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ASYMPTOTIC TREES IN BANACH SPACES WITH SCHAUDER BASES

BY

DAVID MCLAUGHLIN

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ABSTRACT

The notion of an asymptotic tree is introduced and studied within the framework of Banach spaces with a Schauder basis. This concept is related to the geometric and renorming properties of certain Banach spaces. In particular, we demonstrate that spaces renormable uniformly asymptotically smooth along a Schauder basis fail to have such trees whereas they do occur in spaces failing the convex w^*-point of continuity property.

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NOTATION

We list below several symbols which will be used in this thesis.

B_Y = the closed unit ball in the space Y

B_r = the open ball of radius r of a space X

B_r^* = the open ball of radius r for X^*

$B^*(x^*, r)$ = the open ball centered at x^* of radius r

$B_X(\|\cdot\|)$ = the closed unit ball of X with the norm $\|\cdot\|$

ℓ_p = the space of all sequences (x_i) such that $\sum |x_i|^p < \infty$

ℓ_∞ = the space of all sequences (x_i) such that $\sup|x_i| < \infty$

c_0 = the subspace of ℓ_∞ of all sequences converging to 0

INTRODUCTION

In his 1972 paper [7], R.C. James forwarded the notion of the Finite Tree Property and showed that a Banach space is super-reflexive if and only if it fails this property. James has since then used spaces constructed from trees as counter-examples to important problems in Banach space theory. Subsequently variations on this tree theme have been introduced, notably the concept of an ϵ-bush, which completely characterizes the Radon-Nikodým Property.

If we consider the case of a Banach space which is a dual space, then we can make a stronger statement. This important result of Stegall (cf. [2]) states that the space has the Radon-Nikodým Property precisely when it contains no bounded ϵ-trees. For dual spaces we also have the concept of the convex w^*-point of continuity property. This notion, which is strictly weaker than the RNP (cf. [5]), will be the focus of our attention.

Our aim in this thesis is to examine the geometric and renorming implications of a new type of tree (introduced here) which is defined in the dual of a Banach space with Schauder basis.

Chapter One presents the basic definitions of the types of trees we wish to examine and provides examples of spaces which contain or lack these trees. This chapter also gives a brief discussion of the Finite Tree Property.

In Chapter Two, spaces Uniformly Asymptotically Smooth Along a Schauder basis (UASAS) are shown to lack these asymptotic trees under all equivalent renormings. In particular the dual of the Baernstein space is shown to be UASAS thus strengthening the result in [10]. There the Baernstein space is shown to be a reflexive uniformly Kadec-Klee space failing the Banach-Saks property.

We conclude with Chapter Three where we demonstrate that spaces with a Schauder basis which fail the Convex w^*-Point of Continuity Property contain our asymptotic trees.

Chapter 1

TREES IN BANACH SPACES

In this chapter we concern ourselves mainly with trees in Banach spaces. After giving examples and results concerning ordinary trees we will define asymptotic trees and provide examples of these. We start by stating the classical definition of a tree.

1.1 Definition: Let X be a topological vector space. Then $\{x_n\}_{n=1}^{\infty} \subset X$ is said to be a **tree** if for all $n = 1, 2, \ldots,$

$$x_n = \frac{1}{2}(x_{2n} + x_{2n+1}).$$

A tree then is a sequence of points constructed such that given the n^{th} element x_n, the $2n^{th}$ and $(2n+1)^{st}$ elements have x_n as their midpoint. We say that x_{2n} and x_{2n+1} are branches of x_n.

1.2 It is clear that given $x \in X$, then $\{x_n\}_{n=1}^{\infty}$ with $x = x_n$ for all n is a tree. In order to avoid such trivial examples we wish to seperate x_{2n} and x_{2n+1}, i.e. $x_{2n} \neq x_{2n+1}$. To this end we turn to normed linear spaces and introduce the following

Definition: If X is a normed linear space, then a tree $\{x_n\}_{n=1}^{\infty}$ is an ϵ-**tree** if

$$\|x_{2n} - x_n\| > \epsilon$$

Since x_n is the midpoint of x_{2n} and x_{2n+1}, $\|x_{2n+1} - x_n\| = \|x_{2n} - x_n\| = \frac{1}{2}\|x_{2n} - x_{2n+1}\| > \epsilon$.

1.3 Definition: we say a tree $\{x_n\}_{n=1}^{\infty}$ in a normed linear space is a **bounded tree** if there exists a constant C such that $\|x_n\| \leq C$ for all n.

Our concerns will be with bounded ϵ-trees.

1.4 The information found in this section is from [2]. We will be content with defining certain properties and stating two results which show the value of the Finite Tree Property. We begin with a list of definitions.

Definition: A Banach space has the **Finite Tree Property** (FTP) if for each ϵ, $0 < \epsilon < 1$, and each positive integer n there is a finite sequence $\{x_1, \cdots, x_{2^n-1}\}$ in B_X such that

$$x_j = \frac{1}{2}(x_{2j} + x_{2j+1}) \quad \text{and} \quad \|x_{2j} - x_j\| \geq \epsilon$$

for $j = 1, 2, \cdots, 2^{n-1} - 1$.

