A Systematic Construction of Multiwavelets on the Unit Interval

by

Michelle Michelle

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Department of Mathematical and Statistical Sciences
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Abstract

One main goal of this thesis is to bring forth a systematic and simple construction of a multiwavelet basis on a bounded interval. The construction that we present possesses orthogonality in the derivatives of the multiwavelet basis among all scale levels. Since we are mainly interested in Riesz wavelets, we call such wavelets mth derivative–orthogonal Riesz wavelets. Furthermore, we present some necessary and sufficient conditions as to when such a construction can be done. We show that our constructed multiwavelet bases possess many desirable properties such as symmetry, stability, and short support. The second goal of this thesis is to provide some conditions that guarantee a Riesz wavelet in $L_2(\mathbb{R})$ can be adapted so that it forms a Riesz wavelet for $L_2(\mathcal{I})$, where $\mathcal{I}$ is a bounded interval. As the third goal of this thesis, we also evaluate the performance of the newly constructed bases in obtaining the numerical solutions to some differential equations to showcase their potential usefulness. More specifically, we show how the resulting coefficient matrices are sparse and have a low condition number.

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Introduction

1.1 Overview of Wavelet Analysis

The area of wavelet analysis has existed for many decades. The emergence of this field can perhaps be attributed to Haar's discovery in 1909, which these days we recognize as the Haar orthogonal wavelet system. One defining feature of the Haar orthogonal system is its very simple analytic expression, since it is just an indicator function defined on a bounded interval. However, the research work in the area of wavelet analysis did not drastically gain momentum and even accelerate until around the 1980s after the discovery of the famous Daubechies [18], Meyer [46], and Morlet wavelets [29], and the introduction of the concept of multiresolution analysis (MRA) by Mallat and Meyer in [49, 50].

While a wavelet is typically derived from a scalar refinable function, a multiwavelet is derived from a refinable vector function. The concept of multiwavelets was first introduced by the authors of [27, 28]: Geronimo, Hardin, Massopust, Goodman, and Lee in the early 1990s. Even though the focus of this thesis is on wavelets and multiwavelets, it is essential to note that there is also one other counterpart of wavelets, namely framelets. One of the pioneering studies in multiresolution analysis (MRA) and framelets was done by Daubechies, Han, Ron, and Shen in [19]. However, the concept of frames itself was introduced by Duffin and Schaeffer in [25]. Unlike frames and framelets, wavelet and multiwavelet bases do not have any redundancy and are in fact minimal.

Currently, the theory of wavelets and framelets have been well developed. In fact, many wavelets and framelets related theorems and results have been established and now exist in general settings and conditions [33]. From the application standpoint, wavelets and framelets have proved themselves to be a versatile tool. It is impossible to provide an exhaustive list of their applications. However, we describe a handful of the major ones. In general, wavelets and framelets have been used in computer graphics, image processing, signal processing, statistics, machine learning, and numerical analysis. For instance, in computer graphics, wavelets have been used to construct a representation of 3D shapes such as different parts of human brain [23]. Framelets have been used in many facets of image processing such as denoising, inpainting, deblurring, restoration, and compression [38, 48, 53]. In the field of the statistics, one of the earlier uses of wavelets and the self-similarity property of their associated scaling function is in estimating the long range dependence parameter [1], namely the Hurst parameter. Also thanks to wavelets' ability in performing multiscale analysis, wavelets have been used in various types of analysis of time series data [51]. Given how advanced our present technology and society are, we face an increasing necessity to process a lot of information and big data, which is why data science and machine learning have gained so much popularity over the last couple of decades. Wavelets and framelets have played quite an interesting role in some learning algorithms such as pattern recognition [8], artificial neural network [26], and support vector machine [24]. Above all things, wavelets and framelets are first and foremost an approximation tool. Hence, wavelet bases have naturally been used in finding numerical solutions to differential equations. In fact, the application of wavelets in numerical analysis serves as one major focus of this thesis.

1.2 Wavelet Analysis and its Application in Numerical Differential Equations

The amount of literature in the context of wavelet-based numerical solutions to differential equations is very rich. See [5, 11, 13, 14, 16, 15, 34] for some pioneering work. On a related note, since the Sobolev spaces are fundamental in the study of numerical differential equations, we also would like to point out that the theory of Riesz wavelets in the Sobolev spaces has been well studied; for example, see [31, 37, 33]. Not only have wavelets been used in differential equations, they have been employed to provide numerical solutions for integral equations [47, 56]. In regards to adapting a wavelet basis defined on $\mathbb{R}$ to the unit interval, one of the influential papers is [13]. In fact, we shall use a generalized version of the operator introduced by the authors of [13] as a part of our work. In the area of wavelet-based numerical solutions to differential equations, there are typically 2 major directions towards which a research study can go: one is the construction of a wavelet basis on an interval while ensuring the inclusion of as many desirable properties as possible. See [3, 7, 16, 22, 34, 41, 43] for some possible constructions. One imperative issue that arises from constructing a wavelet/multiwavelet basis on the interval is of course the treatment of the boundary elements. In [2] and references therein, the authors have provided us with several ways to deal with the boundary elements for an orthogonal multiwavelet basis. Just like in [41], a common approach is to simply just cut the boundary elements. Still in the subject of the construction of wavelets, many papers only consider specific examples of refinable vector functions, but the methods they use to construct the wavelets differ. Compare [7, 22, 41] for example. Firstly, one common denominator of the 3 papers is the usage of Hermite cubic splines as their refinable vector function. While all three papers impose some sort of orthogonality condition in the first derivative, the first two replace some of the wavelets at even integer shifts with a pair of wavelets with a longer support in an attempt to achieve better sparsity. Researchers have also been interested in the construction of a wavelet basis in a rectangular domain and even a general bounded domain in a higher dimension [6, 17, 22, 54]. One method that has been frequently used is by taking the tensor product of a univariate Riesz wavelet [22, 54]. The general theory of wavelets and framelets on $\mathbb{R}^d$ has been well studied (see [32, 33] and references therein), but for this thesis we simply focus on the construction on $\mathbb{R}$. The other major direction that researchers tend to be invested in is more on the wavelet-based adaptive algorithms [4, 12, 22, 45], which to a certain extent is more concerned with the level of resolution needed to detect and approximate singularities that may be present in the solution. However, this thesis will not deal with these adaptive algorithms. As a last note, a handful of these existing wavelet bases in the literature have been used to solve some engineering problems [9, 21, 55].

Now, we explain why we want to be able to come up with a new type of multiwavelet bases construction on an interval, and why we choose to study multiwavelet bases at the first place. In order to elaborate more on the first point, we discuss some deficiencies that many existing methods seem to have. Roughly speaking, it has been discovered that having a multiwavelet basis may lead to a shorter support. This of course is one desirable feature. Having multiple refinable functions at our disposal in general presents us with more freedom to retain or modify some basis properties. Just because a multiwavelet basis is orthogonal/biorthogonal on $\mathbb{R}$, it does not automatically mean the boundary elements derived from this basis are orthogonal/biorthogonal on a bounded interval. The shorter the support of our basis is the easier time we have in handling the boundaries. Some methods for handling the boundary wavelets can be found in the literature [2, 43] and some (like matrix completion approach [2]) are much harder to implement than others. One of the most convenient to use is perhaps finding a linear combination of boundary crossing elements, but the linear combination has to be carefully selected in a way such that the wavelet basis preserves its original properties on $\mathbb{R}$. Even though finding such a linear combination is not an insurmountable task, it is always preferable if we can simply take the restriction of our wavelet basis on the interval of interest. Furthermore, that way we have more flexibility in preserving orthogonality/biorthogonality property. Some existing basis constructions just like in [16], though mathematically elegant, is extremely complicated to implement; additionally, checking the relevant assumptions is rather difficult. Not only is the construction difficult to follow, the condition number of the basis is very large, which impedes us from performing fast and accurate computations. Also in the context of numerical analysis, many research studies use refinable functions, which do not have analytic expression [9, 21, 52]; thus, it adds a layer of inconvenience to the implementation. The calculations may become less transparent than it should be. Consequently, in many applications, we want to restrict our attention on wavelet bases that have an analytic expression.

Two driving motivations to use a wavelet basis in finding numerical solutions to differential equations is its sparse representation and good localization properties. Thus, we need to be able to exploit its potential so that the linear system coming from the Galerkin formulation is well conditioned. One question that we can ask ourselves is what kind of conditions we can impose to achieve a better sparsity. The construction of a multiwavelet basis on an interval procedures proposed by [7, 22, 41] are perhaps some of the best available methods in the literature due to their simplicity and sparsity. Also, what we observe is that most research papers pick a specific example of scalar refinable function/refinable vector function of interest and afterwards construct the wavelets from it. Thus, the second natural question that still remains unanswered is if there is an underlying theory that guarantees that the construction method of interest can be applied to all refinable vector functions satisfying some given conditions. If the answer to the last question is positive, we are interested in the properties that we are able to retain after adapting a multiwavelet basis on $\mathbb{R}$ to a bounded interval. Overall, there is indeed a lot of room for research in the subject of construction.

1.3 Contribution and Structure of This Thesis

One main goal of this thesis is to bring forth a systematic and simple construction of a multiwavelet basis on a bounded interval. Inspired by [10, 41, 43], the construction that we present possesses orthogonality in the derivatives of the multiwavelet basis across all scale levels, which results in an increased sparsity of the stiffness and mass matrices. Since we are mainly interested in Riesz wavelets, we shall call such wavelets mth derivative–orthogonal Riesz wavelets. Furthermore, we present some necessary and sufficient conditions as to when such a construction can be done. We show that our constructed multiwavelet bases possess many desirable properties such as low condition number, stability, short support, and good approximation order. The second goal of this thesis is to provide some conditions that guarantee a Riesz wavelet on $\mathbb{R}$ can be adapted to a Riesz wavelet on a bounded interval. As the third goal of this thesis, we also evaluate the numerical performance of the newly constructed bases in order to study their potential usefulness. Furthermore, we shall see how easily we can handle the boundaries given the wavelet bases we use.

This thesis has the following organization. The later part of Chapter 1 contains some preliminaries, which will be pertinent to the discussions that ensue. Chapter 2 presents the construction method that we propose including the underlying theoretical justifications. A generalized version of folding operator is introduced in Chapter 3 and we explain how using this operator can transform a Riesz wavelet on $\mathbb{R}$ to a Riesz wavelet on an interval. In Chapter 4, we consider a condition number optimization problem and obtain appropriate scaling coefficients such that our wavelet bases are optimally scaled in some sense; additionally, we evaluate the performance of our newly constructed wavelet bases by considering some numerical examples. Finally, we conclude this thesis by outlining some future work in Chapter 5.

1.4 Preliminaries

This section introduces many definitions that will be used throughout the entire thesis. We shall also review a few basic properties of B-splines and Hermite splines, and state a few basic facts on bases and frames. The materials of this chapter are mostly derived from [33, 44].

Definition 1.1. *Suppose $\phi := (\phi_1, \dots, \phi_r)^{\mathsf{T}}$, where $\phi_\ell : \mathbb{R} \to \mathbb{C}$ for all $\ell = 1, \dots, r$. We call ϕ a refinable vector function if it satisfies the following refinability condition*

$$\phi = 2 \sum_{k \in \mathbb{Z}} a(k) \phi(2 \cdot - k),$$

where $a \in (l_0(\mathbb{Z}))^{r \times r}$.

We have presented the definition of a refinable vector function in the time domain. Once we introduce the definition of Fourier transform, we shall see the definition becomes even easier to understand and write. In fact, the technical details and calculations are much more convenient to handle in the frequency domain.

Definition 1.2. *Let $f \in L_1(\mathbb{R})$. The Fourier transform of f is defined as*

$$\widehat{f}(\xi) := \int_{\mathbb{R}} f(x)e^{-ix\xi}dx, \quad \forall \xi \in \mathbb{R}.$$

Even though we only show the definition for the classical Fourier transform, Fourier transform can actually be extended and applied to tempered distributions (i.e., a continuous linear functional on the space of all $C^{\infty}(\mathbb{R})$ functions φ satisfying the condition: $\|x^{\alpha}\varphi^{(\beta)}(x)\|_{C(\mathbb{R})} < \infty$ for every $\alpha, \beta \in \mathbb{N} \cup \{0\}$) as well. Meanwhile, the space of distributions on $\mathbb{R}$ is the dual of the function space that contains all C^{∞} functions with compact support. Sobolev spaces are a crucial ingredient in our wavelet construction and in the study of differential equations.

Definition 1.3. *A Sobolev space $H^{\tau}(\mathbb{R})$, where $\tau \in \mathbb{R}$, is a Hilbert space that contains all tempered distributions f on $\mathbb{R}$ satisfying*

$$\|f\|^2_{H^{\tau}(\mathbb{R})} := \frac{1}{2}\int_{\mathbb{R}} |\widehat{f}(\xi)|^2(1+|\xi|^2)^{\tau}d\xi < \infty.$$

Assume further that $m \in \mathbb{N} \cup \{0\}$, a function f is in the Sobolev space $H^m(\mathbb{R})$ if and only if $f, f', \dots, f^{(m-1)}$ are all absolutely continuous and f, f', $\dots$, $f^{(m-1)}$, $f^{(m)} \in L_2(\mathbb{R})$.

Also, when $m = \mathbb{N} \cup \{0\}$, recall that the Sobolev norm can also be defined as

$$\|f\|^2_{H^m(\mathbb{R})} := \sum_{k=0}^{m} \|f^{(k)}\|^2_{L_2(\mathbb{R})} = \sum_{k=0}^{m} \int_{\mathbb{R}} |f^{(k)}(x)|^2 dx.$$

Now, we shall move on to the concept of frames, Bessel sequences, and Riesz bases. They are all connected to one another. The last one is in fact a fundamental property that we need for our wavelet bases. Henceforth, let $\mathscr{H}$ be a Hilbert space and Λ be an at most countable index set.

Definition 1.4. *Suppose $\{h_k\}_{k\in\Lambda}$ is a sequence in $\mathscr{H}$. If there exists $C > 0$ such that for all $h \in \mathscr{H}$,*

$$\sum_{k\in\Lambda} |\langle h, h_k\rangle|^2 \leqslant C\|h\|^2,$$

then we call $\{h_k\}_{k\in\Lambda}$ a Bessel sequence.

Below is the definition of a frame. As mentioned before, even though we are not dealing directly with frames in this thesis, it is very important to keep in mind that a Riesz basis is roughly speaking a frame without any redundancy.

Definition 1.5. *Suppose $\{h_k\}_{k\in\Lambda}$ is a sequence in $\mathscr{H}$. $\{h_k\}_{k\in\Lambda}$ is a frame for $\mathscr{H}$ if there exist $C_1, C_2 > 0$ such that for all $h \in \mathscr{H}$, the following inequality is satisfied:*

$$C_1\|h\|^2 \leqslant \sum_{k\in\Lambda} |\langle h, h_k\rangle|^2 \leqslant C_2\|h\|^2.$$

The definition of a dual frame can be found below.

Definition 1.6. *Suppose $\{h_k\}_{k\in\Lambda}$ and $\{\tilde{h}_k\}_{k\in\Lambda}$ are sequences in $\mathscr{H}$. $\{\tilde{h}_k\}_{k\in\Lambda}$ is a dual frame of $\{h_k\}_{k\in\Lambda}$ if each $\{h_k\}_{k\in\Lambda}$ and $\{\tilde{h}_k\}_{k\in\Lambda}$ is a frame for $\mathscr{H}$ and for all $f, g \in \mathscr{H}$, we have*

$$\langle f, g\rangle = \sum_{k\in\Lambda} \langle f, \tilde{h}_k\rangle\langle h_k, g\rangle. \tag{1.1}$$

Finally, we present the definition of a Riesz basis.

Definition 1.7. *Suppose $\{h_k\}_{k\in\Lambda}$ is a sequence in $\mathscr{H}$. Then $\{h_k\}_{k\in\Lambda}$ is a Riesz basis if and only if the linear span of $\{h_k\}_{k\in\Lambda}$ is dense in $\mathscr{H}$ and $\{h_k\}_{k\in\Lambda}$ is a Riesz sequence. I.e., there are $C_3, C_4 > 0$ such that*

$$C_3 \sum_{k\in\Lambda} |c_k|^2 \leqslant \left\| \sum_{k\in\Lambda} c_k h_k \right\|^2 \leqslant C_4 \sum_{k\in\Lambda} |c_k|^2 \tag{1.2}$$

for all finitely supported sequences $\{c_k\}_{k\in\Lambda}$.

The previous definitions build up to the theorem below, which we shall rely heavily on when proving the results in Chapter 3.

Theorem 1.1. *[33, Corollary 4.2.8] Suppose $\{h_k\}_{k\in\Lambda}$ and $\{\tilde{h}_k\}_{k\in\Lambda}$ are sequences in $\mathscr{H}$. Then $(\{\tilde{h}_k\}_{k\in\Lambda}, \{h_k\}_{k\in\Lambda})$ is a pair of biorthogonal bases for $\mathscr{H}$ if and only if $(\{\tilde{h}_k\}_{k\in\Lambda}, \{h_k\}_{k\in\Lambda})$ is a pair of dual frames for $\mathscr{H}$ and $\langle \tilde{h}_j, h_k\rangle = \delta_{j,k}$ for all $j, k \in \Lambda$, where $\delta_{j,k} = 1$ if $j = k$ and $\delta_{j,k} = 0$ if $j \neq k$ (the latter is what we call the biorthogonality condition).*

The following three related definitions will be heavily used in Chapter 2.

Definition 1.8. *Define the wavelet affine system in the Sobolev space $H^\tau(\mathbb{R})$ by*

$$\begin{aligned}\mathsf{AS}_0^\tau(\phi;\psi) := \{\phi_\ell(\cdot - k) : k \in \mathbb{Z}, 1 \le \ell \le r\} \\ \cup \left\{2^{j(1/2-\tau)}\psi_\ell(2^j \cdot -k) : j \in \mathbb{N}_0, k \in \mathbb{Z}, 1 \le \ell \le s\right\}.\end{aligned}$$

By convention, we define $\mathsf{AS}(\phi;\psi) := \mathsf{AS}_0^0(\phi;\psi)$. It is also worth mentioning that

$$\mathsf{AS}_J(\phi;\psi) := \left\{2^{J/2}\phi_\ell(2^J\cdot -k) : k\in\mathbb{Z}, 1\leqslant \ell\leqslant r\right\}$$
$$\cup\left\{2^{j/2}\psi_\ell(2^j\cdot -k) : j\geqslant J, k\in\mathbb{Z}, 1\leqslant\ell\leqslant s\right\},$$

since we shall see $\mathsf{AS}_J(\phi;\psi)$ a few times in Chapter 3. The definition of a Riesz wavelet is stated below.

Definition 1.9. *$\{\phi,\psi\}$ is a Riesz wavelet in $H^\tau(\mathbb{R})$ with $\tau\in\mathbb{R}$ if $\mathsf{AS}_0^\tau(\phi;\psi)$ is a Riesz basis for $H^\tau(\mathbb{R})$. I.e, (1) the linear span of $\mathsf{AS}_0^\tau(\phi;\psi)$ is dense in $H^\tau(\mathbb{R})$, (2) there exist $C_1, C_2>0$ such that*

$$\begin{aligned}
&C_1\Big(\sum_{\ell=1}^r\sum_{k\in\mathbb{Z}}|v_{\ell,k}|^2+\sum_{j=0}^\infty\sum_{\ell=1}^s\sum_{k\in\mathbb{Z}}|w_{\ell,j;k}|^2\Big)\\
&\leqslant\Big\|\sum_{\ell=1}^r\sum_{k\in\mathbb{Z}}v_{\ell,k}\phi_\ell(\cdot-k)+\sum_{j=0}^\infty\sum_{\ell=1}^s\sum_{k\in\mathbb{Z}}w_{\ell,j;k}2^{j(1/2-\tau)}\psi_\ell(2^j\cdot -k)\Big\|^2_{H^\tau(\mathbb{R})}\\
&\leqslant C_2\Big(\sum_{\ell=1}^r\sum_{k\in\mathbb{Z}}|v_{\ell,k}|^2+\sum_{j=0}^\infty\sum_{\ell=1}^s\sum_{k\in\mathbb{Z}}|w_{\ell,j;k}|^2\Big)
\end{aligned}$$

for all finitely supported sequences $\{v_{\ell,k}\}_{k\in\mathbb{Z},1\leqslant\ell\leqslant r}$ and $\{w_{\ell,j;k}\}_{j\in\mathbb{N}_0,k\in\mathbb{Z},1\leqslant\ell\leqslant s}$.

We clarify what we mean by a biorthogonal wavelet in the definition below.

Definition 1.10. *$(\{\tilde\phi;\tilde\psi\},\{\phi;\psi\})$ is a biorthogonal wavelet in $(H^{-\tau}(\mathbb{R}),H^\tau(\mathbb{R}))$ if (1) $\mathsf{AS}_0^\tau(\phi;\psi)$ is a Riesz basis for $H^\tau(\mathbb{R})$ and $\mathsf{AS}_0^{-\tau}(\tilde\phi;\tilde\psi)$ is a Riesz basis for $H^{-\tau}(\mathbb{R})$, (2) $\mathsf{AS}_0^\tau(\phi;\psi)$ and $\mathsf{AS}_0^{-\tau}(\tilde\phi;\tilde\psi)$ are biorthogonal to each other.*

In particular, if $(\{\tilde\phi;\tilde\psi\},\{\phi;\psi\})$ is indeed a biorthogonal wavelet in $(H^{-\tau}(\mathbb{R}),H^\tau(\mathbb{R}))$, then it admits the following representation

$$f=\sum_{\ell=1}^r\sum_{k\in\mathbb{Z}}\langle f,\tilde\phi_\ell(\cdot-k)\rangle\phi_\ell(\cdot-k)$$

$$+\sum_{\ell=1}^{r}\sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}}\langle f, 2^{j(1/2+\tau)}\tilde{\psi}_\ell(2^j\cdot -k)\rangle 2^{j(1/2-\tau)}\psi_\ell(2^j\cdot -k)$$

for all $f \in H^\tau(\mathbb{R})$ and the representation above converges unconditionally in $H^\tau(\mathbb{R})$.

Whenever we analyze a particular wavelet basis, we always want to know the following key characteristics: sum rule (polynomial reproduction capability), vanishing moments, and the smoothness of the filter of the refinable vector function itself. For the sake of convenience, we review the definitions of the three aforementioned terminologies below.

Definition 1.11. *Suppose $a \in (l_0(\mathbb{Z}))^{r\times r}$. The mask a has order m sum rules with a matching filter v if there exists $v \in (l_0(\mathbb{Z}))^{1\times r}$ such that $\widehat{v}(0) \neq 0$, and $\widehat{v}(2\xi)\widehat{a}(\xi) = \widehat{v}(\xi) + \mathscr{O}(|\xi|^m)$ and $\widehat{v}(2\xi)\widehat{a}(\xi+\pi) = \mathscr{O}(|\xi|^m)$ as $\xi \to 0$.*

Since the order of sum rules basically indicates polynomial reproduction ability, it is worth pointing out that these two are directly related to Strang-Fix condition (see [33, Section 5.5] and references therein). In addition to the definition of sum rules, there is also an important result [36, Theorem 2.3] that helps us to study the sum rules order in a multiwavelet setting. More specifically, [36, Theorem 2.3] states that if φ is a compactly supported refinable vector function/distribution and a is its matrix-valued filter satisfying order m sum rules with a matching filter $v \in (l_0(\mathbb{Z}))^{1\times r}$ and $\widehat{v}(0) \neq 1$, then there exists a strongly invertible $r \times r$ matrix $\widehat{U}(\xi)$ (i.e, det $(\widehat{U}(\xi))$ is a nonzero monomial) whose entries consist of 2π-periodic trigonometric polynomials such that

$$(\widehat{U}(2\xi))^{-1}\widehat{a}(\xi)\widehat{U}(\xi) = \begin{bmatrix} 2^{-m}(1+e^{-i\xi})^m\widehat{u_1}(\xi) & 2^{-m}(1-e^{-i2\xi})^m\widehat{u_2}(\xi) \\ \widehat{u_3}(\xi) & \widehat{u_4}(\xi) \end{bmatrix}, \quad (1.3)$$

where $\widehat{u_1}(0) = 1$ and

$$u_1 \in l_0(\mathbb{Z}), \quad u_2 \in (l_0(\mathbb{Z}))^{1\times(r-1)}, \quad (1.4)$$
$$u_3 \in (l_0(\mathbb{Z}))^{(r-1)\times 1}, \quad u_4 \in (l_0(\mathbb{Z}))^{(r-1)\times(r-1)}.$$

In addition, $\widehat{v}(\xi)\widehat{U}(\xi) = (1+\mathscr{O}(|\xi|), \mathscr{O}(|\xi|^m), \dots, \mathscr{O}(|\xi|^m))$ as $\xi \to 0$. Furthermore, if $\widehat{v}(0)\widehat{\phi}(0) = 1$ then the refinable vector function $\phi := (\phi_1, \dots, \phi_r)^\mathsf{T}$, $\widehat{\phi}(\xi) = (\widehat{U}(\xi))^{-1}\varphi(\xi)$ satisfies $\widehat{\phi}(2\xi) = (\widehat{U}(2\xi))^{-1}\widehat{a}(\xi)\widehat{U}(\xi)\widehat{\varphi}(\xi)$ as well as

$$\widehat{\phi_1}(0) = 1 \quad \text{and} \quad \widehat{\phi_1}(\xi + 2\pi k) = \mathscr{O}(|\xi|^m), \quad \xi \to 0, \quad k \in \mathbb{Z}\backslash\{0\}. \quad (1.5)$$

Definition 1.12. *Define $\psi := (\psi_1, \dots, \psi_s)^\mathsf{T}$. For some $\nu \geqslant 0$, a function ψ_ℓ has ν vanishing moments if $\widehat{\psi_\ell}(\xi) = \mathscr{O}(|\xi|^\nu)$ a.e. $\xi \in [-\pi, \pi]$; or equivalently,*

$$\int_{\mathbb{R}} x^k \psi_\ell(x)dx = 0$$

for all $k = 0, \dots, \nu - 1$. Lastly, we define $vm(\psi) := \min_{1\leqslant\ell\leqslant s} \sup\{\nu_\ell : \psi_\ell$ has ν_ℓ vanishing moments$\}$.

Definition 1.13. *Let f be a tempered distribution. We define the smoothness of f as*

$$sm(f) := \sup\{\tau \in \mathbb{R} : f \in H^\tau(\mathbb{R})\}. \quad (1.6)$$

By convention, if the right hand side of (1.6) *is empty, then we have* $sm(f) = -\infty$. *Additionally, if we have a vector function* $f := (f_1, \dots, f_r)^\mathsf{T}$, *then* $sm(f) := \min(sm(f_1), \dots, sm(f_r))$.

Bracket product serves as an important tool to analyze shift-invariant spaces. If $f \in (H^\tau(\mathbb{R}))^{r\times t}$ and $g \in (H^{-\tau}(\mathbb{R}))^{s\times t}$, then the bracket product is defined as

$$[f, g]_\tau(\xi) := \sum_{k\in\mathbb{Z}} f(\xi + 2\pi k)\overline{g(\xi + 2\pi k)}^\mathsf{T}(1 + |\xi|^2)^\tau$$

for all $\xi \in \mathbb{R}$. Define $[f, g](\xi) := [f, g]_0(\xi)$. The integer shifts of a compactly supported refinable vector function/distribution ϕ are stable if and only if $\operatorname{span}\{\widehat{\phi}(\xi + 2\pi k) : k \in \mathbb{Z}\} = \mathbb{C}^r$ for all $\xi \in \mathbb{R}$.

We close this chapter by presenting some basic properties of B-splines and Hermite splines. Even though our construction method applies to a large class of refinable vector functions and we never assume that the refinable vector functions necessarily have an analytic expression, it is worthwhile to review the properties of these splines. One reason is because these splines have been used multitudinously in numerical analysis due to their polynomial expressions. Additionally, in order to avoid being caught in the technicalities of the construction, we shall apply our construction method to these splines. Recall that the B-spline function of order n, B_n, is defined as $B_1 := \chi_{(0,1]}$ and $B_n := B_{n-1} * B_1 = \int_0^1 B_{n-1}(\cdot - t)dt$. Furthermore, the Fourier transform of B-spline of order n satisfies the following refinement equation: $\widehat{B_n}(2\xi) = \widehat{a_n^B}(\xi)\widehat{B_n}(\xi)$, where $\widehat{a_n^B}(\xi) := 2^{-n}(1 + e^{-i\xi})^n$. The filter a_n^B has order n sum rules. Later on, we shall see that Hermite cubic splines serve as an important example of piecewise polynomial refinable vector function with multiplicity $r = 2$. Firstly, recall that Hermite cubic splines take the following form

$$\begin{aligned} \phi_1(x) &= (1-x)^2(1+2x)\chi_{[0,1]} + (1+x)^2(1-2x)\chi_{[-1,0)}, \\ \phi_2(x) &= (1-x)^2 x\chi_{[0,1]} + (1+x)^2 x\chi_{[-1,0)}. \end{aligned} \tag{1.7}$$

The associated filter a of the above refinable vector function is

$$a(-1) = \begin{bmatrix} \frac{1}{4} & \frac{3}{8} \\ -\frac{1}{16} & -\frac{1}{16} \end{bmatrix}, \quad a(0) = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{4} \end{bmatrix}, \quad a(1) = \begin{bmatrix} \frac{1}{4} & -\frac{3}{8} \\ \frac{1}{16} & -\frac{1}{16} \end{bmatrix}, \tag{1.8}$$

with $a(k) = 0$ for all $k \in \mathbb{Z}\backslash\{-1, 0, 1\}$. The filter a has order 4 sum rules. One of the most attractive properties of Hermite cubic splines is its interpolating property: $\phi_1(k) = \delta(k)$, $\phi_1'(k) = 0$, $\phi_2(k) = 0$, and $\phi_2'(k) = \delta(k)$ for all $k \in \mathbb{Z}$ and $\delta(k) = 1$ if and only if $k = 0$. We shall witness how pivotal this interpolating property is in satisfying the boundary condition of differential equation. For $r = 2$, there are 2 other Hermite splines (quadratic and linear) that are of interest to us and will be introduced in due course. Even though this thesis solely considers Hermite splines of multiplicity $r = 2$, we would like to point out that there exist Hermite splines (interpolants) with a higher multiplicity (i.e., with $r > 2$), which have been well studied in [30, 33, 57].

Chapter 2

Construction of mth-Order Derivative-Orthogonal Riesz Wavelets in Sobolev Spaces

In this chapter, we present our proposed construction. Two crucial conditions that we impose, when we generate the wavelets from our choice of refinable vector functions are

$$\langle \psi^{(m)}, \phi^{(m)}(\cdot - k) \rangle = 0, \qquad \forall\, k \in \mathbb{Z}, \tag{2.1}$$

$$\langle \psi^{(m)}(2^j \cdot - k), \psi^{(m)}(2^{j'} \cdot - k') \rangle = 0, \qquad \forall\, k, k' \in \mathbb{Z}, j, j' \in \mathbb{N}_0 \quad \text{with } j \neq j'. \tag{2.2}$$

What (2.1) and (2.2) ultimately mean is that we want our Riesz wavelets to have mth derivative-orthogonality across all levels. For this reason, we call a Riesz wavelet $\{\phi; \psi\}$ in the Sobolev space $H^m(\mathbb{R})$ satisfying conditions (2.1) and (2.2) an mth-order derivative-orthogonal Riesz wavelet in $H^m(\mathbb{R})$.

This type of construction has been used before [7, 10, 22, 41, 43]. However, the authors only consider a specific example of refinable scalar/vector functions. One major goal of this chapter is to provide necessary and sufficient conditions as to when such a construction can be applied. At the same time, the results we present in this chapter essentially give an affirmative answer to the question whether there exists a theory that unify all construction examples in [7, 10, 22, 41, 43]. In order to better understand how the construction works, we apply the mth-order derivative-orthogonal construction to some B-splines and Hermite splines. All of the results and the proofs in this chapter are taken directly from [35].

We first need to state a lemma, which will be used multiple times in the proofs.

Lemma 2.1. *[33, Lemma 5.5.6] Let $m \in \mathbb{N}_0$ and η be a compactly supported distribution on $\mathbb{R}$ such that $\widehat{\eta}(\xi + 2\pi k) = \mathcal{O}(|\xi|^m)$, $\xi \to 0$, for all $k \in \mathbb{Z}$. Define $[n_-, n_+] := fsupp(\eta)$. Then $\eta := \nabla^m g$, $\mathsf{S}(\eta) = \mathsf{S}(g)$, where $\mathsf{S}(\eta) = \mathsf{S}((\eta_1, \dots, \eta_r)^\mathsf{T}) := \{v_1 * \eta_1 + \cdots + v_r * \eta_r : v_1, \dots v_r \in l(\mathbb{Z})\}$, and $fsupp(g) \subseteq [n_-, n_+ - m]$, where g is a compactly supported distribution defined by $g := \sum_{k=0}^{\infty} \frac{(m-1+k)!}{(m-1)!k!} \eta(\cdot - k)$.*

Proposition 2.1. *Let $\phi = (\phi_1, \dots, \phi_r)^\mathsf{T}$ and $\psi = (\psi_1, \dots, \psi_r)^\mathsf{T}$ be vector functions in the Sobolev space $H^\tau(\mathbb{R})$ with $\tau \in \mathbb{R}$ such that the refinable structure:*

$$\widehat{\phi}(2\xi) = \widehat{a}(\xi)\widehat{\phi}(\xi), \quad \widehat{\psi}(2\xi) = \widehat{b}(\xi)\widehat{\phi}(\xi) \tag{2.3}$$

holds for some $a, b \in (l_0(\mathbb{Z}))^{r \times r}$. Suppose that $\{\phi; \psi\}$ is a Riesz wavelet in the

Sobolev space $H^\tau(\mathbb{R})$. Then there exist vector functions $\tilde{\phi} = (\tilde{\phi}_1, \dots, \tilde{\phi}_r)^\mathsf{T}$ and $\tilde{\psi} = (\tilde{\psi}_1, \dots, \tilde{\psi}_r)^\mathsf{T}$ in $H^{-\tau}(\mathbb{R})$ such that $(\{\tilde{\phi}; \tilde{\psi}\}, \{\phi; \psi\})$ is a pair of biorthogonal wavelets in $(H^{-\tau}(\mathbb{R}), H^\tau(\mathbb{R}))$ and

$$\widehat{\tilde{\phi}}(2\xi) = \widehat{\tilde{a}}(\xi)\widehat{\tilde{\phi}}(\xi) \quad \textit{and} \quad \widehat{\tilde{\psi}}(2\xi) = \widehat{\tilde{b}}(\xi)\widehat{\tilde{\phi}}(\xi), \qquad a.e.\, \xi \in \mathbb{R} \tag{2.4}$$

for some $r \times r$ matrices $\widehat{\tilde{a}}$ and $\widehat{\tilde{b}}$ with entries in $L_\infty(\mathbb{T})$ satisfying

$$\begin{bmatrix} \widehat{\tilde{a}}(\xi) & \widehat{\tilde{a}}(\xi+\pi) \\ \widehat{\tilde{b}}(\xi) & \widehat{\tilde{b}}(\xi+\pi) \end{bmatrix} \begin{bmatrix} \overline{\widehat{a}(\xi)}^\mathsf{T} & \overline{\widehat{b}(\xi)}^\mathsf{T} \\ \overline{\widehat{a}(\xi+\pi)}^\mathsf{T} & \overline{\widehat{b}(\xi+\pi)}^\mathsf{T} \end{bmatrix} = I_{2r}, \tag{2.5}$$

for almost every $\xi \in \mathbb{R}$.

Proof. We first assume that $\mathsf{AS}_0^\tau(\phi; \psi)$ is a Riesz basis for $H^\tau(\mathbb{R})$. Since $H^{-\tau}(\mathbb{R})$ is the dual space of $H^\tau(\mathbb{R})$, $\mathsf{AS}_0^\tau(\phi; \psi)$ has a unique dual Riesz basis for $H^{-\tau}(\mathbb{R})$. More specifically, there are vector functions $\tilde{\phi} := (\tilde{\phi}_1, \dots, \tilde{\phi}_r)^\mathsf{T}$ and $\tilde{\psi} := (\tilde{\psi}_1, \dots, \tilde{\psi}_r)^\mathsf{T}$ in $H^{-\tau}(\mathbb{R})$ such that

$$\langle \tilde{\phi}_\ell, \phi_\ell \rangle = 1 \quad \text{and} \quad \langle \tilde{\phi}_\ell, h \rangle = 0 \qquad \forall\, h \in \mathsf{AS}_0^\tau(\phi; \psi) \backslash \{\phi_\ell\}, \ell = 1, \dots, r \tag{2.6}$$

and

$$\langle \tilde{\psi}_\ell, \psi_\ell \rangle = 1 \quad \text{and} \quad \langle \tilde{\psi}_\ell, h \rangle = 0 \qquad \forall\, h \in \mathsf{AS}_0^\tau(\phi; \psi) \backslash \{\psi_\ell\}, \ell = 1, \dots, r. \tag{2.7}$$

By the refinable structure in (2.3), each entry in $\psi(2^j \cdot -k)$ with $j < 0$ and $k \in \mathbb{Z}$ is a finite linear combination of $\phi_\ell(\cdot - k), \ell = 1, \dots, r$ and $k \in \mathbb{Z}$. Now by applying a similar argument as in the proof of [32, Theorem 8] for the case

$\tau = 0$, it follows that $\mathsf{AS}_0^{-\tau}(\tilde{\phi}; \tilde{\psi})$ is biorthogonal to $\mathsf{AS}_0^{\tau}(\phi; \psi)$. Because a given Riesz basis has a unique dual Riesz basis, $(\{\tilde{\phi}; \tilde{\psi}\}, \{\phi; \psi\})$ is a biorthogonal wavelet in the pair of Sobolev spaces $(H^{-\tau}(\mathbb{R}), H^{\tau}(\mathbb{R}))$. In particular, we have the following representation for $\tilde{\phi}(2^{-1}\cdot)$:

$$
\begin{aligned}
\tilde{\phi}(2^{-1}\cdot) = &\sum_{k\in\mathbb{Z}} \langle \tilde{\phi}(2^{-1}\cdot), \phi(\cdot - k)\rangle \tilde{\phi}(\cdot - k) \\
&+ \sum_{j=0}^{\infty} \sum_{k\in\mathbb{Z}} \langle \tilde{\phi}(2^{-1}\cdot), 2^{j(1/2-\tau)}\psi(2^j \cdot - k)\rangle 2^{j(1/2+\tau)}\tilde{\psi}(2^j \cdot - k).
\end{aligned}
\tag{2.8}
$$

Since $\mathsf{AS}_0^{\tau}(\phi; \psi)$ and $\mathsf{AS}_0^{-\tau}(\tilde{\phi}; \tilde{\psi})$ are biorthogonal, it must be the case that $\langle \tilde{\phi}(2^{-1}\cdot), 2^{j(1/2-\tau)}\psi(2^j \cdot - k)\rangle = \langle \tilde{\phi}, 2^{j(1/2-\tau)+1}\psi(2^{j+1} \cdot - k)\rangle = 0$. Hence, (2.8) yields $\tilde{\phi}(2^{-1}\cdot) = \sum_{k\in\mathbb{Z}} \langle \tilde{\phi}(2^{-1}\cdot), \phi(\cdot - k)\rangle \tilde{\phi}(\cdot - k)$. In other words, we have $\widehat{\tilde{\phi}}(2\xi) = \widehat{\tilde{a}}(\xi)\widehat{\tilde{\phi}}(\xi)$ with $\widehat{\tilde{a}}(\xi) := \frac{1}{2}\sum_{k\in\mathbb{Z}} \langle \tilde{\phi}(2^{-1}\cdot), \phi(\cdot - k)\rangle e^{-ik\xi} = [\widehat{\tilde{\phi}}(2\cdot), \widehat{\phi}](\xi)$. Since $\mathsf{AS}_0^{\tau}(\phi; \psi)$ is a Bessel sequence in $H^{\tau}(\mathbb{R})$, each entry in $[\widehat{\phi}, \widehat{\phi}]_{\tau}$ belongs to $L_{\infty}(\mathbb{T})$. Similarly, each entry in $[\widehat{\tilde{\phi}}, \widehat{\tilde{\phi}}]_{-\tau}$ belongs to $L_{\infty}(\mathbb{T})$. We can see that each entry in $\widehat{\tilde{a}}$ belongs to $L_{\infty}(\mathbb{T})$ by applying the Cauchy-Schwarz inequality to the identity $\widehat{\tilde{a}}(\xi) = [\widehat{\tilde{\phi}}(2\cdot), \widehat{\phi}](\xi)$. The second identity in (2.4) can be proved in the same fashion and every entry in $\widehat{\tilde{b}}$ belongs to $L_{\infty}(\mathbb{T})$.

