University of Alberta

STUDENT TEACHERS' CONCEPTIONS OF MATHEMATICAL PROOF

by

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Abstract

The National Council of Mathematics Teachers (2000), in an effort to reform school mathematics programs, has called for a greater emphasis on mathematical poof and proving in the classroom. Student teachers, given their status as novice instructors, are likely to find this directive challenging. Student teachers' conceptions of, confidence in, and attitudes towards proof and proving are sure to influence their instructional choices. In this study, I investigate (1) student teachers' belief about the nature and role of proof in secondary school mathematics and (2) student teacher's ability to complete correct mathematical proofs. Participants in the study included 17 student teachers with Mathematics majors, registered in the final semester of their teacher education— at a large Canadian university. Each participant completed a questionnaire comprised of written response questions, four mathematical proving tasks, and a representational task (concept map). Findings indicate that these student teachers have limited understanding of the nature and role of proof in secondary school mathematics. They would seem to need more experience completing secondary school level mathematical proofs, greater exposure to different functions of proof and the different processes of proving, and deeper understanding of the logic inherent in proving. A greater emphasis on proof and proving as part of the teacher education curriculum may improve the proving skills of future secondary mathematics teachers. Moreover, the findings suggest a need for further study in school mathematics classrooms where students need rich proving experiences in order to understand the value and benefits of developing proving skills.

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CHAPTER 1. A CONTEXT OF REFORM

The Call for Reform

The call for reform in mathematics education is as old as the history of school itself. Over the past three decades, the National Council of Teachers of Mathematics (NCTM) has published several documents (1989, 1991, 1995, & 2000) that have stimulated discussion and debate about reform in mathematics education. In 2000, the NCTM proposed a revision of their own content standards with the purpose of making mathematics more meaningful for both teacher and learner. The NCTM documents remind us that learning mathematics requires more than simply solving exercises by working with symbols, performing desired calculations, and doing routine proofs; rather, learning mathematics is fundamentally about "developing a mathematical view point", "communicating mathematically", "making connections" among mathematical experiences and with "other disciplines", and enhancing "*mathematical reasoning*" (emphasis added; NCTM, 2000, p. 56). It emphasizes that the ability "to reason is essential to understanding": "proof and reasoning," it suggests, are powerful ways of communicating mathematically, developing insights, and making connections between different mathematical domains and among other subjects (p. 56). Hence, developing student proficiency in mathematical proof and reasoning needs to become an integral part of all mathematics education.

The teacher plays a critical role in the development of a student's skill in mathematical reasoning. As the NCTM (2000) points out, "students learn mathematics through the experiences that teachers provide" (p. 16). In other words, teaching shapes students' understandings of math, their ability to use it to solve problems, and their attitude towards mathematics as a discipline. As well, many researchers point out that the teacher's knowledge and beliefs play a critical role in successfully enacting classroom practices (Fennema & Franke, 1992; Thompson, 1992). Teachers must be flexible in their teaching practices, drawing upon their mathematical knowledge appropriately and creatively as they instruct their students (NCTM, 2000). Ultimately, they must accept responsibility for establishing and negotiating an acceptable mathematical explanation and justification within the classroom (Yackel & Cobb, 1996). This goal can best be achieved I feel if the teacher feels secure in his/her own understanding of the concept of "mathematical proof".

Whether or not a person who enjoys math becomes an effective mathematics teacher depends, to a large extent, upon that person's understanding of mathematical concepts and his/her ability to clarify and communicate that understanding. The goal of teacher education programs is to provide the foundation that will enable novice instructors to grow into effective teachers. This can only be achieved by first understanding the student teacher's conceptions, beliefs, and ability with regard to mathematics. Hence, within the current context of reform—a context in which teachers are expected to support their students in achieving the *Standards* set out by the NCTM (2000)—it is critical that teacher educators ensure that future mathematics teachers possess this foundational understanding of proofs and proving.

Proof: An Introduction

Proof is fundamental to mathematics. As Davis and Hersh (1981) point out, mathematical proof has been regarded as one of the key distinguishing characteristics of the discipline of mathematics since the nineteenth century. Indeed, Raman (2002), in her study, *Proof and Justification in Collegiate Calculus*, observes, "since the 6th century BC when Greek mathematicians established the axiomatic method, mathematicians have considered proof to be the *sine qua non* of mathematics" (p. 1). And mathematician Michael Atiyah identifies proof as "the glue that holds mathematics together" (as cited in Dunham, 1994, p. 15). Given its significance to mathematicians themselves, it is important that students and teachers understand what is meant by the term *mathematical proof.*

The term mathematical proof is not limited to a single definition: hence, it can be difficult to know, in any given context, exactly how the term is being used. The Oxford American Dictionary defines proof as "a demonstration of the truth of something" (1980, p. 535). Leddy (2001) offers one of the simplest and most practical definitions of proof: "a reasoned argument from acceptable truths" (p. 13). Yet, once one leaves simplistic definitions behind, the matter becomes more confusing. For example, a proof that is acceptable to a physicist might not be acceptable to a mathematician. Polya (1960) writes that:

> in mathematics as in the physical sciences we may use observation and induction to discover general laws. But there is a difference. In the physical sciences, there is no higher authority than observation and induction, but in mathematics there is such an authority: rigorous proof.
>
> (as cited in Leddy, 2001, pp. 11-12)

In other words, as soon as there is sufficient evidence to support a scientist's hypothesis—and as long as there is no evidence against it—s/he accepts the hypothesis;

but among most mathematicians, a claim to proof involves more stringent criteria. The mathematician reasons that observation cannot prove by itself because eyes can deceive us, measurement cannot prove because the certainty of the conclusion we arrive at depends upon the precision of the measuring instrument and the care of the measurer (both variable factors), and experiment cannot prove because the conclusions can only be considered probable and not invariable (Johnson, 2007).

Even within the mathematical community itself, standards of proof vary due to the autonomous development of mathematical specialties and their subsequent isolation from each other (Almeida, 1996). A number of key words, long used within the mathematics education literature to refer to elements of proof—such as "explanation", "verification", and "justification"—convey different meanings depending upon who is using them. This multiplicity of meaning implies fundamental differences in how mathematicians conceptualize proof. Specifically, definitions tend to vary according to the mathematician's perception of what constitutes an "appropriate formal system" (Hanna, 1991, p. 55).

Since the closing years of the nineteenth century, mathematicians have narrowly defined proof in terms of logic (Davis & Hersh, 1981; Moreira, 1999). Frege (1884/1950), for example, defined proof as a finite sequence of statements such that each statement in the sequence is either an axiom or a valid inference from previous statements. Many decades later, Alonzo Church (1956) demonstrated the same adherence to formal logic. According to Church:

> a finite sequence of one or more well-formed formulas is called a proof if each of the well-formed formulas in the sequence either is an axiom or is immediately

> inferred from proceeding well-formed formulas in the sequence by means of one of the rules of inference. A proof is called a proof of the last well-formed formula in the sequence.
>
> (as cited in Moreira, 1999, p. 93)

Joseph (2000) provides a more recent and succinct take on proof as logical formalism. Proof, he claims, "is a procedure, [an] axiomatic deduction, which follows a chain of reasoning from the initial assumptions to the final conclusion" (p. 127).

Over the years, however, many mathematicians have come to define proof in broader terms. Thirty years ago, Lakotos (1976) described mathematics as an open subject that is constantly being developed and changed through proofs and refutations. He suggested that the definition of proof should be expanded to include explanations, justifications and elaborations of any conjecture subjected to counter examples. Lakotos' view reflects the assumption that proof depends on the insights of the active mathematician and not on mechanistic rules and procedures. Indeed, perceptions of what proof *is* have changed to such a degree that, little more than a decade ago, mathematician William Thurston (1995) claimed, "for the present, formal proofs [in the sense of symbolic logic] are out of reach and mostly irrelevant" (p. 34). Even more inclusive is Hanna's (1995) definition. She insists that the best proof is one that helps us understand the meaning of the theorem that is being proved. She notes that such proofs help us to see, not only *that* a theorem is true, but also *why* it is true. These, Hanna claims, are more convincing and more likely to lead to further discoveries. Hence, in school mathematics, the proofs that explain—narrative proofs—are much more important because they facilitate understanding (Hanna, 1990). In the end, whether one defines proof narrowly or broadly, it is important to remember that proof is an art and the act of proving can "evoke a profound sense of beauty and surprise" (Moreira, 1999, p. 349).

Given that mathematicians differ in their perception of what it is that constitutes a mathematical proof, it follows that they would also differ in their understanding of the role played by proof within mathematics. Indeed, one's view of what it is that proofs *do* typically influences how one defines the term. Listed below are some of the many functions of proof and proving as identified by various mathematics educators (Bell, 1976; de Villiers, 1990, 1999; Hanna, 2000; Hanna & Jahnke, 1996; Lucast, 2003; Luthuli, 1996; Marrades & Gutierrez, 2000):

- verification or justification (concerned with the truth of a statement);
- explanation (providing insight into why a statement is true);
- systemization (the organization of various results into a deductive system of axioms, major concepts and theorems);
- discovery (the discovery or invention of new results);
- communication (the transmission of mathematical knowledge);
- construction of an empirical theory;
- exploration of the meaning of a definition or the consequences of an assumption;
- incorporation of a well-known fact into a new framework, viewing it from a fresh perspective;
- providing an intellectual challenge to the author of the proof.

Obviously, mathematical proof is a complex matter both in terms of the multiplicity of definitions that have been offered to specify the concept and in the variety of functions that have been attributed to it. We see this complexity played out within educational contexts in a number of ways. How educators define proof and expect it to function depends upon the specific factors associated with the educational context including the teacher's understanding of and experience with proof and the student's age, grade level, and mathematical abilities. As Leddy (2001) notes, what is defined and accepted as mathematical proof for a Grade 5 student may very well no longer count as mathematical proof for a high school student.

A Brief History of Proof

If one defines proof broadly, one can find evidence of mathematical proof in the extant computations of various cultural groups that pre-date the ancient Greeks. Of course, few would disagree with Szabo's (1972) assertion that the concept of deductive science was unknown to the eastern people of antiquity before the development of Greek culture. He maintains that:

> in the mathematical documents which came down to us from these [Eastern] people, there are no theorems or demonstrations and the fundamental concepts of deduction, definition and axiom have not yet been formed. These fundamental concepts made their first appearance in Greek Mathematics.
>
> (as cited in Siu, 1993, p. 345)

Indeed, if one defines "mathematical proof as a deductive demonstration of a statement based on clearly formulated definitions and postulates" (Siu, 1993, p. 345), then one must conclude that no proofs can be found in the surviving mathematical texts of the ancient Chinese, Indian, Egyptian or Babylonian peoples (Joseph, 2000). However, one does see within these texts a technical facility with computation, recognition of the applicability of certain procedures to a set of similar problems, and an understanding of the importance of verifying the correctness of a procedure (Joseph, 2000). If one defines proof generally as an explanatory note that serves to convince or enlighten the reader, then one can, in fact, identify an abundance of mathematical proofs and proving within these ancient texts. As Wilder (1978) reminds us, "we must not forget that what constitutes proof varies from culture to culture, as well as age to age" (p. 69).

The Greeks, in an attempt to lay solid foundations for geometry, were the first to introduce a version of the axiomatic method in mathematics (Hanna, 1983). Hence, the deductive approach in mathematics came to be referred to in the nineteenth century as the geometrical or Euclidean method. According to Grabiner (1974; cited in Hanna, 1983), during those years, a desire to focus and narrow mathematical results and avoid errors, as well as a need to formalize mathematical results, all played a part in stimulating a growing interest in formal proof. Over time, this Greek-inspired method of deductive proof came to play a central role in mathematics, though considering the lengthy history of mathematical thought and practices, a greater emphasis on rigor is a relatively recent phenomenon (Hanna, 1983).

Types of Proof

An awareness of the ways in which mathematicians have categorized proof can help in understanding the concept. Many classification systems have been put forth; however, there are four types of proof that mathematicians commonly identify.

1) *Proof by counter-example.* This type of proof involves finding at least one example in which a generalization is false. The counter-example will disprove the generalization or indicate its negation. A student, for instance, may conclude that a

negative number plus a positive number is always a negative number, prompting another student to prove the conjecture wrong by offering a counter-example, say $-3 + 7 = 4$.

2) *Direct/Deductive proof.* In this case, one shows that a given statement is deducible by inferring patterns from given information, previously studied definitions, postulates and theorems. Traditionally, direct proofs have been expressed using two-column or paragraph formats. They can also be presented in the "flow-proof format" suggested by McMurray (1978).

3) *Indirect Proof.* With this type of proof, one assumes that the negation of a statement yet to be proven is true, then shows that this assumption leads to a contradiction. The following situation illustrates the process of an indirect argument. On arriving at the darkened library, Angela thinks, "The library must be closed". The logic behind her thought is this: When libraries are open, patrons and employees require light; thus, the lights are likely to be turned on. Right now, the lights are not on; therefore, the library must be closed. Additionally, the process of proving a proof by proving its contra-positive can be thought of as a special case of indirect proof through contradiction. Paragraph formats are often used to show indirect proofs.

4) *Proof by induction.* According to O'Daffer and Thornquist (1993), this is the most complex type of proof. It is based on the principle of mathematical induction and can be stated as follows: If a given property is true for 1 and if for all $n > 1$, the property being true for n implies it is true for $n + 1$. Thus, we can conclude that the property is true for all natural numbers.

In addition, mathematicians have suggested various other ways of classifying students' justification and thought processes *while* they are involved in proving a mathematical task. One well-known classification—that offered by Balacheff, 1988—serves as a key component in the methodology of this study.

Proof and Curriculum

Gardiner and Moreira (1999) claim that "mathematics is not proof; mathematics is not spotting patterns; mathematics is not calculation. *All are necessary, but none is sufficient*" (emphasis added; p. 19). Thus, they underscore that one cannot teach mathematics without teaching proof. Furthermore, Wu (1996) reminds us that:

> producing a proof of a statement is the basic methodology whereby we can ascertain that the statement is true. Any one who wants to know what mathematics is about must therefore learn how to write down a proof or at least understand what a proof is.
>
> (p. 222)

Wu elaborates:

> in a broader context, mathematics courses are where the students get their rigorous training in logical reasoning; this is where they learn how to cut through deceptive trappings to get at the kernel of truth, where they learn how to distinguish between what is true and what only *seems* [emphasis in original] to be true but is not. They would need all these skills in order to listen to the national debate and make up their minds about such knotty issues as the national deficit and the environment, for example. Learning how to write proofs is a very important component in the acquisition of such skills.
>
> (p. 224)

In mathematics education, as Maher and Martino (1996) have argued, we are interested ultimately in student understanding, not just of mathematical principles but of the world itself, and proof and proving offer a means by which teachers might enhance student understanding. Educators, both practicing teachers and educational researchers, must now address the importance of mathematical proof in the classroom. In fact, Marrades and Gutierrez (2000) insist that helping students "to [come to] a proper understanding of mathematical proof and [so] enhance their proof techniques" has become "one of the most interesting and difficult research fields in mathematics education" (p. 87).

Although proofs "are the guts of mathematics" (Wu, 1996, p. 222), proofs and proving have played a peripheral role at best in secondary school mathematics education (Knuth, 2002a). Unfortunately, many secondary school students have little experience and even less understanding of proof (Bell, 1976; Chazan, 1993; Hadas, Hershkowitz & Schwarz, 2000; Senk, 1985). Knuth (2002a) observes that teachers tend to introduce students to mathematical proof solely through the vehicle of Euclidean geometry. Given this narrow application, it is not surprising that students develop little skill in identifying the objectives or functions of mathematical proof, or that both teachers and students come to perceive mathematical proof as a formal and meaningless exercise (Alibert, 1988; Knuth, 2002a). In general, students learn to imitate and memorize specific proof structures by observing the teacher and studying the textbook, but fail to understand the diverse nature, function, and application of mathematical proof (Hadas, Hershkowitz & Schwarz, 2000). There is no doubt that proving is a complex task that involves a range of student competencies such as identifying assumptions, isolating given properties and structures and organizing logical arguments. If teachers wish to teach students to think for themselves, and not simply fill their minds with facts, then as Hanna and Jahnke (1996) stress, it is essential that they place greater emphasis on communication of meaning rather than on formal derivation. In this respect, the teaching and learning of mathematical proof appears to have failed. (Hadas, Hershkowitz & Schwarz, 2000).

Since 1989, the NCTM has called for substantive change in the nature and role of proof in secondary school mathematics curricula. The NCTM published the *Curriculum and Evaluation Standards for School Mathematics* (1989) at a time when the teaching of mathematical proof—specifically within the US— had almost disappeared from the curriculum or sunk into meaningless ritual (Knuth, 2002a). In that document, the NCTM recommended that less emphasis be given to two-column proofs and to Euclidean geometry as an axiomatic system. In general, recommendations call for a shift in emphasis from (what has often been perceived as) an over-reliance on rigorous proofs to a conception of proof as convincing argument (Hanna, 1990). Unfortunately, the NCTM document encouraged educators and students to think that verification techniques could substitute for proof (Latterell, 2005). In that sense this document failed to utilize the broader perspectives of proof in the teaching and learning of mathematics. In contrast, a more recent NCTM document, *Principles and Standards for School Mathematics* (2000), identifies proof as an actual *standard* and assigns it a much more prominent role within the school mathematics curriculum. Accordingly, curriculum developers and program designers have come to expect that all students experience proof as an integral part of their mathematics education. Notably, the 2000 document recommends that reasoning and proof become a part of the mathematics curriculum *at all levels* from pre-kindergarten through grade 12. The section entitled *Reasoning and Proof* outlines for the reader that students should be able to recognize reasoning and proof as fundamental aspects of mathematics, make and investigate mathematical conjectures, develop and evaluate mathematical arguments and proofs, and select and use various types of reasoning and methods of proof. Given the greater status assigned to proof within the mathematics curriculum, it is essential that teachers plan curricular experiences that can help students develop an appreciation for the value of proof and for those strategies that will assist them in developing proving skills.

Any improvement in mathematics education for students depends upon effective mathematics teaching in the classroom. This includes providing students with opportunities to interact in the classroom, to propose mathematical ideas and conjectures, to evaluate personal thinking, and to develop reasoning skills. Teachers' knowledge and beliefs will play an important role in shaping students' understanding of math and their ability to solve mathematical problems (NCTM, 2000). Again, let me emphasize, mathematics teachers must thoroughly understand the mathematics they are teaching and be able to draw on that knowledge as needed, adapting in appropriate ways to each moment of instruction. Hence, it is crucial that teachers, especially the uninitiated, understand the mathematical concepts that they are expected to teach to adolescents. Pre-service teachers, in particular, must be well-prepared in this respect: if they fail to understand the nature of mathematical proof, they are far less likely to teach the concept in a manner that fosters mathematics reform within the schools. Thus, it becomes critical for teacher educators to assess, and if necessary, address, the understanding and abilities of student teachers in constructing mathematical proof (NCTM, 2000).

Research Purpose and Research Question

The goal of recent reform efforts in mathematics education is to foster the development of *students*' understanding and uses of proof; but success in achieving this goal depends upon *teachers'* understanding of and uses of proof. Jones (1997) notes that the teaching of mathematical proof places significant demands on both the subject matter knowledge and the pedagogical knowledge of secondary mathematics teachers. Furthermore, Knuth (2002a) affirms that a teacher's conception of proof influences both the role that s/he assigns to proof in her/his mathematics classrooms and her/his instructional approach in teaching such a concept. It follows that a student teacher's understanding of and experience working with proof will certainly influence the manner in which he or she approaches teaching the concept, both pedagogically in terms of method and emotionally in terms of confidence. In turn, the poise that student teachers demonstrate or fail to demonstrate in teaching mathematical proof will enhance or impair the soundness of their curricular judgments, the effectiveness of their response to student questions, and the level of skill they demonstrate in making connections, both within the mathematics curriculum and among other academic disciplines. Hence, it is important to examine the prospective secondary teacher's conceptions of proof, for it will be these conceptions and understandings that will shape classroom practice in the future.

The responsibility for enhancing student teachers' conceptions of proof lies with both mathematicians and mathematics educators. Teachers in the field have access to in-service training and professional conferences to help prepare them for enacting the expectations identified within NCTM reform documents; student teachers, on the other hand, must rely, at least initially, upon their own teachers to prepare them for this challenge. University-level educators recognize that there is a need to prepare future mathematics teachers to meet the current challenge of reform (Cuoco, 2001; Ma, 1999; Stigler & Hiebert, 1999). Based on personal experience, I can attest that university educators are attending to the task of introducing student teachers to a deeper understanding of proof, the production of proof, and an appreciation for proof as an instructional goal within the secondary mathematics classroom.

My research examines the degree to which student teachers feel secure in their understanding and conceptions of mathematical proof. The fundamental question that drives this study is, "***What understandings do student teachers possess concerning the notion of mathematical proof?***" In order to answer this question, I must determine what student teachers believe about the nature and function of proof and assess their abilities in applying it. Therefore, I must address two preliminary questions:

1) What do student teachers believe about the nature and role of proof?
2) How able are student teachers when it comes to completing mathematical proofs?

From the perspective of curriculum development, it will be instructive to determine whether or not student teachers have adequately learned mathematical proof. This study will provide data of interest to curriculum developers of teacher education programs in secondary mathematics. My findings should assist them, first, in determining whether or not the program needs revision, and, second, in directing future efforts to revamp and/or revitalize the existing curriculum.

Plan of the Thesis

The remaining chapters of this thesis are organized as follows. Chapter 2 details my research orientation and the basic philosophical assumptions that I bring to the work of research. Chapter 3 offers a literature review which examines the various roles that proof plays in mathematics as well as those aspects of proof that are relevant to its teaching in schools. In Chapter 4, I take a closer look at the literature relevant to my research, especially those publications that have inspired me to carry out my study on prospective secondary school teachers. Chapter 5 provides a description of the research methodology that I employ in the study. Chapter 6 reports and discusses the results of the study with respect to the written responses and interview questions. In Chapter 7, I describe the mathematical tasks that I selected for the study and justify my choices. Chapters 8, 9, 10, and 11 present a task-by-task analysis of the student teachers' work. Chapter 12 takes a close look at the concept maps that were generated. In Chapter 13, I compare my findings with those of Balacheff (1988), Healy and Hoyles (2000), Knuth (2002a), and Jones (1977). These are the primary authors whose work inspired my efforts and the source of some of the mathematical tasks included within my study. I will also revisit my research questions in this chapter. Chapter 14 concludes with a summary of the study and suggestions for further research in this area.

CHAPTER 2. PERSPECTIVES AND PARADIGMS

The Student Becomes a Statistician

My first encounter with a mathematical proof was in Junior High School. I cannot remember the grade in which this happened, but I do recall that it was in relation to Euclidean geometry. Only after memorizing various definitions, properties, and axioms was the class deemed ready for the proof. Proofs were presented in the "two column format", and that is how we were expected to write them. In the first column (on the left) we wrote the statements and in the second column (on the right) we were instructed to write the corresponding reasons. Reasons were all based on the definitions, properties and axioms that we had already committed to memory. I do not recall anyone in the class asking our teacher why it was that we had to write the proofs in this column format; perhaps because at that time, and in that place, we believed that knowledge flowed from above, from the all-knowing teacher, to the lowly student. I recall that we resented how our teacher made us go about the whole exercise of constructing proof and, indeed, about mathematics in general. I slogged through lots of memorization, lots of practice, and lots of homework. It was all a great deal of hard work. Yet, a few years later, when I enrolled as a university student, I was glad that my teacher had exposed me to formal models of proofs, taught me to write proofs, and insisted upon all that hard work. Despite its drudgery, this experience prepared me quite well for my university-level courses.

Understanding the Past—Received Knowledge: Groundings in Positivism

It is clear to me now that my early understandings of knowledge were grounded in a belief system that valued natural science as the sole source of true knowledge

(Newman, 2000). A person discovered knowledge by formulating hypotheses and conducting experiments; when knowledge had been amassed it was passed down to younger generations. Thus, knowledge was something separate from the knower, something taken in, acquired in chunks and then linked into linear chains by means of reason and judgment (Davis, 2004). Later, I would learn that Descartes' analytic method (in his *Discourse on Method*, 1637) sought to reduce all claims to truth to their fundamental assumptions, and became, for many, the means by which all uncertainty could be removed (Davis, 2004). Bacon (in *Novum Organum*, 1620) argued, instead, for careful controlled and replicable experimentation as the means of establishing certainty (Davis, 2004). Although Descartes' rationalism and Bacon's empiricism differed in method, both aimed to uncover the fundamental principles and laws of the universe. The scientific method, as I came to know it, combined deductive logic with precise empirical observations in order to discover and confirm a set of probalistic causal laws that could be used to predict future events (Newman, 2000). The domain of science, governed as it was by logic and rational thought, reigned supreme while the arts and humanities, particularly philosophy, were not considered important. I remember a story that I heard when I was young. When God created the world, he gave people the tools they needed to survive: to the Westerners, He gave science; to the peoples of the Middle East, He gave oil; and to the Indians (Hindustanis), He gave philosophy. Over time, the Westerners prospered because of their science, the people of the Middle East prospered because of their oil, and the Indians sank into poverty because of their philosophy.

When I was a child, it was generally believed that researchers conducting rigorous experiments while working in laboratories were responsible for the discovery of knowledge. Scientists were respected and highly regarded. They conducted studies using precise quantitative data obtained by means of surveys or experiments; then they conducted a complex statistical analysis to uncover Truth. I recall that one of our chemistry teachers at school had received a doctoral degree in Chemistry. Although he taught only the senior students, those of us at the junior high level came to know all about "Dr." Nair because our school headmaster publicly congratulated him on his achievement. I heard from other students that he was a very serious man who seldom talked about anything other than Chemistry, and students never dared to open their mouths once he stepped into the classroom. For me, Dr. Nair's manner reinforced a stereotype of the scientist—someone who was deadly serious, someone who never laughed and never smiled.

Scientists who conducted work as researchers, I reasoned, must be equally somber, particularly because their work was so serious. They had to solve problems, replicate experiments, and verify past results. Validity, reliability, and generalizability were serious matters (Newman, 2000). Their efforts determined what we could know and how well we could address the serious social problems that we lived with. In school, even those teachers who taught arts and humanities stressed that science was the way to get at Truth, the way to understand the world well enough to predict and control it. Consequently, most students, following their secondary school education, tried to get into Medical School, Engineering or other science-based programs. Even if the students themselves had no great interest in science, parents constantly reminded them of the importance of SCIENCE in the "real" world. As a student, I enjoyed reading Malayalam literature, but older and seemingly wiser people in my society reminded me that having a good education in science and/or mathematics would be the key to success in life. So I concentrated on my studies in mathematics and science and all of those "soft" interests of mine slowly disappeared. It was only much later, after I had begun studies in Canada, that I came to understand that I had grown up in a society governed by a system of beliefs known as Positivism (Newman, 2000).

The Statistician Becomes A Teacher

After receiving a doctoral degree in Statistics and teaching at a university for a short time, I left my home country, India, and accepted a position teaching mathematics at a secondary school in Brunei Darussalam. At first, I was not pleased with this career change. In India and other parts of Asia, a move from university lecturer to secondary school teacher was considered a demotion. But, in fact, my experience teaching school was an eye opener. I came to understand the saying, "to learn something, you need to teach it". As a high school mathematics teacher, I realized that mathematics did not need to be compartmentalized into Arithmetic, Algebra, Geometry and Trigonometry: proofs, explanation and justification need not be reserved for Geometry alone. Students and teachers could experience the beauty of proofs—the "Ah ha!" moment that comes with the proving—in all areas of mathematics.

But, as a teacher I did not act upon this realization. Instead, I followed the set curriculum and introduced students to proofs using the traditional two-column format. Like my own teachers, I asked my students to memorize all of the definitions, properties, axioms and theorems thought necessary for solving proofs. And, just as my teachers had done with me, I insisted that my students do lots of examples and a great deal of homework. I continued to act according to the traditional belief that, in mathematics, *practice makes perfect. My contribution to expanding the role of proofs in secondary school mathematics was nil. I simply did my job, followed the curriculum and "helped" my students receive high grades on their final examination. My teaching style was best described as lecture format balanced by classroom discussion and lots of practice: at that time I was unaware of other methods.*

One thing I was very much aware of, however, was what it felt like to struggle and persist. I felt compassion for those students who struggled with mathematics. I established a small-group extra class for further instruction and worked with students one-on-one, and I tried to help all of these students to succeed where they had previously failed. Almost six years of working with adolescent students generated within me a strong passion for secondary school teaching—so strong that I totally forgot about my previous career as a university lecturer of statistics.

Understanding the Past--Constructed Knowledge: Arriving at Constructivism

Although my philosophy of teaching was grounded in an understanding of knowledge as some "thing" passed down from higher levels (teachers) to lower levels (students), I slowly began to recognize that students varied tremendously in how they understood concepts. I still saw myself as the transmitter of knowledge, but my desire to help students learn the subject matter led me to discover that, even as I delivered the same subject matter in the same manner to similar students, what went on in one student's head could be very different from what went on in another's. Still, it never occurred to me that because of differences in individual thinking styles and processes, students could benefit from working together.

I recall that a lecturer from the Universiti Brunei Darussalam once came to our school to conduct an in-service training session. She gave a presentation that introduced us to the concept of cooperative learning and argued passionately for its importance. We were confused: we had come to see group work as something that teachers did when they did not want to teach, or as something that occurred during extracurricular hours while working with the Mathematics Club. For us, group work was allowing students to help each other with a math problem *only* after they had struggled on their own to answer it and, no matter what they had tried, could not find the solution. After many minutes of independent seatwork, we typically *gave* students the correct answer rather than encouraging and making time for them to work together to *discover* the answer. Group work was time-consuming—a strategy used by teachers when they wanted to kill time! We had very little time: constrained by the tight deadlines that were imposed upon us, and the enormous syllabus that we had to cover before the final comprehensive exam, the General Certificate Examination, we had little time to "waste" on discovery learning. Besides, we had been taught that a "good" teacher demonstrates excellent class control; given our value system, to allow the noisy group interaction of exploratory activity was out of the question.

My colleagues and I, therefore, found the idea of approaching mathematical problems through group work totally incomprehensible. We continued to teach according to the principles of positivism, regarding each student as a separate entity, isolated from his/her fellow learners. Nonetheless, I had begun to question the wisdom that taught me to expect from each student the same subjective understanding of the concepts that I taught them: I could not help but wonder if there might be a better way to assist my diverse students in learning mathematics.

The Teacher Becomes a Student

The cycle of life took another turn when, as a family, we decided to emigrate. I resolved that, once in Canada, I would be a compassionate Mathematics schoolteacher, and nothing else. I felt such empathy for students who struggled with Mathematics that I knew this was my calling. Consequently, I enrolled in a B Ed program at a Canadian university and soon discovered those "other" theories of teaching and learning about which I knew nothing when I first began teaching in Brunei. In every education course that I completed I came across the term "constructivism". I did not quite understand this approach nor its methodology, but I was eager to try it out in my own classroom if it could help my struggling students. When I completed my B Ed, I felt that there was still so much I wanted to learn about Mathematics Education. Thus, I decided to continue my studies and pursue a Master's degree. I submitted a brief statement of intent as part of my application and wrote earnestly about the value of different teaching methods in mathematics classrooms. I was passionate about identifying teaching methodologies that would help ensure that I was connecting with every student.

Understanding the Past—Constructed Knowledge: Exploring Constructivism

Those early years in the teacher education program began a process that has been unnerving, yet exciting. As a teacher trainee, I experienced more exploratory mathematics—it was quite a change since I had always associated mathematics with certainty. I began to recognize value in the group work that I had once considered so problematic. I was willing to adopt these methods if they would truly help my students.

But it was difficult to let go of my beliefs, my philosophy of teaching, and assumptions about knowledge and the learning process.

After my initial practicum experience, I came to the conclusion that cooperative learning and discovery approaches to mathematics did have some value, not necessarily intrinsic value, but value based on their usefulness within specific social contexts. I determined that group work and discovery learning could work well in Canadian schools where students had been socialized to be more open in the classroom, more willing to question what they did not think was right. I still questioned the relevance and effectiveness of group work in Indian and Asian contexts where students had long been accustomed to independent learning.

When I began to contemplate what I might focus on in my research on teaching methods in mathematics education, the first idea that sprang to mind came out of my positivist background: I would control for students and teacher, apply different teaching methods, and see which one yielded the best result. As I learned more in my courses on curriculum theory and research methodology, however, I came to understand that educational research is about far more than simply manipulating variables and applying statistical methods. I reconsidered the positivist position. Skinner had argued that researchers could only study positive and negative re-enforcers of behavior. He claimed that what the subject thinks is irrelevant because thoughts cannot be measured. I could no longer accept this. I realized now that no two students would experience social or physical reality in the same way; I needed to shift my focus to what was meaningful and relevant to my students. Cognitive theorist Jean Piaget suggested that the learner constantly revises his or her understandings of the world in accordance with new experience (Davis, 2004). The Constructivists who followed Piaget focused on the learner's attempts to make sense of the world, to construct reality and knowledge (Davis, 2004). I began to wonder if it was even possible, as I had been taught, to study classroom practice as one studied the natural world, with deductive axioms, theorems, and interconnected causal laws (Newman, 2000). It became clear to me that I was now at least equally concerned with achieving an empathetic understanding of the feelings and views of the students as I was with determining effective teaching methods. I did not yet understand that the two were intimately connected.

The Student Becomes a Researcher

During my first semester as a graduate student, I had an opportunity to work as a research assistant for a professor of mathematics education. It was my responsibility to collect materials relating to "proofs and reasoning" in school mathematics. As I carried out this library research, memories of my experiences with proof both as a student and a teacher came back to me. I read research studies indicating that students encounter difficulty with proof no matter what their level of mathematical ability. I also read that it is important for students to develop a solid understanding of mathematical proof—that proof was "the guts of mathematics" (Wu, 1996, p. 222). All of this reading kindled my interest in mathematical proof and changed the direction of my research efforts.

My initial plan for my master's thesis was to conduct a study on the proof experiences of secondary school students. When I read that "students learn mathematics through the experiences that teachers provide" (NCTM, 2000, p. 16), and that a teacher's knowledge can have a major impact upon student learning, my interest shifted towards the student teacher. About that time, I came across Knuth's study on practicing

Mathematics teachers. His finding that these teachers actually had limited experiences with mathematical proof caught my attention. I became concerned that students taught by teachers with "limited" experience in mathematical proof would have an especially difficult time learning the concept. Immediately, I speculated what would be the results if someone conducted a parallel study here in Canada. I realized that I could apply Knuth's methodology to a study, not with practicing teachers, but with student teachers. Although I did recognize elements of positivistic thinking in the idea of replication, I thought that I would aim, not so much to recreate the study, but to apply the research design to a different group of participants and a different context. Such a study would be useful for mathematicians and mathematics educators because it would assist them in identifying problems within the mathematics teacher education curriculum and, if necessary, refining and revamping the program. As an education undergraduate, I had noticed that teacher educators do pay attention to the role of "proof and reasoning" in mathematics as part of their curriculum. I felt it important that a study be conducted examining the degree to which student teachers feel prepared for and confident about teaching "proof and reasoning" to their future students. With this in mind, I requested and completed an independent study course titled "Proof and Reasoning in School Mathematics."

Understanding the Past—Arriving at Interpretive Inquiry: Bridging Two Paradigms

As I continued to take graduate courses in curriculum theory and research methods, I found myself more and more interested in interpretive inquiry. I accepted the idea that people construct their own meanings and exist within shared meaning systems: I had experienced this first hand as an immigrant. Aspects of interpretive inquiry such as this made sense to me. Rather than focusing on quantitative data, applying rigid procedures, and performing a statistical analysis of data, I decided to include qualitative data in the form of semi-structured conversational interviews that I would later transcribe. I would also try to examine those very processes that Skinner said would be impossible to study—the thought processes of student teachers as they worked to prove a number of different types of proof. I found myself moving towards a sensible, yet strange, new paradigm; but I resisted. I was not yet prepared to dismiss the importance of cause and effect relationships.

I moved closer to accepting the social learning aspect of Interpretive Inquiry when I read Lakatos's (1976) "Proofs and Refutations". Lakatos argues that mathematics develops by means of a process of "conscious guessing" and that proofs take a "zig-zag" path originating from conjectures (conscious guesses) and moving forward with the help of counter examples. I found this intriguing. Lakatos' ideas had a great deal in common with what reformists in mathematics education were now advocating—that all students should be making conjectures, abstracting mathematical properties, explaining reasoning, validating assertions, and discussing their own thinking and the thinking of others (NCTM, 1989, 2000).

At that point, I decided I would explore in my study the thought processes of student teachers by examining the written work they produced as they proved various proofs. I determined that concept maps would also help me in assessing student teachers' understandings of the nature and role of proof, and I reasoned that interviews would make it possible for the participants to elaborate upon and explain their thinking processes. As I proceeded I found myself attempting to bridge positivism and more interpretative inquiry: I could not completely let go of all that I had believed and practiced for so much of my life; at the same time, I wanted to move in this new and exciting direction. My research methodology and my philosophical assumptions ultimately reflect this reality. I had learned that interpretative researchers rarely ask objective survey questions or aggregate the answers of many people (Newman, 2000). I did not abandon survey questions but limited the number of them considerably and then prepared for students to expand upon their written responses in interviews. Interpretative researchers recognize that each person's interpretation of a survey question must be placed in a context such as the person's previous experiences or the survey interview situation (Newman, 2000). I acknowledged this as well, unearthing to some extent (though not always sufficiently) my participants' mathematical and school-based experiences as well as considering the impact of my limited experience as an interviewer and my own experiences with mathematics and mathematics education. I recognized that each participant would assign a somewhat different meaning to both the questions that I posed and the answers that were given, and I knew that values, both those of the participants and my own, had to be made explicit. This chapter reflects my desire to make my philosophical assumptions as a researcher clear. As I reflect on the path of the past and the place of the present, I see that I have moved from positivism towards interpretive inquiry, though I currently find myself somewhere between the two paradigms.

CHAPTER 3. LITERATURE REVIEW

The Notion of Proof in the Literature

The literature on mathematical proof is vast and comes from many different sources including the philosophy of mathematics, the history of mathematics, and mathematics education (Raman, 2002, 2003). This explains—in part—the multiplicity of definitions, which in turn, explains—in part—the confusion concerning proof that one often finds among mathematics educators. In Chapter 1, I listed several functions of proof and proving as identified by various mathematics educators (see p. 6). In the following sections, I discuss these in more depth, specifically in relation to what I have discovered in the literature.

Proof as Verification and Convincing

The historical roots of proof, dating back to the time of the ancient Greeks, indicate that the role of proof is to verify mathematical results. As Hanna (1983) observes, the main role of proof in mathematics has been to demonstrate the correctness of a result or the truth of a statement. Proof as verification is often associated with formal proof (mathematicians' proof)—often considered "real proof"—that is, proof that is rigorous and certain. A mathematician uses this mode of proof to communicate her/his new result to the mathematical community; the community then may verify it. Thurston (1995) describes his own experience of working with mathematical proof as a process involving effort on the part of both self and community—the former arrives at the mathematical result, while the latter verifies it. He recalls that:

> mathematicians were actually very quick to accept my proof, and to start quoting it and using it based on what documentation there was, based on their experience and belief in me, and based on acceptance by opinions of experts with whom I spent a lot of time communicating the proof.
>
> (p. 37)

However, if one regards verification as the only function of mathematical proof, one must reconsider the inclusion of proof in the school mathematics curriculum and its presumed value within educational contexts. It is evident from the literature that students are convinced more by examples than rigorous /formal proofs (Coe and Ruthven, 1994; Harel & Sowder, 1998). Students, it would seem, are less convinced by the symbols and operators associated with formal proofs than they are by illustrative instances. This is one of the reasons why many mathematicians and mathematics educators have become advocates for less formalism in school mathematics programs.

Proof as Systemization

Proof also helps to build mathematical knowledge. It functions "as a means of connecting mathematical results into an integrated body of knowledge" (Leddy, 2001, p. 21). As de Villiers (1990) notes, "the main objective clearly is not to check whether certain statements are really true but to organize logically unrelated individual statements which are already known to be true, into a coherent unified whole" (p. 21). Proof functions as a means to systemize known results into a logical system of definitions, axioms and theorems. Proof plays a major role in expanding mathematical knowledge because it is by proving theorems that mathematical knowledge is developed, communicated and advanced (Thurston, 1995). Thus, according to the literature, one important function of mathematical proof is to integrate mathematical knowledge into a coherent body. This process, in turn, develops and strengthens the discipline.

Proof as Explanation

Hanna (1990) insists that mathematicians value proof far more for revealing essential mathematical relationships than for demonstrating the correctness of a result. She uses the terminology *proofs that explain* to signify proofs that not only prove, but also explain *why*. She contrasts this with *proofs that prove*: in other words, proofs that do not have a high degree of explanatory power. Proofs that explain, she argues, have more value for mathematics education than do proofs that merely prove. As Leddy (2001) suggests, not all proofs explain well. Formal proofs are good for verification and/or systemization, but may be of little use in providing explanations. "Proofs that only prove", Leddy adds, "do not hold much value either aesthetically or informatively" (p. 25). Few would disagree with the statement that the main purpose of proof in any school mathematics program is to foster student understanding rather than convincing students of a correct result. Hence, the use of proof to explain becomes extremely important within the school classroom.

Proof as Discovery

Lakatos (1976) identifies proof as a "social process", one used to discover new mathematical results. Obviously, if proofs serve only to verify results, they are not likely to generate new knowledge. In his classic book, *Proofs and Refutations,* Lakatos writes "mathematics does not grow through the monotonous increase in the number of indubitably established theorems but through the incessant improvement of guesses by speculation and criticism, by the logic of proofs and refutations" (p. 5). Lakatos describes proof as explanations, justifications and elaborations that serve to make a conjecture (posed by potential doubters) more convincing. Proof helps one discover whether or not initial and successive conjectures make sense. This notion of proof aligns well with the views of the NCTM (2000): "Doing mathematics involves discovery. Conjecture—that is, informed guessing—is a major pathway to discovery" (p. 57). The NCTM document also reminds mathematics educators that

> Students can learn to articulate their reasoning by presenting their thinking to their groups, their classmates, and to others outside the classroom….The particular format of a mathematical justification or proof, be it narrative argument, "two-column proof", or a visual argument, is less important than a clear and correct communication of mathematical ideas appropriate to the students' grade level.
>
> (p. 57)

Proof as Communication of Mathematical Knowledge

The literature also discusses proof as a particular method for communicating results to others, a method or means that is clear and direct. The NCTM (2000) encourages students to participate in mathematical practices that can lead to new discovery. As one discovers new understandings, one finds it necessary to persuade, first, him/herself, and second, others, that what has been discovered is, indeed, true. This social process necessarily entails that the student communicate the proof to his or her community. The NCTM describes "communication. . . [as] an essential part of mathematics and mathematics education. It is a way of sharing ideas and clarifying understanding" (p. 60). The NCTM document reminds teachers that "the particular format of a mathematical justification or proof, be it narrative argument, two-column proof, or a visual argument, is less important than a clear and correct communication of mathematical ideas appropriate to the students' grade level" (p. 57). Therefore, mathematics educators need to pay close attention to proof as a vehicle to communicate mathematical knowledge.

Proof as Problem Solving

Lucast (2003) argues that proof and problem solving are closely linked. She laments that this conception of proving is too often overlooked in school mathematics. Proving requires that one work towards a solution. Hence, the act of proving can be considered an act of problem solving. Students who demonstrate effective problem-solving skills tend to be skilled at proving as well (Schoenfeld, 1994). Epp (1994) observes that in order to solve a proof, one must be able to use deductive and intuitive reasoning, and both of these mental skills are typically evident in the student who is an effective problem-solver. In problem solving, students must drawn upon their knowledge and skill in the subject area and then apply that to a novel situation. By means of this process, students build new understandings. Proving as part of problem solving requires the same approach. The person who proves a task also draws on past knowledge of the subject and applies it to a new situation. Effective problem solvers, of word-based and formula-based problems such as proofs, analyze the situation carefully, identify assumptions, consider simple case and/or patterns and relationships, and organize logical arguments; then they apply that understanding to future situations. Proving also requires the similar skill- exploration of task, developing mathematical conjectures, and reasoning through the task by providing mathematical arguments.

History of Reform Efforts in Mathematics Education as They Pertain to Proof

In the 1930s, educators identified efficiency in logic and reasoning as a desirable and important goal within the high school mathematics program. In 1938, Fawcett observed that the *in*efficient use of logic and reasoning among students constituted a serious problem, a problem that could only be adequately addressed by reform in mathematics education. His recommendations focused on using mathematics as a means to an end: to develop within the student critical and reflective thinking. Fawcett considered geometry the ideal context for learning critical thinking skills (Williams, 1979). Many mathematics educators welcomed this new direction in mathematics curricula; however, they questioned whether it would be possible to achieve this objective by using traditional approaches to instruction (Latterell, 2005)

From the 1950s until the 1980s, there followed wave after wave of new math initiatives. In the 50s, a "New Math" movement gained momentum, giving birth to the School Mathematics Study Group (SMSG), an organization established, not by teachers, but by mathematicians. As the decade passed, professional mathematicians working under the auspices of the SMSG came to be more and more involved with school mathematics, intent upon improving the school curriculum by emphasizing logic, methods of deductive proof, and the role of axiomatic structures in mathematics. Educators, on the other hand, denounced, on pedagogical grounds, such an extreme emphasis upon axiomatic structures and rigorous proof. Ultimately, the New Math Movement failed. Its focus on exaggerated formalism, the lack of success that teachers experienced teaching students in this way, and the ongoing criticism from various stakeholders of schooling (Hanna & Jahnke, 1993) all contributed to the movement's demise. By the early 1980s the NCTM, responding to the failure of the New Math Movement, proposed another set of reforms for school-based mathematics curricula. Although the specifics of the NCTM's reform plan (1989) had limited success in application, the directions and recommendations outlined in the more recent NCTM document titled may prove more effective.

Proof and the Alberta Curriculum

The Alberta Program of Study (PoS) for Mathematics Education (1996) emerged out of the Common Curriculum Framework developed for the provinces of Alberta, British Columbia, Manitoba, Saskatchewan and the Northwest and Yukon Territories. The PoS recognizes that mathematics is increasingly important in a rapidly advancing, technological society. Philosophically, it rests upon the premise that students learn by attaching meaning to what they do—by constructing their own meaning of mathematics. When students are permitted to learn in this manner, the struggle to learn mathematics—a worldwide phenomenon (Hadas, Hershkowitz & Schwarz, 2000)—will diminish. The Alberta PoS calls for a classroom environment in which learners feel comfortable taking intellectual risks, asking questions and posing conjectures (Alberta Learning, 1996). It encourages classroom procedures that cultivate critical and reflective thought.

Notably, the PoS identifies reasoning as one of the primary goals of a mathematics education, and proof and proving serve as a fundamental means by which to develop critical thinking skills. In fact, the curriculum document does not explicitly address the concept of "proof". The PoS, published in 1996—prior to the NCTM standards document of 2000—obviously does not represent a response to the NCTM recommendations; however, it does cite the NCTM standards document of 1989, and, indeed, builds upon the 1989 standards by highlighting the need for students to develop confidence in their abilities to reason and to justify their thinking. The PoS encourages "testing of conjectures", "formulation of counter examples", and "construction and examination of valid arguments." It underlines the need for students to exercise logical thinking, inductive reasoning and deductive reasoning. The mathematical process R (reasoning) appears in the Alberta PoS, on and off, beginning with Grade 1. However, explicit references to the use of reasoning are very much confined to the Grade 11 Coordinate Geometry section within the mathematics textbooks that are used in Alberta (namely, *Math Power* and *Minds on Math*). Here again, one encounters an emphasis on proof as two-column structure, with statements in one column and reasons in the other. In the Alberta PoS, reasoning and proof seem to be compartmentalized within Euclidean Geometry. Knuth (2002a, 2002b) noted a similar emphasis in the US curriculum, and I observed the same phenomenon teaching secondary mathematics in Brunei. My experience teaching students the concepts of proof and proving have confirmed what many researchers have shown: teaching proof and reasoning in such a rigid manner within such a narrow context tends to generate among students negative attitudes towards mathematical proof (Knuth, 2002a). Had I asked any of my former high school students what they felt about proof and proving, I would have been sure to hear comments like "proof is a necessary evil", "I hate proof" and "I've got to do it anyway." I believe it is time to expand upon the role of proof in the Alberta school mathematics program of study. In line with the NCTM recommendations of 2000, proof needs to be considered throughout the curriculum; but it also needs to be introduced to students in more effective ways. Classroom teachers are far more likely to develop and extend student learning by structuring mathematical experiences so that students have ample opportunity to create and validate proofs. I believe that this needs to become a fundamental goal of mathematics instruction. Although classrooms that function like this will differ significantly from traditional mathematics classrooms, establishing them will be well worth the effort.

Summary

The literature contains multiple definitions of proof, each definition determined on the basis of how a proof can function. Not surprisingly then, mathematics educators seem confused about how best to go about the practice of teaching proof. Over the years, proof has played a peripheral role within the secondary school classroom with teachers limiting discussions of proof to Euclidean geometry (Knuth, 2002a, 2002b). Given this narrow and limited application, it is not surprising that students are rarely able to identify the objectives or functions of mathematical proof (Knuth, 2002a). In fact, teachers themselves have limited knowledge about the role and function of mathematical proof (Knuth, 2002a). It appears, then, that schools and teachers—in North America and the United Kingdom at least—have failed in their efforts to teach students mathematical proof (Hadas, Hershkowitz & Schwarz, 2000).

Recent documents call for a greater emphasis upon proof and proving within school mathematics programs (NCTM, 1991, 2000). Provincial programs of study, for the most part, have taken up the NCTM lead. In Alberta, sweeping curricular changes in the school mathematics program took place in the late nineties (Alberta Learning, 1996). However, since documents were completed prior to the NCTM's explicit recommendations concerning proof (2000), curriculum documents and teaching resources do not emphasis proof to the degree that they could. Following the lead of the NCTM (2000), a number of university educators have revised the teacher education curriculum to incorporate the notion of proof. Now, several years since the NCTM first prompted a greater emphasis upon proof and proving, it will be useful to determine the extent to which student teachers understand mathematical proof and feel confident about teaching proof and proving skills in the classroom.

CHAPTER 4. FINDING DIRECTION

Relevant Literature

In this chapter, I take a closer look at literature that is relevant to my study, especially reports that have inspired my own research. In preceding chapters, I have argued that mathematical proof, given its numerous functions and tendency to develop diverse thinking skills, is particularly important to the school mathematics program. In fact, there are quite a few studies reported within the literature that have examined school students' conceptions of mathematical proof. I will discuss two, one by Balacheff (1988) and the other by Healy and Hoyles (2000). Other researchers have focused on practicing teachers and how they conceive of proof. Since teachers will be instrumental in implementing current reform principles within the classroom, assessments of their competence and proficiency with proof and proving are timely. In this respect, I discuss the work of Knuth (2002a).

Studies such as these in mathematical proof, and their unmistakable findings, have alerted mathematics educators to the difficulties associated with developing, among both teachers and students, proficiency in the construction of mathematical proofs. It is encouraging to see that faculties of education are responding to the NCTM call for reform: mathematics educators have recently incorporated into their teacher education programs topics that are relevant to mathematical justification, principally mathematical proof (Cuoco, 2001). Now is an appropriate time, I suggest, for educators to study the impact that these changes have had on student teachers' conceptions of and facility in working with mathematical proof. Mathematics educators, as well as government bodies, professional associations and the public, should all find the results of such a study helpful in practical ways. Within the literature, however, there are few studies that focus entirely upon student teachers—two exceptions include Cyr (2005) and Jones (1997). I will take a closer look at the study conducted by Jones.

Studies on Secondary School Students' Conceptions of Proof

1. Balacheff (1988)

Nicolas Balacheff (1988) conducted an experimental study of secondary students' conceptions of mathematical proof. He observed students as they were engaged in the process of generating proofs, noting the various ways in which his participants— 28 secondary school students, 13 and 14 years of age—approached a specific mathematical assignment. He permitted students to work in pairs as they set about completing the following task: *"Provide a means of calculating the number of diagonals of a polygon when you know the number of vertices it has"* (p. 221). Balacheff facilitated interaction between the partners by providing only one pen for each pair. Students were allowed to work on the problem until they arrived at a solution; but both students had to agree that their answer did, in fact, provide a solution to the problem *before* they could claim to have finished. In this study, Balacheff focused on both the reasons that students gave for selecting the examples that they did and how they used those examples. He was keen to understand the processes involved in arriving at the product, but he understood that social interaction could either assist or hinder students in arriving at a solution to the proof (p. 222). After analyzing the results, Balacheff classified the student responses into four different types of proofs and argued that these categories represented four increasing sophisticated levels of thinking.