Definition: A Banach space Y is **finitely representable** in X if for each $\epsilon > 0$ and each finite dimensional subspace $W \subset Y$, there is a 1-1 linear operator $T : W \to T(W) \subset X$ with $\|T\|\|T^{-1}\| \leq 1 + \epsilon$.

Definition: X is **super-reflexive** if each Banach space finitely representable in X is reflexive.

Definition: A Banach space is **uniformly convex** if given ϵ, $0 < \epsilon \leq 2$, there exists $\delta = \delta(\epsilon) > 0$ such that if $x, y \in B_X$ with $\|x - y\| \geq \epsilon$, then

$$\frac{1}{2}\|x + y\| \leq 1 - \delta \quad ,$$

We are now in a position to state our two results; the first is due to R. C. James, the second was proven by P. Enflo.

Result 1. X is super-reflexive if and only if X fails the FTP.

Result 2. X is super-reflexive if and only if X has an equivalent norm which is uniformly convex.

These theorems show the relationship between trees and the geometry of the spaces which contain them.

1.5 With our small digression concluded, we turn, as promised, to a host of examples. Our first will be the prototype of the new type of tree to be introduced.

Example 1: For $X = \ell_\infty$, let $x_1 = (1, 0, \cdots)$. If given $x_n^? = (a_1, \ldots, a_k, 0, \cdots)$ where $a_i = \pm 1$, $i = 1, 2, \cdots, k$, then we define $x_{2n} = (a_1, a_2, \cdots, a_k, 1, 0, 0, \cdots)$ and $x_{2n+1} = (a_1, a_2, \cdots, a_k, -1, 0, 0, \cdots)$. By induction this example gives a 1-tree contained in the unit ball of ℓ_∞. On noting that in turn each element in the tree belongs to c_0 we can conclude that bounded ϵ-trees exist in the unit ball of c_0 for all $\epsilon > 0$.

Example 2: $\mathbf{R}$, the real number line, can contain no bounded ϵ-tree for any $\epsilon > 0$.

Example 3: A well known result states that a space containing a bounded ϵ-tree cannot have the Radon-Nikodým Property (RNP)[2]. A class of spaces having this property is the ℓ_p spaces for $1 \leq p < \infty$. As already noted in example 1, we cannot include the ∞ case.

Example 4: Although **R** cannot have bounded ϵ-trees, it can accommodate long finite trees of the type mentioned in connection with the Finite Tree Property. However, **R** does not satisfy the finite tree property, since the long tree need not lie in the unit ball of **R** i.e., the interval $[-1,1]$.

1.6 Having discussed these examples we now wish to define a new type of tree. To this end we first need a definition.

Definition: Let X be a banach space. A **Schauder basis** is a set $\{x_i\}_{i=1}^{\infty} \in X$ such that to each $x \in X$, there corresponds a unique set of scalars $a_1, a_2, \cdots$ such that

$$\lim_{k\to\infty} \|x - \sum_{i=1}^{k} a_i x_i\| = 0.$$

In this case we write

$$x = \sum_{i=1}^{\infty} a_i x_i.$$

If X has a Schauder basis then it is natural to consider elements of X^* restricted to finite dimensional subspaces generated by the elements of this basis. Our trees, which will be called *extended trees*, will be characterized by restrictions to such subspaces. Before the definition is unveiled, we introduce some notation.

Let $S = \{x_k\}_{k=1}^{\infty}$ be a Schauder basis in a Banach space X. For $x =$

$\sum_{k=1}^{\infty} a_k x_k$ define

$$P_n(x) := \sum_{k=1}^{n} a_k x_k.$$

Then $\{P_n\}_{n=1}^{\infty}$ are the basis projections and we define the subspace $P_n(X) \subset X$ as

$$P_n(X) = \left\{ x = \sum_{k=1}^{\infty} a_k x_k : \ a_k = 0, \ k > n \right\}.$$

This notation reflects the fact that P_n maps X onto $P_n(X)$. Although not needed at present, we can set $T_n = I - P_n$ (I being the identity map $I : X \to X$) and define

$$T_n(X) = \left\{ x = \sum_{k=n+1}^{\infty} a_k x_k \right\}.$$

Again this notation stresses the fact that T_n maps X onto $T_n(X)$.

We now consider $f \in X^*$. We denote the restriction of f to the subspace $P_k(X)$ by $f|_k$ and set

$$\|f|_k\| := \|f\|_k$$

. Similarly $\|f\|_{-k}$ will denote the norm of f restricted to $T_k(X)$. We can now proceed to the main definition.

1.7 Definition: If X is a Banach space with Schauder basis, we say $\{f_n\}_{n=1}^{\infty} \subset X^*$ is an **extended ϵ-tree** if there exists $\{k_n\}_{n=1}^{\infty} \subset N$ (k_n increasing) such that

1. $f_n = \frac{1}{2}(f_{2n} + f_{2n+1})$
2. $\|f_{2n} - f_n\| > \epsilon$
3. $\frac{\|f_n\|_{k_n}}{\|f_n\|} \uparrow 1$
4. $f_n|_{k_n} = f_{2n}|_{k_n} = f_{2n+1}|_{k_n}$

1.8 Example 5: In example 4 we noted that **R** could have arbitrarily long bounded ~~ϵ-trees.~~ This is not the case if we consider extended ϵ-trees. In this case, the elements of the tree must be equal on the only non-trivial subspace i.e. **R** itself.

