Abstract

A real-time algorithm to produce correlated random fields on general undirected graphs has been used in CAPTCHA generation and optical character recognition. This algorithm can not simulate all possible joint graph distributions but does match all marginal vertex distributions as well as specified covariance between vertices. Herein, a modified algorithm is given by completing the graph and setting the non-specified covariances to zero. The modified-algorithm graph distribution is derived and the following questions are studied: 1) For which marginal pmfs and covariances will this algorithm work? Are there collections consistent with a random field that this algorithm can not handle? 2) When does the marginal property hold, where the subgraph distribution of a algorithm-simulated field matches the distribution of the algorithm-simulated field on the subgraph? 3) When does the permutation property hold, where the vertex simulation order does not affect the joint distribution?

Quick Simulation Random Field Properties

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Index Terms

Simulation, Markov random field, Graph, Statistical Classification, Statistical Restoration, Permutation Property.

I. Introduction

Correlated random fields are used in science and technology to model spatially distributed random objects. The applications of random fields across the sciences are broad and include computer vision, analysis of gene expression time series, medical image processing, inverse optics and image synthesis, and object detection; see, for example, [1], [2], [3], [4], [5] [6] and [7]. Furthermore, mathematicians often to want to *couple* a collection of random variables with given distributions together on a single probability space while matching some constraint like covariances. In either situation, the complete joint distribution of the field may be unknown or even irrelevant as enough meaningful information is captured by marginal probability mass functions (pmfs) and pairwise covariances between random variables. To meet the diversity of problems in a variety of dimensions, Kouritzin, Newton and Wu [8] considered random fields on a general undirected graph structure and proposed an algorithm for producing a new class of discrete correlated random fields on such graphs by either one-pass simulation or Gibbs-like resampling. The approach has been applied to optical character recognition (OCR) [8] and the generation of both black-and-white [9] and grey-level [10] CAPTCHAs. The class of random fields created by their algorithm incorporate given probability mass functions (pmfs) corresponding to vertices in a graph and specified pairwise covariances corresponding to edges existing in that graph. The joint distribution between pairs of vertices connected by a specified covariance edge is known in terms of two sets of auxiliary parameter pmf collections that can be selected for generality. However, the joint subgraph distribution on an incomplete subgraph is unknown.

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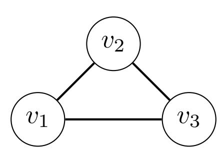
Fig. 1. A grey-level CAPTCHA like those in [10]

The starting point for the simulation is an undirected graph together with the desired marginal vertex pmfs and the collection of non-zero covariances for the graph edges. (This setting is general enough to handle simulation in any dimension for example.) Simulating the graph then amounts to directing the graph in an acyclic manner, fixing a topological sort of the vertices and using Proposition 1 of [8], requoted as Proposition II.1 below, recursively. (See [8] for details.) Our modified algorithm, introduced herein, completes the graph by adding edges of zero covariance wherever necessary before simulation. This completion does not complicate nor slow the simulation yet allows us to derive the complete field distribution in closed form for all possible auxiliary pmf parameters. We call this completed-graph simulation algorithm and resulting random field the *quick simulation algorithm* and *quick simulation field* herein.

This paper focuses on the constraints and properties of the random fields generated by this quick simulation algorithm. Naturally, the algorithm can not work for all possible parameters and might not work for others. We start by giving the joint (field) distribution of the random field generated by this algorithm (when it works). From there, we study *regularity*, meaning when the algorithms does provide a legitimate distribution over the whole space of vertices. This is equivalent to ensuring that the recursive formula (5) of Proposition II.1 produces a conditional pmf in every iteration. It was observed in our CAPTCHA [9] and OCR [8] applications that the occasional illegitimate conditional pmf value outside $[0, 1]$ can be replaced with a value inside without noticeable effect on the simulation. However, it is still important to know when the only possible source of irregularity is numeric and not algorithmic. Next, we establish the *marginality* property that ensures the distribution of a random field on a subgraph projected from the random field constructed on the whole graph is the same as that for a random field constructed directly on this subgraph. Finally, we investigate the *permutation* property that makes sure the random fields simulated from all topological sorts corresponding to the same complete undirected graph are the same in the sense of probability distribution. We establish necessary and sufficient conditions for this permutation property.

Example I.1. *Suppose we have the following complete undirected graph* G *with vertices* v_1, v_2, v_3*,*



probability mass functions $\pi_{v_i}(-1) = \pi_{v_i}(1) = 0.5, i = 1, 2, 3,$ *and covariances* $\beta_{v_i,v_j} = 0.1, i = 1, 2, 3, j =$

$1, 2, 3, i \neq j$. Let us illustrate the marginality and permutation properties of our algorithm.

Looking forward to (9), using the topological sort v_1, v_2, v_3 and setting $\widehat{\pi}_{v_i} = \widetilde{\pi}_{v_i} = \pi_{v_i}$ so $\widetilde{g}(v_i, x_{v_i}) = \frac{x_{v_i}}{2}$ in (2) we assign the joint probabilities as follows.

x_{v_1}	x_{v_2}	x_{v_3}	$\Pi(X_{v_1}, X_{v_2}, X_{v_3})$
1	1	1	0.1625
1	1	-1	0.1125
1	-1	1	0.1125
1	-1	-1	0.1125
-1	1	1	0.1125
-1	1	-1	0.1125
-1	-1	1	0.1125
-1	-1	-1	0.1625

If we change the topologicial sort while maintaining all parameters, then we get the same joint pmf so the permutation property holds. It is straightforward to verify that the pmfs π_{v_i} and covariances β_{v_i,v_j} are as expected. Moreover, if we just simulated two vertices v_i, v_j of the three, then we get

x_{v_i}	x_{v_j}	$\Pi(X_{v_i}, X_{v_j})$
1	1	0.275
1	-1	0.225
-1	1	0.225
-1	-1	0.275

which matches the pmf obtained by summing out a vertex in the previous table so marginality is also maintained.

In this note, we show how to compute these probabilities so that the pmfs and covariances are preserved in general as well as establish the conditions for the marginality and permutation properties above.

The remainder of this note is laid out as follows: Section II contains our notation and background. Next, we give the closed form of correlated random field, discuss regularity and establish the marginality property in Section III. The permutation property is studied in Section IV.

II. Notation and Background

Probabilistic Setup: Let V be a finite set of vertices, $\overrightarrow{V}$ denote this set of vertices with an ordering, and $\mathbf{X}_v$ be a finite state space for each $v \in V$. For any nonempty subsequence $\overrightarrow{B} \subset \overrightarrow{V}$, the space of configurations $x_{\overrightarrow{B}} = (x_v)_{v \in \overrightarrow{B}}$ on $\overrightarrow{B}$ is the Cartesian product $\mathcal{X}_{\overrightarrow{B}} = \prod_{v \in \overrightarrow{B}} \mathbf{X}_v$ and $\overrightarrow{B}^C$ denotes the subsequence so that $\overrightarrow{V} = \overrightarrow{B} \cup \overrightarrow{B}^C$. We abbreviate $\mathcal{X}_{\overrightarrow{V}}$ by $\mathcal{X}$ and x_{v_i} by x_i to ease notation. A *random field* Π is a strictly positive *probability measure* on $\mathcal{X}$. The random vector $X = (X_v)_{v \in \overrightarrow{V}}$ on the probability space $(\mathcal{X}, 2^{\mathcal{X}}, \Pi)$ is also called a random field. For $\overrightarrow{B} \subset \overrightarrow{V}$, the random subfield on $\overrightarrow{B}$ is the projection map $X_{\overrightarrow{B}} : x \to x_{\overrightarrow{B}}$ from $\mathcal{X}$ onto $\mathcal{X}_{\overrightarrow{B}}$.

A *neighborhood system* $\partial = \{\partial(v) : v \in V\}$ is a collection of subsets of V:

1) $v \notin \partial(v)$ for every $v \in V$ and
2) $v \in \partial(u)$ if and only if $u \in \partial(v)$.

