 x_1^\star\{\beta_1\beta_2 x_1^\star x_2^\star + (\beta_1\eta - pq\delta\gamma)x_1^\star y^\star\} \\
&\quad + \beta_2 x^\star\{\beta_1\beta_2 x_1^\star x_2^\star + (\beta_2\eta - \gamma\delta)x_2^\star y^\star + \beta_1\eta x_1^\star y^\star\} \\
&\quad + \eta y^\star\{\beta_1\beta_2 x_1^\star x_2^\star + (\beta_2\eta - \gamma\delta)x_2^\star y^\star + (\beta_1\eta - pq\delta\gamma)x_1^\star y^\star\}.
\end{aligned}
\tag{4.100}
$$

The above results lead us to the following theorem.

Theorem 4.16. *If $\beta_1\beta_2\eta - \gamma\delta(\beta_1 + pq\beta_2) > 0$, then $F^\star(x_1^\star, x_2{}^\star, y^\star, z^\star)$ is locally asymptotically stable.*

Proof:

If

$$\beta_1\beta_2\eta - \gamma\delta(\beta_1 + pq\beta_2) > 0,$$

then $\beta_1\beta_2\eta - pq\gamma\delta\beta_2 > 0$ and $\beta_1\beta_2\eta - \gamma\delta\beta_1 > 0$. This then implies that $b_2 > 0$, $b_3 > 0$ and $b_1b_2 - b_3 > 0$. Hence by Theorem 1.3 (Routh-Hurwitz), all the eigenvalues of J_{33} have negative real parts, which in turn implies all the eigenvalues of $V_F^\star$ have negative real parts. □

4.6 Global stability analysis of $F^\star(x_1^\star, x_2{}^\star, y^\star, z^\star)$

In this section, we assume that all the necessary conditions for the existence of $F^\star(x_1^\star, x_2{}^\star, y^\star, z^\star)$ are satisfied. We also suppose that $F^\star(x_1^\star, x_2{}^\star, y^\star, z^\star)$ exists and is locally asymptotically stable. We now present sufficient conditions under which this locally asymptotically stable equilibrium will be globally (asymptotically) stable.

To do this, we choose a Liapunov function $V = V(x_1, x_2, y, z)$ defined by

$$\begin{aligned} V =& a\{x_1 - x_1^\star - x_1^\star \ln(\frac{x_1}{x_1^\star})\} + b\{x_2 - x_2^\star - x_2^\star \ln(\frac{x_2}{x_2^\star})\} \\ &+ c\{y - y^\star - y^\star \ln(\frac{y}{y^\star})\} + d\{z - z^\star - z^\star \ln(\frac{z}{z^\star})\}, \end{aligned} \tag{4.101}$$

where a,b,c and d are positive constants to be determined.

The derivative of (4.101) along the solution curves of the system given by Equations (4.1)-(4.4) is

$$\begin{aligned} \dot{V} &= \frac{a\dot{x_1}}{x_1}(x_1 - x_1^\star) + b\frac{\dot{x_2}}{x_2}(x_2 - x_2^\star) + \frac{c\dot{y}}{y}(y - y^\star) + \frac{d\dot{z}}{z}(z - z^\star) \\ &= a(f(z) - \beta_1 x_1 + p\gamma y)(x_1 - x_1^\star) + b(g(1-z) - \beta_2 x_2 + \gamma y)(x_2 - x_2^\star) \\ &+ c(-\xi - \eta y + \delta(x_2 + q x_1))(y - y^\star) + d\alpha x_1(1 - \frac{z}{K_z})(z - z^\star) \\ &= -a\beta_1(x_1 - x_1^\star)^2 - b\beta_2(x_2 - x_2^\star)^2 - c\eta(y - y^\star)^2 - \frac{d\alpha x_1}{K_z}(z - z^\star)^2 \\ &+ (ap\gamma + cq\delta)(x_1 - x_1^\star)(y - y^\star) + (b\gamma + c\delta)(x_2 - x_2^\star)(y - y^\star) \\ &+ a\tilde{f}(z, z^\star)(x_1 - x_1^\star)(z - z^\star) + b\tilde{g}(z, z^\star)(x_2 - x_2^\star)(z - z^\star), \end{aligned} \tag{4.102}$$

where $\tilde{f}(z, z^\star)$ and $\tilde{g}(z, z^\star)$ are defined respectively by

$$\tilde{f}(z, z^\star) = \begin{cases} \frac{f(z) - f(z^\star)}{z - z^\star}, & z \neq z^\star \\ f'(z^\star), & z = z^\star \end{cases} \tag{4.103}$$

and

$$\tilde{g}(z, z^\star) = \begin{cases} \frac{g(1-z) - g(1-z^\star)}{z - z^\star}, & z \neq z^\star \\ g'(1 - z^\star), & z = z^\star. \end{cases} \tag{4.104}$$

At this point, we consider two distinct cases:

1. $\gamma < 0$ and

2. $\gamma \geq 0$.

4.6.1 $F^\star(x_1^\star, x_2{}^\star, y^\star, z^\star)$ exists with $\gamma < 0$

In this case, we choose a, b, c and d as follows

$$a = q, \quad b = p, \quad d = K_z \quad and \quad c = \frac{-p\gamma}{\delta}.$$

Equation (4.102) then becomes

$$\begin{aligned}\dot{V} &= -q\beta_1(x_1 - x_1^\star)^2 - p\beta_2(x_2 - x_2^\star)^2 + \frac{p\eta\gamma}{\delta}(y - y^\star)^2 - \alpha x_1(z - z^\star)^2 \\ &+ q\tilde{f}(z, z^\star)(x_1 - x_1^\star)(z - z^\star) + p\tilde{g}(z, z^\star)(x_2 - x_2^\star)(z - z^\star), \\ &= X^t B X,\end{aligned} \tag{4.105}$$

where

$$X = \begin{bmatrix} x_1 - x_1^\star \\ x_2 - x_2^\star \\ y - y^\star \\ z - z^\star \end{bmatrix}, \tag{4.106}$$

$$B = \begin{bmatrix} -q\beta_1 & 0 & 0 & \frac{q\tilde{f}(z,z^\star)}{2} \\ 0 & -p\beta_2 & 0 & \frac{pg(z,z^\star)}{2} \\ 0 & 0 & \frac{p\eta\gamma}{\delta} & 0 \\ \frac{q\tilde{f}(z,z^\star)}{2} & \frac{pg(z,z^\star)}{2} & 0 & -\alpha x_1 \end{bmatrix}, \tag{4.107}$$

and X^t denotes the transpose of X.

Let B_k denote the sequence of principal minors of the matrix B. Then

$$B_1 = -q\beta_1 < 0, \tag{4.108}$$

$$B_2 = det \begin{bmatrix} -q\beta_1 & 0 \\ 0 & -p\beta_2 \end{bmatrix} = pq\beta_1\beta_2 > 0, \tag{4.109}$$

$$B_3 = det \begin{bmatrix} -q\beta_1 & 0 & 0 \\ 0 & -p\beta_2 & 0 \\ 0 & 0 & \frac{p\eta\gamma}{\delta} \end{bmatrix} = \frac{p^2 q\beta_1\beta_2\eta\gamma}{\delta} < 0, \tag{4.110}$$

$$\begin{aligned} B_4 = det &\begin{bmatrix} -q\beta_1 & 0 & 0 & \frac{q\tilde{f}(z,z^\star)}{2} \\ 0 & -p\beta_2 & 0 & \frac{pg(z,z^\star)}{2} \\ 0 & 0 & \frac{p\eta\gamma}{\delta} & 0 \\ \frac{q\tilde{f}(z,z^\star)}{2} & \frac{pg(z,z^\star)}{2} & 0 & -\alpha x_1 \end{bmatrix} \\ &= \frac{-p^2 q\eta\gamma}{4\delta}(4\alpha\beta_1\beta_2 x_1 - p\beta_1\tilde{f}^2(z,z^\star) - q\beta_2\tilde{g}^2(z,z^\star)). \end{aligned} \tag{4.111}$$

Theorem 4.17. *Suppose*

1. $\gamma < 0$,

2. $4\alpha\beta_1\beta_2 x_1 - p\beta_1\tilde{g}^2(z,z^\star) - q\beta_2\tilde{f}^2(z,z^\star) > 0.$

Then $F^\star(x_1^\star, x_2{}^\star, y^\star, z^\star)$ is globally asymptotically stable.

Proof:

Suppose

$$4\alpha\beta_1\beta_2 x_1 - p\beta_1\tilde{f}^2(z,z^\star) - q\beta_2\tilde{g}^2(z,z^\star) > 0.$$

Then from (4.111), we have $B_4 > 0$. That is, we have $B_1 < 0$, $B_2 > 0$, $B_3 < 0$ and $B_4 > 0$ for the real symmetric matrix B. Hence by Frobenius theorem (Theorem 1.5), B is negative definite. □

4.6.2 $F^\star(x_1^\star, x_2{}^\star, y^\star, z^\star)$ exists with $\gamma > 0$

Here, we assume $F^\star(x_1^\star, x_2{}^\star, y^\star, z^\star)$ exists and $\gamma > 0$. We also assume that $F^\star(x_1^\star, x_2{}^\star, y^\star, z^\star)$ is a locally asymptotically stable equilibrium. That is

$$\beta_1\beta_2\eta - \gamma\delta(\beta_1 + pq\beta_2) > 0 \tag{4.112}$$

is satisfied. In this case we choose a,b,c and d as follow:

$$a = \frac{q}{p\gamma},\ b = \frac{1}{\gamma},\ c = \frac{1}{\delta},\ d = K_z.$$

Thus, Equation (4.102) becomes

$$\begin{aligned}\dot{V} &= -\frac{q\beta_1}{p\gamma}(x_1 - x_1^\star)^2 - \frac{\beta_2}{\gamma}(x_2 - x_2^\star)^2 - \frac{\eta}{\delta}(y - y^\star)^2 - \alpha x_1(z - z^\star)^2 \\ &+ 2q(x_1 - x_1^\star)(y - y^\star) + 2(x_2 - x_2^\star)(y - y^\star) \\ &+ \frac{q}{p\gamma}\tilde{f}(z, z^\star)(x_1 - x_1^\star)(z - z^\star) + \frac{1}{\gamma}\tilde{g}(z, z^\star)(x_2 - x_2^\star)(z - z^\star) \\ &= X^t D X,\end{aligned} \tag{4.113}$$

where

$$X = \begin{bmatrix} x_1 - x_1^\star \\ x_2 - x_2^\star \\ y - y^\star \\ z - z^\star \end{bmatrix}, \tag{4.114}$$

$$D = \begin{bmatrix} -\frac{q\beta_1}{p\gamma} & 0 & q & \frac{q\tilde{f}(z,z^\star)}{2p\gamma} \\ 0 & -\frac{\beta_2}{\gamma} & 1 & \frac{g(z,z^\star)}{2\gamma} \\ q & 1 & -\frac{\eta}{\delta} & 0 \\ \frac{q\tilde{f}(z,z^\star)}{2p\gamma} & \frac{g(z,z^\star)}{2\gamma} & 0 & -\alpha x_1 \end{bmatrix}, \tag{4.115}$$

and X^t denotes the transpose of X.