Example 6: If X is ℓ_1 then the standard Schauder basis is $\{e_i\}_{i=1}^{\infty}$ where $e_i = (0,0,\cdots,0,1,0,0,\cdots)$, the 1 occuring in the i^{th} place. Suppose $\{k_n\}_{n=1}^{\infty} \subset \mathbf{N}$ is given. We have $X^* = \ell_\infty$ and we will model our example on the tree given in Example 1. We first set $f_1 = (1,\cdots,1,0,0\cdots)$ where there are k_1 1's. We will show how f_{2n} and f_{2n+1} are found if f_n is known. To do this, we first suppose that $f_n = (a_1,\cdots,a_j,0,0,\cdots)$ where $a_i = \pm 1$ for $i = 1,2,\cdots,j$. Now we choose f_{2n} and f_{2n+1} with k_{2n+1} non-zero entries all occuring in the first k_{2n+1} places. These non-zero entries are chosen so that the first j coincide with the first j of x_n and the remaining $k_{2n+1} - j$ are all 1's for f_{2n} and all -1's for f_{2n+1}. It is clear that $k_{2n+1} > j$ and so we obtain an extended 1-tree.

We should note that this example does not extend to c_0 since c_0 is not a dual space.

CHAPTER TWO

EXTENDED TREES IN SPACES UASAS

In [9], Lewis, Whitfield and Zizler introduced the notion of uniform asymptotic smoothness along a Schauder basis $S = \{x_n\}_{n=1}^{\infty}$ (UASAS) for a Banach space X. This property provides a class of Banach spaces whose duals fail to contain a bounded ϵ-tree for any $\epsilon > 0$.

2.1 Definition: Let X be a Banach space with a Schauder basis S= $\{x_n\}_{n=1}^{\infty}$. We say that X is **Uniformly Asymptotically Smooth Along S** (UASAS) if for every $\epsilon > 0$

$$\delta(\epsilon) = \inf\{1 - \|\frac{f+g}{2}\|_n : f, g \in B_1^*, n \in N, \|f-g\|_n = 0, \|f-g\| \geq \epsilon\} > 0.$$

2.2 Examples

1. In [9] c_0 and ℓ_p, $1 < p < \infty$ are shown to be UASAS.
2. Define the *Baernstein space* as the space of all real sequences $x = (x_1, x_2, \cdots)$ such that

$$\|x\| = \sup\left\{ \left(\sum_{k=1}^{\infty}(\sum_{j\in\gamma_k}|x_j|)^2\right)^{\frac{1}{2}} : (\gamma_1, \gamma_2, \cdots) \in A \right\} < \infty$$

where A is the set of all sequences $(\delta_1, \delta_2, \cdots)$ of finite subsets $\delta_n \subset N$ such that card$\delta_k \leq \min\delta_k$ and $\max\delta_k < \min\delta_{k+1}$ for $k = 1, 2, \cdots$. We claim that the dual of the Baernstein space is UASAS.

It is well known (cf.[11]) that the Baernstein space is reflexive and we will set X to be the dual of the Baernstein space. The usual basis of the sequence spaces (like ℓ_p) is a basis for both the Baernstein space and its dual. For the Baernstein space we denote this basis $\{e_i\}_{i=1}^{\infty}$. In [10] it is stated that for the Baernstein space $x \in sp\{e_1, e_2, \cdots, e_N\}$ and $y \in sp\{e_{N+1}, \cdots\}$ for $N \geq 1$ implies $\|x+y\| \geq (\|x\|^2 + \|y\|^2)^{\frac{1}{2}}$. But this follows since

$$\begin{aligned}
\|x+y\|^2 &= \sup\left\{\sum_{k=1}^{\infty}\Big(\sum_{j\in\gamma_k}|(x+y)_j|\Big)^2 : (\gamma_n) \in A\right\} \\
&\geq \sup\left\{\sum_{k=1}^{\infty}\Big(\sum_{j\in\alpha_k}|(x+y)_j|\Big)^2 : (\alpha_n) \in A,\ \max \alpha_m \leq N,\ \min \alpha_{m+1} > N\right\} \\
&= \sup \sum_{k=1}^{m}\Big(\sum_{j\in\alpha_k}|(x+y)_j|\Big)^2 + \sup \sum_{k=m+1}^{\infty}\Big(\sum_{j\in\alpha_k}|(x+y)_j|\Big)^2 \\
&= \sup \sum_{k=1}^{m}\Big(\sum_{j\in\alpha_k}|x_j|\Big)^2 + \sup \sum_{k=m+1}^{\infty}\Big(\sum_{j\in\alpha_k}|y_j|\Big)^2 \\
&= \sup\left\{\sum_{k=1}^{\infty}\Big(\sum_{j\in\gamma_k}|x_j|\Big)^2 : (\gamma_n) \in A\right\} + \sup\left\{\sum_{k=1}^{\infty}\Big(\sum_{j\in\gamma_k}|y_j|\Big)^2 : (\gamma_n) \in A\right\} \\
&= \|x\|^2 + \|y\|^2
\end{aligned}$$

We are now ready to demonstrate our claim that the dual of the Baernstein space is UASAS.

