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GRAVITATIONAL EQUILIBRIUM AND COLLAPSE

by

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The undersigned certify that they have read and recommend to the Faculty of Graduate Studies and Research for acceptance, a thesis entitled "GRAVITATIONAL EQUILIBRIUM AND COLLAPSE" by Gordon Alexander Wilson in partial fulfillment of the requirements for the degree of Master of Science in Theoretical Physics.

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Date October 4, 1973

ABSTRACT

This thesis is concerned with the study of the gravitational collapse of spherically symmetric thin dust and fluid shells, along with a study of a class of stationary electromagnetic vacuum fields in which "charge" equals "mass".

The features of Schwarzschild and Reissner-Nordström geometries are outlined and the equations for discussing the structure and dynamics of infinitesimal shells are derived. The collapse of thin shells with charge and mass, empty and with interior charge and mass, non-radiating and radiating is discussed; and the collapse paths are traced out on Graves-Brill diagrams - schematic representations of an analytically-extended Reissner-Nordström manifold. In a separate study, it is shown how a new class of stationary electromagnetic vacuum fields can be generated from the source free Einstein-Maxwell equations.

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CHAPTER 1

INTRODUCTION

One of the most peculiar effects to surface from Einstein's equations is the phenomenon of gravitational collapse. The equilibrium of a star is a complex, delicate matter. In its normal state, a star consists of a nearly spherical gas distribution which is in hydrostatic and thermal equilibrium. A condition of hydrostatic equilibrium indicates that the gravitational force balances the pressure force acting on each unit of mass. Thermal equilibrium means that the energy generated per mass unit equals the energy radiated from the surface of each mass unit. A violation of either equilibrium condition leads to structural changes and the "evolution" of the star. Stars with masses much larger than the sun's mass, which cannot shed a critical amount of mass, collapse to states where general relativistic effects dominate. In the following pages, I discuss a number of topics chosen for their significance - because they illustrate the position of "black holes" in relativity.

§1.1 Space-time Structure in General Relativity

In general relativity, physical laws are formulated in terms of geometrical structures - topology, connexion, and metric. A topology defines concepts of nearness, limits, continuity, and connectedness. A connexion provides a means of parallel transport for vectors, and a means of forming derivatives and differential equations. A metric defines inner products, lengths, and causal relations.

On a macroscopic level, space-time is considered to be a four-dimensional smooth manifold. This is a reasonable assumption except, perhaps, at isolated points or singularities. In Wheeler's words [1],

> "Space is like an ocean which looks flat to an aviator who flies high above it. On closer approach, the dynamic structure of the surface is seen (quantum fluctuations) .... The topology of the ocean is recognized to be non-Euclidean."

In this manifold, a physically well-defined metric which defines a pseudo-Riemannian structure is determined. When the line element, ds, is integrated along a particle's ld line, the particle's proper time lapse is given. In this manner, the notion of "clocks" may be tied to general relativity. Space-time is assumed to be locally Minkowskian; and clocks, locally, obey the laws of special relativity. In the framework of special relativity, if two events E and E', are related such that $ds^2 \leq 0$ (+2 signature), one event can causally influence the other. This property should persist globally, that is, in general relativity. From the gravitational potentials, $g_{\mu\nu}$, and using the methods of differential geometry, a Riemann tensor is constructed which is a measure of the "geodesic deviation" of two test particles. The Einstein tensor is formed from contractions of the Riemann tensor as follows:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \quad .$$

From the contracted Bianchi identities, one can show that the divergence of the Einstein tensor vanishes. The energy-momentum tensor, $T_{\mu\nu}$, which enfolds all of the fields except gravitational, also has a vanishing divergence. Einstein's equations equate these two conserved, "energy-like" quantities

$$G_{\mu\nu} = -8\pi \ T_{\mu\nu} \quad .$$

An axiomatic summary of general relativity follows:

(1) Space-time is Riemannian, normal hyperbolic. At any event, E, co-ordinates may be introduced such that

$$(ds^2)_E = dx^2 + dy^2 + dz^2 - d(ct)^2 \quad .$$

(2) The field equations are

$$G_{\mu\nu} = -8\pi \ T_{\mu\nu} \quad .$$

(3) Physical meaning of the metric:

Let E, E' be neighboring events with co-ordinates x^μ, $x^\mu + dx^\mu$. An observer momentarily at E measures the separation of E, E' as

$$[(g_{\mu\nu} + \frac{1}{c^2} v_\mu v_\nu) dx^\mu dx^\nu] \qquad \text{spatial}$$

$$- \frac{1}{c^2} v_\mu dx^\mu \qquad \text{temporal}$$

where $v^\mu = dx^\mu / d\tau$ is his four velocity.

(4) The world line of a free, spinless, uncharged particle is a time-like geodesic.

(5) The world line of a light ray in vacuo is a null geodesic [2].

§1.2 Some Evidence for the Existence of Collapsed Stars - Black Holes

Until a few years ago, general relativistic effects did not seem to be very relevant to those bodies under observation by astronomers. Black holes, or "frozen stars" had been predicted by Oppenheimer and Snyder in 1939 [3], but they proved to be an observational chimera. In order to understand the difficulty involved in pinning down a black hole, I will trace the later stages of stellar evolution.

The evolutionary direction of a star is towards depletion of nuclear fuel and increase of internal temperature and density. The theory of stellar structure in slow evolution has been worked out and supported by observational evidence of luminosity, radii, spectra, and so on. In the very late stages of stellar evolution, one assumes that the nuclear fuel is exhausted, and that the temperature is 0°K; then, the various matter distributions which can have hydrostatic equilibrium are found.

One such equilibrium state is possible for stars with a mass less than 1.2 $M_\odot$ [4]. In this state, nuclei are well-separated, but electron shells are crushed. The pressure of the degenerate electron gas in these "white dwarf" stars balances the gravitational force.

For a mass between 1.2 and 2.0 $M_\odot$ the stable state is that of a neutron star. (This range is quite uncertain, and depends on the collapse model and assumptions of initial states.) [5]. These stars have a radius of about 10 km and a density comparable to that of an atomic nucleus. The magnetic field and rotation of these collapsed stars produce pulsed radio, optical, and x-ray emissions accompanied by acceleration of particles to relativistic velocities. These pulsars most likely generate gravitational waves. The Crab and Vela pulsars are examples of pulsars known to have been produced by supernova explosions.

For a mass larger than about $2 M_{\odot}$, no equilibrium configuration exists, and an observer sees the asymptotic approach of the collapsing star to a certain radius at which it appears to be "frozen". The collapse is accompanied by an exponential decay in luminosity. The observer is a victim of the relativistic retardation of time. From the point of view of a comoving observer, the collapse is not really "frozen" at a certain stage, but continues inwards; however, at a star's "gravitational radius", photons are infinitely red-shifted and any events occurring inside this radius are inaccessible to an external observer.

The evidence for the existence of such frozen stars or black holes is still tentative. Using a one meter long aluminum cylinder which resonates at 1660 hz, Weber [6] claims to have found high frequency pulses of gravitational radiation which have an energy flow of order 10^5 ergs-cm^{-2}/sec. Black holes in formation or collision could be capable of producing such gravitational radiation without causing observable electromagnetic radiation. Weber s results are by no means confirmed.

Recent advances in x-ray astronomy, coupled to precise optical data provide more evidence for collapsed stars, a generic term for white dwarf stars, neutron stars, and black holes. Collapsed stars may be distinguished from normal stars by their pulsations - small objects generate short pulse lengths. Several x-ray sources undergo intensity fluctuations in as short a time as 50 msec. Such fluctuations could come from a collapsed object. Two x-ray sources that can be identified with eclipsing binary star systems in which one of the stars is a collapsed star are Hercules X-1, and Cygnus X-1.

Hercules X-1 has been determined by Crompton and Hutchings [7] to be an eclipsing binary st r system which has a 1.7 day eclipse period. The occulting object has a mass of 2.5 $M_{\odot}$, and the x-ray source a mass of 1.3 $M_{\odot}$ and a 1.24 second pulsation period. Giaconni [8] has shown that rotation cannot be the energy source; rather, accretion of matter onto the x-ray source by gas streaming from the companion star is the most likely energy source.

Cygnus X-1 is more intriguing. The intensity of x-rays emitted often doubles in 50 msec, and there is no predictable periodicity of pulses. Optical observations indicate that Cygnus X-1 may consist of a 12 $M_{\odot}$ supergiant and a 3 $M_{\odot}$ black hole orbiting it every 5.6 days. The aperiodic fluctuations in x-ray intensity provides the theoretical basis for this speculation. The reasons for periodicity and aperiodicity are sketched below.

There is good evidence for "gas-streaming" between binary star companions, but no x-rays usually emerge. A stream of matter falling towards a collapsed object could generate x-rays because the accelerated particles would create "hot spots" on the surface of, say, a white dwarf star, which would then radiate x-rays. Radiation back pressure would temporarily slow the stream, producing roughly periodic variations of intensity. An exact periodicity would be associated with rapid rotation of the object. A black hole has no tangible surface to support these "hot spots"; however, Zel'dovich has calculated that the gravitational field of a black hole would compress the particle stream until it reached temperatures of the order of 10^{11} °K. The resultant radiation could have aperiodic intensity fluctuations. The process of x-ray emission is not so simple as the gas-streaming argument suggests. When matter slowly accretes on a collapsed object from a companion star it is likely that a disk forms around the object. Thorne's and Zel'dovich's associates have calculated the time-averaged behavior of the x-ray spectrum for a disk-type accretion onto black holes, and the results are characteristic of the spectrum from Cygnus X-1 [9].

The diversity of evidence for black holes is not overwhelming; however, the evidence which I have mentioned has been harvested by astronomers in the last few years.

§1.3 Spherical Gravitational Collapse

In this thesis, discussion will be confined mostly to the gravitational collapse of spherically symmetric matter distributions, or to spherical shells which are qualitative analogues to spherical matter distributions in many respects.

Thorne [10] outlines the four stages involved in the collapse of a star:

(1) Instability

Late in its evolution, a star consumes all of its nuclear fuel. This leads to instability through an altered equation of state.

(2) Implosion

Instability leads to a rapid collapse of the dense core with a trailing outer envelope of matter.

(3) Horizon

The stellar surface crosses the gravitational radius in a finite time as measured by a co-moving observer. A distant observer sees the star asymptotically approach its gravitational radius which defines a surface of infinite red shift. This surface acts as an "event horizon" which encloses a region of space-time which cannot communicate with the outside universe.

(4) Singularity

The collapse continues to r = 0 where density and tidal forces are infinite.

The implosion stage of collapse can be stopped by fast exothermic reactions at high temperatures. If a star has enough fuel left to explode during the collapse, it will cast off a shell of matter. A star with a mass greater than around 2 $M_\odot$ cannot cast off enough of its mass to stop the collapse [11].

The singularity stage of the collapse is not inevitable for collapsing shells of matter as will be shown later. The shell may contract to a minimum radius and then "bounce" or re-expand into another region of space-time.

Roger Penrose has considered a number of objections to the collapse picture predicted by general relativity and sketched by Thorne above. They are as follows:

(1) Densities in excess of nuclear density inside the collapsing object could modify the equation of state.

(2) An exact vacuum is assumed outside the collapsing object.

(3) Zero net charge and zero magnetic field are assumed.

(4) Rotation is excluded.

(5) Asymmetries are excluded.

(6) A "cosmological constant" isn't considered.

(7) Quantum effects are ignored.

(8) General relativity is largely untested.

(9) There is no apparent tie-up with observations.

With respect to (1), at the event horizon, $r = 2m$

$$\frac{2m}{r} = 1 = \frac{8}{3}\pi r^2 \rho \quad .$$

For a large enough mass, the density of the collapsing object as it crosses $r = 2m$ could be small. Objections to (2) through (6) are really only objections to handling more complex systems. There are now exact solutions that include angular momentum (Kerr [12]), charge and magnetic moment (Newman [13]) with the addition of a cosmological constant (Carter [14]). It seems likely that asymmetries are radiated away during the collapse (Israel [15]), and that matter in the vicinity of a black hole falls into it.

Gravitational quantum effects would only manifest themselves locally in regions of high density and curvature. These conditions exist well inside a black hole and may be important in the deep interior.

Experimental relativity is still in its infancy; however advances both in observations of collapsed objects and in tests of the validity of relativity are on-going processes (see Dicke [16]).

§1.4 Summary of the Thesis

The preceding introduction to black holes and to spherical gravitational collapse is pursued in detail in the case of the collapse of fluid and dust shells.

Chapter II develops the geometrical background and analytic extensions which are required in discussing the collapse paths of thin shells, concluding with a rather general discussion of the metric which describes the external field of a charged, spheri-symmetric dust cloud.

Chapter III develops the equations which characterize the structure and the dynamics of thin shells.

In Chapter IV, the equations of Chapter III are applied to the case of spherical shells. The equations of motion are derived for both non-radiating and radiating shells, and a continuity equation is introduced.

Finally, some qualitative features of the motion of a collapsing shell are discussed.

Chapter V introduces specific cases of collapsing shells which do not radiate. The cases generally increase in complexity and the chapter ends with a discussion of a charged shell collapsing onto a black hole.

Chapter VI deals qualitatively with the collapse of a radiating dust shell with constant charge. The ways in which the total mass and the proper mass vary in the course of the collapse are used to cast the collapse path into one of the cases considered in Chapter V.

Chapter VII introduces a new topic which applies to static or steadily moving distributions of charged dust with equal charge and mass densities. A class of stationary solutions to the source free Einstein-Maxwell equations is derived.

§1.5 Notation Used in the Thesis

(1) Throughout the thesis, Greek indices run from 1 to 4, and Latin indices run from 1 to 3.

(2) $g_{\mu\nu}$ is variously called the metric tensor, the fundamental tensor, or the gravitational potential.

(3) A signature of +2 is used. This means that in a line element,

$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu$$

the sign of each of the terms is given by (+,+,+,−) when the metric is written in a localized diagonal form. For example,

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2 - dT^2 .$$

(4) For curves, r, θ, ϕ = constant.

For $\begin{cases} ds^2 < 0, \text{ the curves are time-like;} \\ ds^2 > 0, \text{ the curves are space-like.} \end{cases}$

(5) "Geometrized" units are used in which $c = G = 1$.

(6) In Chapters III through VI,

[X] means the jump in X through a hypersurface, $X|^+ - X|^-$.

$\tilde{X}$ means $\frac{1}{2}(X|^+ + X|^-)$.

Any other use should be clear from the context.

(7) R_+ or r_+ and R_- or r_- are used to denote the outer and inner event horizons of a Reissner-Nordström black hole.

(8) On p.33, $f_r \equiv \partial f/\partial r$; $f_{rr} \equiv \partial^2 f/\partial r^2$; $f_t \equiv \partial f/\partial t$. A partial derivative may also be written as, for example, $x^\alpha_{,\mu} \equiv \partial x^\alpha/\partial x^\mu$.

CHAPTER II

SCHWARZSCHILD AND REISSNER-NORDSTRÖM GEOMETRIES

§2.1 Introduction

In this chapter, we consider the properties of the Schwarzschild solution and the Reissner-Nordström solution with the purpose of applying these properties to the gravitational collapse of thin shells. Various analytic extensions of the space-time manifold are derived in order to eliminate co-ordinate singularities.

§2.2 Schwarzschild Geometry

The Schwarzschild solution describes the external vacuum gravitational field of any spherical matter distribution. Since stars exhibit near spherical symmetry, and since nonspherical systems tend to be mathematically intractable, it is important to understand the subtleties enfolded in the Schwarzschild line element. The Schwarzschild solution has the protean property that it may be expressed in many co-ordinate systems, each of which shows the collapse from a different perspective. In its original co-ordinates, the Schwarzschild metric takes the form

$$ds^2 = (1 - \frac{2m}{r})^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta \, d\phi^2) - (1 - \frac{2m}{r}) \, dT^2 \qquad (2.1)$$

where r is the radial co-ordinate; θ and ϕ, the polar angles. The metric is invariant under translation of the time co-ordinate, $T \to T + \text{const.}$ and reflection, $T \to -T$. In geometrized units, m may be identified with the body's mass. Spheres of constant r and T have an intrinsic surface area of $4\pi r^2$. The fantastic nature of some events during the collapse of an object is less mysterious once the external space-time geometry is explained.

When $r = 2m$, g_{44} vanishes, g_{11} becomes infinite, and the metric form (2.1) breaks down. This radius, $r = 2m$, is called the gravitational radius of the body. Outside $r = 2m$, curves of constant r ($ds^2 = g_{44}dT^2 < 0$) are time-like, and curves of constant T ($ds^2 = g_{11}dr^2 > 0$) are space-like. Inside $r = 2m$, their roles are reversed. The pathology of the co-ordinate system at $r = 2m$ is not a physical singularity; rather, it shows the inadequacy of (2.1) in describing the entire manifold. The Riemann tensor does not become infinite at $r = 2m$: tidal gravitational forces are finite. An observer who follows the collapsing body inwards enters a space-time region that is not covered by the (r,θ,ϕ,T) co-ordinates.

In order to solve the problem of the co-ordinate system's not covering all of the manifold, various extensions will be considered. Introduce a new time co-ordinate, u, such that u is constant along the path of any outgoing light ray. In the (r,θ,ϕ,T) system, a radial, outgoing light ray satisfies

$$\begin{cases} d\theta = d\phi = 0 \\ ds^2 = 0 \\ dT > 0 \\ \frac{dr}{dT} > 0 \end{cases}$$

so

$$dT = \frac{dr}{(1 - \frac{2m}{r})} \quad .$$

A retarded time can thus be defined by

$$du = dT - (1 - \frac{2m}{r})^{-1} dr$$

or

$$u = T - \{r + 2m \ \ell n \ (\frac{r}{2m} - 1)\}$$

in terms of which, the Schwarzschild metric is

$$ds^2 = -(1 - \frac{2m}{r}) du^2 - 2du \ dr + r^2 d\Omega^2 \qquad (d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2). \tag{2.2}$$

By introducing an "advanced time", v, such that

$$dv = \frac{dr}{(1 - \frac{2m}{r})} + dT \ ,$$

(2.1) is transformed to the Eddington-Finkelstein form

$$ds^2 = \frac{dr^2}{(1 - \frac{2m}{r})} + r^2 d\Omega^2 - (1 - \frac{2m}{r})\ dT^2$$

$$= (1 - \frac{2m}{r})\left(\frac{dr}{(1 - \frac{2m}{r})} + dT\right)\left(\frac{dr}{(1 - \frac{2m}{r})} - dT\right) + r^2 d\Omega^2$$

$$= 2dv\ dr - (1 - \frac{2m}{r})dv^2 + r^2 d\Omega^2 \quad . \qquad (2.3)$$

These co-ordinate systems cover the range $0 < r < \infty$ without becoming singular; and, taken together, they cover the entire time range.