Let D_k denote the sequence of principal minors of the matrix D. Then

$$D_1 = -\frac{q\beta_1}{p\gamma} < 0, \tag{4.116}$$

$$D_2 = det \begin{bmatrix} -\frac{q\beta_1}{p\gamma} & 0 \\ 0 & -\frac{\beta_2}{\gamma} \end{bmatrix} = \frac{q\beta_1\beta_2}{p\gamma^2} > 0, \tag{4.117}$$

$$D_3 = det \begin{bmatrix} -\frac{q\beta_1}{p\gamma} & 0 & q \\ 0 & -\frac{\beta_2}{\gamma} & 1 \\ q & 1 & -\frac{\eta}{\delta} \end{bmatrix} = \frac{-q}{p\gamma^2\delta}[\beta_1\beta_2\eta - \gamma\delta(\beta_1 + pq\beta_2)] < 0, \tag{4.118}$$

$$D_4 = det \begin{bmatrix} -\frac{q\beta_1}{p\gamma} & 0 & q & \frac{q\tilde{f}(z,z^\star)}{2p\gamma} \\ 0 & -\frac{\beta_2}{\gamma} & 1 & \frac{g(z,z^\star)}{2\gamma} \\ q & 1 & -\frac{\eta}{\delta} & 0 \\ \frac{q\tilde{f}(z,z^\star)}{2p\gamma} & \frac{g(z,z^\star)}{2\gamma} & 0 & -\alpha x_1 \end{bmatrix} = -\alpha D_3 x_1 - \frac{q^2}{2p\gamma^2}\tilde{f}(z,z^\star)\tilde{g}(z,z^\star) - \frac{q(\beta_1\eta - pq\gamma\delta)}{4p\delta\gamma^3}\tilde{g}^2(z,z^\star) - \frac{q^2(\beta_2\eta - \gamma\delta)}{4p^2\delta\gamma^3}\tilde{f}^2(z,z^\star). \tag{4.119}$$

Theorem 4.18. *Suppose*

1. $\gamma > 0$,

2. $D_4 > 0$.

Then $F^\star(x_1^\star, x_2{}^\star, y^\star, z^\star)$ is a globally asymptotically stable equilibrium.

4.7 Criteria for extinction of industrial assets

In this section, we shall derive sufficient conditions for the extinction of industrial assets. This will be done using a Liapunov function to provide criteria for the equilibrium $F_{x_1x_2z}(\frac{f(K_z)}{\beta_1}, \frac{g(1-K_z)}{\beta_2}, 0, K_z)$ to be globally asymptotically stable and hence unique. We assume that this equilibrium exists and is locally asymptotically stable.

That is,

$$-\xi + \delta(\frac{g(1-K_z)}{\beta_2} + q\frac{f(K_z)}{\beta_1}) < 0. \tag{4.120}$$

We now choose a Liapunov function $V = V(x_1, x_2, y, z)$ defined by

$$\begin{aligned} V =& a\{x_1 - \frac{f(K_z)}{\beta_1} - \frac{f(K_z)}{\beta_1}\ln(\frac{\beta_1 x_1}{f(K_z)})\} + b\{x_2 - \frac{g(1-K_z)}{\beta_2} - \frac{g(1-K_z)}{\beta_2}\ln(\frac{\beta_2 x_2}{g(1-K_z)})\} \\ &+ cy + d\{z - K_z - K_z\ln(\frac{z}{K_z})\}, \end{aligned} \tag{4.121}$$

where a,b,c and d are positive constants to be determined. The derivative of (4.121) along the solution curves of the system given by Equations (4.1)-(4.4) is

$$\begin{aligned} \dot{V} =& \frac{a\dot{x_1}}{x_1}(x_1 - \frac{f(K_z)}{\beta_1}) + b\frac{\dot{x_2}}{x_2}(x_2 - \frac{g(1-K_z)}{\beta_2}) + c\dot{y} + \frac{d\dot{z}}{z}(z - K_z) \\ =& a(f(z) - \beta_1 x_1 + p\gamma y)(x_1 - \frac{f(K_z)}{\beta_1}) + b(g(1-z) - \beta_2 x_2 + \gamma y)(x_2 - \frac{g(1-K_z)}{\beta_2}) \\ &+ c(-\xi - \eta y + \delta(x_2 + q x_1))y + d\alpha x_1(1 - \frac{z}{K_z})(z - K_z) \\ =& -a\beta_1(x_1 - \frac{f(K_z)}{\beta_1})^2 - b\beta_2(x_2 - \frac{g(1-K_z)}{\beta_2})^2 - c\eta y^2 - \frac{d\alpha x_1}{K_z}(z - K_z)^2 \\ &+ c\{-\xi + \delta(\frac{g(1-K_z)}{\beta_2} + q\frac{f(K_z)}{\beta_1})\}y + (ap\gamma + cq\delta)(x_1 - \frac{f(K_z)}{\beta_1})y \\ &+ (b\gamma + c\delta)(x_2 - \frac{g(1-K_z)}{\beta_2})y + a\tilde{f}(z, K_z)(x_1 - \frac{f(K_z)}{\beta_1})(z - K_z) \\ &+ b\tilde{g}(z, K_z)(x_2 - \frac{g(1-K_z)}{\beta_2})(z - K_z), \end{aligned} \tag{4.122}$$

where $\tilde{f}(z, z^\star)$ and $\tilde{g}(z, z^\star)$ are defined respectively by

$$\tilde{f}(z, z^\star) = \begin{cases} \frac{f(z)-f(K_z)}{z-K_z}, & z \neq K_z \\ f'(K_z), & z = K_z \end{cases} \tag{4.123}$$

and

$$\tilde{g}(z, K_z) = \begin{cases} \frac{g(1-z)-g(1-K_z)}{z-K_z}, & z \neq K_z \\ g'(1-K_z), & z = K_z. \end{cases} \tag{4.124}$$

We consider two cases:

1. $\gamma < 0$ and

2. $\gamma \geq 0$.

4.7.1 $F_{x_1 x_2 z}(\frac{f(K_z)}{\beta_1}, \frac{g(1-K_z)}{\beta_2}, 0, K_z)$ exists with $\gamma < 0$

In this case we choose a,b,c and d as follows:

$$a = q, \quad b = p, \quad d = K_z \quad and \quad c = \frac{-p\gamma}{\delta}.$$

Equation (4.122) now becomes

$$\begin{aligned}
\dot{V} &= -q\beta_1(x_1 - \frac{f(K_z)}{\beta_1})^2 - p\beta_2(x_2 - \frac{g(1-K_z)}{\beta_2})^2 + \frac{p\gamma\eta}{\delta}y^2 - \alpha x_1(z-K_z)^2 \\
&+ \frac{-p\gamma}{\delta}\{-\xi + \delta(\frac{g(1-K_z)}{\beta_2} + q\frac{f(K_z)}{\beta_1})\}y + q\tilde{f}(z,K_z)(x_1 - \frac{f(K_z)}{\beta_1})(z-K_z) \\
&+ p\tilde{g}(z,K_z)(x_2 - \frac{g(1-K_z)}{\beta_2})(z-K_z) \\
&\leq -q\beta_1(x_1 - \frac{f(K_z)}{\beta_1})^2 - p\beta_2(x_2 - \frac{g(1-K_z)}{\beta_2})^2 + \frac{p\gamma\eta}{\delta}y^2 - \alpha x_1(z-K_z)^2 \\
&+ q\tilde{f}(z,K_z)(x_1 - \frac{f(K_z)}{\beta_1})(z-K_z) + p\tilde{g}(z,K_z)(x_2 - \frac{g(1-K_z)}{\beta_2})(z-K_z) \\
&= X^tBX,
\end{aligned} \tag{4.125}$$

where

$$X = \begin{bmatrix} x_1 - \frac{f(K_z)}{\beta_1} \\ x_2 - \frac{g(1-K_z)}{\beta_2} \\ y \\ z - K_z \end{bmatrix}, \tag{4.126}$$

$$B = \begin{bmatrix} -q\beta_1 & 0 & 0 & \frac{q\tilde{f}(z,K_z)}{2} \\ 0 & -p\beta_2 & 0 & \frac{pg(z,K_z)}{2} \\ 0 & 0 & \frac{p\eta\gamma}{\delta} & 0 \\ \frac{q\tilde{f}(z,K_z)}{2} & \frac{pg(z,K_z)}{2} & 0 & -\alpha x_1 \end{bmatrix}, \tag{4.127}$$

and X^t denotes the transpose of X.

Theorem 4.19. *Suppose*

1. $\gamma < 0$,
2. $4\alpha\beta_1\beta_2 x_1 - p\beta_1\tilde{g}^2(z, K_z) - q\beta_2\tilde{f}^2(z, K_z) > 0$,
3. $-\xi + \delta(\frac{g(1-K_z)}{\beta_2} + q\frac{f(K_z)}{\beta_1}) < 0$.

Then $F_{x_1x_2z}(\frac{f(K_z)}{\beta_1}, \frac{g(1-K_z)}{\beta_2}, 0, K_z)$ *is globally asymptotically stable.*

Proof:

The proof of this theorem is similar to the proof of Theorem 1.17.