Proof: We will show that for $\epsilon \geq 0$ we have $\delta = \delta(\epsilon) \geq 1 - (1 - \frac{\epsilon^2}{4})^{1/2}$. Suppose not. Then there exists $f, g \in B_{X^*}$ with $\|f - g\| \geq \epsilon$, $f = g$ on $P_n(X)$ and $1 - \|f\|_n \leq 1 - (1 - \frac{\epsilon^2}{4})^{1/2}$ i.e. $1 - \|f\|_n^2 \leq \frac{\epsilon^2}{4}$. Now $\|f\| \geq (\|f\|_n^2 + \|f\|_{-n}^2)^{1/2}$. Thus

$$\begin{aligned}
\|f-g\| &= \|f-g\|_{-n} \\
&\leq \|f\|_{-n} + \|g\|_{-n} \\
&\leq (\|f\|^2 - \|f\|_n^2)^{1/2} + (\|g\|^2 - \|g\|_n^2)^{1/2} \\
&\leq (1 - \|f\|_n^2)^{1/2} + (1 - \|g\|_n^2)^{1/2} \\
&\leq 2(1 - \|f\|_n^2)^{1/2} \\
&\leq \epsilon
\end{aligned}$$

This contradicts $\|f - g\| \geq \epsilon$ and our claim is true.

2.3 It should be noted that if X is UASAS then X^* satisfies an important fixed point property. To make this more precise we introduce some definitions.

Definition: Let C be a subset of a Banach space X. A map $T : C \to C$ is called **non-expansive** if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$.

Definition: A dual space X has the **dual fixed point property** (FPP*) if for every w^*-compact convex subset $C \subset X$ and for every non-expansive map $T : C \to C$, there is a point $x \in C$ such that $Tx = x$. We now clarify our initial statement with the following application.

Proposition. *If X is UASAS then X^* has the FPP*.*

Before proving this we give another

Definition: A dual space X^* is said to be **uniformly** w^***-Kadec-Klee** (UW*KK) if for every $\epsilon > 0$ there exists $\delta > 0$ such that if $\{f_n\}_{n=1}^{\infty} \subset X^*$ is a sequence satisfying

1. $f_n \in B_{X^*}$ for $n = 1, 2, \cdots$
2. $\|f_n - f_m\| \geq \epsilon$ for $m \neq n$
3. $f_n \to f$ in the w^*-topology of X^*;

then $\|f\| \leq 1 - \delta$.

We can now complete the proposition.

Proof: This follows immediately from [8], where X UASAS is shown to imply UW*KK for X^*, followed by the result in [4] which states that if X^* is UW*KK then X^* has the FPP*.

2.4 Our main aim in this chapter is to show that these spaces UASAS cannot contain bounded extended ϵ-trees. To this effect, we begin with an elementary observation about extended trees.

Lemma. *Let $\{x_n\}_{n=1}^{\infty}$ be a tree in X and let $x_j \in \{x_n\}_{n=1}^{\infty}$. If $\{x_n\}_{n=1}^{\infty}$ is bounded, then there exists a subsequence $\{x_{j_n}\}_{n=1}^{\infty}$ such that $\|x_{j_n}\| \leq \|x_{j_{n+1}}\|$ and $\|x_{j_n}\| \geq \|x_j\|$ for every n.*

Proof: By choosing $x_{j_1} = x_j$ and $x_{j_{n+1}}$ as the element in the set $\{x_{2j_n}, x_{2j_n+1}\}$ with the larger norm,we see that the desired inequalities are satisfied.

2.5 We now state and prove a result concerning UASAS spaces and bounded extended ϵ-trees.

Theorem. *If X can be renormed by UASAS then for every $\epsilon > 0$ no extended ϵ-tree can live inside the original unit ball of X^*.*

Proof: Suppose X has an equivalent UASAS norm $|\!|\!| \cdot |\!|\!|$ so that for some positive constants θ and ϕ,

$$\frac{1}{\phi}\|x\| \leq |\!|\!|x|\!|\!| \leq \frac{1}{\theta}\|x\| \quad x \in X.$$

Then X^* has an induced equivalent norm $|\!|\!| \cdot |\!|\!|$ with $\theta\|f\| \leq |\!|\!|f|\!|\!| \leq \phi\|f\|$ for all $f \in X^*$.