In tracing the collapsing surface using metric form (2.1), one finds that the light cone of a co-moving observer bends inwards with the collapsing surface, pulled by the intense gravitational field. An external observer is still confronted by the "impenetrability" of the Schwarzschild surface. This surface, an event horizon, is in Penrose's words, "the absolute boundary of all events which can be observed in principle by an outside observer." [17]. The co-moving observer's world-line must remain time-like; however, a time-like world-line transports him to regions of decreasing r and increasing tidal forces that are generated by increasing curvature. The singularity at r = 0 is an intrinsic physical singularity, as is evident from the invariant scalar,

$$R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} = \frac{48m^2}{r^6} \quad .$$

What happens to the matter in the vicinity of the singularity is an interesting problem which will not be considered here since the theory of general relativity is not applicable.

Neither of the co-ordinate systems (r,θ,ϕ,u) or (r,θ,ϕ,v) cover the entire space $r > 0$, $-\infty < T < \infty$ by themselves. For general geometries, the best situation is a system of overlapping co-ordinate patches which allow all of the geometry to be probed. For Schwarzschild geometry, a single global co-ordinate patch was developed by Kruskal [18]. The $(r\theta\phi u)$ form of the metric differs from the $(r\theta\phi v)$ form by the transformation $u \to -v$. $u =$ const. defines a spherically symmetric outgoing or exploding null hypersurface; whereas, $v =$ const. defines a spherically symmetric ingoing or collapsing null hypersurface. To construct a single global patch, first try (u,v,θ,ϕ) as a co-ordinate system.

$$v-u = 2\{r + 2m \ln (\frac{r}{2m} - 1)\} \qquad (2.4)$$

$$v+u = -2T ,$$

and $$ds^2 = -(1 - \frac{2m}{r}) \, du\, dv + r^2 d\Omega^2 .$$

Unfortunately, the determinant of g vanishes at $r = 2m$ and the metric is singular. Rewriting (2.4) as

$$\exp\{\frac{(v-u)}{4m}\} = (1 - \frac{2m}{r})(\frac{r}{2m})\exp(\frac{r}{2m}) \qquad (2.5)$$

we see that (2.1) may be written

$$ds^2 = -(\frac{2m}{r})e^{-r/2m}\, dU\, dV + r^2 d\Omega^2 \qquad (2.6)$$

with $U = -4me^{-u/4m}$, $V = 4me^{v/4m}$.

That this metric form is nonsingular for all $r > 0$ is clear from studying r as a function of U and V [19].

The Kruskal metric (2.6) has the following properties:

(1) Any line of constant $U\theta\phi$ or $V\theta\phi$ is a null line.

(2) The metric is not defined on the extreme top and bottom shaded regions of figs. 1 to 5. In these regions $UV/(4m)^2 > 1$ and r and $g_{\mu\nu}$ are not defined.

(3) The boundary lines $UV/(4m)^2 = 1$ correspond to $r = 0$ and are true singularities of the manifold.

(4) The transformations

$$U = -4me^{-T/4m}\, e^{r/4m}\, (\frac{r}{2m} - 1)^{\frac{1}{2}}$$

$$V = 4me^{T/4m}\, e^{r/4m}\, (\frac{r}{2m} - 1)^{\frac{1}{2}}$$

show that figs.1 to 3 illustrate the domains of regularity of Schwarzschild's co-ordinates, $(r\theta\phi T)$; "retarded" co-ordinates $(r\theta\phi u)$; and "advanced" co-ordinates (r,θ,ϕ,v).

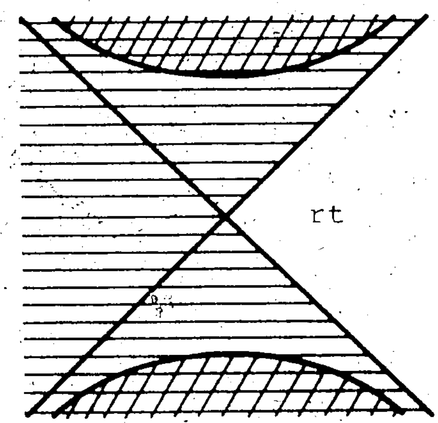
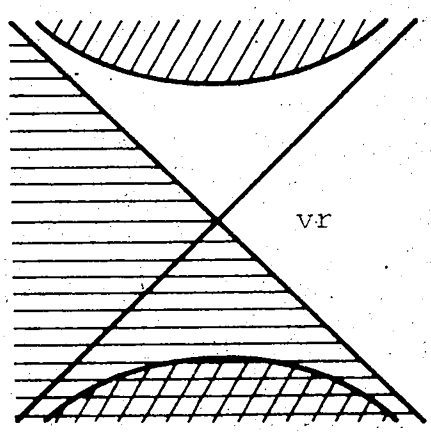


Figure 1. The unshaded region shows the region where Schwarzchild's r,t form a co-ordinate system.



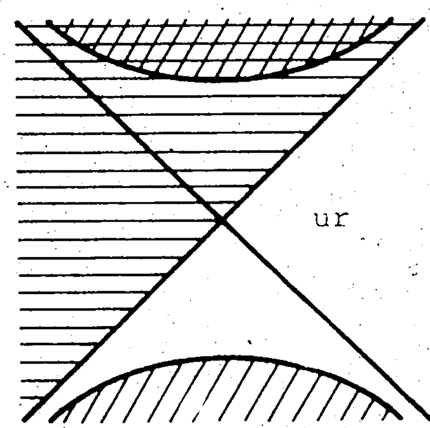


Figure 2. The unshaded region is the "advanced time patch".

Figure 3. The unshaded region is the "retarded time patch".

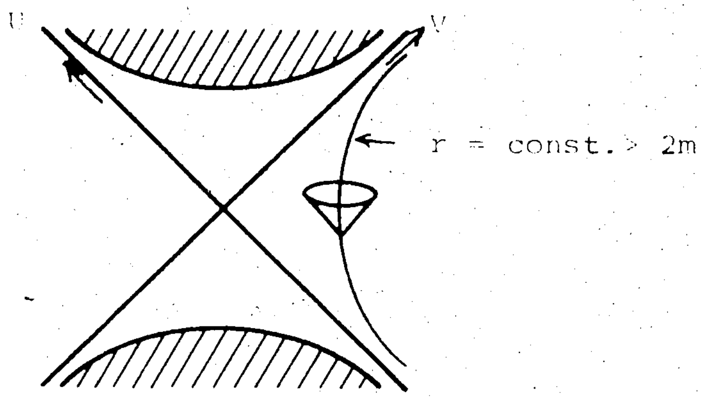


Figure 4. A stationary observer's world line at constant r is shown (UV = const. < 0).

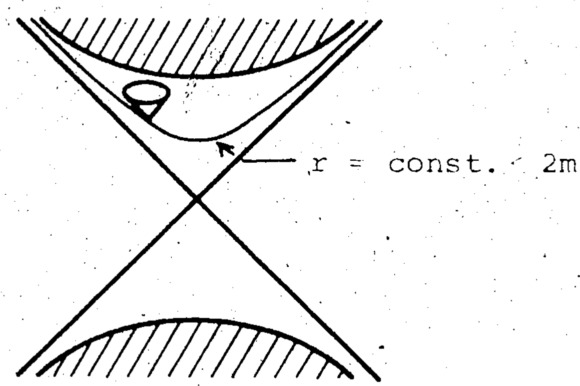


Figure 5. A hypersurface r = const. < 2m is shown (UV = const. > 0). See [19].

Other properties of the Kruskal system are noted in figures 1 to 5. For a more detailed discussion, see Misner [20].

§2.3 Electrodynamics in Relativity

The Reissner-Nordström solution differs from the Schwarzschild solution in that the spherical matter distribution also has an electrical charge. Before discussing Reissner-Nordström geometry, the role of electrodynamics in relativity theory must be examined.

In special relativity, electrodynamics is characterized by the stress tensor

$$T^{\mu}_{\nu} = \frac{1}{4\pi}\left[F^{\mu\alpha}F_{\nu\alpha} - \frac{1}{4}\delta^{\mu}_{\nu}F_{\alpha\beta}F^{\alpha\beta}\right] \qquad (2.7)$$

where the electromagnetic tensor, $F_{\mu\nu}$, may be determined from a vector potential, A_{μ}, by

$$F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu} \qquad (2.8)$$

In terms of familiar field variables, $\underline{E}$ and $\underline{H}$, $F_{\mu\nu}$ is the matrix

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & H_z & -H_y \\ E_x & -H_z & 0 & H_x \\ E_z & H_y & -H_x & 0 \end{pmatrix} \qquad (2.9)$$

The $F_{\mu\nu}$ satisfy Maxwell's equations:

$$F_{\mu\nu,\lambda} + F_{\nu\lambda,\mu} + F_{\lambda\mu,\nu} = 0 \qquad (2.10)$$

$$F^{\mu\nu}{}_{,\nu} = J^{\mu} \text{ where } J^{\mu} \text{ four current .} \qquad (2.11)$$

Extending these relations to apply to general relativity, the $F_{\mu\nu}$ will satisfy

$$F_{\mu\nu|\lambda} + F_{\nu\lambda|\mu} + F_{\lambda\mu|\nu} = 0 \qquad (2.12)$$

$$F^{\mu\nu}{}_{|\nu} = \mathfrak{J}^{\mu} \quad . \qquad (2.13)$$

The covariant derivative, indicated by "| or ;" along with other mathematical operations and entities such as Christoffel symbols are discussed in Chapter III.

One can show that since

$$F_{\mu\nu|\lambda} = F_{\mu\nu,\lambda} - F_{\alpha\nu}\Gamma^{\alpha}_{\mu\lambda} - F_{\mu\alpha}\Gamma^{\alpha}_{\nu\lambda}$$

that (2.12) is equivalent to (2.10).

Using relations

$$F^{\mu\nu}{}_{|\nu} = F^{\mu\nu}{}_{,\nu} + F^{\alpha\nu}\Gamma^{\mu}_{\alpha\nu} + F^{\mu\alpha}\Gamma^{\nu}_{\alpha\nu}$$

$$F^{\alpha\nu}\Gamma^{\mu}_{\alpha\nu} = 0$$

$$\Gamma^{\nu}_{\alpha\nu} = \frac{1}{2g}\, g_{,\alpha}$$

(2.13) may be shown to be equivalent to

$$\{\sqrt{-g}\; F^{\mu\nu}\}_{,\nu} = \sqrt{-g}\; J^{\mu} \quad . \tag{2.14}$$

(2.14), and (2.12) or (2.10) are the general relativistic statement of Maxwell's equations.

§2.4 Reissner-Nordström Geometry

The Reissner-Nordström metric for the external field of a charged spherically symmetric dust cloud is given by

$$ds^2 = (1 - \frac{2m}{r} + \frac{e^2}{r^2})^{-1} dr^2 + r^2 d\Omega^2 - (1 - \frac{2m}{r} + \frac{e^2}{r^2}) dT^2 \tag{2.15}$$

where it is assumed that mass and charge are constant. This co-ordinate system becomes singular at two radii, which are found by setting $g_{44} = 0$

$$\begin{cases} r_+ = m + (m^2 - e^2)^{\frac{1}{2}}, & \text{for} \quad |e| < m \\ r_- = m - (m^2 - e^2)^{\frac{1}{2}} & \end{cases}$$

Thus, where $|e| < m$, there are two event horizons. When $|e| = m$, $g_{44} = -(1 - m/r)^2$, and there is only one event horizon at $r = m$. For $|e| > m$, there are no event horizons.

The metric (2.15) may be transformed to a Kruskal-like system that does not become pathological at r_+ and r_-. The analytic completion of the Reissner-Nordström manifold has been done by Graves and Brill for $e^2 < m^2$ and by Carter for $e^2 = m^2$ [21]. What follows is a summary of their arguments by de la Cruz and Israel [22]:

Case 1: $e^2 = m^2$

In this case, the Reissner-Nordström metric becomes

$$ds^2 = (1 - \frac{m}{r})^{-2} dr^2 + r^2 d\Omega^2 - (1 - \frac{m}{r})^2 dT^2 \quad . \qquad (2.16)$$

The co-ordinate, T, is now time-like ($g_{44} < 0$) for all r. In order to deal with a time range of $-\infty < T < \infty$, introduce an angular time co-ordinate, θ, such that

$$\frac{T}{2m} = \tan\theta \qquad -\frac{\pi}{2} < \theta < \frac{\pi}{2} \quad .$$

(This θ should not be confused with the polar angle θ which arises in $d\Omega^2$). The extended line element

$$ds^2 = (1 - \frac{m}{r})^{-2} dr^2 + r^2 d\Omega^2 - 4m^2 (1 - \frac{m}{r})^2 (d \tan\theta)^2 \qquad (2.17)$$

represents a space-time which has a singularity only at r = 0, and is periodic. The r, θ map breaks down on lines $r = m$, $\theta = (n + \frac{1}{2})\pi$. That $r = m$ is a regular part of the manifold can be verified by expressing the line element in a form that is regular at $r > 0$:

$$ds^2 = 2dv'dr - (1 - \frac{m}{r})^2 dv'^2 + r^2 d\Omega^2 \qquad (2.18)$$

where advanced time v' is related to r and T by

$$dv' = (1 - \frac{m}{r})^{-2} dr + dT \qquad (2.19)$$

In the v', r chart, follow any incoming radial null geodesic v' = const. originating in a region $r > m$ to $r = 0$ (fig. 6). This chart provides a regular mapping of two adjoining regions, Ia and IIIa. By analogous use of a retarded time parameter, a chart for IIIb and Ib can be constructed. A repeating chain of overlapping co-ordinate patches allows one to follow any null or time-like geodesic to $r = 0$.

Case 2 : $e^2 < m^2$ (Note: $r_1 = r_+$; $r_2 = r_-$)

In this case, the quadratic co-efficient $f(r) = (1 - \frac{2m}{r} - \frac{e^2}{r^2})$ in the Reissner-Nordström metric has real, unequal factors

$$f(r) = \frac{(r-r_+)(r-r_-)}{r^2} \qquad 0 < r_- < r_+ \qquad (2.20)$$

Incoming and outgoing radial null geodesic have equations v = const., and u = const. where

$$2ku^{-1}du = f^{-1}dr - dT \qquad (2.21)$$

$$2kv^{-1}dv = f^{-1}dr + dT \qquad (2.22)$$

k is an adjustable constant.

The line element becomes the form

$$ds^2 = \left(\frac{4k^2 f}{uv}\right) du\,dv + r^2 d\Omega^2 \qquad (2.23)$$

Integration of (2.21) and (2.22) yields

$$r + \frac{r_1^2}{r_1 - r_2} \ln\left|\frac{r}{r_1} - 1\right| - \frac{r_2^2}{r_1 - r_2} \ln\left|\frac{r}{r_2} - 1\right| = k \ln|uv| \qquad (2.24)$$

$$T = k \ln\left|\frac{v}{u}\right| \qquad (2.25)$$

Consider the chart u_1, v_1 obtained by setting $k = k_1 = r_1^2/(r_1 - r_2)$. From (2.24)

$$u_1 v_1 = \left(\frac{r}{r_1} - 1\right)\left(\frac{r}{r_2} - 1\right)^{-r_2^2/r_1^2} \exp\left[\frac{r_1 - r_2}{r_1^2} r\right], \qquad (r > r_2). \qquad (2.26)$$

There is no singularity at $r = r_1$. (u_1, v_1) gives a regular mapping of a subregion of the manifold which has $r > r_2$. A co-ordinate singularity does develop at $r = r_2$ however, and it is necessary to go over to another chart before that happens.

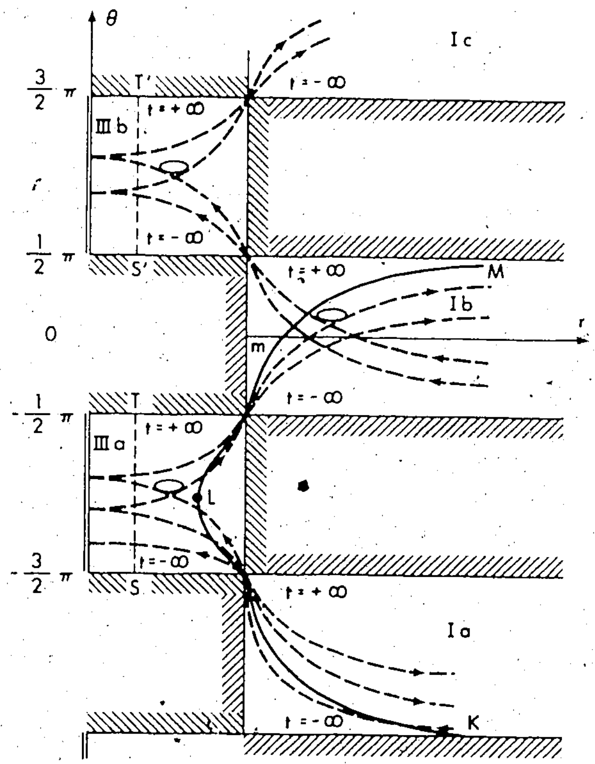


Figure 6. This is a diagram of the extended Reissner-Nordström manifold for e = m. Dashed lines represent radial null geodesics. The time-like curve KLM represents the history of a thin shell which implodes in Ia, reverses its motion at L after passing through the event horizon at r = m, then re-expands into Ib. This diagram is from [21].

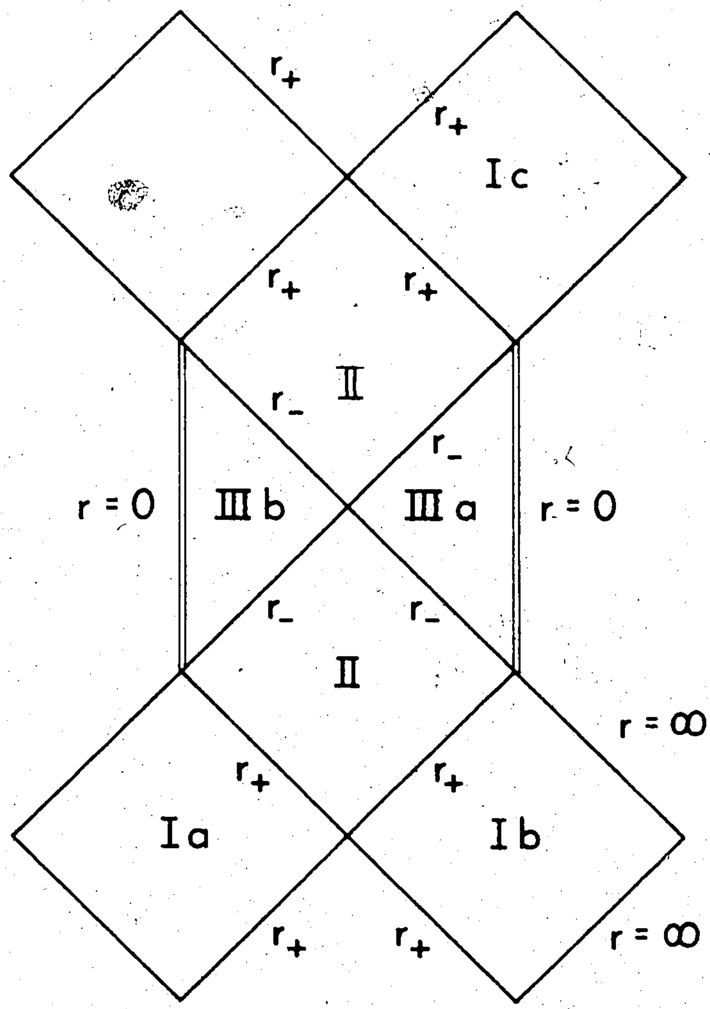


Figure 7. This diagram shows the extended Reissner-Nordström manifold for $e < m$. r_+ and r_- indicate outer and inner event horizons. This may be called a Graves-Brill diagram.