Since $\mathsf{AS}_0^{\tau}(\phi; \psi)$ and $\mathsf{AS}_0^{-\tau}(\tilde{\phi}; \tilde{\psi})$ are biorthogonal, we have $[\widehat{\tilde{\phi}}, \widehat{\phi}] = I_r$, $[\widehat{\tilde{\phi}}, \widehat{\psi}] = 0$ and $[\widehat{\tilde{\psi}}, \widehat{\phi}] = 0$, $[\widehat{\tilde{\psi}}, \widehat{\psi}] = I_r$. By a routine calculation, using (2.3) and (2.4), we deduce that

$$
\begin{aligned}
I_r = [\widehat{\tilde{\phi}}, \widehat{\phi}](2\xi) &= [\widehat{\tilde{a}}(\cdot/2)\widehat{\tilde{\phi}}(\cdot/2), \widehat{a}(\cdot/2)\widehat{\phi}(\cdot/2)](2\xi) \\
&= \widehat{\tilde{a}}(\xi)[\widehat{\tilde{\phi}}, \widehat{\phi}](\xi)\overline{\widehat{a}(\xi)}^{\mathsf{T}} + \widehat{\tilde{a}}(\xi+\pi)[\widehat{\tilde{\phi}}, \widehat{\phi}](\xi+\pi)\overline{\widehat{a}(\xi+\pi)}^{\mathsf{T}} \\
&= \widehat{\tilde{a}}(\xi)\overline{\widehat{a}(\xi)}^{\mathsf{T}} + \widehat{\tilde{a}}(\xi+\pi)\overline{\widehat{a}(\xi+\pi)}^{\mathsf{T}}.
\end{aligned}
$$

By a similar fashion, we conclude from $[\widehat{\tilde{\psi}}, \widehat{\phi}] = 0$ that $\widehat{\tilde{b}}(\xi)\overline{\widehat{a}(\xi)}^{\mathsf{T}} + \widehat{\tilde{b}}(\xi + \pi)\overline{\widehat{a}(\xi+\pi)}^{\mathsf{T}} = 0$; we conclude from $[\widehat{\tilde{\psi}}, \widehat{\psi}] = I_r$ that $\widehat{\tilde{b}}(\xi)\overline{\widehat{b}(\xi)}^{\mathsf{T}} + \widehat{\tilde{b}}(\xi+\pi)\overline{\widehat{b}(\xi+\pi)}^{\mathsf{T}} = I_r$. Thus, (2.5) holds. $\square$

Theorem 2.1. *Let $\phi = (\phi_1, \dots, \phi_r)^{\mathsf{T}}$ be a compactly supported refinable vector function in $H^m(\mathbb{R})$ with $m \in \mathbb{N}_0$ such that the integer shifts of ϕ are stable and $\widehat{\phi}(2\xi) = \widehat{a}(\xi)\widehat{\phi}(\xi)$ for some $a \in (l_0(\mathbb{Z}))^{r\times r}$. Suppose that $\widehat{a}(\xi)$ takes the form in (1.3) with $u_1, \dots, u_4$ in (1.4) and the relation in (1.5) is satisfied. Then there exists a unique compactly supported function $\eta \in L_2(\mathbb{R})$ such that $\phi_1^{(m)} = \nabla^m \eta$, where $\nabla\eta := \eta - \eta(\cdot - 1)$. Define a compactly supported vector function $\mathring{\phi} := (\eta, \phi_2^{(m)}, \dots, \phi_r^{(m)})^{\mathsf{T}}$ in $L_2(\mathbb{R})$. Then*

(1) $\mathring{\phi}$ satisfies the refinement equation $\widehat{\mathring{\phi}}(2\xi) = \widehat{\mathring{a}}(\xi)\widehat{\mathring{\phi}}(\xi)$ with a filter $\mathring{a} \in (l_0(\mathbb{Z}))^{r\times r}$ defined by

$$\widehat{\mathring{a}}(\xi) = 2^m (E_m(2\xi))^{-1}\widehat{a}(\xi)E_m(\xi) \quad \textit{with} \quad E_m(\xi) := \begin{bmatrix} (1-e^{-i\xi})^m & \\ & I_{r-1} \end{bmatrix}; \tag{2.9}$$

(2) The integer shifts of the compactly supported vector function $\mathring{\phi} \in (L_2(\mathbb{R}))^r$ are stable;

(3) Under the extra condition for $r > 1$ that (1.5) holds with m being replaced by $2m$, $H(\xi) := (E_m(\xi))^{-1}[\widehat{\mathring{\phi}}, \widehat{\mathring{\phi}}](\xi)\overline{E_m(\xi)}^{\mathsf{T}}$ and $G(\xi) := E_m(\xi)[\widehat{\mathring{\phi}}, \widehat{\mathring{\phi}}](\xi)\overline{E_m(\xi)}^{-\mathsf{T}}$ are $r\times r$ matrices of 2π-periodic trigonometric polynomials such that $\det(H(\xi)) \neq 0$ and $\det(G(\xi)) \neq 0$ for all $\xi \in \mathbb{R}$.

Proof. Using (1.5), we first show the existence of a unique compactly supported function $\eta \in L_2(\mathbb{R})$ such that $\phi_1^{(m)} = \nabla^m \eta$. Since $\widehat{\phi_1^{(m)}}(\xi) = (i\xi)^m \widehat{\phi}_1(\xi)$, we immediately have $\widehat{\phi_1^{(m)}}(\xi + 2\pi k) = \mathscr{O}(|\xi|^m)$ as $\xi \to 0$ for all $k \in \mathbb{Z}$ from (1.5). Since $\phi_1 \in H^m(\mathbb{R})$, $\phi_1^{(m)}$ is a compactly supported function in $L_2(\mathbb{R})$. By Lemma 2.1, there is a unique compactly supported function $\eta \in L_2(\mathbb{R})$ such that $\phi_1^{(m)} = \nabla^m \eta$, where η is given by $\eta = \sum_{k=0}^{\infty} \frac{(m-1+k)!}{(m-1)!k!} \phi_1^{(m)}(\cdot - k)$.

By the definition of the filter $\mathring{a}$ in (2.9), (1.3) gives us

$$\widehat{\mathring{a}}(\xi) = \begin{bmatrix} \widehat{u_1}(\xi) & \widehat{u_2}(\xi) \\ 2^m(1-e^{-i\xi})^m \widehat{u_3}(\xi) & 2^m \widehat{u_4}(\xi) \end{bmatrix}, \tag{2.10}$$

which means $\mathring{a}$ is a finitely supported filter in $(l_0(\mathbb{Z}))^{r\times r}$. Having $\phi_1^{(m)} = \nabla^m \eta$ means we have $(i\xi)^m \widehat{\phi}_1(\xi) = \widehat{\phi_1^{(m)}}(\xi) = (1-e^{-i\xi})^m \widehat{\eta}(\xi)$. That is,

$$\widehat{\eta}(\xi) = (1-e^{-i\xi})^{-m} \widehat{\phi_1^{(m)}}(\xi) = (1-e^{-i\xi})^{-m} (i\xi)^m \widehat{\phi}_1(\xi), \quad \widehat{\mathring{\phi}}(\xi) = (E_m(\xi))^{-1} \widehat{\phi^{(m)}}(\xi). \tag{2.11}$$

By the definition of $\mathring{\phi}$, we have $\widehat{\mathring{\phi}}(\xi) = (i\xi)^m E_m(\xi)^{-1} \widehat{\phi}(\xi)$ and consequently,

$$\widehat{\mathring{a}}(\xi)\widehat{\mathring{\phi}}(\xi) = 2^m (i\xi)^m (E_m(2\xi))^{-1} \widehat{a}(\xi)\widehat{\phi}(\xi) = (i2\xi)^m (E_m(2\xi))^{-1} \widehat{\phi}(2\xi) = \widehat{\mathring{\phi}}(2\xi).$$

This shows the refinable structure of $\mathring{\phi}$; i.e., $\widehat{\mathring{\phi}}(2\xi) = \widehat{\mathring{a}}(\xi)\widehat{\mathring{\phi}}(\xi)$. Item (1) holds.

We now prove item (2). If $m = 0$, then $\mathring{\phi} = \phi$ and item (2) immediately holds. Assume $m > 0$. Let $\varphi := (\phi_2, \dots, \phi_r)^{\mathsf{T}}$. Suppose that there exist $\xi_0 \in (-\pi, \pi]$, $c_1 \in \mathbb{C}$ and a row vector $c_2 \in \mathbb{C}^{1\times(r-1)}$ such that

$$(c_1, c_2)\widehat{\mathring{\phi}}(\xi_0 + 2\pi k) = c_1 \widehat{\eta}(\xi_0 + 2\pi k) + c_2 \widehat{\varphi^{(m)}}(\xi_0 + 2\pi k) = 0, \qquad \forall\, k \in \mathbb{Z}. \tag{2.12}$$

We want to show that both c_1 and c_2 have to be zero by analyzing two cases.

Case 1: $\xi_0 \in (-\pi, \pi]\backslash\{0\}$. Noting that $1 - e^{-i\xi_0} \neq 0$ and $\xi_0 \neq 0$, we have from (2.11) that $\widehat{\eta}(\xi_0 + 2\pi k) = i^m(\xi_0 + 2\pi k)^m(1 - e^{-i\xi_0})^{-m}\widehat{\phi_1}(\xi_0 + 2\pi k)$ for all $k \in \mathbb{Z}$. We have from (2.12) and $\widehat{\varphi^{(m)}}(\xi) = (i\xi)^m\widehat{\varphi}(\xi)$ that

$$0 = c_1\widehat{\eta}(\xi_0 + 2\pi k) + c_2\widehat{\varphi^{(m)}}(\xi_0 + 2\pi k) = i^m(\xi_0 + 2\pi k)^m(\tilde{c}_1, c_2)\widehat{\phi}(\xi_0 + 2\pi k)$$

with $\tilde{c}_1 := c_1(1 - e^{-i\xi_0})^{-m}$. Due to the stability of the integer shifts of ϕ and $\xi_0 + 2\pi k \neq 0$ for all $k \in \mathbb{Z}$, the above identity yields $\tilde{c}_1 = c_1(1 - e^{-i\xi_0})^{-m} = 0$ and $c_2 = 0$. So, $c_1 = 0$ by $1 - e^{-i\xi_0} \neq 0$.

Case 2: $\xi_0 = 0$. Then (2.12) implies that we have $c_1\widehat{\eta}(0) + c_2\widehat{\varphi^{(m)}}(0) = 0$. Since $\widehat{\varphi^{(m)}}(\xi) = (i\xi)^m\widehat{\varphi}(\xi)$ and $m > 0$, we have $\widehat{\varphi^{(m)}}(0) = 0$. Hence, $c_1\widehat{\eta}(0) = 0$. By (1.5) and (2.11), we must have $\widehat{\eta}(0) = \widehat{\phi_1}(0) = 1$, which means that $0 = c_1\widehat{\eta}(0) = c_1$. This proves $c_1 = 0$. Plugging $c_1 = 0$ back to (2.12), we have $c_2(i2\pi k)^m\widehat{\varphi}(2\pi k) = c_2\widehat{\varphi^{(m)}}(2\pi k) = 0$ for all $k \in \mathbb{Z}$. Hence, $c_2\widehat{\varphi}(2\pi k) = 0$ for all $k \in \mathbb{Z}\backslash\{0\}$. Let $\tilde{c}_1 := -c_2\widehat{\varphi}(0)$. Then $\tilde{c}_1\widehat{\phi_1}(0) + c_2\widehat{\varphi}(0) = \tilde{c}_1 + c_2\widehat{\varphi}(0) = 0$ by $\widehat{\phi_1}(0) = 1$. By (1.5) and $m > 0$, we conclude that $\tilde{c}_1\widehat{\phi_1}(2\pi k) + c_2\widehat{\varphi}(2\pi k) = 0$ for all $k \in \mathbb{Z}\backslash\{0\}$. That is, we proved

$$(\tilde{c}_1, c_2)\widehat{\phi}(2\pi k) = \tilde{c}_1\widehat{\phi_1}(2\pi k) + c_2\widehat{\varphi}(2\pi k) = 0, \qquad \forall\, k \in \mathbb{Z},$$

from which $c_2 = 0$ by the stability of the integer shifts of ϕ. Item 2 holds.

We move on to item (3). Since all the entries in $\mathring{\phi}$ are compactly supported functions in $L_2(\mathbb{R})$, we conclude that $[\widehat{\mathring{\phi}}, \widehat{\mathring{\phi}}](\xi) = \sum_{k\in\mathbb{Z}}\langle\mathring{\phi}, \mathring{\phi}(\cdot - k)\rangle e^{-ik\xi}$ must be an $r \times r$ matrix of 2π-periodic trigonometric polynomials. Item 2 yields

$\det([\widehat{\mathring{\phi}}, \widehat{\mathring{\phi}}](\xi)) \ne 0$ for all $\xi \in \mathbb{R}$. If $r = 1$, then $H(\xi) = (-1)^m e^{im\xi}[\widehat{\mathring{\phi}}, \widehat{\mathring{\phi}}](\xi)$ and $G(\xi) = (-1)^m e^{-im\xi}[\widehat{\mathring{\phi}}, \widehat{\mathring{\phi}}](\xi)$. Item (3) easily holds for $r = 1$.

For the case $r > 1$, we prove item (3) with the extra condition that (1.5) holds with m being replaced by $2m$. By the definition of the matrix H in item (3), we have $\det(H(\xi)) = (-1)^m e^{im\xi} \det([\widehat{\mathring{\phi}}, \widehat{\mathring{\phi}}](\xi)) \ne 0$ for all $\xi \in \mathbb{R}$. Since $\mathring{\phi} = (\eta, (\varphi^{(m)})^\mathsf{T})^\mathsf{T}$ with $\varphi = (\phi_2, \dots, \phi_r)^\mathsf{T}$, we have

$$
\begin{aligned}
&H(\xi) = (E_m(\xi))^{-1}[\widehat{\mathring{\phi}}, \widehat{\mathring{\phi}}](\xi)\overline{E_m(\xi)}^\mathsf{T} \\
&= \begin{bmatrix} (1-e^{-i\xi})^{-m} & \\ & I_{r-1} \end{bmatrix} \begin{bmatrix} [\widehat{\eta}, \widehat{\eta}](\xi) & [\widehat{\eta}, \widehat{\varphi^{(m)}}](\xi) \\ [\widehat{\varphi^{(m)}}, \widehat{\eta}](\xi) & [\widehat{\varphi^{(m)}}, \widehat{\varphi^{(m)}}](\xi) \end{bmatrix} \begin{bmatrix} (1-e^{i\xi})^{m} & \\ & I_{r-1} \end{bmatrix} \\
&= \begin{bmatrix} (-1)^m e^{im\xi}[\widehat{\eta}, \widehat{\eta}](\xi) & (1-e^{-i\xi})^{-m}[\widehat{\eta}, \widehat{\varphi^{(m)}}](\xi) \\ (1-e^{i\xi})^{m}[\widehat{\varphi^{(m)}}, \widehat{\eta}](\xi) & [\widehat{\varphi^{(m)}}, \widehat{\varphi^{(m)}}](\xi) \end{bmatrix}.
\end{aligned}
$$

Since all η and $\varphi^{(m)}$ are compactly supported functions in $L_2(\mathbb{R})$, it follows directly that $[\widehat{\eta}, \widehat{\eta}]$, $[\widehat{\eta}, \widehat{\varphi^{(m)}}]$, $[\widehat{\varphi^{(m)}}, \widehat{\eta}]$ and $[\widehat{\varphi^{(m)}}, \widehat{\varphi^{(m)}}]$ are all 2π-periodic trigonometric polynomials. To prove that $H(\xi)$ is a matrix of 2π-periodic trigonometric polynomials, we still need to show that all entries in $(1-e^{-i\xi})^{-m}[\widehat{\eta}, \widehat{\varphi^{(m)}}](\xi)$ are 2π-periodic trigonometric polynomials. By $\widehat{\varphi^{(m)}}(\xi) = (i\xi)^m \widehat{\varphi}(\xi)$ and the definition of the bracket product,

$$
\begin{aligned}
[\widehat{\eta}, \widehat{\varphi^{(m)}}](\xi) &= \sum_{k\in\mathbb{Z}} \widehat{\eta}(\xi + 2\pi k)\overline{(i(\xi+2\pi k))^m \widehat{\varphi}(\xi+2\pi k)}^\mathsf{T} \\
&= (-1)^m \sum_{k\in\mathbb{Z}} \widehat{D^m\eta}(\xi)\overline{\widehat{\varphi}(\xi+2\pi k)}^\mathsf{T} = (-1)^m[\widehat{D^m\eta}, \widehat{\varphi}](\xi),
\end{aligned}
$$

where $D^m\eta$ denotes the mth-order distributional derivative of η. Since (1.5) holds with m being replaced by $2m$, we have that $\widehat{\eta}(\xi+2\pi k) = \mathscr{O}(|\xi|^m)$ as $\xi \to 0$ for all $k \in \mathbb{Z}\backslash\{0\}$ by (2.11) and calculating the Taylor expansion. Since $\widehat{D^m\eta}(\xi) = (i\xi)^m\widehat{\eta}(\xi)$, we have $\widehat{D^m\eta}(\xi) = \mathscr{O}(|\xi|^m)$ as $\xi \to 0$. Consequently, we proved $\widehat{D^m\eta}(\xi+2\pi k) = \mathscr{O}(|\xi|^m)$ as $\xi \to 0$ for all $k \in \mathbb{Z}$. Noting that $D^m\eta \in H^{-m}(\mathbb{R})$ by $\eta \in L_2(\mathbb{R})$ and Lemma 2.1, we conclude that there is a unique compactly supported function $\mathring{\eta} \in H^{-m}(\mathbb{R})$ such that

$$D^m\eta = \nabla^m\mathring{\eta}, \qquad \text{i.e.,} \quad \widehat{D^m\eta}(\xi) = (1-e^{-i\xi})^m\widehat{\mathring{\eta}}(\xi). \tag{2.13}$$

Consequently,

$$\begin{aligned}
(1-e^{-i\xi})^{-m}[\widehat{\eta}, \widehat{\varphi^{(m)}}](\xi) &= (1-e^{-i\xi})^{-m}(-1)^m[\widehat{D^m\eta}, \widehat{\varphi}](\xi) \\
&= (1-e^{-i\xi})^{-m}(-1)^m(1-e^{-i\xi})^m[\widehat{\mathring{\eta}}, \widehat{\varphi}](\xi) \\
&= (-1)^m[\widehat{\mathring{\eta}}, \widehat{\varphi}](\xi) \\
&= (-1)^m\sum_{k\in\mathbb{Z}}\langle\mathring{\eta}, \varphi(\cdot - k)\rangle e^{-ik\xi}.
\end{aligned}$$

Since both $\mathring{\eta} \in H^{-m}(\mathbb{R})$ and $\varphi \in (H^m(\mathbb{R}))^{r-1}$ have compact support, the above identity proves that $[\widehat{\mathring{\eta}}, \widehat{\varphi}](\xi)$ is indeed a well-defined vector of 2π-periodic trigonometric polynomials. So, $(1-e^{-i\xi})^{-m}[\widehat{\eta}, \widehat{\varphi^{(m)}}](\xi)$ must be a vector of 2π-periodic trigonometric polynomials. Consequently, $H(\xi)$ is a matrix of 2π-periodic trigonometric polynomials. The claim in item (3) for the matrix $G(\xi)$ can be proved similarly. $\square$

Theorem 2.2. *Let $\phi = (\phi_1, \ldots, \phi_r)^\mathsf{T}$ and $\psi = (\psi_1, \ldots, \psi_r)^\mathsf{T}$ be compactly supported vector functions in the Sobolev space $H^m(\mathbb{R})$ with $m \in \mathbb{N}_0$ such that the refinable structure $\widehat{\phi}(2\xi) = \widehat{a}(\xi)\widehat{\phi}(\xi)$ holds a.e. $\xi \in \mathbb{R}$ for some filters*

$a, b \in (l_0(\mathbb{Z}))^{r\times r}$. *If* $\{\phi;\psi\}$ *is a Riesz wavelet in the Sobolev space* $H^m(\mathbb{R})$ *satisfying the mth-order derivative orthogonality conditions in* (2.1) *and* (2.2) *(i.e.,* $\{\phi;\psi\}$ *is an mth-order derivative-orthogonal Riesz wavelet in* $H^m(\mathbb{R})$*), then*

(i) the integer shifts of ϕ *are stable;*

(ii) the high-pass filter b *satisfies*

$$\widehat{b}(\xi)[\widehat{\phi^{(m)}}, \widehat{\phi^{(m)}}](\xi)\overline{\widehat{a}(\xi)}^{\mathsf{T}} + \widehat{b}(\xi+\pi)[\widehat{\phi^{(m)}}, \widehat{\phi^{(m)}}](\xi+\pi)\overline{\widehat{a}(\xi+\pi)}^{\mathsf{T}} = 0 \quad (2.14)$$

and

$$\det(\{\widehat{a};\widehat{b}\})(\xi) := \det\left(\begin{bmatrix}\widehat{a}(\xi) & \widehat{a}(\xi+\pi)\\ \widehat{b}(\xi) & \widehat{b}(\xi+\pi)\end{bmatrix}\right) \neq 0, \qquad \forall\, \xi \in \mathbb{R}. \quad (2.15)$$

Conversely, if items (i) and (ii) are satisfied, then the refinement filter a must have at least order $2m$ *sum rules (i.e.,* $\mathrm{sr}(a) \geqslant 2m$*) and* $\{\phi;\psi\}$ *is an mth-order derivative-orthogonal Riesz wavelet in the Sobolev space* $H^m(\mathbb{R})$ *satisfying both* (2.1) *and* (2.2).

Proof. Necessity ($\Rightarrow$). Since $\mathsf{AS}_0^m(\phi;\psi)$ is a Riesz basis for $H^m(\mathbb{R})$, the sequence $\{\phi_\ell(\cdot - k) \;:\; k \in \mathbb{Z}, \ell = 1, \dots, r\}$ has to be a Riesz sequence in $H^m(\mathbb{R})$. Consequently, the integer shifts of the compactly supported vector function ϕ in $H^m(\mathbb{R})$ must be stable. Item (i) holds.

Proposition 2.1 guarantees the existence of $r \times r$ matrices $\widehat{\tilde{a}}$ and $\widehat{\tilde{b}}$ with all entries from $L_\infty(\mathbb{T})$ such that (2.5) holds. From (2.5), we obtain
$\det(\{\widehat{\tilde{a}};\widehat{\tilde{b}}\})(\xi)\overline{\det(\{\widehat{a};\widehat{b}\})(\xi)} = 1$ for almost every $\xi \in \mathbb{R}$. Since all the entries

of the matrices $\widehat{\tilde{a}}$ and $\widehat{\tilde{b}}$ belong to $L_\infty(\mathbb{T})$, noting that $\det(\{\widehat{a};\widehat{b}\})(\xi)$ is a 2π-periodic trigonometric polynomial, we must have (2.15).

The refinable structure in (2.3) yields

$$\widehat{\phi^{(m)}}(2\xi) = 2^m \widehat{a}(\xi)\widehat{\phi^{(m)}}(\xi) \quad \text{and} \quad \widehat{\psi^{(m)}}(2\xi) = 2^m \widehat{b}(\xi)\widehat{\phi^{(m)}}(\xi). \tag{2.16}$$

Hence, (2.16) gives us

$$\begin{aligned}
&[\widehat{\psi^{(m)}}, \widehat{\phi^{(m)}}](2\xi) \\
&\quad = 2^{2m}[\widehat{b}(\cdot/2)\widehat{\phi^{(m)}}(\cdot/2), \widehat{a}(\cdot/2)\widehat{\phi^{(m)}}(\cdot/2)](2\xi) \\
&\quad = 2^{2m}\widehat{b}(\xi)[\widehat{\phi^{(m)}}, \widehat{\phi^{(m)}}](\xi)\overline{\widehat{a}(\xi)}^{\mathsf{T}} + 2^{2m}\widehat{b}(\xi+\pi)[\widehat{\phi^{(m)}}, \widehat{\phi^{(m)}}](\xi+\pi)\overline{\widehat{a}(\xi+\pi)}^{\mathsf{T}}.
\end{aligned} \tag{2.17}$$

Recall that $[\widehat{\psi^{(m)}}, \widehat{\phi^{(m)}}](\xi) = \sum_{k\in\mathbb{Z}}\langle \psi^{(m)}, \phi^{(m)}(\cdot - k)\rangle e^{-ik\xi}$, which means that (2.1) is equivalent to $[\widehat{\psi^{(m)}}, \widehat{\phi^{(m)}}] = 0$. Thus, (2.14) follows directly from (2.1) and (2.17).

Sufficiency ($\Leftarrow$). By (2.14) and (2.17), we have $[\widehat{\psi^{(m)}}, \widehat{\phi^{(m)}}] = 0$. Hence, (2.1) holds. By the refinable structure in (2.3), we see that each entry in $\psi(2^j\cdot - k)$ with $j < 0$ and $k \in \mathbb{Z}$ is a finite linear combination of $\phi_\ell(\cdot - k), k \in \mathbb{Z}$ and $\ell = 1, \ldots, r$. Thus, (2.2) directly comes from (2.1).

We shall show that the filter a must have at least order $2m$ sum rules, i.e., $\operatorname{sr}(a) \geqslant 2m$. There is nothing to prove for $m = 0$, since any filter a satisfies $\operatorname{sr}(a) \geqslant 0$. Henceforth, let us assume $m > 0$.

By (2.15), we can define $r \times r$ matrices $\widehat{\tilde{a}}$ and $\widehat{\tilde{b}}$ of 2π-periodic continuous functions through

$$\begin{bmatrix} \widehat{\tilde{a}}(\xi) & \widehat{\tilde{a}}(\xi+\pi) \\ \widehat{\tilde{b}}(\xi) & \widehat{\tilde{b}}(\xi+\pi) \end{bmatrix} := \begin{bmatrix} \overline{\widehat{a}(\xi)}^{\mathsf{T}} & \overline{\widehat{b}(\xi)}^{\mathsf{T}} \\ \overline{\widehat{a}(\xi+\pi)}^{\mathsf{T}} & \overline{\widehat{b}(\xi+\pi)}^{\mathsf{T}} \end{bmatrix}^{-1}, \qquad \xi \in \mathbb{R}. \tag{2.18}$$

Then (2.5) trivially holds and all the entries in $\widehat{\tilde{a}}$ and $\widehat{\tilde{b}}$ are continuous functions taking the form

$$\frac{p(\xi)}{q(\xi)} \text{ with } 2\pi\text{-periodic trigonometric polynomials } p \text{ and } q,\ q(\xi) \neq 0\ \forall\, \xi \in \mathbb{R}. \tag{2.19}$$

The first identity in (2.16) gives

$$\widehat{a}(\xi)[\widehat{\phi^{(m)}}, \widehat{\phi^{(m)}}](\xi)\overline{\widehat{a}(\xi)}^{\mathsf{T}} + \widehat{a}(\xi+\pi)[\widehat{\phi^{(m)}}, \widehat{\phi^{(m)}}](\xi+\pi)\overline{\widehat{a}(\xi+\pi)}^{\mathsf{T}} \\ = 2^{-2m}[\widehat{\phi^{(m)}}, \widehat{\phi^{(m)}}](2\xi). \tag{2.20}$$

Combining (2.20) and (2.14) yields

$$\begin{bmatrix} \widehat{a}(\xi) & \widehat{a}(\xi+\pi) \\ \widehat{b}(\xi) & \widehat{b}(\xi+2\pi) \end{bmatrix} \begin{bmatrix} [\widehat{\phi^{(m)}}, \widehat{\phi^{(m)}}](\xi) & 0 \\ 0 & [\widehat{\phi^{(m)}}, \widehat{\phi^{(m)}}](\xi+\pi) \end{bmatrix} \begin{bmatrix} \overline{\widehat{a}(\xi)}^{\mathsf{T}} \\ \overline{\widehat{a}(\xi+\pi)}^{\mathsf{T}} \end{bmatrix} \\ = \begin{bmatrix} 2^{-2m}[\widehat{\phi^{(m)}}, \widehat{\phi^{(m)}}](2\xi) \\ 0 \end{bmatrix}.$$

Applying (2.18), the above identity becomes

$$
\begin{bmatrix} [\widehat{\phi^{(m)}},\widehat{\phi^{(m)}}](\xi) & 0 \\ 0 & [\widehat{\phi^{(m)}},\widehat{\phi^{(m)}}](\xi+\pi) \end{bmatrix} \begin{bmatrix} \overline{\widehat{a}(\xi)}^{\mathsf{T}} \\ \overline{\widehat{a}(\xi+\pi)}^{\mathsf{T}} \end{bmatrix}
= \begin{bmatrix} \overline{\widehat{\mathring{a}}(\xi)}^{\mathsf{T}} & \overline{\widehat{\mathring{b}}(\xi)}^{\mathsf{T}} \\ \overline{\widehat{\mathring{a}}(\xi+\pi)}^{\mathsf{T}} & \overline{\widehat{\mathring{b}}(\xi+\pi)}^{\mathsf{T}} \end{bmatrix} \begin{bmatrix} 2^{-2m}[\widehat{\phi^{(m)}},\widehat{\phi^{(m)}}](2\xi) \\ 0 \end{bmatrix},
$$

from which we obtain

$$
\widehat{a}(\xi)[\widehat{\phi^{(m)}},\widehat{\phi^{(m)}}](\xi) = 2^{-2m}[\widehat{\phi^{(m)}},\widehat{\phi^{(m)}}](2\xi)\widehat{\mathring{a}}(\xi), \tag{2.21}
$$

where we recall that $[\widehat{\phi^{(m)}},\widehat{\phi^{(m)}}](\xi) = \overline{[\widehat{\phi^{(m)}},\widehat{\phi^{(m)}}](\xi)}^{\mathsf{T}}$. Since $\phi \in (H^m(\mathbb{R}))^r$ and the integer shifts of ϕ are stable, the filter $\widehat{a}$ must have at least order m sum rules (see [31, Theorem 4.3]); that is, $\widehat{v}(2\xi)\widehat{a}(\xi) = \widehat{v}(\xi) + \mathscr{O}(|\xi|^m)$ and $\widehat{v}(2\xi)\widehat{a}(\xi+\pi) = \mathscr{O}(|\xi|^m)$ as $\xi \to 0$ with a matching filter $v \in (l_0(\mathbb{R}))^{1\times r}$ and $\widehat{v}(0) \neq 0$. Since the integer shifts of $\phi \in (L_2(\mathbb{R}))^r$ are stable, 1 must be a simple eigenvalue of $\widehat{a}(0)$ by [31, Proposition 3.1]. Consequently, noting that $\widehat{a}(0)\widehat{\phi}(0) = \widehat{\phi}(0)$ and $\widehat{v}(0)\widehat{a}(0) = \widehat{v}(0)$, we must have $\widehat{v}(0)\widehat{\phi}(0) \neq 0$. Without loss of generality, we can assume $\widehat{v}(0)\widehat{\phi}(0) = 1$, which can be achieved by multiplying ϕ with a nonzero constant. By [31, Proposition 2.4] or [33, Theorems 5.6.4 and 5.6.5], without loss of generality, we can assume that $\widehat{a}$ takes the normal form in (1.3) and the relation in (1.5) is satisfied. Define $\mathring{\phi} := (\eta, \phi_2^{(m)}, \dots, \phi_r^{(m)})^{\mathsf{T}}$ as in Theorem 2.1 and the filter $\mathring{a} \in (l_0(\mathbb{Z}))^{r\times r}$ as in (2.9). Then $\mathring{a}$ must take the form in (2.10) and items (1) and (2) in Theorem 2.1 hold. Noting that $\widehat{\phi^{(m)}}(\xi) = E_m(\xi)\widehat{\mathring{\phi}}(\xi)$ by (2.11), we have $[\widehat{\phi^{(m)}},\widehat{\phi^{(m)}}](\xi) = E_m(\xi)[\widehat{\mathring{\phi}},\widehat{\mathring{\phi}}](\xi)\overline{E_m(\xi)}^{\mathsf{T}}$. By (2.9), we have $\widehat{a}(\xi) = 2^{-m}E_m(2\xi)\widehat{\mathring{a}}(\xi)(E_m(\xi))^{-1}$.

Now the identity (2.21) becomes

$$2^{-m}E_m(2\xi)\widehat{\mathring{a}}(\xi)[\widehat{\mathring{\phi}},\widehat{\mathring{\phi}}](\xi)\overline{E_m(\xi)}^{\mathsf{T}} = 2^{-2m}E_m(2\xi)[\widehat{\mathring{\phi}},\widehat{\mathring{\phi}}](2\xi)\overline{E_m(2\xi)}^{\mathsf{T}}\widehat{\tilde{a}}(\xi).$$

That is, we have

$$\widehat{\mathring{a}}(\xi) = [\widehat{\mathring{\phi}},\widehat{\mathring{\phi}}](2\xi)2^{-m}\overline{E_m(2\xi)}^{\mathsf{T}}\widehat{\tilde{a}}(\xi)\overline{E_m(\xi)}^{-\mathsf{T}}([\widehat{\mathring{\phi}},\widehat{\mathring{\phi}}](\xi))^{-1}. \tag{2.22}$$

Note that

$$2^{-m}\overline{E_m(2\xi)}^{\mathsf{T}}\widehat{\tilde{a}}(\xi)\overline{E_m(\xi)}^{-\mathsf{T}} = \begin{bmatrix} 2^{-m}(1+e^{i\xi})^m\widehat{\tilde{a}_{1,1}}(\xi) & 2^{-m}(1-e^{i2\xi})^m\widehat{\tilde{a}_{1,2}}(\xi) \\ (1-e^{i\xi})^{-m}\widehat{\tilde{a}_{2,1}}(\xi) & \widehat{\tilde{a}_{2,2}}(\xi) \end{bmatrix}.$$

Since $\mathring{a} \in (l_0(\mathbb{Z}))^{r\times r}$ and $\det([\widehat{\mathring{\phi}},\widehat{\mathring{\phi}}](\xi)) \neq 0$ for all $\xi \in \mathbb{R}$, we conclude from (2.22) that all the entries in $(1-e^{i\xi})^{-m}\widehat{\tilde{a}_{2,1}}(\xi)$ are continuous and take the form in (2.19).

Since we assumed $m > 0$, we deduce from (1.3) that the first row of $\widehat{a}(\pi)$ is zero and the first row of $\widehat{a}(0)$ is $(1, 0, \ldots, 0)$. Since (2.18) implies that (2.5) holds for all $\xi \in \mathbb{R}$, taking $\xi = 0$ in (2.5) and observing that the first column of the second matrix in (2.5) is $(1, 0, \ldots, 0)^{\mathsf{T}}$, we conclude that $\widehat{\tilde{a}_{1,1}}(0) = 1$, where $\widehat{\tilde{a}_{1,1}}$ is the (1,1)-entry of $\widehat{\tilde{a}}$. Define $\widehat{u}(\xi) := 2^{-m}(1+e^{i\xi})^m\widehat{\tilde{a}_{1,1}}(\xi)$. Then $\widehat{u}(0) = 1$ and $\widehat{u}$ takes the form in (2.19). Consequently, $\widehat{h}(\xi) := \prod_{j=1}^{\infty}\widehat{u}(2^{-j}\xi)$ is a well-defined continuous function satisfying $\widehat{h}(2\xi) = \widehat{u}(\xi)\widehat{h}(\xi)$ and $\widehat{h}(0) = 1$.

Let us define two matching filters $\mathring{v}, v \in (l_0(\mathbb{Z}))^{1\times r}$ by

$$\begin{aligned}
\widehat{\mathring{v}}(\xi) &:= (\widehat{h}(\xi))^{-1}(1,0,\ldots,0)([\widehat{\mathring{\phi}},\widehat{\mathring{\phi}}](\xi))^{-1} + \mathscr{O}(|\xi|^m), \qquad \xi \to 0,\\
\widehat{v}(\xi) &:= \widehat{\mathring{v}}(\xi)\operatorname{diag}((\widehat{B_m}(\xi))^{-1}, (i\xi)^m, \ldots, (i\xi)^m) + \mathscr{O}(|\xi|^{2m}), \qquad \xi \to 0.
\end{aligned} \tag{2.23}$$

Since $\widehat{h}(0) = 1$ and $[\widehat{\mathring{\phi}},\widehat{\mathring{\phi}}](0)$ is invertible, we must have $\widehat{\mathring{v}}(0) = (1,0,\ldots,0)([\widehat{\mathring{\phi}},\widehat{\mathring{\phi}}](0))^{-1} \neq 0$. Now by $(\widehat{h}(2\xi))^{-1}\widehat{u}(\xi) = (\widehat{h}(\xi))^{-1}$ for ξ near 0 and (2.22), as $\xi \to 0$, we have

$$\begin{aligned}
&\widehat{\mathring{v}}(2\xi)\widehat{\mathring{a}}(\xi)\\
&\quad=(\widehat{h}(2\xi))^{-1}(1,0,\ldots,0)\begin{bmatrix} \widehat{u}(\xi) & 2^{-m}(1-e^{i2\xi})^m\widehat{\tilde{a}_{1,2}}(\xi)\\ (1-e^{i\xi})^{-m}\widehat{\tilde{a}_{2,1}}(\xi) & \widehat{\tilde{a}_{2,2}}(\xi)\end{bmatrix}\\
&\qquad([\widehat{\mathring{\phi}},\widehat{\mathring{\phi}}](\xi))^{-1} + \mathscr{O}(|\xi|^m)\\
&\quad=((\widehat{h}(2\xi))^{-1}\widehat{u}(\xi),0,\ldots,0)([\widehat{\mathring{\phi}},\widehat{\mathring{\phi}}](\xi))^{-1} + \mathscr{O}(|\xi|^m)\\
&\quad=(\widehat{h}(\xi))^{-1}(1,0,\ldots,0)([\widehat{\mathring{\phi}},\widehat{\mathring{\phi}}](\xi))^{-1} + \mathscr{O}(|\xi|^m)\\
&\quad=\widehat{\mathring{v}}(\xi) + \mathscr{O}(|\xi|^m), \qquad \xi \to 0.
\end{aligned}$$

Noting that $\widehat{u}(\xi+\pi) = \mathscr{O}(|\xi|^m)$ by $\widehat{u}(\xi) = 2^{-m}(1+e^{i\xi})^m\widehat{\tilde{a}_{1,1}}(\xi)$ and $(1-e^{i\xi})^m = \mathscr{O}(|\xi|^m)$ as $\xi \to 0$, we deduce that

$$\begin{aligned}
&\widehat{\mathring{v}}(2\xi)\widehat{\mathring{a}}(\xi+\pi)\\
&=(\widehat{h}(2\xi))^{-1}(1,0,\ldots,0)\begin{bmatrix} \mathscr{O}(|\xi|^m) & \mathscr{O}(|\xi|^m)\\ (1+e^{i\xi})^{-m}\widehat{\tilde{a}_{2,1}}(\xi+\pi) & \widehat{\tilde{a}_{2,2}}(\xi+\pi)\end{bmatrix}([\widehat{\mathring{\phi}},\widehat{\mathring{\phi}}](\xi+\pi))^{-1}\\
&\quad+\mathscr{O}(|\xi|^m) = (\mathscr{O}(|\xi|^m),\ldots,\mathscr{O}(|\xi|^m)), \qquad \xi \to 0.
\end{aligned}$$

That is, $\mathring{a}$ has order m sum rules with the matching filter $\mathring{\upsilon} \in (l_0(\mathbb{Z}))^{1\times r}$.

