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The Equivariant Degree: Symmetries and Bifurcations

BY

PAOLA VIVI ©

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To Petr and My Parents

ABSTRACT

Bifurcation theory is the study of equations with multiple solutions. Specifically, by bifurcation we mean a change in the number of solutions of an equation as a parameter varies. For a wide variety of differential equations, problems concerning multiple solutions can be reduced to studying how the solutions x of a single scalar equation $f(x, \lambda) = 0$ vary with the parameter λ.

The equivariant degree is a topological tool that can be effectively applied to the study of this equation in the presence of symmetry. Its many properties make it versatile, and it can be especially used to characterize the Hopf bifurcation.

In this thesis, we further develop abstract methods to compute the equivariant degree, we present explicit calculations for certain symmetries and we apply these results to establish the existence of bifurcation.

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Introduction

The recent development of equivariant topological methods, such as equivariant degree theory, provides a powerful tool for the qualitative theory of differential equations with symmetries. Problems involving dynamical systems with symmetries naturally arise in many applications in physics, chemistry, mathematical biology, and engineering. As these new topological methods may be effectively, and relatively simply, used to study nonlinear equations with symmetries, their further development may have an important impact in this area.

The objective of this thesis is to further develop abstract equivariant degree techniques, in particular for some classical non-abelian Lie groups, and to present some applications of the computational results obtained to differential equations and bifurcation theory.

Motivated by the existence problem of periodic solutions of nonlinear differential equations, many researchers have been interested in various techniques involving S^1-equivariant degrees.

As an example, consider the problem of finding a periodic solution, of an unknown period p, to the following system of autonomous ODEs

$$y' = f(y).$$

By introducing the period as a parameter (by using the substitution $x(\tau) = y(\frac{p}{2\pi}\tau)$), we transform this equation into the following parameterized BVProblem:

$$\begin{cases} x' = \frac{p}{2\pi} f(x) \\ x(0) = x(2\pi) \end{cases}$$

This problem can be reformulated as a fixed-point problem in a function space of continuously differentiable 2π-periodic functions, on which there is a natural action of the group S^1 (by shifting of argument). An important observation is that, as the system is autonomous, this fixed point problem is symmetric (equivariant) with respect to this S^1 action. In order to prove the existence of a solution to this problem we employ the S^1-equivariant degree, which is a particular topological degree taking into account the presence of the S^1-symmetry.

The idea of an equivariant degree is not new. Several authors introduced such a tool to study symmetric equations. In particular:

(i) Ulrich (cf. [53]), following the work of Dold ([11]), introduced the notion of a G-equivariant fixed point index for $f : V \to V$ (as an element of the Burnside ring $A(G)$);

(ii) Dylawerski, Gęba, Jodel, Marzantowicz ([14]) introduced a notion of S^1-degree for mappings $f : V \times \mathbf{R} \to V$;

(iii) Ize, Massabó, Vignoli ([34,35,36]) introduced a general G-degree theory, including an S^1-degree for mappings $f : V \to W$, where V and W are possibly two different representations of G.

(iv) Dancer ([9]) introduced a notion of S^1-degree, following the work of Rubinsztain, for S^1-equivariant gradient fields;

(v) Gęba, Massabó, Vignoli ([20]) introduced a general G-degree for gradient fields

(vi) Gęba, Krawcewicz, Wu ([19]) developed an analytic definition of a G-degree which is defined for mappings $f : V \times \mathbf{R}^n \to V$ and is represented as a sequence of integers indexed by orbit types in $V \times \mathbf{R}^n$.

The advantage of the equivariant degree theory introduced by Gęba, Krawcewicz and Wu lies in the fact that it may be defined through an elementary analytic construction (based on the same idea as the analytic construction of the Brouwer degree) therefore it allows for direct calculations in many concrete examples.

In this thesis we develop new abstract methods for computations of the equivariant degree, we use these in explicit calculations for certain non-abelian actions, and we apply the results obtained to bifurcation theory.

Let us now pass to a more detailed description of the individual chapters.

In Chapter One, after introducing the analytic version of the equivariant degree of Gęba, Krawcewicz and Wu, we treat the equivariant degree theory in the absence of the parameter space. In this case the degree is defined as an element of the Burnside ring of the group G. We show that our definition is equivalent to that of Ulrich and we present an alternative proof of the multiplicativity property. This property is of fundamental importance both theoretically and for the computations. We discuss some multiplication formulae for the elements of the Burnside ring and we then give a complete multiplication table for the Burnside ring of $SO(3)$. These results are applied for the computation of the $SO(3)$-degree of $-Id$ on some of its irreducible representations: the importance of this special case stems later on from Proposition 3.4.1.

Chapter Two is mostly computational. We develop several formulae relating the G-equivariant degree to more easily computable topological invariants. This is done in several steps. First we present an Ulrich type formula, in case of a one dimensional parameter space, which reduces the calculations to the well understood S^1-degree. Then, under some assumption on $f : V \times \mathbf{R} \to V$ (condition A), we prove that the G-degree (G abelian) can be computed by means of the restriction of f to certain isotypical components of V. In the last section we show how to apply the previous results to the equivariant Hopf bifurcation. Given the problem

$$x = F(x, \lambda), \quad (x, \lambda) \in W \times \mathbf{R}^2$$

where W is an orthogonal representation of the group $G = \Gamma \times S^1$, Γ a compact Lie group, and F a G-equivariant map. We assume that there exists a 2-dimensional submanifold M of trivial solutions. The existence of a branch of nontrivial solutions bifurcating from a point $(x_0, \lambda_0) \in M$ is then related to the nontriviality of the G-degree of the map $f_\theta : V \times \mathbf{R} \to V$

$$f_\theta(x, \lambda) = (x - F(x, \lambda), \theta(x, \lambda))$$

where $V = W \times \mathbf{R}$ and θ is an auxiliary invariant function. A global result is also established.

In Chapter Three, we extend the concept of normality to equivariant bifurcation problems. The "normality" condition prevents the bifurcation branches of larger orbit types from collapsing into branches with smaller orbit types. We prove that every equivariant bifurcation problem, parameterized by n variables, can be arbitrarily well approximated by a "normal" equivariant map with the same sets of singular points and bifurcation points.

We then use equivariant degree techniques to study normal bifurcation and we establish a Branching Lemma and a global result. With an example of steady state bifurcation with $SO(3)$-symmetry we show that the equivariant degree can detect branches of nontrivial solutions even of submaximal types.

In Chapter Four, we consider the case $G = \Gamma \times G_0$ where Γ is a finite group and G_0 is a compact Lie group. The multiplicativity property of the G-degree (described in Chapter One for G abelian or in absence of the parameter space) is extended to a new case involving the action of the group $\Gamma \times G_0$. We study more extensively the case $D_N \times S^1$ and in particular we present the multiplication tables for the Burnside ring of the dihedral group $A(D_N)$ and a table for $A(D_N) \times A_1(D_N \times S^1)$ (the type needed for the multiplicativity property).

The computational formula presented in Chapter Two is also extended to the case $G = \Gamma \times S^1$ (Γ finite). We again consider $G = D_N \times S^1$ and its irreducible 4-dimensional representations to show the possibility of effective calculations of the topological invariants involved in such formula.

In Chapter Five we apply the computational results from Chapters Two and Four to the Hopf bifurcation theory for functional differential equations. It is known that the problem of looking for bifurcating periodic solutions with prescribed symmetries can be, in the abelian case, always reduced to the case of spatial symmetry group $\mathbb{Z}_N$ or $\mathbb{Z}_\infty := S^1$. Our results in nonabelian case are described in detail for the case of the nonabelian group $D_N \times S^1$. It is shown how the wealth of different irreducible representations implies spontaneous symmetry breaking and appearance of multiple branches of periodic solutions of various types.

In the last section we show how temporal delay in coupling between cells may cause many oscillation patterns that cannot appear in the absence of the delay. This finds applications to chemical or biological oscillators.

Chapter One

The Equivariant Degree and the Burnside Ring

1.1 Introduction

We begin this chapter by briefly recalling some basic notions to present the setting for the definition of the equivariant degree due to Gęba, Krawcewicz and Wu [19]. This degree is defined for continuous functions from $V \times \mathbf{R}^n$ into V, commuting with the action of a Lie group G on V (acting trivially on $\mathbf{R}^n$). As a natural generalization of the Brouwer degree, it possesses several properties, most important of which are the topological invariance with respect to G-equivariant homotopies and the existence of zeros for functions with nontrivial degree. This degree, however, is an element of a complicated algebraic object (free group) $A_n(G)$ derived from the structure of conjugacy classes of subgroups of G. In the case $n = 0$ (or G abelian) it also admits a multiplication operation and $A(G)$ becomes a Burnside ring.

The definition that we present is not the original one from [19], but an equivalent analytical version from [47]. The analytic construction presented in [47] allows for a more direct comprehension and visualization of the degree, thus we prefer it. The degree is built up by means of approximations of a given function by smooth functions satisfying certain 'normality' and regularity conditions, and evaluating relevant determinants corresponding to some orbit types (conjugacy classes of closed subgroups of G). Although the process is in spirit close to the one used for the to a subgroup of H. For a closed subgroup H of G, we use $N(H)$ to denote the *normalizer* of H in G, and $W(H)$ to denote the *Weyl group* $N(H)/H$ of H in G. For every $n \in \mathbf{Z}_+ := \{0, 1, 2, \dots\}$, we put $\Phi_n(G) := \{(H) \in O(G) : \dim W(H) = n\}$.

A compact Lie group is called *bi-orientable* if it has an orientation which is invariant (i.e. under all left and right translations). By definition, every abelian, finite, or connected Lie group is bi-orientable; however, the group $O(2)$ is an example of a compact Lie group which is not bi-orientable. We will also use the following notation:

$$\Phi_n^+(G) := \{(H) \in \Phi_n(G) : W(H) \text{ is bi-orientable}\},$$
$$\Phi_n^-(G) := \{(H) \in \Phi_n(G) : W(H) \text{ is not bi-orientable}\},$$
$$A_n(G) := \mathbf{Z}[\Phi_n^+(G)] \oplus \mathbf{Z}_2[\Phi_n^-(G)].$$

For every element $\alpha = (H)$ from the set Φ_n^+ we choose an invariant orientation of $W(H)$. An element of $\gamma \in A_n(G)$ will be written as a finite sum $\gamma = \sum_\alpha n_\alpha \alpha$, where

$$n_\alpha \in \begin{cases} \mathbf{Z} & \text{if } \alpha \in \Phi_n^+(G); \\ \mathbf{Z}_2 & \text{if } \alpha \in \Phi_n^-(G). \end{cases}$$

Let W be a real finite-dimensional orthogonal representation of the Lie group G. We consider the product space $W \oplus \mathbf{R}^n$, where we will always assume that G acts trivially on the second component. For a given $x \in W \oplus \mathbf{R}^n$, we denote by $G_x := \{g \in G : gx = x\}$ the *isotropy group* of x. According to our previous notation the conjugacy class (G_x) will be called the *orbit type* of x. For an invariant subset

Brouwer degree, the presence of the symmetry (the action of G) complicates the calculations, especially for nonabelian group actions.

It is important to note that the equivariant degree considered throughout this thesis is a proper extension of previously introduced degrees. In Theorem 1.2.4, of Section 1.2, we prove that this degree in the particular case $n = 0$ (i.e. in absence of the parameter space) is equivalent to the well known degree of H. Ulrich [53].

This result will be extensively used later on for the computation of the degree and an extension of this result to the case $n = 1$ will be found in Chapter 2. In Section 1.3 we consider again the case $n = 0$ and we discuss the degree as an element of the Burnside ring. We analyze the structure of the Burnside ring since the multiplicativity property of the degree, which arises naturally in case $n = 0$, is heavily based on it. We compute the multiplicativity table of a particular nonabelian group $SO(3)$: the $SO(3)$ symmetry is one of the most often studied for practical applications [4,25,29]. In Section 1.4, we explicitly compute the $SO(3)$-degree using the results of the previous sections. The function considered is $-\mathrm{Id} : \Omega_i \to V_i$ where Ω_i is the unit ball in the irreducible representation V_i of $SO(3)$ (i=1, ... ,5), and the relevance of the computation of the degree for this particular map will be clear later in Chapter 3 (Example 3.4.2).

1.2 The Equivariant degree

Throughout this thesis we assume that G is a compact Lie group. We say that two closed subgroups H and K are ***conjugate*** in G, denoted by $H \sim K$, if there exists $g \in G$ such that $H = gKg^{-1}$. The relation $\sim$ is an equivalence relation. The equivalence class of H is called a ***conjugacy class*** or ***orbit type*** of H in G and will be denoted by (H). We denote by $O(G)$ the set of all orbit types of closed subgroups of G. The set $O(G)$ is partially ordered: $(H) \leq (K)$ if and only if K is conjugate

$X \subset W \oplus \mathbf{R}^n$, a closed subgroup H of G and an orbit type $\alpha \in O(G)$ we put

$$X^H := \{x \in X : hx = x \text{ for all } h \in H\};$$
$$X_H := \{x \in X : G_x = H\};$$
$$X^{[H]} := \{x \in X : G_x \subsetneq H\};$$
$$X^\alpha := \{x \in X : (G_x) \leq \alpha\};$$
$$X_\alpha := \{x \in X : (G_x) = \alpha\}.$$

We will denote by $\mathcal{J}(X)$ the set of all orbit types of points in X, i.e. $\mathcal{J}(X) := \{(G_x) : x \in X\}$. It is well known (see, for example, [2,37]), that if $\Omega \subset W \oplus \mathbf{R}^n$ is an open invariant subset, then for every $\alpha \in \mathcal{J}(\Omega)$ the set Ω_α is a G-invariant submanifold of $W \oplus \mathbf{R}^n$.

Suppose that M is an invariant submanifold (or simply G-submanifold) of $V := W \oplus \mathbf{R}^n$. Let $N(M) = \{(x, v) \in M \times V : v \perp T_x M\}$ denote the normal bundle of M in V and $N_x(M)$ the normal space to M at $x \in M$. Then $N(M)$ is a G-vector bundle. Moreover, there exists a continuous invariant function $\nu : M \to \mathbf{R}_+$ such that the restriction of the map $\mu : N(M) \to V$, $\mu(x, v) = x + v$, to the set $N(M, \nu) := \{(x, v) \in N(M) : \|v\| < \nu(x)\}$ is a G-diffeomorphism. The image $\mu(N(M, \nu))$ is called a *G-tubular neighbourhood* of M in V.

Let $\Omega \subset W \oplus \mathbf{R}^n$ be an open bounded invariant subset, we say that a *G-equivariant map* $f : \overline{\Omega} \to W$ (i.e. f satisfies $gf(x) = f(gx)$ for every $x \in W \oplus \mathbf{R}^n$ and $g \in G$) is *admissible* if $f(x) \neq 0$ for all $x \in \partial\Omega$. We will also say that an equivariant homotopy $h : \overline{\Omega} \times [0, 1] \to W$ is *admissible* if the map $h_t := h(\cdot, t)$ is admissible for every $t \in [0, 1]$.

Theorem 1.2.1 *To every admissible map $f : \overline{\Omega} \to W$ we can assign an element*

G-Deg $(f,\Omega) \in A_n(G)$ *such that the following properties are satisfied:*

*(P1) (Existence) If G-*Deg $(f,\Omega) = \sum_\alpha n_\alpha \alpha$ *is such that there is* $n_\alpha \neq 0$, *then there exists* $x \in \Omega^H \cap f^{-1}(0)$, *where* $(H) = \alpha$;

(P2) (Homotopy Invariance) If $h : \overline{\Omega} \times [0,1] \to W$ *is an admissible homotopy, then G-*Deg (h_t,Ω) *does not depend on* $t \in [0,1]$;

(P3) (Excision) If $\Omega_o \subset \Omega$ *is an open invariant subset and* $f^{-1}(0) \subset \Omega_o$, *then G-*Deg $(f,\Omega) =$ *G-*Deg (f,Ω_o).

(P4) (Additivity) If Ω_1 *and* Ω_2 *are two open invariant subsets of* Ω *such that* $\Omega_1 \cap \Omega_2 = \varnothing$ *and* $f^{-1}(0) \subset \Omega_1 \cup \Omega_2$, *then*

$$G\text{-Deg}\,(f,\Omega) = G\text{-Deg}\,(f,\Omega_1) + G\text{-Deg}\,(f,\Omega_2);$$

The element *G*-Deg $(f,\Omega) \in A_n(G)$ is called the *G-(equivariant) degree* of the map f with respect to the set Ω. The formal construction of the above *G*-degree theory was presented in [19], but we would like to mention another approach, which can be called *analytic*, to the definition of such equivariant degree. This approach is based on the notion of a normal map and the fact that every admissible map can be approximated by normal maps. More precisely, we introduce the following definition:

Definition 1.2.2 An admissible map $f : \overline{\Omega} \to W$ is called *normal* if for every $x \in f^{-1}(0)$ and $H = G_x$ there exists an $\varepsilon_x > 0$ such that $f(x+h) = h$ for all vectors $h \in N_x(\Omega_\alpha)$ with $\|h\| < \varepsilon_x$, where $\alpha = (H)$. In addition a normal map f is called a *regular normal* map if f is of class C^1 and for every $H = G_x$, where $x \in f^{-1}(0)$, 0 is a regular value of the restricted map $f_H := f_{|\Omega_H} : \Omega_H \to W^H$.

In [47,56] the following *regular approximation theorem* was proven:

Theorem 1.2.3 *For every admissible map $f : \overline{\Omega} \to W$ and for every $\eta > 0$ there exists a regular normal map $\tilde{f} : \overline{\Omega} \to W$ such that $\sup_{x\in\Omega} \|f(x) - \tilde{f}(x)\| < \eta$. The last inequality simply says that $\tilde{f}$ is an η-approximation of f.*

Consequently, we can use this approach to give an analytic definition of G-degree (cf. [47,56]):

Let $f : \overline{\Omega} \to W$ be an admissible map, and put $2\eta := \inf_{x\in\partial\Omega} \|f(x)\|$. By the regular normal approximation theorem, there is a regular normal η-approximation $\tilde{f}$ of f. Then we put $G\text{-Deg}\,(f, \Omega) = \sum_\alpha n_\alpha \alpha$, where for $\alpha = (H)$

$$n_\alpha = \begin{cases} 0 & \text{if } \alpha \notin \mathcal{J}(\tilde{f}^{-1}(0)); \\ \sum_{W(H)x\subset\tilde{f}^{-1}(0)_H} \operatorname{sign} D\tilde{f}_H(x)_{|S_x} & \text{if } W(H) \text{ bi-orientable;} \\ \left|\tilde{f}^{-1}(0)_H/W(H)\right| (\text{mod } 2) & \text{if } W(H) \text{not bi-orientable,} \end{cases}$$

where S_x denotes the linear slice to the orbit $W(H)x$ in the space $W^H \oplus \mathbf{R}^n$ at x, and $\left|\tilde{f}^{-1}(0)_H/W(H)\right|$ denotes the number of all the $W(H)$ orbits in the set $\tilde{f}^{-1}(0)_H$. We choose an orientation of the slice S_x such that the natural orientation of the tangent space $T_xW(H)x$, induced by the chosen invariant orientation of $W(H)$, followed by the orientation of S_x gives the (fixed) orientation of $V^H \oplus \mathbf{R}^n = T_xW(H)x \oplus S_x$.

We refer to [47,56] for all the details and the verification that the above definition indeed leads to an equivariant degree satisfying all the properties (P1)-(P4).

Assume now that V is an infinite dimensional isometric Banach representation of the group G. We consider the space $V \times \mathbf{R}^n$ and denote by $\pi : V \times \mathbf{R}^n \to V$ the natural projection. The G-equivariant degree can be extended to compact fields in V by the standard method of finite dimensional approximations (see [43]). More precisely, let $\pi - F : V \times \mathbf{R}^n \to V$ be an Ω-admissible compact field on Ω, i.e. $\partial\Omega \cap (\pi - F)^{-1}(0) = \varnothing$, then for every $\varepsilon > 0$ there exists an equivariant finite dimensional map $F_\varepsilon : V \times \mathbf{R}^n \to V$ (i.e. $F_\varepsilon(\overline{\Omega}) \subseteq V_0$, where V_0 is a finite dimensional invariant subspace of V) such that

$$\|F_\varepsilon(x) - F(x)\| < \varepsilon, \quad \forall\, x \in \overline{\Omega}.$$

Then we may define

$$G\text{-Deg}\,(f, \Omega) := G\text{-Deg}\,(\pi - F_\varepsilon|_{V_0 \times \mathbf{R}^n},\ \Omega \cap (V_0 \times \mathbf{R}^n)).$$

It turns out that the G-equivariant degree for compact fields, as defined above, satisfies all the standard properties as in Theorem 1.2.1 (see [16,21,46,56]).

A G-equivariant degree for $n = 0$, i.e. when the parameter space $\mathbf{R}^n$ is absent, was introduced by Ulrich (cf. [53]). We now prove that our definition of the G-degree for $n = 0$ coincides with the definition of H. Ulrich.

Theorem 1.2.4 *For every Ω-admissible map $f : \Omega \to V$, where Ω is an open invariant subset of V, we have*

$$G\text{-Deg}\,(f, \Omega) = \sum_{(H) \in \Phi(G)} m_{(H)}(F) \cdot (H), \tag{1.2.1}$$

where $F(x) = x - f(x)$ for $x \in \Omega$ and

$$m_{(H)}(F) = [I(F^H) - I(F^{[H]})]/|W(H)|, \tag{1.2.2}$$

where $I(F^H)$ and $I(F^{[H]})$ denote the fixed point indices of $F^H : \overline{\Omega^H} \to V^H$, and $F^{[H]} : \overline{\Omega^{[H]}} \to V^{[H]}$ respectively.

Proof. The formula (1.2.1) was obtained from the definition of the G-equivariant fixed point index of F, introduced by H. Ulrich in [53], which satisfies all the standard properties of the fixed point index, i.e. existence, additivity, homotopy invariance, excision, multiplicativity and commutativity properties. We refer to [53] for more information about the G-fixed point index.

Due to the results of [47,56] and the equivalent definition of equivariant degree therein, we may assume, without loss of generality, that f is regular normal. Moreover, due to the additivity property of both the equivariant degree and fixed point index, we may assume that $f^{-1}(0) \subset \Omega_{(L)}$ consists of a single orbit, where $(L) \in \Phi(G)$ is the minimal orbit type in Ω. $f^L : \Omega^L \to V^L$ is $W(L)$-equivariant and $W(L)$ is finite thus $(f^L)^{-1}(0) = W(L)x_o$ is also a finite set. Moreover, since f^L is $W(L)$-equivariant, for every $g \in W(L)$ we have $f^L(gx_o) = gf^L(x_o)$, and therefore, $Df^L(gx_o) \circ g = g \circ Df^L(x_o)$. Consequently, $\det Df^L(gx_o) = \det Df^L(x_o)$ for all $g \in W(L)$. This implies that

$$I(F^L) = n_{(L)}|W(L)|,$$

where $n_{(L)}$ is the (L)-component of $G\text{-Deg}(f, \Omega)$. Since $\Omega^{[L]} = \varnothing$, $m_{(L)} = n_{(L)}$. Assume now that $K \subsetneq L$. Since $f = Id - F$ is (L)-normal, it follows from the product property of the fixed point index that $I(F^K) = I(F^L)$. In order to compute $I(F^{[K]})$ we remark that $\Omega^L \subset \Omega^{[K]}$. If $\Omega^{[K]} \setminus \Omega^L = \varnothing$, then clearly $m_{(K)} = 0$. Hence, assume that $\Omega^{[K]} \setminus \Omega^L \neq \varnothing$. Since f is (L)-normal, $F^K(u, v) = (\varphi(u), 0)$ near $f^{-1}(0)$, where $(u, v) \in \overline{\Omega^K}$, $u \in \Omega^L$, v is orthogonal to Ω^L, and $\varphi : \overline{\Omega^L} \to V^L$. Let $r : \overline{\Omega^K} \to \overline{\Omega^{[K]}}$ denote a retraction onto $\overline{\Omega^{[K]}}$. Then by the definition of the fixed point index for ANR-spaces, $I(F^{[K]}) := I(\widetilde{F})$, where $\widetilde{F} = F \circ r : \overline{\Omega^K} \to V^K$. We claim that $H_t := tF^K + (1-t)\widetilde{F}$ is an admissible homotopy (without fixed points on the boundary of Ω^K). Indeed, if $H_t(u, v) = (u, v)$, then it implies that $v = 0$, and thus $(u, v) \in \overline{\Omega^L}$. Since r restricted to $\overline{\Omega^L}$ acts as identity, $\varphi(u) = u$, the homotopy H_t has the same fixed points as F^L. Consequently, by the homotopy property of the fixed point index $I(F^{[K]}) = I(F^K) = I(F^L)$, thus $m_{(K)} = 0$ and the statement follows.

□

1.3 The Burnside Ring and the Multiplicativity Property

In this section we recall the definition of the *Burnside ring* $A(G)$.

Let $\Phi(G)$ denote the set of conjugacy classes (H) such that $N(H)/H$ is finite. We denote by $A(G)$ the free abelian group generated by $(H) \in \Phi(G)$. It is clear that $A(G) = A_0(G)$. There is a *multiplication* operation on $A(G)$ which induces a structure of a ring with identity on $A(G)$. In order to define the multiplication operation, we remark that

$$\begin{aligned}(G/H \times G/K)_{(L)}/G &\cong (G/H \times G/K)_L/N(L)\\ &\subset (G/H \times G/K)^L/N(L)\\ &= (G/H^L \times G/K^L)/(N(L)/L).\end{aligned}$$

Since the spaces G/H^L and G/K^L consist of finitely many $N(L)/L$-orbits and by assumption $N(L)/L$ is finite, G/H^L and G/K^L are finite. Consequently the set $(G/H \times G/K)_{(L)}/G$ is finite.

The multiplication table of the generators (H) is given by the relation

$$(H) \cdot (K) = \sum_{(L)\in\Phi(G)} n_L(L) \tag{1.3.1}$$

where n_L denotes the number of elements in the set $(G/H \times G/K)_{(L)}/G$, i.e.

$$n_L := \left|(G/H \times G/K)_{(L)}/G\right|.$$

The ring $A(G)$ is called the *Burnside Ring of G*.

In order to effectively use the ring structure of $A(G)$ for the computations of G-degree we need to be able to evaluate the multiplication table for $A(G)$, i.e. to compute the numbers n_L in formula (1.3.1).

Some information on the number n_L may be obtained from a purely grouptheoretic argument. Given closed subgroups L and H of the group G, we put.

$$N(L,H) = \{g \in G;\ \ gLg^{-1} \subset H\}.$$

$N(L,H)$ is a closed subset of G, and hence a compact set. If we define a left G-action on G by $(h,g) \mapsto hg$, then $N(L,H)$ is an $N(H)$-invariant subset of G. Since $N(L,H)$ is a compact H-space and the natural projection $\pi : N(L,H) \to N(L,H)/H$ is continuous, the orbit spaces $N(L,H)/H$ and $N(L,H)/N(H)$ are compact. It should be pointed out that the set $N(L,H)$ is not a group in general (cf. [29]). The correspondence $H(g) \mapsto g^{-1}H$ gives a homeomorphism $\Phi : N(L,H)/H \to (G/H)^L$.

Let us introduce, following [29], the number $n(L,H)$, which denotes the number of conjugate copies of L contained in the subgroup H, i.e.

$$n(L,H) = \left|\frac{N(L,H)}{N(H)}\right|.$$

If $\alpha = (L)$ is a minimal orbit type in $G/H \times G/K$, then

$$n_L = n(L,H) \cdot n(L,K)\frac{|W(H)| \cdot |W(K)|}{|W(L)|}. \tag{1.3.2}$$

Assume that $\alpha = (L) \in \Phi(G)$ is not a minimal orbit type in $X := G/H \times G/K$. We denote by Λ_L the set of all closed subgroups M such that $(G/H \times G/K)_L = G/H^L \times G/K^L \setminus \bigcup_{M\in\Lambda_L}(G/H \times G/K)_M$. As $G/H^L \times G/K^L$ is finite, the set Λ_L is also finite. Consequently

$$|(G/H \times G/K)_L| = |G/H^L \times G/K^L| - \sum_{M\in\Lambda_L} k_M|G/H^M \times G/K^M|$$

where the coefficients k_m are integers representing the "repetitions" in the count of the elements of $G/H^M \times G/K^M$. Therefore,

$$|(G/H \times G/K)_L| = |W(H)| \cdot |W(K)| \big[n(L,H)\, n(L,K) - \sum_{M \in \Lambda_L} k_M\, n(M,H)\, n(M,K) \big]. \tag{1.3.3}$$

In the case where G is an abelian group the formula (1.3.1) simplifies to

$$(H) \cdot (K) = n_{H\cap K}(H \cap K)$$

where $n_{H\cap K}$ is equal to the number of all $(H \cap K)$-orbits in $G/H \times G/K$. In this case, the number n_L in the formula (1.3.1) represents the number of elements in $(G/H \times G/K)_{(L)}/G$, i.e. it is the *number of G-orbits in $G/H \times G/K$ of the orbit type* (L), i.e. $n_L = |(G/H \times G/K)/G| = |(G/H \times G/K)|/|G/(H \cap K)|$.

The G-equivariant degree for $n = 0$, i.e. in absence of the parameter space, is an element of the Burnside ring. In this case the G-degree has one more additional important property called the *Multiplicativity Property*:

(P5) **(Multiplicativity)** Let V_1, V_2 be two orthogonal representations of G, $\Omega_i \subset V_i$, $i = 1,2$, two invariant open bounded subsets, $f_i : \overline{\Omega}_i \to V_i$, $i = 1,2$, two equivariant admissible maps. Then

$$G\text{-Deg}\,(f_1 \times f_2, \Omega_1 \times \Omega_2) = G\text{-Deg}\,(f_1, \Omega_1) \cdot G\text{-Deg}\,(f_2, \Omega_2),$$

where the product is taken in the Burnside ring $A(G)$.

Proof. We may assume, without loss of generality, that both maps $f_i = Id - F_i$, $i = 1,2$, are regular normal and $f_i^{-1}(0) = G(x_i)$ for $i = 1,2$, i.e. for both maps the solution sets $f_i^{-1}(0)$ are composed of one single orbit. Assume in addition that $G\text{-Deg}\,(f_1, \Omega_1) = a_1(H)$ and $G\text{-Deg}\,(f_2, \Omega_2) = a_2(K)$, where $a_1, a_2 = \pm 1$ and

$H = G_{x_1}$, $K = G_{x_2}$. Using the same notations as above, we can assume that

$$(G/H \times G/K)_L = G/H^L \times G/K^L \setminus \bigcup_{M \in \Lambda_L} (G/H^M \times G/K^M).$$

We will use the following notation

$$\begin{aligned} F :=& F_1 \times F_2, \\ U :=& (\Omega_1 \times \Omega_2)^L, \;\; Y := (V_1 \times V_2)^L. \end{aligned}$$

It is clear that $F(U_M) \subset Y^M$ for every $M \in \Lambda_L$ and therefore, by additivity and multiplicativity property of the fixed point index, we obtain

$$\begin{aligned} I(F, U) =& I(F_1, \Omega_1^L) \cdot I(F_2, \Omega_2^L) \\ =& a_1 a_2 |G/H^L| \cdot |G/K^L| \\ =& a_1 a_2 n(L, H) n(L, K) |W(H)||W(K)|, \end{aligned}$$

and

$$I(F, U^{[L]}) = \sum_{M \in \Lambda_L} k_M I(F, U^M),$$

where the coefficients k_M are exactly the same as in (1.3.3). On the other hand,

$$\begin{aligned} I(F, U^M) =& a_1 a_2 |G/H^M| \cdot |G/K^M| \\ =& a_1 a_2 n(M, H) n(M, K) |W(H)||W(K)|. \end{aligned}$$

Therefore, using (1.3.3), we obtain

$$\begin{aligned} I(F, U) - I(F, U^{[L]}) =& a_1 a_2 |W(H)||W(K)| \big[n(L, H) n(L, K) - \\ & \sum_{M \in \Lambda_L} k_M n(M, H) n(M, K) \big] \\ =& a_1 a_2 |(G/H \times G/K)_L|, \end{aligned}$$

and consequently, by applying the Ulrich formula, we have

$$G\text{-Deg}(f_1 \times f_2, \Omega_1 \times \Omega_2) = G\text{-Deg}(f_1, \Omega_1) \cdot G\text{-Deg}(f_2, \Omega_2).$$

□

We also recall that when the group G is abelian, for every $n > 0$, the set $A_n(G)$ admits the structure of an $A(G)$-module. The action of $A(G)$ on $A_n(G)$ can be described as follows: if $(H) \in \Phi_0(G)$ and $(K) \in \Phi_n(G)$, then $(H) \cdot (K) = n_{H\cap K}(H \cap K)$ where $n_{H\cap K}$ denotes the number of elements in the set $(G/H \times G/K)/G$. Thus, for G abelian, another kind of multiplicativity property is defined, an analog of (P5) where $\Omega_2 \subset V_2 \times \mathbf{R}^n$.

Computation of the Burnside ring $A(G)$, in some cases, may be quite complicated and require good knowledge of the subgroup structure of G. However, we may use another description of the Burnside ring, due to tom Dieck (cf. [10]), which uses the fact that $A(G)$ may by isomorphically mapped onto a subring of the ring $C(G) := C(\Phi(G); \mathbf{Z})$ of continuous functions from $\Phi(G)$ into the discrete space $\mathbf{Z}$. In order to present this description we need first to explain some facts about the topology on the space $\Phi(G)$.

Let $S(G)$ denote the set of closed subgroups of G. As the group G is a metric space, we may equip $S(G)$ with the usual Hausdorff metric, so $S(G)$ becomes a compact metric space such that the action $G \times S(G) \to S(G)$ defined by $(g, H) \mapsto gHg^{-1}$ is continuous. Moreover, the orbit space $S(G)/G$, which is exactly $O(G)$ is countable and thus a totally disconnected Hausdorff space such that $\Phi(G)$ is a compact subspace of $O(G)$ (see [9]). It can be shown that for every $(H) \in \Phi(G)$ the function $z_H : \Phi(G) \to \mathbf{Z}$, defined by

$$z_H((L)) = |(G/H)^L| = |N(L, H)/H| = n(L, H) \cdot |W(H)|, \tag{1.3.4}$$

for $(L) \in \Phi(G)$, is continuous and thus belongs to $C(G)$. Therefore, we can define a $\mathbb{Z}$-homomorphism $\varphi : A(G) \to C(G)$, which is defined on the generators $(H) \in \Phi(G)$ by $\varphi((H)) = z_H$. The map φ is in fact a well defined injective ring homomorphism, and $C(G)$ is a free abelian group with basis $\{|W(H)|^{-1} z_H;\ (H) \in \Phi(G)\}$.

Since the image $\varphi(A(G))$ may be identified with $A(G)$, it is possible to describe the ring structure of $A(G)$ just by computing the generators z_L, for $(L) \in \Phi(G)$, and then to express the products $z_H \cdot z_K$ as a linear combination of the generators z_L. We will illustrate this process in our example showing how to compute the ring structure in $A(SO(3))$.

Example 1.3.1 We consider the group $SO(3)$ of all 3×3 orthogonal matrices of determinant 1. For all details and additional information we refer to [4,25]. It may be shown that every proper closed subgroup of $SO(3)$ is conjugate to one of the following subgroups:

(i) The subgroup of $SO(3)$, consisting of all 3×3 matrices $\begin{bmatrix} A & 0 \\ 0 & \operatorname{sign}\det A \end{bmatrix}$, $A \in O(2)$, which may be identified with the orthogonal group $O(2)$.

