Topological Design for the Control of Connected Disperse Systems

by

Ahmad Wail Al-Dabbagh

A thesis submitted in partial fulfillment of the requirements for the degree of

Doctor of Philosophy

in

Control Systems

Department of Electrical and Computer Engineering
University of Alberta

Abstract

Today, the world witnesses a growing demand for the connectivity between systems. The connectivity can span vast distances and information can be transferred between the systems in a relatively insignificant time. This is evident in different applications, such as process facilities, remote medical operations, traffic networks, as well as power generation, transmission, and distribution. Thus, there is a need for the advancement of approaches for the control of connected disperse systems.

In a typical control system topology, a plant system is regulated using a controller system. In this thesis, an alternative control system topology is proposed and its associated design is presented. In the proposed topology, a plant system can be connected to a controller system and/or a set of distributed and inter-connected nodes that form a network system. The set of distributed and inter-connected nodes are capable of routing information between all the nodes of the topology, in addition to performing computational tasks. Thus, the plant system is regulated using the controller and network systems, in an individual or a cooperative manner. This introduces both centralized and decentralized control paradigms in the proposed control system topology.

In this thesis, the proposed topology and its associated design are addressed from different perspectives, while delivering modelling frameworks, condition requirements, and design procedures as well as under ideal oper-

ating scenarios, failures and cyber attacks, induced connectivity constraints, and additional specifications in terms of model reduction and segregation of the nodes of the network system into two disjoint sets of nodes with different objectives.

Preface

This thesis is based on a number of articles that are either published, accepted, or submitted for consideration for publication. Its contents, such as texts and figures, are also in the articles. The articles are listed in [2–7] and are also listed below.

- Ahmad W. Al-Dabbagh and Tongwen Chen. Modelling and control of wireless networked control systems: a fixed structure approach. *In the Proceedings of the 2015 IEEE Conference on Control Applications (Part of the 2015 IEEE Multi-Conference on Systems and Control)*, pages 1051 – 1056, Sydney, Australia, 2015.

- Ahmad W. Al-Dabbagh and Tongwen Chen. Design considerations for wireless networked control systems. *IEEE Transactions on Industrial Electronics*, 63(9):5547 – 5557, 2016.

- Ahmad W. Al-Dabbagh and Tongwen Chen. A fixed structure topology for wireless networked control systems. *In the Proceedings of the 55th IEEE Conference on Decision and Control*, pages 3450–3455, Las Vegas, USA, 2016.

- Ahmad W. Al-Dabbagh, Yuzhe Li, and Tongwen Chen. An intrusion detection system for cyber attacks in wireless networked control systems. *IEEE Transactions on Circuits and Systems II: Express Briefs (In Press with Early Access).*

- Ahmad W. Al-Dabbagh. Design of a wireless control system with unreliable nodes and communication links. *IEEE Transactions on Cybernetics (In Press with Early Access).*

- Ahmad W. Al-Dabbagh, Aryan Saadat Mehr, and Tongwen Chen. Strategic topological formation for wireless control systems. *Submitted to a*

Journal.

More specifically, Chapters 1 and 6 are based on several of the listed articles, Chapter 2 is based on the first, second, and sixth listed articles, Chapter 3 is based on the fourth and fifth listed articles, Chapter 4 is based on the sixth listed article, and Chapter 5 is based on the second and third listed articles. The contents of the articles are my original work, where I formulated the problems, carried out the research, derived the results, conducted the simulations, as well as delivered their presentation and writing. My co-authors, Drs. Tongwen Chen, Yuzhe Li, and Aryan Saadat Mehr contributed in providing some feedback for the work, and for the presentation and writing of the articles.

Dedication

To my mother Dr. Aala R. Ali and my father Dr. Wail Y. Al-Dabbagh: Your endless love, motivation, and determination will always inspire me.

Acknowledgements

First, I thank my PhD supervisor Dr. Tongwen Chen for over four years of support and guidance, and for providing a perfect environment for learning and growth. To me, Dr. Chen is truly exemplary; he is a perfect supervisor. I also thank my supervisory committee members Drs. Qing Zhao and Mahdi Tavakoli for their constructive feedback as well as the examiners Drs. Amir G. Aghdam and Jinfeng Liu.

During my PhD studies, I collaborated with several researchers. As a result, several articles were produced, though some of which are not related to this thesis. I thank all my collaborators, Drs. Sirish L. Shah, Wenkai Hu, Aryan Saadat Mehr, Yuzhe Li, Aibing Qiu, Shiqi Lai, Muhammad Shahzad Afzal, and Mr. David Li. In Summer 2016, I visited the University of Toronto Institute for Aerospace Studies, and I therefore thank Dr. Hugh H.T. Liu for being a very welcoming host.

Also during my PhD studies, I received generous financial support to pursue my research and to travel to conferences in Sydney (Australia), Trondheim (Norway), Las Vegas (USA), Toulouse (France), and the Big Island of Hawai'i (USA). I therefore thank the University of Alberta and its Faculty of Graduate Studies and Research as well as the Graduate Students' Association, the Government of Alberta, the Natural Sciences and Engineering Research Council of Canada, the Alberta Innovates – Technology Futures and Alberta Innovation & Advanced Education, the IEEE Control Systems Society, the International Society of Automation, and Shell Canada.

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List of Symbols

$\mathbb{B}$	Boolean Domain
$\mathbb{R}$	Set of Real Numbers
$\mathbb{P}(.)$	Probability Operator of a Random Variable
$\mathbb{E}(.)$	Expectation Operator of a Random Variable
$\vee$	Logic OR Operator
$\mathbf{I}$	Identity Matrix
$\mathbf{0}$	Zero Matrix or Zero Vector
$\mathbf{1}$	One Matrix or One Vector
M^T	Transpose of Matrix M or Vector M
M^{-1}	Inverse of Matrix M
$M_{[i,j]}$	Entry in Row i and Column j of Matrix M
$*$	Symmetric Off-diagonal Term in a Symmetric Matrix
$> (\geq)$	Positive Definiteness (Semi-definiteness) of a Matrix
$<$	Negative Definiteness of a Matrix
diag(.)	Diagonalization Operator of a Matrix
$\| . \|$	Euclidean-based Norm
$\tilde{\lambda}(.)$	Set of Eigenvalues
Dim(.)	Dimension/Number of Elements of a Vector
$\vert . \vert$	Cardinality/Number of Elements of a Set
v_{G_i}	Node i of System G
$\mathscr{N}_{v_i}$	Neighbourhood of Node v_i

$\rightarrow$	Uni-directional Connection Between Two Nodes or Sets
$\leftrightarrow$	Bi-directional Connection Between Two Nodes or Sets
$d(v_i, v_j)$	Distance/Number of Links Between Nodes v_i and v_j
$\bigcup$	Union of Sets
$\bigcap$	Intersection of Sets
$U \setminus Y$	Elements in Set U but not in Set Y

List of Acronyms

CDS	Connected Disperse System
DCS	Decentralized Control System
DoS	Denial of Service
IDS	Intrusion Detection System
LMI	Linear Matrix Inequality
LTI	Linear Time-Invariant
PBH	Popov-Belevitch-Hautus
QTP	Quadruple Tank Process
WCN	Wireless Control Network
WCS	Wireless Control System
WNCS	Wireless Networked Control System
WSAN	Wireless Sensor and Actuator Network

Chapter 1

Introduction

1.1 Background and Motivation

In a standard control system topology, a plant system is regulated using a controller system*. The plant system has sensor nodes to provide measurements and actuator nodes to implement control commands. The controller system has input nodes to receive measurements and output nodes to send control commands, and is capable of making control decisions. An example standard control system topology is depicted in Figure 1.1.

Suppose the plant system has a set of sensor nodes $\mathcal{S} = \{s_1, \ldots, s_p\}$ and a set of actuator nodes $\mathcal{A} = \{a_1, \ldots, a_m\}$. It is modelled as a linear time-invariant (LTI) system in discrete time as

$$\begin{aligned} \mathbf{x}_G(k+1) &= A\mathbf{x}_G(k) + B\mathbf{u}(k), \\ \mathbf{y}(k) &= C\mathbf{x}_G(k) + D\mathbf{u}(k), \end{aligned} \tag{1.1}$$

where the vectors $\mathbf{x}_G(k) \in \mathbb{R}^n$, $\mathbf{u}(k) \in \mathbb{R}^m$, and $\mathbf{y}(k) \in \mathbb{R}^p$ denote its state, controlled input, and measured output, respectively, and all of its system matrices have suitable dimensions. Then, suppose the controller system has a set of input nodes $\Gamma = \{\gamma_1, \ldots, \gamma_p\}$ and a set of output nodes $\Theta = \{\theta_1, \ldots, \theta_m\}$. It is modelled as a LTI system in discrete time as

$$\begin{aligned} \mathbf{x}_K(k+1) &= A_K\mathbf{x}_K(k) + B_K\mathbf{y}(k), \\ \mathbf{u}(k) &= C_K\mathbf{x}_K(k) + D_K\mathbf{y}(k), \end{aligned} \tag{1.2}$$

where the vector $\mathbf{x}_K(k) \in \mathbb{R}^r$ denotes its state, and all of its system matrices have suitable dimensions†. Further, the standard control system topology in

*The notion of topology refers to the setup of a closed-loop control system.
†Note that $|\mathcal{S}| = |\Gamma|$ and $|\mathcal{A}| = |\Theta|$, and that typically $n = r$.

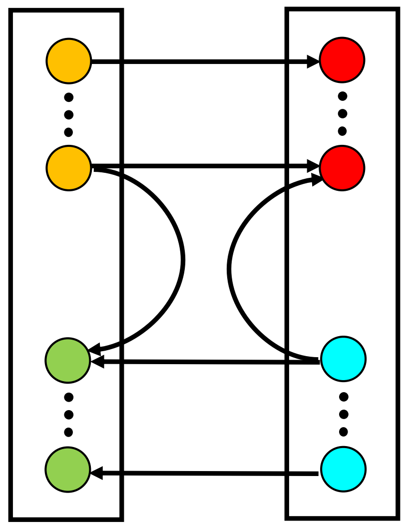

Figure 1.1: Standard control system topology with a plant system (left block) and a controller system (right block), where the sensor nodes (yellow circles) and actuator nodes (green circles) of the plant system as well as the input nodes (red circles) and output nodes (cyan circles) of the controller system are connected using wired and/or wireless communication links (black arrows).

Figure 1.1 is that of a centralized control system paradigm, where the control decisions are made in a centralized manner in the controller system.

In addition, the use of a communication network to connect the nodes of the plant system in (1.1) and the controller system in (1.2) forms a closed-loop networked control system[‡]. More specifically, the use of a wireless communication network forms a closed-loop wireless networked control system (WNCS) (or wireless control system (WCS)). The use of a wireless communication network offers additional advantages, compared to its wired counterpart. The advantages include an associated reduction in the amount of wiring, troubleshooting, and maintenance as well as an enhanced flexibility in the deployment, mobility, configuration, and connectivity of largely disperse nodes and systems. However, challenges also arise; for example, when the nodes are to become more geographically distributed as well as connected, when the information is to be transferred between the nodes in a relatively short time and with a high accuracy, when the control decisions are to be made in a decen-

‡The notion of communication network refers to a set of communication links/channels that are used to transfer information between communicating nodes.

tralized manner, and when the closed-loop control system is to be robust to failures in the operation of the nodes and in the transfer of information as well as to cyber and physical attacks.

1.2 Literature Survey

1.2.1 Control of Connected Systems

Recently in the literature, there has been a growing interest in studying the control of connected systems. The earlier studies investigated the use of communication networks in control systems (for example, see [20, 22, 53]). The more recent studies specifically investigated the use of wireless communication networks in control systems. The investigated challenges that are directly and indirectly related to a WCS include the following:

i. The effects and limitations of using wireless communication;

ii. The improvement of the communication;

iii. The existence of cyber attacks;

iv. The formation of topologies; and

v. The application in industrial settings.