On the other hand, noting that $\widehat{B_m}(\xi) = (1-e^{-i\xi})^m/(i\xi)^m$, we have $\widehat{\upsilon}(\xi) = (i\xi)^m\widehat{\mathring{\upsilon}}(\xi)(E_m(\xi))^{-1} + \mathscr{O}(|\xi|^{2m})$ as $\xi \to 0$. Consequently, by $\widehat{a}(\xi) = 2^{-m}E_m(2\xi)\widehat{\mathring{a}}(\xi)(E_m(\xi))^{-1}$, as $\xi \to 0$, we have

$$\begin{aligned}
\widehat{\upsilon}(2\xi)\widehat{a}(\xi) &= (i2\xi)^m\widehat{\mathring{\upsilon}}(2\xi)(E_m(2\xi))^{-1}2^{-m}E_m(2\xi)\widehat{\mathring{a}}(\xi)(E_m(\xi))^{-1} + \mathscr{O}(|\xi|^{2m})\\
&= (i\xi)^m\widehat{\mathring{\upsilon}}(2\xi)\widehat{\mathring{a}}(\xi)(E_m(\xi))^{-1} + \mathscr{O}(|\xi|^{2m})\\
&= (i\xi)^m(\widehat{\mathring{\upsilon}}(\xi) + \mathscr{O}(|\xi|^m))(E_m(\xi))^{-1} + \mathscr{O}(|\xi|^{2m})\\
&= (i\xi)^m\widehat{\mathring{\upsilon}}(\xi)(E_m(\xi))^{-1} + \mathscr{O}(|\xi|^{2m}) = \widehat{\upsilon}(\xi) + \mathscr{O}(|\xi|^{2m}), \quad \xi \to 0;
\end{aligned}$$

in addition,

$$\begin{aligned}
\widehat{\upsilon}(2\xi)\widehat{a}(\xi+\pi) =&(i2\xi)^m\widehat{\mathring{\upsilon}}(2\xi)(E_m(2\xi))^{-1}2^{-m}E_m(2\xi)\widehat{\mathring{a}}(\xi+\pi)(E_m(\xi+\pi))^{-1}\\
&+ \mathscr{O}(|\xi|^{2m})\\
=&(i\xi)^m\widehat{\mathring{\upsilon}}(2\xi)\widehat{\mathring{a}}(\xi+\pi)(E_m(\xi+\pi))^{-1} + \mathscr{O}(|\xi|^{2m})\\
=&(i\xi)^m\mathscr{O}(|\xi|^m)(E_m(\xi+\pi))^{-1} + \mathscr{O}(|\xi|^{2m}) = \mathscr{O}(|\xi|^{2m}), \quad \xi \to 0.
\end{aligned}$$

This proves that $\widehat{\upsilon}(2\xi)\widehat{a}(\xi) = \widehat{\upsilon}(\xi) + \mathscr{O}(|\xi|^{2m})$ and $\widehat{\upsilon}(2\xi)\widehat{a}(\xi+\pi) = \mathscr{O}(|\xi|^{2m})$ as $\xi \to 0$ with the matching filter υ. To prove that the filter a has order $2m$ sum rules, we still need to show $\widehat{\upsilon}(0) \neq 0$. Since $\mathring{\phi} \in (L_2(\mathbb{R}))^r$ has stable integer shifts and $\widehat{\mathring{\phi}}(2\xi) = \widehat{\mathring{a}}(\xi)\widehat{\mathring{\phi}}(\xi)$, by [31, Proposition 3.1], we see that one must be a simple eigenvalue of $\widehat{\mathring{a}}(0)$. Since $\widehat{\mathring{\upsilon}}(0)\widehat{\mathring{a}}(0) = \widehat{\mathring{\upsilon}}(0)$ and $\widehat{\mathring{a}}(0)\widehat{\mathring{\phi}}(0) = \widehat{\mathring{\phi}}(0)$, by $\widehat{\mathring{\upsilon}}(0) \neq 0$ and $\widehat{\mathring{\phi}}(0) = (1,0,\dots,0)^{\mathsf{T}}$, we must have $\widehat{\mathring{\upsilon}}(0)\widehat{\mathring{\phi}}(0) \neq 0$. That is, we proved $\widehat{\mathring{\upsilon}}(0)(1,0,\dots,0)^{\mathsf{T}} \neq 0$. By the definition of $\widehat{\upsilon}$ in (2.23) and $m > 0$, we conclude that

$$\widehat{v}(0) = \widehat{\mathring{v}}(0)\mathrm{diag}(1, 0, \ldots, 0) = (\widehat{\mathring{v}}(0)(1, 0, \ldots, 0)^{\mathsf{T}}, 0, \ldots, 0) \neq 0.$$

This completes the proof of $\mathrm{sr}(a) \geqslant 2m$.

By $\widehat{\phi_1}(0) = 1$ in (1.5) and the above identity, we have $\widehat{v}(0)\widehat{\phi}(0) = \widehat{\mathring{v}}(0)\widehat{\mathring{\phi}}(0)$. Without loss of generality, we can assume $\widehat{v}(0)\widehat{\phi}(0) = 1$. That is, we can assume that $\widehat{a}$ takes the following form:

$$\begin{bmatrix} 2^{-2m}(1+e^{-i\xi})^{2m}\widehat{v_1}(\xi) & 2^{-2m}(1-e^{-i2\xi})^{2m}\widehat{v_2}(\xi) \\ \widehat{v_3}(\xi) & \widehat{v_4}(\xi) \end{bmatrix} \quad \text{with} \quad \widehat{v_1}(0) = 1 \tag{2.24}$$

and (1.5) holds with m being replaced by $2m$, where $v_1 \in l_0(\mathbb{Z})$, $v_2 \in (l_0(\mathbb{Z}))^{1\times(r-1)}$, $v_3 \in (l_0(\mathbb{Z}))^{(r-1)\times 1}$, and $v_4 \in (l_0(\mathbb{Z}))^{(r-1)\times(r-1)}$. By Theorem 2.1, item (3) of Theorem 2.1 must hold for $H(\xi) := (E_m(\xi))^{-1}[\widehat{\mathring{\phi}}, \widehat{\mathring{\phi}}](\xi)\overline{E_m(\xi)}^{\mathsf{T}}$ and $G(\xi) := (E_m(\xi))[\widehat{\mathring{\phi}}, \widehat{\mathring{\phi}}](\xi)\overline{E_m(\xi)}^{-\mathsf{T}}$. We need use this fact in the next part of the proof.

We are ready to prove that $\mathsf{AS}_0^m(\phi; \psi)$ is a Riesz basis for the Sobolev space $H^m(\mathbb{R})$. Recall that $\varphi = (\phi_2, \ldots, \phi_r)^{\mathsf{T}}$. Define $\breve{\phi} := (\mathring{\eta}, D^{2m}\varphi^{\mathsf{T}})^{\mathsf{T}}$, where $\mathring{\eta}$ is given in (2.13) and D^{2m} is the $(2m)$th distributional derivative. Note that $D^{2m}\phi_1 = D^m\phi_1^{(m)} = \nabla^m D^m \eta = \nabla^{2m}\mathring{\eta}$. Hence, $\widehat{\breve{\phi}}(\xi) = (E_{2m}(\xi))^{-1}\widehat{D^{2m}\phi}(\xi) = (E_{2m}(\xi))^{-1}(i\xi)^{2m}\widehat{\phi}(\xi)$. Define

$$\widehat{\tilde{\phi}}(\xi) := (-1)^m (H(\xi))^{-1}\widehat{\breve{\phi}}(\xi) = (-1)^m (H(\xi))^{-1}(E_{2m}(\xi))^{-1}\widehat{D^{2m}\phi}(\xi). \tag{2.25}$$

Since $\det(H(\xi)) \neq 0$ for all $\xi \in \mathbb{R}$ and $\breve{\phi}$ is a compactly supported vector distribution in $H^{-m}(\mathbb{R})$, we conclude that $\tilde{\phi}$ in (2.25) is a well-defined vector distribution in $H^{-m}(\mathbb{R})$. From (2.22) and the fact that $\widehat{\mathring{a}}(\xi) = 2^m (E_m(2\xi))^{-1}\widehat{a}(\xi)E_m(\xi)$, we have

$$\begin{aligned}\widehat{\tilde{a}}(\xi) =& 2^m \overline{E_m(2\xi)}^{-\mathsf{T}}([\widehat{\mathring{\phi}}, \widehat{\mathring{\phi}}](2\xi))^{-1}\widehat{\mathring{a}}(\xi)[\widehat{\mathring{\phi}}, \widehat{\mathring{\phi}}](\xi)\overline{E_m(\xi)} \\ =& 2^{2m}(H(2\xi))^{-1}(E_{2m}(2\xi))^{-1}\widehat{a}(\xi)E_{2m}(\xi)H(\xi)\end{aligned} \tag{2.26}$$

and by $\widehat{\phi}(2\xi) = \widehat{a}(\xi)\widehat{\phi}(\xi)$ and (2.25),

$$\begin{aligned}\widehat{\tilde{a}}(\xi)\widehat{\tilde{\phi}}(\xi) =& 2^{2m}(H(2\xi))^{-1}(E_{2m}(2\xi))^{-1}\widehat{a}(\xi)E_{2m}(\xi)H(\xi)(-1)^m(H(\xi))^{-1} \\ & (E_{2m}(\xi))^{-1}(i\xi)^{2m}\widehat{\phi}(\xi) \\ =& (-1)^m 2^{2m}(i\xi)^{2m}(H(2\xi))^{-1}(E_{2m}(2\xi))^{-1}\widehat{a}(\xi)\widehat{\phi}(\xi) \\ =& (-1)^m(i2\xi)^{2m}(H(2\xi))^{-1}(E_{2m}(2\xi))^{-1}\widehat{\phi}(2\xi) = \widehat{\tilde{\phi}}(2\xi).\end{aligned}$$

That is, we showed $\widehat{\tilde{\phi}}(2\xi) = \widehat{\tilde{a}}(\xi)\widehat{\tilde{\phi}}(\xi)$ (the refinable structure of $\widehat{\tilde{\phi}}$). Since H and E_{2m} are 2π-periodic, by the definition of $\tilde{\phi}$ in (2.25), we have

$$\begin{aligned}[\widehat{\tilde{\phi}}, \widehat{\phi}](\xi) =& (-1)^m(H(\xi))^{-1}(E_{2m}(\xi))^{-1}[\widehat{D^{2m}\phi}, \widehat{\phi}](\xi) \\ =& (H(\xi))^{-1}(E_{2m}(\xi))^{-1}[\widehat{\phi^{(m)}}, \widehat{\phi^{(m)}}](\xi).\end{aligned}$$

By the second identity in (2.11), we have $\widehat{\phi^{(m)}}(\xi) = E_m(\xi)\widehat{\mathring{\phi}}(\xi)$ and hence we further deduce that

$$\begin{aligned}[\widehat{\tilde{\phi}}, \widehat{\phi}](\xi) &= (H(\xi))^{-1}(E_{2m}(\xi))^{-1}[\widehat{\phi^{(m)}}, \widehat{\phi^{(m)}}](\xi) \\ &= (H(\xi))^{-1}(E_{2m}(\xi))^{-1}E_m(\xi)[\widehat{\mathring{\phi}}, \widehat{\mathring{\phi}}](\xi)\overline{E_m(\xi)}^{\mathsf{T}} \\ &= \overline{E_m(\xi)}^{-\mathsf{T}}([\widehat{\mathring{\phi}}, \widehat{\mathring{\phi}}](\xi))^{-1}E_m(\xi)(E_m(\xi))^{-1}[\widehat{\mathring{\phi}}, \widehat{\mathring{\phi}}](\xi)\overline{E_m(\xi)}^{\mathsf{T}} = I_r,\end{aligned}$$

where we used $(H(\xi))^{-1} = \overline{E_m(\xi)}^{-\mathsf{T}}([\widehat{\mathring{\phi}}, \widehat{\mathring{\phi}}](\xi))^{-1} E_m(\xi)$ and $(E_{2m}(\xi))^{-1} E_m(\xi) = (E_m(\xi))^{-1}$. Note that we have showed $\langle \tilde{\phi}, \phi \rangle = I_r$ and $\langle \tilde{\phi}, \phi(\cdot - k) \rangle = 0$ for all $k \in \mathbb{Z} \backslash \{0\}$. Since (2.5) holds, it is a routine calculation to check that $\mathsf{AS}_0^m(\phi; \psi)$ and $\mathsf{AS}_0^{-m}(\tilde{\phi}; \tilde{\psi})$ are biorthogonal to each other. Since $\mathring{\phi}$ is a compactly supported vector function in $L_2(\mathbb{R})$ and $\mathring{\phi}$ is a refinable vector function with a finitely supported refinement filter $\mathring{a}$, by [31, Theorem 2.2] or [33, Lemma 6.3.2 and Corollary 5.8.2], there exists $\varepsilon > 0$ such that all entries in $[\widehat{\mathring{\phi}}, \widehat{\mathring{\phi}}]_\varepsilon$ belong to $L_\infty(\mathbb{T})$. Since $\widehat{\phi^{(m)}}(\xi) = E_m(\xi) \widehat{\mathring{\phi}}(\xi)$, all entries in $[\widehat{\phi}, \widehat{\phi}]_{m+\varepsilon}$ belong to $L_\infty(\mathbb{T})$. By the definition of $\tilde{\phi}$ in (2.25) and the fact that all entries in $(H(\xi))^{-1}$ are 2π-periodic continuous functions taking the form in (2.19), we conclude that all the entries in $[\widehat{\tilde{\phi}}, \widehat{\tilde{\phi}}]_{-m+\varepsilon}(\xi) = (H(\xi))^{-1} [\widehat{\breve{\phi}}, \widehat{\breve{\phi}}]_{-m+\tau} \overline{H(\xi)}^{-\mathsf{T}}$ belong to $L_\infty(\mathbb{T})$. Meanwhile, since a has order $2m$ sum rules and (2.5) holds, it is a routine calculation to check that all entries in $\tilde{\psi}$ have order $2m$ vanishing moments. In fact, by (2.26) we have $\operatorname{sr}(\tilde{a}) = \operatorname{sr}(a) - 2m$ and ψ has $\operatorname{sr}(\tilde{a})$ vanishing moments. Now by [37, Theorem 2.3] or [33, Theorem 4.6.5], $\mathsf{AS}_0^m(\phi; \psi)$ must be a Bessel sequence in $H^m(\mathbb{R})$ and $\mathsf{AS}_0^{-m}(\tilde{\phi}; \tilde{\psi})$ is a Bessel sequence in $H^{-m}(\mathbb{R})$.

By (2.5) and [32, Theorem 17], $(\mathsf{AS}_0^{-m}(\tilde{\phi}; \tilde{\psi}), \mathsf{AS}_0^m(\phi; \psi))$ forms a pair of dual frames in the frequency domain. Consequently, we conclude that $(\mathsf{AS}_0^{-m}(\tilde{\phi}; \tilde{\psi}), \mathsf{AS}_0^m(\phi; \psi))$ forms a pair of biorthogonal Riesz bases in the pair of the Sobolev spaces $(H^{-m}(\mathbb{R}), H^m(\mathbb{R}))$. More specifically, $\mathsf{AS}_0^m(\phi; \psi)$ must be a Riesz basis in the Sobolev space $H^m(\mathbb{R})$. That is, $\{\phi; \psi\}$ is a Riesz wavelet in the Sobolev space $H^m(\mathbb{R})$. $\square$

Theorem 2.3. *Let $\phi = (\phi_1, \ldots, \phi_r)^\mathsf{T}$ be a compactly supported vector function in the Sobolev space $H^m(\mathbb{R})$ with $m \in \mathbb{N}_0$ such that the integer shifts of ϕ are stable and $\widehat{\phi}(2\xi) = \widehat{a}(\xi)\widehat{\phi}(\xi)$ for some $a \in (l_0(\mathbb{Z}))^{r\times r}$. Then there exists a finitely supported filter $b \in (l_0(\mathbb{Z}))^{r\times r}$ satisfying* (2.14) *and* (2.15) *if and only if the refinement filter a has at least order $2m$ sum rules; i.e.,* $\operatorname{sr}(a) \geqslant 2m$.

Proof. The necessity part ($\Rightarrow$) has been stated and proved in Theorem 2.2. We now deal with the sufficiency part ($\Leftarrow$). Since ϕ is a compactly supported vector function in $H^m(\mathbb{R})$ and has stable integer shifts, by [39, Theorem 1] and [33, Theorem 5.2.4], there exists a compactly supported vector function $g := (g_1, \ldots, g_r)^\mathsf{T}$ in $H^m(\mathbb{R})$ such that

(i) The integer shifts of g are linearly independent, i.e., $\operatorname{span}\{\widehat{g}(\xi + 2\pi k) : k \in \mathbb{Z}\} = \mathbb{C}^r$ for all $\xi \in \mathbb{C}$;

(ii) $\widehat{\phi}(\xi) = \widehat{\Theta}(\xi)\widehat{g}(\xi)$ for some $\Theta \in (l_0(\mathbb{Z}))^{r\times r}$ satisfying $\det(\widehat{\Theta}(\xi)) \neq 0$ for all $\xi \in \mathbb{R}$;

(iii) $\widehat{g}(2\xi) = \widehat{c}(\xi)\widehat{g}(\xi)$ for some $c \in (l_0(\mathbb{Z}))^{r\times r}$.

Since the integer shifts of g are linearly independent, the rank of the $r \times (2r)$ matrix $[\widehat{c}(\xi)\ \widehat{c}(\xi + \pi)]$ must be r for all $\xi \in \mathbb{C}$. The Quillen-Suslin Theorem guarantees the existence of $d \in (l_0(\mathbb{Z}))^{r\times r}$ such that

$$\det(\{\widehat{c}; \widehat{d}\})(\xi) := \det\left(\begin{bmatrix} \widehat{c}(\xi) & \widehat{c}(\xi+\pi) \\ \widehat{d}(\xi) & \widehat{d}(\xi+\pi) \end{bmatrix}\right) \quad \text{must be a nonzero monomial,} \tag{2.27}$$

that is, the above $(2r)\times(2r)$ matrix is strongly invertible. Note that

$$\widehat{a}(\xi)\widehat{\phi}(\xi)=\widehat{\phi}(2\xi)=\widehat{\Theta}(2\xi)\widehat{g}(2\xi)=\widehat{\Theta}(2\xi)\widehat{c}(\xi)\widehat{g}(\xi)=\widehat{\Theta}(2\xi)\widehat{c}(\xi)\widehat{\Theta}(\xi)^{-1}\widehat{\phi}(\xi).$$

Since the integer shifts of ϕ are stable, the above identity gives us $\widehat{a}(\xi)=\widehat{\Theta}(2\xi)\widehat{c}(\xi)(\widehat{\Theta}(\xi))^{-1}$. Hence, we have

$$\begin{bmatrix}\widehat{c}(\xi) & \widehat{c}(\xi+\pi)\\ \widehat{d}(\xi) & \widehat{d}(\xi+\pi)\end{bmatrix}=\begin{bmatrix}(\widehat{\Theta}(2\xi))^{-1} & \\ & I_{r-1}\end{bmatrix}M(\xi)\begin{bmatrix}\widehat{\Theta}(\xi) & \\ & \widehat{\Theta}(\xi+\pi)\end{bmatrix}$$

with

$$M(\xi):=\begin{bmatrix}\widehat{a}(\xi) & \widehat{a}(\xi+\pi)\\ \widehat{d}(\xi)(\widehat{\Theta}(\xi))^{-1} & \widehat{d}(\xi+\pi)(\widehat{\Theta}(\xi+\pi))^{-1}\end{bmatrix}.$$

Since $\det(\widehat{\Theta}(\xi))\neq 0$ for all $\xi\in\mathbb{R}$, we conclude from (2.27) and the above identity that $\det(M(\xi))\neq 0$ for all $\xi\in\mathbb{R}$. Define $\mathring{b}$ by

$$\widehat{\mathring{b}}(\xi):=\det(\widehat{\Theta}(\xi))\det(\widehat{\Theta}(\xi+\pi))\widehat{d}(\xi)(\widehat{\Theta}(\xi))^{-1}.$$

Since all the entries in $\widehat{\Theta}$ are 2π-periodic trigonometric polynomials, observing that $\det(\widehat{\Theta}(\xi))(\widehat{\Theta}(\xi))^{-1}=\operatorname{adj}(\widehat{\Theta}(\xi))$, where $\operatorname{adj}(\cdot)$ is the adjoint of a matrix, we conclude that $\mathring{b}\in(l_0(\mathbb{Z}))^{r\times r}$ is a finitely supported sequence. Moreover, we have

$$\begin{bmatrix}\widehat{a}(\xi) & \widehat{a}(\xi+\pi)\\ \widehat{\mathring{b}}(\xi) & \widehat{\mathring{b}}(\xi+\pi)\end{bmatrix}=\begin{bmatrix}I_r & \\ & \det(\widehat{\Theta}(\xi))\det(\widehat{\Theta}(\xi+\pi))I_r\end{bmatrix}M(\xi).$$

Consequently, we have

$$\det(\{\widehat{a};\widehat{\mathring{b}}\})(\xi) := \det\left(\begin{bmatrix} \widehat{a}(\xi) & \widehat{a}(\xi+\pi) \\ \widehat{\mathring{b}}(\xi) & \widehat{\mathring{b}}(\xi+\pi) \end{bmatrix}\right)$$
$$= \det(\widehat{\Theta}(\xi))\det(\widehat{\Theta}(\xi+\pi))\det(M(\xi)) \neq 0. \quad (2.28)$$

Since a has order $2m$ sum rules, without loss of generality, we can assume that $\widehat{a}$ takes the normal form in (2.24) and (1.5) holds with m being replaced by $2m$. Define $\mathring{a} \in (l_0(\mathbb{Z}))^{r\times r}$ as in (2.9) and $\breve{a}$ by $\widehat{\breve{a}}(\xi) = 2^{2m}(E_{2m}(2\xi))^{-1}\widehat{a}(\xi)E_{2m}(\xi)$. Since $\widehat{a}$ takes the normal form in (2.24), both $\mathring{a}$ and $\breve{a}$ are finitely supported. Let $\mathring{\phi}$ be defined in Theorem 2.1 and $G(\xi) := E_m(\xi)[\widehat{\mathring{\phi}},\widehat{\mathring{\phi}}](\xi)\overline{E_m(\xi)}^{-\mathsf{T}}$ as in item (3) of Theorem 2.1. Since (1.5) holds with m being replaced by $2m$, G is an $r\times r$ matrix of 2π-periodic trigonometric polynomials with $\det(G(\xi)) \neq 0$ for all $\xi \in \mathbb{R}$ by Theorem 2.1. Now we define

$$\widehat{b}(\xi) := \det(G(2\xi))\widehat{\breve{b}}(\xi) \quad \text{with} \quad \widehat{\breve{b}}(\xi) := \widehat{\mathring{b}}(\xi) - F(2\xi)(G(2\xi))^{-1}\widehat{a}(\xi), \quad (2.29)$$

where

$$F(2\xi) := \widehat{\mathring{b}}(\xi)G(\xi)\overline{\widehat{\breve{a}}(\xi)}^{\mathsf{T}} + \widehat{\mathring{b}}(\xi+\pi)G(\xi+\pi)\overline{\widehat{\breve{a}}(\xi+\pi)}^{\mathsf{T}}. \quad (2.30)$$

We observe that $F(\xi)$ is a well-defined matrix of 2π-periodic trigonometric polynomials. By $\det(G(\xi))(G(\xi))^{-1} = \operatorname{adj}(G(\xi))$ and $G(\xi)$ is a matrix of 2π-periodic trigonometric polynomials, $\widehat{b}$ is a well-defined matrix of 2π-periodic trigonometric polynomials, that is, $b \in (l_0(\mathbb{Z}))^{r\times r}$. We show that b is a desired filter satisfying both (2.14) and (2.15).

Since $\widehat{\phi^{(m)}}(\xi) = E_m(\xi)\widehat{\mathring{\phi}}(\xi)$ and $[\widehat{\mathring{\phi}}, \widehat{\mathring{\phi}}](\xi) = (E_m(\xi))^{-1}G(\xi)\overline{E_m(\xi)}^{\mathsf{T}}$, we have

$$\begin{aligned}
[\widehat{\phi^{(m)}}, \widehat{\phi^{(m)}}](\xi) &= E_m(\xi)[\widehat{\mathring{\phi}}, \widehat{\mathring{\phi}}](\xi)\overline{E_m(\xi)}^{\mathsf{T}} \\
&= E_m(\xi)(E_m(\xi))^{-1}G(\xi)\overline{E_m(\xi)}^{\mathsf{T}}\overline{E_m(\xi)}^{\mathsf{T}} = G(\xi)\overline{E_{2m}(\xi)}^{\mathsf{T}}.
\end{aligned}$$

Therefore, by $\widehat{\phi^{(m)}}(2\xi) = 2^m\widehat{a}(\xi)\widehat{\phi^{(m)}}(\xi)$, we have (2.20), that is,

$$\begin{aligned}
\widehat{a}(\xi)[\widehat{\phi^{(m)}}, \widehat{\phi^{(m)}}](\xi)\overline{\widehat{a}(\xi)}^{\mathsf{T}} &+ \widehat{a}(\xi+\pi)[\widehat{\phi^{(m)}}, \widehat{\phi^{(m)}}](\xi)\overline{\widehat{a}(\xi+\pi)}^{\mathsf{T}} \\
&= 2^{-2m}[\widehat{\phi^{(m)}}, \widehat{\phi^{(m)}}](2\xi) = 2^{-2m}G(2\xi)\overline{E_{2m}(2\xi)}^{\mathsf{T}},
\end{aligned}$$

from which and the fact that both F and G are 2π-periodic lead to

$$\begin{aligned}
F(2\xi)&(G(2\xi))^{-1}\widehat{a}(\xi)[\widehat{\phi^{(m)}}, \widehat{\phi^{(m)}}](\xi)\overline{\widehat{a}(\xi)}^{\mathsf{T}} \\
&\quad + F(2\xi)(G(2\xi))^{-1}\widehat{a}(\xi+\pi)[\widehat{\phi^{(m)}}, \widehat{\phi^{(m)}}](\xi)\overline{\widehat{a}(\xi+\pi)}^{\mathsf{T}} \\
&= F(2\xi)(G(2\xi))^{-1}2^{-2m}G(2\xi)\overline{E_{2m}(2\xi)}^{\mathsf{T}} = 2^{-2m}F(2\xi)\overline{E_{2m}(2\xi)}^{\mathsf{T}}. \quad (2.31)
\end{aligned}$$

On the other hand, by $\widehat{a}(\xi) = 2^{-2m}E_{2m}(2\xi)\widehat{\breve{a}}(\xi)(E_{2m}(\xi))^{-1}$ and $[\widehat{\phi^{(m)}}, \widehat{\phi^{(m)}}](\xi) = G(\xi)\overline{E_{2m}(\xi)}^{\mathsf{T}}$, we have

$$\begin{aligned}
\widehat{\mathring{b}}(\xi)[\widehat{\phi^{(m)}}, \widehat{\phi^{(m)}}](\xi)\overline{\widehat{a}(\xi)}^{\mathsf{T}} &= \widehat{\mathring{b}}(\xi)G(\xi)\overline{E_{2m}(\xi)}^{\mathsf{T}}2^{-2m}\overline{E_{2m}(\xi)}^{-\mathsf{T}}\overline{\widehat{\breve{a}}(\xi)}^{\mathsf{T}}\overline{E_{2m}(2\xi)}^{\mathsf{T}} \\
&= 2^{-2m}\widehat{\mathring{b}}(\xi)G(\xi)\overline{\widehat{\breve{a}}(\xi)}^{\mathsf{T}}\overline{E_{2m}(2\xi)}^{\mathsf{T}}.
\end{aligned}$$

Doing a similar calculation for $\xi + \pi$ and summing up the terms, we have

$$\widehat{\mathring{b}}(\xi)[\widehat{\phi^{(m)}}, \widehat{\phi^{(m)}}](\xi)\overline{\widehat{a}(\xi)}^{\mathsf{T}} + \widehat{\mathring{b}}(\xi+\pi)[\widehat{\phi^{(m)}}, \widehat{\phi^{(m)}}](\xi+\pi)\overline{\widehat{a}(\xi+\pi)}^{\mathsf{T}}$$

$$= 2^{-2m}\left(\widehat{\mathring{b}}(\xi)G(\xi)\overline{\widehat{\breve{a}}(\xi)}^{\mathsf{T}} + \widehat{\mathring{b}}(\xi+\pi)G(\xi+\pi)\overline{\widehat{\breve{a}}(\xi+\pi)}^{\mathsf{T}}\right)\overline{E_{2m}(2\xi)}^{\mathsf{T}}$$

$$= 2^{-2m}F(2\xi)\overline{E_{2m}(2\xi)}^{\mathsf{T}}.$$

Hence, combining the above identity with (2.31), by the definition of $\breve{b}$, we eventually get

$$\widehat{\breve{b}}(\xi)[\widehat{\phi^{(m)}},\widehat{\phi^{(m)}}](\xi)\overline{\widehat{a}(\xi)}^{\mathsf{T}} + \widehat{\breve{b}}(\xi+\pi)[\widehat{\phi^{(m)}},\widehat{\phi^{(m)}}](\xi+\pi)\overline{\widehat{a}(\xi+\pi)}^{\mathsf{T}} = 0.$$

Since $\widehat{b}(\xi) = \det(G(2\xi))\widehat{\breve{b}}(\xi)$ and G is 2π-periodic, the identity (2.14) follows trivially from the above identity.

On the other hand, by the definition of $b \in (l_0(\mathbb{Z}))^{r\times r}$ in (2.29), we have

$$\begin{bmatrix} \widehat{a}(\xi) & \widehat{a}(\xi+\pi) \\ \widehat{b}(\xi) & \widehat{b}(\xi+\pi) \end{bmatrix}$$
$$= \begin{bmatrix} I_r & 0 \\ -\det(G(2\xi))F(2\xi)(G(2\xi))^{-1} & \det(G(2\xi))I_r \end{bmatrix}\begin{bmatrix} \widehat{a}(\xi) & \widehat{a}(\xi+\pi) \\ \widehat{\mathring{b}}(\xi) & \widehat{\mathring{b}}(\xi+\pi) \end{bmatrix}.$$

The claim in (2.15) indeed holds by (2.28) and the fact that $\det(G(\xi)) \neq 0$ for all $\xi \in \mathbb{R}$. $\square$

The previous three theorems finally give us the main result of this chapter, which states the existence of a Riesz wavelet in $H^m(\mathbb{R})$ that satisfies the mth derivative orthogonality conditions in (2.1) and (2.2) if and only if the corresponding filter a has at least order $2m$ sum rules and the integer shifts of ϕ are stable. For the scalar case (i.e., when $r = 1$), we can even do a step further and present an explicit formula that gives all possible high-pass filters satisfying (2.14) and (2.15). This will be further discussed in Section 2.1.

Theorem 2.4. *Let $\phi = (\phi_1, \dots, \phi_r)^\mathsf{T}$ be a compactly supported refinable vector function in $H^m(\mathbb{R})$ with $m \in \mathbb{N}_0$ such that $\widehat{\phi}(2\xi) = \widehat{a}(\xi)\widehat{\phi}(\xi)$ for some $a \in (l_0(\mathbb{Z}))^{r\times r}$. Then*

(i) there exists a finitely supported high-pass filter $b \in (l_0(\mathbb{Z}))^{r\times r}$ such that $\{\phi; \psi\}$ with $\widehat{\psi}(\xi) := \widehat{b}(\xi/2)\widehat{\phi}(\xi/2)$ is an mth-order derivative-orthogonal Riesz wavelet in the Sobolev space $H^m(\mathbb{R})$ satisfying (2.1) *and* (2.2) *if and only if the integer shifts of ϕ are stable and the filter a has at least order $2m$ sum rules (i.e., $\mathrm{sr}(a) \geqslant 2m$).*

(ii) Under the condition that the integer shifts of ϕ are stable, for any $b \in (l_0(\mathbb{Z}))^{r\times r}$, $\{\phi; \psi\}$ with $\widehat{\psi}(\xi) := \widehat{b}(\xi/2)\widehat{\phi}(\xi/2)$ is an mth-order derivative-orthogonal Riesz wavelet in the Sobolev space $H^m(\mathbb{R})$ satisfying (2.1) *and* (2.2) *if and only if* (2.14) *and* (2.15) *are satisfied. Moreover, $\mathsf{AS}_0^\tau(\phi; \psi)$ is a Riesz basis in the Sobolev space $H^\tau(\mathbb{R})$ for all τ in the nonempty open interval $(2m - \mathrm{sm}(\phi), \mathrm{sm}(\phi))$.*

We are ready to present some examples of mth derivative-orthogonal Riesz wavelets, where $m = 0, 1, 2$. The scalar case is presented first before moving on to the matrix-valued case.

2.1 Derivative-orthogonal Riesz Wavelets from Scalar Refinable Functions

Let us first deal with the special scalar case, i.e., $r = 1$. The following theorem provides an explicit formula for the derivative-orthogonal Riesz wavelets derived from scalar filters.

Theorem 2.5. *Let $m \in \mathbb{N}_0$ and $a \in l_0(\mathbb{Z})$ such that $\widehat{a}(\xi) = 2^{-2m}(1+e^{-i\xi})^{2m}\widehat{\breve{a}}(\xi)$ with $\breve{a} \in l_0(\mathbb{Z})$ and $\widehat{\breve{a}}(0) = 1$. Define $\widehat{\phi}(\xi) := \prod_{j=1}^{\infty} \widehat{a}(2^{-j}\xi)$ and $\widehat{\mathring{\phi}}(\xi) := \prod_{j=1}^{\infty} \widehat{\mathring{a}}(2^{-j}\xi)$ with $\widehat{\mathring{a}}(\xi) := 2^{-m}(1+e^{-i\xi})^m\widehat{\breve{a}}(\xi)$. Then $\widehat{\phi}(2\xi) = \widehat{a}(\xi)\widehat{\phi}(\xi)$ and $\widehat{\mathring{\phi}}(2\xi) = \widehat{\mathring{a}}(\xi)\widehat{\mathring{\phi}}(\xi)$. Suppose that the compactly supported scalar refinable function $\phi \in H^m(\mathbb{R})$ and the integer shifts of ϕ are stable. Then a finitely supported high-pass filter $b \in l_0(\mathbb{Z})$ satisfies (2.14) and (2.15) if and only if*

$$\widehat{b}(\xi) = e^{i(m-1)\xi}\overline{\widehat{\breve{a}}(\xi+\pi)}[\widehat{\mathring{\phi}}, \widehat{\mathring{\phi}}](\xi+\pi)\widehat{\theta}(2\xi)/\widehat{d}(2\xi), \tag{2.32}$$

where $\theta \in l_0(\mathbb{Z})$ satisfying $\widehat{\theta}(\xi) \neq 0$ for all $\xi \in \mathbb{R}$ and $d \in l_0(\mathbb{Z})$ is defined by

$$\widehat{d}(2\xi) := \gcd\left(\overline{\widehat{\breve{a}}(\xi)}[\widehat{\mathring{\phi}}, \widehat{\mathring{\phi}}](\xi), \overline{\widehat{\breve{a}}(\xi+\pi)}[\widehat{\mathring{\phi}}, \widehat{\mathring{\phi}}](\xi+\pi)\right). \tag{2.33}$$

Moreover, $\{\phi; \psi\}$ with $\widehat{\psi}(\xi) := \widehat{b}(\xi/2)\widehat{\phi}(\xi/2)$ and the high-pass filter b in (2.32) is a derivative-orthogonal Riesz wavelet in $H^m(\mathbb{R})$ satisfying (2.1) and (2.2) and $\mathsf{AS}_0^{\tau}(\phi; \psi)$ is a Riesz basis for $H^{\tau}(\mathbb{R})$ for all $\tau \in (2m - \mathrm{sm}(\phi), \mathrm{sm}(\phi))$.

Proof. Let $\widehat{u}(\xi) := \overline{\widehat{\breve{a}}(\xi)}[\widehat{\mathring{\phi}}, \widehat{\mathring{\phi}}](\xi)/\widehat{d}(2\xi)$, which we know is a 2π periodic trigonometric polynomial from (2.33). We know that $\phi^{(m)} = \nabla^m\mathring{\phi}$ and the integer shifts of $\mathring{\phi}$ are stable. Taking the Fourier transform of $\phi^{(m)}$, we have $\widehat{\phi^{(m)}}(\xi) = (1-e^{-i\xi})^m\widehat{\mathring{\phi}}(\xi)$. With that in mind,

$$\begin{aligned}
\widehat{b}(\xi)[\widehat{\phi^{(m)}}, \widehat{\phi^{(m)}}](\xi)\overline{\widehat{a}(\xi)} &= 2^{-2m}\widehat{b}(\xi)|1-e^{-i\xi}|^{2m}[\widehat{\mathring{\phi}}, \widehat{\mathring{\phi}}](\xi)(1+e^{i\xi})^{2m}\overline{\widehat{\breve{a}}(\xi)} \\
&= 2^{-2m}(-1)^m e^{-i\xi m}(1-e^{i2\xi})^{2m}\widehat{d}(2\xi)\widehat{b}(\xi)\widehat{u}(\xi).
\end{aligned}$$

Carrying out a similar calculation for $\widehat{b}(\xi+\pi)[\widehat{\phi^{(m)}}, \widehat{\phi^{(m)}}](\xi+\pi)\overline{\widehat{a}(\xi+\pi)}$ and summing up the two terms, we have

$$\widehat{b}(\xi)[\widehat{\phi^{(m)}},\widehat{\phi^{(m)}}](\xi)\overline{\widehat{a}(\xi)}+\widehat{b}(\xi+\pi)[\widehat{\phi^{(m)}},\widehat{\phi^{(m)}}](\xi+\pi)\overline{\widehat{a}(\xi+\pi)}$$
$$=2^{-2m}e^{-i\xi m}(1-e^{i2\xi})^{2m}[(-1)^m\widehat{b}(\xi)\widehat{u}(\xi)+\widehat{b}(\xi+\pi)\widehat{u}(\xi+\pi)].$$

For (2.14) to be fulfilled, it must be the case that

$$(-1)^m\widehat{b}(\xi)\widehat{u}(\xi)+\widehat{b}(\xi+\pi)\widehat{u}(\xi+\pi)=0. \tag{2.34}$$

We first prove the ($\Leftarrow$) part of the theorem. Given our assumption for $\widehat{b}(\xi)$ in (2.32) and the definition of $\widehat{u}(\xi)$, it follows that

$$(-1)^m\widehat{b}(\xi)\widehat{u}(\xi)+\widehat{b}(\xi+\pi)\widehat{u}(\xi+\pi)$$
$$=(-1)^m e^{i(m-1)\xi}\widehat{u}(\xi+\pi)\widehat{\theta}(2\xi)+e^{i(m-1)(\xi+\pi)}\widehat{u}(\xi)\widehat{\theta}(2\xi)=0.$$

So, (2.34) and consequently (2.14) are fulfilled. We show that $\det(\{\widehat{a},\widehat{b}\})\neq 0$.

$$\begin{aligned}\widehat{a}(\xi)\widehat{b}(\xi+\pi)&=2^{-m}(1+e^{-i\xi})^m\widehat{\mathring{a}}(\xi)e^{i(m-1)(\xi+\pi)}(\xi)\widehat{u}(\xi)\theta(2\xi)\\&=2^{-m}(1+e^{-i\xi})^m\widehat{\mathring{a}}(\xi)e^{i(m-1)(\xi+\pi)}\overline{\widehat{\mathring{a}}(\xi)}[\widehat{\mathring{\phi}},\widehat{\mathring{\phi}}](\xi)\widehat{\theta}(2\xi)/\widehat{d}(2\xi)\\&=(-1)^{m-1}e^{-i\xi}|\widehat{\mathring{a}}(\xi)|^2[\widehat{\mathring{\phi}},\widehat{\mathring{\phi}}](\xi)\widehat{\theta}(2\xi)/\widehat{d}(2\xi),\end{aligned}$$

where we have applied $\widehat{a}(\xi)=2^{-m}(1+e^{-i\xi})^m\widehat{\mathring{a}}(\xi)$ and $\widehat{\mathring{a}}(\xi)=2^{-m}(1+e^{-i\xi})^m\widehat{\mathring{a}}(\xi)$. This gives

$$\det(\{\widehat{a};\widehat{b}\})(\xi)=\widehat{a}(\xi)\widehat{b}(\xi+\pi)-\widehat{a}(\xi+\pi)\widehat{b}(\xi)$$
$$=(-1)^{m-1}e^{-i\xi}\Big(|\widehat{\mathring{a}}(\xi)|^2[\widehat{\mathring{\phi}},\widehat{\mathring{\phi}}](\xi)+|\widehat{\mathring{a}}(\xi+\pi)|^2[\widehat{\mathring{\phi}},\widehat{\mathring{\phi}}](\xi+\pi)\Big)\widehat{\theta}(2\xi)/\widehat{d}(2\xi).$$

$$=(-1)^{m-1}e^{-i\xi}[\widehat{\mathring{\phi}},\widehat{\mathring{\phi}}](2\xi)\widehat{\theta}(2\xi)/\widehat{d}(2\xi), \tag{2.35}$$

where we have applied the following identity

$$\widehat{\mathring{a}}(\xi)[\widehat{\mathring{\phi}},\widehat{\mathring{\phi}}](\xi)\overline{\widehat{\mathring{a}}(\xi)}+\widehat{\mathring{a}}(\xi+\pi)[\widehat{\mathring{\phi}},\widehat{\mathring{\phi}}](\xi+\pi)\overline{\widehat{\mathring{a}}(\xi+\pi)}=[\widehat{\mathring{\phi}},\widehat{\mathring{\phi}}](2\xi)$$

to go from the second to the third equality. From our definitions of $\widehat{\mathring{a}}(\xi)$ and $d(2\xi)$, and the fact that $\overline{\widehat{\mathring{a}}(\xi)} = 2^{-m}(1+e^{i\xi})^m\overline{\widehat{\breve{a}}(\xi)}$, we can deduce right away that $\widehat{d} \mid [\widehat{\mathring{\phi}},\widehat{\mathring{\phi}}]$. The stability of integer shifts of $[\widehat{\mathring{\phi}},\widehat{\mathring{\phi}}]$ gives us $[\widehat{\mathring{\phi}},\widehat{\mathring{\phi}}](\xi) \neq 0$ for all $\xi \in \mathbb{R}$. Since $\widehat{d} \mid [\widehat{\mathring{\phi}},\widehat{\mathring{\phi}}]$, this implies that $\widehat{d}(\xi) \neq 0$ for all $\xi \in \mathbb{R}$. Given that $\widehat{\theta}(\xi) \neq 0$ for all $\xi \in \mathbb{R}$, we have that the determinant in (2.35) is nonzero for all $\xi \in \mathbb{R}$. Thus, (2.15) holds.