(ii) The subgroups of $O(2)$: $SO(2)$, D_n, $n = 2, 3, \ldots$, $\mathbb{Z}_n$, $n = 1, 2, 3, \ldots$. The subgroup D_1 is conjugate to $\mathbb{Z}_2$, and therefore is not included in the list. On the other hand, the group $D_2 \simeq \mathbb{Z}_2 \oplus \mathbb{Z}_2$ is traditionally called the *Klein group* of order 4, and is denoted by V_4.

(iii) The *exceptional* groups: *tetrahedral* $\boldsymbol{T}$, *octahedral* $\boldsymbol{O}$, and *icosahedral* $\boldsymbol{I}$, which are "rotational" symmetry groups of the regular tetrahedron, octahedron (or cube), and icosahedron (or dodecahedron), respectively. It is well known, that the group $\boldsymbol{T}$ is isomorphic to the alternating group A_4, the group $\boldsymbol{O}$ is isomorphic to the symmetric group S_4, and the group $\boldsymbol{I}$ is isomorphic to the alternating group A_5.

The subgroups of $SO(3)$ can be represented in a lattice of conjugacy classes of

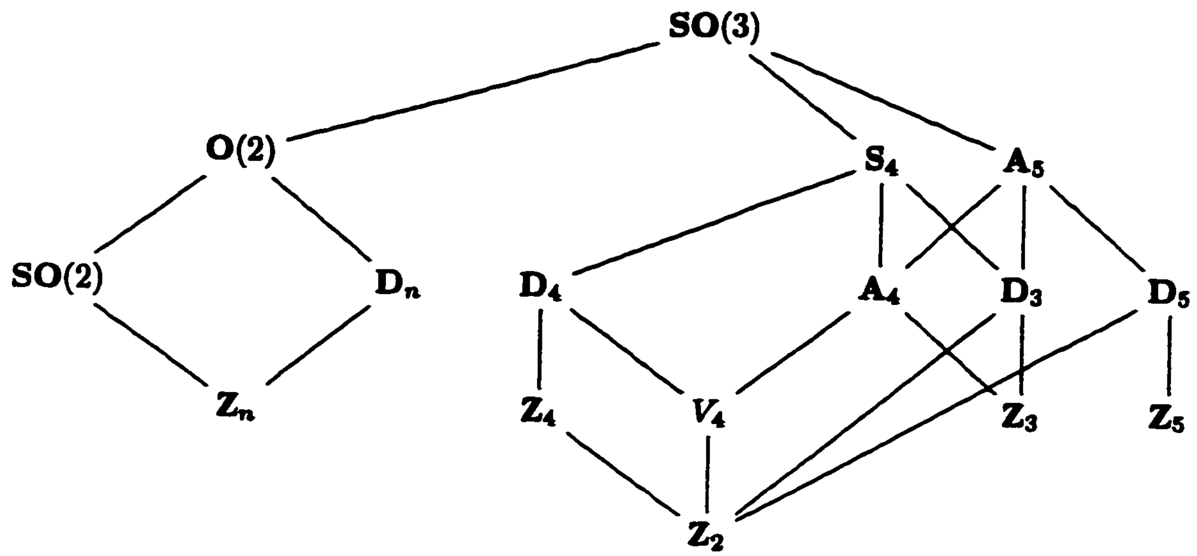


Figure 1.3.1

subgroups of $SO(3)$ (see Figure 1.3.1).

We will also need to identify the subgroups with finite Weyl group (see Table 1.1).

Weyl groups of subgroups in $SO(3)$

H	$N(H)$	$W(H)$
$O(2)$	$O(2)$	$\mathbf{Z}_1$
$SO(2)$	$O(2)$	$\mathbf{Z}_2$
D_n, $n \geq 3$	D_{2n}	$\mathbf{Z}_2$
V_4	S_4	S_3
$\mathbf{Z}_n$, $n \geq 2$	$O(2)$	$O(2)$
$\mathbf{Z}_1$	$SO(3)$	$SO(3)$
S_4	S_4	$\mathbf{Z}_1$
A_5	A_5	$\mathbf{Z}_1$
A_4	S_4	$\mathbf{Z}_2$

Table 1.1

We have the following generators of $A(SO(3))$:

$$\Phi(SO(3)) = \{(SO(3)), (O(2)), (A_4), (A_5), (S_4), (D_n);\ n = 2, 3, \dots\}.$$

We will compute the multiplication table for the ring $A(SO(3))$. We need Table 1.2, borrowed from [4], showing the numbers $n(L, H)$ for the subgroups of $SO(3)$.

Values of $n(L, H)$

L	H	$n(L,H)$	condition
$SO(2)$	$O(2)$	1	
D_m	D_n	1	$n/m \in \mathbb{Z},\ m \geq 3$
V_4	D_n	3	n even
D_m	$O(2)$	1	$m \geq 3$
V_4	$O(2)$	3	
D_3	S_4	2	
D_4	S_4	2	
D_3	A_5	2	
D_5	A_5	2	
V_4	A_4	1	
V_4	S_4	4	
V_4	A_5	2	
A_4	S_4	1	
A_4	A_5	2	

Table 1.2

Since all the computations of the multiplication table for $A(SO(3))$ follow the same pattern, we will show only, as an example, the computation for the product $(H) \cdot (K)$, where $H = S_4$ and $K = A_5$. It is clear from the subgroup lattice that the only possible orbit types in $G/H \times G/K$, which belong to $A(SO(3))$, are $(L_1) = (A_4)$, $(L_2) = (D_3)$ and $(L_3) = (V_4)$. Consequently, $(S_4) \cdot (A_5) = n_1(A_4) + n_2(D_3) + n_3(V_4)$. Since (A_4) and (D_3) are evidently the minimal orbit types in $G/H \times G/K$, we may use the formula (1.3.2) to compute the numbers n_1 and n_2. We have

$$\begin{aligned} n_1 &= n(A_4, S_4) \cdot n(A_4, A_5) \frac{|W(S_4)| \cdot |W(A_5)|}{|W(A_4)|} \\ &= 2 \cdot 1 \frac{1 \cdot 1}{2} = 1 \\ n_2 &= n(D_3, S_4) \cdot n(D_3, A_5) \frac{|W(S_4)| \cdot |W(A_5)|}{|W(D_3)|} \\ &= 2 \cdot 2 \frac{1 \cdot 1}{2} = 2 \end{aligned}$$

In order to compute the coefficient n_3 we will use the ring inclusion $\varphi : A(SO(3)) \to C(SO(3))$, which is defined on generators $(H) \in A(SO(3))$ by $\varphi((H)) = z_H$, where z_H is given by the formula (1.3.4). Since we want to compute the product $(S_4) \cdot (A_5)$, it follows from the fact that φ is a ring homomorphism that

$$\begin{aligned} \varphi((S_4) \cdot (A_5)) &= z_{S_4} \cdot z_{A_5} = n_1 z_{A_4} + n_2 z_{D_3} + n_3 z_{V_4} \\ &= z_{A_4} + 2 z_{D_3} + n_3 z_{V_4}. \end{aligned}$$

Consequently, we need to compute the values of the functions z_H for $(H) \in \Phi(SO(3))$, which we will express in a form of a table, and find the number n_3 by inspection. For two positive integers n and k we define $[n|k]$ by

$$[n|k] = \begin{cases} 1 & \text{if } k \text{ divides } n; \\ 0 & \text{otherwise,} \end{cases}$$

and $[n||k_1, \ldots, k_r] = 1$ if $n \in \{k_1, \ldots, k_r\}$, or zero otherwise.

The values of z_H can be computed directly from (1.3.4) using Table 1.2 (see Table 1.3).

Now, by using the information from the table we obtain

Values of $z_H((L))$

$z_H \setminus (L)$	$SO(3)$	$O(2)$	$SO(2)$	A_4	A_5	S_4	V_4	$D_n,\ n \geq 3$
$z_{SO(3)}$	1	1	1	1	1	1	1	1
$z_{O(2)}$	0	1	1	0	0	0	3	1
$z_{SO(2)}$	0	0	2	0	0	0	0	0
z_{S_4}	0	0	0	1	0	1	4	$2[n\|\|3,4]$
z_{A_5}	0	0	0	2	1	0	2	$2[n\|\|3,5]$
z_{A_4}	0	0	0	2	0	0	2	0
z_{D_5}	0	0	0	0	0	0	0	$2[n\|5]$
z_{D_4}	0	0	0	0	0	0	6	$2[n\|4]$
z_{D_3}	0	0	0	0	0	0	0	$2[n\|3]$
z_{V_4}	0	0	0	0	0	0	6	0
z_{D_m}	0	0	0	0	0	0	$6[n\|2]$	$2[m\|n]$

where $m \geq 3$

Table 1.3

Computation of n_3

	$SO(3)$	$O(2)$	$SO(2)$	A_4	A_5	S_4	V_4	D_3	$D_n,\ n \geq 3$
z_{A_4}	0	0	0	2	0	0	2	0	0
$2z_{D_3}$	0	0	0	0	0	0	0	4	0
z_{V_4}	0	0	0	0	0	0	6	0	0
$z_{S_4} \cdot z_{A_5}$	0	0	0	2	0	0	8	4	0

(the sum of middle rows is equal to the last row, thus $n_3 = 1$)

and consequently we obtain

$$(S_4) \cdot (A_5) = (A_4) + 2(D_3) + (V_4).$$

In a similar simple way one can compute the complete multiplication tables for $A(SO(3))$ (see Tables 1.4 and 1.5).

We have

$$(D_n) \cdot (D_k) = 2[l](D_l) + 2(3 - [l])[l|2](V_4),$$

First multiplication table for $A(SO(3))$

	$(O(2))$	(S_4)	(A_5)	(A_4)
$(O(2))$	$(O(2))+(V_4)$	$(D_4)+(D_3)+(V_4)$	$(D_5)+(D_3)+(V_4)$	(V_4)
(S_4)	$(D_4)+(D_3)+(V_4)$	$(S_4)+(D_4)+(D_3)$ $+(V_4)$	$(A_4)+2(D_3)+(V_4)$	$(A_4)+(V_4)$
(A_5)	$(D_5)+(D_3)+(V_4)$	$(A_4)+2(D_3)+(V_4)$	$(A_5)+(A_4)+(D_3)$ $+(D_5)$	$2(A_4)$
(A_4)	(V_4)	$(A_4)+(V_4)$	$2(A_4)$	$2(A_4)$
(D_5)	(D_5)	0	$2(D_5)$	0
(D_4)	$(D_4)+2(V_4)$	$2(D_4)+2(V_4)$	$2(V_4)$	$2(V_4)$
(D_3)	(D_3)	$2(D_3)$	$2(D_3)$	0
(V_4)	$3(V_4)$	$4(V_4)$	$2(V_4)$	$2(V_4)$
(D_n)	$(D_n)+2[n\|2](V_4)$	$2[n\|4](D_4)+2[n\|3](D_3)$ $+2(2-[n\|4])[n\|2](V_4)$	$2[n\|5](D_5)+2[n\|3](D_3)$ $+2[n\|2](V_4)$	$2[n\|2](V_4)$

(where $n \geq 3$)

Table 1.4

Second multiplication table for $A(SO(3))$

	$SO(2)$	(D_5)	(D_4)	(D_3)	(V_4)
$(O(2))$	$(SO(2))$	(D_5)	$(D_4)+2(V_4)$	(D_3)	$3(V_4)$
$SO(2)$	$2(SO(2))$	0	0	0	0
(S_4)	0	0	$2(D_4)+2(V_4)$	$2(D_3)$	$4(V_4)$
(A_5)	0	$2(D_5)$	$2(V_4)$	$2(D_3)$	$2(V_4)$
(A_4)	0	0	$2(V_4)$	0	$2(V_4)$
(D_5)	0	$2(D_5)$	0	0	0
(D_4)	0	0	$2(D_4)+4(V_4)$	0	$6(V_4)$
(D_3)	0	0	0	$2(D_3)$	0
(V_4)	0	0	$6(V_4)$	0	$6(V_4)$
(D_n)	0	$2[n\|5](D_5)$	$2[n\|4](D_4)+2(3-[n\|4])[n\|2](V_4)$	$2[n\|3](D_3)$	$6[n\|2](V_4)$

(where $n \geq 3$)

Table 1.5

where $l = gcd(n,k)$ is the greatest common divisor of n and k, and

$$[l] = \begin{cases} 0 & \text{if } l < 3, \\ 1 & \text{if } l \geq 3. \end{cases}$$

1.4 The SO(3)-Degree

Let us now consider the case where $G = SO(3)$. Using the results from the previous sections we will compute the degree of a particular function. This computations will be used later on in Chapter 3 to introduce an example of bifurcation.

It is a classical result that $SO(3)$ has precisely one real irreducible representation, up to isomorphism, in each odd dimension $2k+1$. These irreducible representations may be described with the help of vector spaces W_k of homogeneous polynomials $p : \mathbf{R}^3 \to \mathbf{R}$ of degree k. The group $SO(3)$ acts on W_k by $(Ap)(x) := p(A^{-1}x)$, where $A \in SO(3)$. For $k = 0$ the representation W_0 is a one-dimensional trivial representation of $SO(3)$; for $k = 1$ the representation W_1 is the natural representation of $SO(3)$ in $\mathbf{R}^3$. However, for $k \geq 2$ the representation W_k is reducible. In order to describe the irreducible representations of $SO(3)$, we denote by ρ the polynomial $\rho(x) = x_1^2 + x_2^2 + x_3^2$, where $x = (x_1, x_2, x_3)$, and we define a linear (injective) operator $j_k : W_{k-2} \to W_k,\ \ k > 2$, by $j_k(p) = p\rho$, where $p \in W_{k-2}$. Put $J_k = j_k(W_{k-2})$ and let V_k be the space spanned by the spherical harmonics of degree k, that is, the elements $p \in W_k$ for which $\Delta p = 0$, where Δ denotes the Laplacian. The subspace V_k is also invariant under $SO(3)$. We have the following result (see for example [3] and [25]):

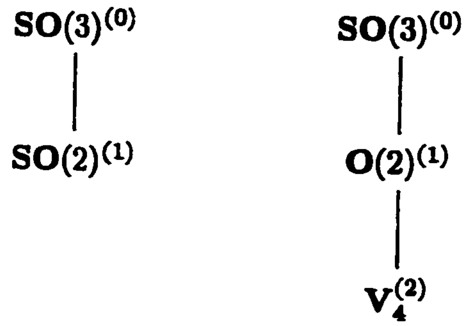
Theorem 1.4.1 *The representations V_k are precisely the irreducible representations of $SO(3)$ such that:*

(i) $W_k = J_k \oplus V_k$, i.e. V_k is an $SO(3)$-invariant complement to J_k;

(ii) $\dim V_k = 2k+1$;

(iii) The representations V_k are absolutely irreducible.

We will compute $SO(3)$-Deg$(-Id, \Omega_i)$, where Ω_i denotes the unit ball in V_i, $i = 1, 2, 3, 4, 5$. For the purpose of computations we will need the following lattice (reduced tó the case of those subgroups H such that $(H) \in \Phi(SO(3))$) of the isotropy groups in V_i, where each isotropy group H is written in the form $H^{(\dim V_i^H)}$.



(a) $i = 1$ (b) $i = 2$

For more details regarding the computation of isotropy lattices for the representations V_i we refer to Thm. 8.1 in [25]. We begin with the computations of $SO(3)$-Deg$(-Id, \Omega_1)$. We use the function $\gamma(t)$ given by

$$\gamma(t) = \begin{cases} 1 & \text{if } t \le \frac{1}{3}; \\ -3t+2 & \text{if } \frac{1}{3} \le t \le \frac{2}{3}; \\ 0 & \text{if } \frac{2}{3} \le t, \end{cases}$$

and we correct the map $f = -Id$ to a normal map $g(x) = (2\gamma(|x|) - 1)x$. The restriction of g to the one-dimensional subspace V^H, where $H = SO(2)$, has exactly one $W(H) = \mathbb{Z}_2$-orbit of critical points. Therefore, by direct computation we obtain $SO(3)$-Deg$(-Id, \Omega_1) = SO(3)$-Deg$(g, \Omega_1) = (SO(3)) - (SO(2))$.

In a similar way, we can compute the degree $SO(3)$-Deg$(-Id, \Omega_2)$. Again, we correct the map $-Id$ to a normal map $g(x) = (2\gamma(|x|) - 1)x$. The space V^H, where

$H = O(2)$, is again one-dimensional, and since $W(O(2)) = \mathbf{Z}_1$, the mapping g^H has exactly two critical points (more precisely, two orbits of critical points). In the case $H = V_4$, we have $\dim V^H = 2$, $F(x) = 2x$. Therefore, in this case, $I(F^H) = 1$. In order to determine the set $\Omega^{[H]}$, we need to find all the subgroups $K \in (O(2))$ such that $V_4 \subset K$. It is easy to verify that for the group

$$V_4 = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \right\}$$

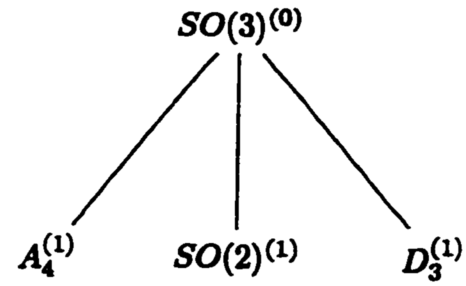
there are exactly three subgroups $K_1, K_2, K_3 \in (O(2))$ such that $V_4 \subset K_i$, $i = 1, 2, 3$. Thus $I(F^{[H]}) = 5$ and $m_{(H)} = (1+5)/6 = 1$. Consequently, we obtain that $SO(3)\text{-Deg}(-Id, \Omega_2) = SO(3)\text{-Deg}(g, \Omega_2) = (SO(3)) - 2(O(2)) + (V_4)$.

We have obtained

$$SO(3)\text{-Deg}\,(-Id, \Omega_1) = (SO(3)) - (SO(2))$$
$$SO(3)\text{-Deg}\,(-Id, \Omega_2) = (SO(3)) - 2(O(2)) + (V_4),$$

where the subgroups $SO(2)$ and $O(2)$ were maximal isotropy groups for V_1 and V_3 respectively.

In what follows, we will compute $SO(3)\text{-Deg}\,(-Id, \Omega_i)$ also for $i = 3$, 4, and 5. For this purpose we will need the lattice of isotropy subgroups of V_i (reduced to subgroups H such that $(H) \in \Phi(SO(3))$). We refer to [4] for more details concerning these lattices of isotropy groups. We start with the lattice of isotropy subgroups for V_3:



Isotropy Lattice for V_3

We may compute $SO(3)$-Deg$\,(-Id, \Omega_3)$, in a similar way as above, i.e. we correct the map $-Id$ to a normal map $g(x) = (2\gamma(|x|) - 1)x$ and compute the degree as follows. The space V^H, for $H = A_4$, $SO(2)$, and D_3 is, in this case, one-dimensional, and since $W(H) = \mathbf{Z}_2$ we know that g^H has exactly two critical points (i.e. one orbit of critical points). Consequently, $SO(3)$-Deg$\,(-Id, \Omega_3) = (SO(3)) - (A_4) - (SO(2)) - (D_3)$. We may also apply the formula (1.2.2) to compute this degree $SO(3)$-Deg$\,(-Id, \Omega_3)$. Since the only orbit types (H) in Ω_3 which belong to $\Phi(SO(3))$ are $(SO(3))$, (A_4), $(SO(2))$, and (D_3), we have that

$$SO(3)\text{-Deg}\,(-Id, \Omega_3) = m_0(SO(3)) + m_1(A_4) + m_2(SO(2)) + m_3(D_3),$$

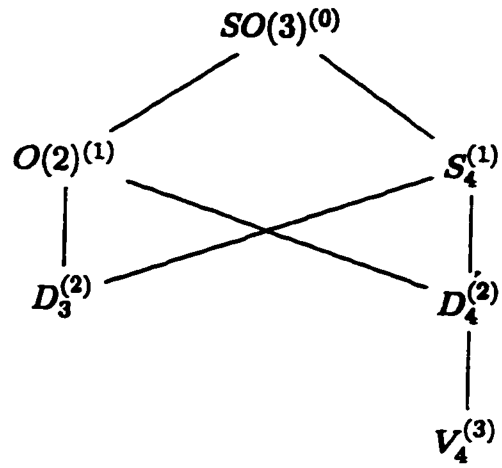
where for $F(x) = 2x$, we have

$$m_0 = I(F^H), \quad H = SO(3),$$
$$m_i = [I(F^H) - I(F^{[H]})]/|W(H)|, \quad i = 1, 2, 3,$$

where $H = A_4$, $SO(2)$, and D_3 respectively. For $H = SO(3)$, we have that $F^H : \{0\} \to \{0\}$, thus $I(F^H) = 1$ and so $m_0 = 1$. For $H = A_4$, $SO(2)$ or D_3, we have $F^H : \Omega^H \to V^H$, and since V^H is one dimensional space, $I(F^H) = -1$. On the other hand, $\Omega^{[H]} = \{0\}$, thus $F^{[H]} : \{0\} \to \{0\}$ and therefore $I(F^{[H]}) = 1$. Consequently, $m_i = (-1-1)/|W(H)| = -2/2 = -1$. So again we obtain

$$SO(3)\text{-Deg}\,(-Id, \Omega_3) = (SO(3)) - (A_4) - (SO(2)) - (D_3).$$

For the representation V_4, we have the following reduced lattice of isotropy groups (i.e. we omit all the orbit types (H) which do not belong to $A(G)$):



Isotropy Lattice for V_4

In order to compute $SO(3)$-Deg $(-Id, \Omega_4)$, we apply formula (1.2.1). In this case

$$SO(3)\text{-Deg}\,(-Id, \Omega_4) = \sum_{(H)} m_{(H)} \cdot (H),$$

where $(H) \in \{(SO(3)),\ (O(2)),\ (S_4),\ (D_3),\ (D_4),\ (V_4)\}$. For $H = SO(3)$, we obtain trivially $m_{(H)} = 1$. For $H = O(2)$ or S_4, since $\dim V^H = 1$, we obtain $I(F^H) - I(F^{[H]}) = -1 - 1 = -2$. In this case, $W(H) = \mathbf{Z}_1$, thus $m_{(H)} = -2$. Now we have to compute $m_{(H)}$ for $H = D_3$ or D_4. Since, in this case, $\dim V^H = 2$, and since $F(x) = 2x$, we obtain that $I(F^H) = 1$. Now we need to determine the set $\Omega^{[H]}$. First, we need to find all the subgroups K such that $H \subset K$ and K is conjugated to $O(2)$ (resp. to S_4). We will use here some well known facts about the maximal torus of a compact Lie group. Let us recall that a subgroup Γ of a Lie group G is called a *torus* if it is isomorphic to a product group $S^1 \times \ldots \times S^1$. A maximal subgroup of G isomorphic to some torus is called a ***maximal torus*** (see [3] for more details). Any two maximal tori are conjugate and if G is a connected compact Lie group, then any element is contained in at least one maximal torus. In $SO(3)$, each nontrivial element is contained in precisely one maximal torus isomorphic to $SO(2)$. This property may be seen from the fact that such an element has a unique eigenvector with eigenvalue +1. The rotations about this eigenvector form the maximal torus. Clearly, the maximal torus also contains all powers of this element. In the case of the group D_3 or D_4, it is clear that there is only one maximal torus, namely $SO(2)$ which contains $\mathbf{Z}_3$ and $\mathbf{Z}_4$. Consequently, if $K \in (O(2))$ and D_3 or D_4 are contained in K, then $K = O(2)$. In order to determine all the subgroups $K \in (S_4)$, which in fact are symmetry groups of an octahedron (or cube), such that K contains D_3 and D_4, we notice that D_3 or D_5 completely determines the position of the octahedron, and thus there is only one such group K, namely S_4. This implies that

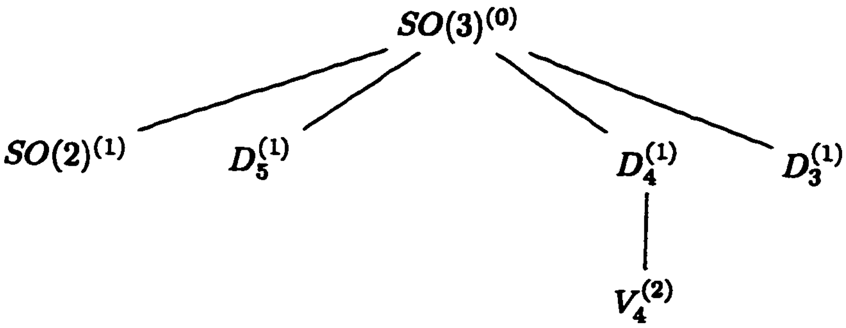
$$\Omega^{[H]} = \Omega^H \cap [V^{O(2)} \cup V^{S_4}].$$

We can identify $\Omega^{[H]}$ with the subset of a plane composed of two lines transversally intersecting at the origin. We have $F(x) = 2x$, and since $\Omega^{[H]}$ is an ENR's, we can compute that $I(F^{[H]}) = -3$. Consequently, for $H = D_3$ or D_4 $m_{(H)} = (1 + 3)/|W(H)| = 4/2 = 2$.

In the case $H = V_4$, we have $\dim V^H = 3$, $F(x) = 2x$. Therefore, in this case, $I(F^H) = -1$. In order to determine the set $\Omega^{[H]}$, we need to find all the subgroups $K \in (D_4)$ such that $V_4 \subset K$. Since the rotations of K belong to a unique maximal torus, it is easy to verify that for the group V_4 there are exactly three different planes of rotations, which implies that there are exactly three subgroups K_1, K_2, $K_3 \in (D_4)$ such that $V_4 \subset K_i$, $i = 1, 2, 3$. It may be verified that $V^{K_1} \cap V^{K_2} \cap V^{K_3} = V^{S_4}$. Indeed, the group S_4 of the orientation-preserving symmetries of the octahedron (the cube) contains three subgroups of symmetries of the parallel faces in the octahedron, which are exactly K_1, K_2 and K_3. On the other hand, it is clear from the lattice of subgroups of $SO(3)$ that any two of the subgroups K_1, K_2 and K_3 generate S_4, therefore $V^{K_i} \cap V^{K_j} = V^{S_4}$ for $i \neq j$, $i, j \in \{1, 2, 3\}$. Consequently, $\Omega^{[H]}$ may be identified with the subset of $\mathbf{R}^3$ consisting of three planes intersecting along a single line (containing the origin). We need to compute $I(F^{[H]})$, where $F^{[H]}:\Omega^{[H]}\to\Omega^{[H]}$ is given by $F^{[H]}(x)=2x$. Let X denote the subspace of $\mathbf{R}^2$ consisting of three lines intersecting at the origin. Then $\Omega^{[H]}$ may be identified with the product $X\times\mathbf{R}$ and $F^{[H]}=F_1\times F_2$, where $F_i(x)=2x$, $i=1,2$, $F_1:X\to X$, and $F_2:\mathbf{R}\to\mathbf{R}$. Then by the product property, $I(F^{[H]})=I(F_1)\cdot I(F_2)$. Since $I(F_1)=-5$, and $I(F_2)=-1$, we obtain $I(F^{[H]})=5$, and consequently $m_{(H)}=(-1-5)/6=-1$. Finally

$$SO(3)\text{-Deg}\,(-Id,\Omega_4)=(SO(3))-2(O(2))-2(S_4)+2(D_4)+2(D_3)-(V_4).$$

Let us now compute $SO(3)\text{-Deg}\,(-Id,\Omega_5)$. We have the following reduced lattice of isotropy subgroups for V_5:



Isotropy Lattice for V_5

It is easy to see for $H=SO(2)$, D_5, D_4, or D_3, we have $m_{(H)}=1$. For $H=V_4$, we have already computed that there are exactly three subgroups K_1, K_2, $K_3\in(D_4)$. We have $I(F^H)=1$. Since $\Omega^{[H]}$ may be identified with the set of three lines in a plane passing through the origin and since $F(x)=2x$, we have $I(F^{[H]})=-5$. Consequently, $m_{(H)}=(1-(-5))/6=1$. This implies

$$SO(3)\text{-Deg}\,(-Id,\Omega_5)=(SO(3))-(SO(2))-(D_5)-(D_4)-(D_3)+(V_4).$$

Chapter Two

Computation of the Degree and Applications to Symmetric Bifurcations

2.1 Introduction

We consider a G-equivariant map $f : V \times \mathbf{R}^n \to V$, where V is a finite dimensional orthogonal representation of a compact Lie group G. In this chapter we provide further development of the equivariant degree with emphasis on the Ulrich-type computational formula. The detailed computations are carried out only in the case where G is a compact abelian group. We will discuss in Chapter 4 the case $\Gamma \times S^1$, for a finite group Γ, and the method could in principle be used for more general non-abelian groups, in particular, for the group $SO(3) \times S^1$.

The formulae obtained are essential in the study of the equivariant bifurcation problem

$$x = F(x, \lambda), \qquad (x, \lambda) \in W \times \mathbf{R}^2, \tag{2.1.1}$$

where W is an orthogonal representation of the product group $G = \Gamma \times S^1$, Γ is a compact Lie group and F is a G-equivariant map. We assume that there exists a 2-dimensional submanifold $M \subseteq W^{S^1} \times \mathbf{R}^2$ such that every point $(x, \lambda) \in M$ satisfies (2.1.1). Points in M are therefore called *trivial solutions* of (2.1.1). We are interested in locating *bifurcation points* of (2.1.1), i.e. all the points $(x_o, \lambda_o) \in M$ such that every neighbourhood of (x_o, λ_o) in $W \times \mathbf{R}^2$ contains a nontrivial solution.

This type of abstract bifurcation problem arises naturally from the Hopf bifurcation theory of differential equations with spatial symmetries, where the action of S^1 comes from the usual shifting of the temporal argument and the action of Γ represents the spatial symmetry of the equation under consideration. For details, we refer to [21,32,33,43,46,56]. In the absence of spatial symmetry ($\Gamma = \{\mathrm{Id}\}$), the above bifurcation problem has been extensively studied. It has been shown that there is a close relation between the (local) bifurcation and the nontriviality of certain topological invariants related to homotopy classes of parametrized linear (or nonlinear) operators acting on W, and that various topological degrees can be employed to prove the nontriviality of those homotopy classes. For details, we refer to [25,33,43] and references therein.

The impact of the presence of symmetries on the existence of bifurcation points has been studied by many authors, see [4,20,21,30-36,39-46] and references therein. Our approach to the equivariant bifurcation problem was inspired by [21], where an S^1-equivariant bifurcation problem is studied with the use of S^1-degree constructed in [14]. We relate the existence of a branch of nontrivial solutions of (2.1.1) bifurcating from (x_o, λ_o) to the nontriviality of the G-degree of the mapping $f_\theta : V \times \mathbf{R} \to V$ defined by

$$f_\theta(x, \lambda) = (x - F(x, \lambda), \theta(x, \lambda)), \qquad (x, \lambda) \in V \times \mathbf{R},$$

where $V = W \times \mathbf{R}$ and θ is an invariant function defined in a sufficiently small neighbourhood U of (x_o, λ_o) and with negative values on $U \cap M$. The computation of G-degree of f_θ is a problem of formidable mathematical complexity. Nevertheless, we will show, in the case where Γ is a finite (Chapter 4) or an abelian group, that the G-degree of f_θ can be completely determined by the information of linear approximation of f_θ on certain *isotypical* components of W with respect to the group action $\Gamma \times S^1$.

In Section 2.2, we consider the case $n = 1$ (one dimensional parameter space) and prove an Ulrich type formula (cf. [43,49,53]) for the equivariant degree. This formula relates the G-degree of the map f to the more easily computable S^1-degree (cf. [14,43]). The generalization to the case $n = 2$ is also briefly discussed. In Section 2.3, we apply these formulae to the computation of G-degree in the case where $f : V \times \mathbf{R} \to V$ has regular zeros in $V^G \times \mathbf{R}$ and G is abelian. In Theorem 2.3.2, the G-degree is expressed, with the use of the Multiplicativity Property, as a product of a certain element $\nu(\omega)$ and a *winding element* $\mu(\omega)$. Both $\nu(\omega)$ and $\mu(\omega)$ can be computed by using linearization of the original map restricted to appropriate isotypical components of the space V. In section 2.4, we discuss the equivariant bifurcation problem related to Hopf bifurcation problems with symmetries. We develop some bifurcation invariants to detect bifurcation points and the global continuation of local branches of bifurcation points.

2.2 Ulrich Type Formula

We would like to begin with a brief discussion on the S^1-degree introduced by Dylawerski *et al.* in [14] and the related index. This degree is a special case of 1-parameter (i.e. $n = 1$) G-degree, where $G = S^1$. It leads to the definition of a corresponding G-fixed point index I_G of G-ANR's in a standard way. To be more precise, assume that Y is an isometric Banach representation of S^1, and $\Omega \subset Y \oplus \mathbf{R}$ is an open S^1-invariant subset. Let $f := \mathrm{Id} - F : \Omega \to Y$ be an S^1-equivariant map such that F is a compact map and $\mathrm{Fix}\,(F) = \{(x, \lambda) \in \Omega;\;\; F(x, \lambda) = x\}$ is a compact subset of Ω. Then the S^1-*fixed point index* of F in Ω is simply defined as

$$I_{S^1}(F, \Omega) := S^1\text{-Deg}\,(\mathrm{Id} - F, \Omega) = \sum_k \deg_k(\mathrm{Id} - F, \Omega) \cdot (\mathbf{Z}_k)$$

where $\deg_k(\mathrm{Id} - F, \Omega) \cdot (\mathbf{Z}_k)$ is the k-th component of the S^1-degree characterizing the zeros of $\text{Id} - F$ with orbit type $(\mathbf{Z}_k)$ (cf. [43]). We will also denote $I_{S^1}(F)_{\mathbf{Z}_k} := \deg_k(\text{Id} - F, \Omega)$. Let X be an S^1-ANR space, $U \subset X \times \mathbf{R}$ an open S^1-invariant subset. We consider a compact S^1-map $F : U \to X$ such that $\text{Fix}(F)$ is a compact subset of U. Then, by using the fact that every S^1-ANR is equivariantly dominated by an open S^1-subset of a Banach S^1-representation (cf. [37]), we can define the S^1-fixed point index $I_{S^1}(F, U) = S^1\text{-Deg}(\text{Id} - i \circ F \circ r, \Omega)$, where $i : X \times \mathbf{R} \to Y \oplus \mathbf{R}$ is an S^1-imbedding, $r : \Omega \to U$ is an S^1-retraction, and $\Omega \subset Y \oplus \mathbf{R}$ is an S^1-invariant open subset. It is well known (cf. [37]) that, for a compact Lie group G, if X is a G-ANR then X^H and $X^{[H]}$ are $W(H)$-ANRs.

Let's now consider V an orthogonal representation of a compact Lie group G and $(H) \in \Phi_1(G)$. Since $W(H)$ is a one dimensional manifold, the connected component of $1 \in W(H)$ can be identified with S^1. In the case when $(H) \in \Phi_1^+(G)$, i.e. the group $W(H)$ has a fixed invariant orientation, the imbedding $S^1 \subset W(H)$ is defined in a unique way. Let $\Omega \subset V \oplus \mathbf{R}$ be an open bounded G-invariant subset and $f := \text{Id} - F : V \oplus \mathbf{R} \to V$ be an Ω-admissible map. Then $f^H = \text{Id} - F^H : V^H \oplus \mathbf{R} \to V^H$ is Ω^H-admissible and $f^{[H]} = \text{Id} - F^{[H]} : V^{[H]} \oplus \mathbf{R} \to V^{[H]}$ is $\Omega^{[H]}$-admissible. Since both F^H and $F^{[H]}$ are $W(H)$-equivariant, they must be S^1-equivariant and hence, the S^1-fixed point indices $I_{S^1}(F^H)$ and $I_{S^1}(F^{[H]})$ can be defined. According to the definition of the S^1-fixed point index, $I_{S^1}(F^{[H]})$ can be defined as $S^1\text{-Deg}(\text{Id} - i \circ F^{[H]} \circ r, \Omega_o^H)$, where Ω_o^H is an S^1-neighbourhood of $\Omega^{[H]}$ in $V^H \oplus \mathbf{R}$, $r : \Omega_o^H \to \Omega^{[H]}$ an S^1-retraction, and $i : V^{[H]} \to V^H$ is the natural inclusion. It can be verified, by using the commutativity property of S^1-fixed point index, that both $I_{S^1}(F^H)$ and $I_{S^1}(F^{[H]})$ do not depend on G-equivariant homotopies and satisfy the excision and additivity properties.