Almost a decade later, Simon and Blume (1996) argued that Balacheff's hierarchy of proofs was, in fact, an extension of van Dormolen's (1977; as cited in Simon & Blume, 1996) taxonomy of proof. According to Simon and Blume, van Dormolen differentiated among proofs by establishing three distinct categories: proofs that (1) focus on a particular example, (2) use an example as a generic embodiment of a concept, and (3) use general and deductive argument. Balacheff, subsequently, identified four categories of proofs: (1) naïve empiricism, (2) crucial experiment, (3) generic example and (4) thought experiment. Balacheff also situated his taxonomy within a developmental model of proving skills. In addition, Balacheff argued that each of these four levels of mathematical proof could be classified within one of two broad categories that he termed *pragmatic* justifications and *conceptual* justifications. He called all justifications *pragmatic* when they focused on the use of examples, actions or showings. He called justifications *conceptual* when they demonstrated abstract formulations of properties and relationships among properties.

The first three levels in Balacheff's proof scheme are all examples of pragmatic justifications. In the case of *naïve empiricism*, the first level in Balacheff's taxonomy of proofs, the student arrives at a conclusion concerning the validity of an assertion on the basis of only a small number of particular cases. Balacheff exemplifies this level in his description of the efforts of school students Pierre and Mathieu. Working together, these boys examined a square, a hexagon and then an octagon. They concluded that they could arrive at the number of diagonals by dividing the number of vertices by two. In this example of naive empiricism, the boys checked the statement to be proved against a few particular examples and, on this basis, made a universal assertion. With *crucial* *experiment*—the second level in Balacheff's taxonomy of proofs—the student deals with the question of generalization after generating a claim based on a few examples by examining a case that is not very particular. If the assertion holds in the considered case, the student will conclude that it is valid. Balacheff illustrated critical experiment by referring to the efforts of Nadine and Elisabeth. These girls chose a polygon of many sides (15) believing that the assertion they came up with could be proved in this instance, then the assertion would be universally true. In other words, at the level of crucial experiment, the student checks the statement by means of a carefully selected example. A defining characteristic of crucial experiment is the intentionality of the student: deliberate choices must be made (Knuth & Elliot, 1998). Notably, both naïve empiricism and crucial experiment deal with actual actions or showings; the main difference between the two rests with the status of the specific example that is selected to validate the assertion—the example used in crucial experiment proof is often based on carefully selected extreme cases. I came to know through my work that one is more able to distinguish between crucial experiment and generic example while observing the student as s/he actually works through the task.

In the case of *generic example*—the third level in Balacheff's taxonomy of proofs—the proof rests upon the properties. Here, the example is a generalization of a class, not a specific example. Although the focus is once again a particular case, it is not used as a particular case, but as an example of a class of objects. The student selects such an example as representative of the class and performs operations/transformations on the example in order to arrive at a justification. Then, the student applies these operations and transformations to the whole class. Balacheff mentions Georges' exploration of the proposition f(n) = n*s(n) (where s(n) is the number of diagonals at each vertex) as an example corresponding to this category. However, it is quite interesting to note that Balacheff does not give an explicit reason as to why this represents a generic example (see Balacheff, 1988, pp. 224-225).

Only with the fourth and highest level of proof in Balacheff's taxonomy do students move from the practical—pragmatic justification—to the intellectual—conceptual justification. At the level of *thought experiment,* students are able to distance themselves from action and make logical deductions based only upon an awareness of the properties and the relationships characteristic of the situation. At this level, actions are internalized and dissociated from the specific examples considered. The justification is based on the use of and transformation of formalized symbolic expressions. Balacheff provides an example of thought experiment in his description of Olivier. This student asserted that *"In a polygon if you have x vertices there are automatically y diagonals from each point because in a boundary of the polygon there are two points which join it; in conclusion there are x-3 which are the diagonals".* Olivier was able to express the properties of a polygon by observing one specific example. It is important to note that Balacheff categorized all assertions that de-contextualize themselves from the traces of formulation of their arguments, even if they are not but are not necessarily fully correct, as thought experiments. It is the students' *approach* to the task of proving that he is categorizing not the validity of the outcome.

Knuth and Elliott (1988) later expanded on Balacheff's taxonomy. They set about providing examples that would demonstrate each of the four levels of thinking. They used *power chord theorem* in their efforts to show how students thinking at any one of these levels might approach the task of proving the proof. In examining both Balacheff's original study and the work of Knuth and Elliott, it has become clear to me that distinguishing between naive empiricism and crucial experiment is a difficult task, especially if one looks only at the end product of the student's writing. I also observe that Knuth and Elliott could *not* provide a concrete instance of generic example. Hence, despite the neatness of the Balacheff model, in practical application one may have some difficulty both in distinguishing between naive empiricism and crucial experiment and in coming across instances of generic example.

The four levels in Balacheff's taxonomy represent a hierarchy through which students are *expected* to progress as their notions of mathematical justification develop. Balacheff reasoned that students' understandings of mathematical justification are likely to proceed from the inductive toward the deductive and toward greater generality. Hence, those with increased mathematical maturity are most likely to be the students who generate deductive proofs. Balacheff also stressed that students will move back and forth between inductive and deductive reasoning depending on the task that they are completing. In other words, a student capable of thought experiment in one situation may regress to naïve empiricism in another. This observation will be verified by instances from my study.

Balacheff's framework provides a way to analyze and classify both the various processes involved in generating conjectures and producing justifications as well as the justifications themselves. A complete assessment of students' justification skills has to take into consideration both products (the justifications produced by students) and processes (the ways by which students produce justifications). Thus, in my study I selected three students and conduct interviews with each following the task completion stage. My goal was to acquire better understanding of the mental processes used in solving the mathematical tasks (discussed further in Chapters 7, 8, 9, 10 and 11).

2. Healy and Hoyles (2000)

A more recent article published by Healy and Hoyles (2000) details the procedure and results of a study that the authors conducted exploring students' conceptions of proof. Healy and Hoyles undertook a massive study examining the mathematical abilities of 2459 high school students from England and Wales. Although all of these 16 year olds had been classified as high achievers (within their schools, all placed within the top 25% in student achievement scores), nonetheless they performed poorly in a proof construction task (and the few who were successful presented their proofs in everyday, explanatory language). Healy and Hoyles observed that those participants who, despite being identified as high achievers, found it difficult to produce proofs, commonly took an approach based on empirical verification, *even though* they understood that they could not generalize from the results obtained in this manner. When presented with problems that challenged their ability to generate empirical examples, these students were unable to move on to a different approach in solving the proof. Overall, student responses were influenced by curricular factors, their views of proof, and their gender, but *mainly* by their mathematical competence (Healy & Hoyles, 2000).

Studies on Elementary School Students' Conceptions of Proof

I also draw on a study that Harding (1999) conducted with her elementary school students. After completing a "Proof and Justification" module as part of a university course, Harding realized that she could introduce logical and deductive reasoning into the lower-level mathematics curriculum. Accordingly, she incorporated a number of justification activities into classroom instruction, both at the primary and elementary levels. She writes that when she "tried out some activities related to proof" she was "amazed" at the level of reasoning that some of her 9- and 10-year old students were capable of (p. 12). Her study focuses on the mathematical abilities of Year 5 students (ages nine and ten), although her analysis includes references to some of the tasks she completed as a student in her University class. I selected, for the purposes of my study, one of the tasks that she used with her Year 5 students. It was a number theory task that required at least a junior high school level of mathematical competence; hence, I thought that my participants should certainly find it accessible. In her analysis, Harding notes that the ways in which her students' grappled with mathematical justification could be categorized in the following manner:

1) one group of children established the rules using specific examples, but provided no generalization;
2) a second group inferred that the rules were generally true since they did work for a variety of examples; and
3) a third group was able to show without endless checking of specific examples that the rules were generally true.

While most of the students tackled the task by using examples, one student came up with a proof that involved algebraic manipulation. It is interesting to note that similarities exist between Harding's categorization of students' proofs and Balacheff's (1988) taxonomy of proof. Harding's work illustrates that primary and elementary school students are capable of logical reasoning; therefore, primary and elementary teachers must ensure that children develop these logical skills by structuring specific and appropriate classroom activities for these students.

Studies on Practicing Teachers' Conceptions of Proof

When Healy and Hoyles (2000) confirmed Senk's earlier findings (1985, 1989) that secondary school students rely, for the most part, on what Balacheff identifies as the lowest level in a developmental taxonomy of proving skills, mathematics educators began to take notice. Soon the focus turned from students to their teachers. In order to help students master mathematical proof, a teacher must diagnose the level at which the student is currently operating; then, s/he must structure diverse curricular experiences that will work to develop higher-level skill within each student. Clearly, this is a challenging task and an important responsibility. The question now becomes: Do teachers possess the understanding and skills that are essential for this task? Given the limited skills demonstrated by secondary-level students, one must examine the particular conceptions and beliefs about mathematical proof held by practicing teachers.

In an effort to address this question, Knuth (2002a) conducted a research study with 16 in-service teachers, using written response and interview as data collection method. Knuth gave the participants two take-home assignments. The first—what Knuth referred to as the first round—required teachers to complete three mathematical tasks. In the case of the first task, only 11 of the 16 teacher-participants succeeded in completing a valid proof; five teachers constructed invalid proofs. (Knuth considered all empirically based proofs to be invalid.) In the second case, only four of the 16 generated a complete and valid proof. However, in completing the third task, the majority was able to generate a valid proof. The second assignment—the second round—required teachers to prove a statement using indirect proof. Knuth reports that only four of 14 teachers who attempted this task were able to produce a valid justification. It is interesting to note that Knuth used two different frameworks—those proposed by Balacheff (1988) and Harel and Sowder (1998)—in analyzing his data. He also observed that these teachers did not recognize proof as a means for promoting mathematical understanding.

Knuth discovered that most of his teacher-participants demonstrated a limited understanding of proof in terms of its role and function in the classroom and a limited facility with the different approaches to proof (1999, 2002a, 2002b). As a result of these findings, Knuth concluded that such limited proficiency in applying mathematical proof would make it challenging for these teachers to implement, with any degree of reasonable success, the recommendations for reform. Knuth suggested that the most likely explanation for this limited skill with mathematical proof was the nature and limitations of the mathematics education received by these teachers when *they* were students. Many researchers now agree that changes in current school practice can best be achieved, not by targeting teachers in the field, but by reforming teacher education programs at post-secondary institutions (Cuoco, 2001; Ma, 1999; Stigler & Hiebert, 1999). University mathematics educators have examined both the findings of studies such as those mentioned above and the recommendations put forth by the NCTM. Consequently, they have taken steps to address this concern by changing the mathematics curriculum for prospective secondary-level mathematics teachers (Cuoco, 2001). What remains unclear at this point is the extent to which these efforts have succeeded.

Studies on Student Teachers' Conceptions of Proof

1. Jones (1997)

Among those studies documented within the literature that examine secondary-level student teachers and their conceptions of mathematical proof, the work of Jones (1997) is directly relevant to my own research. Jones grounded his study in the belief that success in teaching depends upon the subject knowledge of the teacher. Secondary-level teachers are typically viewed as subject specialists. Course requirements for students enrolled in a secondary education program, regardless of the academic discipline involved, usually include a number of courses in the subject area itself. These courses are mandatory even though they are not, strictly speaking, courses dealing with pedagogy. Student teachers who will eventually be teaching challenging concepts at the secondary level are expected to have acquired considerable understanding of and experience working with these difficult concepts. Within the context of school mathematics programs, one such challenging concept is mathematical proof. As Jones suggests in this paper, teaching proof at the secondary level requires extensive knowledge on the part of the teacher, both subject knowledge and pedagogical knowledge. Noting that the limited subject knowledge of both pre-service and in-service mathematics teachers is cause for concern, Jones undertook a small-scale investigation of pre-service secondary school mathematics teachers within the United Kingdom and the ways in which they conceived of mathematical proof and proving. He wanted to know just how confident student teachers felt about the prospect of teaching mathematical proof.

Of particular interest to me was the fact that, Jones utilized the concept maps both as a research and an evaluation tool. A concept map is an explicit, graphical representation of domain material understanding as generated by the learner. The mapmaker uses nodes to represent key domain concepts and links to denote the relationships between these concepts. Jones invited the group of student teacher participants to brainstorm a list of key terms that they associated with the concept of mathematical proof. The students produced a list of 24 terms. Jones then asked each student teacher to create a concept map that represented his/her understanding of mathematical proof. Students were permitted to use any or all of the key terms previously brainstormed. Using a blank sheet of paper, participants arranged the terms as each saw fit, joining terms in what each perceived as a meaningful way. Then each student indicated the relationships among the key terms by drawing lines and/or writing descriptive words on the map. Jones analyzed student maps in terms of three criteria: the specific terms used, the frequency of terms used, and the nature of the relationship (if specified) between any two terms. According to Jones, the higher the student's Grade Point Average (GPA), the more terms the student was likely to use in constructing the map. Furthermore, the student teacher with the highest GPA produced the most sophisticated map: this student added terms that were not on the original list of 24. Jones concludes that the degree of confidence a student teacher experiences, and the likelihood of his/her future success in teaching mathematical proof, depends upon the construction of sound knowledge at the all-important undergraduate level both in terms of subject area and pedagogy.

2. Martin and Harel (1989)

Martin and Harel (1989) also studied prospective *elementary* teachers' conceptions of mathematical proof. They enlisted the assistance of 101 pre-service elementary teachers who were enrolled in a required sophomore-level mathematics course at Northern Illinois University. These students had gained considerable experience working with mathematical proof, both in the university course that they were then taking, and in a prerequisite course, a high school geometry class. During the tenth week of the university semester, the researchers assessed the student teachers' skills in dealing with mathematical proof by asking them to judge the verification of a familiar and an unfamiliar mathematical generalization.

Martin and Harel found that these future elementary teachers failed to distinguish between inductive and deductive reasoning when it came to mathematical proof. Many of them accepted as true what the researchers called "false proof". Martin and Harel define false proof as "a fallacious proof of the generalization, including statements purporting to justify each step [. . .] Although not a deductive argument, it may be incorrectly viewed by students as a deductive argument based on its ritualistic aspects" (p. 44). They provided the following example of "false proof": "Let a be any whole number such that the sum of its digits is divisible by 3. Assuming its digits are x, y and z, then $a = xyz$. Since $x + y + z$ is divisible by 3, also xyz is divisible by 3. Therefore, a is divisible by 3" (p. 45). I observed something similar with the participants in my study: secondary-level student teachers (who apparently did not know how to produce a valid proof) tried to illustrate their proficiency in mathematical proof by including irrelevant algebraic expressions or some other 'ritualistic' format (for example, a two column format). I provide examples of this in Chapters 8, 9, 10 and 11.

Significantly, Martin and Harel found that pre-service teachers accepted as true *both* inductive and deductive arguments; both types of proofs, it seems, were equally persuasive. Students who were apparently convinced by deductive proof wanted further empirical verification. It seems that they were more influenced by the quantity of evidence than by the type of reasoning called upon to prove the proof. According to Martin and Harel, the everyday experience of forming and then evaluating hypotheses by using evidence to support or refute them serves to reinforce inductive reasoning among people, particularly among those who lack high-level mathematical skill. Again, I observed a similar tendency among the student teachers in my study. Students who at first devised some form of deductive argument in response to a task felt the need to confirm the result by subsequently conducting further empirical verification.

Summary

Studies indicate that a vast majority of secondary school students, pre-service mathematics teachers, and in-service math teachers lack a deep understanding of, and proficiency in dealing with, mathematical proof. What all of these groups seem to have in common is the experience of learning proof by means of imitation and memorization within the narrow context of the geometry lesson. Students would develop greater proficiency in mathematical proof if mathematics educators could abandon the perception that the teacher and the textbook must be the sole source of truth in the classroom, and instead encourage in students a willingness to explore and seek truth by means of conjecture and discussion. This will not be an easy shift. In particular, it will require tremendous effort on the part of teachers, for it demands nothing less than rejecting long-standing classroom tradition.

CHAPTER 5. RESEARCH METHODOLOGY

Why Student Teachers?

Since the teacher's conception of the subject matter significantly impacts instructional practice (Ball, 1990; Fennema & Franke, 1992; Thompson, 1984), a teacher who possesses a narrow understanding of a key concept like proof is unlikely to provide satisfactory instruction. Indeed, as Knuth (2002a) has shown, most of the participants in his study of practicing secondary school teachers possessed limited understanding of mathematical proof. This may be why, in part, traditional approaches to teaching mathematical proof at the high school level have not been very successful (Senk, 1989). Moreover, Knuth (2002a) contends that practicing teachers are not likely to spearhead large-scale reform given the strong tradition of compartmentalizing proof within school classrooms. In-service training can be intense but brief (often a one or two day workshop scheduled at irregular intervals) and, therefore, unlikely to challenge firmly-established practices. Thus, expecting practicing teachers with traditional teaching styles and methods, limited in-service training, and heavy teaching loads to lead reform efforts in mathematics education seems impractical. Professional documents and research articles that promote an increased emphasis on mathematical reasoning will be of little benefit if teachers are unwilling or unable to comply.

It may be far more productive to focus on future teachers whose experiences with traditional approaches have been typically limited to their years as elementary and secondary school students. Student teachers, given their recent experience with post-secondary mathematics courses, will have become familiar with a new approach to mathematics instruction that places proof front and center. As well, their recent mathematics education courses will, no doubt, have prompted considerable discussion around issues of current reform. Thus, student teachers who are transitioning into the career role of practicing teacher may be less likely to restrict application of mathematical proof within the mathematics curriculum and more likely to take on the challenge of implementing curricular reform; hence, university teacher preparation programs seem an ideal place to determine the results of early efforts at implementing mathematical reform. This is one reason why, in my study, I focus on prospective teachers who are at the end of their teacher education program.

Proposed Study

Numerous studies related to mathematical proof, undertaken with participants at different levels of schooling and from quite different perspectives, have been reported in the literature. Studies extend from the university-level perspective of students and teachers (Raman, 2003; Housman & Porter, 2003) to the secondary-level point of view of students and teachers (Balacheff, 1988; Healy and Hoyles, 2000; Knuth, 2002a, 2002b) to the perspective of prospective elementary-level student teachers (Martin & Harel, 1989). With the exception of Jones (1997) and Cyr (2005), little work has been done on secondary-level student teachers' conceptions of proof. I aim to contribute to reducing this gap in the literature: the purpose of this study is to report on the conceptions of mathematical proof held by student teachers who will soon be teaching mathematics at junior high school or high school level. I take a case-study approach in this research, using written response and interviews as methods for data collection. The study itself involves two-phases: Phase 1: Participants' responses to Written Tasks, word-based, mathematical, and representational; and Phase 2: Interview of selected students.

Why Case Study?

A case study involves the "exploration of a 'bounded system' or a case (or multiple cases) over time through detailed in-depth data collection involving multiple sources of information rich in context" (Creswell, 1998, p. 61). Yin (1984) identifies five components within a case study research design: (1) research question; (2) its propositions, if any; (3) its unit of analysis; (4) the logic linking the data to the propositions; and (5) the criteria for interpreting findings. The fourth and fifth components, namely the "Logic linking the data to the propositions" and "Criteria for interpreting findings", represent the data analysis steps of this research design. A case can consist of an event, a process, a program or a group of people. A case-study strategy is useful in my situation because my study entails "an examination of [a] contemporary event" (Yin, 1984, p. 14) and involves the exploration of a particular system bounded by time and geography.

Advantages of Case Study

A case study approach offers many advantages. Cohen and Manion (1995) list the following:

1) Case studies are drawn from people's experiences and practices and so represent what is perceived to be strong reality.
2) Case studies allow for generalizations from a specific instance to a more general issue.

3) Case studies allow the researcher to show the complexity of social life. Good case studies built on this can explore alternative meanings and explorations.
4) Case studies can provide a data source from which further analysis can be linked to action.
5) The data collected by means of a case study are close to people's experiences and typically more persuasive and accessible.

Furthermore, a case-study approach offers the unique strength of being able to deal with a variety of evidence including such items as documents, artifacts, and interviews (Yin, 1984; Creswell, 1998). It is also ideally suited to the needs and resources of a small-scale researcher (Blaxter, Hughes & Tight, 2001). Finally, case study is often the best approach when one poses a "how or "why" question about a contemporary set of events over which the researcher has little or no control (Yin, 1984). Since one aspect of my study involves a how question—how able are student teachers when it comes to completing mathematical proof—a case study approach seems appropriate.

Research Question

The fundamental question that drives this study is "***What understandings do student teachers possess concerning the notion of mathematical proof?***" Two primary research questions focus and guide this examination:

1) What do student teachers believe about the nature and role of proof?

2) How able are student teachers when it comes to completing mathematical proofs?

Early on, I determined that in order to reach my target subjects at an appropriate time in their education program, I would need to seek permission of the instructor of the final professional term to enter into this person's classroom in order to collect data collection. The professional term, as it is called at this university, offers methodology courses in a student's teaching major over an intensive full-day, six-week period. I went into the classroom on two occasions towards the end of this six-week period. On my first visit, I explained to the student teachers both the nature of my study and my reasons for conducting the research. I stressed that there is no obligation on their part to participate in this study and they could withdraw at any time. I also outlined the nature of the tasks that I would be asking them to complete. One week later, I made a second visit. At that point, I gave the consent form first and asked them to fill in the form if they were willing to participate in the study. After that I administered a questionnaire involving written responses: three word-based; four mathematical; and one representational (concept map). provided students with adequate time to complete the tasks. The student teachers used their time effectively and made a sincere effort to complete questionnaire. All of the student teachers who were present that day participated in my study.

Data Collection: Questionnaire

I developed a questionnaire that would enable me to collect data to answer my central research question, ***"What understandings do student teachers possess concerning the notion of mathematical proof?"*** My aim was to assess both student teachers' conceptions of proof and their ability to prove mathematical tasks. Hence, the Questionnaire (Appendix I) consists of two parts. Section A is designed to provide answers to the question "What do student teachers believe about the nature and role of proof?" Section B of the Questionnaire is designed to yield data that addresses the question, "How able are student teachers when it comes to completing mathematical proof?" I administered this questionnaire to my 17 student teacher participants at a point in time when most of them were completing the final semester in their teacher education program; in two weeks, they would be off to begin their long practicum in the field. As intended, the data collected did provide information about student teachers' conceptions of, and their proficiency with, mathematical proofs.

I posed three questions in Section A:

a. Describe what the notion of proof means to you.

b. In your opinion, what is the best way to develop students' abilities when it comes to writing to proof?

c. In your opinion, after all, is it important for high school students to learn how to write proof? Why?

In Section B, I asked students to complete five different tasks. Of these, four were mathematical (proving tasks) and the fifth representational (students were to generate a concept map that represented their understanding of mathematical proof). I identify the mathematical tasks below:

1) Prove that when you multiply any 3 consecutive numbers, your answer is always a multiple of 6.

(Taken from Healy and Hoyles [2000])

2) Provide a means of calculating the number of diagonals of a polygon when you know how many vertices it has.

(Taken from Balacheff [1988])

3) Prove that the sum of the exterior angles of a polygon is 360^0.

(Selected from memory; one of the unit-test tasks that I gave my Form 5 students [Grade 11])

4) How do you know whether there exists a two-digit number *ab* such that the difference between *ab* and *ba* is a prime number?

(Taken from Harding [1999])

Rather than generating all of the tasks myself, I decided to incorporate three tasks from past research studies. Since highly skilled researchers had approved these tasks for purposes that were similar to my own, I felt confident that these tasks would be appropriate and effective choices for my study. I wanted to ensure that my data would provide representative and relevant measures of what I intended to assess. Moreover, since these tasks had been tested with other subject groups, results would be more reliable and any comparison between the studies more valuable. I selected both number theory and geometrical tasks drawing upon secondary school content. I reasoned that student teachers, having now moved well beyond the secondary level in mathematics, should have little difficulty completing these tasks, particularly since they were now nearing the completion of their teacher education program. What I was hoping to see was different categories of proof- specifically whether they will use any particular format in their proofs or not. If they use any particular format and will that be consistent across the tasks.

In addition, I selected the concept map task to study the student teachers' depth of knowledge in the area of mathematical proof because a concept map can provide a graphic and compelling representation of knowledge. I provided each of my participants with a list of the 24 terms generated by the student teachers involved in Jones' (1997) study, then asked each to create a concept map that represented his or her personal understanding of mathematical proof. Students could use any or all of the terms on the list and/or other terms that seemed appropriate. I reproduce the list below.

Euclidean	Observation
Logic	General Case
Trail and Improvement	Theorem
Graphical	Assumptions
Axioms	Irrefutable
Syllogism	Deduction
Definitive	Postulate
Lemma	By contradiction
Explanation	Hypothesis
Examples	Implies
Precision	Proposition
Reasoning	Abstraction

Using this method, Jones (1997) successfully identified those student teachers possessing deeper understandings of mathematical proof. As noted in Chapter 4, Jones correlated student GPAs with the level of sophistication that was evident in the maps. Since I did not have access to GPA information, I was unable to establish that correlation in my study. It is interesting to note that, in general, my participants generated concept maps that were less sophisticated than those created by Jones' subjects

Data Collection: Interviews

My aim for this second phase in the research design was to gain more insight into the participants understanding of proof. Following the completion of Sections A and B of the Questionnaire, students left campus for their teaching practicum. They were engaged in the field for almost two months. Hence, I found it difficult to contact participants and, consequently, to schedule interviews within a week or two of the completion of the first phase of the study. Thus, approximately two months passed between the initial task work and the interviews intended to discuss the task work. During this period, I extended an invitation by e-mail to all student participants. Although I had hoped to interview all of those students whose responses warranted further exploration and explanation, only three students in total responded, even after repeated requests via email. I called the three respondents and conducted a semi-structured interview with each. The following questions were typical of those posed: What does the notion of proof mean to you? How would you introduce proof to students? Is proof necessary in school mathematics? Why? I included these questions in an effort to prompt students to elaborate upon their written responses in Section A. Questions varied somewhat from participant to participant.

Analysis

I then analyzed all of the data: written responses to Section A of the questionnaire; mathematical responses to the proof tasks in Section B; graphical responses to the representation task (that is, the concept maps) in Section B; and verbal responses to the interview questions. A detailed account of the data analysis for Section A and the Interviews follows in Chapter 6. A detailed account of the data analysis for Section B of the Questionnaire follows in Chapters 7, 8, 9, 10, 11, and 12. Before I started analyzing student data, pseudonyms were assigned to students. However, I preserved sex and the ethnicity of the students in accordance with research protocol.

Assumptions

The student teachers that I worked with had completed at least 12 advanced-level mathematics courses. Hence, I assumed that each had acquired a broad base of mathematical knowledge and an in-depth understanding of mathematical proof that would enable him or her to complete high school level mathematical tasks without much difficulty. Notably, these student teachers received their education—both in secondary school and at university—at a time when reform efforts in mathematics education were a curricular issue and an instructional concern. Since participants were in the final stage of their teacher education programs, it was reasonable to assume that they had developed a fair understanding of the school mathematics curriculum via their classroom lectures, assignments, and discussions with peers, professors and guest speakers. Given all of these circumstances, I assumed that it would be appropriate to administer high school level tasks drawn from two mathematical areas, geometry and number theory. I expected to see each of the student teachers demonstrate his/her knowledge in these different areas of mathematics and apply that knowledge to solving the tasks. I believed that the manner in which each participant approached the proof tasks would also suggest his/her level of confidence in dealing with mathematical proof. I assumed that *all* of the student teachers would have a knowledge base enabling them to complete successfully all four of the secondary-level proof tasks. I chose not to include mathematical tasks aligned with postsecondary level work because I did not want to cause the participants additional difficulties such as a concern with "not getting correct answers". Moreover, knowing that they would be facing secondary school students in two weeks time, I wanted to keep them focused on the high school mathematics curriculum and its expectations.

Limitations of the Study

In this study, I intended to look at, first, my participants' belief systems by asking what they thought about proofs, second, their proof making skills by examining the product they produced, and third, their specific approaches to solving the proof tasks by discussing the strategies that they used. All of this, I hoped, would help me to discover the extent to which the student teachers *understood* the nature, role, and function of mathematical proof in the classroom. I achieved my aims to a large degree; however, in the end, I was unable to adequately uncover the student teachers' thinking processes due, in part, to the factors identified below.

1. Scheduling

Scheduling proved to be a major challenge in this study. In order to access a suitable group of student teacher participants at an appropriate time (given my aims for the study), I decided that the most effective course of action would be to obtain permission from the professional term instructor to enter into the students' classroom and use some of that instructional time. The instructor explained that s/he would be able to give me some time but only near the end of the term. I realized that this timing might have a negative impact upon the interview process, but I understood the concerns and needs of the primary instructor and so made the best of the situation

Following the completion of Sections A and B of the Questionnaire, students left campus for their teaching practicum. They were engaged in the field for almost two months. Hence, I found it difficult to contact participants and, consequently, to schedule interviews within a week or two of the first phase of the study. Thus, approximately two months passed between the initial task work and the interviews that focused on the task work. During this period, I extended an invitation by e-mail to all student participants. Although I had hoped to interview all of those students whose responses warranted further exploration and explanation, only three students in total responded, even after repeated requests via e-mail. All three students mentioned that they had forgotten what they had written. They also admitted that they had forgotten the thought processes that had prompted them to complete the mathematical tasks in the manner that they had.

Hence, during the interview they could not talk much about what made them prove the tasks the way that they did. (However, immediately following completion of the tasks themselves, I made a point to note the verbal comments that students exchanged while informally discussing their work. I include this data in Chapters 6, Results and Analysis of Data: Written and Interview Responses).

2. Data Collection

I failed to collect information about the specific number of mathematics courses taken by each student. This would have been relevant and useful data. Although I did not record each student's previous course history in mathematics, I did notice that one participant, calling himself a "mathematics geek" and claiming more experience with mathematics courses than any other member of the research group, also produced the most complex concept map. However, because I did not collect past course data including GPA results, I cannot establish a correlation, as does Jones (1997), between the number of mathematics courses taken/GPA and the level of sophistication evident in the construction of the map.

As concerns the interview data, my final collection was insufficient in quantity, limited in depth, and restricted to male perspectives. All three of the students that responded to my request for an interview were male. I realized that this could be problematic: I did, in fact, intend to collect perspectives from both males and females. However, scheduling difficulties impacted my data collection and this was not to be the case. In the end, I felt that given the circumstances I would need to consider the interview data as supplementary, and not primary, data.

3. Concept Maps

One of the limitations in this study is that I did not review or remind the group what a "concept map" was. In fact, I deemed this unnecessary because these students had already received instruction about concept maps and had applied their learning within, at least one compulsory educational psychology course.

4. Student Interaction

Balacheff (1988) facilitated interaction between the partners by providing only one pen for each pair. Students were allowed to work on the problem until they arrived at a solution; but both students had to agree that their answer did, in fact, provide a solution to the problem. My aims, however, were diagnostic rather than instructional. I isolated the student teachers, reasoning that at this stage in their teacher education program they should be able to demonstrate basic proficiency with mathematical proof. Constructing understanding by means of interaction was not the focus; demonstrating current understanding was. The fact that I deviated from Balacheff's research design in this respect may be regarded as problematic, particularly because I analyze the data collected in the study against that collected by Balacheff in his study.

Summary

By and large, despite the difficulties I experienced with the interview process, I collected sufficient data to take a diagnostic look at student teachers' abilities to construct mathematical proof. My research design appropriately utilized a case study format and my data collection methods were diverse in their focus upon word-based written responses, mathematical tasks, a representational—concept map—task, and semi-structured interviews. The 17 student teachers involved in this case study cooperated well on sections A and B of the Questionnaire. Although the two-month interval between task work and interview resulted in less useful interview data than I had anticipated, I nevertheless, collected enough data of interest during the three interviews to supplement the students' written responses. My assumptions concerning the student teachers' knowledge and ability in proving and in mathematics in general were reasonable. Even so, as subsequent chapters detailing my analysis will show, these student teachers, for the most part, failed to demonstrate the degree of knowledge and skill that one would expect of mathematics majors nearing the completion of a teacher education program.

CHAPTER 6. RESULTS AND ANALYSIS OF DATA: WRITTEN AND INTERVIEW RESPONSES

Questionnaire

1. Student Teachers' Conceptions about the Meaning of Proof

This chapter reports and discusses the results of the study with respect to student teachers' beliefs about mathematical proof. The first part of my analysis draws primarily upon the written responses to questions posed in Section A of the Questionnaire. Once again, the questions were as follows:

a. Describe what the notion of proof means to you.

b. In your opinion, what is the best way to develop student's abilities when it comes to writing mathematical proofs?

c. In your opinion, after all, is it important that students in high school learn how to write proof? Why?

I categorized my findings into three key areas: (1) meanings/definitions of proof; (2) ideas concerning how one should go about teaching proof; and (3) ideas concerning the value of teaching proof at the high school level.

As one delves into the literature it becomes immediately apparent that educators of mathematics cannot settle upon one clear meaning of proof; indeed, Reid (2002) claims that mathematical proof *cannot be defined* because it will inevitably mean different things to different people given different contexts of use. My data supports this claim. The meanings assigned to "proof and proving" depend upon specific and varied experiences and perceived uses. As Reid notes, a definition can be based on the *concept* of proof, the *purpose* of teaching proof, the kinds of *reasoning* involved, or the *needs* that the process of proving is seen to address. The participants in my study considered meaning in terms of how proof seems to function within a mathematics classroom. (Notably, Knuth's [2002a] research group also defined proof in terms of its function.) Hence, the student teachers' definitions are broadly similar, differing only in specific details. The table below provides a break down of how students defined proof.

Table 6.1. Student teachers' meanings of proof.

#	Categories	# of student teachers
1	Proof as Verification	9
2	Proof as Derivation	2
3	Proof as Logical Argument	2
4	Proof as Justification	2
5	Proof as Discovery	1
6	Proof as Explanation	1

1) Proof as Verification

As noted in Chapter 3, from the time of the ancient Greeks, the fundamental role of proof has been to verify mathematical results. The findings noted in Table 6.1. indicate that the student teachers in this study understand proof primarily in terms of this function. Of 17 student teachers, nine—approximately half—defined proof as the verification of a statement or an algorithm. For these students, the main role of proof in mathematics is to demonstrate the correctness of a result or truth of a statement. Some of these responses are noted below.

"Proof means that you have shown the relationship you are exploring is true, no matter what." (George)

"Verifying that an algorithm or mathematical statement is correct." (Clare)

"A proof is showing that the concept is true for <u>all</u> cases." (Chandelle)
(emphasis in the original)

I am not surprised that more than half of these student teachers understand proof as something used for *verification*; this is, after all, its most commonly used meaning. Reid (1995) terms verification the 'traditional' concept of proof precisely because proof as verification goes back so far in time both in general use and in the research literature. That this perception of proof is so prevalent among the student teachers of my study, I contend, reflects "traditional" mathematics experiences at both the high school and college/ university levels.

2) Proof as Derivation

Only two student teachers explicitly used the terms *"derivation"* and *"derive"* in their definitions of proof.

> "detailed (though not necessary) derivation." (Tahira)
>
> "a proof is a system of equations/relationships that are used to derive another equation /relationship from a previous understanding." (Brandon)

Likely, these two participants had in mind the derivation of the quadratic equation that they had experienced in their mathematics classroom (Knuth [2002a] suggests this possibility with his own participants). They might have used the term to show that proof is something that demands abstraction, formal vocabulary, and symbols, and that it is founded on a set of formal axioms that satisfy the requirements of a professional mathematician (Hersh, 1993; Marrades & Gutierrez, 2000). I infer from this that they had "rigorous proof" in mind.

3) Proof as Logical Argument

For research mathematicians, a proof of a statement is synonymous with the logical reasoning that makes the statement true. One needs only a series of logical arguments for a proof to function in this manner. The responses below demonstrate an understanding of proof as logical argument:

> "A proof is a solution to why a theorem is true." (Deanna)

> "A statement, based on logic, that indisputably 'proves' a theorem to be true or un-true, so long as there is no fault in underlying logic or assumption." (Philip)

4) Proof as Justification

When one considers proof, what comes to mind immediately is its role in ensuring the truth of some fact. Proof demonstrates to us the truth of intuitively correct facts (Lucast, 2003). The following responses suggest an understanding of proof as justification:

> "A mathematical justification of a statement using previously proven facts." (Sara)

> "The mathematical process that justifies a formula or fact." (Grace)

Most students find examples more convincing than rigorous proofs (Coe and Ruthven, 1994); therefore, it seems that, for students, examples would provide justification more effectively than would proof. Formal proof, then, does not effectively justify or convince students. Besides, students are often confused about what it means to justify. As Rodd (2000) notes, *justification* is often used in a non-technical way to mean "a rationale for a belief". Therefore, considering proof in terms of justification alone seriously limits one's use of proof within the classroom

5) Proof as Discovery

Proof plays an important role in the discovery or creation of new mathematics (Knuth, 2002a). Indeed, one may argue that the "discovery function" of proof is especially important in the mathematics classrooms. The NCTM (2000), for example, stresses the importance of using discovery to learn mathematics. Interestingly, according to Chazan and Yerushalmy (1998), proof plays a greater role as discovery in secondary classrooms where students are able to utilize geometry software. The research data, unfortunately, suggest that student teachers do not commonly understand proof in terms of discovery. Only one of 17 student teachers recognized that proof could also be used for this purpose:

> "Discover of a concept, formula, identity, theory." (Gita)

6) Proofs as Explanation

In school mathematics, the main purpose behind introducing proof to students, most would agree, must be to enhance understanding. Hence, the explanatory value of proof is of great importance in the school context (Leddy, 2001; Hanna, 1990). Yet, only one student alluded to this function. The participant, Spencer, writes that "... You can do a proof for a law, algorithm or a relationship [...] It is meant to be an explanation of why this law, algorithm or relationship is the way it is." I find it extraordinary that only one student teacher out of 17 considered this meaning of proof. I am prompted to ask, "Does this reflect reality within school classrooms?" and "What emphasis do classroom teachers place on explanatory proofs?" Based on the responses of my participants, I would suggest very little.

7) General Comments

Since the student teachers generated definitions that were based on their experiences and practice, and because the majority of those definitions involved verification, derivation, logical argument or justification, my data would seem to confirm that even after introducing reform in mathematics teacher education courses, these student teachers continue to understand proof largely in terms of verification and not so much as a tool for teaching and learning mathematics (see Table 6.1). Just as Knuth

(2002a) found with his practicing teachers, none of my student teacher participants saw proof as a means to *promote* understanding. As Knuth (2002a, 2002b) notes, students are rarely able to identify the main objectives or functions of mathematical proof, likely because they are only exposed to proof within the realm of geometry as the verification of an already known result. My study validates this finding.

2. Student Teachers' Conceptions About the Best Way to Develop Students' Ability to Write Proof

Table 6.2. Student teachers' suggestions about the way to write up a proof.

#	Categories	# of student teachers
1	Step by Step Procedure (or teacher-assisted examples)	13
2	Constructivist	3
3	Social Process	1

A considerable majority of the students thought it best to teach proof by mean of teacher-assisted examples and step-by-step procedures. I quote some of the responses below:

"Practice, show examples of proofs, give steps for determining what direction to take the proof in." (Grace)

"Assisted stage by stage implementation and simple individual proofs." (Cathy)

"Give a few examples of a proof and explain. Give them the structure of a proof. Then give them a very simple proof they can do on their own so they can get the feel of them." "Going through numerous examples in class." (John)

"The best way to develop students' ability to write proof is to start with easy proofs that they understand." (Daniel)

"First develop a good understanding of what constitutes a proof. Second – understand some common strategies like: Third – practice deductive reasoning. Fourth – practice proving identities using axioms." (Brian)

"Exposure to some basic proofs; Description of step by step process of proofs. Practice, Practice, Practice." (Brandon)

> "Have them practice with simple versions then work their way to more complicated questions." (Sara)

> "Practice. Teach them how to reason." (Gita)

> "To introduce the idea early using simple examples. Challenge students to ask how they know something is true, rather than always just assuming it is. Ask students to proceed from one step in the logic to the next" (Philip)

The only method for learning proof suggested in all of these responses is "more practice." In traditional mathematics classrooms, teachers, and, therefore, students, consider mathematics synonymous with the memorization of procedures and mechanistic answer finding. Students who have experienced math in this manner often believe that one can only master the subject by treating it as a body of isolated concepts and procedures that have to be committed to memory.

Yet, as student teachers reflected on possible teaching methods for introducing mathematical proof, their responses suggested that, on some level, they sensed something lacking in this age-old approach. Even though they were unsure about how "to do it better", I found it encouraging that they were questioning the value of what was familiar. This is reflected in Deanna's suggestion that

> "Students should be exposed to different ways that theorems can be proved, so they realize there is more than one way to write a proof. In this sense students will not be anxious or nervous when they need to prove something."
>
> (Deanna)

Some of the students were familiar with the word "Constructivism" and the phrase "Constructivist approach," though they were often quite vague in their use of these terms. A few of them, such as Gita, were not sure what "constructivist" teaching entailed: "Using constructivism, guiding them to discover idea [sic]." While Clare's idea of a constructivist approach is a

> Step by step procedure
> (1) Write what you know

(2) What are you trying to show
(3) What do you know that can help simplify or make the statement easier and so on.

Not only did she specifically use the phrase "constructivist approach" in her written response, but she also explained constructivism in terms of step-by-step procedures.

In general, teachers teach mathematics to the individual within the group; most students learn mathematics by watching and listening to the teacher, reflecting upon the ideas and engaging in plenty of practice. Not surprisingly then, few of the participants viewed mathematical understanding as something that develops out of the collective energy of *groups* of individuals. Only one participant, Sara, understood proof as arising from, or as a product of, social inter-action: "I think the best way to get students to come up with a proof [is] as a group". Sara's perspective has something in common with the complexivist view. Complexivists suggest that learning takes place, not within the individual mind, but within "an ongoing *structural dance*—a complex choreography—of events which, even in retrospect, cannot be fully disentangled and understood, let alone reproduced" (emphasis in the original; Davis, et al., 1996, p. 153). By framing the classroom as a "complex, emergent system, as an individual collective learner rather than as a collection of individual learners" (Davis & Upitis, 2004, p. 126), teachers are able to observe the "thinking" of the collective, . . . [that is,] the interactions and prompts that trigger new possibilites and insights for the collective" (Davis & Simmt, 2003, p. 144). However, given that only one of 17 students stressed co-construction of knowledge, I infer that few of them have experienced cooperative and/or discovery-based mathematics instruction in their secondary classrooms. Although social acceptability is an important aspect of mathematical proof—as Manin (1977) writes, "a proof becomes a proof after the social act of accepting it as a proof" (as cited by Knuth, 2002b, p. 64)—co-creation of proof through social problem solving seems not to be something that these student teachers have deeply considered.

3. Student Teachers' Conceptions About the Importance of Learning Proof in High School

Student responses indicate three lines of thought concerning this point. In response to the question, "Is it important to learn proof in high school?" a few students responded with an emphatic "No." A second group of students, also low in number, said that proof should be taught in every mathematics class and then elaborated by offering reasons for their belief. A third group of students believed that proof should be part of the curriculum; however, they also expressed reservations about making it a dominant element. Most of those in this third group also mentioned that mathematical proof should only be introduced to select groups of students, including those who plan to study advanced mathematics. As Hanna (2000) notes, the idea that proof should be reserved for a "certain group of students" is highly problematic. This attitude implies that it is possible to learn mathematics, without ever needing to prove a theorem. Hanna writes that:

> the basis for this idea seems to be the erroneous assumption that proof is, in fact, a specialized branch of mathematics, even an arcane branch so complicated that it is best avoided by all who do not absolutely require it. [...] Reserving proof for those planning post secondary studies cannot but send the message to the bulk of the students that for them there is really no point in proof at all.
>
> (p. 25)

I list below some of the responses that reflect this belief:

> "... formal proofs are very complicated and mostly confusing and frustrating to high school students." (George)
>
> "in geometry proof is a must, but trigonometric proof[s] are above most students." (Sara)
>
> "Not formal proofs, but proofs that explain." (Spencer)
>
> "Only students going to university should be taught proof." (Daniel)
>
> "It should only be used in pure stream." (Gita)

"No unless they are planning to take advanced mathematical courses." (Beth)

" Introduce proof – let them not be asked to write. High school topics are difficult even without proof." (John)

"No – it is far too complex." (Tahira)

"It is important for verification purpose. They [students] should not necessarily be [with able to come up proof] on their own." (Chandelle)

Overall, these students consider "mathematical proof" a waste of time at the secondary level. Most participants regard proof as something written in two columns by the teacher on a black/white board for students to copy and memorize for an exam. They also believe that adolescents are unlikely to use mathematical proof outside of a classroom. As Terrance stated: "No—because after high school the large [emphasis in the original] majority will never use the knowledge of the proof ever again".

Of particular interest are the responses of those student teachers that *want* to retain proof as a part of the high school curriculum. What is especially intriguing is their ambiguity. They do not wholeheartedly endorse the teaching of proofs at the high school level, but they argue that inclusion will improve fluency in problem solving, and for this reason, as Spencer stated, "proofs should be retained." They are also aware of the fact that, in post-secondary education, mathematical proof is a must, and low-level skill in completing mathematical proofs will be costly for students, especially if their major is in mathematics or related subject areas. George mentioned that "[i]t is important to teach our students to be able to logically discern information much like proof"; Clare said students should be introduced to"[v]ery basic proof such as trigonometric proofs . . . otherwise no"; and Sara thought that trigonometric proofs "are above most students." By trigonometric proofs, Clare might have meant trigonometric identities. She continues: "I feel students should be able to reassure themselves that what they are doing will lead to a correct result . . . I think proofs allow practice with problem solving, except only with the basic proof." Additional responses are listed below:

> "Yes students should be able to write proofs so that it will aid in developing their understanding." (Deanna)
>
> "It [proof] helps not to memorize a theory or concept." (Gita)
>
> "proofs should be taught as it increases confidence in mathematics." (Gita)
>
> "students going to university should be taught proof." (Daniel)
>
> "Important as it develops helps in deductive reasoning skills which can be applied to every day logic problems. ....gives insight to how an algorithm was developed" (Daniel)
>
> "Yes- because students should not look at some pattern and believe the pattern is universal." (Philip)
>
> "Yess [*sic*], helps understanding prepares them for post secondary school. Helps in deeper thinking. Helps to make connection." (Grace)

Finally, some student teachers suggested that proof should be taught in high school because, once it has been learned in geometry, the method could be transferred to other contexts. Reid (2005) disagrees with this idea, arguing that the criterion of acceptability of explanation that defines mathematics as a domain of explanation cannot be transferred to other domains. He suggests that the motivation for teaching/learning proof must be a better understanding of the nature of mathematics itself, not better reasoning in other domains.

The responses of the student teachers in this study substantiate Knuth's (2002a) observation that the teaching of proof is restricted to certain areas and topics in the secondary school curriculum. In the secondary school curriculum, proof is most commonly relegated to the role of *verification*. Since proofs are, therefore, largely about verifying an already known mathematical result, students find proof of little use in school mathematics. Responses provided by the participants in this study suggest that these prospective teachers are unsure about what would constitute an effective strategy for learning and/or teaching mathematical proofs. One of the complications associated with teaching proof is the difficulty of communicating an effective strategy for proving (Hadas, Hershkowitz & Schwarz, 2000). Yet, it is important to assist student teachers in learning how to make proof accessible to high school students. Knuth (2002c) notes that teachers should experience proof as a learning tool so that they can then use proof as an effective teaching tool. University educators who take this into consideration will make mathematics more meaningful for both teacher and student.

Interviews: Clarification and Elaboration

I made phone calls to the three students who agreed to meet with me, scheduled appropriate times, and conducted semi-structured interviews with each. I structured the interview around the following questions: What is proof? Is proof essential in schools? What experiences did you have with proof as a secondary student and as a classroom teacher-in-training? Phrasing of the questions varied somewhat from interview to interview. Some variations on the three guiding questions that I constructed include the following: What does the notion of proof mean to you? How would you introduce proof to students? Is proof necessary in school mathematics? Why? I posed these questions in an effort to prompt students to elaborate upon the written responses that they had provided in Section A.

In the sections below, I provide an analysis of each of the three interviews. I decided to paraphrase rather than to transcribe the conversations verbatim to assist in readability and to highlight the points relevant to the study. A paraphrase approach allowed me to cut the unnecessary parts of the conversation and to focus on critical information. However, in paraphrasing I also quote brief words and phrases just as they were spoken.

1. George

George believes that one can consider anything used to "prove" as a proof (even a picture, he says, can be considered a proof in this "general sense"); but in mathematics (no matter what level), mathematical proof should be something that is "systematically laid on". His comment suggests that he is separating the general proof (proof term that is commonly used) from the mathematical proof. He states that the best way to write a proof is to use a systematic approach: that is, a step-by-step approach in which one begins with an already proven theorem or an accepted statement or axiom, uses this as the basis for step one, and then bases each subsequent step upon the one that preceded it. If one does this, George reasons, it will difficult to find fault in the work. He considers proof by induction to be the best method of writing proof since it is well displayed through its systemization. He recognizes that there is no one single way to write proofs. He also believes that once proven, a proof is proved forever: results will never change. Such an attitude reflects what Ernest (1998) calls an absolutist philosophy of mathematics: the view that mathematics is a body of absolute, certain and infallible truths. Non-absolutists, according to Ernest, regard mathematics as a "corrigible, fallible, and changing social product" (p. 2).

George believes that proof has little importance in high school. He reasons that most students struggle with basic arithmetic, so why introduce proof and make them even more likely to develop low-self esteem when it comes to mathematics. He believes that the role of proof within school mathematics programs should be to teach students that mathematics is not mysterious, that proof is simply the medium through which students learn that results can be proved, that there are steps involved in arriving at a result. George thinks that most students find proof a difficult topic because they receive little exposure to it. Furthermore, George suggests that both secondary and junior high mathematics curricula convey the idea that proofs are of little importance to the student. As a secondary student, he had little experience with proofs. His math teachers presented proof as a special topic within the Grade 10 and 11 Math curricula. They taught him only one proof format, the two-column approach.

George regards the ability to learn and understand proof as a "sign of excellence" in mathematics. From the perspective of a prospective teacher, George strongly recommends that proof be reserved for the "smarter ones" in the class. Students who demonstrate excellence will likely be the ones who become the mathematicians in society. Therefore, teachers should only introduce proof to those who excel in math and who plan to take advanced mathematics courses at university. None of his preliminary teaching experiences in various classrooms have involved teaching proof.

Finally, when I asked George what could be done to improve the teaching and learning of mathematical proof within the school system, he suggested that more mathematics courses directly related to the curricular content that he and his colleagues will be teaching in the schools should be incorporated into the teacher education program. He believes that mathematics is not given its due role in mathematics education courses: "all they talk about is how to teach mathematics and that too in a very limited time of five weeks". The professional term, as it is called at this university, offers methodology courses in a student's teaching major over an intensive six-week period. It is to this that

George refers. It is interesting to note that Wu (1997) made a similar observation about teacher preparation programs in the U.S.. Quoting from *A Nation at Risk,* Wu notes that, "The teacher preparation curriculum is weighted heavily with courses in 'education methods' at the expense of courses in subjects to be taught." (p. 1). Although the student teacher participants in my study were required to take twelve 3-credit courses in mathematics, these were advanced classes dealing with content that limited direct relevance to the secondary mathematics curriculum. Perhaps what student teachers would most appreciate are courses offering an extensive and expansive application of the very concepts that they will be expected to teach in the near future.

2. Spencer

Spencer fails to see how proof is relevant to students studying high school mathematics. He argues that students tend to dislike mathematics precisely because of "complications like proof". As a high school student, he experienced proof in geometry class, most likely grade 11 (though he can not recall with certainty), and only within one unit. Spencer believes that proofs are important only to those who specialize in math at university. Thus, he concludes that proof should not be emphasized at the secondary level.

Spencer claimed that proofs must follow a format in order to be valid, though he could not explain what he meant by the term 'format'. However, he was certain that simple pictures and examples could not serve as proofs. Like George, Spencer admitted that, in the two-month interim between completing the task and discussing his work, he had forgotten much about the tasks and the questionnaire; consequently, he could not say a great deal about how he had approached the task, nor explain why he had taken that approach.

3. Brian

Brian was excited about my research, and quite happy that he would be participating in the study. He was interested in the possibility of an article and its future publication, and keen to read about his contribution to the research within its pages. As with both George and Spencer, Brian had no recollection of what he had written in response to the questions, nor how he had approached the mathematical tasks. I handed him the questionnaire document and he briefly skimmed through it to refresh his memory prior to our discussion.

Brian defines proof as a mathematical process showing that something is done in certain way. Brian believes that proof is "why we do a certain process or application or formula [and] how we come about a result and why it works all the time". He believes that there should be a mathematical authority to decide the status of proof, and stated that for the student, this authority resides in the teacher. He added that among university professors and mathematicians, authority resides in the collective formed by all of one's colleagues. Brian thinks that proofs can be changed, but since mathematics is an ancient subject, chances are that proofs will not undergo significant change

He admits that many students have problems completing proofs. Brian refers to the "way it is now in schools" meaning the way in which proof is currently introduced as a separate topic within the Grade 11 Geometry unit. Proof, therefore, he adds, needs to be introduced to every one. Brian had no memory of working on proofs at any other time during his secondary schooling other than in Grade 11 and within the context of geometry. Brian thinks that proofs should be introduced at an earlier stage of schooling, even, if possible, in Grade 1. Students, he insists, should be focused on *why* not *how*. This will help them later in post-secondary school. He confessed that his greatest struggle with completing a proof is that he cannot always determine a starting point.

