Carving Curriculum out of Chaos: Exploring Teacher Interventions and the Patterning of Small Groups in Mathematics Class

by

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A thesis submitted in partial fulfillment of the requirements for the degree of

Master of Education

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ABSTRACT

Small group work is becoming an increasingly popular structure for promoting communication, sense-making, and reasoning in the mathematics class. This increased sense of student autonomy must co-exist with the curriculum, the expectation to provide opportunities for students to master certain mathematical ideas. This study is an exploration into the patterns that emerge when a teacher of mathematics establishes small groups in their classroom and attempts to balance the inherent complexity of student interaction with the ultimate goal of affecting action toward curricular outcomes.

The analytical framework of complexity theory and the epistemology of enactivism are used to frame the mathematics curriculum as a landscape, and the process of learning mathematics as one of emergence with the mathematical environment while operating in certain curriculum spaces. The image of the curriculum space is introduced as an interpretive tool to visualize the dynamic, drifting nature of the problem deemed relevant to a group, and their movement as they work at varying levels of sophistication with targeted outcomes.

Through this lens, I offer illustrative episodes of group action and an analysis of patterns of teacher action to provide a language to observe small groups as complex systems, inform the work of teachers by presenting viable images of complex systems of learners making sense of their experience, and explore the tendencies of teacher actions when consciously balancing complexity and curriculum. My research suggests that the complex action of small groups can generate curriculum, and that a teacher of mathematics can both prompt action with curricular outcomes and honour the complexity of small groups if they attune themselves to the dynamic movement of a group's problem drift.

PREFACE

This thesis is an original work by Nathaniel J. A. Banting. The research project, of which this thesis is a part, received research ethics approval from the University of Alberta Research Ethics Board 1, Study Title: "Carving Curriculum out of Chaos", Study ID: Pro00064660, June 2, 2016.

"Enjoy thinking sideways"

- *The Big Show*

ACKNOWLEDGEMENTS

Things like this don't happen but for the support of others.

Most glaringly that of my wife, Andra. Your steadying presence and support during this process was always exactly what I needed. This degree has subsumed many long nights, countless moments of indecision, and two complete pregnancies. You are nothing short of spectacular.

Henry, for the many times you wandered into my office and asked, "Daddy, play hockey wif me?" You allowed me to find peace while still somehow providing me with an ever-present image of chaos.

My family, who shaped a legacy on which I simultaneously stand and constantly aim to live up to.

My supervisor, Dr. Elaine Simmt, for always being able to wade through the cacophony of my thought and focus my action.

My colleagues with whom I learned, and continue to learn, to teach.

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LIST OF ABBREVIATIONS

ABBREVIATION	DEFINITION
FPC10	Foundations and Pre-calculus 10
VRG	Visibly Random Grouping
WNCP	Western and Northern Canadian Protocol

CHAPTER 1: INTRODUCTION

"Mathematics education is much more complicated than you expected even though you expected it to be more complicated than you expected."

E. G. Begle, 1971, p. 30

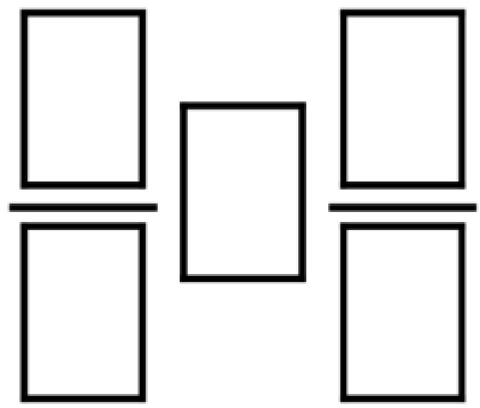

Figure 1.1. An opening classroom episode. The task asked students to place four different digits (2-9) and one operation in the boxes to create an expression with the largest result possible.

Teacher:	How do you know this is the largest?
Student 1:	Because I made the bottoms as small as I could.
Student 2:	Denominators.
Teacher:	Right, denominators. So you chose 2 and 3?
Student 1:	Yes, because they were the smallest, the best.
Student 2:	Because you said we couldn't use 1.
Student 1:	We would have used 1, because it is the best.
Teacher:	1 is the best denominator?
Student 3:	Yes, well no, zero would be the best. It is as small as possible.
Classmates:	No! Can't divide by zero! Zero doesn't work! etc.
Student 3:	I know you can't, but if you could, it would make the largest number.

Western Canada is still feeling the wave of mathematics curriculum and pedagogy reform prompted by the publication of the National Council of Teachers of Mathematics' curriculum and evaluation standards for school mathematics (1989). Now over twenty-five years later, mathematics education in Saskatchewan has undergone a distinct shift in policy. The blanket renewal of secondary school mathematics curricula under the umbrella of the Western and Northern Canadian Protocol (WNCP) has shifted the content students are expected to master, the sequence in which it is encountered, and the pedagogies underpinning their mathematical experiences (WNCP, 2008, 2011). In my experience, the shift in pedagogy has been by far the more tenuous for teachers. The tone of curriculum materials has shifted, and teachers are expected to cultivate student curiosity through investigative activities, productive math communication, and multiple lines of reasoning (WNCP, 2011). The central classroom structure valorized to uphold these practices is the collaborative group, and the curricula and educational policies provide significant pressure on classroom teachers to operate efficiently within this structure (Towers, Martin, & Heater, 2013) which does not share the same affordances of direct instruction with regards to efficient progression through a curriculum by fixing the learning goals prior to instruction (Doll, 1993). Consider the opening vignette of a group's discussion of the task posed in Figure 1.1. What is the teacher to do? Do they entertain the idea of dividing by zero and steer action toward the ideas of division of fractions and infinity? Or perhaps choose to dampen the discourse, define the problem away, and move into the intended lesson on comparing the size of fractions? The teacher is placed in the precarious position of providing space for student sense making, but still retaining a sense of order. This pressure grows in magnitude at the secondary level where—historically—most of the program of schooling can be characterized as a lockstep march toward calculus through a close adherence to curriculum documents organized as hierarchies of skills, each elaborating on the previous level. It seems as though the nature of group work is not well fit to the project of schooling in this regard, yet the world for which schooling is preparing students to live only scarcely contains situations where individuals act in isolation. Teachers attempt to create learning groups in their classrooms with the understanding that groups of students can be unpredictable, and unpredictability has been cast as the diametric opposite of efficiency. It is out of the pragmatic needs of practitioners that the patterns of action from student groups be explored.

Coming to Know My Research Interest

In an attempt to frame the movement toward my research interest, I wanted to stay away from metaphors of linearity. Such images not only betray the convoluted landscape of becoming, but give the environment a static feel and the journey a sense of polished completion. Instead, I wanted to place the interactions with my many influences in the foreground. The idea of mutual influence with one's environment is central to enactivist thought; this mutual influence is explained in greater detail in chapter two, and continues as a major theme throughout the study. It is a continual movement towards coherence in the face of constant change. This study represents, in some sense, the history of my becoming to this point, but lags perpetually in the past tense (Varela, Thompson, & Rosch, 1991). Not only does this document reflect on, and speak to, my countless influences, it has become an influence itself. Like other research conducted in the area of enactivism, the study not only attempts to elaborate on the complex process of education, it is a complex process itself (Reid, 1996). My pragmatic obsession with *carving curriculum out of chaos* has emerged as a result of a variety of influences.

Classroom influences. Of the countless features that influenced my path, the most prominent are the years spent in classrooms employing collaborative group work. From the onset of my career, I was driven by my mandate to *think sideways*—that is, to provide a space for students to encounter divergent mathematical thinking. This began firmly within my control, presenting and playing with curiosities of my own choosing. As my career progressed, the small group became a stalwart structure to allow students to pursue divergence. I was able to design tasks to deepen my focus on divergent thinking through classroom discourse and open problem solving. Operating from an underdeveloped hybridity of social and cultural theory of collaborative work, my goal was to scaffold students toward curricular outcomes through encounters with peers at varying levels of understanding. My actions as the teacher were included in the scaffolding, and a shred of Tayloristic efficiency remained because I was able to intervene with groups that were completely lost while other groups had students that could take on the role of teacher (Taylor, 1911). I developed efficient patterns of interaction in the classroom, and was comfortable networking collections of learners.

As the classroom environment opened to student actions, I began to notice two phenomena that did not synchronize with my (then) current theory of how collaboration in mathematics class functioned. The first was the emergence of sudden moments of enlightenment among a group of learners. A group would establish a novel solution pathway or a unique mode of conceptualizing a problem, but would not be able to attribute it to any one student. These ‘aha’ moments had no single author, no expert toward which to scaffold. Rather, they seemed to emerge out of the interactions of the members, constructed piecemeal from the members’ contributions but belonging to none of them. The second was the frequent activation of my own personal enlightenment while interacting within a group of learners. While attempting to scaffold their learning toward a pre-determined goal, I would be occasioned down a much different path—one I never envisioned (Davis, 1996; Davis, Sumara, & Kieren, 1996). There seemed to be ideas emerging in action that had no author, no scaffold. Mathematical knowing was emerging in erratic and, through the lens of the curriculum, irresponsible ways. I loved it, but could not explain it.

Academic influences. The coursework in my graduate program of study introduced me to the work of William E. Doll, Jr. in the field of complexity and education. His notion of the culture of curriculum described a project of schooling steeped in the ideas of linearity and capitalism, where programs of study are tightly controlled, well-known, and executable in spite of human participants. The awareness of this culture spurned my search for a worldview that could help me explain the nature of mathematical understanding that was evolving through the work of my students (Doll, 2012). Doll’s work toward a pragmatic picture of complexity in classrooms combined explanatory potency and simple elegance. His work shifted my lens away from seeking *divergent* mathematical thinking, and toward a sensitivity for *emergent* mathematical thinking. His influence can be felt throughout this work. Doll theorized a mathematics education “where play, poiesis, and possibility reign” (Doll, 2008, p. 20) in which teachers and learners alike are “embedded, embodied, [and] emboldened” (Doll, 2012, p. 175). Doll’s tendency to employ these alliterative triads can be found in the title itself.

My focus began to hone in on my actions—as the teacher—that occasioned instances of effective group work. It became a process of explicating my influence on learners, recognizing when student actions influenced me, and noticing when moments of complex organization could be harnessed. I began to develop my lens for complexity through the theory of enactivism. Enactivism enabled me to reconcile my two worlds: the world of teaching in a curriculum-based education system, and the world of complex human organization. It theorized complexity in action.

Situating the Study

Collaboration in a culture of curriculum. The project of mathematics education has come a long way since the assembly line oriented ideas of Taylor (1911), the deficit model of Bobbitt (1918), and the input-output notions of the "lawful and determined" behaviorist psychology (Skinner, 1953, p. 6) were hallmarks of the educative process. Mainstream theory has moved beyond the conceptualization that learning is a tightly-controlled individual effort, and now recognizes the important role that social context plays in education. Socio-culturalists have written about various issues and themes ranging from the processes involved in using groups in the classroom (e.g. Cohen & Lotan, 2014; Horn, 2012), the enhancement of reasoning skills that group problem solving affords (e.g. Boaler & Staples, 2008), and equity issues arising in group problem solving (e.g. Esmonde, 2009a, 2009b). In this framework, members of the group interact in a series of scaffolded relationships to move along a continuum of sophistication. In other words, it focuses on using the activity of groups on a mathematical task to construct a personal understanding of the mathematics.

Research establishing the classroom as a complex system (e.g. Davis & Simmt, 2003; Davis & Sumara, 2006; Doll, 1993; Hurford, 2010) and enactivism as a theory of learning (e.g. Kieren, 1995; Kieren, Calvert, Reid, & Simmt, 1995; Proulx & Simmt, 2013; Proulx, Simmt, & Towers, 2009; Reid & Mgombelo, 2015) expands the perspective that the role of the group is simply a vehicle for individual appropriation of meaning. Complexity theory treats the interactions of groups of students as having more potential than the sum of their individual capacities. Moreover, from the perspective of enactivism, curricular mathematics knowledge is an act of in the moment creation. Knowledge is not something that individuals hold after interactions; rather, it is brought forth through interactions. This orientation has led some researchers to merging the idea of collaboration and the theory of enactivism in education. In their studies they explore the structure of collective action and the markers of collective activity in the classroom through the lens of enactivism (e.g. Kieren & Simmt, 2002, 2009; Namukasa & Simmt, 2003; Towers & Martin, 2015). It is here that student work in problem solving groups is analyzed for attributes that can indicate or describe emergent paths of mathematical knowledge toward curricular outcomes. The primacy placed on collaborative classroom design and its trademark structure—the small group—modifies the role of the teacher as they attend to the collective and curriculum simultaneously. This theme will be expanded on and referenced often throughout the study.

Role of the teacher. Curriculum driven courses are derived from sequential order (Doll, 1989, 1993), but complexity does not operate on metaphors of linearity (Davis & Sumara, 2006), and enactivism treats cognition as an emergent, present-tense phenomenon. What, then, are the implications for the role of the teacher if the classroom is conceptualized through these lenses? Enactivism disposes the neatness of the idea of cause and effect (teaching causes learning) and positions the teacher as another member in a system of interaction (teaching can trigger learning) (Proulx, 2010). The teacher still holds major influence in the collective, but not in a prescriptive sense whereby the teacher's actions directly determine the learners' reactions. Metaphorically, the teacher does not stand at the front of the room causing learning through careful presentation. They also do not stand as an overseer of action or a guide listening for opportunities to help a group. The teacher stands in the middle, as a participant in the action (Kieren, 1995), shaping possibilities throughout the inter-action of the task. The teacher is therefore placed in a position where the goal is the delicate balance between commentating the possibilities with the small group and highlighting the curricular outcomes utilized through their actions. Research has begun to investigate the role of teachers in the collective with regards to their mode of listening and the classification of teacher actions, but further exploration into the role of the teacher in operating through an enactivist lens in a culture of curriculum is warranted (Davis, 1996; Kieren, 1995; Towers & Proulx, 2013). If curriculum is understood as a list of predetermined mathematical ends (Doll, 1993), but complexity and enactivism problematize predictability, we need to know more about how the actions of a teacher influence the collective action of small groups of mathematics learners.

Research questions. It is through this theoretical re-casting of complexity thinking and the corresponding epistemology of enactivism that we can examine the classroom at a different level—the level of the small group. Rather than postulate on the ways in which the collaborative group can co-exist with mathematics curriculum, teachers need images of collectivity generating curriculum. It is through this lens that this study explores the following questions:

In what ways can a teacher of mathematics influence the actions of small groups of learners toward curricular goals while working together mathematically on tasks?

- **What patterns of mathematical action emerge from collectives when teachers intentionally offer interventions with curricular goals in mind?**
- **What are the implications for teachers when seeing small group work through an enactivist lens?**

Rationale for the Study

This study does not set out to establish a set of teacher interventions that will trigger predictive, collective responses. Such a conceptualization would not be aligned with the thinking of complexity, the epistemology of enactivism, or the evolutionary path of groups. In fact, there is a distinct effort to avoid the "modern tendencies of exorcizing ambiguity and mechanizing complexity, in effect reducing a fluid form to a static formula" (Davis, 1996, p. 59). While strengthening the ontological pathway for theorizing classroom groups as complex systems, the main contribution of this study is to the growing scholarship revolving around the pragmatic functioning of teachers in complex systems. That being said, it is important to be clear that the study does not attempt to offer a generalizable complex pedagogy. Paralleling the warning from Davis and Sumara (2006) regarding the notion of 'constructivist teaching', the notion of 'complex teaching' also has an oxymoronic feel because the first half of the term forefronts chaotic sense-making and the second half insinuates predictable, "deliberate and generalizable action" to affect the complex system (p. 115). The study illustrates how classroom actions and interactions inform teachers operating with(in) complex systems, but does not claim to provide a general and robust way of teaching complexity.

This means that, despite the pragmatic focus of the work, the tasks presented, actions described, and analyses of the contained classroom episodes are not meant to establish replicable structures for optimizing classroom collectivity or a list of suitable and generalizable teacher actions. Rather, the motivation behind the work is the re-casting of the role of teacher when viewing groups of learners through the lens of complexity thinking. It is, perhaps, best summarized by Kieren and Simmt (2002) when they describe how a shift in theoretical lens affects the teaching and learning of mathematics:

> Because the teacher is both leader of and part of the collective in the classroom, she can use her observations to change her perception of mathematics as occurring in her classroom or of the nature of the mathematical task or prompt as lived out in the actions and inter-actions in her classrooms, but also to change elements of classroom environment

> in ways that she thinks might affect the collective understanding. … More generally, if collective understanding is a coemerging feature of a self-organizing system, the teacher might be prompted to probe her role as both a learning member of that system and as a special 'catalyst' in it. (p. 873)

The study also expands upon the explanation of classroom happenings through the lens of enactivism. As mentioned earlier, and to be extended upon on in a review of the literature, enactivism does not operate on a logic of cause and effect. The shift in action of a collective (if any) is not caused by the teacher's intervention, but rather by the interactions with the new possibilities. The teacher is then balancing multiple goals of instruction with the incoming flow of information. They can control what type of intervention is offered and the tools provided to take up the trigger, but the collective organizes itself dynamically, interpreting the intervention in an emergent fashion. This phenomenon rules out the possibility of a predictive certainty where specific teacher actions are said to cause particular collective re-actions. This creates a novel stance for the teacher, a stance that is not governed by right and wrong interventions, but complex judgements about what learners need in the moment. This study presents an actionable step toward extending the theorization of enactivism into the realm of the practitioner by illuminating patterns of collective interactions so that teachers might attune themselves to the complex nature of groups working with curricular outcomes.

The pragmatic focus of this research is a response to calls from the literature on teaching and learning mathematics with enactivist sympathies.[1] For instance, Towers and Proulx (2013) call for the documentation of how teacher actions and re-actions might occasion student learning. Towers et al. (2013) call for a continuance of this pragmatic line of inquiry where researchers document teaching activities in the context of classrooms to "show what is possible within the structures of regular schools and programmes" (p. 431). The close ties to classroom practice is crucial to the pragmatic emphasis throughout the work. Complexity thinking in mathematics education is moving away from descriptive activities and into a pragmatic discipline of research. The question has shifted from *if* classroom groups can be conceptualized as part of a nested, complex structure to *how* this nature can be occasioned and influenced. In short, the call is to expand the corpus of "viable accounts of how specific learners make sense of specific

[1] See the April 2015 issue of ZDM: Mathematics Education for an entry point into the literature on enactivism and mathematics education.

experiences" (Davis & Sumara, 2006, p. 116). The situated contexts of the episodes used in this work are chosen purposefully to honour the complexity encountered daily by the classroom teacher and continue the pragmatic theme in recent research in complexivist and enactivist mathematics education. The pattern seeking emphasis—firmly positioned within the teacher's reality of classroom practice—places the study at the juncture between the growing body of literature on enactivist thought in mathematics teaching and the shifting curricular landscape. It is an attempt at attuning the teacher to their role in complexities through the vehicle of small groups in mathematics class. Stated succinctly, "Its principal orienting question is neither the fact seeking 'What is?' nor the interpretation-seeking 'What might be?'", although both of these are addressed, "but the practice-oriented 'How should we act?'" (Davis & Sumara, 2006, p. 25).

Contribution to the teaching of mathematics. For the mathematics teacher, the study elaborates on a language—or lens—of complexity at the classroom level. By focusing on the level of the small group, a classroom structure familiar to mathematics teachers, the study aims to transform the analysis of a familiar situation by providing a new possibility—that of complexity thinking and enactivism—from which to act. By means of illustration, imagine viewing a 3D movie without the accompanying glasses. The pictures will appear blurry and offset from one another, and, while it is not impossible to make out a storyline, it forces the viewer to constantly 'squint' in order to fit the action on the screen with their expectations. The 3D glasses resolve these tensions and allow the viewer to see previously unavailable features of the film. Providing a lens through which to observe classroom action and its complexity allows teachers to better take up their role within it. Providing a new and self-consistent language that teachers can use to observe and discuss classroom events is also of critical importance to working with complex systems of learners. Seeing a language develop in context "helps all stakeholders to fabricate their own internal models of dynamical learning systems", which makes the work of immediate significance to the work of teachers of mathematics (Hurford, 2010, p. 583). The study does not claim to equip teachers with the ability to predict the knowing that evolves though collaborative work, but rather to develop the recognition that mathematical understandings are enacted in a complex ecology and can provide possibilities for the generation of curriculum alongside collectivity. The study contributes to the professional discourse on teaching groups in the mathematics classroom because teachers can benefit from the continued synthesis of enactivist theory within the reality of curriculum, a close analysis of the patterns of influence a teacher's actions have on groups of learners, and episodes that model the possibility of the complex group working productively with curriculum.

A Way Forward

This thesis is presented in a fairly conventional style, with the exception of some of the traditional chapter names being replaced with more descriptive alternatives to parallel the language of the research questions. Chapter two presents a review of the literature. This includes complexity theory and its adoption from the natural sciences into educational literature. The two dominant lines of educational inquiry resulting from complexity thinking are described, and a complementary epistemology of enactivism is expanded upon. The result is a re-defined image of classroom collectives from which I am operating. Chapter three describes how the design-based methodology pairs with enactive inquiry. It also frames the research site, participants, data collection tasks and other logistical design decisions. Chapter four details the need for, and creation of, the observational tool of the curriculum space—an image to interpret the patterns of group knowing. Chapter five uses the curriculum space to explicate patterns of group action in three illustrative episodes and analyze the nature teacher interventions during the data collection tasks. Chapter six discusses the precipitates that emerged, for me, from the study, and the implications of the study on the teaching and learning of mathematics. On the whole, the work represents an attempt to invite the complexity of human being into my mathematics classroom, and situate myself alongside it.

CHAPTER 2: REVIEW OF THE LITERATURE

"The past as a reference to interactions gone by and the future as a reference of interactions yet to come are valuable dimensions for us to communicate with each other as observers."

H. R. Maturana & F. J. Varela, 1987, p. 124

Complexity and Education[2]

A foothold in natural science. Complexity theory—long before it was used to theorize classroom action—emerged from a series of curious developments in the natural sciences. For many years previous, the linear metaphors and reductionist approach of Newtonian science encapsulated the whole of scientific pursuit. According to this school of thought, the world is ultimately knowable and the job of science is to uncover the nature of reality. But Newton himself knew of the limitations of the linearity (Stewart, 1989). Maybe the earliest encounter with the non-linear nature of the cosmos was encountered in the *three body problem* in which the movement of three large bodies, each with mutually perturbing gravitational forces, needed to be mapped. Linear approximations provided some semblance of a solution but the problem remained inaccessible to linear mathematics. Long after the three body problem was posed, scientists in numerous domains of inquiry continue to explore natural phenomena that seem to organize themselves internally in non-linear ways. Chemists study dissipative structures in which reactions seem to cycle and re-organize themselves in time and space (Prigogine & Stengers, 1984), physicists study the patterns of action in complex adaptive systems (Holland, 1995), biologists study the bottom-up organizations of auto-poietic systems (Camazine et al., 2001; Maturana & Varela, 1987), and mathematicians use non-linear dynamics alongside a ton of computing power to attempt to understand the systems of differential equations emerging from these situations (Strogatz, 2003). Despite the technical areas from which the theory of complexity emerged, accounts on everything from cellular genetics to urban planning have been penned in approachable prose (e.g. Cohen & Stewart, 1994; Gleick, 1987; Johnson, 2001; Kauffman, 1995; Waldrop, 1992). These accounts began to move the counterintuitive and

[2] Heading used in a deliberate nod to the incomparable influence of Brent Davis and Dennis Sumara's, *Complexity and Education* (2006).

curious study of complexity into the mainstream. Emerging from natural science was a new paradigm, one where "order does not need to be imposed externally—by God, scientific laws, or teachers. Order emerges internally, through interaction" (Doll, 1986, p. 14).

Complexity thinking. Complexity theory does not manifest itself the same way, with the same goals, in educational theory as it does in the natural sciences. Davis and Sumara (2006) describe the evolution of complexity theory research into, what they term, complexity thinking. They represent the work on complexity as *three bodies*[3]—hard (or reductionist) complexity science, soft complexity science, and complexity thinking. Hard complexity science is concerned with uncovering the nature of reality through rigorous testing. This is the approach that is most prevalent in physics. Soft complexity science draws on metaphors developed by hard complexity science to describe living systems. It can be thought of as "*a way of seeing the world*" (p. 18). Complexity thinking, a middle ground, is concerned with the implications of assuming complexity—it is "*a way of thinking and acting*" (p. 18). Complexity thinking places the observer firmly within the system and is concerned with the pragmatic implications of interpreting a phenomenon as complex. In terms of education, complexity thinking views the structures of learning, at various levels of organization, as complex. In this study, the level of organization under analysis is the small group.

The literature on complexity thinking in education can be organized into two major categories: Learning *about* complex systems and learning *as* a complex system (Hurford, 2010). The literature on learning about complex systems defends the benefits of complexity as a curricular topic—gaining accessibility through the rapid development of computing potency (Jacobsen & Wilensky, 2006; Wilensky & Resnick, 1999). The two foci are mutually constitutive, to a point, but the focus of the current work in on examining small groups *as* complex systems. In other words, on observing small groups with a theoretical frame of complexity thinking.

Levelling in complex systems. The shape of complexity is nested, recursive, and fractal in nature which requires those attempting to interpret complex organization to think in levels (Wilensky & Resnick, 1999). The idea of levels is particularly important to research on classroom groups, because, in complex terms, collectives take on their character through the interactions of agents (in this case, students) at lower levels. These interactions are not simple,

[3] Pun intended.

cumulative actions as in the case of, say, days composing weeks and weeks composing months. Rather, they are transformative interactions that see organization emerge at new, outer—to use the language of nestedness—levels. The image of nested layers is found throughout the literature with regards to learning systems in the classroom (Davis, Sumara, & Luce-Kapler, 2008), categories of knowing and knowledge (Davis & Sumara, 2006), and the growth of mathematical understanding both individually (Kieren & Pirie, 1992; Pirie & Kieren, 1994) and collectively (Martin, Towers, & Pirie, 2006). Resnick and Wilensky (1999) illustrate this emergent view of levelling by asking readers to imagine a traffic flow pattern where each car is moving forward, but a traffic jam emerges from the interactions. The traffic jam is an emergent level of organization in the cars' individualistic actions. As the traffic moves, the jam actually moves backward through the flow of vehicles despite each individual vehicle's forward movement. It seems to take on a new pattern of life. Resnick and Wilensky contend that thinking in these levels is not an innate human ability, but must be done through a practiced lens. The levels share a crucial relationship not present in simple interactions.

> These levels might seem similar to the part/whole levels: just as a year is made up of months, traffic jams are made up of cars. But the jam/car relationship is different in some very important ways. For one thing, the composition of the jam keeps changing; some cars leave the jam and other cars enter it. Moreover, the jam arises from interactions among cars. Months do not interact to form a year; they simply accumulate or 'add up'. (p. 5)

Not all levels of organization evolve on the same timescale. The more central the layer, the faster a significant transformation can occur. At the level of a bodily subsystem, changes can occur in milliseconds. Moving outward through the layers of the person, the collective, the society, the species, and, ultimately, the ecosphere, transformation takes significantly longer (Davis & Simmt, 2006; Davis et al., 2008). The project of education then requires researchers—and teachers—to recognize these levels of organization and jump across them fluidly. Davis & Sumara (2006) call this the act of "*transphenomenal* hopping" (p. 157). The focus of the current work is on the level of the small group because organization as a collective can emerge within a timescale observable in the scope of the data collection, and small group work is already a common framework around which teachers organize classroom experiences. Focusing on the patterns of action at one level of organization is a necessary choice, but this choice by no means signals the belief that small groups are the only complex systems in classrooms. Rather, key to the work is the awareness that "education simultaneously affects and is affected by many overlapping, intertwining, and nested learning systems" (Davis et al., 2008, p. 110).

Establishing order in chaos. Complex systems—in short—are systems that learn, and complexity thinking presents that learning, both inside and outside the confines of a classroom, as a non-linear phenomenon. The term 'learning' is not meant to evoke an image of attainment or transaction where an actor stores some fact that can later be called upon. Rather, learning, in the sense of complex systems, is the process by which an agent and their environment are continually redefined, each continuing to mutually specify the potential to action of the agent's structure and the constraints in which it must act. In other words, knowledge is not possessed; it is a structured potential to action. Systems adapt (and learn) because of the presence of dynamic tension. A complex system cannot survive at a state of homeostasis—inaction results in extinction. Instead, it needs enough imbalance to maintain a creative dynamism (Doll, 2008). Cognitive imbalance is necessary for the learning system to thrive. It is on the back of these ripples of imbalance that the system's learning is furthered. Incongruence between learner and environment creates the need for further action, and these perturbations provide "the driving force of development" (Doll, 1989, p. 246). Doll (1993) illustrates a complex system as a magnet swinging between three poles. While it may seem like the magnet falls into a regular pattern of oscillation, the slightest perturbation in that movement creates wild and erratic movement. Eventually, the magnet re-orients itself to the environment and a new pattern of movement begins. For the system to survive, it needs to be impacted; for a collaborative group to act complexly, it needs to be perturbed. No linear pathway can be pre-traced, and careful planning on the part of the teacher can be interrupted by instances of recursion, wandering down a path of interest, or emergent issues that pop up continuously (Proulx, 2010; Varela, 1987). On the surface, a system built around imbalance and unpredictability seems to have no order at all, and establishing collaborative groups as the vehicle for such anarchy is irresponsible; however, complexity theory establishes the inherent chaos in *any* human system. For the most part, human interaction remains in an organized state, what complexity theory terms the attractor space. However, even the most prescriptive environments have the ability to divulge into disorder. In short, complexity in the classroom cannot be avoided, but it is in conditioning the reciprocal—order embedded in chaos—where complexity gains standing as an analytical framework for classroom action, and provides new footing for the use of collaborative groups (Lesh, 2010). This has implications for knowing, learning, and teaching (Davis et al., 2008), and places new emphasis on the affordances of collaborative groupings. The call to recognize complexity is a far cry from viewing the mathematics classroom as a linear, ordered picture of a typical mathematics classroom, but in order to harness complexity, a classroom-based picture of complexity is required.