Let G be a compact Lie group and let V be an orthogonal representation of G. Assume $(H) \in \Phi_1(G)$. Then $S^1 \subset W(H)$ and we have the following result:

Theorem 2.2.1 (ULRICH TYPE FORMULA FOR G-DEGREE) *Let $\Omega \subset V \oplus \mathbf{R}$ be an invariant bounded open subset and $f := Id - F : V \oplus \mathbf{R} \to V$ be a G-equivariant Ω-admissible map. Then*

$$G\text{-Deg}\,(f, \Omega) = \sum_{(H)\in\Phi_1(G)} m_H \cdot (H), \tag{2.2.1}$$

where for $(H) \in \Phi_1(G)$

$$m_H = \left[I_{S^1}(F^H)_{\mathbf{Z}_1} - I_{S^1}(F^{[H]})_{\mathbf{Z}_1}\right] / \left|\frac{W(H)}{S^1}\right| \quad ((\text{mod } 2) \text{ if } H \in \Phi_1^-(G)).$$

Proof. Observe that if f is a regular normal map, then for $(H) \in \mathcal{J}(f^{-1}(0))$ and $(H) = (G_x) \in \Phi_1(G)$, every $W(H)$-orbit of solutions $W(H)x$, of the equation

$$f^H(x) = 0, \quad x \in \Omega^H, \tag{2.2.2}$$

forms a closed submanifold of dimension at most 1. On the other hand, $S^1 \subset W(H)$ acts on $V^H \oplus \mathbf{R}$ and any 1-dimensional orbit $W(H)x$ can be considered as a finite number of S^1-orbits of solutions to (2.2.2).

Since the definitions of $I_{S^1}(F^H)$ and $I_{S^1}(F^{[H]})$ do not depend on Ω-admissible homotopies and satisfy the excision and additivity properties we may assume, without loss of generality, that f is a regular normal map such that $f^{-1}(0) \cap \Omega$ consists of a single orbit Gx_o with $G_{x_o} = H$, $(H) \in \Phi_1(G)$. In addition, we may assume that Ω is a tubular neighbourhood of Gx_o. Thus, (H) is the minimal orbit type in Ω (cf. [37]). We observe that $f^H : \Omega^H \to V^H$ is $W(H)$-equivariant and since $W(H)$ is 1-dimensional, the orbit $W(H)x_o$ is a 1-dimensional submanifold of $V^H \oplus \mathbf{R}$. Denote by S the linear slice to the orbit $W(H)x_o$ at x_o. If $W(H)$ is bi-orientable, the slice S has a natural orientation induced by a fixed orientation of V^H. By definition of G-degree, $m_H = \text{sign}\det Df^H_{|S}(x_o)$. Considering $W(H)x_o$ as composed of S^1-orbits, we obtain from the definition of S^1-degree (cf. [14,43]) that

$$\deg_1(f^H,\Omega^H) = \sum_{i=1}^{k} \operatorname{sign}\det Df^H_{|S_i}(x^i_o), \tag{2.2.3}$$

where $k = |W(H)/S^1|$ and S_i is the slice to the i-th S^1-orbit in $W(H)x_o$ and $x^i_o = g_i x_o$ for some $g_i \in W(H)$, $i = 1,2,\dots,k$. Since f^H is $W(H)$-equivariant,

$$Df^H(gx)\circ g = D(f^H\circ g)(x) = D(g\circ f^H)(x) = g\circ Df^H(x).$$

Moreover, since $g_{i|S} : S \to S_i$ is an isomorphism, we have

$$Df^H(g_i x_o) = g_i \circ Df^H(x_o)\circ g_i^{-1}$$

and

$$Df^H_{|S_i}(g_i x_o) = g_{i|S}\circ Df^H_S(x_o)\circ g_i^{-1}{}_{|S_i}.$$

Therefore, for any choice of a basis in S_i, we have

$$\begin{aligned}
\operatorname{sign}\det Df^H_{|S_i}(g_i x_o) &= \operatorname{sign}\det [g_{i|S}\circ Df^H_{|S}(x_o)\circ g_i^{-1}{}_{|S_i}] \\
&= \operatorname{sign}\det [g_{i|S}]\cdot \operatorname{sign}\det Df^H_{|S}(x_o)\cdot \operatorname{sign}\det [g_{i|S_i}]^{-1} \\
&= \operatorname{sign}\det Df^H_{|S}(x_o) = m_H.
\end{aligned}$$

Consequently, it follows from (2.2.3) that

$$\deg_1(f^H,\Omega^H) = \left|\frac{W(H)}{S^1}\right|\cdot \operatorname{sign}\det Df^H_{|S}(x_o) = \left|\frac{W(H)}{S^1}\right|\cdot m_H.$$

Since (H) is the minimal orbit type in Ω, we have $\Omega^{[H]} = \varnothing$ and $n_H := I_{S^1}(F^H) = m_H\cdot |W(H)/S^1|$. Therefore, formula (2.2.1) holds for the orbit type (H).

Now consider $K \subsetneq H$. Since f is (H)-normal, from the product property of S^1-degree it follows that $I_{S^1}(F^H)_{\mathbf{Z}_1} = I_{S^1}(F^K)_{\mathbf{Z}_1}$. Now, we compute $I_{S^1}(F^{[K]})$. Observe that $\Omega^H \subset \Omega^{[K]}$. If $\Omega^{[K]}\setminus\Omega^H = \varnothing$, then clearly $m_K = 0$ and the validity of formula (2.2.1) follows. Assume therefore that $\Omega^{[K]} \setminus \Omega^H \neq \emptyset$. Since f is (H)-normal, we may assume without loss of generality that $F^K(u,v) = (\varphi(u), 0)$, where $(u,v) \in \Omega^K$, $u \in \Omega^{H_i}$ and $W(K)\Omega^H = \Omega^{H_1} \cup \cdots \cup \Omega^{H_j}$, $j = |W(K)/S^1|$. We notice that the subsets Ω^{H_i} are disjoint (since (H) is the minimal orbit type in Ω). Suppose that $r : \Omega^K \to \Omega^{[K]}$ is an S^1-retraction onto $\Omega^{[K]}$. Then $I_{S^1}(F^{[K]}) = I_{S^1}(F^{[K]} \circ r) = S^1\text{-Deg}(Id - F^{[K]} \circ r, \Omega^K)$. Put $F_t := tF^K + (1-t)F^{[K]} \circ r$. Then F_t is an Ω^K-admissible homotopy. Indeed, if $F_t(u,v) = (u,v)$, then $v = 0$ and $\varphi(u) = u$, and consequently F_t has the same fixed points as F^H. Thus,

$$I_{S^1}(F^{[K]}) = I_{S^1}(F^K) = I_{S^1}(F^H),$$

and $m_K = 0$. This completes the proof.

□

Remark 2.2.2 We point out that the above Ulrich Type Formula for the one-parameter G-degree could be generalized to the G-degree with 2-dimensional parameter space. In this case, we have that if $(H) \in \Phi_2(G)$ then the connected component of $1 \in W(H)$ can be naturally identified with the torus $T^2 = S^1 \times S^1$ and consequently, we can have an analogue of the formula (2.2.1) with the S^1-fixed point index I_{S^1} replaced by the T^2-fixed point index I_{T^2}. However, this approach will fail in the case of 3-dimensional parameter space, since for $(H) \in \Phi_3(G)$ the group $W(H)$ may contain the non-abelian group S^3.

2.3 Some Computational Formulae for the G-Degree

We begin this section with a technical Lemma which we will need later.

Lemma 2.3.1 *Let U be an orthogonal irreducible representation of real type (cf.*

[3,25]) of a compact Lie group G and $\mathcal{U} = U \otimes_{\mathbb{R}} \mathbb{C}$ be the complexification of U. Assume that $\Omega := \{v \in \mathcal{U};\ \frac{1}{4} < \|v\| < 1\}$ and $f : \overline{\Omega} \to \mathcal{U}$ is given by

$$f(v) = 2(1 - 2\|v\|)v, \quad v \in \overline{\Omega}.$$

Then G-Deg$(f, \Omega) = 0$, where G-Deg(f, Ω) is considered to be an element of the Burnside ring $A(G)$.

Proof. Let $(H) \in \mathcal{J}(\Omega)$. Consider the map $f^H = \mathrm{Id} - F^H : \overline{\Omega^H} \to \mathcal{U}^H$, $f^H := f_{|\overline{\Omega^H}}$. It can be verified that

$$\begin{aligned}\mathrm{Deg}(f^H, \Omega^H) &= \mathrm{Deg}(-\mathrm{Id}, B_1(0)) - 1\\ &= (-1)^{2n_H} - 1 = 0,\end{aligned}$$

where $B_1(0)$ denotes the unit ball in $\mathcal{U}^H$ and $n_H = \dim U^H$. Therefore, $I(F^H) = I(F^{[H]}) = 0$ and the conclusion follows from the Ulrich formula (cf. 1.2.4). □

Let V be an orthogonal representation of a compact Lie group G. Consider the following isotypical decomposition of V

$$V = V^G \oplus \bigoplus_{\beta \in \mathfrak{B}} V_\beta,$$

where each component V_β is a direct sum of all subrepresentations of V which are equivalent to a fixed irreducible representation U_β of the group G. We will denote by $\mathfrak{R}$ the set of all $\beta \in \mathfrak{B}$ such that U_β is of real type, by $\mathfrak{C}$ the set of all $\beta \in \mathfrak{B}$ such that U_β is of complex type, and by $\mathfrak{H}$ the set of all $\beta \in \mathfrak{B}$ such that U_β is of quaternionic type. Then $\mathfrak{B} = \mathfrak{R} \cup \mathfrak{C} \cup \mathfrak{H}$. Denote by d_β the number of irreducible components of V_β, i.e. $V_\beta = [U_\beta]^{d_\beta}$. Then $GL^G(V_\beta) \simeq GL(\mathbb{K}, d_\beta)$, where $\mathbb{K}$ denotes $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$, if U_β is respectively of real, complex or quaternionic type. However, since we assume that V is a real representation of G, the complex structure on V_β for

$\beta \in \mathfrak{C}$ is not determined uniquely. Assume, therefore, that this complex structure is chosen. Consequently, $\pi_1(GL^G(V_\beta)) \simeq \mathbb{Z}$ for $\beta \in \mathfrak{C}$, where the isomorphism $\nabla : \pi_1(GL^G(V_\beta)) \to \mathbb{Z}$ is given by

$$\nabla([\sigma]) = \deg(\det_{\mathbf{C}}(\sigma)),$$

for $\sigma : S^1 \to GL^G(V_\beta)$ and $[\sigma]$ is the homotopy class of σ. If $\beta \in \mathfrak{H}$, then $\pi_1(GL^G(V_\beta)) = \pi_1(GL(\mathbb{H}, d_\beta)) = 0$.

For a given $\omega : S^1 \to GL^G(V)$ and $\beta \in \mathfrak{C}$, we define the *β-winding number* $\mu_\beta(\omega)$ of ω by

$$\mu_\beta(\omega) := \nabla([\omega_\beta]),$$

where $\omega_\beta(\sigma) = \omega(\sigma)_{|V_\beta}$.

Let us assume for the rest of this section that G is a compact abelian Lie group. Then every nontrivial isotypical component V_β corresponds to a subgroup H_β of G such that $G_v = H_\beta$ for $v \in V_\beta \setminus \{0\}$. It is well-known (cf.[3]) that if $(H_\beta) \in \Phi_1(G)$, i.e. $\dim G/H_\beta = 1$, then $\beta \in \mathfrak{C}$ and the space V_β has a natural complex structure uniquely induced by the chosen orientation of G/H_β. We put $\mathfrak{C}_1 = \{\beta \in \mathfrak{C};\ \dim G/H_\beta = 1\}$, and we define the *winding element* $\mu(\omega) \in A_1(G)$ for $\omega : S^1 \to GL^G(V)$ by

$$\mu(\omega) := \sum_{\beta \in \mathfrak{C}_1} \mu_\beta(\omega) \cdot (H_\beta). \tag{2.3.1}$$

We may describe the element $\mu(\omega)$ in a different way. Put

$$\Omega_\beta := \{(v, z) \in U_\beta \oplus \mathbf{C};\ \|v\| < 1,\ \frac{1}{2} < |z| < 2\},$$

and let $f_\beta : \overline{\Omega_\beta} \to U_\beta \oplus \mathbf{R}$ be given by

$$f_\beta(v, z) = (z \cdot v, |z|(\|v\| - 1) + \|v\| + 1), \tag{2.3.2}$$

for $(v,z)\in\Omega_\beta$. Since $\beta\in\mathfrak{C}_1$, the map f_β is well defined. It is also clear that f_β is Ω_β-admissible. One can verify that $G\text{-Deg}(f_\beta,\Omega_\beta)=(H_\beta)$. Consequently, we have

$$\mu(\omega):=\sum_{\beta\in\mathfrak{C}_1}\mu_\beta(\omega)\cdot G\text{-Deg}(f_\beta,\Omega_\beta). \tag{2.3.3}$$

Suppose now that $\beta\in\mathfrak{B}$ is such that $G/H_\beta\simeq\mathbb{Z}_2$. Then clearly $\beta\in\mathfrak{R}$. We will denote by $\mathfrak{R}_2$ the set of all $\beta\in\mathfrak{R}$ such that $G/H_\beta\simeq\mathbb{Z}_2$. (Since we assume that G is abelian, it follows ([3]) that $\mathfrak{R}=\mathfrak{R}_2$). For a given $A\in GL^G(V)$, we define the element $\nu(A)\in A(G)$ by

$$\nu(A):=\prod_{\beta\in\mathfrak{R}_2}G\text{-Deg}(A_\beta,\Omega_\beta), \tag{2.3.4}$$

where $A_\beta=A_{|V_\beta}$ and Ω_β denotes the unit ball in V_β. Then $G\text{-Deg}(A_\beta,\Omega_\beta)=G\text{-Deg}(\nu_\beta(A)\mathrm{Id},B_\beta)=(G)-\nu_\beta(H_\beta)$, where $\nu_\beta=\frac{1}{2}(-\varepsilon_\beta(A)+1)$ and $\varepsilon_\beta(A):=\operatorname{sign}\det_{\mathbf{R}}(A_\beta)$.

Assume now that $f:V\oplus\mathbf{R}\to V$ is a G-equivariant C^1-map satisfying the following hypothesis:

(A) *There is an open bounded invariant set $\Omega\subset V\oplus\mathbf{R}$ such that f is Ω-admissible,* 0 *is a regular value for $f|_\Omega$ and*

$$\Sigma:=f^{-1}(0)\cap\Omega\subset V^G\oplus\mathbf{R}$$

is diffeomorphic to S^1.

It is clear from the assumption (A) that $V^G\neq\{0\}$. We choose an orientation of V^G and we orient $V^G\oplus\mathbf{R}$ with the product orientation. Since 0 is a regular value of $f|_\Omega$, for $x\in\Sigma$ the derivative $Df(x)$ maps the orthogonal complement of $T_x\Sigma$ in $V^G\oplus\mathbf{R}$, i.e. $N_x:=(T_x\Sigma)^\perp\cap V^G\oplus\mathbf{R}$, isomorphically onto V^G. Consequently, it induces an orientation of N_x. We fix a diffeomorphism $\eta:S^1\to\Sigma$ such that the orientation of $T_x\Sigma$ (induced by η) followed by the orientation of N_x, gives the chosen orientation of $V^G \oplus \mathbf{R}$.

By assumption (A), we can define $\omega : S^1 \to GL^G(V^*)$, where $V^* := \bigoplus_{\beta\in\mathfrak{B}} V_\beta$, by

$$\omega(\lambda)v := Df(\eta(\lambda))v, \qquad v \in \bigoplus_{\beta\in\mathfrak{B}} V_\beta,\ \lambda \in S^1. \tag{2.3.5}$$

Now, we are in the position to present the first main result of this section.

Theorem 2.3.2 *Let G be a compact abelian Lie group. Suppose that $f : V\oplus\mathbf{R} \to V$ is a G-equivariant mapping satisfying assumption (A). Then*

$$G\text{-Deg}\,(f,\Omega) = \nu(\omega)\cdot\mu(\omega) = \Big(\prod_{\beta\in\mathfrak{R}_2}((G)-\nu_\beta(H_\beta)\Big)\Big(\sum_{\beta\in\mathfrak{C}_1}\mu_\beta(H_\beta)\Big),$$

where ω is defined by (2.3.5), $\mu(\omega)$ by (2.3.1), and $\nu(\omega)$ by (2.3.4).

Proof. We may assume, without loss of generality, that

(i) $\Omega_0 := \Omega\cap(V^G\oplus\mathbf{R})$ is a tubular neighbourhood of Σ in $V^G\oplus\mathbf{R}$. We denote by $\pi : \Omega_0 \to \Sigma$ the natural projection of Ω_0 onto Σ so that every element $x\in\Omega_0$ can be written uniquely as $x = \pi(x) + w$, where w is normal to Σ at $\pi(x)$.

(ii) $\Omega = \Omega_0\times\widetilde{\Omega}$, where $\widetilde{\Omega} := \Omega_{02}\times\widehat{\Omega}$, $\Omega_{02} := \prod_{\beta\in\mathfrak{R}_2} B(V_\beta)$, $B(V_\beta)$ is the unit ball in V_β, $\widehat{\Omega}$ is the unit ball in $\widehat{V} = \bigoplus_{\beta\in\mathfrak{B}\setminus\mathfrak{R}_2} V_\beta$.

(iii) $f(x,v) = f_0(x) + Df(\pi(x))v$, where $f_0 := f|_{\overline{\Omega_0}}$ and $(x,v)\in\overline{\Omega_0}\times\overline{\widetilde{\Omega}}$.

For $\beta\in\mathfrak{R}_2$, we can choose a (real) basis for the space V_β and identify it with $\mathbf{R}^{d_\beta}$, where $d_\beta = \dim_{\mathbf{R}} V_\beta$. Clearly, every $\mathbf{R}$-linear operator $A : V_\beta \to V_\beta$ is G-equivariant. If the homotopy class of $\omega_\beta : S^1 \to GL^G(V_\beta) \cong GL(\mathbf{R}, d_\beta)$ is trivial, then it contains a representative $B_\beta : S^1 \to GL(\mathbf{R}, d_\beta)$ defined by

$$B_\beta(\lambda) = B_\beta = \begin{bmatrix} (-1)^{\nu_\beta} & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix} : \mathbf{R}^{d_\beta} \longrightarrow \mathbf{R}^{d_\beta}. \tag{2.3.6}$$

If the homotopy class of ω_β is not trivial, then it contains a representative B_β : $S^1 \to GL(\mathbf{R}, d_\beta)$ given by

$$B_\beta(\lambda) = \begin{bmatrix} (-1)^{\nu_\beta}\cos(k\delta) & -\sin(k\delta) & \dots & 0 \\ (-1)^{\nu_\beta}\sin(k\delta) & \cos(k\delta) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix} : \mathbf{R}^{d_\beta} \longrightarrow \mathbf{R}^{d_\beta}$$

where $\lambda = \cos\delta + i\,\sin\delta$, and k is an integer.

In our next step we will show that, without loss of generality, we may always assume that for all $\beta \in \mathfrak{R}_2$ the homotopy class of $\omega_\beta : S^1 \to GL^G(V_\beta)$ is trivial and therefore, we may use the form (2.3.6) of B_β. To achieve this, let us consider a fixed $\beta \in \mathfrak{R}_2$ such that the homotopy class of $\omega_\beta : S^1 \to GL^G(V_\beta)$ is not trivial. For simplicity, we may assume (by using the product property) that $V_\beta \cong \mathbf{C}$ and $\nu_\beta = 0$. Consequently, $B_\beta(\lambda)z = \lambda^k z$, $z \in \mathbf{C}$.

We introduce the following piecewise linear function

$$q(t) = \begin{cases} 1 & \text{if } 0 \le t < \frac{1}{4} \\ -2t + \frac{3}{2} & \text{if } \frac{1}{4} \le t < \frac{3}{4} \\ 0 & \text{if } \frac{3}{4} \le t. \end{cases}$$

Define the map $C_\beta : S^1 \times \mathbf{C} \to \mathbf{C}$ by

$$C_\beta(\lambda, z) = q(|z|)z + (1 - q(|z|))B_\beta(\lambda)z.$$

It is easy to verify that $C_\beta(\lambda, z) = 0$ if and only if $z = 0$ or $|z| = \frac{1}{2}$ and $\lambda^k = -1$.

Put $\Omega^{**} = \Omega_0 \times \prod_{\beta' \neq \beta} B(V_{\beta'})$, $U = B(V_\beta)$. Thus $\Omega = U \times \Omega^{**}$. Define $f' : \overline{\Omega} \to$

V by

$$f'(z,x,v) = (C_\beta(\eta^{-1}(\pi(x)), z), f(x,v)),$$

where $z \in \overline{U}$, $x \in \overline{\Omega_0}$, $v \in \prod_{\beta' \neq \beta} \overline{B(V_{\beta'})}$. By the homotopy invariance, we have

$$G\text{-Deg}(f, \Omega) = G\text{-Deg}(f', \Omega).$$

The set $f'^{-1}(0) \cap \Omega = \Gamma_0 \cup \bigcup_{j=1}^k \Gamma_j$, where $\Gamma_0 = \{0\} \times \Sigma \times \{0\} \subset U \times \Omega_0 \times \prod_{\beta' \neq \beta} B(V_{\beta'})$, and

$$\Gamma_j = \{(z,x,v);\ |z| = \tfrac{1}{2},\ x \in \Sigma,\ \eta^{-1}(x) = \lambda_j,\ v = 0\}$$

for $\lambda_j = e^{i\frac{2j-1}{k}\pi}$ and $j = 1, \dots, k$. Since on a small tubular neighbourhood Ω_j of Γ_j, $j = 1, 2, \dots, k$, the mapping f' is G-homotopic to the mapping

$$f''(z,x,v) = (2(1-2|z|z), f(x,v)) = (\varphi(z), f(x,v)),$$

it follows from the product property and Lemma 2.3.1 that $G\text{-Deg}(f'', \Omega_j) = 0$, and consequently

$$G\text{-Deg}(f, \Omega) = G\text{-Deg}(f', \Omega'),$$

where Ω' is a small neighbourhood of Γ_0. It is clear that the mapping f' still satisfies assumption (A) with respect to $\Gamma_0 = \{0\} \times \Sigma \times \{0\}$, but in this case, for $\beta \in \mathfrak{R}_2$, the corresponding homotopy class of ω_β is trivial. This implies that for the purpose of our computations we may assume, without loss of generality, that for every $\beta \in \mathfrak{R}_2$ the homotopy class of ω_β is trivial, and consequently the mapping B_β has the form (2.3.6).

We may define a mapping $\widetilde{f} : \overline{\Omega} \to V$ by

$$\widetilde{f}(x,v) = f_0(x) + A(\pi(x))v, \quad v \in \overline{\widetilde{\Omega}},\ x \in \overline{\Omega_0},$$

where for $\sigma \in \Sigma$

$$A(\pi(x)) = \bigoplus_{\beta \in \mathfrak{R}_2} B_\beta \oplus Df(\pi(x))_{|\widehat{V}}.$$

It follows from the homotopy invariance that

$$G\text{-Deg}(f, \Omega) = G\text{-Deg}(\widetilde{f}, \Omega),$$

and consequently, using the multiplicativity property for G abelian,

$$G\text{-Deg}\,(f, \Omega) = \nu(\omega) \cdot G\text{-Deg}\,(\widehat{f}, \Omega^o),$$

where $\Omega^o := \Omega_0 \times \widehat{\Omega} \subset V^o := V^G \times \widehat{V}$, $\widehat{f} : \overline{\Omega^o} \to V^o$ and $\widehat{f}(x, v) = f_0(x) + Df(\pi(x))v$ for $v \in \widehat{V}$. In order to compute $G\text{-Deg}\,(\widehat{f}, \widehat{\Omega})$, we use Theorem 2.2.1 and a similar computational formula for S^1-degree (see Thm.6.3.5 in [43]). We notice that for every $\beta \in \mathfrak{C}_1$, we have $G/H_\beta \simeq S^1$. Thus

$$m_{H_\beta} = I_{S^1}(\widehat{F}^{H_\beta})_{\mathbb{Z}_1} - I_{S^1}(\widehat{F}^{[H_\beta]})_{\mathbb{Z}_1} = \mu_\beta,$$

and hence $G\text{-Deg}\,(\widehat{f}, \Omega^o) = \mu(\omega)$. This completes the proof. □

Assume now that $G = \Gamma \times S^1$, where Γ is a compact abelian Lie group. We choose the natural orientation of S^1 and assume that for every orbit type $(H) \in \Phi_1(G)$ there has been chosen an invariant *concordant* orientation of G/H i.e. we assume that the map $\{1\} \times S^1 \hookrightarrow \Gamma \times S^1 \to G/H$ preserves the orientations.

Assume again that V is a finite dimensional orthogonal representation of G and let

$$V = V_0 \oplus V_1 \oplus V_2 \oplus \cdots \oplus V_k \tag{2.3.7}$$

be the isotypical decomposition of V with respect to the restricted action of S^1, i.e.

for $x \in V_j \setminus \{0\}$, $j = 1, 2, \ldots, k$, we have $G_x \cap (\{1\} \times S^1) = \mathbf{Z}_j$, and for $x \in V_0 \setminus \{0\}$ we have $G_x \cap (\{1\} \times S^1) = S^1$. For every $j = 1, 2, \ldots, k$, the subspace V_j has a natural complex structure given by

$$(a + ib) \cdot x := ax + b \exp(i\tfrac{\pi}{2j})x, \quad a + ib \in \mathbf{C},\ x \in V_j.$$

Since G is the product of Γ and S^1, every G-isotypical component V_β of V is contained in some V_j. Moreover, if $V_\beta \subset V_j$, where $j \neq 0$, then it can be verified that there is a homomorphism $\theta_\beta : \Gamma \to S^1/\mathbf{Z}_j$ such that

$$H_\beta = \{(\gamma, z) \in \Gamma \times S^1;\ z \in \theta_\beta(\gamma)\}. \tag{2.3.8}$$

We will call such orbit type (H_β) a *basic orbit type,* and θ_β the *associated homomorphism* of the G-isotypical component V_β.

Definition 2.3.3 Let $G = \Gamma \times S^1$ be a compact abelian Lie group. We denote by $A_1^*(G)$ the $\mathbf{Z}$-submodule of $A_1(G)$ generated by all non-basic orbit types of G, and by $\widetilde{A_1}(G)$ the $\mathbf{Z}$-submodule of $A_1(G)$ generated by all basic orbit types of G.

We can identify the quotient module $A_1(G)/A_1^*(G)$ with $\widetilde{A_1}(G)$.

We denote

$$\mathfrak{B}^* := \{\beta \in \mathfrak{B};\ V_\beta \subset V_0\}$$
$$\widetilde{\mathfrak{B}} := \{\beta \in \mathfrak{B};\ V_\beta \subset V_j,\ j \neq 0\}.$$

It follows from (2.3.8) that if $\beta \in \widetilde{\mathfrak{B}}$ then $\dim G/H_\beta = 1$. Therefore, $\widetilde{\mathfrak{B}} \subset \mathfrak{C}_1$.

Corollary 2.3.4 *Suppose $G = \Gamma \times S^1$, where Γ is a compact abelian Lie group.*

Under the same assumptions as in Theorem 2.3.2, we have

$$G\text{-Deg}(f,\Omega) = \xi^*(\omega) + \widetilde{\xi}(\omega),$$

where $\xi^*(\omega) \in A_1^*(G)$, $\widetilde{\xi}(\omega) \in \widetilde{A_1}(G)$, *and*

$$\widetilde{\xi}(\omega) = \sum_{\beta \in \widetilde{\mathfrak{B}}} \mu_\beta(H_\beta).$$

Proof. Assume that (K) is a generator of $A(G)$ such that $G \neq K$. Then K is an open and closed subgroup of $G = \Gamma \times S^1$. Therefore, there exists an open and closed subgroup D of Γ, $D \neq \Gamma$, such that $K = D \times S^1$. Since for every $\beta \in \mathfrak{C}_1$, $(K \cap H_\beta)$ can not be a basic orbit type and

$$(K) \cdot (H_\beta) = n(K \cap H_\beta),$$

where $n = |(G/K \times G/H_\beta)/G|$, we have $(K) \cdot (H_\beta) \in A_1^*(G)$. As $A_1(G) = A_1^*(G) \oplus \widetilde{A_1}(G)$, we get

$$\Big(\prod_{\beta \in \mathfrak{R}_2} ((G) - \nu_\beta(H_\beta))\Big)\Big(\sum_{\beta \in \mathfrak{C}_1} \mu_\beta(H_\beta)\Big) = \xi^*(\omega) + \widetilde{\xi}(\omega),$$

where $\widetilde{\xi}(\omega) = \sum_{\beta \in \widetilde{\mathfrak{B}}} \mu_\beta(H_\beta)$. □

2.4 Equivariant Bifurcation Problems

We begin this section with the following example of an equivariant Hopf bifurcation problem.

Example 2.4.1 Suppose that $V := \mathbf{R}^N$ is an orthogonal representation of a compact Lie group Γ and let $f : V \times \mathbf{R} \to V$ be a Γ-equivariant C^1-map. We are interested in branches of periodic solutions of the autonomous system

$$\dot{x} = f(x, \alpha) \tag{2.4.1}$$

bifurcating from stationary points of (2.4.1). The problem of finding periodic solutions of (2.4.1) can be transformed into the fixed-point problem with two parameters (α, p)

$$z - (L-P)^{-1}[\frac{p}{2\pi}N_f(j(z), \alpha) - Pj(z)] = 0, \tag{2.4.2}$$

where $z \in C^1(S^1; V)$, $S^1 = \mathbf{R}/2\pi\mathbf{Z}$, $Lz = \dot{z}$, $j : C^1(S^1; V) \hookrightarrow C(S^1; V)$, $N_f(u, \alpha)(t) = f(u(t), \alpha)$ for $u \in C(S^1; V)$, and $Pz = \frac{1}{2\pi}\int_0^{2\pi} z(t)dt$. Assume that all the stationary points of (2.4.1) (i.e. the solutions of the equation $f(x, \alpha) = 0$) are nondegenerate (i.e. $D_x f(x, \alpha)$ is an isomorphism for every stationary solution (x, α) of (2.4.1)). In addition, we assume that every center of (2.4.1) (i.e. a stationary solution (x, α) of (2.4.1) having a pair of purely imaginary characteristic values) is isolated. Hopf bifurcation deals with a bifurcation of periodic solutions from a stationary solution. Since equation (2.4.1) is Γ-symmetric, equation (2.4.2) is $\Gamma \times S^1$-symmetric. In this case all stationary solutions $(x, \alpha, p) \in C^1(S^1; V) \times \mathbf{R} \times \mathbf{R}$ of (2.4.2) form a two-dimensional submanifold M of $V_0 \times \mathbf{R}^2$, where V_0 denotes the space V^{S^1}. We are interested in maximum continuation of periodic solutions bifurcating from M. We will show that the relations between stationary points belonging to a maximal but bounded branch of periodic solutions can be characterized by the fact that the sum of $\Gamma \times S^1$-degrees computed in neighbourhoods of bifurcation points is equal to zero. Since the equivariant degree describes (generically) symmetry properties of the solutions, it may be used to justify or to explain certain pattern formations observed in specific dynamical systems with symmetries. This example will serve as a motivation for our abstract setting.

Suppose that Γ is a compact Lie group, $G = \Gamma \times S^1$, and W is a real Banach isometric representation of G.

Recall that W_0 denotes the set of all fixed points of W, called ***stationary points,*** with respect to the restricted S^1-action, i.e. $W_0 = \{x \in W;\ \xi x = x \text{ for all } \xi \in S^1\}$.

We consider the nonlinear problem

$$x = F(x, \lambda), \quad (x, \lambda) \in W \times \mathbf{R}^2 \tag{2.4.3}$$

where $F : W \times \mathbf{R}^2 \to W$ is a given G-equivariant completely continuous map and satisfies the following condition:

(H1) There exists a 2-dimensional G-invariant submanifold $M \subset W_0 \times \mathbf{R}^2$ such that for every $(x, \lambda) \in M$,

(i) $x = F(x, \lambda)$, and

(ii) $F|_{W_0 \times \mathbf{R}^2} : W_0 \times \mathbf{R}^2 \to W_0$ is continuously differentiable and $\mathrm{Id}_{W_0} - D_x F(x, \lambda)|_{W_0} \in GL(W_0)$ (*Nondegeneracy Condition*).

Note that in assumption (H1), we only assume M to be a subset of $W_0 \times \mathbf{R}^2$ (not $W^G \times \mathbf{R}^2$).

Each point in M is a solution of (2.4.3), which will be called a ***trivial solution.*** Other solutions will be called ***non-trivial.*** A point in M is called a ***bifurcation point*** if every neighbourhood of the point contains a non-trivial solution of (2.4.3). Our main goal in this section is to use the G-degree to determine which elements of M are the bifurcation points of (2.4.3). It follows from the implicit function theorem that for every $(x_0, \lambda_0) \in M$, there exist an open neighbourhood U_{x_0} of x_0 in W_0, an open neighbourhood U_{λ_0} of λ_0 in $\mathbf{R}^2$ and a C^1-map $\eta : U_{\lambda_0} \to W_0$ so that

$$M \cap (U_{x_0} \times U_{\lambda_0}) = \{(\eta(\lambda), \lambda);\ \lambda \in U_{\lambda_0}\}. \tag{2.4.4}$$

Assumption (H1) excludes the existence of bifurcation of stationary solutions.

Lemma 2.4.2 *Under assumption (H1), we have the following*

(i) *For every* $(x_0, \lambda_0) \in M$, *the isotropy group* $\Gamma_{(x_0,\lambda_0)}$ *of* (x_0, λ_0) *with respect to the action of* Γ *is a closed subgroup of* Γ *such that* $\dim \Gamma = \dim \Gamma_{(x_0,\lambda_0)}$;

(ii) *Isotropy groups of points in the same connected component of* M *are identical.*

Proof. (i) Using the uniqueness guaranteed by the implicit function theorem, we can see that if $\gamma \in \Gamma$ is such that $\gamma(x_0, \lambda_0) = (\gamma x_0, \lambda_0) \in (\mathcal{U}_{x_0} \times \mathcal{U}_{\lambda_0}) \cap M$, then $(\gamma x_0, \lambda_0) = (x_0, \lambda_0)$. Therefore, the orbit $\Gamma(x_0, \lambda_0)$ must be finite. Consequently, $\dim \Gamma = \dim \Gamma_{(x_0,\lambda_0)}$. (ii) It follows from (i) and the compactness of Γ that there are only a finite number of orbit types in M. Let M_0 denote a connected component of M. Assume that $\alpha = (K)$ is a minimal Γ-orbit type in M_0. Then $(M_0)_K = M_0^K = M_0 \cap (W_0^K \times \mathbb{R}^2)$. So, M_0^K is a nonempty closed subset of M_0.