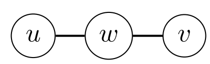
A random field Π is *Markov* with respect to ∂ if for all $x \in \mathcal{X}$

$$\Pi(X_v = x_v | X_u = x_u, u \neq v) = \Pi(X_v = x_v | X_{\partial(v)} = x_{\partial(v)}). \tag{1}$$

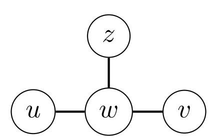
Problem Statement: Let E be a set of edges where each $(u, v) \in E$ with $u, v \in V$ has no orientation but indicates u, v are *neighbors* of each other. Then, $G = (V, E)$ is an *undirected graph*. If for every pair of vertices $u, v \in V$, there is a path of edges in E connecting u and v, then G is *connected*. If every vertex in G has a neighbor with at least two neighbors, then we say G is *sufficiently connected*. If for every pair of non-neighbor vertices z, u there is a neighbor of z and a neighbor of u that are distinct, then we say G is *disjoint pair rich*. The *open neighborhood* of $v \in V$ is $\partial_G(v) = \{u : u \neq v, (u, v) \in E\}$, and its *closed neighborhood* $\partial_G[v] = \partial_G(v) \cup \{v\}$. $\{\partial_G(v), v \in V\}$ is the neighborhood system implied by G. For any nonempty set $B \subset V$, the open neighborhood of B is $\partial_G(B) = \cup_{v \in B} \partial_G(v) \setminus B$ and the closed neighborhood $\partial_G[B] = \partial_G(B) \cup B$. We set $\partial_G(\emptyset) = V$ for convenience.

We illustrate the new concepts of sufficiently connected and disjoint pair rich.

Example II.1. *Consider the following graphs.*



(a) Three vertex example

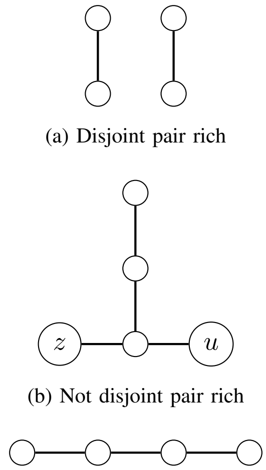


(b) Four vertex example

Fig. 2. Not sufficiently connected

The graphs are connected but not sufficiently connected since none of the neighbors of w have two neighbors.

Example II.2. *The following graphs illustrate the definition of "disjoint pair rich".*



(a) Disjoint pair rich

(b) Not disjoint pair rich

(c) Sufficiently and disjoint pair rich

Fig. 3. Disjoint pair rich

Example II.3. *If every vertex in a graph G has two neighbors, then it is disjoint pair rich. It is also sufficiently connected.*

We are interested in creating a random field over V, where random variable X_v at a vertex $v \in V$ has a predescribed pmf π_v and random vectors (X_u, X_v) have a predescribed *non-zero* covariance β_{uv} $(= \beta_{vu})$ for each $(u, v) \in E$. Naturally, this problem could be ill-posed in the sense that there are mathematically incompatible collections of pmfs and covariances. Also, there often are multiple solutions with some being more efficient to simulate and others having nice properties like the marginal and permutation properties defined above.

Directed Graph: The random variables in the field are simulated in sequence. The first step towards sequencing is directing the graph. Let A be a set of ordered vertex pairs, called *arcs*, (indicating the first vertex is simulated prior to the later). Then, $D = (V, A)$ is a *directed graph*. If $(u, v) \in A$ for $u, v \in V$, then, there is an *arc* from u to v; u is a *parent* of v and v is a *child* of u. The set of parents of v is denoted $\mathrm{pa}(v)$. u is an ancestor of v if there is a sequence of arcs from u to v. D is *acyclic* if there is no $v \in V$ that is an ancestor of itself.

Graph Completion: If $G = (V, E)$ is an undirected graph, then $\overline{G} = (V, \overline{E})$ denotes its *completion*, where there is an edge between every pair of vertices. Similarly, if $D = (V, A)$ is a directed graph, then $\overline{D} = (V, \overline{A})$ denotes its *completion*, where there is an arc between every pair of vertices and the direction of an arc that is also in A matches that of A. [8] gives one possible algorithm to construct an acyclic complete directed graph $\overline{D} = (V, \overline{A})$ from a complete undirected graph $\overline{G} = (V, \overline{E})$ and a topological sort on V, i.e. a simulation order $\overrightarrow{V} = \{v_i\}_{i=1}^{N}$ where $N = |V|$ is the number of vertices. Our new Quick Simulation Algorithm works on a completed acyclic directed graph. Zero covariances are placed along any added arc i.e. $\beta_{v,u} = \mathrm{cov}(X_u, X_v) = 0$ when (v, u) or (u, v) is in $\overline{A} \setminus A$.

Conditional Probability Update: The Quick Simulation Random Fields match a collection of pmfs $\{\pi_v, v \in V\}$ and a collection of covariances $\{\beta_{uv}, (u, v) \in \overline{E}\}$. However, there are also two auxiliary pmf parameter sets

$\{\widehat{\pi}_v, v \in V\}$ and $\{\widetilde{\pi}_v, v \in V\}$ that provide flexibility in the choice of field distribution as well as simulation. (See [8] for examples of choices for these auxiliary pmfs.) They also appear in the conditional probability update through functions:

$$\tilde{g}(v, x_v) = \frac{\tilde{\pi}_v(x_v)(x_v - \tilde{\mu}_v)}{\tilde{\sigma}_v^2}, \tag{2}$$

$$\widehat{h}(u, v) = \prod_{w \in \mathrm{pa}(v) \setminus \{u\}} \widehat{\pi}_w(x_w), \tag{3}$$

for $u \in \mathrm{pa}(v)$, $v \in V$ and $x_v \in \mathbf{X}_v$, where $\tilde{\mu}_v = \sum_{x_v \in \mathbf{X}_v} \tilde{\pi}_v(x_v) x_v$ and $\tilde{\sigma}_v^2 = \sum_{x_v \in \mathbf{X}_v} \tilde{\pi}_v(x_v)(x_v - \tilde{\mu}_v)^2$.

Let $\{v_i\}_{i=1}^N$ be a topological sort of directed graph $D = (V, A)$. For any $B \subset V$, we let $j = \max\{i : 1 \le i \le N, v_i \in B\}$ and find:

$$\Pi(X_B = x_B) = \sum_{x_{v_k}: 1 \le k \le j, v_k \notin B} \prod_{i=1}^{j} \Pi(X_{v_i} = x_{v_i} | X_{\mathrm{pa}(v_i)} = x_{\mathrm{pa}(v_i)}), \tag{4}$$

where $\Pi(X_{v_1} = x_{v_1} | X_{\mathrm{pa}(v_1)} = x_{\mathrm{pa}(v_1)}) = \Pi(X_{v_1} = x_{v_1})$. The main proposition in [8] is:

Proposition II.1. *Assume that $D = (V, A)$ is a directed acyclic graph with N vertices, $\{v_i\}_{i=1}^N$ is a topological sort of the vertices V and $\{\tilde{\pi}_v(x_v) : x_v \in \mathbf{X}_v, v \in V\}$, $\{\widehat{\pi}_v(x_v) : x_v \in \mathbf{X}_v, v \in V\}$ are sets of auxiliary non-trivial pmfs. Suppose further that $\{\pi_v(x_v) : x_v \in \mathbf{X}_v, v \in V\}$ are pmfs and $\{\beta_{v,u} : (u, v) \in A$ or $(v, u) \in A\}$ are numbers such that the right hand side of*

$$\Pi(X_{v_i} = x_i | X_{\mathrm{pa}(v_i)} = x_{\mathrm{pa}(v_i)}) = \pi_{v_i}(x_i) + \frac{\tilde{g}(v_i, x_i)}{\Pi(X_{\mathrm{pa}(v_i)} = x_{\mathrm{pa}(v_i)})} \sum_{u \in \mathrm{pa}(v_i)} \beta_{u,v_i} \tilde{g}(u, x_u) \widehat{h}(u, v_i) \tag{5}$$

is non-negative for each $x_i \in \mathbf{X}_{v_i}$ and $x_{\mathrm{pa}(v_i)} \in \mathbf{X}_{\mathrm{pa}(v_i)}$ ($1 \le i \le N$), where $\Pi(X_{\mathrm{pa}(v_i)} = x_{\mathrm{pa}(v_i)})$ is computed according to (4). Form the conditional probabilities recursively using (5), starting with $\Pi(X_{v_1} = x_1) = \pi_{v_1}(x_1)$. Then, the random field X, defined by

$$\Pi(X = x) = \prod_{i=1}^{N} \Pi(X_{v_i} = x_i | X_{\mathrm{pa}(v_i)} = x_{\mathrm{pa}(v_i)}), \tag{6}$$

has marginal probabilities $\{\pi_v\}$ and covariances $\mathrm{cov}(X_u, X_v) = \beta_{v,u}$ for all $u \in \mathrm{pa}(v)$.