Research into Complex Systems

The education research that has been conducted with the ontological premise that a classroom of students is an inherently complex system can be divided into two categories (Davis & Sumara, 2006). First, descriptive complexity research attempts to recognize and describe complex organization in the classroom. Second, pragmatic complexity research is concerned with how complex organization can be occasioned and manipulated. Because all educationalists have the fundamental responsibility to deliberately impact learners, complexity research with regards to classroom action is evolving from a descriptive enterprise, concerned with recognizing characteristics of complex organization, into a pragmatic one concerned with impacting that same organization by providing conditions necessary for its occurrence. Both categories are explained in greater detail below.

Descriptive complexity research: Characteristics of complex systems. The most common technique for deciding if a system is complex is to look for particular characteristics (Davis & Simmt, 2003; Davis et al., 2008). Educationalists and scientists alike have undertaken the work of describing the characteristics of complex action. The resultant qualities can be organized into two categories. The first provide descriptions in broad strokes, attempting to encapsulate the complexity by widening the scope of their lens. Examples of these are Lesh's dictum that you can brand systems as complex if "when you act on them, they act back" (2010, p. 564), or the findings of Camazine et al. (2001) which contend that complex self-organization is driven by feedback loops. A positive feedback loop is the clustering of action around a specific focus causing that action to gain momentum; a negative feedback loop dampens the reaction of a phenomena causing it to level off or extinguish its effects altogether. For Camazine et al. (2001), the existence of feedback loops signals a complex character.

The second category aims to describe a complex system's character with more thorough lists of characteristics. Self-organization remains the hallmark of these lists, but greater detail is given to the structure and behaviour of self-organizing systems. These lists do not claim to be exhaustive, but do provide a more intricate image of self-organization. In an early attempt, Casti (1994) provides the ingredients for what he terms "the science of surprise" (p. 274). Complex systems, according to Casti, are marked by feedback and feedforward loops, a diffusion of real authority, and an irreducible character. In an elaboration of these initial efforts, Holland (1995) details the seven basics of complex adaptive systems. The seven basics are separated into two categories. The first is a list of four properties that complex systems exhibit: aggregation, non-linearity, flows, and diversity. The second is a list of three mechanisms that complex systems use to build coherence: tagging, internal models, and building blocks. Together, these seven features (summarized in Table 2.1) are seen by Casti to make up the nature of self-organization. Both of these lists are developed within the study of the natural sciences, yet the characteristics proposed by research in the area of education echo many of the same features. According to Davis and Sumara (2006), a complex system exhibits self-organization, a bottom-up structure, scale-free networks, nested organization, ambiguous boundaries that remain organizationally closed, structure determinism, far from equilibrium behaviour, and many short range relationships. This list is helpful in translating the concerns of natural science into the work of teachers in complex settings—their classrooms. A concurrent review of all the proposed characteristics reveals common ground; self-organized emergence contains free movement of ideas and ways to track the results of interactions. Perhaps the over-arching message from descriptive complexity research is "even though these complex systems differ in detail, the question of coherence under change is the central enigma of each" (Holland, 1995, p. 4).

Table 2.1

Holland's seven basics of complex adaptive systems

Seven Basics	
Properties	*Mechanisms*
Aggregation Non-linearity Flows Diversity	Tagging Internal Models Building Blocks

Note. Table 2.1 is adapted from Holland, J. H. (1995). *Hidden order: How adaptation builds complexity*. Copyright 1995 by John H. Holland.

Pragmatic complexity research: Conditions necessary for emergence. If complexity thinking is to influence the work of teachers, it must move beyond the description of complex action and into the study of how the complexity can be occasioned and influenced. This growing body of pragmatic complexity research in mathematics education moves away from a description of the characteristics of complex systems and into the analysis of conditions that make it possible (Hurford, 2010). The necessary conditions can be organized into the three dynamic tensions: *specialization* (internal diversity and internal redundancy), *trans-level learning* (neighbour interactions and distributed control), and *enabling constraints* (randomness and coherence) (Davis et al., 2008). The presence of these characteristics does not guarantee complex action, but complex action cannot occur without them. Addressing them in order, specialization requires that there be enough diversity among agents to enable novel action but still enough redundancy from which to enable habitual movement. Trans-level learning requires that there be structures in place "to allow ideas to stumble across one another" as well as the ability to shift focus and control amongst the agents as the moment dictates (Davis et al., 2008, p. 199). Finally, enabling constraints focus on the necessary creative mix of coherent movement through established knowledge and the (seemingly) random movement of establishing knowledge. Making space for these dynamic tensions results in what Doll (2008) calls an open system, and, in pragmatic terms, "ones important for education, closed systems transfer and transmit, open systems transform" (p. 187).

At a classroom level, the collaborative group provides the necessary conditions for complex action to emerge, and thus heightens its importance as a classroom structure beyond the social and cultural notions of mutual scaffolding. A small group can distribute control, offer close neighbour interactions, and highlight the redundancy and diversity of the learners. The structure of small groups contains the necessary ingredients for complex emergence, and the act of teaching creates the possibility of commentating that collectivity. Through the lens of complexity, the classroom (and classroom teaching) is the constant unfolding of a tension between order (establishing redundancy, managing control, and establishing coherence) and chaos (marketing diversity, distributing control, and navigating randomness). The characteristics of complex organization established through the descriptive literature are cultivated through the necessary conditions established through the pragmatic literature and provide a solid foundation for the theorization of small groups as complex systems. From this position, teaching, learning, and knowing become a complex act.

Enactivism: A Complex Way of Knowing

The non-linear and evolutionary character of complexity gains traction as a theory of learning through the expanding body of literature on enactivism (e.g. Glanfield, Martin, Murphy, & Towers, 2009; Kieren, 1995; Reid & Mgombelo, 2015; Towers et al., 2013). Enactivism conceptualizes knowledge and sense making as evolutionary and biological processes, and largely stems from the work of Maturana and Varela (1987) and Varela et al. (1991). It is centered around two main points. First, human action is perceptually guided. This is the assertion that a living system's reaction to an outside trigger is determined by the structure of the system as it interacts with the perturbation, not by the nature of the trigger alone. We say that the system's response is *structurally determined* (Maturana & Varela, 1987). Second, cognitive structures emerge from these recurrent patterns of perceptually guided action; in other words, they are embodied. This is the assertion that a system does not act on an environment and build a perception to fit within its invariants. Rather, the system and environment reciprocally specify one another through inter-action, a process we call *bringing forth a world of significance* (Kieren & Simmt, 2009). Both of these features are crucial to observing the evolutionary character of knowing and learning, but not only does an observer read these phenomena into the process of learning, they must recognize that they are observing through an enactive lens. That is, their observations are determined by their structure and their observations participate in the bringing forth a world of significance. The role of enactivism on research methodology is expanded upon in chapter three. For now, a further explanation of the notions of structural determinism and world of significance is necessary.

Structural determinism. In the context of teaching and learning, any "changes that result from the interaction between the living being and its environment are brought about by the disturbing agent but *determined by the structure of the disturbed system*" (Maturana & Varela, 1987, p. 96). In other words, the reaction of a complex system is not wholly prescribed by the nature of the perturbation on the system. Reid and Mgombelo (2015) use the example of a billiard ball being struck. We are prone to think of the forces acting on the billiard ball resulting in movement in a particular way. Enactivists, however, attend to the understanding that the collision provides energy, but "the structure of the ball being struck determines what happens to that energy" (p. 173). Enactivists are interested in complex systems of learners which contain far more elaborate structures than billiard balls, making the recognition of their structural determinism all the more important. To understand how a learning system is (re)acting, we must look at the complexity of its structure.

Actions of students (entwined fully in their environments) are embodied by the resources of biology and a direct product of human being (Lakoff & Núñez, 2000). In other words, student cognition is a structurally determined phenomenon whereby a response to a trigger is determined by the complex infrastructure of the agent and not by a deterministic locus of external control. The student is "considered an organism evolving with/in his or her environment in an adapted fashion" (Maheux & Proulx, 2015, p. 212). The environment, including other students, is a source of perturbations, and "the learner's structure allows the environment to be problematic—to occasion learning" (Towers & Proulx, 2013, p. 8). Moving beyond a single learner, if the system is a small group of learners working together on a mathematical problem and the perturbation comes in the form of a teacher's intervention, it is the structural makeup of the group that determines the response to the intervention. The intervention doesn't have an instructive effect in the sense of causation. Any changes in the internal dynamics of the learning system are determined by their own dynamics as they interact with the possibilities provided through the teacher's intervention. It is through this structural determinism that the analytical frame of complexity and the epistemological theory of enactivism are tethered. The complex structure of groups is fully implicated in our knowing.

Structural determinism also has implications on how we observe knowing and learning because it is the structure of the observer that determines their interaction with an observation. It places the researcher in an active stance of sense-making where classroom occurrences do not exist outside of interaction with an observer, but rather are brought forth through the interaction of the observer's structure and the environment. It is the structural determinism of the system that determines how agents will inter-act with their mathematical environment, and, as is signalled by the term enactivism, this inter-action is the key to knowing, learning, and teaching systems of learners within a complex structure.

World of significance. Enactivism has biological and evolutionary foundations through which it views the world as not pre-given but as continually shaped by the actions that the learners engage in. We have already seen that these actions are not wholly determined by the environment, but are mitigated through learners' complex structure. As is the case in other accounts introducing the reciprocal emergence of environment and agent through the lens of enactivism (Kieren, 1995; Proulx & Simmt, 2013; Simmt, 2000), the distinction between the role of the environment in social constructivism is useful to highlight the similarities and differences between the two theories.

Social constructivism[4], used here as an umbrella term for the many branches of constructivism that highlight the importance of social interaction to the process of learning (Ernest, 2010), views the individual as the seat of cognition. The environment plays a large role in cognition, but the learner appropriates the meaning of the environment through interaction, and then transforms it into individualized knowledge of the environment (Ernest, 2010). Through this theoretical lens, individual knowing is a product of social interaction. The environment remains static and the learners fit themselves to invariants in the environment to construct individualized perceptions of its nature. There may be many divergent perceptions of the nature of the environment, but the learners are thought to be using social interaction as a mechanism of internalizing subjective meanings of a shared environment.

I am not claiming that social constructivism is disjoint from the evolutionary theory of enactivism (Cobb, Yackel, & Wood, 1992; Proulx & Simmt, 2013; Reid, 1996; Simmt, 2000), because both theories explore the ways in which knowing is a social act, occurring in context. The key difference lies in the relationship between the learner and the environment. Social constructivism is concerned with the creation of knowledge in a situated environment. As collaborative groups operate in the environment, their created understandings are mitigated through its character. These understandings are negotiated collaboratively, but housed individually. In contrast, evolutionary theories of learning, such as enactivism, are "not so much about the invariants within the environment, but about the coordination of the knower and the environment" (Proulx & Simmt, 2013, p. 66). As the learners interact with the environment, the environment becomes part of a non-linearity, of an emerging world of significance. Learning, then, is a "dynamic co-emergence of knowing agent-and-known world, of self-and-collective" (Davis, 1995, p. 8).

[4] For a closer analysis of constructivism and its various representational and radical forms, see Simmt, 2000.

The environment is not a static set of features that learners must fit into, but a dynamic process of mutual fitting, a process called structural coupling (Kieren et al., 1995; Maturana & Varela, 1987; Reid, 1996). According to enactivism, *knowing is doing* (Maturana & Varela, 1987), and knowledge grows through the learners' interaction with the environment. As the coupling continues, the learner and environment begin a process of coming to know that involves the natural drift in both learner and environment (Maturana & Varela, 1987; Proulx, Simmt, & Towers, 2009). In their co-adaptation, the environment does not select a groups' action, but triggers possible action as determined by their structure; likewise, an agent's action triggers the evolution of the environment. Acting fuels a "fundamental circularity" between knowers and their environment as they mutually specify one another (Davis, 1996, p. 11).

A system learns as their structural response to triggers widens the scope of possible action, and this wider scope triggers changes in the environment. This constant interaction between system and environment results in what Kieren and Simmt (2009) call bringing forth a world of significance, where the evolutionary nature of knowing continually unlocks further possibilities. Learners are not constructing subjective, and possibly divergent, perceptions of the nature of the environment; learners are bringing forth—enacting—the mathematical environment together in an emergent fashion. For the enactivist, "the world is *not preformed*, but *performed*" (Davis, 1996, p. 13). Enactivism theorizes a sphere of possibilities where knowing and coming to know are structurally determined and environmentally constrained. Succinctly stated, "mathematics cognition is seen as an activity fully determined by a person's structure in which he or she brings a world of mathematical significance with others within a sphere of behavioural possibilities" (Kieren, 1995, p. 7). Cognition becomes inseparable from the embodied, interactive process co-emergent with the mathematical environment. Small groups afford interaction and structural coupling; they are a critical mechanism to bring forth a world of significance.

As was the case with structural determinism, viewing knowing as bringing forth a world of significance has implications on the role of observer in a research project. Just as the observer cannot claim objectivity because all observations are made through their structure, the observer cannot claim to observe from outside the system. They become a source of triggering, of bringing forth mathematical significance. This is discussed further in chapter three. What still remains in this chapter is to use the verbiage of enactivism to define the difference, in this study, between a collaborative group and a collective, and to explore the role of the teacher in a classroom that recognizes that all knowing is a structurally determined process of bringing forth a world of mathematical significance.

Collaborative Group and Collective

Throughout the early chapters of this work, I have used both the terms 'collaborative group' and 'collective' to describe the small groups that are the focus of this study. It is important to clarify my intentions, in this study, when using both terms, and to explain why the difference is important. Others have offered definitions to delineate between the two notions. For instance, Armstrong (2013) distinguished between collaborative and collective in terms of the coordination of group members.

> When a group is working *collaboratively*, everyone in the group is working on the same task at the same time. Finally, a group that is working *collectively* has such a high degree of coordinated interaction that it appears to be behaving as a single unit. (Armstrong, 2013, p. 8)

The theme of density of interaction is a common thread within the research viewing groups through the theories of complexity and enactivism, the two underpinning theories of my study. Furthermore, much of the research of classroom group knowing studied through the lens of enactivism refers to the knowing systems as collectives (e.g. Kieren & Simmt, 2002; Martin & Towers, 2015; Namukasa & Simmt, 2003; Towers et al., 2013). Studies have explored the various markers of collectivity as their action brings forth a mutual, or collective, world of significance (e.g. Martin & Towers, 2009, 2015; Thom, 2004; Towers & Martin, 2009, 2015).

In this study, the two terms are used to intentionally demarcate between a particular classroom structure (the small group) and the action of students within such a structure. The term 'collaborative group' is popular in the lexicon of the practitioner; here, it refers to the organization of students in close proximity with the intention that they focus their joint efforts on a common task. For the purposes of the study, a group is simply "a certain number of people who are working together on a designated task for a designated amount of time" (Armstrong, 2013, p. 10). When referring to a collaborative group, I am referring to the students and their organization in the classroom. The term 'collective' has strong ties to the research in complexity and enactivism. Here, it refers to the knowing and learning of a collaborative group—the bringing forth of their world of mathematical significance. For the enactivist, knowing and learning with others is the norm (Kieren, 1995), and it is with a specific awareness toward these knowing relationships that I use the term ‘collective’. When I refer to the collective, I am referring to the group as a knowing system in a deliberate attempt to recognize knowing as inherently communal—that is, we bring forth meaning together. Phrased differently, I use the term ‘collaborative group’ to refer to the *human beings*, and the term ‘collective’ to refer to the *humans being*.

Teaching With(in) the Collective

Teaching takes on a new nature for the enactivist. The structure of the teacher remains the same as that of the student, fully and reciprocally coupled to their environment. The teacher’s structure, however, may be considerably more adept at acting within the mathematical environment of the classroom (Kieren, 1995). This does not mean that the teacher stands outside the system; however, from within this system, the teacher serves as a crucial source of triggering for the rest of its agents. Teaching is not an art of facilitation; teaching is a process of participation (Proulx, 2010; Sumara & Davis, 1997). Teachers provoke, get in the way, orient, influence, and couple with the action of the learning system. Instead of being an interested observer listening for moments to re-shape student action if it begins to stray off course, the teacher listens for the emerging character of the action as a wholly coupled piece of the learning system. Teaching becomes a process of “improvisational competence” (Towers et al., 2013) as the systems of learners move through and expand their sphere of behavioural possibilities. Included in this sphere is the curricular space composed of opportunities for students to interact with the topics and ideas of a program of study.

Teacher interventions with(in) the complex system prompt action. The role of the teacher is redefined as the growing body of literature on classroom collectives re-casts the nature of collaborative groups as a structure which is inherently complex. Towers and Proulx (2013) propose informing practices, orienting practices, and shepherding practices as three broad categories of teacher interactions, each of which can present themselves as optimal in the contexts of learning. Teacher action, then, becomes less about ‘good’ and ‘bad’ interventions, and more about a dynamic fitting with the group action as teachers balance their deliberate and emergent instructional goals (Hoong & Chick, 2007/2008). Here the emphasis is placed on the tension between the goals of working with curriculum outcomes and honouring the action of the group. The teacher’s role then emerges from their setting. In other words, teaching “is not about the application of rules and principles; it is more about judgement in context, about adapting constantly to the context—a context where learners, teacher, and subject matter are structurally coupled.” (Towers & Proulx, 2013, p. 24). The role of the researcher is also recast if one begins to see all observing as structurally determined and part of an emerging world of significance. As such, careful attention needs to be paid in outlining a methodology that honours the complexity of the classroom as well as the enactive theory of knowing.

CHAPTER 3: METHODOLOGY AND DATA COLLECTION

"Always keep one hand in the dirt."

B. Sriraman, personal communication, June 24, 2016

Methodological Approach

The pragmatic focus of this study situates the research problem directly within the complex environment of the classroom. Complexity thinking and enactivism, the two guiding theories of the research, provide grounding for the complex character of student groups and the evolutionary way in which they bring forth a world of significance while operating with a mathematical task. The study required a methodology where the complexity of the classroom was not collapsed, but allowed the patterns of emerging worlds of significance to be explored. In this chapter, I detail the tenants that enactivism places on a program of research and tether these to design based research methods. After establishing the methodology, I detail the logistics of data collection, research site, task creation, and student workspace. The result is a research design that encourages the complex structure of student groups and honours the implications of viewing learning—and the study of learning—through an enactive lens.

Enactivism as methodology. There are three tenants of enactivist methodology that consistently appear in the literature. First is the role of the observer in enactivist research. For the enactivist, "all research is observer dependent, whether enactivist or not", but enactivist research forefronts this mindfulness (Reid & Mgombelo, 2015, p. 180). As the researcher enters a research site, they begin a relationship of reciprocal interdependence (Reid, 1996; Reid & Mgombelo, 2015). Some studies try to minimize this influence by taking a research stance of first among equals, but do not claim to be unobtrusive or objective (Martin & Towers, 2015). Participation in the active unfolding of mathematical events becomes an enfolding of researcher, participants, and research site through the process of structural coupling. All research establishes a relationship with the constituents—components of complex systems—and triggers change in their structure by the very presence of a program of research. The researcher becomes a source of mutual perturbation amongst the students, classroom teacher, and problem environment. Their influence, while claimed to be minimized by some, must be recognized because "there are no observerless observations or measureless measurements. Any and every identification entails and implicates an identifier" (Davis & Sumara, 2006, p. 70). In other words, all observations are structurally determined. In this study, the demarcation between researcher and teacher was intentionally blurred from the onset of the study. It was established with the participating teachers that none of us could stand outside the system as an orchestrator, but, rather, we were all key players in the triggering of the problem environment. As such, we adopted the same mode of operating in the classroom. As the groups worked on the tasks, we moved around the room, engaged with the groups' emerging knowing, and offered our commentary as classroom events evolved.

The second tenant is the nested nature of enactivist research (Reid, 1996; Simmt & Kieren, 2015). As the researcher collects data by participating in the research site, the process of data collection is influenced by the researcher's coupling with the site. The focus of where critical mathematical action occurs can be shifted by the inter-action of researcher and research environment. This means that research questions may drift, with some that began at the periphery potentially emerging as crucial to the inquiry. As interaction with the research changes the structure of the researcher (observer), this new structure provides interpretive drift. Simmt (2000) termed these *fractal research cycles* where each experience with a research site generates flux in researcher attention until the "specific questions that … [guide] inquiry co-emerged with the inquiry" (p. 37). This process continues throughout the analysis of the data collected. In essence, the re-viewing of events influences the theoretical lens being employed, and this new, drifted lens is then used to analyze subsequent data. This means that the writing of a research report, such as this thesis, is an activity in constant recursion as the events are re-searched (Miranda, 2004).

The third tenant of enactivist methodology is the embracing of multiplicity as strength. Unlike the process of triangulation, enactivist research values multiple perspectives because they provide a mosaic of observer impressions, not because they narrow in on a verifiable result. The process of enactivist research is perceived to be a community event, and thus, multiplicity is valued (Brown, 2015; Reid & Mgombelo, 2015). Multiplicity is encouraged in the collection of data as well as its review and analysis, with the research striving to honour all stakeholders in the research site. Classroom teachers working alongside researchers are valued as important meaning makers. They are prompted to comment on the significance of classroom happenings all the while opening up possibilities and connections. Interpreting data influences the ideas that develop from viewing, and this new lens is then in play when interpreting further data. A researcher also may choose to use a variety of techniques to analyze and present the data each of which may provide new understanding; Towers and Martin (2015) call this a "willingness to play with data" (p. 255). Diversity in an enactivist program of research is used as a tool to elucidate possibility, not to prescribe verifiability. The multiple lenses mean that the intent of the inquiry is to provide *models for* rather than *theories of* the complex structure (Reid, 1996). Descriptions of effective action from multiple perspectives open the space for continued synthesis of ideas. The variety of interpretive sources—media types, stakeholder viewpoints, and analysis techniques—attunes the research to the emergent possibilities and adds to the quality of the research.

Design research methods. Design research methods synchronize well with the tenants of enactivist methodology. The methods pair a search for understanding with the consideration of application to practice, and are situated in the middle space between guiding theories—such as complexity and enactivism—and pragmatic application. Design research embraces complex settings in a deliberate move away from artificial laboratory settings. In doing so, a design research study recognizes that the inherent complexity of the classroom is pertinent to inquiry if it is to inform theory as well as remain salient to practice. Although design research studies vary in scope (Gravemeijer & Cobb, 2013; van den Akker, 2013), they all involve designing conditions within a situated context where the actions, reactions, and interactions of participants can be explored. Prediger, Gravemeijer, and Confrey (2015) succinctly summarize the process and mandate of design research methods.

> [Design researchers] design and create classrooms where students are provided rich tasks to work with and ample opportunities to participate, individually and collectively. Once these conditions are met, the research concentrates on the emergence of students' thinking over time and seeks to identify both, productive moments and moments of failure, refining the relevant designs in light of them. (p. 881)

Design research as a research method has been referred to by several different names. Some authors refer to the methodology as a design experiment (Brown, 1992; Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003); others have referred to it as a process of engineering particular contexts (Cobb, Jackson, & Dunlap, 2014). I have chosen to refer to the method as design research in keeping with the recent literature in the field (e.g. Cobb et al., 2014; Prediger et al., 2015; van den Akker, 2013), as well as a distancing from the connotations associated with the classical experiment or process of engineering where the variables can be tightly controlled and the influence of the researcher is pre-planned and exterior to the context.

Educational design research organizes itself into two archetypes (Prediger et al., 2015). The first are studies primarily focused on developing curriculum innovations (often in the form of materials) and the second are studies primarily focused on developing theories of the learning process. The word primarily is used intentionally to signal the dual focus of all design research studies. Within these archetypes, studies often occur in one of two settings: classroom design studies collaborate with a teacher in their classroom context, and professional development design studies work with larger groups of teachers outside of the classroom context to develop instructional practices (Cobb et al., 2014). The present study is primarily focused on developing theories of the learning process of groups through a classroom design, but the dual concerns of theory and practice are embedded within its structure as it is in all design experiments.

Theory informs both the classroom design as well as the actions within it, and so background theories “act as a fundamental core of design research approaches in mathematics education” (Prediger et al., 2015, p. 881). The theory provides an explanatory and advisory aim. Not only does theory attempt to understand classroom events, it aims to inform how productive models of teaching and learning can be promoted. Thus, it has a theoretical-pragmatic blend. The two attentions do not stand at opposite poles, but amalgamate. Design studies build contexts to demonstrate that a phenomenon exists, but the principal focus is on the ability to harness the phenomenon to improve teaching and learning. One aim cannot be parsed from the other.

The literature often captures the spirit of design research methods by listing key characteristics (e.g. Bakker & van Eerde, 2015; Cobb et al., 2003; Cobb et al., 2014). All of the lists contain significant overlap, but Prediger et al. (2015) discern five common characteristics that were particularly informative to my research design. Design research studies are interventionist, theory generative, prospective and reflective, iterative, and ecologically valid and practice oriented. Design research is interventionist in the sense that “researchers deliberately manipulate a condition or teach according to particular theoretical ideas” (Bakker & van Eerde, 2015, p. 6). The deliberate nature does not insinuate that the process of bringing forth new educative possibilities is a simple matter of cause-effect. Rather, interventionist design is the commitment to possibilities for educational improvement. Design research is theory generative in the sense that it builds "local theories and paradigm cases that meant to inform practitioners and researchers" (Prediger et al., 2015, p. 880). The learning context is theorized and theorizing; design based research produces theory from practice to ensure it does real work. Design research is prospective and reflective in the sense that the relationship between theory and practice is reflexive. Theory influences the design context prospectively, and context's action influences theory reflectively. Design research is iterative in the sense that the structures for investigation remain mutable throughout a process that spans several cycles of learning events. An iterative design provides the conduit through which the prospective and reflective analysis is performed. Finally, design research is ecologically valid and practice-oriented in the sense that participants are treated "as epistemic agents of their own who bring to bear their own experience and resources" (Prediger et al., 2015, p. 881). Design research places participants in an active sense-making position in order to take an honest look at implications for improving practice. The emerging theme from the common characteristics is that design research includes a flexible attentiveness to "a learning ecology—a complex, interacting system involving multiple elements of different types and levels" (Cobb et al., 2003, p. 9).

Integrating enactivist methodology and design research. Design research aligns well with an enactivist methodology because it is guided by theory but explores practice as both informed by the theory and theory informing. The combination of enactivist (and complexivist) sensibilities and design research can be further co-implicated through a closer association of the tenants of enactivist methodology and crosscutting characteristics of design research.

In enactivist research the researcher does not act as an external observer, but is fully complicit in the unfolding classroom events. Design research methods honour this with their interventionist, ecologically valid, and practice-oriented characteristics. The researcher is meant to deliberately influence the actions of the system, and, as established previously, this is the necessary stance of the observer in a complex system. This awareness of the influence of researcher along with the desire to act and re-act in synchronicity with the learning ecology knits enactivist theory with design research methods.

Enactivist research has a nested nature where the attention of the researcher is reciprocally influenced by participating in the research process. Design methods also encourage this process through a prospective and reflective stance as well as an iterative design. The iterative structure provides opportunity for events to be examined in light of theory, but also for theory to be re-examined in light of classroom events. An attunement to mutual coupling between environment and agent is a central tenant of enactivist theory and facilitated through reflective design.

Finally, enactivist research encourages multiplicity in order to inform theory. Design research aims to develop theories from practice that are "modest in scope" (Cobb et al., 2014, p. 4), humble in the sense that they are "accountable to the activity of design" (Cobb et al., 2003, p. 10). Multiplicity acts as a check and balance between the theoretical and pragmatic. The theory generating process of design research shares the goals of enactivist research—to develop *models for* and not *theories of*. The features of this study were designed to honour the tenants of enactivist methodology as they intersected with the characteristics of design research. An account of this design process is detailed below.

Data Collection Activities

To explore my research questions, I worked with two teachers over the course of three weeks. The research was conducted at a school in a middle class neighbourhood in an urban community in Saskatchewan. The school was of average size for the area, serving approximately 850 students from grades nine to twelve. The school provided a full offering of curricular and extra-curricular programming in academics, athletics, and the arts. Both classrooms involved in the research were designed, by myself, with intentions of encouraging complex organization. Specific structures in the classroom ecology were established in the hopes that they would encourage the manifestations of complex activity while still retaining the structure of group work familiar to many mathematics classrooms. In particular, group size, grouping procedure, task introduction, and instructional flow were all designed intentionally with these aims in mind.

The study took place three weeks into the school year and consisted of two phases. During the first phase, the study was introduced to the students and permissions were distributed. While we waited for parent consent, I participated in the classrooms as a support and resource for the teachers. During this time, Mrs. Murray and Mrs. Hudson[5] planned the lessons, and I familiarized myself with the environment and the students within it. The second phase, the data collection phase, consisted of six days designed around small group instruction. Data was collected from student action during data collection tasks (see Appendix A) on three separate days in Mrs. Murray's class and two separate days in Mrs. Hudson's class. The data collection

[5] The names of all participants in the study, including the teachers, have been changed to preserve their anonymity.

tasks ranged between fifteen and thirty minutes in length. After the data collection tasks were complete, I remained available to both teachers throughout the semester.

Mrs. Murray. Mrs. Murray had been teaching high school mathematics for eight years, and was entering her second year at this particular school. She had taught the ninth grade mathematics course nine times, giving her a wealth of experience from which to draw. During the study, Mrs. Murray was paired with a student teacher, Ms. Becker, who was completing an internship at the local university. Ms. Becker was included as a full participant in the study's activities. During a typical day, Mrs. Murray's classroom was organized in rows of desks facing the front of the room which featured a digital projector and whiteboard space. Small groups were typically not used during instructional events in her room, but collaboration was encouraged when working on problem sets.