Choose $\epsilon > 0$. Then $f, g \epsilon B_{X^*}(|\!|\!| \cdot |\!|\!|)$, $|\!|\!|f - g|\!|\!| \geq \frac{\theta\epsilon}{\phi}$ and $|\!|\!|f - g|\!|\!|_n = 0$ implies that there exists $\delta = \delta(\epsilon) > 0$ such that $\frac{1}{2}|\!|\!|f + g|\!|\!|_n \leq (1 - \delta)$. Equivalently $f, g \epsilon \phi B_{X^*}(|\!|\!| \cdot |\!|\!|)$, $|\!|\!|f - g|\!|\!| \geq \theta\epsilon$ and $|\!|\!|f - g|\!|\!|_n = 0$ implies there exists $\delta = \delta(\epsilon) > 0$ such that $\frac{1}{2}|\!|\!|f + g|\!|\!|_n \leq \phi(1 - \delta)$.

Suppose the original unit ball $B_{X^*}(\| \cdot \|)$ contains an extended $\frac{\epsilon}{2}$-tree $\{f_n\}_{n=1}^{\infty}$

and let

$$\sup_n \|f_n\| = c.$$

Since $f_n \in B^*(\|\cdot\|)$, $\|f_n\| \le \phi$. By definition, $\|f_{2n} - f_{2n+1}\| \ge \theta\epsilon$ and $\|f_{2n} - f_{2n+1}\|_{k_n} = 0$. Pick n such that

$$\|f_n\| > c(1 - \frac{\delta}{2}).$$

Now, as in lemma, choose $\{f_{n_i}\}_{i=1}^{\infty}$ with $f_{n_1} = f_n$. Find f_{n_m} such that

$$\|f_{n_m}\|_{k_{(n_m)}} \ge \|f_{n_m}\|(1 - \frac{\delta}{2}).$$

Let $m = \max\{\|f_{2n_m}\|, \|f_{2n_m+1}\|\}$ and note that $m \le c$. Then $\|\frac{\phi}{m} f_{2(n_m)}\| \le \phi$, $\|\frac{\phi}{m} f_{2(n_m)+1}\| \le \phi$ and $\|\frac{\phi}{m} f_{n_m}\| \le \phi$ and thus

$$\begin{aligned}
\frac{1}{2}\|\frac{\phi}{m}(f_{2n_m} + f_{2n_m+1})\|_{k_{n_m}} &= \frac{\phi}{m}\|f_{n_m}\|_{k_{n_m}} \\
&\ge \frac{\phi}{m}\|f_{n_m}\|(1 - \frac{\delta}{2}) \\
&\ge \phi\frac{c}{m}(1 - \frac{\delta}{2})(1 - \frac{\delta}{2}) \\
&\ge \phi(1 - \frac{\delta}{2})^2 \\
&\ge \phi(1 - \delta + \frac{\delta^2}{4}) \\
&> \phi(1 - \delta)
\end{aligned}$$

But this is a contradiction and our proof is complete.

CHAPTER THREE

C*PCP AND EXTENDED TREES

In Chapter two we examined a class of spaces which failed to have extended ϵ-trees for any $\epsilon > 0$. Now we wish to present a definition which will provide spaces containing many extended ϵ-trees for fixed $\epsilon > 0$. To this end we introduce the definition of C*PCP spaces.

3.1 Definition: Let X be a Banach space. We say X^* has the **Convex w^*-Point of Continuity Property** (C*PCP) if for every convex bounded $C \subset X^*$ and every $\epsilon > 0$ there exists a w^*-open set $N \subset X^*$ satisfying $C \cap N \neq \emptyset$ and $\text{diam}(C \cap N) < \epsilon$.

Example: RNP implies C*PCP.

3.2 To express the opposite of C*PCP we will say X^* fails C*PCP with C and ϵ if for every w^*-open set $N \subset X^*$ with $C \cap N \neq \emptyset$, $\text{diam}(C \cap N) \geq \epsilon$.

It will be these spaces that fail C*PCP that we will be interested in and which will give us our desired result i.e. spaces containing extended ϵ-trees.

3.3 Proving that spaces which fail C*PCP do have an abundance of extended ϵ-trees requires a great deal of technical theory. Thus an attempt will be made to break this long proof into several manageable segments. The first of these is proved in [3]. We give the proof here for completeness.

Lemma 1. *Let $\{K_n\}_{n=1}^{\infty}$ be a collection of nonempty, compact, convex subsets of a linear-topological space E with $K_{2n} \cup K_{2n+1} \subset K_n$ for each n. Then there exists a tree $\{x_n\}_{n=1}^{\infty}$ in E such that $x_n \in K_n$ for each n.*

Proof: Let

$$Q = \prod_{n=1}^{\infty} K_n$$

and for each n let

$$A_n = \left\{ q \in Q : \frac{1}{2}\Big(q(2n) + q(2n+1)\Big) = q(n) \right\}.$$

We will have our tree if $\bigcap_{n=1}^{\infty} A_n \neq \emptyset$. Now A_n is closed for all n and by Tychonoff's Theorem Q is compact so by the finite intersection property it is sufficient to show that $A_1 \cap \cdots \cap A_k \neq \emptyset$ for each k.

Fix k and define $p \in Q$ by first choosing $p(n) \in K_n$ arbitrarily for $n > k$. Then from $n = k$ to $n = 1$ we suppose that $p(m) \in K_m$ has been defined for $m > n$ and set $p(n) = \frac{1}{2}(p(2n) + p(2n+1))$. K_{2n} and K_{2n+1} are contained in K_n and K_n is convex so $p(n) \in K_n$. By induction $p \in A_1 \cap \cdots \cap A_k$. This completes the proof.