Brian thinks that there should be a certain format for a proof, though he is not sure what the format is—he thinks something like "two-column proofs". A simple show of examples does not make a proof as far as Brian is concerned; however, if the examples include a line of reasoning then that work can be considered a proof. Brian believes that a proof must show why it works for a general case before it can be considered complete. A complete form must also demonstrate algebraic symbols and more detailed mathematics; simply noting patterns does not constitute proving.

Brian suggests that working at mathematical proof enhances critical thinking. He believes that teachers emphasize critical thinking more in social studies and language arts than in mathematics. In mathematics classes, students depend a lot on calculators. Technology, he argues, has handicapped critical thinking in the math classroom. Student depend too much on calculators; they do not visualize or think. He stresses that the more often teachers expose kids to different ways of proving, the more likely that kids will develop proving skills. Most students, he says, are "not ready" for university because university requires that students think "outside the box". Brian thinks that proof is one way to encourage students to think "outside the box".

Brian initially stated that teachers should teach proof at all levels, to all students. After reflecting on the frustration that students experience with even basic mathematics, he changed his position. Proof, he suggested, need not be mandatory in high school, though it could be included in the curriculum for students who are proficien in mathematics. Brian shared that he had introduced proof to students while student-teaching in the schools, but his cooperating teacher told him not to "go into those areas".

4. The Interviews Considered Together

All three participants were frustrated with the nature of their limited secondary-level exposure to proofs. Each recalls working with mathematical proof only in Grade 11: they agree that proof tends to exist only as a part of the geometry curriculum. Even though they regret not having spent more time on proof in high school, they remember well their resentment in having to do proofs, and so conclude that proofs should not be made compulsory in secondary school mathematics, although those who feel comfortable with proof and proving should be given opportunities to work with it. All admit that in university math classes, skill with proof comes in quite handy. If proof is to be "reserved" for the "smart kids" in school, however, I suggest that it may eventually disappear from the lived-curriculum. Denying *all* students experience with proof also suggests that it is possible to learn mathematics without ever needing to prove a theorem. As Hanna (2000a) notes, these views are based on an incorrect assumption, the idea that proof is some sort of specialized and unknowable branch of mathematics so tricky that, if one does not specifically need to know it, one would be better off avoiding it entirely.

Summary

My analysis of the data collected indicates that student teachers' conceptions of mathematical proof are narrowly grounded in traditional understandings of its nature, function, and value. Given that reform efforts in mathematics education have been underway for some time, this is problematic. Most of the participants in my study regarded proof as a means of verification; only one person understood proof as a means of explanation. Even though the teacher education program emphasizes the explanatory value of proof in school mathematics curricula, student teachers continue to conceptualize proof primarily as a means of verifying an already known mathematical result. I consider this a serious concern.

Many student teachers also believe that proof should only be introduced to select groups of students such as those who plan to study advanced mathematics. The fact that student teachers today would take such a position raises a red flag, for it sends the message that there is no need for the majority of students to study proof and proving in the classroom, and so little need for future teachers to teach proof to all of their future students. For the most part, their responses suggest that these student teachers continue to regard proof as a formal, and often meaningless, exercise performed by the teacher. (Interestingly, Alibert [1988] conducted a study twenty years earlier with a similar result: those participants also regarded proof as a formal, and often meaningless, exercise performed by the teacher.) All three of the student teachers that I interviewed spoke with some frustration about their high school experiences with proof and proving.

My conclusions, though they pertain to student teachers, support Knuth's (2002a, 2002b) findings concerning practicing teachers. Most student teachers believe that teacher-assisted practice is the best way to learn proof. Although some students referred to "constructivist approaches," most were unclear about what a "constructivist approach" entails. My sense is that students included the term in their responses to prove that they were aware of current educational trends. As students within post-secondary education classrooms, they are likely to have heard about the benefits of using "constructivist approaches" to learning; even so, they seem confused about what "constructivism" actually means.

CHAPTER 7. THE MATHEMATICAL TASKS

Context

As already noted, I administered the tasks during the final weeks of the professional term. Students were given an hour to complete the entire questionnaire, with additional time available to whomever wanted it. Participants were not allowed to discuss the tasks while attempting them. They could ask me any question regarding the questionnaire: one or two students asked me questions about concept maps, but no one asked questions about the mathematical tasks. Once students were done, they formed small informal groups and began to discuss their work, sharing the different ways in which they had approached the tasks, and seeking clarification on the meaning of key concepts (for example, I overheard students asking each other for the definition of an exterior angle).

Once again, the specific tasks as they were presented to the students in Section B of the Questionnaire are listed below.

1) Prove that when you multiply any 3 consecutive numbers, your answer is always a multiple of 6.

(Taken from Healy and Hoyles [2000])

2) Provide a means of calculating the number of diagonals of a polygon when you know how many vertices it has.

(Taken from Balacheff [1988])

3) Prove that the sum of the exterior angles of a polygon is 360^0.

(Selected from memory; one of the unit-test tasks that I gave my Form 5 students [Grade 11])

4) How do you know whether there exists a two digit number *ab* such that the difference between *ab* and *ba* is a prime number?

(Taken from Harding [1999])

These four tasks challenged the students in two particular mathematical domains: geometry (questions 2 and 3) and number theory (questions 1 and 4).

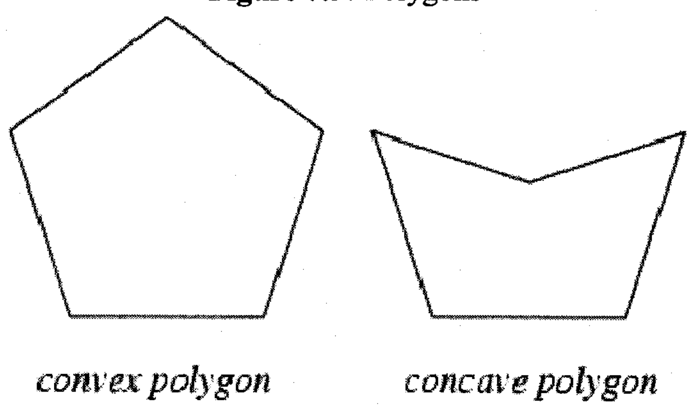
Classifying The Tasks On the Basis of Mathematical Domain

1. Geometry

I chose geometry tasks for my study because proofs are usually seen in connection with geometry. In secondary schools, students commonly learn to write rigorous mathematical proofs within the context of a geometry unit. They are typically taught to solve this type of proof using a two-column format, the column on the left consisting of statements, and the column on the right consisting of reasons. The two geometry tasks that I selected require an understanding of the properties of polygons. Both tasks—2) "provide a means of calculating the number of diagonals of a polygon when you know many vertices it has" and 3) "prove that the sum of the exterior angles of a polygon is always 360^0"—represent, in fact, two well-known properties of polygons. I phrased the one task as "prove that . . . " and deliberately selected, for the other, a task already worded as "provide a means of . . ." (from Balacheff, 1988). In essence, the wording in Task # 2 requires that the student approach this task in the reverse order that he/she approached Task # 3. I wanted to see how the difference in wording would affect the results, specifically, whether or not students would prove both geometry tasks using a two-column format. I assumed that students would explore the tasks in various ways, but then come up with a *correct* formal proof.

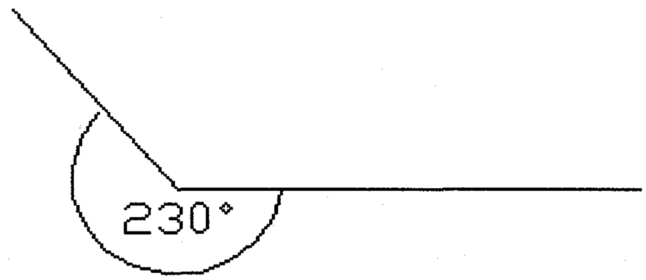
In order to complete these proofs, students would need some basic mathematical knowledge. Clearly, for both tasks, they would need to understand the concept of "polygon." The word "polygon" derives from the Greek *poly*, meaning "many," and *gonia*, meaning "angle." A polygon is defined as a closed plane figure with *n* straight sides. If all sides and angles are equivalent, the polygon is called a *regular* polygon A polygon can be classified as concave or convex. A polygon is convex if it contains *all* of the line segments connecting any pair of its points. A polygon that is not convex is termed a concave polygon. At least one of the internal angles of a concave polygon must be greater than 180^0. See the following figure.

Figure 7.1: Polygons



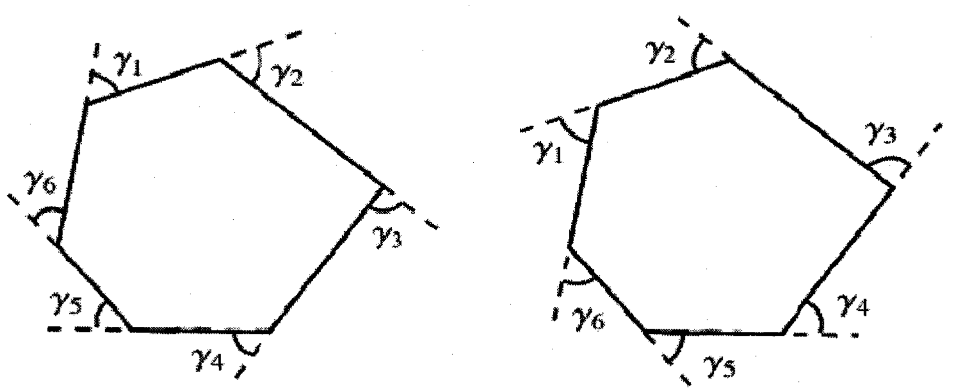
Interestingly, all of the students opted for the convex polygon when it came to providing examples. In order to complete Task # 2, students also had to be able to define "diagonal". A diagonal is defined as "a line joining two nonadjacent vertices of a polygon". With Task # 3, students also needed to know and understand the concepts of "exterior angle" and "interior angle". An exterior angle of a polygon is defined as the angle between one side of a polygon and the extension of an adjacent side. Angles that are greater than 180^0 and less than 360^0 are called reflex angles. (See the diagram below.)

Figure 7.2: Reflex Angles



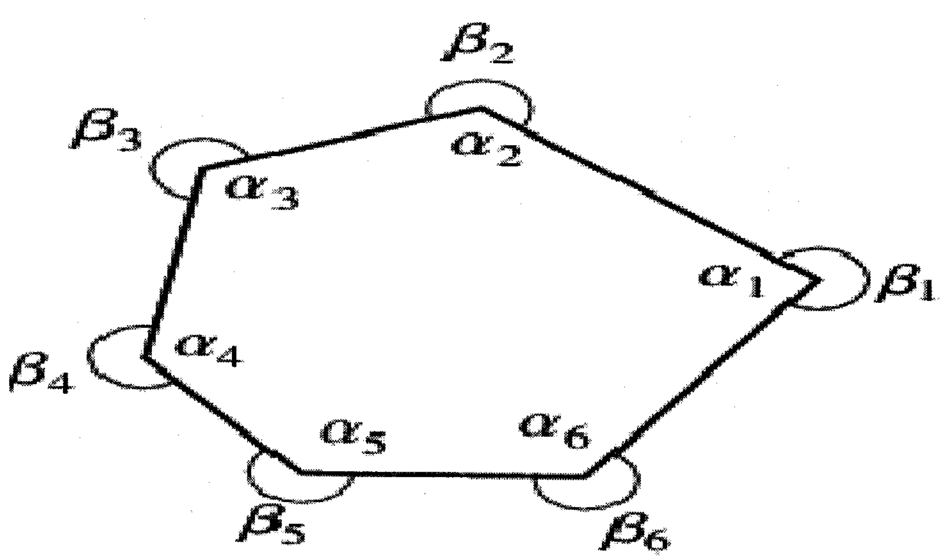
The interior angle of a polygon is defined as "the angles on the inside of a polygon formed by each pair of adjacent sides". In addition, students needed to be familiar with the following result: an "interior angle is always supplementary to the exterior angle at that vertex". Once again, just as with Task # 2, students worked on Task # 3 using examples of convex polygons. Not a single student referred to non-convex polygons.

Figure 7.3: Exterior Angles(Definition 1)



Notably, in Task # 3 there are two exterior angles per vertex. Yet, almost all geometry textbooks state that "the sum of the exterior angles of a polygon equals 360^0". This statement, however, can only be true if (a) only one exterior angle per vertex is considered and (b) all the exterior angles that point in the same direction around the polygon are considered. I assumed that the student teachers would consider these conditions since geometry textbooks commonly assume them. If the participants assume exterior angles as noted below—as angles formed externally between two adjacent sides (definition 2)—then the sum of the exterior angles will be 720^0, and not 360^0.

Figure 7.4: Exterior Angles (Definition 2)



If one were to answer the question "What is the sum of the exterior angles of a polygon?" without taking into account these conditions, an answer of 360° would be incorrect. The question as I posed it specifically reads "prove that the sum of the exterior angles of a polygon is *always* 360°" (emphasis added). Hence, the question implies that the sum of the exterior angles will never be anything other than 360° and that 360° is an already known result—one just needs to *prove* it by drawing on the necessary assumptions.

Since the interior angles of a polygon are those angles at each vertex that are on the inside of the polygon, there is one interior angle per vertex. The interior angle is always supplementary to an exterior angle at that vertex. For a polygon with "*n*" sides, there are "*n*" vertices and "*n*" interior angles. Students also need to know the "sum of the interior angles of a polygon theorem". This theorem is as follows: If a polygon has n sides, the sum of its interior angles is $180\times(n - 2)^0$. If the student was unable to remember the theorem, s/he could still derive the total interior angles by applying the following reasoning: A polygon with "n" sides can be partitioned into $(n - 2)$ triangles; the sum of the angles in a triangle is 180^0; therefore, the total angles in a polygon is $(n - 2) \times 180^0$.

Both Knuth (2002a) and Zazkis and Campbell (2006) note that teachers and textbooks typically use geometry as the sole context and vehicle for introducing the concept of proof. Indeed, Wu (1996) argues " [when] learning to prove something for the first time, most people find it easier to look at a picture than to close their eyes and think abstractly" (p. 228). Thus, it makes sense that I use geometry tasks in my study: the chances that participants will successfully solve this type of proof would seem to be quite high both because of the students' familiarity with it and because of its concrete, visual aspect.

2. Number Theory

Number theory is another area rich in proofs and proving. It is the branch of mathematics that studies the properties of whole numbers. Primes and prime factorization are very important in number theory. Zakis and Campbell (2006) note that the basic concepts of number theory provide a rich venue for learners of all ages to explore mathematical patterns, formulate conjectures, test generalizations, provide justifications and prove theorems. They also observe that topics dealing with such items as factors, divisors and multiples can serve as "springboard[s] toward appreciation and ability to construct formal mathematical proof" (p. 9). Over the years, Zakis has conducted considerable research on number theory with prospective elementary school teachers (Zazkis & Khoury, 1994; Zazkis, 1999; Zazkis & Campbell, 1996; Zazkis & Campbell, 2006). The Conference Board on Mathematical Sciences' (CBMS, 2001) report on *The mathematical education of teachers* states that one can easily *disprove* the idea that proof is relevant only in geometry when one begins to explore proof in relation to number theory, specifically in conjecturing and proving simple theorems about numbers. Granting this position, I include two number theory tasks in my study: Task # 1: "Prove that when you multiply any three consecutive numbers your answer is always a multiple of 6" and Task # 4: How do you know whether there exists a two-digit number *ab* such that the difference between *ab* and *ba* is a prime?

A person attempting to solve these two number theory tasks would require some basic mathematical knowledge. In the case of Task # 1, students would benefit by knowing the following facts:

a) One whole number is divisible by another if, after dividing, the remainder is zero.
b) If one whole number is divisible by another number, then the second number is a factor of the first number.
c) A number is divisible by 2 if the last digit is 0, 2, 4, 6 or 8.
d) A number is divisible by 3 if the sum of the digits is divisible by 3.
e) A number is divisible by 6 if it is divisible by 2 *and* it is divisible by 3.

In the case of Task # 4, students would benefit by knowing the following facts:

a) A two digit number ab is in fact $10*a + b$ and ba is $10*b + a$.
b) The digits of a two-digit number ab are reversed to form a second two-digit number ba and the lesser of the two numbers is subtracted from the greater.
c) a and b must be comprised of only counting numbers 0, 1, 2, . . . , 9.
d) A prime number (often simply called a "prime" for short) is a whole number that has no whole number divisors other than 1 and the number itself.
e) A composite number is a positive integer that has positive integer divisors other than 1 and the number itself.

Classifying the Tasks On the Basis of Mathematical Function

1. Verification

In addition to conceptualizing the tasks in terms of relevant mathematical domain knowledge, I also consider the tasks in terms of their function. The four tasks selected for my study serve as either verification or exploration tasks (see Chapter 3, Literature Review). The *verification* tasks are as follows: Task # 1: "Prove that when you multiply any three consecutive numbers your answer is always a multiple of 6" and Task # 3: "Prove that the sum of the exterior angles of a polygon is always 360^0. I use the same argument for including proofs that verify as I did in selecting proofs that are geometric. Since mathematicians and educators concur that proofs have been used mostly for purposes of verification (Hanna, 1983), my participants were likely to be most familiar with proofs with this function. In other words, I expect that students will feel comfortable dealing with proof tasks that ask them to verify already known results. Specifically, as pertains to Tasks # 1 and # 3, the known results include a) the sum of the exterior angles of a polygon is always 360^0, and b) the product of three consecutive numbers is always a multiple of 6. Since students generally apply the two-column format when proving for the purpose of verification (Hanna, 1983), I was keen to observe whether or not the verification tasks would yield two-column proofs.

2. Exploration

The tasks that ask the students to prove for the purpose of exploration include Task # 2: "Provide a means of calculating the number of diagonals of a polygon when you know how many vertices it has"; and Task # 4: "How do you know whether there exists a two-digit number *ab* such that the difference between *ab* and *ba* is a prime? I selected exploration tasks with the aim of determining whether or not this type of question would yield formats for solving/proving proofs other than the standard two-column format.

Classifying The Tasks On the Basis of Mathematical Reasoning

Generally, the participants are familiar with two methods of proving: inductive and deductive. Inductive reasoning begins with particular cases and produces a generalization from these cases. In other words, inductive reasoning goes from the small to the large. One makes observations about individual parts and those observations lead to subsequent conclusions about the whole. In deductive reasoning one starts with the general major premises and from them deduces the truth about individual parts of the whole.

1. Inductive Justifications

The first two stages in Balacheff's (1988) taxonomy of proofs are inductive because justification is based on checking by means of particular examples. In their discussion concerning empirical justifications, Marrades and Gutierrez (2000) distinguish two classes of proofs based on the way that examples are selected:

i) When the conjecture is justified by showing that it is true in one or several examples, usually selected without a specific criterion, it falls under *naïve empiricism.* The checking may involve visual or tactile perception of examples only (perceptual type) or may also involve the use of mathematical elements or relationships found in examples (inductive type).

ii) If the conjecture is justified by showing that it is true in a specific, carefully selected example, it falls under *crucial experiment.* Students are aware of the need for generalization, so they choose the example as non-particular as possible. Although it is not considered as representative of any other example, students assume that the conjecture is always true if it is true in this example.

2. Deductive justifications

These are characterized by the de-contextualization of the arguments used. In other words, the proving is detached from the specific examples that the student uses to make sense of the task. They are based on generic aspects of the problem, mental operations, and logical deductions that aim to validate the conjecture in a general way. Balacheff's final stage in his taxonomy of proof, the thought experiment provides an example of deductive justification.

General Observations

Data collected by means of written responses and interviews demonstrated that students could define proof reasonably well, in the process referring to various functions of proof (see Chapter 6, Analysis of Written and Interview Responses); yet, data detailing student efforts to complete proving tasks suggests that theoretical understanding is not always deployed in practice (Rodd, 2000). Many of the student teachers found the four mathematical tasks challenging. Almost all of the participants attempted direct proofs by inferring patterns from the information and/or previously studied definitions, postulates and theorems. It would be well, at this point, to recall O'Daffer's and Thornquist's (1993) observation that deductive proofs are less complex than inductive proofs. Although many students could start a proof, and knew the *form* required for a proof, they frequently were unable to complete it correctly (see pages: 150, 152, 154). The work of a number of participants, in fact, provided little evidence of a chain of reasoning that could lead to a valid proof. There were five correct proofs (out of 17) for Task # 1 and 6 (out of 17) for Task # 2. There was only 1 correct proof (of 14 attempted) for Task # 3; and 4 correct proofs (of 9 attempted) for Task # 4. The written responses containing correct proofs include both pragmatic and conceptual justifications. I found it surprising that not a single student teacher was able to provide complete and correct proofs for all four tasks. The table below provides further information. A detailed analysis of each of the four tasks follows in Chapters 8, 9, 10, and 11.

Table 7.1. Observations of the tasks taken together

Categories	Task 1	Task 2	Task 3	Task 4
Correct Proof	5	6	1	4
Partially Correct	12	7	5	5
Formula/Diagram only	0	4	6	0
No Attempt	0	0	3	8
Total	**17**	**17**	**17**	**17**

Exemplars

I begin each of those chapters by providing examples of how one might successfully complete the proof if approaching it in a particular manner. I have devised these exemplars in line with Balacheff's hierarchy of proofs: naïve empiricism, crucial experiment, generic example, and thought experiment. I divide the thought experiment proofs into two categories a) verbal and b) symbolic. In the category of thought experiment-symbolic, I place efforts that are especially sophisticated: in other words, the proof is enriched by the use of algebra or axioms or any other type of formalism and there is a minimal use of natural language. Given their age and their limited mathematical exposure, the examples of thought experiment in Balacheff's (1988) study are verbal. As this group of students are adults and are mathematically sophisticated they seem to have an image of a proof form in mind. Hence, I included an additional category: thought experiment-symbolic.

I found it quite challenging to generate examples that would illustrate the various levels of thinking in Balacheff's taxonomy of proofs, even for the problem that Balacheff himself used in his 1988 study (Task # 2: Provide a means of calculating the number of diagonals of a polygon when you know how many vertices it has.) In particular, I found it difficult to generate illustrative examples of the approach to proving categorized as generic example. My research has taught me that unless the student uses some extreme examples (like a very large number or a polygon with a very large number of sides), or the student specifically mentions (aloud during the process or, less likely , in writing) that the example is *intentionally* selected, it is quite difficult to differentiate between naïve empiricism and crucial experiment on the basis of written work alone. Balacheff, aided by a research assistant, studied thirteen- and fourteen-year-olds *as* they worked on the proving tasks; whereas I analyzed the work *after* it had been completed. Balacheff also facilitated students' verbalization of their thinking processes by grouping participants in pairs, providing only one pencil and written copy per pair, and insisting that both individuals agree on the solution prior to handing their work in. I had only the participants' written responses and mathematical calculations to work with.

Notably, Balacheff's categories focus on the type of argument that the student presents rather than on whether or not the argument itself is correct; however, I take both factors into consideration in my analysis. First, I examine the type of argument that the student has used; and second, I note the success, or lack of success, that the student has achieved in applying that argument.

Summary

The mathematical task data demonstrates that the student teacher participants had difficulty with both the geometry and number theory tasks. I cannot speculate as to why student teachers, all Math majors with at least twelve university-level math courses behind them, would have had so much difficulty proving these high school level mathematical proof tasks. Further study, I suggest, is required. A systematic exploration of when, where, and how proofs associated with different mathematical domains are introduced in both mathematics text books and classrooms within the province of Alberta would appear to be a valuable endeavor.

CHAPTER 8. RESULTS AND DATA ANALYSIS: TASK # 1.

Task # 1.

Prove that when you multiply any three consecutive numbers your answer is always a multiple of 6.

(Taken from Healy and Hoyles [2000])

Exemplars

1. Approach: Naïve Empiricism

1*2*3 = 6   6 = 6*1
2*3*4 = 24   24 = 6*4
3*4*5 = 60   60 = 6*10

Since these all work, when I multiply any three consecutive numbers, my answer will always be a multiple of 6.

2. Approach: Crucial Experiment

1*2*3 = 6   6 = 6*1
2*3*4 = 24   24 = 6*4
3*4*5 = 60   60 = 6*10

Since these all work, I will try once more using a larger trio of numbers. If it works for this case, then it must be true for every case.

Using a calculator,

1136*1137*1138 = 1469877216   1469877216 = 6*244979536

Since the assertion works in this case, too, whenever I multiply any three consecutive numbers, the answer will always be a multiple of 6.

3. Approach: Generic Example

I will try this assertion for 8, 9 and 10, which is of the form even, odd, even.

$$8*9*10 = 720 \qquad 720 = 6*120$$

6 is 2x3. The factor of 2 comes from the 8 (or the 10) while the factor of 3 comes from the 9.

I will try this assertion for 9, 10 and 11, which represents the form odd, even, odd.

$$9*10*11 = 990 \qquad 990 = 6*165$$

The factor 2 comes from the 10 and the factor 3 comes from the 9 and 6 = 2*3.

It will be like one of these in any case.

So when you multiply any three consecutive numbers together, your answer is always a multiple of 6.

4. Approach: Thought Experiment

Let "n", "$(n + 1)$" and "$(n + 2)$" be 3 consecutive numbers. These numbers can take the form odd, even, odd *or* even, odd, even. Therefore, in any three consecutive numbers there will be at least one number divisible by 2, as the factor of 2 comes from the even number. Every third number is a multiple of 3; hence, at least one of any three consecutive numbers will be divisible by 3. Since two and three are factors of 6, the product of three consecutive numbers is always a multiple of 6.

Analysis

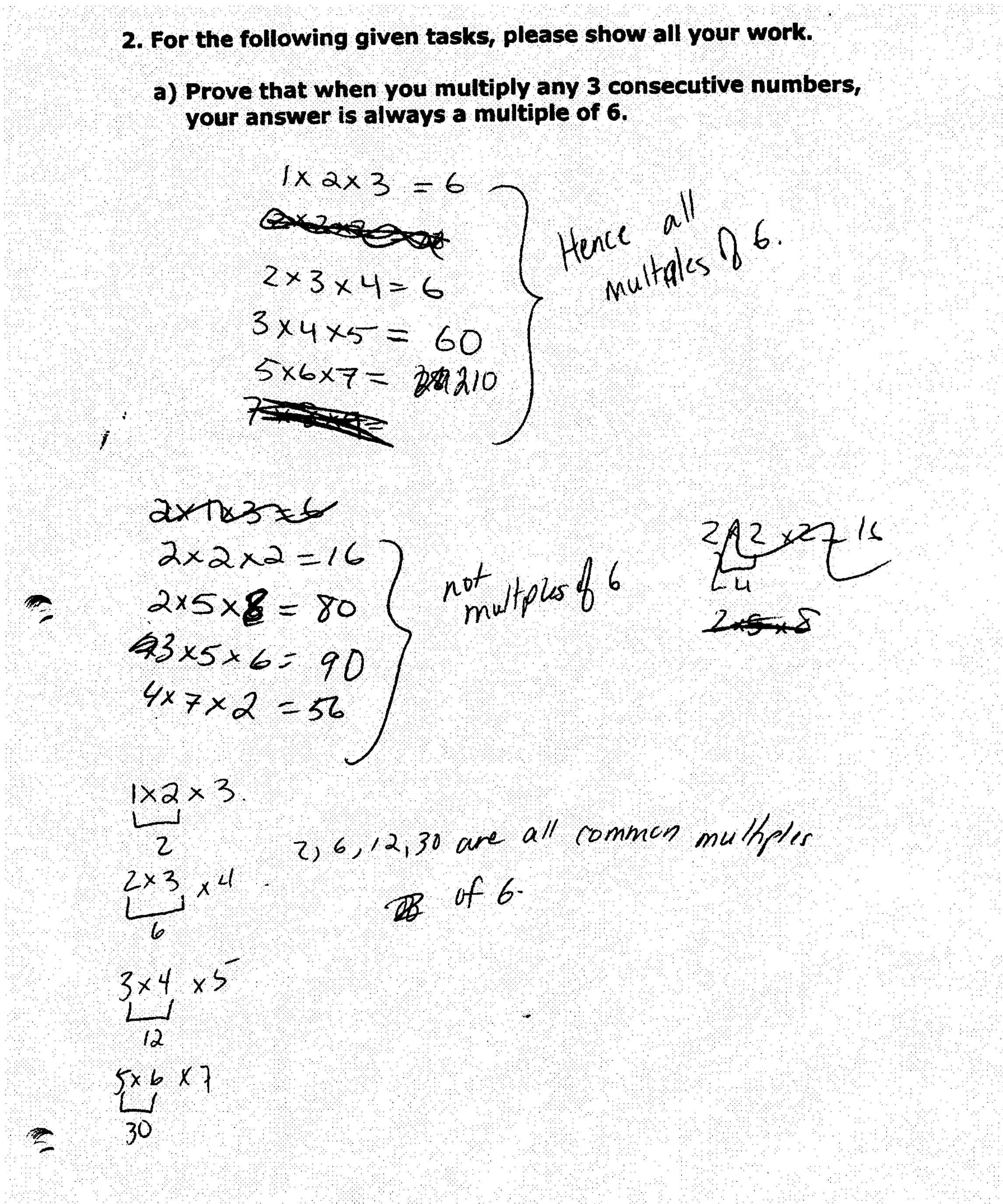
Table 8.1. Observations of student work Task # 1.

Categories *	Correct	Failed	Partially Correct	Total
Naïve Empiricism	0	0	2	**2**
Thought Experiment – verbal	2	2	5	**9**
Thought Experiment – symbolic	3	3	0	**6**
Total	**5**	**5**	**7**	**17**

* In his study, Balacheff (1988) did not consider the correctness or incorrectness of the mathematical proof. Nevertheless, no matter what approach the student takes, the work that results can be classified in one of three ways: as fully mathematically correct; as partially mathematically correct; and as mathematically incorrect. In this study, I categorize the students' efforts as correct only if the work is fully mathematically correct. In the case of Task # 1, as noted above, the work of the two students who took the approach of naive empiricism can only be categorized as partially correct. As noted before, Knuth (1999) in his study considered all proofs under the category of "naïve empiricism" as invalid. I maintain these distinctions throughout my analysis (see Chapters 9, 10, and 11).

1. Naïve Empiricism

Gita



2. For the following given tasks, please show all your work.

a) Prove that when you multiply any 3 consecutive numbers, your answer is always a multiple of 6.

1x2x3 = 6
2x3x4 = 6
3x4x5 = 60
5x6x7 = 210
Hence all multiples of 6.

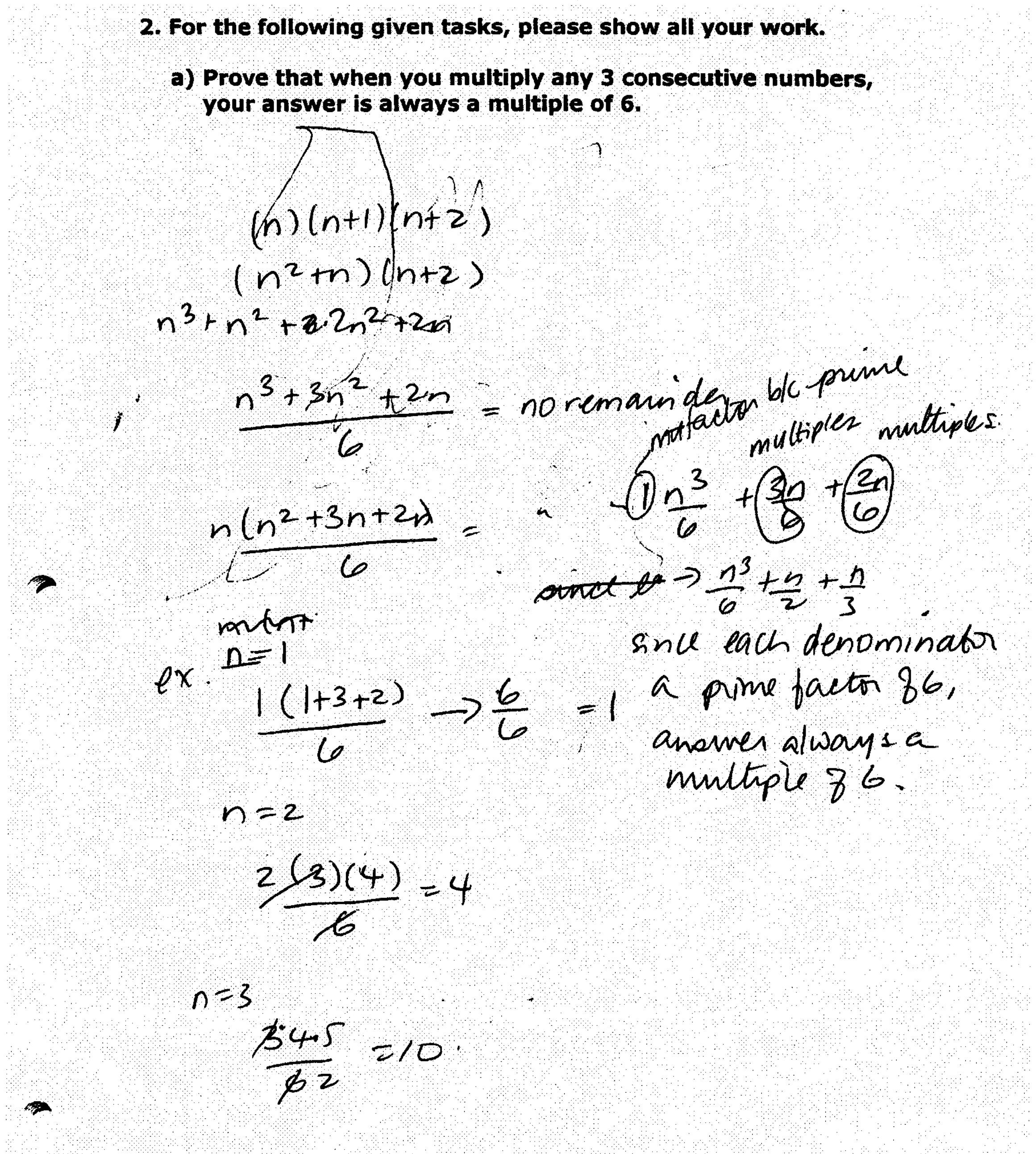
2x2x2 = 16
2x5x8 = 80
3x5x6 = 90
4x7x2 = 56
not multiples of 6

1x2 x 3 → 2
2x3 x4 → 6
3x4 x5 → 12
5x6 x7 → 30
2, 6, 12, 30 are all common multiples of 6.

Gita explored this task using three different groups of numbers. In the first group, she considered 4 sets of three consecutive numbers and found their product; then she showed that for such a set of numbers, the product will be a multiple of 6. I see an error in the product of the second set of three numbers: 2x3x4 should have been 24 not 6. In the second group of numbers, I see her multiplying 3 numbers which are not consecutive. In the first set, she considers three same numbers (2x2x2) and illustrates that the product is not a multiple of 6. In the second set of numbers, she selects three numbers with a difference of 3 between them (2x5x8) and illustrates that the product is not a multiple of 6. In the third (3x5x6) and fourth sets (4x7x2), she selects numbers randomly and illustrates that they are not multiples of 6. Based on the manipulation of the two groups of numbers noted above, I conjecture that she is trying to prove that the product of any three consecutive numbers will be a multiple of 6 and, if that is not the case, then the product will not be a multiple of six.

With the last group of numbers I see her trying to illustrate further her argument pertaining to the first group of numbers. She seems to be trying to show that if three consecutive numbers are considered, there will be multiples of 6 in their product. I infer this from the reversed square bracket she drew below two selected numbers. This can be easily verified from the third and the fourth set of numbers (3x4x5 and 5x6x7). 60 is shown as 12x5 and 210 is shown as 30x7. Here, she also notes that "2, 6, 12 and 30 are all *common multiples* of 6". Common multiples are defined as multiples that are common to two or more numbers; consequently, her reference to "common multiple of 6" is *incorrect*; it should be common multiples of 2 and 3 or common multiples of 2, 3 and 6. I categorize this proof as naïve empiricism because I see an assertion of the truth "of a result after verifying several cases" (Balacheff, 1988, p. 218). Even though she tries to further justify her observations regarding the first group of numbers, using the third group of numbers, she fails to give "explicit reasons for the truth of an assertion by means of operations or transformations on an object [example] that is not there in its own right, but as a characteristic representative of its class" (Balacheff, 1988. p 219). Hence, this cannot be seen as a generic example.

Tahira



2. For the following given tasks, please show all your work.

a) Prove that when you multiply any 3 consecutive numbers, your answer is always a multiple of 6.

$(n)(n+1)(n+2)$

$(n^2+n)(n+2)$

$n^3+n^2+2n^2+2n$

$\frac{n^3+3n^2+2n}{6}$ = no remainder b/c prime factor multiples multiples.

$\frac{n(n^2+3n+2n)}{6}$ = $\frac{n^3}{6}+\frac{3n}{6}+\frac{2n}{6}$

~~since~~ $\rightarrow \frac{n^3}{6}+\frac{n}{2}+\frac{n}{3}$.

since each denominator a prime factor of 6, answer always a multiple of 6.

ex. $n=1$

$\frac{1(1+3+2)}{6} \rightarrow \frac{6}{6} = 1$

$n=2$

$\frac{2(3)(4)}{6} = 4$

$n=3$

$\frac{3 \cdot 4 \cdot 5}{6} = 10$

She defined the three numbers as n, $n + 1$ and $n + 2$ and found the product. She made sense of the question by rephrasing it as $(n^3 + 3n^2 + 2n) / 6$ = no remainder which indicates that for a number to be a multiple of 6, it should be divisible by 6 evenly (with no remainder). Then she factorized $(n^3 + 3n^2 + 2n)$ as $n\,(n^2 + 3n + 2n)$. I think the "$2n$" term is a simple mistake: it should be 2. Even though she provided some algebraic manipulation, her proof is ultimately based on examples. It seems that she decided to substitute for $n = 1$, $n = 2$ and $n = 3$ and then simplified to show that the product is divisible by 6. I conjecture that when she got stuck with this algebraic approach, she decided to show three examples ($n = 1$, $n = 2$ and $n = 3$). This proof demonstrates a great deal of algebraic manipulation and, therefore, looks very "mathematical".

There are other mathematical flaws in this work as well. Tahira substituted n as 1 in the equation n $(n^2 + 3n + 2n)/6$. Her idea seems to be to substitute for the n that is outside the bracket. However, she multiplied the n outside of the bracket with the first term inside the bracket, n^2; then she post-multiplied the product, n^3, by 1 and retained it as $n^3/6$. For the other two terms, she substituted 1 for the "n" outside the bracket and retained the rest. The she simplified the equation and came up with $n^3/6 + n/2 + n/3$. Her argument "since [in] each denominator [there is a] prime factor of 6, [the] answer [is] always a multiple of 6" is incorrect. I categorize this as **naïve empiricism** because, even though she provides algebraic manipulations at the beginning of the proof, she bases her argument on three examples.

2. Thought Experiment - Verbal

a. failed thought experiment-verbal

Deanna

6

2. For the following given tasks, please show all your work.

a) Prove that when you multiply any 3 consecutive numbers, your answer is always a multiple of 6.

3 consecutive numbers...

$a, a+1, a+2$

added together $(a)+(a+1)+(a+2) = ?$

sum of 3 consec. #s. $= 3a+3$

sum is a multiple of 6 because

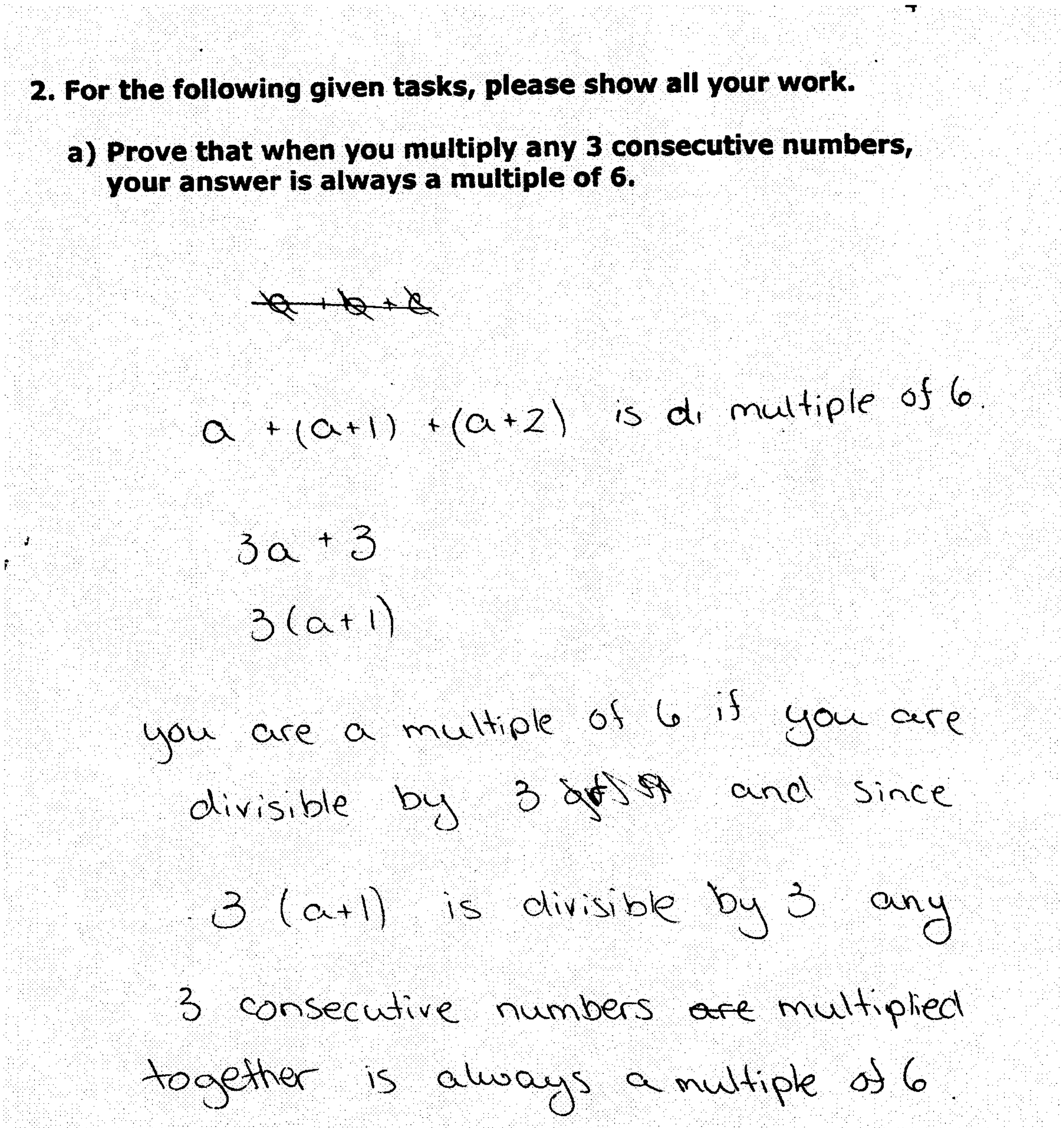
$\frac{3a+3}{6} = I$ (where I is an integer.)

$\updownarrow$

$\frac{3(a+1)}{6} = I$

Deanna misread the question: instead of multiplying her numbers, she adds them and then tries to prove the result for the sum. However, she could not complete the proof because she got stuck. It is interesting to note that for the set of integers her notation is "I". It is difficult to determine whether she used "I" as a variable or as a symbol to denote the set of integers. If she did, in fact, use "I" to denote the set of integers then she has demonstrated a common misconception among students. I have come across many students who use the notation "I" for the set of integers. The set of all integers is usually denoted in mathematics by a boldface **Z** or $\mathbb{Z}$. I categorize this as **failed verbal thought experiment.**

Clare



2. For the following given tasks, please show all your work.

a) Prove that when you multiply any 3 consecutive numbers, your answer is always a multiple of 6.

~~a + a + a~~

$a + (a+1) + (a+2)$ is a multiple of 6.

$3a + 3$

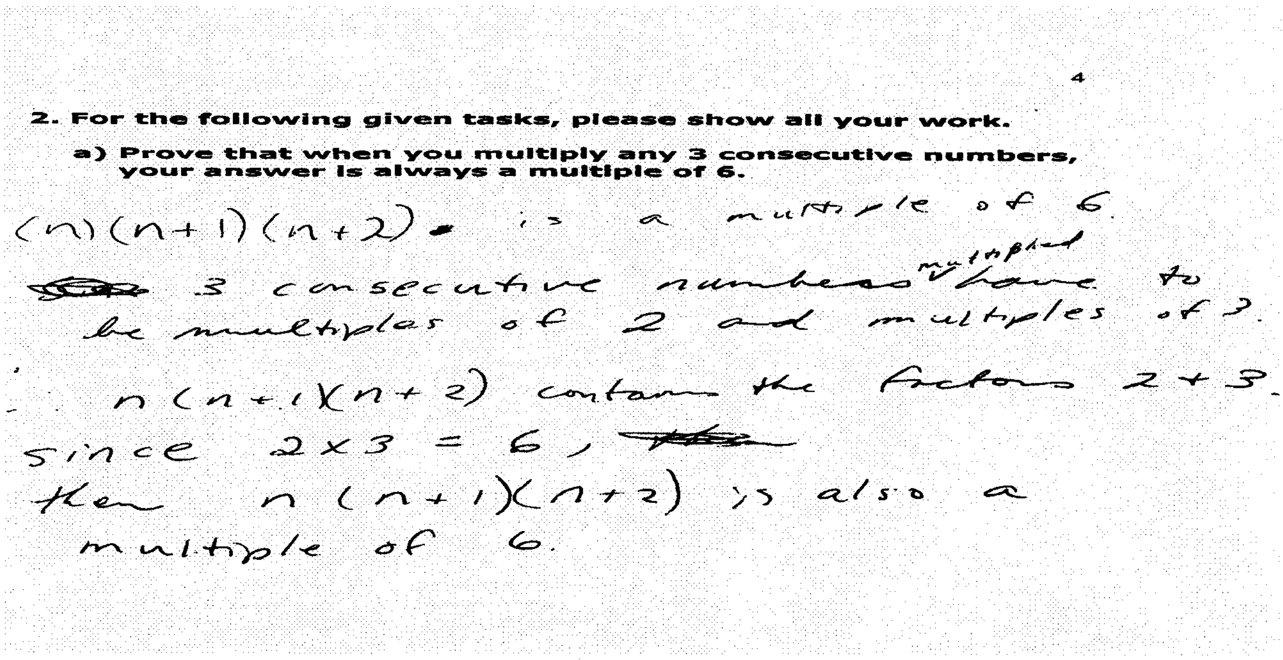
$3(a+1)$

you are a multiple of 6 if you are divisible by 3 ~~[illegible]~~ and since $3(a+1)$ is divisible by 3 any 3 consecutive numbers ~~are~~ multiplied together is always a multiple of 6.

It seems that Clare also misread the question. Instead of multiplying the three numbers *a*, $a + 1$ and $a + 2$, she added them and then argued that to be a multiple of 6 the product should be divisible by 3. She forgot to mention that, in addition to being divisible by 3, the product must also be divisible by 2, if that product is to be divisible by 6. I categorize this as a **failed verbal thought experiment**

b. partially correct thought experiment-verbal

Daniel



4

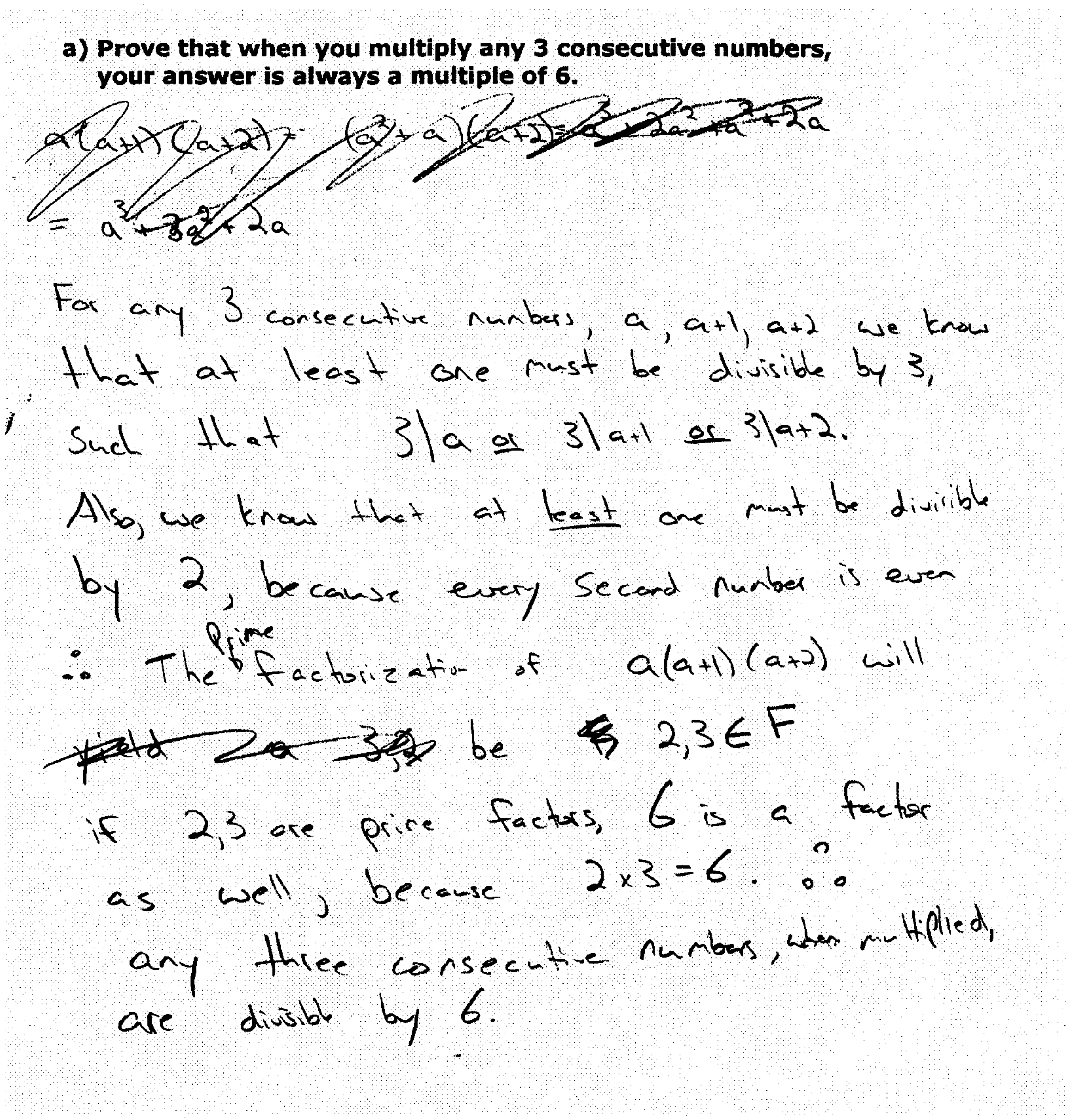
2. For the following given tasks, please show all your work.

a) Prove that when you multiply any 3 consecutive numbers, your answer is always a multiple of 6.

(n)(n+1)(n+2) is a multiple of 6.
3 consecutive numbers multiplied have to be multiples of 2 and multiples of 3.
∴ n(n+1)(n+2) contain the factors 2 + 3.
since 2 × 3 = 6,
then n(n+1)(n+2) is also a multiple of 6.

Daniel defines the three numbers as n, $n + 1$ and $n + 2$. He paraphrases the task as "$n(n + 1)(n + 2)$ is a multiple of 6". His proof can be considered a **partially correct verbal thought experiment** since he fails to state explicitly why the three consecutive numbers that are multiplied have to be multiples of 2 *and* 3. It is evident from his justification that he understands that in order to be a multiple of 6, the number should be a multiple of 2 and also 3.

Philip



a) Prove that when you multiply any 3 consecutive numbers, your answer is always a multiple of 6.

~~$a(a+1)(a+2) = (a^2+a)(a+2) = a^3+2a^2+a^2+2a$~~

$= a^3 + 3a^2 + 2a$

For any 3 consecutive numbers, a, a+1, a+2 we know that at least one must be divisible by 3, Such that 3\a or 3\a+1 or 3\a+2.