More specifically, the studies that addressed challenge (i) include the stabilization of the control system with communication channels subject to fading [15], the design of a predictive control and a self-triggered sampling scheme for networked systems with communication subject to delays and data loss [30, 43], and the use of delay impulsive systems to model WNCSs with variable sampling intervals, delays, and packet dropouts [36].

The studies that addressed challenge (ii) include the minimization of the power consumption of the communication system [46], the design of optimal control and communication power management policies [19], the adjustment of the probability of successful communication using redundant transmission in communication protocols [34], and the investigation of the maximum area coverage while accounting for the convergence of the estimator [26].

The studies that addressed challenge (iii) include the control and estimation of linear systems under corrupted sensor and actuator nodes [16], the development of techniques to detect integrity attacks on the sensor nodes [35], the resiliency of the control system under replay attacks [62], and the characterization and modelling of a control system under different types of cyber attacks [55].

The studies that addressed challenge (iv) include the placement of an adaptive controller system in a wireless sensor and actuator network (WSAN) under the presence of erasure channels [45], the development of a decentralized event-triggered approach over WSANs [33], the characterization of the controllability of complex networks [31], the investigation of strategic approaches for multi-layer network formations [48–50], and the development of a wireless control network (WCN) that delivers a distributed control strategy and eliminates the need for a centralized controller system [32, 37–42, 52].

Finally, the studies that addressed challenge (v) include the use of wireless communication networks at the field level for factory automation [13], the deployment of a wireless fieldbus for plastic machineries [17], the evaluation of a distributed estimation and collaborative control scheme for WSANs in industrial control systems [9], the integration of a wireless interface for sensor and actuator nodes into wired fieldbus networks for factory automation [27], and the investigation of an $\mathcal{H}_\infty$ fault estimation scheme for industrial applications [57].

1.2.2 The Wireless Control Network

As previously discussed, the development of the WCN [32, 37–42, 52] delivers a distributed control strategy and eliminates the need for a centralized controller system. This makes it a promising topology for the control of connected systems. More specifically, in the control system topology that utilizes the WCN, a plant system with sensor nodes $\mathcal{S} = \{s_1, \ldots, s_p\}$ and actuator nodes $\mathcal{A} = \{a_1, \ldots, a_m\}$ is regulated by only using a set of distributed and inter-connected nodes $\mathcal{V} = \{v_1, \ldots, v_N\}$ that collectively form the WCN. The nodes are capable of having connections with each other as well as with the sensor and actuator nodes of the plant system through the use of wireless communication links, in addition to performing computational tasks. An example control system topology that utilizes the WCN is depicted in Figure 1.2.

Further, suppose the WCN is utilized to regulate the plant system in (1.1). Each node of the WCN updates its state based on its current state as well as

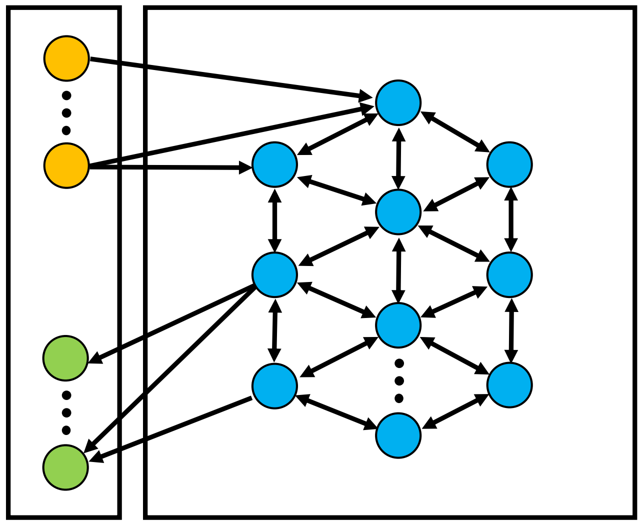

Figure 1.2: Control system topology that utilizes the WCN [42] with a plant system (left block) and the WCN (right block), where the sensor nodes (yellow circles) and the actuator nodes (green circles) of the plant system as well as the distributed and inter-connected nodes (blue circles) of the WCN are connected using wireless communication links (black arrows).

the current states of its neighbouring nodes (namely, those of the WCN and the sensor nodes of the plant system)[§]. Similarly, each actuator node of the plant system updates its state based on the current states of its neighbouring nodes. The update procedure is modelled for each node of the WCN with state x_{N_i} and for each actuator node of the plant system with state u_i in discrete time as

$$x_{N_i}(k+1) = \omega_{ii} x_{N_i}(k) + \sum_{v_j \in \mathcal{N}_{v_i}} \omega_{ij} x_{N_j}(k) + \sum_{s_j \in \mathcal{N}_{v_i}} \lambda_{ij} y_j(k),$$

$$u_i(k) = \sum_{v_j \in \mathcal{N}_{a_i}} \upsilon_{ij} x_{N_j}(k) + \sum_{s_j \in \mathcal{N}_{a_i}} \xi_{ij} y_j(k),$$

where the coefficients ω_{ij}, λ_{ij}, υ_{ij}, and ξ_{ij} denote the weights assigned to the states received by node i from node j, and ω_{ii} denotes the weight assigned to the self-connectivity link. The states of the nodes of the WCN and the actuator nodes of the plant system in an augmented manner (namely, in a

§The notion of neighbouring nodes refers to the set of nodes that directly transfer information to the receiving node.

stacked format) provide a LTI system in discrete time as

$$\mathbf{x}_N(k+1) = \Omega\mathbf{x}_N(k) + \Lambda\mathbf{y}(k),$$
$$\mathbf{u}(k) = \Upsilon\mathbf{x}_N(k) + \Xi\mathbf{y}(k),$$

where the matrices Ω, Λ, Υ, and Ξ contain the coefficients denoting the weight assignments. The values of the matrices are determined to deliver a control system configuration that satisfies specific objectives (e.g., stability and performance), and they are therefore considered as the design variables.

The topology that utilizes the WCN in Figure 1.2 offers several advantages, compared to its standard control system topology counterpart in Figure 1.1. The advantages include the following:

i. An enhanced compositionality that accommodates for the scalability of the control system;

ii. A utilization of simple scheduling schemes to transfer information between the nodes; and

iii. A requirement of low computation and communication overhead.

Further, the control system topology that utilizes the WCN in Figure 1.2 is that of a decentralized control system paradigm, where the control decisions are made in a decentralized manner by the set of distributed and inter-connected nodes of the WCN.

1.3 Contribution and Organization

First, consider nodes and systems that are physically distributed over a wide geographical area and that are required to transfer information over communication networks. In this thesis, such an apparatus is referred to as a connected disperse system (CDS). Further, this thesis addresses the topological design for the control of connected disperse systems; it proposes an alternative control system topology and presents its design. In the proposed topology, a plant system can be connected to a controller system and/or a set of distributed and inter-connected nodes that form a network system. The set of distributed and inter-connected nodes are capable of routing information between all the nodes of the topology, in addition to performing computational tasks. More specifically, the proposed topology is a hybrid combination of the standard control system topology in Figure 1.1 and the control system topology that utilizes the WCN in Figure 1.2. Thus, the plant system is regulated using the controller and network systems, in an individual or a cooperative manner. This allows for the introduction of both centralized and decentralized control paradigms in the control system topology. The contribution and organization of this thesis are as follows.

In Chapter 2, the definition of the operation of the nodes and the connectivity between the nodes of the proposed topology are discussed; the modelling of the closed-loop control system and the modelling framework to facilitate the design of the proposed topology are presented; conditions required to characterize the existence of the design of the proposed topology are provided; and the design procedure of the proposed topology by using algorithms for computing its design variables is addressed.

In Chapter 2, the provided results are for the proposed topology under ideal operating scenarios. In Chapter 3, the design of the proposed topology under abnormal operating scenarios is considered. The abnormal operating scenarios can be a result of failures in the nodes and in the transfer of information between the nodes as well as cyber attacks. More specifically, the modelling framework, the condition requirements, and the design procedure of Chapter 2 are extended to accommodate for failures in the nodes and in the transfer of information between the nodes. In addition, a detection scheme is discussed, where an intrusion detection system to detect cyber attacks in the proposed topology is modelled and designed.

In Chapters 2 and 3, the provided results are for the proposed topology with full connectivity, such that all nodes and communication links between all the nodes are utilized. In Chapter 4, the design of the proposed topology under induced connectivity constraints is considered. More specifically, the definition and the modelling of the proposed topology with using a decentralized control system (DCS) setup as well as the modelling framework to facilitate the design of the proposed topology are presented; conditions required to characterize the connectivity between the nodes are provided; a strategic formation of the connectivity between the nodes of the proposed topology is demonstrated; and the design procedure of the proposed topology by using an algorithm for computing its design variables is addressed.

In Chapter 5, the design of the proposed topology under two additional specifications is considered. For the first additional specification, the design of the proposed topology with using a model reduction approach to remove nodes and associated communication links is considered. This offers an alternative approach to that of Chapter 4, in order to result in utilizing a reduced number of nodes and communication links between the nodes of the proposed topology. For the second additional specification, the design of the proposed topology by segregating the set of distributed and inter-connected nodes of the network system into two independent sets of nodes (namely, disjoint sets) is considered. One set of nodes is responsible for the transfer of information from the plant system to the controller system, and the other set of nodes is responsible for the transfer of information from the controller system to the plant system. More specifically, for the design of the proposed topology under each of the two additional specifications, the modelling of the closed-loop control system and the modelling framework to facilitate the design of the proposed topology are presented; and the design procedure of the proposed topology by using algorithms for computing its design variables is addressed.

Finally, in Chapter 6, a conclusion is provided and several possible directions for future work are suggested and discussed.

Chapter 2

Alternative Control System Topology for the Control of Connected Disperse Systems

2.1 Definition of the Topology

As discussed in Section 1.3, the proposed control system topology is a hybrid combination of the standard control system topology in Figure 1.1 and the control system topology that utilizes the WCN in Figure 1.2. It consists of the following systems: a plant system, a controller system, and an intermediate network system (namely, similar to the WCN).

The plant system has a set of sensor nodes $\mathcal{S} = \{s_1, \ldots, s_p\}$ and a set of actuator nodes $\mathcal{A} = \{a_1, \ldots, a_m\}$; the controller system has a set of input nodes $\Gamma = \{\gamma_1, \ldots, \gamma_d\}$ and a set of output nodes $\Theta = \{\theta_1, \ldots, \theta_t\}$; and the intermediate network system has a set of distributed and inter-connected nodes $\mathcal{V} = \{v_1, \ldots, v_N\}$. In the proposed topology, nodes of different sets of nodes are capable of having connectivity with each other with no restriction. More specifically, the sensor nodes of the plant system provide measurements to the actuator nodes of the plant system, the distributed and inter-connected nodes of the network system, and the input nodes of the controller system (namely, to any node from the sets of nodes $\mathcal{A}$, $\mathcal{V}$, and Γ). Similarly, the output nodes of the controller system send control commands and the distributed and inter-connected nodes of the network system transfer information to any node from the sets of nodes $\mathcal{A}$, $\mathcal{V}$, and Γ. The connectivity between the nodes of the proposed topology is achieved using a set of wireless communication links $\mathcal{E} = \{e_1, \ldots, e_{\mathcal{L}}\}$. An example of the proposed control system topology is depicted in Figure 2.1, where the connectivity from the sets of nodes $\mathcal{S}$, $\mathcal{V}$, and Θ to the set the nodes $\mathcal{A}$ is represented using solid, dotted, and dashed red arrows, and is denoted by $\mathcal{E}_{\mathcal{S}\to\mathcal{A}}$, $\mathcal{E}_{\mathcal{V}\to\mathcal{A}}$, and $\mathcal{E}_{\Theta\to\mathcal{A}}$, respectively; that from $\mathcal{S}$, $\mathcal{V}$, and Θ to $\mathcal{V}$ using dashed, solid, and dotted black arrows, and is denoted by $\mathcal{E}_{\mathcal{S}\to\mathcal{V}}$, $\mathcal{E}_{\mathcal{V}\leftrightarrow\mathcal{V}}$, and $\mathcal{E}_{\Theta\to\mathcal{V}}$, respectively; and that from $\mathcal{S}$, $\mathcal{V}$, and Θ to Γ using dashed, dotted, and solid yellow arrows, and is denoted by $\mathcal{E}_{\mathcal{S}\to\Gamma}$, $\mathcal{E}_{\mathcal{V}\to\Gamma}$, and $\mathcal{E}_{\Theta\to\Gamma}$, respectively.