Define the chart (u_2,v_2) by setting $k = k_2 = -r_2^2/(r_1-r_2)$ in (2.24) and (2.25). Then

$$u_2 v_2 = \left(\frac{r}{r_2} - 1\right)\left(1 - \frac{r}{r_1}\right)^{-r_1^2/r_2^2} \exp\left[-\frac{r_1-r_2}{r_2^2} r\right], \quad (r < r_1) \tag{2.27}$$

and this provides a regular covering for any subregion with $r < r_1$. In the domain of overlap, $r_2 < r < r_1$, the charts are related by

$$|u_1|^{r_1^2} = |u_2|^{-r_2^2}; \quad |v_1|^{r_1^2} = |v_2|^{-r_2^2} . \tag{2.28}$$

The complete manifold for $e^2 < m^2$ is a periodic lattice of alternating regions of type I $(r > r_1)$, type II $(r_2 < r < r_1)$ and type III $(r < r_2)$. Fig. 7 shows the results of the preceding transformations.

§2.5 A Radiating Reissner-Nordström Metric

The Reissner-Nordström metric for the external field of a charged, spherically symmetric dust cloud has been given by

$$ds^2 = \left(1 - \frac{2m}{r} + \frac{e^2}{r^2}\right)^{-1} dr^2 + r^2 d\Omega^2 - \left(1 - \frac{2m}{r} + \frac{e^2}{r^2}\right) dT^2 .$$

By making the mass and charge functions of a retarded time, t, the Reissner-Nordström metric is transformed into its radiating counterpart. The general form of such a spheri-symmetric metric is given by [23]

$$ds^2 = -F(r,t)dt^2 - 2drdt + r^2 d\Omega^2$$

$$= -(1 - \frac{f(r,t)}{r})dt^2 - 2drdt + r^2 d\Omega^2 \quad . \qquad (2.29)$$

If $F(r,t) = 1 - 2m(t)/r$, we have the case of a radiating Vaidya metric, which was dealt with by Israel [24].

For a radiating Reissner-Nordström background,

$$F(r,t) = 1 - \frac{2m(t)}{r} + \frac{e^2(t)}{r^2} \qquad (2.30)$$

$$f(r,t) = 2m(t) - \frac{e^2(t)}{r} \quad . \qquad (2.31)$$

For future reference in calculations, we write out explicitly the components of the metric (2.29) which are the gravitational potentials, or fundamental tensor

$$||g_{\mu\nu}|| = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & r^2 & 0 & 0 \\ 0 & 0 & r^2\sin^2\theta & 0 \\ -1 & 0 & 0 & -F \end{pmatrix} \qquad (2.32)$$

$$||g^{\mu\nu}|| = \begin{pmatrix} F & 0 & 0 & -1 \\ 0 & \frac{1}{r^2} & 0 & 0 \\ 0 & 0 & \frac{1}{r^2\sin^2\theta} & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} = ||g_{\mu\nu}||^{-1} \qquad (2.33)$$

A differential form calculation (Appendix 1) gives for components of the Einstein tensor

$$G^{\alpha}{}_{\beta} = \begin{pmatrix} \frac{1}{r^2} f_r & 0 & 0 & \\ 0 & \frac{1}{2r} f_{rr} & 0 & 0 \\ 0 & 0 & \frac{1}{2r} f_{rr} & 0 \\ -\frac{1}{r^2} f_t & 0 & 0 & \frac{1}{r^2} f_r \end{pmatrix} \qquad (2.34)$$

The energy tensor takes the form

$$T^{\mu}{}_{\nu} = \lambda t'^{\mu} t,_{\nu} + \frac{1}{4\pi} [F^{\mu\alpha} F_{\nu\alpha} - \frac{1}{4} \delta^{\mu}{}_{\nu} F_{\alpha\beta} F^{\alpha\beta}] . \qquad (2.35)$$

t'^{μ} is tangent to the outgoing null geodesic t = const. and λ will be determined from (2.34) by applying Einstein's field equations. The second term on the right is the electromagnetic contribution. $T^{\mu}{}_{\nu}$ represents a radiation-filled exterior region to a spherical shell whose mass and charge change as a function of t as a result of radial radiation of particles with the speed of light.

If we assume a vector potential

$$A_\mu(t) = (0,0,0,\frac{e(t)}{r}) \tag{2.36}$$

where

$$F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu} \tag{2.37}$$

and apply Einstein's equations, using (2.34), we get

$$T^1_1 = T^4_4 = -\frac{e^2(t)}{8\pi r^4} \tag{2.38}$$

$$T^2_2 = T^3_3 = \frac{e^2(t)}{8\pi r^4} \tag{2.39}$$

$$T^1_4 = \frac{1}{4\pi}\left(\frac{\dot{m}(t)}{r^2} - \frac{e(t)\dot{e}(t)}{r^3}\right) = -\frac{F_t}{8\pi r} \equiv \lambda \tag{2.40}$$

Once the mechanism for discussing the collapse of a thin shell has been derived in the next Chapter, the results of sections §2.4 and §2.5 will be applied to the case of a charged dust shell in both a static and a radiating Reissner-Nordström background.

CHAPTER III

GENERAL SHELL THEORY

§3.1 Introduction

The features of spherical gravitational collapse are preserved when one considers the collapse of an infinitesimal shell of matter, a singular hypersurface. Compressing a cloud of dust into a shell results in jump discontinuities, and one must consider the junction conditions at the discontinuous surface. In Newtonian theory, the jump discontinuities of the potential and its first derivatives are calculated across the surface. In relativity theory, the continuity of the gravitational potentials depends on the smoothness with which the co-ordinate system being used covers the space-time manifold. In order to avoid the caprices of various co-ordinate systems, we characterize the hypersurface by the extrinsic curvature of its embedding in the four dimensional background Riemannian manifold.

§3.2 A Description of the Shell Hypersurface

Let Σ be a smooth, time-like hypersurface embedded in a four dimensional Riemannian manifold, dividing it into two regions, V_-, the region interior to the surface, and V_+, the region exterior to the surface. V has co-ordinates x^α and metric tensor $g_{\alpha\beta}$. $\sum$ has intrinsic co-ordinates ξ^i and metric tensor g_{ij}. The extrinsic and intrinsic co-ordinates are related by some function

$$x^\alpha = f^\alpha(\xi^1, \xi^2, \xi^3) \quad . \tag{3.1}$$

An infinitesimal displacement in $\sum$ is given by

$$dx^\alpha = \frac{\partial x^\alpha}{\partial \xi^i} d\xi^i \equiv e^\alpha_{(i)} d\xi^i \tag{3.2}$$

or by

$$d\underline{s} = \underline{e}_{(i)} d\xi^i \tag{3.3}$$

where we have introduced three linearly independent tangent base vectors $\underline{e}_{(i)}$ whose components are

$$e^\alpha_{(i)} = \frac{\partial x^\alpha}{\partial \xi^i} \quad . \tag{3.4}$$

The metric of $\sum$ is given by

$$\begin{aligned} ds^2 = d\underline{s}\cdot d\underline{s} &= \underline{e}_{(i)}\cdot\underline{e}_{(j)} d\xi^i d\xi^j \\ &= g_{ij} d\xi^i d\xi^j \end{aligned} \tag{3.5}$$

where

$$g_{ij} = \underline{e}_{(i)}\cdot\underline{e}_{(j)} = g_{ji} \quad .$$

The following entities and operations provide information about the structure of the hypersurface:

(i) <u>Reciprocal or Dual Base Vectors</u>

The dual base vector is defined by

$$\underline{e}^{(i)} = g^{ij}\underline{e}_{(j)} \tag{3.6}$$

where

$$||g^{ij}|| \equiv ||g_{ij}||^{-1}$$

$$g^{ij}g_{kj} = \delta^i_k$$

$$g \equiv \det(g_{ij}) \neq 0 \ .$$

Since $\underline{e}^{(i)} \cdot \underline{e}_{(j)} = \delta^i_j$, $\underline{e}^{(i)} \perp \underline{e}_{(j)}$ $\quad i \neq j$.

The dual base vectors are linearly independent vectors tangent to Σ.

(2) <u>Vectors in Σ</u>

Let $\underline{A}$ be any vector tangent to Σ. We can write

$$\underline{A} = A^p\underline{e}_{(p)} = A_p\underline{e}^{(p)} \tag{3.7}$$

where A_p are the covariant components of $\underline{A}$, and A^p, the contravariant components. The metric tensor, g_{ij} and its inverse, g^{ij}, are used to raise and lower indices.

(3) Affine Connexion (Christoffel Symbols)

Define

$$\Gamma^h_{ij} \equiv \underline{e}^{(h)} \cdot \frac{\partial \underline{e}_{(i)}}{\partial \xi^j} = \underline{e}^{(h)} \cdot \frac{\partial \underline{e}_{(j)}}{\partial \xi^i} = \Gamma^h_{ji} \qquad (3.8)$$

where $\partial \underline{e}_{(i)}/\partial \xi^j$ is the absolute or intrinsic derivative of $\underline{e}_{(i)}$ defined with respect to the four-dimensional affine connexion. Another way of writing Γ^h_{ij} may be found from

$$\frac{\partial}{\partial \xi^j} (\underline{e}_{(i)} \cdot \underline{e}^{(h)}) = \frac{\partial}{\partial \xi^j} \delta^h_i = 0$$

$$\longrightarrow \quad \underline{e}^{(h)} \cdot \frac{\partial \underline{e}_{(i)}}{\partial \xi^j} = -\underline{e}_{(i)} \cdot \frac{\partial \underline{e}^{(h)}}{\partial \xi^j} \quad .$$

Thus

$$\Gamma^h_{ij} = - \underline{e}_{(i)} \cdot \frac{\partial \underline{e}^{(h)}}{\partial \xi^j} \quad . \qquad (3.9)$$

A second Christoffel symbol, $\Gamma_{\ell,ij}$ is defined by

$$\Gamma_{\ell,ij} = \underline{e}_{(\ell)} \cdot \frac{\partial \underline{e}_{(i)}}{\partial \xi^j} \quad . \qquad (3.10)$$

It is related to Γ^h_{ij} by

$$\Gamma^h_{ij} = g^{h\ell} \Gamma_{\ell,ij} \qquad (3.11)$$

and may be written more explicitly in terms of the gravitational potentials (metric tensor) as

$$\Gamma_{\ell,ij} = \frac{1}{2}(\partial_j g_{\ell i} + \partial_i g_{\ell j} - \partial_\ell g_{ij}) \quad . \tag{3.12}$$

(4) Intrinsic Differentiation of a Vector

The intrinsic, covariant derivative of a vector, $\underline{A}$, with respect to ξ^i represents the projection of the derivative of $\underline{A}$ on the hypersurface, Σ.

Let $\underline{A}(y)$ be a vector tangent to Σ and defined on the curve $\xi^i = \xi^i(y)$

$$\underline{A}(y) = A^i(y)\underline{e}_{(i)} \tag{3.13}$$

$$\frac{d\underline{A}(y)}{dy} = \frac{dA^i}{dy}\underline{e}_{(i)} + A^i \frac{d\underline{e}_{(i)}}{dy} \quad . \tag{3.14}$$

Now split $d\underline{A}(y)/dy$ into components tangential to Σ and perpendicular to Σ.

$$\frac{d\underline{A}(y)}{dy} = \left(\frac{d\underline{A}}{dy}\right)_{||} + \left(\frac{d\underline{A}}{dy}\right)_{\perp} \quad . \tag{3.15}$$

Only the parallel component is accessible to an observer on the hypersurface

$$\left(\frac{dA}{dy}\right)^h_{||} = \underline{e}^{(h)} \cdot \left(\frac{d\underline{A}}{dy}\right)_{||} = \underline{e}^{(h)} \cdot \frac{d\underline{A}}{dy} \tag{3.16}$$

$$(3.14),\ (3.16) \longrightarrow \left(\frac{dA^h}{dy}\right)_{||} = \frac{dA^h}{dy} + A^i \underline{e}^{(h)} \cdot \frac{\partial \underline{e}_{(i)}}{\partial \xi^j} \frac{d\xi^j}{dy}$$

$$(3.8) \longrightarrow \left(\frac{dA}{dy}\right)^h_{||} = \frac{dA^h}{dy} + \Gamma^h_{ij} A^i \frac{d\xi^j}{dy}$$

$$\equiv \frac{\delta A^h}{\delta y} \tag{3.17}$$

The intrinsic derivative of a vector $\underline{A}$ with respect to a parameter y is written

$$\frac{\delta A^h}{\delta y} = \frac{dA^h}{dy} + \Gamma^h_{ij} A^i \frac{d\xi^j}{dy} . \tag{3.18}$$

The intrinsic derivative of a nontangent vector may also be defined, and represents the projection of the derivative of the vector on Σ.

(5) Covariant Derivative of a Vector Field

For some value of i, let $\xi^i = y$

$$\frac{\delta A^h}{\delta \xi^i} \equiv A^h{}_{|i} = \frac{\partial A^h}{\partial \xi^i} + \Gamma^h_{pq} A^p \frac{\partial \xi^q}{\partial \xi^i}$$

$$= \frac{\partial A^h}{\partial \xi^i} + \Gamma^h_{pi} A^p \tag{3.19}$$

which is the covariant derivative of the components A^h of vector

$$\underline{A} = A^h \underline{e}_{(h)} .$$

Similarly, it may be shown that

$$A_{h|i} = \frac{\partial A_h}{\partial \xi^i} - \Gamma^p_{hi} A_p . \tag{3.20}$$

(6) Extrinsic 3-Curvature

Let $\underline{n}$ be a space-like normal to Σ pointing from V_- to V_+ with properties

$$n_\alpha n^\alpha = 1$$

$$n_\alpha \frac{\delta n^\alpha}{\delta \xi^i} = 0 \quad \text{by intrinsic differentiation}$$

$$\longrightarrow \quad \frac{\delta n^\alpha}{\delta \xi^i} \perp n_\alpha \, . \tag{3.21}$$

The Christoffel symbols, and the Riemann 3-tensor defined by

$$R^h_{ijk} = \partial_j \Gamma^h_{ik} - \partial_k \Gamma^h_{ij} + \Gamma^\ell_{ik}\Gamma^h_{\ell j} - \Gamma^\ell_{ij}\Gamma^h_{\ell k} \tag{3.22}$$

depend only on g_{ij} and its derivatives - all intrinsic quantities. Properties of a non-intrinsic character are given by the way the hypersurface "bends" in the background manifold; that is, by the way that the normal varies with respect to the intrinsic co-ordinates.

The extrinsic 3-curvature, K_{ij}, is defined by

$$\frac{\delta n^\alpha}{\delta \xi^i} \quad K^j_i e^\alpha_{(j)} \, . \tag{3.23}$$

$$\longrightarrow \quad K_{ij} = e^\alpha_{(j)} \frac{\partial n_\alpha}{\partial \xi^i} = -n_\alpha \frac{\partial e^\alpha_{(j)}}{\partial \xi^i} = -n_\alpha \frac{\partial e^\alpha_{(i)}}{\partial \xi^j} \, . \tag{3.24}$$

(7) <u>Gauss-Weingarten Equations; Gauss-Codazzi Equations</u>

From (3.19), (3.24),

$$\frac{\delta A^\alpha}{\delta \xi^j} = A^i{}_{;j} e^\alpha_{(i)} - A^i K_{ij} n^\alpha \, . \tag{3.25}$$

The Gauss-Weingarten equations (3.26), follow from (3.10), (3.24):

$$\frac{\delta e^{\alpha}_{(i)}}{\delta \xi^{j}} = K_{ij} n^{\alpha} + \Gamma^{h}_{ij} e^{\alpha}_{(h)} \quad . \tag{3.26}$$

Given an $\underline{A}(u,v)$ defined on $x^{\lambda} = x^{\lambda}(u,v)$, the Ricci commutation relations are

$$\left[\left(\frac{\partial}{\partial u} \frac{\partial}{\partial v} - \frac{\partial}{\partial v} \frac{\partial}{\partial u} \right) \underline{A} \right]^{\alpha} = R^{\alpha}_{\beta\lambda\mu} A^{\beta} \frac{\partial x^{\lambda}}{\partial u} \frac{\partial x^{\mu}}{\partial v} \quad . \tag{3.27}$$

Taking $\delta/\delta\xi^{k}$ of (3.26), and using (3.23) and (3.26) along with the Ricci commutation relation yields the Gauss-Codazzi equations

$$R_{\alpha\beta\gamma\delta} e^{\alpha}_{(a)} e^{\beta}_{(b)} e^{\gamma}_{(c)} e^{\delta}_{(d)} = R_{abcd} - (K_{ac}K_{bd} - K_{bc}K_{ad}) \tag{3.28}$$

$$R_{\alpha\beta\gamma\delta} n^{\alpha} e^{\beta}_{(b)} e^{\gamma}_{(c)} e^{\delta}_{(d)} = (K_{bc|d} - K_{bd|c}) \quad . \tag{3.29}$$

Multiplying (3.28) by $g^{bc}g^{ad}$, (3.29) by g^{bd}, and using the relation

$$g^{bc} e^{\beta}_{(b)} e^{\gamma}_{(c)} = g^{\beta\gamma} - n^{\beta} n^{\gamma} \tag{3.30}$$

yields

$${}^{3}R - K_{ab}K^{ab} + K^{2} = -2G_{\alpha\beta} n^{\alpha} n^{\beta} \quad . \tag{3.31}$$

(8) <u>Surface Energy Tensor</u>

If the 3-tensor defined by

$$\gamma_{ij} = K^{+}_{ij} - K^{-}_{ij} \tag{3.32}$$

is nonzero, the hypersurface represents a thin shell.