Remark II.1. *The term non-trivial pmfs can be interpreted as: Each $\widetilde{\pi}_v$ should have non-zero variance and each $\widehat{\pi}_v$ should be strictly positive.*

Remark II.2. *In [8], there was the stronger constraint that the right hand side of (5) is in $[0, 1]$. However,*

$\sum_{x_v \in \mathbb{X}_v} \frac{\tilde{g}(v,x_v)}{\Pi(X_{\mathrm{pa}(v)}=x_{\mathrm{pa}(v)})} \sum_{u\in \mathrm{pa}(v)} \beta_{u,v}\tilde{g}(u,x_u)\widehat{h}(u,v) = 0$ *since*

$$\sum_{x_v \in \mathbb{X}_v} \tilde{g}(v,x_v) = \sum_{x_v \in \mathbb{X}_v} \frac{\tilde{\pi}_v(x_v)(x_v-\tilde{\mu}_v)}{\tilde{\sigma}_v^2} = 0.$$

Hence, if (5) is non-negative, then it is in $[0,1]$ *and (5) defines a legitimate conditional pmf.*

Remark II.3. *Notice that (5) gives the same value, whether we consider the given graph* D *or its completion* $\overline{D}$ *where the added arcs have zero covariance.*

III. Distribution and Marginality of Quick Simulation Fields

Proposition II.1 can be extended to give the full field distribution when the graph is complete.

Proposition III.1. *Assume that* $\overline{D} = (V, \overline{A})$ *is a complete directed acyclic graph with* N *vertices,* $\overrightarrow{V} = \{v_i\}_{i=1}^N$ *is a topological sort of the vertices* V *and* $\{\tilde{\pi}_v(x_v) : x_v \in \mathbf{X}_v, v \in V\}$*,* $\{\widehat{\pi}_v(x_v) : x_v \in \mathbf{X}_v, v \in V\}$ *are auxiliary non-trivial pmf sets. Suppose further that* $\{\pi_v(x_v) : x_v \in \mathbf{X}_v, v \in V\}$ *are pmfs and* $\{\beta_{v,u} : (u,v) \in \overline{A}$ *or* $(v,u) \in \overline{A}\}$ *are numbers such that the right hand side of*

$$\Pi(X_{v_i} = x_i | X_{v_1} = x_1, ..., X_{v_{i-1}} = x_{i-1}) = \\ \pi_{v_i}(x_i) + \frac{\tilde{g}(v_i,x_i)\sum_{j=1}^{i-1}\beta_{v_j,v_i}\tilde{g}(v_j,x_j)\widehat{h}(v_j,v_i)}{\Pi(X_{v_1} = x_1, ..., X_{v_{i-1}} = x_{i-1})} \tag{7}$$

is non-negative for each $x_i \in \mathbf{X}_{v_i}$*,* $i = 1, ..., N$*. Form the conditional probabilities recursively using (7), starting with* $\Pi(X_{v_1} = x_1) = \pi_{v_1}(x_1)$*. Then, the random field* X*, defined by*

$$\Pi(X = x) = \\ \prod_{i=1}^N \Pi(X_{v_i} = x_i | X_{v_1} = x_1, ..., X_{v_{i-1}} = x_{i-1}), \tag{8}$$

a) *has marginal probabilities* $\{\pi_v\}$ *and covariances* $\mathrm{cov}(X_u, X_v) = \beta_{v,u}$ *for all* $u, v \in V$*, and*
b) *has closed form*

$$\Pi_{v_1,...,v_n}(x_1, ..., x_n) = \\ \prod_{i=1}^n \pi_{v_i}(x_i) + \\ \sum_{1\le j<i\le n} \left(\prod_{k=1,k\neq j}^{i-1} \widehat{\pi}_{v_k}(x_k) \times \prod_{k=i+1}^{n} \pi_{v_k}(x_k) \right) \\ \widetilde{g}(v_i,x_i)\beta_{v_i,v_j}\widetilde{g}(v_j,x_j) \tag{9}$$

for each $x_i \in \mathbf{X}_{v_i}$ *and* $n = 1, ..., N$*.*

Remark III.1. *Since the terms in (7) with* $\beta_{v_i,v_j} = 0$ *disappear, the computations are the same as for the algorithm in [8] on the incomplete graph.*

Remark III.2. *Regularity means that the right hand side of (7) is a conditional pmf. As noted in Remark II.2, the right hand side of (7) need only be non-negative, which is equivalent to*

$$-\tilde{\sigma}_{v_i}^2 \pi_{v_i}(x_i)\Pi(X_{v_1} = x_1, ..., X_{v_{i-1}} = x_{i-1}) \leq \\ (x_i - \tilde{\mu}_{v_i}) \sum_{j=1}^{i-1} \beta_{v_j,v_i} \tilde{g}(v_j, x_j) \hat{h}(v_j, v_i), \tag{10}$$

and can be checked during the iteration. Notice:

1) *There is no constraint on β_{v_i,v_j} when $x_i = \tilde{\mu}_{v_i}$ or $x_j = \tilde{\mu}_{v_j}$.*
2) *$\beta_{v_i,v_j} = 0$ automatically satisfies the constraint.*

By (2), we need only check

$$-\tilde{\sigma}_{v_i}^2 \pi_{v_i}(x_i)\Pi(X_{v_1} = x_1, ..., X_{v_{i-1}} = x_{i-1}) \leq \\ (x_i - \tilde{\mu}_{v_i}) \sum_{u \in \mathrm{pa}(v_i)} \beta_{u,v_i} \tilde{g}(u, x_u) \hat{h}(u, v_i), \tag{11}$$

where $\mathrm{pa}(v_i)$ denotes parents in the original (not-completed) graph. If $\mathrm{pa}(v_i) = \{v_{i-1}\}$ is a singleton, then (11) further simplifies by (2) to

$$-\tilde{\sigma}_{v_i}^2 \tilde{\sigma}_{v_{i-1}}^2 \frac{\pi_{v_i}(x_i)\pi_{v_{i-1}}(x_{i-1})}{\tilde{\pi}_{v_i}(x_i)\tilde{\pi}_{v_{i-1}}(x_{i-1})} \leq \\ (x_i - \tilde{\mu}_{v_i})\beta_{v_{i-1},v_i}(x_{i-1} - \tilde{\mu}_{v_{i-1}}) \tag{12}$$

for $x_i \in \mathbf{X}_{v_i}, x_{i-1} \in \mathbf{X}_{v_{i-1}}$. One can check (11) or (12) iteratively to ensure the Quick Simulation algorithm is producing a field with the desired pmfs and covariances. Now, we show how equality in (12) is hit:

$$\begin{aligned} & -\tilde{\sigma}_{v_i}^2 \tilde{\sigma}_{v_{i-1}}^2 \frac{\pi_{v_i}(x_i)\pi_{v_{i-1}}(x_{i-1})}{\tilde{\pi}_{v_i}(x_i)\tilde{\pi}_{v_{i-1}}(x_{i-1})} = \\ & \qquad (x_i - \tilde{\mu}_{v_i})\beta_{v_{i-1},v_i}(x_{i-1} - \tilde{\mu}_{v_{i-1}}) \\ \Leftrightarrow \quad & \pi_{v_i}(x_i) = \pi_{v_i}(x_i)(1 - \pi_{v_{i-1}}(x_{i-1})) - \\ & \qquad \widetilde{g}(v_{i-1}, x_{i-1})\beta_{v_i,v_{i-1}}\widetilde{g}(v_i, x_i) \\ \Leftrightarrow \quad & \pi_{v_i}(x_i) = \sum_{y \neq x_{i-1}} \big(\pi_{v_i}(x_i)\pi_{v_{i-1}}(y) + \\ & \qquad \widetilde{g}(v_{i-1}, y)\beta_{v_i,v_{i-1}}\widetilde{g}(v_i, x_i)\big) \\ \Leftrightarrow \quad & \pi_{v_i}(x_i) = \Pi(X_{v_{i-1}} \neq x_{i-1}, X_{v_i} = x_i), \end{aligned} \tag{13}$$

since $\sum_{y \neq x_{i-1}} \widetilde{g}(v_{i-1}, y) = 0$ and it is shown in Proposition 1 of [8] that $\Pi(X_{v_{i-1}} = x_{i-1}, X_{v_i} = x_i) = \pi_{v_i}(x_i)\pi_{v_{i-1}}(x_{i-1}) + \widetilde{g}(v_{i-1}, x_{i-1})\beta_{v_i,v_{i-1}}\widetilde{g}(v_i, x_i)$. Hence, we hit this bound when we have a singleton parent and one value of X_{i-1} precludes another value of X_i.