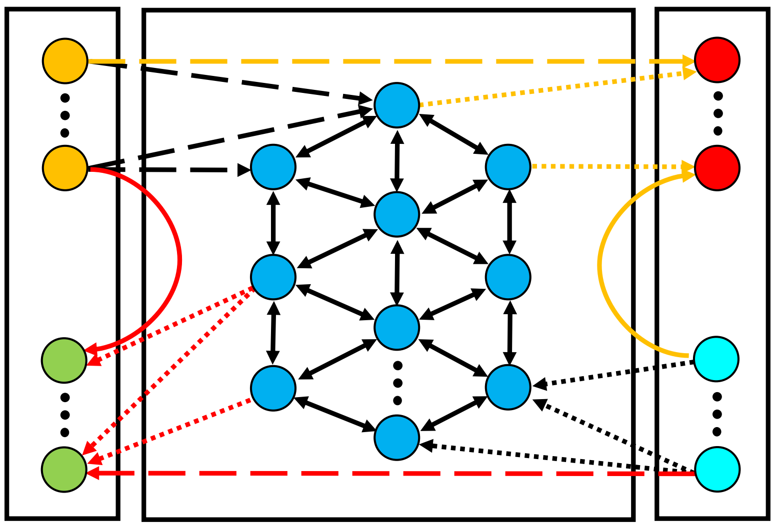

Figure 2.1: An example of the proposed control system topology with a plant system (left block), a network system (middle block), and a controller system (right block), where the sensor nodes (yellow circles) and actuator nodes (green circles) of the plant system, the distributed and inter-connected nodes (blue circles) of the network system, and the input nodes (red circles) and output nodes (cyan circles) of the controller system are connected using wireless communication links (different colors and shapes of arrows).

Suppose the plant system, denoted by G, is modelled as a LTI system in discrete time as

$$\begin{aligned} \mathbf{x}_G(k+1) &= A\mathbf{x}_G(k) + B_1\mathbf{w}(k) + B_2\mathbf{u}(k), \\ \mathbf{z}(k) &= C_1\mathbf{x}_G(k) + D_{11}\mathbf{w}(k) + D_{12}\mathbf{u}(k), \\ \mathbf{y}(k) &= C_2\mathbf{x}_G(k) + D_{21}\mathbf{w}(k), \end{aligned} \tag{2.1}$$

where the vectors $\mathbf{x}_G(k) \in \mathbb{R}^n$, $\mathbf{w}(k) \in \mathbb{R}^s$, $\mathbf{u}(k) \in \mathbb{R}^m$, $\mathbf{z}(k) \in \mathbb{R}^q$, and $\mathbf{y}(k) \in \mathbb{R}^p$ denote its state, external input, controlled input, output to be controlled, and the measured output, respectively, and all of its system matrices have suitable dimensions. Then, suppose the controller system, denoted by K, is modelled as a LTI system in discrete time as

$$\begin{aligned} \mathbf{x}_K(k+1) &= A_K\mathbf{x}_K(k) + B_K\mathbf{f}(k), \\ \mathbf{g}(k) &= C_K\mathbf{x}_K(k) + D_K\mathbf{f}(k), \end{aligned} \tag{2.2}$$

where the vectors $\mathbf{x}_K(k) \in \mathbb{R}^r$, $\mathbf{f}(k) \in \mathbb{R}^d$, and $\mathbf{g}(k) \in \mathbb{R}^t$ denote its state, input, and output, respectively, and all of its system matrices have suitable dimensions. Further, similar to the update procedure of the WCN as discussed in Section 1.2.2, each node of the network system updates its state based on its current state as well as the current states of its neighbouring nodes (namely, those of the network system, sensor nodes of the plant system, and output nodes of the controller system). Similarly, each actuator node of the plant system and input node of the controller system updates its state based on the current states of its neighbouring nodes. The update procedure is modelled for each node of the network system with state x_{N_i}, actuator node of the plant system with state u_i, and input node of the controller system with state f_i in discrete time as

$$\begin{aligned} x_{N_i}(k+1) &= \omega_{ii}x_{N_i}(k) + \sum_{v_j\in\mathcal{N}_{v_i}} \omega_{ij}x_{N_j}(k) + \sum_{s_j\in\mathcal{N}_{v_i}} \lambda_{ij}y_j(k) + \sum_{\theta_j\in\mathcal{N}_{v_i}} \psi_{ij}g_j(k), \\ u_i(k) &= \sum_{v_j\in\mathcal{N}_{a_i}} \upsilon_{ij}x_{N_j}(k) + \sum_{s_j\in\mathcal{N}_{a_i}} \xi_{ij}y_j(k) + \sum_{\theta_j\in\mathcal{N}_{a_i}} \delta_{ij}g_j(k), \\ f_i(k) &= \sum_{v_j\in\mathcal{N}_{\gamma_i}} \pi_{ij}x_{N_j}(k) + \sum_{s_j\in\mathcal{N}_{\gamma_i}} \sigma_{ij}y_j(k) + \sum_{\theta_j\in\mathcal{N}_{\gamma_i}} \phi_{ij}g_j(k), \end{aligned}$$

where the coefficients ω_{ij}, λ_{ij}, ψ_{ij}, υ_{ij}, ξ_{ij}, δ_{ij}, π_{ij}, σ_{ij}, and ϕ_{ij} denote the weights assigned to the states received by node i from node j, and ω_{ii} denotes the weight assigned to the self-connectivity link. The states of the network system, the actuator nodes of the plant system, and the input nodes of the controller system in an augmented manner provide the network system, denoted by N, which is modelled as a LTI system in discrete time as

$$\begin{aligned} \mathbf{x}_N(k+1) &= \Omega\mathbf{x}_N(k) + \Lambda\mathbf{y}(k) + \Psi\mathbf{g}(k), \\ \mathbf{u}(k) &= \Upsilon\mathbf{x}_N(k) + \Xi\mathbf{y}(k) + \Delta\mathbf{g}(k), \\ \mathbf{f}(k) &= \Pi\mathbf{x}_N(k) + \Sigma\mathbf{y}(k) + \Phi\mathbf{g}(k), \end{aligned} \tag{2.3}$$

where the matrices Ω, Λ, Ψ, Υ, Ξ, Δ, Π, Σ, and Φ contain the coefficients denoting the weight assignments. The values of the matrices, in addition to those of the controller system in (2.2), are determined to deliver a control system configuration that satisfies specific objectives (e.g., stability and performance), and they are therefore the design variables.

Further, in the proposed topology, the plant system can be connected to the controller system and/or the network system. If the plant system is only connected to the controller system, the proposed topology reduces to the standard control system topology in Figure 1.1. In contrast, if the plant system is only connected to the network system, the proposed topology reduces to the control system topology that utilizes the WCN in Figure 1.2. Thus, the plant system is regulated using the controller and network systems, in an individual or a cooperative manner. Moreover, the proposed topology shares the same advantages as those of the control system topology that utilizes the WCN in Figure 1.2 (namely, as discussed in Section 1.2.2). Further, the proposed topology offers several additional advantages, compared to its control system topology counterparts in Figures 1.1 and 1.2. The advantages are as follows.

- Connectivity: In a connected disperse system, the plant system may be physically far away from the controller system. The inclusion of the network system allows for bridging the plant and the controller systems (namely, through its set of distributed and inter-connected nodes). This provides additional flexibility in the deployment, mobility, configuration, and connectivity of the nodes of the plant, controller, and network systems.

- Practicality: In practice, some applications require the control decisions to be made in a centralized manner (namely, using a dedicated and centralized controller system). In contrast, some other applications require the control decisions to be made in a decentralized manner. The inclusion of the controller and network systems in the control system topology allows for simultaneously having centralized and decentralized control system paradigms. More specifically, the use of the controller system provides a centralized control system paradigm, and the use of the network system provides a decentralized control system paradigm.

- Availability: The inclusion of the controller and network systems allows for having two independent systems in the control system topology that can stabilize the plant system. Thus, if one of the two systems fails, the resulting control system topology reduces to either the standard control system topology in Figure 1.1 or the control system topology that utilizes the WCN in Figure 1.2. In either of the two topologies, the plant system can be stabilized. In addition, the use of the distributed and inter-connected nodes of the network system facilitates the transfer of information between the nodes of the plant, controller, and network systems using several paths. Thus, if one path becomes unavailable, an alternative path can be utilized to maintain the connectivity between nodes of the systems.

- Generality: The inclusion of the controller and network systems in the control system topology delivers a more generic topology. Namely, it captures the standard control system topology in Figure 1.1 as well as the control system topology that utilizes the WCN in Figure 1.2. In addition, the modelling of the plant, controller, and network systems is also generic. Namely, the plant system in (2.1) is modelled as a standard system with external input and output vectors (namely, $\mathbf{w}$ and $\mathbf{z}$, respectively), the controller system in (2.2) is modelled as a dynamic output feedback system, and the network system in (2.3) captures all possible connectivity scenarios between the nodes of the plant, controller, and network systems.

2.2 Modelling Framework of the Topology

The plant system in (2.1), the controller system in (2.2), and the network system in (2.3) are connected through feedback (namely, through the communication links between their nodes). This results in the feedback setup of the closed-loop control system depicted in Figure 2.2.