The data collection in Mrs. Murray's classroom took place during a unit on fractions in the Mathematics 9 course (Saskatchewan Ministry of Education, 2009) with twenty-seven enrolled students. Of the twenty-seven, four chose not to participate in the study. Mathematics 9 is the only math course for grade nine students in the regular pathway in Saskatchewan high schools. Although instructional hours vary from building to building, this school offered the course over a semester and a half, amounting to approximately 150 hours of instruction.

Mrs. Hudson. Mrs. Hudson had been teaching high school mathematics for nine and a half years, and was transferred to this particular school at the end of the last school year. She had taught this grade ten mathematics course twice before. Like Mrs. Murray, Mrs. Hudson's classroom was primarily organized in rows of desks facing a digital projector and whiteboard space at the front of the room, but the room was reorganized for small group work which occurred nearly every week. The room also had a small area on one side of the room organized around mathematical puzzles and games.

The data collection in Mrs. Hudson's classroom took place during a unit on surface area and volume of various geometric solids in a Foundations and Pre-calculus 10 (FPC10) course (Saskatchewan Ministry of Education, 2010). The class had thirty-one students enrolled. Of the thirty-one students, fourteen chose not to participate in the study. FPC10 is one of two courses offered at the grade ten level in the regular pathway for Saskatchewan students. It is offered in a single semester (approximately 100 hours of instruction), and designed to prepare students for the study of calculus.

Forming Groups. The vast majority of class time during the study was spent working in small groups. While formal video and audio data was not collected during every class period, every task where video data was collected (henceforth called data collection tasks) was completed in small groups. It is important to note that creating small groups encouraged intra-action as the primary source of meaning making. This, however, did not preclude groups from communication at the inter-group level—a property that Liljedahl (2014) has termed the "porosity of groups" (p. 14). The formation of groups for the purpose of this study does not pretend to eliminate meaning-making between groups, but rather attempts to encourage meaning-making within them. Group porosity once again calls attention to the levels present in any complex system of interaction.

Group size. Studies on classroom collectivity that focus on the whole class as the unit of analysis can result in such high density of action that documentation and analysis quickly becomes unwieldly even with multiple researchers and sources of video (Towers et al., 2013). The focus on small groups (somewhat) alleviates this pressure, but the descriptor *small* remains vague. Assuming a single person cannot be considered a group in a social sense[6], the next available size would be groups of two. Yackel (1991) found that more sophisticated forms of explanation were not present when students worked in pairs or alone. Perhaps the balance of redundancy and diversity is unstable or perhaps a pair lacks significant opportunity for neighbour interactions, but the study of complex groupings often is focused on groups that contain more than one dyadic link (Arrow, McGrath, & Berdahl, 2000). Fear of the Ringelmann Effect, where addition of co-workers leads to a decrease in the average performance of each member (Steiner, 1972), coupled with personal experiences with small groups in mathematics classes caused me to avoid groups larger than or equal to four members. Groups of three were chosen as the ideal size to facilitate the conditions of complexity as well as to allow each student to maintain an influential voice in the group's action.

Grouping strategy. Students were divided into groups each day using the strategy of visibly random grouping (VRG) (Liljedahl, 2014). Employing VRG meant that student groups were created randomly in real time and in plain sight of the students so as to avoid any controversy regarding goals of the groupings. With VRGs, a student could no longer assume that

[6] Viewing the body as a complex organization of bodily systems opens this up for debate. See Davis et al., 2008.

the teacher had some sort of personal vendetta against them (pedagogically or socially) and the teacher could no longer assume that students were grouped solely for social reasons, as is often the case when student self-select their groupings. VRG provided an interesting caveat into the classification of groupings provided by Arrow et al. (2000). These groups would have been classified as *concocted* because they were imposed from an external control and were planned—not driven by extenuating circumstance or need. However, the transparency created through randomness served to alleviate the sense of external control often held by a teacher in the classroom. In this regard, VRG began to decentralize control. Although most effective if used over stretches longer than the scope of the data collection, VRG has been found to result in several desirable classroom affordances such as increased engagement, increased mobility of knowledge, and increase in intra-group interactions (Liljedahl, 2014).

Creating daily VRGs began to establish other conditions for complex emergence alongside decentralized control. Randomness was a natural one; students may (or may not) have been paired with new partners daily, creating a new group dynamic each day as their actions coalesced. On the other side of the dynamic tension, a sense of coherence was maintained because the tasks continued along a common content thread. Strategies, structures, and patterns of action from previous group tasks were brought by each new member in the form of action potential on the new task. The common content and history of action provided threads of coherence in the randomness. Varied histories also created a great deal of redundancy and diversity in a classroom. Instead of engineering groups that, in the impression of the observer, contained a balance of diversity and redundancy, VRG used the random mechanism to create a dynamic equilibrium. This minimized the social tensions, provided opportunity for combinations that may have been overlooked altogether, and reaffirmed that "one cannot impose diversity from the top down… Diversity cannot be assigned or legislated" (Davis & Sumara, 2006, p. 138).

Group workspace. To encourage productive interaction, mechanisms were established to encourage ideas to stumble across one another. The group workspace was designed to facilitate that stumbling. Each group sat in a pod composed of three desks and a large, dry-erase board to be shared between them. The entirety of their work was archived on this communal space. Their physical proximity caused their work to remain more-or-less self-contained, and the non-permanence of the workspace had been shown to provide affordances such as eagerness to start, discussion, participation, and non-linearity of work (Liljedahl, 2016). The workspace was designed to encourage a high density of neighbour interactions—a key to complex organization.

Other classroom features. Other facets of the classroom environment were purposefully established. Each class period during the study began with a starter question linked to the theme of the unit of study. The questions encouraged student decisions and allowed for multiple ways of reasoning. The starter was shown on the projector at the front of the room; students detailed their reasoning by themselves for three minutes, shared with a partner for two minutes, and then volunteered solutions and reasoning to the entire group for a short discussion. The method of introducing the data collection tasks was also varied intentionally. Some tasks were given as visuals projected at the front of the room, some were given as handouts at the groups' workspaces, and some were provided in stages. All contained some degree of verbal explanation and the invitation for further clarification as needed. The goal of this practice was to de-centralize focus in the room.

Building Data Collection Tasks

Design researchers dedicate a large amount of time building the classroom tasks and anticipating how the features of the task might occasion student action. This attention is heightened for enactivist research because agent interaction is the impetus for learning; even with the conditions for complex manifestation present in small groups, it is unlikely that collective action will organize if the task dampens any or all of the conditions by being trivial or prescriptive (Davis & Sumara, 2006). Numerous titles have been coined for tasks that provide occasion for multiplicity in student patterns of action and conceptualization. Tasks that encourage a process of open conjecturing have been called nonroutine (Papert, 1972; Thom, 2004). Problems of this variety are not immediately solvable and typically require some intervention (Towers & Martin, 2014). Tasks specifically designed to be approachable at a variety of levels have been called low floor, high ceiling (Boaler, 2016) or low threshold high ceiling (McClure, 2011). These tasks offer a relatively simple mathematical entry point, but can be expanded with more sophisticated approaches. Tasks that have multiple, viable starting points have been called variable entry (Simmt, 2000), and tasks that call on resources from several members of a group and offer multiple solution strategies have been called groupworthy (Boaler & Staples, 2008; Horn, 2005, 2012). There is no denying that all of these contain considerable overlap in definition. Instead of defining a specific set of requirements, I chose tasks[7] that, in my opinion, reflected two guiding principles from the literature. First, the students "need to perceive of the tasks as both relevant and do-able—that is, as coherent. At the same time, there must be sufficient play in the questions to open spaces for broader discussion" (Davis & Sumara, 2006, p. 150). Second, the problems should "maintain a delicate balance between sufficient structure, to limit a pool of virtually limitless possibilities, and sufficient openness, to allow for flexible and varied responses" (Davis et al., 2008, p. 193). A brief synopsis of each of the five data collection tasks is given in Table 3.1 and a more thorough explanation, including materials provided to the student groups, is provided in Appendix A.

Table 3.1

Brief descriptions of the five data collection tasks

Task name	**Description of the task**
The Surface Area Doubling Task	Groups were given the dimensions of a house and asked to design an addition so that the surface area became exactly double that of the original house.
The Tile Design Task	Groups were given coloured, square tiles and asked to build shapes that met various specifications provided on a sequence of stage cards.
The Fill in the Blanks Task	Groups were asked to satisfy an expression structure that contained inequality and equality statements by filling in the blanks using the digits one to nine.
The Solid Fusing Task	Groups were given six solids and asked to construct a composite solid that had a surface area as close to identical as possible to its volume.
The Number Line Cards Task	Groups were given cards that represented pieces of number lines and asked to place them end-to-end to build coherent number lines.

Sources of Data

The study contained two major sources of data. The first—video data—was used to capture the action of the groups as they worked on the tasks. The second—teacher interviews—

[7] Of the five data collection tasks, I built the Tile Design task, the Solid Fusing task, and the Number Line Cards task specifically for this study, and the Surface Area Doubling task and the Fill in the Blanks task were adapted from my previous work in classrooms.

were used as a reflective device to discuss critical moments in the classroom action, and as a generative device to build connections between critical events and theories for their emergence.

Video data was collected during each data collection task from three classroom groups chosen at random. These recordings were captured by tablets positioned above the workspace and resulted in a close-range account of the conversations, actions, and artefacts of the group's doing. No attempt was made to diminish the fact that the classroom action was being recorded. Every member of the study—including teachers—were made aware of the tablets as well as the purpose of the recordings. Video recording has an inescapable effect on human behaviour and the type of data collected (Armstrong, 2013; Pirie, 1996), but the attempt was made to background concerns by inviting all questions from stakeholders regarding the recordings.

Data from teacher interviews is a cornerstone of both enactivist inquiry and design based methods. In an iterative design study (in this case, each enactment of a data collection task served as an iteration) it is crucial to hold debriefing sessions where events are interpreted and future events are attended to (Cobb et al., 2014). This is the essence of the reflective-prospective duality. The interviews became "the sites where the intelligence of the study [was] generated and communicated" (Cobb et al., 2003, p. 12). For the enactivist, interviews are linked closely to the doing of the study. They are chances to experience the classroom action as another has experienced it, and, in doing so, occasion a larger interpretive breadth. As such, a loosely structured interview was conducted with the classroom teacher(s) after each data collection task. The teachers were asked to describe what we called critical events in the action—moments when a member of the research team felt a particular episode of group work warranted more investigation. Together, we talked about what we saw, why we intervened the way we did, and what we felt the result of the intervention was. The goal of the interviews was not to analyze the teachers' thought pattern in the moments of teaching and provide a more productive alternative. Rather, the interviews provided an occasion to get the participants "talking about the detail of their actions in their classrooms, what happened from their perspectives. What then happens is that out of this space they then report new connections as ideas come to them" (Brown, 2015, p. 193).

Having collected the two cornerstone sources of data for the study, my attention turned to the creation of a framework that could image the complexity of the theoretical underpinnings, research design, and subsequent data. The creation of the interpretive device—the curriculum space—is detailed in chapter four.

CHAPTER 4: IMAGING A CURRICULUM SPACE

"We shape our tools and afterwards our tools shape us."
M. McLuhan, 1994, p. xxi

Framework for Analysis

The interaction between the complexity of the research design and the outcomes of the mathematics curriculum needed to be intertwined in order to image the patterns of complexity in the midst of curriculum. That is, I needed a structure through which to view the non-linear, far-from-equilibrium nature of groups' knowing in relation to the outcomes of their program of study. Within this interaction, the role of the teacher needed to be explored. In this chapter, I establish the two conceptual frames used in the analysis of data. First, the notion of problem drift as a way of viewing the complex shifting of a group's curricular attention, and second, the classification of teacher actions provided by Towers and Proulx (2013). From these two frames, the process of establishing an explanatory tool to interpret the evolutionary nature of bringing forth a world of significance is detailed. The result is a tool of analysis, termed the curriculum space, through which patterns of group action with the given tasks are analyzed in chapter five.

Problem drift. The notion of problem drift is rooted in the enactivist idea of the dynamic nature of reciprocal coupling between environment and agent.[8] Drawing from evolutionary imagery, Maturana and Varela (1987) use the term *natural drift* to describe the process of knowing and coming to know. In the process of natural drift, species and environment co-adapt to one another, each influencing the other through a process of structural coupling (Proulx, Simmt, & Towers, 2009). For a coupling to be viable, it "must simply facilitate the continuing integrity of the system" (Varela et al., 1991, p. 205). As this co-adaptation unfolds, the structure of the organism determines which changes occur. Framed in the context of a group of students working on a task, it is the structure of the group that chooses "the relevant issues that need to be addressed at each moment…where what counts as relevant is contextually determined" (Varela et al., 1991, p. 145). The result is an image of problem drift where a group's attention is paid to what it determines is worthy of mathematical attention. Returning to the enactivist notion of

[8] This process of bringing forth a world of significance was discussed in greater detail in Chapter 2.

structural determinism (see chapter two), attention is not forced through nuanced cues or structures in a problem or classroom milieu. Rather, it is determined by the structure of the group as it interacts with the triggers it encounters. Students do not act on pre-existing situations. Their "co-determination and continual interaction with the environment creates, enables and specifies the possible situations…to act upon" (Proulx & Simmt, 2013, p. 69). This metaphor allows the environment—the problem focusing group action—to be interpreted as a process, constantly emerging as the group redefines relevance. In this active language, problems are not solved—they are posed.

The curricular environment—the group's world of mathematical significance—is enacted as the topics that need to be addressed are deemed as productive ways forward. Maheux and Proulx (2015) provide an example of problem drift where a student working on reproducing a shape with tangrams abruptly changes her course of action. Her decision seemed to relate to "the identification of a 'shape of interest,'" and the proposition that the search for a mathematical solution involved the "endorsement of new, *yet unseen* figures" (p. 218). Over the course of pursuing a solution to the task, the learner posed a yet unseen problem of interest, drifted the problem of relevance, and re-defined the inquiry.

As a conceptual frame for data analysis, problem drift allows us to interpret a problem as plural, as that which focuses action in the moment. The task provided serves as a source of perturbations, but it is through the taking up and acting on these triggers that the problem gains mathematical relevance. The actions on the problem subsumed by the curriculum are a part of the larger sphere of behavioural possibilities available to the group. A framework of analysis was needed to be able to communicate the image of problem drift in relation to the curricular goals of the teacher/researcher, a tool that must begin to capture the complex nature of problems drifting in and out of prominence.

Teacher actions. The classification of teacher actions by Towers and Proulx (2013), the second conceptual frame for data analysis, was used to classify each intervention offered to a group by a teacher during the data collection tasks. Consistent with the authors' interpretation, I did not intend the framework as a trajectory from unproductive to productive or novice to expert teacher moves. Instead, in line with enactivism, they describe a productive teacher intervention as one which fits with the context in which it is offered. The teacher acts with the problems relevant to groups of learners as a fully coupled agent in an attempt "to do the right thing in context—for enriching students' learning…in a landscape of complex goals and commitments" (p. 10).

The framework consists of the three broad categories of informing, orienting, and shepherding actions, each with several sub-categories. I used the three broad categories, and ten sub categories, to classify the teachers' actions as we operated with groups of learners with the intentional focus on occasioning encounters with curricular outcomes. They are not, however, meant to be isolated from one another, and often intertwine at different times during teaching. The first category, informing actions, give information. They consist of three sub-categories: enculturing actions, which induct students into the wider customs of the mathematics community; reinforcing actions, which place emphasis on certain ideas; and telling actions, which explain and correct. The second category, orienting actions, direct students' attention. Towers and Proulx describe four sub-categories of orienting actions: Clue-giving actions use hints to orient students toward specific pathways; blocking actions prevent students from following certain solution pathways; pretending actions take a position in order to elicit argumentation; and anticipating actions remove challenge or make the problem more accessible. The third and final category, shepherding actions, support or coordinate the possible. They consist of three sub-categories: Inviting actions offers an avenue of possibility for exploration; rug-pulling actions destabilize student thinking in order to provide new possibilities; and retreating actions give students the space to consider the current problem's possibilities. The distinctions are summarized in Table 4.1.

Table 4.1

Mathematics teaching actions

Category	Sub-Category	Explanation
Informing	Enculturating Reinforcing Telling	Giving information
Orienting	Clue-giving Blocking Pretending Anticipating	Directing students' attention
Shepherding	Inviting Rug-pulling Retreating	Supporting and coordinating the possible

Note. Table 4.1 is reproduced from Towers, J., & Proulx, J. (2013). An enactivist perspective on teaching mathematics: Reconceptualizing and expanding teaching actions. *Mathematics Teacher Education and Development, 15*(1), 5-28. Copyright 2013 by the Mathematics Education Research Group of Australasia.

Process of Analysis

It is important to detail the process I went through when analyzing the data. Equally important as the description of the tools of analysis are the ways in which these tools, interpreted through my structure as an observer, became ones with explanatory power. No model can perfectly capture the complex activity of human interaction, but a tool for observing the action of a group within the intended curricular outcomes was needed. It also needed to balance a sensitivity to complexity with the ability to remain relevant to the work of the teacher. To assist in the explanation of the model's development, artefacts created from a session of data collection where Cohen, Anne, and Lucas were working with the Surface Area Doubling task (Appendix A) are provided throughout this chapter.

Viewing the data. Each of the five recorded classroom tasks contained video data from three groups for a total of fifteen sessions of group action. Analysis began with the viewing of all sessions in order to timestamp every time a teacher intervened with a group. Audio recordings of the post-session teacher interviews were used to identify the moments that the team felt were critical or interesting. Two of the three groups from each task were chosen for further analysis based on the frequency of critical events (as perceived by myself or the classroom teachers) and their cooperative structure. For instance, if a group worked with little interaction or was dominated by a single student, the conditions for complexity were not deemed adequate and that session was not chosen for closer analysis. A total of ten sessions were selected and transcribed.

Identifying problem drift. In order to gain an image of problem drift, the transcript of a session was dissected into four columns. The first was a recording of the time elapsed in the session, and the other three focused on the constituents of the classroom: the teacher, the students, and the problem. The utterances of the teacher were separated out from the transcript and placed in the second column. The third column contained the transcript of the group members, anonymized to focus on the theme of the discourse rather than the source. The fourth column contained an idealized form of what I interpreted as the relevant problem around which the group was operating. Read vertically, this column contained a sort of record of the problem drift. Two excerpts of this modified transcript of Cohen, Anne, and Lucas working with the Surface Area Doubling task are given in Figure 4.1 and Figure 4.2.

In this format, two relationships emerged as relevant through my interpretation. The first was the relationship between the student action and the nature of the relevant problem. Comparing the third and fourth columns brought forth an image of the evolutionary character of group knowing—how the coupling of group and task resulted in problem drift. The second relationship was between the teacher interventions and the nature of the relevant problem. Comparing the second and fourth columns brought forth an image of the various influences on the evolutionary character of group knowing.

Time	Intervention	Transcript	Relevant Problem
3:50		Alright, so I guess start with these, so 3 times 10 is 30 times 2 is 60. So now both these sides are done. So we just need 3 times 6. Wait, so which sides do you have? These ones? So these two are done, so now we need these two, so 3 times 6 is a number. 18. 18. Are we allowed to use calculators? Times 2. He didn't say not to. 36. So now these two, oh wait, no, we have to minus 1 times 2 so that's. We don't want the door, so. Which is 2, right? Yeah. It'd be 16. So 16 plus 18. 34.	What is the total surface area of the house?
5:02	So what are you doing currently? What are you trying to do?		
		We found. Surface area of this part. Yeah, we've done these two and these.	
5:09	So you're finding the surface area of the original building?		
		Yeah.	
5:12	Okay and what would be the point of that?		
		So we know how big to make the other one. Yeah, how big to make the other part of the house.	
5:17		…	
5:22		And now we need to figure out. Triangle. These two are still rectangles. Yeah, so 5 that's 50 so 100. 100, and then the triangles are. Base. Times height divided by 2. So base times. We don't know the height.	What is the total surface area of the house?

Figure 4.1. An expanded excerpt of transcript of Cohen, Anne, and Lucas while working with the Surface Area Doubling task.

Time	Intervention	Transcript	Relevant Problem
15:54		Okay we just have to do the same thing, so 21. Yeah okay. And then.	What is the total surface area of the new house?
17:05		…	
17:18		Equals 36 minus 2 is, oh wait we have to go. For both sides, which we have, yeah so 36 minus 2 which is 34. These are exactly as tall as each other, right? Then 5, 5 times, 5 times 21, 105. 210. Then the, plus the 30, so 126, 210, 34, and 30 is… only 400, dang it. What did we do wrong? Do we know what the surface area of this is? The entire thing? The single house. Yeah that was. 224. Like, in total it was 224. And I'm pretty sure this is 236 for this part of the house. 236? Alright.	
18:51		…	
20:12	Did it work out when you found the surface area?		
		We were short 48. Life sucks.	
20:22	What did you decide to do with that 48? There's got to be an easy answer.		
		We could add a little thing off the end. Yeah, let's add, let's add a doghouse. Yeah. Ok, so how is that? A doghouse. Oh! Under, oh. No that's taking away surface area. No, it's adding. K. I think we should attach it with a string so that we don't lose any surface area.	How can we add something to the house without losing more surface area?

Figure 4.2. Another excerpt of expanded transcript of Cohen, Anne, and Lucas while working with the Surface Area Doubling task.

Defining a curriculum space. Problem drift conceptualizes curriculum as lived through interactions at varying levels of sophistication, not as something encountered or collected. The drifting nature of the group's action necessitated a way for myself, as a researcher, to observe the movement in relation to the intended curricular outcomes. An image of the curricular landscape was needed to display the evolutionary nature of problem drift and the various strengths of association with curricular outcomes. The Cartesian grid, a stalwart frame for visualizing location and movement, was used as the framework to house the curriculum landscape. Ironically, Cartesian images are set up as juxtaposed to complexity thinking because they are part of a larger system designed to prescribe order and hierarchy between places and not the examine the features of the places themselves (Davis, 1996). In this instance, however, the grid was an approachable and familiar medium, and a catalyst for further examination of the curricular places the groups visited along their way. I do not propose the curriculum space as an ending point to whitewash complexity, but a tool through which the classroom and curriculum can be interpreted.

The horizontal and vertical axes were each designated with a curricular outcome from the program of study intended to become a focus through the day's task. These outcomes were established during pre-conference meetings with the classroom teachers. The Surface Area Doubling task had "Calculating the surface area of a composite shape" as the outcome on the horizontal axis and "Understanding the effect of overlap on the surface area of a composite shape" as the outcome on the vertical axis. The image of the curriculum space for this particular task can be seen in Figure 4.3.

The result was an elementary image of the facets of curriculum that captured the attention of the groups as they completed a task; it is an attempt to conceptualize the curriculum as a landscape. Traditionally, curriculum outcomes are seen as a list of endpoints to be mastered in a tight sequence. The curriculum space still uses these objectives to label the axes, but they are now imaged in the middle space of sense-making, as processes for addressing relevant issues posed in their world of mathematical significance. The curriculum outcomes are not completed at the end of action, they are the process used to resolve perturbation and prompt further action. The curriculum space, then, is then an image of a productive orientation toward mathematics. I use the word productive to mean that a group, in moments of productivity, is using the intended curricular outcomes to address the problem deemed relevant. That is, the action of a group is seen as productive when the curriculum outcomes emerge as useful ways to address the problem deemed relevant to the group. In other words, the group has identified (what the researcher and teacher understand as) the curricular outcome as containing potential to resolve the tension provided by the current instantiation of problem drift.

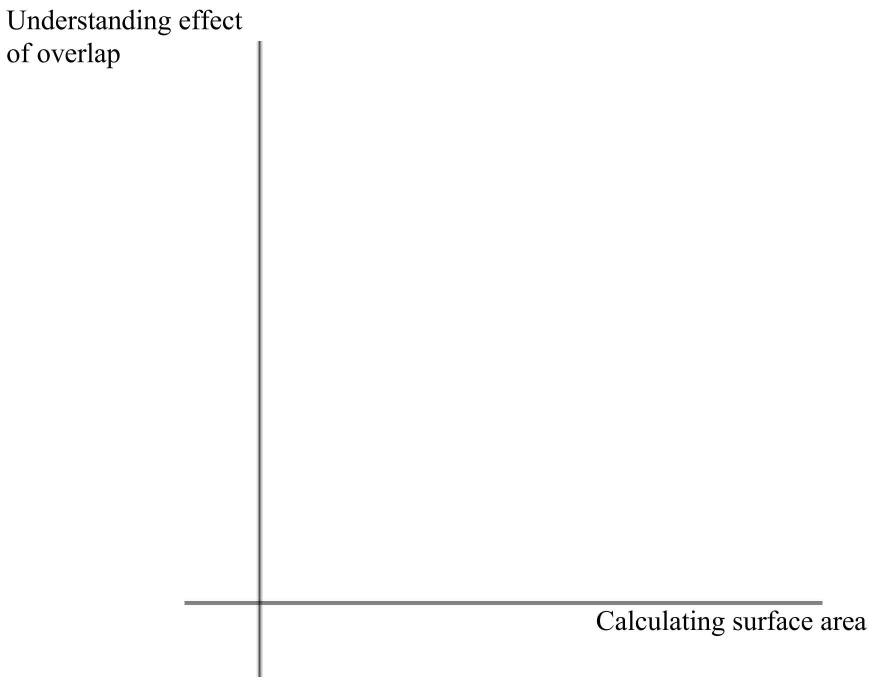

Figure 4.3. The curriculum space for the Surface Area Doubling task.

The curriculum space's greatest attribute is its simplicity. As mentioned earlier, a mathematics classroom is complex, and the activities of groups are extremely dense, filled with social and logistical factors. The curriculum space is established as a subset of the sphere of behavioral possibilities available to the group, concerned only with the nature of the problem deemed relevant to the solution of the task and its relationship to the intended curricular outcomes. It treats curriculum as enacted and the curriculum outcomes as processes to engage the relevant problem, rather than a list of products that result from learning. It is a structure to provide an image of a group's drifting curricular action—their problem drift.

Translating data into the curriculum space. The process of converting the group's action, in the form of video data, into a location in a curricular space was necessarily a qualitative one. The relevant problem was defined as the one which focused group action, the theme of the mathematical interactions. Some of the problems remained relevant for long periods of time, and others held group attention for a short while and then gave way to other possibilities.

A group's coordinates in the curriculum space were coded every time a teacher interacted with the group, or the group action signaled to the researcher/observer that significant problem drift had occurred. The word 'significant' is used in the recognition that the relevant problem is continually being re-made for the knower through structural coupling, but instances can only be coded when they become large enough to be viewed by an observer. That is to say, I can only recognize problem drift at the level of inter-action, and cannot ascribe intentions to learners based on inference. Also, teacher interventions were not interpreted in a deterministic fashion. That is to say, coding the problem drift after a teacher's intervention with a group was not intended to claim that the teacher caused the group to focus on a new feature of the task. Instead, teacher interventions provided triggers and possibilities around which group action focused. Once again, the result of the intervention was not within the intervention itself, but in the group's interaction with it (Towers et al., 2013). A teacher's intervention may have proven potent much later in a solution pathway, but I was limited to coding its immediate impact and not making inferences about lingering effects.

Prior to coding a session, a list of indicators was established for each curricular outcome. These lists contained expected and unexpected actions of a group of learners operating with the targeted curricular outcomes. The word "unexpected" is not used in the sense that I, or the classroom teachers, did not anticipate the strategy, but rather that it signaled a sophisticated utilization of the outcome that was not expected from every group. Unexpected actions were those that, in my interpretation, signaled a deep understanding of a targeted curricular outcome. Both expected and unexpected actions indicated that students were using curricular processes to address the relevant problem of the group. The list of indicators for expected and sophisticated action established for the Surface Area Doubling task appear in Table 4.2. These indicators of expected and unexpected action were established prior to coding classroom sessions, but were open to amendment when student groups did enact sophisticated, yet unanticipated, strategies. For instance, during their work with the Surface Area Doubling task, Cohen, Anne, and Lucas wondered if it would be possible to attach pieces to the structure without producing any overlap. They brainstormed the use of string, but they decided, with the help of the teacher, that a string does, in fact, have some surface area and this strategy would not suffice. Later on in the task, one member realized that if they built an extension over the open doorway, there would be no loss in overlapping surface area. I did not anticipate this strategy emerging as relevant, but it showed a deep understanding of the effect of overlap on a composite shape, the vertical curricular outcome in the curriculum space. The new indicator was then added to the list of those actions that signaled an outer level of sophistication with the curricular outcomes. It can be seen in parentheses in Table 4.2.

Table 4.2

Expected and unexpected indicators for the Surface Area Doubling task

Inside Benchmark (Expected actions)	**Outside Benchmark** (Unexpected actions)
Calculating the surface area of a composite shape	
• Using formulae to calculate surface area of entire solids. • Recognizing a composite shape as composed of 2-dimensional shapes.	• Dissecting the formulae to correspond to 2-dimensional shapes. • Using the symmetry of the shape to combine calculations.
Understanding the effect of overlap on the surface area of a composite shape	
• Recognizing that overlap results in losing the overlapped area twice. • Making general compensations for anticipated surface area loss. • Making an estimation, calculating the surface area, and then adjusting addition to add or subtract area as needed.	• Calculating and accounting for the specific amount of surface area lost. • Building an addition with an exact surface area corresponding to the loss due to overlap. • (Constructing the addition in a way where no overlap will occur.)