4.7.2 $F_{x_1x_2z}(\frac{f(K_z)}{\beta_1}, \frac{g(1-K_z)}{\beta_2}, 0, K_z)$ exists with $\gamma \geq 0$

Here, we assume $F_{x_1x_2z}(\frac{f(K_z)}{\beta_1}, \frac{g(1-K_z)}{\beta_2}, 0, K_z)$ exists and $\gamma > 0$. We also assume that $F_{x_1x_2z}(\frac{f(K_z)}{\beta_1}, \frac{g(1-K_z)}{\beta_2}, 0, K_z)$ is a locally asymptotically stable equilibrium. In this case we choose a,b,c and d as follow:

$$a = \frac{q}{p\gamma}, \; b = \frac{1}{\gamma}, \; c = \frac{1}{\delta}, \; d = K_z.$$

Thus, Equation (4.122) becomes

$$
\begin{aligned}
\dot{V} &= -\frac{q\beta_1}{p\gamma}(x_1 - \frac{f(K_z)}{\beta_1})^2 - \frac{\beta_2}{\gamma}(x_2 - \frac{g(1-K_z)}{\beta_2})^2 - \frac{\eta}{\delta}y^2 - \alpha x_1(z-K_z)^2 \\
&+ \frac{1}{\delta}\{-\xi + \delta(\frac{g(1-K_z)}{\beta_2} + q\frac{f(K_z)}{\beta_1})\}y + 2q(x_1 - \frac{f(K_z)}{\beta_1})y \\
&+ 2(x_2 - \frac{g(1-K_z)}{\beta_2})y + \frac{q}{p\gamma}\tilde{f}(z,K_z)(x_1 - \frac{f(K_z)}{\beta_1})(z-K_z) \\
&+ \frac{1}{\gamma}\tilde{g}(z,K_z)(x_2 - \frac{g(1-K_z)}{\beta_2})(z-K_z) \\
&\le -\frac{q\beta_1}{p\gamma}(x_1 - \frac{f(K_z)}{\beta_1})^2 - \frac{\beta_2}{\gamma}(x_2 - \frac{g(1-K_z)}{\beta_2})^2 - \frac{\eta}{\delta}y^2 - \alpha x_1(z-K_z)^2 \\
&+ 2q(x_1 - \frac{f(K_z)}{\beta_1})y + 2(x_2 - \frac{g(1-K_z)}{\beta_2})y + \frac{q}{p\gamma}\tilde{f}(z,K_z)(x_1 - \frac{f(K_z)}{\beta_1})(z-K_z) \\
&+ \frac{1}{\gamma}\tilde{g}(z,K_z)(x_2 - \frac{g(1-K_z)}{\beta_2})(z-K_z) \\
&= X^t D X,
\end{aligned}
\tag{4.128}
$$

where

$$
X = \begin{bmatrix} x_1 - \frac{f(K_z)}{\beta_1} \\ x_2 - \frac{g(1-K_z)}{\beta_2} \\ y \\ z - K_z \end{bmatrix}, \tag{4.129}
$$

$$
D = \begin{bmatrix} -\frac{q\beta_1}{p\gamma} & 0 & q & \frac{q\tilde{f}(z,z^*)}{2p\gamma} \\ 0 & -\frac{\beta_2}{\gamma} & 1 & \frac{g(z,z^*)}{2\gamma} \\ q & 1 & -\frac{\eta}{\delta} & 0 \\ \frac{q\tilde{f}(z,z^*)}{2p\gamma} & \frac{g(z,z^*)}{2\gamma} & 0 & -\alpha x_1 \end{bmatrix}, \tag{4.130}
$$

and X^t denotes the transpose of X.

Let D_k denote the sequence of principal minors of the matrix D. Then

$$
D_1 = -\frac{q\beta_1}{p\gamma} < 0, \tag{4.131}
$$

$$D_2 = det \begin{bmatrix} -\frac{q\beta_1}{p\gamma} & 0 \\ 0 & -\frac{\beta_2}{\gamma} \end{bmatrix} = \frac{q\beta_1\beta_2}{p\gamma^2} > 0, \tag{4.132}$$

$$\begin{aligned} D_3 = det &\begin{bmatrix} -\frac{q\beta_1}{p\gamma} & 0 & q \\ 0 & -\frac{\beta_2}{\gamma} & 1 \\ q & 1 & -\frac{\eta}{\delta} \end{bmatrix} \\ &= \frac{-q}{p\gamma^2\delta}[\beta_1\beta_2\eta - \gamma\delta(\beta_1 + pq\beta_2)], \end{aligned} \tag{4.133}$$

$$\begin{aligned} D_4 = det &\begin{bmatrix} -\frac{q\beta_1}{p\gamma} & 0 & q & \frac{q\tilde{f}(z,z^\star)}{2p\gamma} \\ 0 & -\frac{\beta_2}{\gamma} & 1 & \frac{g(z,z^\star)}{2\gamma} \\ q & 1 & -\frac{\eta}{\delta} & 0 \\ \frac{q\tilde{f}(z,z^\star)}{2p\gamma} & \frac{g(z,z^\star)}{2\gamma} & 0 & -\alpha x_1 \end{bmatrix} \\ &= -\alpha D_3 x_1 - \frac{q^2}{2p\gamma^2}\tilde{f}(z,z^\star)\tilde{g}(z,z^\star) - \frac{q(\beta_1\eta - pq\gamma\delta)}{4p\delta\gamma^3}\tilde{g}^2(z,z^\star) \\ &- \frac{q^2(\beta_2\eta - \gamma\delta)}{4p^2\delta\gamma^3}\tilde{f}^2(z,z^\star). \end{aligned} \tag{4.134}$$

The above results and Frobenius Theorem lead to the following theorem:

Theorem 4.20. *Suppose*

1. $\gamma > 0$,
2. $-\xi + \delta(\frac{g(1-K_z)}{\beta_2} + q\frac{f(K_z)}{\beta_1}) < 0$,
3. $D_3 < 0$,
4. $D_4 > 0$.

Then $F_{x_1x_2z}(\frac{f(K_z)}{\beta_1}, \frac{g(1-K_z)}{\beta_2}, 0, K_z)$ *is a globally asymptotically stable equilibrium.*

4.8 Criteria for extinction of agricultural assets x_2

In this section, we shall derive sufficient conditions for the extinction of agricultural assets of farming group B, that is $x_2(t)$. At the end of the section, we will give similar conditions for the extinction of agricultural assets for farming group A, that is $x_1(t)$.

We derive sufficient conditions for the extinction of agricultural assets of farming group B, that is $x_2(t)$ by using a Liapunov function to provide criteria for the equilibrium $F_{x_1yz}(\breve{x}_1, 0, \breve{y}, K_z)$ to be globally asymptotically stable. We assume that this equilibrium exists and is locally asymptotically stable. That is,

$$g(1-K_z)+\gamma\breve{y}<0. \tag{4.135}$$

We now choose a Liapunov function $V=V(x_1,x_2,y,z)$ defined by

$$V = a\{x_1-\breve{x}_1-\breve{x}_1\ln(\frac{x_1}{\breve{x}_1})\}+bx_2 \quad +c\{y-\breve{y}-\breve{y}\ln(\frac{y}{\breve{y}})\}+d\{z-K_z-K_z\ln(\frac{z}{K_z})\}, \tag{4.136}$$

where a,b,c and d are positive constants to be determined. The derivative of (4.136) along the solution curves of the system (4.1)-(4.4) is

$$\begin{aligned}
\dot{V} &= \frac{a\dot{x}_1}{x_1}(x_1-\breve{x}_1)+b\dot{x}_2+\frac{c\dot{y}}{y}(y-\breve{y})+\frac{d\dot{z}}{z}(z-K_z)\\
&= a(f(z)-\beta_1x_1+p\gamma y)(x_1-\breve{x}_1)+b(g(1-z)-\beta_2x_2+\gamma y)x_2\\
&+c(-\xi-\eta y+\delta(x_2+qx_1))(y-\breve{y})+d\alpha x_1(1-\frac{z}{K_z})(z-K_z)\\
&= -a\beta_1(x_1-\breve{x}_1)^2-b\beta_2x_2^2-c\eta(y-\breve{y})^2-\frac{d\alpha x_1}{K_z}(z-K_z)^2\\
&+(ap\gamma+cq\delta)(x_1-\breve{x}_1)(y-\breve{y})+a\tilde{f}(z,K_z)(x_1-\breve{x}_1)(z-K_z)\\
&+(b\gamma+c\delta)x_2(y-\breve{y})++b\tilde{g}(z,K_z)x_2(z-K_z)+b(g(1-K_z)+\gamma\breve{y})x_2,
\end{aligned} \tag{4.137}$$

where $\tilde{f}(z, K_z)$ and $\tilde{g}(z, K_z)$ are defined respectively by

$$\tilde{f}(z, K_z) = \begin{cases} \frac{f(z)-f(K_z)}{z-K_z}, & z \neq K_z \\ f'(K_z), & z = K_z \end{cases} \tag{4.138}$$

and

$$\tilde{g}(z, K_z) = \begin{cases} \frac{g(1-z)-g(1-K_z)}{z-K_z}, & z \neq K_z \\ g'(1-K_z), & z = K_z. \end{cases} \tag{4.139}$$

At this point, we choose a,b,c and d as follows:

$$a = q, \quad b = p, \quad c = \frac{-p\gamma}{\delta} \quad and \quad d = \frac{K_z}{\alpha}.$$

Equation (4.137) now becomes

$$\begin{aligned} \dot{V} &= -q\beta_1(x_1 - \breve{x}_1)^2 - p\beta_2 x_2^2 + \frac{p\gamma\eta}{\delta}(y - \breve{y})^2 - x_1(z - K_z)^2 \\ &+ q\tilde{f}(z, K_z)(x_1 - \breve{x}_1)(z - K_z) + p\tilde{g}(z, K_z)x_2(z - K_z) + p(g(1 - K_z) + \gamma\breve{y})x_2 \\ &\leq -q\beta_1(x_1 - \breve{x}_1)^2 - p\beta_2 x_2^2 + \frac{p\gamma\eta}{\delta}(y - \breve{y})^2 - x_1(z - K_z)^2 \\ &+ q\tilde{f}(z, K_z)(x_1 - \breve{x}_1)(z - K_z) + p\tilde{g}(z, K_z)x_2(z - K_z) \\ &= X^t B X, \end{aligned} \tag{4.140}$$

where

$$X = \begin{bmatrix} x_1 - \breve{x}_1 \\ x_2 \\ y - \breve{y} \\ z - K_z \end{bmatrix}, \tag{4.141}$$

$$B = \begin{bmatrix} -q\beta_1 & 0 & 0 & \frac{q\tilde{f}(z,K_z)}{2} \\ 0 & -p\beta_2 & 0 & \frac{p\tilde{g}(z,K_z)}{2} \\ 0 & 0 & \frac{p\eta\gamma}{\delta} & 0 \\ \frac{q\tilde{f}(z,K_z)}{2} & \frac{p\tilde{g}(z,K_z)}{2} & 0 & -x_1 \end{bmatrix}, \tag{4.142}$$

and X^t denotes the transpose of X.