The closed-loop control system is modelled as a LTI system in discrete time as

$$\begin{aligned} \mathbf{x}(k+1) &= \underbrace{\begin{bmatrix} \mathscr{A}_{11} & \mathscr{A}_{12} & \mathscr{A}_{13} \\ \mathscr{A}_{21} & \mathscr{A}_{22} & \mathscr{A}_{23} \\ \mathscr{A}_{31} & \mathscr{A}_{32} & \mathscr{A}_{33} \end{bmatrix}}_{\mathscr{A}} \mathbf{x}(k) + \underbrace{\begin{bmatrix} \mathscr{B}_1 \\ \mathscr{B}_2 \\ \mathscr{B}_3 \end{bmatrix}}_{\mathscr{B}} \mathbf{w}(k), \\ \mathbf{z}(k) &= \underbrace{\begin{bmatrix} \mathscr{C}_1 & \mathscr{C}_2 & \mathscr{C}_3 \end{bmatrix}}_{\mathscr{C}} \mathbf{x}(k) + \mathscr{D}\mathbf{w}(k), \end{aligned} \tag{2.4}$$

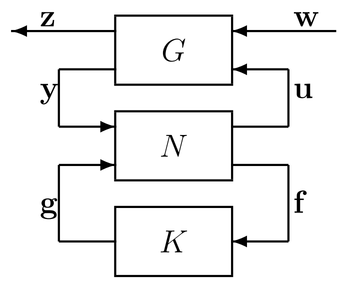

Figure 2.2: Feedback setup of the closed-loop control system consisting of the plant, controller, and network systems.

where the vector $\mathbf{x} = [\mathbf{x}_G^T \ \mathbf{x}_N^T \ \mathbf{x}_K^T]^T$ denotes its state, and the system matrices are defined as

$$
\begin{aligned}
\mathscr{A}_{11} &= A + B_2(\Xi + \Delta(\mathbf{I} - D_K\Phi)^{-1}D_K\Sigma)C_2,\\
\mathscr{A}_{12} &= B_2(\Upsilon + \Delta(\mathbf{I} - D_K\Phi)^{-1}D_K\Pi),\\
\mathscr{A}_{13} &= B_2\Delta(\mathbf{I} - D_K\Phi)^{-1}C_K,\\
\mathscr{A}_{21} &= (\Lambda + \Psi(\mathbf{I} - D_K\Phi)^{-1}D_K\Sigma)C_2,\\
\mathscr{A}_{22} &= \Omega + \Psi(\mathbf{I} - D_K\Phi)^{-1}D_K\Pi,\\
\mathscr{A}_{23} &= \Psi(\mathbf{I} - D_K\Phi)^{-1}C_K,\\
\mathscr{A}_{31} &= B_K(\mathbf{I} - \Phi D_K)^{-1}\Sigma C_2,\\
\mathscr{A}_{32} &= B_K(\mathbf{I} - \Phi D_K)^{-1}\Pi,\\
\mathscr{A}_{33} &= A_K + B_K(\mathbf{I} - \Phi D_K)^{-1}\Phi C_K,\\
\mathscr{B}_1 &= B_1 + B_2(\Xi + \Delta(\mathbf{I} - D_K\Phi)^{-1}D_K\Sigma)D_{21},\\
\mathscr{B}_2 &= (\Lambda + \Psi(\mathbf{I} - D_K\Phi)^{-1}D_K\Sigma)D_{21},\\
\mathscr{B}_3 &= B_K(\mathbf{I} - \Phi D_K)^{-1}\Sigma D_{21},\\
\mathscr{C}_1 &= C_1 + D_{12}(\Xi + \Delta(\mathbf{I} - D_K\Phi)^{-1}D_K\Sigma)C_2,\\
\mathscr{C}_2 &= D_{12}(\Upsilon + \Delta(\mathbf{I} - D_K\Phi)^{-1}D_K\Pi),\\
\mathscr{C}_3 &= D_{12}\Delta(\mathbf{I} - D_K\Phi)^{-1}C_K,\\
\mathscr{D} &= D_{11} + D_{12}(\Xi + \Delta(\mathbf{I} - D_K\Phi)^{-1}D_K\Sigma)D_{21}.
\end{aligned}
$$

As can be observed from the system matrices of the closed-loop control system in (2.4), the system matrices of the plant, controller, and network systems are coupled together in a nonlinear manner. Thus, the computation of the system matrices of the controller and network systems (namely, the design variables) becomes more difficult. In order to facilitate the computation of the design variables, the approach presented in [21, 51] is applied. More specifically, the system matrices are expressed in terms of matrices affine on the system matrices of the controller system and the network system.

First, consider the controller system K in (2.2). It regulates the augmented plant and network system, denoted by P, which is modelled as

$$\begin{aligned}\mathbf{x}_P(k+1) &= \mathscr{A}_P\mathbf{x}_P(k) + \mathscr{B}_{P_1}\mathbf{w}(k) + \mathscr{B}_{P_2}\mathbf{g}(k),\\ \mathbf{z}(k) &= \mathscr{C}_{P_1}\mathbf{x}_P(k) + \mathscr{D}_{P_{11}}\mathbf{w}(k) + \mathscr{D}_{P_{12}}\mathbf{g}(k),\\ \mathbf{f}(k) &= \mathscr{C}_{P_2}\mathbf{x}_P(k) + \mathscr{D}_{P_{21}}\mathbf{w}(k) + \mathscr{D}_{P_{22}}\mathbf{g}(k),\end{aligned} \tag{2.5}$$

where the vector $\mathbf{x}_P = [\mathbf{x}_G^T \ \mathbf{x}_N^T]^T$ denotes its state, and the system matrices are defined as

$$\mathscr{A}_P = \begin{bmatrix} A + B_2\Xi C_2 & B_2\Upsilon \\ \Lambda C_2 & \Omega \end{bmatrix}, \mathscr{B}_{P_1} = \begin{bmatrix} B_1 + B_2\Xi D_{21} \\ \Lambda D_{21} \end{bmatrix}, \mathscr{B}_{P_2} = \begin{bmatrix} B_2\Delta \\ \Psi \end{bmatrix},$$
$$\mathscr{C}_{P_1} = \begin{bmatrix} C_1 + D_{12}\Xi C_2 & D_{12}\Upsilon \end{bmatrix}, \mathscr{C}_{P_2} = \begin{bmatrix} \Sigma C_2 & \Pi \end{bmatrix}, \mathscr{D}_{P_{11}} = \ D_{11} + D_{12}\Xi D_{21},$$
$$\mathscr{D}_{P_{12}} = \ D_{12}\Delta,\ \mathscr{D}_{P_{21}} = \ \Sigma D_{21},\ \mathscr{D}_{P_{22}} = \ \Phi.$$

Then, let $\Phi = \mathbf{0}$ (namely, $\mathscr{D}_{P_{22}}$ in (2.5)) to simplify the derivation, and suppose the associated controller parameter is denoted by $\mathcal{K}$ and defined as

$$\mathcal{K} = \begin{bmatrix} D_K & C_K \\ B_K & A_K \end{bmatrix}. \tag{2.6}$$

The system matrices of the closed-loop control system in (2.4) are expressed in terms of the controller parameter as

$$\begin{aligned}\mathscr{A}(\mathcal{K}) &= \mathcal{A}_P + \mathcal{B}_{P_2}\mathcal{K}\mathcal{C}_{P_2},\\ \mathscr{B}(\mathcal{K}) &= \mathcal{B}_{P_1} + \mathcal{B}_{P_2}\mathcal{K}\mathcal{D}_{P_{21}},\\ \mathscr{C}(\mathcal{K}) &= \mathcal{C}_{P_1} + \mathcal{D}_{P_{12}}\mathcal{K}\mathcal{C}_{P_2},\\ \mathscr{D}(\mathcal{K}) &= \mathcal{D}_{P_{11}} + \mathcal{D}_{P_{12}}\mathcal{K}\mathcal{D}_{P_{21}},\end{aligned}$$

where the matrices are defined as

$$\mathcal{A}_P = \begin{bmatrix} \mathscr{A}_P & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \mathcal{B}_{P_1} = \begin{bmatrix} \mathscr{B}_{P_1} \\ \mathbf{0} \end{bmatrix}, \mathcal{B}_{P_2} = \begin{bmatrix} \mathscr{B}_{P_2} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}, \mathcal{C}_{P_1} = \begin{bmatrix} \mathscr{C}_{P_1} & \mathbf{0} \end{bmatrix},$$
$$\mathcal{C}_{P_2} = \begin{bmatrix} \mathscr{C}_{P_2} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}, \mathcal{D}_{P_{11}} = \mathscr{D}_{P_{11}}, \mathcal{D}_{P_{12}} = \begin{bmatrix} \mathscr{D}_{P_{12}} & \mathbf{0} \end{bmatrix}, \mathcal{D}_{P_{21}} = \begin{bmatrix} \mathscr{D}_{P_{21}} \\ \mathbf{0} \end{bmatrix}.$$

Now, consider the network system N in (2.3). It regulates the plant system, with or without the controller system. Then, let $D_K = \mathbf{0}$ in (2.4) to simplify the derivation, and suppose the associated network parameter is denoted by $\mathcal{N}$ and defined as

$$\mathcal{N} = \begin{bmatrix} \Xi & \Upsilon & \Delta \\ \Lambda & \Omega & \Psi \\ \Sigma & \Pi & \Phi \end{bmatrix}. \tag{2.7}$$

The system matrices of the closed-loop control system in (2.4) are expressed in terms of the network parameter as

$$\begin{aligned} \mathscr{A}(\mathcal{N}) &= \mathcal{A} + \mathcal{B}\mathcal{N}\mathcal{C}, \\ \mathscr{B}(\mathcal{N}) &= \mathcal{B}_1 + \mathcal{B}\mathcal{N}\mathcal{D}_{21}, \\ \mathscr{C}(\mathcal{N}) &= \mathcal{C}_1 + \mathcal{D}_{12}\mathcal{N}\mathcal{C}, \\ \mathscr{D}(\mathcal{N}) &= \mathcal{D}_{11} + \mathcal{D}_{12}\mathcal{N}\mathcal{D}_{21}, \end{aligned}$$

where the matrices are defined as

$$\mathcal{A} = \begin{bmatrix} A & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & A_K \end{bmatrix}, \mathcal{B} = \begin{bmatrix} B_2 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & B_K \end{bmatrix}, \mathcal{C} = \begin{bmatrix} C_2 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & C_K \end{bmatrix},$$

$$\mathcal{B}_1 = \begin{bmatrix} B_1 \\ \mathbf{0} \end{bmatrix}, \mathcal{C}_1 = \begin{bmatrix} C_1 & \mathbf{0} \end{bmatrix}, \mathcal{D}_{11} = D_{11}, \mathcal{D}_{12} = \begin{bmatrix} D_{12} & \mathbf{0} \end{bmatrix}, \mathcal{D}_{21} = \begin{bmatrix} D_{21} \\ \mathbf{0} \end{bmatrix}.$$

2.3 Condition Requirements of the Topology

First, consider the conditions required to characterize the existence of the design of the controller system. The existence of such a design depends on the stabilizability and detectability of the augmented plant and network system P in (2.5). Further, suppose the set of eigenvalues of the matrix $\mathscr{A}_P$ are denoted and defined as $\tilde{\Lambda}_{\mathscr{A}_P} \triangleq \{\tilde{\lambda}_l^{\mathscr{A}_P} : l = 1, \ldots, n + N\}$, of matrix A are denoted and defined as $\tilde{\Lambda}_A \triangleq \{\tilde{\lambda}_i^A : i = 1, \ldots, n\}$, and finally, of matrix Ω are denoted and defined as $\tilde{\Lambda}_\Omega \triangleq \{\tilde{\lambda}_j^\Omega : j = 1, \ldots, N\}$. Then, applying the approach presented in [11], the controllability and observability of the system P is characterized in the following two theorems; and from which, the stabilizability and detectability can be easily deduced.

Theorem 2.1. Suppose any of the following scenarios holds:
(i) $\Lambda = \mathbf{0}$ and $\Xi = \mathbf{0}$;
(ii) $\Delta = \Xi$ and $\Lambda = \Psi$ (with $p = t$); or
(iii) $\Lambda = \Psi\Delta^{-1}\Xi$ (with $m = t$ and Δ is square and invertible).

Then, the system P is controllable if and only if the pair (Ω, Ψ) is controllable and

$$\text{rank}\begin{bmatrix} A - \tilde{\lambda}_i^A \mathbf{I} & B_2 \Upsilon & B_2 \Delta \\ \mathbf{0} & \Omega - \tilde{\lambda}_i^A \mathbf{I} & \Psi \end{bmatrix} = n + N$$

for all $\tilde{\lambda}_i^A \in \tilde{\Lambda}_A$.