We prove the ($\Rightarrow$) part of the theorem. Due to (2.14), we must have (2.34). By (2.33) and the definition of $\widehat{u}$, we observe that $\gcd(\widehat{u}(\xi),\widehat{u}(\xi+\pi)) = 1$. Since (2.34) holds, we must have $\widehat{u}(\xi+\pi) \mid \widehat{b}(\xi)$, or equivalently $\widehat{b}(\xi) = \widehat{v}(2\xi)\widehat{u}(\xi+\pi)$ for some $v \in l_0(\mathbb{Z})$. Now, equation (2.34) brings us to $(-1)^m\widehat{v}(\xi)+\widehat{v}(\xi+\pi) = 0$. So, $\widehat{v}(\xi) = e^{i(m-1)\xi}\widehat{\theta}(2\xi)$ for some $\theta \in l_0(\mathbb{Z})$. We have arrived at (2.32). Since we have shown that $\widehat{d}(\xi) \neq 0$ and $[\widehat{\mathring{\phi}},\widehat{\mathring{\phi}}](\xi) \neq 0$ for all $\xi \in \mathbb{R}$, as well as we have assumed that $\det(\{\widehat{a};\widehat{b}\}) \neq 0$ (from (2.15)), we conclude from (2.35) that $\widehat{\theta}(\xi) \neq 0$ for all $\xi \in \mathbb{R}$. Therefore, $\{\phi,\psi\}$ is an mth order derivative-orthogonal Riesz wavelet in $H^m(\mathbb{R})$ from Theorem 2.2. $\square$

For the following examples, we set $\widehat{d}(\xi) = 1$ if we observe that the greatest common divisor of (2.33) is trivial (a monomial). From our experience, the greatest common divisor of $\overline{\widehat{\breve{a}}(\xi)}[\widehat{\mathring{\phi}},\widehat{\mathring{\phi}}](\xi)$ and $\overline{\widehat{\breve{a}}(\xi+\pi)}[\widehat{\mathring{\phi}},\widehat{\mathring{\phi}}](\xi+\pi)$ is in most cases a monomial.

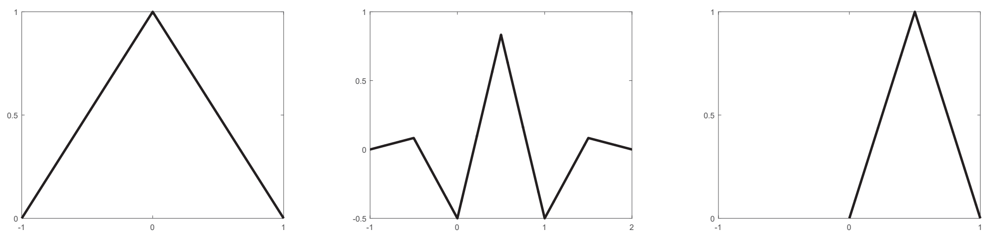
Example 2.1. Let $a \in l_0(\mathbb{Z})$ be given by $\widehat{a}(\xi) = 2^{-2}(1+e^{-i\xi})^2 e^{i\xi}$, i.e., $a = \left\{\frac{1}{4}, \boldsymbol{\frac{1}{2}}, \frac{1}{4}\right\}_{[-1,1]}$. Define $\widehat{\phi}(\xi) := \prod_{j=1}^{\infty} \widehat{a}(2^{-j}\xi)$. Then $\phi = B_2(\cdot - 1)$ is the centered B-spline of order 2. Note that $\phi \in H^1(\mathbb{R})$ with $\operatorname{sm}(\phi) = 3/2$, the integer shifts of ϕ are stable, and a has order 2 sum rules with $\operatorname{sr}(a) = 2$.

For $m = 0$, by $\breve{a} = a$, $\mathring{\phi} = \phi$, and $[\widehat{\mathring{\phi}}, \widehat{\mathring{\phi}}](\xi) = \frac{1}{3}(2+\cos(\xi)) \neq 0$ for all $\xi \in \mathbb{R}$, we have $\widehat{d}(\xi) = 1$ in (2.33). Choosing $\widehat{\theta}(\xi) = 1$ in (2.32), we have $\widehat{b_0}(\xi) = \frac{1}{24}(1-e^{-i\xi})^2(e^{-i\xi} + e^{i\xi} - 4)$; i.e., $b_0 = \left\{\frac{1}{24}, -\boldsymbol{\frac{1}{4}}, \frac{5}{12}, -\frac{1}{4}, \frac{1}{24}\right\}_{[-1,3]}$. Note that $\det(\{\widehat{a}; \widehat{b_0}\})(\xi) = -\frac{1}{3}e^{-i\xi}(2+\cos(2\xi)) \neq 0$ for all $\xi \in \mathbb{R}$. Define $\widehat{\psi}(\xi) := \widehat{b_0}(\xi/2)\widehat{\phi}(\xi/2)$. Then $\{\phi; \psi\}$ is a semi-orthogonal Riesz wavelet in $L_2(\mathbb{R})$ with $\|\phi\|_{L_2(\mathbb{R})} = \sqrt{2/3}$ and $\|\psi\|_{L_2(\mathbb{R})} = 1/2$. Moreover, $\mathsf{AS}_0^{\tau}(\phi; \psi)$ is a Riesz basis for $H^{\tau}(\mathbb{R})$ for all $\tau \in (-3/2, 3/2)$.

For $m = 1$, by $\widehat{\breve{a}}(\xi) = e^{i\xi}$ and $[\widehat{\mathring{\phi}}, \widehat{\mathring{\phi}}](\xi) = 1$, we have $\widehat{d}(\xi) = 1$ in (2.33). Choosing $\widehat{\theta}(\xi) = -\frac{1}{2}$ in (2.32), we have $\widehat{b_1}(\xi) = \frac{1}{2}e^{-i\xi}$; i.e., $b_1 = \left\{\frac{1}{2}\right\}_{[1,1]}$. Note that $\det(\{\widehat{a}; \widehat{b_1}\})(\xi) = -\frac{1}{2}e^{-i\xi} \neq 0$ for all $\xi \in \mathbb{R}$. Define $\widehat{\psi}(\xi) := \widehat{b_1}(\xi/2)\widehat{\phi}(\xi/2)$. Then $\{\phi; \psi\}$ is a first-order derivative-orthogonal Riesz wavelet in $H^1(\mathbb{R})$ with $\|\phi'\|_{L_2(\mathbb{R})} = \sqrt{2}$ and $\|\psi'\|_{L_2(\mathbb{R})} = 2$. Moreover, $\mathsf{AS}_0^{\tau}(\phi; \psi)$ is a Riesz basis for the Sobolev space $H^{\tau}(\mathbb{R})$ for all $\tau \in (1/2, 3/2)$. This example recovers the Riesz basis constructed in [10]. See Figure 2.1 to visualize the associated wavelet functions ψ associated with filters b_0 and b_1, respectively.

Example 2.2. Let $a \in l_0(\mathbb{Z})$ be given by $\widehat{a}(\xi) = 2^{-3}(1+e^{-i\xi})^3 e^{2i\xi}$, i.e., $a = \left\{\frac{1}{8}, \frac{3}{8}, \boldsymbol{\frac{3}{8}}, \frac{1}{8}\right\}_{[-2,1]}$. Define $\widehat{\phi}(\xi) := \prod_{j=1}^{\infty} \widehat{a}(2^{-j}\xi)$. Then $\phi = B_3(\cdot - 2)$ is the shifted B-spline of order 3. Note that $\phi \in H^2(\mathbb{R})$ with $\operatorname{sm}(\phi) = 5/2$, the integer shifts of ϕ are stable, and a has order 3 sum rules with $\operatorname{sr}(a) = 3$.

For $m = 0$, by $\breve{a} = a$, $\mathring{\phi} = \phi$, and $[\widehat{\mathring{\phi}}, \widehat{\mathring{\phi}}](\xi) = \frac{11}{20} + \frac{13}{30}\cos(\xi) + \frac{1}{60}\cos(2\xi) \neq 0$

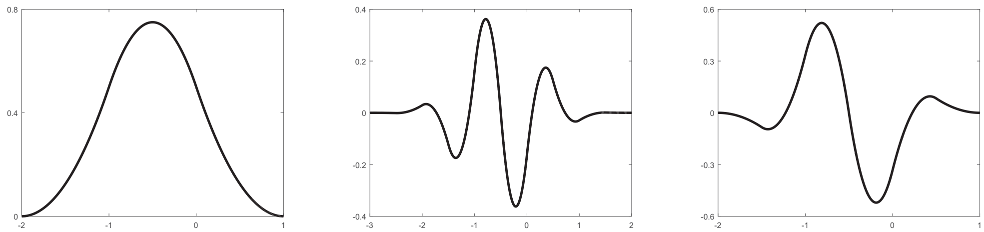


(a) (b) (c)

Figure 2.1: (a) is the refinable function ϕ in Example 2.1, which is the centered piecewise linear B-spline of order 2. (b) is the wavelet function ψ associated with the filter b_0 such that $\{\phi; \psi\}$ is a semi-orthogonal wavelet in $L_2(\mathbb{R})$ with $\mathrm{supp}(\psi) = [-1, 2]$. (c) is the wavelet function ψ associated with the filter b_1 such that $\{\phi; \psi\}$ is a first-order derivative-orthogonal Riesz wavelet for $H^1(\mathbb{R})$ with $\mathrm{supp}(\psi) = [0, 1]$.

for all $\xi \in \mathbb{R}$, we have $\widehat{d}(\xi) = 1$ in (2.33). Choosing $\widehat{\theta}(\xi) = e^{i\xi}$ in (2.32), we have $\widehat{b_0}(\xi) = -\frac{1}{960}(1 - e^{-i\xi})^3(66e^{2i\xi} - 26e^{i\xi} - 26e^{3i\xi} + e^{4i\xi} + 1)$; i.e., $b_0 = \left\{-\frac{1}{960}, \frac{29}{960}, -\frac{49}{320}, \frac{101}{320}, -\mathbf{\frac{101}{320}}, \frac{49}{320}, -\frac{29}{960}, \frac{1}{960}\right\}_{[-4,3]}$. Note that $\det(\{\widehat{a}; \widehat{b_0}\})(\xi) = -\frac{1}{120}e^{-i\xi}(66 + 52\cos(2\xi) + 2\cos(4\xi)) \neq 0$ for all $\xi \in \mathbb{R}$. Define $\widehat{\psi}(\xi) := \widehat{b_0}(\xi/2)\widehat{\phi}(\xi/2)$. Then $\{\phi; \psi\}$ is a semi-orthogonal Riesz wavelet in $L_2(\mathbb{R})$ with $\|\phi\|_{L_2(\mathbb{R})} = \sqrt{11/20}$ and $\|\psi\|_{L_2(\mathbb{R})} = \sqrt{23939/8640}$. Moreover, $\mathsf{AS}_0^\tau(\phi; \psi_0)$ is a Riesz basis for the Sobolev space $H^\tau(\mathbb{R})$ for all $\tau \in (-5/2, 5/2)$.

For $m = 1$, by $\widehat{\mathring{a}}(\xi) = 2^{-1}(1 + e^{-i\xi})e^{2i\xi}$ and $[\widehat{\mathring{\phi}}, \widehat{\mathring{\phi}}](\xi) = \frac{1}{3}(2 + \cos(\xi)) \neq 0$ for all $\xi \in \mathbb{R}$, we have $\widehat{d}(\xi) = 1$ in (2.33). Choosing $\widehat{\theta}(\xi) = -e^{i\xi}$ in (2.32), we have $\widehat{b_1}(\xi) = -\frac{1}{12}(1 - e^{-i\xi})(1 - 4e^{i\xi} + e^{2i\xi})$; i.e., $b_1 = \left\{-\frac{1}{12}, \frac{5}{12}, -\mathbf{\frac{5}{12}}, \frac{1}{12}\right\}_{[-2,1]}$. Note that $\det(\{\widehat{a}; \widehat{b_0}\})(\xi) = -\frac{1}{3}e^{-i\xi}(2 + \cos(2\xi)) \neq 0$ for all $\xi \in \mathbb{R}$. Define $\widehat{\psi}(\xi) := \widehat{b_1}(\xi/2)\widehat{\phi}(\xi/2)$. Then $\{\phi; \psi\}$ is a first-order derivative-orthogonal Riesz wavelet in $H^1(\mathbb{R})$ with $\|\phi'\|_{L_2(\mathbb{R})} = 1$ and $\|\psi'\|_{L_2(\mathbb{R})} = 2$. Moreover, $\mathsf{AS}_0^\tau(\phi; \psi_1)$ is a Riesz basis for the Sobolev space $H^\tau(\mathbb{R})$ for all $-1/2 < \tau < 5/2$. See Figure 2.2 to visualize the associated wavelet functions ψ associated with filters b_0 and b_1, respectively.



(a) (b) (c)

Figure 2.2: (a) is the refinable function ϕ in Example 2.2, which is the shifted B-spline of order 3. (b) is the wavelet function ψ associated with the filter b_0 such that $\{\phi;\psi\}$ is a semi-orthogonal Riesz wavelet in $L_2(\mathbb{R})$ with $\mathrm{supp}(\psi)=[-3,2]$. (c) is the wavelet function ψ associated with the filter b_1 such that $\{\phi;\psi\}$ is a first-order derivative-orthogonal Riesz wavelet for $H^1(\mathbb{R})$ with $\mathrm{supp}(\psi)=[-2,1]$.

.

Though $\phi\in H^m(\mathbb{R})$ with $m=2$, since its filter a has no more than 3 sum rules, Theorem 2.5 tells us that there does not exist any finitely supported filter b_2 such the associated wavelet system satisfies the second-order derivative-orthogonal property, i.e., conditions (2.1) and (2.2) with $m=2$.

Example 2.3. Let $a\in l_0(\mathbb{Z})$ be given by $\widehat{a}(\xi)=2^{-4}(1+e^{-i\xi})^4e^{2i\xi}$, i.e., $a=\left\{\frac{1}{16},\frac{1}{4},\boldsymbol{\frac{3}{8}},\frac{1}{4},\frac{1}{16}\right\}_{[-2,2]}$. Define $\widehat{\phi}(\xi):=\prod_{j=1}^{\infty}\widehat{a}(2^{-j}\xi)$. Then $\phi=B_4(\cdot-2)$ is the centered B-spline of order 4. Note that $\phi\in H^3(\mathbb{R})$ with $\mathrm{sm}(\phi)=7/2$, the integer shifts of ϕ are stable, and a has order 4 sum rules with $\mathrm{sr}(a)=4$.

For $m=0$, by $\breve{a}=a$, $\mathring{\phi}=\phi$, and $[\widehat{\mathring{\phi}},\widehat{\mathring{\phi}}](\xi)=\frac{151}{315}+\frac{397}{840}\cos(\xi)+\frac{1}{21}\cos(2\xi)+\frac{1}{2520}\cos(3\xi)\neq 0$ for all $\xi\in\mathbb{R}$, we have $\widehat{d}(\xi)=1$ in (2.33). Choosing $\widehat{\theta}(\xi)=\frac{315}{32}$ in (2.32), we have $\widehat{b_0}(\xi)=\frac{1}{8192}(1-e^{-i\xi})^4(120e^{-i\xi}+2416e^{i\xi}-1191e^{2i\xi}-e^{-2i\xi}+120e^{3i\xi}-e^{4i\xi}-1191)$; i.e.,

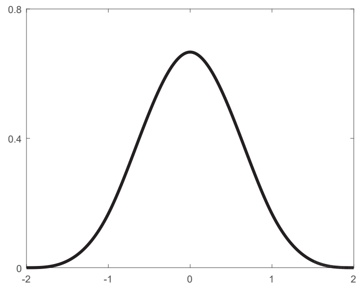
$$b_0=\left\{-\frac{1}{8192},\frac{31}{2048},-\frac{1677}{8192},\frac{247}{256},-\boldsymbol{\frac{9241}{4096}},\frac{3033}{1024},-\frac{9241}{4096},\frac{247}{256},-\frac{1677}{8192},\frac{31}{2048},\right.$$
$$\left.-\frac{1}{8192}\right\}_{[-4,6]}.$$

Also, $\det(\{\widehat{a};\widehat{b_0}\})(\xi) = -\frac{1}{512}e^{-i\xi}(2416+2382\cos(2\xi)+240\cos(4\xi)+2\cos(6\xi)) \neq 0$ for all $\xi \in \mathbb{R}$. Define $\widehat{\psi}(\xi) := \widehat{b_0}(\xi/2)\widehat{\phi}(\xi/2)$. Then $\{\phi;\psi\}$ is a semi-orthogonal Riesz wavelet in $L_2(\mathbb{R})$ wavelet in $L_2(\mathbb{R})$ with $\|\phi\|_{L_2(\mathbb{R})} = \sqrt{151/315}$ and $\|\psi\|_{L_2(\mathbb{R})} = \sqrt{21100677/5242880}$. Moreover, $\mathsf{AS}_0^\tau(\phi;\psi_0)$ is a Riesz basis for the Sobolev space $H^\tau(\mathbb{R})$ for all $\tau \in (-7/2, 7/2)$.

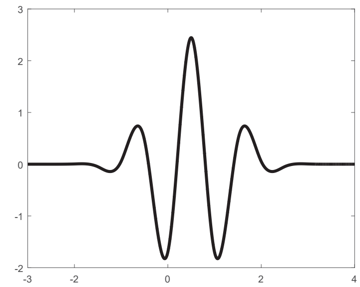
For $m = 1$, by $\widehat{\mathring{a}}(\xi) = 2^{-2}(1+e^{-i\xi})^2 e^{2i\xi}$ and $[\widehat{\mathring{\phi}},\widehat{\mathring{\phi}}](\xi) = \frac{1}{120}(66+52\cos(\xi)+2\cos(2\xi)) \neq 0$ for all $\xi \in \mathbb{R}$, we have $\widehat{d}(\xi) = 1$ in (2.33). Choosing $\widehat{\theta}(\xi) = -\frac{15}{4}$, in (2.32), we have $\widehat{b_1}(\xi) = \frac{1}{128}(1-e^{-i\xi})^2(26e^{-i\xi}+26e^{i\xi}-e^{2i\xi}-e^{-2i\xi}-66)$; i.e., $b_1 = \left\{-\frac{1}{128}, \frac{7}{32}, -\frac{\mathbf{119}}{\mathbf{128}}, \frac{23}{16}, -\frac{119}{128}, \frac{7}{32}, -\frac{1}{128}\right\}_{[-2,4]}$. Note that $\det(\{\widehat{a};\widehat{b_1}\})(\xi) = -\frac{1}{32}e^{-i\xi}(66+52\cos(2\xi)+2\cos(4\xi)) \neq 0$ for all $\xi \in \mathbb{R}$. Define $\widehat{\psi}(\xi) := \widehat{b_1}(\xi/2)\widehat{\phi}(\xi/2)$. Then $\{\phi;\psi\}$ is a first-order derivative-orthogonal Riesz wavelet in $H^1(\mathbb{R})$ with $\|\phi'\|_{L_2(\mathbb{R})} = \sqrt{2/3}$ and $\|\psi'\|_{L_2(\mathbb{R})} = \sqrt{17011/68}$. Moreover, $\mathsf{AS}_0^\tau(\phi;\psi_1)$ is a Riesz basis for $H^\tau(\mathbb{R})$ for all $\tau \in (-3/2, 7/2)$.

For $m = 2$, by $\widehat{\mathring{a}}(\xi) = e^{2i\xi}$ and $[\widehat{\mathring{\phi}},\widehat{\mathring{\phi}}](\xi) = \frac{1}{3}(2+\cos(\xi)) \neq 0$ for all $\xi \in \mathbb{R}$, we have $\widehat{d}(\xi) = 1$ in (2.33). Choosing $\widehat{\theta}(\xi) = \frac{3}{2}$ in (2.32), we have $\widehat{b_2}(\xi) = -\frac{1}{4}+e^{-i\xi}-\frac{1}{4}e^{-2i\xi}$; i.e., $b_2 = \left\{-\frac{\mathbf{1}}{\mathbf{4}}, 1, -\frac{1}{4}\right\}_{[0,2]}$. Note that $\det(\{\widehat{a};\widehat{b_2}\})(\xi) = -e^{-i\xi}(1+\frac{1}{2}\cos(2\xi)) \neq 0$ for all $\xi \in \mathbb{R}$. Define $\widehat{\psi}(\xi) := \widehat{b_2}(\xi/2)\widehat{\phi}(\xi/2)$. Then $\{\phi;\psi\}$ is a second-order derivative-orthogonal Riesz wavelet in $H^2(\mathbb{R})$ with $\|\phi''\|_{L_2(\mathbb{R})} = \sqrt{8/3}$ and $\|\psi''\|_{L_2(\mathbb{R})} = 12$. Moreover, $\mathsf{AS}_0^\tau(\phi;\psi_2)$ is a Riesz basis for the Sobolev space $H^\tau(\mathbb{R})$ for all $\tau \in (1/2, 7/2)$. See Figure 2.3 to visualize the associated wavelet functions ψ associated with filters b_0, b_1, b_2, respectively.

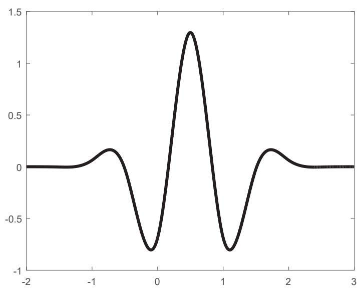
The filter b_2 and its associated wavelet function ψ that we get are exactly the same filter and wavelet function that are used in [43]. In other words, Jia and Zhao in [43] are actually using a second-order derivative-orthogonal Riesz wavelet in their paper, even though it is not explicitly stated there. We refer



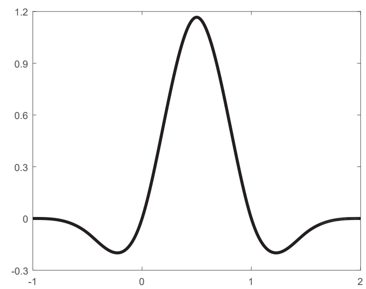
(a)



(b)



(c)



(d)

Figure 2.3: (a) is the refinable function ϕ in Example 2.3, which is the centered B-spline of order 4. (b) is the wavelet function ψ associated with the filter b_0 such that $\{\phi; \psi\}$ is a semi-orthogonal Riesz wavelet in $L_2(\mathbb{R})$ with $\text{supp}(\psi) = [-3, 4]$. (c) is the wavelet function ψ associated with the filter b_1 such that $\{\phi; \psi\}$ is a first-order derivative-orthogonal Riesz wavelet for $H^1(\mathbb{R})$ with $\text{supp}(\psi) = [-2, 3]$. (d) is the wavelet function ψ associated with the filter b_2 such that $\{\phi; \psi\}$ is a second-order derivative-orthogonal Riesz wavelet for $H^2(\mathbb{R})$ with $\text{supp}(\psi) = [-1, 2]$.

interested readers to [43] for the condition numbers of the two-dimensional tensor product wavelet using B_4 and the filter b_2 at various scales.

2.2 Derivative-orthogonal Riesz Wavelets from Refinable Vector Functions

As we move away from scalar filters and into the realm of matrix-valued filters, we would like to emphasize that generating derivative-orthogonal Riesz wavelets from refinable function vectors is significantly harder. Unlike the scalar case, we do not have any explicit formula that gives all derivative-orthogonal Riesz wavelets generated from a matrix-valued filter. Currently, we are not sure if it is even possible to find such a formula. Thankfully, their existence is already guaranteed by Theorem 2.2. In this section, we are interested in constructing derivative-orthogonal wavelets with short supports. The next few examples deal with Hermite splines, which possess multiplicity of $r = 2$. They will play a pivotal role in finding the numerical solutions to certain differential equations, which we shall discuss in Chapter 4. The resulting wavelets generally have a shorter support compared to the generated scalar wavelets, particularly as m gets larger. The fact that we are able to obtain a Riesz wavelet with a shorter support simply illustrates one of the benefits in using multiwavelet bases.

Example 2.4. Let $\phi = (\phi_1, \phi_2)^\mathsf{T}$ with the Hermite cubic splines ϕ_1, ϕ_2 given in (1.7) and $\mathrm{sm}(\phi) = 5/2$. Recall that the refinement filter a for ϕ takes the form of (1.8) with $\mathrm{sr}(a) = 4$ and the integer shifts of ϕ are stable. In the following, we denote by b_0, b_1, and b_2 the wavelet filters we obtain by imposing the mth-order derivative-orthogonality condition in (2.14) on the original function ($m = 0$), the first-order derivative ($m = 1$), and the second-order derivative ($m = 2$).

For $m = 0$, the filter b_0 is supported on $[-3, 2]$ and is given by

$$b_0 = \left\{ \begin{bmatrix} \frac{79}{4440} & \frac{1}{10} \\ -\frac{27}{2960} & -\frac{1}{20} \end{bmatrix}, \begin{bmatrix} -\frac{419}{1110} & -\frac{11}{10} \\ \frac{87}{740} & \frac{1}{4} \end{bmatrix}, \begin{bmatrix} -\frac{3871}{4440} & -\frac{2731}{185} \\ \frac{879}{1480} & \frac{18793}{2960} \end{bmatrix}, \begin{bmatrix} \frac{1367}{555} & 0 \\ 0 & \frac{1621}{148} \end{bmatrix}, \right.$$
$$\left. \begin{bmatrix} -\frac{3871}{4440} & \frac{2731}{185} \\ -\frac{879}{1480} & \frac{18793}{2960} \end{bmatrix}, \begin{bmatrix} -\frac{419}{1110} & \frac{11}{10} \\ -\frac{87}{740} & \frac{1}{4} \end{bmatrix}, \begin{bmatrix} \frac{79}{4440} & -\frac{1}{10} \\ \frac{27}{2960} & -\frac{1}{20} \end{bmatrix} \right\}_{[-3,3]}.$$

Note that $\det(\{\widehat{a}; \widehat{b_0}\})(\xi) = \frac{1}{102675}(-7214350 + 4097408\cos(2\xi) - 97370\cos(4\xi) + 592\cos(6\xi)) \neq 0$ for all $\xi \in \mathbb{R}$. Define $\widehat{\psi}(\xi) := \widehat{b_0}(\xi/2)\widehat{\phi}(\xi/2)$. Then $\{\phi; \psi\}$ is a semi-orthogonal Riesz wavelet in $L_2(\mathbb{R})$ with $\|\psi_1\|_{L_2(\mathbb{R})} = \sqrt{166496636/10780875}$, $\|\psi_2\|_{L_2(\mathbb{R})} = \sqrt{21439343/7187250}$ and

$$[\widehat{\phi}, \widehat{\phi}](\xi) = \begin{bmatrix} \frac{9}{70}e^{-i\xi} + \frac{9}{70}e^{i\xi} + \frac{26}{35} & -\frac{13}{420}e^{i\xi}(e^{-2\xi i} - 1) \\ \frac{13}{420}e^{i\xi}(e^{-2\xi i} - 1) & \frac{2}{105} - \frac{1}{140}e^{-i\xi} - \frac{1}{140}e^{i\xi} \end{bmatrix}. \tag{2.36}$$

Moreover, $\mathsf{AS}_0^\tau(\phi; \psi)$ is a Riesz basis for the Sobolev space $H^\tau(\mathbb{R})$ for all $\tau \in (-5/2, 5/2)$.

For $m = 1$, the filter b_1 is supported on $[-1, 1]$ and is given by

$$b_1 = \left\{ \begin{bmatrix} \frac{2}{21} & 1 \\ \frac{1}{9} & 1 \end{bmatrix}, \begin{bmatrix} -\frac{4}{21} & 0 \\ 0 & \frac{4}{3} \end{bmatrix}, \begin{bmatrix} \frac{2}{21} & -1 \\ -\frac{1}{9} & 1 \end{bmatrix} \right\}_{[-1,1]}.$$

Note that $\det(\{\widehat{a}; \widehat{b_1}\})(\xi) = \frac{8}{63}(6 - 2\cos(2\xi)) \neq 0$ for all $\xi \in \mathbb{R}$. Define $\widehat{\psi}(\xi) := \widehat{b_1}(\xi/2)\widehat{\phi}(\xi/2)$. Then $\{\phi; \psi\}$ is a first-order derivative-orthogonal Riesz wavelet in $H^1(\mathbb{R})$ with

$$
\|\psi_1'\|_{L_2(\mathbb{R})} = \sqrt{\tfrac{4864}{735}}, \quad \|\psi_2'\|_{L_2(\mathbb{R})} = \tfrac{32\sqrt{15}}{105},
$$

$$
[\widehat{\phi'}, \widehat{\phi'}](\xi) = \begin{bmatrix} -\frac{6}{5}e^{i\xi}(e^{-i\xi}-1)^2 & \frac{1}{10}e^{i\xi}(e^{-2i\xi}-1) \\ -\frac{1}{10}e^{i\xi}(e^{-2i\xi}-1) & \frac{4}{15}-\frac{1}{30}e^{i\xi}-\frac{1}{30}e^{-i\xi} \end{bmatrix}. \quad (2.37)
$$

Note that $[\widehat{\phi'}, \widehat{\phi'}](\xi) = \sum_{k\in\mathbb{Z}}\langle \phi', \phi'(\cdot - k)\rangle e^{-ik\xi}$. Moreover, $\mathsf{AS}_0^{\tau}(\phi;\psi)$ is a Riesz basis for the Sobolev space $H^{\tau}(\mathbb{R})$ for all $\tau \in (-1/2, 5/2)$. This recovers the Riesz wavelet obtained in [41].

For $m = 2$, we have $b_2 = \{\mathrm{diag}(\frac{1}{2}, \frac{1}{2})\}_{[1,1]}$. Note that $\det(\{\widehat{a}; \widehat{b_2}\})(\xi) = \frac{1}{8}e^{-2i\xi} \neq 0$ for all $\xi \in \mathbb{R}$. Define $\widehat{\psi}(\xi) := \widehat{b_2}(\xi/2)\widehat{\phi}(\xi/2)$. Then $\{\phi;\psi\}$ is a second-order derivative-orthogonal Riesz wavelet for $H^2(\mathbb{R})$ with

$$
\|\psi_1''\|_{L_2(\mathbb{R})} = 8\sqrt{3}, \quad \|\psi_2''\|_{L_2(\mathbb{R})} = 8,
$$

$$
[\widehat{\phi''}, \widehat{\phi''}](\xi) = \begin{bmatrix} -12e^{i\xi}(e^{-i\xi}-1)^2 & 6e^{i\xi}(e^{-2i\xi}-1) \\ -6e^{i\xi}(e^{-2i\xi}-1) & 2e^{-i\xi}+2e^{i\xi}+8 \end{bmatrix}. \quad (2.38)
$$

Moreover, $\mathsf{AS}_0^{\tau}(\phi;\psi)$ is a Riesz basis for the Sobolev space $H^{\tau}(\mathbb{R})$ for all $\tau \in (3/2, 5/2)$. As we shall see in Chapter 4, our choice for the second-order derivative-orthogonal Riesz wavelet in this example is particularly interesting, since it has a condition number exactly equal to 1 after introducing a simple linear combination at the coarsest level. See Figure 2.4 to visualize the wavelet functions ψ associated with filters b_0, b_1, b_2, respectively.

Example 2.5. Let $\phi := (\phi_1, \phi_2)^{\mathsf{T}}$, where the Hermite quadratic splines ϕ_1, ϕ_2

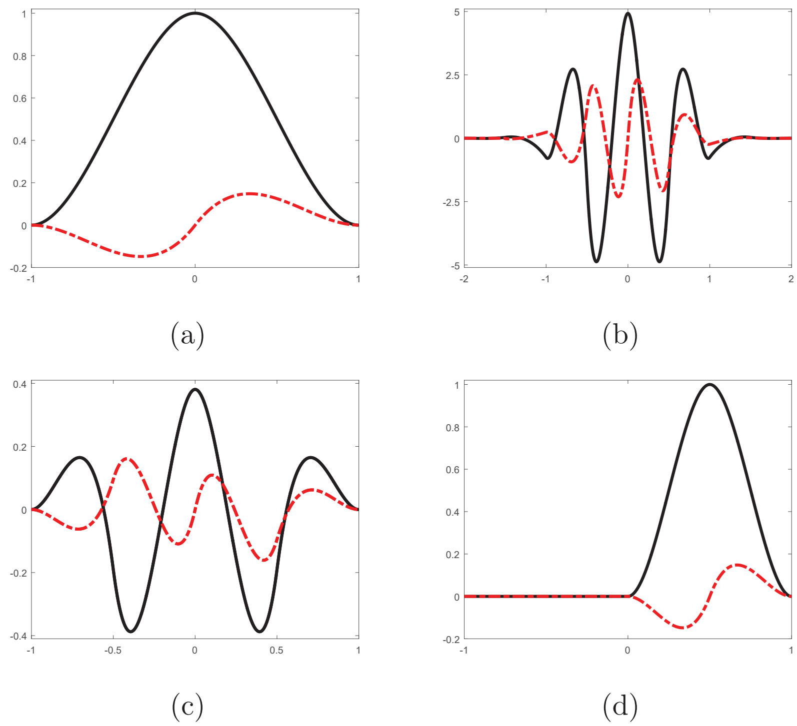


Figure 2.4: (a) is the refinable vector function $\phi := (\phi_1, \phi_2)^\mathsf{T}$ in Example 2.4. (b) is the wavelet vector function ψ associated with the filter b_0 such that $\{\phi; \psi\}$ is a semi-orthogonal Riesz wavelet in $L_2(\mathbb{R})$ with $\operatorname{supp}(\psi) = [-2, 2]$. (c) is the wavelet vector function ψ associated with the filter b_1 such that $\{\phi; \psi\}$ is a first-order derivative-orthogonal Riesz wavelet in $H^1(\mathbb{R})$ with $\operatorname{supp}(\psi) = [-1, 1]$. (d) is the wavelet vector function ψ associated with the filter b_2 such that $\{\phi; \psi\}$ is a second-order derivative-orthogonal Riesz wavelet in $H^2(\mathbb{R})$ with $\operatorname{supp}(\psi) = [0, 1]$. The solid line is for the first component and the dotted line is for the second component in a 2×1 vector function.

are given by:

$$
\begin{aligned}
\phi_1(x) &= 2(x-1)^2\chi_{[\frac{1}{2},1]} + (1-2x^2)\chi_{(-\frac{1}{2},\frac{1}{2})} + 2(1+x)^2\chi_{[-1,-\frac{1}{2}]},\\
\phi_2(x) &= \frac{1}{2}(x-1)^2\chi_{[\frac{1}{2},1]} + (x-\frac{3}{2}x^2)\chi_{[0,\frac{1}{2})} + (x+\frac{3}{2}x^2)\chi_{(-\frac{1}{2},0)}\\
&\quad - \frac{1}{2}(x+1)^2\chi_{[-1,\frac{1}{2}]}.
\end{aligned}
\tag{2.39}
$$

We can check that this particular refinable vector function ϕ satisfies the refinement equation $\widehat{\phi}(2\xi) = \widehat{a}(\xi)\widehat{\phi}(\xi)$ with the filter $a \in (l_0(\mathbb{Z}))^{2\times 2}$ given by

$$a = \left\{ \begin{bmatrix} \frac{1}{4} & \frac{1}{2} \\ -\frac{1}{16} & -\frac{1}{8} \end{bmatrix}, \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{4} \end{bmatrix}, \begin{bmatrix} \frac{1}{4} & -\frac{1}{2} \\ \frac{1}{16} & -\frac{1}{8} \end{bmatrix} \right\}_{[-1,1]}. \tag{2.40}$$

This filter a has only order 3 sum rules with $\mathrm{sr}(a) = 3$. In the following, we denote by b_0 and b_1 the wavelet filters we obtain by imposing the mth-order derivative-orthogonality condition on the original function ($m = 0$) and the first-order derivative ($m = 1$), respectively.

For $m = 0$, the filter b_0 is supported on $[-2, 2]$ and is given by

$$b_0 = \left\{ \begin{bmatrix} -\frac{7}{156} & -\frac{5}{26} \\ \frac{7}{450} & \frac{1}{15} \end{bmatrix}, \begin{bmatrix} -\frac{8}{39} & -\frac{35}{13} \\ \frac{23}{225} & \frac{16}{15} \end{bmatrix}, \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} -\frac{8}{39} & \frac{35}{13} \\ -\frac{23}{225} & \frac{16}{15} \end{bmatrix}, \begin{bmatrix} -\frac{7}{156} & \frac{5}{26} \\ -\frac{7}{450} & \frac{1}{15} \end{bmatrix} \right\}_{[-2,2]}.$$

Note that $\det(\{\widehat{a}; \widehat{b_0}\})(\xi) = -\frac{8}{8775}(3006 - 1088\cos(2\xi) + 2\cos(4\xi)) \neq 0$ for all $\xi \in \mathbb{R}$. Define $\widehat{\psi}(\xi) := \widehat{b_0}(\xi/2)\widehat{\phi}(\xi/2)$. Then $\{\phi; \psi\}$ is a semi-orthogonal Riesz wavelet in $L_2(\mathbb{R})$ with $\|\psi_1\|_{L_2(\mathbb{R})} = \sqrt{87712/1521}$,
$\|\psi_2\|_{L_2(\mathbb{R})} = \sqrt{2777696/151875}$ and

$$[\widehat{\phi}, \widehat{\phi}](\xi) = \begin{bmatrix} \frac{7}{60}e^{-i\xi} + \frac{7}{60}e^{i\xi} + \frac{23}{30} & -\frac{13}{480}e^{i\xi}(e^{-2\xi i} - 1) \\ \frac{13}{480}e^{i\xi}(e^{-2\xi i} - 1) & \frac{1}{48} - \frac{1}{160}e^{-i\xi} - \frac{1}{160}e^{i\xi} \end{bmatrix}. \tag{2.41}$$

Moreover, $\mathsf{AS}_0^\tau(\phi; \psi)$ is a Riesz basis for the Sobolev space $H^\tau(\mathbb{R})$ for all $\tau \in (-5/2, 5/2)$.

For $m = 1$, the filter b_1 is supported on $[-1, 1]$ and is given by

$$b_1 = \left\{ \begin{bmatrix} \frac{1}{6} & 1 \\ \frac{1}{6} & 1 \end{bmatrix}, \begin{bmatrix} -\frac{1}{3} & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} \frac{1}{6} & -1 \\ -\frac{1}{6} & 1 \end{bmatrix} \right\}_{[-1,1]}.$$

Note that $\det(\{\widehat{a}; \widehat{b_1}\})(\xi) = \frac{1}{9}(14 - 2\cos(2\xi)) \neq 0$ for all $\xi \in \mathbb{R}$. Define $\widehat{\psi}(\xi) := \widehat{b_1}(\xi/2)\widehat{\phi}(\xi/2)$. Then $\{\phi; \psi\}$ is a first-order derivative-orthogonal Riesz wavelet in $H^1(\mathbb{R})$ with

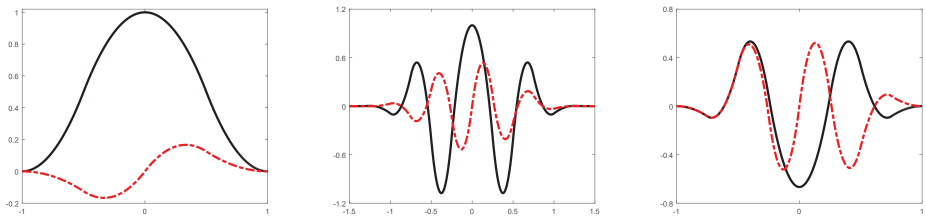
$$\begin{aligned} &\|\psi_1'\|_{L_2(\mathbb{R})} = \sqrt{\tfrac{352}{27}}, \quad \|\psi_2'\|_{L_2(\mathbb{R})} = \sqrt{\tfrac{512}{27}}, \\ &\qquad [\widehat{\phi'}, \widehat{\phi'}](\xi) = \begin{bmatrix} -\frac{4}{3}e^{i\xi}(e^{-i\xi} - 1)^2 & \frac{1}{6}e^{i\xi}(e^{-2i\xi} - 1) \\ -\frac{1}{6}e^{i\xi}(e^{-2i\xi} - 1) & \frac{1}{3} \end{bmatrix}. \end{aligned} \tag{2.42}$$

Moreover, $\mathsf{AS}_0^\tau(\phi; \psi)$ is a Riesz basis for the Sobolev space $H^\tau(\mathbb{R})$ for all $\tau \in (-1/2, 5/2)$. See Figure 2.5 to visualize the wavelet functions ψ associated with filters b_0, b_1, respectively. Though $\phi \in (H^2(\mathbb{R}))^2$, since $\mathrm{sr}(a) = 3$, according to Theorem 2.4, one cannot obtain a compactly supported second-order derivative-orthogonal Riesz wavelet in $H^2(\mathbb{R})$ from ϕ.