Proof: Proposition III.1 a) This follows immediately from Proposition II.1 and the fact that the parents of v_i are all $v_1, ..., v_{i-1}$ when the graph is complete.

b) Note (9) holds for $n=1$. Now, we assume it is true for $n-1$ with some $n \in \{2,....,N\}$ and show it for n. (7) is equivalent to:

$$\Pi_{v_1,...,v_n}(x_1,...,x_n) = \pi_{v_n}(x_n)\Pi_{v_1,...,v_{n-1}}(x_1,...,x_{n-1}) + \sum_{j=1}^{n-1}\left(\widetilde{g}(v_n,x_n)\beta_{v_j,v_n}\widetilde{g}(v_j,x_j)\prod_{k=1,k\neq j}^{n-1}\widehat{\pi}_{v_k}(x_k)\right) \tag{14}$$

so by (14) and (9) with $n-1$

$$\begin{aligned}
&\Pi_{v_1,...,v_n}(x_1,...,x_n)\\
=&\pi_{v_n}(x_n)\prod_{i=1}^{n-1}\pi_{v_i}(x_i)\\
&+\sum_{j=1}^{n-1}\left(\widetilde{g}(v_n,x_n)\beta_{v_n,v_j}\widetilde{g}(v_j,x_j)\prod_{k=1,k\neq j}^{n-1}\widehat{\pi}_{v_k}(x_k)\right)\\
&+\pi_{v_n}(x_n)\\
&\times\sum_{1\le j<i\le n-1}\left[\left(\prod_{k=1,k\neq j}^{i-1}\widehat{\pi}_{v_k}(x_k)\times\prod_{k=i+1}^{n-1}\pi_{v_k}(x_k)\right)\right.\\
&\left.\times\widetilde{g}(v_i,x_i)\beta_{v_i,v_j}\widetilde{g}(v_j,x_j)\right]\\
=&\prod_{i=1}^{n}\pi_{v_i}(x_i)\\
&+\sum_{1\le j<i\le n-1}\left[\left(\prod_{k=1,k\neq j}^{i-1}\widehat{\pi}_{v_k}(x_k)\times\prod_{k=i+1}^{n}\pi_{v_k}(x_k)\right)\right.\\
&\left.\times\widetilde{g}(v_i,x_i)\beta_{v_i,v_j}\widetilde{g}(v_j,x_j)\right]\\
&+\sum_{j=1}^{n-1}\left(\prod_{k=1,k\neq j}^{n-1}\widehat{\pi}_{v_k}(x_k)\right)\widetilde{g}(v_n,x_n)\beta_{v_n,v_j}\widetilde{g}(v_j,x_j)
\end{aligned}$$

so the result follows by induction. ■

It follows immediately from Proposition III.1 that the field produced on $\{v_1,...,v_{N-1},v_N\}$ extends the field produced on $\{v_1,...,v_{N-1}\}$. However, it is natural to wonder if the distribution of a subfield is the same as the distribution of the quick simulation field on the corresponding subgraph. Considering the marginal distribution with vertex v_l removed, using (9) and recalling $\sum_{x_l}\widetilde{g}(v_l,x_l)=0$, we break sum below into: $i,j\neq l$, $i=l$, $j=l$ to find

$$\sum_{x_l\in X_{v_l}}\Pi_{v_1,...,v_{l-1},v_l,v_{l+1},...,v_N}(x_1,..., x_{l-1},x_l,x_{l+1},...,x_N) \tag{15}$$

$$
\begin{aligned}
=&\prod_{\substack{i=1\\ i\neq l}}^{N}\pi_{v_i}(x_i)+\sum_{\substack{1\le j<i\le N\\ i,j\neq l}}\Bigg[\\
&\left(\sum_{x_l}\prod_{k=1,k\neq j}^{i-1}\widehat{\pi}_{v_k}(x_k)\times\prod_{k=i+1}^{N}\pi_{v_k}(x_k)\right)\\
&\times\widetilde{g}(v_i,x_i)\beta_{v_i,v_j}\widetilde{g}(v_j,x_j)\Bigg]\\
&+\sum_{1\le j<l}\left(\prod_{k=1,k\neq j}^{i-1}\widehat{\pi}_{v_k}(x_k)\times\prod_{k=i+1}^{N}\pi_{v_k}(x_k)\right)\\
&\times\sum_{x_l}\widetilde{g}(v_l,x_l)\beta_{v_l,v_j}\widetilde{g}(v_j,x_j)\\
&+\sum_{l<i\le N}\left(\prod_{k=1,k\neq j}^{i-1}\widehat{\pi}_{v_k}(x_k)\times\prod_{k=i+1}^{N}\pi_{v_k}(x_k)\right)\\
&\times\widetilde{g}(v_i,x_i)\beta_{v_i,v_l}\sum_{x_l}\widetilde{g}(v_l,x_l)\\
=&\prod_{\substack{i=1\\ i\neq l}}^{N}\pi_{v_i}(x_i)\\
&+\sum_{\substack{1\le j<i\le N\\ i,j\neq l}}\left(\prod_{\substack{k=1\\ k\neq j,l}}^{i-1}\widehat{\pi}_{v_k}(x_k)\times\prod_{\substack{k=i+1\\ k\neq l}}^{N}\pi_{v_k}(x_k)\right)\\
&\times\widetilde{g}(v_i,x_i)\beta_{v_i,v_j}\widetilde{g}(v_j,x_j).
\end{aligned}
$$

This is just the distribution we would have arrived at if we had just simulated $\{v_1,...,v_{l-1},v_{l+1},...,v_N\}$ in order. Using (15) repeatedly, we have proved the following *marginality* lemma.

Lemma III.1. *Suppose the conditions of Proposition III.1 hold and $\overrightarrow{B}\subset\overrightarrow{V}$. Then,*

$$\Pi_{\overrightarrow{B}}(x_{\overrightarrow{B}})=\sum_{x_{\overrightarrow{B}^C}\in\mathbf{X}_{\overrightarrow{B}^C}}\Pi_{\overrightarrow{V}}(x). \tag{16}$$

Example III.1. *The closed form on $\overrightarrow{V}=\{1,2\}$ is*

$$\Pi_{(1,2)}(x_1,x_2)=\pi_1(x_1)\pi_2(x_2)+\widetilde{g}(2,x_2)\beta_{2,1}\widetilde{g}(1,x_1) \tag{17}$$

and the closed form on $\overrightarrow{V}=\{1,2,3\}$ is

$$
\begin{aligned}
&\Pi_{(1,2,3)}(x_1,x_2,x_3)=\\
&\pi_1(x_1)\pi_2(x_2)\pi_3(x_3)\\
&\quad+\pi_3(x_3)\widetilde{g}(2,x_2)\beta_{2,1}\widetilde{g}(1,x_1)\\
&\quad+\widehat{\pi}_2(x_2)\widetilde{g}(3,x_3)\beta_{3,1}\widetilde{g}(1,x_1)\\
&\quad+\widehat{\pi}_1(x_1)\widetilde{g}(3,x_3)\beta_{3,2}\widetilde{g}(2,x_2).
\end{aligned}
$$

Now, suppose $\beta_{3,1} = \beta_{3,2} = 0$ so X_1 and X_2 are both uncorrelated from X_3 by Proposition III.1 a) and

$$\begin{aligned}&\Pi_{(1,2,3)}(x_1,x_2,x_3)\\ =&\big(\pi_1(x_1)\pi_2(x_2)+\widetilde{g}(2,x_2)\beta_{2,1}\widetilde{g}(1,x_1)\big)\pi_3(x_3)\\ =&\Pi_{(1,2)}(x_1,x_2)\pi_3(x_3)\end{aligned}$$

by the previous two equations so X_3 is actually independent of X_1, X_2. The situation is less simple when not considering the last vertex simulated. If $\beta_{3,1} = \beta_{2,1} = 0$ so X_3 and X_2 are both uncorrelated from X_1 by Proposition III.1 a), then by Lemma III.1 $\Pi_{(2,3)}(x_2,x_3) = \pi_2(x_2)\pi_3(x_3) + \widetilde{g}(3,x_3)\beta_{3,2}\widetilde{g}(2,x_2)$ and

$$\begin{aligned}&\Pi_{(1,2,3)}(x_1,x_2,x_3) =\\ &\quad\pi_1(x_1)\Pi_{(2,3)}(x_2,x_3)\\ &\quad+\left(\widehat{\pi}_1(x_1)-\pi_1(x_1)\right)\widetilde{g}(3,x_3)\beta_{3,2}\widetilde{g}(2,x_2).\end{aligned}$$

Hence, since each $\widetilde{\pi}_v$ is non-trivial we must either have $\widehat{\pi}_1 = \pi_1$ or $\beta_{3,2} = 0$ for X_1 to be independent of X_2, X_3. The case of X_2 being independent of X_1, X_3 similarly requires $\widehat{\pi}_2 = \pi_2$ or $\beta_{3,1} = 0$ in addition to $\beta_{3,2} = \beta_{2,1} = 0$.