Proof. The system P is controllable if and only if the pair

$$\left(\begin{bmatrix} A + B_2 \Xi C_2 & B_2 \Upsilon \\ \Lambda C_2 & \Omega \end{bmatrix}, \begin{bmatrix} B_2 \Delta \\ \Psi \end{bmatrix} \right)$$

is controllable. Consider the Popov-Belevitch-Hautus (PBH) test for controllability. The system P is controllable if and only if

$$\text{rank}\begin{bmatrix} A + B_2 \Xi C_2 - \tilde{\lambda}_l^{\mathscr{A}_P} \mathbf{I} & B_2 \Upsilon & B_2 \Delta \\ \Lambda C_2 & \Omega - \tilde{\lambda}_l^{\mathscr{A}_P} \mathbf{I} & \Psi \end{bmatrix} = n + N$$

for all $\tilde{\lambda}_l^{\mathscr{A}_P} \in \tilde{\Lambda}_{\mathscr{A}_P}$. Now, consider scenario (i) and setting $\Lambda = \mathbf{0}$ and $\Xi = \mathbf{0}$. The system P is controllable if and only if

$$\text{rank}\begin{bmatrix} A - \tilde{\lambda}_l \mathbf{I} & B_2 \Upsilon & B_2 \Delta \\ \mathbf{0} & \Omega - \tilde{\lambda}_l \mathbf{I} & \Psi \end{bmatrix} = n + N \tag{2.8}$$

for all $\tilde{\lambda}_l \in \tilde{\Lambda}_A \cup \tilde{\Lambda}_\Omega$. Assuming the pair (Ω, Ψ) is controllable; namely, $\text{rank}[\Omega - \tilde{\lambda}_j^\Omega \mathbf{I} \ \Psi] = N$ for all $\tilde{\lambda}_j^\Omega \in \tilde{\Lambda}_\Omega$, then

$$\text{rank}\begin{bmatrix} A - \tilde{\lambda}_i^A \mathbf{I} & B_2 \Upsilon & B_2 \Delta \\ \mathbf{0} & \Omega - \tilde{\lambda}_i^A \mathbf{I} & \Psi \end{bmatrix} = n + N$$

for all $\tilde{\lambda}_i^A \in \tilde{\Lambda}_A$. This proves sufficiency. For the proof of necessity, the controllability condition of (Ω, Ψ) must hold. For scenario (ii), by post-multiplying the matrix used in the PBH test by

$$\begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} \\ -C_2 & \mathbf{0} & \mathbf{I} \end{bmatrix}$$

and setting $\Delta = \Xi$ and $\Lambda = \Psi$ (given that $p = t$), condition (2.8) is obtained and the proof follows the same steps. Similarly, for scenario (iii), by post-multiplying the matrix used in the PBH test by

$$\begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} \\ -\Delta^{-1} \Xi C_2 & \mathbf{0} & \mathbf{I} \end{bmatrix}$$

and setting $\Lambda = \Psi\Delta^{-1}\Xi$ (given that $m = t$ and Δ is square and invertible), condition (2.8) is obtained and the proof follows the same steps. $\square$

Theorem 2.2. Suppose any of the following scenarios holds:
(i) $\Upsilon = \mathbf{0}$ and $\Xi = \mathbf{0}$; or
(ii) $\Sigma = \Xi$ and $\Upsilon = \Pi$ (with $m = d$).
Then, the system P is observable if and only if the pair (Π, Ω) is observable and

$$\text{rank}\begin{bmatrix} A - \tilde{\lambda}_i^A \mathbf{I} & \mathbf{0} \\ \Lambda C_2 & \Omega - \tilde{\lambda}_i^A \mathbf{I} \\ \Sigma C_2 & \Pi \end{bmatrix} = n + N$$

for all $\tilde{\lambda}_i^A \in \tilde{\Lambda}_A$.

Proof. The system P is observable if and only if the pair

$$\left(\begin{bmatrix} \Sigma C_2 & \Pi \end{bmatrix}, \begin{bmatrix} A + B_2 \Xi C_2 & B_2 \Upsilon \\ \Lambda C_2 & \Omega \end{bmatrix} \right)$$

is observable. Consider the PBH test for observability. The system P is observable if and only if

$$\text{rank}\begin{bmatrix} A + B_2 \Xi C_2 - \tilde{\lambda}_l^{\mathscr{A}_P} \mathbf{I} & B_2 \Upsilon \\ \Lambda C_2 & \Omega - \tilde{\lambda}_l^{\mathscr{A}_P} \mathbf{I} \\ \Sigma C_2 & \Pi \end{bmatrix} = n + N$$

for all $\tilde{\lambda}_l^{\mathscr{A}_P} \in \tilde{\Lambda}_{\mathscr{A}_P}$. Now, consider scenario (i) and setting $\Upsilon = \mathbf{0}$ and $\Xi = \mathbf{0}$. The system P is observable if and only if

$$\text{rank}\begin{bmatrix} A - \tilde{\lambda}_l \mathbf{I} & \mathbf{0} \\ \Lambda C_2 & \Omega - \tilde{\lambda}_l \mathbf{I} \\ \Sigma C_2 & \Pi \end{bmatrix} = n + N \tag{2.9}$$

for all $\tilde{\lambda}_l \in \tilde{\Lambda}_A \cup \tilde{\Lambda}_\Omega$. Assuming the pair (Π, Ω) is observable; namely,

$$\text{rank}\begin{bmatrix} \Omega - \tilde{\lambda}_j^\Omega \mathbf{I} \\ \Pi \end{bmatrix} = N$$

for all $\tilde{\lambda}_j^\Omega \in \tilde{\Lambda}_\Omega$, then

$$\text{rank}\begin{bmatrix} A - \tilde{\lambda}_i^A \mathbf{I} & \mathbf{0} \\ \Lambda C_2 & \Omega - \tilde{\lambda}_i^A \mathbf{I} \\ \Sigma C_2 & \Pi \end{bmatrix} = n + N$$

for all $\tilde{\lambda}_i^A \in \tilde{\Lambda}_A$. This proves sufficiency. For the proof of necessity, the observability condition of (Π, Ω) must hold. For scenario (ii), by pre-multiplying the third row by $-B_2$ and adding it to the first row of the matrix used in the PBH test, and setting $\Sigma = \Xi$ and $\Upsilon = \Pi$ (given that $m = d$), condition (2.9) is obtained and the proof follows the same steps. $\square$

Next, consider the conditions required to characterize the existence of the design of the network system. The existence of such a design depends on the stabilizability and detectability of the plant system G in (2.1), and is independent of the controller system.

2.4 Design Procedure of the Topology

The design procedure of the proposed topology is addressed for the following three scenarios:

i. The design of the controller system K for a given plant system G and a given network system N (namely, designing the system in (2.2) for the systems in (2.1) and (2.3));

ii. The design of the network system for a given plant system and a given controller system; and

iii. The joint design of the controller and network systems for a given plant system.

The design procedure of the proposed topology is achieved by extending the approach presented in [32] and [21]. First, consider the following definition from [51] and [21].

Definition 2.1. For a discrete-time system with input vector $\mathbf{w}$ and output vector $\mathbf{z}$, the energy-to-peak and energy-to-energy gains are denoted and defined as $\nabla_{ep} \triangleq \sup_{\|\mathbf{w}\|_{l_2} \leqslant 1} \| \mathbf{z} \|_{l_\infty}$ and $\nabla_{ee} \triangleq \sup_{\|\mathbf{w}\|_{l_2} \leqslant 1} \| \mathbf{z} \|_{l_2}$, respectively. The l_2-norm and the l_∞-norm of a discrete-time signal are expressed as $\| \,.\, \|_{l_2} \triangleq \sqrt{\sum_{k=0}^{\infty} \| \,.\, \|^2}$ and $\| \,.\, \|_{l_\infty} \triangleq \sup_{k \geq 0} \| \,.\, \|$, respectively.

Then, suppose the set containing the controller and network parameters is denoted and defined as $Q \triangleq \{\mathcal{K}, \mathcal{N}\}$, and consider the following lemma and theorem from [51] and [21].

Lemma 2.1. Consider the closed-loop control system in (2.4). Suppose there exist matrices X, Z, Q, and Y such that the following matrix inequality holds:

$$\begin{bmatrix} X & Z & \mathscr{A}(Q) & \mathscr{B}(Q) \\ * & Y & \mathscr{C}(Q) & \mathscr{D}(Q) \\ * & * & X^{-1} & \mathbf{0} \\ * & * & * & \mathbf{I} \end{bmatrix} > \mathbf{0}. \tag{2.10}$$

Then, the closed-loop control system is asymptotically stable.

The proof of Lemma 2.1 can be obtained by observing Lemma 6.1.2 and Corollary 2.3.7 from [51] and Lemma 1 from [21].

Theorem 2.3. For an asymptotically stable closed-loop control system and a given positive scalar κ, the following statements are true:
(i) $\nabla_{ep} < \kappa$ if and only if there exist matrices X, Z, Q, and Y such that $\kappa\mathbf{I} - Y > \mathbf{0}$ and condition (2.10) holds.
(ii) $\nabla_{\mathcal{H}_2} = ||\mathscr{C}(Q)(z\mathbf{I} - \mathscr{A}(Q))^{-1}\mathscr{B}(Q) + \mathscr{D}(Q)||_2 < \kappa$ if and only if there exist matrices X, Z, Q, and Y such that $\kappa^2 -$ trace$(Y) > \mathbf{0}$ and condition (2.10) holds.
(iii) $\nabla_{ee} = ||\mathscr{C}(Q)(z\mathbf{I} - \mathscr{A}(Q))^{-1}\mathscr{B}(Q) + \mathscr{D}(Q)||_\infty < \kappa$ if and only if there exist matrices X, Q, and Y such that $\kappa^2\mathbf{I} - Y > \mathbf{0}$ and condition (2.10) holds when $Z = \mathbf{0}$.

For Lemma 2.1 and Theorem 2.3, the non-convex term X^{-1} in (2.10) can be linearized as presented in [21], by utilizing a linearization operator of a matrix X^{-1} at a positive definite value X_k denoted and defined as

$$\begin{aligned} \mho(X^{-1}, X_k) &= X_k^{-1} - X_k^{-1}(X - X_k)X_k^{-1} \\ &= X_k^{-1}(2\mathbf{I} - XX_k^{-1}). \end{aligned}$$

Thus, with the linearized term, the condition (2.10) becomes a linear matrix inequality (LMI). Then, using Theorem 2.3, the controller parameter $\mathcal{K}$ in (2.6) and the network parameter $\mathcal{N}$ in (2.7) are computed to provide a closed-loop control system that is stable as well as optimal (namely, according to the performance measures specified in Theorem 2.3). First, consider scenarios (i) and (ii), the design procedure of the proposed topology is achieved using the following two algorithms.

In Algorithm 2.1, initial values are computed for the matrix X. Also, the computed values of the parameter Q result in a closed-loop control system

Algorithm 2.1 Algorithm for the Design of Initial Controller and Network Systems

Step 1. Specify the feasibility parameter $\eta > 0$, $k = 0$, and X_0 as an arbitrary symmetric matrix.
Step 2. Compute X by solving the convex optimization problem,

$$[X] = \arg \min_{X,Z,Q,Y} \eta$$

subject to

$$\begin{bmatrix} X & Z & \mathscr{A}(Q) & \mathscr{B}(Q) \\ * & Y & \mathscr{C}(Q) & \mathscr{D}(Q) \\ * & * & X_k^{-1}(2\mathbf{I} - XX_k^{-1}) & \mathbf{0} \\ * & * & * & \mathbf{I} \end{bmatrix} > \mathbf{0}. \tag{2.11}$$

Step 3. If $\eta < 0$, exit; else, set $k = k + 1$ and return to Step 2.

Algorithm 2.2 Algorithm for the Separate Design of Optimal Controller and Network Systems

Step 1. Specify the convergence threshold $\epsilon > 0$ and $k = 0$, and use the initial values for X computed using Algorithm 2.1.
Step 2. Compute Q, Y, and κ by solving the convex optimization problem,

$$[Q, Y, \kappa] = \arg \min_{X,Z,Q,Y,\kappa} \kappa$$

subject to

condition (2.11), and
$\kappa\mathbf{I} - Y > \mathbf{0}$ if solving for ∇_{ep},
$\kappa^2 - \text{trace}(Y) > \mathbf{0}$ if solving for $\nabla_{\mathcal{H}_2}$, or
$\kappa^2\mathbf{I} - Y > \mathbf{0}$ (with $Z = \mathbf{0}$) if solving for ∇_{ee}.