On the other hand, the mapping $g : W_0 \times \mathbb{R}^2 \to W_0$, $g(x, \lambda) = x - F(x, \lambda)$ for $(x, \lambda) \in W_0 \times \mathbb{R}^2$, is K-equivariant and the derivative $D_x g(x, \lambda)$ is an isomorphism for every $(x, \lambda) \in M$. Therefore, for a given $(x_0, \lambda_0) \in M_0^K$, $D_x g^K(x_0, \lambda_0) = D_x g(x_0, \lambda_0)|_{W_0^K}$ is also an isomorphism, where $g^K = g|_{W_0^K \times \mathbb{R}^2}$. By the implicit function theorem, there is a neighbourhood $\mathcal{U} \subseteq \mathcal{U}_{\lambda_0}$ of λ_0 in $\mathbb{R}^2$ and a mapping $p : \mathcal{U} \to W_0^K$ such that

$$g(p(\lambda), \lambda) = g^K(p(\lambda), \lambda) = 0 \quad \text{for} \quad \lambda \in \mathcal{U}.$$

So,

$$M_0 \cap (\mathcal{U}_{x_0} \times \mathcal{U}) \subseteq \{(p(\lambda), \lambda);\ \lambda \in \mathcal{U}\} \cap M_0 \subseteq M_0^K$$

and $\Gamma_{(p(\lambda),\lambda)} = K$ for $\lambda \in \mathcal{U}$. This implies that M_0^K is open in M_0. Consequently, $M_0^K = M_0$. This completes the proof. □

We consider the set of *singular points* $\Lambda := \{(x,\lambda) \in M;\ \text{Id} - D_xF(x,\lambda) \notin GL(W)\}$. Suppose that $(x_0,\lambda_0) \in M$ is an isolated singular point and $\eta : U_{\lambda_0} \to W_0$ is the C^1-map defined in (2.4.4). Let $U(r,\rho) = \{(x,\lambda) \in W \times \mathbf{R}^2; \|x - \eta(\lambda)\| < r, |\lambda - \lambda_0| < \rho\}$ be a *special neighbourhood* of (x_0,λ_0) with respect to the action of $G_0 := \Gamma_0 \times S^1$, where $\Gamma_0 = \Gamma_{(x_o,\lambda_o)} \subset \Gamma$ denotes the isotropy group of (x_o,λ_o), r and ρ are sufficiently small positive numbers. Suppose that $\theta : \overline{U(r,\rho)} \to \mathbf{R}$ is a G_0-invariant *complementing function* (with respect to $U(r,\rho)$), i.e. $\theta(\eta(\lambda),\lambda) < 0$ and $\theta(x,\lambda) > 0$ for $\|x - \eta(\lambda)\| = r$. As a particular example of a complementing function, we can take

$$\theta(x,\lambda) := \|\lambda - \lambda_0\|\Big(\|x - \eta(\lambda)\| - r\Big) + \|x - \eta(\lambda)\|. \tag{2.4.5}$$

Consequently, G_0-Deg $(f_\theta, U(r,\rho))$ is well defined.

Note that for every $\gamma \in \Gamma\backslash\Gamma_0$, $\gamma U(r,\rho)$ is a special neighbourhood of the singular point $(\gamma x_0,\lambda_0)$. Therefore, $GU(r,\rho)$ is composed of a finite disjoint union of special neighbourhoods of $G(x_0,\lambda_0)$. We extend the G_0-invariant function θ to a G-invariant function $\bar\theta : GU(r,\rho) \to \mathbf{R}$.

To associate the orbit of singular points $G(x_0,\lambda_0)$ with a local invariant, we need the formula for G-Deg $(f_{\bar\theta}, GU(r,\rho))$ given in the following:

Proposition 2.4.3 *Let $(x_0,\lambda_0) \in M$ be an isolated singular point such that $G_{(x_0,\lambda_0)} = G_0$, $U(r,\rho)$ a special neighbourhood of (x_0,λ_0) and $\theta : \overline{U}(r,\rho) \to \mathbf{R}$ a complementing function. Suppose $\bar\theta : GU(r,\rho) \to \mathbf{R}$ is a G-invariant extension of the complementing function θ. Then*

$$G\text{-}Deg\,(f_{\bar\theta}, GU(r,\rho)) = \Upsilon(G_0\text{-Deg}(f_\theta, U(r,\rho)), \tag{2.4.6}$$

where $\Upsilon : A_1(G_0) \to A_1(G)$ is the natural homomorphism given by $\Upsilon((H)) = (H)$.

Proof. We may assume, without loss of generality, that the G-mapping $f_{\bar{\theta}}$ is regular normal in $GU(r,\rho) = U(r,\rho) \cup \gamma_1 U(r,\rho) \cup \cdots \cup \gamma_l U(r,\rho)$ for some $\gamma_1, \ldots, \gamma_l \in \Gamma \setminus \Gamma_0$. We claim that the G_0-mapping f_θ is regular normal in $U(r,\rho)$. Indeed, let $p \in U(r,\rho)$ be such that $f_\theta(p) = 0$ and let $G_p = H$. Then $H_0 = (G_0)_p = G_0 \cap H$. Let S denote the slice at p to the orbit $G(p)$. The G-normality condition for $f_{\bar{\theta}}$ means that for every vector $v \in S$ such that $\|v\|$ is sufficiently small and $v \perp S^H$, we have $f_{\bar{\theta}}(v) = f_\theta(v) = v$. Since $H_0 \subseteq H$, $S^H \subseteq S^{H_0}$, every vector v perpendicular to S^{H_0} is also perpendicular to S^H, and thus f_θ is G_0-normal. Regarding the regularity condition, we notice that $Df_\theta^{H_0}(p) = Df_\theta^H(p) \oplus Id$. Therefore, $\operatorname{sign} Df_\theta^{H_0}(p) = \operatorname{sign} Df_\theta^H(p)$. This shows that f_θ satisfies the regularity condition.

We need to show that formula (2.4.6) is well defined. Indeed, if $(G_0)_p = H_0$ then it follows from the assumptions that $H_0 = \Gamma_0 \times S^1$, where Γ/Γ_0 is a finite set. It is clear that if $\gamma \in \Gamma \setminus \Gamma_0$, then for every $\tau \in S^1$, $(\gamma,\tau)p \notin U(r,\rho)$, and consequently $(\gamma,\tau) \notin H = G_p$. Therefore, $H = G_p = (G_0)_p = H_0$ and the the formula (2.4.6) follows. □

Now we can state and prove the following global bifurcation theorem.

Theorem 2.4.4 *Suppose that every singular point in M is isolated and M is complete. Let S denote the closure of the set of all nontrivial solutions to (2.4.3). Then for each bounded connected component C of S the set $GC \cap M$ is finite and is composed of a finite number of disjoint Γ-orbits*

$$GC \cap M = \bigcup_{i=1}^{q} \Gamma(x_i, \lambda_i).$$

Moreover, we have

$$\sum_{i=1}^{q} G\text{-Deg}(f_{\bar{\theta}_i}, GU_i) = 0, \tag{2.4.7}$$

where U_i denotes the special neighbourhood of (x_i, λ_i) and $\tilde{\theta}_i$ is an auxiliary function on GU_i.

Proof. The proof is standard. Nevertheless, we present it here for the sake of completeness. Since every point of $GC \cap M$ is a bifurcation point and singular points of M are isolated, the set $GC \cap M$ is finite. Put $GC \cap M = \Gamma(x_1, \lambda_1) \cup \cdots \cup \Gamma(x_q, \lambda_q)$ for some integer $q > 0$. Choose $r > \rho > 0$ sufficiently small so that for each $i = 1, 2, \ldots, q$, we can choose a special neighbourhood $U_i = U_i(r, \rho)$ of the point (x_i, λ_i) and $G\overline{U}_i \cap G\overline{U}_j = \varnothing$ if $i \neq j$. Let $U = GU_1 \cup GU_2 \cup \cdots \cup GU_q$. The set U is G-invariant and we can find $\Omega_1 \subset W \oplus \mathbf{R}^2$, an open bounded G-invariant subset such that $\overline{\Omega}_1 \cap M = \varnothing$, $\quad GC \backslash U \subset \Omega_1$ and $\partial\Omega_1 \setminus U \cap S = \varnothing$.

We put $\Omega = U \cup \Omega_1$. We construct $\theta : \overline{\Omega} \to \mathbf{R}$ such that

(i) $\theta(x, \lambda) = -|\lambda - \lambda_i|$ if $(x, \lambda) \in U_i \cap M$,

(ii) $\theta(x, \lambda) = r$ if $(x, \lambda) \in \overline{\Omega} \backslash U$.

Let $f_\theta : \overline{\Omega} \to W \times \mathbf{R}$ be defined by

$$f_\theta(x, \lambda) \triangleq \big(x - F(x, \lambda), \theta(x, \lambda)\big), \quad (x, \lambda) \in \overline{\Omega}.$$

Then $f_\theta^{-1}(0) \subseteq GC$ and hence G-Deg (f_θ, Ω) is well defined.

We now consider the homotopy $H : \overline{\Omega} \times [0, 1] \to W \times \mathbf{R}$ given by

$$H(x, \lambda, t) = \big(x - F(x, \lambda), (1 - t)\theta(x, \lambda) + t\rho\big), \quad (x, \lambda, t) \in \overline{\Omega} \times [0, 1].$$

By (i)–(ii), $H(x, \lambda, t) \neq 0$ for all $(x, \lambda, t) \in \partial\Omega \times [0, 1]$, thus H is an Ω-admissible homotopy. Since $H(x, \lambda, 0) = f_\theta(x, \lambda)$ and $H(x, \lambda, 1) = \big(x - F(x, \lambda), \rho\big) \neq 0$ for all $(x, \lambda) \in \overline{\Omega}$, we have G-Deg $(f_\theta, \Omega) = 0$. By (ii), $F_\theta^{-1}(0) \subset GC \cap U$. Therefore, by the excision and additivity properties, we have

$$G\text{-Deg}\,(f_\theta, \Omega) = \sum_{i=1}^{q} G\text{-Deg}(f_{\theta_i}, GU_i).$$

□

Let us remark that the local invariants G-Deg(f_{θ_i}, GU_i) in formula (2.4.7) can be computed by using Proposition 2.4.3 and Theorem 2.3.2 (in the case where Γ is an abelian Lie group) (or Theorem 4.3.1 in Chapter 4, in the case where Γ is a finite group). Indeed, assume that $p_i = (x_i, \lambda_i)$ is an isolated singular point of (2.4.3) such that $G_i = G_p$, $U(r_i, \rho_i)$ is a special neighbourhood of p_i, and $\theta_i : U(r_i, \rho_i) \to \mathbf{R}$ is the complementing function given by (2.4.5). Then by taking the map $\theta_i + \varepsilon$, where $\varepsilon > 0$ is a sufficiently small number, we obtain a G_i-mapping $f_{\theta_i} : U(r_i, \rho_i) \to \mathbf{R}$ which satisfies hypothesis (A) of Section 3. Consequently, it is possible to compute the degree G_i-Deg$(f_{\theta_i}, U(r_i, \rho_i))$ by using the winding elements $\mu(x_i, \lambda_i)$ given by (2.3.3) and the elements $\nu(x_i, \lambda_i)$ given by (2.3.4). It is also important to notice that G-Deg$(f_{\theta_i}, GU_i) \neq 0$ if and only if there is a nonzero winding number $\mu_{\beta i}(x_i, \lambda_i)$, but the relations between these winding numbers can be quite complicated. In the case where G is an abelian group and the manifold M is contained in $W^G \times \mathbf{R}^2$, these relations become relatively simple. Suppose that we have the following bifurcation points $(x_1, \lambda_1), \ldots, (x_q, \lambda_q)$ belonging to the closure of a bounded branch of nontrivial solutions of (2.4.3). Then we can express these relations in the following Corollary:

Corollary 2.4.5 *Let G be an abelian group. Suppose that $M \subset W^G \oplus \mathbf{R}^2$, all the singular points in M are isolated and M is complete. Let S denote the closure of the set of all non-trivial solutions to (2.4.3). Then for each bounded connected* *component $\mathcal{C}$ of $\mathcal{S}$, we have $G\mathcal{C} = \mathcal{C}$, $G\mathcal{C} \cap M$ is finite:*

$$\mathcal{C} \cap M = \{(x_1, \lambda_1), \ldots, (x_q, \lambda_q)\},$$

and

$$\sum_{i=1}^{q} \varepsilon(x_i, \lambda_i)\mu_{ni}(x_i, \lambda_i) = 0$$

for every $n > 0$, $i \geq 1$.

Proof. By Theorem 2.4.4, we have

$$\sum_{i=1}^{q} G\text{-Deg}(f_{\theta_i}, U_i) = 0,$$

where U_i is a special neighbourhood of (x_i, λ_i) and θ_i is an auxiliary function on U_i. It follows from Corollary 2.3.4 that

$$0 = \sum_{i=1}^{q} G\text{-Deg}(f_{\theta_i}, U_i) = \sum_{i=1}^{q} \varepsilon(x_i, \lambda_i)\big(\xi^*(x_i, \lambda_i) + \tilde{\xi}(x_i, \lambda_i)\big).$$

Consequently

$$0 = p\Big(\sum_{i=1}^{q} \varepsilon(x_i, \lambda_i)\big(\xi^*(x_i, \lambda_i) + \tilde{\xi}(x_i, \lambda_i)\big)\Big) = \sum_{i=1}^{q} \varepsilon(x_i, \lambda_i)\tilde{\xi}(x_i, \lambda_i),$$

where $p : A_1(G) \to \tilde{A}_1(G)$ is the natural homomorphism, and the conclusion follows. □

Corollary 2.4.5 can be further refined as follows:

Corollary 2.4.6 *Suppose that $M \subseteq W^G \otimes \mathbf{R}^2$, all singular points in M are isolated and M is complete. Let $\mathcal{S}^{ni}$ denote the closure of the set of all nontrivial solutions in $W^{G_{ni}} =: W^{ni}$. Then for each bounded connected component $\mathcal{C}^{ni}$ of $\mathcal{S}^{ni}$,* *we have $GC^{ni} = C^{ni}, C^{ni} \cap M$ is a finite set:*

$$C^{ni} \cap M = \{(v_1, \lambda_1), (v_2, \lambda_2), \ldots, (v_q, \lambda_q)\},$$

and

$$\sum_{k=1}^{q} \varepsilon(v_k, \lambda_k) \cdot \mu_{ni}(v_k, \lambda_k) = 0.$$

Proof. Let $f_\theta^{ni} = f_\theta|_{W^{ni} \times \mathbf{R}^2}$. Then $f_\theta^{ni} : W^{ni} \times \mathbf{R}^2 \to W^{ni}$, W^{ni} is G-invariant and $M \subseteq (W^{ni})^G \times \mathbf{R}^2$. Therefore, by Theorem 2.4.5, we have

$$\sum_{k=1}^{q} G\text{-Deg}\,(f_\theta^{ni}, U_k^{ni}) = 0,$$

where U_k^{ni} is a special neighbourhood in $W^{ni} \times \mathbf{R}^2$ of (v_k, λ_k). It is easy to observe from the construction of the G-degree that the (G_{ni})-component of G-Deg $(f_\theta^{ni}, U_k^{ni})$ is the same as that of G-Deg (f_θ, U_k). Therefore, we have

$$\sum_{k=1}^{q} \varepsilon(v_k, \lambda_k)\mu_{ni}(v_k, \lambda_k) = 0.$$

This completes the proof. □

In the case where Γ is a finite group, global relations between the winding numbers of various bifurcation points can be established in a similar way, using our computations in Chapter 4.

Finally we point out that the symmetric bifurcation theory outlined in this section can be extended, in principle, to the case of higher dimensional parameter spaces.

Chapter Three

Normal Bifurcation Problems

3.1 Introduction

In this chapter we propose a concept of ***normality*** for equivariant bifurcation problems. Our approach is based on the idea of approximating an equivariant map with a map having all the orbit types of its zeros separated. More precisely, let W be an orthogonal representation of a compact Lie group G, and $\tilde{f} : W \times \mathbf{R}^n \to W$ be a continuous G-equivariant map such that it has no zeros on a boundary of an invariant open bounded subset Ω of W. Then we recall from Chapter 1 that $\tilde{f}$ is called ***normal*** in Ω if it satisfies the following normality condition:

for every $x \in \tilde{f}^{-1}(0) \cap \Omega$ and $H = G_x$ there exist an $\varepsilon_x > 0$ such that $\tilde{f}(x+h) = h$ for all vectors h normal to the submanifold $(W \times \mathbf{R}^n)_{(H)}$ at the point x with $\|h\| < \varepsilon_x$.

This condition imposes some kind of obstacle between different types of zeros of $\tilde{f}$ preventing orbits of zeros with larger orbit type from collapsing onto orbits with smaller orbit type.

We will extend the notion of normality to a class of equivariant bifurcation problems. We consider a complete submanifold M of the representation $W^G \oplus \mathbf{R}^n$ of dimension n, and let $f : W \oplus \mathbf{R}^n \to W$ be an equivariant map such that $M \subset f^{-1}(0)$. We are interested in describing branches of solutions, and their possible orbit types, bifurcating from points in M. For this purpose we impose on f an additional *normality* condition which will prevent, in a similar way as in the case of a normal map, the branches of solutions with different orbit types, bifurcating from M, from mixing together. This condition can be expressed as follows:

for every $x \in f^{-1}(0) \setminus M$ and $H = G_x$ there exists $\varepsilon_x > 0$ such that $f(x+h) = h$, where the vector h is normal to the submanifold $(W \oplus \mathbf{R}^n)_{(H)}$ at the point x with $\|h\| < \varepsilon_x$.

One of the main results of this chapter is that the set of such normal bifurcation maps is in fact dense in the set of all bifurcation maps. In other words a bifurcation problem may be "corrected" slightly to a *normal bifurcation problem* which will produce the largest possible number of bifurcation branches with different orbit types.

In order to study the normal bifurcation we apply the concept of equivariant degree developed in the previous chapters to establish several branching results for local and global normal bifurcation problems.

In Section 3.2 we introduce the notion of *normal bifurcation map* and present a proof of the Normal Approximation Theorem, which is next used in Section 3.3 to define the notion of *α-essential bifurcation*. Next we prove the Equivariant Branching Lemma and the global bifurcation results for isolated compact M-singular sets. Finally, in Section 3.4 we applied the obtained results to a problem of steady-state bifurcation with $SO(3)$ symmetry. We show that the equivariant degree detects the orbit types of normal bifurcation and it may also be applied in the case of reducible representations of $SO(3)$. It should be emphasized that this method permits detection of branches of nontrivial solutions not only of maximal orbit type but also those with submaximal orbit types.

3.2 Normal Approximation for Equivariant Bifurcation

We again assume that W is an orthogonal representation of the compact Lie group G. We denote by V the representation $W \oplus \mathbf{R}^n$.

In what follows let M be an n-dimensional smooth complete submanifold of $W^G \oplus \mathbf{R}^n$. For an element $x \in M$ we denote by T_xM the tangent space to M at x and by N_xM the normal space to M at x, i.e. we have $W \oplus \mathbf{R}^n = T_xM \oplus N_xM$.

We denote by $\mathcal{C}_M$ the class of all continuous equivariant maps $f : W \oplus \mathbf{R}^n \to W$ such that

(i) f is differentiable at every point $x \in M$ and the derivative $Df(x)$ depends continuously on $x \in M$;

(ii) $M \subset f^{-1}(0)$, i.e. for all $x \in M$ we have $f(x) = 0$.

Let $f \in \mathcal{C}_M$, we define the set

$$\Lambda(f) := \{x \in M : D_v f(x) := Df(x)_{|N_xM} : N_xM \to W \text{ is not an isomorphism}\},$$

which we will call the set of *M-singular points* of f.

By the class of *equivariant bifurcation maps* on M we mean

$$\mathcal{C}^1_M := \{f \in \mathcal{C}_M : f \text{ is of class } C^1 \text{ on the set } W \oplus \mathbf{R}^n \setminus \Lambda(f)\}.$$

Suppose that $f \in \mathcal{C}^1_M$. The condition (ii) implies that the manifold M is contained in the solution set of the equation

$$f(x) = 0, \quad x \in W \oplus \mathbf{R}^n, \tag{3.2.1}$$

and therefore we will call the points from M the *trivial solutions* of the equation (3.2.1). All other solutions of (3.2.1) will be called *nontrivial.* We will also say that a point $x_o \in M$ is a *bifurcation point* of (3.2.1), if in every neighbourhood of x_o there exists a nontrivial solution of (3.2.1). We will denote be $\mathcal{B}(f)$ the subset of M

of all bifurcation points of (3.2.1).

It follows from the implicit function theorem that if $x_o \in M$ is a bifurcation point of (3.2.1), then $x_o \in \Lambda(f)$, i.e. $\mathcal{B}(f) \subset \Lambda(f)$. We are interested in describing the problem of bifurcation of branches of nontrivial solutions for (3.2.1) from $x_o \in \Lambda(f)$, where $f \in \mathcal{C}^1_M$.

We introduce the following definition of an equivariant normal bifurcation map on M:

Definition 3.2.1 Assume that $A \subset W \oplus \mathbf{R}^n$ is a compact invariant subset and let $f \in \mathcal{C}^1_M$. We say that f is an *(equivariant) normal bifurcation map*, or just simply a *normal b-map*, if for every $x \in [f^{-1}(0) \setminus M] \cap A$ (i.e. for every nontrivial solution x of (3.2.1) from A), there exists $\varepsilon_x > 0$ such that the following α-normality condition, for $\alpha = (G_x)$, is satisfied:

$f(x+h) = h$ for all $h \in N_x(V_\alpha)$ with $\|h\| < \varepsilon_x$.

We will denote by $\mathcal{G}_M(A)$ the set of all normal b-maps.

Since the space $\mathcal{C}^1_M$ is a subset of the complete locally convex space $C(W \oplus \mathbb{R}^n; W)$ of continuous maps φ from $W \oplus \mathbf{R}^n$ to W, equipped with the topology induced by seminorms $p_K(\varphi) = \sup_{x\in K} \|\varphi(x)\|$, where K is a compact subset of $W \oplus \mathbf{R}^n$, $\mathcal{C}^1_M$ has the induced topology.

The following result is the main result of this section:

Theorem 3.2.2 (Normal Approximation Theorem) *Assume that $A \subset W \oplus \mathbb{R}^n$ is a compact invariant subset and let $f \in \mathcal{C}^1_M$. Then for every $\eta > 0$ there exists $\widetilde{f} \in \mathcal{G}_M(A)$ such that*

(i) $\sup_{x\in W\oplus\mathbb{R}^n} \|\widetilde{f}(x) - f(x)\| < \eta$,

(ii) $\Lambda(f) = \Lambda(\widetilde{f})$,

(iii) $\mathcal{B}(f) = \mathcal{B}(\widetilde{f})$.

Consequently, the set $\mathcal{G}_M(A)$ is dense in C^1_M.

Proof. In our proof we use a finite inductive procedure on the orbit types in $V \setminus M$. We construct an approximation of f that at each step "increases" its property of being normal. At the n-th step the approximation that we construct is α-normal for every $\alpha \leq \alpha_n$ (α_n being the orbit type considered at the n-th step).

We define the following continuous function $\varepsilon : V \to \mathbf{R}$, $\varepsilon(x) := [\text{dist}\,(x, M)]^2$. Since M is an invariant manifold, the function ε is also invariant. We put

$$\varepsilon_1(x) = \frac{\varepsilon(x)}{n_{11}},$$

where n_{11} is a positive integer which will be specified later. We put $Z_1 := [f^{-1}(0) \setminus M] \cap A$ and we define

$$D_1 = \{x \in V \setminus M : \text{dist}\,(x, Z_1) < \varepsilon_1(x)\},$$
$$L_1 = \{x \in V \setminus M : \text{dist}\,(x, Z_1) < 2\varepsilon_1(x)\},$$
$$A_1 = \{x \in V \setminus M : \text{dist}\,(x, Z_1) < \frac{\varepsilon_1(x)}{2}\}.$$

It is clear that D_1, L_1 and A_1 are open subsets of $V \setminus M$. Moreover, if $x \in \overline{L_1} \cap M$, then x is a bifurcation point of (3.2.1). Indeed, let $x_n \in L_1$ be a sequence such that $x_n \to x$. By definition $\text{dist}\,(x_n, Z_1) < \frac{2[\text{dist}\,(x_n, M)]^2}{n_{11}} \to 0$, hence there is a sequence $y_n \in Z_1$ such that $\|x_n - y_n\| \to 0$, and therefore $\|x - y_n\| \leq \|x - x_n\| + \|x_n - y_n\| \to 0$, which implies that x is a bifurcation point.

We extend the partial order in the set $\mathcal{J}(V \setminus M)$ of all possible orbit types in $V \setminus M$ to a total order $\alpha_1 < \alpha_2 < \cdots < \alpha_k < \alpha_{k+1} < \cdots < \alpha_m$, and consider the minimal orbit type $\alpha = \alpha_1$. Note that due to [37] the number of possible orbit types is finite.

Put $\widetilde{D}_1 := D_1 \cap V_\alpha \subset L_1$ and $\widetilde{A}_1 = A_1 \cap V_\alpha \subset L_1$. If $\widetilde{A}_1 = \varnothing$, then we put $f_1(x) = f(x)$. Assume therefore $\widetilde{A}_1 \neq \varnothing$. Since the function $x \mapsto \text{dist}\,(x, Z_1)$ is an invariant function, the open sets D_1, L_1 and A_1 are also invariant, and consequently $\widetilde{D}_1$ is an open invariant subset of V_α. Thus, there exists a continuous invariant function $\nu_1 : \widetilde{D}_1 \to \mathbf{R}_+$ such that $\nu_1(x) \leq \frac{\varepsilon(x)}{2}$ and the mapping $\mu : N(V_\alpha) \to V$, $\mu(v,w) = v + w$, restricted to the set $N(\widetilde{D}_1, \nu_1)$ is a G-imbedding into L_1. We put $N_1 := \mu(N(\widetilde{D}_1, \nu_1))$. Let $\gamma_1 : V \setminus M \to [0,1]$ be an invariant C^∞-function such that $\gamma_1(x) = 1$ for $x \in \mu(N(\widetilde{A}_1, \frac{\nu_1}{2}))$ and $\gamma_1(x) = 0$ for $x \in V \setminus (M \cup N_1)$. We define $f_1 : V \to W$ by

$$f_1(x) := \begin{cases} f(x), & \text{for } x \in V \setminus N_1; \\ \gamma_1(x)(f(v) + w) + (1 - \gamma_1(x))f(x), & \text{for } x = v + w \in N_1, \end{cases}$$

where $x = v + w$ denotes the decomposition $v \in \widetilde{D}_1$ and $w \in N_v(\widetilde{D}_1)$).

We put $\delta_1 := \max_{x \in \widetilde{D}_1} \nu_1(x)$. We may now assume that the number n_{11} is chosen to be sufficiently large such that the corresponding function $\nu_1(x)$ may also be chosen in such a way that

$$\sup_{v+w \in N_1} \|f(v) - f(v+w)\| \leq \frac{\eta}{m} - \delta_1.$$

This is evidently possible, because the above estimation is done on a relatively compact (in V) neighbourhood of the set Z_1, on which the function f is zero, therefore the existence of such a number n_{11} follows from the continuity of f. Consequently,

$$\begin{aligned} \sup_{x \in V} \|f_1(x) - f(x)\| &= \sup_{x \in N_1} \|f_1(x) - f(x)\| \\ &= \sup_{x = v+w \in N_1} \|\gamma_1(x)(f(v) - f(x)) + \gamma_1(x)w\| \\ &\leq \sup_{x = v+w \in N_1} \|f(v) - f(x)\| + \delta_1 \\ \leq \frac{\eta}{m} - \delta_1 + \delta_1 = \frac{\eta}{m}. \end{aligned}$$

This implies that $f_1 : V \to W$ is a well defined $\frac{\eta}{m}$-approximation of f. It is also clear that f_1 is a C^1-function on $V \setminus M$. Moreover f_1 is invariant and satisfies the following conditions; *for every* $x \in [f^{-1}(0) \setminus M] \cap A$ *such that* $(G_x) = \alpha$, *for every*

$w \in N_x V_\alpha$, $\|w\| \leq \frac{\nu_1(x)}{2}$, *we have* $f_1(x+w) = f_1(x) + w$. On the other hand, the set $[f_1^{-1}(0) \setminus M] \cap V_\alpha \cap A$ is contained in A_1. That means that in the first step of our construction we have "corrected" the map f to a new map f_1, such that, f_1 satisfies α_1-normality condition.

We need only to check that $f_1 \in C^1_M$. We will show that f_1 is differentiable at every point $x \in M$ and that the derivative $Df_1(x)$ depends continuously on $x \in M$. It is clear that $f_1(x) = f(x)$ for all $x \in \overline{Z_1} \cap M$, and therefore we need only to check the differentiability at the points $x \in \overline{Z_1} \cap M$. We will show that for $x \in \overline{Z_1} \cap M$ we have $Df_1(x) = Df(x)$. For this purpose, assume that $x + h \in N_1$ and let $x + h = v + w$, where $w \in N_v \widetilde{D}_1$. Then we have

$$
\begin{aligned}
&\|f_1(x+h) - Df(x)h\| \\
&= \|\gamma_1(x+h)\big(f(v) + w\big) + (1 - \gamma_1(x+h)f(x+h) - Df(x)h\| \\
&\leq \|f(x+h) - Df(x)h\| + \|\gamma_1(x+h)\big(f(v) + w - f(x+h)\big)\| \\
&\leq \|f(x+h) - Df(x)h\| + \|f(v) + w - f(x+h)\| \\
&\leq \|f(x+h) - Df(x)h\| \\
&+ \|f(v) - Df(x)(v-x) - \big(f(x+h) - Df(x)h\big) + Df(x)(v - x - h)\| + \|w\| \\
&\leq 2\|f(x+h) - Df(x)h\| + \|f(v) - Df(x)(v-x)\| + \|Df(x)\|\|w\| + \|w\| \\
&\leq 2o(\|h\|) + o(\|v - x\|) + \|Df(x)\|\|w\| + \|w\|.
\end{aligned}
$$

Let $H = G_v$. Since $x \in V^G$, $x - v \in V_H \subset V_\alpha$, thus $x - v$ is orthogonal to w. That implies that $\|x - v\|^2 + \|w\|^2 = \|h\|^2$. We may assume that $\|h\| < 1$, then $\operatorname{dist}(v, M) \leq \|x - v\| < 1$, as well as $\|w\| < 1$. Let $y \in \overline{M}$ be such that $\operatorname{dist}(x+h, M) = \|x + h - y\|$. Then

$$\|v - y\| \leq \|x + h - y\| + \|w\| = \operatorname{dist}(x+h, M) + \|w\|,$$

thus

$$\operatorname{dist}(v,M) \le \inf_{y\in M} \|v-y\| \le \operatorname{dist}(x+h,M) + \|w\|.$$

Since $\|w\| \le \frac{[\operatorname{dist}(v,M)]^2}{2} \le \frac{\operatorname{dist}(v,M)}{2}$, therefore $\operatorname{dist}(v,M) \le \operatorname{dist}(x+h,M) + \frac{\operatorname{dist}(v,M)}{2}$, hence $\operatorname{dist}(v,M) \le 2\operatorname{dist}(x+h,M)$. Consequently

$$\|w\| \le \frac{[\operatorname{dist}(v,M)]^2}{2} \le 2[\operatorname{dist}(x+h,M)]^2 \le 2\|h\|^2,$$

and thus

$$\|f_1(x+h) - Df(x)h\| = o(\|h\|).$$

This means that f_1 is differentiable at every point $x \in M$. Since $Df_1(x) = Df(x)$ and $f \in \mathcal{C}^1_M$, thus $f_1 \in \mathcal{C}^1_M$. Moreover, it is clear that $\Lambda(f_1) = \Lambda(f)$.

We put $\Omega_1 := \mu(N(\widetilde{A}_1, \frac{\mu_1}{2}))$.

In the k-th step we consider the set

$$Z_k := \left(f_{k-1}^{-1}(0) \setminus (M \cup \Omega_1 \cup \cdots \cup \Omega_{k-1})\right) \cap A,$$

which has to be disjoint from all the sets $\overline{\Omega_1}$, $\overline{\Omega_2}$, ..., $\overline{\Omega_{k-1}}$. Moreover, all the orbit types in Z_k are bigger than α_{k-1}. Next, we define the function

$$\varepsilon_k(x) := \frac{\varepsilon(x)}{n_{k1}} + \sum_{i=2}^{k} \frac{\operatorname{dist}(x, \Omega_{i-1})}{n_{ki}},$$

where the integers $n_{k1}, \ldots, n_{kk+1}$ are chosen to be sufficiently large. We put

$$D_k := \{x \in V \setminus M : \operatorname{dist}(x, Z_k) < \varepsilon_k(x)\},$$

$$L_k := \{x \in V \setminus M : \operatorname{dist}(x, Z_k) < 2\varepsilon_k(x)\},$$

$$A_k := \{x \in V \setminus M : \operatorname{dist}(x, Z_k) < \frac{\varepsilon_k(x)}{2}\},$$

and for $\alpha = \alpha_k$ we consider the sets $\widetilde{D}_k := D_k \cap V_\alpha \subset L_k$ and $\widetilde{A}_k := A_k \cap V_\alpha \subset$

L_k. If $\widetilde{A}_k = \varnothing$ then we put $f_k = f_{k-1}$, otherwise we find a continuous invariant function $\nu_k : \widetilde{D}_k \to \mathbb{R}_+$ such that $\nu_k(x) \le \frac{\varepsilon(x)}{2}$ and the mapping $\mu : N(V_\alpha) \to V$, $\mu(v,w) = v + w$, restricted to the set $N(\widetilde{D}_k, \nu_k)$ is an G-imbedding into L_k. We put $N_k := \mu(N(\widetilde{D}_k, \nu_k))$. Let $\gamma_k : V \setminus M \to [0,1]$ be an invariant C^∞-function such that $\gamma_k(x) = 1$ for $x \in \mu(N(\widetilde{A}_k, \frac{\nu_k}{2}))$ and $\gamma_k(x) = 0$ for $x \in V \setminus (M \cup N_k)$. We define $f_k : V \to W$ by

$$f_k(x) := \begin{cases} f_{k-1}(x), & \text{if } x \in V \setminus N_k; \\ \gamma_k(x)\big(f_{k-1}(v) + w\big) + \big(1 - \gamma_k(x)\big) f_{k-1}(x), & \text{if } x = v + w \in N_k. \end{cases}$$

We put $\delta_k = \max_{x \in \widetilde{D}_k} \nu_k(x)$, and we may assume that

$$\sup_{v+w \in N_k} \|f_{k-1}(v) - f_{k-1}(v+w)\| < \frac{\eta}{m} - \delta_k.$$

Consequently,

$$\begin{aligned} \sup_{x \in V} \|f_k(x) - f(x)\| &\le \sup_{x \in V} \|f_k(x) - f_{k-1}(x)\| + \tfrac{(k-1)\eta}{m} \\ &\le \sup_{x = v+w \in N_k} \|f_{k-1}(v) - f_{k-1}(x)\| + \delta_k + \tfrac{(k-1)\eta}{m} \\ &\le \tfrac{k\eta}{m}. \end{aligned}$$

Next, by similar arguments as before, $f_k \in C^1_M$ and by induction assumption f_k satisfies the α-normality condition on A for all $\alpha = \alpha_1, \alpha_2, \ldots, \alpha_k$. Moreover, $Df_k(x) = Df(x)$ for all $x \in M$ and thus $\Lambda(f_k) = \Lambda(f)$.