The Lanczos equations

$$\gamma_{ij} - g_{ij}\gamma = -8\pi S_{ij} \qquad (\gamma = g^{ij}\gamma_{ij}) \tag{3.33}$$

$$\longleftrightarrow \quad \gamma_{ij} = -8\pi \left(S_{ij} - \frac{1}{2} g_{ij} S\right) \tag{3.34}$$

define a symmetric tensor, S_{ij}, called the surface energy tensor. The surface energy tensor is the integral of the Einstein tensor across the surface layer. This may be shown as follows:

Let the surface layer defined by boundary surfaces $\sum^-$ and $\sum^+$ have thickness ε where $\varepsilon \to 0$. Using Gaussian co-ordinates based on $\sum^-$, $\sum^-$ is characterized by $x^1 \stackrel{*}{=} 0$, and $\sum^+$ by $x^1 \stackrel{*}{=} \varepsilon$. The extrinsic 3-curvature is given by

$$K_{ij} \stackrel{*}{=} \frac{1}{2} \frac{\partial\, {}^4g_{ij}}{\partial x^1} \tag{3.35}$$

and

$${}^4R_{ij} \stackrel{*}{=} \frac{\partial K_{ij}}{\partial x^1} + Z_{ij} \tag{3.36}$$

where

$$Z_{ij} = {}^3R_{ij} - KK_{ij} + 2K_i^p K_{pj} \tag{3.37}$$

Integrating $R_{\alpha\beta} = -8\pi (T_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} T)$ (a variant of Einstein's field equation) through the layer gives

$$-8\pi \int_0^{\varepsilon} (T_{ij} - \frac{1}{2} g_{ij} T)\, dx^1 = \int_0^{\varepsilon} \frac{\partial K_{ij}}{\partial x^1}\, dx^1 + \int_0^{\varepsilon} Z_{ij}\, dx^1 . \qquad (3.38)$$

Both K_{ij} and ${}^3R_{ij}$ are finite as $\varepsilon \to 0$, so the last integral on the right side tends to zero.

$$\therefore \qquad -8\pi \int_0^{\varepsilon} (T_{ij} - \frac{1}{2} g_{ij} T)\, dx^1 \triangleq K^+_{ij} - K^-_{ij} = \gamma_{ij} . \qquad (3.39)$$

By comparing (3.39) and (3.38), one sees that

$$S_{ij} = \lim_{\varepsilon \to 0} \int_0^{\varepsilon} T_{ij}\, dx^1 . \qquad [25] \qquad (3.40)$$

(9) Jump Condition for the Electromagnetic Field

The method just demonstrated of integrating through a surface layer will now be used to determine the jump condition of the electromagnetic field produced by a charged surface through the surface. Again, let the surface layer defined by Σ^- and Σ^+ have thickness ε where $\varepsilon \to 0$, and use Gaussian co-ordinate based on Σ^-. The jump condition on the field is found as follows:

For a charged surface,

$$F^{\mu\nu}{}_{|\nu} = J^{\mu} \quad \text{inside the surface} \qquad (3.41)$$

Σ^- is characterized by $x^1 \triangleq 0$; Σ^+ by $x^1 \triangleq \varepsilon$. Integrating from Σ^- to Σ^+,

$$\int_0^{\varepsilon} F^{\mu\nu}{}_{|\nu}\, dx^1 = \int_0^{\varepsilon} J^{\mu}\, dx^1 \qquad (3.42)$$

$$F^{\mu\nu}{}_{|\nu} = F^{\mu\nu}{}_{,\nu} + F^{\alpha\nu}\Gamma^{\mu}_{\alpha\nu} + F^{\mu\alpha}\Gamma^{\nu}_{\alpha\nu} \quad . \tag{3.43}$$

The last two terms on the right are finite as $\varepsilon \to 0$, so (3.42) is equivalent to

$$\int_0^{\varepsilon} F^{\mu\nu}{}_{,\nu} dx^1 = \int_0^{\varepsilon} J^{\mu} dx^1 \tag{3.44}$$

$$\longleftrightarrow \int_0^{\varepsilon} (F^{\mu 1}{}_{,1} + F^{\mu 2}{}_{,2} + F^{\mu 3}{}_{,3} + F^{\mu 4}{}_{,4}) dx^1 = \int_0^{\varepsilon} J^{\mu} dx^1 \quad . \tag{3.45}$$

The last three terms on the left are all finite across the surface - the field and hence its tangential derivatives are finite through the surface - therefore, their integrals tend to zero.

$$\therefore \quad \int_0^{\varepsilon} F^{\mu 1}{}_{,1} \, dx^1 = \int_0^{\varepsilon} J^{\mu} dx^1 \tag{3.46}$$

$$\longrightarrow \quad [F^{\mu 1}] = \int_0^{\varepsilon} J^{\mu} dx^1 = j^{\mu} \tag{3.47}$$

where $[F^{\mu 1}]$ means $F^{\mu 1}$ evaluated at Σ^{+} minus $F^{\mu 1}$ evaluated at Σ^{-}. Since $F^{\mu\nu}$ is skew symmetric, $j^1 = 0$; therefore $j^{\mu} = j^a e^{\mu}$ $(a=2,3,4)$.

Also, in this co-ordinate system

$$n_{\alpha} \stackrel{*}{=} (1,0,0,0) \quad . \tag{3.48}$$

$$\therefore \quad [F^{\mu\alpha} n_{\alpha}] = j^{\mu} \quad . \tag{3.49}$$

$e_{\mu(a)}$ are components of unit tangent vectors (dual base vectors) to the hypersurface. Multiplying both sides of (3.49) by $e_{\mu(a)}$ yields

$$[F^{\mu\alpha}e_{\mu(a)}n_{\alpha}] = j^{\mu}e_{\mu(a)} \quad . \tag{3.50}$$

In this form of the equations, both sides are 4-scalars which can be evaluated in arbit y 4-dimensional co-ordinate systems.

§3.3 The Shell in Motion: Equations of Motion; Continuity Equation

The concepts introduced in §3.2 will be used to determine the equations of motion of a collapsing shell and to derive a continuity equation describing the energy flow from the surface layer.

The jump of (3.31) across the hypersurface is

$$-2[G_{\alpha\beta}n^{\alpha}n^{\beta}] = (-K_{ab}K^{ab}+K^2)|^{+} - (-K_{ab}K^{ab}+K^2)|^{-} \quad \text{from (3.31)}$$

$$= [K_{ab}]\tilde{K}_{ab} + [K]\tilde{K} \qquad (\text{where } \tilde{K}_{ab} = \tfrac{1}{2}(K^{+}_{ab} + K^{-}_{ab}))$$

$$= -\gamma_{ab}\tilde{K}^{ab} + \gamma\tilde{K}$$

$$= -(\gamma_{ab} - \gamma g_{ab})\tilde{K}^{ab}$$

$$= 8\pi S_{ab}\tilde{K}^{ab} \quad . \tag{3.51}$$

By comparing (3.51) with $G_{\alpha\beta} = -8\pi T_{\alpha\beta}$,

$$S^{ab}\tilde{K}_{ab} = [T_{\alpha\beta}n^{\alpha}n^{\beta}] \quad . \tag{3.52}$$

From (3.29), (3.32), (3.33)

$$S^{b}_{a|b} = -[T_{\alpha\beta}e^{\alpha}_{(a)}n^{\beta}] \quad . \tag{3.53}$$

An ideal fluid shell has a surface energy tensor

$$S^{ab} = (\sigma + P)u^{a}u^{b} + Pg^{ab} \tag{3.54}$$

where σ is the surface energy density; P, the surface pressure; and u^{a}, the 4-velocity of an observer on the surface.

$$(3.52),(3.54) \longrightarrow \tilde{K}_{ab}u^{a}u^{b} = \frac{1}{(\sigma + P)}([T_{\alpha\beta}n^{\alpha}n^{\beta}] - P\tilde{K}) \tag{3.55}$$

$$(3.33),(3.54) \longrightarrow \gamma_{ab}u^{a}u^{b} = -8\pi(P + \frac{\sigma}{2}) \quad . \tag{3.56}$$

The 4-acceleration of an element of the shell is (from 3.25),

$$\frac{\delta u^{\alpha}}{\delta\tau} = e^{\alpha}_{(i)}u^{i}_{|j}u^{j} - n^{\alpha}K_{ij}u^{i}u^{j} \quad , \tag{3.57}$$

Since motions of the shell are in a direction normal to the shell

$$n_{\alpha}\frac{\delta u^{\alpha}}{\delta\tau} = -u^{i}u^{j}K_{ij} \quad . \tag{3.58}$$

from which,

$$n_\alpha \frac{\delta u^\alpha}{\delta\tau}\bigg|^+ - n_\alpha \frac{\delta u^\alpha}{\delta\tau}\bigg|^- = -u^a u^b \gamma_{ab} \tag{3.59}$$

$$n_\alpha \frac{\delta u^\alpha}{\delta\tau}\bigg|^+ + n_\alpha \frac{\delta u^\alpha}{\delta\tau}\bigg|^- = -2u^a u^b \tilde{K}_{ab} \quad . \tag{3.60}$$

From (3.55), (3.56), (3.59), (3.60)

$$[n_\alpha \frac{\delta u^\alpha}{\delta\tau}] = 8\pi(P + \frac{\sigma}{2}) \tag{3.61}$$

$$n_\alpha \frac{\delta u^\alpha}{\delta\tau} = \frac{1}{(\sigma+P)}(P\tilde{K} - [T_{\alpha\beta}n^\alpha n^\beta]) \quad . \tag{3.62}$$

Either (3.32) or (3.61) and (3.62) may be used to derive the equations of motion of the shell. Multiplying (3.53) by u^a yields a continuity equation:

$$(\sigma u^a)_{|a} + P u^a_{\ |a} = [T_{\alpha\beta}u^\alpha n^\beta] \quad . \tag{3.63}$$

This equation will be evaluated for a charged, radiating shell later.

CHAPTER IV

SPHERICAL SHELLS

§4.1 Introduction

The equations which were derived in the last chapter will be applied to the case of a spherical fluid shell. We then allow the shell to have charge and allow for the possibility of radiation. In other words, the shell is embedded in a radiating Reissner-Nordström background. The equations of motion of the shell in such a background are determined, and the qualitative features of the motion of the collapsing shell are discussed using the Graves-Brill diagram introduced in §2.4.

§4.2 Equations of Motion for a Spherical Shell

A shell of radius $R(\tau)$ has metric

$$ds_{\Sigma}^{2} = R^{2}(\tau)\,d\Omega^{2} - d\tau^{2} \tag{4.1}$$

with components

$$g_{ij} = \text{diag}\,(R^{2}, R^{2}\sin^{2}\theta, -1)\ . \tag{4.2}$$

The intrinsic co-ordinates are

$$\xi^{a} \equiv (\theta, \phi, \tau) \tag{4.3}$$

$$\begin{cases} u^a = (0,0,1) \quad ; \quad u_a = (0,0,-1) \\ u^a u_a = -1 \quad . \end{cases} \tag{4.4}$$

For the background space-time metric, we shall use the general form of a spheri-symmetric radiating metric which was introduced in §2.5.

$$ds^2 = -F(r,t)dt^2 - 2drdt + r^2 d\Omega^2 \tag{4.5}$$

with components (2.32) and (2.33).

On the shell,

$$r = R(\tau)$$

$$t = t(\tau)$$

$$ds^2 = -Fdt^2 - 2dt\dot{R}d\tau + R^2 d\Omega^2 \quad . \tag{4.6}$$

But also

$$ds^2 = R^2 d\Omega^2 - d\tau^2 \quad . \tag{4.7}$$

Thus, equating (4.6) and (4.7),

$$d\tau^2 = Fdt^2 + 2dtd\tau\dot{R} \quad . \tag{4.8}$$

Dividing by $d\tau^2$ and solving for $dt/d\tau$:

$$\frac{dt}{d\tau} = \frac{-\dot{R} \pm \sqrt{\dot{R}^2 + F}}{F} \quad . \tag{4.9}$$

The unit tangent and normal vectors to the hypersurface are

$$u^\alpha = (\dot{R}(\tau), 0, 0, X) \quad , \quad u_\alpha = (-X, 0, 0, -FX-\dot{R})$$

$$n_\alpha = (X, 0, 0, -\dot{R}) \quad , \quad n^\alpha = (FX+\dot{R}, 0, 0, -X) \tag{4.10}$$

where X is determined as follows:

$$g_{\alpha\beta}u^\alpha u^\beta = -1 \tag{4.11}$$

$$\longrightarrow \quad FX^2 + 2X\dot{R} = -1 \quad . \tag{4.12}$$

Solving for X

$$X = \frac{-\dot{R} \pm \sqrt{\dot{R}^2 + F}}{F} = \frac{-\dot{R} + r_{,\alpha}n^\alpha}{F} \quad . \tag{4.13}$$

Comparing with (4.9), we see that

$$X = \frac{dt}{d\tau} = \frac{-\dot{R} \pm \sqrt{\dot{R}^2 + F}}{F} \quad . \tag{4.14}$$

If we assume that the shell is an ideal fluid shell with surface energy tensor

$$S_{ij} = (\sigma + P)u_i u_j + Pg_{ij} \tag{4.15}$$

the equations of motion may be found either directly from the Lanczos equations or from (3.61).

From the Lanczos equations,

$$\gamma_{ij} = K^{+}_{ij} - K^{-}_{ij} = -8\pi (S_{ij} - \frac{1}{2} g_{ij} S) \tag{4.16}$$

$$S = g^{ij} S_{ij}$$

$$= (\sigma + P) u_i u^i + g^{ij} g_{ij} P$$

$$= 2P - \sigma \tag{4.17}$$

$$\therefore \quad \gamma_{ij} = -8\pi ((\sigma + P) u_i u_j + \frac{\sigma}{2} g_{ij})$$

$$= K^{+}_{ij} - K^{-}_{ij} \quad . \tag{4.18}$$

The components of the extrinsic 3-curvature are calculated as follows:

$$K_{\theta\theta} = e^{\alpha}_{(\theta)} \frac{\delta n_{\alpha}}{\delta \theta} \tag{4.19}$$

$$e^{\alpha}_{(\theta)} = \frac{\partial x^{\alpha}}{\partial \theta} = \delta^{\alpha}_{2} \tag{4.20}$$

$$\frac{\delta n_{\alpha}}{\delta \theta} = -n_{\mu} \Gamma^{\mu}_{\alpha\beta} \frac{\partial x^{\beta}}{\partial \theta} = -n_{\mu} \Gamma^{\mu}_{\alpha 2} \tag{4.21}$$

$$\therefore \quad K_{\theta\theta} = -n_{\mu} \Gamma^{\mu}_{22} = -n^{\mu} \Gamma_{\mu,22} \quad . \tag{4.22}$$

The only non-vanishing component is $-n^{1}\Gamma_{1,22}$

$$\therefore \quad K_{\theta\theta} = -n^{1} \Gamma_{1,22} = \frac{1}{2} n^{1} \frac{\partial g_{22}}{\partial r} = n^{1} R$$

$$= (r_{,\alpha} n^{\alpha}) R \quad . \tag{4.23}$$

Similarly,

$$K_{\phi\phi} = e^{\alpha}_{(\phi)} \frac{\delta n_{\alpha}}{\delta\phi} = R \sin^2\theta (r_{,\alpha} n^{\alpha}) \qquad (4.24)$$

$$K_{\tau\tau} = u^{\alpha} \frac{\delta n_{\alpha}}{\delta\tau} = -n_{\alpha} \frac{\delta u^{\alpha}}{\delta\tau} \quad . \qquad (4.25)$$

The equations of motion of the shell follow immediately from (4.16), (4.23), (4.25).

$$\gamma_{\theta\theta} = K^{+}_{\theta\theta} - K^{-}_{\theta\theta} = R[r_{,\alpha} n^{\alpha}]$$

$$= -8\pi((\sigma+P)0 + \frac{\sigma}{2} R^2) = -4\pi R^2 \sigma \quad . \qquad (4.26)$$

$$\gamma_{\tau\tau} = K^{+}_{\tau\tau} - K^{-}_{\tau\tau} = -8\pi(P + \frac{\sigma}{2})$$

$$= -[n_{\alpha} \frac{\delta u^{\alpha}}{\delta\tau}] \quad . \qquad (4.27)$$

The first of these equations,

$$R[r_{,\alpha} n^{\alpha}] = -4\pi R^2 \sigma$$

is very useful in characterizing the motion of a collapsing shell, and will be often referred to later.

§4.3 A Shell in a Radiating Reissner-Nordström Background

In §2.5, we introduced the general form of a spheri-symmetric metric which could be applied to a charged, radiating shell. The non-vanishing components of the energy tensor and the Einstein tensor were computed. The qualitative features of the motion of such a shell are examined.

Since the shell is radiating, we will look first at the continuity equation for the shell;

$$(\sigma u^j)_{|j} + Pu^j_{\ |j} = [T_{\alpha\beta} u^\alpha n^\beta] \quad . \tag{4.28}$$

The first term on the left is the rate of increase of the total surface energy/unit area. The second term is the surface pressure times the rate of expansion. The right side of the equation is the radial energy flow from the shell. This interpretation will be clear when (4.28) is evaluated explicitly:

$$u^j_{\ |j} = \frac{1}{\sqrt{-g}}\, \partial_j (\sqrt{-g}\, u^j) = \frac{2\dot{R}}{R} \tag{4.29}$$

$$(\sigma u^j)_{|j} = \frac{1}{R}\, (\sigma R)^{\cdot} + \frac{\sigma \dot{R}}{R} \tag{4.30}$$

$$\begin{aligned} [T_{\alpha\beta} u^\alpha n^\beta] &= [T_\alpha^{\ \beta} u^\alpha n_\beta] \\ &= [T^1_4 X^2] \\ &= \frac{-1}{8\pi r}\, [F_t X^2] \quad . \end{aligned} \tag{4.31}$$

Putting the equation together yields

$$\frac{1}{R}\, (\sigma R)^{\cdot} + \frac{\sigma \dot{R}}{R} + \frac{2P\dot{R}}{R} = \frac{-1}{8\pi R}\, [X^2 F_t]$$

$$\longleftrightarrow \quad \frac{d}{d\tau}(4\pi R^2\sigma) + P\frac{dA}{d\tau} = \frac{1}{2}R[X^2F_t] \qquad (A \equiv 4\pi r^2) \ . \tag{4.32}$$

Now consider a spherical shell, charge $e(t)$, mass $m(t)$ falling in an external field due to a spherical distribution of gravitational mass $m_1(t)$, and charge $e_1(t)$ near its center. We have defined the charge and mass of the shell and its interior as

$$\begin{aligned} e_2 &= e + e_1 \\ m_2 &= [illegible]\ m_1 \ . \end{aligned} \tag{4.33}$$

$r = R(\tau)$ is the equation of the hypersurface $\sum$, and divides space-time into two regions, V_+ and V_- with metrics

$$ds^2 = -F_{\underset{+}{-}}\, dt^2 - 2dtdr + r^2 d\Omega^2 \qquad \begin{matrix} r < R(\tau) \\ r > R(\tau) \end{matrix} \tag{4.34}$$

$$F_- = (1 - \frac{2m_1(t)}{r} + \frac{e_1^2(t)}{r^2}) \tag{4.35}$$

$$F_+ = (1 - \frac{2m_2(t)}{r} + \frac{e_2^2(t)}{r^2}) \ . \tag{4.36}$$

Recall that an equation of motion for a spherical shell is given by

$$R[\dot{r}_{,\alpha}n^\alpha] = -4\pi R^2\sigma$$

$$\longleftrightarrow \quad R[r_{,\alpha}n^\alpha] = -M \qquad (\text{or} = -M(t)) \tag{4.37}$$

where M is the total proper mass of the shell, defined as

$$M = 4\pi R^2\sigma = 4\pi R^2 S^4_4 = [illegible]^2 \int T^4_4 dx^1 \quad . \tag{4.38}$$

Recall also that $[r_{,\alpha}n^\alpha]$ represents the jump in $\sqrt{\dot{R}^2 + F}$ across the shell. (4.37) may be written as

$$\frac{M}{R} = \sqrt{\dot{R}^2 + F_-} - \sqrt{\dot{R}^2 + F_+} \quad . \tag{4.39}$$

$$\longrightarrow \quad \left(\frac{M}{R} - \sqrt{\dot{R}^2 + F_-}\right)^2 = \dot{R}^2 + F_+ \tag{4.40}$$

$$\longrightarrow \quad \pm\sqrt{\dot{R}^2 + F_-} = \frac{m_2 - m_1}{M} + \frac{M^2 + e_1^2 - e_2^2}{2MR} = r_{,\alpha}n^\alpha\big|^- \tag{4.41}$$

$$\pm\sqrt{\dot{R}^2 + F_+} = \frac{m_2 - m_1}{M} - \frac{M^2 + e_1^2 - e_2^2}{2MR} = r_{,\alpha}n^\alpha\big|^+ \quad . \tag{4.42}$$

(4.41) and (4.42) represent the values of the radial component of the normal to the surface at the inner and outer surface of the shell.