Step 3. If $\kappa < \epsilon$, exit; else, set $k = k + 1$ and return to Step 2.

that is stable, but not optimal. In Algorithm 2.2, the initial values computed in Algorithm 2.1 are used, and the computed values for the parameter Q result in a closed-loop control system that is stable as well as optimal. Moreover, the computation of the initial values are essential for the design procedure of the proposed topology. Thus, if it is not feasible to compute such initial values, changes in the construction of the proposed topology need to be considered (namely, the size of the systems, the number of nodes, and the connectivity between the nodes).

Remark 2.1. The presented design procedure of the proposed topology allows for the design of the controller system and the network system to be of a specified size. In contrast to the standard control system topology in Figure 1.1 that requires the plant and controller systems to be of the same size, the presented design procedure of the proposed topology allows for the size of the designed controller and network systems to be less than that of the plant system. Thus, a simpler design can be achieved.

Algorithms 2.1 and 2.2 deliver the design of the proposed topology for scenarios (i) and (ii). Namely, they compute the controller parameter $\mathcal{K}$ or the network parameter $\mathcal{N}$ separately and for different given systems. Next, consider scenario (iii); the design procedure of the proposed topology is achieved using the following two algorithms.

Algorithm 2.3 Algorithm for the Design of a Restricted Optimal Network System

---

Step 1. Specify $\Phi = \mathbf{0}$, and obtain initial values for X such that condition (2.11) holds.
Step 2. Specify $\epsilon_{\mathcal{N}} > 0$ and $k = 0$.
Step 3. Compute $\mathcal{N}$, Y, and κ by solving the convex optimization problem,

$$[\mathcal{N}, Y, \kappa] = \arg \min_{X,Z,\mathcal{N},Y,\kappa} \kappa$$

subject to

condition (2.11) (with $\Phi = \mathbf{0}$) and

$\kappa \mathbf{I} - Y > \mathbf{0}$.

Step 4. If $\kappa < \epsilon_{\mathcal{N}}$, exit; else, set $k = k + 1$ and return to Step 3.

In Algorithm 2.3, Algorithms 2.1 and 2.2 are combined and extended by adding a constraint for the computation of the network parameter $\mathcal{N}$ with $\Phi = \mathbf{0}$. More specifically, Algorithm 2.3 computes the restricted network parameter to satisfy the performance measure defined in terms of the energy-to-

Algorithm 2.4 Algorithm for the Joint Design of Optimal Controller and Network Systems

Step 1. Set the iteration step number $l = 1$, and specify the controller parameter $\mathcal{K}$ as an arbitrarily matrix with $D_K = \mathbf{0}$.
Step 2. With the controller parameter $\mathcal{K}$, compute $\mathcal{N}$ using Algorithm 2.3.
Step 3. If $\epsilon_{\mathcal{N}}(l) = \epsilon_{\mathcal{N}}(l-2)$, exit; else, set $l = l + 1$.
Step 4. With the network parameter $\mathcal{N}$, compute $\mathcal{K}$ using Algorithm 2.3 (while making the appropriate variable replacements).
Step 5. If $\epsilon_{\mathcal{K}}(l) = \epsilon_{\mathcal{K}}(l-2)$, exit; else, set $l = l + 1$ and return to Step 2.

peak gain (namely, by incorporating the condition $\kappa\mathbf{I} - Y > \mathbf{0}$ that corresponds to ∇_{ep} in Theorem 2.3). For the computation of the controller parameter, the variables $\mathcal{N}$, $\epsilon_{\mathcal{N}}$, and $\Phi = \mathbf{0}$ need to be replaced with the variables $\mathcal{K}$, $\epsilon_{\mathcal{K}}$ and $D_K = \mathbf{0}$, respectively. In Algorithm 2.4, the computation of the controller and network parameters is implemented iteratively until there is no significant change in the performance measures. Namely, the network parameter $\mathcal{N}$ is computed first to result in an optimal network system; then, using the optimal network system, the controller parameter is computed to result in an optimal controller system. Then, the computation continues iteratively until the optimal controller and network systems do not result in lower performance measures (namely, in the values of the convergence thresholds $\epsilon_{\mathcal{N}}$ and $\epsilon_{\mathcal{K}}$). As can be noted, Algorithm 2.4 starts with an arbitrary controller parameter. For starting with an arbitrary network parameter, the replacement of the respective variables is needed.

Remark 2.2. Suppose the plant system G in (2.1) is stabilizable and detectable, and there exists initial values of X for computing the controller parameter $\mathcal{K}$ in (2.6) and the network parameter $\mathcal{N}$ in (2.7). Algorithm 2.4 delivers a local optimal design of the controller system K in (2.2) and the network system N in (2.3) to control the plant system.

2.5 Simulations

The design of the proposed topology is demonstrated by applying the design procedure discussed in Section 2.4. First, consider the second-order plant system given as

$$\mathbf{x}_G(k+1) = \begin{bmatrix} 1.1 & 0 \\ 0.3 & 0.2 \end{bmatrix} \mathbf{x}_G(k) + \begin{bmatrix} 0.1 & 0.2 \\ 2 & 1 \end{bmatrix} \mathbf{w}(k) + \begin{bmatrix} 1.8 \\ 0 \end{bmatrix} \mathbf{u}(k),$$
$$\mathbf{z}(k) = \begin{bmatrix} 1 & 0.1 \\ 0.2 & 0 \end{bmatrix} \mathbf{x}_G(k) + \begin{bmatrix} 0.1 & 0.6 \\ 0.2 & 0.1 \end{bmatrix} \mathbf{w}(k) + \begin{bmatrix} 0 \\ 0.2 \end{bmatrix} \mathbf{u}(k),$$
$$\mathbf{y}(k) = \begin{bmatrix} 0.8 & 1 \end{bmatrix} \mathbf{x}_G(k) + \begin{bmatrix} 0 & 0.2 \end{bmatrix} \mathbf{w}(k).$$

Algorithms 2.3 and 2.4 are applied to compute the controller parameter $\mathcal{K}$ in (2.6) and the network parameter $\mathcal{N}$ in (2.7), in order to design the controller system in (2.2) and the network system in (2.3), respectively. In the simulation example, the controller and network systems are specified as second- and fourth-order systems, respectively. In addition, the convergence thresholds $\epsilon_{\mathcal{N}}$ and $\epsilon_{\mathcal{K}}$ are obtained using a bisection approach. The implementation of the algorithms is achieved using MATLAB's Robust Control Toolbox, and is performed for two case studies. In the first case study, Algorithm 2.4 begins with an arbitrary initial controller system, and in the second case study, it begins with an arbitrary initial network system.

In the first case study, the computation of the controller and network parameters led to the results presented in Table 2.1. The use of an arbitrary initial controller system or an optimal controller system did not affect the performance measure (namely, the value of the convergence threshold $\epsilon_{\mathcal{N}}$).

Table 2.1: Results of the joint design of the controller and network systems in the first case study

Iteration Step	$\epsilon_{\mathcal{N}}$	$\epsilon_{\mathcal{K}}$
1	1.096556	–
2	–	1.096556
3	1.096556	–

In the second case study, the computation of the controller and network parameters led to the results presented in Table 2.2. The use of an optimal design of the network system did indeed affect the performance measure (namely, the value of the convergence threshold $\epsilon_{\mathcal{K}}$).

For the second case study, the computed network and controller parameters result in the system matrices of the network and controller systems given as

Table 2.2: Results of the joint design of the controller and network systems in the second case study

Iteration Step	$\epsilon_{\mathcal{N}}$	$\epsilon_{\mathcal{K}}$
1	–	2.956359
2	1.096556	–
3	–	1.096556
4	1.096556	–

$$\Omega = \begin{bmatrix} 0.0326 & -0.0367 & 0.0480 & 0.0331 \\ -0.0367 & -0.0271 & 0.0214 & -0.0612 \\ 0.0480 & 0.0214 & -0.0230 & 0.0836 \\ 0.0331 & -0.0612 & 0.0836 & -0.0545 \end{bmatrix},$$

$$\Lambda = \begin{bmatrix} 0.5416 \\ 4.3967 \\ -6.3649 \\ 6.5458 \end{bmatrix}, \Psi = \begin{bmatrix} 0.0089 \\ -0.0137 \\ 0.0357 \\ -0.0096 \end{bmatrix},$$

$$\Upsilon = \begin{bmatrix} -0.2181 & -0.1231 & 0.0523 & 0.1576 \end{bmatrix},$$

$$\Xi = -0.1094, \Delta = 0.0001, \Sigma = -5.1254, \Phi = 0,$$

$$\Pi = \begin{bmatrix} -18.2765 & -10.5233 & 4.6238 & 12.9098 \end{bmatrix},$$

$$A_K = \begin{bmatrix} -0.5479 & 0.3894 \\ 0.3894 & -0.1950 \end{bmatrix}, B_K = \begin{bmatrix} 0.5308 \\ 1.2614 \end{bmatrix},$$

$$C_K = \begin{bmatrix} 9.4716 & -5.7718 \end{bmatrix}, D_K = 0.$$

The computed system matrices of the network system N contain the coefficients denoting the weights assigned to the states received from the neighbouring nodes (namely, ω_{ij}, λ_{ij}, ψ_{ij}, υ_{ij}, ξ_{ij}, δ_{ij}, π_{ij}, σ_{ij}, and ϕ_{ij}) as well as those of the self-connectivity links (namely, ω_{ii}). For example, in the computed matrix Ω, $\omega_{12} = -0.0367$ is the weight assigned to each state value which node υ_1 receives from node υ_2 of the network system at any time instant k.

Remark 2.3. In the implementation of Algorithms 2.3 and 2.4, the use of the same initial values for the controller parameter $\mathcal{K}$ and the convergence threshold $\epsilon_{\mathcal{N}}$ resulted in different optimal designs of the network system. Similarly, the use of the same initial values for the network parameter $\mathcal{N}$ and the convergence threshold $\epsilon_{\mathcal{K}}$ resulted in different optimal designs of the controller system. Further, the use of different optimal designs of the network and controller systems as initial systems did not lead to lower values of the convergence thresholds $\epsilon_{\mathcal{K}}$ and $\epsilon_{\mathcal{N}}$, respectively. Finally, the value of κ that decreases below the values of $\epsilon_{\mathcal{K}}$ and $\epsilon_{\mathcal{N}}$ was different in different implementations of the algorithms.

2.6 Summary

In this chapter, the following were delivered:

- The definition of the operation of the nodes and the connectivity between the nodes of the proposed topology;
- The modelling of the closed-loop control system and the modelling framework to facilitate the design of the proposed topology;
- The conditions required to characterize the existence of the design of the proposed topology; and
- The design procedure of the proposed topology by using algorithms for computing its design variables.

Chapter 3

Design of the Topology Under Abnormal Operating Scenarios

In Chapter 2, the provided results are for ideal operating scenarios. Namely, the behaviour of the proposed topology was assumed to always be reliable and is not subject to any failure or intrusion. In this chapter, the design of the proposed topology under abnormal operating scenarios is considered. The abnormal operating scenarios are the failures in the nodes and in the transfer of information between the nodes as well as cyber attacks.

3.1 Design of the Topology Under Unreliable Nodes

The sensor nodes $\mathcal{S}$ and actuator nodes $\mathcal{A}$ of the plant system G in (2.1), the input nodes Γ and output nodes Θ of the controller system K in (2.2), and the nodes $\mathcal{V}$ of the network system N in (2.3) can become unavailable. Their unavailability may result from several factors, including scheduled maintenance, malfunction, battery drainage, and disconnectivity. In this section, the design of the proposed topology is addressed when a subset of its nodes becomes unavailable during operation time.