Theorem 4.21. *Suppose*

1. $g(1-K_z)+\gamma\breve{y}<0$
2. $4\beta_1\beta_2x_1-p\beta_1\tilde{g}^2(z,K_z)-q\beta_2\tilde{f}^2(z,K_z)>0.$

Then $F_{x_1yz}(\breve{x}_1,0,\breve{y},K_z)$ *is globally asymptotically stable.*

Proof:

The proof of this theorem is the same as the proof of Theorem 4.19.

Similarly, one can prove the following Theorem:

Theorem 4.22. *Suppose*

1. $f(z_c)+p\gamma\breve{y}<0$
2. $4\beta_1\beta_2x_1-p\beta_1\tilde{g}^2(z,z_c)-q\beta_2\tilde{f}^2(z,z_c)>0.$

Then $F_{x_2yz}(0,\breve{x}_2,\breve{y},z_c)$ *is globally asymptotically stable.*

4.9 Numerical examples

In this section, we describe some numerical examples to illustrate some of our results. In all our examples, numerical values are chosen for numerical convenience to illustrate mathematical results and may not represent any actually agricultural-industrial system for two farming groups.

However, a study of Hutterites and the Alberta economy by Frank X. Kehoe [48] in 1972 revealed among other things that, in 1971

1. much of the expenses on Hutterites farms just like the farms registered in the Alberta Farm Business Analysis, (FBA) program went into industry, such as machinary repairs and parts, fuel, fertilizers and sprays etc,

2. there was very little difference between the economics of the Hutterite farms and the FBA farms,

3. the Hutterites colonies had a per acre income of \$30.87 while the FBA farms had an average income of \$31.21 per acre,

4. the FBA farms showed a total expenses of \$ 24.79 per acre including \$2.14 per acre of hired labor expense while the Hutterites farms showed a total expenses of \$ 21.52 with no hired labour expense,

5. the profit of the two groups were \$ 9.35 per acre for the Hutterite farm and \$ 7.42 per acre for the FBA farms.

Based on the above study, the following assumptions will be made while choosing the model parameters and the function $f(z)$ and $g(1-z)$.

1. The functions $f(z)$ and $g(1-z)$ have a similar form.

2. $p \approx 1$.

3. $q \approx 1$.

4. $\beta_1 \approx \beta_2$.

There are many functions which satisfy the properties of f given in this model. For simplicity, we choose only two of such functions. One of these functions will be when $f(z) = \beta X_{1K} z$ which in general would be appropriate if x_1 were not subject to saturation levels. The other is when $f(z)$ is a Michaelis-Menten function, i.e. $f(z) = \frac{2\beta_1 X_{1K} z}{1+z}$. This choice is appropriate for the case where x_1, due to limited resources, is subject to saturation levels. Michaelis-Menten functions are the simplest which exhibit that property and are the prototypes of all such functions. We shall demonstrate with the above two functions, that no matter what function one chooses to be f, as long as it satisfies the conditions given in this chapter, will give the same qualitative result. We note that the quantitative results may vary but that is not the subject of study in this thesis.

4.9.1 $f(z)$ is a linear function of z

In our first set of examples, we describe $f(z)$ and $g(1-z)$ by linear functions of z. Hence our model equations assume the form

$$\begin{aligned} \dot{x_1} &= x_1[\beta_1 X_{1K} z - \beta_1 x_1 + p\gamma y] \\ \dot{x_2} &= x_2[\beta_2 X_{2K}(1-z) - \beta_2 x_2 + \gamma y] \\ \dot{y} &= y[-\xi - \eta y + \delta(x_2 + q x_1)] \\ \dot{z} &= \alpha z x_1 (1 - \frac{z}{K_z}), \end{aligned} \tag{4.143}$$

with $x_{10} = 2, \quad x_{20} = 8, \quad y_0 = 1, \quad z_0 = 0.1.$

Example 4.1

In this example, we set

$$\alpha = 0.2, \quad \beta_1 = 6, \quad \beta_2 = 6.1, \quad \gamma = 0.3, \quad X_{1K} = 10, \quad X_{2K} = 9,$$
$$\xi = 1, \quad \eta = 0.25, \quad \delta = 0.2, \quad K_z = 0.5, \quad p = q = 0.95.$$

The graphs of numerical solutions for $t \in (0, 50)$, are exhibited in Figure 4.1. This example illustrates the global asymptotic stability of $F^\star(x_1^\star, x_2{}^\star, y^\star, z^\star)$ for $\gamma > 0$ (cf. Theorem 4.18).

Example 4.2

In this example, the coefficients are given by the following:

$$\alpha = 0.2, \quad \beta_1 = 6, \quad \beta_2 = 6.1, \quad \gamma = 0.3, \quad X_{1K} = 10, \quad X_{2K} = 9,$$

$$\xi = 4, \quad \eta = 0.25, \quad \delta = 0.2, \quad K_z = 0.5, \quad p = q = 0.95.$$

The graphs of numerical solutions for $t \in (0, 50)$, are exhibited in Figure 4.2. In this example industrial assets go extinct after a finite time (cf. Theorem 4.20).

Example 4.3

In this example, we set

$$\alpha = 0.2, \quad \beta_1 = 6, \quad \beta_2 = 6.1, \quad \gamma = -2, \quad X_{1K} = 10, \quad X_{2K} = 9,$$

$$\xi = 1, \quad \eta = 0.25, \quad \delta = 0.25, \quad K_z = 0.90, \quad p = q = 0.95.$$

The graphs of numerical solutions for $t \in (0, 50)$, are presented in Figure 4.3. In this example agricultural assets $x_2(t)$ goes extinct after $t > 0$, thus, making $F_{x_1yz}(\breve{x}_1, 0, \breve{y}, K_z)$ a globally asymptotically stable equilibrium (cf. Theorem 4.21).

Example 4.4

In this example, we set

$$\alpha = 0.2, \quad \beta_1 = 6, \quad \beta_2 = 6.1, \quad \gamma = -2, \quad X_{1K} = 10, \quad X_{2K} = 9,$$

$$\xi = 1, \quad \eta = 0.25, \quad \delta = 0.33, \quad K_z = 0.50, \quad p = q = 0.95.$$

The graphs of numerical solutions for $t \in (0, 50)$, are presented in Figure 4.4. In this example agricultural assets $x_1(t)$ goes extinct, thus, making $F_{x_2yz}(0, \breve{x}_2, \breve{y}, z_c)$ a globally asymptotically stable equilibrium (cf. Theorem 4.22).

4.9.2 $f(z)$ is a Michaelis-Menten function

In the next set of examples, $f(z)$ and $g(1-z)$ are assumed to be Michaelis-Menten or Holling type II functions. Hence our model equations assume the form

$$\begin{aligned}
\dot{x_1} &= x_1[2\beta_1 X_{1K}\frac{z}{1+z} - \beta_1 x_1 + p\gamma y] \\
\dot{x_2} &= x_2[2\beta_2 X_{2K}\frac{1-z}{2-z} - \beta_2 x_2 + \gamma y] \\
\dot{y} &= y[-\xi - \eta y + \delta(x_2 + q x_1)] \\
\dot{z} &= \alpha z x_1(1 - \frac{z}{K_z}),
\end{aligned} \tag{4.144}$$

with $x_{10} = 2, \quad x_{20} = 8, \quad y_0 = 1, \quad z_0 = 0.1.$

Example 4.5

In this example, we choose

$$\alpha = 0.2, \quad \beta_1 = 6, \quad \beta_2 = 6.1, \quad \gamma = 0.3, \quad X_{1K} = 10, \quad X_{2K} = 9,$$

$$\xi = 1, \quad \eta = 0.25, \quad \delta = 0.2, \quad K_z = 0.5, \quad p = q = 0.95.$$

The graphs of numerical solutions for $t \in (0, 50)$, are exhibited in Figure 4.6. This example illustrate the global asymptotic stability of $F^\star(x_1^\star, x_2{}^\star, y^\star, z^\star)$ for $\gamma > 0$ (cf. Theorem 4.18).

Example 4.6

In this example, the coefficients are given by the following:

$$\alpha = 0.2, \quad \beta_1 = 6, \quad \beta_2 = 6.1, \quad \gamma = 0.3, \quad X_{1K} = 10, \quad X_{2K} = 9,$$

$$\xi = 4, \quad \eta = 0.25, \quad \delta = 0.2, \quad K_z = 0.5, \quad p = q = 0.95.$$

The graphs of numerical solutions for $t \in (0, 50)$, are exhibited in Figure 4.7. In this example industrial assets go extinct after a finite time (cf. Theorem 4.20).

Example 4.7

In this example, we set

$$\alpha = 0.2, \quad \beta_1 = 6, \quad \beta_2 = 6.1, \quad \gamma = -2, \quad X_{1K} = 10, \quad X_{2K} = 9,$$

$$\xi = 1, \quad \eta = 0.25, \quad \delta = 0.33, \quad K_z = 0.90, \quad p = q = 0.95.$$

The graphs of numerical solutions for $t \in (0, 50)$, are presented in Figure 4.8. In this example agricultural assets $x_2(t)$ goes extinct after $t > 10$, thus, making $F_{x_1yz}(\breve{x}_1, 0, \breve{y}, K_z)$ a globally asymptotically stable equilibrium (cf. Theorem 4.21).

Example 4.8

In this example, we choose

$$\alpha = 0.2, \quad \beta_1 = 6, \quad \beta_2 = 6.1, \quad \gamma = -2, \quad X_{1K} = 10, \quad X_{2K} = 9,$$

$$\xi = 1, \quad \eta = 0.25, \quad \delta = 0.25, \quad K_z = 0.50, \quad p = q = 0.95.$$

The graphs of numerical solutions for $t \in (0, 50)$, are presented in Figure 4.9. In this example agricultural assets $x_1(t)$ goes extinct, thus, making $F_{x_2yz}(0, \breve{x}_2, \breve{y}, z_c)$ a globally asymptotically stable equilibrium (cf. Theorem 4.22).