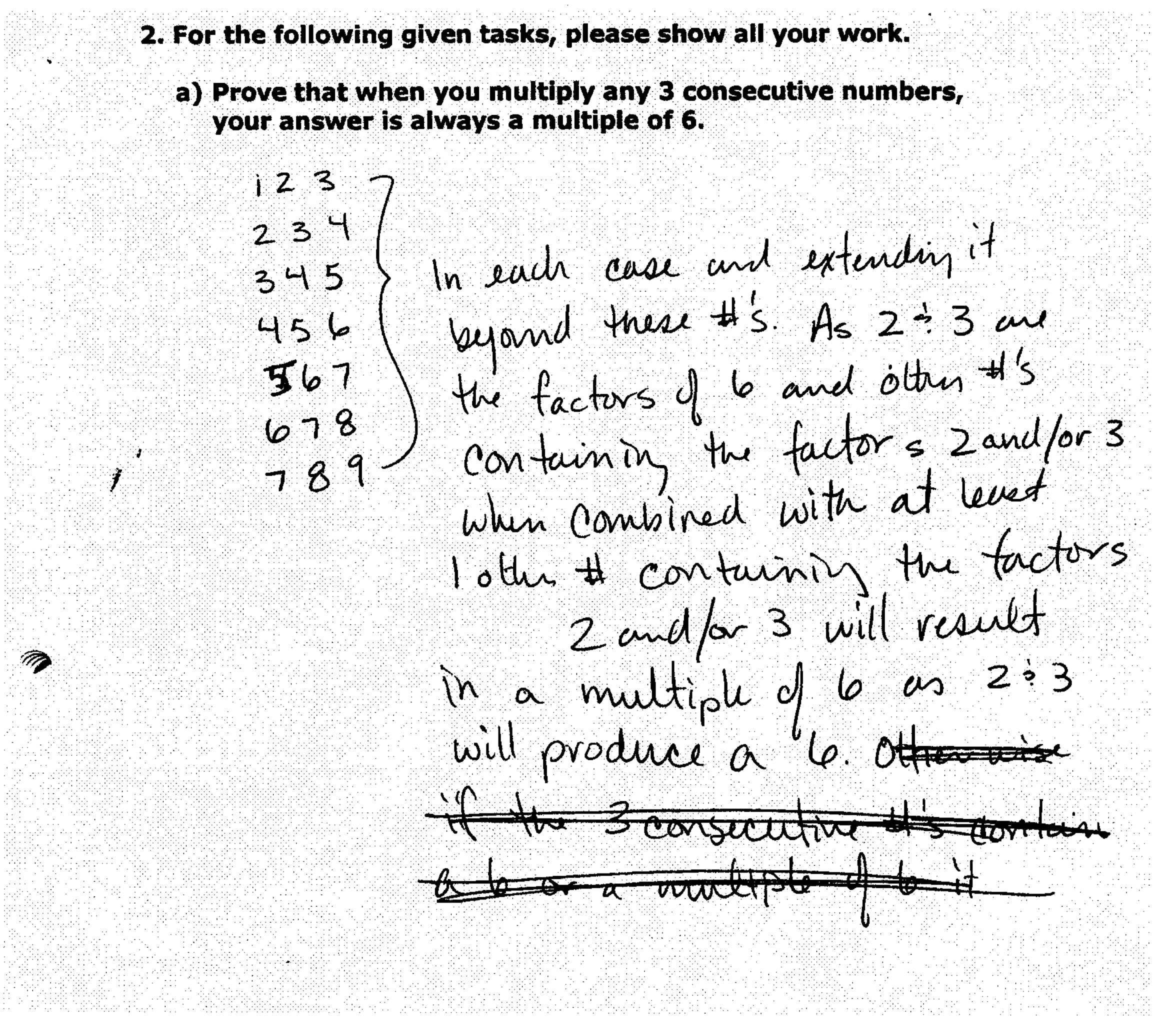
Also, we know that at least one must be divisible by 2, because every second number is even

∴ The prime factorization of a(a+1)(a+2) will ~~yield 2a 3a~~ be 2,3 ∈ F

if 2,3 are prime factors, 6 is a factor as well, because 2 × 3 = 6. ∴ any three consecutive numbers, when multiplied, are divisible by 6.

Philip initially explored the problem by defining the three numbers as *a* (*a* + 1) and (*a* + 2), and then multiplying them together. Although he failed to define the numbers as such, his scribbles (later crossed through) show that he was trying to explore the task in an algebraic way. I conjecture that when he realized that he would not succeed with this algebraic approach, he resorted to a thought experiment. This makes his proof similar to Daniel's; Philip, however, uses more symbols than Daniel. In this proof, Philip mentions that, for any three consecutive numbers, "at least one must be divisible by 3" and "at least one must be divisible by 2"; but he does not explicitly state the reason why. Hence, I categorize it as a **partially correct verbal thought experiment.**

Terrence



2. For the following given tasks, please show all your work.

a) Prove that when you multiply any 3 consecutive numbers, your answer is always a multiple of 6.

1 2 3
2 3 4
3 4 5
4 5 6
5 6 7
6 7 8
7 8 9

In each case and extending it beyond these #'s. As 2 ÷ 3 are the factors of 6 and other #'s containing the factors 2 and/or 3 when combined with at least 1 other # containing the factors 2 and/or 3 will result in a multiple of 6 as 2 ÷ 3 will produce a 6. ~~Otherwise if the 3 consecutive #'s contain a 6 or a multiple of 6 it~~

Terrence wrote 7 sets of 3 consecutive numbers. In each set he noticed factors of 2 and 3 and then made the argument that " in each case and extending it beyond these #s" there are factors of 2 and 3. Even though he made sense of his proof with several examples, his understanding of the concept of divisibility by 6 in terms of being divisible by 2 and 3 is "detached from particular examples"; in other words, we can see his movement "from practical to intellectual proofs" (Simon & Blume, 1996, p. 8). However, he fails to state explicitly the reason why the product of three consecutive numbers has factors of 2 and 3; hence, I categorize this proof as a **partially correct verbal thought experiment**.

Chandelle

2. For the following given tasks, please show all your work.

a) Prove that when you multiply any 3 consecutive numbers, your answer is always a multiple of 6.

Let n = a number.

$\therefore n, (n+1), (n+2)$ are three consecutive numbers

$n(n+1)(n+2)$

$(n^2+n)(n+2)$

$n^3+2n^2+n^2+2n$

n^3+3n^2+2n

- to be divisible by six, the number must be divisible by 2 & 3.

Divisible by 2:

(#1) if n is even, then (n+1) is odd and (n+2) is even

or (#2) n is odd, then (n+1) is even and (n+2) is odd

these are the only two cases (#1 & #2)

In both (#1) and (#2) there is a factor that is even, and we know that all even #s are divisible by two, we know that n^3+3n^2+2n will always be divisible by two.

Divisible by 3:

Her proof is well written, but incomplete. All of her arguments for the divisibility by two are clearly suggested, but the second part of the proof remains incomplete. I place this proof in the category of **partially correct verbal thought experiment.**

Sara

4 JS

2. For the following given tasks, please show all your work.

a) Prove that when you multiply any 3 consecutive numbers, your answer is always a multiple of 6.

Let your 3 numbers be

x, y, z; n = the product.

Let x = first number

y = 2nd number

z = 3rd number

We know

$y = x+1$

$z = x+2$

Prove that

$xyz = 6n$

$x(x+1)(x+2) = 6n$

$x(x^2+3x+2) = 6n$

$x^3+3x^2+2x = 6n$.

For each consecutive number, one of them (at least) must be divisible by 2 (Even number). We also know that every 3 numbers is a multiple of 3. Therefore, the product xyz must contain one number divisible by 2 and one number divisible by 3. For a number to be divisible by 6, it must be divisible by 2 and 3. Since 3 consecutive numbers are always multiples of 2 + 3, their product is always a multiple of 6.

Sara rephrased the proof task with her three numbers "x", "y" and "z" and a product "n". Later she writes $xyz = 6n$, to imply the product is a multiple of 6. She could not go much beyond this point, so she redefined y as $x + 1$ and z as $x + 2$. Again, she made little headway; so then she provided a general argument. I categorize this proof as **partially correct thought experiment** because the proof is incomplete. She makes the claim "we know that every three numbers is a multiple of 3" which is, in itself, incorrect. However, based on the rest of her work it seems that she used the word "every" to mean "every three consecutive numbers"—and not every *possible* three numbers—are a multiple of 3; but, she fails to provide a reason for this argument. She also continues to write that "the product xyz must contain one number divisible by 2 and one number divisible by 3", but does not give an explicit reason why. However, her understanding about the divisibility of 6 such that numbers are evenly divisible by 6 if they are evenly divisible by both 2 *and* 3, is evident from the last sentence of the proof. I categorize this proof as a **partially correct verbal thought experiment** because the proof is not complete.

c. correct thought experiment-verbal

Cathy

2. For the following given tasks, please show all your work.

a) Prove that when you multiply any 3 consecutive numbers, your answer is always a multiple of 6.

If the 3 numbers being multiplied there will always be a multiple of 3, and at least one even number, an the divisibility rule for 6 is that it has a factor of 3 and is even. (factor of 2)

1 · 2 · 3

2 · 3 · 4

4 · 5 · 6

all have a multiple of 3 because every 3rd # is a multiple of 3 & all have 1 or 2 even #'s, so the resulting product of any 3 consecutive is even.

□

John

2. For the following given tasks, please show all your work.

a) Prove that when you multiply any 3 consecutive numbers, your answer is always a multiple of 6.

Let n be a number

∴ n, n+1, n+2 are 3 consecutive numbers

$(n)(n+1)(n+2) = n^3 + 3n^2 + 2n$

Prove: $n^3 + 3n^2 + 2n$ is divisible by 2 and 3 (divisibility rule for 6)

divisible by 2:

If n is an even number, n+2 is an even number and (n^3+3n^2+2n) will be even since at least one of its factors is even. Since even numbers have a common factor of 2, n^3+3n^2+2n will be divisible by 2.

If n is not even, n+1 is even and the same argument holds true.

divisible by 3

If n is divisible by 3, n^3+3n^2+2n is also divisible by 3 (share 3 as a factor)

If n is not divisible by 3, then either n+1 or n+2 is divisible by 3 (multiples of 3 occur every third number so within three consecutive numbers, one must be a multiple of 3)

So n^3+3n^2+2n is divisible by 3

∴ Since n^3+3n^2+2n is divisible by 2 and 3, n^3+3n^2+2n is divisible by 6

John's proof, with its variety of variables and its expression of the product of three numbers (*n*), (*n* + 1) and (*n* + 2) as a polynomial, looks more mathematically rich than

Cathy's proof. However, the language of the argument itself is very plain. I categorize this as a **correct thought experiment.**

3. Thought Experiment - Symbolic

a. failed thought experiment-symbolic

Beth

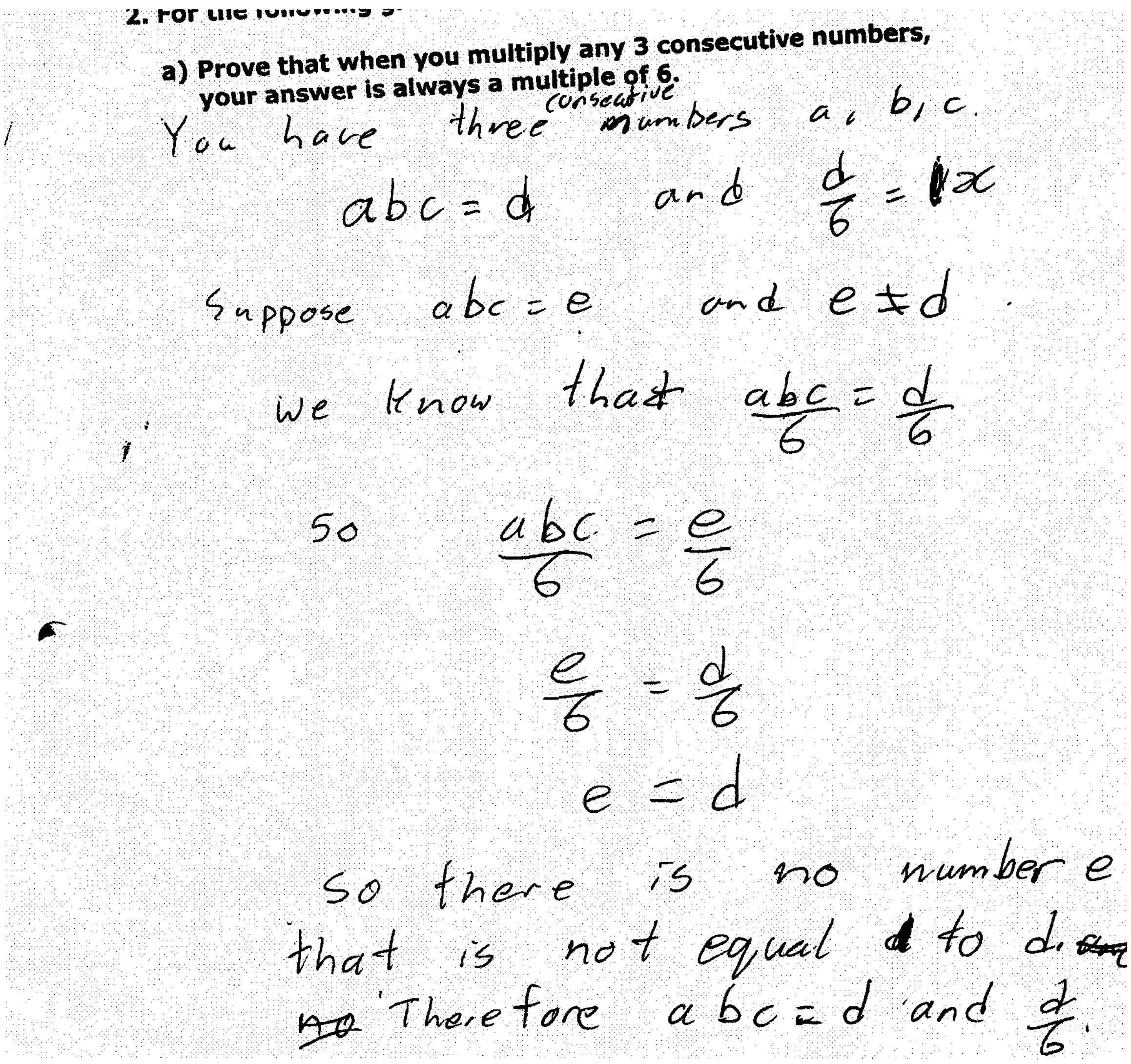
I categorize this as **failed symbolic thought experiment** mainly because of the algebraic manipulation and the absence of natural language.

a) Prove that when you multiply any 3 consecutive numbers, your answer is always a multiple of 6.

numbers = n, $n+1$, $n+2$

$$n(n+1)(n+2) = (n^2+n)(n+2)$$
$$= n^3 + 2n^2 + n^2 + 2n$$
$$= n^3 + 3n^2 + 2n$$
$$n(n^2+3n+2)$$

Spencer



2. For the following ...

a) Prove that when you multiply any 3 consecutive numbers, your answer is always a multiple of 6.

You have three consecutive numbers a, b, c.

$abc = d$ and $\frac{d}{6} = x$

Suppose $abc = e$ and $e \neq d$.

we know that $\frac{abc}{6} = \frac{d}{6}$

So $\frac{abc}{6} = \frac{e}{6}$

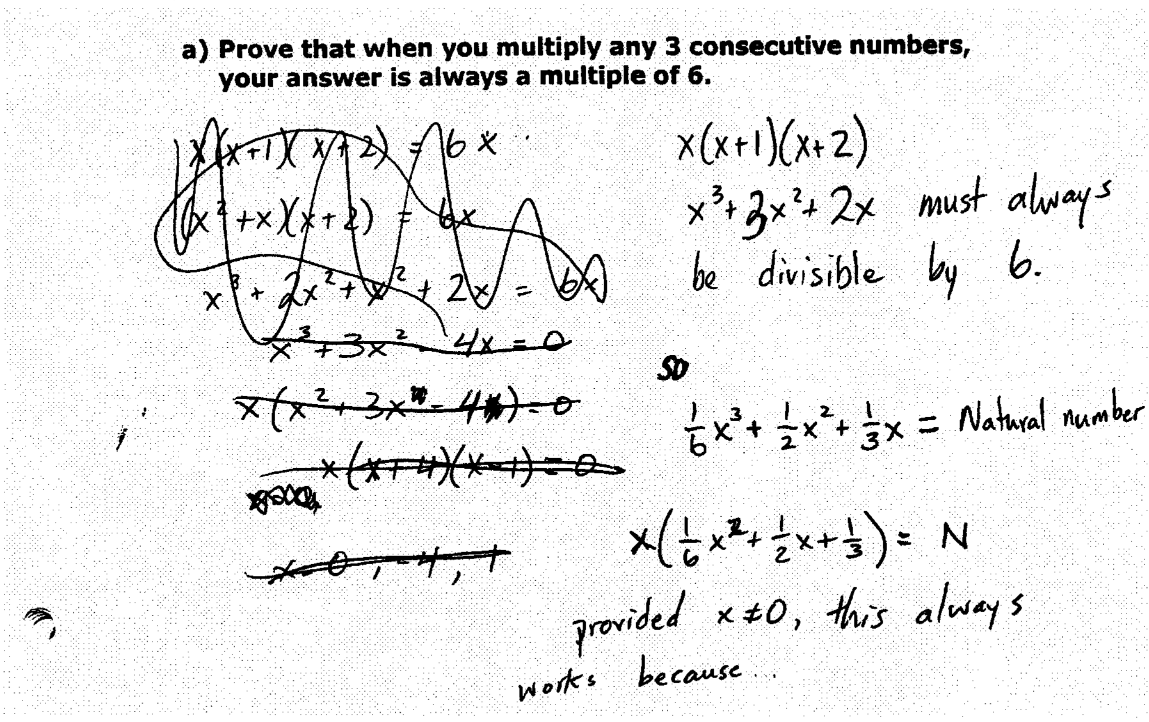
$\frac{e}{6} = \frac{d}{6}$

$e = d$

So there is no number e that is not equal to d. Therefore $abc = d$ and $\frac{d}{6}$.

Spencer's work looks like a formal mathematical proof with all the variables and the mathematical operators evident. However, this proof does not yield much information. It seems that Spencer is trying to prove by contradiction. Proof by contradiction is also known as *reductio ad absurdum,* Latin for "*reduced to the point of absurdity*". This proof establishes the truth of a given proposition by the supposition that it is false and then draws a conclusion that is contradictory to it to prove the proposition (Koshy, 2002). I conjecture that this is proof by contradiction because Spencer writes that "suppose $abc = e$ and $e \neq d$" then goes on to say, in the end, that "there is no number e that is not equal to d. Therefore $abc = d$ and $d/6$". These statements do not agree since they imply that $abc = d = d/6$. Possibly what he had in mind is what he had started with: "$abc = d$ and $d/6 = x$". Despite the formal look of this proof, it reveals little. I have classified this as a **failed symbolic thought experiment.**

Brandon



a) Prove that when you multiply any 3 consecutive numbers, your answer is always a multiple of 6.

$x(x+1)(x+2)$

$x^3 + 3x^2 + 2x$ must always be divisible by 6.

so

$\frac{1}{6}x^3 + \frac{1}{2}x^2 + \frac{1}{3}x$ = Natural number

$x(\frac{1}{6}x^2 + \frac{1}{2}x + \frac{1}{3}) = N$

provided $x \neq 0$, this always works because . . .

Brandon's scribbles (later crossed out) indicate that he initially multiplied x, $(x + 1)$ and $(x + 2)$ and noted the product as $6x$. The symbols x, $x + 1$ and $x + 2$ represent three consecutive numbers. However, the use of "x" with 6 ($6x$) on the right hand side of the equation does not mean the same as the "product of the three consecutive numbers is a multiple of 6". He may have used "$6x$" to remind himself that the *product* is a *multiple of 6.* In this regard, Brandon's attempt is similar to Sara's proof (see pp. 114 -115). Brandon used algebraic manipulations including the factorization of the polynomial, but had little success in arriving at a solution. It seems that he then rephrased the question as "must always be divisible by 6". This is correct reasoning. To be a multiple of 6, the number should be divisible by 6. Despite its formal look due to algebraic manipulations and symbols, this proof reveals little. He also failed to come up with a general argument. I categorize this as **failed symbolic thought experiment.**

b. correct symbolic thought experiment

George

a) Prove that when you multiply any 3 consecutive numbers, your answer is always a multiple of 6.

We are given 3 consecutive numbers. So we have

$x, x+1, x+2$

Now, we know that at least one of these have to be even, as numbers progress even, odd, even in natural numbers, so thusly:

$2 \mid (x)(x+1)(x+2)$

Also, seeing as we have 3 consecutive numbers, then one of them must be divisible by 3. This is evident using the division algorithm: $x = 3a + b$ $a \in$ Whole numbers, $b \in \{0, 1, 2\}$

So if $b = 0$ for x, $3 \mid x$

if $b = 1$ for x, $3 \mid x+2$

if $b = 2$ for x, $3 \mid x+1$

Thusly $3 \mid (x)(x+1)(x+2)$

So, given $2 \mid (x)(x+1)(x+2)$

and $3 \mid (x)(x+1)(x+2)$

then $3 \cdot 2 \mid (x)(x+1)(x+2)$ as 3 and 2 are mutually prime

thus $6 \mid (x)(x+1)(x+2)$ □

George defines the three numbers as x, $x + 1$ and $x + 2$. His first argument implies that there will be an even number in any three consecutive numbers and even numbers are divisible by 2. He also introduces division algorithm in his argument concerning divisibility by 3. Since he uses more symbols and provides a proof that is more mathematical than the rest of the group members, I categorize this as a **correct symbolic thought experiment**. George ended his proof with a hollow black square. This is a simple way of stating that the proof is complete. This symbol is usually used to end proof when it is done in a formal way. In other words, this hollow black square or sometimes a dark black square (also called tombstone) is used instead of writing ***Q.E.D.*** which is an abbreviation of theLatin phrase *"quod erat demonstrandum"* (literally, "which was to be demonstrated").

Grace

2. For the following given tasks, please show all your work.

a) Prove that when you multiply any 3 consecutive numbers, your answer is always a multiple of 6.

$x = n(n+1)(n+2)$

$x = (n^2+n)(n+2)$

$x = n^3 + 3n^2 + 2n$

↓ even ↓ even ↓ even

since all are even, the solution must be even; thus divisible by 2.

→ must prove $x \equiv 0 \bmod 3$

$x = n(n+1)(n+2)$

2mod3 0mod3 1mod3

or

1mod3 2mod3 0mod3

or

0mod3 1mod3 2mod3

out of 3 consecutive numbers, one will always be a multiple of 3.

⇒ if x is a multiple of 2 and 3 ⇒ x is a multiple of 6.

The **mod** operator is often used in discrete mathematics and computer science to divide two numbers and retain only the remainder. So, a **mod** b = remainder when a is divided by b. I classify Grace's work as a **correct symbolic thought experiment** because of the formal language.

Brian

2. For the following given tasks, please show all your work.

a) Prove that when you multiply any 3 consecutive numbers, your answer is always a multiple of 6.

$(n)(n+1)(n+2)$

$n^3 + 3n^2 + 2n$

$n(n^2 + 3n + 2)$

case 1:
odd
n^3 is odd
$3n^2$ is odd
$2n$ is even
$\Rightarrow$ sum is even

case 2:
even
n^3 is even
$3n^2$ is even
$2n$ is even
$\Rightarrow$ sum is even

division by 3.

$\frac{1}{3}(n^3 + 3n^2 + 2n)$

Cases:

$n \bmod 3 = 0$
$(n+1) \bmod 3 = 1$
$(n+2) \bmod 3 = 2$
3

$n \bmod 3 = 1$
$(n+1) \bmod 3 = 2$
$(n+2) \bmod 3 = 0$
3

$n \bmod 3 = 2$
$(n+1) \bmod 3 = 0$
$(n+2) \bmod 3 = 1$
3

$\therefore$ it is divible by 3.

$\Rightarrow$ it is a multiple of 6 because it is divisible by 2 and 3

Brian's proof can also be categorized as a **correct symbolic thought experiment.**

Summary

Of the 17 student teachers that attempted this task, 11 produced either correct or partially correct proofs. Most of those 11 (10 students) opted for conceptual justification: only one person preferred a pragmatic justification. Even though a great many of the students approached this task at the level of thought experiment, example-based mathematical proofs were also evident. Six student teachers found this task especially difficult. I categorized their work as "failed" proofs. Each of these failed proofs made little sense. Since only six people achieved a fully correct proof, I conclude that this group of student teachers experienced considerable difficulty when it comes to proving this type of mathematical proof task.

CHAPTER 9. RESULTS AND DATA ANALYSIS: TASK # 2.

Task # 2.

Provide a means of calculating the number of diagonals of a polygon when you know how many vertices it has.

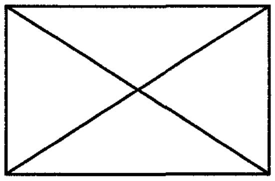
(Taken from Balacheff [1988])

Exemplars

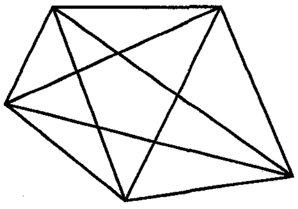
1. Approach: Naïve Empiricism

A rectangle has four vertices and two diagonals.

Vertices = v = 4
Diagonals = d = 2



A pentagon has five sides and five diagonals.



Vertices = v = 4
Diagonals = d = 2

Hence, if "v" is even, the number of diagonals $d = v / 2$.

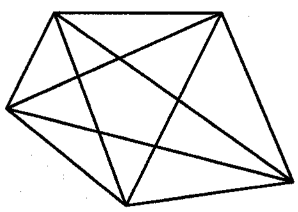
And if "v" is odd, the number of diagonals $d = v$.

2. Approach: Crucial Experiment

Balacheff (1988) distinguishes between crucial experiment and naïve empiricism on the basis of the student's selection of the example. Both Balacheff (p. 224) and Knuth and Elliot (1998) note that students at the level of crucial experiment intentionally select an extreme case, and if the proof works for that example, they will then conclude that their conjecture is correct and the proof proved. Thus, I approach Task # 2 task at the level of crucial experiment in the following manner.

I conjecture that the # of diagonals = # of vertices and will use the extreme case of the pentagon to verify my conjecture. I use the pentagon (as an extreme case) because it is the polygon with the greatest number of sides that I can still draw with relative ease.

The pentagon has five sides and five diagonals

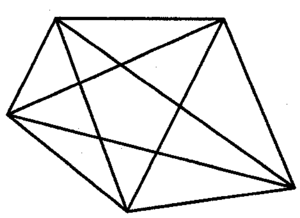


vertices = $v = 5$
diagonals = $d = 5$

Hence $d = v$

3. Approach: Generic Example

The pentagon has five sides and five diagonals.



Vertices = $v = 5$
Diagonals = $d = 5$

A pentagon has 5 sides ($n = 5$) and so 5 vertices ($v = 5$). From each vertex, I can draw *only* 2 diagonals because there are no diagonals from a vertex back to itself and there are no diagonals to the vertices on either side. Thus, there will be three fewer diagonals than the total number of sides (namely 2 at each vertex). Since there are 5 sides and 5 vertices, I can draw 5* 2 (= 10) diagonals in total.

Diagonals have two ends; counting both ends of the same diagonal I would arrive at a total of 10. However, I need only count one end; therefore, I must divide 10 by 2 to get the correct number of diagonals, which equals the number of vertices.

This exemplar illustrates generic reasoning because the calculations and answers are specific to the fact that one is considering a pentagon, although the same reasoning would apply whatever the number of sides involved. (The conjecture of course is false).Arguing from the specific to the general distinguishes the generic example from the thought experiment. Let me reiterate that Balacheff also failed to provide a concrete example of generic example (see Balcheff 1988 & 1991). When Knuth and Elliot (1998) elaborated on Balacheff's taxonomy of proofs, they did not provide a concrete example either. One reason why none of these researchers could provide a ready example of generic example may be because it is difficult to find a representative polygon that will consider all the different polygons of the same number of sides and all the different polygons simultaneously.

4. Approach: Thought Experiment

Consider a polygon with "v" sides. If there are "v" sides, there are "v" vertices. Beginning with each vertex, I will draw (v - 3) (again, recall that there is no diagonal from a vertex back to itself and there are no diagonals to the vertices on either side). Thus, there will be three fewer diagonals than the total number of sides—that is, (v - 3) diagonals from each vertex. As there are "n" sides, there will be a total of $v(v - 3)$ diagonals. My approach, however, counts both ends of the diagonal. That means I am counting each diagonal twice. Hence, to get the correct number of diagonals, I will divide the product by 2. Therefore, the formula for the number of diagonals is $d = v(v - 3) / 2$.

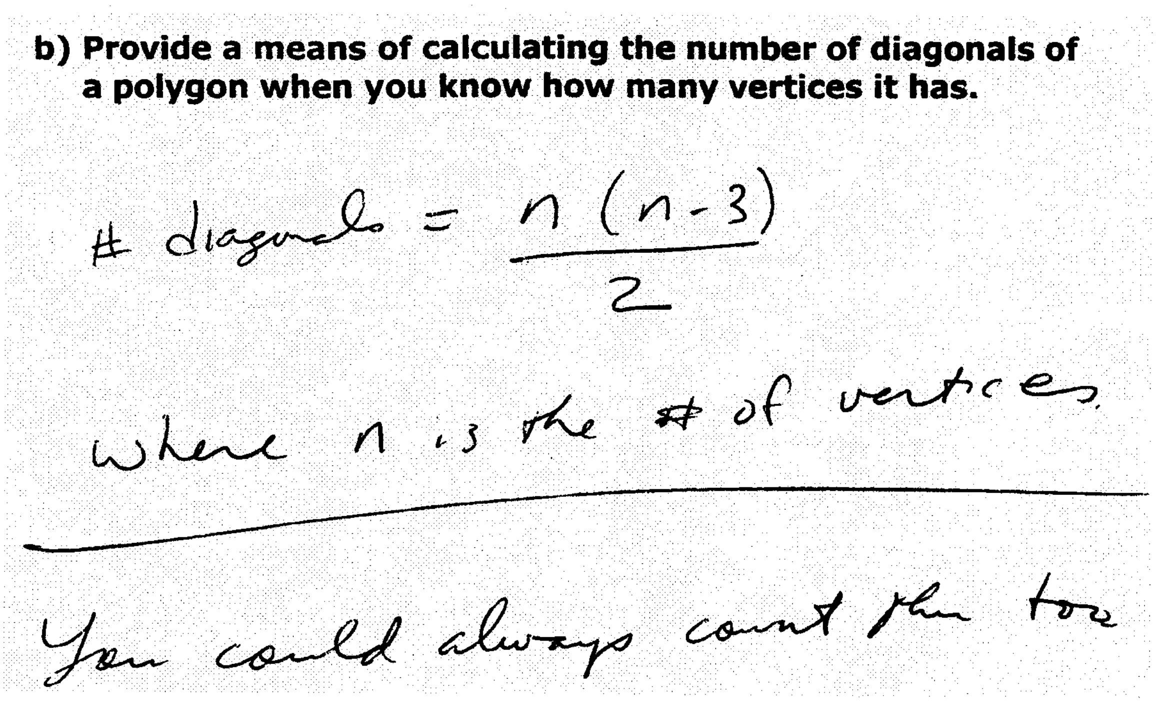
Analysis

Table 9.1. Observations of student work Task # 2.

Categories	Correct	Partially Correct	Failed	Total
Formula Only	4	0	0	4
Naïve Empiricism	0	2	0	2
Generic Example	2	1	0	3
Thought Experiment- verbal	3	0	3	6
Thought Experiment -symbolic	1	1	0	2
Total	**10**	**2**	**3**	**17**

1. **Formula Only**

Daniel


b) Provide a means of calculating the number of diagonals of a polygon when you know how many vertices it has.

diagonals = $\frac{n(n-3)}{2}$

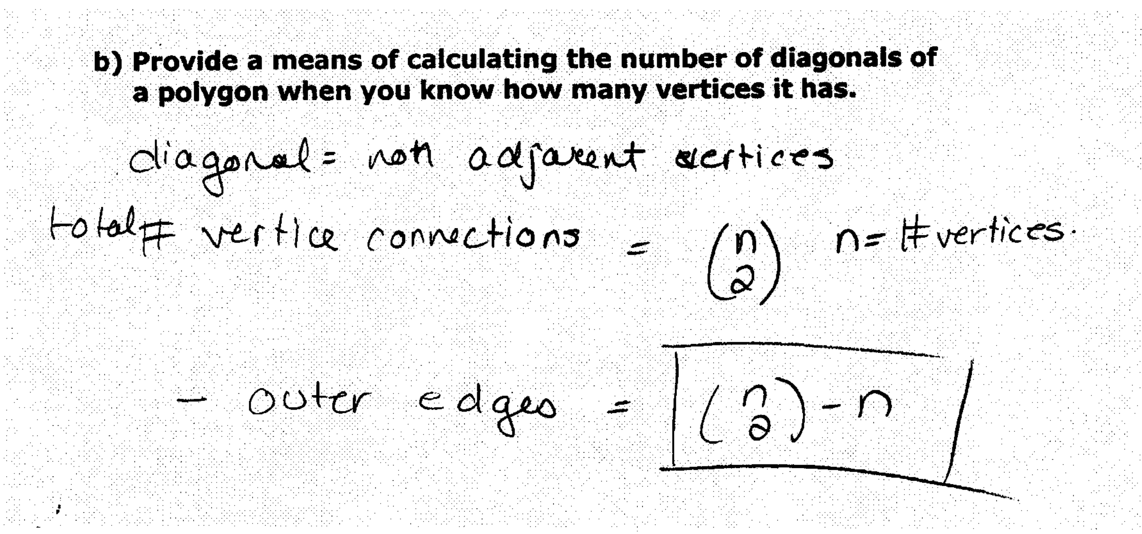
where n is the # of vertices.

You could always count them too

In addition to providing a formula for his work, Daniel also mentioned that one could count the number of diagonals in any given polygon. His choice of word—"counting"—suggests that he is responding to the wording of the task—"provide a means of calculating….": in other words, one other means of calculating is by "counting".

Beth

Beth came up with a formula and minimal justification using natural language. She calculated the total number of connections between vertices using $_nC_2$ and then subtracted the number of outer edges (n). Her justification also correctly implies that the number of vertices = number of sides. This is a possible thought experiment. However, I decided to categorize her work under "formula only" because of the minimal use of language and explanation.

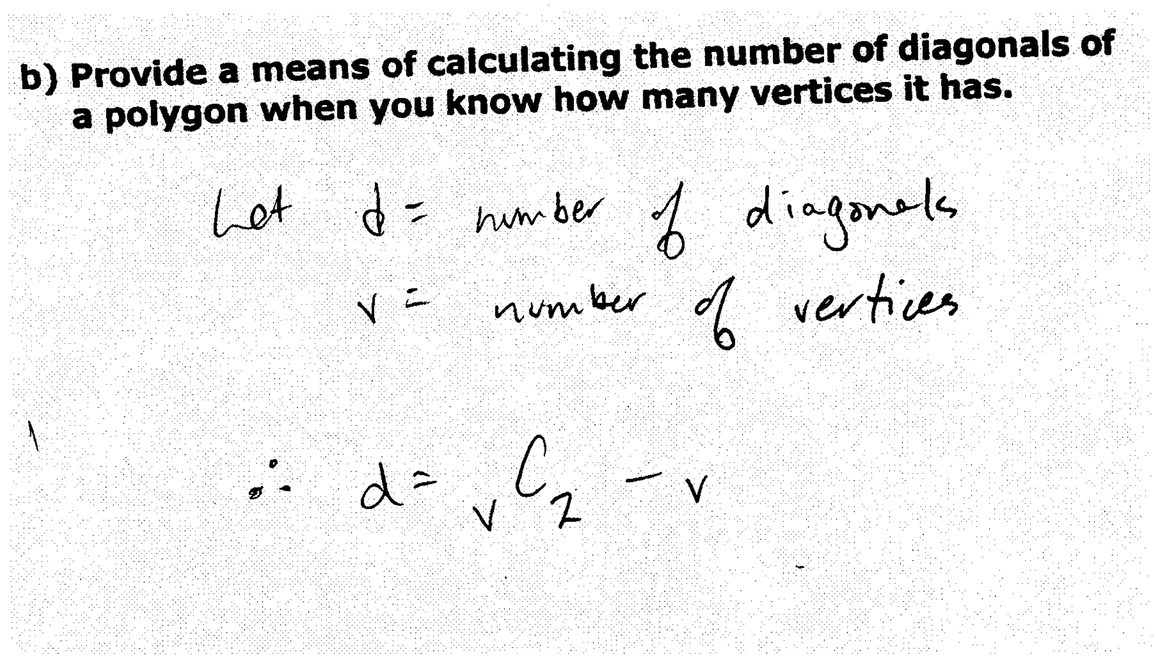

b) Provide a means of calculating the number of diagonals of a polygon when you know how many vertices it has.

diagonal = non adjacent vertices

total # vertice connections = $\binom{n}{2}$ n = # vertices.

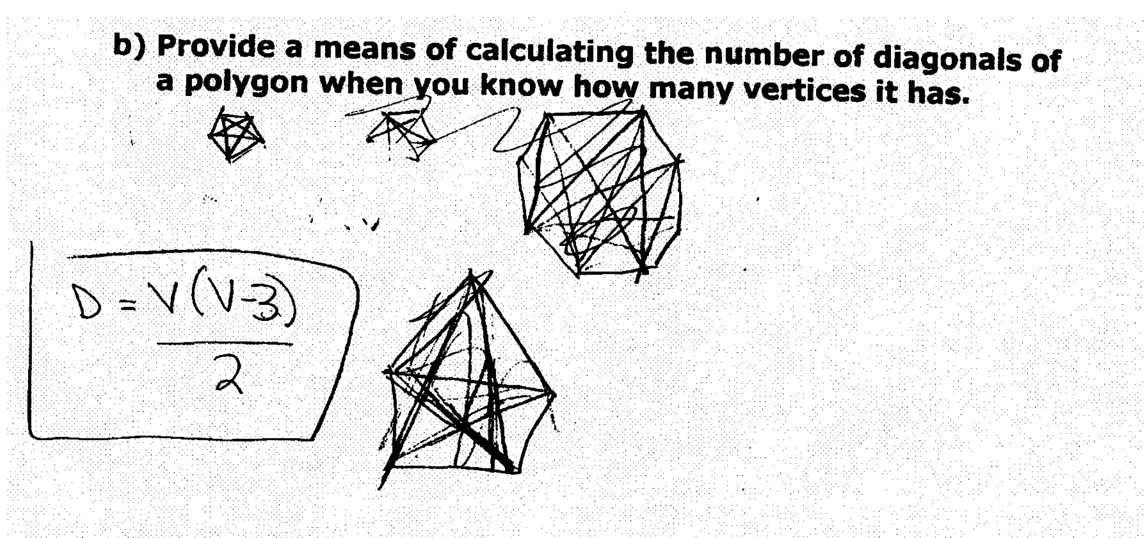
− outer edges = $\binom{n}{2} - n$

John



John's formula below is similar to that of Beth's (above).

Philip



Philip evidently explored the task using four different polygons. It seems clear that he counted the number of diagonals in the pentagon (twice, in fact, with two differently sized polygons), the hexagon, and the heptagon. Again, one could argue that

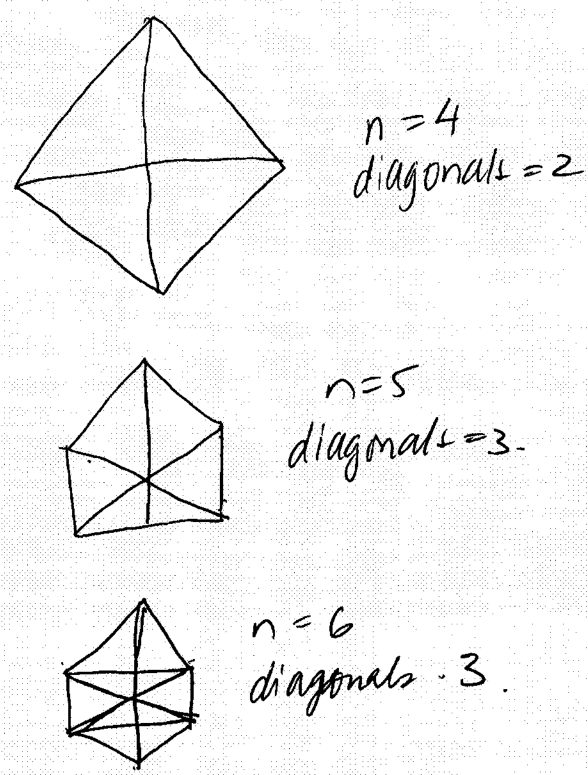
Philip might have arrived at this correct formula through a possible thought experiment. However, I decided to include it in the "formula only" category because of the minimal use of language and explanation. Balacheff (1991) indicated that his student participants were able to provide a correct formula for this task because they correctly understood the concepts of polygon and diagonal. He assumed that students were able to produce these formulas by means of a deductive process.

2. Naive Empiricism

Tahira

5

b) Provide a means of calculating the number of diagonals of a polygon when you know how many vertices it has.

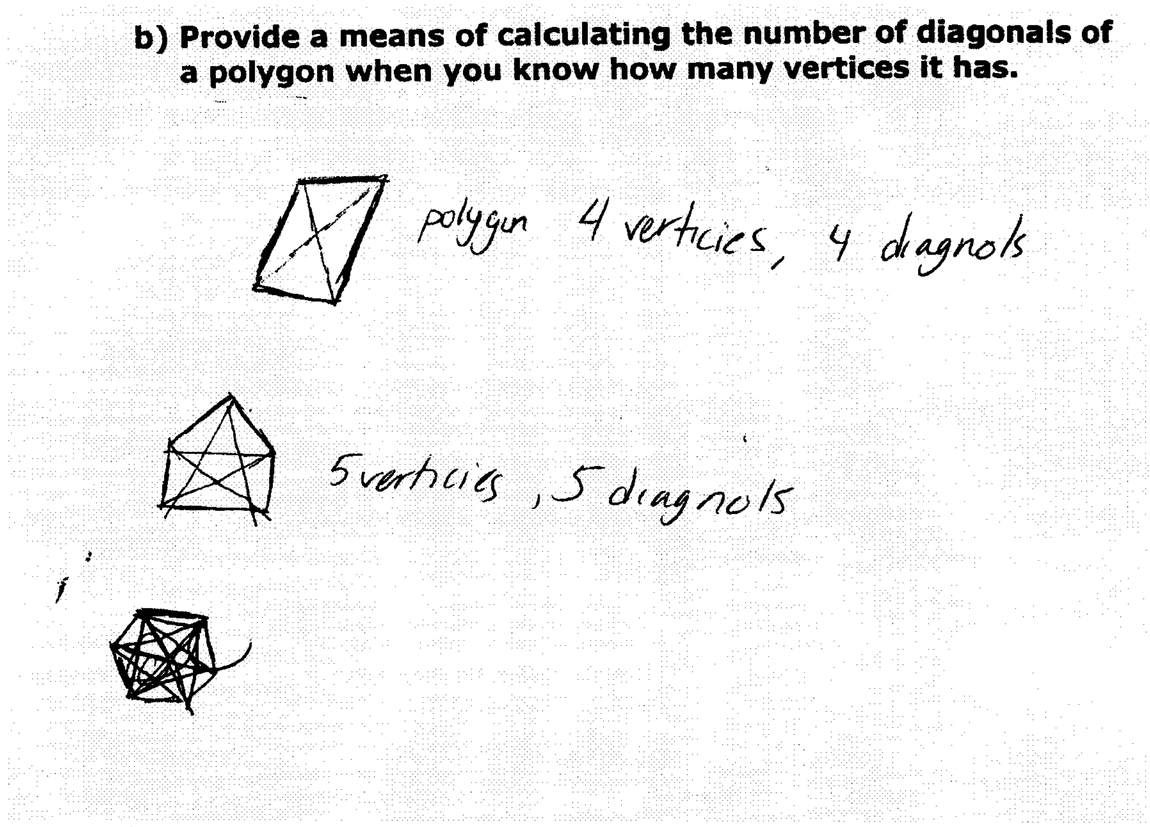


if even $\frac{n}{2}$; if odd then $\frac{n+1}{2}$ would be # vertices

Tahira explored three polygons—a square, a pentagon and a hexagon. She wrongly counted the number of diagonals for the pentagon and the hexagon; however, her incorrect counting fitted her conjecture very well since "If even *n*/2, if odd then

$n + 1/2$ would be # [of] vertices". There is another error in her conjecture. The question asks that she provide a means for calculating the number of diagonals, given the vertices. However, it seems that in her answer she provides a means for calculating the number of vertices. Her approach to proof suggests the first category in Balacheff's taxonomy as she explores a few polygons and comes up with a conjecture that she assumes to be true for all cases. I classify this as **naïve empiricism.**

Gita

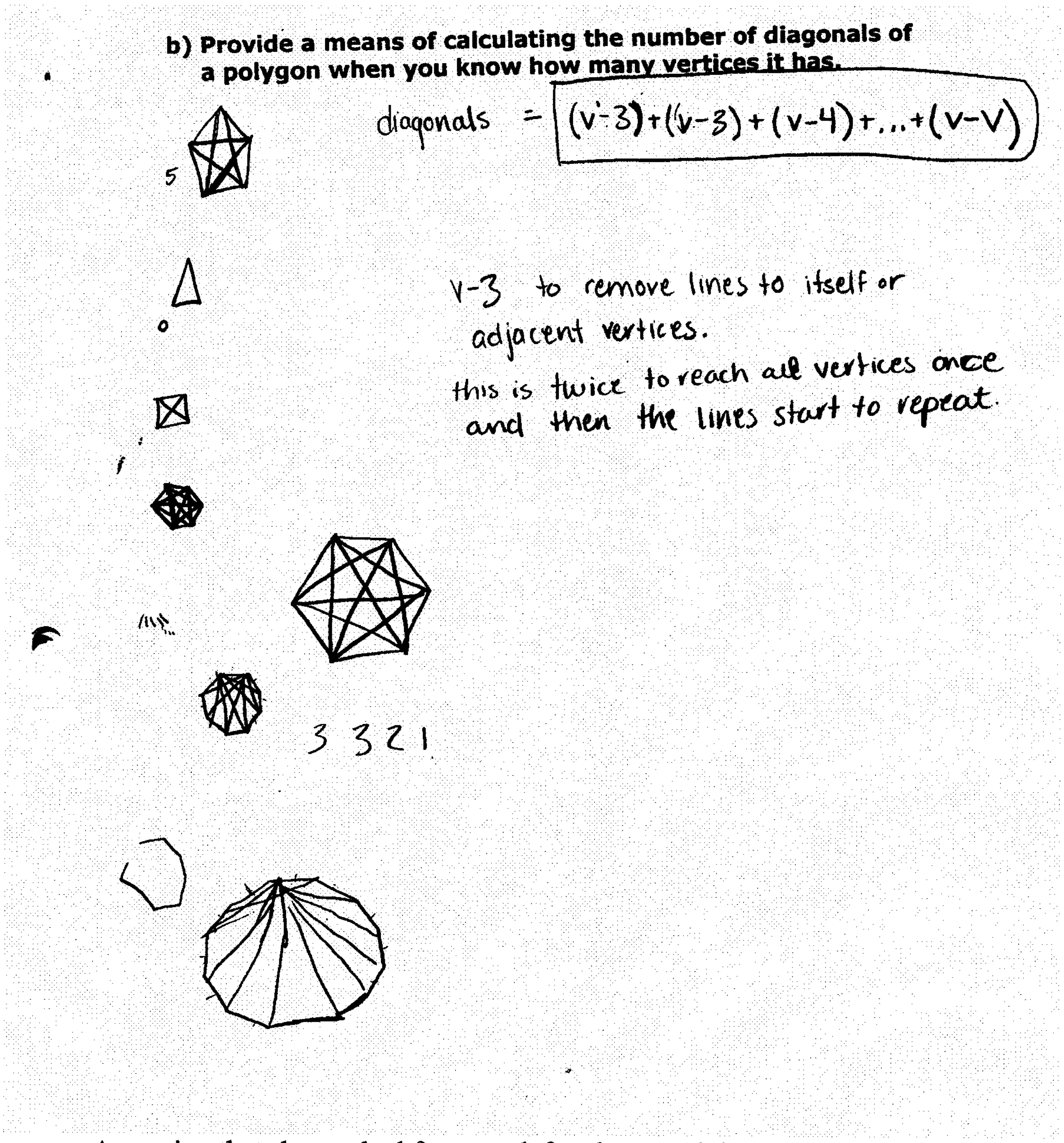


Gita explored the task using three polygons—a square, a pentagon, and a hexagon. Other than the exploratory particulars, there is not any conjecture as such. While exploring the quadrilateral, she arrived at 4 diagonals for 4 vertices. I am not sure how she arrived at 4 diagonals for a quadrilateral. Later she explored the pentagon and came up with the correct number of diagonals. However, with these two polygons, I suspect that she came up with a conjecture such that the number of vertices = number of diagonals. Again, I conjecture that she explored an additional polygon—the hexagon—when she realized that the number of diagonals is more than the sides (6) because she strikes through the diagram of the hexagon and leaves the task at that point. I see that she counted more than 6 diagonals, and when she realized this polygon does not conform to her conjecture (number of vertices = number of diagonals), she "adjusted the monster" (Lakatos, 1988) by simply ignoring it; or, possibly she "surrendered" after not being able to fit her conjecture with what she found through exploration. I classify this as **naïve empiricism.**

3. Generic Example

a. Generic Example with correct generalization

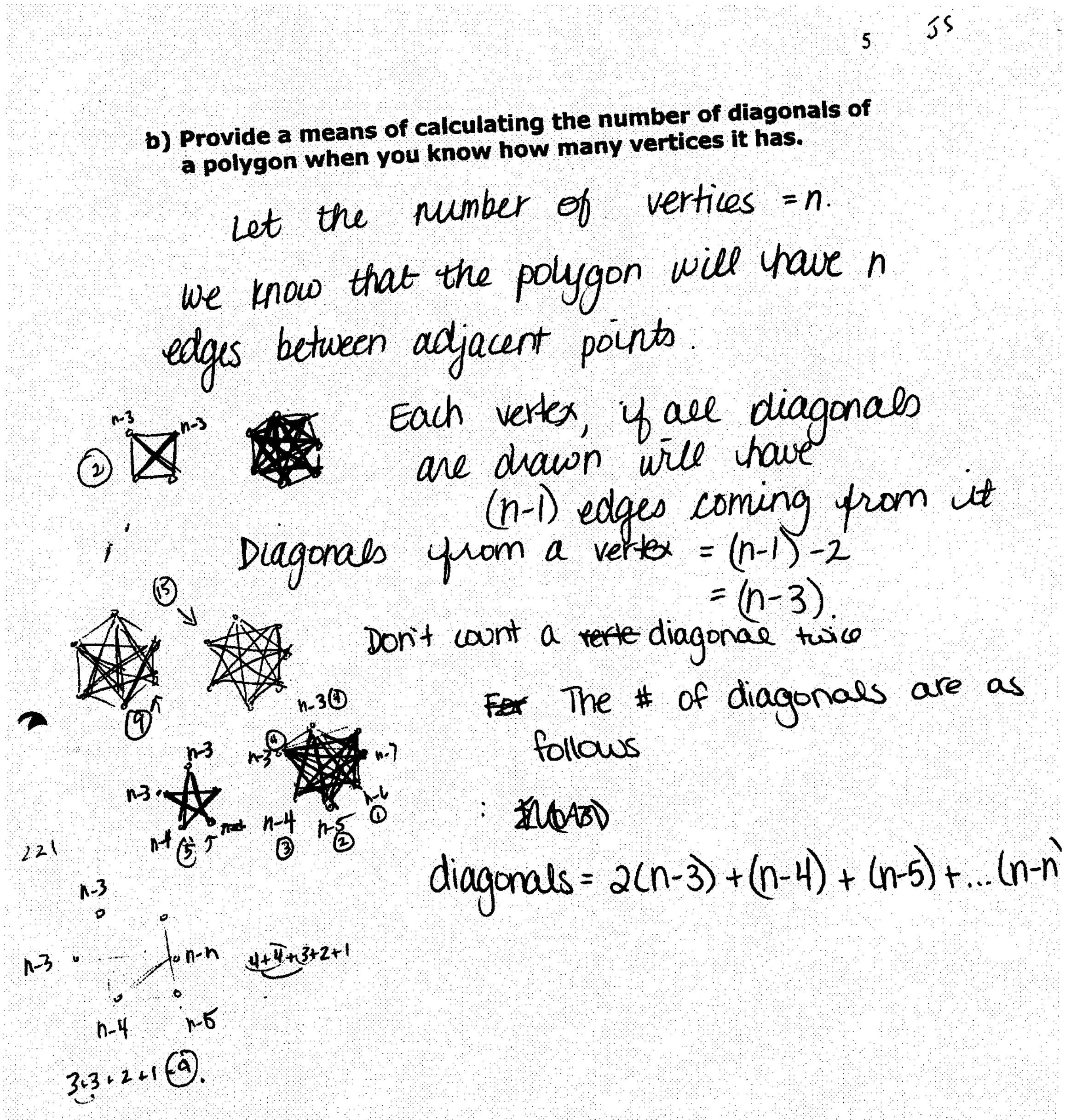
Grace



Assuming that she worked from top left to bottom right, Grace counted the number of diagonals for the pentagon and found it to be 5. She notes on the page the number of diagonals next to the pentagon figure. Then, she seems to have moved to the triangle, noting that the number of diagonals is 0. Next, she likely drew a square and counted the number of diagonals, but did not bother to note the number. Then she moved to a hexagon and counted the number of diagonals correctly. She drew a heptagon and tried to count the number of diagonals, but left it undone. At that point, she seems to have tried an "extreme" polygon with 12 sides. However, she may have realized that it is not an easy job to count the number of diagonals of a 12-sided polygon. Whatever the reason, she left it undone after drawing only a few diagonals. It is apparent that she was trying to spot a pattern that would allow her to predict further results. Generating examples, looking for regularities in the data, making and articulating conjectures are the first steps towards generalization (Rowland, 2001).