For $m = k$, we put $\widetilde{f} := f_m$, and it is now clear that

$$\sup_{x \in V} \|\widetilde{f}(x) - f(x)\| \le m \frac{\eta}{m} = \eta,$$

and that $\widetilde{f}$ satisfies all the required properties. □

3.3 Equivariant Branching Lemma

We assume that $f \in \mathcal{C}^1_M$ and we consider the following bifurcation equation on M

$$f(x) = 0, \quad x \in W \oplus \mathbf{R}^n. \tag{3.3.1}$$

As it was already pointed out in Section 3.2 the set of bifurcation points of the equation (3.3.1) is contained in the set $\Lambda(f)$ of M-singular points of f. Suppose that $K \subset \Lambda(f)$ is a compact component of $\Lambda(f)$. We want to use the equivariant degree to determine the existence of a bifurcation point in K. For this purpose we consider (using Tietze-Gleason thearem) an open bounded neighbourhood D of K in M such that $\overline{D} \cap \Lambda(f) = K$. Since $\overline{D}$ is a compact subset of M, there exists a sufficiently small $\varepsilon > 0$ such that the restriction of the map $\mu : N(M) \to W \oplus \mathbf{R}^n$, $\mu(x, v) = x + v$, to the set $N(\overline{D}, \varepsilon) = \{(x, v) \in N(M);\ x \in \overline{D},\ \ \|v\| \le \varepsilon\}$ is an equivariant imbedding. As we may choose $\varepsilon > 0$ to be an arbitrary small positive number, we may also assume that for every $x \in \partial D$ and $0 \le \|v\| \le \varepsilon$ we have $f(x + v) \neq 0$. Indeed, $\partial D \cap \Lambda(f) = \varnothing$ thus all the points x from ∂D are not M-singular points of (3.3.1). We put $\overline{\mathcal{U}} := \mu(N(\overline{D}, \varepsilon))$, and we will call the set $\overline{\mathcal{U}}$ a *special neighbourhood* of the compact component K of $\Lambda(f)$, which we will also call an *isolated compact M-singular set.* Let $\varphi : \overline{\mathcal{U}} \to \mathbf{R}$ be an invariant continuous function such that $\varphi(x) < 0$ for all $x \in \overline{D}$ and $\varphi(x) > 0$ for all $x = u + v$ satisfying $u \in \overline{D}$ and $\|v\| = \varepsilon$. We will call such function φ, following [30,31], a *complementing function.*

In order to study the problem of existence of a bifurcation point in the set K we define the map $f_\varphi : \overline{\mathcal{U}} \to W \oplus \mathbf{R}$ by

$$f_\varphi(x) = (f(x), \varphi(x)), \quad x \in \overline{\mathcal{U}}.$$

It is clear that $f_\varphi(x) \neq 0$ for all $x \in \partial\mathcal{U}$, therefore, the G-degree $G\text{-Deg}\,(f_\varphi, \mathcal{U})$ is well defined. We have the following result.

Proposition 3.3.1 *The G-degree* $G\text{-Deg}(f_\varphi, \mathcal{U})$ *does not depend on the choice of the special neighbourhood* $\mathcal{U}$ *of* K *nor on the choice of the complementing function* φ*. Moreover, if* $G\text{-Deg}(f_\varphi, \mathcal{U}) \neq 0$ *then the set* K *contains a bifurcation point of (3.3.1).*

Proof. Suppose that $\varphi_0, \varphi_1 : \overline{\mathcal{U}} \to \mathbf{R}$ are two complementing functions. Then for every $t \in [0,1]$ the function

$$\varphi_t(x) = t\varphi_1(x) + (1-t)\varphi_0(x)$$

is also a complementing function, thus it follows from the homotopy property that

$$G\text{-Deg}(f_{\varphi_0}, \mathcal{U}) = G\text{-Deg}(f_{\varphi_t}, \mathcal{U}) = G\text{-Deg}(f_{\varphi_1}, \mathcal{U}).$$

Suppose now that $\mathcal{U}_0$, $\mathcal{U}_1$ are two special neighbourhoods of K such that $\overline{\mathcal{U}}_i := \mu(N(\overline{D}_i, \varepsilon_i))$, where $\overline{D}_0 \subset D_1$. Since the compact set $\overline{D}_1 \setminus D_0$ contains no M-singular point, there exists $\varepsilon > 0$ such that $\varepsilon < \min\{\varepsilon_0, \varepsilon_1\}$ and the set $A := \mu(N(\overline{D}_1 \setminus D_0, \varepsilon))$ contains no nontrivial solution of (3.3.1). Let $\varphi : W \oplus \mathbf{R}^n \to \mathbf{R}$ be an invariant function such that $\varphi(u+v) > 0$ for $x \in \overline{\mathcal{U}}_1$, $x = u + v$, $u \in \overline{D}_1$ and $\|v\| > \frac{\varepsilon}{2}$, and $\varphi(x) < 0$ for $x \in \overline{D}_1$. Then φ is a good complementing function for both special neighbourhoods $\mathcal{U}_0$ and $\mathcal{U}_1$. Therefore, by using twice the excision property we obtain the following equalities

$$\begin{aligned} G\text{-Deg}(f_\varphi, \mathcal{U}_1) &= G\text{-Deg}\Big(f_\varphi, \mu(N(D_1, \varepsilon))\Big) \\ &= G\text{-Deg}\Big(f_\varphi, \mu(N(D_0, \varepsilon))\Big) \\ &= G\text{-Deg}(f_\varphi, \mathcal{U}_0), \end{aligned}$$

thus $G\text{-Deg}(f_\varphi, \mathcal{U})$ does not depend on the choice of a special neighbourhood $\mathcal{U}$ of the set K.

Suppose now that $G\text{-Deg}(f_\varphi, \mathcal{U}) \neq 0$ and let $\mathcal{V}$ be an arbitrary open neighbourhood of the set K in $W \oplus \mathbf{R}^n$. Then there exists another special neighbourhood $\mathcal{U}_1$ of K such that $\mathcal{U}_1 \subset \mathcal{U} \cap \mathcal{V}$. Assume that $\varphi_1 : \overline{\mathcal{U}}_1 \to \mathbf{R}$ is a complementing function. Then $0 \neq G\text{-Deg}(f_\varphi, \mathcal{U}) = G\text{-Deg}(f_{\varphi_1}, \mathcal{U}_1)$, hence by the existence property there exists a solution to the system

$$\begin{cases} f(x) = 0; \\ \varphi_1(x) = 0, \end{cases}$$

which is evidently a nontrivial solution of (3.3.1). This means that in an arbitrary small neighbourhood of K there is a nontrivial solution of (3.3.1). Since K is compact, there exists a bifurcation point in K. □

Definition 3.3.2 Let $\alpha \in \mathcal{J}(W \oplus \mathbf{R}^n)$. We say that an isolated compact M-singular set $K \subset \Lambda(f)$ is an *α-essential bifurcation set of (3.3.1)* if for every bounded open neighbourhood U of K in $W \oplus \mathbf{R}^n$, there is $\eta > 0$ such that for all $\tilde{f} \in \mathcal{G}_M(\overline{U})$ satisfying

(i) $\sup_{x \in W \oplus \mathbf{R}^n} \|f(x) - \tilde{f}(x)\| < \eta$;

(ii) $\Lambda(\tilde{f}) = \Lambda(f)$;

the equation

$$\tilde{f}(x) = 0, \quad x \in W \oplus \mathbf{R}^n \tag{3.3.2}$$

has a sequence of nontrivial solutions $\{x_n\}$ such that

(1) $\text{dist}\,(x_n, K) \to 0$ as $n \to \infty$;

(2) $(G_{x_n}) = \alpha$ for all $n = 1, 2, \ldots$.

Lemma 3.3.3 *If $K \subset \Lambda(f)$ is an α-essential bifurcation set of (3.3.1), then K contains a bifurcation point.*

Proof. If K contains no bifurcation points, then there exists a neighbourhood U of K in $W \oplus \mathbf{R}^n$ which does not contain a nontrivial solution of (3.3.1). Then the mapping $\widetilde{f} = f$ belongs to $\mathcal{G}_M(\overline{U})$, and therefore K is not an α-essential bifurcation set of (3.3.1) for all $\alpha \in \mathcal{J}(W \oplus \mathbf{R}^n)$. □

Theorem 3.3.4 (Branching Lemma) *If G-Deg$(f_\varphi, \mathcal{U}) = \sum_\alpha n_\alpha \alpha$ is such that $n_\alpha \neq 0$, where $\alpha \in \Phi_{n-1}(G)$, then K is an α-essential bifurcation set of (3.3.1), i.e there exists $\eta > 0$ such that every $\widetilde{f} \in \mathcal{G}_M(\overline{\mathcal{U}})$ satisfying $\sup_{x \in W\oplus\mathbf{R}^n} \|f(x) - \widetilde{f}(x)\| < \eta$ and $\Lambda(\widetilde{f}) = \Lambda(f)$, has a continuum C_α of nontrivial solutions bifurcating from the set K, such that $(G_x) = \alpha$ for all $x \in C_\alpha$. We will call C_α an α-branch of nontrivial solutions bifurcating from K.*

Proof. Let $\mu(N(D,\varepsilon))$ be the special neighbourhood $\mathcal{U}$ of the isolated compact M-singular set K and let $\varphi : \mathcal{U} \to \mathbf{R}$ be a complementing function on $\mathcal{U}$. Assume that

$$0 < 2\eta < \inf\{\|f(x)\| : x = u + v,\ u \in \partial D,\ \|v\| \le \varepsilon,\ \varphi(x) = 0\}.$$

Then for every $\widetilde{f} \in \mathcal{G}_M(\mathcal{U})$ such that $\Lambda(\widetilde{f}) = \Lambda(f)$ and $\sup_{x\in\mathcal{U}} \|\widetilde{f}(x) - f(x)\| < \eta$, the homotopy

$$h_\varphi(x,t) := (t\widetilde{f}(x) + (1-t)f(x), \varphi(x)), \quad x \in \overline{\mathcal{U}},\ t \in [0,1],$$

has no zero in $\partial\mathcal{U}$. Indeed, for all $x = u+v$ such that $u \in \partial D$, $\|v\| \le \varepsilon$ and $\varphi(x) = 0$ we have

$$\begin{aligned}\|t\widetilde{f}(x) + (1-t)f(x)\| &\ge \|f(x)\| - t\|\widetilde{f}(x) - f(x)\| \\ &\ge 2\eta - \eta > 0,\end{aligned}$$

thus

$$G\text{-Deg}(f_\varphi,\mathcal{U}) = G\text{-Deg}(\widetilde{f}_\varphi,\mathcal{U}).$$

The map $\widetilde{f}_\varphi$ is not normal but it may be "corrected" to a normal map $\widetilde{f}_{\widetilde{\varphi}}$ such that $\widetilde{f}_\varphi^{-1}(0) = \widetilde{f}_{\widetilde{\varphi}}^{-1}(0)$. For this purpose we consider the set $Z = \widetilde{f}_\varphi^{-1}(0) = \widetilde{f}^{-1}(0) \cap \varphi^{-1}(0)$. Since $\widetilde{f}$ is a normal bifurcation map, thus for every $x \in Z$ with $\alpha = (G_x)$ we have

$$\widetilde{f}(x+h) = \widetilde{f}(x) + h, \tag{3.3.3}$$

where $h \in N_x(V_\alpha)$, with $\|h\| < \varepsilon_x$. We may assume without loss of generality that the α-normality condition is satisfied for $x \in U_\alpha$, where U_α is an open invariant neighbourhood of $Z \subset V_\alpha$ in V_α for all $\alpha \in \mathcal{J}(Z)$. We may assume that all the orbit types in $\mathcal{J}(Z)$ are totally ordered, i.e. we have an order

$$\alpha_1 < \alpha_2 < \cdots < \alpha_m,$$

and let $\alpha = \alpha_1$. We consider the set $Z_\alpha = Z \cap V_\alpha$. We claim that Z_α is compact. Indeed, if $\{x_n\} \subset Z_\alpha$ is a sequence such that $x_n \to x \notin Z_\alpha$, then by the slice theorem (cf. [37]) $(G_x) < (G_{x_n}) = \alpha$, and this is a contradiction with the assumption that α was the minimal orbit type in $\mathcal{J}(Z)$.

Consequently, we may find two relatively compact open invariant neighbourhoods A_1 and B_1 of Z_α in V_α such that

$$Z_\alpha \subset A_1 \subset \overline{A}_1 \subset B_1 \subset \overline{B}_1 \subset U_\alpha.$$

Then there is an $\delta_1 > 0$ such that $\delta_1 < \varepsilon_x$ for all $x \in B_1$, and we can put $N_1 = \mu(N(B_1,\delta_1))$. We denote by $\gamma_1 : V \to [0,1]$ an invariant C^∞-function such that

$\gamma_1(x) = 1$ for $x \in \mu(N(A_1, \frac{\delta_1}{2}))$, and $\gamma_1(x) = 0$ for $x \in V \setminus N_1$. Then we may define

$$\widetilde{f}^1_\varphi(x) = \begin{cases} \widetilde{f}_\varphi(x), & \text{if } x \in V \setminus N_1; \\ \gamma_1[\widetilde{f}_\varphi(v) + w] + (1 - \gamma_1(x))\widetilde{f}_\varphi(x), & \text{for } x = v + w \in N_1, \end{cases}$$

where $x = v + w$ denotes the decomposition $v \in B_1$, $w \in N_v(V_\alpha)$. Let us notice that for $x = v + w \in N_1$ we have

$$\begin{aligned} \widetilde{f}^1_\varphi(x) &= (\gamma_1(x)(\widetilde{f}(v) + w) + (1 - \gamma_1(x))\widetilde{f}(x), \gamma_1(x)\varphi(v) + (1 - \gamma_1(x))\varphi(x)) \\ &= (\widetilde{f}(v) + w, \gamma_1(x)\varphi(v) + (1 - \gamma_1(x))\varphi(x)) \\ &= (\widetilde{f}(x), \gamma_1(x)\varphi(v) + (1 - \gamma_1(x)\varphi(x)). \end{aligned}$$

Since for $x \in N_1$ we have that $\widetilde{f}^1_\varphi(x) = 0$ if and only if $w = 0$, $\widetilde{f}(v) = 0$, and $\varphi(v) = 0$, thus it implies that $\widetilde{f}^1_\varphi(x) = 0$ if and only if $\widetilde{f}_\varphi(x) = 0$. We put $\Omega_1 = \mu(N(A_1, \frac{\delta_1}{2}))$. Consequently, the mapping $\widetilde{f}^1_\varphi$ is normal in the set Ω_1.

Assume for the purpose of induction, that we have constructed the mapping $\widetilde{f}^k_\varphi$ such that it is normal in the open invariant sets $\Omega_1, \ldots, \Omega_k$ such that $Z \cap V_{\alpha_i} \subset \Omega_i$, and that $\widetilde{f}^k_\varphi(x) = 0$ if and only if $\widetilde{f}_\varphi(x) = 0$. Then we consider the orbit type $\alpha = \alpha_{k+1}$ and the set $Z_\alpha = Z \cap V_\alpha$. We claim that the set Z_α is compact and $Z_\alpha \cap \Omega_i = \varnothing$ for all $i = 1, 2, \ldots, k$. Indeed, suppose $\{x_n\} \subset Z_\alpha$ is a sequence such that $x_n \to x \notin Z_\alpha$. Then $\beta = (G_x) \in \{\alpha_1, \ldots, \alpha_k\}$, and consequently $\widetilde{f}^k_\varphi$ satisfies β-normality condition in a neighbourhood of x in V_β. But this condition excludes the possibility of the existence of other zeros of $\widetilde{f}^k_\varphi$, with the orbit type larger than β, in a neighbourhood of x, and this is a contradiction with the assumption that $x_n \to x$.

Now we may define relatively compact open invariant neighbourhoods A_{k+1} and B_{k+1} of Z_α in V_α such that

$$Z_\alpha \subset A_{k+1} \subset \overline{A}_{k+1} \subset B_{k+1} \subset \overline{B}_{k+1} \subset U_\alpha.$$

Then there is $\delta_{k+1} > 0$ such that $\delta_{k+1} < \varepsilon_x$ for all $x \in B_{k+1}$ and we can put $N_{k+1} := \mu(N(B_{k+1}, \delta_{k+1}))$. Let $\gamma_{k+1} : V \to [0,1]$ be an invariant C^∞-function such that $\gamma_{k+1}(x) = 1$ for $x \in \mu(N(A_{k+1}, \frac{\delta_{k+1}}{2}))$ and $\gamma_{k+1}(x) = 0$ for $x \in V \setminus N_{k+1}$. Then we may define

$$\widetilde{f}_\varphi^{k+1}(x) = \begin{cases} \widetilde{f}_\varphi^k(x), & \text{if } x \in V \setminus N_{k+1}; \\ \gamma_{k+1}[\widetilde{f}_\varphi^k(v) + w] + (1 - \gamma_{k+1}(x))\widetilde{f}_\varphi^k(x), & \text{if } x = v + w \in N_{k+1}, \end{cases}$$

and again

$$\widetilde{f}_\varphi^{k+1}(x) = \left(\widetilde{f}(x), \gamma_{k+1}(x)\varphi(v) + (1 - \gamma_{k+1}(x)\varphi(x)\right).$$

Thus, for $k = m - 1$ we obtain $\widetilde{f}_{\widetilde{\varphi}}(x) = \widetilde{f}_\varphi^m(x)$, and $\widetilde{f}_{\widetilde{\varphi}}$ is the required normal mapping such that $\widetilde{f}_{\widetilde{\varphi}}^{-1}(0) = \widetilde{f}_\varphi^{-1}(0)$.

In the above construction, we have constructed a new complementing function $\widetilde{\varphi}$ such that $\widetilde{f}_{\widetilde{\varphi}}(x) = (\widetilde{f}(x), \widetilde{\varphi}(x))$ is a normal mapping.

In order to conclude the proof we need only to remark that it follows from the normality of $\widetilde{f}_{\widetilde{\varphi}}$ that for every $n_\alpha \neq 0$ there is an x such that $\widetilde{f}_{\widetilde{\varphi}}(x) = 0$ and $(G_x) = \alpha$. Since $\widetilde{f}_{\widetilde{\varphi}}^{-1}(0) = \widetilde{f}_\varphi^{-1}(0)$, the conclusion follows from the fact that we may use an arbitrary complementing function φ. □

Now we shall discuss the global bifurcation problem.

Definition 3.3.5 We say that the sequence $\{K_i\}$ of compact subsets of $\Lambda(f)$ is an *admissible decomposition* of $\Lambda(f)$ if

(i) $\Lambda(f) = \bigcup_{i=1}^\infty K_i$,

(ii) for every K_i there is an open subset U_i of M such that $K_i \subset U_i$ and $U_i \cap K_j = \varnothing$ for all $j \neq i$.

In other words, the existence of an admissible decomposition $\{K_i\}$ implies that

$\Lambda(f)$ is a disjoint union of isolated M-singular sets K_i. Consequently, for every subset K_i there exists a special neighbourhood $\mathcal{U}_i$, together with a complementing function $\varphi_i : \overline{\mathcal{U}}_i \to \mathbf{R}$ such that $\overline{\mathcal{U}}_i \cap \overline{\mathcal{U}}_j = \emptyset$ for $i \neq j$. Since for every subset K_i the G-degree does not depend on the special neighbourhood $\mathcal{U}_i$, either on the complementing function φ_i, for every $\alpha \in \Phi_{n-1}(G)$ we may define the integer $n_\alpha(K_i)$ by

$$G\text{-Deg}\,(f_{\varphi_i}, \mathcal{U}_i) = \sum_\alpha n_\alpha(K_i)\alpha,$$

and we will call $n_\alpha(K_i)$ the ***α-bifurcation invariant*** of the set K_i.

Let us denote by $\mathfrak{S}(f)$ the closure of the set of all nontrivial solutions of (3.3.1), i.e.

$$\mathfrak{S}(f) := \overline{\{x \in V \setminus M : f(x) = 0\}}.$$

Definition 3.3.6 Let $\mathfrak{C}$ be a component of the set $\mathfrak{S}(f)$. We say that $\mathfrak{C}$ is $\{K_i\}$-*compatible* if for every set K_i such that $\mathfrak{C} \cap K_i \neq \emptyset$ the following condition is satisfied

- if $x \in \mathfrak{S}(f) \cap K_i$ then $x \in \mathfrak{C} \cap K_i$.

In other words, the above condition says that if the component $\mathfrak{C}$ intersects the set K_i then there is no other component of $\mathfrak{S}(f)$ connected to K_i, or simply $\mathfrak{C}$ contains all the connected components of $\mathfrak{S}(f)$ intersecting K_i.

Let us remark that if $\Lambda(f)$ admits an admissible decomposition $\{K_i\}$ such that $K_i = \{x_i\}$, i.e. there are only isolated M-singular points in $\Lambda(f)$ and the sets K_i contain only one point each, or if the sets K_i are connected, then every component $\mathfrak{C}$ of the set $\mathfrak{S}(f)$ is $\{K_i\}$-compatible. Moreover, if $\mathfrak{C}$ is a connected component of

$\mathfrak{S}(f)$ such that there is K_i such that $\mathfrak{C} \cap K_i \neq \varnothing$, then $\mathfrak{C}$ is an invariant set. Therefore, if $\mathfrak{C}$ is $\{K_i\}$-compatible component of $\mathfrak{S}(f)$, and if it contains only connected components intersecting $\Lambda(f)$, then $\mathfrak{C}$ is necessarily an invariant set.

Now we are in the position to state our global bifurcation result.

Theorem 3.3.7 *Let $\{K_i\}$ be an admissible decomposition of $\Lambda(f)$ and let $\mathfrak{C}$ be a bounded invariant $\{K_i\}$-compatible component of the set $\mathfrak{S}(f)$. Then there exists only a finite collection of sets K_i such that $\mathfrak{C} \cap K_i \neq \varnothing$, which we denote by $\{K_{i_1}, \dots, K_{i_r}\}$. Moreover, for all $\alpha \in \Phi_{n-1}(G)$ we have the following relation:*

$$\sum_{j=1}^{r} n_\alpha(K_{i_j}) = 0. \tag{3.3.4}$$

Proof. We claim that the set $\{K_i : K_i \cap \mathfrak{C} \neq \varnothing\}$ is finite. Indeed, suppose there is a sequence $\{x_k\} \subset \mathcal{B}(f) \cap \mathfrak{C}$. Since $\mathfrak{C}$ is bounded, the sequence $\{x_k\}$ is also bounded, and therefore it contains a convergent subsequence. We may assume that $x_k \to x_o$ as $k \to \infty$. Since the set of all bifurcation points $\mathcal{B}(f)$ is closed, $x_o \in \mathcal{B}(f)$. Let K_{i_o} be such that $x_o \in K_{i_o}$. Then there is an open neighbourhood U_{i_o} of K_{i_o} in M such that $U_{i_o} \cap K_j = \varnothing$ for $j \neq i_o$. Therefore, for k sufficiently large $x_k \in K_{i_o}$, and consequently the set $\{K_i : K_i \cap \mathfrak{C} \neq \varnothing\}$ is finite.

We put $\{K_i : K_i \cap \mathfrak{C} \neq \varnothing\} = \{K_{i_1}, \dots, K_{i_r}\}$. We can choose special neighbourhoods $\mathcal{U}_{i_j}$ of K_{i_j} such that $\overline{\mathcal{U}}_{i_j} \cap \overline{\mathcal{U}}_{i_k} = \varnothing$ for $i_j \neq i_k$. We also assume that $\mathcal{U}_{i_j} = \mu(N(D_{i_j}, \varepsilon))$, where $D_{i_j} = \{x \in M : \operatorname{dist}(x, K_{i_j}) < \rho\}$, and the numbers $\varepsilon > 0$ and $\rho > 0$ are chosen sufficiently small. Let $\mathcal{V} = \bigcup_{j=1}^{r} \mathcal{U}_{i_j}$. Since $\mathfrak{C}$ is bounded, we may choose a bounded invariant open subset $\mathcal{V}_1 \subset W \oplus \mathbf{R}^n$ such that $\mathfrak{C} \setminus \mathcal{V} \subset \mathcal{V}_1$, $\mathcal{V}_1 \cap M = \varnothing$ and $(\partial \mathcal{V}_1 \setminus \mathcal{V}) \cap \mathfrak{C} = \varnothing$. We put $\mathcal{U} = \mathcal{V} \cup \mathcal{V}_1$.

Let $\varepsilon : \overline{\mathcal{U}} \to \mathbf{R}$ be an invariant continuous function such that $\varepsilon(x) < 0$ for all

$x \in \overline{\mathcal{U}} \cap M$ and $\varepsilon(x) = a > 0$ for $x \in \overline{\mathcal{U}} \setminus \mathcal{V}$. We define $f_\varepsilon : \overline{\mathcal{U}} \to W \oplus \mathbf{R}$ by

$$f_\varepsilon(x) = (f(x), \varepsilon(x)), \quad x \in \overline{\mathcal{U}}.$$

Then $f_\varepsilon^{-1}(0) \subset \mathcal{U}$ and hence $G\text{-Deg}\,(f_\varepsilon, \mathcal{U})$ is well defined. We consider the following homotopy $h : \overline{\mathcal{U}} \times [0,1] \to W \oplus \mathbf{R}^n$ given by

$$h(x,t) = (f(x), (1-t)\varepsilon(x) - t\rho), \quad (x,t) \in \overline{\mathcal{U}} \times [0,1],$$

where $\rho > 0$. We claim that h is an $\mathcal{U}$-admissible homotopy. Indeed, suppose by contradiction that $h(x,t) = 0$ for some $t \in [0,1]$ and $x \in \partial\mathcal{U}$. Since $f(x) \neq 0$ for $x \in (\partial\mathcal{U} \setminus \mathcal{V})$, thus $x \in \partial\mathcal{V} = \bigcup_{j=1}^{r} \partial\mathcal{U}_{i_j}$. On the other hand, $f(x) = 0$ implies that $x \in \mathfrak{C} \cup M$. Since $x \in \mathfrak{C} \cap \partial\mathcal{V}$ implies $x \in \mathcal{V}_1$, we have $x \in M$, but $\varepsilon(x) < 0$ for $x \in M \cap \overline{\mathcal{U}}$, thus $(1-t)\varepsilon(x) - t\rho < 0$, which is a contradiction to the assumption that $h(x,t) = 0$.

Clearly $h_0 = f_\varepsilon$, thus $G\text{-Deg}\,(f_\varepsilon, \mathcal{U}) = G\text{-Deg}\,(h_1, \mathcal{U})$. However, since $h_1(x) = (f(x), -\rho) \neq 0$ for all $x \in \overline{\mathcal{U}}$, we have $G\text{-Deg}\,(f_\varepsilon, \mathcal{U}) = G\text{-Deg}\,(h_1, \mathcal{U}) = 0$. Moreover, since $f_\varepsilon(x) \neq 0$ for $x \in \mathcal{V}_1 \setminus \mathcal{V}$, thus

$$\begin{aligned} 0 &= G\text{-Deg}\,(f_\varepsilon, \mathcal{U}) \\ &= G\text{-Deg}\,(f_\varepsilon, \mathcal{V}) \\ &= G\text{-Deg}\,(f_\varepsilon, \bigcup_{j=1}^{r} \mathcal{U}_{i_j}) \\ &= \sum_{j=1}^{r} G\text{-Deg}\,(f_\varepsilon, \mathcal{U}_{i_j}) \\ &= \sum_{\alpha} \Big(\sum_{j=1}^{r} n_\alpha(K_{i_j}) \Big) \alpha, \end{aligned}$$

and consequently

$$\sum_{j=1}^{r} n_\alpha(K_{i_j}) = 0$$

for all $\alpha \in \Phi_{n-1}(G)$. □

3.4 Steady-State Bifurcation – Spontaneous Symmetry Breaking

Let G be a compact Lie group and $W = \mathbf{R}^n$ an orthogonal representation of G. Assume that $F : W \times \mathbf{R} \to W$ is a smooth G-equivariant map, i.e. $F(gx, \lambda) = gF(x, \lambda)$ for all $g \in G$, $x \in W$, $\lambda \in \mathbf{R}$. Consider the ordinary differential equation

$$\frac{dx}{dt} = F(x, \lambda). \tag{3.4.1}$$

We assume that $F(0, \lambda) = 0$, and $L := D_x F(0,0)$ is singular. Thus L has a zero eigenvalue. Due to the presence of symmetries, $\lambda = 0$ is not a simple eigenvalue in general.

In this section, we shall present, as an example, some particular classes of steady-state bifurcations for the action of the group $SO(3)$. We believe that similar arguments should be applied to other groups, in particular, to $O(3)$ and possibly to $O(n)$.

There is much interest in finding solutions of (3.4.1) with special symmetry such as the axisymmetric solutions. The lattice of isotropy subgroups of G, for irreducible representations of G, provides a way to classify solutions by their symmetry. Let $x \in W$ and $H = G_x$ be the isotropy group of x. Then the mapping F preserves the fixed-point subspace W^H, i.e. W^H is flow-invariant, that is, any trajectory of (3.4.1) starting in W^H remains in W^H. We would like to know when there exists a *branch of solutions*, with isotropy H, bifurcating from the solution $x = 0$. Since this branch of solutions have less symmetry than the solution $x = 0$, this phenomenon is called *spontaneous symmetry breaking*.

Since the isotropy subgroups of points on the same orbit are conjugate, we will talk about orbit types (H) rather than isotropy groups H when we describe the symmetry property of the orbit.

In this section, we will study the case where $V = Ker(L)$ is an absolutely irreducible representation of G. So the only linear maps on $Ker(L)$ that commute with G are scalar multiples of the identity, and therefore the multiplicity of the zero eigenvalue of L is equal to the dimension m of $Ker(L)$. Let us denote by $\sigma(L_\lambda)$ the spectrum of the operator $L_\lambda := D_xF(0, \lambda)$, and let σ_- (resp. σ_+) denote the number of negative eigenvalues of L_λ for $\lambda = -\rho$ (resp. $\lambda = \rho$) for some sufficiently small $\rho > 0$. We will call the integer $\sigma = \sigma_- - \sigma_+$ the *crossing number* for the problem (3.4.1). We assume that $\sigma = \pm m$, i.e. the eigenvalue of L changes sign by passing through zero.

Since we have assumed that $V = Ker(L)$ is an absolutely irreducible representation of G, $V^G = \{0\}$ and for all $x \in V \setminus \{0\}$ the isotropy group G_x is not equal to G. If (H) is a minimal orbit type in $V \setminus \{0\}$, then the isotropy group H is called *maximal*. All other isotropy groups in $V \setminus \{0\}$ are called *submaximal*.

The following version of the equivariant branching lemma was first proved by Cicogna [6]. Here we present a proof of this result, based on an application of the equivariant degree.

Proposition 3.4.1 *Suppose that W is an orthogonal representation of the Lie group G, $F : W \times \mathbb{R} \to W$ is a smooth G-equivariant map such that $F(0, \lambda) = 0$, $L_\lambda := D_xF(0, \lambda)$ is nonsingular for $\lambda \neq 0$, and singular for $\lambda = 0$. Moreover, assume that $Ker(L_0) = V$ is an absolutely irreducible representation of G such that the crossing number σ associated with (3.4.1) is equal $\pm m$, where $m = \dim V$. Then for every maximal isotropy group H in V such that $\dim V^H$ is odd, there exists a branch of stationary points of (3.4.1) bifurcating from $(0, 0)$ with the orbit type exactly (H).*

Proof. Without loss of generality, we can assume that a reduction has been carried out and that $Ker(L) = \mathbb{R}^n = W$. We will apply Proposition 3.3.1. Therefore, we need to compute

$$G\text{-Deg}\,(F_\varphi, U(r,\rho)) = G\text{-Deg}\,(F_-, \Omega^-(r,\rho)) - G\text{-Deg}\,(F_+, \Omega^+(r,\rho)),$$

where $\Omega^\pm(r,\rho) = \{x \in W;\ \ \|x\| < r\}$, $U(r,\rho)$ is a special neighbourhood of $(0,0)$, $F_\pm : W \to W$ is defined by $F_\pm(x) = F(x, \pm\rho)$, $x \in W$, and φ is a complementary function. By using the linearization at the points $(0, \pm\rho)$, we may assume that $F_\pm(x) = \pm x$ (or $F_\pm = \mp x$). Since $G\text{-Deg}\,(Id, \Omega^+(r,\rho)) = (G)$, we need only to prove that (H)-component $m_{(H)}$ of $G\text{-Deg}\,(F_-, \Omega^-(r,\rho)) = G\text{-Deg}\,(-Id, \Omega^-(r,\rho))$ is non-zero.

For this purpose, we will apply formula (1.2.2), that is

$$m_{(H)}(F_-) = [I(F_-^H) - I(F_-^{[H]})]/|W(H)|.$$

Since W^H is an odd-dimensional space and we can replace F_- by $-Id$, we have $I(F_-^H) = -1$. By the assumption that H is a maximal isotropy group, it follows that $W^{[H]} = \{0\}$. Thus $I(F_-^{[H]}) = 1$, and consequently $m_{(H)} = (-1-1)/|W(H)| = -2$ or -1 (since in this case $|W(H)|$ has to be 1 or 2). This shows that the (H)-component of $G\text{-Deg}\,(F_\varphi, U(r,\rho))$ is not equal to 0, and the conclusion follows from Proposition 3.3.1.

□

The methods employed in the proof of Proposition 3.4.1 illustrate the importance of computing $G\text{-Deg}\,(-Id, \Omega)$, where Ω denotes the unit ball in $V = Ker(L)$, in order to determine the occurrence of bifurcations. In particular, for every non-zero (H)-component $m_{(H)}$ of $G\text{-Deg}\,(-Id, \Omega) - (G)$ there exists a branch of solutions with orbit type (K) such that $(G) < (K) \le (H)$. Unfortunately, due to the topological nature of the problem, we are unable to determine if for a submaximal isotropy group H there is a bifurcating branch with the orbit type (H). However, it is possible to construct a kind of a parametrized 'normal' mapping F such that, as with normal mappings constructed for the purpose of the definition of G-degree, such a mapping satisfies the property that if $m_{(H)} \neq 0$, then there exists a branch of solutions with orbit type (H). More precisely, if $G\text{-Deg}\,(F_\varepsilon, U(r,\rho)) = \sum_\alpha m_\alpha \alpha$ is such that $m_\alpha \neq 0$, then $(0,0)$ is an α-essential bifurcation point in the sense of Definition 3.3.2.

We should also mention that our approach may be used to study more complex situations, such as the case where $Ker(L) = V$ is a direct sum of two or more absolutely irreducible representations of G. In this case the multiplicativity property as well as the multiplication table for $A(G)$ can provide the information necessary to compute the degree $G\text{-Deg}\,(F_\varphi, U(r,\rho)) = G\text{-Deg}\,(F_-, \Omega^-(r,\rho)) - G\text{-Deg}\,(F_+, \Omega^+(r,\rho))$. We will illustrate this situation by an example.