Based on the established indicators, I marked a benchmark space in the curriculum space that created a reference point between expected and unexpected group action. The space is shaded to hint at the ability of group's problems to drift between, what I interpreted as, inner and outer levels of sophistication. Groups operating in productive, but less-sophisticated ways with the curricular outcomes had their problem drift coded at locations in the bottom-left of the curriculum space, inside the benchmark. Groups operating at sophisticated levels would have their problem drift coded outside of the benchmark in the curriculum space resulting in ‘outside the box’ thinking in both the literal and figurative sense. This reference point was established to provide more explanatory power to the tool of curriculum spaces as well as provide more stability in the coding system. Figure 4.4 shows the benchmark included in the curriculum space of the Surface Area Doubling task.

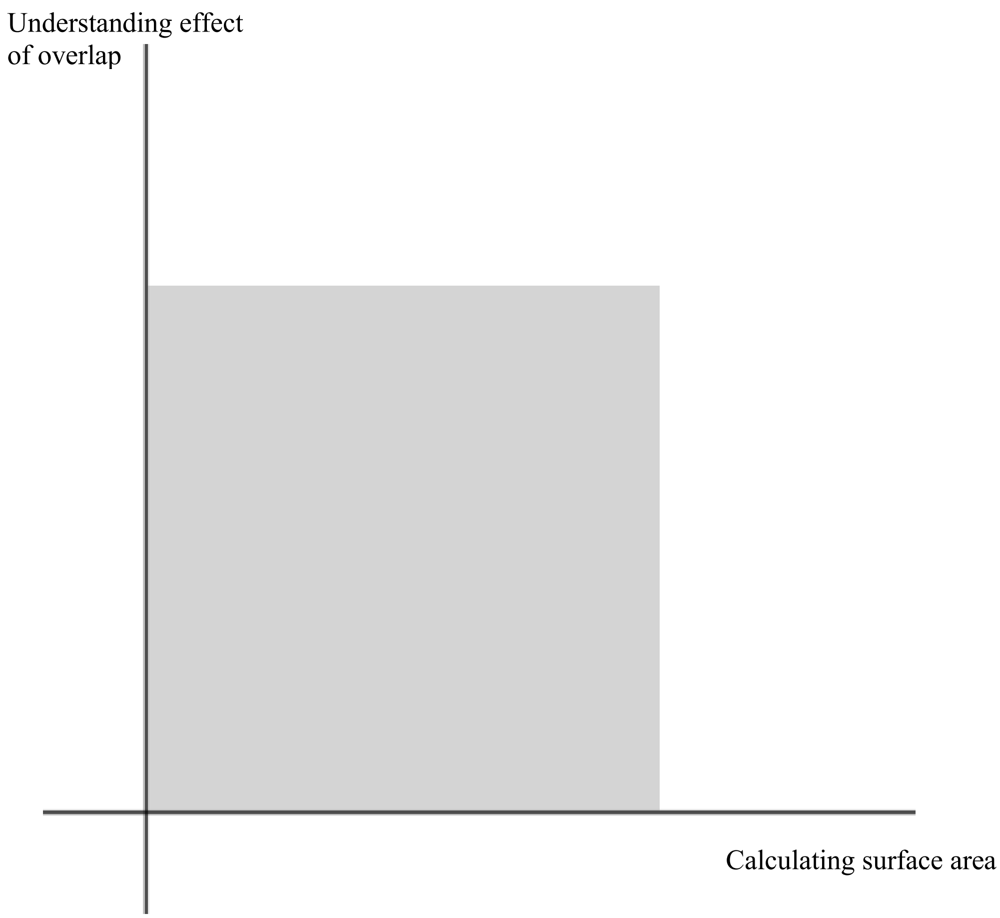

Figure 4.4. The benchmark space imaged in the curriculum space of the Surface Area Doubling task.

In the curriculum space, an axis represents the demarcation between relevance and irrelevance based on my observation of group action. When a node was coded on the horizontal axis, there was no indication of the vertical curricular outcome proving relevant to the group’s current instance of problem drift. When a node was coded on the vertical axis, there was no indication from the group’s action that the horizontal curricular outcome was relevant to the current problem focusing action. The interpretive tool of the curriculum space contains no negative values on the axes because it did not make sense to talk about curricular outcomes in various degrees of irrelevance; either the outcome was relevant in the group's actions (to some degree) or it was not. Therefore, no node was coded as negative on either axis, and higher degrees of relevant action with the curricular outcomes appear as nodes further away from the horizontal and vertical axes into positive space.

It is important to note that the coded location in the curriculum space was not based on the correctness of a group's work, but on the reasoning behind their action. A group may establish a fairly sophisticated way of reasoning with a particular outcome, but make an error in the process of calculation. Alternatively, a group may interact with the outcome in a less-sophisticated manner but execute a solution method perfectly. The former group would be coded as having a more distal location, in relation to the origin, in the curriculum space with that particular outcome. Phrased differently, the indicators for coding were based on actions that indicate that a group is operating in sophisticated ways with the curricular outcome, and not that a group arrived at a correct solution. The curriculum space is an image of process, not of product—an image of relevance, not an image of competence.

A node was placed in the curriculum space every time a group was interpreted as acting on a new problem of relevance. That is, a new node was placed after every interpreted instance of problem drift. A node was also placed after each time a teacher offered an intervention to a group, whether it triggered problem drift or not. The location of the node was determined by the interpreted degree that each outcome played in focusing the group's action. That is, greater evidence of the number or extent of an outcome's anticipated actions being used to address the relevant problem resulted in the instance of problem drift being coded higher on that particular outcome's axis. In order to breach the shaded benchmark space, the group needed to show evidence of using the outcome in an unexpected (or unanticipated, yet sophisticated) fashion while addressing the relevant problem. Again, the number or extent of these actions, as interpreted by me, determined where in space the node was coded.

The nodes were connected with arrows to give a dynamic feel to the task session. This resulted in every teacher intervention and occurrence of problem drift having a starting node and an ending node. An arrow between the two nodes represented the re-posing process of problem drift. Solid arrows symbolized problem drift occurring directly after a teacher intervention, and dashed arrows represented problem drift that emerged out of the interactions of the group without immediate teacher intervention. The timing and sequence of nodes were left off of the curriculum space purposefully to accentuate where groups have been—emphasizing curriculum as landscape—and not for how long or in what order they arrived—which may emphasize curriculum as sequence.

Due to the density of the emerging images, certain notational features were adopted. If a group was deemed to be working at a specific location in the curriculum space and that location was unchanged after a teacher intervention, a hollow node was used. A node was bolded if a group drifted away from a location in the curriculum space but returned later. On one occasion, no discernable problem drift occurred after a teacher intervention, but the node was returned to at a later time in the session. This location was coded as a bolded, hollow node. On another occasion, two consecutive teacher interventions triggered no problem drift, but the group eventually drifted away from the location and returned at a later time. This location was coded with concentric, hollow rings, the outermost of which was bolded. One group completely stopped all curricular action to prepare their solutions for presentation, but during their preparation, a curricular problem re-emerged. The interaction between the two nodes was not coded with an arrow (signaling problem drift), but was connected with a bolded line containing flat ends instead of arrowheads. Refer to Table 4.3 for a complete legend of the coding notation.

Throughout the coding process, problem drift arrows were broken to avoid intersection with co-linear points and to aid in clarity. Examples of the coding from the excerpts provided in Figure 4.1 and Figure 4.2 appear in Figure 4.5 and Figure 4.6 respectively. Figure 4.7, Figure 4.8, and Figure 4.9 isolate the three other instances of teacher triggered problem drift coded in the curriculum space of Cohen, Anne, and Lucas working with the Surface Area Doubling task. Sections of transcript corresponding to the location of each node are provided on the curriculum space to give a sense of the student action that caused the node to appear in that particular location in the curriculum space. Alongside the coding in curriculum space, each teacher intervention was also coded as belonging to one of the categories and sub-categories of mathematics teaching actions (see Table 4.1). The descriptions and illustrative examples offered by Towers and Proulx (2013) were consulted heavily in these determinations.

Table 4.3

Legend of coding notation

Notation	Meaning
	A standard node denotes a coded instance of problem drift.
	A hollow node denotes a teacher intervention triggered no problem drift.
	A bolded node denotes a group drifted away from but later returned to the same location.
	A hollow node with a bolded ring denotes a teacher action triggered no direct problem drift, but action later returned to this location.
	A node with concentric rings with outer ring bolded denotes two consecutive teacher interventions that triggered no problem drift, but the group later returned to this location.
	A solid line denotes teacher triggered problem drift.
	A dashed line denotes group triggered problem drift.
	A bolded line without arrowheads denotes group action stopped but re-organized at a much later time.

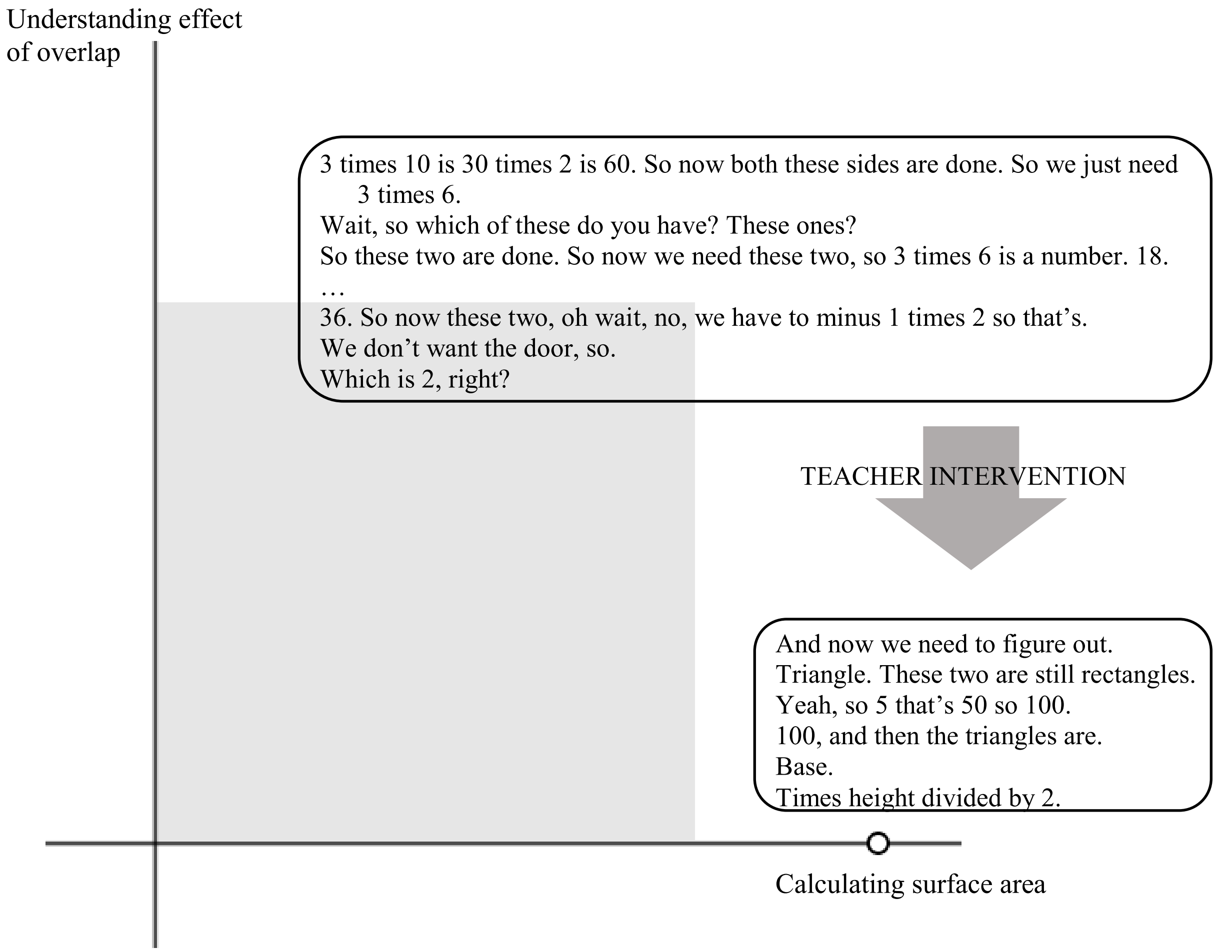

Figure 4.5. The problem drift from the expanded transcript of Figure 4.1 coded into the curriculum space of the Surface Area Doubling Task.

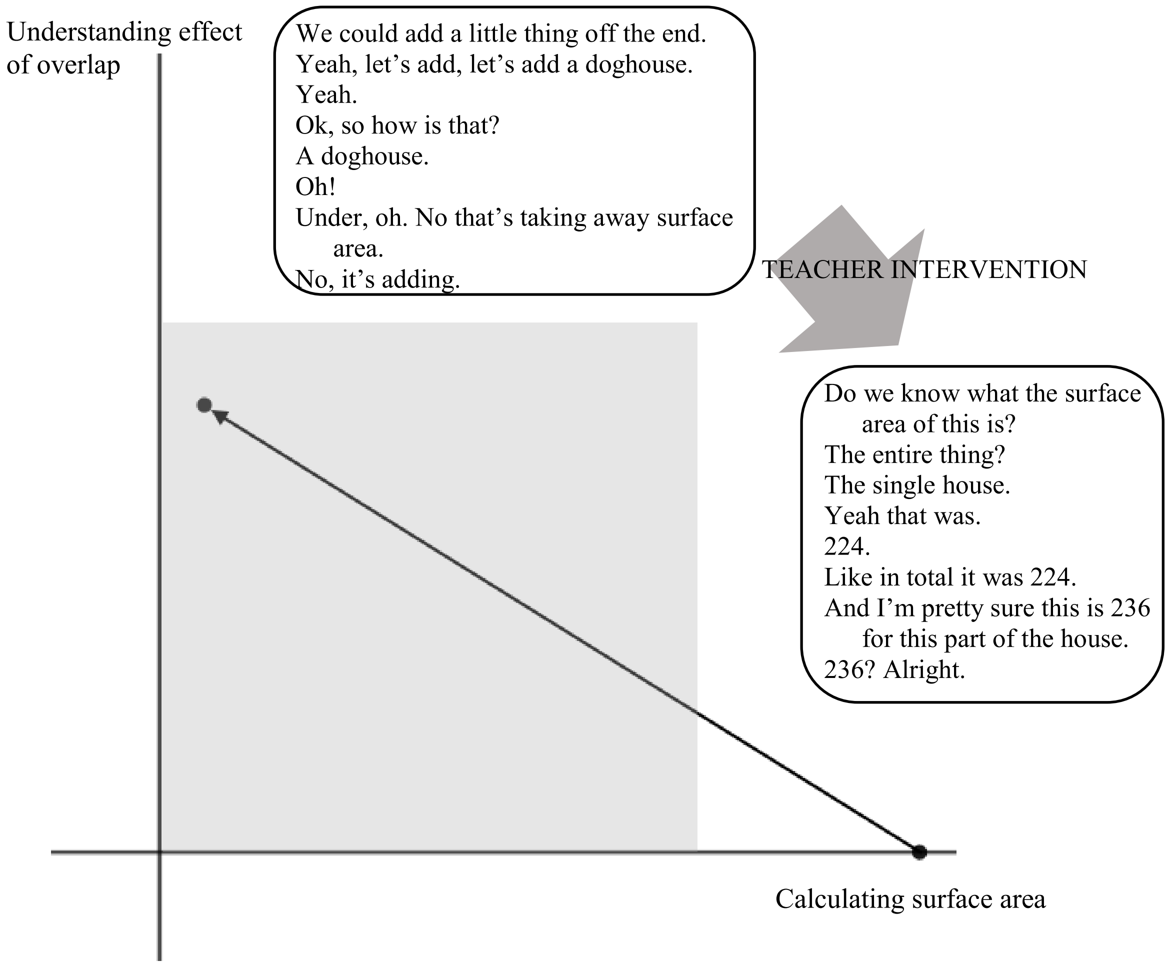

Figure 4.6. The problem drift from the expanded transcript of Figure 4.2 coded into the curriculum space of the Surface Area Doubling task.

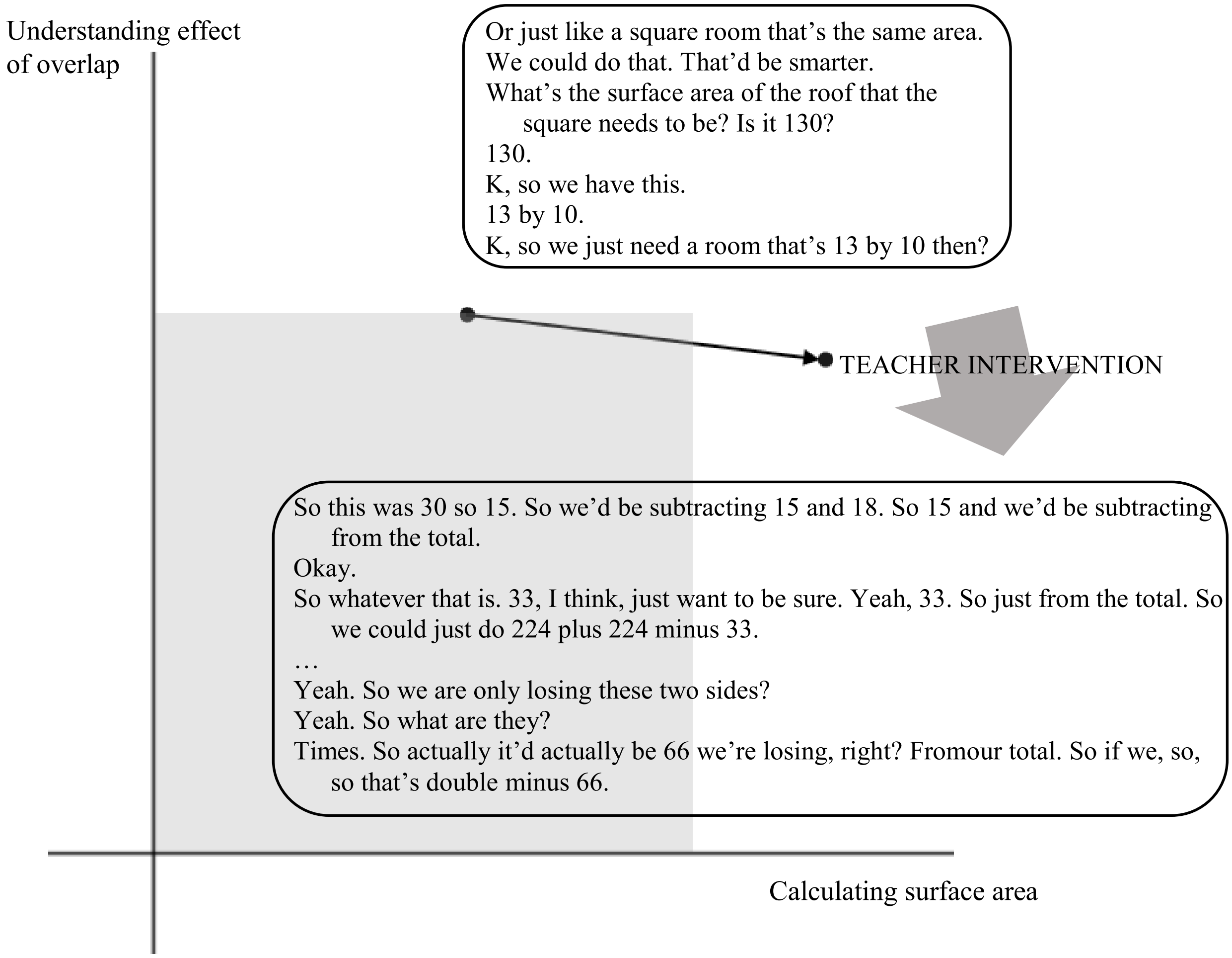

Figure 4.7. A third isolated coding of problem drift in the action of Cohen, Anne, and Lucas working with the Surface Area Doubling task.

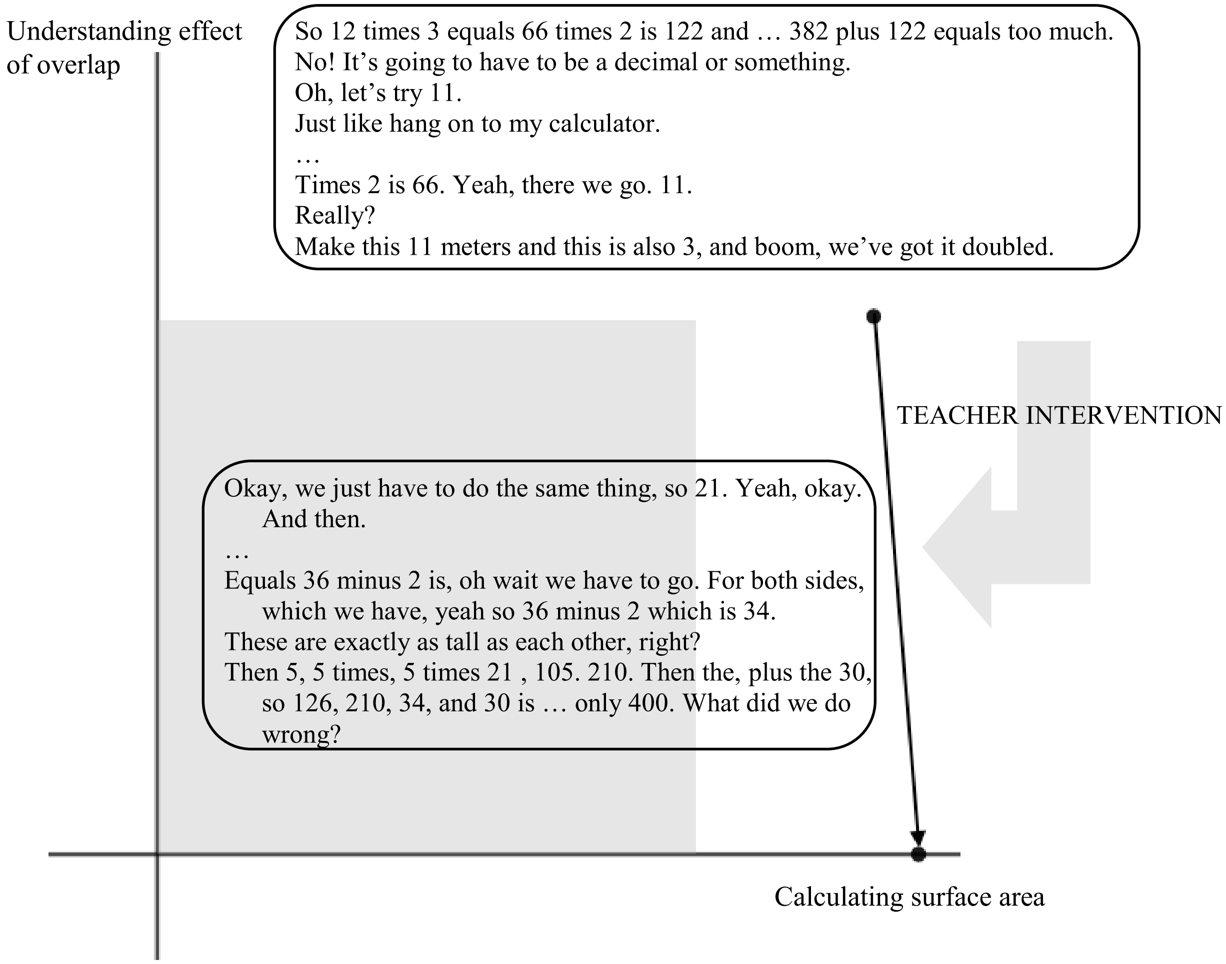

Figure 4.8. A fourth isolated coding of problem drift in the action of Cohen, Anne, and Lucas working with the Surface Area Doubling task.

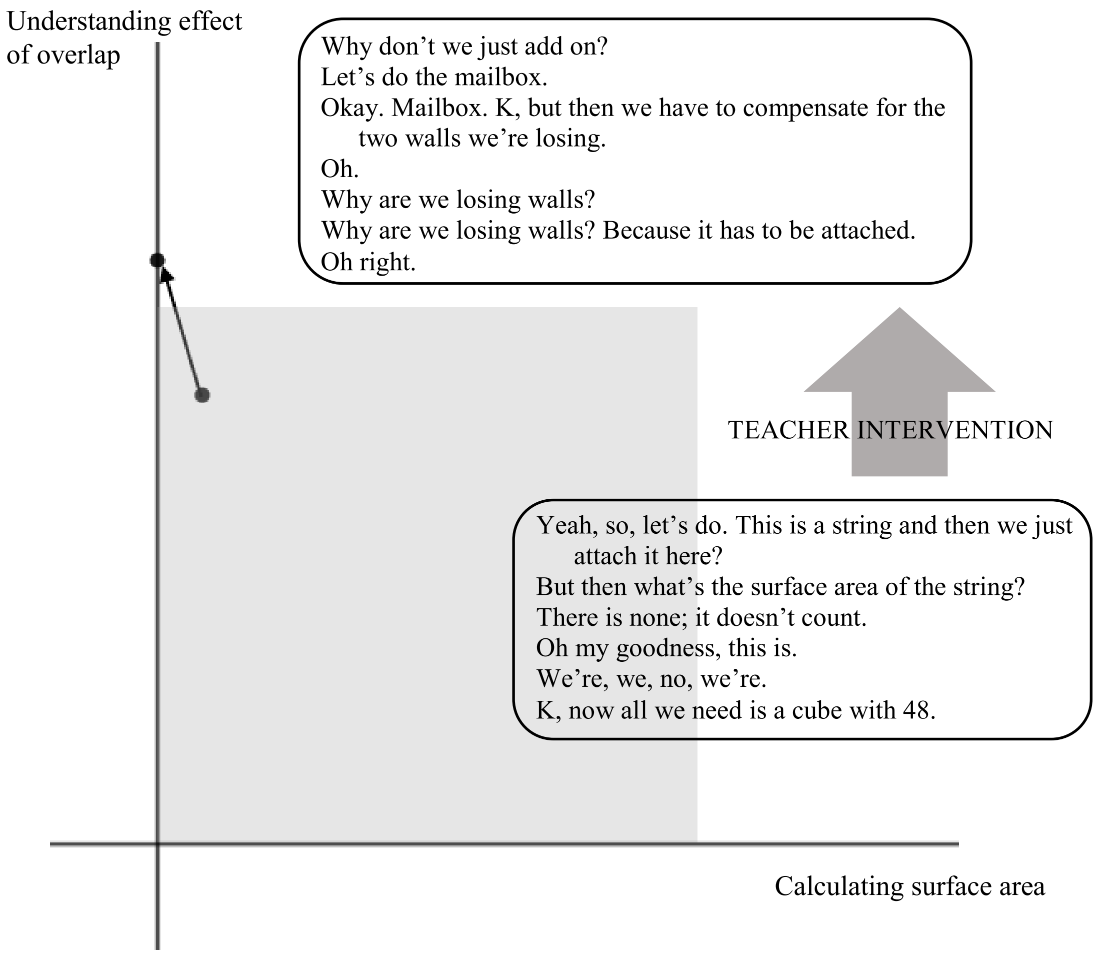

Figure 4.9. A fifth isolated coding of problem drift in the action of Cohen, Anne, and Lucas working with the Surface Area Doubling task.

Curriculum Space

The product of the analysis process is an image of curriculum as landscape where the human interaction couples with the given task to continually pose problems of relevance at varying levels of curricular significance. The images tell stories, develop personalities, and capture critical and interesting moments. They represent the interaction of complexity, curriculum, and the process of coming to know that weaves the two together. The complete curriculum space from the action of Cohen, Anne, and Lucas used throughout this chapter is shown in Figure 4.10. Appendix B contains the curriculum spaces from all ten group sessions chosen for close analysis.

The curriculum spaces are meant to reflect an image of a teacher acting fully implicated in the learning context—a context that is not static, but drifting amidst relevant problems. The image provided by a curriculum space is meant to honour the coupling of group, teacher, and problem, and the delicate nature of commentating collectivity and curriculum. It is through patterns in curriculum space that the problem drift of the groups working toward curricular goals, as well as the pattern in teacher interventions, is observed and interpreted. Using the curriculum space to analyze classrooms provides a glimpse into the emerging worlds of mathematical significance brought forth by groups of learners and interpreted and acted on by teachers while working together on mathematical tasks.