We return to (4.37) to get the equation of motion for a dust shell as a quadratic equation:

$$(4.37) \longleftrightarrow \quad r_{,\alpha}n^\alpha\big|^+ - r_{,\alpha}n^\alpha\big|^- = n^1_+ - n^1_- = -\frac{M}{R} \tag{4.43}$$

where $M = 4\pi R^2\sigma$.

$$n_+^1 + n_-^1 = \frac{2(m_2-m_1)}{4\pi R^2\sigma} - \frac{(e_2^2-e_1^2)}{4\pi R^3\sigma} \quad . \tag{4.44}$$

From (4.43) and (4.44) we get

$$M = n_-^1 R - n_+^1 R = \{R^2(1+\dot{R}^2) - 2m_1 R + e_1^2\}^{\frac{1}{2}} - \{R^2(1+\dot{R}^2) - 2m_2 R + e_2^2\}^{\frac{1}{2}} \tag{4.45}$$

$$n_-^1 R + n_+^1 R = \{2R(m_2 - m_1) - (e_2^2 - e_1^2)\}/M \quad . \tag{4.46}$$

Adding (4.45) and (4.46) and squaring the result gives

$$\left\{\begin{array}{l} 1 + \dot{R}^2 = A + \frac{B}{R} + \frac{C}{R^2} \\ A = \frac{(m_2 - m_1)^2}{M^2} \\ B = m_1 + m_2 - (m_2 - m_1)(e_2^2 - e_1^2)/M^2 \\ C = (e_2^2 - e_1^2)^2/4M^2 + \frac{M^2}{4} - (e^2 + e_2^2)/2 \quad [26] \end{array}\right. \tag{4.47}$$

This equation is the relativistic analogue of the Newtonian equation which expresses energy conservation of the system:

$$\left\{\begin{array}{l} T + U + \frac{(e_2-e_1)^2}{2R} - \frac{(m_2-m_1)^2}{2R} - \frac{(m_2-m_1)m_1}{R} = \text{const.} \\ dU = -P d(4\pi R^2) \\ T = \text{kinetic energy} \\ U = \text{thermal energy} \quad . \end{array}\right. \tag{4.48}$$

§4.4 Some Qualitative Features of the Motion of a Collapsing Shell

We will now examine the Graves-Brill diagram of fig. 7 more closely. The following conditions are important in determining the path of the collapsing shell on the diagram:

(1) The normal is directed perpendicularly to the collapsing shell, from the inside to the outside.

(2) $\dot{R} < 0$ means that the shell is moving inwards.

(3) $r_{,\alpha}n^{\alpha} > 0$ implies that the radius increases as one moves outwards from the shell. This quantity is just n^1, the radial component of n^{α}, which is given by

$$n^1 = FX + \dot{R} = \pm\sqrt{\dot{R}^2 + F} \; .$$

(4) The path of the shell must be time-like during the collapse.

(5) Regions of constant radius are variously time-like and space-like in different portions of the Graves-Brill diagram.

For curves $r, \theta, \phi = \text{const.}$,
$ds^2 = g_{44}dT^2 < 0$ implies that such curves are time-like. If $ds^2 > 0$, the curves are space-like.

$$\therefore \quad (1 - \frac{2m}{r} + \frac{e^2}{r^2}) < 0 \longrightarrow \text{space-like curves}$$

$$\longrightarrow \quad (r-r_-)(r-r_+) < 0 \longrightarrow \text{space-like curves}$$

where r_- is the inner event horizon and r_+, the outer. This condition holds for $r_- < r < r_+$. In the cases where $r > r_+$ and where $r < r_-$, the curves r, θ, ϕ = const. are time-like. Since $\underline{\nabla} r$ is orthogonal to curves of constant r, and since $\underline{n}$ is orthogonal to and directed outwards from the path of collapse, fig. 8 can be sketched to show some possible collapse paths for the shell.

(6) For paths 1 and 2, fig. 8, we shall find a condition on the turning points of the motion: $r_{,\alpha} n^\alpha = \pm\sqrt{\dot{R}^2 + F}$ is real since $r_{,\alpha} n^\alpha$ is the radial normal component. At a turning point,

$$\dot{R} = 0 \longrightarrow F > 0$$

$$\longleftrightarrow (1 - \frac{2m}{r} + \frac{e^2}{r^2}) > 0$$

$$\longleftrightarrow (r - (m - \sqrt{m^2 - e^2}))(r - (m + \sqrt{m^2 - e^2})) > 0$$

$$\longleftrightarrow (r-r_-)(r-r_+) > 0$$

is a condition on the turning points. What this means is that curves r = const. must be time-like at a stationary point and no such point

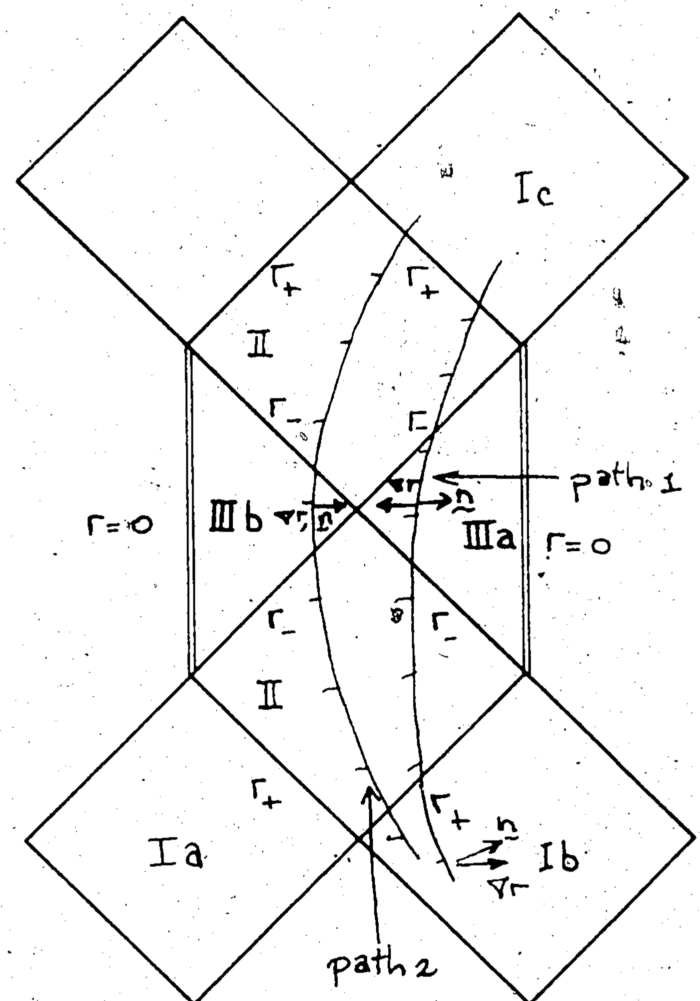


Figure 8. Two collapse paths on a Graves-Brill diagram. This is an illustrative case to show collapse from "external spacetime", Ib, across both event horizons, and re-expansion in a distinct "external spacetime" Ic.

exists in the range $r_- < r < r_+$. Thus, the possibility of having a turning point and a "gravitational bounce" or re-expansion of the shell exists in regions IIIa and IIIb. These cases will be considered in the next chapter.

CHAPTER V

A CHARGED DUST SHELL IN A STATIC REISSNER-NORDSTRÖM BACKGROUND

§5.1 Introduction

A space-time is called stationary if a system of co-ordinates exists so that $g_{\mu\nu,4} = 0$, $\mu,\nu = 1,2,3,4$. If also $g_{i4} = 0$, $i = 1,2,3$, so that the metric is cast into the form

$$ds^2 = g_{ij}dx^i dx^j + g_{44}(dx^4)^2 \qquad (5.1)$$

the space-time is called static. Any spherically symmetric field in vacuo is static. In this chapter, we consider the collapse of a dust shell, without radiation, in such a static background metric. A proof will be given to show that a charged shell collapsing onto an uncharged black hole cannot "unlock" the black hole by collapsing to a singularity which is not shielded by an event horizon.

§5.2 Collapse of an Empty Shell [27]

Consider an empty shell with mass m, charge e, for which

$P = 0$

$e_1 = m_1 = 0$

proper mass M = const.

An equation of motion is obtained from (4.39)

$$R \left(\sqrt{1 - \frac{2m}{R} + \frac{e^2}{R^2} + \dot{R}^2} - \sqrt{1 + \dot{R}^2} \right) = -M$$

$$\longleftrightarrow \quad \frac{m}{M} + \frac{M^2 - e^2}{2MR} = \sqrt{1 + \dot{R}^2} \quad . \qquad (5.2)$$

Let us look at the Newtonian analogue of this equation. Solving for m:

$$m = M\sqrt{1 + \dot{R}^2} - \frac{M^2 - e^2}{2R}$$

$$\simeq M + \frac{1}{2} M\dot{R}^2 - \frac{1}{2}\frac{M^2}{R} + \frac{1}{2}\frac{e^2}{R} \quad . \qquad (5.3)$$

This is simply a statement that the total energy, m, equals the rest mass plus thermal energy of the shell, M, plus the kinetic energy, $\frac{1}{2}M\dot{R}^2$, plus the gravitational potential energy, $-\frac{1}{2}\frac{M^2}{R}$, plus the electrostatic field energy, $\frac{1}{2}\frac{e^2}{R}$.

Define a potential function U(R) such that

$$1 + U(R) = \frac{m}{M} + \frac{M^2 - e^2}{2MR} = \sqrt{1 + \dot{R}^2} \geq 1 \quad . \qquad (5.4)$$

If $m > M > |e|$, there are no turning points, and the collapse is irreversible. This is evident from a graph plotting U(R) versus R for $\dot{R} = 0$ (fig. 9).

Case 1 : $m > M > |e|$

In this case, there are two event horizons (real solutions to $1 - \frac{2m}{r} + \frac{e^2}{r^2} = 0$), and no turning points. From (5.2), setting $\dot{R} = 0$ yields

$$R_{turn} = \frac{M^2 - e^2}{2(M-m)} < 0 \quad . \tag{5.5}$$

Hence, the shell collapses to zero radius. The normal to the shell

$$n^1_+ = r_{,\alpha} n^\alpha \Big|^+ = \frac{m}{M} - \frac{M^2 + e^2}{2MR} \tag{5.6}$$

changes sign at $R = M^2 + e^2/2M$, which lies between event horizons

$$\begin{cases} r_+ = m + \sqrt{m^2 - e^2} \\ r_- = m - \sqrt{m^2 - e^2} \end{cases} .$$

Since the normal is positive in region IIIb, and negative in IIIa, the shell must pass into IIIa. The collapse path is shown as path 1 of fig.10.

Case 2 : $m > M \quad |e| > M$

Again from (5.2), setting $\dot{R} = 0$,

$$R_{turn} = \frac{M^2 - e^2}{2(M-m)} \quad . \tag{5.7}$$

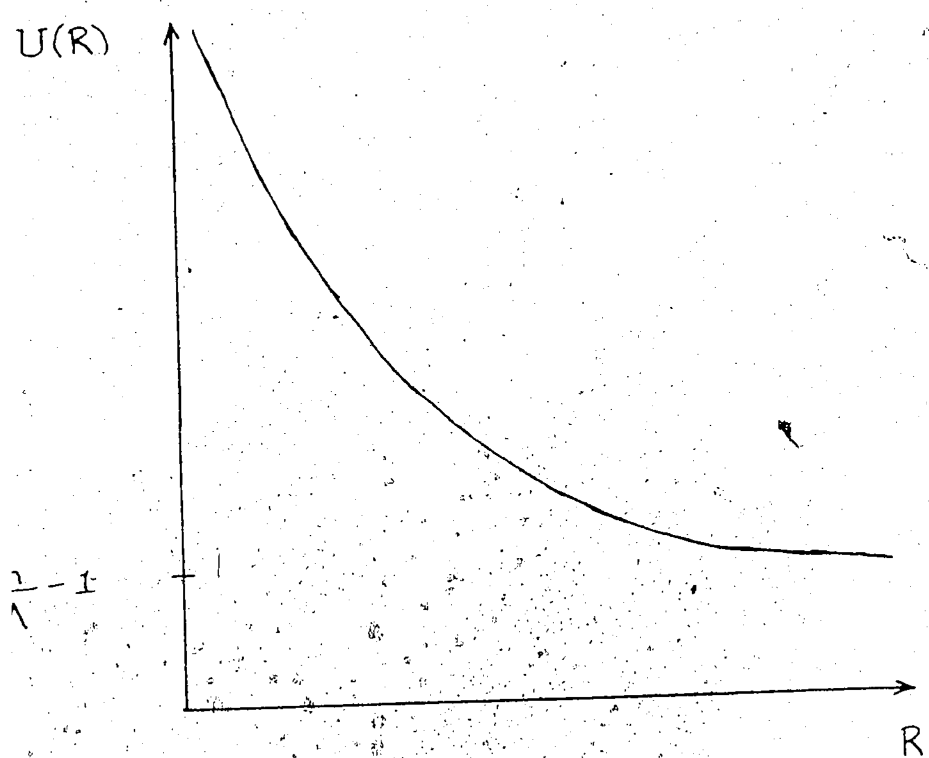


Figure 9

and $$m = M + \frac{(e^2-M^2)}{2R_{turn}} > 0 \quad . \tag{5.8}$$

Now we have three subcases to consider:

a) $m > |e| > M$

In this case, from (5.8), $0 < R_{turn} < M+|e|/2$. The turning point lies inside the inner event horizon since

$$r_- - R_{turn} = m - \sqrt{m^2-e^2} - \frac{M^2 - e^2}{2(M-m)} > 0$$

$$\longleftrightarrow \quad \frac{1}{2} e^2M^2 - mM(M^2-e^2) + m^2M^2 + \frac{M^4+e^4}{4} > 0 \tag{5.9}$$

which is true for $m > |e| > M$. (Henceforth, $R_{turn} \equiv R_T$).

If $R_T > M$, the radial component of the normal at the turning point is positive:

$$n^1(R_T) = \frac{m}{M} - \frac{M^2+e^2}{2R_T} = \frac{M^2+e^2-M^2/2R_T}{M} - \frac{M^2+e^2}{2MR_T} = 1 - \frac{M}{R_T}. \tag{5.10}$$

The shell passes through region IIIb and re-emerges in I(path 2).

If $R_T < M$, $n^1(R_T) < 0$, and the shell passes through IIIa (path 3).

If $R_T = M$, $n^1(R_T) = 0$ at the point where regions IIIa and IIIb meet. The shell passes through this point and re-expands into I (path 4).

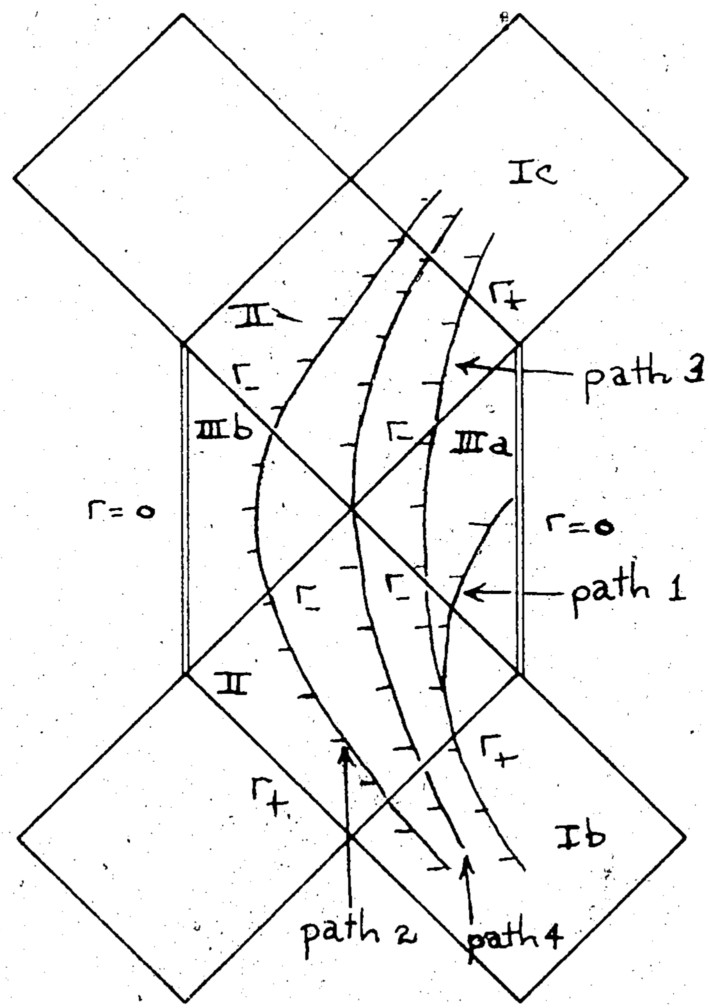


Figure 10. Various collapse paths. The shaded area denotes the interior side of the shell

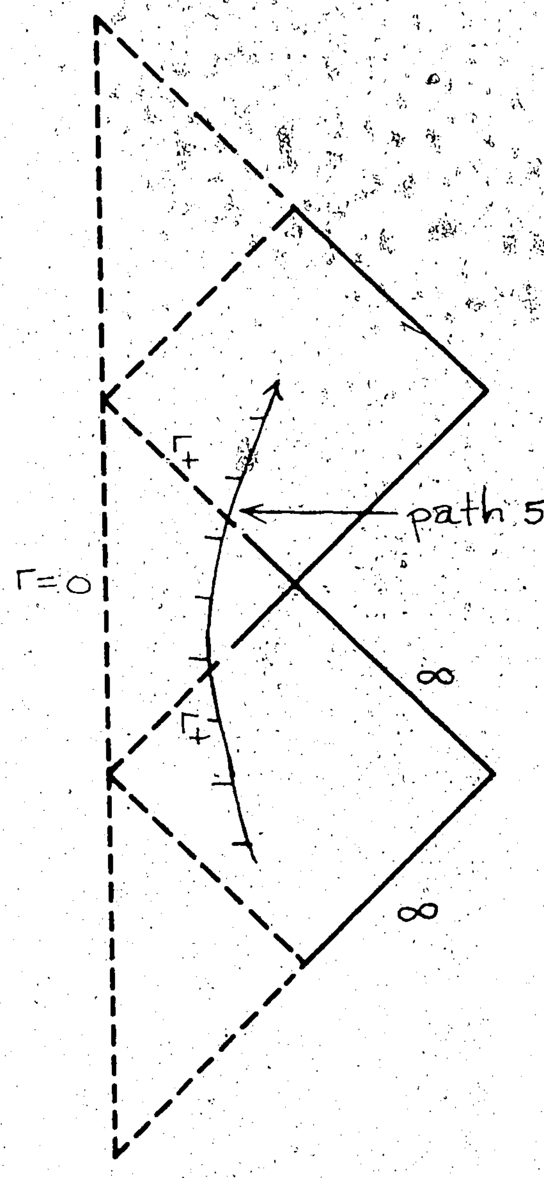


Figure 11. A modified Graves-Brill diagram for $|e| = m$. Path 5 shows the collase path for Case 2c.

b) $|e| > m > M$

In this case, there is no event horizon. The shell collapses to R_T and "bounces" by re-expanding.

c) $(m = |e|) > M$

In this case, there is an event horizon at $r_+ = m$, and a real turning point at

$$R_T = \frac{M^2 - m^2}{2(M-m)} = \frac{M+m}{2} = \frac{M+|e|}{2} \quad . \tag{5.11}$$

This turning point lies inside the event horizon since

$$r_+ - R_T = m - \frac{M+m}{2} > 0 \tag{5.12}$$

and the normal at the turning point is positive. A path for the shell is shown as path 5 in fig. 11. This figure is a modified Graves-Brill diagram for which $|e| = m$.