Let us consider the case where $G = SO(3)$. In Section 4 of Chapter 1 we computed $SO(3)\text{-Deg}\,(-Id, \Omega_i)$, where Ω_i denotes the unit ball in the $(2i+1)$-dimensional absolutely irreducible representation V_i $(i = 1,2,3,4,5)$ of the group $SO(3)$. We obtained the following results:

$$SO(3)\text{-Deg}\,(-Id, \Omega_1) = (SO(3)) - (SO(2))$$

$$SO(3)\text{-Deg}\,(-Id, \Omega_2) = (SO(3)) - 2(O(2)) + (V_4)$$

$$SO(3)\text{-Deg}\,(-Id, \Omega_3) = (SO(3)) - (A_4) - (SO(2)) - (D_3)$$

$$SO(3)\text{-Deg}\,(-Id, \Omega_4) = (SO(3)) - 2(O(2)) - 2(S_4) + 2(D_4) + 2(D_3) - (V_4)$$

$$SO(3)\text{-Deg}\,(-Id, \Omega_5) = (SO(3)) - (SO(2)) - (D_5) - (D_4) - (D_3) + (V_4)$$

Using the results from Section 1.3 and Section 1.4 of Chapter 1 we now present the following

Example 3.4.2 Consider the steady-state bifurcation problem (3.4.1) in the case where $G = SO(3)$ and $Ker(L) = V_3 \oplus V_5$. Assume for simplicity that $V = V_3 \oplus V_5$, and that $L_{\lambda|V_3} : V_3 \to V_3$ (resp. $L_{\lambda|V_5} : V_5 \to V_5$) changes its eigenvalue from positive to negative (resp. from negative to positive), as λ crosses 0 passing from negative to positive values. Then, in order to determine the 'essential' orbit types for bifurcating branches of solutions, i.e. the orbit types that have orbit types (H) corresponding to a non-zero component $m_{(H)}$ of the degree $SO(3)\text{-Deg}\,(F_\varphi, U(r,\rho))$, it is sufficient to notice that

$$\begin{aligned} SO(3)\text{-Deg}\,(F_\varphi, U(r,\rho) &= SO(3)\text{-Deg}\,(-Id,\Omega) - SO(3)\text{-Deg}\,(Id,\Omega) \\ &= \big(SO(3)\text{-Deg}\,(-Id,\Omega_3)\big)\cdot\big(SO(3)\text{-Deg}\,(-Id,\Omega_5)\big) - (SO(3)) \\ &= -(A_4) - (D_5) - (D_4) + (V_4) \end{aligned}$$

where Ω denotes the unit ball in V. Consequently, we can detect the existence of branches with the orbit types H, where $H = A_4, D_5, D_4$ (which reflects the fact that the corresponding fixed point spaces V^H are one-dimensional) and $H = V_4$, which is not a maximal orbit type. The interaction between two representation, due to nonlinear coupling, destroys the essential bifurcation of the branches with the orbit types $SO(2)$ and D_3; thus detecting them is difficult. Possibly, there is no normal bifurcation of this type at all. The method developed in this section, although it can not detect exact orbit types of bifurcating branches, can be effectively applied to those situations where there are several irreducible components in $Ker(L)$, to determine what type of bifurcation can surely take place.

Chapter Four

An Extension of the Computational Formulae

4.1 Introduction

This chapter may be considered as an appendix of Chapter 2 and an introduction to Chapter 5.

In Section 4.2 the multiplicativity formula for an equivariant degree, so far investigated only in the case $n = 0$ or G being abelian, is extended to $G = \Gamma \times G_0$, where Γ is finite and G_0 is a compact Lie group. We illustrate the general results with the group $D_N \times S^1$. This gives us the opportunity to introduce the dihedral symmetries which will be extensively studied again later in Chapter 5.

In Section 4.3, we apply these formulae to the computation of G-degree in the case where $f : V \times \mathbf{R} \to V$ has regular zeros in $V^G \times \mathbf{R}$ and $G = \Gamma \times S^1$ with Γ being a finite group.

The obtained results are studied for $\Gamma = D_N$ and complete computations are carried out in this case.

4.2 An Extension of the Multiplicativity Formula

Our next goal is to develop a specific multiplicativity formula for the G-degree with non-abelian group G, similar to the multiplicativity formula for the G-degree with an abelian compact Lie group G established in [19]. In the following result we assume that $G = \Gamma \times G_o$, where Γ is a finite group.

Lemma 4.2.1 *Let Γ be a finite group and G_o a compact Lie group. Then there is a natural structure of $A(\Gamma)$-module on $A_n(\Gamma \times G_o)$, where the product $A(\Gamma) \times A_n(\Gamma \times G_o) \to A_n(\Gamma \times G_o)$ is defined on generators by the formula:*

$$(K) \cdot (H) = \sum_{(L) \in \Phi_n(\Gamma \times G_o)} n_L \cdot (L),$$

and

$$n_L = \left| \left(\frac{\Gamma \times G_o}{K \times G_o} \times \frac{\Gamma \times G_o}{H} \right)_{(L)} / (\Gamma \times G_o) \right|, \tag{4.2.1}$$

where $K \subset \Gamma$ is a subgroup, $(H) \in \Phi_n(\Gamma \times G_o)$. The integer n_L denotes the number of (L)-orbits in $\frac{\Gamma \times G_o}{K \times G_o} \times \frac{\Gamma \times G_o}{H}$.

Proof. Since the group Γ is finite and $\frac{\Gamma \times G_o}{H}$ has a finite number of (L)-orbits (see [2]), formula (4.2.1) defines a finite number n_L. Since (K) and (H) are free generators of $A(\Gamma)$ and $A_n(\Gamma \times G_o)$, the statement follows. □

Theorem 4.2.2 (MULTIPLICATIVITY PROPERTY) *Suppose that $G = \Gamma \times G_o$ is a compact Lie Group, where Γ is a finite group. Let W be an orthogonal representation of Γ and V be an orthogonal representation of G. Assume that $\Omega_o \subset W$ is a bounded open invariant subset, $f_o : W \to W$ is a Γ-equivariant Ω_o-admissible map, $\Omega_1 \subset V$ is an open bounded invariant subset, and $f_1 : V \oplus \mathbf{R}^n \to V$ is a G-equivariant Ω_1-admissible map. Then the map $f : W \oplus V \oplus \mathbf{R}^n \to W \oplus V$ defined by $f(w, v, \lambda) = (f_o(w), f_1(v, \lambda))$, $(w, v, \lambda) \in W \oplus V \oplus \mathbf{R}^n$, is an $\Omega = \Omega_o \times \Omega_1$-admissible G-map (where W is a representation of $G = \Gamma \times G_o$ with the action of G_o trivial),* *and*

$$G\text{-Deg}(f,\Omega) = \Gamma\text{-Deg}(f_o,\Omega_o)\cdot G\text{-Deg}(f_1,\Omega_1).$$

Proof. We may assume, without loss of generality, that f_o and f_1 are both regular normal, $f_o^{-1}(0)\cap\Omega_o$ consists of exactly one orbit Γw_o such that $\Gamma_{w_o} = K$, and $f_1^{-1}(0)\cap\Omega_1$ consists of exactly one orbit Gx_1 such that $G_{x_1} = H$ and $(H)\in\Phi_n^+(G)$. It is clear that $f^{-1}(0)\cap\Omega = \Gamma(w_o)\times G(x_1)$, therefore it is G-diffeomorphic to the space

$$\frac{\Gamma\times G_o}{K\times G_o}\times\frac{\Gamma\times G_o}{H}.$$

Since Γ is finite, we claim that the map f is regular normal in Ω. Indeed, let (L) be an orbit type in $\frac{\Gamma\times G_o}{K\times G_o}\times\frac{\Gamma\times G_o}{H}$. Then $L=\gamma K\gamma^{-1}\times G_o\cap H$ for some $\gamma\in\Gamma$. We may assume for simplicity that $\gamma = 1$. Thus $L\subset K\times G_o$ and $L\subset H$, and hence $W^K\subset W^L$ and $V^H\subset V^L$. By the assumption, $\dim W(H) = \dim W(L)$. So the orbit $N(H)x_1$ in $V^H\oplus\mathbb{R}^n$ has the same dimension as the orbit $W(L)(w_o,x_1)$ in $W^L\oplus V^L\oplus\mathbb{R}^n$. As both f_o and f_1 are normal, f_1 (and similarly f_o) acts as identity operator on small vectors v in the linear slice to the orbit $G(x_1)$ at x_1 which are orthogonal to $V^H\oplus\mathbb{R}^n$. Consequently, f also acts as identity operator on small vectors (w,v) in the linear slice to the orbit $G(w_o,x_1)$ at the point (w_o,x_1) orthogonal to $W^L\oplus V^L\oplus\mathbb{R}^n$. This implies that f is normal and regular. So the conclusion follows from the analytic formula of G-degree. □

Example 4.2.3 Suppose that Γ is a finite group and $G=\Gamma\times S^1$. It is easy to verify that the generators of $A_1(\Gamma\times S^1)$ are the so called *l-folded θ-twisted* subgroups $K^{(\theta,l)}$ of $\Gamma\times S^1$, where K is a subgroup of Γ, $\theta: K\to S^1$ a homomorphism, l a non-negative integer, and

$$K^{(\theta,l)} := \{(\gamma, z) \in K \times S^1;\ \ \theta(\gamma) = z^l\}.$$

For certain groups $G = \Gamma \times S^1$, it is possible to compute the multiplication table for $\cdot : A(\Gamma) \times A_1(G) \to A_1(G)$ directly from the multiplication table of the Burnside ring $A(\Gamma)$. Let $H = M^{(\theta,l)} \subset \Gamma \times S^1$ and $K \subset \Gamma$ be two subgroups. Then we have that

$$(K) \cdot (M^{(\theta,l)}) = \sum_{(L) \leq (M)} n_{(L)} \cdot (L^{(\theta,l)}),$$

where the numbers $n_{(L)}$ are given by (4.2.1). Assume that the subgroup $M^{(\theta,l)}$ satisfies the following properties:

(i) for every subgroup $L \subset M$ and for every $g \in N(L, M) := \{g \in \Gamma;\ \ g^{-1}Lg \subset M\}$, we have $\theta(g^{-1}\gamma g) = \theta(\gamma)$ for $\gamma \in L$;

(ii) $N(M^{(\theta,l)}) = N(M) \times S^1$.

Then $n_{(L)} = m_{(L)}$, where the coefficients $m_{(L)}$ are given in the formula

$$(K) \cdot (M) = \sum_{(L) \leq (M)} m_{(L)} \cdot (L).$$

To justify this, it is enough to notice that we have the following

$$\begin{aligned}
\left(\frac{\Gamma}{K} \times \frac{\Gamma \times S^1}{M^{(\theta,l)}}\right)^{L^{(\theta,l)}} / W(L^{(\theta,l)}) &= \left(\left(\frac{\Gamma}{K}\right)^L \times \frac{N(L,M) \times S^1}{M^{(\theta,l)}}\right) / W(L^{(\theta,l)}) \\
&= \left(\left(\frac{\Gamma}{K}\right)^L \times \frac{N(L,M) \times S^1}{M^{(\theta,l)}}\right) / (W(L) \times S^1) \\
&= \left(\left(\frac{\Gamma}{K}\right)^L \times \frac{N(L,M) \times S^1}{M^{(\theta,l)}} / \frac{M \times S^1}{M^{(\theta,l)}}\right) / W(L) \\
&= \left(\frac{\Gamma}{K} \times \frac{\Gamma}{M}\right) / W(L).
\end{aligned}$$

The above formula also shows that if $(L^{(\theta,l)})$ is a minimal orbit type in the product

$\frac{\Gamma}{K} \times \frac{\Gamma \times S^1}{M^{(\theta,l)}}$ then $n_{(L)} = m_{(L)}|N(L)/M_L|$, where $N(L^{(\theta,l)}) = N_L \times S^1$.

We now derive the multiplication table where $\Gamma = D_N$, $N > 2$. We start with the classification of conjugacy classes of subgroups of D_N. Note that if n is an odd number, then

$$\Phi(D_N) = \{(D_k),\ (\mathbf{Z}_k);\ \ k|N\};$$

and if n is even then

$$\Phi(D_N) = \{(D_k),\ (\widetilde{D}_k),\ (\mathbf{Z}_k);\ \ k \mid N\},$$

where $\widetilde{D}_k = \mathbf{Z}_k \cup \kappa\xi_N\mathbf{Z}_k$, $\xi_N = e^{\frac{2i\pi}{N}}$ and $\kappa = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. We have the following table

Table 4.1.

K	$N(K)$	$W(K)$
D_k, $2k \nmid N$	D_k	$\mathbf{Z}_1$
D_k, $2k \mid N$	D_{2k}	$\mathbf{Z}_2$
$\widetilde{D}_k$, $2k \nmid N$	$\widetilde{D}_k$	$\mathbf{Z}_1$
$\widetilde{D}_k$, $2k \mid N$	$\widetilde{D}_{2k}$	$\mathbf{Z}_2$
$\mathbf{Z}_k$	D_N	$D_{\frac{N}{k}}$

where $k \mid N$.

Using Table 4.1, together with the results discussed in Section 1.3 of Chapter 1 and [4], we are able to compute the multiplication table for the Burnside ring $A(D_N)$.

Table 4.2. Multiplication Table for $A(D_N)$

	$(\mathbf{D}_m)$ $2m \nmid N$	$(\mathbf{D}_m)$ $2m \mid N$	$(\tilde{\mathbf{D}}_m)$ $2m \nmid N$	$(\tilde{\mathbf{D}}_m)$ $2m \mid N$	$(\mathbf{Z}_m)$
$(\mathbf{D}_k)$ $2k \nmid N$	$(D_l)+$ $\frac{Nl-mk}{2mk}(\mathbf{Z}_l)$	$(D_l)+$ $\frac{Nl-mk}{2mk}(\mathbf{Z}_l)$	$\frac{lN}{2mk}(\mathbf{Z}_l)$	$\frac{lN}{2mk}(\mathbf{Z}_l)$	$\frac{Nl}{km}(\mathbf{Z}_l)$
$(\mathbf{D}_k)$ $2k \mid N$	$(D_l)+$ $\frac{Nl-mk}{2mk}(\mathbf{Z}_l)$	$2(D_l)+$ $\frac{Nl-2mk}{2mk}(\mathbf{Z}_l)$	$\frac{lN}{2mk}(\mathbf{Z}_l)$	$\frac{lN}{2mk}(\mathbf{Z}_l)$	$\frac{Nl}{km}(\mathbf{Z}_l)$
$(\tilde{\mathbf{D}}_k)$ $2k \nmid N$	$\frac{lN}{2mk}(\mathbf{Z}_l)$	$\frac{lN}{2mk}(\mathbf{Z}_l)$	$(\tilde{D}_l)+$ $\frac{Nl-mk}{2mk}(\mathbf{Z}_l)$	$(\tilde{D}_l)+$ $\frac{Nl-mk}{2mk}(\mathbf{Z}_l)$	$\frac{Nl}{km}(\mathbf{Z}_l)$
$(\tilde{\mathbf{D}}_k)$ $2k \mid N$	$\frac{lN}{2mk}(\mathbf{Z}_l)$	$\frac{lN}{2mk}(\mathbf{Z}_l)$	$(\tilde{D}_l)+$ $\frac{Nl-mk}{2mk}(\mathbf{Z}_l)$	$2(\tilde{D}_l)+$ $(\frac{Nl-2mk}{2mk}(\mathbf{Z}_l)$	$\frac{Nl}{km}(\mathbf{Z}_l)$
$(\mathbf{Z}_k)$	$\frac{Nl}{km}(\mathbf{Z}_l)$	$\frac{Nl}{km}(\mathbf{Z}_l)$	$\frac{Nl}{km}(\mathbf{Z}_l)$	$\frac{Nl}{km}(\mathbf{Z}_l)$	$\frac{2Nl}{km}(\mathbf{Z}_l)$

where $l = gcd(m,k)$, $m \mid N$ and $k \mid N$

We have the following classification (up to conjugacy classes) of the non-trivial l-folded θ-twisted subgroups of $D_N \times S^1$:

- The subgroups $D_k^{(c,l)}$ and $\tilde{D}_k^{(c,l)}$, where $c : D_k \to \mathbf{Z}_2$ is a homomorphism such that Ker $c = \mathbf{Z}_k$,
- The subgroup $D_k^{(d,l)}$ (resp. $\tilde{D}_k^{d,l}$) (when k is even), where $d : D_k \to \mathbf{Z}_2$ (resp. $d : \tilde{D}_k \to \mathbf{Z}_2$) is a homomorphism such that Ker $d = D_{\frac{k}{2}}$ (resp. Ker $d = \tilde{D}_{\frac{k}{2}}$).
- If k is divisible by 4 then there exists the conjugacy class of the subgroup $D_k^{(\hat{d},l)}$, where Ker $\hat{d} = \hat{D}_{\frac{k}{2}} := \mathbf{Z}_{\frac{k}{2}} \cup \kappa\xi_k\mathbf{Z}_{\frac{k}{2}}$ with $\xi_k = e^{\frac{2i\pi}{k}}$.
- The subgroups $\mathbf{Z}_k^{(\varphi,l)}$, where the homomorphism φ is given by $\varphi(z) = z^\nu$, ν is an integer and $z \in \mathbf{Z}_k \subset S^1 \subset \mathbf{C}$.
- If k is an even number, we have also the homomorphism $d : \mathbf{Z}_k \to \mathbf{Z}_2$ such that Ker $d = \mathbf{Z}_{\frac{k}{2}}$, for which we have the l-folded d-twisted subgroup $\mathbf{Z}_k^{(d,l)}$.

One can verify that, except the subgroups $D_k^{(d,l)}$ for $2k \mid N$, all other l-folded θ-twisted subgroups satisfy conditions (i) and (ii). Therefore, we can use the multiplication formula in the Burnside ring $A(D_N)$ to compute the multiplication table for $\cdot : A(D_N) \times A_1(D_N \times S^1) \to A_1(D_N \times S^1)$. For example, we have Table 4.3.

Table 4.3. Table of Multiplication

	$(\mathbf{D}_k^{(c,l)}$ $2k \nmid N$	$(\mathbf{D}_k^{(c,l)})$ $2k \mid N$	$(\mathbb{Z}_k^{(d,l)})$ $2 \mid k$
$(\mathbf{D}_r)$ $2r \nmid N$	$(D_m^{(c,l)}) + \frac{Nm-kr}{2kr}(\mathbb{Z}_m \times \mathbb{Z}_l)$	$(D_m^{(c,l)}) + \frac{Nm-kr}{2kr}(\mathbb{Z}_m \times \mathbb{Z}_l)$	$\frac{Nm}{rk}(\mathbb{Z}_m^{(d,l)})$
$(\mathbf{D}_r)$ $2r \nmid N$	$(D_m^{(c,l)}) + \frac{Nm-kr}{2kr}(\mathbb{Z}_m \times \mathbb{Z}_l)$	$2(D_m^{(c,l)}) + (\frac{Nm-2kr}{2kr}(\mathbb{Z}_m \times \mathbb{Z}_l)$	$\frac{Nm}{rk}(\mathbb{Z}_m^{(d,l)})$
$(\mathbb{Z}_r)$	$\frac{Nm}{kr}(\mathbb{Z}_m \times \mathbb{Z}_l)$	$\frac{Nm}{kr}(\mathbb{Z}_m \times \mathbb{Z}_l)$	$\frac{2Nm}{kr}(\mathbb{Z}_m^{(d,l)})$

where $m = gcd(k, r)$, $r \mid N$ and $k \mid N$

Suppose now that k and r are even numbers such that $k|N$ and $r|N$. Consider the subgroup $D_k^{(d,l)}$ of $D_N \times S^1$. We have $(D_r) \cdot (D_k^{(d,l)}) = n_1(D_m^{(d,l)}) + n_2(\mathbb{Z}_m^{(d,l)})$. The only problem with the computations of the numbers n_1 and n_2 occurs when $2l|N$. However, for $L^{(d,l)} = D_m^{(d,l)}$, the orbit type (L) is maximal in $\frac{D_N}{D_r} \times \frac{D_N \times S^1}{D_k^{(d,l)}}$. Therefore, $n_1 = 2$ whenever exactly one of the numbers $2k$ or $2r$ divides N, and $n_1 = 4$ if both numbers $2k$ and $2r$ divide N. We can compute n_2 just by counting the orbits. In the first case, we obtain $n_2 = \frac{ml}{2kr} - 1$ and in the second case, $n_2 = \frac{ml}{2kr} - 2$. That gives us the following multiplication table

Table 4.4.

	$(\mathbf{D}_k^{(d,l)})$, $2\mid k$ $2k \nmid N$	$(\mathbf{D}_k^{(d,l)})$, $2\mid k$ $2k\mid N$
$(\mathbf{D}_r)$, $2\mid r$ $2r \nmid N$	excluded	$2(D_m^{(d,l)}) + \frac{lm-2kr}{2kr}(\mathbf{Z}_m^{(d,l)})$
$(\mathbf{D}_r)$, $2\mid r$ $2r \mid N$	$2(D_m^{(d,l)}) + \frac{ml-2kr}{2kr}(\mathbf{Z}_m^{(d,l)})$	$4(D_m^{(d,l)}) + \frac{ml-4kr}{2kr}(\mathbf{Z}_m^{(d,l)})$

where we assume $m = gcd(k, r)$ is such that $2m\mid N$

4.3 An Extension of the Computational Formula for $\Gamma \times S^1$, Γ finite

We will now consider a special but important case of a non-abelian Lie group G. More precisely, suppose that $G = \Gamma \times S^1$, where Γ is a finite group not necessarily abelian. Let V be an orthogonal representation of G such that

$$V = V^G \oplus \bigoplus_{\beta\in\mathfrak{B}} V_\beta$$

is the isotypical decomposition of V. Denote by V_n, $n = 0, 1, 2, \dots k$, the isotypical components of V with respect to the action of $S^1 \subset \Gamma \times S^1$, where $V_0 := V^{S^1}$ and for $n > 0$ we have that the S^1-isotropy group of a non-zero element in V_n is exactly $\mathbf{Z}_n$. Then, if V_β is of complex type and there is an orbit type $(H) \in \Phi_1(G)$ in V_β, then $V_\beta \subseteq V_n$ for some $n > 0$. Since V_n for $n > 0$ has a natural complex structure induced by S^1-action, it is clear that all the components V_β contained in V_n are of complex or quaternionic type. We denote by $\mathfrak{C}_1$ the set of all β such that $V_\beta \subseteq V_n$ for some $n > 0$ and V_β is of complex type, and by $\mathfrak{R}$ the set of all β such that V_β is of real type.

Assume that $f : V \oplus \mathbf{R} \to V$ is a G-equivariant C^1-map satisfying condition (A) in Section 2.3 of Chapter 2. If we use the same notation as before, we can define $\omega : S^1 \to GL^G(V^*)$, where $V^* := \bigoplus_{\beta \in \mathfrak{B}} V_\beta$, by $\omega(\lambda)v := Df(\eta(\lambda))v$, $v \in V^*$. Then we can define $\mu(\omega)$ by (2.3.3) and $\nu(\omega)$ by

$$\nu(\omega) := \prod_{\beta \in \mathfrak{R}} \Gamma\text{-Deg}\,(\omega_\beta(\lambda_o), B_\beta), \tag{4.3.1}$$

where B_β denotes the unite ball in V_β and $\lambda_o \in S^1$. Since for $\beta \in \mathfrak{R}$ we have $GL^G(V_\beta) \simeq GL(n_\beta, \mathbf{R})$, we can use this identification to define the number $\nu_\beta(\omega) :=$ sign det $(\omega(\lambda_o))$, where det $: GL(n_\beta, \mathbf{R}) \to \mathbf{R}^*$. Then we have

$$\Gamma\text{-Deg}\,(\omega_\beta(\lambda_o), B_\beta) = \Gamma\text{-Deg}\,(\nu_\beta(\omega)\text{Id}, B_\beta).$$

Theorem 4.3.1 *Let $G = \Gamma \times S^1$, where Γ is a finite group, and let V be an orthogonal representation of G. Suppose that $f : V \oplus \mathbf{R} \to V$ is a G-equivariant mapping satisfying assumption (A). Then*

$$G\text{-Deg}\,(f, \Omega) = \nu(\omega) \cdot \mu(\omega),$$

where $\nu(\omega)$ is given by (4.3.1) and $\mu(\omega)$ by (2.3.3).

Proof. The proof is analogous to that of Theorem 2.3.2.

We may assume, without loss of generality, that the following conditions are satisfied:

(i) $\Omega_0 := \Omega \cap (V^G \oplus \mathbf{R})$ is a tubular neighbourhood of Σ in $V^G \oplus \mathbf{R}$. We denote by $\pi : \Omega_0 \to \Sigma$ the natural projection of Ω_0 onto Σ, so that every element $x \in \Omega_0$ can be written uniquely as $x = \pi(x) + w$, where w is normal to Σ at $\pi(x)$.

(ii) $\Omega = \Omega_0 \times \widetilde{\Omega}$, where $\widetilde{\Omega} := \Omega_r \times \widehat{\Omega}$, $\Omega_r := \prod_{\beta \in \mathfrak{R}} B(V_\beta)$, $B(V_\beta)$ is the unit ball in V_β, $\widehat{\Omega}$ is the unit ball in $\widehat{V} = \bigoplus_{\beta \in \mathfrak{B} \setminus \mathfrak{R}} V_\beta$.

(iii) $f(x,v) = f_0(x) + Df(\pi(x))v$, where $f_0 := f|_{\overline{\Omega_0}}$ and $(x,v) \in \overline{\Omega_0} \times \overline{\overline{\Omega}}$.

For $\beta \in \mathfrak{R}$, we may identify $GL^G(V_\beta)$ with $GL(\mathbf{R}, d_\beta)$. By a similar argument to those of Theorem 2.3.2 and Lemma 2.3.1, we may assume that the homotopy class of ω_β is trivial and thus contains a representative $B_\beta : S^1 \to GL(\mathbf{R}, d_\beta)$ given by

$$B_\beta(\lambda) = B_\beta = \begin{bmatrix} (-1)^{\nu_\beta} & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix} : \mathbf{R}^{d_\beta} \longrightarrow \mathbf{R}^{d_\beta}. \tag{4.3.2}$$

We may define a mapping $\widetilde{f} : \overline{\Omega} \to V$ by

$$\widetilde{f}(x,v) = f_0(x) + A(\pi(x))v, \quad v \in \overline{\overline{\Omega}}, \ \ x \in \overline{\Omega_0},$$

where for $\sigma \in \Sigma$

$$A(\pi(x)) = \bigoplus_{\beta \in \mathfrak{R}_2} B_\beta \oplus Df(\pi(x))_{|\widehat{V}}.$$

It follows from the homotopy invariance that

$$G\text{-Deg}(f, \Omega) = G\text{-Deg}(\widetilde{f}, \Omega),$$

and consequently by Theorem 4.2.2, we have

$$G\text{-Deg}\,(f, \Omega) = \nu(\omega) \cdot G\text{-Deg}\,(\widehat{f}, \Omega^o),$$

where $\Omega^o := \Omega_0 \times \widehat{\Omega} \subset V^G \oplus \widehat{V}$, $\widehat{f} : \overline{\Omega^o} \to V^o$ and $\widehat{f}(x,v) = f_0(x) + Df(\pi(x))v$ for $v \in \widehat{V}$. We will compute $G\text{-Deg}\,(\widehat{f}, \Omega^o)$.

We choose a total order in $\mathfrak{B} \backslash \mathfrak{R}$ so that we can write $\mathfrak{B} \backslash \mathfrak{R} = \{\beta_1, \dots, \beta_n, \beta_{n+1}, \dots \dots, \beta_{n+r}\}$, where $\mathfrak{C}_1 = \{\beta_1, \dots, \beta_n\}$. For a vector $v \in \widehat{V}$, we denote by v_i the V_{β_i}-component of v, $i = 1, \dots, n+r$, and put

$$v^i = v_1 + v_2 + \dots + v_i, \qquad 1 \le i \le n+r.$$

We will simplify the form of the linear mapping $Df(\pi(x))$. For all $i = 1, 2, \dots, n$, we identify $GL^G(V_{\beta_i})$ with $GL(\mathbb{C}, d_{\beta_i})$ and choose a representative of the homotopy class of $\omega_{\beta_i} : S^1 \to GL(\mathbb{C}, d_{\beta_i})$ which has the following matrix form

$$b_i(\lambda) = \begin{bmatrix} \lambda^{\mu_i} & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix} : \mathbb{C}^{d_i} \to \mathbb{C}^{d_i} \tag{4.3.3}$$

where $\mu_i = \mu_{\beta_i} := \Delta([\omega_{\beta_i}])$ and $d_i := d_{\beta_i}$. If U_{β_i} is of quaternionic type, then we can assume that $b_i(\lambda) = \mathrm{Id}_{V_{\beta_i}}$, for others β_i we put $b_i = \omega_{\beta_i}$.

We can define a mapping $g : \overline{\Omega^o} \to \widehat{V}$ by

$$g(x, v) = f_0(x) + a(\pi(x))v, \quad v \in \widehat{V}, \ x \in \overline{\Omega_0},$$

where for $\sigma \in \Sigma$

$$a(\sigma) = \sum_{i=1}^{n+r} a_i(\sigma),$$
$$a_i(\sigma) := (b_i \circ \eta^{-1})(\sigma), \quad a_i(\sigma) : V_{\beta_i} \to V_{\beta_i}, \quad i = 1, \dots, n + r.$$

It follows from the homotopy invariance that

$$G\text{-Deg}(\widehat{f}, \Omega^o) = G\text{-Deg}(g, \Omega^o).$$

We now introduce the following piecewise linear functions

$$q_j(t) = \begin{cases} 1 & \text{if } 0 \le t < s_j \\ -\frac{1}{\varepsilon_j}(t - t_j) & \text{if } s_j \le t < t_j \\ 0 & \text{if } t_j \le t, \end{cases} \tag{4.3.4}$$

where $j = 1, 2, \ldots$, and

$$s_j = \frac{j}{j+1} - \frac{1}{2(j+2)^2}$$
$$t_j = \frac{j}{j+1} + \frac{1}{2(j+2)^2}$$
$$\varepsilon_j = t_j - s_j = \frac{1}{(j+2)^2}.$$

Clearly, $\frac{t_j+s_j}{2} = \frac{j}{j+1}$, $q_j(\frac{j}{j+1}) = \frac{1}{2}$, and $q_j(t) \neq 0$ for $t \in (s_j, t_j)$.

Our next step is to further deform g to a new mapping h. We define

$$c_i : \overline{\Omega^o} \to V_{\beta_i}, \quad i = 1, \ldots, n+r$$

by the following formula

$$c_i(x, v) = q_{n-i+1}(\|v^i\|)v_i + (1 - q_{n-i+1}(\|v^i\|))a_i(\pi(x))v_i, \tag{4.3.5}$$

where $i = 1, 2, \ldots, n$, $v \in \widehat{V}$, $x \in \overline{\Omega_0}$ and $c_i = a_i$ for $i = n+1, \ldots, n+r$. Let $h : \overline{\Omega^o} \to V^o$ be defined by

$$h(x, v) = f_0(x) + c(x, v), \quad x \in \overline{\Omega_0},\ v \in \widehat{V},$$

where $c(x, v) = \sum_{i=1}^{n+r} c_i(x, v)$, $v \in \widehat{V}$. Then again, by the homotopy property, we have

$$G\text{-Deg}\,(\widetilde{f}, \Omega^o) = G\text{-Deg}\,(h, \Omega^o).$$

It is clear that $\Sigma \subset h^{-1}(0)$. In order to describe other zeros of h in Ω, we note that $h(x, v) = 0$ if and only if $x \in \Sigma$ and $c_i(x, v) = 0$ for all $i = 1, \ldots, n+r$. It implies that for $i = 1, \ldots, n$

$$q_{n-i+1}(\|v^i\|)v_i + (1 - q_{n-i+1}(\|v^i\|))a_i(\sigma)v_i = 0, \quad x = \sigma + w, \sigma = \pi(x).$$

If for some $i \in \{1, \ldots, n\}$, $v_i \neq 0$, then it follows from the definition of $a_i(\sigma)$ that

$v_i = (z_i, 0, \ldots, 0) \in \mathbf{C}^{d_i}$, $z_i \neq 0$, and

$$q_{n-i+1}(\|v^i\|) + (1 - q_{n-i+1}(\|v^i\|))\lambda^{\mu_i} = 0, \quad \lambda = \eta^{-1}(\sigma).$$

Therefore,

$$q_{n-i+1}(\|v^i\|) = \frac{1}{2}, \qquad \|v^i\| = \frac{n-i+1}{n-i+2}, \qquad \lambda^{\mu_i} = -1. \tag{4.3.6}$$

We claim that if (x, v) is a solution of the equation $h(x, v) = 0$, then the vector v can not have more than one non-zero component v_i. Indeed, suppose on the contrary that v has two nonzero components v_i and v_j, where $1 \le i < j \le n$. Then it follows from the definition that $\|v^i\| \le \|v^j\|$. However, by (2.12), we have

$$\|v^i\| = \frac{n-i+1}{n-i+2} > \frac{n-j+1}{n-j+2} = \|v^j\|,$$

a contradiction.

Consequently, we can classify all the solutions of the equation $h(x, v) = 0$ as follows:

(I) the set Σ;

(II) for every $i \in \{1, \ldots, nr\}$ there are $|\mu_i|$ sets of solutions $\Sigma_{i,\ell} := \{x_\ell\} \times \{v, 0, \ldots, 0\} \in U^{d_i}_{\beta_i}$, $|v| = \frac{n-i+1}{n-i+2} \ge \frac{1}{2}$ and $\lambda_1, \ldots, \lambda_{|\mu_i|}$ denote the complex roots of the equation $\lambda^{\mu_i} = -1$ with $x_\ell = \eta(\lambda_\ell)$.

We now compute G-Deg(h, Ω). We first compute G-Deg(h, U) for a neighbourhood U of Σ. Note that if $\|v\| < \frac{1}{2}$, then $\|v^i\| < \frac{1}{2}$ for all i, and thus $q_{n-i+1}(\|v^i\|) = 1$. Therefore, the mapping h transforms vectors $v \in \widehat{V}$, $\|v\| < \frac{1}{2}$, identically into themselves, i.e. $h(x, v + v_0) = h(x, v_0) + v$. Thus, it is normal and G-Deg$(h, U) = 0$.

We choose an invariant neighbourhood $\mathcal{U}_{i,\ell}$ of $\Sigma_{i,\ell}$ such that

$$q_{n-j+1}(\|v^j\|) = \begin{cases} 1 & \text{for } j < i, \ (x, v) \in \mathcal{U}_{i,\ell}, \\ 0 & \text{for } j > i, \ (x, v) \in \mathcal{U}_{i,\ell}. \end{cases}$$

Therefore, the map h on $\mathcal{U}_{i,\ell}$ is homotopic (by an $\mathcal{U}_{i,\ell}$-admissible homotopy) to a map $h_{(i,\ell)}(x,v) = f_0(x) + \widetilde{c}(v)$, where $\widetilde{c} = \bigoplus \widetilde{c}_j$ with $\widetilde{c}_j = \mathrm{Id}_{V_{\beta_j}}$ for $j \neq i$ and

$$\widetilde{c}_i(v_i^1, v_i^2, \dots, v_i^{d_i}) = (q_{n-i+1}(\|v_i^1\|)v_i^1 + (1 - q_{n-i+1}(\|v_i^1\|)v_i^1, v_i^2, \dots, v_i^{d_i}).$$

In addition, by changing the coordinate system in $V^G \oplus \mathbf{R}$, we may assume that $f_0(u,z) = (u, 1 - |z|)$, where $(u,z) = (u,t,s) \in (V_o^G \times \mathbf{R}) \oplus \mathbf{R} = V^G \oplus \mathbf{R}$, and that $x_{i,\ell} = (0, \dots, 1, 0)$. It is now clear that $G\text{-Deg}\,(h, U_{i,\ell}) = G\text{-Deg}\,(f_{\beta_i}, \Omega_{\beta_i})$, where $f_{\beta_i}(v,z) = (z \cdot v, |z|(\|v\| - 1) + \|v\| + 1)$ for $(v,z) \in \overline{\Omega_{\beta_i}} := \{(v,z) \in U_{\beta_i} \oplus \mathbf{C};\ \ \|v\| \leq 1,\ \frac{1}{2} < |z| < 2\}$. Now, by using the additivity property, we obtain that

$$G\text{-Deg}\,(\widehat{f}, \Omega^o) = G\text{-Deg}\,(h, \Omega^o) = \sum_{i=1}^{n} \mu_i \cdot G\text{-Deg}\,(f_{\beta_i}, \Omega_{\beta_i}).$$

This completes the proof.

□

Finally, we present the following example for the application of the previous theorem and of Theorem 4.2.2 (multiplicativity property) to the computations of the G-degree of a certain function f equivariant under the action of the non-abelian group $D_N \times S^1$.