Example 2.6. Let $\phi := (\phi_1, \phi_2)^\mathsf{T}$, where the Hermite linear splines ϕ_1, ϕ_2 are given by:

$$\begin{aligned} \phi_1(x) &= (2 - 3x)\chi_{[\frac{1}{3}, \frac{2}{3}]} + \chi_{(-\frac{1}{3}, \frac{1}{3})} + (2 + 3x)\chi_{[-\frac{2}{3}, -\frac{1}{3}]}, \\ \phi_2(x) &= (\frac{2}{3} - x)\chi_{[\frac{1}{3}, \frac{2}{3}]} + x\chi_{(-\frac{1}{3}, \frac{1}{3})} - (\frac{2}{3} + x)\chi_{[-\frac{2}{3}, -\frac{1}{3}]}. \end{aligned} \tag{2.43}$$

Note that $\mathrm{sm}(\phi) = 3/2$. We can check that this particular refinable vector



(a) (b) (c)

Figure 2.5: (a) is the refinable vector function $\phi := (\phi_1, \phi_2)^\mathsf{T}$ in Example 2.5. (b) is the wavelet vector function ψ associated with the filter b_0 such that $\{\phi; \psi\}$ is a semi-orthogonal Riesz wavelet in $L_2(\mathbb{R})$ with $\text{supp}(\psi) = [-3/2, 3/2]$. (c) is the wavelet vector function ψ associated with the filter b_1 such that $\{\phi; \psi\}$ is a first-order derivative-orthogonal Riesz wavelet in $H^1(\mathbb{R})$ with $\text{supp}(\psi) = [-1, 1]$. The solid line is for the first component and the dotted line is for the second component in a 2×1 vector function.

function ϕ satisfies $\widehat{\phi}(2\xi) = \widehat{a}(\xi)\widehat{\phi}(\xi)$ with the filter $a \in (l_0(\mathbb{Z}))^{2\times 2}$ given by

$$a = \left\{ \begin{bmatrix} \frac{1}{4} & \frac{3}{4} \\ -\frac{1}{12} & -\frac{1}{4} \end{bmatrix}, \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{4} \end{bmatrix}, \begin{bmatrix} \frac{1}{4} & -\frac{3}{4} \\ \frac{1}{12} & -\frac{1}{4} \end{bmatrix} \right\}_{[-1,1]}. \tag{2.44}$$

This filter has only order 2 sum rules with $\text{sr}(a) = 2$. In the following, let b_0 and b_1 be the wavelet filters we obtain by imposing mth-order derivative-orthogonal condition on the original function ($m = 0$) and the first derivative ($m = 1$), respectively.

For $m = 0$, the filter b_0 is supported on $[-2, 2]$ and is given by

$$b_0 = \left\{ \begin{bmatrix} -\frac{113}{1411} & -\frac{1}{2} \\ \frac{59}{459} & 1 \end{bmatrix}, \begin{bmatrix} -\frac{1049}{1411} & -\frac{405}{83} \\ \frac{7}{34} & \frac{5}{2} \end{bmatrix}, \begin{bmatrix} \frac{28}{17} & 0 \\ 0 & \frac{11}{2} \end{bmatrix}, \begin{bmatrix} -\frac{1049}{1411} & \frac{405}{83} \\ -\frac{7}{34} & \frac{5}{2} \end{bmatrix}, \begin{bmatrix} -\frac{113}{1411} & \frac{1}{2} \\ -\frac{59}{459} & 1 \end{bmatrix} \right\}_{[-2,2]}.$$

Note that $\det(\{\widehat{a};\widehat{b_0}\})(\xi) = -\frac{20}{38097}(49132+8478\cos(2\xi)-964\cos(4\xi) \neq 0$ for all $\xi \in \mathbb{R}$. Define $\widehat{\psi}(\xi) := \widehat{b_0}(\xi/2)\widehat{\phi}(\xi/2)$. Then $\{\phi;\psi\}$ is a semi-orthogonal Riesz wavelet in $L_2(\mathbb{R})$ with $\|\psi_1\|_{L_2(\mathbb{R})} = \sqrt{1569359008/161264601}$, $\|\psi_2\|_{L_2(\mathbb{R})} = \sqrt{6927776/1896129}$ and

$$[\widehat{\phi},\widehat{\phi}](\xi) = \begin{bmatrix} \frac{1}{18}e^{-i\xi} + \frac{1}{18}e^{i\xi} + \frac{8}{9} & -\frac{1}{54}e^{i\xi}(e^{-2\xi i}-1) \\ \frac{1}{54}e^{i\xi}(e^{-2\xi i}-1) & \frac{4}{81} - \frac{1}{162}e^{i\xi} - \frac{1}{162}e^{-i\xi} \end{bmatrix}.$$

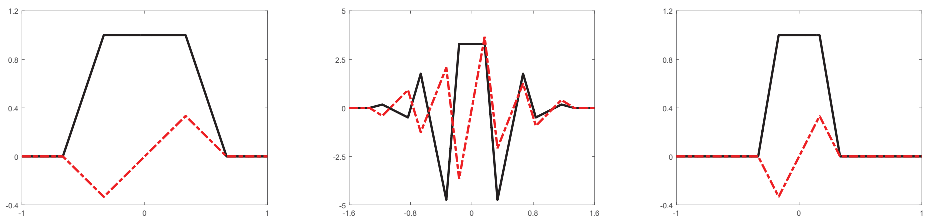
Moreover, $\mathsf{AS}_0^{\tau}(\phi;\psi)$ is a Riesz basis for the Sobolev space $H^{\tau}(\mathbb{R})$ for all $\tau \in (-3/2, 3/2)$.

For $m = 1$, the filter b_2 is given by $b_2 = \{\operatorname{diag}(1/6, \sqrt{2}/4)\}_{[0,0]}$. Note that $\det(\{\widehat{a};\widehat{b_1}\})(\xi) = -\frac{\sqrt{2}}{24} \neq 0$ for all $\xi \in \mathbb{R}$. Define $\widehat{\psi}(\xi) := \widehat{b_1}(\xi/2)\widehat{\phi}(\xi/2)$. Then $\{\phi;\psi\}$ is a first-order derivative-orthogonal Riesz wavelet in $H^1(\mathbb{R})$ with

$$\|\psi_1'\|_{L_2(\mathbb{R})} = \|\psi_2'\|_{L_2(\mathbb{R})} = \frac{2\sqrt{3}}{3},$$
$$[\widehat{\phi'},\widehat{\phi'}](\xi) = \begin{bmatrix} -3e^{i\xi}(e^{-i\xi}-1)^2 & e^{-i\xi} - e^{i\xi} \\ e^{i\xi} - e^{-i\xi} & \frac{1}{3}e^{-i\xi} + \frac{1}{3}e^{i\xi} + \frac{4}{3} \end{bmatrix}. \quad (2.45)$$

Moreover, $\mathsf{AS}_0^{\tau}(\phi;\psi)$ is a Riesz basis for the Sobolev space $H^{\tau}(\mathbb{R})$ for all $\tau \in (1/2, 3/2)$. As we shall also see in the next section of this paper, our choice for the first-order derivative-orthogonal wavelet in this example is particularly interesting, since it has a condition number exactly equal to 1 after introducing a simple linear combination at the coarsest level. See Figure 2.6 to visualize the wavelet vector functions ψ associated with filters b_0, b_1, respectively.

We conclude this section with two remarks. We would like to point out



(a) (b) (c)

Figure 2.6: (a) is the refinable vector function $\phi := (\phi_1, \phi_2)^\mathsf{T}$ in Example 2.6. (b) is the wavelet vector function ψ associated with the filter b_0 such that $\{\phi; \psi\}$ is a semi-orthogonal Riesz wavelet in $L_2(\mathbb{R})$ with $\operatorname{supp}(\psi) = [-4/3, 4/3]$. (c) is the wavelet vector function ψ associated with the filter b_1 such that $\{\phi; \psi\}$ is a first-order derivative-orthogonal Riesz wavelet in $H^1(\mathbb{R})$ with $\operatorname{supp}(\psi) = [-1/3, 1/3]$. The solid line is for the first component and the dotted line is for the second component.

that all of the above examples possess a symmetry (not necessarily about the y-axis). As we shall see in Chapter 3, symmetry is of paramount importance in showing that a Riesz wavelet on $L_2(\mathbb{R})$ indeed leads to a Riesz wavelet on $L_2(\mathcal{I})$. For the purpose of numerical computations, we can scale differently the components in our wavelet vector functions from any of the examples above accordingly to achieve a better condition number hence a more well-behaved linear system arising from the Galerkin formulation. We shall elaborate more on this subject in Chapter 4.

Chapter 3

Riesz Wavelets on a Bounded Interval

In this chapter, we introduce a generalized version of the folding operator used in [33]. The general idea of the argument follows [33] with some modifications. We aim to show that a Riesz wavelet on $L_2(\mathbb{R})$ can be adapted in such a way that it becomes a Riesz wavelet on $L_2(\mathcal{I})$, where $\mathcal{I}$ is a bounded interval on $\mathbb{R}$. The use of the folding operator to prove that a Riesz wavelet on $\mathbb{R}$ is in fact a Riesz wavelet on an interval was first introduced in [13]. The difference between the argument that we are about to lay out and [33] lies on the fact that instead of having just a single type of symmetry/anti-symmetry applied to both left and right endpoints of the intervals, we have one for each endpoint. E.g. we may have symmetry at the left endpoint, and anti-symmetry at the right endpoint of the bounded interval.

Definition 3.1. *Suppose $f \in L_2(\mathbb{R})$. Let f either be a compactly supported function or a function that has a nice decay. A two-sided folding operator*

$F_{c,\epsilon_1,\epsilon_2}$ *is defined as*

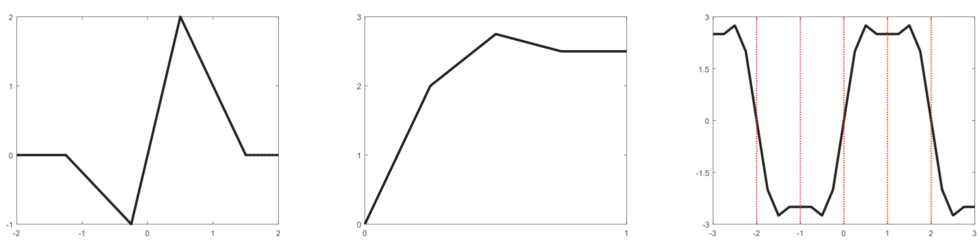
$$F_{c,\epsilon_1,\epsilon_2}(f) = \sum_{k\in\mathbb{Z}} f(\cdot - 4k) + \epsilon_1 f(c - \cdot - 4k) + \epsilon_2 f(c + 2 - \cdot - 4k) + \epsilon_1\epsilon_2 f(\cdot + 2 - 4k), \tag{3.1}$$

where $\epsilon_1, \epsilon_2 \in \{-1, 1\}$, *and* $c \in \mathbb{R}$.

Example 3.1. We present a simple example to illustrate how the folding operator works. Consider the following piecewise linear function:

$$f(x) = (-x - \frac{5}{4})\chi_{[-\frac{5}{4},-\frac{1}{4})} + (3 - 2x)\,\chi_{[\frac{1}{2},\frac{3}{2}]} + (4x)\,\chi_{[-\frac{1}{4},\frac{1}{2})}. \tag{3.2}$$

See Figure 3.1 to see how the folding operator works visually.



(a) (b) (c)

Figure 3.1: (a) shows the original function f in (3.2). (b) shows the function we obtain from symmetrically folding $f|_{[1,2]}$ and anti-symmetrically folding $f|_{[-\frac{5}{4},0]}$ onto the interval $[0, 1]$, and summing up each folded piece. (c) shows the resulting function after we apply the folding operator with $\epsilon_1 = -1$, $\epsilon_2 = 1$, and $c = 0$. The red dotted lines correspond to axes of symmetry and anti-symmetry.

Right from the definition of the folding operator itself, we can deduce that $F_{c,\epsilon_1,\epsilon_2}(f)$ is a 4 periodic function. Next, we have 2 propositions, which cater to the calculations in the subsequent proofs.

Proposition 3.1. *Let* $f_{2^j;k} := 2^{j/2} f(2^j \cdot -k)$, $f(c_f - \cdot) = \epsilon_f f$, *and* $f_{2^j;m} :=$

$\epsilon_f f_{2^j;-m-c_f}(-\cdot)$. *The following identities hold:*

$$F_{c,\epsilon_1,\epsilon_2}(f_{2^j;m}) = \sum_{k\in\mathbb{Z}} f_{2^j;m+2^{j+1}(2k)} + \epsilon_1 f_{2^j;m-2^jc+2^{j+1}(2k)}(-\cdot)$$

$$+ \epsilon_2 f_{2^j;m-2^jc+2^{j+1}(2k-1)}(-\cdot) + \epsilon_1\epsilon_2 f_{2^j;m+2^{j+1}(2k-1)}, \quad (3.3)$$

$$F_{c,\epsilon_1,\epsilon_2}(f_{2^j;m}) = \epsilon_1\epsilon_f F_{c,\epsilon_1,\epsilon_2}(f_{2^j;2^jc-c_f-m}), \quad (3.4)$$

$$F_{c,\epsilon_1,\epsilon_2}(f_{2^j;m}) = \epsilon_2\epsilon_f F_{c,\epsilon_1,\epsilon_2}(f_{2^j;2^jc-c_f-m+2^{j+1}}), \quad (3.5)$$

$$F_{c,\epsilon_1,\epsilon_2}(f_{2^j;m}) = \epsilon_1\epsilon_2 F_{c,\epsilon_1,\epsilon_2}(f_{2^j;m-2^{j+1}}). \quad (3.6)$$

Proof. The proof relies entirely on definitions and direct calculation. Subsituting $f_{2^j;k} = 2^{j/2}f(2^j\cdot -k)$ and $f_{2^j;m} = \epsilon_f f_{2^j;-m-c_f}(-\cdot)$ to (3.1), we automatically get (3.3). Next, we know that

$$\begin{aligned}
\epsilon_f F_{c,\epsilon_1,\epsilon_2}(f_{2^j;2^jc-c_f-m}) =&\epsilon_f \sum_{k\in\mathbb{Z}} f_{2^j;2^jc-c_f-m+2^{j+1}(2k)} + \epsilon_1 f_{2^j;-c_f-m+2^{j+1}(2k)}(-\cdot) \\
&+ \epsilon_2 f_{2^j;-c_f-m+2^{j+1}(2k-1)}(-\cdot) + \epsilon_1\epsilon_2 f_{2^j;2^jc-c_f-m+2^{j+1}(2k-1)} \\
=&\sum_{k\in\mathbb{Z}} f_{2^j;m-2^jc-2^{j+1}(2k)}(-\cdot) + \epsilon_1 f_{2^j;m-2^{j+1}(2k)} \\
&+ \epsilon_2 f_{2^j;m-2^{j+1}(2k-1)} + \epsilon_1\epsilon_2 f_{2^j;m-2^jc-2^{j+1}(2k-1)}(-\cdot) \\
=&\epsilon_1 F_{c,\epsilon_1,\epsilon_2}(f_{2^j;m}),
\end{aligned} \quad (3.7)$$

from which we get (3.4). Identity (3.5) is obtained by firstly substituting m for $2^jc - c_f - m + 2^{j+1}$ in (3.3) and following a similar calculation as in (3.7). Identity (3.6) is obtained from combining (3.4) and (3.5), and from observing that $F_{c,\epsilon_1,\epsilon_2}(f_{2^j;m}) = F_{c,\epsilon_1,\epsilon_2}(f_{2^j;m+2^{j+2}k})$ for all $k, m \in \mathbb{Z}$. $\square$

The following function space was first introduced by [42] and is also used in [33].

Definition 3.2. *Let $1 \leqslant p \leqslant \infty$. Define $\mathcal{L}_p$ to be the linear space of all measurable functions f on $\mathbb{R}$ such that if $f \in \mathcal{L}_p$, then*

$$\|f\|_{\mathcal{L}_p(\mathbb{R})} := \left\| \sum_{k\in\mathbb{Z}} |f(\cdot - k)| \right\|_{L_p([0,1])} < \infty.$$

Note that the $\mathcal{L}_p$ space contains L_p functions, which have either compact supports, or fast enough decay. The decay here can be exponential or even polynomial among other things. We also know for a fact that the nestedness property holds: $\mathcal{L}_p \subseteq L_p$ for all $1 \leqslant p \leqslant \infty$.

Proposition 3.2. *Suppose $f, g \in \mathcal{L}_2$. Then, the following equalities hold*

$$\langle f, F_{c,\epsilon_1,\epsilon_2}(g)\rangle = \langle F_{c,\epsilon_1,\epsilon_2}(f), g\rangle = \langle F_{c,\epsilon_1,\epsilon_2}(f), F_{c,\epsilon_1,\epsilon_2}(g)\rangle_{L_2([\frac{c}{2},\frac{c}{2}+1])}, \tag{3.8}$$

where $c \in \mathbb{R}$. The first equality means that the folding operator $F_{c,\epsilon_1,\epsilon_2}$ is self-adjoint.

Proof. We use (3.1) to show the self-adjoint property. Now, let f, g, then

$$\begin{aligned}\langle f, F_{c,\epsilon_1,\epsilon_2}(g)\rangle &= \int_{\mathbb{R}} f(x) \sum_{k\in\mathbb{Z}} \overline{g(x-4k)}dx + \epsilon_1 \int_{\mathbb{R}} f(x) \sum_{k\in\mathbb{Z}} \overline{g(c-x-4k)}dx \\ &\quad + \epsilon_2 \int_{\mathbb{R}} f(x) \sum_{k\in\mathbb{Z}} \overline{g(c+2-x-4k)}dx \\ &\quad + \epsilon_1\epsilon_2 \int_{\mathbb{R}} f(x) \sum_{k\in\mathbb{Z}} \overline{g(x+2-4k)}dx.\end{aligned}$$

We shall interchange the order of summation and integral numerous times. Firstly, we justify why this operation can be done. Now, we know for a fact that

$$\begin{aligned}
\int_{\frac{c}{2}+\ell}^{\frac{c}{2}+1+\ell} |F_{c,\epsilon_1,\epsilon_2}(g)|^2\, dx =& \int_{\frac{c}{2}+\ell}^{\frac{c}{2}+1+\ell} |F_{c,\epsilon_1,\epsilon_2}(g)|^2 dx \\
\leqslant& 16\int_{\frac{c}{2}+\ell}^{\frac{c}{2}+1+\ell} \left|\sum_{k\in\mathbb{Z}} g(x-4k)\right|^2 + \left|\sum_{k\in\mathbb{Z}} g(c-x-4k)\right|^2 \\
&+ \left|\sum_{k\in\mathbb{Z}} g(c+2-x-4k)\right|^2 + \left|\sum_{k\in\mathbb{Z}} g(x+2-4k)\right|^2 dx \\
\leqslant& 16\int_0^1 \left|\sum_{k\in\mathbb{Z}} g(y+\frac{c}{2}+\ell-4k)\right|^2 + \left|\sum_{k\in\mathbb{Z}} g(\frac{c}{2}-y-\ell-4k)\right|^2 \\
&+ \left|\sum_{k\in\mathbb{Z}} g(\frac{c}{2}+2-y-\ell-4k)\right|^2 + \left|\sum_{k\in\mathbb{Z}} g(y+\frac{c}{2}+\ell+2-4k)\right|^2 dy \\
\leqslant& 64\|g\|_{\mathcal{L}_2}^2
\end{aligned}$$

for all $\ell \in \mathbb{Z}$. Consequently,

$$\begin{aligned}
\langle |f|, |F_{c,\epsilon_1,\epsilon_2(g)}|\rangle =& \int_{\mathbb{R}} \left| f\overline{F_{c,\epsilon_1,\epsilon_2}(g)}\right| dx = \sum_{\ell\in\mathbb{Z}} \int_{\frac{c}{2}+\ell}^{\frac{c}{2}+1+\ell} \left| f\overline{F_{c,\epsilon_1,\epsilon_2}(g)}\right| dx \\
\leqslant& \sum_{\ell\in\mathbb{Z}} \left(\int_{\frac{c}{2}+\ell}^{\frac{c}{2}+1+\ell} |f|^2 dx\right)^{1/2} \left(\int_{\frac{c}{2}+\ell}^{\frac{c}{2}+1+\ell} |F_{c,\epsilon_1,\epsilon_2}(g)|^2 dx\right)^{1/2} \\
\leqslant& 8\|g\|_{\mathcal{L}_2} \sum_{\ell\in\mathbb{Z}} \left(\int_{\frac{c}{2}+\ell}^{\frac{c}{2}+1+\ell} |f|^2 dx\right)^{1/2} \\
\leqslant& C_1\|f\|_{L_2} < \infty,
\end{aligned}$$

which means that $f\overline{F_{c,\epsilon_1,\epsilon_2}(g)} \in L_1$. We can change the order of summation and the integral by appealing to Fubini's Theorem. Keeping in mind that

$f, g \in \mathcal{L}_2$ (thus, f, g should have some nice decay property in $L_2(\mathbb{R})$) and using a change of variables, we have

$$\begin{aligned}
\epsilon_1 \int_{\mathbb{R}} f(x) \sum_{k\in\mathbb{Z}} \overline{g(c-x+4k)} dx &= \epsilon_1 \int_{\mathbb{R}} \sum_{k\in\mathbb{Z}} f(x) \overline{g(c-x+4k)} dx \\
&= \epsilon_1 \int_{\mathbb{R}} \sum_{k\in\mathbb{Z}} f(c-x+4k) \overline{g(x)} dx \\
&= \epsilon_1 \int_{\mathbb{R}} \left(\sum_{k\in\mathbb{Z}} f(c-x+4k) \right) \overline{g(x)} dx.
\end{aligned} \tag{3.9}$$

The other three terms can be found by doing a similar calculation.

$$\int_{\mathbb{R}} f(x) \sum_{k\in\mathbb{Z}} \overline{g(x-4k)} dx = \int_{\mathbb{R}} \left(\sum_{k\in\mathbb{Z}} f(x-4k) \right) \overline{g(x)} dx, \tag{3.10}$$

$$\epsilon_2 \int_{\mathbb{R}} f(x) \sum_{k\in\mathbb{Z}} \overline{g(c+2-x-4k)} dx = \epsilon_2 \int_{\mathbb{R}} \left(\sum_{k\in\mathbb{Z}} f(c+2-x-4k) \right) \overline{g(x)} dx, \tag{3.11}$$

$$\epsilon_1\epsilon_2 \int_{\mathbb{R}} f(x) \sum_{k\in\mathbb{Z}} \overline{g(x+2-4k)} dx = \epsilon_1\epsilon_2 \int_{\mathbb{R}} \left(\sum_{k\in\mathbb{Z}} f(x+2-4k) \right) \overline{g(x)} dx. \tag{3.12}$$

Summing up equations (3.9) to (3.12), we have (3.8). We basically use brute force to prove the second equality. In fact, we apply change of variables numerous times. The fact that f, g have nice decay property (or even possess compact supports) gives us an abundant of freedom to switch the order of the integral and the two summations.

$$
\begin{aligned}
&\langle F_{c,\epsilon_1,\epsilon_2}(f), F_{c,\epsilon_1,\epsilon_2}(g)\rangle_{L_2([\frac{c}{2},\frac{c}{2}+1])}\\
=&\Bigg(\int_{\frac{c}{2}}^{\frac{c}{2}+1}\sum_{k,l\in\mathbb{Z}} f(x-4k)\overline{g(x-4l)}dx\\
&+\int_{\frac{c}{2}}^{\frac{c}{2}+1}\sum_{k,l\in\mathbb{Z}} f(c-x-4k)\overline{g(c-x-4l)}dx+\int_{\frac{c}{2}}^{\frac{c}{2}+1}\sum_{k,l\in\mathbb{Z}} f(c+2-x-4k)\overline{g(c+2-x-4l)}dx\\
&+\int_{\frac{c}{2}}^{\frac{c}{2}+1}\sum_{k,l\in\mathbb{Z}} f(x+2-4k)\overline{g(x+2-4l)}dx\Bigg)+\epsilon_1\Bigg(\int_{\frac{c}{2}}^{\frac{c}{2}+1}\sum_{k,l\in\mathbb{Z}} f(x-4k)\overline{g(c-x-4l)}dx\\
&+\int_{\frac{c}{2}}^{\frac{c}{2}+1}\sum_{k,l\in\mathbb{Z}} f(c-x-4k)\overline{g(x-4l)}dx+\int_{\frac{c}{2}}^{\frac{c}{2}+1}\sum_{k,l\in\mathbb{Z}} f(c+2-x-4k)\overline{g(x+2-4l)}dx\\
&+\int_{\frac{c}{2}}^{\frac{c}{2}+1}\sum_{k,l\in\mathbb{Z}} f(x+2-4k)\overline{g(c+2-x-4l)}dx\Bigg)+\epsilon_2\Bigg(\int_{\frac{c}{2}}^{\frac{c}{2}+1}\sum_{k,l\in\mathbb{Z}} f(x-4k)\overline{g(c+2-x-4l)}dx\\
&+\int_{\frac{c}{2}}^{\frac{c}{2}+1}\sum_{k,l\in\mathbb{Z}} f(c-x-4k)\overline{g(x+2-4l)}dx+\int_{\frac{c}{2}}^{\frac{c}{2}+1}\sum_{k,l\in\mathbb{Z}} f(c+2-x-4k)\overline{g(x-4l)}dx\\
&+\int_{\frac{c}{2}}^{\frac{c}{2}+1}\sum_{k,l\in\mathbb{Z}} f(x+2-4k)\overline{g(c-x-4l)}dx\Bigg)+\epsilon_1\epsilon_2\Bigg(\int_{\frac{c}{2}}^{\frac{c}{2}+1}\sum_{k,l\in\mathbb{Z}} f(x-4k)\overline{g(x+2-4l)}dx\\
&+\int_{\frac{c}{2}}^{\frac{c}{2}+1}\sum_{k,l\in\mathbb{Z}} f(c-x-4k)\overline{g(c+2-x-4l)}dx+\int_{\frac{c}{2}}^{\frac{c}{2}+1}\sum_{k,l\in\mathbb{Z}} f(c+2-x-4k)\overline{g(c-x-4l)}dx\\
&+\int_{\frac{c}{2}}^{\frac{c}{2}+1}\sum_{k,l\in\mathbb{Z}} f(x+2-4k)\overline{g(x-4l)}dx\Bigg)\\
=&\sum_{k,l\in\mathbb{Z}}\int_{\frac{c}{2}-4k-1}^{\frac{c}{2}-4k+3} f(y)\overline{g(y+4(k-l))}dy+\epsilon_1\sum_{k,l\in\mathbb{Z}}\int_{\frac{c}{2}-4k-1}^{\frac{c}{2}-4k+3} f(y)\overline{g(c-y-4(k+l))}dy\\
&+\epsilon_2\sum_{k,l\in\mathbb{Z}}\int_{\frac{c}{2}-4k-1}^{\frac{c}{2}-4k+3} f(y)\overline{g(c+2-y-4(k+l))}dy+\epsilon_1\epsilon_2\sum_{k,l\in\mathbb{Z}}\int_{\frac{c}{2}-4k-1}^{\frac{c}{2}-4k+3} f(y)\overline{g(y+2+4(k-l))}dy\\
=&\sum_{k,l'\in\mathbb{Z}}\int_{\frac{c}{2}-4k-1}^{\frac{c}{2}-4k+3} f(y)\overline{g(y-4l')}dy+\epsilon_1\sum_{k,l'\in\mathbb{Z}}\int_{\frac{c}{2}-4k-1}^{\frac{c}{2}-4k+3} f(y)\overline{g(c-y-4l')}dy\\
&+\epsilon_2\sum_{k,l'\in\mathbb{Z}}\int_{\frac{c}{2}-4k-1}^{\frac{c}{2}-4k+3} f(y)\overline{g(c+2-y-4l')}dy+\epsilon_1\epsilon_2\sum_{k,l'\in\mathbb{Z}}\int_{\frac{c}{2}-4k-1}^{\frac{c}{2}-4k+3} f(y)\overline{g(y+2-4l')}dy\\
=&\langle f, F_{c,\epsilon_1,\epsilon_2}(g)\rangle.
\end{aligned}
$$

This completes the proof of (3.8). □

We arrive at an important theorem, which basically says if we have a Riesz wavelet in $L_2(\mathbb{R})$, it still remains a Riesz wavelet for $L_2(\mathcal{I})$.

Theorem 3.1. *Let* $\Phi = \{\phi^1, \dots, \phi^r\}$, $\Psi = \{\psi^1, \dots, \psi^s\}$, $\tilde{\Phi} = \{\tilde{\phi}^1, \dots, \tilde{\phi}^r\}$, *and* $\tilde{\Psi} = \{\tilde{\psi}^1, \dots, \tilde{\psi}^s\}$ *be finite subsets of compactly supported functions in* $L_2(\mathbb{R})$. *Suppose that for* $\ell = 1, \dots, r$,

$$\phi^\ell(c_\ell^\phi - \cdot) = \epsilon_\ell^\phi \phi^\ell, \quad \tilde{\phi}^\ell(c_\ell^\phi - \cdot) = \epsilon_\ell^\phi \tilde{\phi}^\ell, \quad \textit{with} \quad c_\ell^\phi \in \mathbb{Z}, \epsilon_\ell^\phi \in \{-1, 1\}, \tag{3.13}$$

and for $\ell = 1, \dots, s$,

$$\psi^\ell(c_\ell^\psi - \cdot) = \epsilon_\ell^\psi \psi^\ell, \quad \tilde{\psi}^\ell(c_\ell^\psi - \cdot) = \epsilon_\ell^\psi \tilde{\psi}^\ell, \quad \textit{with} \quad c_\ell^\psi \in \mathbb{Z}, \epsilon_\ell^\psi \in \{-1, 1\}. \tag{3.14}$$

Let $\epsilon_1, \epsilon_2 \in \{-1, 1\}$ *and* $c \in \mathbb{Z}$. *Define* $\mathcal{I} := [\frac{c}{2}, \frac{c+1}{2}]$. *For* $j \in \mathbb{N}_0$, *define*

$$\begin{gathered} d_{j,\ell}^\phi := \left\lfloor 2^{j-1}c - 2^{-1}c_\ell^\phi \right\rfloor, \quad d_{j,\ell}^\psi := \left\lfloor 2^{j-1}c - 2^{-1}c_\ell^\psi \right\rfloor \\ o_{j,\ell}^\phi := \mathit{odd}(2^j c - c_\ell^\phi), \quad o_{j,\ell}^\psi := \mathit{odd}(2^j c - c_\ell^\psi), \end{gathered} \tag{3.15}$$

where $\mathit{odd}(m) := 1$ *if* m *is an odd integer and* $\mathit{odd}(m) := 0$ *if* m *is an even integer. Let* $\chi_{\mathcal{I}}$ *denote the characteristic function of the set* $\mathcal{I}$. *For* $j \in \mathbb{N}_0$, *define*

$$
\Psi_j^\ell := \begin{cases}
\{F_{c,\epsilon_1,\epsilon_2}(\psi^\ell_{2^j;k})\chi_I : k = d^\psi_{j,\ell}+1,\ldots,d^\psi_{j,\ell}+2^j\}, \quad \textit{if} \quad o^\psi_{j,\ell} = 1,\\
\{F_{c,\epsilon_1,\epsilon_2}(\psi^\ell_{2^j;k})\chi_I : k = d^\psi_{j,\ell}+1,\ldots,d^\psi_{j,\ell}+2^j-1\}, \quad \textit{if} \quad o^\psi_{j,\ell} = 0, \quad \epsilon_1=\epsilon_2\ne\epsilon^\psi_\ell,\\
\{F_{c,\epsilon_1,\epsilon_2}(\psi^\ell_{2^j;k})\chi_I : k = d^\psi_{j,\ell}+1,\ldots,d^\psi_{j,\ell}+2^j-1\} \cup \{\frac{1}{\sqrt{2}}F_{c,\epsilon_1,\epsilon_2}(\psi^\ell_{2^j;d^\psi_{j,\ell}})\chi_I\}\\
\cup\{\frac{1}{\sqrt{2}}F_{c,\epsilon_1,\epsilon_2}(\psi^\ell_{2^j;d^\psi_{j,\ell}+2^j})\chi_I\} \quad \textit{if} \quad o^\psi_{j,\ell} = 0, \quad \epsilon_1=\epsilon_2=\epsilon^\psi_\ell,\\
\{F_{c,\epsilon_1,\epsilon_2}(\psi^\ell_{2^j;k})\chi_I : k = d^\psi_{j,\ell}+1,\ldots,d^\psi_{j,\ell}+2^j-1\} \cup \{\frac{1}{\sqrt{2}}F_{c,\epsilon_1,\epsilon_2}(\psi^\ell_{2^j;d^\psi_{j,\ell}})\chi_I\},\\
\textit{if} \quad o^\psi_{j,\ell} = 0, \quad \epsilon_1=\epsilon^\psi_\ell\ne\epsilon_2,\\
\{F_{c,\epsilon_1,\epsilon_2}(\psi^\ell_{2^j;k})\chi_I : k = d^\psi_{j,\ell}+1,\ldots,d^\psi_{j,\ell}+2^j-1\} \cup \{\frac{1}{\sqrt{2}}F_{c,\epsilon_1,\epsilon_2}(\psi^\ell_{2^j;d^\psi_{j,\ell}+2^j})\chi_I\},\\
\textit{if} \quad o^\psi_{j,\ell} = 0, \quad \epsilon_2=\epsilon^\psi_\ell\ne\epsilon_1,
\end{cases} \tag{3.16}
$$

and define $\tilde{\Psi}^\ell_j$, Φ^ℓ_j, $\tilde{\Phi}^\ell_j$ *similarly. For* $J \in \mathbb{N}_0$, *define*

$$
\mathcal{B}_J := (\cup_{\ell=1}^r \Phi^\ell_J) \cup \cup_{j=J}^\infty (\cup_{\ell=1}^s \Psi^\ell_j), \quad \tilde{\mathcal{B}}_J := (\cup_{\ell=1}^r \tilde{\Phi}^\ell_J) \cup \cup_{j=J}^\infty (\cup_{\ell=1}^s \tilde{\Psi}^\ell_j). \tag{3.17}
$$

Then for every $J \in \mathbb{N}_0$, *the following statements hold:*

1. *If* $(\{\tilde{\Phi};\tilde{\Psi}\},\{\Phi;\Psi\})$ *is a biorthogonal wavelet in* $L_2(\mathbb{R})$, *then* $(\tilde{\mathcal{B}}_J,\mathcal{B}_J)$ *is a biorthogonal basis for* $L_2(\mathcal{I})$. *In particular,* $\mathcal{B}_J$ *and* $\tilde{\mathcal{B}}_J$ *are Riesz bases for* $L_2(\mathcal{I})$.

2. *If* $\{\Phi;\Psi\}$ *is an orthogonal wavelet in* $L_2(\mathbb{R})$, *then* $\mathcal{B}_J$ *is an orthonormal basis for* $L_2(\mathcal{I})$.

Proof. Let $f \in L_2(\mathcal{I})$. Since we assume that $\Phi,\Psi,\tilde{\Phi},\tilde{\Psi}$ are finite subsets of compactly supported functions in $L_2(\mathbb{R})$, we know that $\Phi,\Psi,\tilde{\Phi},\tilde{\Psi} \in \mathcal{L}_2$. Furthermore, following the argument as in the proof of [33, Theorem 7.5.3], we conclude that there exists $N \in \mathbb{N}$ such that all elements in $\Phi,\Psi,\tilde{\Phi},\tilde{\Psi}$ vanish outside $[-N,N]$, $\mathcal{I} \subseteq [-N,N]$, $|c| \leqslant N$, $\max_{\ell=1,\ldots,r} |c_\ell^\phi| \leqslant N$, $\max_{\ell=1,\ldots,r} |c_\ell^\psi| \leqslant N$, and for all $j \in \mathbb{N}_0$,

$$
\begin{aligned}
&h_{2^j;k} = 0, \quad \forall x \in [-N,N], \quad k \in \mathbb{Z}\backslash[-N(2^j+1), N(2^j+1)], \\
&h_{2^j;k} = 0, \quad \forall x \in \mathbb{R}\backslash[-3N,3N], \quad k \in [-N(2^j+1), N(2^j+1)],
\end{aligned} \tag{3.18}
$$

for all $h \in \Phi \cup \Psi \cup \tilde{\Phi} \cup \tilde{\Psi}$. Also, since $k = d_{j,\ell}^\psi, \ldots, d_{j,\ell}^\psi + 2^j$, we have that $|k| \leqslant |d_{j,\ell}^\psi| + 2^j \leqslant 2^{j-1}N + 2^{-1}N + 2^j = 2^j(2^{-1}N + 2^{-j-1}N + 1)$. For $x \notin [-2N-1, 2N+1]$ and $j \in \mathbb{N}_0$, we have $|2^j x - k| \geqslant 2^j|x| - |k| \geqslant 2^j(2N+1) - (2^{j-1}N + 2^{-1}N + 2^j) \geqslant N$. As a result, we conclude that $\psi_{2^j;k}^\ell$ has to vanish outside $[-2N-1, 2N+1]$.

First, we start by proving that if $\mathsf{AS}_J(\Phi;\Psi)$ is a Bessel sequence in $L_2(\mathbb{R})$, then $\mathcal{B}_J$ and $\tilde{\mathcal{B}}_J$ are Bessel sequences in $L_2(\mathcal{I})$. Note that

$$
\begin{aligned}
\sum_{h \in \Psi_j^\ell} |\langle f, h\rangle|^2_{L_2(\mathcal{I})} &\leqslant \sum_{k=d_{j,\ell}^\psi}^{d_{j,l}^\psi + 2^j} |\langle f\chi_{\mathcal{I}}, F_{c,\epsilon_1,\epsilon_2}(\psi_{2^j;k}^\ell)\rangle|^2 \\
&= \sum_{k=d_{j,\ell}^\psi}^{d_{j,\ell}^\psi + 2^j} |\langle F_{c,\epsilon_1,\epsilon_2}(f), \psi_{2^j;k}^\ell\rangle|^2 \\
&= \sum_{k=d_{j,\ell}^\psi}^{d_{j,\ell}^\psi + 2^j} |\langle F_{c,\epsilon_1,\epsilon_2}(f)\chi_{[-2N-1,2N+1]}, \psi_{2^j;k}^\ell\rangle|^2
\end{aligned} \tag{3.19}
$$

where we apply (3.18) to move from the second to the third inequality. This helps us to proceed with the following argument

$$
\sum_{h \in \mathcal{B}_J} |\langle f, h\rangle|^2_{L_2(\mathcal{I})} = \sum_{\ell=1}^{r} \sum_{h \in \Phi_J^\ell} |\langle f, h\rangle|^2 + \sum_{j=J}^{\infty} \sum_{\ell=1}^{s} \sum_{h \in \Psi_j^\ell} |\langle f, h\rangle|^2
$$

$$
\begin{aligned}
&\leqslant \sum_{\ell=1}^{r}\sum_{k\in\mathbb{Z}}|\langle F_{c,\epsilon_1,\epsilon_2}(f)\chi_{[-2N-1,2N+1]},\phi^{\ell}_{2^j;k}\rangle|^2\\
&\quad+\sum_{j=J}^{\infty}\sum_{\ell=1}^{s}\sum_{k\in\mathbb{Z}}|\langle F_{c,\epsilon_1,\epsilon_2}(f)\chi_{[-2N-1,2N+1]},\psi^{\ell}_{2^j;k}\rangle|^2\\
&\leqslant C\|F_{c,\epsilon_1,\epsilon_2(f)\chi_{[-2N-1,2N+1]}}\|^2_{L_2(\mathbb{R})}\\
&\leqslant C(4N+2)\|f\|^2_{L_2(\mathbb{R})}.
\end{aligned}
$$

The transition from the second line to the third is made possible from the fact that $\mathsf{AS}_J(\Phi;\Psi)$ is a Bessel sequence in $L_2(\mathbb{R})$. We can apply a similar argument to show that that $\tilde{\mathcal{B}}_J$ is a Bessel sequence in $L_2(\mathcal{I})$.