Case 3 : $M > m$

Again we consider three subcases:

a) $M > m > |e|$

In this case, there are event horizons, but no turning point. The solution for the turning point represents the initial radius from which the collapse begins. This radius lies outside the outer event horizon:

$$R_T = \frac{M^2 - e^2}{2(M-m)} > 0$$

$$= \frac{m^2 - M^2}{2(M-m)} + R_T > 0$$

$$= R_T - m - \frac{(M-m)}{2} > 0$$

$$\therefore \quad R_T - m > 0 \; .$$

Also, $(R_T - m)^2 - (m^2 - e^2) = \frac{(M-M^2-e^2)}{2(M-m)} > 0$

$$\therefore \quad R_T - m > \sqrt{m^2 - e^2}$$

$$R_T > m + \sqrt{m^2 - e^2} = R_+ \qquad . \tag{5.13}$$

At the initial values of collapse, R_i,

$$m = \text{total energy} = M + \frac{e^2}{2R_i} - \frac{M^2}{2R_i} \tag{5.14}$$

$$R_T = \frac{M^2 - e^2}{2(M-m)} = \frac{M^2 - e^2}{2\left(M - M - \frac{e^2}{2R_i} + \frac{M^2}{2R_i}\right)} = R_i \; . \tag{5.15}$$

Now we must determine in which region the collapse begins. From (5.8), we see that for $R_T > M+|e|/2$, $M > m > e$. If $R_T > M$, then $R_T > R_+$ and $n^1(R_T) > 0$. The collapse thus begins in a Ib region. n^1 changes sign at $M^2+e^2/2M$ which lies between the event horizons, and the collapse proceeds to $r = 0$ in the IIIa region. An

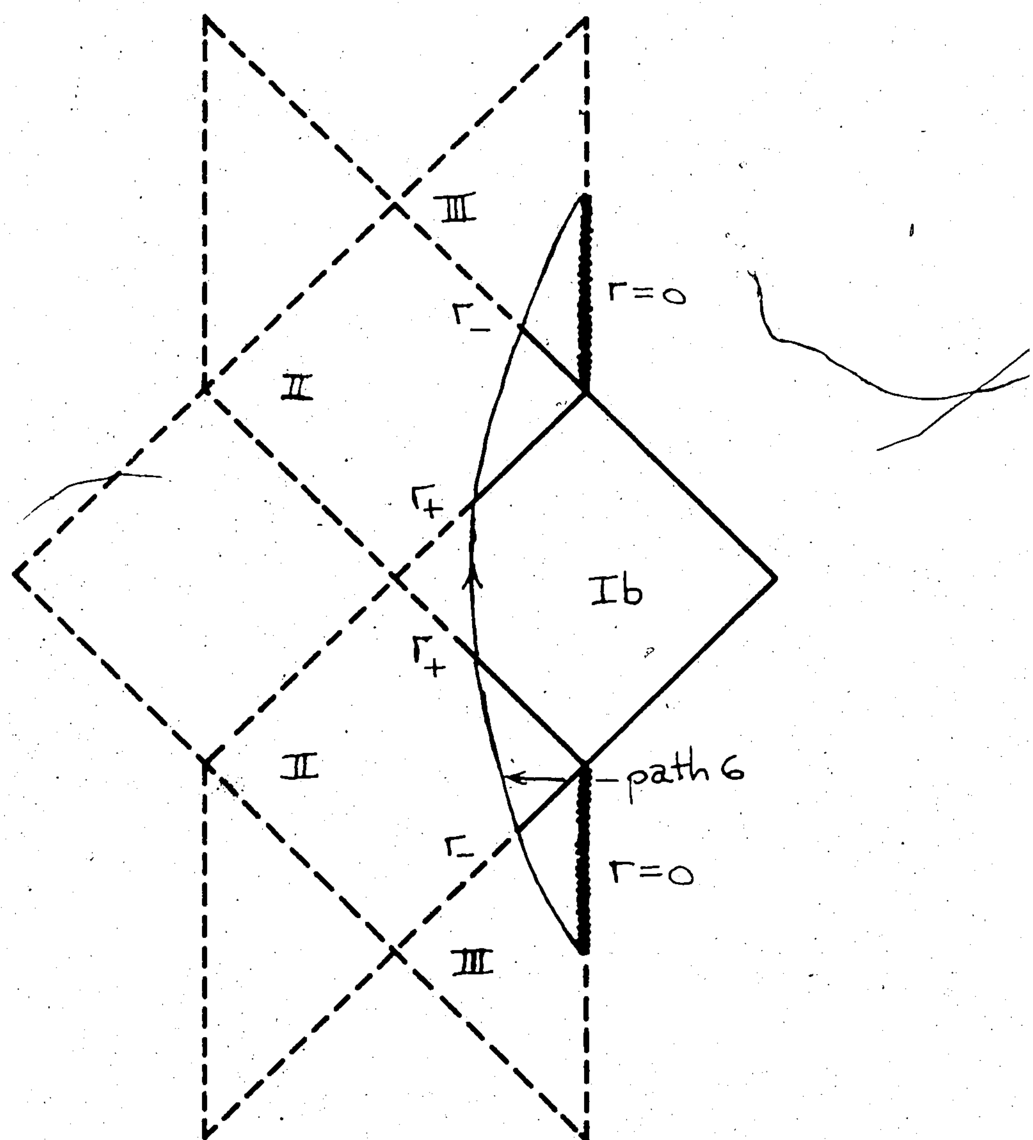


Figure 12. Collapse path for Case 3a. The shell explodes from a singularity, emerges into an "external spacetime", and collapses to a singularity.

interesting way of looking at this case is to consider that the shell has expanded f om a point, producing an external singularity at r 0. It briefly expands into an exterior space, and a R_T, it again collapses to a point in IIIa (path 6, fig. 12).

If $M > R_T > M + |e|/2$, $R_T > R_+$ and $n^1(R_T) < 0$. The collapse begins in the Ia region and proceeds into IIa and IIIa.

b) $M > |e| > m$

In this case, $0 \leq R_T < M+|e|/2$ and the shell starts from rest and collapses to $R = 0$ with $\dot{R} = -\infty$. This case is not physically realizable, however, since n^1_+ is less than zero for values of R less than $M^2+e^2/2m$, which is less than R_i. This means that the radius decreases along the outward pointing normal, which is clearly not the case in our external space.

c) $M > (m = e)$

There is one event horizon at $R_+ = m$. The turning point, $R_T = M+m/2$ lies outside R_+ and represents the initial collapse radius. Again, however, $n^1(R_T) < 0$.

§5.3 Can a Black Hole be Unlocked? Collapse of a Charged Shell with Interior Mass

Consider a shell with mass m and charge e for which

$P = 0$

$e_1 = 0$

$m_1 =$ interior mass at the shell's centre

proper mass $M =$ const.

$m_2 = m + m_1$.

We return to (4.47) to use the quadratic form of the equations of motion, which, in this case become

$$(R\dot{R})^2 = (\theta^2 - 1)R^2 + BR + C \tag{5.16}$$

$$\theta \equiv \frac{m_2 - m_1}{M} \tag{5.17}$$

$$B \equiv m_1 + m_2 - \frac{(m_2 - m_1)e_2^2}{M^2} \tag{5.18}$$

$$4C \equiv \frac{(e_2^2 - M)^2}{M} \quad . \tag{5.19}$$

From (4.42), the normal to the outer surface of the shell is given by

$$r_{,\alpha}n^\alpha\Big|^+ = \pm\sqrt{\dot{R}^2 + 1 - \frac{2m_2}{R} + \frac{e_2^2}{R^2}} = \theta - \frac{(M^2 + e_2^2)}{2MR} \quad . \tag{5.20}$$

If we consider the interior mass, m_1, to represent a black hole with an event horizon at $R = 2m_1$, the collapsing shell would "unlock" the black hole if it could collapse to a naked singularity without forming an event horizon. The following theorem shows that this is not possible:

<u>Theorem</u>: There are no solutions to (5.16) with collapse to R = 0 satisfying

$$\begin{cases} e_2 > m_2 \geq m_1 \geq 0 \quad , \quad e_1 = 0 & (5.21) \\ r_{,\alpha} n^{\alpha} \Big|_{R_o}^{+} > 0 \quad . & (5.22) \end{cases}$$

The condition $|e_2| > m_2$ assures that the total charge of the black hole and shell is greater than the total mass of the black hole and shell, so no event horizon will form during the collapse. The condition of a positive radial component of the outward pointing normal, $r_{,\alpha} n^{\alpha} > 0$, is necessary for any collapse without an event horizon. The proof of the theorem depends on three cases:

$$\begin{cases} \theta < 1 \\ \theta = 1 \\ \theta > 1 \end{cases}$$

(i) $\theta < 1$ (positive binding energy) (fig.13)

Let $a \equiv \dfrac{M^2 - e_2^2}{4Mm_1}$

$x = \dfrac{2m_1}{R_o}$ R_o is the radius for which $\dot{R} = 0$

$(R_o \equiv R_i \equiv R_T)$

$\beta \equiv \dfrac{m_1}{M}$

$$(5.16) \longleftrightarrow (\theta^2 - 1)R_o^2 + BR_o + C = 0$$

$$\longleftrightarrow (\theta^2 - 1)\frac{1}{x^2} + (1 + 2a\theta)\frac{1}{x} + a^2 = 0$$

$$\longrightarrow x = \frac{-(1 + 2a\theta) + \sqrt{1 + 4a\theta + 4a^2}}{2a^2} \qquad (5.23)$$

Rewriting (5.20) in terms of a, β, θ and evaluating at R_o,

$$[illegible] \frac{1}{1 + 2a\theta + \sqrt{1 + 4a\theta + 4a^2}} E(a,\beta,\theta) \qquad (5.24)$$

$$E(a,\beta,\theta) = \frac{\theta^2 - 1}{\beta} + \theta(1 + \sqrt{1 + 4a\theta + 4a^2} + 2a) \qquad (5.25)$$

We want to show that $E(a,\beta,\theta) < 0$ for all a, β, θ satisfying (5.21). For fixed values of β, θ, E is an increasing function of a and hence is largest when a is largest compatible with (5.21).

$$a_{max} = \frac{M^2 - m_2^2}{4Mm_1} = \frac{1 - (\theta + \beta)^2}{4\beta} \qquad (5.26)$$

Thus

$$E(a,\beta,\theta) < E_{max}(\beta,\theta) \equiv E(a_{max},\beta,\theta) \qquad (5.27)$$

$$(5.26),\ (5.27) \longrightarrow$$

$$E_{max}(\beta,\theta) = -\frac{1}{2\beta}(1 - \theta^2 + \beta^2) + \theta\sqrt{1 + 4a\theta + 4a^2} \qquad (5.28)$$

If we can show that E_{max} is negative, then $r_{,\alpha}n^{\alpha}|^{+} < 0$. Multiply (5.28) by the positive expression,

$$+\frac{1}{2\beta}\,(1-\theta^2+\beta^2) + \theta\sqrt{1+4a\theta+4a^2}$$

and set $\beta = y - \theta$:

$$E_{max}\left[\frac{1}{2\beta}\,(1-\theta^2+\beta^2) + \theta\sqrt{1+4a\theta+4a^2}\,\right] =$$

$$-\frac{1}{4(y-\theta)^2}\,(1-\theta^2)(1-2y\theta+y^2)^2 < 0 \qquad \text{for } \theta < 1 \tag{5.29}$$

Thus, $E_{max} < 0$ and hence, $r_{,\alpha}n^{\alpha}| < 0$ for $\theta < 1$, violating (5.22).

(ii) $\theta > 1$.

Collapse to $R = 0$ is only possible in two cases:

a) In (5.16), $B > 0$, $B^2 > 4C(\theta^2-1)$ (fig. 14).
These conditions imply $\theta e_2 \le \sqrt{m_2^2 - m_1^2}$ or that $e_2 < m_2$, which contradicts (5.21).

b) $B^2 \le 4C(\theta^2-1)$ (fig. 15)

Let $B^2 - 4C(\theta^2-1) = Q(\theta^2)/\theta^2$, where $Q(\theta^2)$ is a quadratic in θ^2 defined by

$$Q(\theta^2) = e_2^2\theta^4\left[\frac{e_2^2}{(m_2-m_1)^2} - \frac{2m_1+m_2}{m_2-m_1} + 2\right] +$$

$$\theta^2(4m_1m_2 - 2e_2^2) + (m_2-m_1)^2. \tag{5.30}$$

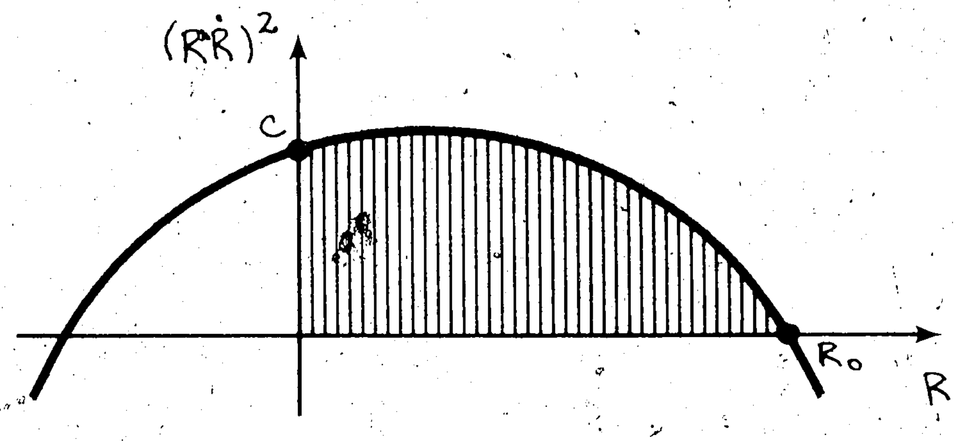


Figure 13. Graph of equation 5.16 for $\theta < 1$.

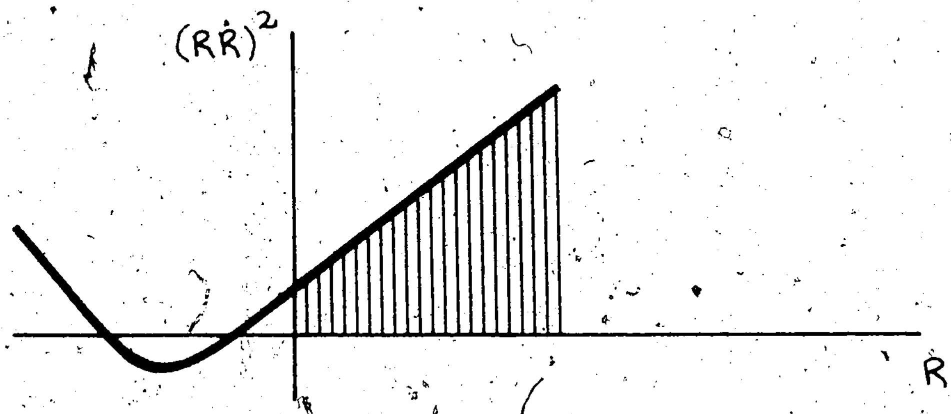


Figure 14. Graph of 5.16 for (ii) a.

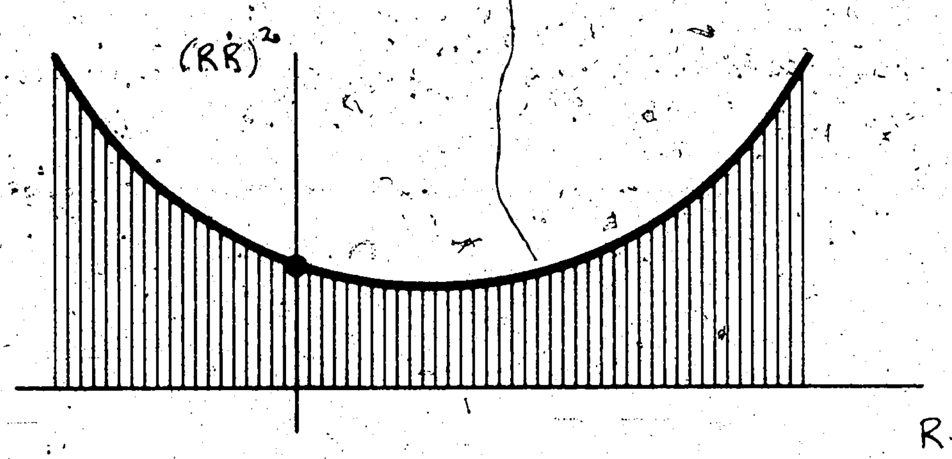


Figure 15. Graph of 5.16 for (ii) b.

The discriminant of $Q(\theta^2)$ is

$$16m_1^2(m_2^2 - e_2^2) < 0 \qquad \text{for} \quad e_2 > m_2 \quad . \qquad (5.31)$$

Hence, $Q(\theta^2) > 0$ for all θ; therefore case (b) is impossible.

iii) $\theta = 1$

In this case, $E_{max} = 0$; hence $r_{,\alpha}n^{\alpha}|^{+} < 0$ contradicting (5.22).

There are thus, no physically realizable situatoons in which a charged infinitesimally thin dust shell collapsing onto an interior black hole "unlocks" the black hole by removing the event horizon and proceeds to form a naked singularity. The theorem supports both Penrose's "Cosmic Censorship" hypothesis [28], and the Hawking statement of the third law of black hole mechanics [29]. The "Cosmic Censorship" hypothesis states that all naked singularities must be clothed by event horizons. The third law, in the form which confirms to the material in this chapter states that it is impossible to convert a black hole with $|e| < m$ into an "extremal" black hole (with $|e| = m$) in a finite number of steps.

CHAPTER VI

A RADIATING DUST SHELL WITH CONSTANT CHARGE

§6.1 Introduction

In this chapter, we shall examine the behaviour of a charged shell which radiates entities such as photons - mass but not charge. The way in which t, m(t), M(t), m(τ) and M(τ) vary from $R > r_+$ to $R < r_-$ will be examined. Finally, the effect of these varying quantities on the collapse path of the shell will be studied.