Example 4.3.2 We consider the group $G = D_N \times S^1$, where $N \geq 3$. The irreducible 4-dimensional representations of G (such that both the action of D_N and S^1 are nontrivial) can be described as the action of $G = D_N \times S^1$ on the space $\mathbf{R}^2 \oplus \mathbf{R}^2 = \mathbf{C} \oplus \mathbf{C}$ given by:

(a1) $(\gamma, \tau)(z_1, z_2) := (\gamma^j \tau^l z_1, \gamma^{-j} \tau^l z_2)$ for $(\gamma, \tau) \in \mathbf{Z}_N \times S^1$;

(a2) $(\kappa\gamma, \tau)(z_1, z_2) := (\gamma^{-j} \tau^l z_2, \gamma^j \tau^l z_1)$ for $(\kappa\gamma, \tau) \in \kappa\mathbf{Z}_N \times S^1$,

where $(z_1, z_2) \in \mathbf{C} \oplus \mathbf{C}$, $l = 1, 2, 3, \dots$, $j = 1, 2, \dots, [\frac{N}{2}]$ and $h = gcd(j, N)$ is such that $m := \frac{N}{h} > 2$ (otherwise the action of G is abelian and the representation $\mathbf{C} \oplus \mathbf{C}$ is reducible).

If N is odd then we have nontrivial irreducible two-dimensional representations on $\mathbf{R}^2 = \mathbf{C}$ of $D_N \times S^1$ which are given by:

(b1) $(\gamma, \tau)z = \tau^l z$, $(\gamma, \tau) \in \mathbf{Z}_N \times S^1$,

(b2) $(\kappa\gamma, \tau)z = -\tau^l z$, $(\kappa\gamma, \tau) \in \kappa\mathbf{Z}_N \times S^1$,

where $l = 1, 2, 3, \dots$.

If N is even then there are additional two-dimensional irreducible representations on $\mathbf{R}^2 = \mathbf{C}$ (non-trivial with respect to the action of D_N) of $D_N \times S^1$ given by

(c1) $(g, \tau)z = \tau^l z$, if $(g, \tau) \in D_{\frac{N}{2}} \times S^1$;

(c2) $(g, \tau)z = -\tau^l z$, if $(g, \tau) \in (D_N \setminus D_{\frac{N}{2}}) \times S^1$.

Finally, we have two dimensional representations on $\mathbf{R}^2 = \mathbf{C}$ of $D_N \times S^1$, where $2|N$, given by

(d1) $(\gamma, \tau)z = \gamma^{\frac{N}{2}} \tau^l z$, where $(\gamma, \tau) \in \mathbf{Z}_N \times S^1$,

(d2) $(\kappa\gamma, \tau)z = -\gamma^{\frac{N}{2}} \tau^l z$, where $(\kappa\gamma, \tau) \in \kappa\mathbf{Z}_N \times S^1$.

We consider a representation $V = \mathbf{C} \oplus \mathbf{C}$ of G given by (a1) and (a2), where $h = gcd(j, N)$ is such that $m := \frac{N}{h} > 2$. Denote

$$\Omega := \{(v, z) \in V \oplus \mathbf{C};\ \ \|v\| < 1,\ \frac{1}{2} < |z| < 2\}.$$

In what follows, we will consider the map

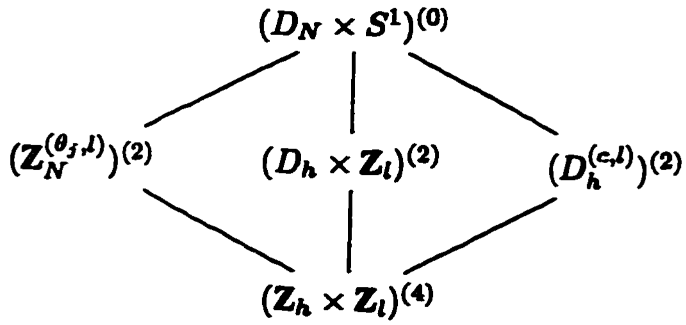
$$f : \overline{\Omega} \to V \oplus \mathbf{R},$$

previously encountered in Chapter 2 (2.3.2), given by

$$f(v, z) = f(z \cdot v, |z|(\|v\| - 1) + \|v\| + 1). \tag{4.3.7}$$

The map f is evidently G-equivariant and Ω-admissible. We will compute G-degree of f in Ω.

Case m is odd: In this case, we have the following lattice of the isotropy groups in V:



where $\theta_j : \mathbf{Z}_N \to S^1$ is given by $\theta_j(\gamma) = \gamma^{-j}$, $\gamma \in \mathbf{Z}_N$, j is an integer, and the numbers in brackets denote the dimension of the corresponding fixed-point space. We also have the following table:

Table 4.5. Normalizers and Weyl's Groups of Isotropy Subgroups

H	$N(H)$	$W(H)$	# of conjugated subgroups
$D_N \times S^1$	$D_N \times S^1$	$\mathbb{I}$	1
$\mathbf{Z}_N^{(\theta_j,l)}$	$\mathbf{Z}_N \times S^1$	S^1	2
$D_h \times \mathbf{Z}_l$	$D_h \times S^1$	S^1	m
$D_h^{(c,l)}$	$D_h \times S^1$	S^1	m
$\mathbf{Z}_h \times \mathbf{Z}_l$	$D_N \times S^1$	$D_m \times S^1$	1

Now we are in the position to compute the G-Deg(f, Ω) for m odd. By using the results for S^1-degree (cf. [21,43]), we obtain immediately that for $H = \mathbf{Z}_N^{(\theta_j,l)}$, $D_h \times \mathbf{Z}_l$, or $D_h^{(c,l)}$, the coefficient $n_{(H)}$ of the degree is equal to 1. In order to compute the coefficient $n_{(H)}$ for $H = \mathbf{Z}_h \times \mathbf{Z}_l$, we remark that

$$S^1\text{-Deg}\,(f^H, \Omega^H) = 2(\mathbf{Z}_n \times \mathbf{Z}_l).$$

It follows from the above table that

$$\sum_{K \supsetneq H} \Big(S^1\text{-Deg}\,(f^K, \Omega^K) \Big) = 2 + m + m.$$

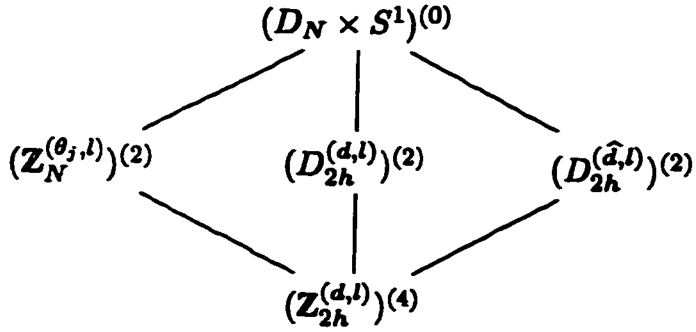
Therefore,

$$n_H = \Big(2 - (2 + m + m) \Big) / (2m) = -1.$$

Consequently,

$$G\text{-Deg}\,(f, \Omega) = (\mathbf{Z}_N^{(\theta_j, l)}) + (D_h \times \mathbf{Z}_l) + (D_h^{(c,l)}) - (\mathbf{Z}_h \times \mathbf{Z}_l).$$

Case $m \equiv 2 \pmod 4$**:** In this case, we have the following lattice of isotropy subgroups:



and the following table:

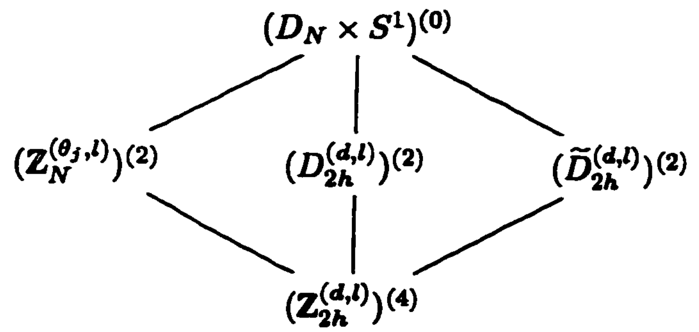
Table 4.6. Normalizers and Weyl's Groups of Isotropy Subgroups

H	$N(H)$	$W(H)$	# of conjugated subgroups
$D_N \times S^1$	$D_N \times S^1$	$\mathbb{I}$	1
$\mathbf{Z}_N^{(\theta_j,l)}$	$\mathbf{Z}_N \times S^1$	$\dot{S}^1$	2
$D_{2h}^{(d,l)}$	$D_{2h} \times S^1$	S^1	$\frac{m}{2}$
$D_{2h}^{(\hat{d},l)}$	$D_{2h} \times S^1$	S^1	$\frac{m}{2}$
$\mathbf{Z}_{2h}^{(d,l)}$	$D_N \times S^1$	$D_{\frac{m}{2}} \times S^1$	1

By arguments similar to those above, we obtain:

$$G\text{-Deg}\,(f,\Omega) = (\mathbf{Z}_N^{(\theta_j,l)}) + (D_{2h}^{(d,l)}) + (D_{2h}^{(\hat{d},l)}) - (\mathbf{Z}_{2h}^{(d,l)}).$$

Case $m \equiv 0 \pmod 4$**:** In this case, we have the following lattice of isotropy subgroups:



and the following table:

Table 4.7. Normalizers and Weyl's Groups of Isotropy Subgroups

H	$N(H)$	$W(H)$	# of conjugated subgroups
$D_N \times S^1$	$D_N \times S^1$	$\mathbb{I}$	1
$\mathbf{Z}_N^{(\theta_j,l)}$	$\mathbf{Z}_N \times S^1$	S^1	2
$D_{2h}^{(d,l)}$	$D_{2h} \times S^1$	S^1	$\frac{m}{2}$
$\widetilde{D}_{2h}^{(d,l)}$	$D_{2h} \times S^1$	S^1	$\frac{m}{2}$
$\mathbf{Z}_{2h}^{(d,l)}$	$D_N \times S^1$	$D_{\frac{m}{2}} \times S^1$	1

It is now easy to verify that

$$G\text{-Deg}\,(f, \Omega) = (\mathbf{Z}_N^{(\theta_j,l)}) + (D_{2h}^{(d,l)}) + (\widetilde{D}_{2h}^{(d,l)}) - (\mathbf{Z}_{2h}^{(d,l)}).$$

It is now clear that in the case of an arbitrary orthogonal representation V of $G = D_N \times S^1$ and a G-equivariant mapping $\hat{f} : V \oplus \mathbf{R} \to V$ satisfying assumption (A), the winding element $\mu(\omega)$, as defined in (2.3.3), can be computed using the above formulas. Consequently, the G-degree of $\hat{f}$ is fully computable.

Chapter Five

Hopf Bifurcations of FDE's with Dihedral Symmetries

5.1 Introduction

In this chapter, we develop a Hopf bifurcation theory for functional differential equations (FDEs) with dihedral symmetries with emphasis on the description and classification of all possible types of Hopf bifurcations with the dihedral group. We will also discuss the joint impact of the time delay and the spatial dihedral symmetry on the multiplicity of bifurcating solutions in coupled cells.

Our main technical tools are the computational formulae obtained in Chapters 2 and 4 ([42]) for the equivariant degree. In principle, the computation is a key and crucial issue in the application of the equivariant degree. However, in the case where the involved nonlinearity is a family of equivariant maps parametrized by a single real parameter, the Ulrich type computation formula obtained in Chapter 2 [42] shows that the computation of the equivariant degree can be significantly simplified to the calculation of the relatively simpler S^1-degree [14,16] and to the counting of certain orbits of the zeros of the nonlinear map restricted to certain subspaces. We will show that the aforementioned formula together with the idea of complementing functions and the standard linear homotopy technique enables the development of the degree theoretical approach to the Hopf bifurcation theory for FDEs with dihedral symmetries. The existence of a large number of branches of periodic solutions can be obtained by using information about the subgroup structure and the irreducible representations of $D_N \times S^1$ and by using certain local Hopf bifurcation invariants which can be completely calculated from the classical Brouwer degree of an analytic function arising from the linearization of the FDEs at equilibria.

This approach, based on equivariant degree and complementing functions, provides an alternative method for the study of symmetric Hopf bifurcation problems. It does not require genericity conditions on vector fields and dimension restrictions on some point spaces. Moreover, the procedure for the calculation of the local bifurcation invariants is quite standard and involves only elementary algebraic computations of subgroups and irreducible representations of the involved groups. We are presenting a comprehensive example in the case of $D_N \times S^1$, but the method can be applied in a similar (but, of course, more complicated) fashion to one-parameter symmetric bifurcation problems involving more complex spatial groups such as $O(2)$, $SO(3)$ and $O(3)$.

Symmetric bifurcation problems have been extensively studied and the monographs [17,24,25,33] provide some detail account of the subject for ordinary differential equations and partial differential equations. As for FDEs, an analytic (local) Hopf bifurcation theorem was obtained in [58] as an analogue of the Golubitsky-Stewart theorem [23]. Moreover, a topological Hopf bifurcation theory was developed in [45] for FDEs in the case where the spatial symmetry group is the abelian group $\mathbf{Z}_N$ or $\mathbf{Z}_\infty$. While the problem of looking for bifurcations of periodic solutions with prescribed symmetries in a general case can always be reduced to the one where the spatial symmetry group is $\mathbf{Z}_N$ or $\mathbf{Z}_\infty$ (see [17,23]), examining the global interaction of all bifurcated periodic solutions requires the consideration of the full symmetry. Our results, especially the presented application to coupled cells arising from neural networks with memory [55,57], illustrate that a non-abelian action, due to the fact that its irreducible representations may contain many different orbit types, can cause spontaneous bifurcations of multiple branches of periodic solutions with various types. For example, if a coupled cell consists on N cells with N being a prime number, then at certain critical values of the parameter (usually the delay) the system possess at least $2(N+1)$ distinct branches of non-constant periodic solutions with certain spatio-temporal patterns.

Section 5.2 contains the general results on (local) Hopf bifurcations of FDEs with dihedral group symmetry and Section 5.3 presents some applications of the general results to coupled cells.

5.2 Hopf D_N-Symmetric Bifurcation Theorems

Let $\tau \geq 0$ be a given constant, n a positive integer and $C_{n,\tau}$ the Banach space of continuous functions from $[-\tau, 0]$ into $\mathbf{R}^n$ equipped with the usual supremum norm

$$\|\varphi\| = \sup_{-\tau\leq\theta\leq 0} |\varphi(\theta)|, \qquad \varphi \in C_{n,\tau}.$$

Also, if $x : [-\tau, A] \to \mathbf{R}^n$ is a continuous function with $A > 0$ and if $t \in [0, A]$, then $x_t \in C_{n,\tau}$ is defined by

$$x_t(\theta) = x(t+\theta), \quad \theta \in [-\tau, 0].$$

In what follows, for any $x \in \mathbf{R}^n$ we will use $\bar{x}$ to denote the constant mapping from $[-\tau, 0]$ into $\mathbf{R}^n$ with the value $x \in \mathbf{R}^n$.

Consider the following one parameter family of retarded functional differential equations

$$\dot{x} = f(x_t, \alpha), \tag{5.2.1}$$

where $x \in \mathbf{R}^n$, $\alpha \in \mathbf{R}$, $f : C_{n,\tau} \times \mathbf{R} \to \mathbf{R}^n$ is a continuously differentiable and completely continuous mapping. Assume there is an orthogonal representation Θ :

$\Gamma \to O(n)$ of $\Gamma := D_N$, $N > 2$, on $\mathbf{R}^n$, which naturally induces an isometric Banach representation of Γ on the space $C_{n,\tau}$ with the action $\cdot : \Gamma \times C_{n,\tau} \to C_{n,\tau}$ given by:

$$(\gamma\varphi)(\theta) := \Theta(\gamma)(\varphi(\theta)), \qquad \gamma \in \Gamma, \ \theta \in [-\tau, 0].$$

We make the following assumptions

(A1): The mapping f is Γ-equivariant, i.e.

$$f(\gamma\varphi, \alpha) = \gamma f(\varphi, \alpha), \quad \varphi \in C_{n,\tau}, \ \alpha \in \mathbf{R}, \ \gamma \in \Gamma.$$

(A2): $f(0, \alpha) = 0$ for all $\alpha \in \mathbb{R}$, i.e. $(0, \alpha)$ is a ***stationary solution*** of (5.2.1) for every $\alpha \in \mathbf{R}$.

Since $\mathbf{R}^n$ is an orthogonal representation of the group D_N, we have the following unique isotypical decomposition of $\mathbf{R}^n$ with respect to the action of D_N

$$V := \mathbb{R}^n = V_0 \oplus V_1 \oplus \cdots \oplus V_k, \tag{5.2.2}$$

where $k = (N+1)/2$ if N is odd, $k = (N+4)/2$ if N is even [25], and

(i) V_0 denotes the fixed-point space of the action of Γ, i.e. $V_0 := V^\Gamma = \{v \in V; \ \forall \gamma \in \Gamma \ \gamma v = v\}$.

(ii) Each one of the subrepresentations V_j, $(j = 1, \ldots, k)$ called an ***isotypical component***, is a direct sum of all subrepresentations of V equivalent to a certain irreducible orthogonal representation on D_N, and all the irreducible subrepresentations ρ of D_N are described as follows:

(a1) For every integer number $1 \le j < [\frac{N}{2}]$ there is an orthogonal representation ρ_j (of real type) of D_N on $\mathbf{C}$ given by:

$$\gamma z := \gamma^i \cdot z, \quad \gamma \in \mathbf{Z}_N, \ z \in \mathbf{C};$$
$$\kappa z := \bar{z},$$

where $\gamma^j \cdot z$ denotes the usual multiplication of the complex numbers;

(a2) There is a representation $c : D_N \to \mathbf{Z}_2 \subset O(1)$, such that Ker $c = \mathbf{Z}_N$;

(a3) If N is even, there is an irreducible representation $d : D_N \to \mathbf{Z}_2 \subset O(1)$, and Ker $d = D_{\frac{N}{2}}$, and

(a4) If N is even and $j = \frac{N}{2}$, there is an irreducible representation $\hat{d} : D_N \to \mathbf{Z}_2 \subset O(1)$, and Ker $\hat{d} = \hat{D}_{\frac{N}{2}}$.

We will denote by $U := \mathbf{C}^n$ the standard complexification of $V = \mathbf{R}^n$. It is not difficult to see that the isotypical decomposition (5.2.2) induces the following isotypical decomposition of the complex representation U:

$$U = U_0 \oplus U_1 \oplus \cdots \oplus U_k, \tag{5.2.3}$$

where $U_0 := U^\Gamma$ and each of the isotypical components U_j is characterized by complex representation of the following types:

(b1) For $1 \le j < [\frac{N}{2}]$ the representation η_j on $\mathbf{C} \oplus \mathbf{C}$ is given by

$$\gamma(z_1, z_2) := (\gamma^j \cdot z_1, \gamma^{-j} \cdot z_2), \quad \gamma \in \mathbf{Z}_N,\ z_1,\ z_2 \in \mathbf{C},$$
$$\kappa(z_1, z_2) := (z_2, z_1);$$

(b2) The representation $c : D_N \to \mathbf{Z}_2 \subset U(1)$, such that Ker $c = \mathbf{Z}_N$;

(b3) In the case when N is even, the representation $d : D_N \to \mathbf{Z}_2 \subset U(1)$, such that Ker $d = D_{\frac{N}{2}}$; and

(b4) In the case when N is even, the representation $\hat{d} : D_N \to \mathbf{Z}_2 \subset U(1)$ such that Ker $\hat{d} = \hat{D}_{\frac{N}{2}}$.

An element $(x, \alpha) \in \mathbf{R}^n \times \mathbf{R}$ is called a stationary solution of (5.2.1) if $f(\bar{x}, \alpha) = 0$. A complex number $\lambda \in \mathbf{C}$ is said to be a *characteristic value* of the stationary solution

(x, α) if it is a root of the following ***characteristic equation***

$$\det_{\mathbf{C}} \Delta_{(x,\alpha)}(\lambda) = 0, \tag{5.2.4}$$

where

$$\Delta_{(x,\alpha)}(\lambda) := \lambda \mathrm{Id} - D_x f(\bar{x}, \alpha)(e^{\lambda \cdot}\mathrm{Id}).$$

A stationary solution (x_0, α_0) is called ***nonsingular*** if $\lambda = 0$ is not a characteristic value of (x_0, α_0), and a nonsingular stationary point (x_0, α_0) is called a ***center*** if it has a purely imaginary characteristic value. We will call (x_0, α_0) an ***isolated center*** if it is the only center in some neighborhood of (x_0, α_0) in $\mathbf{R}^n \times \mathbf{R}$.

We now make the following assumption:

(A3): There is a stationary solution $(0, \alpha_0)$ which is an isolated center such that $\lambda = i\beta_0$, $\beta_0 > 0$, is a characteristic value of $(0, \alpha_0)$.

Let $\Omega_1 := (0, b) \times (\beta_0 - c, \beta_0 + c) \subset \mathbf{C}$. Under assumption (A3), the constants $b > 0$, $c > 0$ and $\delta > 0$ can be chosen such that the following condition is satisfied:

(*) For every $\alpha \in [\alpha_0 - \delta, \alpha_0 + \delta]$ if there is a characteristic value $u + iv \in \partial\Omega_1$ of $(0, \alpha)$ then $u + iv = i\beta_0$ and $\alpha = \alpha_0$.

Note that $\Delta_{(0,\alpha)}(\lambda)$ is analytic in $\lambda \in \mathbf{C}$ and continuous in $\alpha \in [\alpha_0 - \delta, \alpha_0 + \delta]$. It follows that $\det_{\mathbf{C}} \Delta_{(0,\alpha_0 \pm \delta)}(\lambda) \neq 0$ for $\lambda \in \partial\Omega_1$.

Since the mapping f is Γ-equivariant, for every $\alpha \in \mathbf{R}$ and $\lambda \in \mathbf{C}$ the operator $\Delta_{(0,\alpha)}(\lambda) : \mathbf{C}^n \to \mathbf{C}^n$ is Γ-equivariant and consequently for every isotypical component U_j of $U = \mathbf{C}^n$ we have $\Delta_{(0,\alpha)}(\lambda)(U_j) \subseteq U_j$ for $j = 0, 1, \ldots, k$.

We put

$$\Delta_{\alpha,j}(\lambda) := \Delta_{(0,\alpha)}(\lambda)|_{U_j} : U_j \to U_j.$$

Solutions $\lambda \in \mathbf{C}$ of the equation

$$\det{}_{\mathbf{C}}\Delta_{\alpha,j}(\lambda) = 0$$

where $j = 0, 1, \ldots, k$, will be called the j-th ***isotypical characteristic values*** of $(0, \alpha)$. It is clear that λ is a characteristic value of the solution $(0, \alpha)$ if and only if it is a j-th isotypical characteristic value of $(0, \alpha)$ for some $j = 0, 1, \ldots, k$.

Following the idea of a crossing number in a non-equivariant case (cf. [16,21,43,46]), we define

$$c_{1,j}(\alpha_0, \beta_0) := \deg_B(\det{}_{\mathbf{C}}\Delta_{\alpha_0-\delta,j}(\cdot), \Omega) - \deg_B(\det{}_{\mathbf{C}}\Delta_{\alpha_0+\delta,j}(\cdot), \Omega)$$

for $0 \le j \le k$. The number $c_{1,j}(\alpha_0, \beta_0)$ will be called the j-th ***isotypical crossing number***, for the isolated center $(0, \alpha_0)$ corresponding to the characteristic value $i\beta_0$. The crossing number $c_{1,j}(\alpha_0, \beta_0)$ indicates how many j-th characteristic values (counted with algebraic multiplicity) of the stationary points $(0, \alpha)$ "escape" from the region Ω_1 when α crosses the value α_0.

Since an integer multiple of $i\beta_0$ can also be a j-th isotypical characteristic value of $(0, \alpha_0)$, we define for $l > 1$

$$c_{l,j}(\alpha_0, \beta_0) := \deg_B(\det{}_{\mathbf{C}}\Delta_{\alpha_0-\delta,j}(\cdot), \Omega_l) - \deg_B(\det{}_{\mathbf{C}}\Delta_{\alpha_0+\delta,j}(\cdot), \Omega_l)$$

where $\Omega_l := (0, b) \times (l\beta_0 - c, l\beta_0 + c) \subset \mathbf{C}$ and the constants $b > 0$, $c > 0$ and $\delta > 0$ can be chosen to be sufficiently small so that there are no characteristic values of $(0, \alpha)$ in $\partial\Omega_l$ except perhaps $il\beta_0$ for $\alpha = \alpha_0$. In other words, $c_{l,j}(\alpha_0, \beta_0) = c_{1,j}(\alpha_0, l\beta_0)$. If $il\beta$ is not a j-th isotypical characteristic value of $(0, \alpha_0)$ then clearly $c_{l,j}(\alpha_0, \beta_0) = 0$.

In order to establish the existence of small amplitude periodic solutions bifurcating from the stationary point $(0, \alpha_0)$, i.e. the existence of Hopf bifurcations at the stationary point $(0, \alpha_0)$, we will use the standard degree-theoretical approach (cf. [16,21,43,45,46]). We reformulate the Hopf bifurcation problem for the equation (5.2.1) as a $\Gamma \times S^1$-equivariant bifurcation problem (with two parameters) in an appropriate Hilbert isometric representation of $G = \Gamma \times S^1$. For this purpose we make the following change of variable $x(t) = z(\frac{\beta}{2\pi}t)$ for $t \in \mathbf{R}$. We obtain the following, equivalent to (5.2.1), equation

$$\dot{z}(t) = \tfrac{2\pi}{\beta} f(z_{t,\beta}, \alpha), \tag{5.2.5}$$

where $z_{t,\beta} \in C_{n,\tau}$ is defined by

$$z_{t,\beta}(\theta) = z\big(t + \tfrac{\beta}{2\pi}\theta\big), \quad \theta \in [-\tau, 0].$$

Evidently, $z(t)$ is a 1-periodic solution of (5.2.5) if and only if $x(t)$ is a $\frac{2\pi}{\beta}$-periodic solution of (5.2.1).

Let $S^1 = \mathbf{R}^1/\mathbf{Z}$, $W = L^2(S^1; \mathbf{R}^n)$ and define

$$L : H^1(S^1; \mathbf{R}^n) \to W, \quad Lz(t) = \dot{z}(t), \quad z \in H^1(S^1; \mathbf{R}^n), \quad t \in S^1;$$

$$K : H^1(S^1; \mathbf{R}^n) \to W, \quad Kz(t) = \int_0^1 z(s)\, ds, \quad z \in H^1(S^1; \mathbf{R}^n), \quad t \in S^1.$$

It can be easily shown that $(L + K)^{-1} : W \to H^1(S^1; \mathbf{R}^N)$ exists and the map $F : W \times (\alpha_0 - \delta, \alpha_0 + \delta) \times (\beta_0 - c, \beta_0 + c) \to W$ defined by

$$F(z, \alpha, \beta) = (L + K)^{-1}\Big[Kz + \tfrac{2\pi}{\beta} N_f(z, \alpha, \beta)\Big]$$

is completely continuous, where $N_f : W \times (\alpha_0 - \delta, \alpha_0 + \delta) \times (\beta_0 - c, \beta_0 + c) \to W$ is defined by

$$N_f(z, \alpha, \beta)(t) = f(z_{t,\beta}, \alpha), \quad t \in S^1, \ (z, \alpha, \beta) \in W \times (\alpha_0 - \delta, \alpha_0 + \delta) \times (\beta_0 - c, \beta_0 + c).$$

Moreover, (z, α, β) is an 1-periodic solution of (5.2.5) if and only if $z = F(z, \alpha, \beta)$.

The space W is an isometric Hilbert representation of the group $G = D_N \times S^1$ with the action being given by

$$(\gamma, \theta)x(t) = \gamma(x(t+\theta)), \quad \theta, t \in S^1,\ \gamma \in D_N,\ x \in W.$$

The nonlinear operator F is clearly G-equivariant.

With respect to the restricted S^1-action on W, we have the following isotypical decomposition of the space W

$$W = \overline{\bigoplus_{l=0}^{\infty} W_l},$$

where W_0 is the space of all constant mappings from S^1 into $\mathbf{R}^n$, and W_l, with $l > 0$ is the vector space of all mappings of the form $x \sin 2l\pi \cdot + y \cos 2l\pi \cdot$, $x,\ y \in \mathbf{R}^n$. For $l > 0$, the subspace W_l can be endowed with a complex structure by

$$i \cdot (x \sin 2l\pi \cdot + y \cos 2l\pi \cdot) = x \cos 2l\pi \cdot - y \sin 2l\pi \cdot, \quad x,\ y \in \mathbf{R}^n.$$

Since the above multiplication by i induces an operator $J : W_l \to W_l$ such that $J^2 = -\mathrm{Id}$, it follows that $\mathrm{Id} + J$ is an Γ-equivariant isomorphism and every function in W_l can be uniquely represented as $e^{i2l\pi \cdot}(x+iy)$, $x, y \in \mathbf{R}^n$. In particular, we notice that the above defined complex structure on W_l coincides with the complex structure given by $x + iy \in \mathbf{C}^n$. In addition, the complex isomorphism $A_l : W_l \to U := \mathbf{C}^n$ given by $A_l(e^{i2l\pi \cdot}(x+iy)) = x + iy$, $x + iy \in \mathbf{C}^n$ is Γ-equivariant. Thus, as a complex Γ-representation, W_l is equivalent to U. Consequently, the isotypical Γ-decomposition (5.2.3) of U induces the following isotypical Γ-decomposition of W_l

$$W_l := W_{0,l} \oplus W_{1,l} \oplus \cdots \oplus W_{k,l},$$

where the isotypical components $W_{j,l}$, $l > 0$ can be described exactly by the same conditions (b1)–(b4). On the other hand the component W_0 is exactly the representation $V = \mathbf{R}^n$, which admits the isotypical decomposition (5.2.2). To unify the notations, we denote this isotypical decomposition by $W_0 = W_{0,0} \oplus \cdots \oplus W_{k,0}$, where for every j we have $W_{j,0} := V_j$. As the complex structure on W_j, with $j > 0$ was defined using the S^1-action, and all the subspaces $W_{j,l}$, with $l > 0$ are complex Γ-invariant subspaces, $W_{j,l}$ with $l > 0$ are also S^1-invariant. Therefore, $W_{j,l}$ are the isotypical G-components of the representation W.

For every j and l, we define

$$a_{j,l}(\alpha,\beta) := \mathrm{Id} - (L+K)^{-1}\left[K + \tfrac{2\pi}{\beta} D_z N_f(0,\alpha,\beta)\right]\Big|_{W_{j,l}},$$

where $(\alpha,\beta) \in (\alpha_0 - \delta, \alpha_0 + \delta) \times (\beta_0 - c, \beta_0 + c)$.

We observe that

$$(L+K)^{-1}\Big(e^{i2l\pi\cdot}(x+iy)\Big) = \tfrac{1}{i2l\pi} e^{i2l\pi\cdot}(x+iy) \tag{5.2.6}$$

for every x, $y \in \mathbf{R}^n$, and since

$$a_{j,l}(\alpha,\beta) = (L+K)^{-1}\left[L - \tfrac{2\pi}{\beta} D_z N_f(0,\alpha,\beta)\right]\Big|_{W_{j,l}},$$

we obtain

$$a_{j,l}(\alpha,\beta)e^{i2l\pi\cdot}z = (L+K)^{-1}\left[i2l\pi\, e^{i2l\pi\cdot}z - \tfrac{2\pi}{\beta} e^{i2l\pi\cdot} D_x f(0,\alpha)(e^{il\beta\cdot})z\right]$$

$$= e^{i2l\pi\cdot}\tfrac{1}{il\beta}\Delta_{(0,\alpha)}(il\beta)(z).$$

Consequently,

$$a_{j,l}(\alpha,\beta) = \tfrac{1}{il\beta}\Delta_{\alpha,j}(il\beta).$$

A standard idea of using topological degrees to study the existence of Hopf bifurcations and the various symmetry properties of the solutions is based, as we have seen in the previous chapters, on the notion of a *complementing function*. More precisely, let $\lambda = \alpha + i\beta = (\alpha, \beta) \in \mathbf{R}^2 = \mathbf{C}$ and $\lambda_0 = \alpha_0 + \beta_0$. In this case, we define a special neighborhood $U(r, \rho)$ of the solution $(0, \lambda_0) \in W \times \mathbf{R}^2$ by

$$U(r, \rho) := \{(z, \lambda) \in W \times \mathbf{C};\ \|z\| < r, \text{ and } |\lambda - \lambda_0| < \rho\}.$$

By taking sufficiently small $r > 0$ and $\rho > 0$, we may assume that the equation

$$F(z, \lambda) = 0, \quad z \in W,\ \lambda \in \mathbf{C} = \mathbf{R}^2, \tag{5.2.7}$$

has no solution (z, λ) such that $(z, \lambda) \in \partial U(r, \rho)$, $z \neq 0$ and $|\lambda - \lambda_0| = \rho$. A G-invariant function $\xi : \overline{U(r, \rho)} \to \mathbf{R}$ define by

$$\xi(z, \lambda) := |\lambda - \lambda_0|(\|z\| - r) + \|z\|$$

is called a *complementing function* with respect to $U(r, \rho)$. Define the mapping $F_\xi : \overline{U(r, \rho)} \to W \times \mathbf{R}$ by

$$F_\xi(z, \lambda) := (F(z, \lambda), \xi(z, \lambda)), \text{ where } (z, \lambda) \in \overline{U(r, \rho)}. \tag{5.2.8}$$

The mapping F_ξ is a compact G-equivariant field. It is well known that the G-equivariant degree $G\text{-Deg}\,(F_\xi, U(r, \rho))$ does not depend on the numbers $r > 0$ and $\rho > 0$ (if r and ρ are sufficiently small), thus the standard properties of G-degree imply that if $G\text{-Deg}\,(F_\xi, U(r, \rho)) \neq 0$ then $(0, \lambda_0)$ is a bifurcation point of (5.2.7), i.e. there exists a continuum $\mathcal{C} \subset U(r, \rho)$ of non-constant periodic solutions of (5.2.7) such that $(0, \lambda_0) \in \overline{\mathcal{C}}$. We can consider the G-degree $G\text{-Deg}\,(F_\xi, U(r, \rho))$ as a local bifurcation invariant.

The computations in Chapter 2 and Chapter 4 provide us with a complete information needed to evaluate the exact value of $G\text{-Deg}\,(F_\xi, U(r, \rho))$. To illustrate this point, we need a more detailed description of the G-isotypical components $W_{j,l}$.

For every isotypical component $W_{j,l}$, we denote by $Y_{j,l}$ the corresponding irreducible representation of G (i.e. $Y_{j,l}$ is equivalent to every irreducible subrepresentation of $W_{j,l}$). We describe the action of the group $G = D_N \times S^1$, $N \geq 3$, on $Y_{j,l}$ with $j > 0$ as follows:

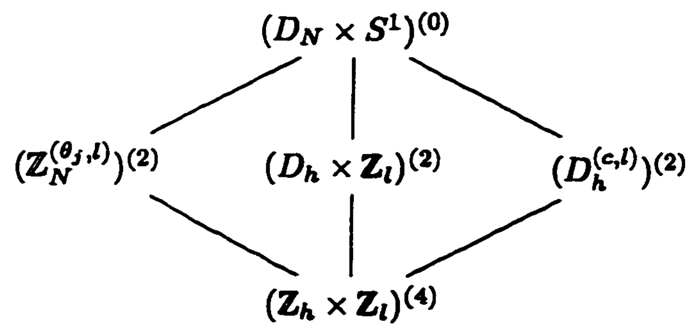
The first type of the isotypical components $W_{j,l}$ corresponds to the irreducible 4-dimensional representations $Y_{j,l}$ of G (described as complex D_N-representations in (b1)), where the action of $G = D_N \times S^1$ on the space $\mathbf{R}^2 \oplus \mathbf{R}^2 = \mathbf{C} \oplus \mathbf{C}$ is given by:

$$(\gamma, \tau)(z_1, z_2) := (\gamma^j \tau^l z_1, \gamma^{-j} \tau^l z_2) \text{ for } (\gamma, \tau) \in \mathbf{Z}_N \times S^1;$$

$$(\kappa\gamma, \tau)(z_1, z_2) := (\gamma^{-j} \tau^l z_2, \gamma^j \tau^l z_1) \text{ for } (\kappa\gamma, \tau) \in \kappa\mathbf{Z}_N \times S^1,$$

where $(z_1, z_2) \in \mathbf{C} \oplus \mathbf{C}$, $l = 1, 2, 3, \ldots$, $1 \leq j < [\frac{N}{2}]$. We put $h = gcd(j, N)$, $m = \frac{N}{h}$.