Now, let us proceed to the proof of item (1). Since $(\{\tilde{\Phi};\tilde{\Psi}\},\{\Phi;\Psi\})$ is a biorthogonal wavelet in $L_2(\mathbb{R})$, by Theorem 1.1, we know that $(\{\tilde{\Phi};\tilde{\Psi}\},\{\Phi;\Psi\})$ is a pair of dual frames for $L_2(\mathbb{R})$ and the biorthogonality condition is satisfied. Hence, from [33, Theorem 4.3.5], we know that $(\mathsf{AS}_J(\tilde{\Phi};\tilde{\Psi}),\mathsf{AS}_J(\Phi;\Psi))$ is a pair of dual frames for $L_2(\mathbb{R})$ for all $J\in\mathbb{Z}$. As a first step, we need to show that $\{\tilde{\mathcal{B}}_J,\mathcal{B}_J\}$ is a pair of dual frames for $L_2(\mathcal{I})$. Since $(\mathsf{AS}_J(\tilde{\Phi};\tilde{\Psi}),\mathsf{AS}_J(\Phi;\Psi))$ is a pair of dual frames for $L_2(\mathbb{R})$, by the second identity of (3.18), we must have the following representation for all $f,g\in L_2(\mathcal{I})$

$$
\begin{aligned}
\langle f,g\rangle =&\langle f,F_{c,\epsilon_1,\epsilon_2}(g)\rangle=\langle f,F_{c,\epsilon_1,\epsilon_2}(g)\chi_{[-3N,3N]}\rangle\\
=&\sum_{\ell=1}^{r}\sum_{k=-N(2^J+1)}^{N(2^J+1)}\langle f,\phi^{\ell}_{2^J;k}\rangle\langle\tilde{\phi}^{\ell}_{2^J;k},F_{c,\epsilon_1,\epsilon_2}(g)\chi_{[-3N,3N]}\rangle\\
&+\sum_{\ell=1}^{s}\sum_{j=J}^{\infty}\sum_{k=-N(2^j+1)}^{N(2^j+1)}\langle f,\psi^{\ell}_{2^j;k}\rangle\langle\tilde{\psi}^{\ell}_{2^j;k},F_{c,\epsilon_1,\epsilon_2}(g)\chi_{[-3N,3N]}\rangle\\
=&\sum_{\ell=1}^{r}\sum_{k\in\mathbb{Z}}\langle f,\phi^{\ell}_{2^J;k}\rangle\langle F_{c,\epsilon_1,\epsilon_2}(\tilde{\phi}^{\ell}_{2^J;k}),g\rangle
\end{aligned}
$$

$$+\sum_{\ell=1}^{s}\sum_{j=J}^{\infty}\sum_{k\in\mathbb{Z}}\langle f,\psi^{\ell}_{2^j;k}\rangle\langle F_{c,\epsilon_1,\epsilon_2}(\tilde{\psi}^{\ell}_{2^j;k}),g\rangle,$$

where we have taken into account of (3.18). Recall that $F_{c,\epsilon_1,\epsilon_2}(h_{2^j;m+2^{j+2}k}) = F_{c,\epsilon_1,\epsilon_2}(h_{2^j;m})$ for all $j,k,m\in\mathbb{Z}$. We employ the following decomposition, let

$$\begin{aligned}\mathbb{Z} = \{&d^{\psi}_{j,\ell}+1-2^j,\ldots,d^{\psi}_{j,\ell},d^{\psi}_{j,\ell}+1,\ldots,d^{\psi}_{j,\ell}+2^j,d^{\psi}_{j,\ell}+2^j+1,\ldots,d^{\psi}_{j,\ell}+2^{j+1},\\ &d^{\psi}_{j,\ell}+1+2^{j+1},\ldots,d^{\psi}_{j,\ell}+2^j+2^{j+1}\}+2^{j+2}\mathbb{Z}.\end{aligned}\tag{3.20}$$

So, we have

$$\begin{aligned}&\sum_{k\in\mathbb{Z}}\langle f,\psi^{\ell}_{2^j;k}\rangle\langle F_{c,\epsilon_1,\epsilon_2}(\tilde{\psi}^{\ell}_{2^j;k}),g\rangle\\ &=\sum_{m=d^{\psi}_{j,\ell}+1-2^j}^{d^{\psi}_{j,\ell}+2^j+2^{j+1}}\sum_{k\in\mathbb{Z}}\langle f,\psi^{\ell}_{2^j;m+2^{j+2}k}\rangle\langle F_{c,\epsilon_1,\epsilon_2}(\tilde{\psi}^{\ell}_{2^j;m}),g\rangle\\ &=\sum_{m=d^{\psi}_{j,\ell}+1}^{d^{\psi}_{j,\ell}+2^j}\sum_{k\in\mathbb{Z}}\langle f,\psi^{\ell}_{2^j;m+2^{j+2}k}\rangle\langle F_{c,\epsilon_1,\epsilon_2}(\tilde{\psi}^{\ell}_{2^j;m}),g\rangle\\ &\quad+\sum_{m=d^{\psi}_{j,\ell}+1-2^j}^{d^{\psi}_{j,\ell}}\sum_{k\in\mathbb{Z}}\langle f,\psi^{\ell}_{2^j;m+2^{j+2}k}\rangle\langle\epsilon_1\epsilon^{\psi}_{\ell}F_{c,\epsilon_1,\epsilon_2}(\tilde{\psi}^{\ell}_{2^j;2^jc-c^{\psi}_{\ell}-m}),g\rangle\\ &\quad+\sum_{m=d^{\psi}_{j,\ell}+1+2^j}^{d^{\psi}_{j,\ell}+2^{j+1}}\sum_{k\in\mathbb{Z}}\langle f,\psi^{\ell}_{2^j;m+2^{j+2}k}\rangle\langle\epsilon_2\epsilon^{\psi}_{\ell}F_{c,\epsilon_1,\epsilon_2}(\tilde{\psi}^{\ell}_{2^j;2^jc-c^{\psi}_{\ell}-m+2^{j+1}}),g\rangle\\ &\quad+\sum_{m=d^{\psi}_{j,\ell}+2^{j+1}+1}^{d^{\psi}_{j,\ell}+2^j+2^{j+1}}\sum_{k\in\mathbb{Z}}\langle f,\psi^{\ell}_{2^j;m+2^{j+2}k}\rangle\langle\epsilon_1\epsilon_2F_{c,\epsilon_1,\epsilon_2}(\tilde{\psi}^{\ell}_{2^j;m-2^{j+1}}),g\rangle\end{aligned}\tag{3.21}$$

by applying (3.4), (3.5), and (3.6) to the second, third, and fourth terms.

Consider the case where $o^{\psi}_{j,\ell} = 1$ by keeping in mind that $2d^{\psi}_{j,\ell} + o^{\psi}_{j,\ell} = 2^j c - c^{\psi}_{j,\ell}$. By (3.21), we have that

$$
\begin{aligned}
&\sum_{k\in\mathbb{Z}} \langle f, \psi^{\ell}_{2^j;k}\rangle \langle F_{c,\epsilon_1,\epsilon_2}(\tilde{\psi}^{\ell}_{2^j;k}), g\rangle \\
&\quad= \sum_{m=d^{\psi}_{j,\ell}+1}^{d^{\psi}_{j,\ell}+2^j} \sum_{k\in\mathbb{Z}} \Big(\langle f, \epsilon_1\epsilon^{\psi}_{\ell}\psi^{\ell}_{2^j;2^jc-c^{\psi}_{\ell}-m+2^j+2^jk}\rangle + \langle f, \epsilon_2\epsilon^{\psi}_{\ell}\psi^{\ell}_{2^j;2^jc-c^{\psi}_{\ell}-m+2^{j+1}+2^{j+2}k}\rangle \\
&\qquad + \langle f, \epsilon_1\epsilon_2\psi^{\ell}_{2^j;m-2^{j+1}+2^{j+2}k}\rangle + \langle f, \psi^{\ell}_{2^j;m+2^{j+2}k}\rangle\Big) \langle F_{c,\epsilon_1,\epsilon_2}(\tilde{\psi}^{\ell}_{2^j;m}), g\rangle \\
&\quad= \sum_{m=d^{\psi}_{j,\ell}+1}^{d^{\psi}_{j,\ell}+2^j} \sum_{k\in\mathbb{Z}} \Big(\langle f, \psi^{\ell}_{2^j;m+2^{j+2}k} + \epsilon_1\psi^{\ell}_{2^j;m-2^jc-2^{j+2}k}(-\cdot) \\
&\qquad + \epsilon_2\psi^{\ell}_{2^j;m-2^jc-2^{j+1}(1-2k)}(-\cdot) + \epsilon_1\epsilon_2\psi^{\ell}_{2^j;m-2^{j+1}(1-2k)}\rangle\Big) \langle F_{c,\epsilon_1,\epsilon_2}(\tilde{\psi}^{\ell}_{2^j;m}), g\rangle \\
&\quad= \sum_{h\in\Psi^{\ell}_j} \langle f, h\chi_{\mathcal{I}}\rangle\langle \tilde{h}\chi_{\mathcal{I}}, g\rangle.
\end{aligned}
$$

Consider the case where $o^{\psi}_{j,\ell} = 0$ and $\epsilon_1 = \epsilon^{\psi}_{\ell} \neq \epsilon_2$. We employ the same decomposition as in (3.20). Our argument then follows (3.21). At this point, we would like to pay extra attention on terms $d^{\psi}_{j,\ell} + 2^j$ and $d^{\psi}_{j,\ell} + 2^j + 2^{j+1}$. Let us consider the term $d^{\psi}_{j,\ell} + 2^j$ first. Note that this term can be included in the summation that involves the terms $m = d^{\psi}_{j,\ell} + 2^j + 1, \ldots, d^{\psi}_{j,\ell} + 2^{j+1} - 1$ in (3.21). Now,

$$
\begin{aligned}
&\langle f, \psi^{\ell}_{2^j;d^{\psi}_{j,\ell}+2^j+2^{j+2}k}\rangle \langle F_{c,\epsilon_1,\epsilon_2}(\tilde{\psi}^{\ell}_{2^j;d^{\psi}_{j,\ell}+2^j}), g\rangle \\
&\qquad\qquad = \langle f, \psi^{\ell}_{2^j;d^{\psi}_{j,\ell}+2^j+2^{j+2}k}\rangle \langle \epsilon_2\epsilon^{\psi}_{\ell} F_{c,\epsilon_1,\epsilon_2}(\tilde{\psi}^{\ell}_{2^j;2^jc-c^{\psi}_{\ell}-(d^{\psi}_{j,\ell}+2^j)+2^{j+1}}), g\rangle \\
&\qquad\qquad = \langle f, \psi^{\ell}_{2^j;d^{\psi}_{j,\ell}+2^j+2^{j+2}k}\rangle \langle \epsilon_2\epsilon^{\psi}_{\ell} F_{c,\epsilon_1,\epsilon_2}(\tilde{\psi}^{\ell}_{2^j;d^{\psi}_{j,\ell}+2^j}), g\rangle,
\end{aligned} \tag{3.22}
$$

since $2d^{\psi}_{j,\ell}+o^{\psi}_{j,l}=2^jc-c^{\psi}_{\ell}$. Since $\epsilon^{\psi}_{\ell}\neq\epsilon_2$ means $\epsilon^{\psi}_{\ell}\epsilon_2=-1$, (3.22) yields

$$F_{c,\epsilon_1,\epsilon_2}(\tilde{\psi}^{\ell}_{2^j;d^{\psi}_{j,\ell}+2^j})=0. \tag{3.23}$$

Next, let us look at the term $d^{\psi}_{j,\ell}+2^j+2^{j+1}$. Note that this term can be included in the summation that involves the terms $m=d^{\psi}_{j,\ell}+1-2^j,\ldots,d^{\psi}_{j,\ell}-1$ due to the 4-periodic property of $F_{c,\epsilon_1,\epsilon_2}$. That is, $F_{c,\epsilon_1,\epsilon_2}(h_{2^j;m+2^{j+2}k})=F_{c,\epsilon_1,\epsilon_2}(h_{2^j;m})$ for all $m,k\in\mathbb{Z}$. Now,

$$\begin{aligned}
&\langle f,\psi^{\ell}_{2^j;d^{\psi}_{j,\ell}+2^j+2^{j+1}+2^{j+2}k}\rangle\langle\epsilon_1\epsilon_2F_{c,\epsilon_1,\epsilon_2}(\tilde{\psi}^{\ell}_{2^j;d^{\psi}_{j,\ell}+2^j}),g\rangle,\\
&\quad=\langle f,\psi^{\ell}_{2^j;d^{\psi}_{j,\ell}+2^j+2^{j+1}+2^{j+2}k}\rangle\langle\epsilon_1\epsilon^{\psi}_{\ell}F_{c,\epsilon_1,\epsilon_2}(\tilde{\psi}^{\ell}_{2^j;2^jc-c^{\psi}_{\ell}-(d^{\psi}_{j,\ell}+2^j+2^{j+1})}),g\rangle\\
&\quad=\langle f,\psi^{\ell}_{2^j;d^{\psi}_{j,\ell}+2^j+2^{j+1}+2^{j+2}k}\rangle\langle\epsilon_1\epsilon^{\psi}_{\ell}F_{c,\epsilon_1,\epsilon_2}(\tilde{\psi}^{\ell}_{2^j;d^{\psi}_{j,\ell}-2^j-2^{j+1}+2^{j+2}}),g\rangle\\
&\quad=\langle f,\psi^{\ell}_{2^j;d^{\psi}_{j,\ell}+2^j+2^{j+1}+2^{j+2}k}\rangle\langle\epsilon_1\epsilon^{\psi}_{\ell}F_{c,\epsilon_1,\epsilon_2}(\tilde{\psi}^{\ell}_{2^j;d^{\psi}_{j,\ell}+2^j}),g\rangle.
\end{aligned} \tag{3.24}$$

Since $\epsilon_1\epsilon_2=-1$ and $\epsilon^{\psi}_{\ell}\epsilon_1=1$, (3.24) yields

$$F_{c,\epsilon_1,\epsilon_2}(\tilde{\psi}^{\ell}_{2^j;d^{\psi}_{j,\ell}+2^j})=0, \tag{3.25}$$

which is consistent with (3.23). Let us turn our attention to the terms $d^{\psi}_{j,\ell}$, $d^{\psi}_{j,\ell}+2^{j+1}$, and observe that

$$\sum_{k\in\mathbb{Z}}\psi^{\ell}_{2^j;d^{\psi}_{j,\ell}+2^{j+1}(2k)}=\sum_{k\in\mathbb{Z}}\psi^{\ell}_{2^j;2^jc-c^{\psi}_{\ell}-d^{\psi}_{j,\ell}+2^{j+1}(2k)}=\epsilon^{\psi}_{\ell}\sum_{k\in\mathbb{Z}}\psi^{\ell}_{2^j;d^{\psi}_{j,\ell}-2^jc+2^{j+1}(2k)}(-\cdot), \tag{3.26}$$

$$\sum_{k\in\mathbb{Z}}\psi^{\ell}_{2^j;d^{\psi}_{j,\ell}+2^{j+1}(2k-1)}=\sum_{k\in\mathbb{Z}}\psi^{\ell}_{2^j;2^jc-c^{\psi}_{\ell}-d^{\psi}_{j,\ell}+2^{j+1}(2k-1)}$$

$$= \epsilon_\ell^\psi \sum_{k\in\mathbb{Z}} \psi^\ell_{2^j;d^\psi_{j,\ell}-2^jc+2^{j+1}(2k-1)}(-\cdot), \quad (3.27)$$

which means that

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}} \epsilon_1\epsilon_\ell^\psi \psi^\ell_{d^\psi_{j,\ell}+2^{j+1}(2k)} + \epsilon_2\epsilon_\ell^\psi \psi^\ell_{2^j;d^\psi_{j,\ell}+2^{j+1}(2k-1)} \\
&= \frac{1}{2}\epsilon_1\epsilon_\ell^\psi \left(\sum_{k\in\mathbb{Z}} \psi^\ell_{d^\psi_{j,\ell}+2^{j+2}k} + \epsilon_1 \psi^\ell_{2^j;d^\psi_{j,\ell}-2^jc+2^{j+2}k}(-\cdot) \right. \\
&\quad \left. + \epsilon_1\epsilon_2 \psi^\ell_{2^j;d^\psi_{j,\ell}+2^{j+1}(2k-1)} + \epsilon_2 \psi^\ell_{2^j;d^\psi_{j,\ell}-2^jc+2^{j+1}(2k-1)}(-\cdot) \right) \\
&= \frac{1}{2} F_{c,\epsilon_1,\epsilon_2}(\psi^\ell_{2^j;d^\psi_{j,\ell}}), \qquad (3.28)
\end{aligned}$$

since $\epsilon_\ell^\psi = \epsilon_1 = 1$, $\epsilon_1\epsilon_2 = -1$, and $\epsilon_\ell^\psi\epsilon_1 = 1$. In order to put things in a concrete manner for the case $o^\psi_{j,\ell} = 0$ and $\epsilon_1 = \epsilon_\ell^\psi \neq \epsilon_2$, by (3.21) we have

$$\begin{aligned}
\sum_{k\in\mathbb{Z}} \langle f, \psi^\ell_{2^j;k}\rangle \langle F_{c,\epsilon_1,\epsilon_2}(\tilde{\psi}^\ell_{2^j;k}), g\rangle = &\sum_{m=d^\psi_{j,\ell}+1}^{d^\psi_{j,\ell}+2^j-1} \langle f, F_{c,\epsilon_1,\epsilon_2}(\psi^\ell_{2^j;m})\rangle \langle F_{c,\epsilon_1,\epsilon_2}(\tilde{\psi}^\ell_{2^j;m}), g\rangle \\
&+ \langle f, \frac{1}{\sqrt{2}} F_{c,\epsilon_1,\epsilon_2}(\psi^\ell_{2^j;d^\psi_{j,\ell}})\rangle \langle \frac{1}{\sqrt{2}} F_{c,\epsilon_1,\epsilon_2}(\tilde{\psi}^\ell_{2^j;d^\psi_{j,\ell}}), g\rangle.
\end{aligned}$$

Consider the case where $o^\psi_{j,\ell} = 0$ and $\epsilon_1 \neq \epsilon_\ell^\psi = \epsilon_2$. We employ the same decomposition as in (3.20). Our argument then follows (3.21). At this point, we would like to pay extra attention on terms $d^\psi_{j,\ell}$ and $d^\psi_{j,\ell} + 2^{j+1}$. We again follow the same arguments as in (3.22) and (3.24) in that we consider 2 representations of the term $d^\psi_{j,\ell}$ by including it in the summation that involves the terms $d^\psi_{j,\ell}+1, \ldots, d^\psi_{j,\ell}+2^j-1$ and the term $d^\psi_{j,\ell}+2^{j+1}$ by including it in the summation that involves the terms $d^\psi_{j,\ell}+2^{j+1}+1, \ldots, d^\psi_{j,\ell}+2^j+2^{j+1}-1$.

That is we have

$$\langle f, \psi^{\ell}_{2^j; d^{\psi}_{j,\ell}+2^j+2k}\rangle \langle F_{c,\epsilon_1,\epsilon_2}(\tilde{\psi}^{\ell}_{2^j; d^{\psi}_{j,\ell}}), g\rangle$$
$$= \langle f, \psi^{\ell}_{2^j; d^{\psi}_{j,\ell}+2^j+2k}\rangle \langle \epsilon_1 \epsilon^{\psi}_{\ell} F_{c,\epsilon_1,\epsilon_2}(\tilde{\psi}^{\ell}_{2^j; 2^j c - c^{\psi}_{\ell} - d^{\psi}_{j,\ell}}), g\rangle,$$

and

$$\langle f, \psi^{\ell}_{2^j; d^{\psi}_{j,\ell}+2^{j+1}+2^{j+2}k}\rangle \langle \epsilon_2 \epsilon^{\psi}_{\ell} F_{c,\epsilon_1,\epsilon_2}(\tilde{\psi}^{\ell}_{2^j; 2^j c - c^{\psi}_{\ell} - d^{\psi}_{j,\ell} - 2^{j+1} + 2^{j+1}}), g\rangle$$
$$= \langle f, \psi^{\ell}_{2^j; d^{\psi}_{j,\ell}+2^{j+1}+2^{j+2}k}\rangle \langle \epsilon_1 \epsilon_2 F_{c,\epsilon_1,\epsilon_2}(\tilde{\psi}^{\ell}_{2^j; d^{\psi}_{j,\ell}+2^{j+1}-2^{j+1}}), g\rangle.$$

What we shall observe is that

$$F_{c,\epsilon_1,\epsilon_2}(\tilde{\psi}^{\ell}_{2^j; d^{\psi}_{j,\ell}}) = 0. \tag{3.29}$$

We may use a similar argument in (3.26), (3.27), and (3.28), which helps us to arrive at the following equation

$$\sum_{k\in\mathbb{Z}} \langle f, \psi^{\ell}_{2^j;k}\rangle \langle F_{c,\epsilon_1,\epsilon_2}(\tilde{\psi}^{\ell}_{2^j;k}), g\rangle = \sum_{m=d^{\psi}_{j,\ell}+1}^{d^{\psi}_{j,\ell}+2^j-1} \langle f, F_{c,\epsilon_1,\epsilon_2}(\psi^{\ell}_{2^j;m})\rangle \langle F_{c,\epsilon_1,\epsilon_2}(\tilde{\psi}^{\ell}_{2^j;m}), g\rangle$$
$$+ \langle f, \frac{1}{\sqrt{2}} F_{c,\epsilon_1,\epsilon_2}(\psi^{\ell}_{2^j; d^{\psi}_{j,\ell}+2^j})\rangle \langle \frac{1}{\sqrt{2}} F_{c,\epsilon_1,\epsilon_2}(\tilde{\psi}^{\ell}_{2^j; d^{\psi}_{j,\ell}+2^j}), g\rangle.$$

Consider the case where $o^{\psi}_{j,\ell} = 0$ and $\epsilon_1 = \epsilon^{\psi}_{\ell} = \epsilon_2$. Then, by (3.3), we have

$$F_{c,\epsilon_1,\epsilon_2}(\psi^{\ell}_{2^j;k}) = \sum_{k\in\mathbb{Z}} \psi^{\ell}_{m+2^{j+1}k} + \epsilon^{\psi}_{\ell} \psi^{\ell}_{2^j; m-2^j c+2^{j+1}k}(-\cdot),$$

which boils down to one of the cases considered in [33]. Having $\epsilon_1 = \epsilon_2$ allows us to take advantage of $F_{c,\epsilon_1,\epsilon_2}(h_{2^j;m}) = F_{c,\epsilon_1,\epsilon_2}(h_{2^j;m+2^{j+1}k})$. For this case, particularly for the boundary elements, notice that for all $q \in \mathbb{Z}$

$$\sum_{k\in\mathbb{Z}} \psi^{\ell}_{2^j;d^{\psi}_{j,\ell}+2^jq+2^{j+1}(2k)} = \epsilon^{\psi}_{\ell} \sum_{k\in\mathbb{Z}} \psi^{\ell}_{2^j;d^{\psi}_{j,\ell}+2^jq-2^jc+2^{j+1}(2k)}(-\cdot),$$

$$\sum_{k\in\mathbb{Z}} \psi^{\ell}_{2^j;d^{\psi}_{j,\ell}+2^jq+2^{j+1}(2k-1)} = \epsilon^{\psi}_{\ell} \sum_{k\in\mathbb{Z}} \psi^{\ell}_{2^j;d^{\psi}_{j,\ell}+2^jq-2^jc+2^{j+1}(2k-1)}(-\cdot). \tag{3.30}$$

Keeping in mind that $2^{j+1}k$ can decomposed into a set of terms that involves $2^{j+1}(2k)$ and another that involves $2^{j+1}(2k+1)$, we have by (3.30)

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}} \psi^{\ell}_{2^j;d^{\psi}_{j,\ell}+2^jq+2^{j+1}k} \\
&\quad= \sum_{k\in\mathbb{Z}} \psi^{\ell}_{2^j;d^{\psi}_{j,\ell}+2^jq+2^{j+1}(2k)} + \psi^{\ell}_{2^j;d^{\psi}_{j,\ell}+2^jq+2^{j+1}(2k-1)} \\
&\quad= \frac{1}{2}\Bigg(\sum_{k\in\mathbb{Z}} \psi^{\ell}_{2^j;d^{\psi}_{j,\ell}+2^jq+2^{j+1}(2k)} + \epsilon^{\psi}_{\ell}\psi^{\ell}_{2^j;d^{\psi}_{j,\ell}+2^jq-2^jc+2^{j+1}(2k)}(-\cdot) \\
&\qquad + \psi^{\ell}_{2^j;d^{\psi}_{j,\ell}+2^jq+2^{j+1}(2k-1)} + \epsilon^{\psi}_{\ell}\psi^{\ell}_{2^j;d^{\psi}_{j,\ell}+2^jq-2^jc+2^{j+1}(2k-1)}(-\cdot)\Bigg) \\
&\quad= \frac{1}{2} F_{c,\epsilon_1,\epsilon_2}(\psi^{\ell}_{2^j;d^{\psi}_{j,\ell}+2^jq}).
\end{aligned}$$

Hence,

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}} \langle f, \psi^{\ell}_{2^j;k}\rangle \langle F_{c,\epsilon_1,\epsilon_2}(\tilde{\psi}^{\ell}_{2^j;k}), g\rangle \\
&\quad= \sum_{q=0}^{1}\sum_{k\in\mathbb{Z}} \langle f, \psi^{\ell}_{2^j;d^{\psi}_{j,\ell}+2^jq+2^{j+1}k}\rangle \langle F_{c,\epsilon_1,\epsilon_2}(\tilde{\psi}^{\ell}_{2^j;d^{\psi}_{j,\ell}+2^jq}), g\rangle \\
&\qquad + \sum_{m=d^{\psi}_{j,\ell}+1}^{d^{\psi}_{j,\ell}+2^j-1} \langle f, \sum_{k\in\mathbb{Z}} \psi^{\ell}_{2^j;m+2^{j+1}k} + \epsilon^{\psi}_{\ell}\psi^{\ell}_{2^j;m-2^jc+2^{j+1}k}(-\cdot)\rangle \langle F_{c,\epsilon_1,\epsilon_2}(\tilde{\psi}^{\ell}_{2^j;m}), g\rangle
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{q=0}^{1} \langle f, \frac{1}{\sqrt{2}} F_{c,\epsilon_1,\epsilon_2}(\psi^\ell_{2^j;d^\psi_{j,\ell}+2^jq})\rangle \langle \frac{1}{\sqrt{2}} F_{c,\epsilon_1,\epsilon_2}(\tilde{\psi}^\ell_{2^j;d^\psi_{j,\ell}+2^jq}), g\rangle \\
&\quad + \sum_{m=d^\psi_{j,\ell}+1}^{d^\psi_{j,\ell}+2^j-1} \langle f, F_{c,\epsilon_1,\epsilon_2}(\psi^\ell_{2^j;m})\rangle \langle F_{c,\epsilon_1,\epsilon_2}(\tilde{\psi}^\ell_{2^j;m}), g\rangle \\
&= \sum_{h\in\Psi^\ell_j} \langle f, h\chi_{\mathcal{I}}\rangle \langle \tilde{h}\chi_{\mathcal{I}}, g\rangle. \qquad (3.31)
\end{aligned}
$$

Next, consider the case when $o^\psi_{j,\ell} = 0$ and $\epsilon_1 = \epsilon_2 \neq \epsilon^\psi_\ell$. We simply refer back to (3.31) and analyze the first summation. Having $\epsilon_1 = \epsilon_2$ allows us to take advantage of the following periodicity $F_{c,\epsilon_1,\epsilon_2}(h_{2^j;m+2^{j+1}k}) = F_{c,\epsilon_1,\epsilon_2}(h_{2^j;m})$. By (3.4) and the fact that $2^jc - c^\psi_\ell = 2d^\psi_{j,\ell}$ and $F_{c,\epsilon_1,\epsilon_2}(\psi^\ell_{2^j;d^\psi_{j,\ell}+2^jq+2^{j+1}k}) = F_{c,\epsilon_1,\epsilon_2}(\psi^\ell_{2^j;d^\psi_{j,\ell}+2^jq})$, we have

$$
\begin{aligned}
F_{c,\epsilon_1,\epsilon_2}(\psi^\ell_{2^j;d^\psi_{j,\ell}+2^jq}) &= \epsilon_1\epsilon^\psi_\ell F_{c,\epsilon_1,\epsilon_2}(\psi^\ell_{2^j;2^jc-c^\psi_\ell-d^\psi_{j,\ell}-2^jq}) = -F_{c,\epsilon_1,\epsilon_2}(\psi^\ell_{2^j;d^\psi_{j,\ell}-2^jq+2^{j+1}}) \\
&= -F_{c,\epsilon_1,\epsilon_2}(\psi^\ell_{2^j;d^\psi_{j,\ell}+2^jq})
\end{aligned}
$$

for $q \in \{0, 1\}$. Hence, we have

$$
F_{c,\epsilon_1,\epsilon_2}(\psi^\ell_{2^j;d^\psi_{j,\ell}+2^jq}) = 0,
$$

and (3.31) boils down to

$$
\begin{aligned}
\sum_{k\in\mathbb{Z}} \langle f, \psi^\ell_{2^j;k}\rangle \langle F_{c,\epsilon_1,\epsilon_2}(\tilde{\psi}^\ell_{2^j;k}), g\rangle &= \sum_{m=d^\psi_{j,\ell}+1}^{d^\psi_{j,\ell}+2^j-1} \langle f, F_{c,\epsilon_1,\epsilon_2}(\psi^\ell_{2^j;m})\rangle \langle F_{c,\epsilon_1,\epsilon_2}(\tilde{\psi}^\ell_{2^j;m}), g\rangle \\
&= \sum_{h\in\Psi^\ell_j} \langle f, h\chi_{\mathcal{I}}\rangle \langle \tilde{h}\chi_{\mathcal{I}}, g\rangle.
\end{aligned}
$$

Even though the arguments above explicitly involve the wavelets, we can tailor them so that they remain valid for each component of the refinable vector function. We can summarize the above 5 cases as an equation stated below

$$\langle f, g\rangle_{L_2(\mathcal{I})} = \sum_{\ell=1}^{r} \sum_{h\in\Phi_J^\ell} \langle f, h\rangle\langle \tilde{h}, g\rangle + \sum_{\ell=1}^{s} \sum_{j=J}^{\infty} \sum_{h\in\Psi_j^\ell} \langle f, h\rangle\langle \tilde{h}, g\rangle.$$

for all $f, g \in L_2(\mathcal{I})$. With that, we have proved that $(\tilde{\mathcal{B}}_J, \mathcal{B}_J)$ is a pair of dual frames for $L_2(\mathcal{I})$.

In order to complete the proof of item (1), the next is to prove that the biorthogonality condition for each of the 5 cases. In fact, all of them have the same starting point, which is stated below

$$\begin{aligned}
&\int_{\mathcal{I}} F_{c,\epsilon_1,\epsilon_2}(\psi^\ell_{2^j;m})\overline{F_{c,\epsilon_1,\epsilon_2}(\tilde{\psi}^p_{2^q;n})}dx\\
=&\langle F_{c,\epsilon_1,\epsilon_2}(\psi^\ell_{2^j;m}), \tilde{\psi}^p_{2^q;n}\rangle\\
=&\sum_{k\in\mathbb{Z}}\langle \psi^\ell_{2^j;m+2^{j+1}(2k)} + \epsilon_1\psi^\ell_{2^j;m-2^jc+2^{j+1}(2k)}(-\cdot) + \epsilon_2\psi^\ell_{2^j;m-2^jc+2^{j+1}(2k-1)}(-\cdot)\\
&+ \epsilon_1\epsilon_2\psi^\ell_{2^j;m+2^{j+1}(2k-1)}, \tilde{\psi}^p_{2^q;n}\rangle.\\
=&\sum_{k\in\mathbb{Z}}\langle \psi^\ell_{2^j;m+2^{j+1}(2k)}, \tilde{\psi}^\ell_{2^j;n}\rangle + \epsilon_1\langle \psi^\ell_{2^j;m-2^jc+2^{j+1}(2k)}(-\cdot), \tilde{\psi}^\ell_{2^j;n}\rangle\\
&+ \epsilon_2\langle \psi^\ell_{2^j;m-2^jc+2^{j+1}(2k-1)}(-\cdot), \tilde{\psi}^\ell_{2^j;n}\rangle + \epsilon_1\epsilon_2\langle \psi^\ell_{2^j;m+2^{j+1}(2k-1)}, \tilde{\psi}^\ell_{2^j;n}\rangle\\
=&1 + \epsilon_1\epsilon^\psi_\ell\sum_{k\in\mathbb{Z}}\delta(2^jc - c^\psi_\ell - 2m - 2^{j+2}k)\\
&+ \epsilon_2\epsilon^\psi_\ell\sum_{k\in\mathbb{Z}}\delta(2^jc - c^\psi_\ell - 2m - 2^{j+1}(2k-1)) + \epsilon_1\epsilon_2\sum_{k\in\mathbb{Z}}\delta(2^{j+1}(2k-1)),
\end{aligned} \tag{3.32}$$

whereby the biorthogonality of $(\mathsf{AS}_J(\tilde{\Psi};\tilde{\Phi}),\mathsf{AS}_J(\Psi;\Phi))$ essentially forces us to select $p=\ell$, $q=j$, and $m=n$. We have $m=n$ because $d^{\psi}_{j,\ell}\leqslant m,n\leqslant d^{\psi}_{j,\ell}+2^j$ and $m-n\equiv 0 \mod 2^{j+2}$ forces $k=0$ for the first summation.

Recall that for the case $o^{\psi}_{j,\ell}=1$, $d^{\psi}_{j,\ell}+1\leqslant m,n\leqslant d^{\psi}_{j,\ell}+2^j$. Hence, by (3.32), we have

$$\int_{\mathcal{I}} F_{c,\epsilon_1,\epsilon_2}(\psi^{\ell}_{2^j;m})\overline{F_{c,\epsilon_1,\epsilon_2}(\tilde{\psi}^{p}_{2^q;n})}dx=1,$$

because $\delta(2^jc-c^{\psi}_{\ell}-2m-2^{j+2}k)=\delta(2d^{\psi}_{j,\ell}+o^{\psi}_{j,\ell}-2m-2^{j+2}k)$ but $m=d^{\psi}_{j,\ell}+1/2-2^{j+1}k\notin\{d^{\psi}_{j,\ell}+1,\ldots,d^{\psi}_{j,\ell}+2^j\}$ for all $k\in\mathbb{Z}$; and similarly, $\delta(2^jc-c^{\psi}_{\ell}-2m-2^{j+1}(2k-1))=\delta(2d^{\psi}_{j,\ell}+o^{\psi}_{j,\ell}-2m-2^{j+1}(2k-1))$ but $m=d^{\psi}_{j,\ell}+1/2-2^j(2k-1)\notin\{d^{\psi}_{j,\ell}+1,\ldots,d^{\psi}_{j,\ell}+2^j\}$ for all $k\in\mathbb{Z}$.

Consider the case when $o^{\psi}_{j,\ell}=0$ and $\epsilon_1=\epsilon^{\psi}_{\ell}\neq\epsilon_2$. Refer to (3.32) again. For $m=\{d^{\psi}_{j,\ell}+1,\ldots,d^{\psi}_{j,\ell}+2^j-1\}$,

$$\int_{\mathcal{I}} F_{c,\epsilon_1,\epsilon_2}(\psi^{\ell}_{2^j;m})\overline{F_{c,\epsilon_1,\epsilon_2}(\tilde{\psi}^{p}_{2^q;n})}dx=1,$$

because $\delta(2^jc-c^{\psi}_{\ell}-2m-2^{j+2}k)=\delta(2d^{\psi}_{j,\ell}-2m-2^{j+2}k)$ but $m=d^{\psi}_{j,\ell}-2^{j+1}k\notin\{d^{\psi}_{j,\ell}+1,\ldots,d^{\psi}_{j,\ell}+2^j-1\}$ for all $k\in\mathbb{Z}$ and $j\geqslant 1$; and similarly, $\delta(2^jc-c^{\psi}_{\ell}-2m-2^{j+1}(2k-1))=\delta(2d^{\psi}_{j,\ell}-2m-2^{j+1}(2k-1))$ but $m=d^{\psi}_{j,\ell}-2^j(2k-1)\notin\{d^{\psi}_{j,l}+1,\ldots,d^{\psi}_{j,l}+2^j-1\}$ for all $k\in\mathbb{Z}$ and $j\geqslant 1$. For $j=0$, recall from (3.16) that we only have a boundary term, $m=d^{\psi}_{j,\ell}$. Now, when $m=d^{\psi}_{j,\ell}$,

$$\int_{\mathcal{I}} F_{c,\epsilon_1,\epsilon_2}(\psi^{\ell}_{2^j;m})\overline{F_{c,\epsilon_1,\epsilon_2}(\tilde{\psi}^{p}_{2^q;n})}dx=1+\epsilon_1\epsilon^{\psi}_{\ell}=2. \tag{3.33}$$

Consider the case when $o^{\psi}_{j,\ell}=0$ and $\epsilon_1\neq\epsilon^{\psi}_{\ell}=\epsilon_2$. The only difference between this case and the one where $o^{\psi}_{j,\ell} = 0$ and $\epsilon_1 = \epsilon^{\psi}_{\ell} \neq \epsilon_2$ is that now we have the (boundary) term $m = d^{\psi}_{j,\ell} + 2^j$, but we no longer have $m = d^{\psi}_{j,\ell}$. Refer to (3.32) again. For $m = \{d^{\psi}_{j,\ell} + 1, \ldots, d^{\psi}_{j,\ell} + 2^j - 1\}$,

$$\int_{\mathcal{I}} F_{c,\epsilon_1,\epsilon_2}(\psi^{\ell}_{2^j;m})\overline{F_{c,\epsilon_1,\epsilon_2}(\tilde{\psi}^{p}_{2^q;n})}dx = 1,$$

by arguments made for the case $o^{\psi}_{j,\ell} = 0$ and $\epsilon_1 = \epsilon^{\psi}_{\ell} \neq \epsilon_2$. For $m = d^{\psi}_{j,\ell} + 2^j$, noting the summation that involves $\epsilon_2\epsilon^{\psi}_{\ell}$ is equal to 1, when $k = 0$, we have

$$\int_{\mathcal{I}} F_{c,\epsilon_1,\epsilon_2}(\psi^{\ell}_{2^j;m})\overline{F_{c,\epsilon_1,\epsilon_2}(\tilde{\psi}^{p}_{2^q;n})}dx = 1 + \epsilon_2\epsilon^{\psi}_{\ell} = 2. \tag{3.34}$$

Consider the case when $o^{\psi}_{j,\ell} = 0$ and $\epsilon_2 \neq \epsilon_1 = \epsilon^{\psi}_{\ell}$; or, $o^{\psi}_{j,\ell} = 0$ and $\epsilon_1 \neq \epsilon_2 = \epsilon^{\psi}_{\ell}$. Then, the only terms we have are simply $m = \{d^{\psi}_{j,\ell}+1, \ldots, d^{\psi}_{j,\ell}+2^j-1\}$. We just apply the same argument as above. Lastly, consider the case when $o^{\psi}_{j,\ell} = 0$ and $\epsilon_1 = \epsilon_2 = \epsilon^{\psi}_{\ell}$. The interior terms that involve $m = \{d^{\psi}_{j,\ell}+1, \ldots, d^{\psi}_{j,\ell}+2^j-1\}$ have been handled before. For $m = d^{\psi}_{j,\ell}$, we are back at (3.33). Meanwhile, we are back at (3.34) for $m = d^{\psi}_{j,\ell} + 2^j$. Therefore, we proved that $\tilde{\mathcal{B}}_J$ and $\mathcal{B}_J$ are biorthogonal. So, item (1) holds. Item (2) follows directly from item (1). $\square$

Next, we discuss the refinable structure of the folded wavelets on the interval. Due to the more sophisticated structure of the folding operator, we need to modify the argument used in [33, Proposition 7.5.4], but the general idea remains the same.