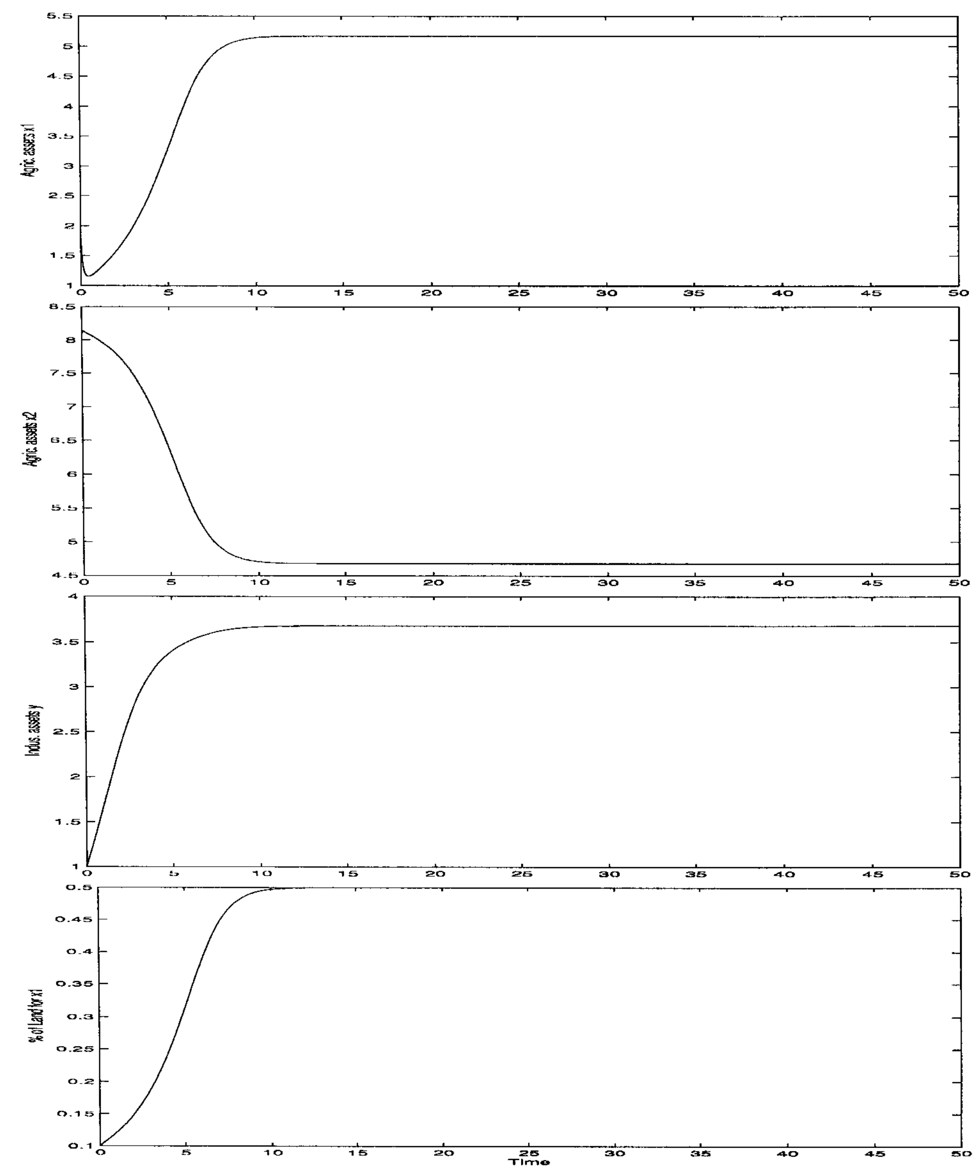

Figure 4.1: The system persists, that is all the assets peacefully co-exist.

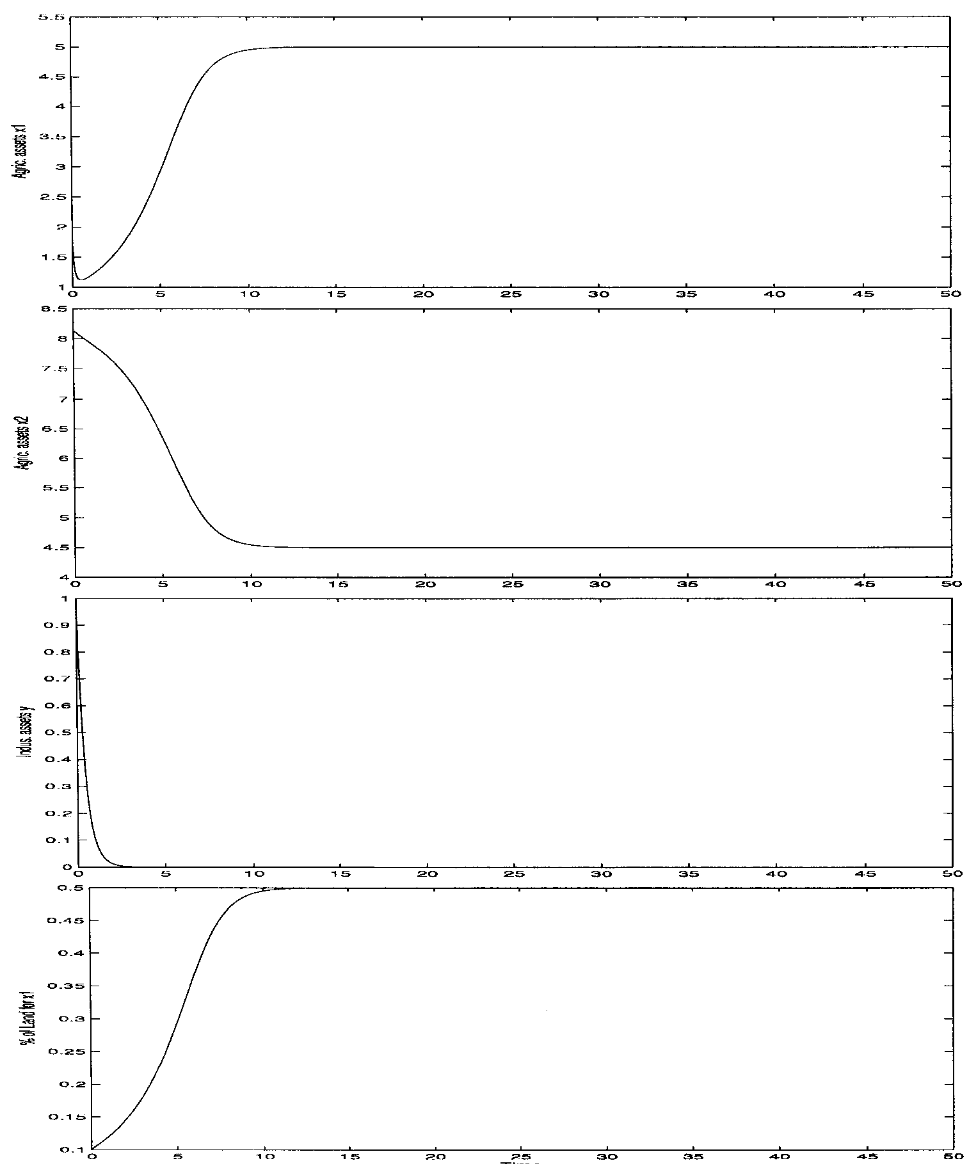

Figure 4.2: $F_{x_1x_2z}(\frac{f(K_z)}{\beta_1}, \frac{g(1-K_z)}{\beta_2}, 0, K_z)$ is globally asymptotically stable, that is industry is driven extinct.

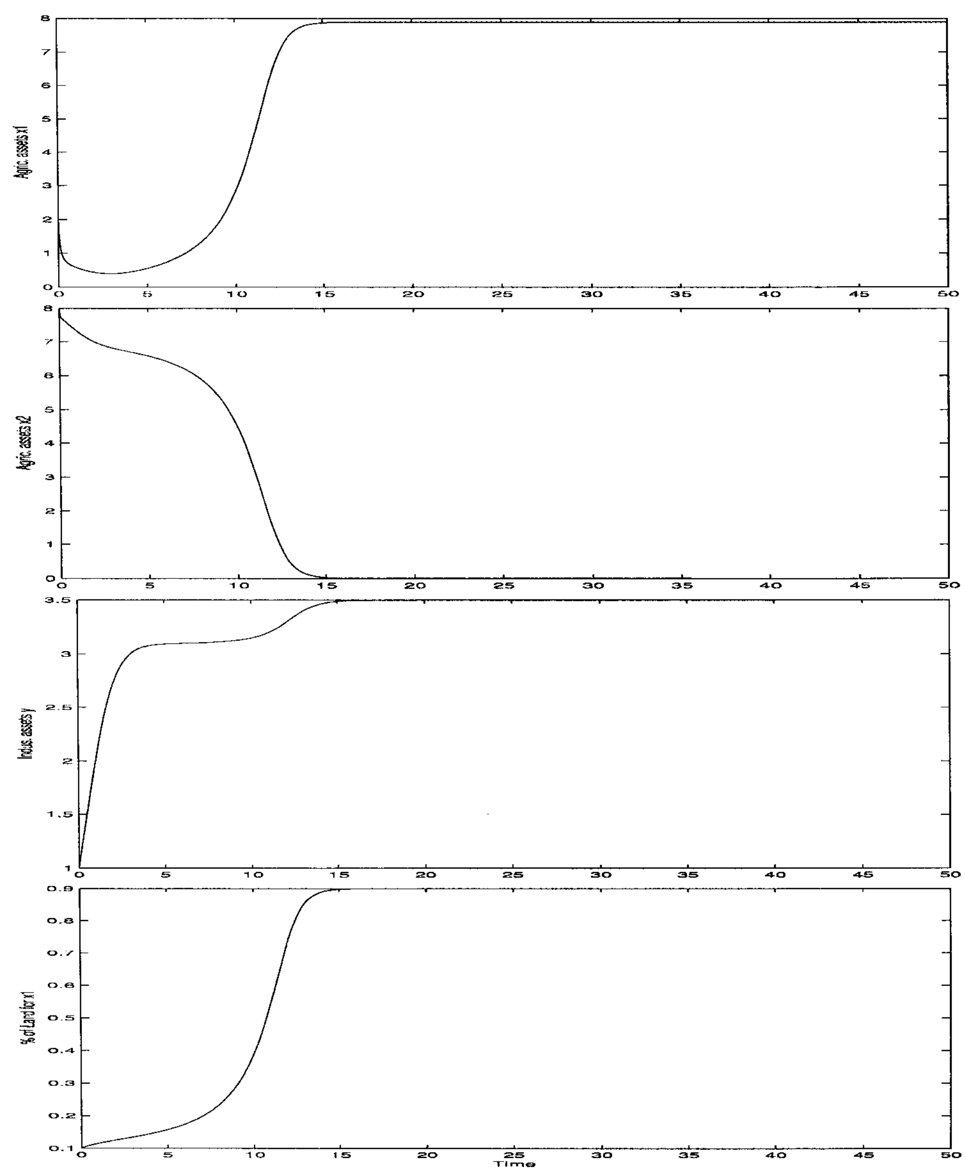

Figure 4.3: $F_{x_1yz}(\breve{x}_1, 0, \breve{y}, K_z)$ is globally asymptotically stable, that is agriculture assets x_2 is driven extinct.