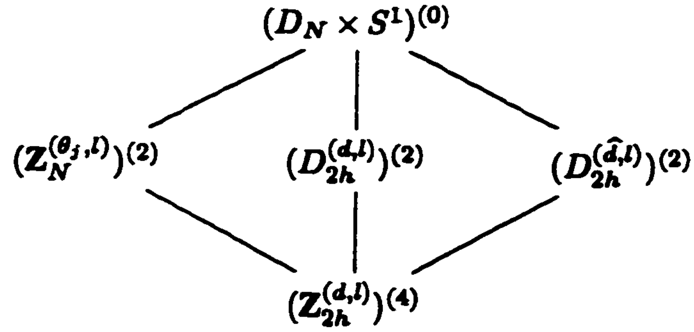
(i1): *m is odd*: In this case, we have the following lattice of the isotropy groups in $Y_{j,l}$:



where $\theta_j : \mathbf{Z}_N \to S^1$ is given by $\theta_j(\gamma) = \gamma^{-j}$, $\gamma \in \mathbf{Z}_N$, j integer and the numbers in brackets denote the dimension of the corresponding fixed-point space. We define the following element of $A_1(D_N \times S^1)$

$$\deg_{j,l} := (\mathbf{Z}_N^{(\theta_j,l)}) + (D_h \times \mathbf{Z}_l) + (D_h^{(c,l)}) - (\mathbf{Z}_h \times \mathbf{Z}_l).$$

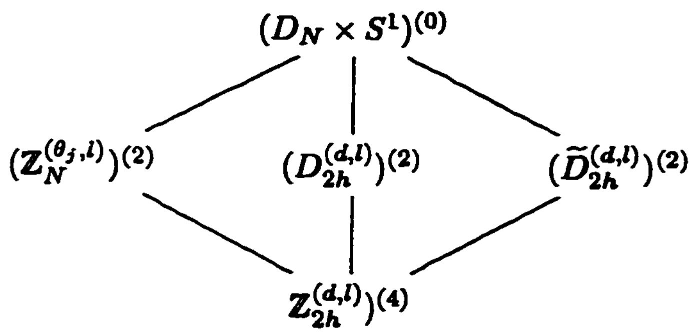
(i2): $m \equiv 2$ *(mod 4)*: In this case, we have the following lattice of isotropy subgroups in $Y_{j,l}$:



and we define

$$\deg_{j,l} = (\mathbf{Z}_N^{(\theta_j,l)}) + (D_{2h}^{(d,l)}) + (D_{2h}^{(\hat{d},l)}) - (\mathbf{Z}_{2h}^{(d,l)}).$$

(i3): $m \equiv 0$ *(mod 4)*: In this case, we have the following lattice of isotropy subgroups in $Y_{j,l}$:



and we put:

$$\deg_{j,l} := (\mathbf{Z}_N^{(\theta_s,l)}) + (D_{2h}^{(d,l)}) + (\widetilde{D}_{2h}^{(d,l)}) - (\mathbf{Z}_{2h}^{(d,l)}).$$

(i4): For an isotypical component $W_{j,l}$ corresponding to irreducible two-dimensional representation $Y_{j,l}$ on $\mathbb{R}^2 = \mathbb{C}$ of $D_N \times S^1$ which is given by

$(\gamma, \tau)z = \tau^l z$, $(\gamma, \tau) \in \mathbf{Z}_N \times S^1$;

$(\kappa\gamma, \tau)z = -\tau^l z$, $(\kappa\gamma, \tau) \in \kappa\mathbf{Z}_N \times S^1$,

where $l = 1, 2, 3, \ldots$ and we have the following lattice of the isotropy groups in $Y_{j,l}$

$$(D_N \times S^1)^{(0)}$$

$$|$$

$$(\widetilde{D}_N^{(c,l)})^{(2)}$$

We define

$$\deg_{j,l} := (\widetilde{D}_N^{(c,l)}).$$

(i5): If N is even then there is a two-dimensional irreducible representation on $Y_{j,l} = \mathbf{R}^2 = \mathbf{C}$ of $D_N \times S^1$ given by

$(g,\tau)z = \tau^l z$, if $(g,\tau) \in D_{\frac{N}{2}} \times S^1$;

$(g,\tau)z = -\tau^l z$, if $(g,\tau) \in (D_N \setminus D_{\frac{N}{2}}) \times S^1$.

We have the following lattice of the isotropy subgroups in $Y_{j,l}$

$$(D_N \times S^1)^{(0)}$$

$$|$$

$$(D_N^{(d,l)})^{(2)}$$

In this case, we put

$$\deg_{j,l} := (D_N^{(d,l)}).$$

(i6): Finally, for N even and $j = \frac{N}{2}$, there may also be an isotypical component $W_{\frac{N}{2},l}$ corresponding to the two dimensional representation on $Y_{\frac{N}{2},l} := \mathbf{R}^2 = \mathbf{C}$ of $D_N \times S^1$ given by

$(\gamma,\tau)z = \gamma^{\frac{N}{2}}\tau^l z$, where $(\gamma,\tau) \in \mathbf{Z}_N \times S^1$,

$(\kappa\gamma,\tau)z = -\gamma^{\frac{N}{2}}\tau^l z$, where $(\kappa\gamma,\tau) \in \kappa\mathbf{Z}_N \times S^1$.

In this case, we have the following lattice of the isotropy groups in $Y_{\frac{N}{2},l}$

$$(D_N \times S^1)^{(0)}$$

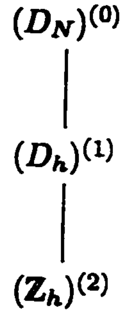
$$|$$

$$(D_N^{(\hat{d},l)})^{(2)}$$

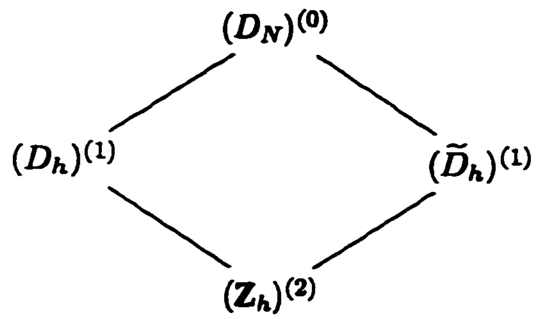
and we define

$$\deg_{j,l} := (D_N^{(\hat{d},l)}).$$

(j1): For the isotypical component corresponding to the type (a1) of the irreducible representations of D_N i.e. $W_{j,0} := V_j$, where $1 \le j < [\frac{N}{2}]$, we have the following lattice of isotropy groups of $Y_{j,0}$



For m odd



For m even

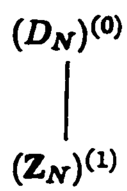
where $h = gcd(j, N)$ and $m := \frac{N}{h}$. If m is odd, we put

$$\deg_j := (D_h) + (\mathbf{Z}_h)$$

and if m is even, we put

$$\deg_j := (D_h) + (\widetilde{D}_h) - (\mathbf{Z}_h).$$

(j2): For the isotypical component $W_{j,l}$ corresponding to the irreducible representation $Y_{j,0}$ of type (a2), we have the following lattice of isotropy groups in $Y_{j,l}$



and we put

$$\deg_j := (\mathbb{Z}_N).$$

(j3): For $W_{j,l}$ corresponding to the irreducible representation $Y_{j,0}$ of type (a3), we have the following lattice of isotropy groups in $Y_{j,l}$

$$\begin{array}{c} (D_N)^{(0)} \\ | \\ (D_{\frac{N}{2}})^{(1)} \end{array}$$

and we put

$$\deg_j := (D_{\frac{N}{2}}).$$

(j4): In the case $j = \frac{N}{2}$, $W_{j,0} = V_j$ corresponds to a one-dimensional irreducible representation $Y_{j,0}$ of type (a4), having the following lattice of isotropy groups

$$\begin{array}{c} (D_N)^{(0)} \\ | \\ (\hat{D}_{\frac{N}{2}})^{(1)} \end{array}$$

and we put

$$\deg_j := (\hat{D}_{\frac{N}{2}}).$$

We also define, for every $j = 0, 1, \dots, k$, the number

$$\nu_j(\alpha_0, \beta_0) = \begin{cases} 1 & \text{if sign} \det a_{j,0}(\alpha_0, \beta_0) = -1, \\ 0 & \text{if sign} \det a_{j,0}(\alpha_0, \beta_0) = 1. \end{cases}$$

We have the following result:

Theorem 5.2.1 *Under the above assumptions, the degree of the function F_ξ in*

(5.2.8) is given by

$$G\text{-Deg}\,(F_\xi, U(r,\rho)) = \nu_0\Big(\prod_{j=1}^{k}((D_N) - \nu_j(\alpha_0,\beta_0)\,\deg_j)\Big)\Big(\sum_{j,l;l>0} c_{l,j}(\alpha_0,\beta_0)\,\deg_{j,l}\Big).$$

where $\nu_0 := \nu_0(\alpha_0, \beta_0)$ *and the products are given by the multiplication in the Burnside ring* $A(D_N)$ *and by the multiplication* $A(D_N) \times A_1(D_N \times S^1) \to A_1(D_N \times S^1)$*, respectively.*

Proof. Let Ω_j denote the unit ball in the isotypical component $W_{j,0}$. We denote by $Y_{j,l}$ the irreducible representation (except for these (j,l) such that $j = \frac{N}{2}$) corresponding to the isotypical component $W_{j,l}$, where $l > 0$. We also denote by

$$\Omega_{j,l} := \{(v,z) \in Y_{j,l} \oplus \mathbb{C};\ \ \|v\| < 1,\ \ \tfrac{1}{2} < |z| < 2\}.$$

By Theorem 4.3.1 in Chapter 4, we have

$$\begin{aligned} &G\text{-Deg}\,(F_\xi, U(r,\rho)) = \\ &= \nu_0\Big(\prod_{j=1}^{k} D_N\text{-deg}\,(a_{j,0}(\alpha_0,\beta_0),\Omega_j)\Big)\Big(\sum_{j,l;l>0} c_{l,j}(\alpha_0,\beta_0)\,G\text{-Deg}\,(f_{j,l},\Omega_{j,l})\Big), \end{aligned}$$

where $f_{j,l} : Y_{j,l} \oplus \mathbb{C} \to Y_{j,l} \oplus \mathbb{R}$ is define by

$$f(v,z) = (z\cdot v, |z|(\|v\| - 1) + \|v\| + 1).$$

The computations of G-Deg$\,(f_{j,l}, \Omega_{j,l})$ were essentially done in Example 4.3.2 in Chapter 4 ([43]), where the Ulrich Type Formula (Theorem 2.2.1) was applied to show that for every (j,l) such that $l > 0$ we have $G\text{-Deg}\,(f_{j,l},\Omega_{j,l}) = \deg_{j,l}$. In order to compute $D_N\text{-deg}\,(a_{j,0}(\alpha_0,\beta_0),\Omega_j)$ we can use the properties of the Ulrich equivariant degree (the case $n = 0$) (cf. [53]) and the standard computations based mostly on the evaluation of appropriate fixed point indices (see [43]) In particular, we can verify that $D_N - \deg\,(a_{j,0}(\alpha_0,\beta_0)) = (D_N) - \nu(\alpha_0,\beta_0)\deg_j$. □

Theorem 5.2.2 *Under the above assumptions, for every nonzero crossing number $c_{l,j}(\alpha_0,\beta_0)$ there exist, bifurcating from $(0,\alpha_0,\beta_0)$, branches of non-constant periodic solutions of (5.2.5) such that:*

(i1) if the element $\deg_{j,l}$ corresponding to the index (j,l) is $(\mathbf{Z}_N^{(\theta_j,l)}) + (D_h \times \mathbf{Z}_l) + (D_h^{(c,l)}) - (\mathbf{Z}_h \times \mathbf{Z}_l)$, i.e. $m \equiv 1 \pmod 2$, then there are 2 branches of periodic solutions with the orbit type $(\mathbf{Z}_N^{(\theta_j,l)})$, $m = \frac{N}{h}$ branches with the orbit type $(D_h \times \mathbf{Z}_l)$, and $m = \frac{N}{h}$ branches with the orbit type $(D_h^{(c,l)})$;

(i2) if $\deg_{j,l} = (\mathbf{Z}_N^{(\theta_j,l)}) + (D_{2h}^{(d,l)}) + (D_{2h}^{(\hat{d},l)}) - (\mathbf{Z}_{2h}^{(d,l)})$ (i.e. $m \equiv 2 \pmod 4$), then there are 2 branches of periodic solutions with orbit type $(\mathbf{Z}_N^{(\theta_j,l)})$, $\frac{N}{2h}$ branches with the orbit type $(D_{2h}^{(d,l)})$, and $\frac{N}{2h}$ branches with the orbit type $(D_{2h}^{(\hat{d},l)})$;

(i3) if $\deg_{j,l} = (\mathbf{Z}_N^{(\theta_*,l)}) + (D_{2h}^{(d,l)}) + (\tilde{D}_{2h}^{(d,l)}) - (\mathbf{Z}_{2h}^{(d,l)})$ (i.e. $m \equiv 0 \pmod 4$), then there are 2 branches of periodic solutions of type $(\mathbf{Z}_N^{(\theta_j,l)})$, $\frac{N}{2h}$ branches of orbit type $(D_{2h}^{(d,l)})$, and $\frac{N}{2h}$ branches of the orbit type $(\tilde{D}_{2h}^{(d,l)})$;

(i4) if $\deg_{j,l} = (\tilde{D}_N^{(c,l)})$, then there is one branch of periodic solutions of the orbit type $(\tilde{D}_N^{(c,l)})$;

(i5) if $\deg_{j,l} = (D_N^{(d,l)})$, then there exists one branch of periodic solutions of the orbit type $(D_N^{(d,l)})$;

(i6) if $\deg_{j,l} = (D_N^{(\hat{d},l)})$, then there exists one branch of periodic solutions of the orbit type $(D_N^{(\hat{d},l)})$.

Proof. Using the fact that all the orbit types mentioned in Theorem 5.2.2 are maximal, it follows from Theorem 5.2.1 that if the crossing number $c_{l,j}(\alpha_0,\beta_0)$ is nonzero, then there is a non-zero component $c_{l,j}(\alpha_0,\beta_0)\deg_{j,l}$ of the degree $G\text{-Deg}(F_\xi, U(r,\rho))$. Consequently, from the existence property of the G-degree, it follows that to every maximal orbit type (H) contained in $\deg_{j,l}$ corresponds to at least $|G/H|$ branches of bifurcating from $(0,\alpha_0,\beta_0)$ non-constant periodic solutions of the orbit type exactly equal to (H). □

Remark 5.2.3 Note that in Theorem 5.2.2, for a sequence of non-constant periodic solutions $x(t)$ of (5.2.1) corresponding to the 1-periodic solutions $(z_k(t), \alpha_k, \beta_k)$ of (5.2.5) such that $(z_k(t), \alpha_k, \beta_k) \to (0, \alpha_0, \beta_0)$ in $W \oplus \mathbf{R}^2$ as $k \to \infty$, $\frac{2\pi}{\beta_k}$ is not necessarily the minimal period of $x_k(t)$. However, by applying the same idea as in [45], one can show that if p_k is a minimal period of $x_k(t)$ such that $\lim_{k\to\infty} p_k = p_0$ then there exists an integer r such that $\frac{2\pi}{\beta_0} = rp_0$ and $ir\beta_0$ is a characteristic value of $(0, \alpha)$. In particular, if other pure imaginary characteristic values of $(0, \alpha_0)$ are not integer multiples of $\pm i\beta_0$, then $\frac{2\pi}{\beta_k}$ is the minimal period of $x_k(t)$.

Remark 5.2.4 We should emphasize that the computational formula for G-Deg$(F_\xi, U(r, \rho))$ gives more information than what was used in the proof of Theorem 5.2.2. For example, we did not refer to the factor

$$\nu_0\Big(\prod_{j=1}^{k}\Big((D_N) - \nu_j(\alpha_0, \beta_0)\deg_j\Big)\Big) \in A(D_N)$$

which provides additional information about the type of symmetries involved in this Hopf bifurcation. At this stage, we are unable to predict the existence of branches of periodic solutions corresponding to sub-maximal orbit types but the computational formula for G-Deg$(F_\xi, U(r, \rho)$ indicates that there is a potential for this type of branches. It was shown in Chapter 3 [40] that in the case of a finite dimensional symmetric bifurcation, a non-zero component of the equivariant degree corresponding to an orbit type (H) (possibly sub-maximal) implies that the existence of a branch of solutions with the orbit type (H) can be achieved using arbitrarily small perturbations of the original equation. Finally, we point out that the degree G-Deg$(F_\xi, U(r, \rho))$ can be regarded as a local invariant characterizing the Hopf bifurcation from $(0, \alpha_0, \beta_0)$. If we are interested in the global behavior of the branches of periodic solutions, we can use the standard method to show that for a bounded component of non-constant periodic solutions the sum of the above local invariants has to be 0. As the G-degree is fully computable for the D_N-symmetric Hopf bifurcation, this type of global bifurcation result would provide a set of relations which can be used to gain more information about the existence of large amplitude periodic solutions. We refer to papers [43] and [45] for more details and examples.

5.3 Hopf Bifurcations in a Ring of Identical Oscillators

In this section we consider a ring of identical oscillators with identical coupling between adjacent cells. Such a ring was modeled by Turing (cf. [52]) and provides models for various situations in biology, chemistry and electrical engineering. The local Hopf bifurcation of this Turing ring has been extensively studied in the literature, see [1,22,28,45,51,57] and references therein.

There are many reasons to emphasize the importance of temporal delays in coupling between cells, for example in many chemical or biological oscillators the time needed for transport or processing of chemical components or signals may be of considerable length (see [45]).

We will analyze how the temporal delay in the kinetic and in the coupling of cells together with the dihedral symmetries of the system may cause various types of oscillations in the case when each cell is described by only one state variable. It has been shown in [25] that such oscillations can not occur if the temporary delay is neglected.

We consider a ring of N identical cells coupled symmetrically by diffusion along the sides of an N-gon (see Figure 5.1). Each cell may be regarded as a chemical system with m distinct chemical species. In what follows we will assume, for the sake of simplicity, that $m = 1$. However, our method based on the use of the G-equivariant degree can also be effectively applied to more complex systems including the case where $m > 1$. We denote by $u^j(t)$ the concentration of the chemical species in the j-th cell, $0 \le j \le N-1$. We assume that the coupling is "nearest-neighbour" and symmetric in the sense that the interaction between any neighbouring pair of cells takes the same form. For simplicity, we also assume that the coupling between adjacent cells is linear. Thus, we have the following system of retarded functional differential equations

$$\frac{d}{dt}u^j(t) = f(u_t^j, \alpha) + K(\alpha)(u_t^{j-1} - 2u_t^j + u_t^{j+1}), \quad 0 \le j \le N-1, \tag{5.3.1}$$

where $t \in \mathbf{R}$ denotes the time, $\alpha \in \mathbf{R}$ is a parameter, $u_t^j(\theta) = u^j(t+\theta)$, $0 \le j \le N-1$, $f : C([-\tau, 0]; \mathbf{R}) \times \mathbf{R} \to \mathbf{R}$ is continuously differentiable, and $K(\alpha) : C([-\tau, 0]; \mathbf{R}) \to \mathbf{R}$ is a bounded linear operator and the mapping $K : \mathbf{R} \to L(C([-\tau, 0]; \mathbf{R}), \mathbf{R})$ is continuously differentiable. In (5.3.1) we assume that the integer $j+1$ is taken modulo N.

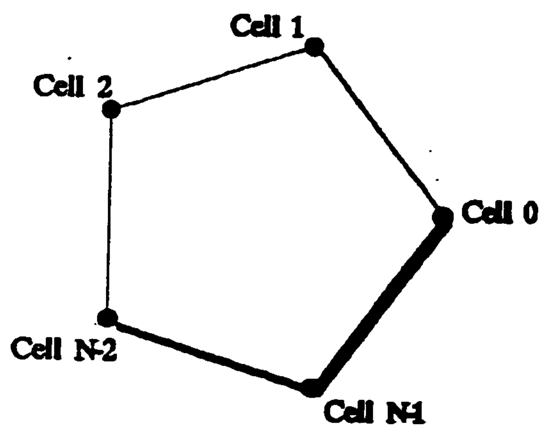


Figure 5.1

The function f describes the kinetic law obeyed by the concentrations u^j in every cell, and $K(\alpha)$ represents coupling strength, where the additional term

$$K(\alpha)(u_t^{j-1} - u_t^j) + K(\alpha)(u_t^{j+1} - u_t^j), \quad 0 \le j \le N-1$$

in (5.3.1) is usually supported by the ordinary law of diffusion, i.e. the chemical substance moves from region of greater concentration to region of less concentration, at a rate proportional to the gradient of the concentration. We refer to [1,52] for more details.

We assume that

$$f(0,\alpha) = 0. \tag{5.3.2}$$

Then $(0,0,\ldots,0,\alpha) \in \mathbb{R}^N \times \mathbb{R}$ is a stationary solution of (5.3.1) and the linearization of (5.3.1) at $(0,0,\ldots,0,\alpha)$ is

$$\frac{d}{dt}x^j(t) = D_x f(0,\alpha)x_t^j + K(\alpha)[x_t^{j-1} - 2x_t^j + x_t^{j+1}], \quad 0 \le j \le N-1. \tag{5.3.3}$$

Therefore, a number $\lambda \in \mathbb{C}$ is a characteristic value of the stationary solution $(0,\alpha) \in \mathbb{R}^N \times \mathbb{R}$ if there exists a non-zero vector $z = (z_0, \ldots, z_{N-1}) \in \mathbb{C}^N$ such that

$$\operatorname{diag}\Big(\lambda \mathrm{Id} - D_x f(0,\alpha)(e^{\lambda\cdot}\mathrm{Id})\Big)z + r(\alpha,\lambda)z = 0,$$

where $\operatorname{diag}\Big(\lambda \mathrm{Id} - D_x f(0,\alpha)(e^{\lambda\cdot}\mathrm{Id})\Big)$ denotes the diagonal $N \times N$ matrix and $r(\alpha,\lambda) : \mathbb{C}^N \to \mathbb{C}^N$ is defined by

$$\{r(\alpha,\lambda)z\}_j = K(\alpha)[e^{\lambda\cdot}(z_{j-1} - 2z_j + z_{j+1})]; \quad 0 \le j \le N-1.$$

We put $\{\delta z\}_j = z_{j-1} - 2z_j + z_{j+1}$, $0 \le j \le N-1$. The operator δ is the discretized Laplacian. Therefore, a number λ is a characteristic value if and only if the matrix

$$\Delta_\alpha(\lambda) = \operatorname{diag}(\lambda \mathrm{Id} - D_x f(0,\alpha)(e^{\lambda\cdot}\mathrm{Id})) - r(\alpha,\lambda)$$

is singular, i.e. the following characteristic equation is satisfied

$$\det{}_{\mathbb{C}}\Delta_\alpha(\lambda) = 0.$$

We have the following

Proposition 5.3.1 *A number $\lambda \in \mathbf{C}$ is a characteristic value of the stationary solution if and only if*

$$\det{}_{\mathbf{C}}\Delta_\alpha(\lambda) = \prod_{r=0}^{N-1}[\lambda - D_x f(0,\alpha)e^{\lambda\cdot} + 4\sin^2\tfrac{\pi r}{N}\,K(\alpha)e^{\lambda\cdot}] = 0.$$

Proof. For every $z \in \mathbf{C}$ and $r \in \{0, 1, \ldots, N-1\}$, we have

$$\begin{aligned}
&(\Delta_\alpha(\lambda)(1, \xi^r, \ldots, \xi^{(N-1)r})z)_{j+1}\\
=&[\lambda\xi^{jr} - D_x f(0,\alpha)(e^{\lambda\cdot})\xi^{jr} - K(\alpha)e^{\lambda\cdot}(\xi^{(j+1)r} - 2\xi^{jr} + \xi^{(j-1)r}]z\\
=&[\lambda - D_x f(0,\alpha)e^{\lambda\cdot} - K(\alpha)e^{\lambda\cdot}(\xi^r - 2 + \xi^{-r})]\xi^{jr}z\\
=&[\lambda - D_x f(0,\alpha)e^{\lambda\cdot} - K(\alpha)e^{\lambda\cdot}(2\mathrm{Re}\,\xi^r - 2)]\xi^{jr}z\\
=&[\lambda - D_x f(0,\alpha)e^{\lambda\cdot} - 2(\cos\tfrac{2\pi r}{N} - 1)K(\alpha)e^{\lambda\cdot}]\xi^{jr}z\\
=&[\lambda - D_x f(0,\alpha)e^{\lambda\cdot} + 4\sin^2\tfrac{\pi r}{N}K(\alpha)e^{\lambda\cdot}]\xi^{jr}z.
\end{aligned}$$

□

It is well known (see [25]) that a Hopf bifurcation from a stationary solution $(0,\alpha)$ can not occur if the equation (5.3.1) has no temporal delay. However, the temporal delay in the coupling cells may cause various types of oscillations in the system (5.3.1) as will be illustrated in the following.

It is clear that the system (5.3.1) is equivariant with respect to the action of the dihedral group D_N, where the subgroup $\mathbf{Z}_N$ of rotations acts on $\mathbb{R}^N$ in such a way that the generator $\xi := e^{\frac{2\pi i}{N}}$ sends the j-th coordinate of the vector $x = (x_0, x_1, \ldots, x_{N-1}) \in \mathbb{R}^N$ to the $j+1$ (mod N) coordinate, and the flip κ sends the j-th coordinate of x to the $-j$ (mod N) coordinate. We assume that $N > 2$ and denote this representation by $\Theta : D_N \to O(N)$.

First, we consider the action of $\mathbf{Z}_N$ on the complexification $U := \mathbf{C}^N$ of the representation Θ. It is clear that the $\mathbf{Z}_N$-isotypical decomposition of U is given by

$$U = \tilde{U}_0 \oplus \tilde{U}_1 \oplus \cdots \oplus \tilde{U}_{N-1},$$

where $\tilde{U}_r := \{z(1, \xi^r, \xi^{2r}, \ldots, \xi^{(N-1)r}); \quad z \in \mathbf{C}\}$. The flip κ sends $\tilde{U}_r$ onto $\tilde{U}_{-r}$, where $-r$ is taken (mod N), thus $U_0 := \tilde{U}_0$ and $U_r := \tilde{U}_r \oplus \tilde{U}_{-r}$ for $0 < r < [\frac{N}{2}]$ are the isotypical components of U with respect to the action of D_N. If N is even, there is one more isotypical component $U_{\frac{N}{2}} := \tilde{U}_{\frac{N}{2}}$. It is easy to see that the isotypical components U_r, $0 < r < [\frac{N}{2}]$, correspond to the representations of D_N on $\mathbf{C} \oplus \mathbf{C}$ of the type (b1) given by

$$\gamma(z_1, z_2) = (\gamma^{-r} \cdot z_1, \gamma^r \cdot z_2) \text{ if } \gamma \in \mathbf{Z}_N;$$

$$\kappa(z_1, z_2) = (z_2, z_1),$$

where $(z_1, z_2) \in \mathbf{C} \oplus \mathbf{C}$.

If N is an even number, the isotypical component $U_{\frac{N}{2}}$ is equivalent to the D_N-representation on $\mathbf{C} = \mathbb{R}^2$ of the type (b3), where

$$gz = \begin{cases} -z & \text{if } g \in D_N \setminus D_{\frac{N}{2}}, \\ z & \text{if } g \in D_{\frac{N}{2}}. \end{cases}$$

We make the following hypothesis:

(H1) *There exists* $(0, \alpha_0) \in \mathbb{R}^N \times \mathbb{R}$ *such that* $(0, \alpha_0)$ *is an isolated center of* (5.3.1) *such that* $\det_{\mathbf{C}} \Delta_{(0,\alpha_0)}(i\beta_0) = 0$, $\beta_0 > 0$.

It is straightforward to obtain the next two technical results:

Corollary 5.3.2 *A complex number* $\lambda \in \mathbf{C}$ *is a* j*-th isotypical characteristic value of* $(0, \alpha)$, *where* $0 < j < [\frac{N}{2}]$, *if and only if*

$$p_{\alpha,j}(\lambda) := \lambda - D_x f(0, \alpha) e^{\lambda \cdot} + 4 \sin^2 \tfrac{\pi j}{N} \, K(\alpha) e^{\lambda \cdot} = 0.$$

Corollary 5.3.3 *Under the assumption (H1), the j-th isotypical crossing number for the isolated center $(0, \alpha_0)$ corresponding to the value $il\beta_0$ is equal to*

(i) for $0 < j < [\frac{N}{2}]$

$$c_{l,j}(\alpha_0, \beta_0) = 2\Big(\deg_B(p_{\alpha_0-\delta,j}(\cdot), \Omega_l) - \deg_B(p_{\alpha_0+\delta,j}(\cdot), \Omega_l)\Big)$$

(ii) for $j = 0$ or $j = \frac{N}{2}$ (if N is even)

$$c_{l,j}(\alpha_0, \beta_0) = \deg_B(p_{\alpha_0-\delta,j}(\cdot), \Omega_l) - \deg_B(p_{\alpha_0+\delta,j}(\cdot), \Omega_l),$$

where $\Omega_l := (0, b) \times (l\beta_0 - c, l\beta_0 + c) \subset \mathbf{C}$ and the constants $b > 0$, $c > 0$ and $\delta > 0$ are sufficiently small.

Using Theorem 5.2.2, we can establish the following

Theorem 5.3.4 *Assume the hypothesis (H1) is satisfied. If $c_{1,j}(\alpha_0, \beta_0) \neq 0$, then the stationary point $(0, \alpha_0)$ is a bifurcation point of (5.3.1). Moreover,*

(i) if $1 < j < \frac{N}{2}$, $h = gcd(j, N)$ and $\frac{N}{h}$ is odd, then there are at least 2 branches of periodic solutions corresponding to the orbit type $(\mathbf{Z}_N^{(\theta_j,1)})$, $\frac{N}{h}$ branches of periodic solutions corresponding to the orbit type $(D_h \times \mathbf{Z}_1)$, and $\frac{N}{h}$ branches of periodic solutions corresponding to the orbit type $(D_h^{(c,1)})$;

(ii) if $1 < j < \frac{N}{2}$, $h = gcd(j, N)$ and $\frac{N}{h} \equiv 2 \pmod 4$, then there are at least 2 branches of periodic solution corresponding to the orbit type $(\mathbf{Z}_N^{(\theta_j,1)})$, $\frac{N}{2h}$ branches of periodic solutions corresponding to the orbit type $(D_{2h}^{(d,1)})$, and $\frac{N}{2h}$ branches of periodic solutions corresponding to the orbit type $(D_{2h}^{(\hat{d},1)})$;

(iii) if $1 < j < \frac{N}{2}$, $h = gcd(j, N)$ and $\frac{N}{h} \equiv 0 \pmod 4$, then there are at least 2 branches of periodic solution corresponding to the orbit type $(\mathbf{Z}_N^{(\theta_j,1)})$, $\frac{N}{2h}$ branches of periodic solutions corresponding to the orbit type $(D_{2h}^{(d,1)})$, and $\frac{N}{2h}$ branches of periodic solutions corresponding to the orbit type $(\tilde{D}_{2h}^{(d,1)})$;

(iv) if $j = \frac{N}{2}$, then there exists at least one branch of periodic solutions corresponding

to the orbit type $(D_N^{(\hat{d},l)})$*;*

(v) if $j=0$*, then there exists at least one branch of periodic solutions corresponding to the orbit type* $(D_N \times \mathbf{Z}_1)$*.*

Example 5.3.5 We consider the following system of retarded functional differential equations (cf. [45])

$$\dot{x}_j(t) = -\alpha x_j(t) + \alpha h(x_j(t))[g(x_{j-1}) - 2g(x_j(t-1)) + g(x_{j+1}(t-1))], \qquad (5.3.4)$$

where $0 \le j \le N-1$ and we use the convention that $j+1$ is always taken (mod N), $\alpha > 0$, h, $g : \mathbb{R} \to \mathbb{R}$ are continuously differentiable, h does not vanish and $g(0) = 0$, $g'(0) > 0$. By using an appropriate change of variables and rescaling the time, the equation (5.3.4) can be transformed into an equation of the same type as the equation (5.3.1). In addition we have

$$p_{\alpha,j}(\lambda) = \lambda + \alpha + 4\sin^2 \tfrac{\pi r}{N} \alpha\mu e^{-\lambda},$$

where $\mu = h(0)g'(0)$. Assume that there exists j, $0 < j < \frac{N}{2}$, such that $\mu > \frac{1}{4\sin^2 \frac{\pi j}{N}}$, then the number $i\beta_{0,j}$, where $\beta_{0,j} \in (\frac{\pi}{2}, \pi)$, is the unique solution of $\cos\beta_{0,j} = -\frac{1}{4\sin^2 \frac{\pi j}{N}}$ is a j-th isotypical characteristic value corresponding to the stationary solution $(0, \alpha_{0,j})$, where $\alpha_{0,,j} = -\beta_{0,j}\cot\beta_{0,j}$. It can be computed (see [45]) that $(0, \alpha_{0,j})$ satisfies the assumption (H1) and we have $c_{1,j}(\alpha_{0,j}, \beta_{0,j}) < 0$. Consequently, by Theorem 5.3.4 we have

Proposition 5.3.6 *Let* $h = gcd(j, N)$*. If there exists* j*,* $0 < j < \frac{N}{2}$ *such that* $\mu > \frac{1}{4\sin^2 \frac{\pi j}{N}}$*, then the stationary solution* $(0, \alpha_0)$ *is a bifurcation point for the equation (5.3.4). In particular,* *(i) if $\frac{N}{h}$ is odd, then there are at least 2 branches of periodic solution corresponding to the orbit type* $(\mathbf{Z}_N^{(\theta_j,1)})$, *$\frac{N}{h}$ branches of periodic solutions corresponding to the orbit type* $(D_h \times \mathbf{Z}_1)$, *and $\frac{N}{h}$ branches of periodic solutions corresponding to the orbit type* $(D_h^{(c,1)})$;

(ii) if $\frac{N}{h} \equiv 2 \pmod 4$, then there are at least 2 branches of periodic solution corresponding to the orbit type $(\mathbf{Z}_N^{(\theta_j,1)})$, *$\frac{N}{2h}$ branches of periodic solutions corresponding to the orbit type* $(D_{2h}^{(d,1)})$, *and $\frac{N}{2h}$ branches of periodic solutions corresponding to the orbit type* $(D_{2h}^{(\hat{d},1)})$;

(iii) if $\frac{N}{h} \equiv 0 \pmod 4$, then there are at least 2 branches of periodic solution corresponding to the orbit type $(\mathbf{Z}_N^{(\theta_j,1)})$, *$\frac{N}{2h}$ branches of periodic solutions corresponding to the orbit type* $(D_{2h}^{(d,1)})$, *and $\frac{N}{2h}$ branches of periodic solutions corresponding to the orbit type* $(\tilde{D}_{2h}^{(d,1)})$.

Corollary 5.3.7 *Assume that N is a prime number. If there exists j, $0 < j < \frac{N}{2}$, such that $\mu > \frac{1}{4\sin^2\frac{\pi j}{N}}$, then there are at least $2(N+1)$ different branches of non-constant periodic solutions of (5.3.4) bifurcating from the stationary solution* $(0, \alpha_0)$.

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