I notice that Grace also draws another hexagon, bigger in size than the other, and drawn with all the diagonals correct. This suggests that she returned to the hexagon and drew a larger diagram in order to make sense of the structure. This time she did not simply count the number of diagonals arriving at a sum of 9; instead, she wrote the structure as 3, 3, 2 1. Since there are six sides in a hexagon there will be six vertices ($v = 6$); hence $(v - 3) = (6-3) = 3$ (there are two "3s" in the formula, "$(v - 3) + (v - 3)$"). The next number in her structure is "2", which agrees with her proposition "$v - 4$" = 6-4 and so on. "3 + 3 + 2 + 1" will yield the same result as "3 + 3 + 2 + 1 + 0"; hence, the formula is $(v - 3) + (v - 3) + (v - 4) + \ldots + (v - v)$ where "$v - v$" is 0. I conjecture that she was attending to the hexagon when she wrote the formula. In other words, she uses the hexagon as a **generic example** in order to reach the general structure and the formula $(v - 3) + (v - 3) + (v - 4) + \ldots + (v - v)$. I believe I can reasonably assume that after she generated the formula for the number of diagonals, she wrote a partially complete general argument "v-3 to remove lines to itself or adjacent vertices. This is twice to reach all vertices once and then the lines start to repeat". Her argument is quite unclear to me. I suspect that she was trying to explain how she arrived at the formula.

Sara



b) Provide a means of calculating the number of diagonals of a polygon when you know how many vertices it has.

Let the number of vertices = n.

we know that the polygon will have n edges between adjacent points.

Each vertex, if all diagonals are drawn will have (n-1) edges coming from it

Diagonals from a vertex = (n-1) -2
= (n-3)

Don't count a ~~verte~~ diagonal twice

~~For~~ The # of diagonals are as follows

diagonals = 2(n-3) + (n-4) + (n-5) + ... (n-n)

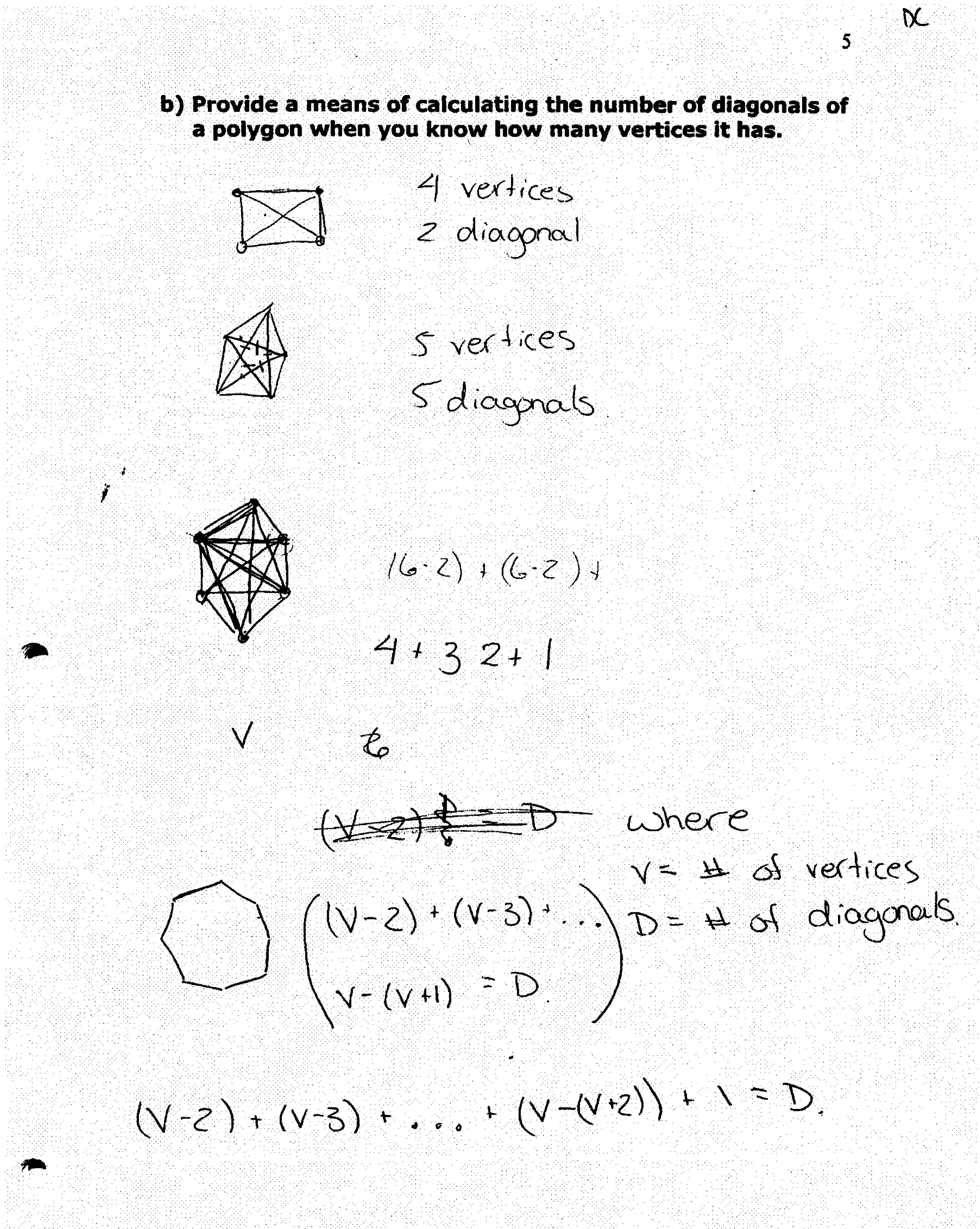
Sara clearly states in her proof that the variable "*n*" refers to the number of vertices. After defining the variable she writes: "we know that the polygon will have "*n*" edges between adjacent points". This is an incorrect statement: there can only be one edge between two adjacent vertices. A further analysis of her justification suggests that what she, in fact, meant was "if there are "n" vertices then there are "n" edges. It seems at this point, she considers drawing the number of diagonals from each vertex. She notes that "if *all* diagonals are drawn will have n -1 edges coming from it". Again, this is an incorrect statement because a diagonal is defined as "a line joining two nonadjacent vertices of a polygon". Here, what she seems to mean by the phrase "all diagonals" is all possible lines that can be drawn from a vertex.

Eventually, she came up with the correct formula for the number of diagonals that can be drawn from a vertex as "n -3". She also correctly reminds herself "not to count a diagonal twice". If she had developed this idea, she could have arrived at $n(n - 3) /2$. However, she arrives at a different formula. This indicates that her intital thought proces did not help her to arrive at the correct formula; consequently, she had to provide another means of justification. Her actual formula proved to be quite different from the formula that could have developed out of her initial thought process. Her new forumula is "diagonals = $2(n - 3) + (n - 4) + \ldots + (n - n)$".

Sara possibly arrived at this formula with the help of the polygons that she had drawn (square, hexagon, heptagon, and pentagon). I notice that she drew the hexagon twice and the heptagon thrice. I suspect that she did this not just to count the diagonals, but so as to understand the structure. She did not use one generic example, but rather several generic examples. By writing "n - 3", "n - 4" and so on at the vertices of the polygons that she explored. Her work suggests that she is searching for a structure that conforms to the number of diagonals that can be drawn from each vertex. Sara is clearly looking "generally" at the particular examples; hence, I categorize her proof under generic example. However, she uses not one, but several generic examples.

b. Generic Example with incorrect generalization

Clare



Clare 's work is similar to that of both Sara and Grace (noted above). Clare also tries to spot a pattern by using the square, the pentagon and the hexagon. I suspect she uses the hexagon both to discover the number of diagonals that can be drawn given that polygon and to visualize the structure in which the diagonals could be drawn. Notably, she incorrectly counts the number of diagonals: instead of 9 diagonals for a hexagon, she ends up with 10. I infer this from her notation alongside the structure: "4 + 3 2 + 1" (though she seems to have forgotten to put a "+" between numbers 3 and 2). She defines her variables: "v" as # of vertices and D as # of diagonals. Even though she drew an octagon, Clare left it unexplored, which suggests that, when she was writing the generalized formula, she was referring back to the structure for the hexagon"4+3 2+1". I suspect that she was attending to the hexagon, even though the formula is placed next to the octagon because the term "v-2" in her formula corresponds to the first term of 4+3 2+1 which is "4". If the number of vertices is 6, then $(6 - 2) = 4$ agrees with the first term of the structure noted alongside the hexagon. Her final formula is $(v - 2) + (v - 3) + (v - 4) + \ldots + (v - (v+1)) = D$. There is an error in this generalization. She does not realize that $(v - (v + 1))$ will not yield 1. When she recorded the generalization a second time, she decided to write "1" instead of "$(v - (v + 1))$", and rewrote the formula as $(v - 2) + (v - 3) + (v - 4) + \ldots + (v - (v + 2) + 1 = D$. However, "$(v - (v + 2))$" will not yield 2. This suggests that she got her algebra wrong when she translated her observation into a formula.

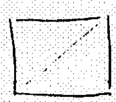
4. Thought Experiment – Verbal

a. correct verbal thought experiment

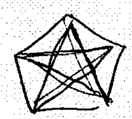
Chandelle

b) Provide a means of calculating the number of diagonals of a polygon when you know how many vertices it has.

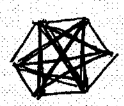
d = # of diagonals.
v = # of vertices.



d = 2
v = 4



d = 5
v = 5



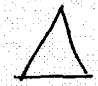
d = 9
v = 6.

- each vertex has (v − 3) diagonals.
(doesn't have diagonal with itself or two adjacent vertices)
Therefore:

$$d = \frac{v(v-3)}{2}$$

(must divide by 2 b/c a diagonal connects two vertices if you do not divide by 2 you will be counting the same diagonal twice).

ie/ for v = 3. $d = \frac{v(v-3)}{2} = \frac{3(3-3)}{2} = \frac{0}{2} = 0$ ✓



triangle has no diagonals

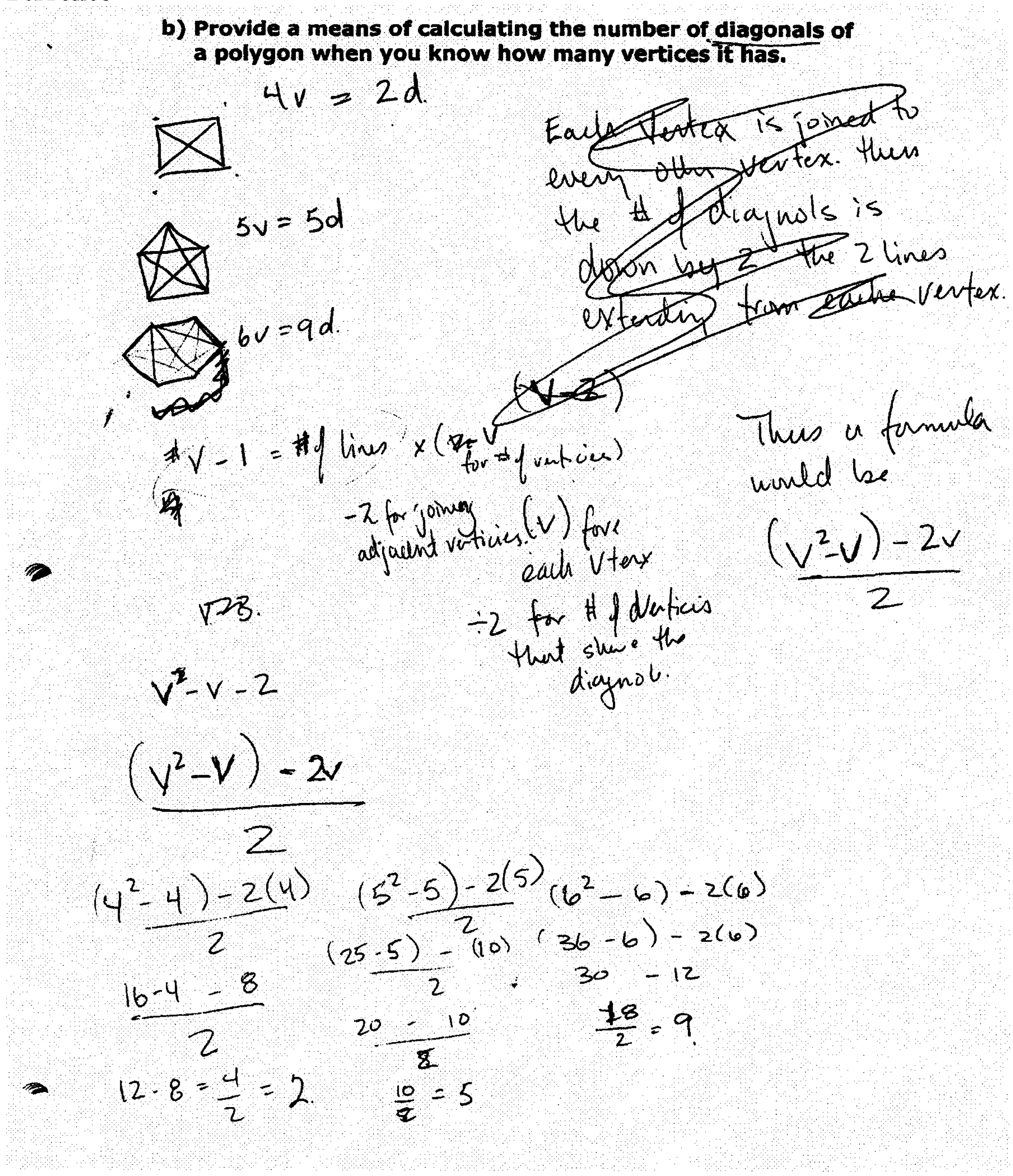
v = 4: $d = \frac{4(4-3)}{2} = \frac{4(1)}{2} = \frac{4}{2} = 2$ ✓

v = 5: $d = \frac{5(5-3)}{2} = \frac{5(2)}{2} = \frac{10}{2} = 5$ ✓

v = 6: $d = \frac{6(6-3)}{2} = \frac{6(3)}{2} = \frac{18}{2} = 9$ ✓

Chandelle tried 3 different polygons—square, pentagon and hexagon—in her efforts to make sense of the problem. She counted the vertices and the number of diagonals for each of these three polygons. Once she acquired a sense of the problem and the structure for generalization, she arrived at and justified a formula. She then verified her formula to determine whether or not she had arrived at the correct one. Since Chandelle uses the same examples that she had used earlier to explore the problem, I infer that she is now engaged in verification. Jahnke (2005) notes that some students will verify a statement, even after it has been proved, by means of examples (an observation, he claims, made also by Fishbein [1982]). I categorized this proof as a **correct verbal thought experiment** because Chandelle had developed a general explanation detached from the specifics of all her individual examples.

Terrence



b) Provide a means of calculating the number of diagonals of a polygon when you know how many vertices it has.

4v = 2d

5v = 5d

6v = 9d.

Each vertex is joined to every other vertex. thus the # of diagnols is ~~divided by 2~~ the 2 lines extending from each vertex.

v - 1 = # of lines x (v for # of vertices)

-2 for joining adjacent verticies (v) for each Vtex

÷2 for # of verticies that share the diagnol.

Thus a formula would be

$\frac{(v^2 - v) - 2v}{2}$

$v^2 - v - 2$

$\frac{(v^2 - v) - 2v}{2}$

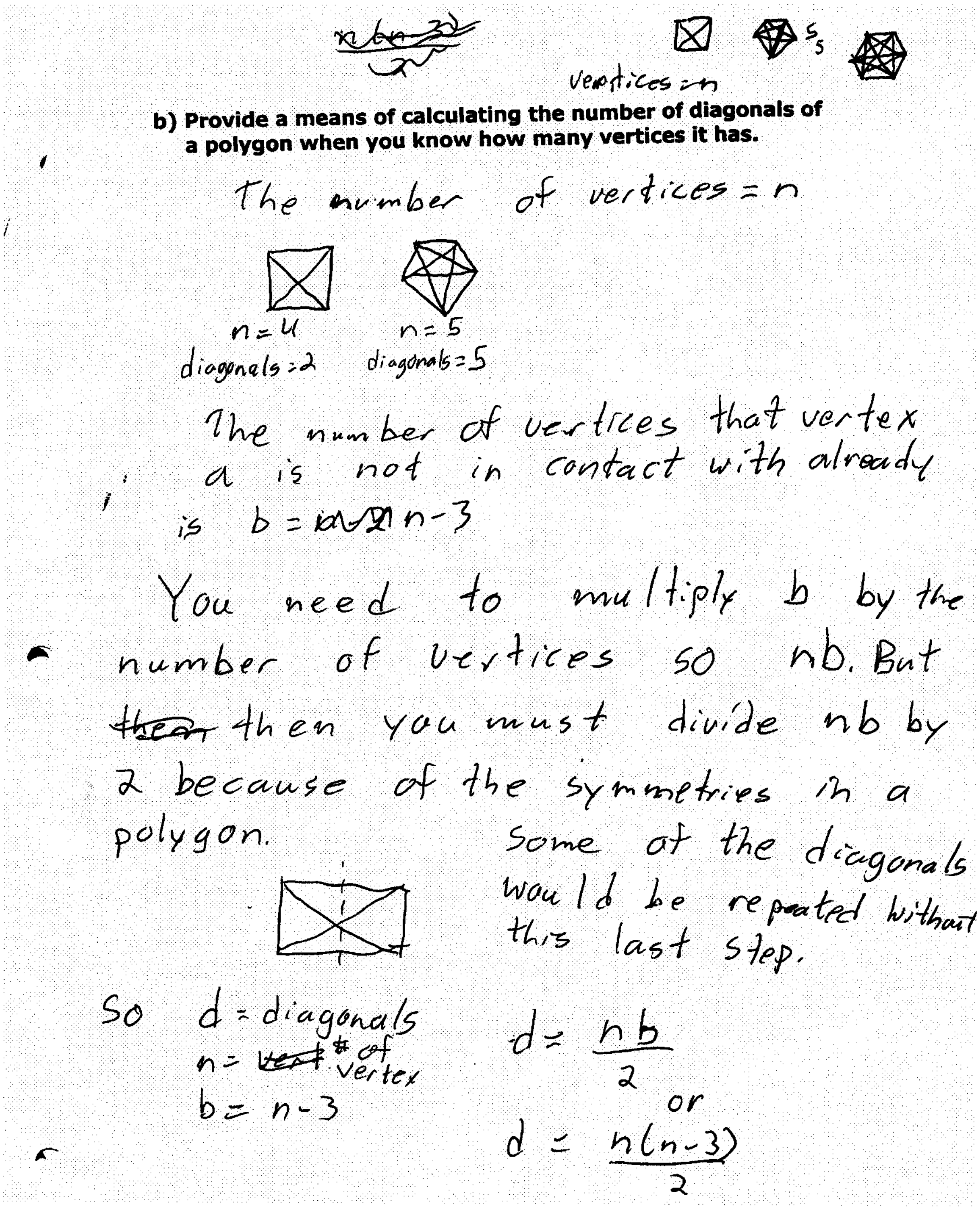
$\frac{(4^2 - 4) - 2(4)}{2}$ $\frac{16 - 4 - 8}{2}$ $12 - 8 = \frac{4}{2} = 2.$

$\frac{(5^2 - 5) - 2(5)}{2}$ $\frac{(25 - 5) - (10)}{2}$ $\frac{20 - 10}{2}$ $\frac{10}{2} = 5$

$(6^2 - 6) - 2(6)$ $(36 - 6) - 2(6)$ $30 - 12$ $\frac{18}{2} = 9.$

Terrence explored the task using the square, the pentagon and the hexagon. He examined the number of vertices and diagonals and provided a formula representing the relationship between the vertices and the diagonals: “$4v = 2d$”, “$5v = 5d$” and “$6v = 9d$”. His formula and argument, though, are not related to any of the specific examples that he provided or to the initial way in which he wrote the relationship. He notices from his drawings of the polygon that from each vertex "there are v-1 # of lines". However, it is difficult to identify which polygon he was looking at when he made that observation. He distinguishes the "number of diagonals" from the "number of lines" by subtracting 2 from v-1 (he does not use the term number of diagonals explicitly). Later he multiplies v-1-2 with v and gets $(v^2 - v - 2v)$. Finally, it seems that he divided $(v^2 - v - 2v)$ by 2 to arrive at "the # of vertices that share the diagonals". It is interesting to note that he did not simplify the formula, but left it as $(v^2 - v - 2v) / 2$. I infer from his paperwork that he then applied his formula to the square, pentagon and hexagon; hence, it seems that he engaged in a process of verification in an effort to establish the "correctness" of the formula for these additional examples. In that respect, his work is similar to Chandelle's. I categorized this proof as a **correct verbal thought experiment** because Terrence developed a general explanation detached from the specifics of any individual example.

Spencer



~~n(n-3)/2~~

vertices = n

b) Provide a means of calculating the number of diagonals of a polygon when you know how many vertices it has.

The number of vertices = n

n = 4
diagonals = 2

n = 5
diagonals = 5

The number of vertices that vertex a is not in contact with already is b = ~~n-2~~ n-3

You need to multiply b by the number of vertices so nb. But ~~then~~ then you must divide nb by 2 because of the symmetries in a polygon.

Some of the diagonals would be repeated without this last step.

So d = diagonals
n = ~~vert~~ # of vertex
b = n-3

$d = \frac{nb}{2}$

or

$d = \frac{n(n-3)}{2}$

Spencer defines the number of vertices as "*n*". He evidently explored the task by working with the square, the pentagon and the hexagon. Based upon the notations in the top corner, it appears that he worked the problem out mentally, arriving at the formula

$n(n - 3)/ 2$ and, at that point, decided to write a "formal" proof. In writing up the proof, he specifically uses two examples, the square and the pentagon. Spencer counted the vertices and the number of diagonals and noted them clearly. In the next step he tries to establish an explicit relationship between the vertices and the diagonals. He introduces two new variables, "*a*" and "*b*", though he fails to define them. He writes "The number of vertices that vertex *a* is not in contact with already is $b = n - 3$". Although the statement is awkwardly constructed, his idea is correct. I think he is trying to specify a particular vertex by using the variable *"a"* since he said "you need to multiply *b* by the number of vertices" which turns out to be *"nb"*. I do notice an error in this proof in the following statement: "But then you must divide "*nb*" by 2 because of the symmetries in a polygon. *Some* of the diagonals would be repeated without this last step". His last step is dividing the product *"nb"* by 2. He seems aware that each diagonal has two ends, so he should have referred to counting *each* (not some) diagonal twice. However, he has the idea—that one must divide the product by 2 in order to adjust for counting each diagonal twice. So the formula for the number of diagonals is $d = nb/ 2 = n (n - 3) / 2$. I categorize this as a **correct verbal thought experiment.**

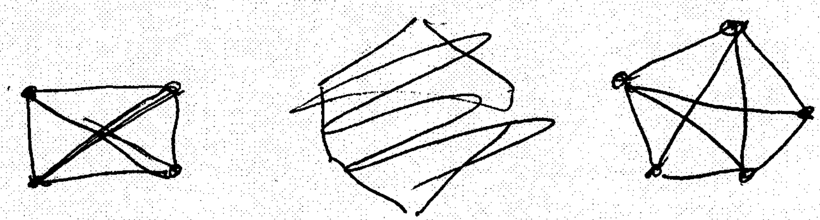
b. failed thought experiment - verbal

Brian

1 2 3 4
4 3 2 1

$\frac{n+1}{2}$ 5

b) Provide a means of calculating the number of diagonals of a polygon when you know how many vertices it has.



each vertex cannot have a diagonal to the adjacent vertex. also, for each remaining vertex, you cannot have a diagonal to a vertex that has already been consider.

~~# of diagonals (vertices~~

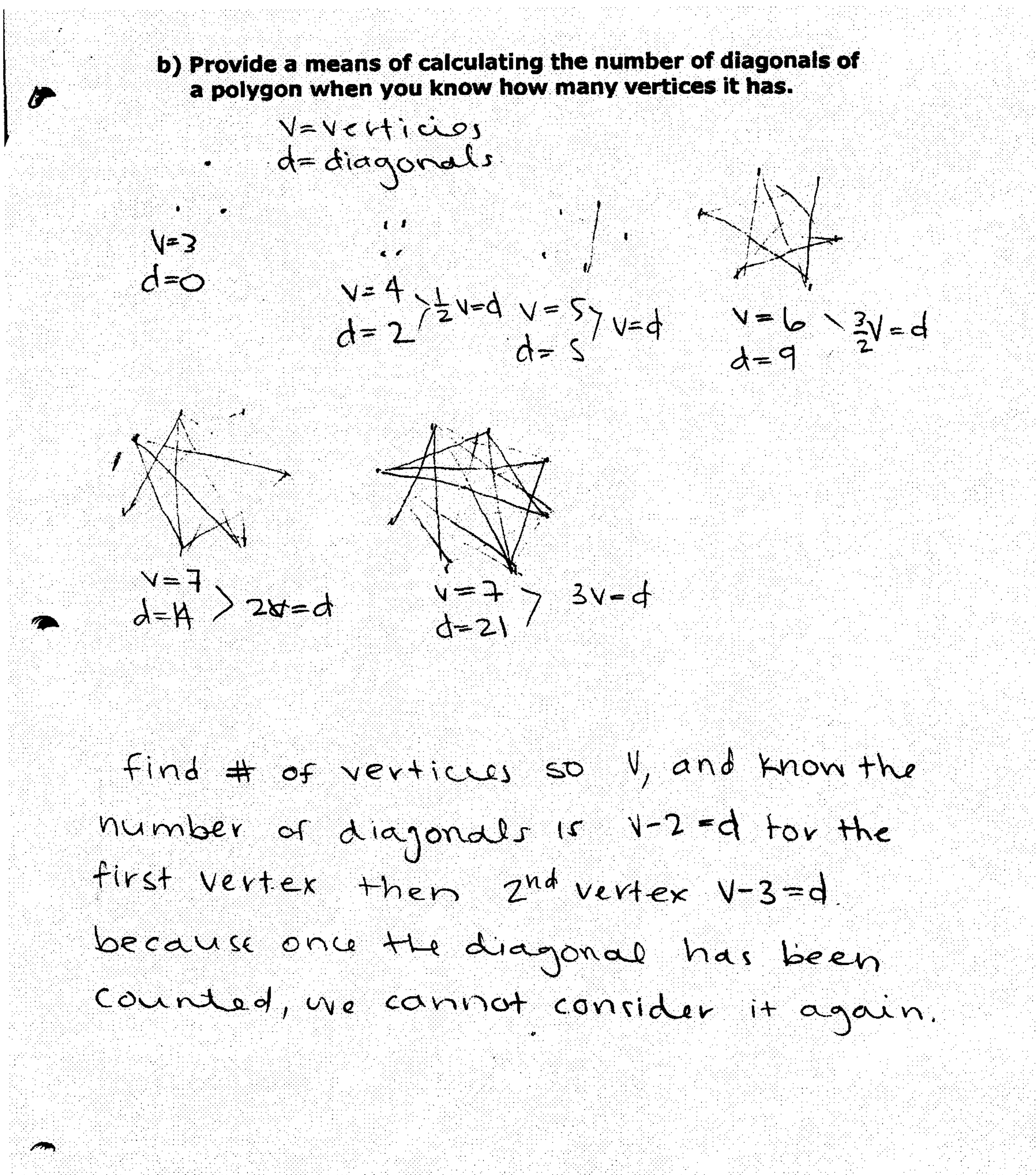
let x by the number of vertices and ~~f(x) be~~ d(x) be the number of diagonals.

$$d(x) = \sum_{i=1}^{x-2} i = \frac{x-2+1}{2}$$

$$= \frac{x-1}{2}$$

I sense Brian trying to come up with a formula for the first "n" natural numbers with 1, 2, 3 and 4. Based on Brian's scribbles above the question (writing the four numbers in reverse order below 1, 2, 3 and 4), I infer that he tried the "Gaussian" approach, but arrived at the wrong formula $(n + 1) / 2$. His formula suggests that the sum of the first "n" numbers is (first number + last number) / 2. He uses this formula to arrive at a general expression for the number of diagonals. He also explores the task with the rectangle and the pentagon. He incorrectly draws the pentagon, then strikes through his diagram (see the middle portion of the diagram). He uses his diagrams both to count the number of diagonals and to see how the diagonals are distributed from each vertex. I note that he incorrectly counts the number of diagonals for the pentagon: I see one diagonal missing. Brian then phrases his explanation as follows: "each vertex cannot have a diagonal to the adjacent vertex. Also, for each remaining vertex you cannot have a diagonal to a vertex that has already been considered". His statement demonstrates that he understands how diagonals should be counted. However, he does not go much further with this. He arrives at an expression for calculating the number of diagonals as $d(x) = \sum i = (x - 2 + 1) / 2 = (x - 1) / 2$ (i = takes values from 1 to x - 2, and x is the number of vertices); however, this expression does not really connect with what he observed about how diagonals are distributed in a polygon. Thus, I categorize Brian's effort as a **failed verbal thought experiment.**

Deanna



b) Provide a means of calculating the number of diagonals of a polygon when you know how many vertices it has.

V=verticios
d=diagonals

V=3
d=0

V=4
d=2 > $\frac{1}{2}V=d$

V=5
d=5 > V=d

V=6
d=9 > $\frac{3}{2}V=d$

V=7
d=14 > 2V=d

V=7
d=21 > 3V=d

find # of verticies so V, and know the number of diagonals is V-2=d for the first vertex then 2nd vertex V-3=d because once the diagonal has been counted, we cannot consider it again.

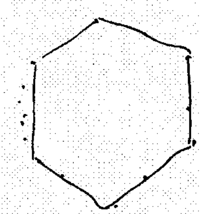
Deanna clearly defined the variables that she proposed to use, "*v*" for vertices and "*d*" for diagonals. Deanna's exploration pattern is similar to that of both George (pp. 152-154) and Brandon (pp. 155-156). She did not draw polygons but only the polygon's vertices. She explored the number of diagonals for the triangle, square, pentagon, hexagon, heptagon and octagon. She expresses the relation between the diagonals and vertices with the following "ratio pattern":

For triangle – $0v = d$
Square – $½v = d$
Pentagon – $v = d$
Hexagon – $3/2v = d$
Heptagon – $2v = d$
Octagon – $3v = d$. I notice an error in this last step, this should have been $2.5v = d$.

The last item in the pattern is obviously a mistake: she incorrectly counted the number of vertices and diagonals for the octagon as 7 and 21, respectively. However, her incorrect counting resulted in her conjecture $3v = d$. Also of interest, there is a very clear pattern here: 0, ½, 1, 3/2, 2, . . . I wonder why the inconsistency in this pattern did not strike the student teacher when she wrote $3v = d$ in the last step. However, in arriving at her justification she failed to use this pattern. It is interesting to note that the argument she provides is detached from the pattern that she developed using the diagrams. Deanna eventually came up with a general argument. If I elaborate upon her argument (which is "the number of diagonals are "$v - 2$" for the first vertex, "$v - 3$" for the second vertex and so on…"), I would conjecture that her formula is: $d = (v - 2) + (v - 3) +$ and so on.

Cathy

b) Provide a means of calculating the number of diagonals of a polygon when you know how many vertices it has.



Given n vertizes, there are already 2 edges connecting each vertice to another (sides of the polygon). The joint of each vertize to a non connected vertize which remains will give a diagonal, so $(n-2)$ is the number of diagonals.

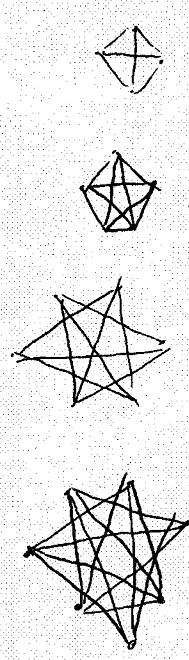
Cathy examined a particular polygon—a hexagon—in order to organize her proof. Even though she used the hexagon to get a sense of the problem, she presented a general argument, one detached from the specifics of the polygon that she observed. Since her argument is detached from the polygon she observed and she arrives at an incorrect formula, I categorize this as a **failed verbal thought experiment.**

5. Thought Experiment - Symbolic

a. correct symbolic thought experiment

George

b) Provide a means of calculating the number of diagonals of a polygon when you know how many vertices it has.



$V = 3$, $d = 0$) +2   0

$V = 4$, $d = 2$ $(4-3)+(4-3)$ × 0

) +3 $(5-3)+(5-3)+1$ × .5

$V = 5$, $d = 5$

) +4 $(6-3)+(6-3)+3$ × 1

$V = 6$, $d = 9$

) +5 $(7-3)+(7-3)+6$ × 1.5

$V = 7$, $d = 14$

General relationship

$d_n = d_{n-1} + n - 2$

initial case 4

$d_4 = 2$

$d_4 = \frac{4}{2}(4-3) = 2 \cdot 1 = 2$

$d_{n+1} = d_n + (n+1) - 2$
$= d_n + n - 1$
$= \frac{n}{2}(n-3) + n - 1$
$= \frac{n^2 - 3n}{2} + n - 1$
$= \frac{n^2 - 3n + 2n - 2}{2} = \frac{n^2 - n - 2}{2} = \frac{(n+1)^2 - 3(n+1)}{2}$ □

and $2(n-3) + \frac{n-4}{2}(n-3)$
$= \left(2 + \frac{n-4}{2}\right)(n-3)$
$= \frac{n}{2}(n-3) = \frac{n^2 - 3n}{2}$

George made sense of the problem with the help of a triangle, square, pentagon, hexagon and heptagon. Even though he fails to specify his variables, it is evident that "*v*" stands for the vertices and "*d*" for the diagonals. The triangle does not have a diagonal; hence, $v = 3$ and $d = 0$. Regarding the square, $v = 4$ and $d = 2$. The curved arrow pointing downwards and the +2 next to it indicate the difference in the number of diagonals between the triangle above and the square below. This pattern repeats with each new figure as the mathematical work proceeds down the page. When he arrives at the pentagon, he represents the number of vertices as $v = 5$ and the number of diagonals as $d = 5$. The difference in the number of diagonals between the square and the pentagon is 3; hence, he wrote +3. In the case of the hexagon, $v = 6$ and $d = 9$. The difference in the number of diagonals between the pentagon and hexagon is 4; hence, he wrote +4. In the case of the heptagon, $v = 7$ and $d = 14$. The difference in the number of diagonals between the hexagon and heptagon is 5; hence, he wrote +5.

At the right hand side he writes "0", "0.5", "1", and "1.5" in an effort to establish a relation between vertices and diagonals. This is, in fact, the ratio format, and is similar to Deanna's work (pp. 149-150). The pattern is provided below:

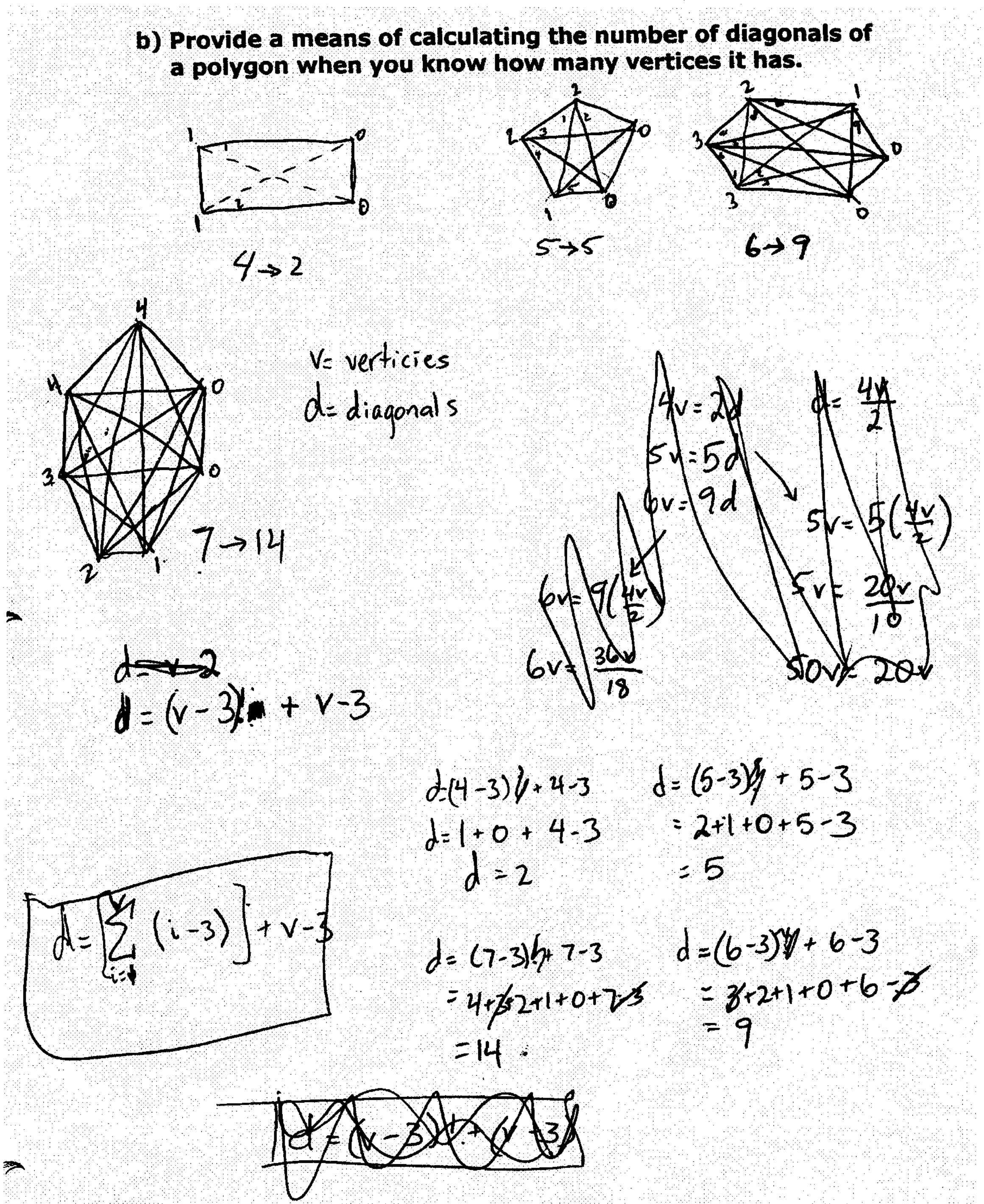
For triangle – $0v = d$
Square – $\frac{1}{2} v = d$
Pentagon – $v = d$
Hexagon – $3/2\, v = d$
Heptagon – $2v = d$

George did not go any further with the ratio relation between "vertices" and "diagonals". He could not arrive at a generalized formula with the ratio pattern, so he introduced a recursive relation $d_n = d_{n-1} + n - 2$. This relation is identical to that noted by Balacheff as $f(n - 1) + n - 2$. It is interesting that George used "*v*" and "*d*" all the while, and then suddenly switched to "*d*" and "*n*". The way in which he spotted the pattern and the formula that he later developed from the pattern both represent sophisticated thinking. George put into play a number of different ideas in arriving at the general formula. At that point, it appears that he left that particular formula and, on the left side of the page, derived a formula for d_{n+1}. His derivation of $d_{n+1} = [(n + 1)^2 - 3(n + 1)] / 2$ is, in fact, correct and does yield the correct number of diagonals, for we substitute (n -1) for "*n*". I found it quite interesting that George made great efforts to derive d_{n+1}, but then did not simplify this complicated expression. If simplified, the expression yields $(n + 1)(n - 2)/ 2$.

On the right side of the recursive formula, I also see $n/2\ (n\text{-}3) = n^2\text{-}3n / 2$, which is the correct formula for calculating the number of diagonals. He seems to have developed this as another expression for the number of diagonals. I conjecture that $2(n\text{-}3)$ comes from what he initially observed (as noted in the upper right side)—that is, (as (4-3) +(4-3), , (5-3) (5-3) and so on). The ratios 0, 0.5, 1, and 1.5 are transformed into and expressed as $(n\text{-}4) / 2$ and the increment in the number of diagonals that occurs each time is $(n\text{-}3)$. Thus, his formula is $2(n\text{-}3) + (n\text{-}4)/2 * (n\text{-}3)$, which will yield $n/2\ (n\text{-}3) = n^2\text{-}3n / 2$ when simplified. I consider this high level thought, indeed. George generates a mathematical proof based on induction and uses examples to demonstrate how the number of diagonals increases as the number of sides increase. His expressions demonstrate high-level mathematical thinking. Hence, I categorize this as fully **correct mathematical proof.** I also notice the hollow black square in this proof also.

b. partially correct symbolic thought experiment

Brandon



Brandon did not provide a general verbal explanation for the proof, but he did generate a general formula that is very mathematical because of its symbolism:

$d = [\sum(i - 3)] + v - 3$. This formula would have been the correct generalization of the pattern he observed if "*i*" in the formula could take values from 4, instead of 1. He correctly noted the relation between the number of vertices and the number of diagonals. In exploring the task, he used 4 polygons: the rectangle, the pentagon, the hexagon, and the heptagon. He noted the relation between the vertices "*v*" and diagonals "*d*" for these different polygons as $4v = d$; $5v = 5d$; $6v = 9d$.

Using these three equations, he tried to generate several other equations in an effort to find one general equation that could give the number of diagonals from the number of vertices. However, he could not arrive at a successful equation in terms of the ratios, so he scribbled them out. He seems then to have provided another formula based on factorial notation: $d = (v - 3)! + v - 3$. I also see him testing this formula for each of the four different polygons in which he counted diagonals. Even though he used the factorial notation, what he seems to have had in mind is the summation because he eventually strikes through the factorial notation as he worked out the examples. This realization—that it is the sum of the numbers and not the product—might have led him later to change the factorial notation to summation. When he used factorial notation he must have had in mind the need to add; if not, why would he have taken the factorial notation off the formula? Interestingly, at the bottom of the page, he also strikes through another formula with factorial notation. Evidently, Brandon was playing with a number of mathematical ideas. I categorize his work as **partially correct symbolic thought experiment.**

Summary

Not all participants attempted Tasks # 3, and # 4; but, here, with Task #2, as in the case of Task # 1 all of the student teachers attempted the task. Four of 17 students provided the correct formula alone while a further six of 17 student teachers produced correct proofs. Of the 13 participants who showed their work, five used pragmatic justifications and eight used conceptual justifications. Interestingly, a number of student teachers used an explanatory way of proving this task. Task # 2 proved to be a successful exploration task since most of the students explored and made sense of it by using different polygons. Furthermore, most of the proofs were of explanatory nature, rich in natural language. It is generally assumed that once students realize that a task is geometrical, they will opt for a two-column format; my data suggests that this is a false assumption.

CHAPTER 10. RESULTS AND DATA ANALYSIS: TASK # 3.

Task # 3.

Prove that the sum of the exterior angles of a polygon is always 360^0.

(Selected from memory; one of the unit-test tasks that I gave my Form 5 students [Grade 11])

Exemplars

1. Approach: Naïve Empiricism

Consider an equilateral triangle. The interior angles are 60^0 each. The exterior angle at each vertex is 120^0 each. Therefore, the sum of the exterior angles of a triangle is $120^0 + 120^0 + 120^0 = 360^0$

Consider a square. The interior angles are 90^0 each. The exterior angle at each vertex is also 90^0 each. The total of the exterior angle is $90^0 + 90^0 + 90^0 + 90^0 = 360^0$.

Since both these work, the sum of the exterior angles of a polygon is always 360^0

2. Approach: Crucial Experiment

Consider an equilateral triangle. The interior angles are 60^0 each. The exterior angle at each vertex is 120^0. Therefore, the sum of the exterior angles of a triangle is $120^0 + 120^0 + 120^0 = 360^0$

Consider a square. The interior angles are 90^0 each. The exterior angle at each vertex is also 90^0 each. The sum of the exterior angle is $90^0 + 90^0 + 90^0 + 90^0 = 360^0$.

I will try one more polygon, a regular hexagon. If I get the same answer with this example, I can conclude that the answer will always be the same. The interior angles are

120^0 each. The exterior angle at each vertex is 60^0. Therefore, the sum of the exterior angles of a triangle is $60^0 + 60^0 + 60^0 + 60^0 + 60^0 + 60^0 = 360^0$

Since all three cases worked, the sum of the exterior angles of a polygon is always 360^0.

3. Approach: Generic Example

Consider a triangle. Let "*a*", "*b*", and "*c*" represent the interior angles of a triangle: $a + b + c = 180^0$. The the exterior angles at each vertex will be $(180 - a)^0$, $(180 - b)^0$, $(180 - c)^0$, respectively, as the interior and exterior angles are supplementary at each vertex. The sum of the exterior angles is $(180 - a)^0 + (180 - b)^0 + (180 - c)^0 = 540 - (a + b + c)^0$. But $(a + b + c)^0 = 180^0$; therefore, the sum of the exterior angles = $540^0 - 180^0 = 360^0$.

OR

At each vertex, there is an interior angle and an exterior angle. The interior angle is always supplementary to an exterior angle at that vertex. Since there are 3 sides, the total of interior and exterior angles is $180*3 = 540^0$. The sum of the interior angles in a triangle is 180^0. Therefore, the sum of the exterior angles is 360^0.

OR

In a regular pentagon, the interior is made up of five triangles. The total of all the angles in 5 triangles is $180*5 = 900^0$. The sum of the angles of the triangles includes all the angles where the common vertex is a point in the interior of the polygon. The angles add up to 360^0 as they encircle a point. So the sum of the interior angles of a pentagon is $900-360 = 540^0$.

Since there are 5 interior angles there will be 5 exterior angles. The interior and exterior angle form a linear pair. Hence, the total of all the interior and exterior angles is

$180*5 = 900^0$. The sum of interior angles alone is 540^0. Therefore, the sum of all the exterior angles = $900^0 - 540^0 = 360^0$ and, hence, proved.

NOTE: All of the above arguments are partially general within one particular class of figures (namely triangles in the first two cases and pentagons in the third case).

4. Approach: Thought Experiment

Consider a convex polygon of "*n*" sides. A convex polygon of "*n*" sides can be divided into "*n*" triangles. The sum of the angles in a triangle is 180^0. Therefore, the sum of the angles of "*n*" triangles in the polygon is $180n^0$.

The sum of the angles of the triangles includes all the angles where the common vertex is a point in the interior of the polygon. The angles add up to 360^0 as they encircle a point. So the sum of the interior angles of a polygon is $180n^0$-360^0.

An exterior angle of a polygon is the angle formed by the side of a polygon and an extended adjacent side. The exterior angle and the corresponding interior angle together form a linear pair; hence, the sum of all the interior angles with all the exterior angles is $180n$. (At each vertex, there is an interior angle and an exterior angle. The interior angle is always supplementary to an exterior angle at that vertex).

There are "*n*" vertices as the polygon is of "*n*" sides. Hence, the total of all the interior and exterior angles is $180n^0$.

Sum of interior angles + Sum of exterior angles $= 180n$

Sum of interior angles = $180n^0 - 360^0$.

Hence, the sum of all the exterior angles is $180n^0$-$(180n^0$-$360^0) = 360^0$. And so it is *proved* that the sum of the exterior angles of a polygon is always 360^0.

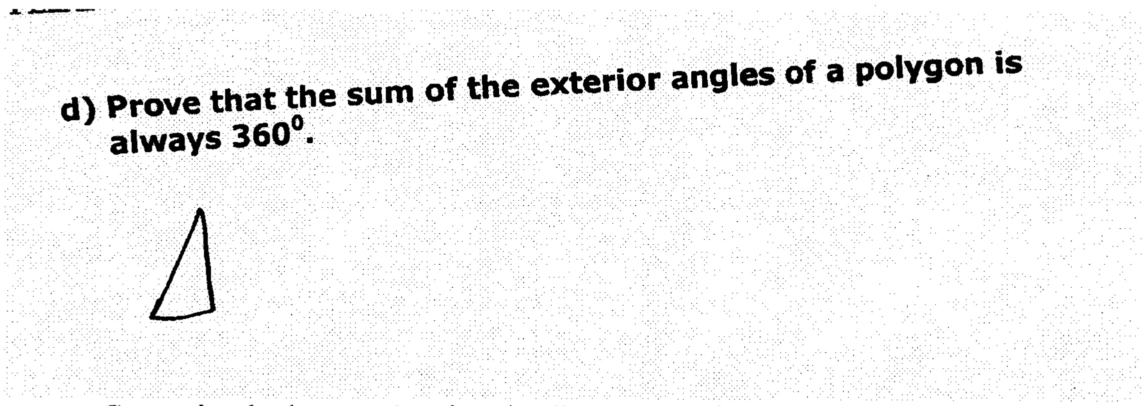
Analysis

Table 10. 1. Observations of student work Task # 3.

Categories	**Correct**	**Incorrect**	**Almost Correct**	**Total**
Diagram Only	1	5	0	6
Generic Example	0	1	0	1
Thought Experiment- Verbal	1	2	0	3
Thought Experiment-Symbolic	0	2	2	4
Total	**2**	**10**	**2**	**14**

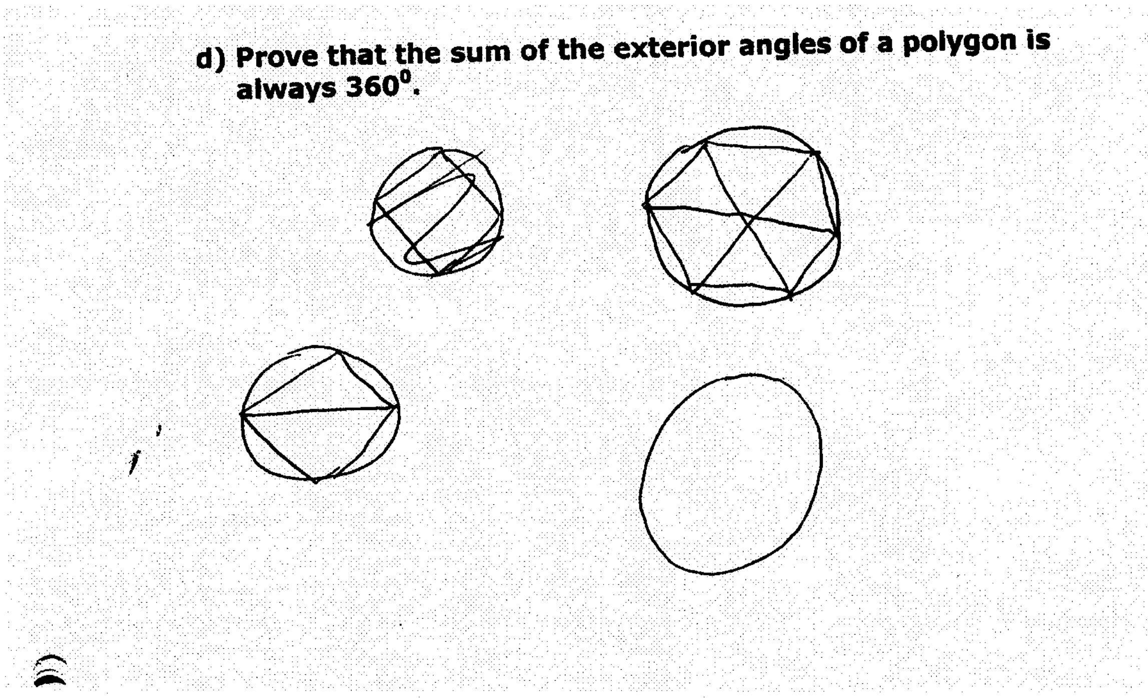
1. Diagram Only

Grace



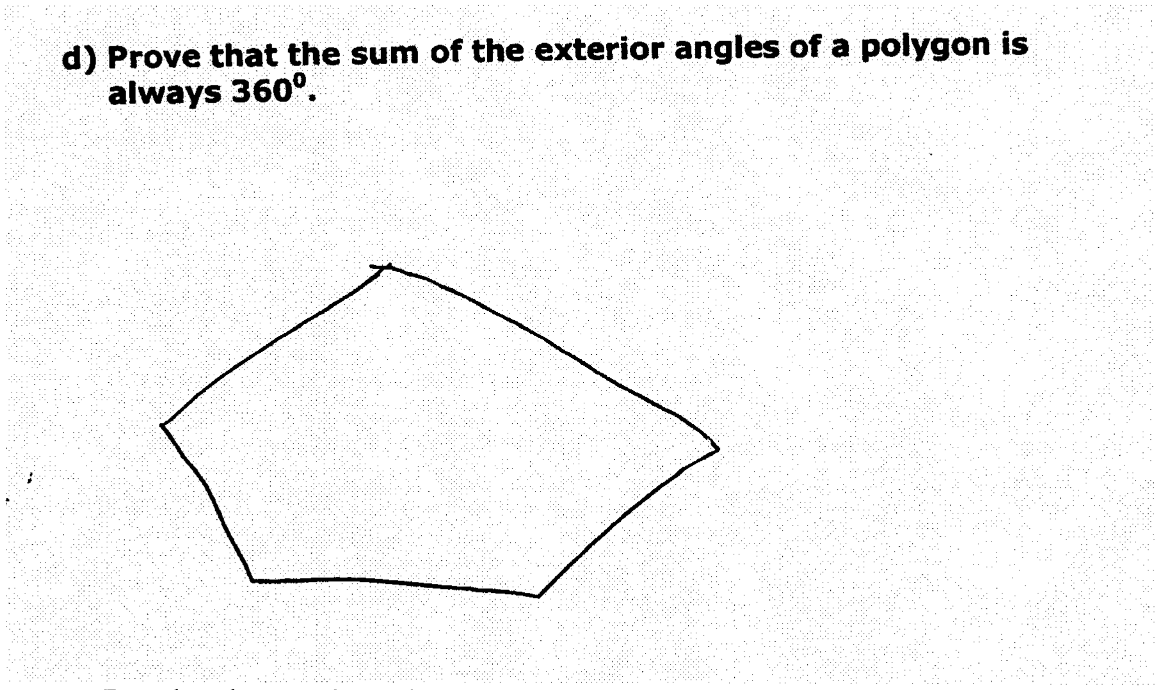
Grace simply drew a triangle. The diagram itself provides little information. However, one may assume that she drew the smallest polygon in order to get a sense of the problem.