§6.2 Behaviour of the Total Mass and Proper Mass

A shell with a mass m(t) and a constant charge e will have event horizons set at

$$r_+(t_o) = m(t_o) + (m^2(t_o) - e^2)^{\frac{1}{2}} \qquad (6.1)$$

$$r_-(t_1) = m(t_1) - (m^2(t_1) - e^2)^{\frac{1}{2}} \qquad (6.2)$$

where t_o and t_1 are the retarded times at which the shell crosses the outer and inner horizons. We shall examine the behaviour of t, m'(t), M'(t), m'(τ), and M'(τ) in its regions delimited by the event horizons.

Case 1 : $R \geq r_+$

In this region,

$$F = \left(1 - \frac{2m(t)}{r} + \frac{e^2}{r^2}\right) > 0$$

$$\dot{R} < 0$$

$$r_{,\alpha} n^\alpha = n^1 > 0$$

$$X = -\frac{\dot{R} + r_{,\alpha} n^\alpha}{F} = \frac{dt}{d\tau} > 0 \quad .$$

Since τ is the proper time on the shell, $d\tau > 0$ in all regions. Thus, X and t tend to infinity as R tends to r_+.

The proper mass of the shell as a function of proper time is related to the proper mass as a function of retarded time by

$$\dot{M}'(\tau) = M'(t)\frac{dt}{d\tau} = XM'(t) \quad . \tag{6.3}$$

A shell radiating photons loses proper mass, so the quantity $\dot{M}'(\tau)$ is negative in all regions.

From the continuity equation, (4.32), in the case where P = 0, $m_1 = e_1 = 0$, e = constant,

$$\dot{M}'(\tau) = 4\pi R^2[T^1_4 X^2] = m'(t)X^2 \quad . \tag{6.4}$$

Since

$$m'(\tau) = Xm'(t) \quad , \tag{6.5}$$

$$M'(\tau) = Xm'(\tau) \quad . \tag{6.6}$$

Collecting these equations:

$$M'(\tau) = \begin{cases} XM'(t) < 0 \\ X^2 m'(t) < 0 \\ Xm'(\tau) < 0 \end{cases} \tag{6.7}$$

$X > 0 \longrightarrow$

$$\begin{cases} M'(t) < 0 \\ m'(t) < 0 \\ m'(\tau) < 0 \end{cases} \tag{6.8}$$

<u>Case 2 : $r_- \leq R \leq r_+$</u>

In this region,

$$F < 0$$

$$\dot{R} < 0$$

$$\longrightarrow \quad \pm\sqrt{\dot{R}^2 + F} \pm r_{,\mu} n^{\mu} < |\dot{R}|$$

$$X = -\frac{\dot{R} \pm \sqrt{\dot{R}^2 + F}}{F} < 0$$

$$t \rightarrow -\infty \quad \text{as} \quad R \rightarrow r_-$$

$$\begin{cases} M'(t) > 0 \\ m'(t) < 0 \\ m'(\tau) > 0 \end{cases} \quad (6.9)$$

Case 3 : $R \leq r_-$

There are two subcases to consider:

In region IIIb of the Graves-Brill diagram,

$r_{,\alpha} n^{\alpha} > 0$

$F > 0$

$\dot{R} > 0$

$X > 0 \longrightarrow t \to +\infty$ as $R \to r_-$

$$\begin{cases} M'(t) < 0 \\ m'(t) < 0 \\ m'(\tau) < 0 \end{cases} \quad (6.10)$$

In region IIIa,

$r_{,\alpha} n^{\alpha} < 0$

$F > 0$

$\dot{R} < 0$

$X < 0 \longrightarrow t \to -\infty$ as $R \to r_-$

$$\begin{cases} M'(t) > 0 \\ m'(t) < 0 \\ m'(\tau) > 0 \end{cases} \quad (6.11)$$

From the above calculations, we can sketch the graph of fig. 16. Inside the outer event horizon, t, the retarded time, or, "the external observer's time" has no meaning. However τ, the proper time of a co-moving observer increases towards the future, and is useful in characterizing the motion of the shell. We see from the graph that while the proper mass decreases in all regions, the total mass (gravitational mass), $m(\tau)$, oscillates, and its behaviour determines the collapse path followed by the shell. $m(\tau)$ increases in the region between the event horizons because the radiation from the collapsing shell is "dragged back" by the gravitational field and piles up on the singularity.

§ 6.3 Some Collapse Paths

Qualitatively, it is possible to cast collapse paths into one of the cases of §5.2. We consider the equations of motion to be a function of proper time

$$\frac{m(\tau)}{M(\tau)} + \frac{M^2(\tau) - e^2}{2M(\tau)R(\tau)} = \sqrt{1 + \dot{R}^2} \qquad (6.12)$$

and look for turning points. The difference between the case and those of §5.2 is that in §5.2, m, M, and e had fixed values. From fig.16 we see that $M(\tau)$ constantly decreases and $m(\tau)$ oscillates. Thus in the region $R > r_+$,

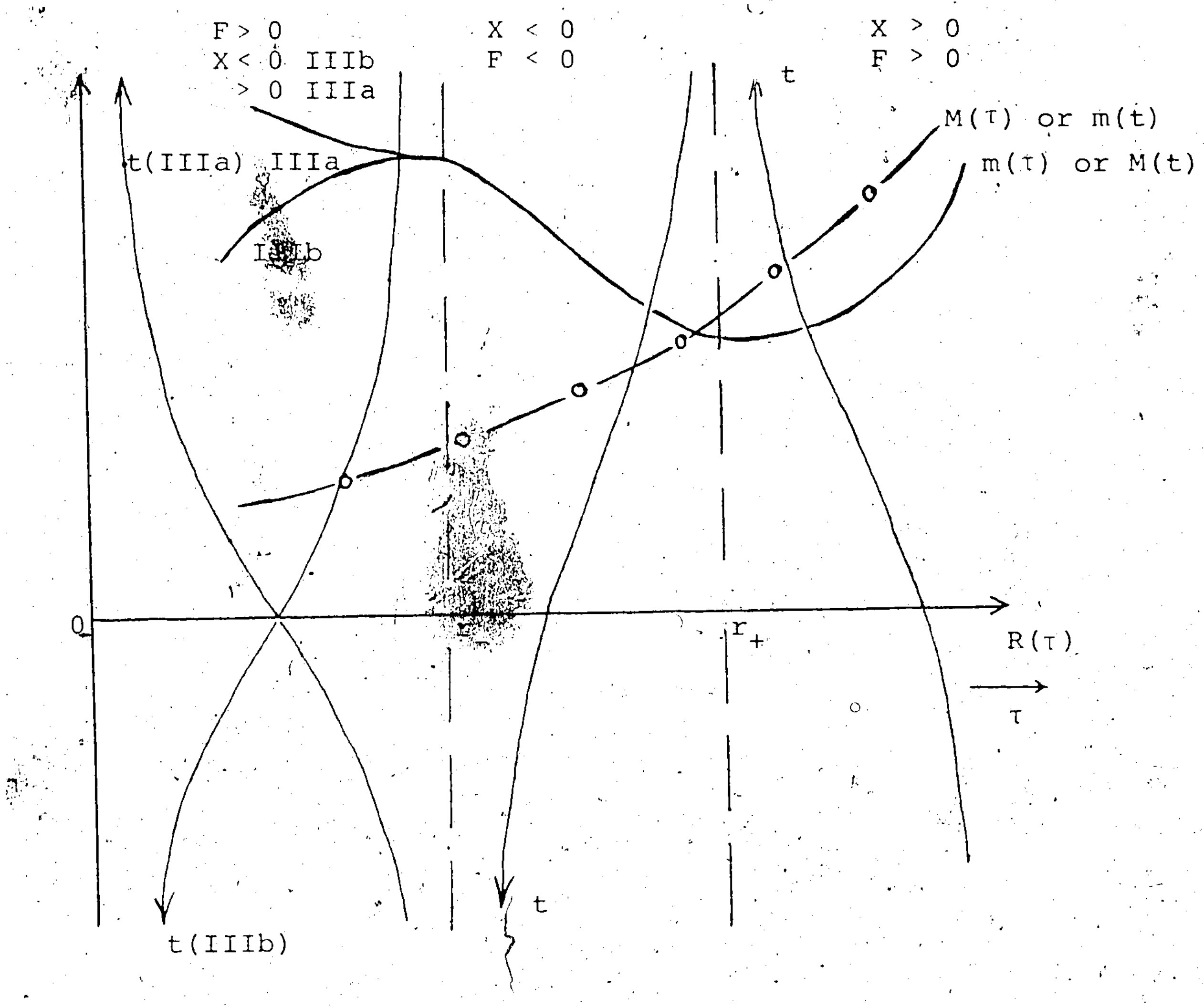


Figure 16. Behavior of M(τ), m(τ) for a radiating shell. The notation IIIa or IIIb refers to the regions in a Graves-Brill diagram.

it is possible for, say, $M(\tau) > m(\tau) > e$; however, in the region between the horizons, the total energy may have grown larger than the proper mass, so $m(\tau) > M(\tau) > e$ and we are looking at case 1, §5.2.

From fig. 16 we see that $m(\tau)$ is a minimum at r_+. At r_-, $m(\tau)$ may reach a maximum, or may continue increasing, depending on whether or not $n^1(R_-)$ is positive or negative. The same sort of analysis as in §5.2 may be carried out for a variety of cases. One line of investigation which could be pursued would be to set conditions on the radiation process, for example, have $m(\tau)$ change as a step function due to a sudden burst of radiation.

CHAPTER VII

A CLASS OF STATIONARY ELECTROMAGNETIC VACUUM FIELDS[30]

§7.1 Introduction

A stationary space-time allows for consideration of systems which exhibit steady motions. A space-time is stationary if the metric is time independent in some system of co-ordinates; that is, if $g_{\mu\nu,4} = 0$. In this chapter, it is shown how a new class of stationary electromagnetic vacuum fields can be generated from solutions of Laplace's equation. These fields are a stationary generalization of the static electromagnetic vacuum fields of Weyl, Majumdar, and Papapetrou, and are plausibly interpreted as exterior fields of static or steadily moving distributions of charged dust having numerically equal charge and mass densities.

Coulomb's law and Newton's law of gravity are formally identical apart from a sign. Hence, classically, any unstressed distribution of matter can, if suitably charged, be maintained in neutral equilibrium under a balance between the gravitational attraction and electrical repulsion of its parts.

Indications that this obvious Newtonian fact has a relativistic analog first emerged when Weyl [31] obtained a particular class of static electromagnetic vacuum fields later generalized by Majumdar [32] and Papapetrou [33] to remove Weyl's original restriction to axial symmetry and further studied by Bonnor [34] and Synge [35]. The Papapetrou-Majumdar fields are to all appearances the external fields of static sources whose charge and mass are numerically equal (in relativistic units: G = C = 1). That they are indeed interpretable as external fields of static distributions of charged dust having equal charge and mass densities has been shown by Das [36], who has examined the corresponding interior fields.

Astrophysical bodies are electrically neutral to a good approximation, and the Papapetrou-Majumdar solutions have up to now received little attention. It seems, however, that they can play a useful, if limited, astrophysical role in providing simple quasistatic analogues for complex dynamical processes like the disappearance of asymmetries in gravitational collapse or the collision of black holes. In reality, such a process always involves large kinetic energies and at present can only be handled by elaborate numerical integrations under the assumption of small departures from spherical symmetry [37], [38]. However, for charged bodies in neutral equilibrium the process can be made arbitrarily slow, and the details easily followed as a sequence of stationary configurations. While this procedure prevents the consideration of features of undeniable observational importance, such as the emission of gravitational waves, it is, for that very reason, ideally suited for isolating and elucidating certain basic issues of principle relating to the final phases of the process. Some of these questions are pursued in detail elsewhere [39]. The purpose here is to demonstrate that the Papapetrou-Majumdar class can be extended straightforwardly from the static to the stationary realm. In other words, a class of stationary solutions to the source free Einstein-Maxwell equations are derived.

§7.2 Stationary Fields

The metric of an arbitrary stationary field is conveniently expressed in the form [40]

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -f^{-1}\gamma_{ij} dx^i dx^j + f(dx^4 + \omega_i dx^i)^2. \quad (7.1)$$

f is any time independent function. ω_i and γ_{ij} are also independent of the time co-ordinate, x^4.

$$g_{\mu\nu} = \begin{pmatrix} -f^{-1}\gamma_{ij}+f\omega_i\omega_j & f\omega_j \\ f\omega_i & f \end{pmatrix}, \tag{7.2}$$

$$g^{\mu\nu} = \begin{pmatrix} -f\gamma^{ij} & f\omega^i \\ f\omega^j & f^{-1}-f\omega^2 \end{pmatrix}. \tag{7.3}$$

$$\begin{cases} \det g^{\mu\nu} = -f^{+2} \det \gamma^{ij} & g \equiv \det g_{\mu\nu} \\ \det g_{\mu\nu} = -f^{-2} \det \gamma_{ij} & \gamma \equiv \det \gamma_{ij} \\ \therefore \quad -g = f^{-1}\sqrt{\gamma} \end{cases} \tag{7.4}$$

The 3-vector ω_m in (7.1) is arbitrary up to an additive gradient $\partial_m\lambda(x^1,x^2,x^3)$ corresponding to the possibility of making arbitrary time translations $x^4 \rightarrow x^{4'} = x^4 - \lambda(x^1,x^2,x^3)$. It may be interpreted as an angular velocity component. We can derive from it an invariant "torsion vector"

$$f^{-2}\tau^m = -\sqrt{g}\;\varepsilon^{mpq}\partial_p\omega_q$$

$$\text{or} \qquad f^{-2}\tau = -\text{curl}\;\omega \tag{7.5}$$

in terms of a three-dimensional vector calculus employing $\gamma_{mn}dx^m dx^n$ as base metric.

We next consider a stationary electromagnetic field. $F_{\mu\nu} = \partial_\nu A_\mu - \partial_\mu A_\nu$ in the space-time (7.1). The condition of time independence $\partial_4 A_\mu = 0$ yields for the "electric" components

$$F_{4n} = \partial_n A_4 \quad (7.6)$$

Now

$$\begin{aligned} F^{mn} &= g^{ma}g^{nb}F_{ab} + g^{m4}g^{nb}F_{4b} + g^{ma}g^{n4}F_{a4} \\ &= f^2 F^{mn} - f^2 (A_4|^n \omega^m - A_4|^m \omega^n) \end{aligned} \quad (7.7)$$

$$\begin{aligned} F^{m4} &= g^{ma}g^{4b}F_{ab} + g^{m4}g^{4b}F_{4b} + g^{ma}g^{44}F_{a4} \\ &= -f^2 \omega^b F^m{}_b + (f^2 \omega^m \omega^b + g^{mb})\partial_b A_4 - f^2 \omega^2 A_4|^m \end{aligned} \quad (7.8)$$

The source free Maxwell equations

$$\partial_n [(-g) \; {}^{(4)}F^{\mu n}] = 0 \quad (7.9)$$

for $\mu = m$ give the "magnetic" components,

(7.4), (7.9) $\longrightarrow$

$$\partial_n (\sqrt{\gamma}\, f^{-1} \; {}^{(4)}F^{mn}) = 0 \quad (7.10)$$

$$\therefore \quad {}^{(4)}F^{mn} = f \frac{1}{\sqrt{\gamma}} \varepsilon^{mnp} \partial_p \phi \quad (7.11)$$

in terms of a magnetic scalar potential ϕ where

$$\phi_i = f^2 \sqrt{\gamma}\, \varepsilon_{ijk}\omega^{j|k} \quad . \tag{7.12}$$

All remaining components (7.6), (7.11) are then conveniently expressed in terms of these six; for example,

$$^{(4)}F^{n4} = \omega_m {}^{(4)}F^{mn} + F_{4m}\gamma^{mn} \tag{7.13}$$

an identity which follows readily from (7.2) or (7.3). Equation (7.10) with $\mu = 4$ now yields, on substituting (7.13), (7.12), (7.6), and (7.5)

$$\mathrm{div}(f^{-1}\nabla A_4) = -f^{-2}\tau\cdot\nabla\phi \quad . \tag{7.14}$$

Next, writing F_{mn} $(=\partial_n A_m - \partial_m A_n)$ in terms of (7.6) and (7.11) and expressing the cyclic identity;

$$\varepsilon^{mnp}\partial_p F_{mn} = 0 \ ,$$

we obtain

$$\mathrm{div}(f^{-1}\nabla\phi) = f^{-2}\tau\cdot\nabla A_4 \quad . \tag{7.15}$$

If we now introduce the complex scalar potential

$$\psi = A_4 + i\phi \tag{7.16}$$

then (7.14) and (7.15) combine to give

$$\mathrm{div}(f^{-1}\nabla\psi) = if^{-2}\tau\cdot\nabla\psi \quad . \tag{7.17}$$

We have thus reduced the entire set of Maxwell's equations to the single complex equation (7.17).

§7.3 Gravitational Field Equations

The Ricci tensor

$$R_{\mu\nu} = \partial_{\mu}\Gamma^{\alpha}_{\nu\alpha} - \partial_{\alpha}\Gamma^{\alpha}_{\mu\nu} + \Gamma^{\alpha}_{\beta\mu}\Gamma^{\beta}_{\alpha\nu} - \Gamma^{\alpha}_{\beta\alpha}\Gamma^{\beta}_{\mu\nu} \tag{7.18}$$

for the general stationary metric (7.1) is conveniently expressed in terms of a complex 3-vector $\underline{G}$, defined

$$2f\underline{G} = \nabla f + i\underline{\tau} \quad . \tag{7.19}$$

Then [41],

$$-f^{-2}R_{44} = \mathrm{div}\ \underline{G} + (\underline{G}^* - \underline{G})\cdot\underline{G} \tag{7.20a}$$

$$-2if^{-2}\ {}^4R^m_4 = \gamma^{-\frac{1}{2}}\varepsilon^{mpq}(\partial_q G_p + G_p G^*_q) \tag{7.21a}$$

$$f^{-2}(\gamma_{pm}\gamma_{qn}\ {}^{(4)}R^{mn} - \gamma_{pq}R_{44}) = R_{pq}(\gamma)$$

$$+ G_p G^*_q + G^*_p G_q \quad . \tag{7.22a}$$

Here $R_{pq}(\gamma)$ denotes the Ricci tensor formed from the 3-metric $\gamma_{mn}dx^m dx^n$.

For the electromagnetic energy tensor

$$-4\pi T_{\mu\nu} = g^{\alpha\beta}F_{\mu\alpha}F_{\nu\alpha} - \frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}$$

one derives from the formulas of the previous section

$$\frac{1}{2}F_{\mu\nu}F^{\mu\nu} = (\nabla\phi)^2 - (\nabla A_4)^2$$

$$8\pi f^{-1}T_{44} = (\nabla\phi)^2 + (\nabla A_4)^2 \tag{7.20b}$$

$$4\pi f^{-1}\ {}^{(4)}T^{m}{}_{4} = g^{-\frac{1}{2}}\epsilon^{mpq}\partial_p\phi\partial_q A_4 \qquad (7.21b)$$

$$-4\pi f^{-1}\ {}^{(4)}T^{mn} = \partial^m\phi\partial^n\phi + \partial^m A_4\partial^n A_4 -$$

$$\frac{1}{2}g^{mn}[(\nabla\phi)^2 + (\nabla A_4)^2] \qquad (7.22b)$$

with $\partial^m = g^{mn}\partial_n$.