3.4 This next lemma is a slight generalization of a result stated for CPCP spaces (which are defined analogously in a space X) in [1].

Lemma 2. *Suppose X^* fails C^*PCP with A and ϵ. If $A_r = A + B_r^*$, then $diam(A_r \cap (x + E)) \geq \epsilon$ for each $x^* \in A_r$ and each cofinite dimensional subspace E of X^*.*

Proof: We will show by induction that if $x^* \in A_r$, $E \subset X^*$, codim$E \leq n$ and N is a w^*-open neighborhood of x^*, then diam$(A_r \cap (x^* + E) \cap N) \geq \epsilon$.

If $n = 0$, then $E = X^*$ and the condition C*PCP gives the result.

Now assume that the statement is true for n and let $\text{codim}E = n+1$. Take $x^* \in A_r$ and $N = N(x^*, x_1, \cdots, x_p, \delta)$ such that $\|x_i\| \leq 1$ for $1 \leq i \leq p$. We will show $\text{diam}(A_r \cap (x^* + E) \cap N) \geq \epsilon - \gamma$, where $0 < \gamma < \delta$.

There exists a subspace $F \subset X^*$ with $\text{codim}F = n$ and there exists $x \in X$ of norm 1 such that $E = F \cap \ker x$ (where x is considered an element of X^{**}). Clearly x is not zero on F so taking $\rho > 0$ with $B^*(x, \rho) \subset A_r$ we can find $\alpha > 0$ such that

$$x(F \cap B^*_\rho) \supset [-\alpha, \alpha].$$

Set $\beta = \min(\frac{\alpha\gamma}{2D}, \frac{\delta}{2})$ where $D = \text{diam}A_r$. Now let $M = N(x^*, x_1, \cdots, x_p, x, \beta)$.

By induction, $\text{diam}(A_r \cap (x^* + F) \cap M) \geq \epsilon$. In addition $(x^* + E) \subset (x^* + F)$ and $M \subset N$ thus $(A_r \cap (x^* + E) \cap M) \subset (A_r \cap (x^* + F) \cap N)$. It follows then that if we show $\text{dist}(y^*, A_r \cap (x^* + E) \cap N) \leq \frac{\gamma}{2}$ for all $y^* \in (A_r \cap (x^* + F) \cap M)$, then $\text{diam}(A_r \cap (x^* + E) \cap N) \geq \epsilon - \gamma$. Fix y^*.

Take $h^* \in F \cap B^*_\rho$ in the following manner; if $x(x^*) \geq x(y^*)$ choose h such that $x(h^*) = \alpha$; if $x(x^*) < x(y^*)$ choose h such that $x(h^*) = -\alpha$. Set

$$\lambda = \frac{x(x^*) - x(y^*)}{x(x^*) - x(y^*) + x(h^*)}.$$

Since $y^* \in M$ and $\|x\| = 1$, $|x(x^* - y^*)| < \beta$ and $\lambda \in [0, \frac{\beta}{\alpha}]$.

Finally we set $z^* = (1-\lambda)y^* + \lambda(x^* + h^*)$. We claim $z^* \in A_r \cap (x^* + E) \cap N$. Since A_r is convex and $y^*, (x^* + h^*) \in A_r$, $z^* \in A_r$. To see $z^* \in x^* + E$ note that $z^* - x^* = (1-\lambda)(y^* - x^*) + \lambda h^* \in F$ by convexity. It is a routine calculation to show $x(z^*) = x(x^*)$ which gives $z^* - x^* \in \ker x$. Thus $z^* - x^* \in E$ or $z^* \in x^* + E$.

In addition, $y^* - z^* = \lambda y^* - \lambda(x^* + h^*) = \lambda(y^* - (x^* + h^*))$ so $\|y^* - z^*\| \leq \frac{\gamma}{2}$. $\|x^* - z^*\| \leq \|x^* - y^*\| + \|y^* - z^*\| < \delta$ which implies $z^* \in N$. Thus $z^* \in (A_r \cap (x^* + E) \cap N)$ and $\|y^* - z^*\| \leq \frac{\gamma}{2}$ yields $\text{dist}(y^*, A_r \cap (x^* + E) \cap N) \leq \frac{\gamma}{2}$ which completes the proof.

3.5 We now are in a position to prove the final lemma. This lemma will play a crucial role in the proof of the theorem which will follow and will be referred to frequently. Throughout the lemma the quantifier i will appear. In all instances this is to be interpreted as $i = 1, 2$. Now to the lemma.