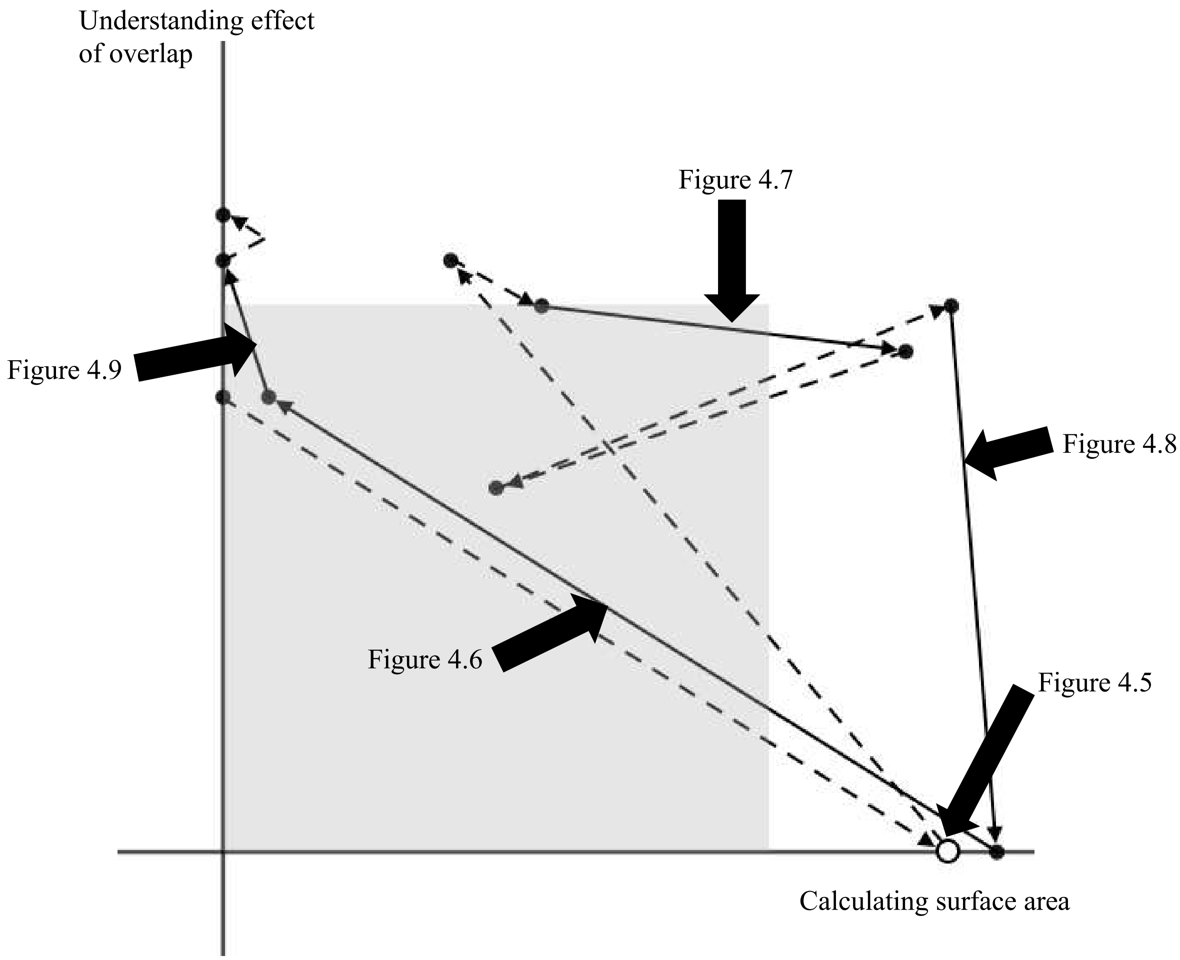

Figure 4.10. Cohen, Anne, and Lucas' complete curriculum space while working with the Surface Area Doubling task.

Chapter 5: Patterns in Curriculum Space

"Looking for the patterns in static / They start to make sense the longer I'm at it."
—Benjamin Gibbard, *Lightness*, 2003

Patterns in Classroom Action

The results of using the tool of curriculum space to interpret the mathematical action of groups was both informative and generative. I began the study with the intention of investigating the patterns of mathematical action that emerged from collectives of students. The curriculum space provided an accessible image of problem drift; it granted the group a sort of *personality* in relation to the curriculum they enacted. Important understandings about teaching with a sensitivity for problem drift emerged for me (and they will be documented in the remaining chapters), but alongside these were questions emerging from the interactions with participant teachers. Classroom sessions ended with an interrogation of the teachers' own interventions. Many of their questions had a meta-pedagogical tone where they were not necessarily asking, "Was that a profitable intervention at the time?", but rather, "Why did I think that was a profitable intervention at the time?". As is the intention of iterative design research, the work began to theorize at the edge of new boundaries. The notion of collective pushed outward to include the teacher, and the idea of pattern began to include how the teacher felt they should act based on the character of the group. Analysis was then extended to include an exploration into the tendencies of teacher action that emerged over the scale of the entire project. The result was a holistic view of collective action where, through the image of curriculum space, the observed patterns offered implications for teaching in a complex classroom.

The results of the study presented here lay the groundwork for developing a lens for teaching groups in a complex classroom. They do not prescribe or tell, but rather image and imagine. They are not presented as archetypes into which all group action fits or should be fit, but as explanatory episodes that might help teachers recognize possibilities in their own practice. Returning to the heart of enactivist inquiry, the results are intended as *models for* and not *theories of* (Reid, 1996).

The chapter is broken into three illustrative episodes of the patterns of group actions followed by an analysis of teacher actions as they occurred in context. The three episodes of group action are presented to observe key patterns that emerged, allow teachers to interpret their own practice into the data, and draw from them advice that we, as teachers, might carry into our own classroom practice. The patterns of teacher intervention are included to create an image of how teachers operated in a classroom that aimed to elicit the conditions for complexity and carve a curriculum from its action. Taken together, the image of classroom complexity is given a character. To be clear, this is not an ontological mission to prove the existence of complexity in classrooms; rather, my aim is to allow the images of classroom patterns to begin to describe viable ways in which curriculum and complexity might co-exist. In the same way that Davis (1996) speaks of allowing a lesson to unfold, I present these results in the recognition that "we cannot make others think the way we think or know what we know, but we can create those openings where we can interactively and jointly move toward a deeper understanding of a shared situation" (p. 239).

Episode 1: The Tile Design Task

The Tile Design task asked students to create a series of shapes with coloured square tiles to satisfy requirements given to them by a stage card. A sample stage card appears in Figure 5.1, and the complete set used in the classroom episode is included in Appendix A. Mrs. Murray, Ms. Becker, and I each had copies of the stage cards which were organized into four general stages. As the classroom session unfolded, the three of us visited the groups, offered interventions, and, when we felt the group was ready, gave them a new challenge. Sometimes the new stage card was at the same stage as the previous one, and sometimes it introduced them to the next stage. Students worked on the task for approximately twenty-five minutes, after which an entire group de-briefing was held to share strategies and distill curricular competencies.

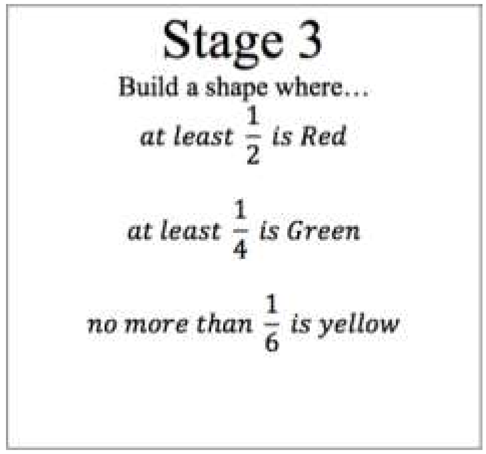

Figure 5.1. A sample stage card used in the Tile Design task.

During the pre-session meeting with the teachers, it was established that the task was aimed to occasion two curricular outcomes. In particular, we wanted to collect evidence of students creating equivalent fractions as well as comparing and reasoning about fractions in a part-whole model. These two outcomes became the metrics on the curriculum space's horizontal and vertical axes respectively. Episode 1 contains the action of Brock, Ria, and Sharla as they enacted the task. Their resultant curriculum space is shown in Figure 5.2.

Reading the curriculum space. In the post-session interview, all three teachers described this group as being thorough but stubborn. Their action was habitual in the sense that they returned to a familiar space to think about new stages and did not appreciate teacher attempts to disturb this pattern. They often complained about interventions offered by the teacher, because they did not want to jeopardize their solutions. Teacher reflection on the session categorized the group's movement as polished; the group initially rejected many interventions because they posed some difficulty with their methods. They responded to many of the perturbations by rationalizing why the intervention did not make sense, and returning their workspace to its original organization. This tendency is imaged as a dense aggregation of nodes in the central portion of the curriculum space. The following excerpt of the group working on the stage from Figure 5.1 typifies the group's interaction with a teacher's intervention:

Teacher: What was the first thing you focused on?

Brock: Red.

Teacher: Why red?

Ria: Because most of it is red.

Teacher: Oh, the biggest section.

Ria:	So we need to have half the shape red, so if we have that, we can kind of go from there.
Teacher:	Okay. Can I ask why you used 12 total tiles?
Brock:	Because these all have a common denominator of 12.

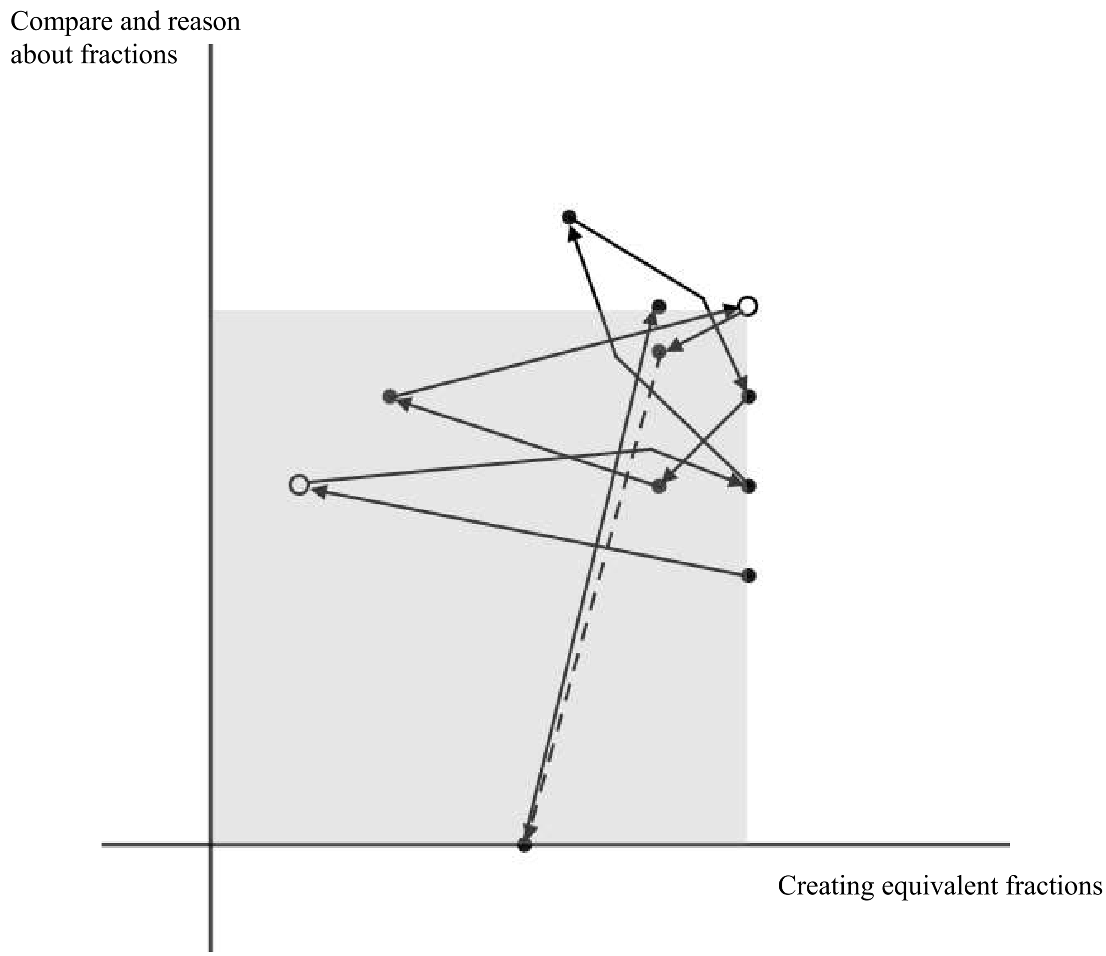

Figure 5.2. The curriculum space of Brock, Ria, and Sharla working with the Tile Design task.

The top of Figure 5.3 shows the group's solution to the stage. The red tiles are arranged in such a manner so as to quickly identify the one-half requirement. The bottom of Figure 5.3 shows the intervention offered by the teacher; a green tile is removed from the arrangement with the intention that students might establish what fraction of the shape is red, yellow, and green and check the sizes of each fraction against the requirements. This would prompt vertical movement in the curriculum space as the students compared the sizes of the new sections. Instead, the group rejects the intervention, and replaces the tile that was removed.

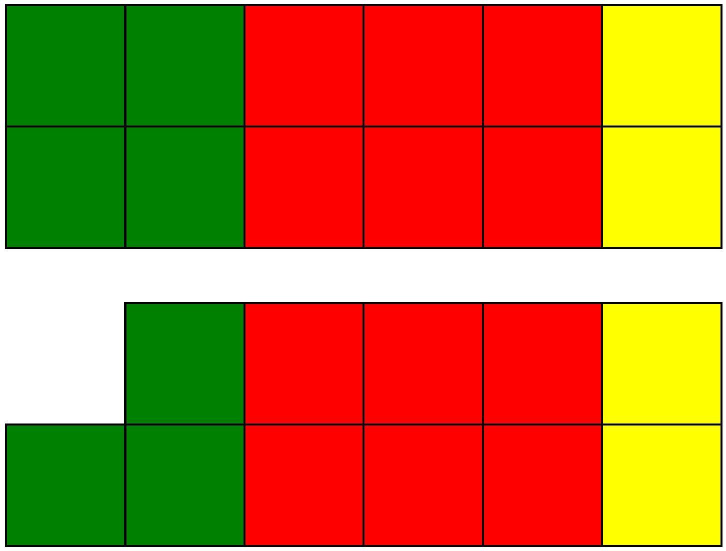

Figure 5.3. Top: Brock, Ria, and Sharla's solution to a stage in the Tile Design task. Bottom: A green block is removed by the teacher.

Teacher: What if I did this? Does that still work?

Sharla: No, because he.

Ria: No, because this is less than a quarter now.

Sharla: No, it's not.

Brock: No, it's a quarter.

Ria: No, but it's still not a quarter of this.

Brock: But it doesn't equal 12.

Ria: Exactly, so you need that. [*Ria replaces the tile to restore the arrangement from Figure 5.3 (top)*].

If the group would have reasoned about the new sizes of each of the three colours (now each with a denominator of eleven), and then compared them with the requirements set by the stage card, they would have drifted into the outer region of the curriculum space. Instead, the group chose not to reason about the new fractions. This tendency resulted in pattern resembling a knot in the curriculum space. We see that almost the entirety of the group's action falls within the benchmark space (Figure 5.2), even though teachers attempted to trigger movement outside of the benchmark numerous times throughout the classroom session. Often, this took the form of adding or removing tiles in hopes that the group would begin to compare sizes of fractions with different denominators. The tendency to cluster along the right hand border of the benchmark signals that the group had an understanding of creating common denominators, but insisted that certain denominators be used.

Lessons in the curriculum space. Several interventions did not have the intended effect, and the curriculum space offers an image of group action that is habitual and knotted in nature. However, the image of the group's problem drift directs attention to two features. First, as Figure 5.2 shows, the group—although stubborn—is operating in a productive curricular space. They recognize and create common denominators, reason about the size of sections when pieces are added or subtracted, and show understanding of inequalities and fraction size. Just because their action was densely knotted does not mean this group did not use the curricular outcomes in productive ways. Second, the curriculum space begs for further exploration into the moment where the group action breaches the benchmark. The moment may seem to stand out as occasioned by a particularly novel teacher intervention, but that instance of problem drift was instigated by a teacher asking a simple question while the group was working with a stage requiring half of the arrangement to be made up of blue tiles:

Teacher: Is that half blue?

The flurry of group sense-making serves as testament to the unpredictable nature of problem drift. Sometimes, interventions anticipated to prompt sophisticated responses do not trigger the desired curricular movement, while simple clues may result in an outbreak of action. For the teacher, attuning to the interaction with the trigger is key for teaching in the collective. For this group, well-designed interventions attempted to occasion the comparison of fractions size, but were not effective because the structure of the group would not allow them to be. Instead it was a simple fact-checking intervention that triggered movement outside the benchmark.

Brock: This is only 14!

Ria: We need 20 tiles.

Sharla: Because the common denominator is 20?

Teacher: What were you going to say, Ria?

Ria: You need 20 tiles because the denominator, the common denominator is 20.

Brock: Yeah. It doesn't equal 20.

…

Ria: You add other blues.

Sharla: If you add, then you have to add in more. But then you have to add in another colour. Add another colour.

Brock: Wait, how many green do we have?

Sharla: We need to add in more green then if you added more blues.

Ria: Which means we have to add more yellow because it has to be equal.

Episode 2: The Fill in the Blanks Task

The Fill in the Blanks task provided students with an expression structure containing several blank squares (Figure 5.4). The students were asked to fill in the blanks with the digits 1 through 9 to make the expression true. Each digit could be used at most once each. Worthy of note, one of the fractions in the expression contained two adjacent blanks. The class was told that the two digits placed in these blanks would take on the appropriate place value. As the classroom session unfolded, Mrs. Murray, Ms. Becker, and I visited the groups and offered interventions with our curricular outcomes in mind. The session lasted approximately fifteen minutes and was followed, as was our custom, by a whole-class debriefing to share strategies and distill curricular competencies.

$$\frac{\square}{\square} > \frac{\square}{\square\,\square} = \frac{\square}{\square} > \frac{\square}{\square}$$

Figure 5.4. The expression structure given to the groups in the Fill in the Blanks task.

At the pre-session meeting, the target outcomes were established. We wanted group action to focus on creating equivalent fractions and reasoning about the size of fractions. Creating equivalent fractions was chosen as the curriculum outcome for the horizontal axis to mimic the curriculum space from the Tile Design task (see Figure 5.2). Reasoning about the size of fractions was coded on the vertical axis. Episode 2 contains the action of two groups as they enacted the task. The curriculum space of Amina, Elliot, and Makalia is shown in Figure 5.5, and the curriculum space of Colin, Duncan, and Madlyn is shown in Figure 5.6.

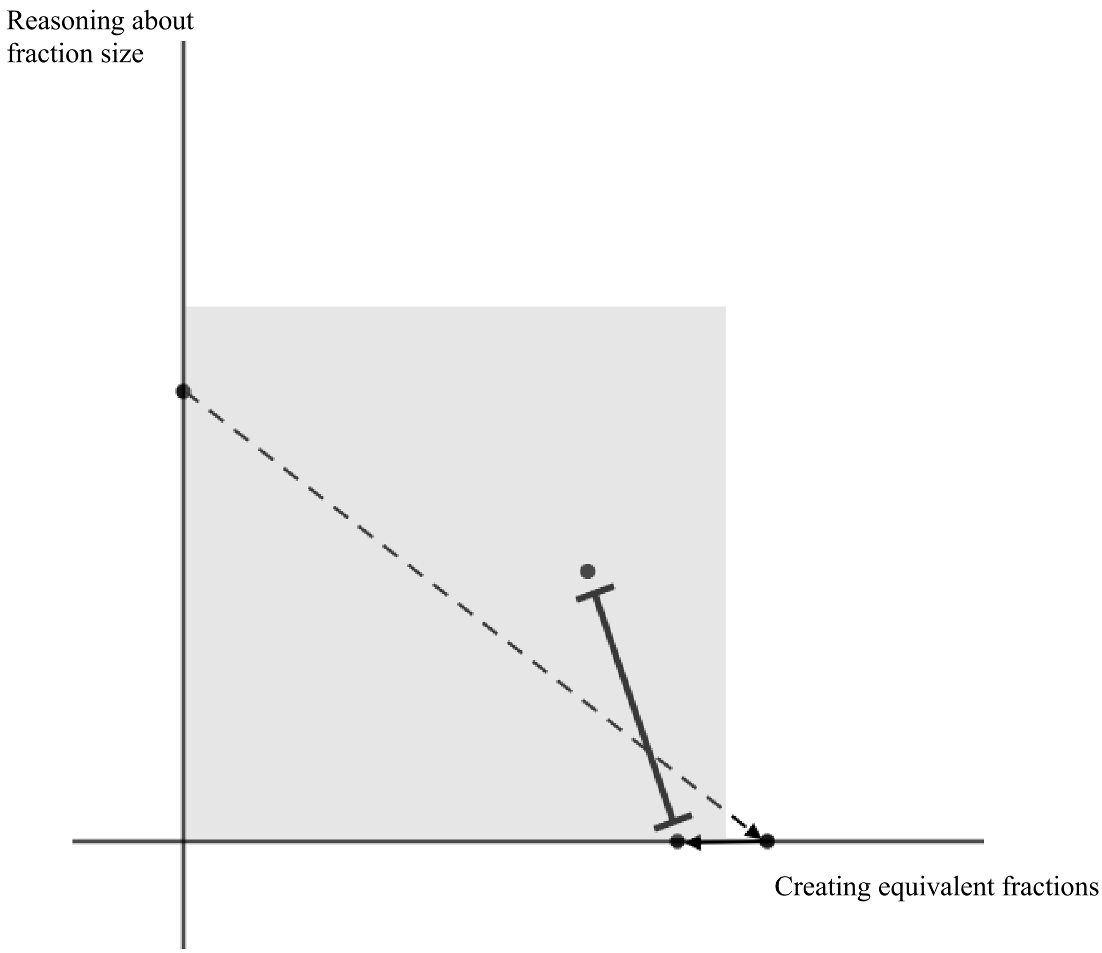

Figure 5.5. The curriculum space of Amina, Elliot, and Makalia working with the Fill in the Blanks task.

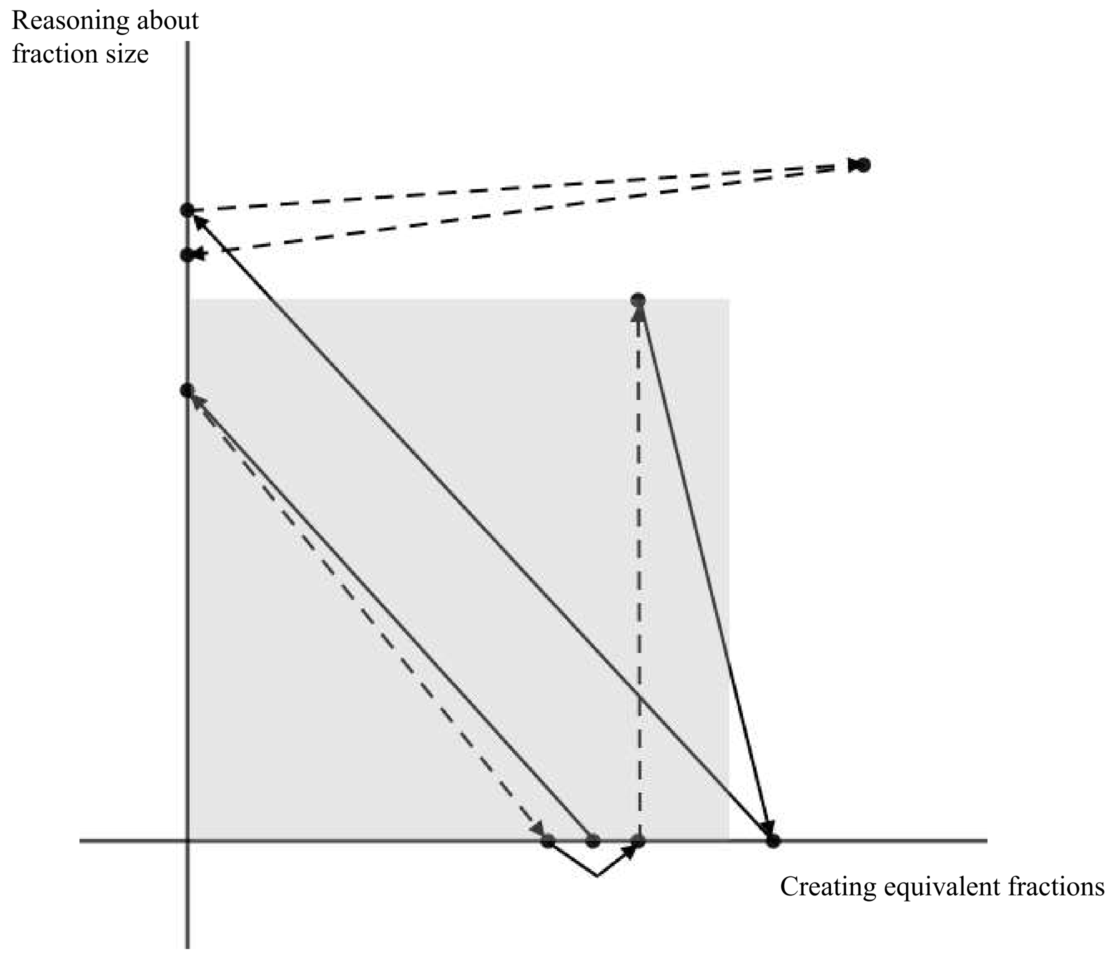

Figure 5.6. The curriculum space of Colin, Duncan, and Madlyn working with the Fill in the Blanks task.

Reading the curriculum space. The curriculum spaces of these two groups clearly shows the difference in personality enacted by the two groups, but there are similarities in their structure as well. The majority of their trace is spent on the axes of the outcomes—that is, both group's action was often coded directly on one of the two axes. Again, these are not unproductive spaces, but do signal that the two curricular outcomes were often acted on in isolation. Consider the following exchange where Amina, Elliot, and Makalia were trying to choose which digits will make up the middle section of the expression:

Amina: Then what if we did 2 over 4? No, let's do 4 over 8 and 3 over 6.

Makalia: Yeah.

Amina: 4 over 8 and 3 over 6, and then. Wait, this one has to have two digits.

Makalia: Do this first then.

…

Amina: What if we did like 6 over 12.

Makalia: Yes.

Elliot: You can't use 12.

Makalia: You can't use 12?

Amina: Yeah we can. That's be two digits. They are just right next to each other.

The group decided to choose two fractions from a generated list of familiar fractions equivalent to one-half. The structure of the task caused a problem with their choices, but they quickly adjusted to meet all of the requirements. At no moment in their doing does the size of the fraction become a relevant problem; the result is action contained on the horizontal axis in the lower-right portion of the curriculum space. Contrast that exchange with the relevant problem evident in the following action from Colin, Duncan, and Madlyn as they tried to choose the digits to place in the leftmost fraction of the expression:

Madlyn: Okay.

Colin: Does 9 make that big?

Duncan: Is that a 9?

Madlyn: We need a fraction.

…

Duncan: It has to be 1 to 9.

Colin: It does?

Duncan: Yes.

Madlyn: Yes.

Colin: 9?

Duncan: 9. Put a 9 and then a 1.

Madlyn: Now look, it's a really big fraction.

This group also ran into a problem provided by the structure of the problem, but they quickly imbued it into the relevant problem they were pursuing: How can we make the largest possible fraction? In the end, creating the improper fraction seems to hold the answer to that problem. After a short time questioning whether nine can be classified as a fraction, they end at a point of clarity, feeling confident they have created the largest possible fraction without creating equivalent fractions ever becoming relevant to their action. The result is drift along the vertical axis through the upper-left portion of the curriculum space.

There are, however, significant differences in the character of the two curriculum spaces. Based on the observed problem drift, the action of Amina, Elliot, and Makalia (Figure 5.5) appears to be at a location much more proximal to the origin than that of Colin, Duncan, and Madlyn (Figure 5.6). In fact, the relevant problem for Amina, Elliot, and Makalia never enters the upper-right portion of the curriculum space. It would be easy to assign some type of competence to the students based on this image, but the curriculum space is not an image of competence—it is an image of relevance. Amina, Elliot, and Makalia's strategy was to insert the four leftover numbers into the blanks comprising the first and last fractions after using the first five to establish equivalency in the middle of the expression. This strategy worked both times the group solved the task. After two successful solutions, the group was comfortable that their strategy would always work, and began to prepare their answers for presentation. The bolded line denotes the termination of action on the task, but in the process of recalling their solutions, they re-engaged the curricular outcomes. Amina, Elliot, and Makalia never acted on a relevant problem that forced them to reason at a high level about fraction size, but this says nothing about their ability to reason about fraction size. In fact, the reason for the stability in their problem drift was that they were able to quickly and efficiently arrive at correct answers without ever having fraction size prove relevant.

Colin, Duncan, and Madlyn have an extremely active pattern of problem drift and a resultant curriculum space filled with sweeping, volatile movements. In the post-session interview, one teacher described the group as sensitive, always receptive to possibilities afforded by the teacher interventions. This is evidenced by the long arrows making up the problem drift. Colin, Duncan, and Madlyn spend the large majority of their time in a very productive space because they made an error during their initial attempt at the problem (Figure 5.7). After they built the equivalent fractions, they realized that they could not create a fraction to place in the rightmost side of the expression that was smaller than one-ninth.

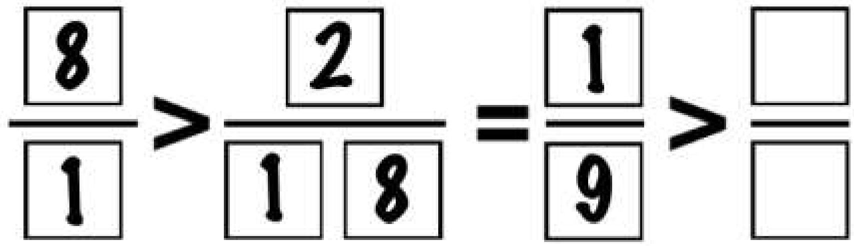

Figure 5.7. Colin, Duncan, and Madlyn's initial attempt at the Fill in the Blanks task.

Madlyn: What's 18 divided by 2?

Duncan: 9. No! We already have 9.

Madlyn: Let's change it to an 8. Look, it's equal.

Duncan: But.

Madlyn: Oh, but this has to be big. [*Pointing to the one-ninth*]

Duncan: No, that has to be small.

Madlyn: No, but like this has to be big but it's small. So we have to make it big.

Colin: Change it.

This allowed the curricular ideas of creating equivalent fractions and reasoning about the size of fractions to combine into a hybrid problem: How can we create equivalent fractions that are big enough so we can still create a fraction smaller than them? This new, drifted problem caused the group to be coded at very peripheral locations in the curricular outcomes as evidenced through their action.

Duncan: Okay. Find even numbers for these two and then we can fill these two ones after.

Madlyn: But make sure these two are.

Duncan: Are using small numbers.

Madlyn: No, they should use our big numbers.

Duncan: Yeah.

Madlyn: Okay, we got this. Big numbers.

…

Madlyn: 6 and 3. Because that's all we can do.

Duncan: What would we do below it?

Madlyn: So then the bottom.

Colin: 7?

Duncan: Why is it 7?

Colin: Why not?

Duncan: Because why would it be 7?

Madlyn: We need to have bigger numbers in the middle.

…

Colin: I'm saying we could do 14.

…

Madlyn: Is it equivalent?