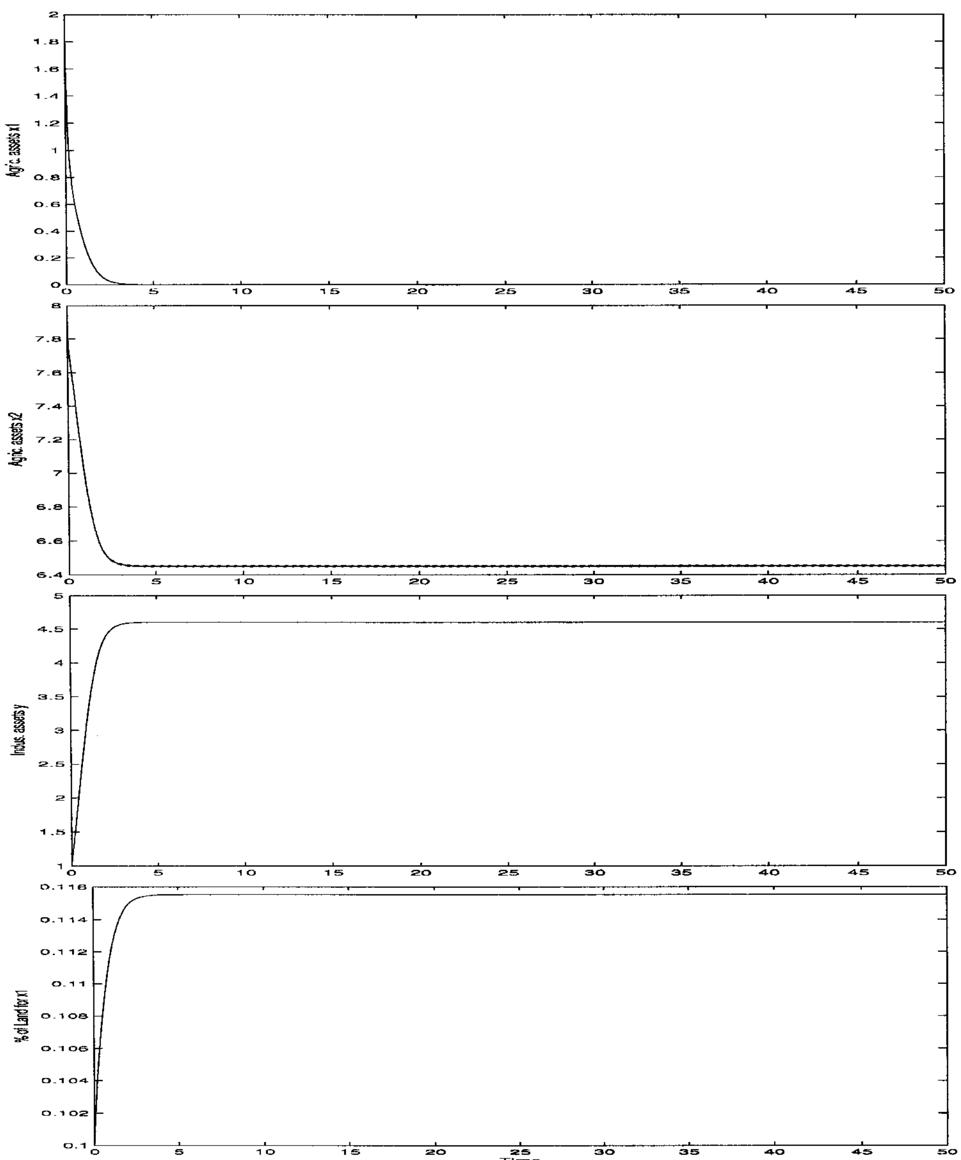

Figure 4.4: $F_{x_2yz}(0, \breve{x}_2, \breve{y}, z_c)$ is globally asymptotically stable, that is agriculture assets x_1 is driven extinct.

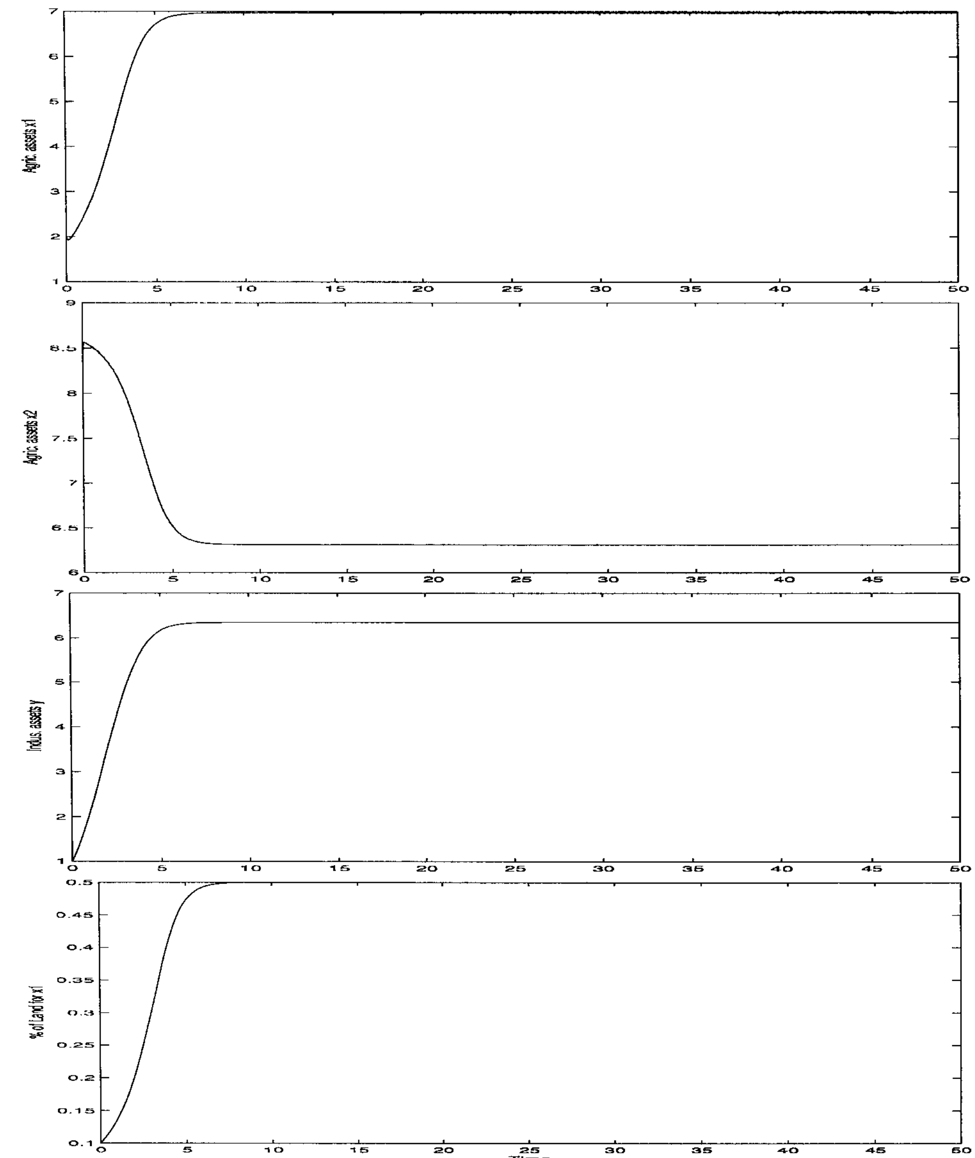

Figure 4.5: The system persists, that is all the assets peacefully co-exist.

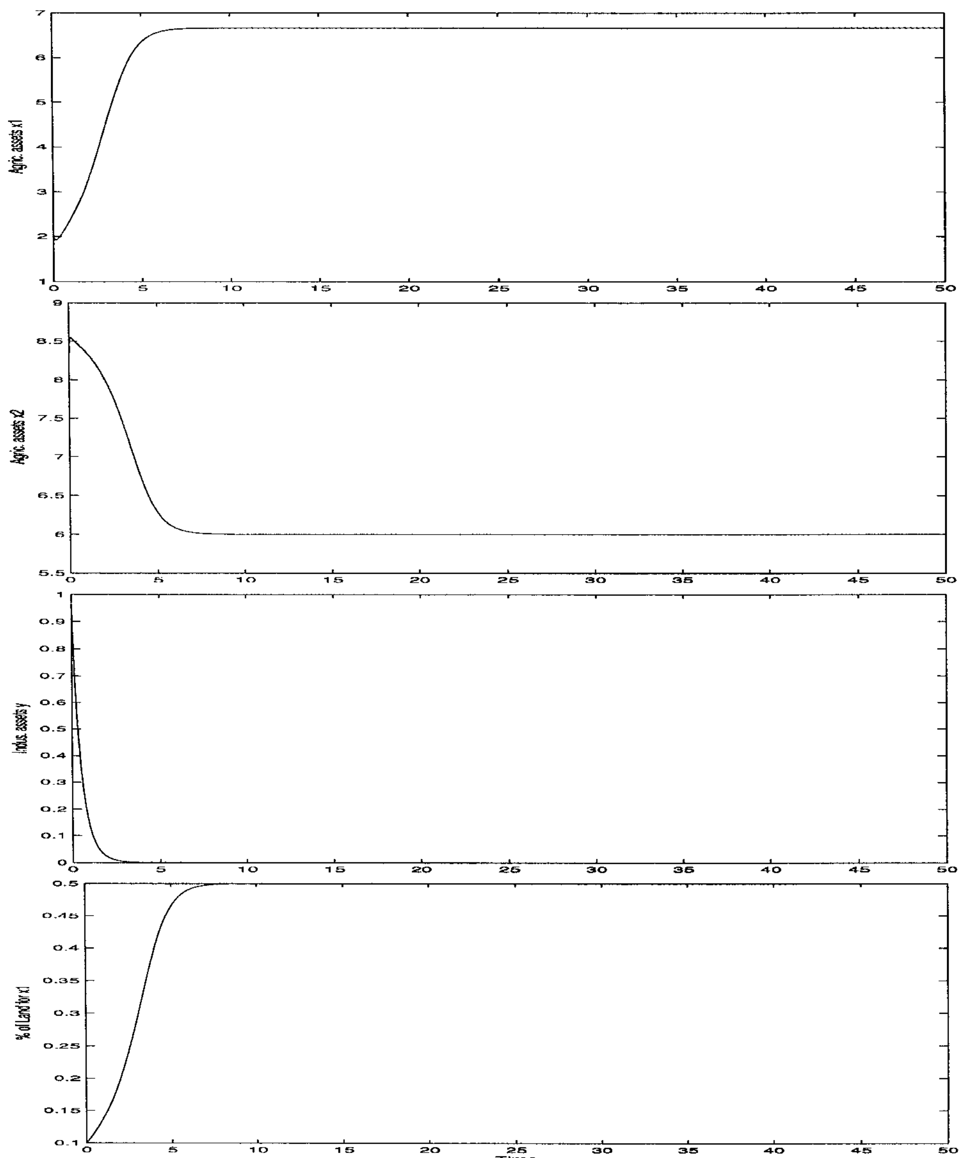

Figure 4.6: $F_{x_1x_2z}(\frac{f(K_z)}{\beta_1}, \frac{g(1-K_z)}{\beta_2}, 0, K_z)$ is globally asymptotically stable, that is industry is driven extinct.