Proposition 3.3. *Let $\phi = (\phi^1, \ldots, \phi^r)^{\mathsf{T}}$ and $\psi = (\psi^1, \ldots, \psi^s)^{\mathsf{T}}$ be vectors of* *compactly supported distributions satisfying*

$$\phi = 2\sum_{k\in\mathbb{Z}} a(k)\phi(2\cdot -k), \quad \psi = 2\sum_{k\in\mathbb{Z}} b(k)\phi(2\cdot -k) \tag{3.35}$$

for some $a \in (l_0(\mathbb{Z}))^{r\times r}$ *and* $b \in (l_0(\mathbb{Z}))^{s\times r}$*. Define*

$$\begin{aligned} \mathsf{S}(\xi) &:= diag(\epsilon_1^{\phi} e^{-ic_1^{\phi}\xi}, \dots, \epsilon_r^{\phi} e^{-ic_r^{\phi}\xi}) \\ \mathsf{T}(\xi) &:= diag(\epsilon_1^{\psi} e^{-ic_1^{\psi}\xi}, \dots, \epsilon_s^{\psi} e^{-ic_s^{\psi}\xi}) \end{aligned} \tag{3.36}$$

with $c_1^{\phi}, \dots, c_r^{\phi}, c_1^{\psi}, \dots, c_s^{\psi} \in \mathbb{Z}$ *and* $\epsilon_1^{\phi}, \dots, \epsilon_r^{\phi}, \epsilon_1^{\psi}, \dots, \epsilon_s^{\psi} \in \{-1, 1\}$.

1. *If* $\widehat{a}(\xi) = \mathsf{S}(2\xi)\widehat{a}(-\xi)\mathsf{S}^{-1}(\xi)$ *for all* $\xi \in \mathbb{R}$ *and if 1 is a simple eigenvalue of* $\widehat{a}(0)$ *and* $\det(2^j I_r - \widehat{a}(0)) \neq 0$ *for all* $j \in \mathbb{N}$*, then* $\widehat{\phi}(\xi) = \mathsf{S}(\xi)\widehat{\phi}(-\xi)$*, that is, the first identity in* (3.13) *holds.*

2. *If* $\widehat{\phi}(\xi) = \mathsf{S}(\xi)\widehat{\phi}(-\xi)$ *and* $\widehat{b}(\xi) = \mathsf{T}(2\xi)\widehat{b}(-\xi)\mathsf{S}^{-1}(\xi)$*, then* $\widehat{\psi}(\xi) = \mathsf{T}(\xi)\widehat{\psi}(-\xi)$*, that is, the first identity in* (3.14) *holds.*

3. *If* $\widehat{\phi} = \mathsf{S}(\xi)\widehat{\phi}(-\xi)$ *and* $\widehat{\psi}(\xi) = \mathsf{T}(\xi)\widehat{\psi}(-\xi)$ *(that is, the first identities in both* (3.13) *and* (3.14) *hold) with* $c_1^{\phi}, \dots, c_r^{\phi}, c_1^{\psi}, \dots, c_s^{\psi} \in \mathbb{Z}$*, then there exist* $(\#\vec{\Phi}_{j-1}) \times (\#\vec{\Phi}_j)$ *matrices* A_j *and* $(\#\vec{\Psi}_{j-1}) \times (\#\vec{\Phi}_j)$ *matrices* B_j *such that*

$$\vec{\Phi}_{j-1} = A_j\vec{\Phi}_j \quad \vec{\Psi}_{j-1} = B_j\vec{\Phi}_j, \quad j \in \mathbb{N}, \tag{3.37}$$

where $\vec{\Phi}$ *and* $\vec{\Psi}$ *are column vectors by listing all the elements in* $\cup_{\ell=1}^{r}\Phi_j^{\ell}$ *and* $\cup_{\ell=1}^{s}\Psi_j^{\ell}$*, respectively. Here,* Ψ_j^{ℓ} *is defined in* (3.16) *and* Φ_j^{ℓ} *is defined similarly as in* (3.16).

Proof. In order to prove item (1), we start by defining $\widehat{\eta}(\xi) := \mathsf{S}(\xi)\widehat{\phi}(-\xi)$.

Recall that (3.35) is simply equivalent to having $\widehat{\phi}(2\xi) = \widehat{a}(\xi)\widehat{\phi}(\xi)$ and $\widehat{\psi}(2\xi) = \widehat{b}(\xi)\widehat{\phi}(\xi)$ by Fourier transform. Next, $\widehat{a}(\xi)\mathsf{S}(\xi) = \mathsf{S}(2\xi)\widehat{a}(-\xi)$ implies that

$$\widehat{a}(\xi)\widehat{\eta}(\xi) = \widehat{a}(\xi)\mathsf{S}(\xi)\widehat{\phi}(-\xi) = \mathsf{S}(2\xi)\widehat{a}(-\xi)\widehat{\phi}(-\xi) = \mathsf{S}(2\xi)\widehat{\phi}(-2\xi) = \widehat{\eta}(2\xi).$$

By the refinability of η, the fact that $\widehat{\eta}(0) = \mathsf{S}(0)\widehat{\phi}(0)$, our assumption in that 1 is a simple eigenvalue of $\widehat{a}(0)$ and $\det(2^j I_r - \widehat{a}(0)) \neq 0$ for all $j \in \mathbb{N}$, and the uniqueness of solution to compactly supported refinable distribution [33, Theorem 5.1.3], we have $\eta = \phi$. Thus $\widehat{\phi}(\xi) = \mathsf{S}(\xi)\widehat{\phi}(-\xi)$.

From item (1), we know that $\widehat{\phi}(\xi) = \mathsf{S}(\xi)\widehat{\phi}(\xi)$. Rearranging our assumption, we have $\mathsf{T}(2\xi)\widehat{b}(-\xi) = \widehat{b}(\xi)\mathsf{S}(\xi)$. With a straightforward calculation, we get

$$\mathsf{T}(2\xi)\widehat{\psi}(-2\xi) = \mathsf{T}(2\xi)\widehat{b}(-\xi)\widehat{\phi}(-\xi) = \widehat{b}(\xi)\mathsf{S}(\xi)\widehat{\phi}(-\xi) = \widehat{b}(\xi)\widehat{\phi}(\xi) = \widehat{\psi}(\xi),$$

which brings us to item (2). We shall do the proof for the wavelets. The argument for each component of the refinable vector function proceeds identically, we just need to change ψ to ϕ and the filter b to filter a. Let us start with

$$\psi_{2^{j-1};n} = \sqrt{2}\sum_{k\in\mathbb{Z}} b(k)\phi_{2^j;k+2n}, \quad j \in \mathbb{N} \tag{3.38}$$

Let us denote $[b(k)]_{p,\ell}$ the p-th row and and ℓ-th column of matrix $b(k)$. Then, applying the folding operator to (3.38), we can rewrite (3.38) further as

$$F_{c,\epsilon_1,\epsilon_2}(\psi^p_{2^{j-1};n}) = \sqrt{2}\sum_{\ell=1}^{r}\sum_{k\in\mathbb{Z}}[b(k)]_{p,\ell}F_{c,\epsilon_1,\epsilon_2}(\phi^\ell_{2^j;k+2n})$$

$$= \sqrt{2}\sum_{\ell=1}^{r}\sum_{m=d^{\phi}_{j,\ell}+1-2^j-2n}^{d^{\phi}_{j,\ell}+2^j+2^{j+1}-2n}\sum_{k\in\mathbb{Z}}[b(m+2^{j+2}k)]_{p,\ell}F_{c,\epsilon_1,\epsilon_2}(\phi^{\ell}_{2^j;m+2^{j+2}k+2n})$$

$$= \sqrt{2}\sum_{\ell=1}^{r}\sum_{m=d^{\phi}_{j,\ell}+1-2^j-2n}^{d^{\phi}_{j,\ell}+2^j+2^{j+1}-2n}\sum_{k\in\mathbb{Z}}[b(m+2^{j+2}k)]_{p,\ell}F_{c,\epsilon_1,\epsilon_2}(\phi^{\ell}_{2^j;m+2n}) \tag{3.39}$$

Recall the decomposition that we have in (3.21).

$$\begin{aligned}
&\sum_{m=d^{\psi}_{j,\ell}+1-2^j}^{d^{\psi}_{j,\ell}+2^j+2^{j+1}}\sum_{k\in\mathbb{Z}}[b(m+2^{j+2}k-2n)]_{p,\ell}F_{c,\epsilon_1,\epsilon_2}(\phi^{\ell}_{2^j;m})\\
&\quad=\sum_{m=d^{\phi}_{j,\ell}+1}^{d^{\phi}_{j,\ell}+2^j}\sum_{k\in\mathbb{Z}}[b(m+2^{j+2}k-2n)]_{p,\ell}F_{c,\epsilon_1,\epsilon_2}(\phi^{\ell}_{2^j;m})\\
&\qquad+\epsilon_1\epsilon^{\phi}_{\ell}\sum_{m=d^{\phi}_{j,\ell}+1-2^j}^{d^{\phi}_{j,\ell}}\sum_{k\in\mathbb{Z}}[b(m+2^{j+2}k-2n)]_{p,\ell}F_{c,\epsilon_1,\epsilon_2}(\phi^{\ell}_{2^j;2^jc-c^{\phi}_{\ell}-m})\\
&\qquad+\epsilon_2\epsilon^{\phi}_{\ell}\sum_{m=d^{\phi}_{j,\ell}+1+2^j}^{d^{\phi}_{j,\ell}+2^{j+1}}\sum_{k\in\mathbb{Z}}[b(m+2^{j+2}k-2n)]_{p,\ell}F_{c,\epsilon_1,\epsilon_2}(\phi^{\ell}_{2^j;2^jc-c^{\phi}_{\ell}-m+2^{j+1}})\\
&\qquad+\epsilon_1\epsilon_2\sum_{m=d^{\phi}_{j,\ell}+2^{j+1}+1}^{d^{\phi}_{j,\ell}+2^j+2^{j+1}}\sum_{k\in\mathbb{Z}}[b(m+2^{j+2}k-2n)]_{p,\ell}F_{c,\epsilon_1,\epsilon_2}(\phi^{\ell}_{2^j;m-2^{j+1}}).
\end{aligned} \tag{3.40}$$

For the case when $o^{\phi}_{j,\ell}=1$, we have

$$\sum_{m=d^{\psi}_{j,\ell}+1-2^j}^{d^{\psi}_{j,\ell}+2^j+2^{j+1}}\sum_{k\in\mathbb{Z}}[b(m+2^{j+2}k-2n)]_{p,\ell}F_{c,\epsilon_1,\epsilon_2}(\phi^{\ell}_{2^j;m})$$

$$
\begin{aligned}
&= \sum_{m=d^{\phi}_{j,\ell}+1}^{d^{\phi}_{j,\ell}+2^j} \sum_{k\in\mathbb{Z}} \big([b(m+2^{j+2}k-2n)]_{p,\ell} \\
&\quad + \epsilon_1\epsilon^{\phi}_{\ell}[b(2^jc-c^{\phi}_{\ell}-m+2^{j+2}k-2n)]_{p,\ell} \\
&\quad + \epsilon_2\epsilon^{\phi}_{\ell}[b(2^jc-c^{\phi}_{\ell}-m+2^{j+1}(2k+1)-2n)]_{p,\ell} \\
&\quad + \epsilon_1\epsilon_2[b(m+2^{j+1}(2k+1)-2n)]_{p,\ell}\big)\, F_{c,\epsilon_1,\epsilon_2}(\phi^{\ell}_{2^j;m}) \\
&= \sum_{m=d^{\phi}_{j,\ell}+1}^{d^{\phi}_{j,\ell}+2^j} \sum_{k\in\mathbb{Z}} [b^n_{j,\ell}(m)]_{p,\ell} F_{c,\epsilon_1,\epsilon_2}(\phi^{\ell}_{2^j;m}),
\end{aligned}
$$

where

$$
\begin{aligned}
b^n_{j,\ell}(m) :=& b(m+2^{j+2}k-2n) + \epsilon_1\epsilon^{\phi}_{\ell} b(2^jc-c^{\phi}_{\ell}-m+2^{j+2}k-2n) \\
&+ \epsilon_2\epsilon^{\phi}_{\ell} b(2^jc-c^{\phi}_{\ell}-m+2^{j+1}(2k+1)-2n) \\
&+ \epsilon_1\epsilon_2 b(m+2^{j+1}(2k+1)-2n),
\end{aligned}
\tag{3.41}
$$

where $m \in \mathbb{Z}$.

For the case when $o^{\phi}_{j,\ell} = 0$, recall that we have 4 subcases. In the following, we deal with a large part that all 4 subcases share. This consists of all the terms $m = \{d^{\phi}_{j,\ell}+1, \ldots, d^{\phi}_{j,\ell}+2^j-1\}$. In order to find the new filter $b^n_{j,\ell}$, we simply apply similar decomposition and the identical change of variables as in (3.40), which allow us to reach (3.41) for $m = \{d^{\phi}_{j,\ell}+1, \ldots, d^{\phi}_{j,\ell}+2^j-1\}$. Let us proceed to the boundary terms, which are $m = \{d^{\phi}_{j,\ell}, d^{\phi}_{j,\ell}+2^j\}$. Suppose $\epsilon_2 \neq \epsilon_1 = \epsilon^{\phi}_{\ell}$. Then, since (3.23) holds, we shall only consider $m = d^{\phi}_{j,\ell}, d^{\phi}_{j,\ell}+2^{j+1}$ from (3.40),

$$
\sum_{k\in\mathbb{Z}} \Big(\epsilon_1\epsilon^{\phi}_{\ell}[b(d^{\phi}_{j,\ell}+2^{j+2}k-2n)]_{p,\ell}
$$

$$
\begin{aligned}
&+\epsilon_2\epsilon_\ell^\phi[b(d_{j,\ell}^\phi+2^{j+1}(2k+1)-2n)]_{p,\ell}\Big) F_{c,\epsilon_1,\epsilon_2}(\phi^\ell_{2^j;d^\phi_{j,\ell}})\\
=&\frac{1}{2}\epsilon_1\epsilon_\ell^\phi\sum_{k\in\mathbb{Z}}\Bigg([b(d_{j,\ell}^\phi+2^{j+2}k-2n)]_{p,\ell}F_{c,\epsilon_1,\epsilon_2}(\phi^\ell_{2^j;d^\phi_{j,\ell}})\\
&+\epsilon_1\epsilon_\ell^\phi[b(d_{j,\ell}^\phi+2^{j+2}k-2n)]_{p,\ell}F_{c,\epsilon_1,\epsilon_2}(\phi^\ell_{2^j;2^jc-c^\phi_\ell-d^\phi_{j,\ell}})\\
&+\epsilon_2\epsilon_\ell^\phi[b(d_{j,\ell}^\phi+2^{j+1}(2k+1)-2n)]_{p,\ell}F_{c,\epsilon_1,\epsilon_2}(\phi^\ell_{2^j;2^jc-c^\phi_\ell-d^\phi_{j,\ell}})\\
&+\epsilon_1\epsilon_2[b(d_{j,\ell}^\phi+2^{j+1}(2k+1)-2n)]_{p,\ell}F_{c,\epsilon_1,\epsilon_2}(\phi^\ell_{2^j;d^\phi_{j,\ell}})\Bigg)\\
=&\sum_{k\in\mathbb{Z}}\frac{1}{2}[b^n_{j,\ell}(d^\phi_{j,\ell})]_{p,\ell}F_{c,\epsilon_1,\epsilon_2}(\phi^\ell_{2^j;d^\phi_{j,\ell}})
\end{aligned}
\tag{3.42}
$$

where we have applied identity (3.4) and a change of variables to move from the second inequality to the third inequality.

Suppose $\epsilon_1 \neq \epsilon_2 = \epsilon_\ell^\phi$. Then, since (3.29) holds, we shall only consider $m = d^\phi_{j,\ell} + 2^j, d^\phi_{j,\ell} + 2^j + 2^{j+1}$ from (3.40),

$$
\begin{aligned}
&\sum_{k\in\mathbb{Z}}\Big([b(d_{j,\ell}^\phi+2^j+2^{j+2}k-2n)]_{p,\ell}\\
&\quad+\epsilon_1\epsilon_2[b(d_{j,\ell}^\phi+2^j+2^{j+1}(2k+1)-2n)]_{p,\ell}\Big) F_{c,\epsilon_1,\epsilon_2}(\phi^\ell_{2^j;d^\phi_{j,\ell}+2^j})\\
&=\frac{1}{2}\sum_{k\in\mathbb{Z}}\Bigg([b(d_{j,\ell}^\phi+2^j+2^{j+2}k-2n)]_{p,\ell}F_{c,\epsilon_1,\epsilon_2}(\phi^\ell_{2^j;d^\phi_{j,\ell}+2^j})\\
&\quad+\epsilon_1\epsilon_\ell^\phi[b(d_{j,\ell}^\phi+2^j+2^{j+1}+2^{j+2}k-2n)]_{p,\ell}F_{c,\epsilon_1,\epsilon_2}(\phi^\ell_{2^j;2^jc-c^\phi_\ell-d^\phi_{j,\ell}-2^j-2^{j+1}})\\
&\quad+\epsilon_2\epsilon_\ell^\phi[b(d_{j,\ell}^\phi+2^j+2^{j+2}k-2n)]_{p,\ell}F_{c,\epsilon_1,\epsilon_2}(\phi^\ell_{2^j;2^jc-c^\phi_\ell-d^\phi_{j,\ell}+2^j})\\
&\quad+\epsilon_1\epsilon_2[b(d_{j,\ell}^\phi+2^j+2^{j+1}(2k+1)-2n)]_{p,\ell}F_{c,\epsilon_1,\epsilon_2}(\phi^\ell_{2^j;d^\phi_{j,\ell}+2^j})\Bigg)\\
&=\sum_{k\in\mathbb{Z}}\frac{1}{2}[b^n_{j,\ell}(d^\phi_{j,\ell}+2^j)]_{p,\ell}F_{c,\epsilon_1,\epsilon_2}(\phi^\ell_{2^j;d^\phi_{j,\ell}+2^j}),
\end{aligned}
\tag{3.43}
$$

where we observe that $F_{c,\epsilon_1,\epsilon_2}(\phi^{\ell}_{2^j;2^jc-c^{\phi}_{\ell}-d^{\phi}_{j,\ell}-2^j-2^{j+1}}) = F_{c,\epsilon_1,\epsilon_2}(\phi^{\ell}_{2^j;d^{\phi}_{j,\ell}+2^j})$ by the periodicity of the folding operator; additionally,

$$\sum_{k\in\mathbb{Z}}[b(d^{\phi}_{j,\ell}+2^j+2^{j+1}+2^{j+2}k-2n)]_{p,\ell} = \sum_{k\in\mathbb{Z}}[b(d^{\phi}_{j,\ell}-2^j+2^{j+2}(k+1)-2n)]_{p,\ell}.$$

These two items help us to arrive at (3.41), where $m = d^{\phi}_{j,\ell}+2^j$.

Lastly, since the case $\epsilon_1 = \epsilon_2 = \epsilon^{\phi}_{\ell}$ requires two boundaries, we naturally include (3.42) and (3.43). Due to (3.39), the existence of a $\#\vec{\Psi}_{j-1}\times\#\vec{\Phi}_j$ matrix B_j such that $\vec{\Psi}_{j-1} = B_j\vec{\Phi}_j$ is guaranteed from which item (3) follows. $\square$

Before we present our last theorem for this section, we state two significant results, which are [33, Corollary 6.4.7] and [33, Corollary 6.4.8] respectively. The formal definition of $\mathrm{sm}(a)$ is defined in [33]. However, by [33, Theorem 6.3.3], we have a result that says $(\phi) = \mathrm{sm}(a)$ given that the integer shifts of ϕ are stable, and ϕ is an $r\times 1$ compactly supported distribution satisfying $\widehat{\phi}(0)\neq 0$ and $\widehat{\phi}(2\xi) = \widehat{a}(\xi)\widehat{\phi}(\xi)$. Also, recall that $\mathrm{sm}(\phi) := \sup\{\tau\in\mathbb{R} : \phi\in H^{\tau}(\mathbb{R})\}$, where ϕ here is assumed to be a tempered distribution.

Lemma 3.1. *[33, Corollary 6.4.7] Let $a, b, \tilde{a}, \tilde{b} \in (l_0(\mathbb{Z}))^{r\times r}$ such that the following statement holds for a and $\tilde{a}$: 1 is a simple eigenvalue of $\widehat{a}(0)$ and $\det(2^jI_r-\widehat{a}(0))\neq 0$ for all $j\in\mathbb{N}$. Let ϕ and $\tilde{\phi}$ be $r\times 1$ vectors of compactly supported distributions satisfying $\widehat{\phi}(2\xi) = \widehat{a}(\xi)\widehat{\phi}(\xi)$ and $\widehat{\tilde{\phi}}(2\xi) = \widehat{\tilde{a}}(\xi)\widehat{\tilde{\phi}}(\xi)$. Define $\widehat{\psi}(\xi) = \widehat{b}(\xi/2)\widehat{\psi}(\xi/2)$ and $\widehat{\tilde{\psi}}(\xi) = \widehat{\tilde{b}}(\xi/2)\widehat{\tilde{\psi}}(\xi/2)$. For $\tau\in\mathbb{R}$, $(\{\tilde{\phi};\tilde{\psi}\},\{\phi;\psi\})$ is a biorthogonal wavelet in $(H^{\tau}(\mathbb{R}), H^{-\tau}(\mathbb{R}))$ if and only if*

1. $(\{\tilde{a};\tilde{b}\},\{a;b\})$ *is a biorthogonal wavelet filter bank satisfying*

$$\begin{bmatrix} \widehat{\tilde{a}}(\xi) & \widehat{\tilde{a}}(\xi+\pi) \\ \widehat{\tilde{b}}(\xi) & \widehat{\tilde{b}}(\xi+\pi) \end{bmatrix} \begin{bmatrix} \overline{\widehat{a}(\xi)}^{\mathsf{T}} & \overline{\widehat{a}(\xi+\pi)}^{\mathsf{T}} \\ \overline{\widehat{b}(\xi)}^{\mathsf{T}} & \overline{\widehat{b}(\xi+\pi)}^{\mathsf{T}} \end{bmatrix} = I_{2r}; \qquad (3.44)$$

2. $sm(a) > \tau$ *and* $sm(\tilde{a}) > -\tau$*;*

3. $\widehat{\tilde{\phi}}(0)^{\mathsf{T}}\overline{\widehat{\phi}(0)} = 1.$

Lemma 3.2. *[33, Corollary 6.4.8] Let* $a, b \in (l_0(\mathbb{Z}))^{r\times r}$*. Let* ϕ *be an* $r \times 1$ *a vector of compactly supported distributions satisfying* $\widehat{\phi}(2\xi) = \widehat{a}(\xi)\widehat{\phi}(\xi)$ *for all* $\xi \in \mathbb{R}$*. Define* ψ *by* $\widehat{\psi}(\xi) = \widehat{b}(\xi/2)\widehat{\phi}(\xi/2)$*. Then* $\{\phi;\psi\}$ *is an orthogonal wavelet in* $L_2(\mathbb{R})$ *if and only if*

1. $\{a;b\}$ *is an orthogonal wavelet filter bank; that is,* (3.44) *holds with* $\tilde{a} = a$ *and* $\tilde{b} = b$*;*

2. $sm(a) > 0$*;*

3. $\|\widehat{\phi}(0)\|_{l_2}^2 = \widehat{\phi}(0)^{\mathsf{T}}\overline{\widehat{\phi}(0)} = 1.$

Theorem 3.1 and Proposition 3.3 culminate in the following theorem. The argument is pretty much verbatim from [33, Theorem 7.5.5] or [3, Theorem 2.1], since most of the calculations have been handled in Theorem 3.1 and Proposition 3.3.

Theorem 3.2. *Let* $\phi := (\phi^1,\ldots,\phi^r)^{\mathsf{T}}$*,* $\psi := (\psi^1,\ldots,\psi^s)^{\mathsf{T}}$*,* $\tilde{\phi} := (\tilde{\phi}^1,\ldots,\tilde{\phi}^r)^{\mathsf{T}}$*,* $\tilde{\psi} := (\tilde{\psi}^1,\ldots,\tilde{\psi}^s)^{\mathsf{T}}$*. Assume that* $\|\widehat{\phi}(0)\|_{L_2} = 1$ *and* $\overline{\widehat{\phi}(0)}^{\mathsf{T}}\widehat{\tilde{\phi}}(0)^{\mathsf{T}} = 1$*. Suppose that there exist* $a, \tilde{a} \in (l_0(\mathbb{Z}))^{r\times r}$ *and* $b, \tilde{b} \in (l_0(\mathbb{Z}))^{s\times r}$ *such that*

$$\widehat{\phi}(2\xi) = \widehat{a}(\xi)\widehat{\phi}(\xi), \quad \widehat{\tilde{\phi}}(2\xi) = \widehat{\tilde{a}}(\xi)\widehat{\tilde{\phi}}(\xi), \widehat{\psi}(2\xi) = \widehat{b}(\xi)\widehat{\psi}(\xi), \quad \widehat{\tilde{\psi}}(2\xi) = \widehat{\tilde{b}}(\xi)\widehat{\tilde{\psi}}(\xi).$$

Then, there exist matrices $A_j, B_j, \tilde{A}_j, \tilde{B}_j$ such that

$$\vec{\Phi}_{j-1} = A_j \vec{\Phi}_j, \quad \vec{\Psi}_{j-1} = B_j \vec{\Phi}_j, \quad \vec{\tilde{\Phi}}_{j-1} = \tilde{A}_j \vec{\tilde{\Phi}}_j, \quad \vec{\tilde{\Psi}}_{j-1} = \tilde{B}_j \vec{\tilde{\Phi}}_j, \tag{3.45}$$

for all $j \in \mathbb{N}$. Moreover, for every $J \in \mathbb{N}$,

1. *if $(\{\tilde{a}; \tilde{b}\}, \{a; b\})$ is a biorthogonal wavelet filter bank satisfying* (3.44) *and if $sm(a) > 0$ and $sm(\tilde{a}) > 0$, then $(\{\tilde{\Phi}; \tilde{\Psi}\}, \{\Phi; \Psi\})$ is a biorthogonal wavelet in $L_2(\mathbb{R})$, $(\tilde{\mathcal{B}}_J, \mathcal{B}_J)$ is a pair of biorthogonal bases for $L_2(\mathcal{I})$, and the square matrix $\begin{bmatrix} A_j \\ B_j \end{bmatrix}$ has the inverse $\begin{bmatrix} \overline{\tilde{A}_j}^{\mathsf{T}}, \overline{\tilde{B}_j}^{\mathsf{T}} \end{bmatrix}$; i.e.,*

$$\overline{\tilde{A}_j}^{\mathsf{T}} A_j + \overline{\tilde{B}_j}^{\mathsf{T}} B_j = Id; \tag{3.46}$$

2. *if $\{a; b\}$ is an orthogonal wavelet filter bank and $sm(a) > 0$, then Ψ is an orthogonal wavelet in $L_2(\mathbb{R})$ and $\mathcal{B}_J$ is an orthonormal basis for $L_2(\mathcal{I})$.*

Proof. We know that $(\{\tilde{\Phi}; \tilde{\Psi}\}, \{\Phi; \Psi\})$ is a biorthogonal wavelet in $L_2(\mathbb{R})$ due to Lemma 3.1 with $\tau = 0$. Hence, $\{\tilde{\mathcal{B}}_J, \mathcal{B}_J\}$ is a pair of biorthogonal bases for $L_2(\mathcal{I})$ by Theorem 3.1. Furthermore, for all $f \in L_2(\mathcal{I})$,

$$\langle f, \vec{\tilde{\Phi}}_{j-1} \rangle \vec{\Phi}_{j-1} + \langle f, \vec{\tilde{\Psi}}_{j-1} \rangle \vec{\Psi}_{j-1} = \langle f, \vec{\tilde{\Phi}}_j \rangle \vec{\Phi}_j. \tag{3.47}$$

Inserting (3.45) to the identity above, we have

$$\langle f, \vec{\tilde{\Phi}}_j \rangle \left(\overline{\tilde{A}_j}^{\mathsf{T}} A_j + \overline{\tilde{B}_j}^{\mathsf{T}} B_j \right) \vec{\Phi}_j = \langle f, \vec{\tilde{\Phi}}_j \rangle \vec{\Phi}_j. \tag{3.48}$$

In order to show $\overline{\tilde{A}_j}^\top A_j + \overline{\tilde{B}_j}^\top B_j = \mathrm{Id}$ is indeed true, we note that the entries of Φ_j and the entries of $\tilde{\Phi}_j$ have to be linearly independent due to the biorthogonality of $\tilde{\Phi}_j$ and Φ_j. Since the entries in $\tilde{\vec{\Phi}}_j$ are linearly independent, the mapping $f \in L_2(\mathcal{I}) \mapsto \langle f, \tilde{\vec{\Phi}}_j \rangle \in (l_2)^{1 \times \# \tilde{\vec{\Phi}}_j}$ is onto. Identity (3.46) then is obtained, because the entries of $\vec{\Phi}_j$ are also linearly independent. From (3.47), we have that $\#\vec{\Phi}_j \leqslant \#\vec{\Phi}_{j-1} + \#\vec{\Psi}_{j-1}$. At the same time, we know that $\vec{\Phi}_{j-1} \cup \vec{\Psi}_{j-1}$ is biorthogonal to $\tilde{\vec{\Phi}}_{j-1} \cup \tilde{\vec{\Psi}}_{j-1}$. So, (3.47) again gives us $\#\vec{\Phi}_j \geqslant \#\vec{\Phi}_{j-1} + \#\vec{\Psi}_{j-1}$. We can conclude that $\#\vec{\Phi}_j = \#\vec{\Phi}_{j-1} + \#\vec{\Psi}_{j-1}$. Therefore, $\begin{bmatrix} A_j \\ B_j \end{bmatrix}$ and $\begin{bmatrix} \overline{\tilde{A}_j}^\top, \overline{\tilde{B}_j}^\top \end{bmatrix}$ have to be square matrices.

Lemma 3.2 and item (1) finally give us item (2). $\square$

Chapter 4

Assessing the Numerical Performance

The goal of this chapter is twofold: firstly, we show how we can introduce a simple scaling factor to our wavelet bases in order to obtain a quasi-optimal condition number for the energy norm. Secondly, by taking into account this scaling factor, we evaluate the numerical performance of our wavelet bases. The examples discussed in Section 4.2 are taken directly from [35].

4.1 Minimizing the Condition Numbers of Wavelet Bases

For the purpose of finding the numerical solution to a differential equation, we certainly want to have a Riesz wavelet that has many nice features. One feature that we desire is a small condition number in the energy norm. Since we have a Riesz wavelet, this condition number is simply the ratio of the upper bound and lower constants. More practically speaking, if A is a matrix of inner products (i.e. A is symmetric positive definite), the condition number is just the ratio of the maximum and minimum eigenvalues of A.

When generating a Riesz wavelet that satisfies the mth order derivative–orthogonality in Chapter 2, there is one degree of freedom that typically remains. Indeed, we can scale our wavelet basis in a way such that we obtain a quasi-optimal condition number. Scaling is probably the most effortless operation that we can employ to reduce the condition number, while preserving the shift-invariant and localization properties. It should not come as a surprise that the optimization problem we are headed is almost the same as finding the optimal pre-conditioner.

Proposition 4.1. *Suppose* $\det([\widehat{\psi^{(m)}}, \widehat{\psi^{(m)}}](\xi)) \neq 0$ *for all* $\xi \in \mathbb{R}$*, where* $m \in \mathbb{N} \cup \{0\}$ *and* $\psi^{(m)} := (\psi_1^{(m)}, \psi_2^{(m)})^\mathsf{T}$*. Let* $\theta > 0$*. If*

$$[diag(\theta, 1)\widehat{\psi^{(m)}}, diag(\theta, 1)\widehat{\psi^{(m)}}](\xi) := \begin{bmatrix} \theta^2 \eta_{11}(\xi) & \theta \eta_{12}(\xi) \\ \theta \eta_{21}(\xi) & \eta_{22}(\xi) \end{bmatrix},$$

then the condition number of $[diag(\theta, 1)\widehat{\psi^{(m)}}, diag(\theta, 1)\widehat{\psi^{(m)}}](\xi)$ *can be reduced by considering the following problem*

$$\sup_{\theta \in \mathbb{R}, \theta > 0} \inf_{\xi \in \mathbb{R}} \frac{4\theta^2 \det([\widehat{\psi^{(m)}}, \widehat{\psi^{(m)}}](\xi))}{(\theta^2 \eta_{11}(\xi) + \eta_{22}(\xi))^2}. \tag{4.1}$$

Proof. We first note that $[\widehat{\psi^{(m)}}, \widehat{\psi^{(m)}}](\xi) = \overline{[\widehat{\psi^{(m)}}, \widehat{\psi^{(m)}}](\xi)}^\mathsf{T}$ for all $\xi \in \mathbb{R}$, which means that $[\widehat{\psi^{(m)}}, \widehat{\psi^{(m)}}](\xi)$ is a Hermitian matrix. Hence, all eigenvalues are real.

Now, $\det([\operatorname{diag}(\theta,1)\widehat{\psi^{(m)}},\operatorname{diag}(\theta,1)\widehat{\psi^{(m)}}](\xi)-\lambda I)=\lambda^2-\lambda(\theta^2\eta_{11}(\xi)+\eta_{22}(\xi))+\theta^2\det([\widehat{\psi^{(m)}},\widehat{\psi^{(m)}}](\xi))=0$ if and only if

$$\lambda_{1,2}=\frac{\theta^2\eta_{11}(\xi)+\eta_{22}(\xi)\pm\sqrt{(\theta^2\eta_{11}(\xi)+\eta_{22}(\xi))^2-4\det([\widehat{\psi^{(m)}},\widehat{\psi^{(m)}}](\xi))}}{2}.$$

Since the condition number (denoted by $\kappa(\cdot)$) is just the ratio of the largest and smallest singular values and the singular values in our case can be found by taking the absolute value of the eigenvalues, we have that

$$\begin{aligned}
&\kappa([\operatorname{diag}(\theta,1)\widehat{\psi^{(m)}},\operatorname{diag}(\theta,1)\widehat{\psi^{(m)}}](\xi))\\
&=\frac{\theta^2\eta_{11}(\xi)+\eta_{22}(\xi)+\sqrt{(\theta^2\eta_{11}(\xi)+\eta_{22}(\xi))^2-4\theta^2\det([\widehat{\psi^{(m)}},\widehat{\psi^{(m)}}](\xi))}}{\theta^2\eta_{11}(\xi)+\eta_{22}(\xi)-\sqrt{(\theta^2\eta_{11}(\xi)+\eta_{22}(\xi))^2-4\theta^2\det([\widehat{\psi^{(m)}},\widehat{\psi^{(m)}}](\xi))}},\qquad(4.2)\\
&=\frac{1+\sqrt{1-\frac{4\theta^2\det([\widehat{\psi^{(m)}},\widehat{\psi^{(m)}}](\xi))}{(\theta^2\eta_{11}(\xi)+\eta_{22}(\xi))^2}}}{1-\sqrt{1-\frac{4\theta^2\det([\widehat{\psi^{(m)}},\widehat{\psi^{(m)}}](\xi))}{(\theta^2\eta_{11}(\xi)+\eta_{22}(\xi))^2}}}.
\end{aligned}$$

We note that the current condition number can be found by setting $\theta=1$ and taking the supremum of (4.2) with respect to $\xi\in\mathbb{R}$. In order to reduce the condition number, we want to consider the following problem

$$\inf_{\theta\in\mathbb{R},\theta>0}\sup_{\xi\in\mathbb{R}}\kappa([\operatorname{diag}(\theta,1)\widehat{\psi^{(m)}},\operatorname{diag}(\theta,1)\widehat{\psi^{(m)}}](\xi)).\qquad(4.3)$$

That is for a given scaling θ, we want to find the associated condition number, which entails finding the supremum with respect to $\xi\in\mathbb{R}$. Given $\xi(\theta)$, we want to find the infimum of all possible condition numbers with respect to θ. We note that $\frac{1+\sqrt{1-x}}{1-\sqrt{1-x}}$ is a decreasing function of x, which means we want x to be as large as possible. Hence, (4.3) and (4.2) give (4.1). $\square$

Our experiments suggest that a good choice for the scaling factor is $\theta^* = \sqrt{\frac{\eta_{22}(\pi/2)}{\eta_{11}(\pi/2)}}$. The next section requires us to find the Galerkin formulation of a differential equation, which gives rise to a linear system that needs to be solved. Now, in order for such a linear system to have a small condition number, Proposition 4.1 and the preceding discussion suggest that we should normalize the wavelet basis such that the L_2 norm of the mth order derivative of the wavelet basis is equal to 1.

4.2 Numerical Results

In this section, we present some examples to demonstrate the performance of our constructed derivative-orthogonal Riesz wavelets constructed in Section 2. In order to showcase the flexibility of our constructed wavelet bases, we use different types of Riesz wavelets to solve one-dimensional Sturm-Liouville differential equations and biharmonic equations with various types of boundary conditions. That is, we do not first transform the differential equations to make their boundary conditions homogeneous. This shows the flexibility of our constructed wavelets.

We shall use the mth-order derivative-orthogonal Riesz wavelets constructed from Hermite linear splines with $m = 1$, Hermite quadratic splines with $m = 1$, and Hermite cubic splines with $m = 2$. Since all of those wavelets have symmetry and are supported inside $[-1, 1]$, we shall mainly use Theorem 3.1 for building Riesz wavelet bases on the interval $[0, 1]$ derived from wavelets on the real line. As can be seen in a moment, the boundary elements are obtained by a simple restriction operation on the interval $[0,1]$. This simple restriction operation is primarily induced by the folding operator in Theorem 3.1 and the short support of our Riesz wavelets. Referring back to the notations used in Theorem 3.1, we have $c = 0$, $c_1^\phi = c_1^\psi = c_2^\phi = c_2^\psi = 0$, $\epsilon_1^\phi = \epsilon_1^\psi = 1$, and $\epsilon_2^\phi = \epsilon_2^\psi = -1$ for the Riesz wavelets constructed from Hermite linear and quadratic splines we shall use momentarily. Additionally, for the Riesz wavelet constructed from Hermite cubic splines, we have $c = 0$, $c_1^\phi = c_2^\phi = 0$, $c_1^\psi = c_2^\psi = 1$, $\epsilon_1^\phi = \epsilon_1^\psi = 1$, and $\epsilon_2^\phi = \epsilon_2^\psi = -1$.

We shall follow the same set-up as in [41] in that the coarsest scale level that we will use is 1 instead of 0. Let n be the scale or resolution level in our numerical scheme. At the coarsest scale, we have only four elements $\Phi_0 := \{\phi^L(2\cdot), \phi_1(2\cdot -1), \phi_2(2\cdot -1), \phi^R(2\cdot)\}$ and at every scale level $j \geqslant 1$ we have only one left boundary wavelet function ψ^L and one right boundary wavelet function ψ^R such that all the boundary elements

$$\begin{aligned} &\phi^L \in \{\phi_1|_{[0,1]}, \phi_2|_{[0,1]}\}, \qquad \phi^R \in \{\phi_1(\cdot - 1)|_{[0,1]}, \phi_2(\cdot - 1)|_{[0,1]}\}, \\ &\psi^L \in \{\psi_1|_{[0,1]}, \psi_2|_{[0,1]}\}, \qquad \psi^R \in \{\psi_1(\cdot - 1)|_{[0,1]}, \psi_2(\cdot - 1)|_{[0,1]}\} \end{aligned} \tag{4.4}$$

are chosen according to the boundary conditions of a given differential equation. To simplify our notation, we enumerate our basis as follows using n scale level in our numerical scheme to find the weak solution $u \in H^m(\mathbb{R})$. Define $g_1, \ldots, g_4$ at the coarse scale such that $\operatorname{span}\{g_1, \ldots, g_4\} = \operatorname{span}\Phi_0$ and $\{g_1^{(m)}, \ldots, g_4^{(m)}\}$ is an orthonormal system in $L_2(\mathbb{R})$. For $1 \leqslant j \leqslant n$, we define

$$\begin{aligned} g_{2^{j+1}+1} &:= 2^{j(1/2-m)}\psi^L(2^j\cdot)/\|[\psi^L]^{(m)}\|_{L_2(\mathbb{R})}, \\ g_{2^{j+1}+2^j} &:= 2^{j(1/2-m)}\psi^R(2^j\cdot - 2^j)/\|[\psi^R]^{(m)}\|_{L_2(\mathbb{R})}, \end{aligned} \tag{4.5}$$

and for all $k = 1, \ldots, 2^j - 1$,

$$\begin{aligned} g_{2^{j+1}+(2k-1)} &:= 2^{j(1/2-m)}\psi_1(2^j \cdot -k)/\|\psi_1^{(m)}\|_{L_2(\mathbb{R})}, \\ g_{2^{j+1}+2k} &:= 2^{j(1/2-m)}\psi_2(2^j \cdot -k)/\|\psi_2^{(m)}\|_{L_2(\mathbb{R})}. \end{aligned} \tag{4.6}$$

Keeping Theorem 4.1 in mind, we have used the normalization $\|g_k^{(m)}\|_{L_2(\mathbb{R})} = 1$ for all $k = 1, \ldots, 2^{n+2}$. Since ϕ has symmetry, if $\phi^L = \phi_\ell|_{[0,1]}$ for either $\ell = 1$ or 2, then it is trivial to have $\|[\phi^L]^{(m)}\|_{L_2(\mathbb{R})} = \|\phi_\ell^{(m)}\|_{L_2(\mathbb{R})}/\sqrt{2}$ and the same relation holds for $\|[\phi^R]^{(m)}\|_{L_2(\mathbb{R})}$, $\|[\psi^L]^{(m)}\|_{L_2(\mathbb{R})}$, and $\|[\psi^R]^{(m)}\|_{L_2(\mathbb{R})}$.