This example illustrates several things about Quick Simulation Fields: order matters in general, there are dependent uncorrelated fields, and independence generally does not happen when $\widehat{\pi}_v \neq \pi_v$. Indeed, we explain below there is usually dependence even when $\widehat{\pi}_v = \pi_v$.

Example III.2. *In the important special case where $\widehat{\pi}_v = \pi_v$ for all v the closed form becomes:*

$$\begin{aligned}&\Pi_{v_1,...,v_n}(x_1,...,x_n) =\\ &\prod_{l=1}^{n}\pi_{v_l}(x_l)\left(1+\sum_{1\le j<i\le n}\frac{\widetilde{g}(v_i,x_i)}{\pi_{v_i}(x_i)}\beta_{v_i,v_j}\frac{\widetilde{g}(v_j,x_j)}{\pi_{v_j}(x_j)}\right)\end{aligned}\tag{18}$$

for each $x_i \in \mathbf{X}_{v_i}$ and $n = 1,...,N$.

Now, suppose that $l \in \{1,...,N-1\}$ and $\beta_{v_i,v_j} = 0$ when $j \le l < i$. Then,

$$\begin{aligned}&\frac{\Pi_{v_1,...,v_N}(x_1,...,x_N)}{\Pi_{v_1,...,v_l}(x_1,...,x_l)\Pi_{v_{l+1},...,v_N}(x_{l+1},...,x_N)} =\\ &1-\frac{\left(\sum\limits_{1\le j<i\le l}\frac{\widetilde{g}(v_i,x_i)}{\pi_{v_i}(x_i)}\beta_{v_i,v_j}\frac{\widetilde{g}(v_j,x_j)}{\pi_{v_j}(x_j)}\right)}{\left(1+\sum\limits_{1\le j<i\le l}\frac{\widetilde{g}(v_i,x_i)}{\pi_{v_i}(x_i)}\beta_{v_i,v_j}\frac{\widetilde{g}(v_j,x_j)}{\pi_{v_j}(x_j)}\right)}\times\\ &\frac{\left(\sum\limits_{l+1\le j<i\le N}\frac{\widetilde{g}(v_i,x_i)}{\pi_{v_i}(x_i)}\beta_{v_i,v_j}\frac{\widetilde{g}(v_j,x_j)}{\pi_{v_j}(x_j)}\right)}{\left(1+\sum\limits_{l+1\le j<i\le N}\frac{\widetilde{g}(v_i,x_i)}{\pi_{v_i}(x_i)}\beta_{v_i,v_j}\frac{\widetilde{g}(v_j,x_j)}{\pi_{v_j}(x_j)}\right)}\end{aligned}$$

so one requires

$$\left(\sum_{1\le j<i\le l}\frac{\widetilde{g}(v_i,x_i)}{\pi_{v_i}(x_i)}\beta_{v_i,v_j}\frac{\widetilde{g}(v_j,x_j)}{\pi_{v_j}(x_j)}\right)=0 \text{ or}$$
$$\left(\sum_{l+1\le j<i\le N}\frac{\widetilde{g}(v_i,x_i)}{\pi_{v_i}(x_i)}\beta_{v_i,v_j}\frac{\widetilde{g}(v_j,x_j)}{\pi_{v_j}(x_j)}\right)=0$$

for independence.

IV. Permutation Property

Let $V=\{v_i\}_{i=1}^N$, $M_k=\{1,2,\cdots,k\}$ for $k\le N$ and $\mathbb{G}_k$ be the symmetric group of permutations on M_k with composition, denoted $\circ$, as group operation, identity permutation $e(i)=i$, $\forall i\in M_k$ and generators $(i\ \ i+1)$ in cyclic form for $1\le i\le k-1$. $\{\{v_{a(i)}\}_{i=1}^N : a\in\mathbb{G}_N\}$ gives the possible simulation orders. We are interested in when the distribution is unchanged.

Definition IV.1. *Random field Π on $V=\{v_i\}_{i=1}^N$ satisfies the* ***Permutation Property*** *if:*

$$\Pi_{v_{a(1)},...,v_{a(N)}}(x_{a(1)},...,x_{a(N)})=\Pi_{v_1,...,v_N}(x_1,...,x_N)$$

for every $a\in\mathbb{G}_N$.

Marginality then gives

$$\Pi_{v_{i_{a(1)}},...,v_{i_{a(k)}}}(x_{i_{a(1)}},...,x_{i_{a(k)}})=\Pi_{v_{i_1},...,v_{i_k}}(x_{i_1},...,x_{i_k})$$

for every $1\le i_1<i_2<\cdots<i_k\le N$ and $a\in\mathbb{G}_k$ when the permutation property holds.

Theorem IV.1. *Suppose $N\ge 3$, $\overline{G}=(V,\overline{E})$ is the completion of* connected *undirected graph $G=(V,E)$ and $\{\pi_v\}_{v\in V}$, $\{\widetilde{\pi}_v\}_{v\in V}$ and $\{\widehat{\pi}_v\}_{v\in V}$ are non-degenerate pmfs. Then, in the following 1) and 2) are equivalent, and 3) implies 1) and 2).*

1) *The permutation property of the $(\pi_v,\widehat{\pi}_v,\widetilde{\pi}_v,\beta)$-Quick Simulation field Π on $\overline{G}$ holds.*
2) *For each distinct $u,v,w\in V$:*

$$\widetilde{\sigma}_w^2\widetilde{\pi}_v(x_v)(x_v-\widetilde{\mu}_v)\beta_{v,u}(\widehat{\pi}_w(x_w)-\pi_w(x_w)) = \widetilde{\sigma}_v^2\widetilde{\pi}_w(x_w)(x_w-\widetilde{\mu}_w)\beta_{w,u}(\widehat{\pi}_v(x_v)-\pi_v(x_v)) \tag{19}$$

for all $x_v\in\mathbf{X}_v$, $x_w\in\mathbf{X}_w$.

3) *For each $w\in V$:*

$$\pi_w(x_w)=\widehat{\pi}_w(x_w)+c_w(x_w-\widetilde{\mu}_w)\widetilde{\pi}_w(x_w), \tag{20}$$

where for each distinct $u,v,w\in V$ the constants satisfy:

$$\widetilde{\sigma}_u^2\beta_{wv}c_u=\widetilde{\sigma}_v^2\beta_{wu}c_v=\widetilde{\sigma}_w^2\beta_{vu}c_w. \tag{21}$$

Remark IV.1. *The following proof reveals the equivalence of 1) and 2) holds even if the original graph G is not connected.*

Proof: To ease notation, we let $\mathbf{X}_i = \mathbf{X}_{v_i}$, $\pi_i = \pi_{v_i}$, $\widehat{\pi}_i = \widehat{\pi}_{v_i}$, $\widetilde{\pi}_i = \widetilde{\pi}_{v_i}$, $\beta_{i,j} = \beta_{v_i,v_j}$ and

$$y_i(x_i) = \widehat{\pi}_i(x_i) - \pi_i(x_i) \quad \forall x_i \in \mathbf{X}_i, 1 \le i \le N. \tag{22}$$

For $a \in \mathbb{G}_N$, one has by commutativity and (9) that $X_{a(1)}, ..., X_{a(N)}$ has joint pmf

$$\begin{aligned}\Pi_{a(1),...,a(N)}(x_{a(1)}, ..., x_{a(N)}) = \prod_{i=1}^{N} \pi_i(x_i) + \\ \sum_{1 \le j < i \le N} \Big[\Big(\prod_{k=1, k \ne j}^{i-1} \widehat{\pi}_{a(k)}(x_{a(k)}) \times \prod_{k=i+1}^{N} \pi_{a(k)}(x_{a(k)}) \Big) \\ \times \widetilde{g}(a(i), x_{a(i)}) \beta_{a(i),a(j)} \widetilde{g}(a(j), x_{a(j)}) \Big]\end{aligned} \tag{23}$$

for each $x_i \in \mathbf{X}_i$ ($1 \le i \le N$). By (23) the permutation property is equivalent to

$$\begin{aligned}\sum_{1 \le j < i \le N} \Big(\prod_{k=1, k \ne j}^{i-1} \widehat{\pi}_{b(k)}(x_{b(k)}) \times \prod_{k=i+1}^{N} \pi_{b(k)}(x_{b(k)}) \Big) \times \\ \widetilde{g}(b(i), x_{b(i)}) \beta_{b(i),b(j)} \widetilde{g}(b(j), x_{b(j)}) = \\ \sum_{1 \le j < i \le N} \Big(\prod_{k=1, k \ne j}^{i-1} \widehat{\pi}_{a(k)}(x_{a(k)}) \times \prod_{k=i+1}^{N} \pi_{a(k)}(x_{a(k)}) \Big) \times \\ \widetilde{g}(a(i), x_{a(i)}) \beta_{a(i),a(j)} \widetilde{g}(a(j), x_{a(j)})\end{aligned} \tag{24}$$

for any two permutations a, b.