Lemma 3. *Let X be a Banach space with Schauder basis $\{x_n\}_{n=1}^{\infty}$. For $U \subset X^*$ and $f^* \in X^*$ suppose $diam U > \epsilon$ and that $U = B_r^* \cap (f + E)$ where E is a cofinite dimensional subspace of X^* and*

$$E = \{x^* \in X^* : x^*(y) = 0, y \in sp\{x_1, x_2, \cdots, x_m\}\}$$

$$\cap \{x^* \in X^* : x^*|_k = 0\}.$$

Then there exists $x \in X$, $\|x\| = 1$ and $f_i \in U$ with $(f_1 - f_2)(x) \geq \epsilon$. If given $k < k_1 < k_2$ and $r_i < r$, then there exist U_i, each contained in U, such that

$$U_i = B_{r_i}^* \cap (f_i + E_i)$$

with $E_i \subset E$ and

$$E_i = \{x^* \in X^* : x^*(y) = 0, y \in sp\{x_1, x_2, \cdots, x_m, x_{m+1}, \cdots, x_j, x\}$$

$$\cap \{x^* \in X^* : x^*|_{k_1} = 0\}.$$

Furthermore, if $g \in U_1$ and $h \in U_2$ then $(g - h)(x) \geq \epsilon$.

Proof: Diam $U > \epsilon$ guarantees that we can find f_i in U with $\|f_1 - f_2\| > \epsilon$. Pick $x \in X$, $\|x\| = 1$ such that $(f_1 - f_2)(x) \geq \epsilon$.

First we show that the E_i's are contained in E. But this follows since we are considering elements of X^* restricted to larger subspaces of X. In particular, those elements belonging to E_i are equal to zero on $P_k(X)$ and $sp\{x_1, \cdots, x_m\}$ and so belong to E.

To show $U_i \subset U$ notice that the elements of $(f + E)$ are precisely those which equal f on the subspace $P_k(X) \cup sp\{x_1, \cdots, x_m\}$ while $(f_i + E_i)$ is the set whose elements are equal to f_i on $P_{k_n}(X) \cup sp\{x_1, \cdots, x_m, x_{m+1}, \cdots, x_j, x\}$. But $f_i \in U$ so $f_i = f$ on $P_k(X) \cup sp\{x_1, \cdots, x_m\}$ and it follows that $x^* \in (f_i + E_i)$ implies $x^* = f_i = f$ on $P_k(X) \cup sp\{x_1, \cdots, x_m\}$. This shows $(f_i + E_i) \subset (f + E)$. Now if $r_i < r$ then $B^*_{r_i} \subset B^*_r$ and $U_i = B^*_{r_i} \cap (f_i + E_i) \subset U$ is obtained.

The last statement follows since $h(x) = f_2(x)$ and $g(x) = f_1(x)$.

3.6 We now prove the theorem.

Theorem 2. *Let X be a Banach space with a Schauder basis $\{x_n\}$. If X^* fails C^*PCP with the unit ball and δ, then for every $0 < \epsilon < \delta$, the unit ball of X^* contains an extended ϵ-tree.*

Proof: Choose $0 < \epsilon < \delta$. It is sufficient to find a sequence of points $\{y_n\}_{n=1}^{\infty} \subset X$ with $\|y_n\| = 1$ for every n, and non-empty, bounded, convex subsets U_n of X^* such that

1. $U_{2n} \cup U_{2n+1} \subset U_n$ for every n.
2. $(g - h)(y_n) \geq \epsilon$ for all $g \in U_{2n}$, and $h \in U_{2n+1}$.
3. $\|f\|_{k_n} \geq \rho_n \|f\|$ for all $f \in U_n$ where ρ_n is arbitrary and $\rho_n \uparrow 1$.
4. $f \in U_n$, $h \in U_{2n}$ and $g \in U_{2n+1}$ implies $f = g = h$ on $P_{k_n}(X)$.

To see the sufficiency of these statements, we let K_n be the w^*-closure of U_n and note that K_n is nonempty, convex and bounded. Since K_n is w^*-closed and bounded it is w^*-compact for all n, thus by lemma 1 there exists a tree $f_n \in B^*_1$ such that $f_n \in K_n$. By the definition of w^*-closure, we see that the conditions in 2.,3., and 4. hold for K_n in place of U_n so f_n is an extended ϵ-tree.

Our task then is to find $\{k_n\}_{n=1}^{\infty}$, $\{x_n\}_{n=1}^{\infty}$ and $\{U_n\}_{n=1}^{\infty}$ satisfying the above conditions. Towards this aim we will require further that

5. diam$U_n \geq \epsilon + \gamma$ for some $\gamma > 0$ with $\epsilon + \gamma < \delta$

6. $U_n = B^*_{r_n} \cap (\phi_n + E_n)$ where $\phi_n \in B^*_{r_n}$, $E_n = \{h \in X^* : f|_{k_n} = 0\} \cap \{f \in$ $f = 0$ on $sp\{y_1, \cdots, y_n\}\}$, and $r_n > \alpha > 0$ for some α constant. The r_n's and α will be determined as we progress. We proceed inductively.

Our strategy is to construct each U_n in a manner that will allow us to use lemma 2 repeatedly for the same set C. This set will be an open ball of radius $\alpha < 1$, and by homogeneity will fail C*PCP with the constant $\delta\alpha$. To determine α pick γ such that $\epsilon + \gamma < \delta$ and choose $\alpha < 1$ so that $\delta\alpha \geq \epsilon + \gamma$. Now find $\phi_1 \in B^*_1$ and $\rho_1 < 1$ with $\rho_1\|\phi_1\| \geq \alpha$. Next choose k_1 such that $\|\phi_1\|_{k_1} \geq \rho_1\|\phi_1\|$. Let $C = B^*_\alpha$.