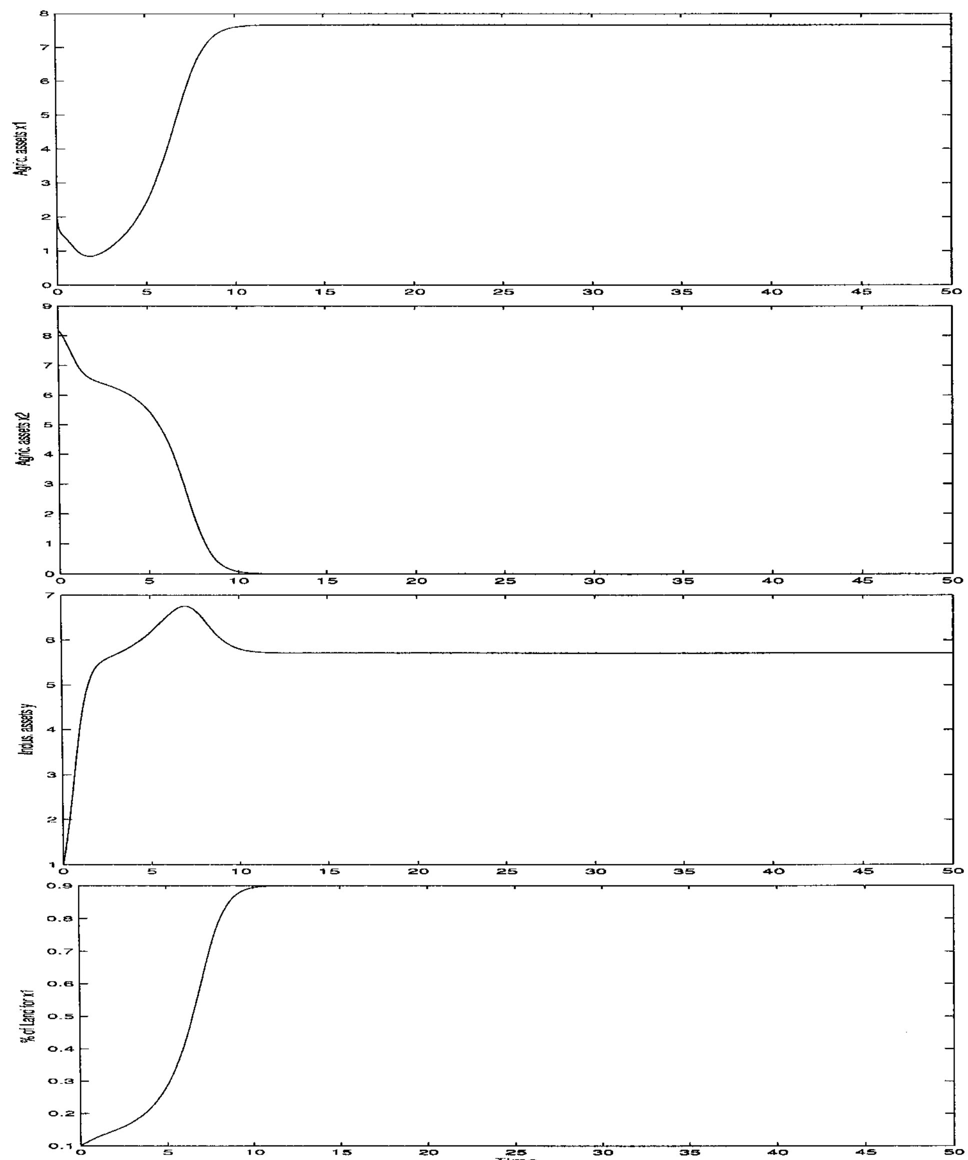

Figure 4.7: $F_{x_1yz}(\breve{x}_1, 0, \breve{y}, K_z)$ is globally asymptotically stable, that is agriculture assets x_2 is driven extinct.

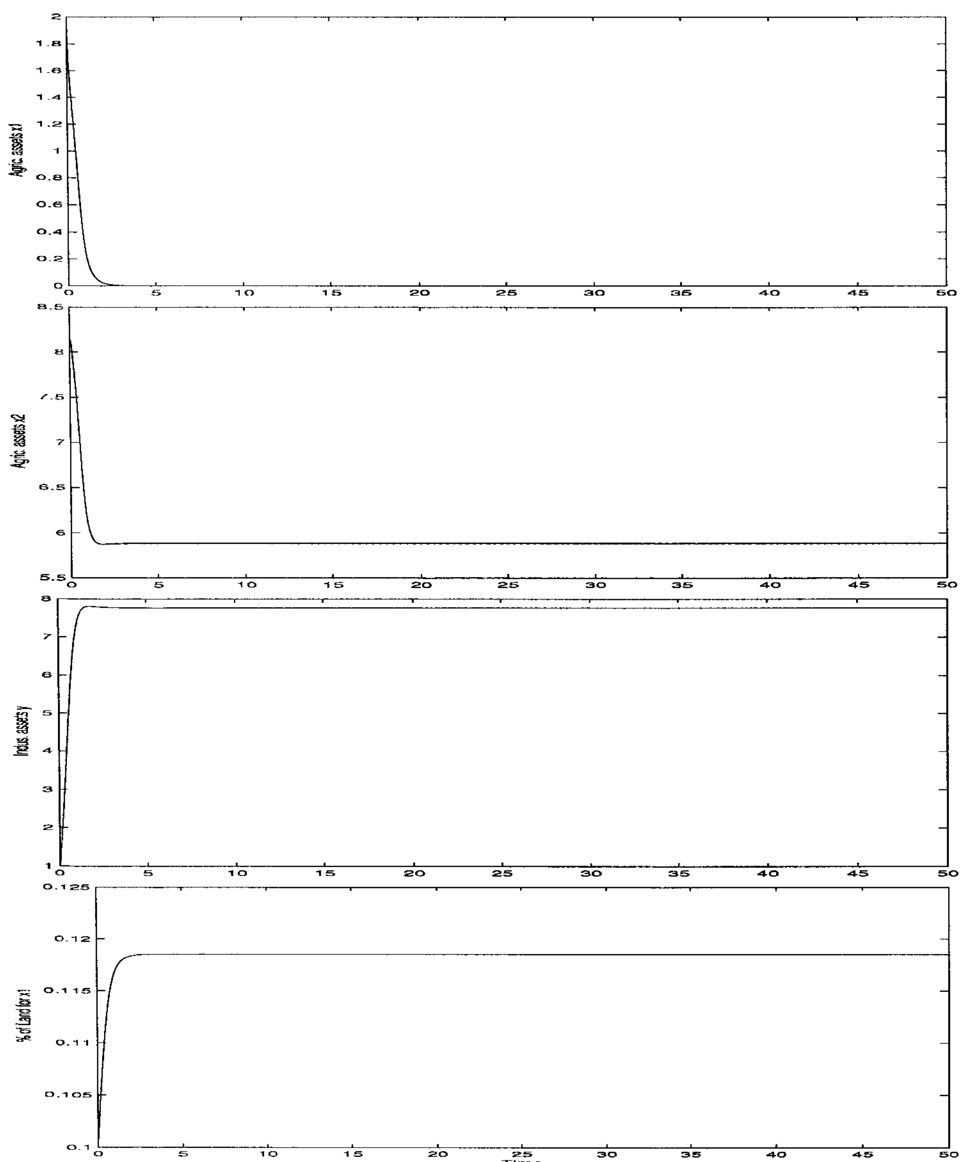

Figure 4.8: $F_{x_2yz}(0, \breve{x}_2, \breve{y}, z_c)$ is globally asymptotically stable, that is agriculture assets x_1 is driven extinct.

4.10 Summary and conclusions

In the preceding sections of this chapter, we used mathematical models to discuss and study the interaction between agriculture of two farming groups, industry as related to agriculture and land. By making some reasonable assumptions and using mathematical tools such as differential equations and Liapunov functions, we are able to give generalized conditions for the persistence of the system.

When γ is non-positive, we derived criteria for the system to be dissipative. All the possible set of equilibria for the system were determined in §4.4. We noted that some of the equilibria always exists while other may exist under certain conditions, and these conditions were determined explicitly.

In §4.5, we derived criteria for the local stability of all the hyperbolic equilibria using the Hartman-Grobman theorem in local qualitative theory of ordinary differential equations. In the case of the non hyperbolic equilibria, where the Hartman-Grobman theorem could not be applied, we used Liapunov functions to derive sufficient conditions for such equilibria to be globally stable.

In §4.6, a Liapunov function was used to establish the sufficient conditions under which $F^{\star}(x_1^{\star}, x_2{}^{\star}, y^{\star}, z^{\star})$ will be globally asymptotically stable. A Liapunov function is also used to established sufficient criteria for $F_{x_1x_2z}(\frac{f(K_z)}{\beta_1}, \frac{g(1-K_z)}{\beta_2}, 0, K_z)$ to be globally asymptotically stable is §4.7. Conditions for the globally asymptotic stability of $F_{x_1yz}(\breve{x}_1, 0, \breve{y}, K_z)$ and $F_{x_2yz}(0, \breve{x}_2, \breve{y}, z_c)$ were given in §4.8.

Finally numerical examples were given in section §4.9 to illustrate some of the mathematical results we obtained. By comparing Figure 4.1 to 4.5, 4.2 to 4.6, 4.3 to 4.7, and 4.4 to 4.8, we observe that the qualitative behavior of results obtained for both choices of f and g are similar but quantitatively different.

Chapter 5

Discussion and Future Research

This chapter reviews and discusses the core concepts and results obtained in this thesis. Future extension of our model to involve competition for land by renewable agriculture, normal agriculture and industry in general is mentioned.