Duncan: Yeah.

Colin: Yeah

Duncan: Because 3 fits into 6, two times. 7 fits into 14, two times.

After all their work, they still arrive at a solution that doesn't quite satisfy all the requirements of the task, because the group uses the digit "1" in two different blanks. (Figure 5.8).

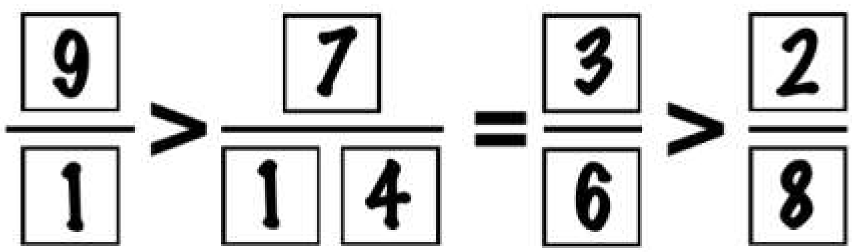

Figure 5.8. Colin, Duncan, and Madlyn's final answer for the Fill in the Blanks task.

Lessons in the curriculum space. Viewing the classroom action through the lens of a curriculum space shifts the perception of right answers. The goal of the task was to have groups operate in sophisticated ways with the targeted curricular outcomes. One group efficiently generated correct answers with a problem drift categorized as stable; another group worked with a volatile problem drift pattern as misunderstandings allowed numerous productive problems to be posed as relevant. If the curriculum is to be envisioned as a landscape, then learning emerges from effective action on the problems posed in the moment, and very productive action occurs when the relevant problem opens up possibilities for students to act in connected ways in the curricular outcomes. Here, we see that stable patterns of action caused by efficient pathways to right answers may restrict the conditions for complex emergence, streamline the phenomenon of problem drift, and result in a lower level interaction with the intended curricular outcomes. Lively patterns in a curriculum space do not always signal correct answers and vice versa. However, if we conceptualize curriculum as *travelled through* instead of completed, correctness and efficiency might become barriers as completing tasks takes a back seat to sense making.

Episode 3: The Solid Fusing Task

The Solid Fusing task gave groups a set of six geometric solids and asked the students to combine them to create a composite shape that had a surface area (measured in square units) as identical as possible to its volume (measured in cubic units). Two of the solids are pictured in Figure 5.9, and the complete set can be found in Appendix A. Students were given further parameters, all of which their new, composite solid had to suffice. First, the new composite shape had to include at least two of the six solids provided. Second, solids could only have their faces fused; vertices and edges could not be 'fused' to faces or to each other. Lastly, the faces on both solids could not be fused partially; sides could not *hang off* one another. The image used to illustrate this point is shown in Figure 5.10. The first configuration pictures illegal fusing where the cylinder hangs off the cube. The next two images show legal fusing. In the middle, the base of the pyramid and the side of the cube fuse perfectly, and the last image shows a cube completely fused to the top of a cylinder. This fusing is legal because the face of the cube is entirely fused to the cylinder's face. Mrs. Hudson and I decided to provide the groups with a formula sheet (Appendix C) so that the focus was on employing the formulas under the threat of overlap and not simply on recalling the structure of a particular formula.

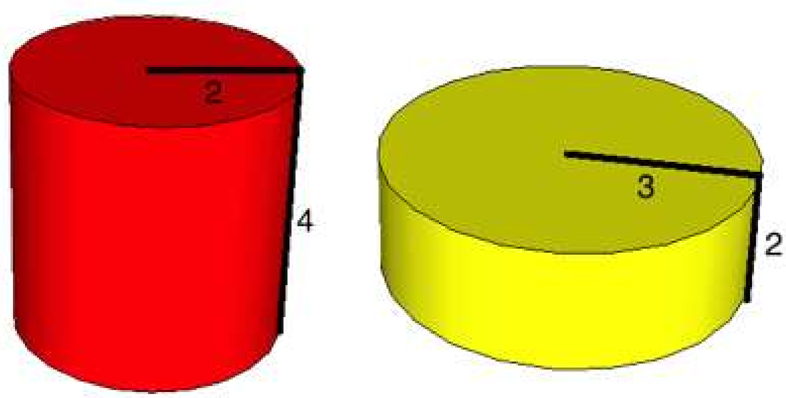

Figure 5.9. Two of the six solids from the Solid Fusing task.

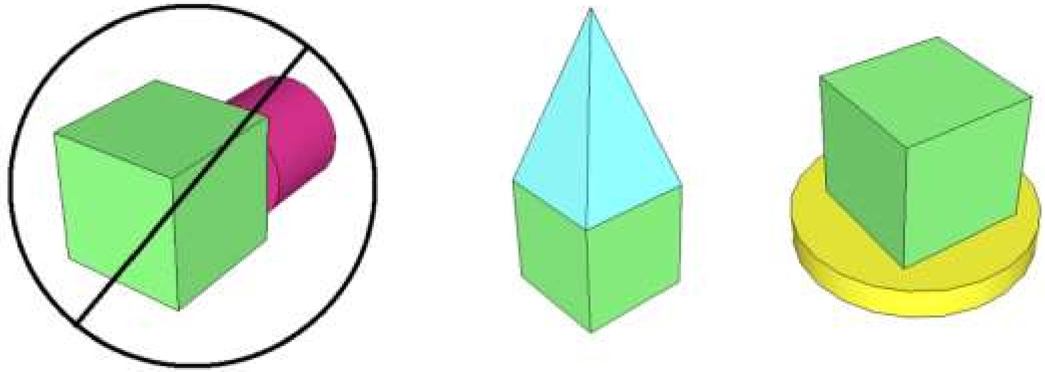

Figure 5.10. Possible types of solid fusings in the Solid Fusing task.

At the pre-session meeting, the target outcomes were established. We wanted the task to encourage students to calculate the surface area and volume of various solids and to understand the effect on surface area and volume when fusing solids. These curricular outcomes closely mimic those from the first classroom session in Mrs. Hudson's room, the Surface Area Doubling task (see Figure 4.8). The Solid Fusing task was the final task enacted with her class. The whole-class debrief focused on strategy, but also on the actions throughout the unit of study. Paralleling the target curriculum outcomes with the Surface Area Doubling task was done intentionally to give the unit a sense of finality. Episode 3 contains the action of Justin, Marin, and Ben during the approximately 30 minutes they worked with the task. Their curriculum space is shown in Figure 5.11.

Reading the curriculum space. The curriculum space supports the teachers' assessment of this group's action. Both teachers described this group as autonomous and efficient, and the benchmark in the curriculum space illustrates the sophisticated manner of their thinking because the group is never coded as operating within it. Aside from this, two characteristics readily emerge from the curriculum space. First is the group's autonomy and the second is the clustering of action around distinct attractor spaces (to borrow a term from complexity science).

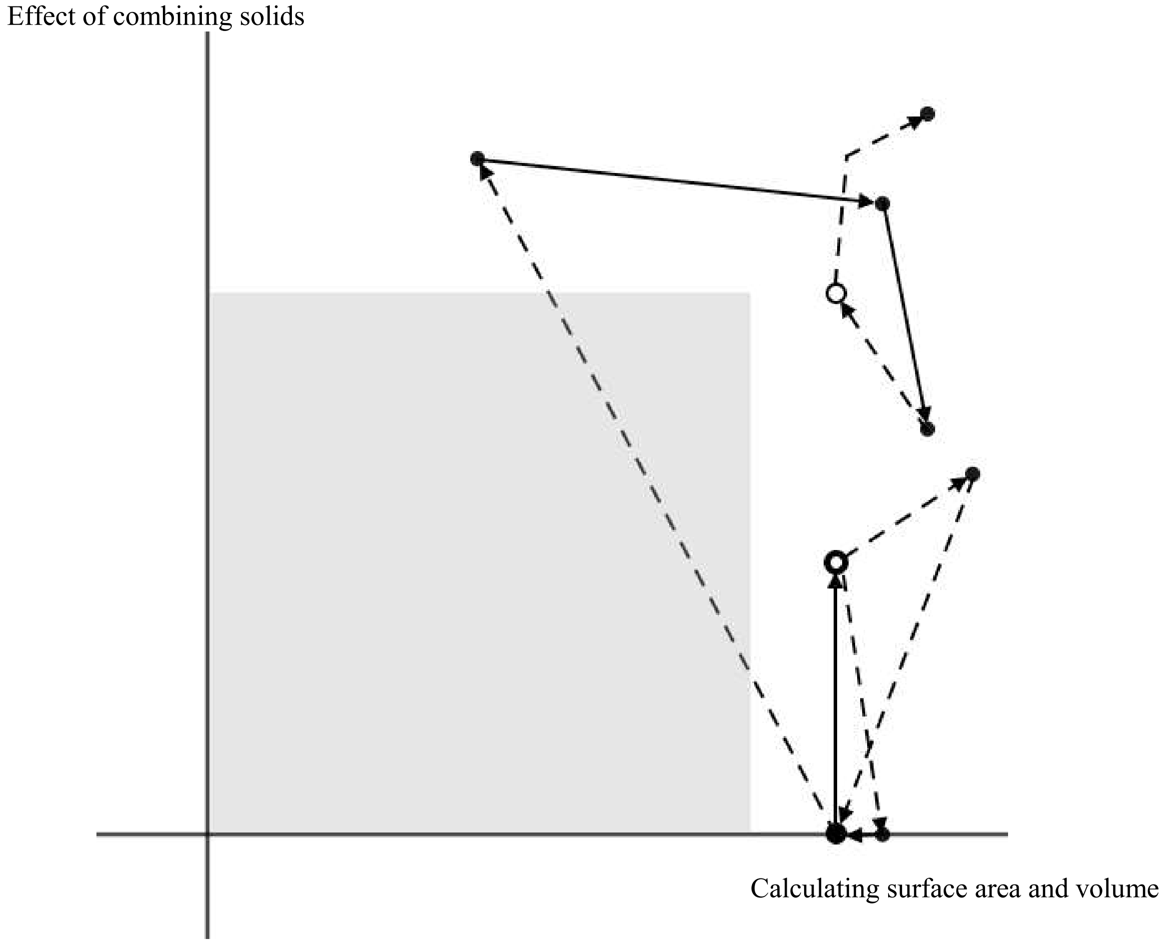

Figure 5.11. The curriculum space of Justin, Marin, and Ben working with the Solid Fusing task.

This image of group action displays a different type of autonomy than that of the group in Episode 1. Brock, Ria, and Sharla acted habitually and often tried to unseat a teacher's intervention rather than entertain it. The result was a curricular space knotted in familiar places and dominated by solid lines as the teachers attempted to perturb this pattern (see Figure 5.2). Here, Justin, Marin, and Ben act at very high levels of sophistication and often trigger problem drift in their own interactions, demonstrated by the prevalence of dashed problem drift lines in their curriculum space. Recall that operating at a high level does not always mean correct answers (see Episode 2). This group did make mistakes, but the structure of the group allowed the task to be consistently problematic, and the problems emerging in action moved the group through sophisticated problems around two central clusters, posed in three distinct phases.

The first is an entry phase, where the group's attention is focused on the calculation of the surface areas and volumes of the solids. The action is coded at a high level because the group consistently used the provided formulae flexibly in order to match the expressions in the formula to the specific faces of the solid. When the formula for the surface area of a cube was missing, the group attempted to reason about which calculations would make sense.

Ben: Well you've got 6 sides.

Justin: I'll do 4 squared times 6. I think that would be correct.

Ben: Um.

Justin: 4 squared times six because you have 6 sides.

Ben: Yeah.

Justin: Wait, that might be the volume?

Marin: Maybe I should try, like … it would be 4 times 4, plus 4 times 4, plus 4 times 4. Right, because all the measurements are 4?

The surface area formula for a rectangular prism was given (see Appendix C), but the group never connected the relationship between the rectangular prism and cube. Despite that, they are able to arrive at a solution that makes conceptual sense using a sophisticated method. The second phase is a transition phase, triggered by the realization that they could have made the task simpler if they would have anticipated some of the sides being lost in the process of fusing.

Marin: I think we forgot to subtract the sides. Did you subtract a side when you were doing the surface area? Because I forgot to.

Justin: Like, this?

Marin: Yeah, like one of the sides.

Justin: No, I didn't.

Marin: Okay. Well.

Ben: We could just do it after.

Marin: Well, I mean, the numbers we'll have to get are pretty similar anyway, so, if they look about the same then we can subtract it. So.

Ben: Yeah, the surface area for this is 25.13. Alright, so now we're going to have to figure out all the different combinations and see which ones will be closer to each other.

Marin: Okay, we should look at similar volumes and stuff. So like, these volumes are pretty similar.

During this phase, the problem becomes the intelligent calculation of the surface area and volume with an anticipation toward potential fusings. They carry this awareness into the third phase where the flexible calculations from phase one are now combined with some type of expectation of surface area loss based on how much the faces of the new solid have overlapped.

Marin: How'd we get 138? Because it should be less.

Justin: Um, 150.

Marin: Try just subtracting this from, like, just the top.

Ben: Yeah, okay so what's that? Which side?

Marin: So 12.56 is just the top of this.

Ben: That? Okay.

Marin: So then we have to calculate just the top of that, and subtract this from it. So that'd be pi times 3 squared.

Justin: 28.27.

Marin: So we add that.

The group eventually arrived at the conjecture that they needed to overlap the faces of solids as much as possible to achieve the best solution, and posed the problem, "How can we overlap the faces as much as possible?" Justin, Marin, and Ben acted elegantly on the problems as they become relevant. Their smooth and efficient movements create an image of classroom complexity quite similar to the image of complex organization offered by the Lorenz attractor (Figure 5.12)—one of the most widespread images of order in chaos. Despite its path being governed by unpredictable equations, the Lorenz attractor patterns its action in two dense areas known as attractors. Here, the mathematical action of Justin, Marin, and Ben reveals certain areas of attraction in the curriculum space and creates an image of harmony between complexity and curriculum.

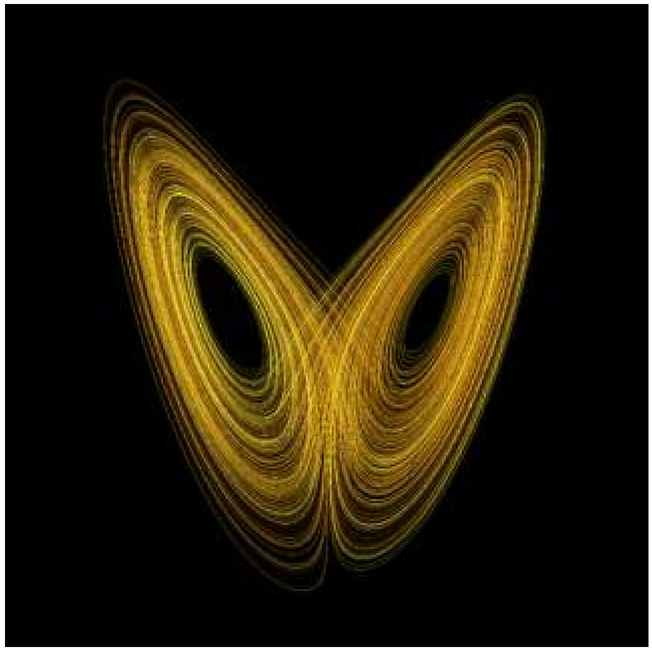

Figure 5.12. The Lorenz attractor exhibiting two attractor spaces.

Lessons in the curriculum space. Teaching in a curriculum space requires the recognition of when groups have posed productive problems and which intended curriculum outcomes have become attractor spaces for group action. During this episode, both teachers seemed to recognize this and not a single intervention was coded as shepherding—an attempt to support or coordinate the possible. This is not a mark of failure for the teachers, but rather a pedagogical decision made in context. We did not need to coordinate the possible, because the group's structure allowed them to interact on its perpetual cusp. This did not mean that the teacher was left without a purpose. The group did not anticipate the ring of surface area remaining when fusing the bases of the two cylinders pictured in Figure 5.9. The process of teaching became one of perturbing the group to avoid inaction, because in complexity, inaction is extinction. In this sense, the job of the teacher was not to provide an extravagant possibility, it was to be available.

Teacher:	How did you get the surface area, sorry?
Marin:	Um, I just calculated the surface area without the side, and he calculated without this side.
Teacher:	Ah, okay, it's like. But you don't lose this whole side?
Marin:	Oh right! Because it's bigger.

Patterns in Teacher Action

An analysis of teacher actions is not an attempt to see when and where in the curriculum space a teacher acted most competently. Rather, it is done to add to the interpretation of group work provided in the previous three episodes. Attention is now turned to the teachers' patterns, intuitions, and tendencies when attempting to act as a fully coupled agent in a complex system. The exploration included enumerating all instances of the three categories of teacher actions provided by Towers and Proulx (2013), as well as analyzing their aggregate curriculum spaces to determine whether a character emerged.

Teacher intervention statistics. Every intervention was coded with a category as well as a timing relative to its occurrence in the classroom session. The breakdown of the frequency of interventions by type is organized in Table 5.1.

Table 5.1

Frequency of teacher intervention organized by category from the ten classroom sessions

					Total
Informing	Enculturating 7	Reinforcing 8	Telling 8		23
Orienting	Clue-giving 10	Blocking 2	Pretending 2	Anticipating 4	18
Shepherding	Inviting 5	Rug-pulling 8	Retreating 4		17
					58

Note. Categories and sub-categories taken from Towers & Proulx (2013).

Perhaps unsurprisingly, each category of intervention appears over the course of the study. Although no single category was used far more often than the rest, the most frequently used category of intervention was clue-giving. These actions "deliberately use clues to orient students toward specific pathways" (Towers & Proulx, 2013, p. 14). Perhaps this is due in part to the teachers not wanting to give direct information, a worry expressed by all three participating teachers. Mrs. Hudson summed up her focus while offering interventions in a post-session interview:

> I just have to be more questioning and [use] less statements, right? I have to ask them like, "So what have you done?" or, "Where are you?" and then see if they can explain their thinking, and that, I imagine, is probably enough to get them started going somewhere else on their own, right? So I think I have to stick to questions versus statements when I'm working.

The timing of teacher actions was not a measurement of when, in elapsed time, the intervention was chosen as relevant to the group's needs. Instead, it labels when, in the relative course of the classroom session, the intervention occurred—beginning, middle, or end. To arrive at accurate plotting, each problem session was divided into nine equal sections and each intervention was timestamped. The interventions were then organized into the nine time intervals. When read from left to right, the area graphs (Figure 5.13) show the number of times a specific type of teacher intervention was used over the course of the data collection task. That is, they report the relative frequency that a particular classification of intervention was deemed appropriate at a particular juncture of the classroom session. The leftmost edge of the graph represents the onset of classroom action, and the rightmost edge represents the conclusion of the data collection task. The taller the area, the more often interventions of that category were offered during that time interval during the task.

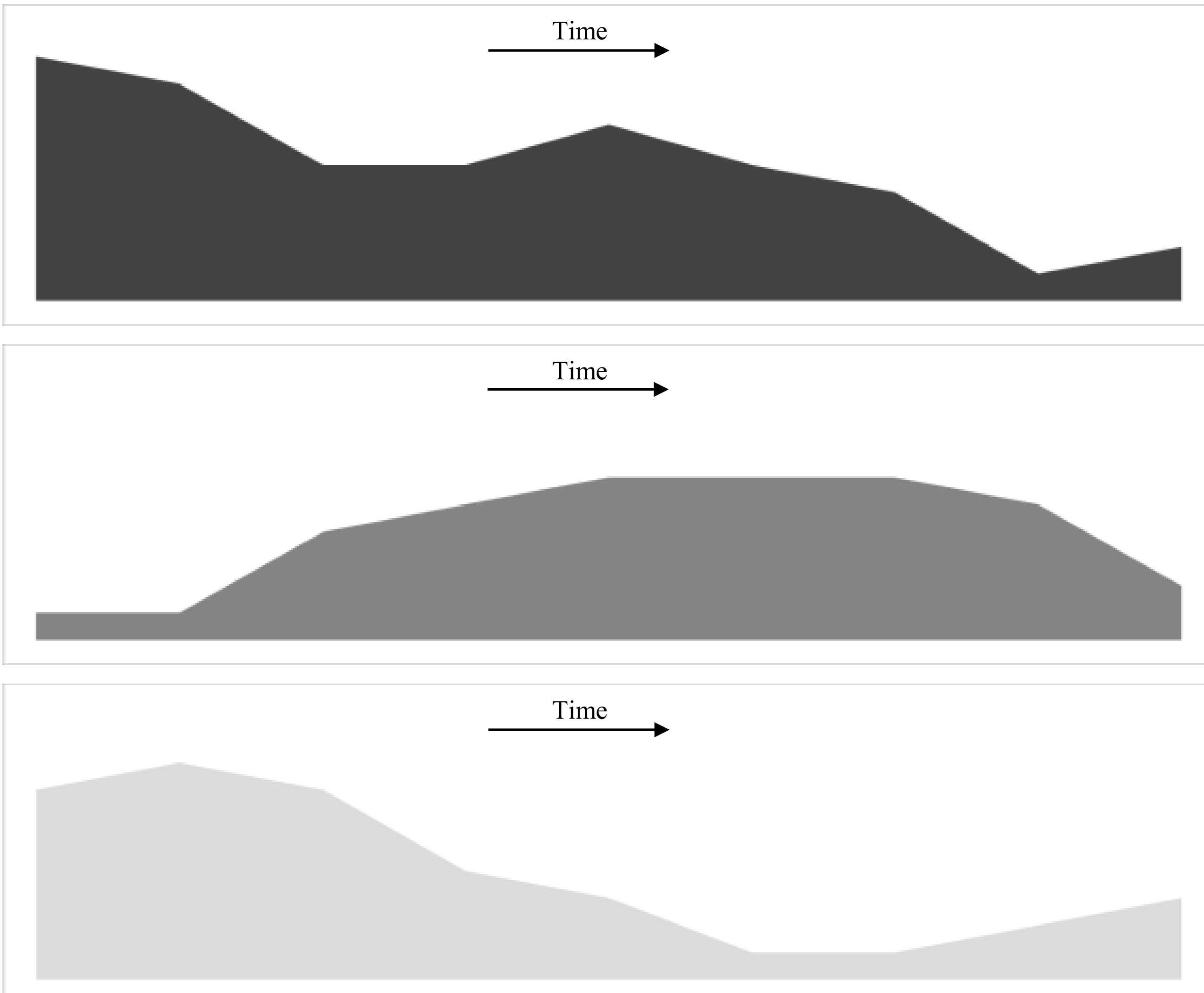

Figure 5.13. Timing of teacher interventions by category. Informing (top), Orienting (middle), and Shepherding (bottom).

Each type of intervention appears as used most frequently during certain portions of the classroom sessions. Overlaying the graphs, we begin to see a global pattern of teacher action during a task (Figure 5.14). If we think of a classroom session as an unfolding context, the image becomes one of teacher intuition—of what type of intervention we, as teacher of mathematics, felt paired best with temporal phases of classroom sessions. Informing actions dominate the introduction of tasks when the giving of information is intended to clarify the task. They taper off through the middle portion of the task, but reemerge as prominent when the teacher checks in to see group solutions. Orienting actions are focused in the middle and end of the classroom sessions. It is during this phase that the group diversity is at its highest, and teachers act to foster numerous mathematical ideas. Shepherding actions have a similar pattern in time as informing actions. Initially, this correspondence was surprising because *giving information* and *coordinating the possible* seemed like wildly different processes, but we will see this congruence re-emerge when the two categories are analyzed in curriculum space as well. Shepherding actions used at the beginning of task sessions were intended to kick-start group thinking, a goal shared with informing actions offered in the early phases of a task. It seemed as though teachers were waiting for groups to understand the constituents of the task, and then attempting to instigate action by presenting a previously unforeseen wrinkle. At the end of classroom sessions, shepherding actions were used to offer extensions to groups who, through their action, had become comfortable that they had arrived at a solution.

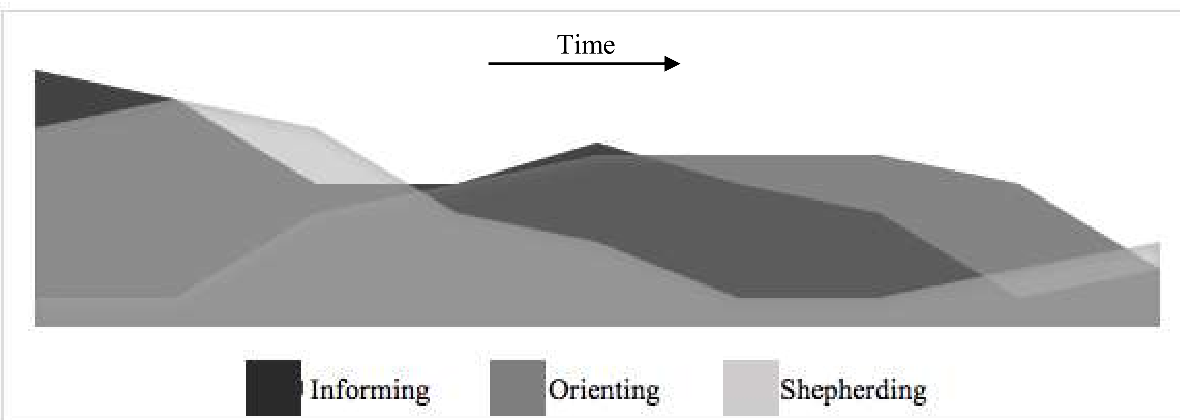

Figure 5.14. Composite graph of the timing of teacher interventions by category.

The distributed nature of interventions and their timings speaks to the important role of context. The common thread that runs throughout every teacher intervention offered during the study is that teachers identified their action as the right way to act in context—to affect the collective.

Teacher interventions in curricular space. Patterns of teacher action were also analyzed using the tool of curriculum space. To do so, the occasions of problem drift instigated by teacher interventions were separated according to the three categories of teacher actions. The result is an image of problem drift in curriculum space triggered by each individual category of teacher intervention. Because the interventions from all ten group sessions are amalgamated into these curriculum spaces, no outcomes appear on the axes. The starting nodes give us an image of where in curriculum space (as opposed to when in curricular time) certain types of interventions were deemed suitable, and the arrows and ending nodes give us an image of the global pattern of effect. That is, where, in space, did the teacher feel it suitable to give information (informing action), direct attention (orienting action), or coordinate the possible (shepherding action), and what problem drift did those interventions sponsor? As was the case with the illustrative episodes, the curriculum spaces tell a story. These curriculum spaces are images of effect—what role did specific types of interventions play in carving curriculum out of chaos?

The curriculum space of informing teacher actions appears in Figure 5.15. Keeping in mind the many, differing contexts in which the interventions were offered, there are some interesting patterns that exist. A large percentage of nodes are arranged on the axes around the periphery of the space. Also, informing interventions at the periphery of the space seem to occasion little problem drift. This is especially apparent along the horizontal axis in Figure 5.15. On the whole, the nodes are fairly evenly distributed across the space signaling no apparent bias to where, in a curricular sense, a teacher feels giving information is most effective or ineffective.

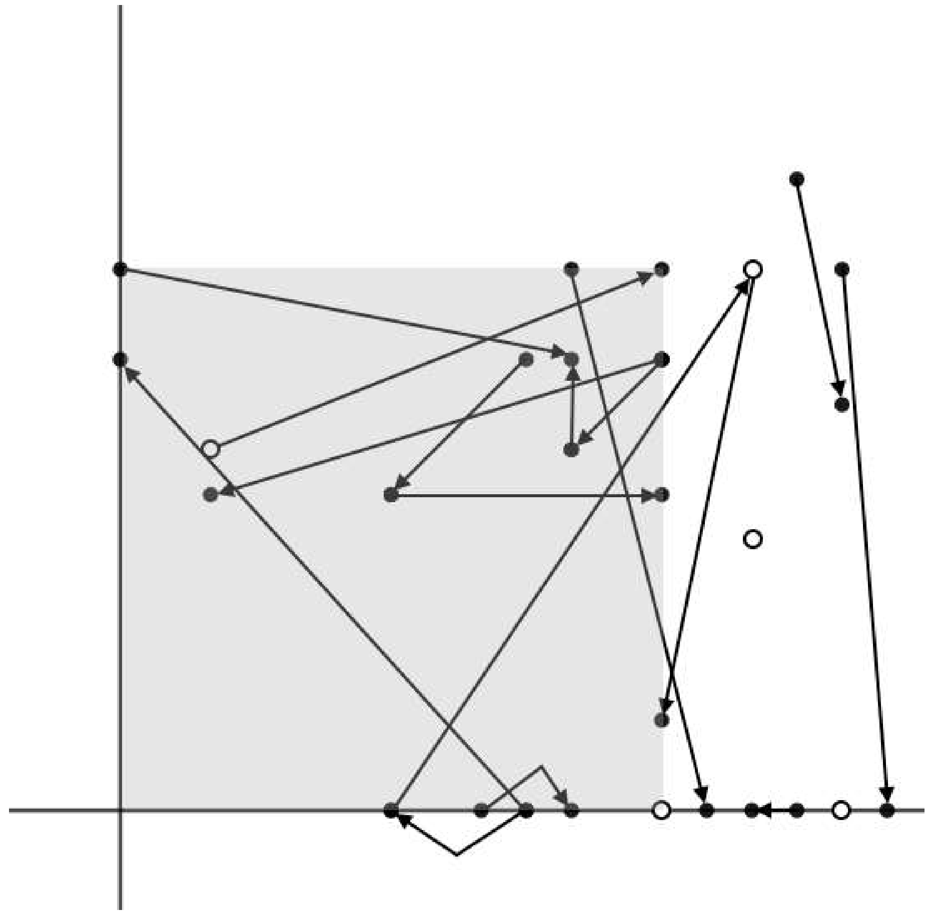

Figure 5.15. The curriculum space containing all informing interventions during the study.