Gita



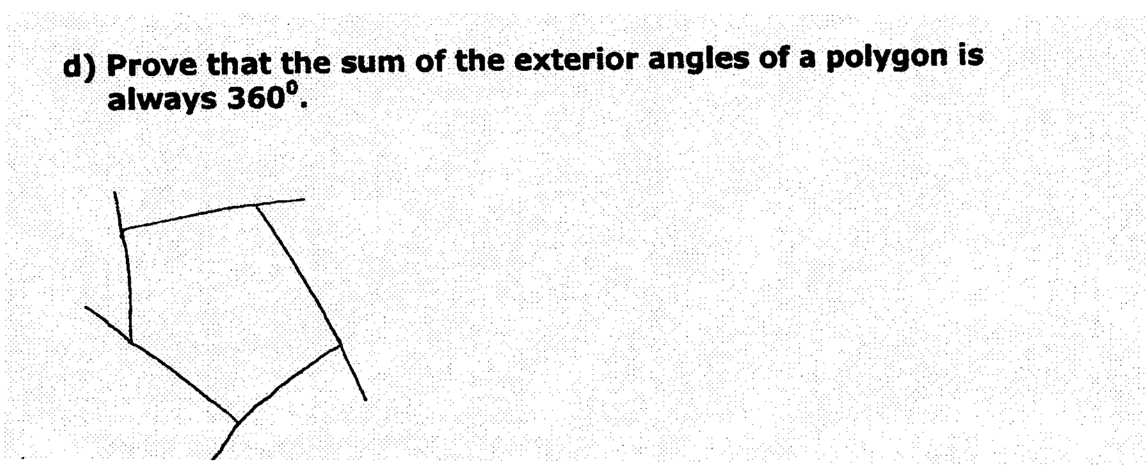
Gita drew four different circles, three of which are inscribed polygons (that is, a polygon placed inside a circle so that each vertex of the polygon touches the circle). The diagram provides little information about how she was planning to prove this task; however, it seems that she was trying to relate polygons and the circle.

Brandon



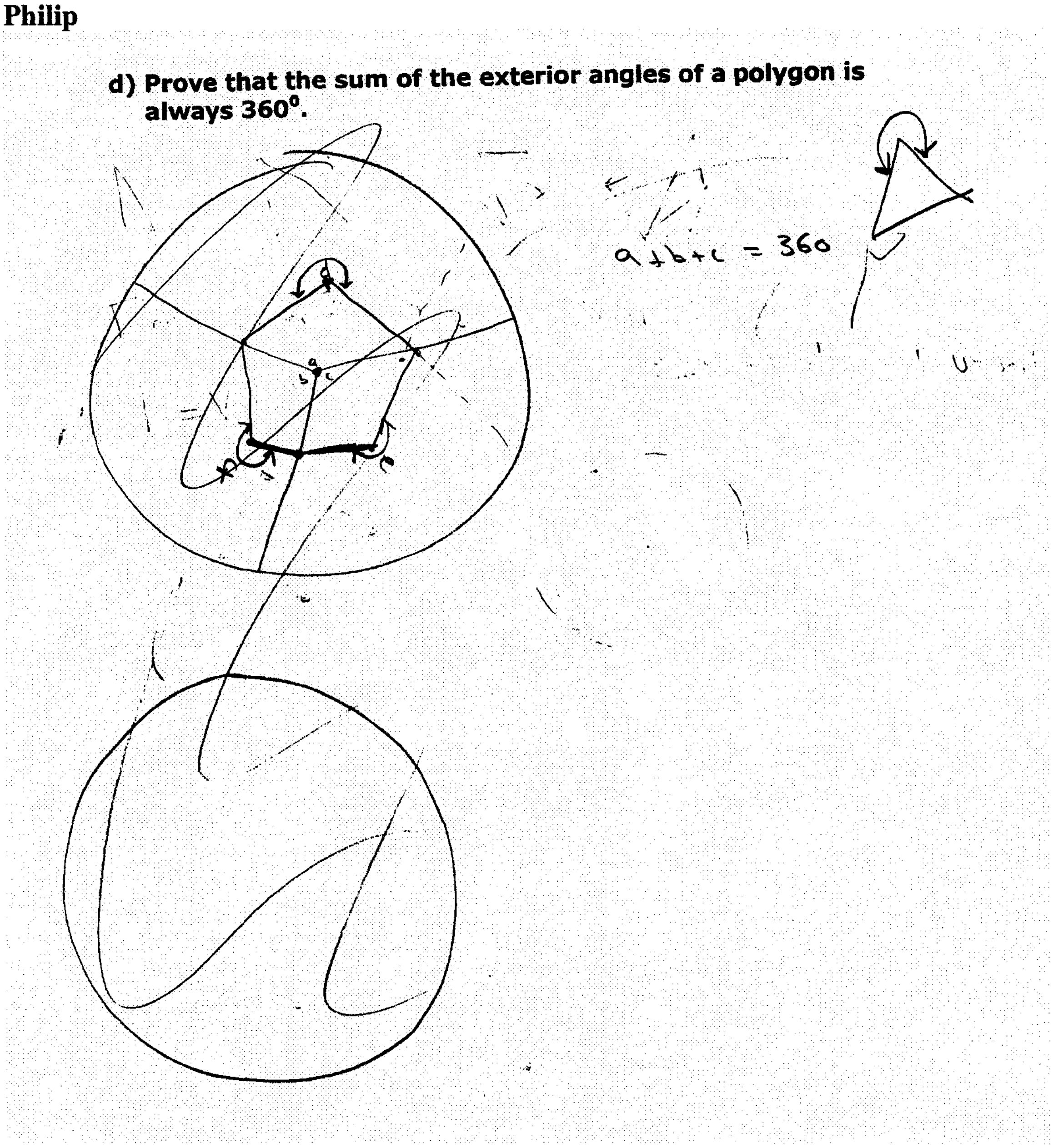
Brandon drew an irregular pentagon. He could not go further. His diagram reveals little information.

Clare



Clare's diagram provides more information than those above. It indicates that she knows what an exterior angle of a polygon is. Clare could go no further.

Philip

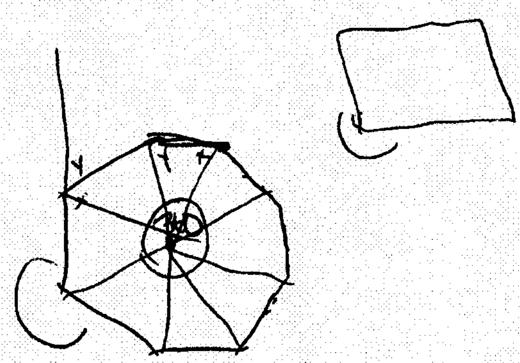


Here, I see three different diagrams. Two of them are circles. In one of the circles I see a pentagon. This diagram indicates that the student brought in the following theorem: the sum of the angles at a center point is 360^0. The two diagrams—the circle with the pentagon at the center and the triangle—indicate that Philip defines an exterior angle as a "360 – interior angle"; in other words, a reflex angle. We form the exterior angle for a polygon by extending one side of the polygon from one of its endpoints.

Exterior and interior angles together form a linear pair of angles. The work produced by others shows that many of them share Philip's understanding/definition of an exterior angle.

Cathy

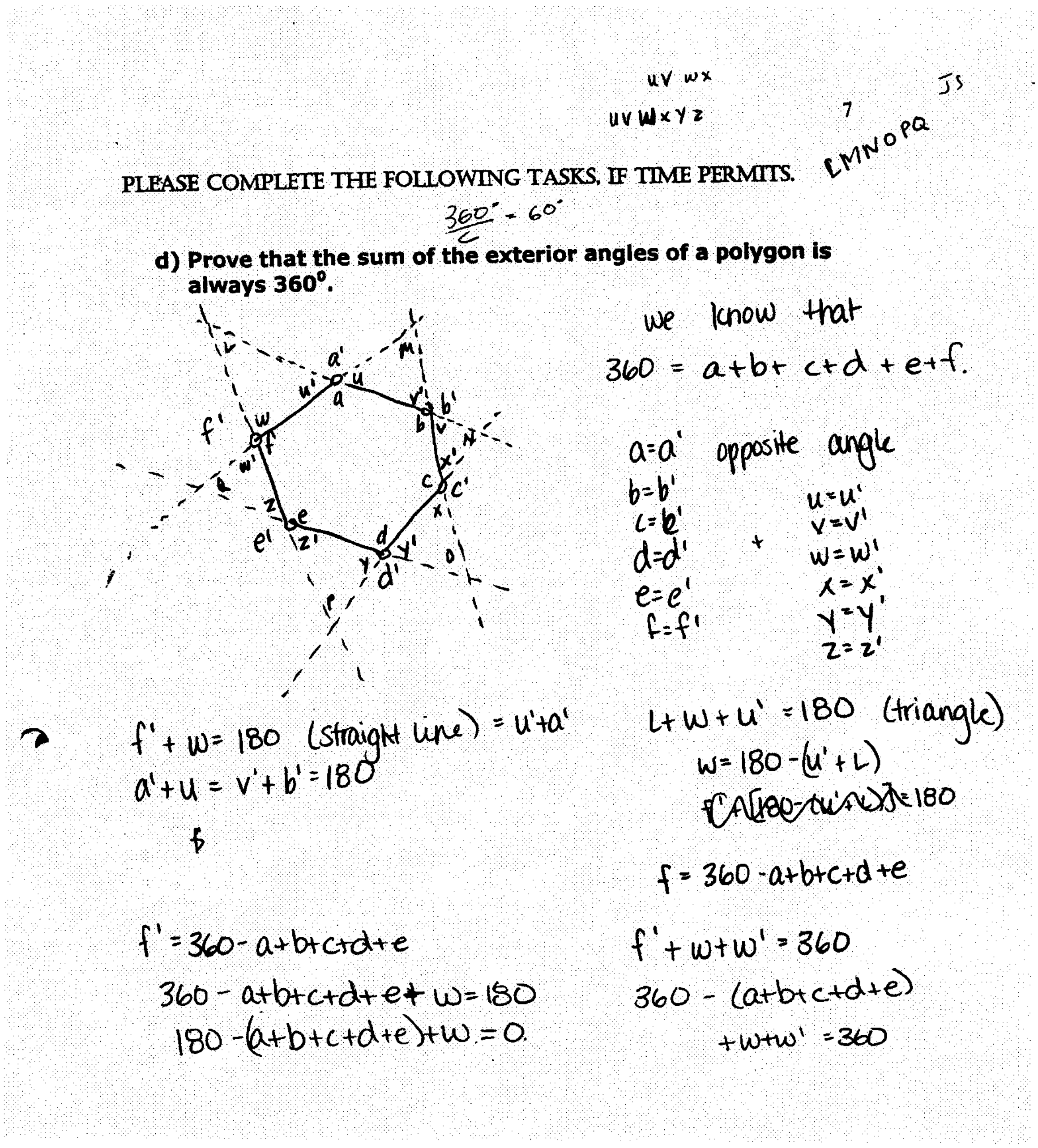
d) Prove that the sum of the exterior angles of a polygon is always 360°.



Cathy's diagram indicates that she wanted to work with an irregular octagon. She divided the octagon and different triangles. The common vertex of all the triangles inside the octagon is marked and 360 is noted there. This indicates that she was trying to find the sum of the interior angles. It also seems to me that she is not sure what an exterior angle is, even though her definition of exterior angle is similar to that of the student teacher noted above. I notice an angle marked as *y* that is an extension of one of the sides of a polygon. This concept is correct. However, in another diagram, a quadrilateral, she marks another angle as the exterior angle when that angle is really "360^0- interior angle". She could go no further with these drawings. The two diagrams in which she marks the exterior angle differently suggest that she was unclear about what precisely the term exterior angle means.

2. Generic Example

Sara



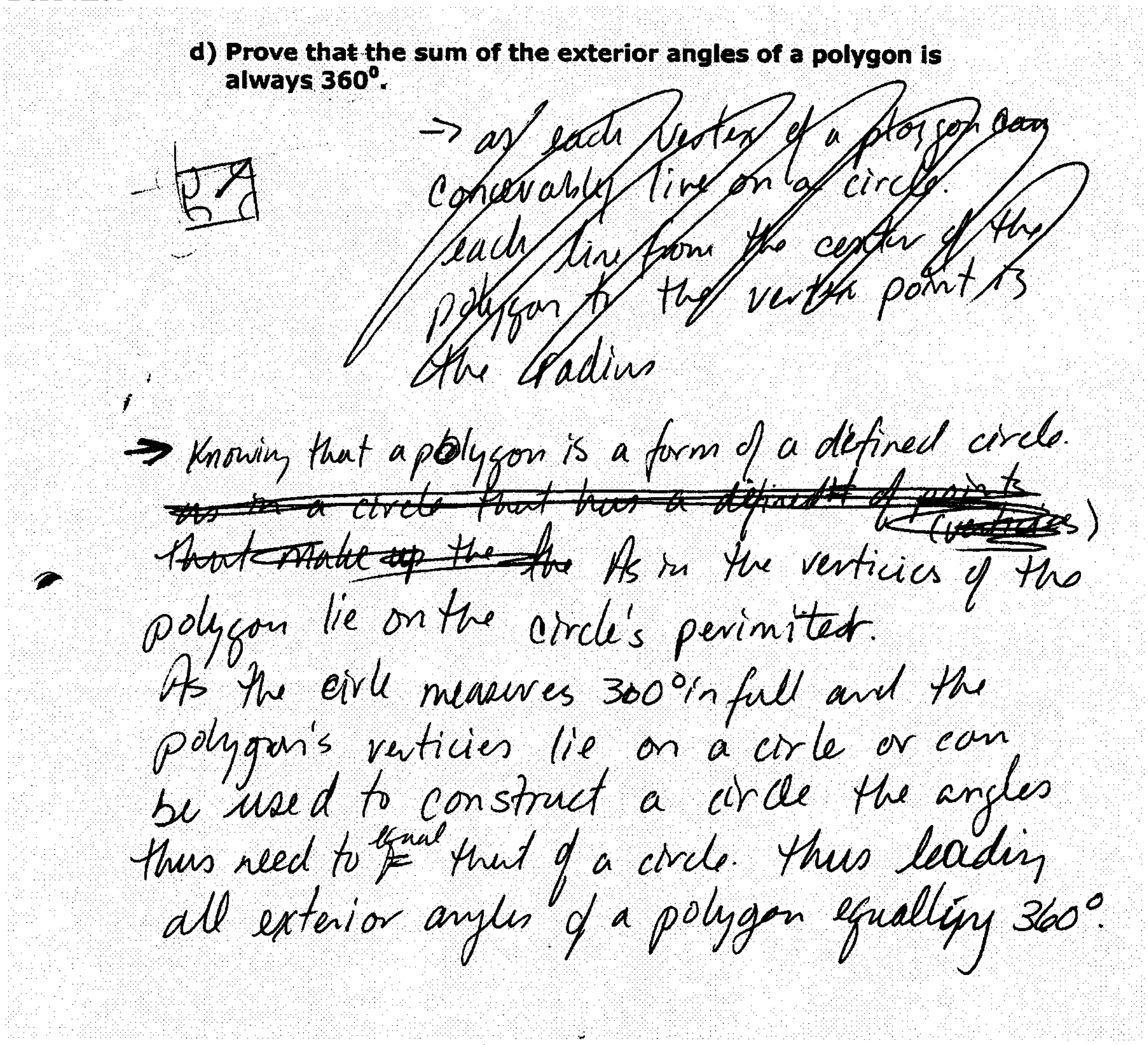
She starts off the proof by saying: we know that "360 = *a* + *b* + *c* + *d* + *e* + *f*". This is an incorrect assumption. In her diagram, *a*, *b*, *c*, *d*, *e* and f are interior angles of a hexagon. Hence, she clearly assumes that the sum of the interior angles of a hexagon is 360. Even though she divided 360 by 6, arriving at 60, suggesting that it would be a regular hexagon, and even though her drawing shows the hexagon as a regular hexagon, Sara used six different letters to show the angles. The other results that she brings in are a) vertically opposing angles are equal, b) adjacent angles, when added, will total 180, and c) the angle sum of a triangle is 180. Although the proof looks very ritualistic in terms of its mathematical manipulations, the student has not provided a meaningful response.

I categorize this as **failed generic example.** The students' perception of the case is important at the level of generic example: students should see the particular example as a representative of all such cases (Knuth & Elliot, 1998). The written work alone may not lead one to conclude that this particular case is generic example, but it does seem that here the student is bringing in several theorems and trying to see the *general* in the particular example. In other words, she is using this hexagon as a generic example

3. Thought Experiment-Verbal

a. failed verbal thought experiment

Terrence


d) Prove that the sum of the exterior angles of a polygon is always 360°.

→ as each vertex of a polygon can conceivably live on a circle. each line from the center of the polygon to the vertex point is the radius

→ Knowing that a polygon is a form of a defined circle. as in a circle that has a defined # of points (vertices) that make up the the As in the verticies of the polygon lie on the circle's perimiter.

As the circle measures 360° in full and the polygon's verticies lie on a circle or can be used to construct a circle the angles thus need to equal that of a circle. thus leading all exterior angles of a polygon equalling 360°.

Terrence's proof pertaining to this task is based on an incorrect argument. He argues the following: "As the circle measures 360 in full and the polygon's vertices lie on a circle's perimiter [sic] As the circle measures 360 in full or can be used to construct a circle the angle thus need to be equal [to] that of a circle. Thus leading all exterior angles of a polygon equaling 360". Since this is a general argument, I categorize the attempt under **failed verbal thought experiment.**

Daniel

d) Prove that the sum of the exterior angles of a polygon is always 360^0.

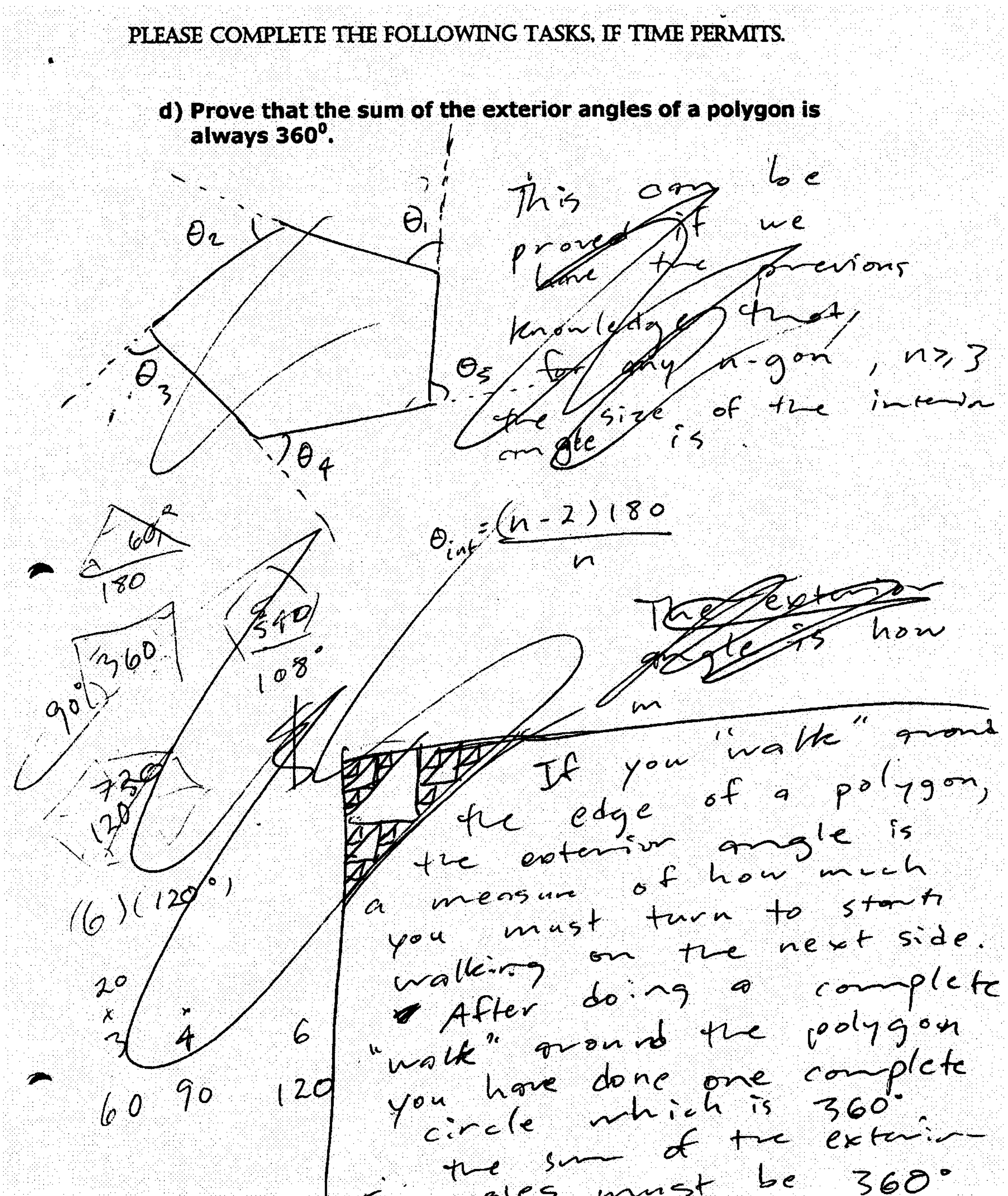
Circle contains 360°

Since polygons are closed, the sum must be 360

Daniel provides an incorrect justification for his proof. He argues that for all closed figures the sum is 360. He also connects his justification to a circle, presumably because he associates the term, 360^0, with the circle. I categorize this as a **failed verbal thought experiment**.

b. correct verbal thought experiment

Brian


PLEASE COMPLETE THE FOLLOWING TASKS, IF TIME PERMITS.

d) Prove that the sum of the exterior angles of a polygon is always 360°.

This can be proved if we have the previous knowledge that, for any n-gon, n≥3 the size of the interior angle is

$\theta_{int} = \frac{(n-2)180}{n}$

The exterior angle is how m

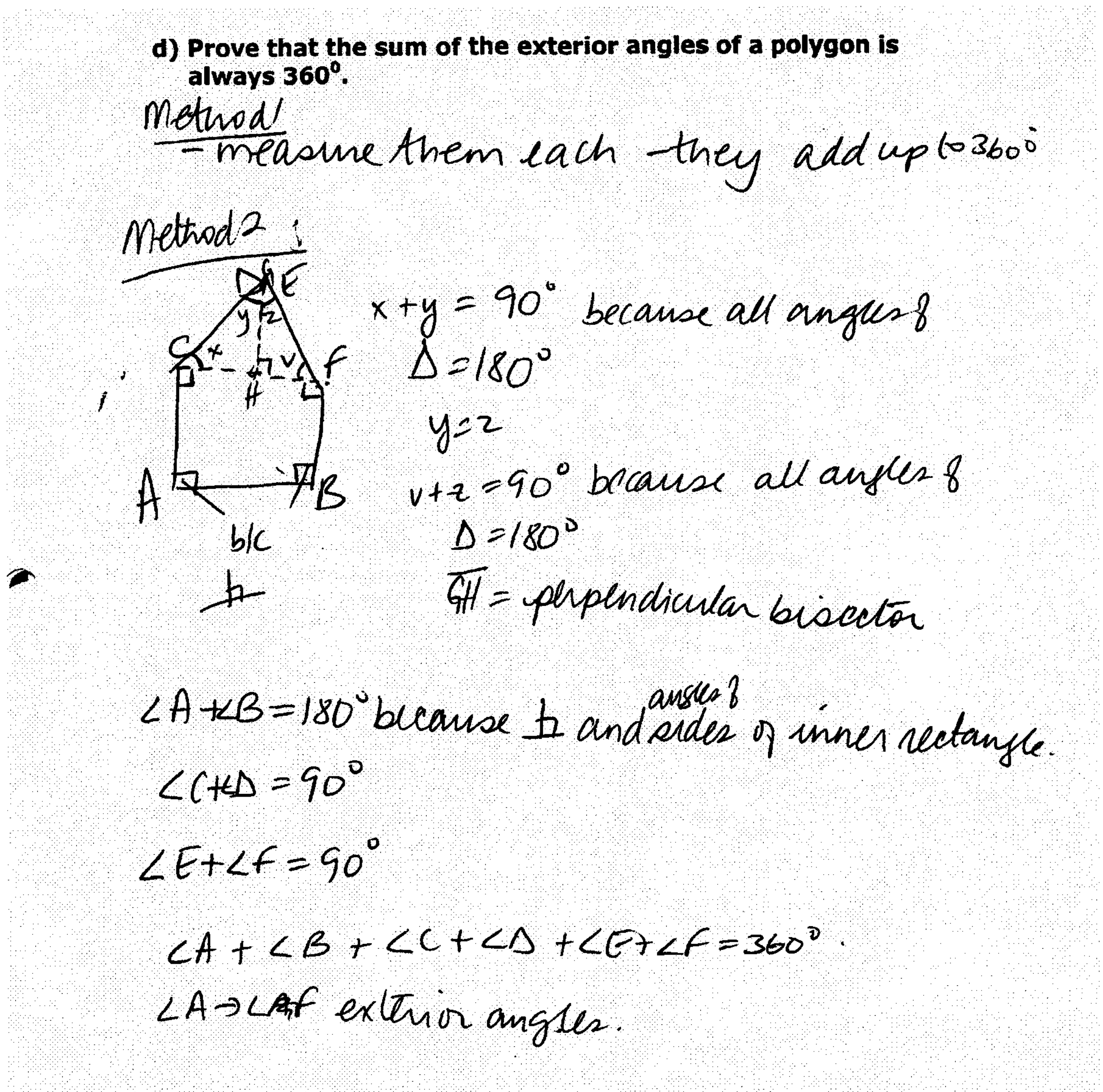
If you "walk" around the edge of a polygon, the exterior angle is a measure of how much you must turn to start you walking on the next side. After doing a complete "walk" around the polygon you have done one complete circle which is 360°. ∴ the sum of the exterior angles must be 360°

It is evident from his diagram that Brian understands the concept of an exterior angle. His scribbles indicate that he wanted to prove the task mathematically, rather than by general argument. It also seems that he tried to prove this task for a regular polygon since he produced drawings for the equilateral triangle, rectangle, regular pentagon, regular hexagon, and so on. He also provided a general formula for the size of an interior angle within a regular polygon. However, he got "stuck" and could not proceed. At that point, he likely resorted to the general argument. I categorize his proof as a **correct verbal thought experiment.**

4. Thought Experiment – Symbolic

a. failed symbolic thought experiment

Tahira



d) Prove that the sum of the exterior angles of a polygon is always 360°.

Method 1
– measure them each – they add up to 360°

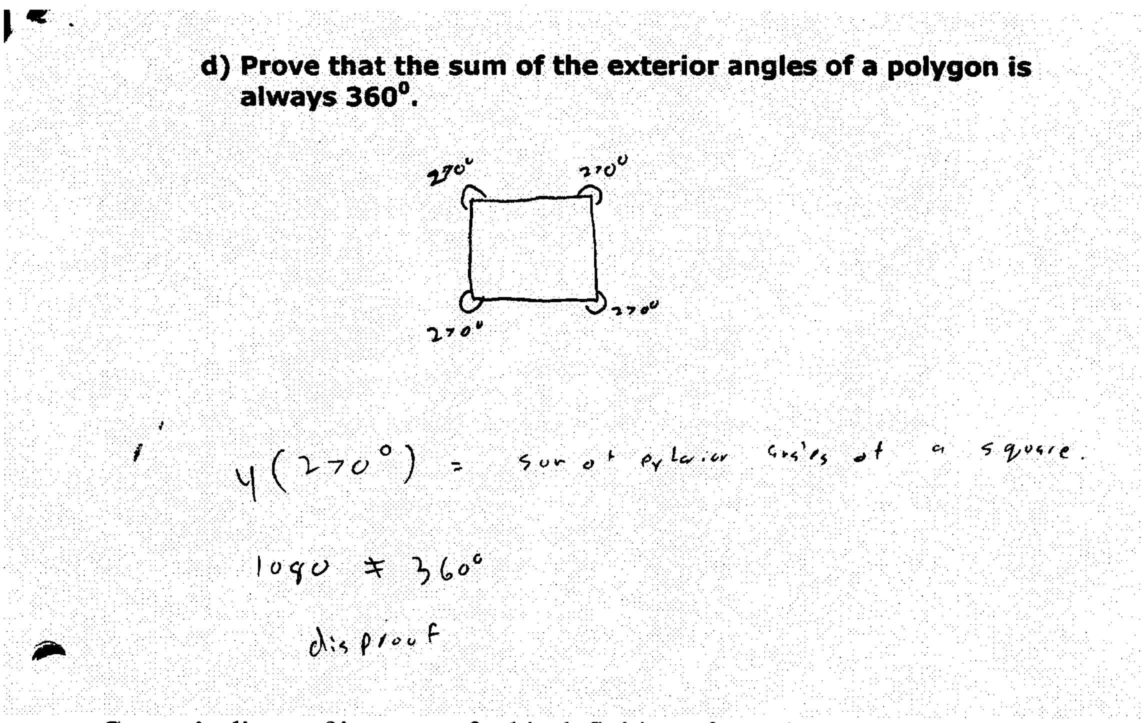
Method 2:

x + y = 90° because all angles of Δ = 180°
y = z
v + z = 90° because all angles of Δ = 180°
GH = perpendicular bisector

∠A + ∠B = 180° because ⊥ and angles of sides of inner rectangle.
∠C + ∠D = 90°
∠E + ∠F = 90°
∠A + ∠B + ∠C + ∠D + ∠E + ∠F = 360°.
∠A → ∠F exterior angles.

Tahira provides two methods of proof. In her first proof, she suggests to "measure them" to prove that the sum of the exterior angles is always 360. Recall that Chazan (1993) claimed students are more convinced by examples (three are more than enough) than by formal proofs. Tahira's second proof looks formal with traces of the two-column format. I recognize, as well, statement and reason in each step. She uses a very particular polygon for her proof. This is a pentagon with an isosceles right triangle placed on the top of a rectangle. She appears to have introduced two different theorems into her proof: a) the interior angles of a square are 90^0 each and b) the sum of the angles of a triangle is 180^0. Even though the task directed the student to develop the proof for exterior angles, Tahira seems to work on the interior angles. Based on the assumptions she brings to the task and the formal look of the proof, I categorize this proof as a **failed symbolic thought experiment.**

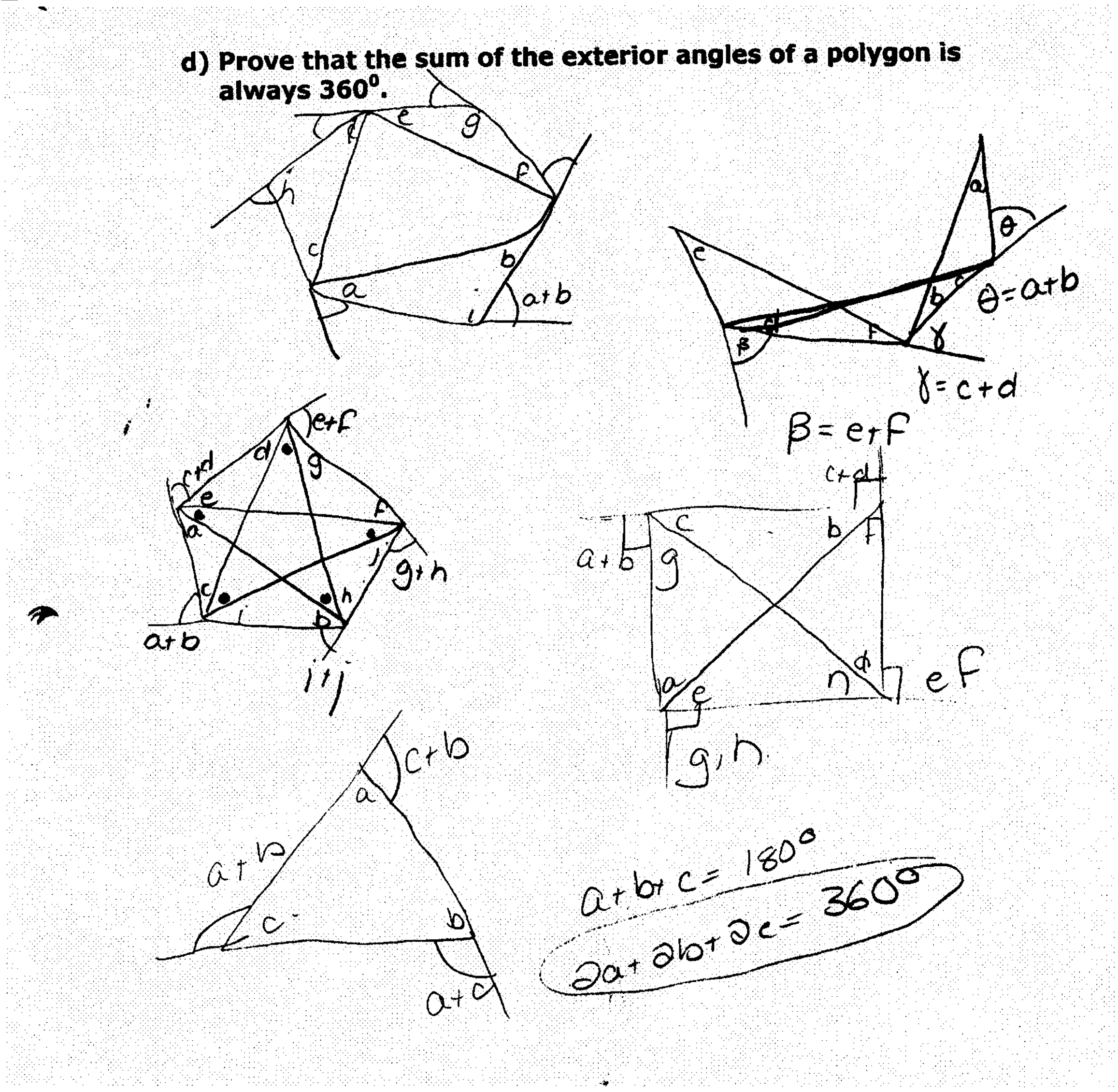
George


d) Prove that the sum of the exterior angles of a polygon is always 360⁰.

George's disproof is correct for his definition of exterior angle. He is well aware that one single example (counter example) is enough to establish that something is false. However, because his definition of an exterior angle is wrong, I categorize this as a **failed symbolic thought experiment.**

b. almost correct symbolic thought experiment

Beth



Beth tried several polygons to prove this task. Her diagrams indicate that she is aware of the assumptions that need to be considered in order to prove this task. She is also one of a small group of participants who could correctly define an exterior angle. I infer from the placement of her work upon the page that she drew the hexagon first (I see that on the top left hand side of the page, and since westerners are taught to write from left to right and top to bottom this seems a logical supposition). Her diagram suggests that she tried to divide the polygons into triangles so as to prove the theorem with the help of triangles. However, this attempt does not succeed in the case of the hexagon, so she moves on to the pentagon and, then, on to the square, though still with no success. In other words, she decreases the number of sides of a polygon until she finds a polygon for which she can prove her strategy. Then she resorts to a triangle and proves the theorem beautifully for that. However, her proof for the triangle does not prove the task for *any polygon.* Beth beautifully uses two results in proving the theorem: a) the sum of the angles in a triangle is 180^0 and b) the exterior angle of a triangle is equal to the sum of the interior angles at the other two vertices. Her proof is correct for any triangle; but a proof for a triangle does not prove the task for *any polygon.* Hence, I categorize her effort as an **almost correct symbolic thought experiment.**

John

d) Prove that the sum of the exterior angles of a polygon is always 360^0.

let n = number of vertices
number of vertices = number of interior angles = number of exterior angl
The sum of the interior angles of a polygon
is 360 (n - 2)
Interior and exterior angles are supplementary
so sum of extrior and interior = 180 n
So 360(n-2) - 180 n = sum of exterior
360n - 720 - 180n

John's proof format is correct. He applied correct assumptions, but unfortunately could not remember the formula for the sum of the interior angles of a polygon. He considered the sum of the interior angles of a polygon of "*n*" sides to be 360 (*n*-2), rather than 180 (n-2). Also of interest, he left the proof incomplete: rather than writing it as 180n-720, he left the sum of the exterior angles as 360n-720-180n. At this point I think it interesting to compare this proof with that completed by George. George persevered, and eventually disproved the task (see p. 176) while John left it incomplete. This suggests that George felt quite confident about his definition of exterior angle (which is, in fact, incorrect) while John, perhaps, felt insecure about his formula for the sum of the interior angles of a triangle. I suspect that if John had the same confidence that George seems to have had in the result that he brought in to prove the task, John would not have left it incomplete. I categorize this as **almost correct symbolic thought experiment.**

Summary

Of the 14 student teachers who attempted Task # 3, only four correctly recalled the concepts (or results) that were needed in order to prove this task correctly. One of those four students was able to provide a fully correct proof, and that was in the form of a general argument. There were, however, two almost correct proofs. Of these, Beth's proof would have been fully correct if the question had referred exclusively to triangles rather than to polygons. I conclude from this analysis that the student teachers had great difficulty proving this task.

CHAPTER 11. RESULTS AND ANALYSIS: TASK # 4.

Task # 4.

How do you know whether there exists a two-digit number *ab* such that the difference between *ab* and *ba* is a prime number?

(Taken from Harding [1999])

Exemplars

1. Approach: Naïve Empiricism

Consider a two-digit number, 13. The reverse is 31.

The difference is 31 - 13 = 18. 18 is not a prime number.

Consider another two-digit number, 25. The reverse is 52.

The difference is 52 - 25 = 27. 27 is not a prime number.

Since both examples worked, the difference between *ab* and *ba* is never a prime number.

2. Approach: Crucial Experiment

Consider a two-digit number, 13. The reverse is 31.

The difference is 31 – 13 = 18. 18 is not a prime number.

Consider another two-digit number, 25. The reverse is 52.

The difference is 52-25 = 27. 27 is not a prime number.

Since, in these two examples, the difference between the number and its reverse is not a prime number, I will try one more example. If the answer is not a prime number here, either, it will never be prime.

Consider another two-digit number, 19. The reverse is 91.

The difference is 91 - 19 = 72. 72 is not a prime.

As the difference between this two-digit number and its reverse is a composite number here too, the difference between *ab* and *ba* is never a prime number.

3. Approach: Generic Example

Consider a number, 36. The reverse of it is 63.

The difference between 63 and 36 is 27. 27 is not a prime number.

63 = 60 + 3
36 = 30 + 6

Subtracting the two gives 30 - 3 = 3x10 – 3 = 9x3 = 27.

It will be the same for any pairs of such numbers.

4. Approach: Thought Experiment

Let "*ab*" be a two-digit number and its reverse be "*ba*".

The number "*ab*" can be written as $10a + b$

The number "*ba*" can be written as $10b + a$

The difference "*ab*" - "*ba*" = $(10a + b) - (10b + a) = 9(a - b)$

As the difference is $9 * (a - b)$, it is never a prime.

Hence it is *proved* that that the difference between "*ab*" and "*ba*" is not a prime number.

Analysis

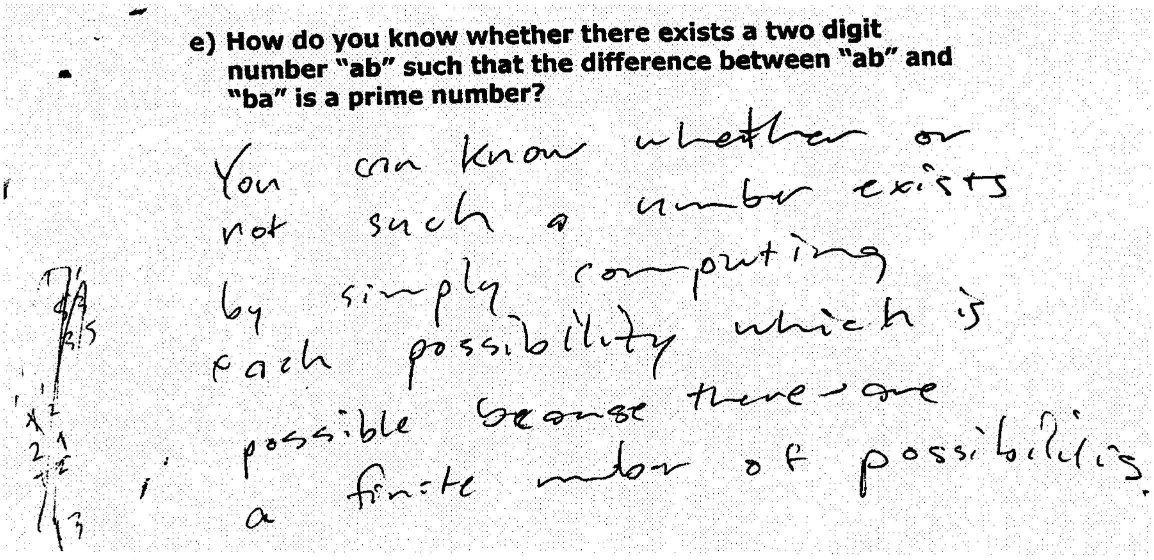
Table 11.1. Observations of student work Task # 4. *

Categories	Correct	Failed	Total
Naïve Empiricism	0	1	1
Crucial Experiment	2	0	2
Generic Example	1	0	1
Thought Experiment- Verbal	0	2	2
Thought Experiment - Symbolic	1	1	2
Other	0	1	1
Total	4	5	9

*NOTE: this is the only task in which I was able to collect student work illustrating all of Balacheff's four categories.

1. Naïve Empiricism

Brian



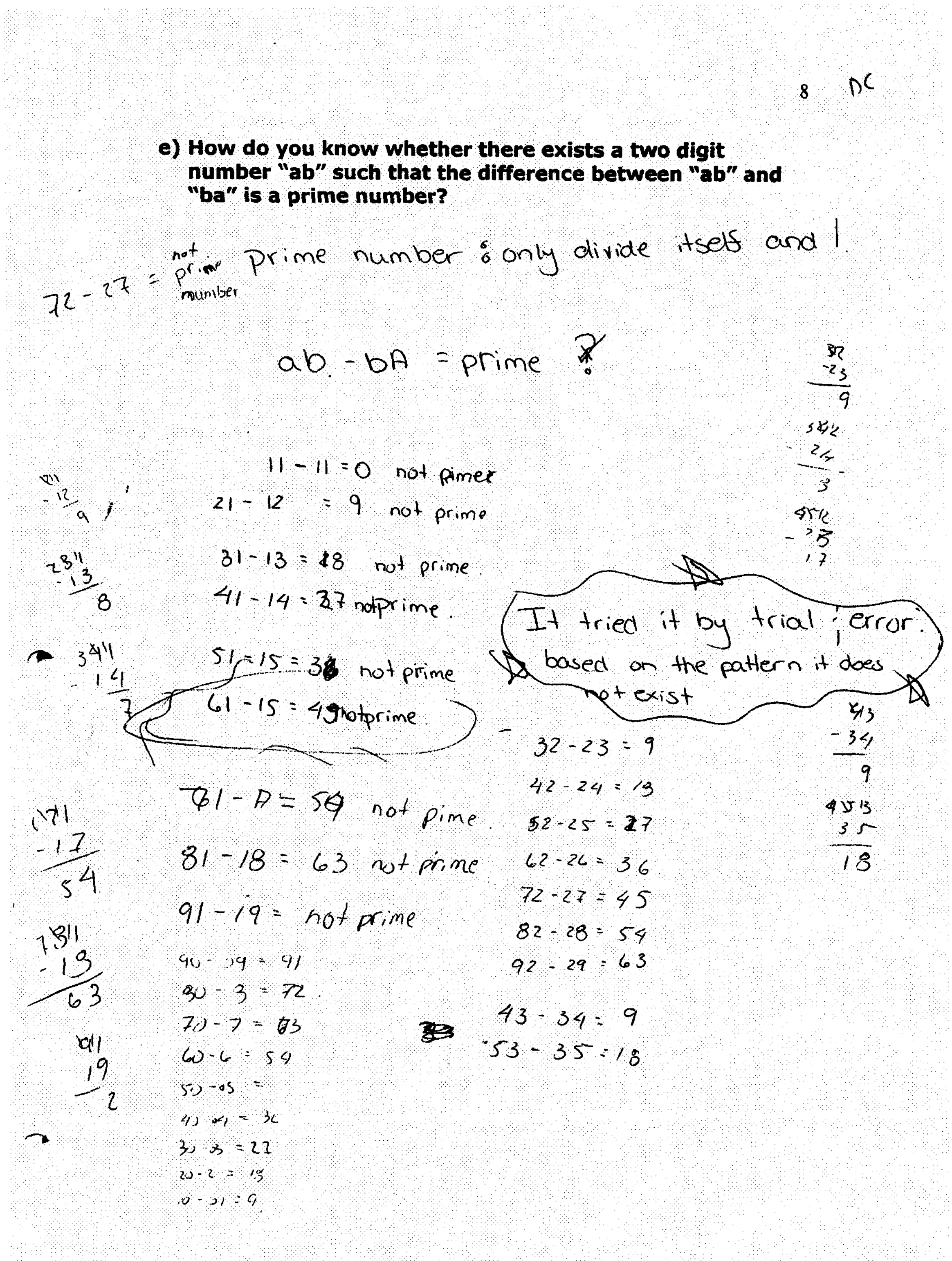
e) How do you know whether there exists a two digit number "ab" such that the difference between "ab" and "ba" is a prime number?

You can know whether or not such a number exists by simply computing each possibility which is possible because there are a finite number of possibilities.

Brian's argument is that "as there are finite number of possibilities, we could count". In other words, try out all the possibilities, if it works then it is proved. This argument places him in the category of naïve empiricism. However, he does not explicitly state whether such a difference exists or not. I see some scribbling on the side of the page, but the observation was not transformed into a result. Hence, I classify it as **naïve empiricism.**

2. Crucial Experiment

Clare



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e) How do you know whether there exists a two digit number "ab" such that the difference between "ab" and "ba" is a prime number?

72 - 27 = not prime number

prime number : only divide itself and 1

ab - bA = prime ?

11 - 11 = 0 not pimer
21 - 12 = 9 not prime
31 - 13 = 18 not prime
41 - 14 = 27 notprime
51 - 15 = 36 not prime
61 - 15 = 45 notprime
71 - 17 = 54 not pime
81 - 18 = 63 not prime
91 - 19 = not prime

It tried it by trial ; error.
based on the pattern it does not exist

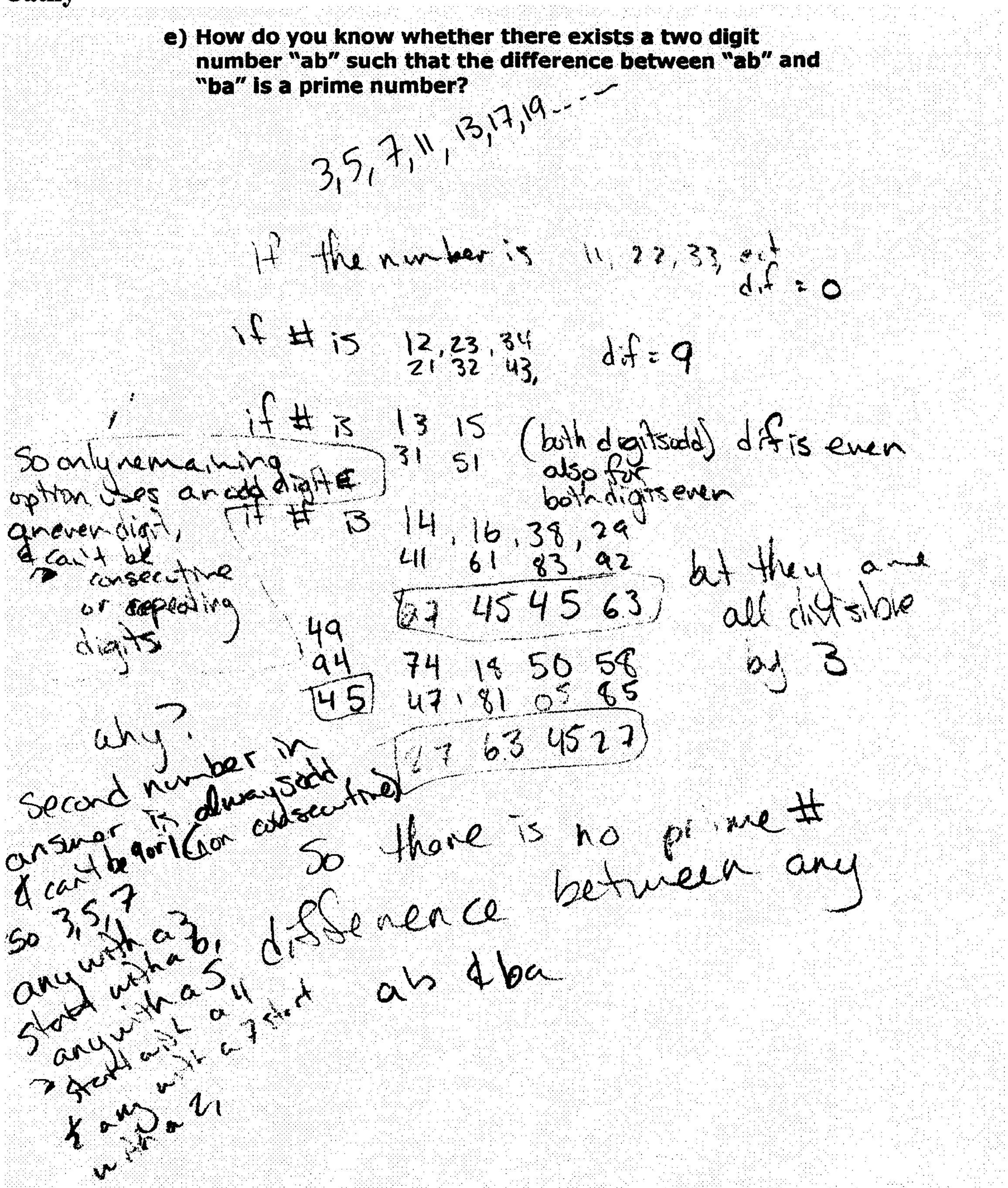
32 - 23 = 9
42 - 24 = 18
52 - 25 = 27
62 - 26 = 36
72 - 27 = 45
82 - 28 = 54
92 - 29 = 63

43 - 34 = 9
53 - 35 = 18

90 - 09 = 81
80 - 8 = 72
70 - 7 = 63
60 - 6 = 54
50 - 05 =
40 - 04 = 36
30 - 03 = 27
20 - 2 = 18
10 - 01 = 9

Clare tried several examples to prove this task. Her conclusion is "tried trial and error [;] based on the pattern it does not exist". Naïve empiricism encompasses the process of testing several examples and considering those examples as a proof. However, I categorize this proof as crucial experiment, not naïve empiricism, because her selection of sets of numbers is not random. The sets are a deliberate choice. I notice one set, where she tries 11, 12, 13, and so on, up to 19. In another set, I see her testing the conjecture with numbers 10, 20, and so on, up to 90. In the third set, I see her trying out numbers like 32, 42, 52, and so, up to 92. I note that Clare carefully selects the numbers, and I see some *extreme* two-digit numbers such as 91 and 11 being tested. Consequently, I categorize this effort as an example of **correct crucial experiment.**

Cathy



e) How do you know whether there exists a two digit number "ab" such that the difference between "ab" and "ba" is a prime number?

3, 5, 7, 11, 13, 17, 19 . . .