1) implies 2): Taking $b = (2\ 3) \circ a$, one finds that the left and right side terms in (24) are the same when $j = 1, i > 3$; $j = 2, i = 3$ or $j > 3$ so, upon cancelling these terms and substituting in for b, the remaining ($j = 1, i = 2$; $j = 1, i = 3$; $j = 2, i > 3$ and $j = 3, i > 3$) terms in (24) become

$$\begin{aligned}&\pi_{a(2)}(x_{a(2)}) \times \\ &\prod_{k=4}^{N} \pi_{a(k)}(x_{a(k)}) \widetilde{g}(a(3), x_{a(3)}) \beta_{a(3),a(1)} \widetilde{g}(a(1), x_{a(1)}) \\ &+ \widehat{\pi}_{a(3)}(x_{a(3)}) \times \\ &\prod_{k=4}^{N} \pi_{a(k)}(x_{a(k)}) \widetilde{g}(a(2), x_{a(2)}) \beta_{a(2),a(1)} \widetilde{g}(a(1), x_{a(1)}) \\ &+ \widehat{\pi}_{a(1)}(x_{a(1)}) \widehat{\pi}_{a(2)}(x_{a(2)}) \times \\ &\sum_{i=4}^{N} \prod_{k=4}^{i-1} \widehat{\pi}_{a(k)}(x_{a(k)}) \times \\ &\prod_{k=i+1}^{N} \pi_{a(k)}(x_{a(k)}) \widetilde{g}(a(i), x_{a(i)}) \beta_{a(i),a(3)} \widetilde{g}(a(3), x_{a(3)})\end{aligned}$$

$$
\begin{aligned}
&+\widehat{\pi}_{a(1)}(x_{a(1)})\widehat{\pi}_{a(3)}(x_{a(3)})\times\\
&\sum_{i=4}^{N}\prod_{k=4}^{i-1}\widehat{\pi}_{a(k)}(x_{a(k)})\times\\
&\prod_{k=i+1}^{N}\pi_{a(k)}(x_{a(k)})\widetilde{g}(a(i),x_{a(i)})\beta_{a(i),a(2)}\widetilde{g}(a(2),x_{a(2)})\\
=&\pi_{a(3)}(x_{a(3)})\times\\
&\prod_{k=4}^{N}\pi_{a(k)}(x_{a(k)})\widetilde{g}(a(2),x_{a(2)})\beta_{a(2),a(1)}\widetilde{g}(a(1),x_{a(1)})\\
&+\widehat{\pi}_{a(2)}(x_{a(2)})\times\\
&\prod_{k=4}^{N}\pi_{a(k)}(x_{a(k)})\widetilde{g}(a(3),x_{a(3)})\beta_{a(3),a(1)}\widetilde{g}(a(1),x_{a(1)})\\
&+\widehat{\pi}_{a(1)}(x_{a(1)})\widehat{\pi}_{a(3)}(x_{a(3)})\times\\
&\sum_{i=4}^{N}\prod_{k=4}^{i-1}\widehat{\pi}_{a(k)}(x_{a(k)})\times\\
&\prod_{k=i+1}^{N}\pi_{a(k)}(x_{a(k)})\widetilde{g}(a(i),x_{a(i)})\beta_{a(i),a(2)}\widetilde{g}(a(2),x_{a(2)})\\
&+\widehat{\pi}_{a(1)}(x_{a(1)})\widehat{\pi}_{a(2)}(x_{a(2)})\times\\
&\sum_{i=4}^{N}\prod_{k=4}^{i-1}\widehat{\pi}_{a(k)}(x_{a(k)})\times\\
&\prod_{k=i+1}^{N}\pi_{a(k)}(x_{a(k)})\widetilde{g}(a(i),x_{a(i)})\beta_{a(i),a(3)}\widetilde{g}(a(3),x_{a(3)}),
\end{aligned}
$$

which simplifies using (22) to

$$
\begin{aligned}
&\widetilde{g}(a(2),x_{a(2)})\beta_{a(2),a(1)}y_{a(3)}(x_{a(3)})=\\
&\widetilde{g}(a(3),x_{a(3)})\beta_{a(3),a(1)}y_{a(2)}(x_{a(2)}).
\end{aligned}
$$

Letting a be such that $a(1)=u$, $a(2)=v$ and $a(3)=w$, we find (19) is necessary.

2) implies 1): Multiplying (19) by $\tilde{g}(u,x_u)$ yields

$$
\begin{aligned}
&\tilde{g}(v,x_v)\beta_{v,u}\tilde{g}(u,x_u)(\widehat{\pi}_w(x_w)-\pi_w(x_w))=\\
&\tilde{g}(w,x_w)\beta_{w,u}\tilde{g}(u,x_u)(\widehat{\pi}_v(x_v)-\pi_v(x_v))
\end{aligned}
\tag{25}
$$

for all $x_u\in\mathbf{X}_u$, $x_v\in\mathbf{X}_v$, $x_w\in\mathbf{X}_w$ and distinct $u,v,w\in M_N$. Take $a\in\mathbb{G}_N$ and let $b=(l\ l+1)\circ a$ for $1\le l\le N-1$. Noting that the transpose operations $(l\ l+1)$ are generators, we just need to show (24) for (arbitrary) a and this b. However, the left hand terms in (24) with $i<l$; $j>l+1$; $j\le l-1, l+2\le i$; and $j=l, i=l+1$ directly cancel with the corresponding right hand terms for this b. Considering the (remaining) terms on the left side of (24) with $j \leq l-1$ and $i = l, l+1$ for this b and using (25) with $u = a(j), v = a(l), w = a(l+1)$, we get upon manipulation:

$$\begin{aligned}
&\sum_{j=1}^{l-1} \prod_{\substack{k=1 \\ k \neq j}}^{l-1} \widehat{\pi}_{a(k)}(x_{a(k)}) \times \\
&\prod_{k=l+2}^{N} \pi_{a(k)}(x_{a(k)}) \widetilde{g}(a(l+1), x_{a(l+1)}) \times \\
&\beta_{a(l+1),a(j)} \widetilde{g}(a(j), x_{a(j)}) \pi_{a(l)}(x_{a(l)}) \\
&+ \sum_{j=1}^{l-1} \prod_{\substack{k=1 \\ k \neq j}}^{l-1} \widehat{\pi}_{a(k)}(x_{a(k)}) \times \\
&\prod_{k=l+2}^{N} \pi_{a(k)}(x_{a(k)}) \widetilde{g}(a(l), x_{a(l)}) \times \\
&\beta_{a(l),a(j)} \widetilde{g}(a(j), x_{a(j)}) \widehat{\pi}_{a(l+1)}(x_{a(l+1)}) \\
= &\sum_{j=1}^{l-1} \prod_{\substack{k=1 \\ k \neq j}}^{l-1} \widehat{\pi}_{a(k)}(x_{a(k)}) \times \\
&\prod_{k=l+2}^{N} \pi_{a(k)}(x_{a(k)}) \widetilde{g}(a(l+1), x_{a(l+1)}) \times \\
&\beta_{a(l+1),a(j)} \widetilde{g}(a(j), x_{a(j)}) \widehat{\pi}_{a(l)}(x_{a(l)}) \\
&+ \sum_{j=1}^{l-1} \prod_{\substack{k=1 \\ k \neq j}}^{l-1} \widehat{\pi}_{a(k)}(x_{a(k)}) \times \\
&\prod_{k=l+2}^{N} \pi_{a(k)}(x_{a(k)}) \widetilde{g}(a(l), x_{a(l)}) \times \\
&\beta_{a(l),a(j)} \widetilde{g}(a(j), x_{a(j)}) \pi_{a(l+1)}(x_{a(l+1)})
\end{aligned} \tag{26}$$