The curriculum space of orienting teacher actions appears in Figure 5.16. In contrast to the peripheral tendencies of informing actions, the orienting nodes tend to cluster in the central portion of the curriculum space. This hints that teachers feel the context for directing attention is most suited for when groups are already working with both target outcomes, what could be classified as a very productive curricular space. The orienting curriculum space also contains fewer instances of dramatic problem drift. This may be explained by their central mandate of *directing attention*. When compared with the *giving information* and *coordinating the possible* of informing and shepherding actions respectively, directing attention has a much less invasive tone.

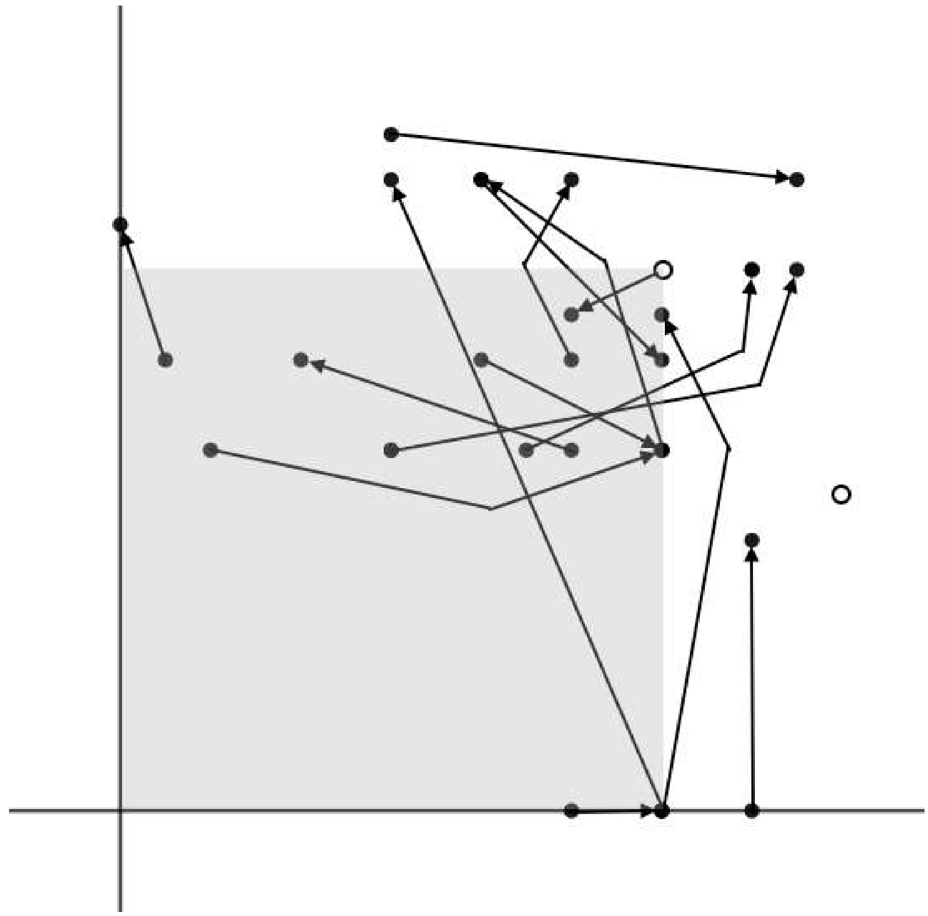

Figure 5.16. The curriculum space containing all orienting interventions during the study.

The curriculum space of shepherding teacher actions appears in Figure 5.17. The clustering of problem drift within the central portion of the curriculum space seems to be consistent between the orienting and shepherding curriculum spaces. Perhaps this is the case because once specific curriculum processes become relevant, they have the tendency to remain relevant when offered an intervention that directs attention or coordinates further possibilities.

As was the case with the timing of the categories of teacher actions, the shepherding and informing categories have some shared characteristics (see Figure 5.13). While the majority of the nodes cluster in its central portion, the curriculum space for the shepherding interventions contains several nodes coded along either axis. The informing intervention curriculum space also contained several nodes coded along the axes; however, unlike informing interventions, the shepherding interventions used at these locations seem to occasion considerable problem drift. This suggests that, when a group is operating fairly exclusively with a curricular outcome, shepherding interventions may trigger more drastic re-orientations of problem drift when compared to informing actions. This dramatic problem drift is not mimicked by the set of nodes in the central portion of the informing and shepherding curriculum spaces. Here, the nodes in the informing curriculum space are less clustered when compared to the nodes in the central portion of the shepherding curriculum space. This signals that the direct giving of information may be better suited for quick changes in group focus when the group is operating with an amalgam of anticipated curriculum outcomes.

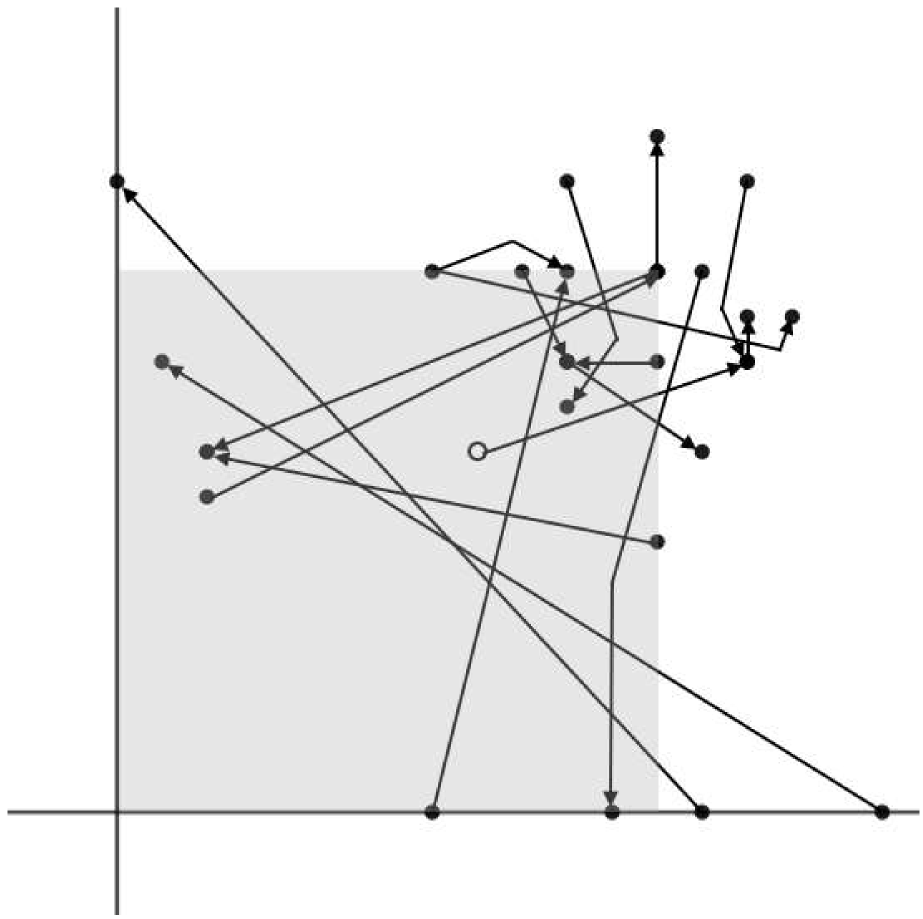

Figure 5.17. The curriculum space containing all shepherding interventions during the study.

In this chapter, the interpretive tool of the curriculum space was used to investigate patterns in group action, teacher action, and the interweaving of the two. The curriculum spaces of illustrative episodes display patterns that emerge from groups, and offer lessons for the teacher operating with both a lens of complexity and a mandate to teach curriculum. The curriculum space also documented how various types of teacher interventions occasioned group action in the form of problem drift—an area that Towers and Proulx (2013) identified as needing further exploration. The categories of giving-directing-coordinating are not inappropriate. It would be a mistake, however, to classify a certain type of intervention as always more advanced, the work of a master teacher. This mistake is illustrated through the observed similarities between the curriculum spaces of informing and shepherding actions. Both types of action appeared in similar patterns in time (see Figure 5.13) and were chosen as appropriate, in context, in similar patterns in space (see starting nodes in Figure 5.15 and Figure 5.17). Despite these similarities, the interventions had differing patterns of effect. Around the periphery of curriculum space, informing actions had a smaller effect on problem drift than shepherding actions, but in the central area, informing actions had a greater effect on problem drift than shepherding actions. The effect of a teacher's intervention was dependent on where (and when) in curricular space (and time) it was offered, and not wholly dependent on the nature of the intervention itself. It follows that equating the use of interventions from a specific category (say, shepherding) as the work of a master teacher is shortsighted, because it does not allow for teacher judgement in context. In order to classify an intervention as a successful one, a teacher must first have some sort of criteria for success. The hallmark of a master teacher is intentionality. That is, an expert intervention is one where the teacher is aware of the purpose of their action. That purpose may align with any of the three categories of teacher action, meaning that any one of giving information, directing attention, or coordinating the possible may prove the most potent action in context. As such, the similarities and differences between informing and shepherding interventions from throughout the study adds to Towers and Proulx' assertion that shepherding interventions should not always be equated with high-level teaching practice. Rather, teaching is the process of dynamic fitting with collectives (Towers & Proulx, 2013). Here, we observe the types of interventions as serving different settings, not on a continuum from stale to innovative. The analysis of patterns within a classroom balancing complexity and curriculum begins to unpack a teacher's tacit modes of classroom operation, places teacher intention at the forefront of effect, and offers valuable implications for the work of teachers.

Chapter 6: Implications in Curriculum Space

"The real voyage of discovery consists not in seeking new landscapes, but in having new eyes."
—Marcel Proust, *In Search of Lost Time*, 1923

Comments on implications seem overly presumptuous. It seems contradictory to explore the nature of human activity as complex and then attempt to delineate the implications of such an inquiry, as if the implications are certain and well-formed for each individual observer. This concluding chapter, then, represents the implications that emerged for me through the process of design, implementation, and analysis. It examines those features that emerge as relevant to my research problem—how can the teacher provide meaningful opportunities for small groups to work with curricular outcomes?

Carving Curriculum

It is my intention to use this chapter as a space to attend to the study as a whole in order to unearth its saliency to the work of mathematics teachers. With that goal in mind, I begin at the beginning—the very beginning, with the first word: carving. Attuning teachers to their role in the complexities of small group work required some way of envisioning the role of the teacher, but there is danger in metaphor because every metaphor eventually breaks down. The metaphor of carving is no different. So what was to gain by comparing the role of teacher to that of sculptor?

The image of sculpting a piece of stone is static, even in its most poetic forms. Michelangelo, widely recognized as one of the greatest sculptors to ever live, has been credited with claiming—in reference to his greatest work, David—that he saw the angel in the marble, and carved until he set him free. The image portrays the sculptor as liberator, one that excavates potential. This image of teacher as sculptor is problematized on two fronts. First, it gives the student groups a static feeling of stone, instead of granting them their dynamic, drifting reality. Second, it places the sculptor outside of the becoming, as one that acts *on* and not *with* the marble. Despite these two issues, this metaphor of sculptor and stone may have been the image called to mind by a reader when reading the word *carving*; this was intentional. My hope was that it serves as a reminder of how easy it is to distance oneself from the students, but also how crucial it is that this distancing does not occur.

Rather than the image of a sculptor carving marble, the metaphor of carving is meant to evoke the way a river carves its way through its surroundings while at the same time becoming inextricable from the landscape. The landscape is enacted as the river encounters obstacles with which it couples to find a productive way forward. In the case of small groups, these sites of interaction could come from a teacher intervention, the task, or a piece of the social environment of school. The role of sculptor (teacher) emerges through the interaction of the features in the environment with the structure of the river (student groups). These interactions are sometimes subtle and sometimes prominent. It is unproductive to visualize the river without taking into consideration the landscape that helped form it; the same goes for the landscape which is then formed by the re-actions of the river. In other words, they are in a continual process of re-defining one another. The image of the riverbed (Figure 6.1) provides balance between directionality, influence, and chaos. In this sense, the image of carving is granted to the inter-action between the river and environment, and is not viewed as the sole responsibility of either one. In fact, the landscape cannot be brought forth without both.

Figure 6.1. The Amazon River landscape. Photo by NASA (NASA WORDL WIND 4.1 Screenshot) [Public domain], via Wikimedia Commons.

Chaos

The image of chaos provided by problem drift has implications for teachers because the teacher must recognize the problem relevant to a group of learners at a given time and place and then act intentionally. Throughout the study, the image of problem drift and curriculum space triggered two important learnings, for me, that attend to the question of how teachers may act with(in) a complex ecology of the collective. The first is the use of what I have come to call acclimatizing actions, and the second is a re-defining of the orienting question while teaching with(in) collective knowing.

Acclimatizing actions. Having record of student action throughout the task sessions made the moments I missed in the course of attending to several groups glaringly obvious. I recognized that, by necessity, a teacher operating with small groups cannot witness the doing of every group, but I was never aware of how the teacher makes up for that deficiency. This periodic absence is heightened within an epistemology where knowing is doing. By implementing a classroom structure of small groups I was not just missing doing—I was missing knowing.

As each teacher intervention was coded, it was often a struggle to classify an intervention. Sometimes, there was no hard and fast delineation between giving information, directing attention, and coordinating possibilities. However, one facet was constant throughout almost every teacher intervention from every participating teacher in the study. Before attempting to shape the group's action, the teacher began by asking the group where they were, in an effort to acclimatize to the group's needs by orienting themselves to the problem drift.

Teacher: So what are you doing currently? What are you trying to do?

…

Teacher: What was the first thing you focused on?

…

Teacher: How did you get the surface area, sorry?

…

Teacher: I think I walked in half-way through your explanation.

…

Teacher: So what are we doing? Where are we going?

I had spent years working with small groups prior to the study, and Mrs. Murray, for example, readily admitted that she had little experience working with this classroom structure. Despite these seemingly large gaps in familiarity, teaching began with listening (Davis, 1996). Of the fifty-eight interventions offered by teachers throughout the study, only seven were offered without an acclimatizing action. Isolating the effect on problem drift in curriculum space of these interventions results in the image in Figure 6.2.

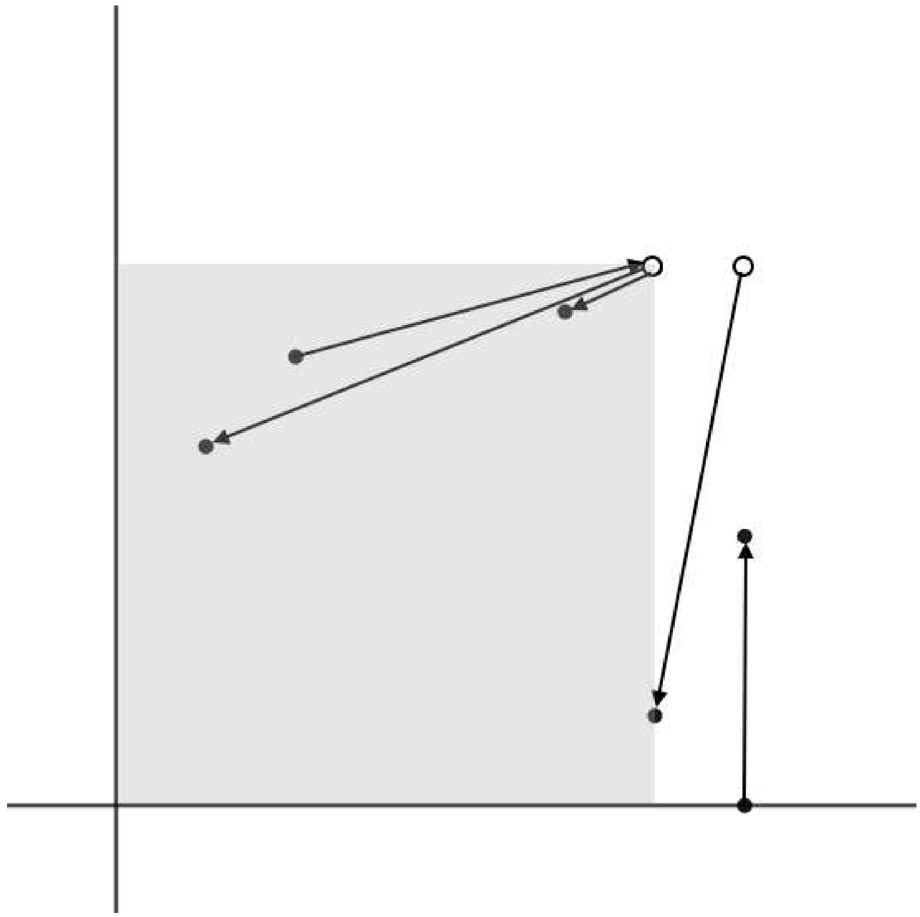

Figure 6.2. The curriculum space containing the seven interventions before which the teacher failed to acclimatize to problem drift.

The majority of the problem drift resulting from non-acclimatized teacher interventions occasion no problem drift or problem drift directed toward the origin of the curriculum space. While this is a very small sample, offering interventions to a group of learners without first acclimatizing to their relevant problem seems to have a negative effect in curriculum space. This is not intended to serve as proof that all non-acclimatized actions have negative effects on learning curriculum; however, it does speak to the importance of acclimatizing to the relevance of the group.

Again, the action of acclimatizing seemed natural to the work of teaching. Maybe so much so that it may be dismissed as a social mechanism, used only to form natural transitions between conversations. I contend here that the action of acclimatizing is a pedagogical one in the sense that it is an intentional teacher action used to adapt to the problem drift of a group of learners. This extends the theorization of teacher actions into a new category of actions that are employed not for the sake of the students, but for the teacher. This is not to claim that the students gain no benefit from acclimatizing. In the same vein that the action is not simply social, the recapitulating of their action is a generative process as the students explicate their world of mathematical significance. For the enactivist, the opportunity to step back and observe the environment as it has shifted through their process of coupling provides the possibility for further coupling with it. It is a learning experience in itself. In other words, acclimatizing actions are not a simple reviewing, but a re-viewing—an occasion to trigger further action.

Teaching in curriculum space. Ultimately, in order for the theorization of complex groups to be useful, it needs to inform the work of teachers. In the previous section, acclimatizing actions have offered a step toward that aim. As the study progressed, the images of problem drift became a part of my operation, and the conceptualization of curriculum space shifted how I interpreted my work with small groups. Re-casting the curriculum as a landscape, as opposed to a list, required that I stop looking for signals that a group could execute certain skills, and began looking for the patterns with which they encountered curriculum outcomes as ways to resolve mathematical tensions. This pattern seeking determined which interventions presented themselves as suitable in context, and provided feedback on whether the intervention had the desired effect. Teaching became a process of dwelling in context. It became clear to me that the task of carving curriculum out of chaos is not oriented by the guiding question, "*How did they solve the problem?*," but rather by asking, "*What problem are they solving?*". Allowing that question to orient the action of teaching became the starting point for influencing the complexity of small groups working toward mathematical, curricular goals.

Concluding Comment

It is my hope that this work prompts teachers to respect and recognize the complexity inherent in the important role all educators have in the amalgamation of knower and known. The work has maintained its pragmatic focus throughout in an attempt to provide a lens through which teachers can interpret classroom action in small groups. Not as one who gazes down as a kind of informed, omniscient observer, but rather as one working in the midst of it. It is intended as “a way of stepping into the current of curriculum” (Davis, 1996, p. 127); not as a sculptor disembodied from the work, but as a feature in its dynamic landscape.

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APPENDIX A: THE FIVE TASKS

The Surface Area Doubling Task

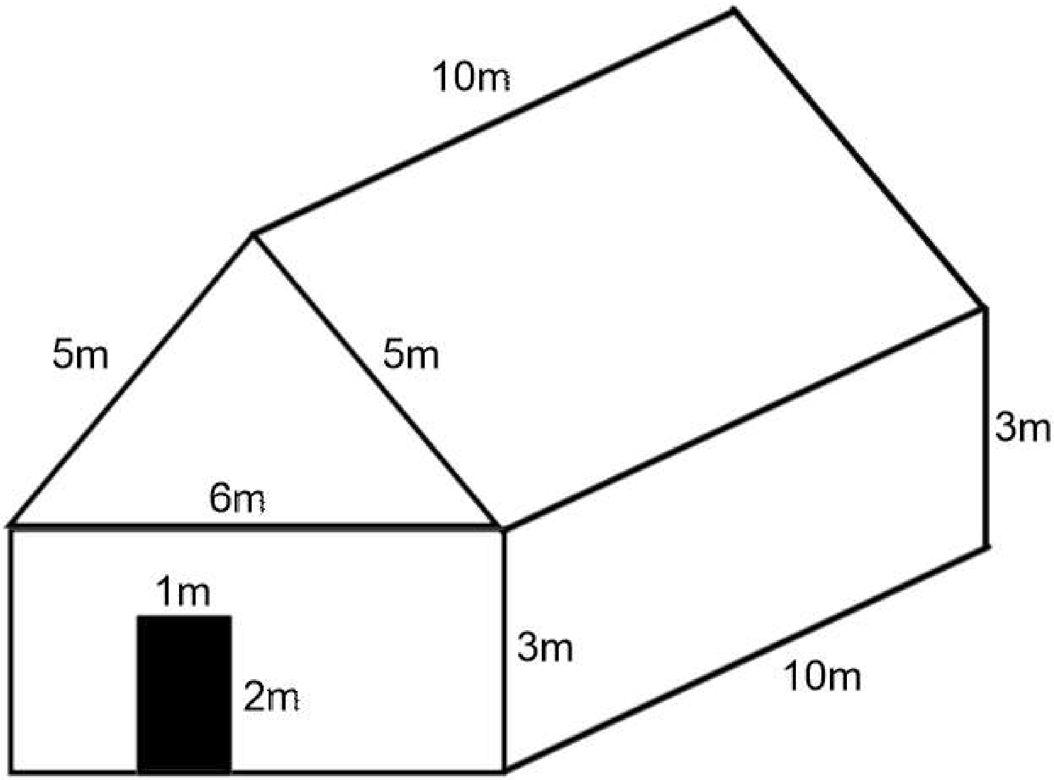


Figure A1. The original house used in the Surface Area Doubling task.

The Surface Area Doubling task asked groups to design an addition to the house pictured in Figure A1. Two conditions had to be met. First, the designed addition for the house had to share at least one common wall with the original house. Second, the surface area of the new house (original house with the addition included) had to be exactly double that of the original house.

The Tile Design Task

Stage 1 Build a shape where… $\frac{1}{4}$ *is Red* $\frac{1}{2}$ *is Blue* $\frac{1}{4}$ *is Green*	Stage 2 Build a shape where… $\frac{1}{3}$ *is Red* $\frac{1}{4}$ *is Yellow* $\frac{1}{12}$ *is Green* The remaining is *Blue*
Stage 3 Build a shape where… *at least* $\frac{1}{2}$ *is Red* *at least* $\frac{1}{4}$ *is Green* *no more than* $\frac{1}{6}$ *is yellow*	Stage 4 Build a shape where… *at least* $\frac{1}{4}$ *is Red* *at least* $\frac{1}{12}$ *is Blue* the number of *Green* squares is no more than double than the number of *Red* squares.
Stage 1 Build a shape where… $\frac{1}{2}$ *is Red* $\frac{1}{3}$ *is Blue* $\frac{1}{6}$ *is Yellow*	Stage 2 Build a shape where… $\frac{1}{20}$ *is Yellow* $\frac{1}{4}$ *is Green* $\frac{1}{2}$ *is Blue* The remaining is *Red*
Stage 3 Build a shape where… *at least* $\frac{1}{3}$ *is Green* *at least* $\frac{1}{6}$ *is Blue* *no more than* $\frac{1}{4}$ *is yellow*	Stage 4 Build a shape where… *more than* $\frac{1}{2}$ *is Yellow* exactly half the number of *Yellow* squares are *Blue squares* $\frac{1}{8}$ *is Green*

Figure A2. The eight stages distributed by teachers during the Tile Design task.

The Tile Design task asked groups to build shapes out of coloured, square tiles that met all the specifications of a stage cards (Figure A2) given in sequence to each group.

The Fill in the Blanks Task

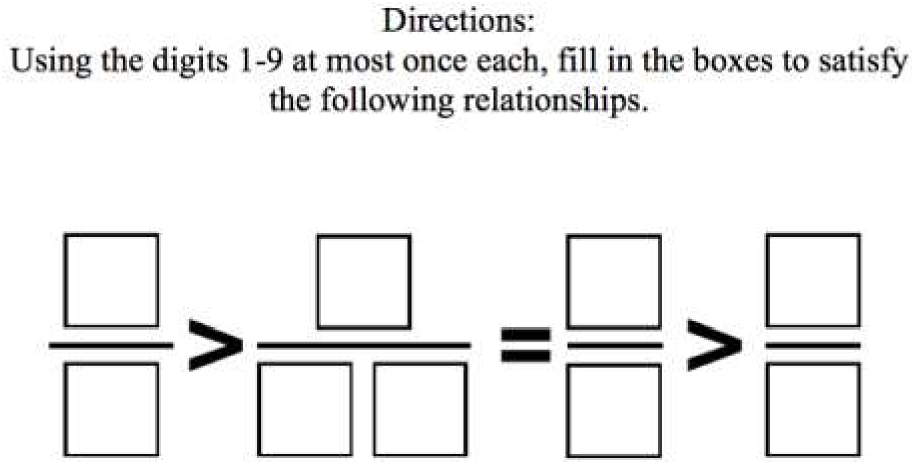


Figure A3. The expression structure given to the groups in the Fill in the Blanks task.

The Fill in the Blanks task asked groups to choose which digits to place in the nine blanks (Figure A3) in order to make the expression true. When finished, groups were asked to find as many arrangements of digits that satisfied the expression as possible.

The Solid Fusing Task

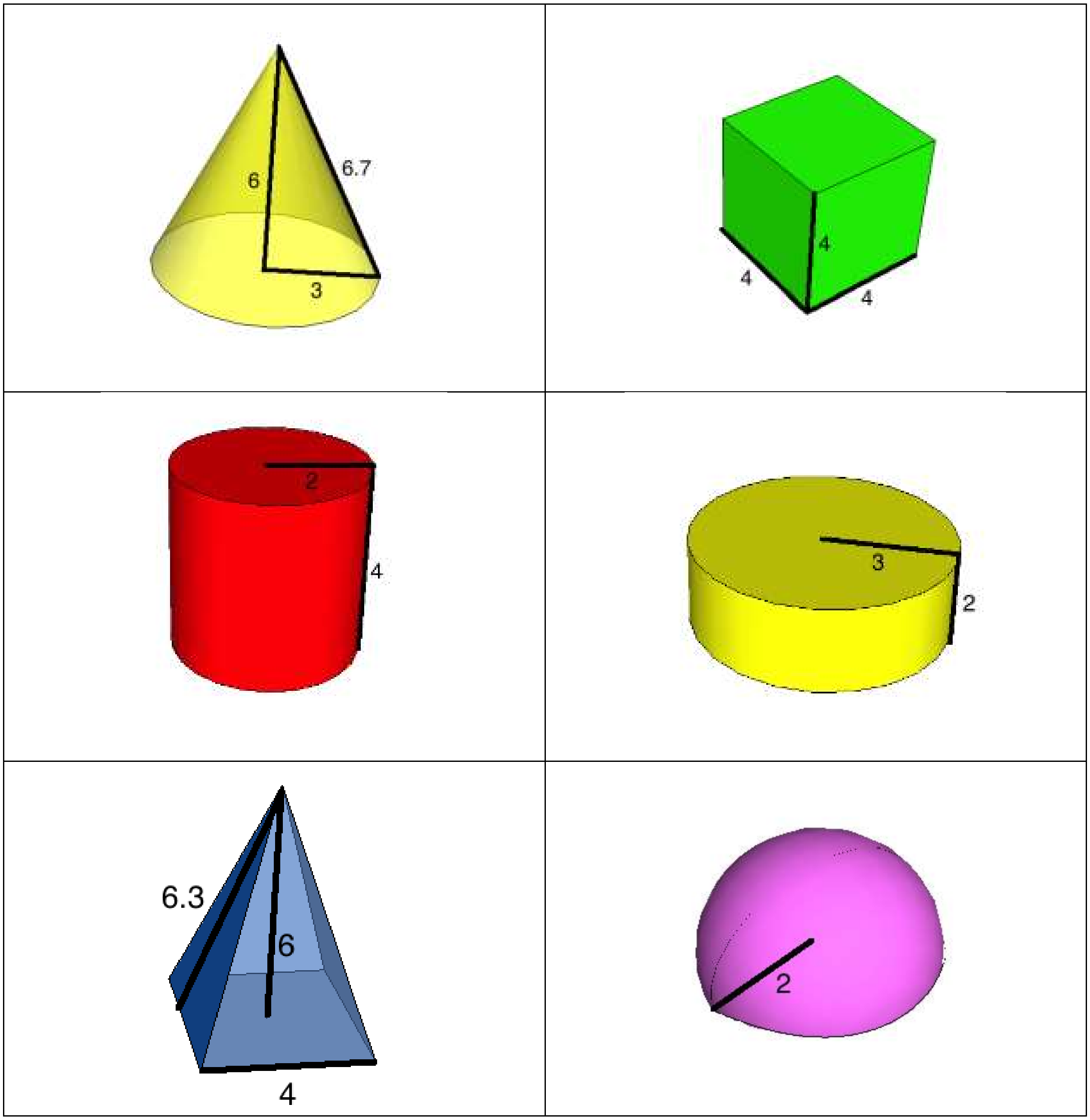


Figure A4. The solids available for use in the Solid Fusing task.

The Solid Fusing task asked groups to build a composite shape by fusing the faces of at least two of the solids in Figure A4. The goal was to create a composite shape that had a surface area (measured in square units) as close to identical to its volume (measured in cubic units).