Let $1 - \alpha = r_1 > 0$ and let $E_1 = \{h \in X^* : h|_{k_1} = 0\}$. Then $B^*_1 = C + B^*_{r_1}$. Letting $U_1 = B^*_1 \cap (\phi_1 + E_1)$ we have diam$U_1 \geq \delta\alpha$ by lemma 2. It is easily seen that U_1 is nonempty, convex, bounded and satisfies 3.,5. and 6.

Following the construction in [3] we now suppose that for $m \geq 1$, U_k is defined for $1 \leq k < 2^m$ with 5. and 6. true for these k and that y_n is defined for $1 \leq n < 2^{m-1}$ so that conditions 1.,2.,3. and 4. hold $1 \leq n < 2^{m-1}$. Let $2^{m-1} \leq s < 2^m$ and inductively suppose that $y_1, \cdots, y_{s-1}$, $k_1, \cdots, k_{2s-1}$, $\rho_1, \cdots, \rho_{2s-1}$ have been found.

Diam$U_s > \epsilon$ so we can apply lemma 3. As in lemma 3, choose $\phi_{2s} \in U_s$ and $\phi_{2s+1} \in U_s$ and $y_s \in X$, $\|y_s\| = 1$ such that

$$(\phi_{2n} - \phi_{2n+1})(y_s) \geq \epsilon.$$

Since the constructions of U_{2s} and U_{2s+1} are nearly identical we shall show only that of U_{2s}.

By induction $U_s = B^*_{r_s} \cap (\phi_s + E_s)$. Let $\omega_{2s} = r_s - \|\phi_{2s}\|$ Since $B^*_{r_s}$ is open $\omega_{2s} > 0$. Take $1 > \rho_{2s} > \max\{1 - \frac{1}{2s}, \rho_{2s-1}\}$ and find α_{2s} and β_{2s} such that

$0 < \beta_{2s} < 1, 0 < \alpha_{2s} < \omega_{2s}$ and

$$\frac{\beta}{1+\alpha_{2s}} = \rho_{2s}$$

Now choose $k_{2s} > k_{2s-1}$ such that

$$\|\phi_{2s}\|_{k_{2s}} \geq \|\phi_{2s}\|\beta_{2s}.$$

Set $r_{2s} = \|\phi_{2s}\|(1+\alpha_{2s})$. Then $r_{2s} < r_s$ and we can construct U_{2s} as in lemma 3, with $E_{2s} = \{f \in X^* : h|_{k_{2s}} = 0\} \cap \{f \in X^* : f = 0 \text{ on } sp\{y_1, \cdots, y_s\}\}$. This completes the construction of U_{2s}. We will assume U_{2s+1} has been constructed in a similiar manner and proceed to verify conditions 1.-6.

By lemma 3 1.,2. and 6. are satisfied, while 4. follows from the fact that $U_{2s} \cup U_{2s+1} \subset U_s$. For 3. we need to show that $\|f\|_{k_{2s}} \geq \rho_{2s}\|f\|$ for all $f \in U_{2s}$. Indeed, since $\|f\| < r_{2s} = \|\phi_{2s}\|(1+\alpha_{2s})$ we have

$$\begin{aligned} \|f\|_{k_{2s}} = \|\phi_{2s}\|_{k_{2s}} &\geq \|\phi_{2s}\|\beta_{2s} \\ &> \frac{\|f\|}{(1+\alpha_{2s})}\beta_{2s} \\ &= \rho_{2s}\|f\|. \end{aligned}$$

We note that since ρ_n is chosen to be greater than $1 - \frac{1}{n}$ and greater than ρ_{n-1}, $\rho_n \uparrow 1$ and so 3. has been established for $n = 2s$.

Once we show $r_{2s} > \alpha$ we can use lemma 2 to prove 5. for U_{2s} i.e. diam$U_{2s} > \epsilon$. To see this note that ϕ_{2s} is equal to ϕ_1 on $P_1(X)$. Hence $\|\phi_{2s}\| \geq \|\phi_{2s}\|_{k_1} \geq \rho_1\|\phi_1\| \geq \alpha$. Thus $r_{2s} > \|\phi_{2s}\| \geq \alpha$ and we have 5. verified.

As s varies through $2^{m-1} \leq s < 2^m$, $n = 2s$ and $n = 2s+1$ exhaust $\{n : 2^m \leq n < 2^{m+1}\}$ and our induction is complete. This concludes the proof.

A well known result in the literature (cf. [6]) states that for all spaces X containing an isomorphic copy of ℓ_1, the dual of X fails C^*PCP with the unit ball and some $\epsilon > 0$. We thus have the following corollary.

Corollary. *If X is a Banach space with a Schauder basis and containing an isomorphic copy of ℓ_1, then X^* contains an extended tree for some $\epsilon > 0$.*

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