3.1.1 Modelling Framework of the Topology

The design of the proposed topology under unreliable nodes is achieved by adopting the approach presented in the fault-tolerant control literature (for example, see [32, 54, 59–61]). Namely, each node of the proposed topology, $a_i \in \mathcal{A}$ and $s_i \in \mathcal{S}$ of the plant system, $\gamma_i \in \Gamma$ and $\theta_i \in \Theta$ of the controller system, and $v_i \in \mathcal{V}$ of the distributed and inter-connected nodes of the network system, is associated with a pre-multiplier coefficient. The coefficient switches between 1 and 0 to capture the availability and unavailability of the associated node, respectively. For example, suppose a node $v_i \in \mathcal{V}$ is associated with a coefficient denoted by $f^{x_{N_i}}$. For the set of nodes $\mathcal{V}$, the coefficients of the nodes are combined in a pre-multiplier matrix denoted by $\tilde{F}^{\mathbf{x}_N}$, such that the coefficient of each node $v_i \in \mathcal{V}$ is placed along its diagonal. Similarly, the sets of the nodes $\mathcal{A}$, $\mathcal{S}$, Γ, and Θ have associated pre-multiplier matrices denoted by $\tilde{F}^{\mathbf{u}}$, $\tilde{F}^{\mathbf{y}}$, $\tilde{F}^{\mathbf{f}}$, and $\tilde{F}^{\mathbf{g}}$, respectively, that contain the pre-multiplier coefficients of the associated nodes.

Next, consider the plant system G in (2.1), the controller system K in (2.2), and the network system N in (2.3), each input vector that represents the nodes of the proposed topology is associated with the respective pre-multiplier matrix. More specifically, the vectors $\mathbf{u}$, $\mathbf{y}$, $\mathbf{x}_N$, $\mathbf{f}$, and $\mathbf{g}$ are pre-multiplied by the pre-multiplier matrices $\tilde{F}^{\mathbf{u}}$, $\tilde{F}^{\mathbf{y}}$, $\tilde{F}^{\mathbf{x}_N}$, $\tilde{F}^{\mathbf{f}}$, and $\tilde{F}^{\mathbf{g}}$, respectively. In addition, in the network system, the matrices Ω, Λ, and Ψ are pre-multiplied by the pre-multiplier matrix $\tilde{F}^{\mathbf{x}_N}$. This leads to the extended models of the plant, controller, and network systems denoted by $\tilde{G}$, $\tilde{K}$, and $\tilde{N}$, respectively. Then, the modelling framework of Section 2.2 is extended to account for unreliable nodes of the proposed topology. Thus, the closed-loop control system (namely, the augmented plant system $\tilde{G}$, controller system $\tilde{K}$, and network system $\tilde{N}$) is modelled as a LTI system in discrete time as

$$\begin{aligned} \mathbf{x}(k+1) &= \tilde{\mathscr{A}}\mathbf{x}(k) + \tilde{\mathscr{B}}\mathbf{w}(k), \\ \mathbf{z}(k) &= \tilde{\mathscr{C}}\mathbf{x}(k) + \tilde{\mathscr{D}}\mathbf{w}(k). \end{aligned} \tag{3.1}$$

The system matrices of the closed-loop control system in (3.1) are expressed in terms of matrices affine on the system matrices of the controller system and the network system. First, let $\Phi = \mathbf{0}$ to simplify the derivation, and consider the controller parameter $\mathcal{K}$ in (2.6). The system matrices of the closed-loop control system in (3.1) are expressed in terms of the controller parameter as

$$\begin{aligned} \tilde{\mathscr{A}}(\mathcal{K}) &= \tilde{\mathcal{A}}_P + \tilde{\mathcal{B}}_{P_2}\mathcal{K}\tilde{\mathcal{C}}_{P_2}, \\ \tilde{\mathscr{B}}(\mathcal{K}) &= \tilde{\mathcal{B}}_{P_1} + \tilde{\mathcal{B}}_{P_2}\mathcal{K}\tilde{\mathcal{D}}_{P_{21}}, \\ \tilde{\mathscr{C}}(\mathcal{K}) &= \tilde{\mathcal{C}}_{P_1} + \tilde{\mathcal{D}}_{P_{12}}\mathcal{K}\tilde{\mathcal{C}}_{P_2}, \\ \tilde{\mathscr{D}}(\mathcal{K}) &= \tilde{\mathcal{D}}_{P_{11}} + \tilde{\mathcal{D}}_{P_{12}}\mathcal{K}\tilde{\mathcal{D}}_{P_{21}}. \end{aligned}$$

Then, let $D_K = \mathbf{0}$ to simplify the derivation, and consider the network parameter $\mathcal{N}$ in (2.7). The system matrices of the closed-loop control system in (3.1) are expressed in terms of the network parameter as

$$\begin{aligned}
\tilde{\mathscr{A}}(\mathcal{N}) &= \mathcal{A} + \mathcal{B}\tilde{F}^{\mathcal{N}}_{\text{pre}}\mathcal{N}\tilde{F}^{\mathcal{N}}_{\text{post}}\mathcal{C}, \\
\tilde{\mathscr{B}}(\mathcal{N}) &= \mathcal{B}_1 + \mathcal{B}\tilde{F}^{\mathcal{N}}_{\text{pre}}\mathcal{N}\tilde{F}^{\mathcal{N}}_{\text{post}}\mathcal{D}_{21}, \\
\tilde{\mathscr{C}}(\mathcal{N}) &= \mathcal{C}_1 + \mathcal{D}_{12}\tilde{F}^{\mathcal{N}}_{\text{pre}}\mathcal{N}\tilde{F}^{\mathcal{N}}_{\text{post}}\mathcal{C}, \\
\tilde{\mathscr{D}}(\mathcal{N}) &= \mathcal{D}_{11} + \mathcal{D}_{12}\tilde{F}^{\mathcal{N}}_{\text{pre}}\mathcal{N}\tilde{F}^{\mathcal{N}}_{\text{post}}\mathcal{D}_{21},
\end{aligned}$$

where the matrices are the same as those presented in Section 2.2, while incorporating the matrices $\tilde{F}^{\mathbf{u}}$, $\tilde{F}^{\mathbf{y}}$, $\tilde{F}^{\mathbf{x}_N}$, $\tilde{F}^{\mathbf{f}}$, $\tilde{F}^{\mathbf{g}}$, as well as

$$\tilde{F}^{\mathcal{N}}_{\text{pre}} = \begin{bmatrix} \tilde{F}^{\mathbf{u}} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \tilde{F}^{\mathbf{x}_N} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \tilde{F}^{\mathbf{f}} \end{bmatrix}, \text{ and } \tilde{F}^{\mathcal{N}}_{\text{post}} = \begin{bmatrix} \tilde{F}^{\mathbf{y}} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \tilde{F}^{\mathbf{x}_N} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \tilde{F}^{\mathbf{g}} \end{bmatrix}.$$

Remark 3.1. In the presented modelling framework, when the coefficients of the pre-multiplier matrices are all set to 1 to capture the availability of the nodes (namely, $\tilde{F}^{\mathbf{u}}$, $\tilde{F}^{\mathbf{y}}$, $\tilde{F}^{\mathbf{x}_N}$, $\tilde{F}^{\mathbf{f}}$, and $\tilde{F}^{\mathbf{g}}$ are identity matrices), the same modelling framework as that presented in Section 2.2 is obtained. Thus, the modelling framework presented in this section is more generic, compared to the modelling framework counterpart of Section 2.2.

3.1.2 Condition Requirements of the Topology

First, consider the conditions required to characterize the existence of the design of the controller system. The existence of such a design is characterized in the following theorem.

Theorem 3.1. Suppose the pairs $(A, B_2\tilde{F}^{\mathbf{u}})$ and

$$\left(\begin{bmatrix} A + B_2\tilde{F}^{\mathbf{u}}\Xi\tilde{F}^{\mathbf{y}}C_2 & B_2\tilde{F}^{\mathbf{u}}\Upsilon\tilde{F}^{\mathbf{x}_N} \\ \tilde{F}^{\mathbf{x}_N}\Lambda\tilde{F}^{\mathbf{y}}C_2 & \tilde{F}^{\mathbf{x}_N}\Omega\tilde{F}^{\mathbf{x}_N} \end{bmatrix}, \begin{bmatrix} \mathbf{0} \\ \tilde{F}^{\mathbf{x}_N}\Psi\tilde{F}^{\mathbf{g}} \end{bmatrix} \right)$$

are stabilizable, and the pairs $(\tilde{F}^{\mathbf{y}}C_2, A)$ and

$$\left(\begin{bmatrix} \mathbf{0} & \tilde{F}^{\mathbf{f}}\Pi\tilde{F}^{\mathbf{x}_N} \end{bmatrix}, \begin{bmatrix} A + B_2\tilde{F}^{\mathbf{u}}\Xi\tilde{F}^{\mathbf{y}}C_2 & B_2\tilde{F}^{\mathbf{u}}\Upsilon\tilde{F}^{\mathbf{x}_N} \\ \tilde{F}^{\mathbf{x}_N}\Lambda\tilde{F}^{\mathbf{y}}C_2 & \tilde{F}^{\mathbf{x}_N}\Omega\tilde{F}^{\mathbf{x}_N} \end{bmatrix} \right)$$

are detectable. Then, there exists a controller system in (2.2) that can provide a stable closed-loop control system in (3.1).

Proof. Consider the following two scenarios: (i) there is no direct connection between the controller system and the plant system (namely, the matrices Δ and Σ are zero matrices), and (ii) there exists a direct connection between the two systems. For scenario (i), the controller system communicates indirectly with the plant system through the intermediate network system. The augmented plant and network system (namely, with state vector $\mathbf{x}_P = [\mathbf{x}_G^T \ \mathbf{x}_N^T]^T$, input vectors $\mathbf{w}$ and $\mathbf{g}$, and output vectors $\mathbf{z}$ and $\mathbf{f}$) has the following system matrices:

$$\mathscr{A}_P \triangleq \begin{bmatrix} A + B_2\tilde{F}^{\mathbf{u}}\Xi\tilde{F}^{\mathbf{y}}C_2 & B_2\tilde{F}^{\mathbf{u}}\Upsilon\tilde{F}^{\mathbf{x}_N} \\ \tilde{F}^{\mathbf{x}_N}\Lambda\tilde{F}^{\mathbf{y}}C_2 & \tilde{F}^{\mathbf{x}_N}\Omega\tilde{F}^{\mathbf{x}_N} \end{bmatrix},$$

$$\mathscr{B}_{P_2} \triangleq \begin{bmatrix} \mathbf{0} \\ \tilde{F}^{\mathbf{x}_N}\Psi\tilde{F}^{\mathbf{g}} \end{bmatrix}, \mathscr{C}_{P_2} \triangleq \begin{bmatrix} \mathbf{0} & \Pi\tilde{F}^{\mathbf{x}_N} \end{bmatrix},$$

for the vectors $\mathbf{x}_P$, $\mathbf{g}$, and $\mathbf{f}$. Thus, if the pair $(\mathscr{A}_P, \mathscr{B}_{P_2})$ and $(\mathscr{C}_{P_2}, \mathscr{A}_P)$ are stabilizable and detectable, respectively, in addition to adding the effect of the availability of the input nodes of the controller system (namely, using the term $\tilde{F}^{\mathbf{f}}\mathscr{C}_{P_2}$), a controller system in (2.2) can be found to provide a stable closed-loop control system. For scenario (ii), the plant system can be reached directly from the controller system (namely, Δ and Σ are not zero matrices). Thus, if the pairs $(A, B_2\tilde{F}^{\mathbf{u}})$ and $(\tilde{F}^{\mathbf{y}}C_2, A)$ are stabilizable and detectable, respectively, a controller system can be found to provide a stable closed-loop control system. Therefore, under any of the two scenarios, or a mix of the two scenarios, a controller system can be found. □

Next, consider the conditions required to characterize the existence of the design of the network system. The existence of such a design is characterized in the following theorem.