Because the linear systems we are dealing with are well conditioned, we simply use the conjugate gradient (CG) method to find the solution for each linear system without any preconditioning. More specifically, we use an error tolerance of 10^{-14} for the MATLAB built-in CG function. Let u be the true solution and u_n be the numerical solution with a scale level n of a given differential equation. We calculate the error L_∞ and L_2 norms of $e_n := u - u_n$ as follows: (1) fix a sufficiently large discretization level $J > n$ (we shall use $J = 19$), create a dyadic grid $\{0, 2^{-J}, \ldots, (1 - 2^{-J}), 1\} = 2^{-J}\mathbb{Z} \cap [0,1]$, (2) calculate $\|e_n\|_\infty := \max_{0 \le k \le 2^J} |u(k/2^J) - u_n(k/2^J)|$ and $\|e_n\|_{L_2} := \left(2^{-J} \sum_{k=0}^{2^J} |u(k/2^J) - u_n(k/2^J)|^2\right)^{1/2}$. Using the central difference of u_n for the first derivative, we also calculate the error in H_1 norm and give the convergence rates $\log_2 (\|e_{n-1}\|_{L_2}/\|e_n\|_{L_2})$ and $\log_2 (\|e_{n-1}\|_{H^1}/\|e_n\|_{H^1})$.

We also would like to point out that the calculation of the stiffness and mass matrices (as we shall see in (4.9) and (4.14)) can be done very efficiently. It suffices for us to find the eigenvector of the transition operator defined below

$$
\begin{aligned}
[\widehat{\phi^{(m)}},\widehat{\phi^{(m)}}](\xi) = 2^{2m}\Big(&\widehat{a}(\xi/2)[\widehat{\phi^{(m)}},\widehat{\phi^{(m)}}](\xi/2)\overline{\widehat{a}(\xi/2)}^{\mathsf{T}} \\
&+\widehat{a}(\xi/2+\pi)[\widehat{\phi^{(m)}},\widehat{\phi^{(m)}}](\xi/2+\pi)\overline{\widehat{a}(\xi/2+\pi)}^{\mathsf{T}}\Big),
\end{aligned}
$$

and use the refinability structure of our wavelet basis. Moreover, the computation of the right hand sides of (4.9) and (4.14) can be similarly carried out in an efficient manner by using the following approximation:

$$
\langle f, \phi_{2^j;k}\rangle \approx \sum_{\ell=0}^{n-1} 2^{-j(\ell+m+1)}\frac{\overline{c_\ell}}{\ell!}\frac{f^{(\ell)}(2^{-j}k)}{\|\phi^{(m)}\|_{L_2(\mathbb{R})}},
$$

where the coefficients c_ℓ can be recovered by computing the Taylor expansion $\widehat{g}(\xi) \approx \sum_{\ell=0}^{n-1}\frac{c_\ell}{\ell!}(-i\xi)^\ell$, and using the refinability structure of the wavelet basis again. Further details can be found in [33]. We can indeed show that the accuracy of the approximation is controlled by the smoothness of the function f and the scale level j.

4.2.1 Applications to Sturm-Liouville Differential Equations

Example 4.1. Consider the following Sturm-Liouville differential equation with nonhomogeneous boundary conditions:

$$
-u''+5u = f \quad \text{on} \quad (0,1) \quad \text{with} \quad u'(0) = 100(1-e^{-1}), \quad u(1) = 200e^{-1}-100, \tag{4.7}
$$

where $f(x) = -100e^{-x} - 500(1-e^{-x}) + 500e^{-1}x$. The exact solution to this differential equation is $u(x) = -(100(1-e^{-x}) - 100e^{-1}x)$. Using the scale level $n > 1$, we have the dimension $N = 2^{n+2}$ and the corresponding Galerkin formulation of (4.7) is just

$$\sum_{l=1}^{N} A_{k,l} c_l = \langle g_k, f \rangle, \quad k = 1, \ldots, N \tag{4.8}$$

with the $N \times N$ coefficient matrix A given by

$$A_{k,l} := \langle g_k', g_l' \rangle + \alpha \langle g_k, g_l \rangle, \qquad k, l = 1, \ldots, N, \tag{4.9}$$

where $\alpha = 5$. We use the first-order derivative-orthogonal Riesz wavelet $\{\phi; \psi\}$ for $H^1(\mathbb{R})$ derived from the Hermite quadratic spline in Example 2.5 with $m = 1$. According to the boundary conditions in (4.7), we choose the boundary elements as follows:

$$\phi^L = \phi_2|_{[0,1]}, \quad \phi^R = \phi_1(\cdot - 1)|_{[0,1]}, \quad \psi^L = \psi_2|_{[0,1]}, \quad \psi^R = \psi_1(\cdot - 1)|_{[0,1]}.$$

Using the norms in (2.41) and (2.42), the first four elements are given by

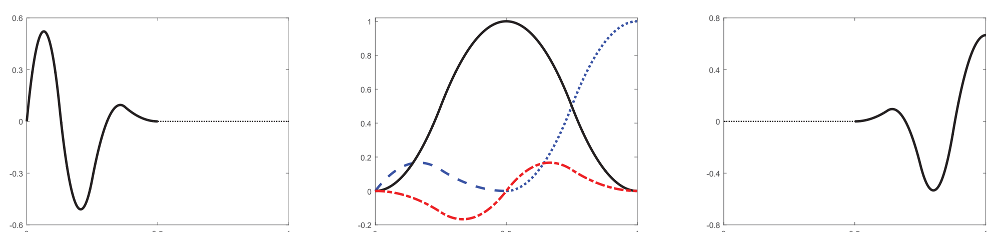
$$g_1 = \frac{\phi^L(2\cdot)}{\sqrt{2}\|[\phi^L]'\|_{L_2(\mathbb{R})}}, \quad g_2 = \frac{\phi_1(2 \cdot -1)}{\sqrt{2}\|\phi_1'\|_{L_2(\mathbb{R})}},$$

$$g_3 = \frac{\phi_2(2 \cdot -1)}{\sqrt{2}\|\phi_2'\|_{L_2(\mathbb{R})}}, \quad g_4 = \frac{\phi^R(2 \cdot -2)}{\sqrt{2}\|[\phi^R]'\|_{L_2(\mathbb{R})}}.$$

As mentioned earlier, the coefficient matrix A can be easily generated using the fast wavelet transform and the bracket products in (2.41) and (2.42), since $[\widehat{\phi^{(m)}}, \widehat{\phi^{(m)}}](\xi) = \sum_{k\in\mathbb{Z}} \langle \phi^{(m)}, \phi^{(m)}(\cdot - k) \rangle e^{-ik\xi}$. For all $n = 3, \ldots, 9$, we notice that the condition numbers of A are approximately 3.2106 and 15 iterations are sufficient by CG scheme with tolerance 10^{-14}.

Level	Size	Iteration	κ	$\|e_n\|_{L_\infty}$	$\|e_n\|_{L_2}$	$\log_2 \frac{\|e_{n-1}\|_{L_2}}{\|e_n\|_{L_2}}$
3	32	15	3.2106	2.7040e-04	1.1825e-05	—
4	64	15	3.2106	3.3781e-06	1.4602e-06	3.0176
5	128	15	3.2106	4.2213e-07	1.8141e-07	3.0089
6	256	15	3.2106	5.2757e-08	2.2607e-08	3.0044
7	512	15	3.2106	6.5948e-09	2.8215e-09	3.0022
8	1024	15	3.2106	8.2152e-10	3.5243e-10	3.0011
9	2048	16	3.2106	1.1396e-10	4.5919e-11	2.9402

Table 4.1: We use Example 2.5 with $m = 1$ to solve the differential equation in (4.7) for Example 4.1 at scale levels $n = 3, \dots, 9$. Size is the dimension of the linear system. Iteration is the number of CG iterations needed with tolerance 10^{-14}. κ is the condition number of the coefficient matrix A in (4.9) with $\alpha = 5$. $\log_2 \frac{\|e_{n-1}\|_{L_2}}{\|e_n\|_{L_2}}$ is the convergence rate which agrees with the sum rule order $\mathrm{sr}(a) = 3$.

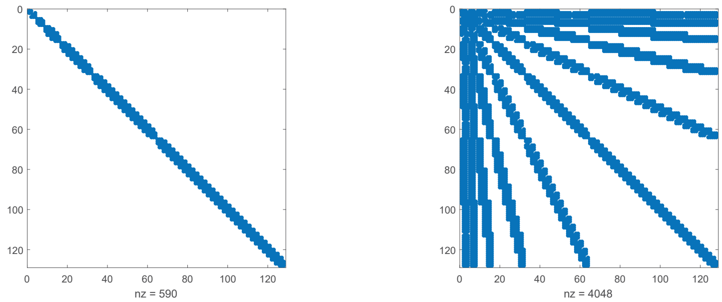


(a) (b) (c)

Figure 4.1: (a) is the left boundary wavelet $\psi_2(2\cdot)|_{[0,1]}$ at the scale level 1 for Example 4.1. (b) is the elements at the coarsest scale level 1, where the black line is $\phi_1(2\cdot -1)$, the red dotted–dashed line is $\phi_2(2\cdot -1)$, the blue dashed line is the left boundary scaling function $\phi_2(2\cdot)|_{[0,1]}$, and the blue dotted line is the right boundary scaling function $\phi_1(2\cdot -2)|_{[0,1]}$. (c) is the right boundary element $\psi_1(2\cdot -2)|_{[0,1]}$ at the scale level 1.

Example 4.2. We revisit [41, Example 1] with the homogeneous boundary conditions:

$$-u'' = f \quad \text{on } (0,1) \quad \text{with} \quad u(0) = u(1) = 0, \tag{4.10}$$



(a) (b)

Figure 4.2: (a) shows the sparsity of the 128×128 stiffness matrix in Example 4.1. The usage of the first derivative–orthogonal Riesz wavelet is clearly reflected by the block diagonal structure of the stiffness matrix. (b) shows the sparsity of the 128×128 mass matrix in Example 4.1 after some thresholding (rounding down entries smaller than 10^{-14} to 0).

where $f(x) = (53.7\pi)^2 \sin(53.7\pi x) + (2.3\pi)^2 \sin(2.3\pi x)$ for $x \in (0,1)$. The exact solution to this differential equation is $u(x) = \sin(53.7\pi x) + \sin(2.3\pi x)$. Instead of using the Riesz wavelet derived from Hermite cubic splines in [41], we use Example 2.5 with $m = 1$ derived from Hermite quadratic splines. Taking the scale level $n > 1$, we have the dimension $N = 2^{n+2}$ and the corresponding Galerkin approximation of this problem is (4.8) with $\alpha = 0$ in (4.9). According to the boundary conditions in (4.10), we choose the boundary elements as follows:

$$\phi^L = \phi_2|_{[0,1]}, \quad \phi^R = \phi_2(\cdot - 1)|_{[0,1]}, \quad \psi^L = \psi_2|_{[0,1]}, \quad \psi^R = \psi_2(\cdot - 1)|_{[0,1]},$$

where $j = 1, \ldots, n$. Using the norms in (2.42), the first four elements are given by

$$g_1 = \frac{\sqrt{210}}{56} g_2 - \frac{\sqrt{14}}{56} g_4 + \sqrt{\frac{15}{14}} \frac{\phi_2(2\cdot)|_{[0,1]}}{\sqrt{2}\|[\phi^L]'\|_{L_2(\mathbb{R})}}, \quad g_2 = \frac{\phi_1(2\cdot -1)}{\sqrt{2}\|\phi_1'\|_{L_2(\mathbb{R})}},$$

$$g_3 = \frac{\phi_2(2\cdot -1)}{\sqrt{2}\|\phi_2'\|_{L_2(\mathbb{R})}}, \quad g_4 = -\frac{1}{\sqrt{15}}g_2 + \frac{4}{\sqrt{15}}\frac{\phi_2(2\cdot -2)|_{[0,1]}}{\sqrt{2}\|[\phi^R]'\|_{L_2(\mathbb{R})}}.$$

The condition numbers of the resulting linear systems are approximately half of the condition numbers of the Riesz wavelet used in [41].

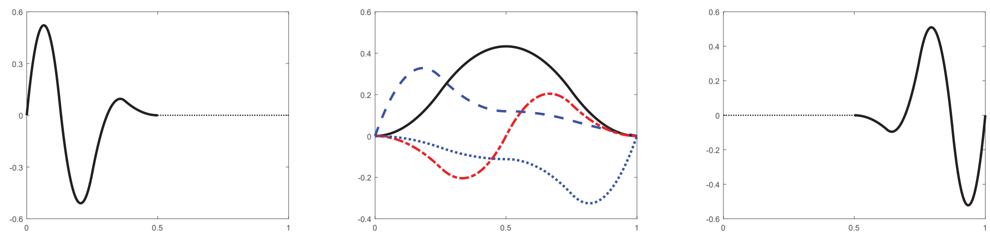
Level	Size	Iteration	κ	$\|e_n\|_{L_\infty}$	$\|e_n\|_{L_2}$	$\log_2 \frac{\|e_{n-1}\|_{L_2}}{\|e_n\|_{L_2}}$
6	256	16	1.7496	2.4874e-03	1.3123e-03	—
7	512	16	1.7499	2.9267e-04	1.4990e-04	3.1300
8	1024	16	1.75	3.6039e-05	1.8302e-05	3.0339
9	2048	16	1.75	4.4881e-06	2.2742e-06	3.0086
10	4096	16	1.75	5.6049e-07	2.8385e-07	3.0022
11	8192	16	1.75	6.9984e-08	3.5468e-08	3.0005
12	16384	16	1.75	8.7469e-09	4.4331e-09	3.0001

Table 4.2: We use Example 2.5 with $m = 1$ to solve the differential equation in (4.10) for Example 4.2 at scale levels $n = 6, \dots, 12$. Size is the dimension of the linear system. Iteration is the number of CG iterations needed with tolerance 10^{-14}. κ is the condition number of the coefficient matrix A in (4.9) with $\alpha = 0$. $\log_2 \frac{\|e_{n-1}\|_{L_2}}{\|e_n\|_{L_2}}$ is the convergence rate which agrees with the sum rule order $\operatorname{sr}(a) = 3$.

Example 4.3. Consider the following differential equation with mixed boundary conditions:

$$-u'' = f \quad \text{on } (0,1) \quad \text{with} \quad u'(0) = 0, \quad u(1) = 0, \tag{4.11}$$

where $f(x) = x\ln(x+1)$. Its exact solution is $u(x) = (5x^3 - 12x^2 - 12x + 19 - 24\ln(2) - 6(x-2)(x+1)^2\ln(x+1))/36$. Let us use the Hermite linear splines in Example 2.6 with $m = 1$. According to the boundary conditions in



(a) (b) (c)

Figure 4.3: (a) is the left boundary element $\psi_2(2\cdot)|_{[0,1]}$ at the scale level 1 for Example 4.2. (b) is the four linear transformed elements at the coarsest scale level 1, where the black line is g_2, the red dotted–dashed line is g_3, the blue dashed line is g_1, and the blue dotted line is g_4. (c) is the right boundary element $\psi_2(2\cdot -2)|_{[0,1]}$ at the scale level 1.

(4.11), we choose the boundary elements as follows:

$$\phi^L = \phi_1|_{[0,1]}, \quad \phi^R = \phi_2(\cdot - 1)|_{[0,1]}, \quad \psi^L = \psi_1|_{[0,1]}, \quad \psi^R = \psi_2(\cdot - 1)|_{[0,1]}.$$

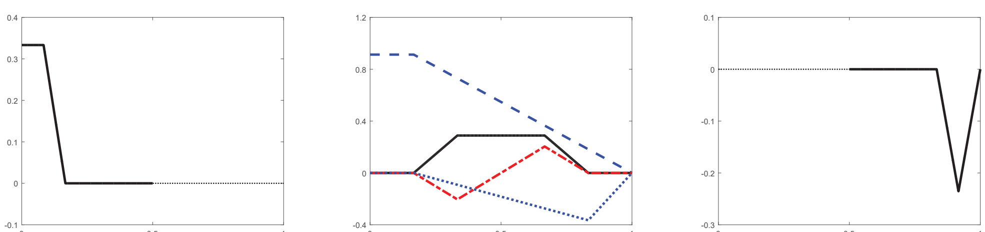
Using the norms in (2.45), the first four elements are given by

$$g_1 = \sqrt{\frac{5}{2}} g_2 - \frac{\sqrt{5}}{2} g_3 - \frac{1}{2} g_4 + \sqrt{5} \frac{\phi_1(2\cdot)|_{[0,1]}}{\sqrt{2}\|[\phi^L]'\|_{L_2(\mathbb{R})}}, \quad g_2 = \frac{\phi_1(2\cdot -1)}{\sqrt{2}\|\phi_1'\|_{L_2(\mathbb{R})}},$$
$$g_3 = \frac{\phi_2(2\cdot -1)}{\sqrt{2}\|\phi_2'\|_{L_2(\mathbb{R})}}, \quad g_4 = -\sqrt{\frac{2}{5}} g_2 - \frac{1}{\sqrt{5}} g_3 + \frac{2}{\sqrt{5}} \frac{\phi_2(2\cdot -2)|_{[0,1]}}{\|[\phi_R]'\|_{L_2(\mathbb{R})}}.$$

Notice that coefficient matrix A is exactly an identity matrix. Hence, there is no need to use any linear solver to compute the solution of (4.8). It is sufficient to just compute $\langle g_k, f\rangle$, where $k = 1, \ldots, N$.

Level	Size	κ	$\|e_n\|_{L_\infty}$	$\|e_n\|_{L_2}$	$\log_2 \frac{\|e_{n-1}\|_{L_2}}{\|e_n\|_{L_2}}$
6	256	1	2.3188e-06	6.6489e-07	—
7	512	1	5.8364e-07	1.6689e-07	1.9942
8	1024	1	1.4640e-07	4.1807e-08	1.9971
9	2048	1	3.6662e-08	1.0462e-08	1.9985
10	4096	1	9.1733e-09	2.6169e-09	1.9995
11	8192	1	2.2943e-09	6.5439e-10	1.9996
12	16384	1	5.7370e-10	1.6363e-10	1.9997

Table 4.3: We use Example 2.6 with $m = 1$ to solve the differential equation in (4.11) for Example 4.3 at scale levels $n = 6, \ldots, 12$. Size is the dimension of the linear system. Iteration is the number of CG iterations needed with tolerance 10^{-14}. κ is the condition number of the coefficient matrix A in (4.9) with $\alpha = 0$. $\log_2 \frac{\|e_{n-1}\|_{L_2}}{\|e_n\|_{L_2}}$ is the convergence rate which agrees with the sum rule order $\mathrm{sr}(a) = 2$.



(a) (b) (c)

Figure 4.4: (a) is the left boundary element $\psi_1(2\cdot)|_{[0,1]}$ at a scale level 1 for Example 4.3. (b) is the four linear transformed elements at the coarsest scale level 1, where the black line is g_2, the red dotted–dashed line is g_3, the blue dashed line is g_1, and the blue dotted line is g_4. (c) is the right boundary element, $\psi_2(2\cdot -2)|_{[0,1]}$, at the scale level 1.

4.2.2 Applications to Biharmonic Equations

For numerical solutions to biharmonic equations, we shall use the second-order derivative-orthogonal Riesz wavelets derived from Hermite cubic splines that are discussed in Example 2.4 with $m = 2$.

Example 4.4. Consider the following differential equation with homogeneous boundary conditions:

$$u^{(4)}+11u = f \quad \text{on} \quad (0,1) \quad \text{with} \quad u(0) = 0, \quad u(1) = 0, \quad u'(0) = 0, \quad u'(1) = 0, \tag{4.12}$$

where $f(x) = -4\pi^4 \cos(2\pi x) + \frac{11}{4}(1 - \cos(2\pi x))$. The exact solution to the differential equation is $u(x) = \frac{1}{4}(1-\cos(2\pi x))$. Using the scale level $n > 1$, we have the dimension $N = 2^{n+2}$ and the corresponding Galerkin approximation of the problem in (4.12) is

$$\sum_{l=1}^{N} B_{k,l} c_l = \langle g_k, f \rangle, \quad k = 1, \ldots, N, \tag{4.13}$$

with the $N \times N$ coefficient matrix B given by

$$B_{k,l} := \langle g_k'', g_l'' \rangle + \alpha \langle g_k, g_l \rangle, \quad k, l = 1, \ldots, N, \tag{4.14}$$

where $\alpha = 11$. As mentioned earlier, the coefficient matrix B can be easily generated using the fast wavelet transform and the bracket product in (2.36) and (2.38). According to the boundary conditions in (4.12), we choose the boundary elements as follows:

$$\phi^L = \emptyset, \quad \phi^R = \emptyset, \quad \psi^L = \emptyset, \quad \psi^R = \emptyset. \tag{4.15}$$

Using the norms in (2.38), the first four elements are given by

$$g_1 = \emptyset, \quad g_2 = 2^{-3/2}\frac{\phi_1(2\cdot -1)}{\|\phi_1''\|_{L_2(\mathbb{R})}}, \quad g_3 = 2^{-3/2}\frac{\phi_2(2\cdot -1)}{\|\phi_2''\|_{L_2(\mathbb{R})}}, \quad g_4 = \emptyset. \tag{4.16}$$

Since both ψ_1 and ψ_2 are supported on $[0,1]$, no boundary wavelets are present. Notice that the matrix B has a condition number around 1.0220 and 4 iterations are sufficient by the CG scheme with tolerance 10^{-14}.

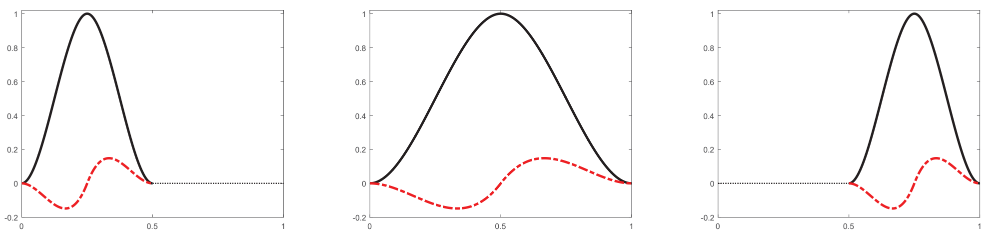
Level	Size	Iteration	κ	$\|e_n\|_{L_\infty}$	$\|e_n\|_{L_2}$	$\log_2 \frac{\|e_{n-1}\|_{L_2}}{\|e_n\|_{L_2}}$	$\|e_n\|_{H^1}$	$\log_2 \frac{\|e_{n-1}\|_{H_1}}{\|e_n\|_{H_1}}$
3	30	4	1.0220	1.5147e-05	6.9273e-06	—	3.8595e-04	—
4	62	4	1.0220	9.6241e-07	4.3380e-07	3.9972	4.8324e-05	2.9976
5	126	4	1.0220	6.0397e-08	2.7126e-08	3.9993	6.0430e-06	2.9994
6	254	4	1.0220	3.7787e-09	1.6956e-09	3.9998	7.5546e-07	2.9999
7	510	4	1.0220	2.3623e-10	1.0596e-10	4.0001	9.4434e-08	3.0000
8	1022	4	1.0220	1.4765e-11	6.6267e-12	3.9991	1.1804e-08	3.0000

Table 4.4: We use Example 2.4 with $m = 2$ to solve the differential equation in (4.12) for Example 4.4 at scale levels $n = 3, \ldots, 8$. Size is the dimension of the linear system. Iteration is the number of CG iterations needed with tolerance 10^{-14}. κ is the condition number of the coefficient matrix B in (4.14) with $\alpha = 11$. $\log_2 \frac{\|e_{n-1}\|_{L_2}}{\|e_n\|_{L_2}}$ is the convergence rate which agrees with the sum rule order $\mathrm{sr}(a) = 4$.

Example 4.5. Consider the following differential equation with homogeneous boundary conditions:

$$u^{(4)} = f \quad \text{on} \quad (0,1) \quad \text{with} \quad u(0)=0, \quad u(1)=0, \quad u'(0)=0, \quad u'(1)=0, \tag{4.17}$$

where $f(x) = -4\pi^4\cos(2\pi x)$. Its exact solution is $u(x) = \frac{1}{4}(1-\cos(2\pi x))$. The corresponding Galerkin approximation of this problem is (4.13) with $\alpha = 0$ in (4.14). Since (4.17) has the same homogeneous boundary conditions as



(a) (b) (c)

Figure 4.5: (a) is the first left interior wavelets $\psi_1(2\cdot)$ and $\psi_2(2\cdot)$ at the scale level 1 for Examples 4.4, 4.5, and 4.7, since they are originally supported on $[0,1]$. (b) is the elements at the coarsest scale level 1, where the black line is $\phi_1(2\cdot-1)$ and the red dotted-dashed line is $\phi_2(2\cdot-1)$ (c) is the last interior wavelets $\psi_1(2\cdot-1)$ and $\psi_2(2\cdot-1)$ at the scale level 1.

in Example 4.4, we use the same Riesz wavelet as in Example 4.4. Notice that matrix B is exactly an identity matrix. Hence, there is no need to use any linear solver to compute the coefficients in (4.13). It is sufficient to just compute $\langle g_k, f\rangle$ for $k = 1,\dots,N$.

Level	Size	κ	$\|e_n\|_{L_\infty}$	$\|e_n\|_{L_2}$	$\log_2 \frac{\|e_{n-1}\|_{L_2}}{\|e_n\|_{L_2}}$	$\|e_n\|_{H^1}$	$\log_2 \frac{\|e_{n-1}\|_{H_1}}{\|e_n\|_{H_1}}$
6	254	1	3.7787e-09	1.7037e-09	—	7.5546e-07	—
7	510	1	2.3623e-10	1.0649e-10	4.0000	9.4434e-08	3.0000
8	1022	1	1.4765e-11	6.6555e-12	4.0000	1.1804e-08	3.0000
9	2046	1	9.2293e-13	4.1600e-13	3.9999	1.4761e-09	2.9995
10	4094	1	5.8009e-14	2.6024e-14	3.9987	1.8864e-10	2.9681

Table 4.5: We use Example 2.4 with $m = 2$ to solve the differential equation in (4.17) for Example 4.5 at scale levels $n = 6,\dots,10$. Size is the dimension of the linear system. κ is the condition number of the coefficient matrix B in (4.14) with $\alpha = 0$. $\log_2 \frac{\|e_{n-1}\|_{L_2}}{\|e_n\|_{L_2}}$ is the convergence rate which agrees with the sum rule order $\mathrm{sr}(a) = 4$.

Example 4.6. Consider the differential equation with nonhomogeneous boundary conditions:

$$u^{(4)} = f \quad \text{on} \quad (0,1) \quad \text{with} \quad u(0) = 16, \quad u(1) = 0, \quad u'(0) = -64, \quad u'(1) = 0, \tag{4.18}$$

where $f(x) = 24(15x^2 - 50x + 41)$. The exact solution to this differential equation is $u(x) = (x-2)^4(x-1)^2$. The corresponding Galerkin formulation of (4.18) is just

$$\sum_{l \neq \{1,\ldots,N\} \setminus \{1,4\}} \langle g_k'', g_l'' \rangle c_l = \langle g_k, f \rangle - u(0)\langle g_k'', g_1'' \rangle - u'(0)\langle g_k'', g_4'' \rangle, \quad k = 2, 3,$$

and (4.13) with $\alpha = 0$ for $k = \{5, \ldots, N\}$. According to the boundary conditions in (4.18), we choose the boundary elements as follows:

$$\phi_1^L = \phi_1|_{[0,1]}, \quad \phi_2^L = \phi_2|_{[0,1]}, \quad \phi^R = \emptyset, \quad \psi^L = \emptyset, \quad \psi^R = \emptyset.$$

where ϕ_1^L and ϕ_2^L are the first and second boundary coarse-scale elements respectively. Using the norms in (2.36) and (2.38), the first four elements are given by

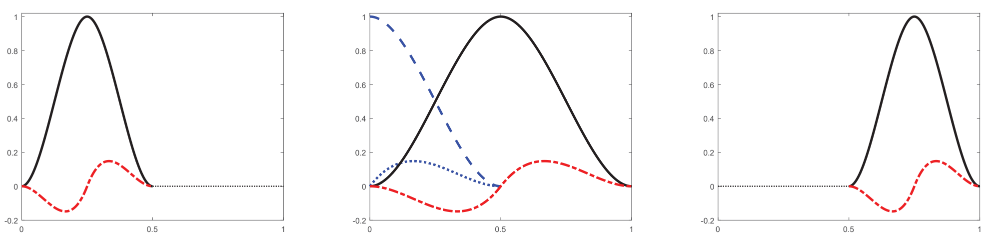
$$g_1 = 2^{-3/2} \frac{\phi_1(2\cdot)|_{[0,1]}}{\|[\phi_1^L]''\|_{L_2(\mathbb{R})}}, \quad g_2 = 2^{-3/2} \frac{\phi_1(2\cdot -1)}{\|\phi_1''\|_{L_2(\mathbb{R})}},$$

$$g_3 = 2^{-3/2} \frac{\phi_2(2\cdot -1)}{\|\phi_2''\|_{L_2(\mathbb{R})}}, \quad g_4 = 2^{-3/2} \frac{\phi_2(2\cdot)|_{[0,1]}}{\|[\phi_2^L]''\|_{L_2(\mathbb{R})}}.$$

Since ψ_1 and ψ_2 are supported on $[0,1]$, no boundary wavelets are present. Notice that the matrix B is exactly an identity matrix. Hence, there is no need to use any linear solver to compute the coefficients in (4.13). It is sufficient to just compute $\langle g_k, f \rangle$ for $k = 1, \ldots, N$.

Level	Size	κ	$\|e_n\|_{L_\infty}$	$\|e_n\|_{L_2}$	$\log_2 \frac{\|e_{n-1}\|_{L_2}}{\|e_n\|_{L_2}}$	$\|e_n\|_{H_1}$	$\log_2 \frac{\|e_{n-1}\|_{H^1}}{\|e_n\|_{H_1}}$
5	126	1	1.5129e-07	5.5402e-08	—	1.2283e-05	—
6	254	1	9.5006e-09	3.4627e-09	4.0000	1.5354e-06	3.0000
7	510	1	5.9521e-10	2.1642e-10	4.0000	1.9193e-07	3.0000
8	1022	1	3.7247e-11	1.3528e-11	3.9998	2.4004e-08	2.9992
9	2046	1	2.3306e-12	8.4696e-13	3.9975	3.1070e-09	2.9497

Table 4.6: We use Example 2.4 with $m = 2$ to solve the differential equation in (4.18) for Example 4.6 at scale levels $n = 5, \dots, 9$. Size is the dimension of the linear system. κ is the condition number of the coefficient matrix B which is the identity matrix. $\log_2 \frac{\|e_{n-1}\|_{L_2}}{\|e_n\|_{L_2}}$ is the convergence rate which agrees with the sum rule order $\mathrm{sr}(a) = 4$.



(a) (b) (c)

Figure 4.6: (a) is the first interior wavelets $\psi_1(2\cdot)$ and $\psi_2(2\cdot)$ at the scale level 1 for Example 4.6. (b) is the elements at the coarsest scale level 1, where the black line is $\phi_1(2\cdot -1)|_{[0,1]}$, the red dotted-dashed line is $\phi_2(2\cdot -1)|_{[0,1]}$, the blue dashed line is the left boundary element $\phi_1(2\cdot)|_{[0,1]}$, and the blue dotted line is the second left boundary element $\phi_2(2\cdot)|_{[0,1]}$. (c) is the last interior wavelets $\psi_1(2\cdot -1)$ and $\psi_2(2\cdot -1)$ at the scale level 1.

Example 4.7. Let us present one last example with a solution having a sharp change/jump. Consider the following differential equation with homogeneous boundary condition:

$$u^{(4)} = f \quad \text{on} \;\; (0,1) \quad \text{with} \quad u(0)=0, \quad u(1)=0, \quad u'(0)=0, \quad u'(1)=0, \tag{4.19}$$

where

$$\begin{aligned} f =& 7\big((2\pi)^4 \sin(2\pi(x-1/2)) + 2(4)^3\pi^4 \sin(4\pi(x-1/2)) \\ &+ 2(6)^3\pi^4 \sin(6\pi(x-1/2)) + 2(8)^3\pi^4 \sin(8\pi(x-1/2)) \\ &+ 2(10)^3\pi^4 \sin(10\pi(x-1/2)) + 2(12)^3\pi \sin(12\pi(x-1/2)) \\ &+ 2(14)^3\pi^4 \sin(14\pi(x-1/2)) + 2(16)^3\pi^4 \sin(16\pi(x-1/2))\big). \end{aligned}$$

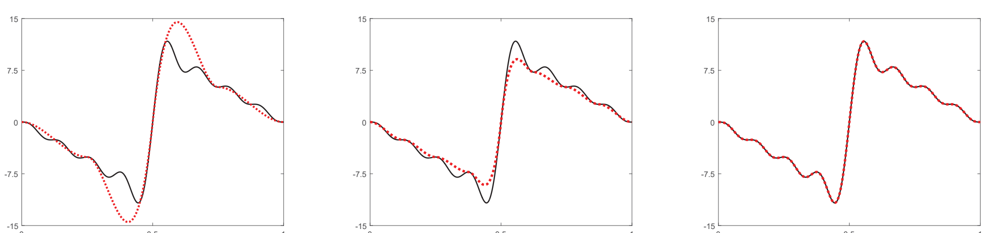
The exact solution to the differential equation is

$$\begin{aligned} u(x) =& 7\sin(2\pi(x-1/2)) + \tfrac{7}{2}\sin(4\pi(x-1/2)) + \tfrac{7}{3}\sin(6\pi(x-1/2)) \\ &+ \tfrac{7}{4}\sin(8\pi(x-1/2)) + \tfrac{7}{5}\sin(10\pi(x-1/2)) + \tfrac{7}{6}\sin(12\pi(x-1/2)) \\ &+ \sin(14\pi(x-1/2)) + \tfrac{7}{8}\sin(16\pi(x-1/2)). \end{aligned}$$

Its corresponding Galerkin approximation is (4.13) with $\alpha = 0$ in (4.14). Since (4.19) has the same homogeneous boundary conditions as in Example 4.5, we use the same Riesz wavelet as in Example 4.5. Notice that the matrix B is exactly an identity matrix. Hence, there is no need to use any linear solver to compute the coefficients in (4.13). It is sufficient to just compute $\langle g_k, f\rangle$, where $k = 1, \ldots, N$. We have also included the plots for the numerical solutions at scale levels 1, 2, and 4 in Figure 4.7. As one can see, even at scale/resolution level 4 - which is considerably low- our wavelet approximation scheme to the theoretical solution with a sharp change/jump behaves pretty well in the sense that it is able to capture the jump and follow the oscillation of the theoretical solution well.

Level	Size	κ	$\|e_n\|_{L_\infty}$	$\|e_n\|_{L_2}$	$\log_2 \frac{\|e_{n-1}\|_{L_2}}{\|e_n\|_{L_2}}$	$\|e_n\|_{H_1}$	$\log_2 \frac{\|e_{n-1}\|_{H_1}}{\|e_n\|_{H_1}}$
6	254	1	1.3056e-04	3.1823e-05	—	1.4114e-02	—
7	510	1	8.1498e-06	1.9923e-06	3.9976	1.7669e-03	2.9978
8	1022	1	5.1071e-07	1.2457e-07	3.9994	2.2094e-04	2.9995
9	2046	1	3.1936e-08	7.7865e-09	3.9998	2.7621e-05	2.9999
10	4094	1	1.9977e-09	4.8667e-10	4.0000	3.4527e-06	2.9999
11	8190	1	1.2635e-10	3.0429e-11	3.9994	4.3194e-07	2.9988
12	16382	1	9.3863e-12	2.1441e-12	3.8270	5.6709e-08	2.9292

Table 4.7: We use Example 2.4 with $m = 2$ to solve the differential equation in (4.19) for Example 4.7 at scale levels $n = 6, \ldots, 12$. Size is the dimension of the linear system. κ is the condition number of the coefficient matrix B in (4.14) with $\alpha = 0$. Note that B is the identity matrix. $\log_2 \frac{\|e_{n-1}\|_{L_2}}{\|e_n\|_{L_2}}$ is the convergence rate which agrees with the sum rule order $\mathrm{sr}(a) = 4$.



(a) (b) (c)

Figure 4.7: (a) is a plot for the numerical solution to Example 4.7, when we use approximation scheme only with the coarsest resolution level 1. (b) is a plot for the numerical solution to Example 4.7, when we use approximation scheme only with the coarsest resolution 1 and highest resolution level 2. (c) is a plot for the numerical solution to Example 4.7, when we use approximation scheme with the coarsest resolution level 1 and highest resolution level 4. The black line is the theoretical solution to Example 4.7 and the red dotted line is the numerical solution.

We close this section with one final remark. The wavelet-based finite element method is actually related to the multigrid method in the sense that they are both multiscale methods and their main goal is to accelerate the computational speed by minimizing the condition numbers of the linear system. The first method requires us to handle different scale levels simultaneously (from the coarsest to the highest). Furthermore, the usage of our mth order derivative-orthogonal Riesz wavelet increases the sparsity of the stiffness matrix so that it is block diagonal. Since the resulting coefficient matrix is well conditioned, we are able to find the numerical solutions efficiently. On the other hand, the multigrid method traverses back and forth between fine and coarse scale levels to refine our approximated solutions. I.e., an approximated solution for the linear system (stemming from the discretization of the problem) is computed at a fine scale level, but the residual is further processed at a coarse scale level by restriction. Afterwards this information is returned to the fine scale level by interpolation and updates the initial approximated solution. The general idea of the multigrid method and some hybrid wavelet multigrid methods have been studied in [20, 40] and references therein.

Chapter 5

Conclusion

Let us recapitulate what we have done in this thesis. Firstly, we propose a systematic Riesz wavelet construction procedure by imposing derivative–orthogonality condition. Furthermore, we present some necessary and sufficient conditions from [35] to help us determine under which circumstances our construction procedure can be applied. Afterwards, we apply this construction to B-splines and Hermite splines, and obtain some examples of Riesz wavelets that have an analytic expression in certain Sobolev spaces. This is because we would like our Riesz wavelet examples to be potentially useful in many applications (e.g., numerical differential equations). Ensuring that our Riesz wavelets have an analytic expression typically leads to a much easier implementation process. Thirdly, we show that a Riesz wavelet on $L_2(\mathbb{R})$ is indeed leading to a Riesz wavelet on $L_2(\mathcal{I})$, where $\mathcal{I}$ is a bounded interval on $\mathbb{R}$, given that it satisfies some symmetry property. Fourthly, we show that normalizing the mth order derivative of the wavelet basis with respect to L_2 norm is important to achieve a well conditioned energy norm. Lastly, we test the numerical performance of our Riesz wavelets by using them to solve some simple differential equations. The results are indeed very encouraging in that the condition numbers of the stiffness matrix (or the sum of stiffness and mass matrices) are very close to 1 and some are even identically equal to 1. More importantly, the implementation can be done easily and quickly. In particular, we see that the boundary construction does not pose any additional difficulty.

There are still many issues that we would like to answer as part of our future work. Firstly, we want to know how the tensor product of a univariate derivative-orthogonal Riesz wavelet performs in finding the numerical solution of higher dimensional differential equations. This is because the observed performance in 1D setting is very encouraging. Secondly, in some cases (after a linear combination), we observe that the stiffness matrix of certain Riesz wavelets is exactly the identity matrix, we want to see if there is a family of Riesz wavelets that exhibits such an interesting property. Thirdly, we still need to come up with a proof for a Riesz wavelet $H^m(\mathbb{R})$ having symmetry is indeed leading to a Riesz wavelet for $H^m(\mathcal{I})$. We hypothesize that the general idea and procedure of the proof remain mostly the same as the one we use to prove the L_2 case. One technical problem that may arise is when the dual basis is a Dirac distribution. Because we rely heavily on the folding operator, we are still unsure as to how to fold a Dirac distribution. Fourthly and still related to the folding operator, even though it is well defined for $f \in \mathcal{L}_2$, we still need to figure out a proper technique (or additional conditions) to show that a folded wavelet basis having an infinite support and satisfying certain decay conditions is still a Bessel sequence on a bounded interval.

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