If the number is 11, 22, 33, ect dif = 0

if # is 12, 23, 34 / 21 32 43, dif = 9

if # is 13 15 / 31 51 (both digits odd) dif is even also for both digits even

So only remaining option uses an odd digit & an even digit, & can't be consecutive or repeating digits

if # is 14, 16, 38, 29 / 41 61 83 92 / 27 45 45 63 but they are all divisible by 3

49 / 94 / 45 / 74 18 50 58 / 47 81 05 85 / 27 63 45 27

why? second number in answer is always odd & can't be 9 or 1 (non consecutive) so 3, 5, 7

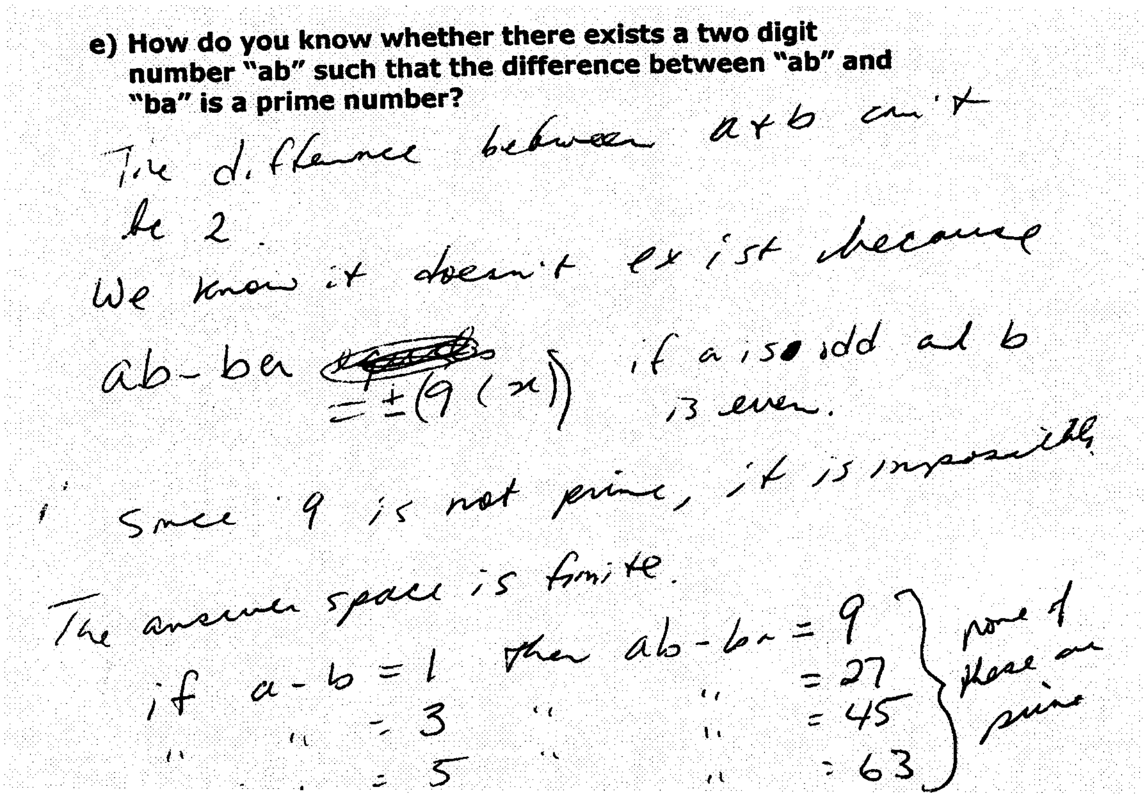
any with a 3, start with a 5, any with a 11 any with a 7 start with a 7 & any with a 11

So there is no prime # difference between any ab & ba

I notice a list of prime numbers at the beginning of Cathy's proof. Much like Clare, Cathy tries to prove this conjecture by trying out several two-digit numbers. With the exception of two sets of numbers (11, 22, 33 and so on, and 13 and 15), the numbers are randomly selected. In other words, the *intentional* selection of numbers (as noted in the case of Clare) can be seen in the two sets of selected numbers that I have noted above. Furthermore, Cathy tests a large number (94). Hence, this proof is classified as an example of **correct crucial experiment.**

3. Generic Example

Daniel



e) How do you know whether there exists a two digit number "ab" such that the difference between "ab" and "ba" is a prime number?

The difference between a & b can't be 2.
We know it doesn't exist because
ab - ba = ±(9 (x)) if a is odd and b is even.
Since 9 is not prime, it is impossible.
The answer space is finite.
if a - b = 1 then ab - ba = 9
" = 3 " " = 27
" = 5 " " = 45
" " = 63
none of these are prime

Daniel starts his proof with the statement "the difference between *a* & *b* can't be 2". He is rephrasing the question to make sense of it. However, his paraphrase does not match the meaning within the question. The difference between *a* and *b* is, in fact, the difference between the digit at the tens place and the digit at the units place. Daniel is saying that this difference cannot be *two* (emphasis added). The rest of the proof indicates that he is using a particular number—"2"—to represent the general term "prime number". Following that statement, he shows *ab-ba* as +/- 9 (*x*), but does not define "*x*". He elaborates upon his argument with three or four examples. However, he is not trying out three specific examples; instead, each example represents a class of two-digit numbers. The first class is the set of numbers where the difference between "*a*" and "*b*" in "*ab*" is 1, then *ab –ba* will be 9. The second class is the set of numbers where the difference between "*a*" and "*b*" in '*ab*' is 3, then *ab –ba* will be 27. The third class is the set of numbers where the difference between "*a*" and "*b*" in '*ab*' is 5, then *ab-ba* will be 63. Daniel only considered the class of two-digit numbers in which the differences between the two digits are the first three odd numbers. I wonder why he did not give an example for even numbers. I categorize this as a **correct generic example** in which the student considers not simply one, but three generic examples.

4. Thought Experiment – Verbal

Deanna

8 J.D

e) How do you know whether there exists a two digit number "ab" such that the difference between "ab" and "ba" is a prime number?

91 = 9-1 = 8 and 1-9 = -8.

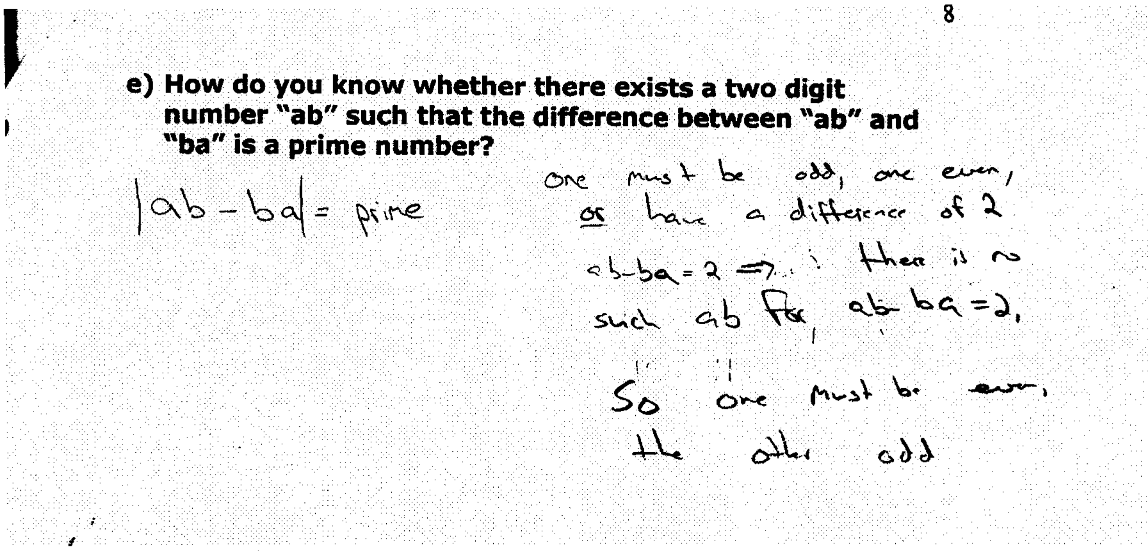
~~(91 = (9)(1) = (1)(9) = 0)~~

72 = 7-2 = 5 prime, 2-7 = -5 not prime b'cuz 1x5 = 5, -1x-5 = 5.

There never exists a number ab such that a-b = prime and b-a = prime. Because where a<b, a-b = -number and a negative number is not prime. Also when a>b, b-a = -number and again this negative number is not prime.

Deanna used two examples to make sense of this task. Unfortunately, she also misread the question. She mistakes the meaning in the words "difference between *ab* and *ba*", and instead reads the phrase to mean "difference between *a* and *b*." Her argument demonstrates traces of thought experiment; hence, I categorize it as **failed verbal thought experiment.**

Philip


8

e) How do you know whether there exists a two digit number "ab" such that the difference between "ab" and "ba" is a prime number?

$|ab - ba| =$ prime

One must be odd, one even, <u>or</u> have a difference of 2

$ab - ba = 2 \Rightarrow$ there is no such ab for $ab - ba = 2$,

So one must be even, the other odd

Philip's proof also demonstrates traces of thought experiment; but, the argument that he uses does not help him much in terms of proving the task. Hence, I categorize this proof as **failed verbal thought experiment**.

5. Thought Experiment - Symbolic

a. correct symbolic thought experiment

George

e) How do you know whether there exists a two digit number "ab" such that the difference between "ab" and "ba" is a prime number?

$$\begin{array}{r} a\ b \\ \underline{b\ a} \end{array}$$

$10(a-b) + b-a$

$10a - 10b + b - a$

$9a - 9b$

$9 \mid 9a - 9b$

$\therefore\ 9 \mid ab - ba$

No such difference.

George's proof is a **correct symbolic thought experiment.**

b. failed symbolic thought experiment

Tahira

8

e) How do you know whether there exists a two digit number "ab" such that the difference between "ab" and "ba" is a prime number?

ab − ba = prime number

~~ab(...)=~~

(10a + b) − (10b + a) = prime #

10a − 10b + b − a = " "

10(a−b) − (a−b) = 9a − 9b = " "

9(a−b) = " "

a−b

(10−1) (a−b)

It should not be divisible by 9.

(a−b) should result in prime number.

Tahira's proof is similar to George however; she was not able to complete the proof as George did. Tahira's argument is that for the task to be true, *a-b* should be a prime number. She doesn't see the difference between ab and ba will always be (10-1)* (*a-b*), hence the difference won't be prime. I categorize this as **failed symbolic thought experiment.**

6. Other

e) How do you know whether there exists a two digit number "ab" such that the difference between "ab" and "ba" is a prime number?

primes = 2, 3, 5, 7, 9, 11, 13, 17, 19, 23, 31,

Beth

Beth listed all the prime numbers up to 31, but she forgot that the number 9 is a composite and not a prime. She also missed number 29. Since she did not do much with this list, I categorize this attempt as **other.**

Summary

Only nine of the 17 participants attempted Task # 4. This may because it was the final task and the student teachers were becoming fatigued or simply tired of mathematical proving. It is interesting to note that this was the only task for which students provided work that illustrated all of Balacheff's (1988) proof categories. Only 4 out of the 9 students who attempted the task were able to prove it. One person had a *fully* correct symbolic thought experiment. The remaining three used pragmatic justification.

CHAPTER 12. RESULTS AND ANALYSIS OF DATA: CONCEPT MAPS

The Concept Map as a Research Tool

It is not an easy task to assess student understanding of specific concepts. When high-level concepts, particularly abstract, mathematical concepts, are involved, the task becomes especially difficult. Researchers have long used paper and pencil tests as tools to assess the learning of mathematical concepts. However, the need for a better way to represent conceptual understanding has led to the development of concept maps as an alternative tool (Novak & Canas, 2006). Within the realm of mathematics education, some researchers (Schimittau, 2004; Afamasa-Fuatai 2004a, b) specifically promote the use of concept maps as an means by which one can accurately assess a student's mathematical knowledge.

A concept map is an explicit, graphical representation of knowledge. Concept maps can effectively map what is *inside* the mind to the *outside* (Tergan, 1988) and reveal conceptual understandings that are not generally identifiable by other assessment tools (Hasemann & Mansfield, 1995). They provide research participants with a useful means for demonstrating understanding and the researcher with an opportunity to witness how the student-participant connects ideas and groups or organizes information. In other words, concept maps effectively reveal the overall integrated knowledge of the learner.

The theoretical foundation of concept mapping rests upon Ausbel's theory of learning. This theory posits that meaningful learning takes place by assimilating new concepts into existing conceptual frameworks held by the learner (Ausbel, 1963; 1968; Ausbel et al., 1978). Learners who are asked to draw a concept map must choose visual symbols and/or details to represent concepts and to clarify the nature and relationships among these concepts. Details and connections between concepts can be added in any order. Generally maps are drawn with concepts contained in oval shapes and linking words noted on the lines connecting concepts/shapes as below:

Figure 12.1. Concept map: shapes.

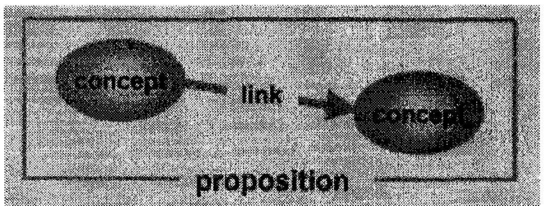


Figure 1. The basic element of a concept map is a proposition, consisting of two concepts connected by a link that shows the relationship between them.

(Source: Rebich and Gautier, 2005, p. 358)

As a visual representation of learning, cognitive maps provide an effective device and opportunity for metacognitive reflection.

My aim in using concept maps was to assess the student teacher participants' levels of conceptual understanding pertaining to the task of proving mathematical proofs. I assume that high levels of conceptual understanding will be associated with high levels of confidence in attacking a proving task and, conversely, low levels of conceptual understanding with low levels of confidence in completing a proving task. I established two primary indicators for high-level understanding. First, I examined the form and structure of the concept map that the student produced. Specifically, I examined three key features: the different forms/shapes of the maps; the number of different key terms used in the maps and the number of specified relationships among key terms as indicated by cross-links. Second, I analyzed the manner in which the construction of the map reflected a hierarchy of 'proving' skill as established by Balacheff (1988). Briefly, once again, his four types of proofs include *naïve empiricism* in which the truth of a result is verified with a few examples, *crucial experiment* in which a result is verified on a particular case which is recognized as typical, *generic example* in which the truth of assertion is made explicit using a prototypical case, and *thought experiment* in which operations and foundational relations of the proof are dissociated from the specific examples considered (in this case, the proofs are based on the use of and transformation of formalized symbolic expressions). Thus, I mapped the conceptual understanding as displayed within the student concept maps onto Balacheff's (1988) taxonomy of proofs.

It is important to repeat at this point that participants produced these maps at the end of their teacher education program (two weeks prior to the start of their final practicum). At least one of their compulsory educational psychology courses includes within its curricular content, the concept, nature and function of concept maps. Hence, all of the study participants came to the study familiar with concept maps and how to construct them. Furthermore, they were Mathematics majors who had completed at least twelve 3-credit courses in math. Since the participants had already taken several university-level mathematics courses dealing with proofs and proving, it was logical to assume that they would recognize and understand most of the concepts/terms provided. It was likely that they had already developed mental models of mathematical proof as a result of their previous exposure to advanced proof and proving within various mathematics courses, models that very likely contained some of the twenty-four terms/concepts presented to them. By engaging in the concept map process, participants were able to embark on a cognitive process of constructing meaning and making sense by integrating unfamiliar terms into their pre-existing conceptual frameworks or models. By providing a list of terms in advance and offering participants an opportunity to apply all or none in their concept maps, I hoped that I would be able to help these student teachers broaden their perceptions and expand their abilities to process information (Rebich & Gautier, 2005).

Considerations Concerning Analysis

There are a number of important considerations when one undertakes to analyze a concept map. First, there is no such thing as a single correct map; rather, there will always be a multitude of possible ways in which one can generate a concept map, with some maps serving as more informative representations of conceptual understanding than others (for example, labels and/or connecting verbs that make relationships explicit and relationships that are clearly appropriate reflect considerable conceptual understanding). Next, in addition to analyzing what has been included in the concept map, one may analyze what is lacking. In other words, analysis can proceed on the basis of the *absence* of essential concepts. Finally, one can analyze a concept map on the basis of its general form and structure.

My analysis focuses mainly on the structure of the map, especially the degree of complexity indicated by the general form. Vanides, Yin, Tomita and Ruiz-Primo (2005) identify four typical structures for concept maps: 1) Linear, 2) Circular 3) Tree and 4) Network. The College of Agricultural, Consumer and Environmental Sciences, at the University of Illinois also offers four general categories of concept maps: 1) Spider Maps 2) Hierarchy Maps, 3) Flow Chart Maps and 4) Systems Map (found on-line at http://classes.aces.uiuc.edu/ACES100/Mind/c-m2.html) According to the College, concept maps that have a central theme or unifying factor placed in the center and subthemes radiating from it can be called a "spider map". The map that presents information in descending order of importance (from top to bottom) is a hierarchy map. A concept map that organizes the concept in a linear format is called a flow chart map. And the map that organizes information in a similar flow-chart format with the addition of "inputs" and "outputs" can be termed a systems map. Classification systems are by no means definitive; nor are they exhaustive. A conceptual map will ultimately take whatever form best serves the cognitive needs of the individual constructing it—hence, structure/form in concept maps is always variable.

More critical than the choice of form is the extent to which the map illustrates complex conceptual relationships. One must analyze concept maps carefully in terms of how the key terms are used and the way in which relationships among them are specified (Jones, 1997). Concept maps that incorporate multiple ideas/concepts in ways that clarify conceptual relationships and cross-relationships (commonly referred to as network maps) demonstrate sophisticated levels of understanding (Jones, 1997; Vanides, Yin, Tomita & Ruiz-Primo, 2005). Vanides, Yin, Tomita and Ruiz-Primo (2005) note that both proficient students and subject experts tend to create highly *interconnected* maps, while novices tend to create simple structures that are linear, circular, or organic. Network maps, that also include important propositions that correctly describe the conceptual relationships that are foundational to the main ideas, demonstrate *extremely* sophisticated conceptual understanding.

Analysis

Three of the 17 students did not attempt the task. Of the remaining 14 who did, not one generated a high-level structure—that is, a complicated structure with extensive interconnectedness among concepts. See Table 12.1 for a detailed analysis of the student teachers' concept maps with regards to their shape, terms, relationships, and links.

Table 12.1 Observations: Concept Maps.

Name	Form/ Shape of the Map	# of key terms used (# of relationships indicated is noted in brackets)	Specific Relationship shown (Yes/No)	# of cross links between concepts	Terms outside the list used
Daniel	Solar system	19 (18)	No	0	Proof
Grace	Tree	21 (20)	Yes (by arrow)	0	Proof
Clare	-	-	-	-	-
Chandelle	Solar system	19 (18)	no	0	Mathematical Proof
Sara	Solar system	11 (10)	No	0	Proof Prior Knowledge Patterns & Relations Puzzle put together
Cathy	Crunched different terms together (4 categories for 17 terms). No identifiable shape	19 (18)	Just once (used the term deduction)	0	Proof
Terrence	Linear	19 (18)	Yes (by arrow)	0	Problem, Grouping, Theory, Special Case
Gita	Linear	7(7)	Yes (by arrow)	0	-
Deanna	Tree	13 (12)	No	0	Proof
Tahira	Solar system	11 (10)	Using verbs	0	Proof
Brian	Tree	29 (29)	0	0	Proof Specific cases Negative Examples Plan Induction
John	Input- Output	17 (17)	Yes (with arrows)	0	Proof
George	No particular shape	10 (5)	Yes With verbs	0	-
Spencer	Spread out	24 (24)	-	0	Proof
Brandon	-	-	-	-	-
Beth	Input-Output	8 (5)	-		-
Philip	-	-	-	-	-

Discussion

The task of generating a concept map as a means of demonstrating conceptual understanding of "mathematical proof" proved a challenge for the pre-service teachers. I anticipated that participants would be familiar with *most* of the 24 terms provided since most of them are commonly used in the study of mathematical proof. This was not necessarily the case. One participant used only seven of the 24 terms listed (the fewest used by any participant); she also neglected to include the key term, proof (see Figure 12.1). This map took a simple linear form. In contrast, another student utilized all 24 terms from the list plus five more of his own selection (see Figure12.2). This map took the form of a branching tree.

Concept maps were typically simple in structure. For the most part, their shape conformed to one of the following structures: a line, branching tree, input-output figure, or solar system (see Figures 12.1, 12.2, 12.3, 12.4, and 12.5 below). In most examples, the student placed the word "proof" or "mathematical proof" at the center of the map and then added other terms around and about it. This formation often reflects unsophisticated thinking because students can place terms around the central word without necessarily considering where they are best positioned in relation to *each other*. However, one may demonstrate complex understanding even with a fairly simple map structure, such as the solar system, as long as one thoughtfully adds connecting verbs (see p. 209).

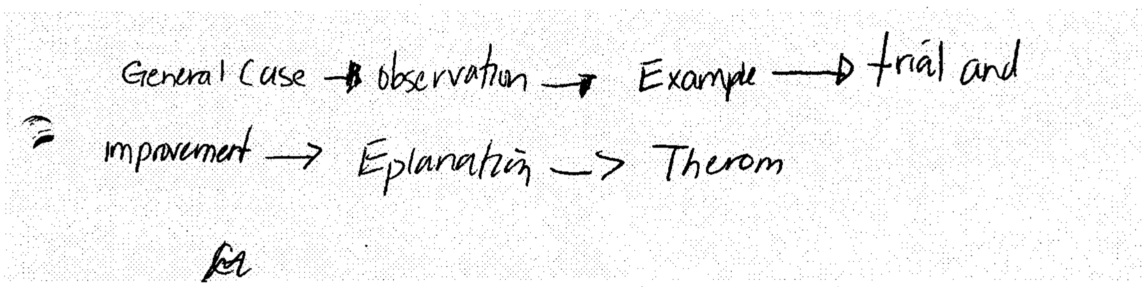
In general, student teachers mapped concepts within oval shapes and then connected the shapes by linking them with lines; however, few students made these connections explicit by using arrows or labels. I found it especially interesting that none of the participants in my study produced maps showing *interconnections* among the terms/concepts. According to Jones and Vanides, et al. (2005), interconnections demonstrate high-level understanding of the concept. Only two student teachers used linking terms to connect the main concepts with the others (see Figure 12.6 for one such example). As well, few students used propositions (verbs) to describe the relationship between the concepts.

In his study, Jones (1997) used concept maps to identify those student teachers possessing extensive subject knowledge of proof. He discovered that student teachers who had completed more mathematics courses than their peers, and who had received higher grades, were able to produce sophisticated maps containing a high number of key terms. I did not collect data on the exact number and type of mathematics courses taken by each participant; however, I did notice that the pre-service teacher who claimed to have taken more mathematics courses than any of his classmates, and who referred to himself as a "mathematics geek," also produced the most complex map. His map took the form of a tree with branches and sub-branches. He utilized all 24 terms from the list; what is more he included five additional terms (see Figure 12.2). This is consistent with Jones' findings (1997): those who excel at mathematics typically produce the most complex conceptual maps.

The concept maps produced by the student teachers can be categorized into four major types: 1) Linear Maps, 2) Tree Maps, 3) Input-Output Maps and 4) Solar system maps. I provide, below, four student generated concept maps, each as an example that will illustrate one of these four basic shapes/forms.

1. Linear Map

Figure 12.1. Linear concept map



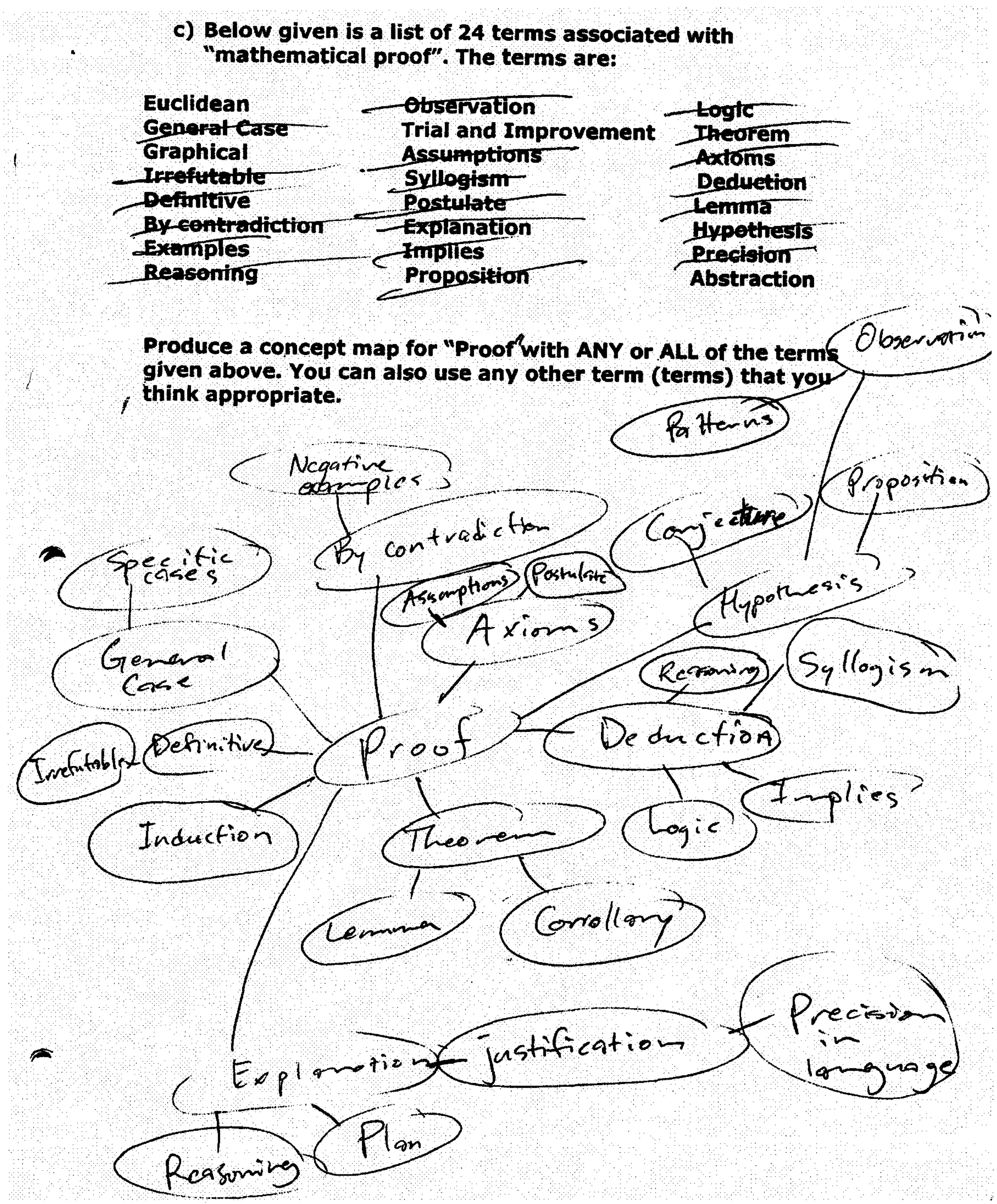
Although the College of Agricultural, Consumer and Environmental Sciences Website identifies this type of concept map as a "Flow Chart," I prefer to call it a linear map because this term more accurately describes the map's structure. Vanides, Yin, Tomita and Ruiz-Primo (2005) also favour the term "linear".

Based on this linear concept map, it seems that the student begins by observing from a general case (probably what the student teacher had in mind is a general observation or a conjecture), and then indicates that one must try another example. If the example works, one must provide an explanation for the result. If it does not work, one must try "trial and improvement". The structure of this map suggests conceptual understanding that is in line with Balacheff's (1988) *crucial experiment.* At this level, one deals with the question of generalization by examining a case that is not very particular. If the assertion holds in the considered case, the student will argue that it is valid. Balacheff (1988) notes an example in which two students, in the process of working out a mathematical proof, decide to test a polygon with many sides (15). They proceed under the assumption that if the proof works in the case of the extreme polygon, it will work for all cases. In other words, at this level the thinker checks the statement with a carefully selected example that is representative of a certain class.

This mapmaker did not include in the map either the term "proof" or "mathematical proof;" rather, s/he substituted the term "explanation." This may be because the words "proof" and "mathematical proof" had not been included in the list of given terms. The inclusion of the "term" explanation may imply that this student teacher believes that *explanation* is one of the functions of mathematical proof. However, it is interesting to note that this mapmaker, when completing the first part of the questionnaire, did not define proof in terms of "explanation".

2. Tree Map

Figure 12.2. Tree concept map

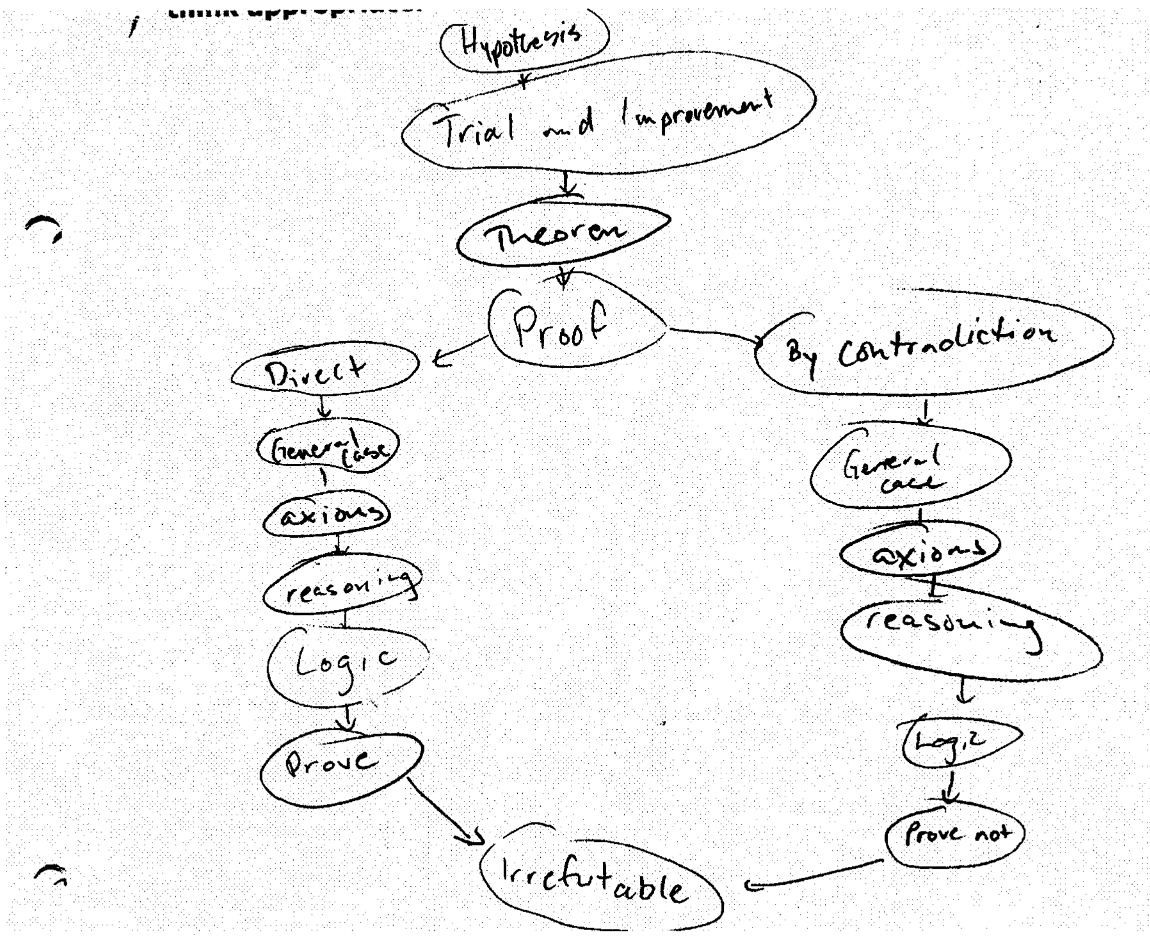


The concept map above (Figure 12.2) closely resembles a "tree" and constitutes the most complex concept map produced by any one participant (as noted earlier, it is the work of the student who self-identified as a "mathematics geek"). Yet despite its relative complexity, the map lacks the verbs needed to link the concepts. One notable feature of this concept map is that it does not progress from an initial stage/start point towards a final stage/end point, as did the example noted above (Figure 12.1). Consequently, it is difficult to position this student's understanding within Balacheff's hierarchy of thinking and proving skill. It is clear that the student carefully and deliberately lays out all of the concepts in a meaningful manner. This suggests that the student teacher has a *holistic* understanding of the concept. This holistic presentation of the concepts as well as the extensive detail of the map suggest that this student's understanding of mathematical proof is akin to the level of thought experiment. As noted before, when students operate at the thought experiment level they are able to distance themselves from action and make logical deductions based upon an awareness of the properties and the relationships inherent within the situation. In other words, holistic understanding is necessary if one is to operate at the level of thought experiment. It is at this level that students move from practical to intellectual proofs.

3. Input–Output Map

The following concept map (see Figure 12.3) takes the form of, what I call, an "Input –Output Map". The inputs are "Hypothesis, "Trial and Improvement" and "Theorem", and the Output is "irrefutable proof." The mapmaker arrives at output as s/he processes the concepts of general case, axioms, reasoning, and logic. I interpret the message conveyed by this concept map as follows: Based on the "hypothesis" and using "trial and error" (or examples) a theorem can be formulated. One can then prove the theorem using either a "Direct Method" or an "Indirect Method". Whether one uses a "direct method" or an "indirect method," axioms, reasoning, and logic will play a major role in the "proving" or the "disproving".

Figure 12.3. Input-output concept map



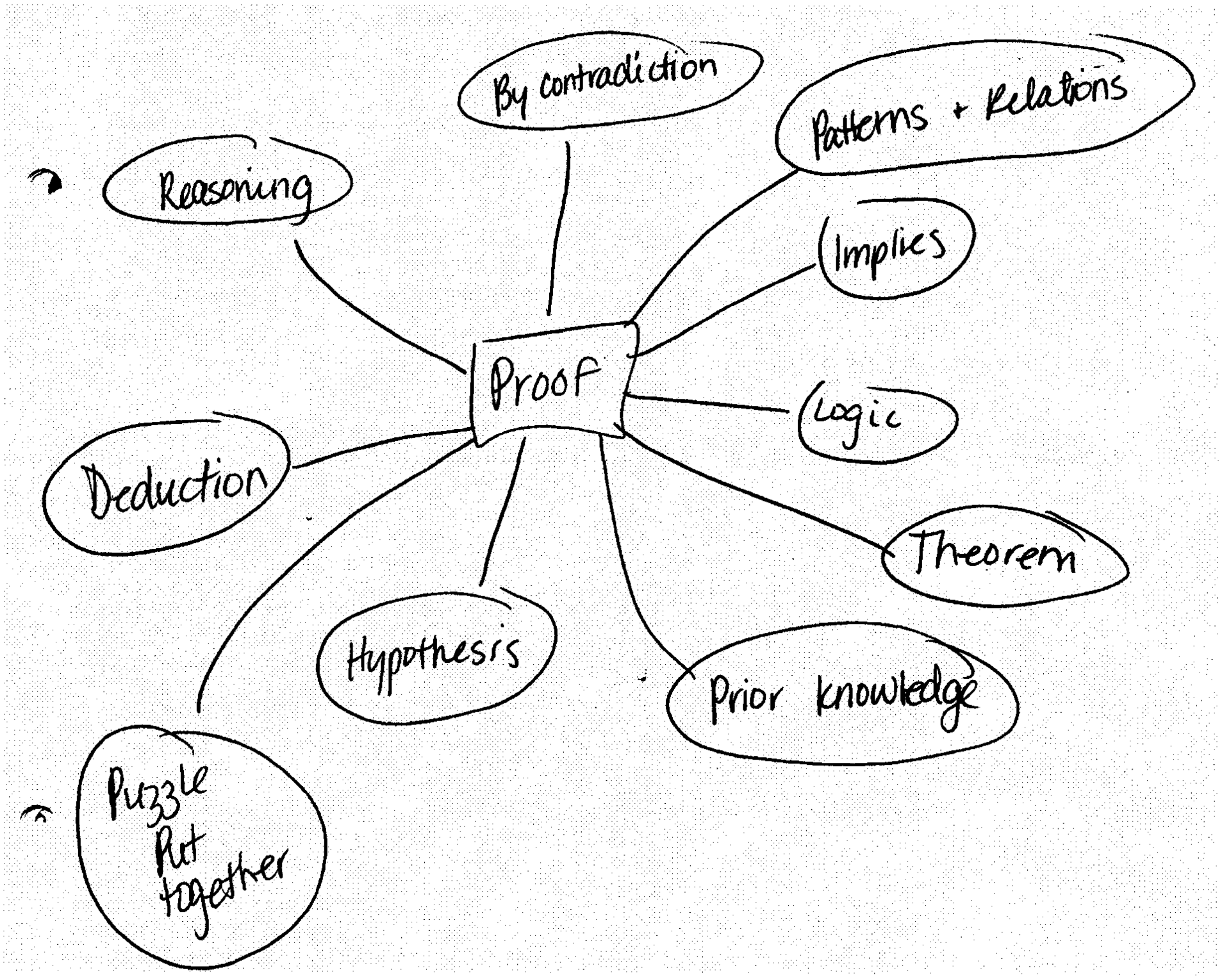
This concept map provides one major insight into this student teacher's conception of mathematical proof: s/he believes that *once a theorem is proved, it is irrefutable*—this belief suggests an absolutist philosophy of mathematics (Ernest, 1999). Even though far fewer concepts are evident in this map than in the more sophisticated 'tree' map noted above, the mapmaker places the concepts in a deliberate manner.

As noted earlier, research indicates that the greater the number of terms in the map and the interconnectedness between them, the deeper the conceptual understanding of the mapmaker. Based on the *number* of terms/concepts used by this student teacher, I conclude that his/her understanding of mathematical proof is fairly unsophisticated. The input section includes the phrase "trial and error" (that is, trying out various examples); but the student teacher also includes the words "general case". I suggest that the student teacher, in his/her use of the term "general case", means a "generalization of several cases observed through trial and error". This would indicate that the student's understanding of mathematical proof likely corresponds to level three in Balacheff's taxonomy of proof —generic example. With generic example, the proof rests upon the properties, and the example, and, rather than being specific, represents a generalized class. The particular case is not used as a particular case but as an example of a class of objects. Justification is based on operations or transformations on the representative example and then made upon the whole class.

4. Solar System Map

The most popular format for constructing a concept map proved to be the "solar system map" (or "spider map", if one prefers the College's nomenclature; see p. 196). Here, the mapmaker places the central theme or unifying factor in the center of the map and then organizes related items around the center (see Figure 12.4). In my view, this sort of mapping typically indicates low-level thinking: any number of terms/concepts can be placed around a main concept in no time at all and with little thought. Researchers consider Linear Maps (see Figure 12.1) to be the most simplistic of all concept maps; however, some degree of careful thought is necessary even at this level since the mapmaker must determine where to place each concept "in relationship". In other words, the mapmaker must place each concept term in relation to what has come before, what *immediately* precedes it, what *immediately* follows it, and what will follow down the linear line of relationship. When constructing a solar system map, however, the mapmaker need not consider relationships as carefully: concepts placed around a given term are usually considered only in relationship to the word placed in the center and not in terms of their relationships with the other items in the circle.

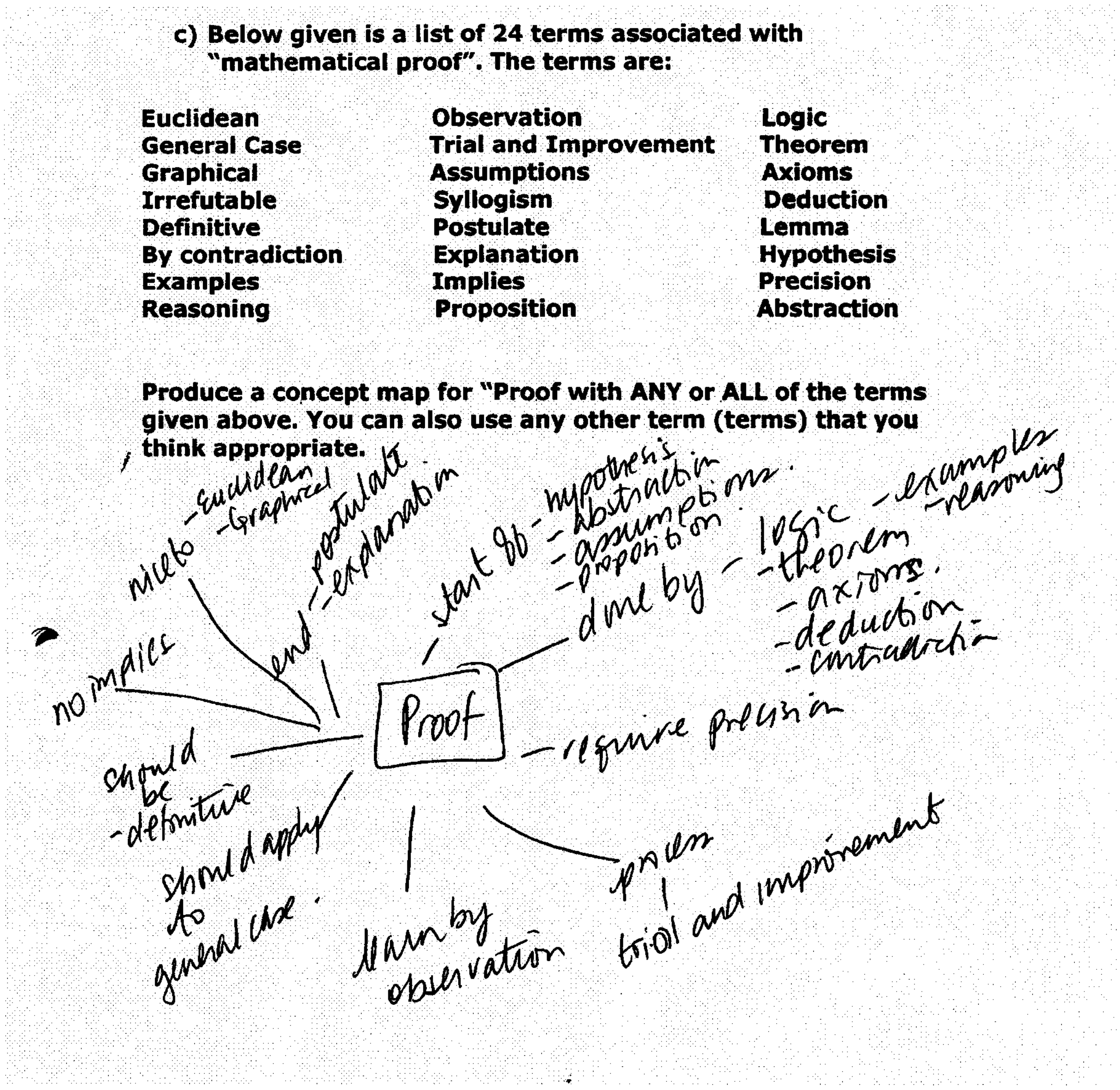
Figure 12.4. Solar system concept map



I suggest then, that solar system maps typically reveal less sophisticated thinking than linear maps. I align this solar system mapmaker's level of understanding with one of the lower levels in Balacheff's (1988) hierarchy of proofs.

However, having said that, I do include one example of a solar system map in which it seems apparent that the mapmaker gave some thought to the relationships among the items in the circle (Figure 12.5). As would be expected with a solar system structure, the central concept, "proof", has been placed in the center of the map with related terms around it in a circle. The mapmaker uses various linking verbs to connect the terms/concepts that s/he has selected. Unlike other examples of the solar system structure, however, here the mapmaker employs a clockwise direction to assist in explaining what a proof is. I infer his/her thinking process from the concept map as follows: Proof always starts off with "hypothesis/ abstraction/ assumption, proposition". Proof is accomplished by "logic"; it uses "theorems", "axioms", "examples" and "reasoning". Proofs can be proved either by "deduction" or by "contradiction". Proofs require "precision". A process that could be used to improve "proof" is "trial and error". Proofs can be learned by "observation". Proofs should apply to the "general case" and should be "definitive". "Nice" proofs are obtained by "Euclidean" and "graphical" methods. Proofs end with "postulates" and "explanation". This example demonstrates that even with a relatively simple solar system form, a mapmaker can generate a sophisticated map as long as he/she possesses a deep understanding of the concept. I suggest that this mapmaker displays high-level understanding of mathematical proof.

Figure 12. 5. Solar system concept map with linking verbs



c) Below given is a list of 24 terms associated with "mathematical proof". The terms are:

Euclidean	**Observation**	**Logic**
General Case	**Trial and Improvement**	**Theorem**
Graphical	**Assumptions**	**Axioms**
Irrefutable	**Syllogism**	**Deduction**
Definitive	**Postulate**	**Lemma**
By contradiction	**Explanation**	**Hypothesis**
Examples	**Implies**	**Precision**
Reasoning	**Proposition**	**Abstraction**

Produce a concept map for "Proof with ANY or ALL of the terms given above. You can also use any other term (terms) that you think appropriate.

General Comments

Among the different structures generated as concept maps, the solar system form proved to be the most popular. The reason, I surmise, is because the solar system format best accommodates a large number of terms, and students were given 24 terms with which to work. As noted earlier, only two student teachers used verbs to specify the relationships. Among those terms added by the mapmakers (that is, terms that were not included in the original list of 24), the most common was "proof" or "mathematical proof'. Most of the student teachers placed, in a logical manner, only a few of the original 24 terms. Those who used a high number of the 24 terms opted for the "solar system" format, which, by virtue of its visual design, can easily accommodate quite a number of terms. If one uses the following criteria—a high number of terms *used in a deliberate and logical way*—to establish high confidence levels in understanding and teaching mathematical proof, then most of the student teachers in this group can be seen to operate within the bounds of *pragmatic* justification.

Summary

Although familiar with concept map structures, these student teachers had difficulty generating sophisticated representations of their understandings of mathematical proof. They could not effectively incorporate into their maps (even with their thinking stimulated by 24 relevant terms) the various sub-concepts and ideas associated with proofs and proving. I offer two possible explanations for this. First, student teachers may have had difficulty, not with the mathematical concepts per se, but with the task of visually representing their understanding of the concepts in the form of a concept map. These student teachers were first introduced to concept maps in a class setting that had little or nothing to do with mathematics. Now they were being asked to apply their learning within a different context, a mathematics classroom. They might have found this quite difficult. If this was, indeed, the case, one must ask: Does this reflect the degree to which mathematics is compartmentalized and set apart from other subjects and basic learning strategies and tools? Second, many of these student teachers may, in fact, lack the deep understanding of mathematical proof that would enable them to generate sophisticated concept maps. My analysis of the mathematical tasks indicates that the student teachers had great difficulty completing the tasks correctly. If one takes the concept maps that these students produced as an accurate indication of their depth of understanding of mathematical proof, then the results of this representation task clearly serve to reinforce my earlier observations pertaining to the mathematical tasks: students have difficulty with the concept of mathematical proof.

CHAPTER 13. LOOKING BACK

Balacheff (1988) in Relation to My Research

Balacheff's (1988) study is one of the most quoted in scholarly and professional publications dealing with mathematical proof. His work has influenced many researchers and his findings have long been a source of interest and debate. For all of these reasons, I wished to conduct a study that would resonate with Balacheff's work. Hence, I adopted his research design, carefully considered his findings, and devised a similar study, though within a different context. Like Balacheff, I place a great deal of importance on the concept of mathematical proof. Also like Balacheff, my interest was to study how participants engage in the proving process. Balacheff gave his students ample time to complete the task and would only accept their work when both students had agreed that they had completed the task; in my case, I also provided sufficient time for the completion of all tasks and permitted participants extra time if they needed it.

Yet despite the similarities between Balacheff's work and my own, it is the distinctions between Balacheff's study and my own study that warrant particular attention. Balacheff conducted his study with junior high school students; in contrast, I conducted my study with secondary-school student teachers. Balacheff asked his participants to complete just one mathematical task, and allowed them to work in pairs; instead, I asked my student teacher participants to complete four different proof tasks and had the students work individually. Balacheff enlisted the aid of a research assistant and closely observed and recorded student comments as the participants were engaged in the process of proving the task; I worked primarily from the mathematical work left on the page, the end product of the proving process, and interpreted, from the evidence, how the process may have unfolded. I also included an element that Balacheff did not: the use of concept maps as a strategy for assessing the depth of conceptual understanding of mathematical proof of my student teacher participants. In this component of my study as well, I focused my efforts on the analysis of an end product. Consequently, my research work includes an interpretative dimension that, one may argue, commonly marks research work situated within a paradigm of interpretative inquiry rather than a paradigm of positivism.

Context is always critical. It is understandable, then, that contextual changes would result in some interesting differences between Balacheff's and my own findings. Based on the efforts of his teenage participants, Balacheff outlined a proof hierarchy reflecting four increasingly more sophisticated levels of thought and skill pertaining to mathematical proof. Moreover, he managed to place all of his students into one of these four levels with relative ease. I adopted Balacheff's taxonomy of proof and attempted to place each of my student participants into one of these four categories. I worked with students who had completed a minimum of 12 university-level mathematics courses: their understandings of and experience with proof, likely, far exceeded that of Balacheff's thirteen and fourteen year old participants. Since the student teachers were familiar with the expected forms for mathematical proof, they tended to try and make their work look 'mathematical.' I found that almost all of the proof work that these student teachers placed on the page of the questionnaire reflected *traces* of thought experiment, the highest level in Balacheff's hierarchy of proof. But traces did not necessarily mean that the students had successfully operated at the sophisticated level of thought experiment as they attempted to complete the proof task. There was also evidence of lower levels of thought. Indeed, the data indicates that most of these young adults did not operate predominately nor successfully at the highest level in Balacheff's taxonomy of proofs. Hence, it was difficult to categorize their work and place their proofs into the neat categories afforded by Balacheff's well-defined taxonomy

Balacheff's four levels of proof—naïve empiricism, the crucial experiment, the generic example, and the thought experiment—are developmental. Implicit in this hierarchy is the notion that students move from one level to the next, progressing to more mature and more sophisticated levels of thinking while embodying what has come before. The role of the teacher is to lead the students, by means of classroom discourse, towards higher and higher levels. Since I administered the questionnaire at one point in time only, my research design did not address student movement from one level to the next. Although I administered the various proof and proving tasks towards the end of the student teachers' course work , and students had already covered their curricular segment on mathematical proof, it is clear that students had considerable difficulty successfully completing proof tasks designed for the secondary school level. This is a concern that merits further investigation.

I thought it enlightening to compare the various formulas generated by my student teacher participants with those generated by Balacheff's much younger and more mathematically inexperienced student participants. (Since Balacheff modified many of his students' formulas, what I include below does not reflect the exact formulas that the students themselves produced.) For example, compare the formulas of Daniel, Spencer, Chandelle, and Philip from my study with the formula arrived at by Balcheff's two students, Christopher and Bertrand (see below).

Daniel

$$\# \text{ diagonals} = \frac{n(n-3)}{2}$$

where n is the # of vertices.

You could always count the too

Spencer

$$d = \frac{n(n-3)}{2}$$

Chandelle

$$d = \frac{v(v-3)}{2}$$

Philip

$$D = \frac{V(V-3)}{2}$$

These formulas are similar to the formula that Christopher and Bertrand arrived at: $n(n\text{-}3)/2$. In discussing his findings, Balacheff refers to this particular solution as the one that reached the "classical formulation".

Note as well the formulas of Grace and Sara and how they are similar to the formula devised by Martine and Laura in Balacheff's study (see below).

Grace

vhen you know how many

$$\text{diagonals} = (v-3)+(v-3)+(v-4)+\ldots+(v-v)$$

Sara

$$\text{diagonals} = 2(n-3)+(n-4)+(n-5)+\ldots(n-n)$$

The formula that Marine and Laura arrived at was

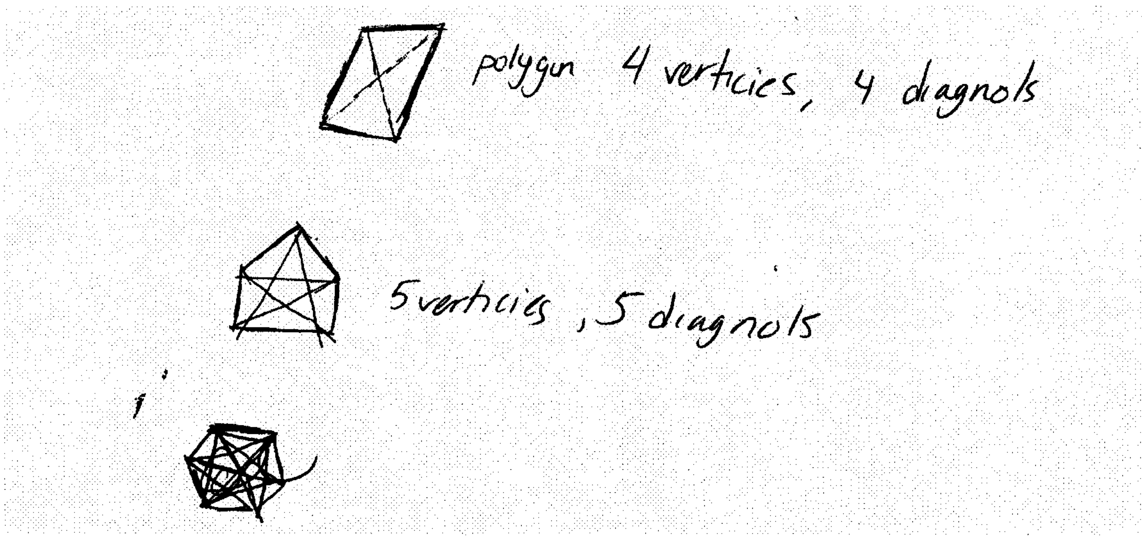
$f_2(n) = (n\text{-}3)+(n\text{-}3)+(n\text{-}4)+(n\text{-}5)+\ldots+2+1$

Although less similar, **Clare's** formula (a student teacher participant in my study) bears some similarity to the formula noted by Marine and Laura above.