so they are equal to the corresponding terms on the right of (24). (Notice the switch of π and $\widehat{\pi}$ in the final factors in (26).) Finally, the terms on the left of (24) with $j = l, i \geq l+2$ and $j = l+1, i \geq l+2$ for $b = (l\ l+1)a$:

$$\begin{aligned}
&\sum_{i=l+2}^{N} \Bigg(\prod_{\substack{k=1 \\ k \neq l+1}}^{i-1} \widehat{\pi}_{a(k)}(x_{a(k)}) \prod_{k=i+1}^{N} \pi_{a(k)}(x_{a(k)}) \Bigg) \\
&\widetilde{g}(a(i), x_{a(i)}) \beta_{a(i),a(l+1)} \widetilde{g}(a(l+1), x_{a(l+1)}) \\
&+ \sum_{i=l+2}^{N} \Bigg(\prod_{\substack{k=1 \\ k \neq l}}^{i-1} \widehat{\pi}_{a(k)}(x_{a(k)}) \prod_{k=i+1}^{N} \pi_{a(k)}(x_{a(k)}) \Bigg) \\
&\widetilde{g}(a(i), x_{a(i)}) \beta_{a(i),a(l)} \widetilde{g}(a(l), x_{a(l)})
\end{aligned}$$

are just the terms on the right of (24) with $j = l+1, i \geq l+2$ and $j = l, i \geq l+2$ i.e. in reverse order. Hence, by breaking the summation up, we have shown (24) holds for arbitrary a and $b = (l\ l+1) \circ a$, which implies (24) holds for arbitrary a, b and sufficiency follows.

3) implies 2): Letting $u, v, w \in V$ be distinct and using (20,21), we have that:

$$\begin{aligned}
&\widetilde{\sigma}_w^2 \widetilde{\pi}_v(x_v)(x_v - \widetilde{\mu}_v)\beta_{v,u}(\widehat{\pi}_w(x_w) - \pi_w(x_w)) \\
=& -\widetilde{\sigma}_w^2 \widetilde{\pi}_v(x_v)(x_v - \widetilde{\mu}_v)\beta_{v,u} c_w \widetilde{\pi}_w(x_w)(x_w - \widetilde{\mu}_w) \\
=& -\widetilde{\sigma}_v^2 \widetilde{\pi}_v(x_v)(x_v - \widetilde{\mu}_v)\beta_{w,u} c_v \widetilde{\pi}_w(x_w)(x_w - \widetilde{\mu}_w) \\
=& \widetilde{\sigma}_v^2 (\widehat{\pi}_v(x_v) - \pi_v(x_v))\beta_{w,u} \widetilde{\pi}_w(x_w)(x_w - \widetilde{\mu}_w)
\end{aligned}$$

for all $x_v \in \mathbf{X}_v$, $x_w \in \mathbf{X}_w$. ■

In Theorem IV.1, 1) and 2) almost imply 3), which would establish equivalence. However, the graph must be sufficiently connected as the following example shows.

Example IV.1. *Suppose G is a connected graph with $N \geq 3$. Suppose further that G is not sufficiently connected. Choose distinct vertices u, v, w so that u and v are neighbors of w, and that u and v are not neighbors. The completion of the graph will set $\beta_{vu} = 0$ in (19), which in turn implies that either $\widehat{\pi}_v(x_v) - \pi_v(x_v) = 0\ \forall x_v \in \mathbf{X}_v$ or $\beta_{u,w} = 0$ on the RHS of (19). In the former, $\widehat{\pi}_v = \pi_v$. In the latter, G is not connected so only the former is possible. If 3) were true, then $c_v = 0$ by (20) and then $c_u = c_w = 0$ by (21) and connectedness.*

Theorem IV.2. *Suppose $N \geq 3$, $\overline{G} = (V, \overline{E})$ is the completion of* sufficiently connected *undirected graph $G = (V, E)$; $\{\pi_v\}_{v\in V}$, $\{\tilde{\pi}_v\}_{v\in V}$ and $\{\widehat{\pi}_v\}_{v\in V}$ are non-degenerate pmfs with $\{\pi_v \neq \widehat{\pi}_v\}_{v\in V}$; and 1), 2), and 3) are as in Theorem IV.1. Then, 1) or 2) imply 3).*

Proof: 1) and 2) are equivalent by Theorem IV.1.

2) implies 3): Let w be the neighbor of u that has a second neighbor v. This means there are non-zero covariances from w to the other two. (19) is (by permuting u, v, w) equivalent to:

$$\begin{bmatrix} \widetilde{g}(v, x_v)\beta_{wv} & -\widetilde{g}(u, x_u)\beta_{wu} & \\ & \widetilde{g}(w, x_w)\beta_{wu} & -\widetilde{g}(v, x_v)\beta_{vu} \\ -\widetilde{g}(w, x_w)\beta_{wv} & & \widetilde{g}(u, x_u)\beta_{vu} \end{bmatrix}
\begin{bmatrix} y_u(x_u) \\ y_v(x_v) \\ y_w(x_w) \end{bmatrix} = 0,$$

which implies all solutions have the form

$$\begin{aligned}
y_u(x_u) &= \frac{\widetilde{g}(u, x_u)\beta_{vu}}{\widetilde{g}(w, x_w)\beta_{wv}} y_w(x_w), \\
y_v(x_v) &= \frac{\widetilde{g}(v, x_v)\beta_{vu}}{\widetilde{g}(w, x_w)\beta_{wu}} y_w(x_w)
\end{aligned}$$

for all $x_u \in \mathbf{X}_u$, $x_v \in \mathbf{X}_v$, $x_w \in \mathbf{X}_w$. This implies that $\frac{y_w(x_w)}{\tilde{g}(w,x_w)}$ is constant, which in turn implies

$$\pi_u(x_u) = \widehat{\pi}_u(x_u) + c_u(x_u - \widetilde{\mu}_u)\widetilde{\pi}_u(x_u), \tag{27}$$

$$\pi_v(x_v) = \widehat{\pi}_v(x_v) + c_v(x_v - \widetilde{\mu}_v)\widetilde{\pi}_v(x_v), \tag{28}$$

where c_u, c_v are constants. Since G is sufficiently connected (by non-zero covariances) every vertex can be included in some connected triple as above and we must have that

$$\pi_w(x_w) = \widehat{\pi}_w(x_w) + c_w(x_w - \widetilde{\mu}_w)\widetilde{\pi}_w(x_w) \tag{29}$$

for all $w \in V$. Now, choosing distinct (not-necessarily connected) $u, v, w \in V$ and using (19), we find that these constants must satisfy: $\widetilde{\sigma}_u^2 \beta_{wv} c_u = \widetilde{\sigma}_v^2 \beta_{wu} c_v = \widetilde{\sigma}_w^2 \beta_{vu} c_w$. ■

Example IV.2. *When $N = 3$, one c, c_u say, can be chosen arbitrarily and the other two can then be solved for by (21).*

Example IV.3. *There is always the trivial solution $\widehat{\pi}_u = \pi_u$ (and $\widetilde{\pi}_u$ arbitrary) for all u. This corresponds to taking all the c_u to be 0.*

The above theorem gives us the necessary relation

$$\pi_w(x_w) = \widehat{\pi}_w(x_w) + c_w(x_w - \widetilde{\mu}_w)\widetilde{\pi}_w(x_w) \quad \forall w \in V, \tag{30}$$

for the permutation property to hold under sufficient connectivity. Below we will consider *completely non-trivial Quick Simulation Fields* meaning $\pi_w \neq \widehat{\pi}_w$, i.e. $c_w \neq 0$, for all $w \in V$.