We can now impose the Einstein field equations $R_{\mu\nu} = -8\pi T_{\mu\nu}$. From (7.21a), (7.21b), we find

$$\text{curl}\ \tau = -4\nabla\phi \times \nabla A_4$$

$$= i\ \text{curl}(\psi\nabla\psi^* - \psi\nabla\psi)$$

so that the equation

$$\tau + i(\psi^*\nabla\psi - \psi\nabla\psi^*) = \nabla\Psi \qquad (7.23)$$

defines a real scalar Ψ up to an additive constant. We next define a complex function [42]

$$\xi = f - \psi\psi^* + i\Psi \quad . \qquad (7.24)$$

By virtue of (7.19) and (7.23),

$$f\underline{G} = \frac{1}{2}\nabla\xi + \psi^*\nabla\psi \quad . \qquad (7.25)$$

Substituting (7.25) into the field equations (7.20a), (7.20b) and employing (7.17) leads to [42]

$$f\nabla^2\xi = \nabla\xi\cdot(\nabla\xi + 2\psi^*\nabla\psi) \qquad (7.26)$$

while (7.17) itself can be written

$$f\nabla^2\psi = \nabla\psi\cdot(\nabla\xi + 2\psi^*\nabla\psi) \tag{7.27}$$

and we note from (7.24) that

$$f = \frac{1}{2}(\xi + \xi^*) + \psi\psi^* \; . \tag{7.28}$$

Finally, the field equations (7.22a), (7.22b) reduce to

$$-f^2 R_{mn}(\gamma) = \frac{1}{2}\xi_{,(m}\xi^*_{,n)} + \psi\xi_{,(m}\psi_{,n)} + \psi^*\xi^*_{,(m}\psi_{,n)}$$

$$- (\xi + \xi^*)\psi_{,(m}\psi^*_{,n)} \tag{7.29}$$

in which, for example,

$$2\xi_{,(m}\xi^*_{,n)} = (\partial_m\xi)\partial_n\xi^* + (\partial_n\xi)(\partial_m\xi^*) \; .$$

The complete system of electromagnetic and gravitational field equations for an arbitrary electromagnetic vacuum field are summed up in (7.26), (7.27) and (7.29).

§7.4 Generalized Papapetrou-Majumdar Solutions

So far, our considerations have been quite general. We now examine whether solutions of the system (7.26), (7.27) and (7.29) exist for which the background metric is flat. In this case equation (7.29) [with $R_{mn}(\gamma) = 0$] is satisfied if and only if there is a linear relation

$$\psi = a + b\xi \quad \text{with} \quad a^*b + ab^* = -\frac{1}{2}$$

(as one easily verifies, for example, by choosing $\xi = x^1$ and $\xi^* = x^2$ as co-ordinates). Both ξ and ψ contain arbitrary additive constants, and it is convenient to adjust these so that $\xi \to 1$ when $\psi \to 0$. We thus obtain

$$\psi = \frac{1}{2} e^{i\alpha}(1 - \xi) \tag{7.30}$$

in which the arbitrary real constant α represents the "complexion" of the electromagnetic field.

If we now substitute (7.30) into (7.26) and (7.27), both reduce to

$$\nabla^2[(1 + \xi)^{-1}] = 0 \tag{7.31}$$

which is Laplace's equation in Euclidean 3-space.

We conclude by summarizing the procedure for obtaining the complete field:

(a) Write down a solution of (7.31) in terms of any convenient co-ordinates x^m. Suppose the Euclidean line element takes the form $\gamma_{mn}dx^m dx^n$ in these co-ordinates.

(b) Obtain f, τ, and ω from the equations

$$f = \frac{1}{4}(1 + \xi)(1 + \xi^*)$$

$$if^{-1}\tau = \nabla\{\ln[(1 + \xi)/(1 + \xi^*)]\}$$

$$\text{curl } \underline{\omega} = -f^{-2}\underline{\tau} \tag{7.32}$$

The space-time metric is given by (7.1).

(c) Obtain $\psi = A_4 + i\phi$ from (7.30). The electromagnetic field can be found from (7.6) and (7.11).

7.5 Example: Charged Kerr-like Solutions

The Kerr-Newman solution with $m^2 = e^2$ corresponds to the simplest complex solution of (7.31). We choose

$$\sqrt{\frac{2}{(1+\xi)}} = 1 + \frac{m}{R} \quad \text{with } R^2 = x^2 + y^2 + (z - ia)^2 \qquad (7.33)$$

where a and m are real constants and x, y, z Cartesian co-ordinates. In terms of oblate spheroidal co-ordinates r, θ, φ defined by

$$\begin{cases} x + iy = [(r-m)^2 + a^2]^{\frac{1}{2}} \sin\theta \, e^{i\phi} \\ z = (r - m) \cos\theta \, , \end{cases} \qquad (7.34)$$

the Euclidean 3-metric becomes

$$\gamma_{mn} dx^m dx^n = [(r-m)^2 + a^2\cos^2\theta] \left[\frac{dr^2}{(r-m)^2 + a^2} + d\theta^2 \right]$$
$$+ [(r-m)^2 + a^2]\sin^2\theta \, d\phi^2 \, . \qquad (7.35)$$

Further, we find

$$R = r - m - ia\cos\theta$$

$$f = \frac{[(r-m)^2 + a^2\cos^2\theta]}{r^2 + a^2\cos^2\theta}$$

$$\psi = e^{i\alpha} m/(r - ia \cos\theta)$$

and, after a somewhat lengthy calculation,

$$\omega_m dx^m = \{[(2mr - m^2)a \sin^2\theta]/(r-m)^2 + a^2\cos^2\theta]\}d\phi.$$

Putting everything together, we recover the charged Kerr metric with $m^2 = e^2$ in its usual form [43].

As a natural generalization of (7.33), one may consider

$$\frac{2}{1+\xi} = 1 + \sum_{k=1}^{n} \frac{m_k}{R_k}$$

where $R_k^2 = (r-c_k)^2$, r is the Euclidean position vector, and c_k an arbitrary set of complex, constant vectors. The resulting metric will represent the field of a set of arbitrarily spinning, charged Kerr-like particles in neutral equilibrium. For the static analog of this solution, representing a set of Reissner-Nordström particles with $e_k = m_k$, see [35]. An extensive analysis of solutions of the Einstein-Maxwell equations, including a discussion of what Hartle and Hawking call the Israel-Wilson metrics may be found in reference [44].

APPENDIX 1

A DIFFERENTIAL FORM CALCULATION OF $G^{\mu}{}_{\nu}$

The Einstein tensor for the spheri-symmetric metric (2.29) is most easily computed using the method of differential forms. The tensor is calculated for the -2 signature form of the metric in order to conform with [45], and the +2 signature form of $G^{\mu}{}_{\nu}$ is given at the end of the calculation.

$$ds^2 = (1 - \frac{f(t,r)}{r})dt^2 + 2drdt - r^2(d\theta^2 + \sin^2\theta d\phi^2) \qquad (1)$$

$$f(t,r) = 2m(t) - \frac{e^2(t)}{r}$$

Let $\theta^1 = dr + \frac{1}{2}(1 - \frac{f}{r})\,dt$

$\theta^2 = rd\theta$

$\theta^3 = r\sin\theta d\phi$

$\theta^4 = dt$

$$ds^2 = 2\theta^1\theta^4 - (\theta^2)^2 - (\theta^3)^2$$

$g_{14} = g_{41} = -g_{22} = -g_{33} = 1 \qquad$ other $g_{ab} = 0$

$g^{ab} = g_{ab}$

$$d\theta^1 = d^2r + \frac{1}{2}d^2t - d(\frac{f(t,r)}{2r}dt) \qquad (2)$$

$$= \frac{1}{2r}(\frac{f}{r} - f_r)\theta^1\wedge\theta^4 ,$$

$$= dr\wedge d\theta = \frac{1}{r}\theta^1\wedge\theta^2 - \frac{1}{2r}\left(1-\frac{f}{r}\right)\theta^4\wedge\theta^2 \qquad (3)$$

$$d\theta^3 = d(r\sin\theta d\phi) = \sin\theta dr\wedge d\phi + r\cos\theta d\theta\wedge d\phi$$

$$= \frac{1}{r}\left[\theta^1 - \frac{1}{2}\left(1-\frac{f}{r}\right)\theta^4\right]\wedge\theta^3 + \frac{1}{r}\cot\theta\ \theta^2\wedge\theta^3 \qquad (4)$$

$$d\theta^4 = d^2t = 0 \qquad (5)$$

$$d\theta^a = -\omega^a{}_b\wedge\theta^b \qquad (6)$$

$$\omega_{ab} = -\omega_{ba} \longrightarrow \text{e.g. } \omega_{44} = 0 \quad \omega^1{}_4 = g^{14}\omega_{44} = 0 \qquad (7)$$

(2), (6) $\longrightarrow$

$$\omega^1{}_1 = \frac{1}{2r}\left(\frac{f}{r} - f_r\right)\theta^4, \quad \omega^1{}_2 = A\theta^2, \quad \omega^1{}_3 = B\theta^3, \quad \omega^1{}_4 = 0$$

(3), (6) $\longrightarrow$

$$\omega^2{}_1 = \frac{\theta^2}{r}, \quad \omega^2{}_2 = 0, \quad \omega^2{}_3 = C\theta^3, \quad \omega^2{}_4 = -\frac{1}{2r}\left(1-\frac{f}{r}\right)\theta^2$$

(4), (6) $\longrightarrow$

$$\omega^3{}_1 = \frac{\theta^3}{r}, \quad \omega^3{}_2 = \frac{\cot\theta}{r}\theta^3, \quad \omega^3{}_3 = 0, \quad \omega^3{}_4 = -\frac{1}{2r}\left(1-\frac{f}{r}\right)\theta^3$$

(7) $\longrightarrow$:

$$\omega^1{}_2 = g^{14}\omega_{42} = -\omega_{24} = -g_{22}\omega^2{}_4 = \omega^2{}_4 = -\frac{1}{2r}\left(1-\frac{f}{r}\right)\theta^2$$

$$\omega^1{}_3 = \omega^3{}_4 = -\frac{1}{2r}\left(1-\frac{f}{r}\right)\theta^3$$

$$\therefore A = -\frac{1}{2r}\left(1-\frac{f}{r}\right), \qquad B = -\frac{1}{2r}\left(1-\frac{f}{r}\right)$$

$$\omega^2{}_3 = -\omega^3{}_2 = -\frac{\cot\theta}{r}\theta^3 \longrightarrow C = -\frac{\cot\theta}{r}$$

$$\omega^4{}_1 = 0, \quad \omega^4{}_2 = \omega^2{}_1 = \frac{\theta^2}{r}, \quad \omega^4{}_3 = \omega^3{}_1 = \frac{\theta^3}{r}$$

$$\omega^4{}_4 = -\omega^1{}_1 = -\frac{1}{2r}\left(\frac{f}{r} - f_r\right)\theta^4$$

$$\Omega^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b$$ Second equation of structure

$$\Omega_{ab} = -\Omega_{ba}$$

$\longrightarrow$

$$\Omega^1{}_1 = \frac{1}{2}\left(-\frac{2f}{r^3} + \frac{2f_r}{r^2} - \frac{f_{rr}}{r}\right)\theta^1\wedge\theta^4$$

$$\Omega^1{}_2 = \frac{1}{2r^2}\left(f_r - \frac{f}{r}\right)\theta^1\wedge\theta^2 - \frac{f_t}{2r^2}\theta^2\wedge\theta^4$$

$$\Omega^1{}_3 = \frac{1}{2r^2}\left(f_r - \frac{f}{r}\right)\theta^1\wedge\theta^3 - \frac{f_t}{2r^2}\theta^3\wedge\theta^4$$

$$\Omega^1{}_4 = 0$$

$$\Omega^2{}_1 = \frac{1}{2r^2}\left(\frac{f}{r} - f_r\right)\theta^2\wedge\theta^4$$

$$\Omega^2{}_2 = 0$$

$$\Omega^2{}_3 = \frac{f}{r^3}\theta^2\wedge\theta^3$$

$$\Omega^2{}_4 = \Omega^1{}_2$$

$$\Omega^3{}_1 = \frac{1}{2r^2}\left(\frac{f}{r} - f_r\right)\theta^3\wedge\theta^4$$

$\Omega^3{}_2 = -\Omega^2{}_3$

$\Omega^3{}_3 = 0$

$\Omega^3{}_4 = \Omega^1{}_3$

$\Omega^4{}_1 = 0$

$\Omega^4{}_2 = \Omega^2{}_1$

$\Omega^4{}_3 = \Omega^3{}_1$

$\Omega^4{}_4 = -\Omega^1{}_1$

$\Omega^a{}_b = \frac{1}{2} R^a{}_{bcd}\theta^c \wedge \theta^d$ curvature 2 forms

e.g. $\Omega^1{}_1 = \frac{1}{2}(R^1{}_{114}\theta^1 \wedge \theta^4 + R^1{}_{141}\theta^4 \wedge \theta^1)$

$\rightarrow R^1{}_{114} = -R^1{}_{141} = \frac{1}{2}\left(-\frac{2f}{r^3} + \frac{2f_r}{r^2} - \frac{f_{rr}}{r}\right)$.

Similarly, we get

$R^1{}_{114} = \frac{1}{2}\left(-\frac{2f}{r^3} + \frac{2f_r}{r^2} - \frac{f_{rr}}{r}\right)$ $\qquad R^1{}_{1cd} = 0$

$R^1{}_{212} = \frac{1}{2r^2}\left(f_r - \frac{f}{r}\right)$ $\qquad R^1{}_{224} = -\frac{f_t}{2r^2}$

$R^1{}_{313} = \frac{1}{2r^2}\left(f_r - \frac{f}{r}\right)$ $\qquad R^1{}_{334} = -\frac{f_t}{2r^2}$

$$R^1{}_{4cd} = 0$$

$$R^2{}_{124} = -\frac{1}{2r^2}\left(f_r - \frac{f}{r}\right)$$

$$R^2{}_{2cd} = 0$$

$$R^2{}_{323} = \frac{f}{r^3}$$

$$R^2{}_{424} = -\frac{f_t}{2r^2} \qquad R^2{}_{412} = \frac{1}{2r^2}\left(f_r - \frac{f}{r}\right)$$

$$R^3{}_{134} = -\frac{1}{2r^2}\left(f_r - \frac{f}{r}\right)$$

$$R^3{}_{223} = -\frac{f}{r^3}$$

$$R^3{}_{3cd} = 0.$$

$$R^3{}_{413} = \frac{1}{2r^2}\left(f_r - \frac{f}{r}\right) \qquad R^3{}_{434} = -\frac{f_t}{2r^2}$$

$$R^4{}_{1cd} = 0$$

$$R^4{}_{224} = -\frac{1}{2r^2}\left(f_r - \frac{f}{r}\right)$$

$$R^4{}_{334} = -\frac{1}{2r^2}\left(f_r - \frac{f}{r}\right)$$

$$R^4{}_{414} = -\frac{1}{2}\left(-\frac{2f}{r^3} + \frac{2f_r}{r^2} - \frac{f_{rr}}{r}\right)$$

where $R^i{}_{jk\ell} = -R^i{}_{j\ell k}$. Other tetrad components of the Riemann tensor vanish.

Contracting to get the Ricci tensor

$$R_{\mu\nu} = R^{i}{}_{\mu\nu i} \longrightarrow$$

$$R_{14} = \frac{f_{rr}}{2r}, \quad R_{22} = -\frac{f_r}{r^2}, \quad R_{33} = -\frac{f_r}{r^2}, \quad R_{44} = \frac{f_t}{r^2},$$

Other $R_{\mu\nu} = 0$.

Converting from tetrad components to actual co-ordinates

$$R_{\alpha\beta}dx^{\alpha}dx^{\beta} = \underline{R}_{ab}\theta^{a}\theta^{b}$$

$$\longrightarrow 2\underline{R}_{14}(drdt + \tfrac{1}{2}(1-\tfrac{f}{r})\,dt^2) + \underline{R}_{22}r^2d\theta^2$$

$$+ \underline{R}_{33}r^2\sin^2\theta d\phi^2 + \underline{R}_{44}dt^2$$

$$= 2R_{14}drdt + R_{22}d\theta^2 + R_{33}d\phi^2 + R_{44}dt^2$$

$$\longrightarrow R_{14} = \frac{f_{rr}}{2r}, \quad R_{22} = -f_r, \quad R_{33} = -f_r\sin^2\theta,$$

$$R_{44} = \frac{f_{rr}}{2r}(1-\frac{f}{r}) + \frac{f_t}{r^2}.$$

Now from metric (1)

$$g_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & -r^2 & 0 & 0 \\ 0 & 0 & -r^2\sin^2\theta & 0 \\ 1 & 0 & 0 & (1-\frac{f}{r}) \end{pmatrix}$$

$$g^{\mu\nu} = \begin{pmatrix} -(1-\frac{f}{r}) & 0 & 0 & 1 \\ 0 & -\frac{1}{r^2} & 0 & 0 \\ 0 & 0 & -\frac{1}{r^2\sin^2\theta} & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$R = g^{\mu\nu}R_{\mu\nu} = 2g^{14}R_{14} + g^{22}R_{22} + g^{33}R_{33}$$

$$= \frac{f_{rr}}{r} + \frac{f_r}{r^2} + \frac{f_r}{r^2}$$

$$= \frac{f_{rr}}{r} + \frac{2f_r}{r^2}$$

$$R^1{}_1 = g^{41}R_{14} = \frac{f_{rr}}{2r}$$

$$R^2{}_2 = g^{22}R_{22} = +\frac{f_r}{r^2}$$

$$R^3{}_3 = g^{33}R_{33} = \frac{f_r}{r^2}$$

$$R^4{}_4 = g^{14}R_{14} + g^{44}R_4 = \frac{f_{rr}}{2r}$$

$$R^1{}_4 = g^{14}R_{44} + g^{11}R_{\ 4} = +\frac{f_r}{r^2}$$

$$G^\mu{}_\nu = R^\mu{}_\nu - \frac{R}{2}\,\delta^\mu{}_\nu$$

$$\therefore\ G^\mu{}_\nu = \begin{pmatrix} -\frac{f_r}{r^2} & 0 & 0 & 0 \\ 0 & -\frac{f_{rr}}{2r} & 0 & 0 \\ 0 & 0 & -\frac{f_{rr}}{2r} & 0 \\ \frac{f_t}{r^2} & 0 & 0 & -\frac{f_r}{r^2} \end{pmatrix}$$

This $G^{\mu}{}_{\nu}$ is for a -2 signature metric. Changing to a +2 signature,

$$G^{\mu}{}_{\nu} = \begin{pmatrix} +\frac{f_r}{r^2} & 0 & 0 & 0 \\ 0 & +\frac{f_{rr}}{2r} & 0 & 0 \\ 0 & 0 & +\frac{f_{rr}}{2r} & 0 \\ -\frac{f_t}{r^2} & 0 & 0 & +\frac{f_r}{r^2} \end{pmatrix}$$

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