The Number Line Cards Task

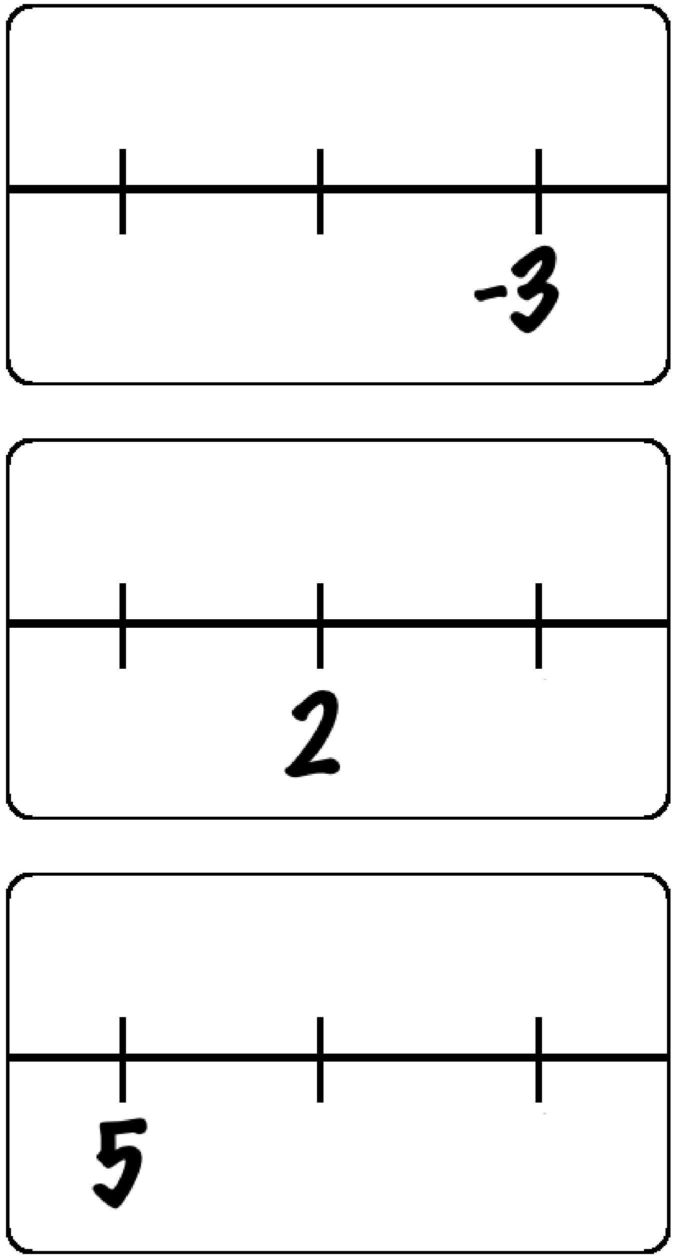


Figure A5. Three sample cards used in the Number Line Cards task.

The Number Line Cards task asked groups to build number lines by arranging customized number line cards side by side (Figure A5). The groups were given cards with the integers between negative five and five (inclusive) placed randomly at one of three tick marks. To begin the task, the groups were asked to create three, separate, correct number lines, each of which had to include at least two cards.

APPENDIX B: THE TEN CURRICULUM SPACES

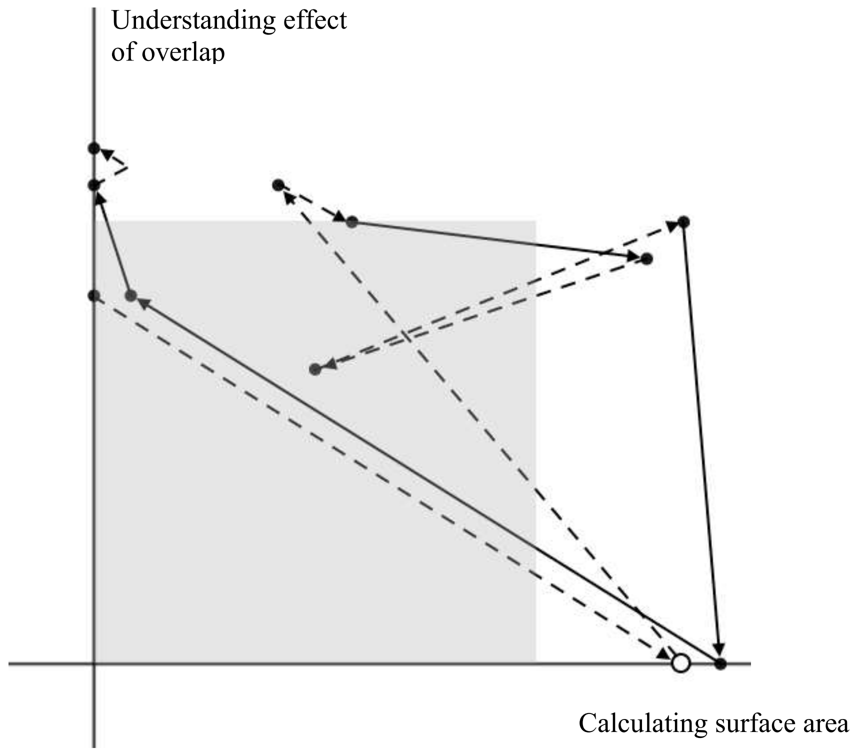


Figure B1. The curriculum space of Cohen, Anne, and Lucas working with the Surface Area Doubling task.

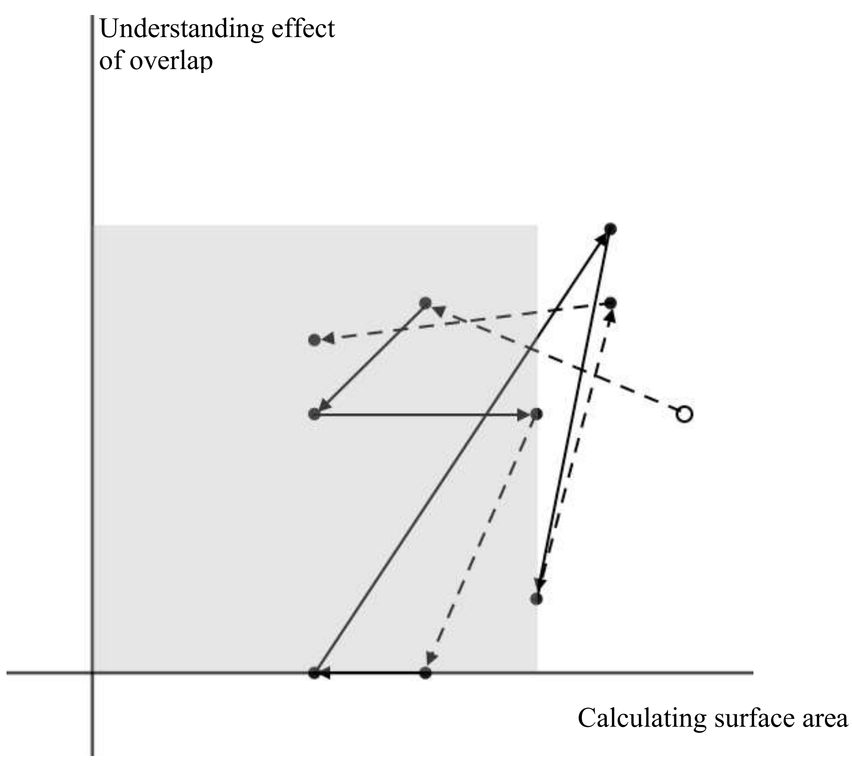


Figure B2. The curriculum space of a group working with the Surface Area Doubling task.

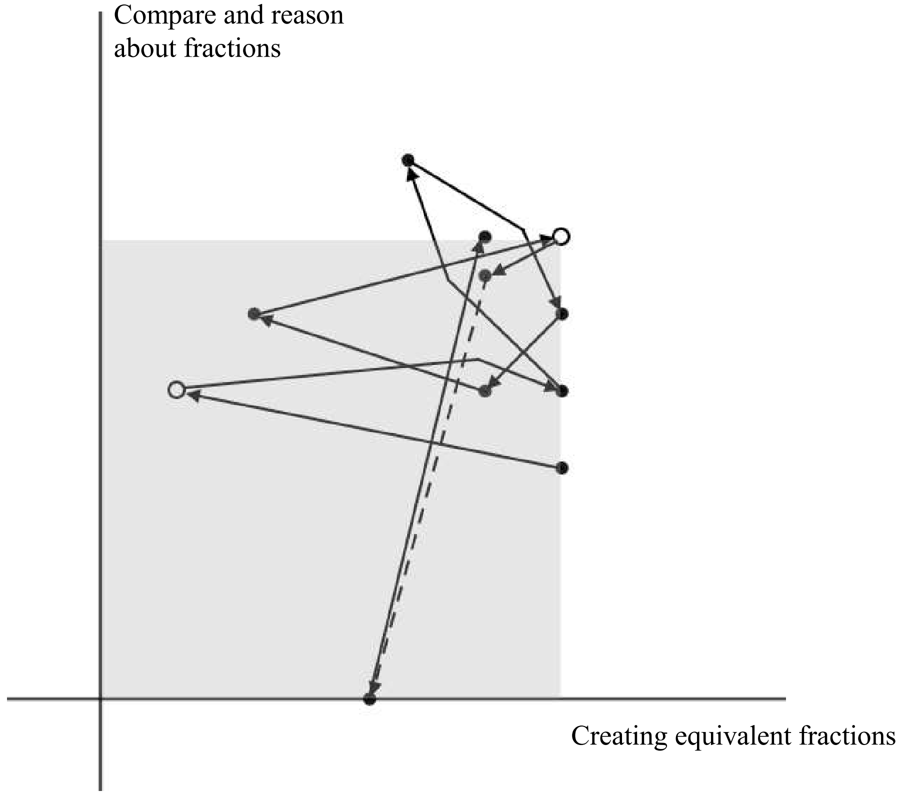


Figure B3. The curriculum space of Brock, Ria, and Sharla working with the Tile Design Task.

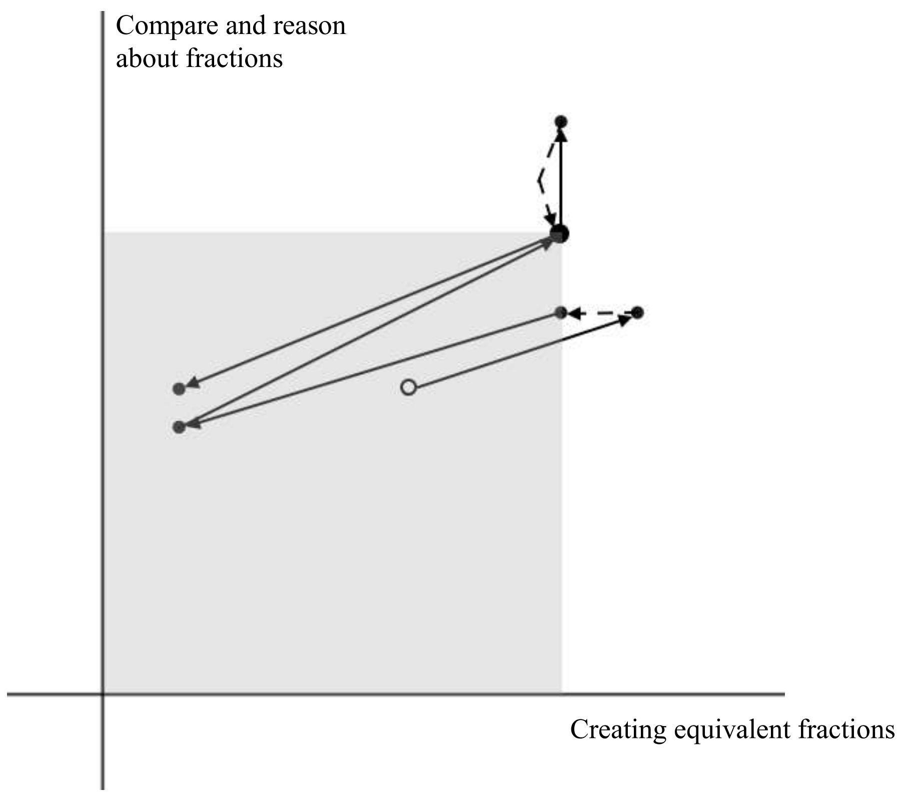


Figure B4. The curriculum space of a group working with the Tile Design task.

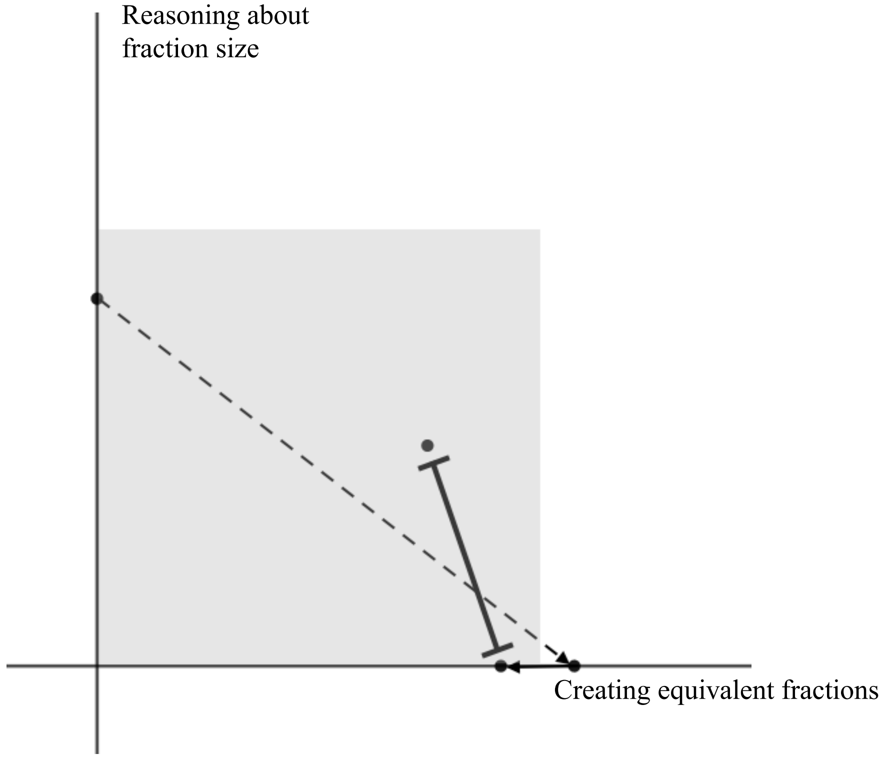


Figure B5. The curriculum space of Amina, Eliot, and Makalia working with the Fill in the Blanks task.

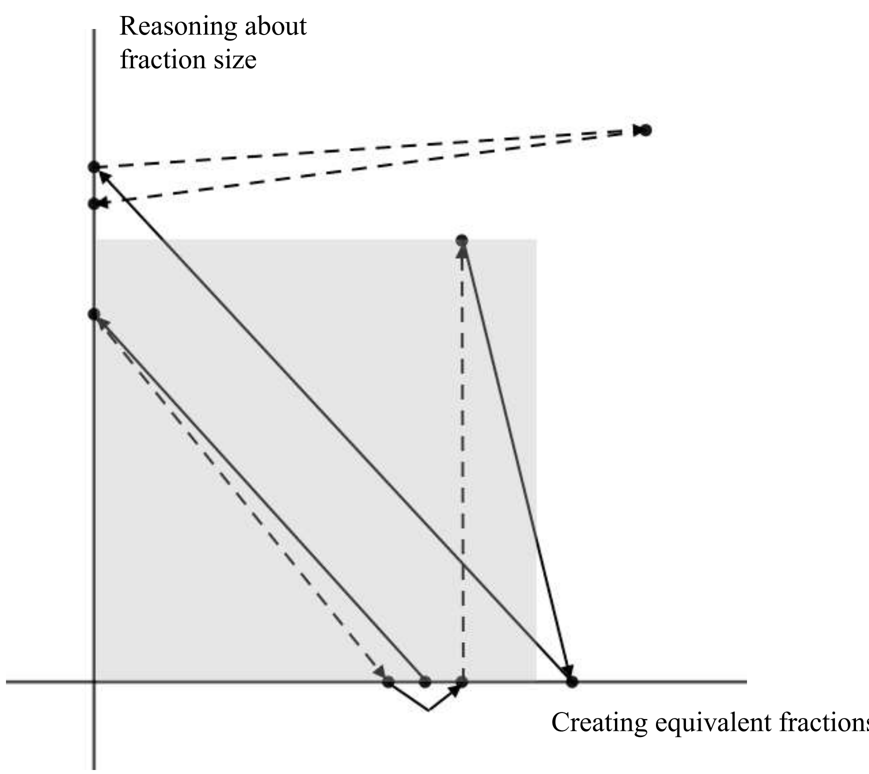


Figure B6. The curriculum space of Colin, Duncan, and Madlyn working with the Fill in the Blanks task.

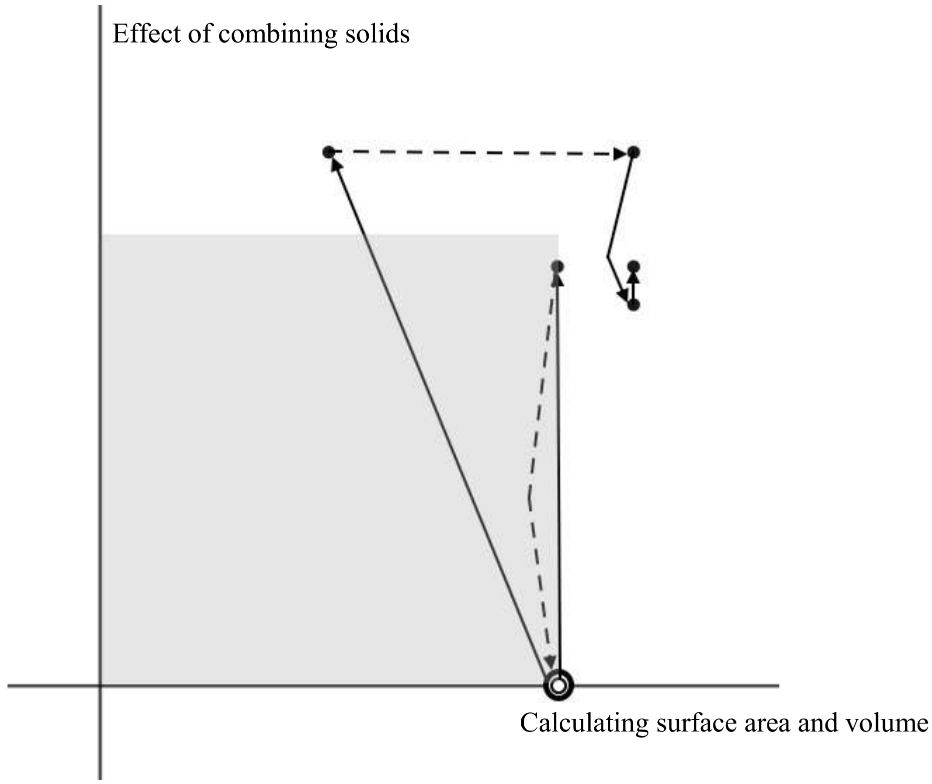


Figure B7. The curriculum space of a group working with the Solid Fusing task.

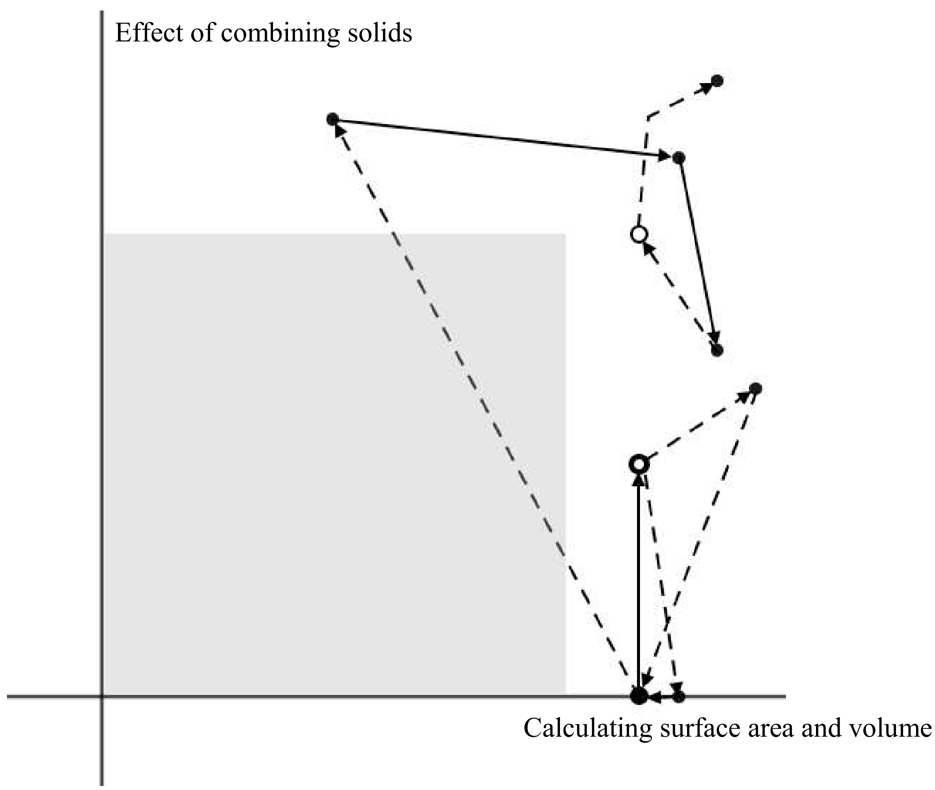


Figure B8. The curriculum space of Justin, Marin, and Ben working with the Solid Fusing task.

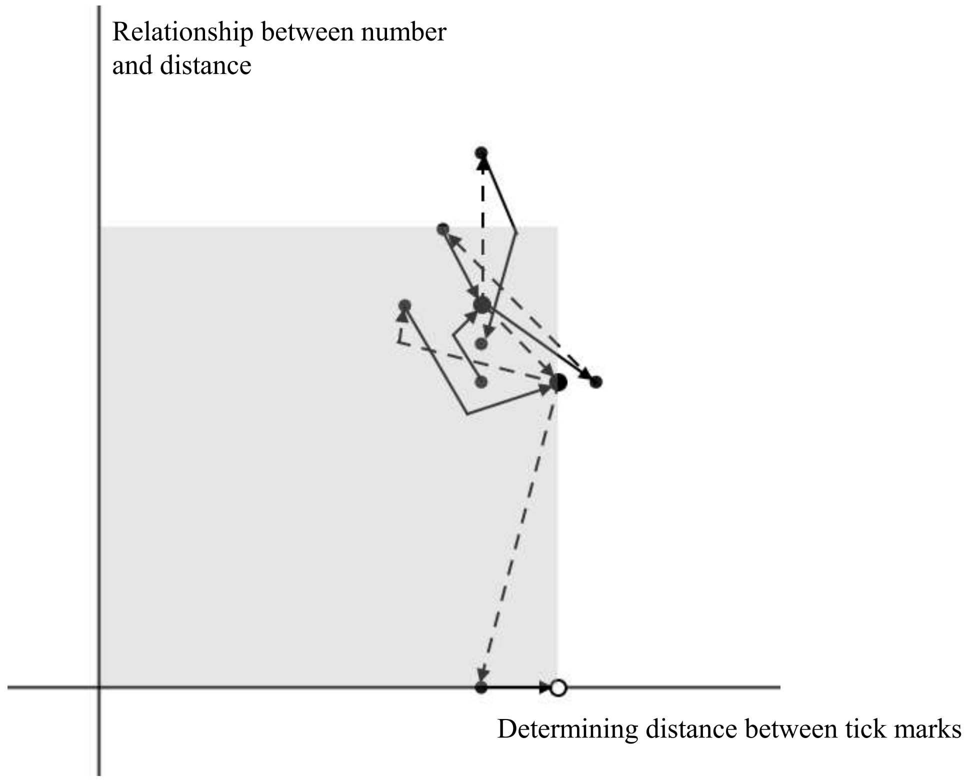


Figure B9. The curriculum space of a group working with the Number Line Cards task.

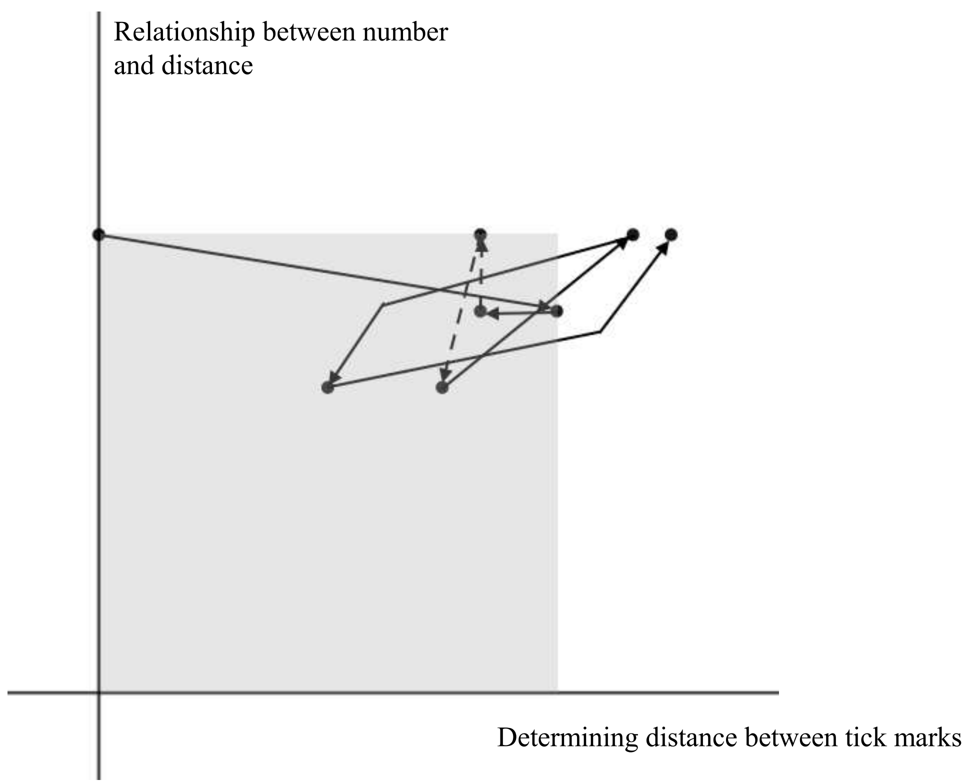
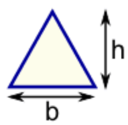


Figure B10. The curriculum space of a group working with the Number Line Cards task.

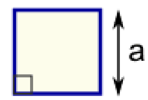
APPENDIX C: THE SOLID FUSING TASK FORMULA SHEET

AREA

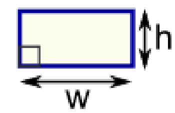
Triangle
Area = ½ × b × h
b = base
h = vertical height



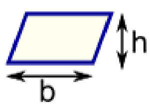
Square
Area = a^2
a = length of side



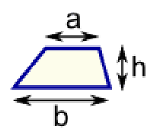
Rectangle
Area = w × h
w = width
h = height



Parallelogram
Area = b × h
b = base
h = vertical height



Trapezoid
Area = ½(a+b) × h
h = vertical height

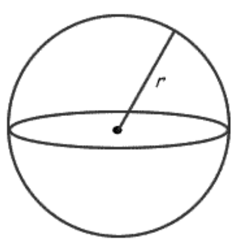


Circle
Area = πr^2
Circumference = $2\pi r$
r = radius

SURFACE AREA and VOLUME

Sphere

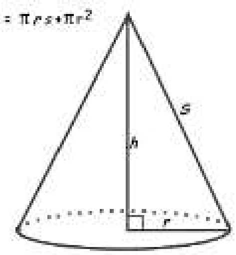
Surface Area $A = 4\pi r^2$



Volume $V = \frac{4}{3}\pi r^3$

Cone

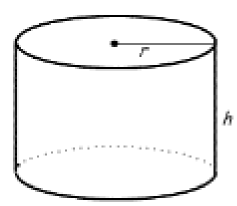
Surface Area: We will need to calculate the surf area of the cone and the base.
Area of the cone is πrs
Area of the base is πr^2
Therefore the Formula is: $SA = \pi rs + \pi r^2$



Volume $V = \frac{1}{3}\pi r^2 h$

Cylinder

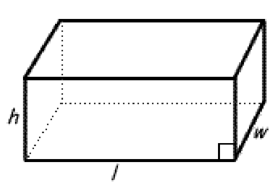
Surface Area: We will need to calculate the sur area of the top, base and sides.
Area of the top is πr^2
Area of the bottom is πr^2
Area of the side is $2\pi rh$
Therefore the Formula is: $A = 2\pi r^2 + 2\pi rh$



Volume $V = \pi r^2 h$

Rectangular Prism

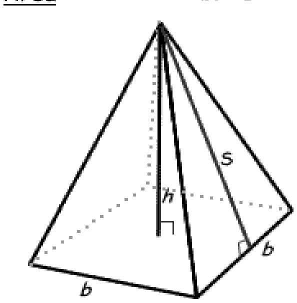
Surface Area $A = 2(wh + lw + lh)$



Volume $V = lwh$

Square Based Pyramid

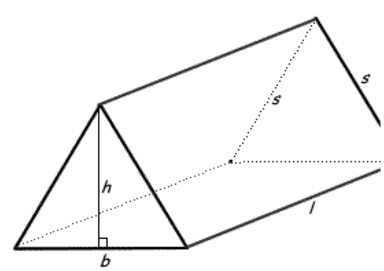
Surface Area $A = 2bs + b^2$



Volume $V = \frac{1}{3}b^2 h$

Isosceles Triangular Prism

Surface Area $A = bh + 2ls + lb$



Volume $V = \frac{1}{2}(bh)l$