D stands for the number of diagonals

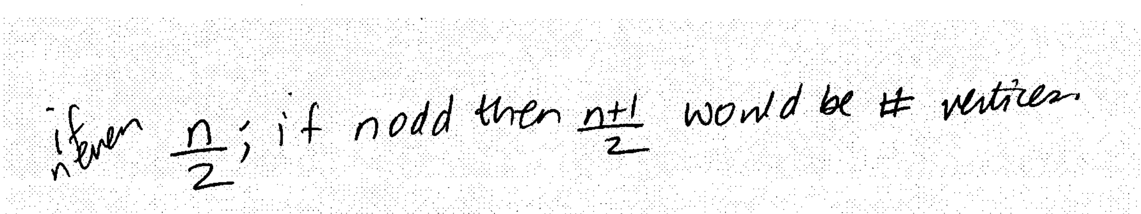
$$(V-2)+(V-3)+\ldots+(V-(V+2))+1=D.$$

Consider another instance. Even though **Gita** (my study) did not give an explicit formula, it can be inferred that what she had in mind is that # of diagonals = # of vertices.



Gita's implicit argument can be compared both to the work of Lionel and Laurent and the efforts of and Pierre and Mathieu (Balacheff's study). Both pairs suggested that the number of diagonals is *n*. Notably, Oliver and Stephane and Georges and Olivier (Balacheff's study) also had a similar answer. As well, **Tahira's** argument (my study), as noted in the formula below, is comparable to that put forth by Blandine and Elisabeth (Balacheff's study) who argued that the answer is "*n*/2 or (*n*-1)/2".

Tahira



One can also draw a parallel between Brian's answer and that of Blandine and Elisabeth.

Brian

$$d(x) = \sum_{i=1}^{x-2} i = \frac{x-2+1}{2} = \frac{x-1}{2}$$

Here, Brian used a functional notation to indicate the number of diagonals rather than using a variable *"n"* or *"d"*. He further explained that x stands for the number of vertices. Compare this again with the work of by Blandine and Elisabeth (Balacheff's study) who argued that the answer is "*n*/2 or (*n*-1)/2".

There aren't any formulas in Balacheff's study that gave similarity with

Cathy

give a diagonal, so (n - 2) is the number of diagonals.

Terrence

Thus a formula would be

$$\frac{(v^2-v)-2v}{2}$$

Of particular interest to me were the following formulas. These were unique to my study group; none of Balacheff's student participants produced formulas like them.

Deanna

find # of vertices so V, and know the number of diagonals is V-2=d for the first vertex then 2nd vertex V-3=d.

Deanna could not come up with a formula, and so could not generalize further.

George

$$= \frac{(n+1)^2 - 3(n+1)}{2} \quad \square$$

George did not define *n*. I assume that he meant *n* as the # of vertices

John

Let d = number of diagonals

v = number of vertices

$$\therefore\ d = {}_vC_2 - v$$

Brandon

He defined *d* as diagonals and *v* as vertices

$$d = \left[\sum_{i=4}^{v} (i-3)\right] + v - 3$$

Beth

diagonal = non adjacent vertices

total # vertice connections $= \binom{n}{2}$ $n =$ # vertices.

$-$ outer edges $= \boxed{\binom{n}{2} - n}$

Beth did not define D and V.

I make these various comparisons to show that even after five or so years of advanced mathematics courses, my student teacher participants produced formulas that in some respects, are quite similar to those produced by the teenage student participants in Balacheff's study. This observation may be interpreted into two ways: first, it may reflect the way in which students embody the lower levels of thinking (as identified in Balacheff's taxonomy of proving skill) as they move on to the higher levels of thinking; second, it may be interpreted as an indication of similar levels of thinking among the participants within the two groups. Given that the participants in my study group were all young adults with training in advanced mathematics and that Balacheff's teenage participants were, apparently, not students noted for being gifted in mathematics, these similarities suggest cause for concern. Are student teachers demonstrating the level of thinking skill that one would expect of them as they enter into the classroom as mathematics teachers who will be responsible for teaching mathematical proof to students much the same age as the young people in Balacheff's study? The data of my study suggests that this question warrants further investigation.

Healy and Hoyles (2000) in relation to My Research

Healy and Hoyles (2000) investigated how high achieving students (top 20% in their school populations in England and Wales) understand and evaluate the effectiveness of various approaches to proving. The researchers examined proof and proving skill as it pertains to only one mathematical domain—that of algebra—whereas my study focused on the domains of both geometry and number theory. In their study, they collected data using three types of instruments:

1) Students were asked to provide *written descriptions* about proof;
2) Students were presented with mathematical conjectures and a range of different types of arguments in support of them; then they were asked *to select from among the arguments* that which was closest to their own approach and that which they believed would receive the best mark from the teacher;
3) Using a *multiple-choice format*, students were presented with two conjectures, one familiar and the other unfamiliar, and the arguments associated with each conjecture; students were then asked to evaluate the various arguments offered, once again by selecting the best answer based on each multiple choice question stem. (In this respect, Healy and Hoyles employed a strategy similar to the one used by Martin and Harel (1989) in their study with pre-service teachers.)

Of the three instruments that I used in the collection of data in my study, one corresponds to the first instrument used by Healy and Hoyles while the other two differed from the those used in that 1989 study. In my study, I posed three questions about the nature, role, and function of proof and asked participants to provide *written responses*. Hence, my instrument, so similar to that employed by Healy and Hoyles, also served to collect similar types of data. However, when it came to assessing the participants' ability to identify and evaluate effective forms of proof, Healy and Hoyles presented students with a variety of already solved proofs and asked them to differentiate among, identifying those that made sense that those that did not. In contrast, I asked the student teachers to complete the proofs themselves; thus, the participants in my study not only needed to identify the best or most effective argument (that which made the most sense), they also had to demonstrate an ability to construct those effective arguments.

Our respective findings also bear some scrutiny. Healy and Hoyles (2000) found that majority of the students were unable to construct valid proofs in the domain of algebra. My study led me to conclude that, in the mathematical domains of both geometry and number theory, the student teacher participants experienced considerable difficulty constructing correct proofs. Also of note, Healy and Hoyles determined that students predominantly use empirical arguments for their own proofs; I noted the same tendency among the participants in my study. This was the case even though a majority of the participants in both studies realized that empirical proofs sit lower on the hierarchy of proof difficulty. Quite a few participants within each of the study groups shared that they believe once a proof is established, it is final and cannot be changed. Healy and Hoyles came to the conclusion that students firmly believe that teachers expect them to demonstrate, in their proof work, complicated algebra. The student teacher participants in my study seemed to share a similar point of view: many of them attempted to incorporate into their work both complicated formats and various algebraic manipulations with the apparent aim of making the proof look "ritualistic". Healy and Hoyles noted that students prefer explanatory arguments most of all; I see a similar tendency among the participants in my study. In instances where students failed to prove the proof by means of "ritualistic" manipulations, those students typically then opted for narrative form.

Knuth (2002a) in Relation to My Research

Knuth's (2000a) study and my own were similar with regards to, first, sample size and attributes, and, second, aims and focus. Knuth worked with 16 in-service teachers and I worked with 17 pre-service teachers. The participants in both studies had close associations with secondary level school mathematics and were former mathematics majors. One of the central concerns in Knuth's study was to determine how teachers perceived the role and function of proof within the context of the school classroom. My study also asked students to consider the role and function of proof as it pertained to mathematics instruction at the secondary level. The participants in both Knuth's study group and my own described various roles for proof in the math classroom. These results suggest that teachers and student teachers the role of mathematical proof to be multiple and diverse. Of particular interest, not one participant in either of the two studies mentioned that one function/role of proof is to promote understanding. Knuth explains this oddity by pointing out that students' experiences with proof typically involve arriving at a correct final product by means of deductive reasoning; hence, when these practicing teachers where students themselves they would have been unlikely to have been introduced to proof as a means for developing and enhancing understanding. It is not surprising, given this context, that Knuth's teacher participants also tended to identify as *correct* arguments, those arguments employing symbolic manipulation or particular expected formats.

Empirical evidence proved to be important to participants in all studies. After having testing their proofs with empirical evidence, a number of participants in Knuth's study developed strong convictions as to the correctness of the conclusion. Both Martin and Harel (1989) and Healy and Hoyles (2000) recorded similar observations, even though their participants had less extensive backgrounds in mathematics than did Knuth's. My student teacher participants, who, like Knuth's teacher participants, had received an extensive education in mathematical concepts, also seemed to derive a great deal of confidence from testing their proofs by means of empirical evidence.

Knuth summarized his study with the statement that teachers' conceptions of proof are somewhat limited; I have come to a similar conclusion. In his doctoral dissertation *The nature of secondary school mathematics teachers' conceptions of proof,* Knuth (1999) noted that "the teachers' facilities with proof, as well as their proof schemes, were in many cases, not what one would expect from individuals who are knowledgeable about mathematics and, in particular, from teachers of secondary school mathematics" (p. 161). He also notes that reform efforts in mathematics education will place serious demands on secondary school mathematics teachers. He concludes: "Their [teachers'] success in responding to these demands depends largely on their own conceptions of proof. …The results of this study suggest that such success may be difficult for teachers given aspects of their current conceptions of proof in secondary school mathematics" (p. 162) The results of my study, suggest that student teachers who are about to begin their teaching careers may experience a similar difficulty.

Jones (1997) in Relation to My Research

The main similarity between my study and that conducted by Jones is the fact that we both used concept maps as a tool to assist us in analyzing prospective secondary school teachers' conception of mathematical proof. Jones administered his task to a group of 25 student teachers. The task consisted of generating concepts maps around the concept of mathematical proof. Following a brainstorming session, the student teachers came up with 24 different terms all related to the idea of mathematical proof. Jones aimed to see how student teachers with varying GPAs (the terminology he uses is Pass, Third Class Honours, and Second Class Honours) differ in their ability to construct concept maps. He based his analysis on the following factors:

(1) number of key terms;

(2) number of relationships;

(3) number of specified relationships;

(4) number of cross links/multiple relationships.

His study led to two findings: first, the higher the student's qualifications, the more terms the student used in constructing the map; and second, the higher the student's qualifications, the more sophisticated the map. Indeed, the most highly qualified student teacher produced the most sophisticated map by adding other relevant terms, terms that were not on the original list of 24. Jones' student teacher participants created their concept maps during week number 16 in a 36 week-long course.

My study, when set next to that conducted by Jones, offers some interesting insights. I opted to use the list of terms as compiled by Jones' student teachers. I shared all 24 terms with my student teacher participants and instructed them to use whatever terms they found helpful in constructing their concept maps: they could use all 24, none of the 24, or any combination of the 24. There were also welcome to incorporate terms of their own choosing, terms that were not present in the original list of 24.

Concept maps, as I noted earlier, are part of the secondary education curriculum at this university; yet, for the most part, student concept maps were fairly simplistic, with most students selecting one term, placing it in the middle of the page, and adding other terms as satellites around it. If one regards complexity of design and extensive detail as indicators of sophisticated understanding, then most of the student teachers that generated concept maps within my study could be said to have demonstrated rather unsophisticated understandings of mathematical proof. If one also regards interconnectedness and clearly specified links among the terms as further indicators of high-level understanding, then most of the student teachers in my study, by not specifying links, could be said to have demonstrated rather simplistic understandings of mathematical proof. Finally, if one considers the clear presence of a high number of terms as an indicator of high-level conceptual understanding, then, on the basis of the various ways in which the student teachers incorporated terms, I suggest that some demonstrated deeper understandings than others. The person who referred to him self as a "mathematical geek" managed to incorporate all 24 terms, plus five more that were not on the list, generating a map in the shape of a tree with branches and sub-branches. This proved to be the most structurally complicated map of any participant. On the other hand, the participant who used the least number of concepts—seven—chose them all from the list. This student generated a fairly simple, linear concept map. The data provided by the concept maps supports my observations based on the data provided by the mathematical tasks: these student teachers have difficulty conceptualizing the complex nature of mathematical proof. In general, my findings also substantiate Jones' observations.

The Research Process: Lessons Learned, Lessons Pending

The literature refers to only a few studies conducted with the aim of examining the conceptions of proof and the proving skills of prospective secondary-level mathematics teachers. Given that student teachers will soon be instructing these very concepts to secondary students in the classroom, it is vital that educational researchers and teacher educators identify the extent to which student teachers are prepared for this challenge. The aim of my research study was to provide a snap shot of future secondary school mathematics teachers' conceptions of proof and their ability to construct proof as they neared the end of their preparation programs. I believe I have achieved my goal.

Some of the most important lessons learned in this study, and some of the critical areas upon which I shall focus my attention in future research efforts, pertain to research methodology. My intention in conducting interviews was to record—in the words of the students themselves—the participants' understandings of *how* and *why* they approached the various proof tasks in the manner that they did. I wanted to provide students with an opportunity to consider the tasks and the completion of the tasks in a meta-cognitive way. I believed that such an exercise would be valuable not only to me as the researcher, but to each of them, especially at a point in time when they were about to take those first steps in establishing their teaching careers in the field.

Unfortunately, the lapse of time between the completion of the task and the interview itself prevented me from achieving my goal. I was limited in both the number of interviews that I was able to conduct (only three male students responded to my request for interviews) and by the lengthy interval between the time when the tasks were completed and the time when I was able to sit down with students to discuss the work that they had completed. My aim was to interview all students whose task work had prompted me to ask questions about what they had done and why; however, this was not achieved.

However, I did learn a great deal from and about the interview process. Most importantly, I learned that in future I must conduct interviews either immediately following the task-work or while the student progresses through the task. Otherwise, participants forget, not only what they have done, but also how and why it was done. When this happens, the interview fails to fulfill its purpose. Transcripts of the conversations reveal my inexperience as an interviewer. I frequently follow the participant as he leads us into other areas of discussion; consequently, the interviews often deviate from my central concerns. While there is value in the meanderings that circle around a topic, at times the interviews moved into clearly irrelevant areas.

Similarly, with the concept map task, it may have been more effective if I had reviewed the concept map process before I administered the task. I learned that the students' experiences with concept maps had been largely limited to non-mathematical areas of study: had I discussed the use of concept maps in a mathematical context, and provided students with examples of how one might use the concept map as means of demonstrating one's understanding of a particular mathematical concept, the students would have been better prepared to tackle the specific concept map representation task that I asked of them—the task dealing with mathematical proof.

Revisiting the Research Question

I began this research study by posing the question **"What understandings do student teachers possess concerning the notion of mathematical proof?"** Now, I have sufficient data to provide an answer. In response to my first of two secondary questions-- "What do student teachers believe about the nature and role of proof"—I have determined that the majority of the student teacher participants in this study view proof as an integral part of mathematics. Most of them defined proof in terms of its function, and, in fact, typically noted both a primary and a secondary function within the scope of their definitions. (Note: I identify only primary responses in Table 6.1: secondary definitions were not included). Such a response is consistent with findings reported within the literature on mathematics education: when asked to define proof, virtually all studies show that the participants defined the concept in terms of its function. This suggests that student teachers *do* understand that proof can play different roles in mathematics, even though they typically identify verification as the most critical role.

However, participants varied in their views of the role that mathematical proof should play in school mathematics. A number of the student teachers regard mathematical proof as something that *only* "smart kids" can comprehend. The goal of reform efforts in mathematics education is to assist *all* students in developing sophisticated understandings of proof (NCTM, 2000); but, according to most of these student teachers, proof should be reserved for those who are mathematically 'bright' and/or intending to enroll in advanced-level university mathematics courses. In this respect, the student teachers' beliefs do not reflect the foundational philosophy of reform in secondary school mathematics. Moreover, many of the participants stated that the best way to learn proof is by "more and more practice". Such a belief, I contend, suggests that, as secondary students themselves, these young people had little, if any, exposure to reform principles. Clearly, they do not conceive of the mathematics classroom as a learning community in which teacher and students collaboratively construct knowledge. In accordance with tradition, quite a few student teachers identified geometry as the true home of mathematical proof. This suggests that traditional approaches to mathematical proof are still commonplace. Certainly, this is consistent with the findings reported in the mathematics education literature.

No matter which of the four mathematical tasks student teachers were engaged in proving, they typically, first tried out examples, and then formulated and investigated conjectures. This is an important mathematical practice for it develops within students a drive to understand why a conjecture is true (Hoyles, 1997). Furthermore, after coming up with a conjecture, students then tried to formalize the argument with algebra or with natural language. However, when they got stuck and were not sure how to proceed with the conceptual argument, they resorted to proving the task by means of empirical evidence. It is interesting to note that student teachers who used empirical evidence in proving one task did not necessarily use that same approach in other tasks. In other words, a student who used empirical evidence to prove one task may have very well have used an algebraic approach, a geometric method or, even, a verbal argument to prove another task. Tahira, for example, illustrates this. She provided empirical justification for the first two tasks (see p. 106 and p. 135); then she took a thought experiment approach in proving the third (see p. 175) *and* the fourth (see p. 191). Similarly, Brian set out to prove the first two tasks by means of thought experiments (see p. 150 and p. 173) and then provided empirical evidence for the final task (see p.182). The fact that students proved these tasks in various ways (and not simply by providing empirical justifications) indicates that students *do understand* that examples are not always sufficient when it comes to proving. (Indeed, I suspect that some students opted for empirical justifications, not because they believed these to be the way to generate a valid proof, but because they felt that examples were 'better than nothing.') It also suggests that students select what they deem to be the most appropriate approach to solving the proof on the basis of the nature of the task itself. Students did not approach these tasks by applying one predetermined, all-purpose proving format.

Student teachers also understand that one may apply different techniques in solving proofs, techniques that reflect different ways of reasoning. A number of the student teachers in my study believe that proving activities can help students develop logical thinking skills that can then be applied to both mathematical and non-mathematical contexts. In solving these four tasks, students generally used either inductive or deductive justification. In one instance, the student teacher employed proof by contradiction, and, in another, the student teacher provided a counter example that disproved the task. I observed that when student teachers did not know how to prove the task, they resorted to using irrelevant algebraic expressions or the traditional two–column format (see p. 175). I also noticed that some students who managed to prove the task by means of deductive argument, later verified their result with empirical justifications (for example, see pp. 144, 146 and 158).

My study also suggests that logical assumptions about proof and proving are not necessarily correct. I included two geometry tasks in my study. Given the fact that, in general, teachers introduce geometry proofs to their students by using the two-column format, one might assume that students would routinely apply a two-column format when asked to complete geometric proofs. Similarly, since teachers introduce proof not only with a two-column format, but also as a means of verifying already proven results, one might assume that students would routinely approach all verification tasks using a two-column format. My study indicates that such assumptions are problematic. Sometimes the assumption holds true; but at other times it does not. Student teachers did not always apply a two-column format when proving a geometry task; nor did they when seeking verification. Instead, they seemed to prefer an explanatory type of proof, one grounded in plain language; what is more, they tended to prefer explanatory formats regardless of the task involved. In addition, students who employed a thought experiment approach tended to favor verbal over symbolic forms. I assumed that most of the students, given their extensive backgrounds in mathematics, would opt for a symbolic type of thought experiment. However, that was not the case. Hanna (1990; 1995), Hanna and Jahnke (1993), and other researchers claim that the explanatory power of proof has more pedagogical value than the formalistic. My study supports their assertion. Yet, this obvious preference, clearly evident in their mathematical work, was not evident in their definitions of proof. It seems there is dis-connect between belief and practice.

The answer that I have arrived at in response to my other secondary research question ("How able are student teachers when it comes to completing mathematical proof?") concerns me a great deal. When it came to constructing secondary school-level mathematical proofs, this group of student teachers lacked proficiency. Neither did their work suggest more proficiency proving one type of task as opposed to another (specifically, geometric over number theory, or vice versa). Although student teachers understand the role of proof and its various functions both within and beyond mathematics, they have great difficulty when it comes to applying that understanding in practical ways. Theoretical and practical competencies are not aligned. My answer to this question, therefore, leads me directly to another: "How can one develop and enhance student teachers' capabilities and skills when it comes to solving mathematical proofs?" Since proof and proving are both central to mathematics and difficult to master, this becomes an important question. Quite simply, the concept of mathematical proof is a challenging one—it does not matter if the 'learner' is a junior high school, secondary, or post-secondary student or perhaps a prospective or even a practicing teacher. How can we make the difficult do-able? Mathematicians and mathematics educators, by establishing a *culture of proving* in *all* classrooms, may eventually assist learners by shifting their perceptions of mathematical proof from 'that which is impossible' to 'that which is positively possible.'

CHAPTER 14. LOOKING AHEAD

Areas for Further Research

The findings of my study raise key questions for teacher educators engaged in the important task of assisting student teachers in learning and implementing new methods and strategies for introducing mathematical proof to secondary students in the classroom. First, what teaching strategies will be most effective when introducing students to proof? Educators must develop instructional alternatives to traditional methods of writing the proof on the white/black board and asking students to copy it down and memorize it. Many of the students in my study used pictorial representation to provide a link between the concrete and the abstract/symbolic. Teacher educators would do well to investigate how visual representations can assist students in deepening their understanding of proof.

Second, how can teachers identify the critical moments in the proving process that are most likely to cause students to become 'stuck'? Although many of my student teacher participants could start a proof, they were unable, in a number of cases, to complete it correctly. When they were only one or two steps away from successfully completing the proof, they dropped their ideas and turned to a very different approach in their effort to prove the task. What signs or indications, evident in the early stages of the proving process, might assist teachers in anticipating subsequent moments of 'mental paralysis'? And, how can students themselves learn to anticipate these moments, work through them and persevere, ultimately moving on to the next step in the proving process and so eventually arrive at a correct proof?

Third, how can teachers ensure that students become aware of the various functions of proof? Clearly, the uses of proof go far beyond verification; yet most student teachers continue to view proof precisely in these terms. The data I collected from both Section A and Section B of my questionnaire suggest a need for a systematic exploration of the *occurrence* of proofs and proving tasks in different mathematical domains both in text books and in classrooms within the province of Alberta. But, an examination of where and when proof shows up in the curriculum is not sufficient. Researchers also need to address *variability*. Are teachers and textbooks introducing proof in terms of its multiple and diverse functions? And, if the answer proves to be 'no', how can we rectify the situation? If proving is to play an instrumental role in the mathematics classroom, teachers must go beyond simply making students write more proof exercises, in the same manner, for the same purpose as they have done in the past. Only then will we prevent what Galbraith (1982) calls a "recycling effect"—when teachers, who lack fundamental skills, fail to teach students those fundamental skills, and those students, becoming teachers themselves, then perpetuate the process.

My findings also identify a need for further scholarly research in mathematics as it relates to cognition. Much of the data collected in this research study consists of failed proof attempts that I found difficult to analyze. The data clearly indicates that mastering proof is far more challenging than, perhaps, educators generally realize. The student teachers in my study often failed to complete a proof correctly because they eventually reached an impasse; at this point, they simply did not know what to do. In order to understand these failures, it will be necessary to study the processes that students use when constructing proofs. More cognitive-based research on proof construction needs to be done. In addition, since the proving tasks asked of these students were all drawn from or reflective of secondary school level mathematics, further study in student teachers' understanding of content areas and mathematical tasks *other* than proof and proving would be beneficial. Only by addressing precisely what it is that student teachers need to understand about mathematics, mathematical thought, and the skills necessary for effective instruction of mathematical concepts will teacher educators be able to prepare a new generation of teachers for the challenge of introducing a new generation of students to the complex mathematical and technological world in which we now live.

Recommendations: Focus on the Student Teacher

Knuth's (1999, 2002a, 2002b) research indicates that many practicing teachers possess limited understanding of and skill in the area of mathematical proof. Other studies—for example, that conducted by Martin & Harel (1989)—report similar findings. Hence, my results should not come as a surprise. However, we must remember that teachers' subject matter conceptions represent one of the most important influences on their instructional practice and, ultimately, on what students learn (Thompson, 1984). If student teachers are to be successful in enhancing the role of proof in secondary mathematics classrooms, they must be assisted in constructing a strong understanding of the nature and role of proof in the school mathematics curriculum. In other words, as Knuth (2002c) notes, teachers/student teachers need to experience proof as a *meaningful* tool for teaching and learning mathematics.

If teacher educators expect student teachers to develop a 'reformed understanding' of what it means to learn and teach mathematics, then student teachers themselves must be offered a wealth of experiences with this mode of learning *and* with this new approach to teaching. I suggest that prospective teachers become involved in the process of learning mathematics *while* they are learning to teach the content: they must be challenged both pedagogically *and* mathematically. However, more mathematics content does not necessarily produce mathematical vitality (Ball, 1990; Galbraith, 1982). Teacher educators need to think about developing a number of mathematical courses within the faculty of education, courses geared specifically to the concepts within the secondary school mathematics curriculum, courses that allow student teachers to explore, extend, and elaborate upon these curriculum concepts so as to construct deep understandings of what it is that they must later teach to children and adolescents. By grounding these math courses in learning approaches and teaching methods that are deliberately aligned with the current reform movement in mathematics education, teacher educators will model what student teachers themselves need to learn as they begin to explore the possibilities within instructional practice: a new way of teaching mathematics that emphasizes the role that math can play in exploring, understanding, and creating the world in which we live. This is how teacher educators in mathematics can best meet the pedagogical needs of future mathematics teachers.

Mathematics Education: Directions for the Future

Reasoning and proof skills form a pivotal fulcrum in any efforts to reform mathematics education. As the NCTM (2000) stresses "Reasoning and proof are not special activities reserved for special times or special topics in the curriculum; [rather they] "should [be] a natural, ongoing part of classroom discussions, no matter what topic is being studied" (p. 342). The effective teaching of "proof and reasoning", as the NCTM (2000) underscores, will develop and promote critical thinking skills in the mathematics classroom, skills that can then be applied to various and diverse contexts. Will student teachers who have not yet refined their own critical thinking skills when it comes to proof and proving be able to teach those skills effectively to their students? I suggest not.

Current reform efforts in mathematics education and curriculum will continue to call for an increased emphasis on "reasoning and proof" as key stepping-stones towards a better understanding of mathematics. But this directive cannot be fulfilled if teachers themselves lack a solid understanding of "reasoning and proof." It should be clear by now that if teachers lack such understanding, so too will most of their students. The role of the teacher is critical since the instruction that students receive in school influences impacts their understandings of mathematics and their ability to solve problems and develop logical reasoning and justification skills. Teacher educators must remember that when student teachers, demonstrating incomplete understandings of proof and proving, return to the educational system as mathematics teachers, they are unlikely to rise to one of the central challenges of mathematics education in the early decades of the twenty-first century: achieving system-wide reform through a greater emphasis upon and wider application of proof and proving in the mathematics curriculum.

Galbraith (1982), some twenty-five years ago, expressed concern about the quality of an educational system in which students, failing to master essential mathematical concepts, eventually return to the system as mathematics teachers. Clearly the quality of education diminishes as incomplete understandings become recycled from teacher to student to a new generation of teachers and students. Few, I am certain, would doubt the destructive impact of such a "recycling effect" (Galbraith, 1982, p. 90). What I find especially distressing, however, is that researchers and educators today continue to voice such concerns (see, for example Jones, 2000). This fact suggests how very much reform in school mathematics is needed. As the current wave of reform in mathematics education begins to flow from the academy into the classroom we must remember that ongoing research can serve to facilitate the process. Indeed, further research addressing specifically the issues identified within my study is not only necessary, but also vital, if teacher educators, practicing teachers, and student teachers are to revitalize the mathematics curriculum and implement reforms that will ensure effective teaching and effective learning for both teachers and students in mathematics education.

References

Afamasa-Fuatai, K (2004a). An undergraduate student's understanding of differential equations through concept maps and vee diagrams. In A.J. Canas, J.D. Novak, &F.M Gonzalez (Eds.) *Proceedings of the First International Conference on Concept Mapping*. Retrieved January 10, 2007 from http://cmc.ihmc.us/papers/cmc2004-208.pdf

Afamasa-Fuatai, K (2004b). Concept maps and vee diagrams as tools for learning newmathematics topics. In A.J. Canas, J.D. Novak, & F.M Gonzalez (Eds.), *Proceedings of the First International Conference on Concept Mapping*. Retrieved January 10, 2007 from http://cmc.ihmc.us/papers/cmc2004-208.pdf

Alberta Learning (1996). *Program of study for mathematics*. Edmonton, Canada: Alberta Learning.

Alibert, D. (1988). Towards new customs in the classroom. *For the Learning of Mathematics,* 8(2), 31-35.

Almeida, D. (1996). Variation in proof standards: Implications for mathematics education. *International Journal of Mathematical Education in Science and Technology,* 27(5), 659-665.

Ausbel, D.P. (1963). *The psychology of meaningful verbal learning.* New York: Grune & Stratton.

Ausbel, D. P. (1968). *Educational psychology: A cognitive view*. New York: Grune & Stratton.

Ausbel, D. P., Novak, J. D., & Hanesian, H. (1978). *Educational psychology: A cognitive view* (2nd ed.). Holt, Rinehart and Winston: New York.

Balacheff, N. (1988). Aspects of proof in pupils' practice of school mathematics. In D. Pimm (Ed.) *Mathematics, teachers and children,* (pp. 216-235). (Trans. By D. Pimm). London: Hodder & Stoughton.

________. (1991). Treatment of refutations: Aspects of the complexity of a constructivist approach to mathematical learning. In E.von Glassersfeld (Ed.) *Radical Constructivism in Mathematics Education*, (pp. 89-110). Dordrecht: Kluwer Academic Publishers.

Ball, D. L. (1990). The mathematical understanding that prospective teachers bring to teacher education. *The Elementary School Journal,* 90(4), 449-466.

Bell, A. (1976). A study of pupils' proof-explanations in mathematical situations. *Educational Studies in Mathematics*, 7, 23-40.Blaxter, L., Hughes, C., & Tight, M. (2001). *How to research* (2nd ed.). Open University Press: Buckingham.

Chazan, D. (1993). High school geometry students' justification for their views of empirical evidence and mathematical proof. *Educational Studies in Mathematics,* 24, 359-387.

College of Agricultural, Consumer and Environmental Sciences, University of Illinois. Retrieved December 15, 2006 from, http://classes.aces.uiuc.edu/ACES100/Mind/c-m2.html

Conference Board of the Mathematical Sciences. (2001). *The mathematical education of teachers*. Providence, RI: American Mathematical Society.

Chazan, D & Yrushalmy, M. (1989). Charting a course for secondary geometry. In R. Lehrer & D. Chazan (Eds.) *Designing learning environments for developing understanding of geometry and space,* (pp. 67-90). Mahwah, NJ: Erlbaum.

Coe, R. & Ruthven, K. (1994). Proof practices and constructs of advanced mathematicsstudents. *British Educational Research Journal, 20, 41-53.*

Cohen. D. (1987). The use of concept maps to represent unique thought process: toward more meaningful learning. *Journal of Curriculum and Supervision,* 2, 285-289.

Cohen, L. & Manion, L. (1994). *Research methods in education.* Routledge: New York.

Creswell, J. W. (1997). *Qualitative inquiry research design: Choosing among five traditions.* Sage Publications: Thousand Oaks.

Cuoco, A. (2001). Mathematics for teaching. *Notices of the American Mathematical Society,* 48(2), 168-174.

Cyr, S. (2004). Conceptions of proof among pre-service high school mathematics teachers. In D. E. McDougall & J. A. Ross (Eds.) *Proceedings of the Twenty-Sixth Annual Meeting of North American Chapter of the International Group for the Psychology of Mathematics Education,* (pp. 569-574). Toronto: Canada.

Davis, B. (2004). *Inventions of teaching: a genealogy.* Mahwah, NJ: Lawrence Erlbaum.

Davis, B., D. J. Sumara, & T.E. Kieren. (1996). Cognition, co-emergence, curriculum. *Journal of Curriculum Studies*, 28, 151-169.

Davis, B. & E. Simmt. (2003). Understanding learning systems: mathematics education and complexity science. *Journal for Research in Mathematics Education*, 34(2), 137-167.

Davis, B. & Upitis. R. (2004). Pending knowledge: on the complexities of teaching and learning. *Journal of Curriculum Theorizing*, 20(3), 113-128.

Davis, P. J. & Hersh, R. (1981). *The mathematical experience.* New York: Viking Penguin Inc.

de Villers, M. (1990). The role and function of proof in mathematics. *Pythagoras,* 24, 17-24.

__________. (1999). *Rethinking proof with the geometer's sketchpad.* Key Curriculum Press: Emeryville, CA.

Dunham, W. (1994). *Mathematical universe.* New York: John Wiley & Sons Inc.

Epp, S. (1994). The role of proof in problem solving. In A.H. Schoenfeld (Ed.) *Mathematical thinking and problem solving,* (pp. 257-269). Hillsdale, NJ: Lawrence Erlbaum Associates Publishers.

Ernest, P. (1998). Social constructivism as a philosophy of mathematics: Radical constructivism rehabilitated. Retrieved May 14, 2007 from www.people.ex.ac.uk/PErnest/soccon.htm

Fawcett, H.P. (1938). *The nature of proof.* Thirteenth yearbook of the National Council of Teachers of mathematics. NY: Columbia University Press.

Fennema, E. & Franke, M. (1992). Teachers' knowledge and its impact. In D. Grouws (Ed.), *Handbook of Research on Mathematics Teaching and Learning,* (pp. 147-164). New York: Macmillan

Fishbein, E. (1982). Intuition and proof. *For the Learning of Mathematics,* 3(2), 9-24.

Frege, G. (1950/1884). *The foundations of arithmetic: a logical-mathematic investigation into the concept of number.* Oxford: Blackwell.

________. (1984). *Collected papers on mathematics, logic, and philosophy.* Brian McGuiness (Ed.) Trans. By Max Black. Oxford: Blackwell.

Hasemann, K. & Mansfield, H. (1995). Concept mapping in research on mathematical knowledge development: Background, methods, findings and conclusions. *Educational Studies in Mathematics,* 29(1), 45-72.

Galbraith, P. (1982). The mathematical vitality of secondary mathematics graduates and prospective teachers: A comparative study. *Educational Studies in Mathematics,* 13(1), 89-112.

Gardiner, T & Moreira, C. (1999). Proof matters. *Mathematics Teaching,* 169, 17-21.Hadas, N., Hershkowitz, R., & Schwarz, B.B. (2000). The role of contradiction and uncertainty in promoting the need to prove in dynamic geometry environments. *Educational Studies in Mathematics*, 44 (1&2), 127-150.

Hanna, G. (1983). *Rigorous proof in mathematics education.* OISE Press: Toronto.

________. (1989). More than formal proof. *For the Learning of Mathematics.* 9(1), 20-23.

________. (1990). Some pedagogical aspects of proof. *Interchange.* 21(1), 6-13.

________. (1991). Mathematical proof. In D. Tall (Ed.) *Advanced mathematical thinking*, (pp. 54-61). Dordrecht: Kluwer Academic Publishers

________. (1995). Challenges to the importance of proof. *For the Learning of Mathematics,* 15(3), 42-49.

________. (2000). Proof, explanation and exploration: An overview. *Educational Studiesin Mathematics*, 44(1), 5-23.

________. (2000a). A critical examination of three factors in the decline of proof. *Interchange,* 31(1), 21-33.

Hanna, G. & Jahnke, H.N. (1993). Proof and application. *Educational Studies in Mathematics*, 24(4), 421-438.

________. (1996). Proof and proving. In A.J. Bishop, et al. (Eds.) *International handbook of mathematics education*, (pp. 877-908). Kluwer Academic Publishers, Dordrecht.

Harding, B. (1999). Proof and justification: A primary teaching perspective. *Mathematics Teaching*. 169, 12-16.

Healy, L. & Hoyles, C. (2000). A study of proof conceptions in algebra. *Journal for Research in Mathematics Education,* 31(4), 396-428.

Hersh, R. (1993). Proving is convincing and explaining. *Educational Studies in Mathematics,* 24, 389-399.

Housman, D. & Potter, M. (2003). Proof schemes and learning strategies of above average mathematics students. *Educational Studies in Mathematics*. 52(3), 139-158.

Hoyles, C. (1997). The curricular shaping of students' approaches to proof. *For the Learning of Mathematics,* 17(1), 7-15.

Jahnke, N.H. (2005). A genetic approach to proof. Retrieved January 10, 2007 from www.lettredelapreuve.it/Newsletter/05Automne/CERME4Jahnke.pdf

Johnson, M. (2007). Number theory and proving real theorems. Retrieved March 27, 2007 www.stanford.edu/class/cs103a/handouts/24.%20Real%20Theorems.pdf.

Jones, K. (1997). Student teachers' conceptions of mathematical proof. *Mathematics Education Review,* 9, 21-32.

Joseph, G.G. (2000). *The crest of the peacock: Non-European roots of mathematics.* Princeton University Press: Princeton.

Koshy, T. (2002). *Elementary number theory with applications.* Harcourt Science and Technology Company: San Diego.

Knuth, E. J. (1999). *The nature of secondary school mathematics teachers' conceptions of proof*. Unpublished doctoral dissertation. University of Colorado.Denver, Colorado.

________. (2002a). Secondary school mathematics teachers' conceptions of proof. *Journal for Research in Mathematics Education,* 33(5), 379-405.

________. (2002b). Teachers' conceptions of proof in the context of secondary school mathematics. *Journal for Mathematics Teacher Education*, 5, 61-88.

________. (2002c). Proof as a learning tool. *Mathematics Teacher,* 95(7), 486-490.

Knuth, E.J. & Elliot, R.L. (1998). Characterizing students' understandings of mathematical proof. *Mathematics Teacher,* 91(8), 714-717.

Lakatos, I. (1976). *Proofs and refutations: The logic of mathematical discovery.* Cambridge, UK: Cambridge University Press.

Latterell, C.M. (2005). *Math wars: A guide for parents and teachers.* Westport: Praeger.

Leddy, J.F.J. (2001). *Justifying and proving in secondary school mathematics.* Unpublished doctoral dissertation. OISE, University of Toronto, Canada.

Lucast, E.K. (2003). *Proof as method: A new case for proof in mathematics curricula.* Unpublished masters' thesis. Carnegie Mellon University, Pittsburgh, United States.

Luthuli, D. (1996). The proof of Euclidean geometry riders as an exercise in mathematical creativity. *Pythagoras,* 41, 17-26.

Ma, L. (1999). *Knowing and teaching elementary mathematics: Teachers understanding of fundamental mathematics in China and the United States*. Mahwah, NJ: Lawrence Erlbaum.

Marrades, R., & Guiterrez, A. (2000). Proofs produced by secondary school students learning geometry in a dynamic computer environment. *Educational Studies in Mathematics,* 44, 87-125.

Maher, C.A. & Martino, A.M. (1996). The development of the idea of mathematical proof: A five year case study. *Journal for Research in Mathematics Education.* 27(2), 194-214.

Manin, Y. (1997). *A course in mathematical logic.* New York: Springer Verlag.

Martin, W.G. & Harel, G. (1989). Proof frames of preservice elementary teachers. *Journal for Research in Mathematics Education,* 20(1), 41-51.

McMurray, R. (1978). Flow proofs in geometry. *Mathematics Teacher,* 71(11), 592-595.

Moreira, C. (1999). *Reflections on proof in mathematics and mathematics education.* In Ahmed, H. Williams, and J.H. Kraemer (Eds.) Cultural diversity in mathematics (education): Cheaem 51, (pp. 347-353). Chichester: Horwood Publishing.

National Council of Teachers of Mathematics. (1989). The *curriculum and evaluation standards for school mathematics.* Reston, VA: National Council of Teachers of Mathematics.

________. (1991). *Professional standards for teaching mathematics.* Reston, VA: National Council of Teachers of Mathematics.

________. (1995). *Assessment standards for school mathematics.* Reston, VA: National Council of Teachers of Mathematics.

________. (2000). *Principles and standards for school mathematics.* Reston, VA: National Council of Teachers of Mathematics.

Newman, L. (2000). The meanings of methodology, Chapter 4. *Social research methods: Qualitative and quantitative approaches*. Needham Heights, MA: Allan & Bacon.

Novak, J.D. (1990). Concept mapping: A useful tool for science education. *Journal of Research in Science Teaching,* 27, 937-949.

Novak, J. D. & Canas, A.J. (2006). The theory underlying concept maps and how toconstruct to construct them. Technical Report IHMC Cmap Tools 2006-01. Florida Institute for Human and Machine Cognition. Retrieved January 10, 2007 from http://cmap.ihmc.us/Publications/ReserachPapers/TheoryUnderlyingConceptMaps.pdf

O'Daffer, P.G. & Thornquist, B.A. (1993). *Critical thinking, mathematical reasoning andproof.* In P.S. Wilson, (Ed.) Research ideas for the classroom: High school mathematics, (pp. 39-56). New York: Macmillan Publishing Company:

Polya, G. (1960). How to solve it. In J. R. Newman (Ed.) *The world of mathematics* (Vol. III). Soho, London: Novello & Co., Ltd.

Philip, D.C. (2000). *An opinionated account of the constructivist landscape.* In D.C. Philip (Ed.). Constructivism in Education: opinions and second opinions on controversial issues, (pp. 1-15). Chicago, IL: National Society for the Study of Education.

Rebich, S., & Gautier, C. (2005). Use of concept mapping to reveal prior knowledge and conceptual change in a mock summit course on global climate change. *Journal of Geosciences Education,* 53(4), 355-366.

Raman, M. J. (2002). *Proof and justification in collegiate calculus.* Unpublished doctoral dissertation. University of California, Berkeley.

________. (2003). Key ideas: what are they and how can they help us understand how people view proof? *Educational Studies in Mathematics*, 52(3), 319-325.

Rodd, M.M. (2000). On mathematical warrants: Proof does not always warrant, and a warrant may be other than proof. *Mathematical Thinking and Learning,* 2(3), 221-244.

Reid, D. A. (1995). *The need to prove.* Unpublished doctoral dissertation. University of Alberta, Canada. (2002). What is proof? *International Newsletter on the teaching and learning of mathematical proof.* Retrieved Nov 23, 2006 from http://www.didactique.imag.fr/preuve.

________. (2005) *The meaning of proof in mathematics education.* Paper presented to working group 4: Argumentation and proof. Fourth annual conference of the European Society for Research in Mathematics Education, (pp. 17 – 21). Sant Feliu de Guíxols, Spain. February 2005.

Rowland, T. (2001). Generic Proofs: Setting a good example. *Mathematics Teaching,* 177, 40-43.

Schimittau, J. (2004). Use of concept mapping to reveal prior knowledge and conceptual change in mock summit course on global climate change. *Journal of Geosciences Education,* 53, 355-366.

Schoenfeld, A.H. (Ed.) (1994). *Mathematical thinking and problem solving.* Hillsdale, NJ: Lawrence Erlbaum Associates Publishers.

Senk, S. L. (1985). How well do students write geometry proofs? *Mathematics Teacher,* 78(6), 448-456.

________. (1989). Van Hiele levels and achievement in writing geometry proofs. *Journal for Research in Mathematics Education,* 20(3), 309-321.

Simon, M. & Blume, G. (1996). Justification in the mathematics classroom: A study of prospective elementary teachers. *Journal of Mathematical Behavior,* 15(1), 3-31.

Stigler, J.W. & Hiebert, J. (1999). *The teaching gap: Best ideas from the world's teachers for improving education in the classroom.* New York, NY: Free Press.

Siu, M. (1993). Proof and pedagogy in ancient china: Examples from liu hui's commentary on jiu zhang suan shu. *Educational Studies in Mathematics,* 24, 345-357.

Sowder, L. & Harel, G. (1998). Types of students' justifications. *Mathematics Teacher,* 91(8), 670-675.

Tergan, S.O. (1988). Qualitative wissenanalyse. Methodogische Grundlage, In H. Mandl and H. Spada (Eds.). Wissenpsychologie. Munchen: Psychologic Verlags Union.

Thompson, A.G. (1984). The relationship of teachers' conceptions of mathematics and mathematics teaching to instructional practice. *Educational Studies in Mathematics,* 15(2), 105-127.

Thurston, W.P. (1995). On proof and progress in mathematics. *For the Learning of Mathematics.* 15(1), 29-37.

Vanides, J., Yin,Y., Tomita, M., & Ruiz-Primo, M.A. (2005). Using concept maps in the science classrooms. *Science Scope,* 28(8), 27-31.

Yackel, E. & Cobb, P. (1996). Socio-mathematical norms, argumentation and autonomy in mathematics. *Journal for Research in Mathematics Education,* 27(3), 458-477.

Yin, R. K. (1984). *Case study research: Design and methods.* Beverly Hills, CA: Sage Publications.

Wilder, R. (1978). *Evolution of Mathematical Concepts*. Milton Keynes: Open University Press (original date of publication: 1968; New York: Wiley).

Williams, E. R. (1979). *An investigation of senior high school students understanding of the nature of mathematical proof.* Unpublished doctoral dissertation. University of Alberta, Canada.

Winchester, I. (1990). Introduction: Creativity, thought and mathematical proof. *Interchange,* 21(1), i-vi.

Wu, H. (1996). The role of euclidean geometry in high school. *Journal of Mathematical Behavior,* 15, 221-237.

________. (1997). On the education of mathematics teachers. Retrieved March, 23, 2007 from math.berkeley.edu/~wu/teacher-education.pdf.

Zazkis, R. (1999). Challenging basic assumptions: Mathematical experience for pre-service teachers. *International Journal of Mathematical Education in Science and Technology,* 30(5), 631-650.

Zazkis, R., & Campbell, S. R. (1996). Divisibility and multiplicative structure of natural numbers: Preservice teachers' understanding. *Journal for Research in Mathematics Education*, 27(5), 540-563.

________. (2006). *Number theory in mathematics education. Perspectives and prospects.* Mahwah, N.J: Lawrence Erlbaum.

Zazkis, R., & Khoury, H. (1994). To the right of the decimal point: Pre-service teachers' concepts of place value and multi-digit structures. *Research in Collegiate Mathematics Education,* 1, 195-224.

Appendix –I
Questionnaire

Student Teachers' Conceptions of Mathematical Proof

General Information

Name:

Sex: M F

E mail:

Phone number (if you prefer to be contacted by phone):

1. For the past two decades, mathematics education circle have witnessed a world wide trend toward a gradual return to the teaching of proof in high school program of studies.

a) Describe what the notion of proof means to you.

b) In your opinion, what is the best way to develop students' ability to write proof?

c) In your opinion after all, is it important for high school students to learn how to write proof? Why?

d) Additional comments on teaching and learning of proof.

2. For the following given tasks, please show all your work.

a) Prove that when you multiply any 3 consecutive numbers, your answer is always a multiple of 6.

b) Provide a means of calculating the number of diagonals of a polygon when you know how many vertices it has.

c) Below given is a list of 24 terms associated with "mathematical proof". The terms are:

Euclidean	Observation	Logic
General Case	Trial and Improvement	Theorem
Graphical	Assumptions	Axioms
Irrefutable	Syllogism	Deduction
Definitive	Postulate	Lemma
By contradiction	Explanation	Hypothesis
Examples	Implies	Precision
Reasoning	Proposition	Abstraction

d) Prove that the sum of the exterior angles of a polygon is always 360^0.

e) How do you know whether there exists a two digit number "ab" such that the difference between "ab" and "ba" is a prime number?

Appendix -II

Sample Information Letter for students involved in the study

Date

Dear

I am inviting you to take part in a small research study entitled "Student Teachers' Conception of Mathematical Proof". The aim of this study is to understand how student teachers perceive the notion of mathematical proof. Investigating and describing how student teachers understand proof will enable university educators to reformat their curricula that focus on enhancing and guiding student conceptions of mathematical proof (if needed). The results of this research will be disseminated as a Thesis submitted to the University of Alberta, at academic and teacher professional conferences and workshops and through academic and professional journals. Knowledge gained from this study can provide student teachers an insight into their own understanding of mathematical proof. This insight will also help student teachers formulate instructional practice that complies with NCTM (2000) standards.

During the course of this research, you will be invited to participate by completing a mathematical task. You will respond to this task in writing. The task will be administered on a week day after your class and it should not take more than 60 minutes of your time. You will be informed of the time and venue in advance. **The mathematical task WILL NOT BE USED to test your mathematical ability, but to simply see the way you respond to a task about mathematical proof.**

All data will be handled in compliance with the standards reflected in the GFC Policy Manual section 66 entitled "Human Research –University of Alberta Standards for the Protection of Human Research Participants". This document is available on the University website at http://www.ualberta.ca/~unisecr/policy/sec66.html. Names of all participants will be changed to ensure anonymity, and the name of the section or class and other identifiers will also be kept anonymous. Information collected –documents, tapes, transcripts etc. will be kept for a minimum of 5 years following completion of this research in a secure locked cabinet.

You may withdraw from this project at any time without any negative consequences. Any information related to your participation would be destroyed and not used within the written report, or any follow up publications or presentations.

This study has been reviewed and approved by the Faculties of Education and Extension Research Ethics Board (EE REB) at the University of Alberta. For questions regarding participant rights and ethical conduct of research, contact the Chair of the EE REB at (780) 492 3751. If you have any questions or concerns about the research study, please feel to contact me (Thomas Varghese: thomasv@ualberta.ca) or my supervisor Dr David Pimm (dpimm@ualberta.ca).

Thank you for your co-operation.

Sincerely,

Thomas Varghese

Appendix-III

Sample Information Letter for those being interviewed

Date

Dear

I am inviting you to take part in a small research study entitled "Student Teachers' Conception of Mathematical Proof". The aim of this study is to understand how student teachers perceive the notion of mathematical proof. Investigating and describing how student teachers understand mathematical proof will enable university educators to reformat their curricula that focus on enhancing and guiding student conceptions of proof (if needed). The results of this research will be disseminated as a Thesis submitted to the University of Alberta, at academic and teacher professional conferences and workshops and through academic and professional journals.

Knowledge gained from this study can provide student teachers an insight into their own understanding of mathematical proof. This insight will also help student teachers formulate instructional practice that complies with NCTM (2000) standards.

During the course of this research, you will be invited to participate in a semi structured interview. The interviews will be audio taped. **The purpose of the interview IS NOT to test your mathematical ability, but to see the way you understand mathematical proof.**

All data will be handled in compliance with the standards reflected in the GFC Policy Manual section 66 entitled "Human Research –University of Alberta Standards for the Protection of Human Research Participants". This document is available on the University website at http://www.ualberta.ca/~unisecr/policy/sec66.html. Names of all participants will be changed to ensure anonymity, and the name of the school and other identifiers will also be kept anonymous. Information collected –documents, tapes, transcripts etc. will be kept for a minimum of 5 years

This study has been reviewed and approved by the Faculties of Education and Extension Research Ethics Board (EE REB) at the University of Alberta. For questions regarding participant rights and ethical conduct of research, contact the Chair of the EE REB at (780) 492 3751. If you have any questions or concerns about the research study, please feel to contact me (Thomas Varghese: thomasv@ualberta.ca) or my supervisor Dr David Pimm (dpimm@ualberta.ca).

Thank you for your co-operation.

Sincerely,

Thomas Varghese

Appendix-IV

Sample Consent Letter for students involved in the study

I,____, herby consent to participate
(print name)
in the research **"Student Teachers' Conception of Mathematical Proof"** conducted by Thomas Varghese at the University of Alberta. By signing this form I agree to:

Take part in the written task regarding the notion of mathematical proof.

I understand that:

- The task is not to test my mathematical ability and will not be used in the course or anywhere else to decide my grades.
- I may withdraw from the research at any time without penalty.
- All information gathered will be treated confidentially and discussed only with the researcher's supervisor(s).
- Any information that identifies me will be destroyed upon completion of this research.
- I will not be identifiable in any documents resulting from this research.

I also understand that the results of this research will be used only in the thesis produced and in presentations, written articles in various scholarly journals

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(Signature)

(Date)

If you have any questions or concerns about the research study, please feel to contact the researcher (Thomas Varghese: thomasv@ualberta.ca) or his supervisor Dr David Pimm (dpimm@ualberta.ca).

Appendix-V

Sample Consent Letter for those being interviewed

I,____, herby consent to participate
(print name)
in the research **"Student Teachers' Conception of Mathematical Proof"** conducted by Thomas Varghese at the University of Alberta.

By signing this form I agree to:

Take part in a one-on-one interview regarding the notion of mathematical proof.

I understand that:

- The task is not to test my mathematical ability and will not be used in the course or anywhere else to decide my grades.
- I may withdraw from the research at any time without penalty.
- All information gathered will be treated confidentially and discussed only with the researcher's supervisor(s).
- Any information that identifies me will be destroyed upon completion of this research.
- I will not be identifiable in any documents resulting from this research.

I also understand that the results of this research will be used only in the thesis produced and in presentations, written articles in various scholarly journals

\-
(Signature)

\-
(Date)

If you have any questions or concerns about the research study, please feel to contact the researcher (Thomas Varghese: thomasv@ualberta.ca) or his supervisor Dr David Pimm (dpimm@ualberta.ca).