5.1 Review of core results

In this section, we discuss and summarize the many important mathematical, agricultural, industrial, ecospheric and ecological results presented in Chapters 2-4.

5.1.1 Review of the competition model for two industrial assets

In Chapter 2, we presented a mathematical model that involved agriculture, in the combined form of renewable and normal agriculture, two different industries associated with agriculture and the ecosphere in the form of quality of land. We assumed that this system acted like a food chain, where agriculture depended on the ecosphere for survival and each of the industries depended on agriculture for survival. We also assumed that sometimes the interaction of industry with agriculture could benefit agriculture by increasing its assets.

We showed that if the interactions between both industries and agriculture are that of parasitism, then the solution will eventually be bounded (dissipative). We also showed that if the interaction between agriculture and industry is that of mutualism, and the gains by agriculture is not too much, then the system will be dissipative. However, if the gain by agriculture is too much even with one industrial asset, then as demonstrated in Figure 2.12, the two industries and agriculture will grow unbounded whiles the ecosphere degrades until such a time that the ecosphere can no longer sustain the agriculture, and the whole system collapses. The obvious implication of this later results is that this has to be taken seriously, because a collapse in agricultural-industrial-ecospheric system could have catastrophic consequences, such as loss of biodiversity, famine, etc for our present generation as well as for future generations.

Explicit conditions were given for the extinction of both industrial assets. These could be generalized for situations where one has only one industrial or more industrial assets. The importance of industry to sustainable agriculture has already been mentioned in the introduction of Chapter 2. Thus, for sustainable agriculture and ecosphere, one would like to avoid a situation in which industry goes extinct. One of the best ways to avoid the occurrence of industrial extinction is to know the conditions under which such extinction occurs and prevent it from happening in terms of the model presented in Chapter 2.

We also presented general and mathematical criteria for the persistence of the system and the existence of a positive interior four dimensional equilibrium. The model we presented and analyzed in Chapter 2 therefore, could serve as a guidance of how to solve the problem of sustainable agriculture and ecosphere if we look at agriculture as a combination of both normal agriculture and renewable agriculture. The model in Chapter 2 does not, however, answer some of the questions of sustainable agriculture that most economists with interest in the environment, environmentalist or environmental scientists are interested in. This is because our model gives conditions for persistence of agricultural assets in general and does not show whether normal agriculture will take over the whole of renewable agriculture or vise versa or that normal agriculture and renewable agriculture will be in equilibrium with each other in the long run. This problem gave rise to the model in Chapter 3.

5.1.2 Review of the Competition between normal and renewable agriculture model

In Chapter 3, we considered agriculture consisting of two independent parts namely, normal agriculture and renewable agriculture competing with each other. As a result we could define the natural environment as the union of renewable agriculture and the ecosphere. We were able to determine the criteria for the extinction of renewable agriculture and normal agriculture separately and also determine conditions for the persistence of both renewable and normal agriculture.

In Theorem 3.21, we gave sufficient criteria for the extinction of both normal agriculture and industry. Under the conditions of Theorem 3.21, the natural environment will be preserved at its highest quality level in the long run. The obvious implication of this theorem is that normal agriculture will go extinct and as such humans within such a locality would have to live on wild plants and animals for survival. This could in turn lead to starvation, hunger and death and subsequently a sharp decline in the human population within such a locality if not extinction, because there will not be enough wild plants and animals to feed the population.

Criteria for the extinction of both renewable agriculture and industry is given in Theorem 3.22. One of the adverse consequence of this theorem is that there will be a total loss of biodiversity. Why is loss of biodiversity or extinction of renewable agriculture so important to human survival and why are people so concerned? Some of the reasons are given in the introduction to Chapter 3 (i.e §3.1)and the books, Nature and the Marketplace by Geoffrey Heal [40] and Nature's Services: Societal Dependence on Natural Ecosystems, edited by Gretchen C. Daily [20]. I couldn't have summed it up better than Gretchen C. Daily [20] who wrote "unless humanity is suicidal, it should want to preserve, at least the natural life-support systems and processes required to sustain its own existence". It is not possible to list all the life-support services supplied by renewable agriculture to humans but I will mention a few here.

1. Most of the developing countries such as South Africa, Namibia, Zimbabwe and Costa Rica get a lot of their national income from ecotourism which relies on renewable agriculture [40].

2. Most of our drugs and other pharmaceuticals on which modern medicine relies were developed in some way from the genetic resources of wild plants (some of which are from weeds considered to be poisonous) and animals, eg. aspirin and taxol. Taxol is a promising cancer drug which was first extracted from a tree in the wild called the Pacific Yew [40,65,71].

3. About 87% of all our current foodstuffs are domesticated and cross-bred from wild stock by farmers and scientists, sometimes by trial and error experimenta-

tion [65].

4. Most of the current world population, especially those in developing countries, such Liberia, Zaire and Nigeria depend on marine fisheries and wild animals for their daily protein needs [58,66].

5. Many insects are very important ecologically not only as pollinators of important plants or as part of the food web, but as predators to destructive pests for normal agriculture and for maintaining soil fertility [20,58].

6. Humanity depends on wood for both fuel and shelter. Most of the wood used in today's world is taken from natural forests rather than from plantations.

Sufficient conditions for the extinction of industrial assets only and that for the extinction of renewable agriculture only were given in Theorem 3.23 and 3.24 respectively. Criteria for coexistence of the assets was given in Theorems 3.18 and the existence of a positive four dimensional interior equilibrium was given in 3.19. In conclusion, we state that our model in Chapter 3 suggests that the interaction between agriculture, industry and the ecosphere can lead to extinction of renewable agriculture (i.e. a total loss of biodiversity), extinction of industry, extinction of normal agriculture and industry, extinction of renewable agriculture and industry depending on agricultural and industrial practices delineated by the parameter values of the model.

5.1.3 Review of the competition for land model

For the first time, we introduce a competition for land model involving two farming groups within a locality. Using the mathematical model (4.1)-(4.4) and the associated assumptions and hypotheses we determined all the possible equilibria for the model. Local and global stability properties of each equilibrium was determined.

In Theorem (4.9), we presented a criterion for $F_{x_2}(0, X_{2K}, 0, 0)$ to be globally asymptotically stable. Also in Theorem (4.12) and (4.13), various criteria were given for the equilibrium $F_{x_2y}(0, \bar{x_2}, \bar{y}, 0)$ to be globally asymptotically stable. In all the above theorems, the percentage of land owned by farming group A, (i.e. z) goes extinct and as such their agricultural assets also go extinct and vise versa. This is just a common sense result which our model just establishes.

Criteria for the global stability of $F^\star(x_1^\star, x_2^\star, y^\star, z^\star)$ were given in Theorems 4.17 and 4.18. These conditions give the most desirable results, because there is coexistence between the farming groups. This is the desirable results from a collective point of view of "component representation" but that may not be the case from each farming group or economic perspective. Conditions under which industrial assets will go extinct were given in Theorems 4.19 and 4.20.

Conditions for the extinction of one of the agricultural assets only were given in Theorems 4.21 and 4.22. These two results were key in the study in Chapter 4. This is because, we wanted to know whether it was possible for one of the agricultural assets to go extinct (more especially x_2), and if this is possible under what conditions. It turns out that the extinction of either x_2 and x_1 critically depends on γ, and the growth functions of the agricultural assets, $f(z)$ and $g(z)$. The interesting part of the results in Theorem 4.21 and 4.22 is that if the terms of trade between agriculture and industry is favorable to agriculture (i.e. $\gamma > 0$) then none of the agricultural assets will go extinct. It is also important to note that the carrying capacity of the percentage of land owned by farming group A (i.e. K_z) plays a role in the extinction or survival of the assets.

5.2 General discussion and conclusions

Normal agriculture plays an essential role in the life of every human. It also plays a multifunctional role in relation to the environment (i.e. renewable agriculture and the ecosphere) and industry in general. This is in part (if not wholly) due to the accelerated demand for meat and useful plant biomass (i.e. normal agriculture) as a result of population growth and human advancement. As a result normal agriculture generates industrial and environmental externalities as observed in modernization of normal agriculture, (i.e. increase in the use of farm machinery, fertilizers, pesticides, developed hybrid strains, e.t.c.) commonly found in developed countries, deforestation, overgrazing, increase in agricultural farm land commonly found in developing countries, degradation of soil fertility and water resources crucial to both farm productivity and human health, loss of biodiversity and various forms of pollution [7]. It is these industrial and environmental externalities generated by normal agriculture that has led to the problems in this study of sustainable agriculture and environment.

In our thesis, we employed mathematical tools of modelling, differential analysis, persistence theory, Liapunov theory and linear systems theory to study and investigate the interactions between agriculture, industry, environment and land. In each of our models we determined conditions for persistence of each of the components of the system involved in the model and conditions for extinctions of some of the components. We considered persistence and existence of a positive interior equilibrium as "desirable" or "good" results in terms of "component representation" of the model. As for policy makers, farmers, farming groups, etc., we derive all the conditions for extinctions and persistence and leave it for them to make the choice of what is "good" or "bad" for them.

5.3 Future research project

The study done in this thesis is the beginning of are more complex study. For example our models contain constant parameters, which imply that technology remains constant. This is certainly not the case in our world now, so assuming that our models parameters are constant is too simplifying an assumption. In future works, we can allow some of the parameters to vary, incorporating technological change.

As mentioned in the introduction to Chapter 4 of this Thesis, the purpose of the model in Chapter 4 was only to serve as an approximation and also give us an insight into a future research project. In that future project, we intend to model the competition for land between normal agriculture, renewable agriculture and industry. Some parallels could be drawn between the model in Chapter 4 and our future model in the sense that before the introduction of agriculture and industry, the whole land belonged to renewable agriculture only. Once a given land is converted for agricultural and industrial purposes, it cannot be or is usually not converted back to renewable agricultural land. Even tough, we sometimes try to convert normal agricultural and industrial lands to renewable agricultural land through afforestation, some of the species lost as a result of the earlier conversion are sometimes usually lost forever. So the competition for land in this case can be seen as renewable agriculture losing land, while industry and normal agriculture gain it. A model of such a complex system will involve six or more differential equations. The purpose of such a model will be to determine under what conditions some of such assets will go extinct and conditions under which those assets will persist and also determine if any, the role the percentage of land owned by each asset combined with growth rate, terms of trade, etc. contribute to its survival or extinction.

Beginning from our work in this thesis, we can continue in the future to the (much more difficult) study of price/yield/demand interaction. That is one can examine the economic parameters relative to sustained-agricultural-yield with maximum returns for the environment.

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