Theorem IV.3. *Suppose $N \geq 4$, $\overline{G} = (V, \overline{E})$ is the completion of* connected *undirected graph $G = (V, E)$ and $\{\pi_v\}_{v\in V}$, $\{\tilde{\pi}_v\}_{v\in V}$ and $\{\widehat{\pi}_v\}_{v\in V}$ are non-degenerate pmfs. Then, the following are equivalent:*

a) *G is sufficiently connected and disjoint pair rich and there is a completely non-trivial $(\pi_v, \widehat{\pi}_v, \widetilde{\pi}_v, \beta)$-Quick Simulation field Π on $\overline{G}$ satisfying the permutation property.*
b) *Original graph G is complete, (30) holds with at least one $c_w \neq 0$, and for each distinct $u, v, w, z \in V$:*

$$\beta_{u,v}\beta_{w,z} = \beta_{u,w}\beta_{v,z}. \tag{31}$$

If a) and b) hold, the constants in (30) can be taken as:

$$c_{v_1} = \frac{\widetilde{\sigma}_{v_2}^2}{\widetilde{\sigma}_{v_1}^2}\frac{\beta_{v_3,v_1}}{\beta_{v_3,v_2}}c_{v_2}, \tag{32}$$

$$c_{v_{i+1}} = \frac{\widetilde{\sigma}_{v_i}^2}{\widetilde{\sigma}_{v_{i+1}}^2}\frac{\beta_{v_{i+1},v_{i-1}}}{\beta_{v_i,v_{i-1}}}c_{v_i} \quad \forall i = 2, ..., N-1,$$

where $c_{v_2} \neq 0$ is arbitrary and $G = \{v_i\}_{i=1}^N$.

Proof: **(a) implies (b):** (30) holds by Theorem IV.2. For distinct $u, v, w, z \in V$, we find by (21) that

$$\widetilde{\sigma}_u^2 \beta_{zu}\beta_{wv}c_u$$
$$= \widetilde{\sigma}_v^2 \beta_{zu}\beta_{wu}c_v = \widetilde{\sigma}_v^2 \beta_{wu}\beta_{zu}c_v = \widetilde{\sigma}_u^2 \beta_{zv}\beta_{wu}c_u$$

so after cancellation ($\widetilde{\pi}_u$ is non-trivial, $c_u \neq 0$)

$$\beta_{zu}\beta_{wv} - \beta_{zv}\beta_{wu} = 0. \tag{33}$$

Now, suppose $z, u \in V$ that are *not* neighbors in G, we choose distinct v and w to be neighbors of z and u respectively (by disjoint pair rich property). Then, (33) implies

$$\beta_{zu}\beta_{wv} = \beta_{zv}\beta_{wu} \neq 0 \;\Rightarrow \beta_{zu} \neq 0 \tag{34}$$

and there is a contradiction. Hence, every $z, u \in V$ are neighbors and G is complete.

(b) implies (a): It follows from completeness and (32) that each $c_i \neq 0$,

$$\widetilde{\sigma}_1^2\beta_{3,2}c_1 = \widetilde{\sigma}_2^2\beta_{3,1}c_2 = \widetilde{\sigma}_3^2\beta_{2,1}c_3$$

and

$$\widetilde{\sigma}_{i+1}^2 c_{i+1}\beta_{i,i-1} = \widetilde{\sigma}_i^2 c_i\beta_{i+1,i-1} = \widetilde{\sigma}_{i-1}^2 c_{i-1}\frac{\beta_{i,i-2}\beta_{i+1,i-1}}{\beta_{i-1,i-2}}$$

for all $i = 3, ..., N-1$. However, it follows by (31) that

$$\beta_{i,i-2}\beta_{i+1,i-1} - \beta_{i-1,i-2}\beta_{i+1,i} = 0 \tag{35}$$

so by the previous two equations

$$\widetilde{\sigma}_{i+1}^2 c_{i+1}\beta_{i,i-1} = \widetilde{\sigma}_i^2 c_i\beta_{i+1,i-1} = \widetilde{\sigma}_{i-1}^2 c_{i-1}\beta_{i+1,i} \tag{36}$$

$\forall i \in \{2, ..., N-1\}$.

We have shown (21) in the case $u = v_{i-1}$, $v = v_i$, and $w = v_{i+1}$. Now, let $u, v, w \in V$ be arbitrary. Then, they correspond to $v_{i_3}, v_{i_2}, v_{i_1}$ respectively and, without loss of generality, we can assume that $1 \leq i_1 < i_2 < i_3 \leq N$. Using the left hand equality in (36) repeatedly, we find

$$\begin{aligned}
&\widetilde{\sigma}_{i_3}^2 c_{i_3}\beta_{i_2,i_2-1} \\
=&\widetilde{\sigma}_{i_3}^2 c_{i_3}\prod_{j=i_2}^{i_3-1}\beta_{j,j-1}\frac{1}{\prod_{j=i_2+1}^{i_3-1}\beta_{j,j-1}} \\
=&\widetilde{\sigma}_{i_2}^2 c_{i_2}\prod_{j=i_2}^{i_3-1}\beta_{j+1,j-1}\frac{1}{\prod_{j=i_2+1}^{i_3-1}\beta_{j,j-1}}.
\end{aligned} \tag{37}$$

However, it follows by repeated use of (31) that

$$\prod_{j=i_2}^{i_3-1}\beta_{j+1,j-1} = \beta_{i_3,i_2-1}\prod_{j=i_2+1}^{i_3-1}\beta_{j,j-1}. \tag{38}$$

Combining (37) and (38), one finds

$$\widetilde{\sigma}_{i_3}^2 c_{i_3}\beta_{i_2,i_2-1} \quad = \quad \widetilde{\sigma}_{i_2}^2 c_{i_2}\beta_{i_3,i_2-1}. \tag{39}$$

Moreover, using (31) again (when $i_2 - 1 \neq i_1$), one has that

$$\frac{\beta_{i_2,i_2-1}}{\beta_{i_3,i_2-1}} = \frac{\beta_{i_2,i_1}}{\beta_{i_3,i_1}} \tag{40}$$

and, substituting (40) into (39) and relabelling, that

$$\widetilde{\sigma}_u^2 c_u \beta_{v,w} \quad = \quad \widetilde{\sigma}_v^2 c_v \beta_{u,w}. \tag{41}$$

Hence, the first equality in (21) holds. The second equality follows in exactly the same manner using the second equality in (36) in lieu of the first. ■

Example IV.4. *Let* $V = \{1,2,3,4\}$*;* $A = \{(1,2),(1,3),(1,4),(2,3),(2,4),(3,4)\}$*;* $\mathbf{X}_i = \{1,-1\}$ *for* $1 \le i \le 4$*;* $\tilde{\pi}_i(x_i) = \frac{1}{2}\ \forall x_i \in \mathbf{X}_i$*; and* $\pi_i(1) = p$ *and* $\pi_i(-1) = 1-p$ *where* $0 < p < 1$*. It follows that* $\tilde{\mu}_i = 0$ *and* $\tilde{\sigma}_i^2 = 1$*. Let* $\beta_{12} = \beta_{13} = \beta_{23} = \beta_{14} = \beta_{24} = \beta_{34} = \beta$ *so (31) holds trivially. It follows by (32) that* $c_1 = c_2 = c_3 = c_4$*. Then,* $\widehat{\pi}_i$ *must satisfy (30) and be a non-trivial pmf for each* $i = 1,2,3,4$ *i.e.*

$$\widehat{\pi}_i(1) = \underbrace{\pi_i(1)}_{p} - c_i \cdot 1 \cdot \underbrace{\widetilde{\pi}_i(1)}_{\frac{1}{2}} > 0$$

and

$$\widehat{\pi}_i(-1) = 1 - p + \frac{c_i}{2} > 0.$$

This translates into the condition $2p > c_i > 2p-2$*. The permutable joint pmf of* $X_1,\ldots,X_4$ *follows from (9) as*

$$\begin{aligned}
&\Pi_{1,2,3,4}(x_1,x_2,x_3,x_4) = \\
&\quad \prod_{i=1}^{4} \pi_i(x_i) \\
&\quad + \frac{1}{4}\beta\big[x_1x_2\pi_3(x_3)\pi_4(x_4) + x_1x_3\widehat{\pi}_2(x_2)\pi_4(x_4) \\
&\quad + x_2x_3\widehat{\pi}_1(x_1)\pi_4(x_4) + x_1x_4\widehat{\pi}_2(x_2)\widehat{\pi}_3(x_3) \\
&\quad + x_2x_4\widehat{\pi}_1(x_1)\widehat{\pi}_3(x_3) + x_3x_4\widehat{\pi}_1(x_1)\widehat{\pi}_2(x_2)\big]
\end{aligned}$$

for each $x_i \in \mathbf{X}$ *(*$1 \le i \le 4$*).*

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