Theorem 3.2. For the design of the closed-loop control system in (3.1), a network system in (2.3) can stabilize the plant system in (2.1) if and only if the pair $(A, B_2\tilde{F}^{\mathbf{u}})$ is stabilizable and the pair $(\tilde{F}^{\mathbf{y}}C_2, A)$ is detectable, for actuator and sensor nodes whose availability are captured by the matrices $\tilde{F}^{\mathbf{u}}$ and $\tilde{F}^{\mathbf{y}}$, respectively.

Proof. Similar to the discussion of Section 2.3, for designing the network system (namely, by computing the parameter $\mathscr{N}$), it is only required that the plant system is stabiliazable and detectable. The controller system does not impact the existence of a network system (namely, in the worst case where the controller system is unavailable, the proposed topology reduces to the control system topology that utilizes the WCN and the closed-loop control system can still be designed to be stable, in a similar manner to the work in [42]). Further, the matrices $\tilde{F}^{\mathbf{u}}$ and $\tilde{F}^{\mathbf{y}}$ capture the availability of the actuator and sensor nodes, respectively, and their effects are therefore added to the system matrices of the plant system B_2 and C_2, respectively. Based on those arguments, the result in Theorem 3.2 is obtained. □

Further, consider the condition requirements that characterize the maximum number of unavailable nodes which the proposed topology can tolerate*. The maximum number of unavailable nodes which the proposed topology can tolerate for stability is characterized in the following corollary.

Corollary 3.1. The controller system in (2.2) and the network system in (2.3) can be designed for a closed-loop control system in (3.1) that is stable and can tolerate a maximum number of unavailable nodes by finding a solution to the optimization problem,

$$\begin{aligned} &\text{minimize} \quad \operatorname{trace}(\operatorname{diag}(\tilde{F}^{\mathbf{u}}\ \tilde{F}^{\mathbf{y}}\ \tilde{F}^{\mathbf{x}_N}\tilde{F}^{\mathbf{f}}\ \tilde{F}^{\mathbf{g}})) \\ &\text{subject to} \quad \text{the pairs } (A, B_2\tilde{F}^{\mathbf{u}}) \text{ and} \\ &\left(\begin{bmatrix} A + B_2\tilde{F}^{\mathbf{u}}\Xi\tilde{F}^{\mathbf{y}}C_2 & B_2\tilde{F}^{\mathbf{u}}\Upsilon\tilde{F}^{\mathbf{x}_N} \\ \tilde{F}^{\mathbf{x}_N}\Lambda\tilde{F}^{\mathbf{y}}C_2 & \tilde{F}^{\mathbf{x}_N}\Omega\tilde{F}^{\mathbf{x}_N} \end{bmatrix}, \begin{bmatrix} \mathbf{0} \\ \tilde{F}^{\mathbf{x}_N}\Psi\tilde{F}^{\mathbf{g}} \end{bmatrix}\right) \end{aligned}$$

being stabilizable, and the pairs $(\tilde{F}^{\mathbf{y}}C_2, A)$ and

$$\left(\begin{bmatrix} \mathbf{0} & \tilde{F}^{\mathbf{f}}\Pi\tilde{F}^{\mathbf{x}_N} \end{bmatrix}, \begin{bmatrix} A + B_2\tilde{F}^{\mathbf{u}}\Xi\tilde{F}^{\mathbf{y}}C_2 & B_2\tilde{F}^{\mathbf{u}}\Upsilon\tilde{F}^{\mathbf{x}_N} \\ \tilde{F}^{\mathbf{x}_N}\Lambda\tilde{F}^{\mathbf{y}}C_2 & \tilde{F}^{\mathbf{x}_N}\Omega\tilde{F}^{\mathbf{x}_N} \end{bmatrix}\right)$$

being detectable, and the maximum number of unavailable nodes is the number of zero entries along the diagonal of the matrix $\operatorname{diag}(\tilde{F}^{\mathbf{u}}\ \tilde{F}^{\mathbf{y}}\ \tilde{F}^{\mathbf{x}_N}\tilde{F}^{\mathbf{f}}\ \tilde{F}^{\mathbf{g}})$.

Proof. By observing Theorems 3.1 and 3.2, and that the number of nonzero entries along the diagonal of the matrices $\tilde{F}^{\mathbf{u}}, \tilde{F}^{\mathbf{y}}, \tilde{F}^{\mathbf{x}_N}, \tilde{F}^{\mathbf{f}}$, and $\tilde{F}^{\mathbf{g}}$ specifies the number of available nodes of the plant, controller, and network systems, the result in Corollary 3.1 is obtained. □

*The notion of tolerance refers to the proper operation of the proposed topology under abnormal operating scenarios.

3.1.3 Design Procedure of the Topology

First, consider again Lemma 2.1 and Theorem 2.3 of Section 2.4 (namely, based on the results from [21, 51]), $\mathcal{Q} = \{\mathcal{K}, \mathcal{N}\}$ denoting the set containing the controller and network parameters, and ∇_{ep} denoting the energy-to-peak gain. The extension of the lemma and the theorem for the design of the proposed topology under unreliable nodes is presented in the following lemma.

Lemma 3.1. Suppose the closed-loop control system in (3.1) is asymptotically stable for a given set of matrices $\tilde{F}^{\mathbf{u}}$, $\tilde{F}^{\mathbf{y}}$, $\tilde{F}^{\mathbf{x}_N}$, $\tilde{F}^{\mathbf{f}}$, and $\tilde{F}^{\mathbf{g}}$. Given a positive scalar κ, the energy-to-peak gain satisfies $\nabla_{ep} < \kappa$ if and only if there exist matrices X, Z, $\mathcal{Q}$, and Y such that $Y < \kappa\mathbf{I}$ and the following matrix inequality holds:

$$\begin{bmatrix} X & Z & \tilde{\mathscr{A}}(\mathcal{Q}) & \tilde{\mathscr{B}}(\mathcal{Q}) \\ * & Y & \tilde{\mathscr{C}}(\mathcal{Q}) & \tilde{\mathscr{D}}(\mathcal{Q}) \\ * & * & X^{-1} & \mathbf{0} \\ * & * & * & \mathbf{I} \end{bmatrix} > \mathbf{0}. \tag{3.2}$$

Next, the maximum number of unavailable nodes which the proposed topology can tolerate for a prescribed performance is characterized in the following theorem.

Theorem 3.3. Suppose the closed-loop control system in (3.1) is both asymptotically stable and satisfies the energy-to-peak gain $\nabla_{ep} < \kappa$ for a given positive scalar κ. The closed-loop control system can tolerate a maximum number of unavailable nodes by finding a solution to the optimization problem,

$$\begin{aligned} &\text{minimize} && \operatorname{trace}(\operatorname{diag}(\tilde{F}^{\mathbf{u}}\ \tilde{F}^{\mathbf{y}}\ \tilde{F}^{\mathbf{x}_N}\ \tilde{F}^{\mathbf{f}}\ \tilde{F}^{\mathbf{g}})) \\ &\text{subject to} && \text{condition (3.2) and } \kappa\mathbf{I} - Y > \mathbf{0}, \end{aligned}$$

for given matrices $X, Z, \mathcal{Q}$, and Y, and the maximum number of unavailable nodes is the number of zero entries along the diagonal of the matrix $\operatorname{diag}(\tilde{F}^{\mathbf{u}}\ \tilde{F}^{\mathbf{y}}\ \tilde{F}^{\mathbf{x}_N}\tilde{F}^{\mathbf{f}}\ \tilde{F}^{\mathbf{g}})$.

Proof. By observing Lemma 3.1 and that the number of nonzero entries along the diagonal of the matrices $\tilde{F}^{\mathbf{u}}$, $\tilde{F}^{\mathbf{y}}$, $\tilde{F}^{\mathbf{x}_N}$, $\tilde{F}^{\mathbf{f}}$, and $\tilde{F}^{\mathbf{g}}$ specifies the number of available actuator and sensor nodes of the plant system, nodes of the network system, and input and output nodes of the controller system, respectively, the result in Theorem 3.3 is obtained. □

Then, Algorithms 2.1, 2.2, 2.3, and 2.4 of Section 2.4 are extended and generalized to design the proposed topology under different operating scenarios. Namely, in each scenario, a subset of the nodes of the proposed topology can become unavailable during operation time. This is presented in the following algorithm, where ρ denotes the number of operating scenarios.

Algorithm 3.1 Algorithm for the Design of the Controller and Network Systems Under Unreliable Nodes

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Step 1. Specify A_K, B_K, C_K as arbitrary nonzero matrices, $D_K = \mathbf{0}$, $\eta = \eta_{\mathcal{N}} > 0$, $\epsilon = \epsilon_{\mathcal{N}} > 0$, $\mathcal{Q} = \mathcal{N}$, $k = 0$, X_0 as a symmetric matrix; and further, for each operating scenario $i \in \mathcal{R} = \{1, \ldots, \rho\}$, specify the set $\tilde{F}_i \triangleq \{\tilde{F}_i^{\mathbf{u}},\ \tilde{F}_i^{\mathbf{y}},\ \tilde{F}_i^{\mathbf{x}_N}, \tilde{F}_i^{\mathbf{f}},\ \tilde{F}_i^{\mathbf{g}}\}$ to reflect the unavailable nodes.
Step 2. Compute X by solving the convex optimization problem,

$$[X] = \arg \min_{X,Z,\mathcal{Q},Y} \eta$$
$$\text{subject to}$$

$$\begin{bmatrix} X & Z & \tilde{\mathscr{A}}_i(\mathcal{Q}) & \tilde{\mathscr{B}}_i(\mathcal{Q}) \\ * & Y & \tilde{\mathscr{C}}_i(\mathcal{Q}) & \tilde{\mathscr{D}}_i(\mathcal{Q}) \\ * & * & X_k^{-1}(2\mathbf{I} - XX_k^{-1}) & \mathbf{0} \\ * & * & * & \mathbf{I} \end{bmatrix} > \mathbf{0}\ \forall i \in \mathcal{R} = \{1, \ldots, \rho\}.$$

Step 3. If $\eta < 0$, set $j = 0$ and go to Step 4; else, set $k = k + 1$ and return to Step 2.
Step 4. Compute $\mathcal{Q}$, Y, and κ by solving the following convex optimization problem,

$$[\mathcal{Q}, Y, \kappa] = \arg \min_{X,Z,\mathcal{Q},Y,\kappa} \kappa$$
$$\text{subject to}$$
$$\text{the LMI conditions in Step 2 and } \kappa\mathbf{I} - Y > \mathbf{0}.$$

Step 5. If solving for $\mathcal{N}$ and $\kappa < \epsilon$, go to Step 6; if solving for $\mathcal{K}$ and $\kappa < \epsilon$, exit; else, set $j = j + 1$ and return to Step 4.
Step 6. Set $\eta = \eta_{\mathcal{K}} > 0, \epsilon = \epsilon_{\mathcal{K}} > 0$, $\mathcal{Q} = \mathcal{K}$, $k = 0$, and X_0 as a symmetric matrix, and go to Step 2.

In Algorithm 3.1, the first optimization problem of Step 2 delivers a stable closed-loop control system, and the second optimization problem of Step 4 delivers a stable as well as optimal closed-loop control system. The matrices $\tilde{F}_i^{\mathbf{u}}$, $\tilde{F}_i^{\mathbf{y}}$, $\tilde{F}_i^{\mathbf{x}_N}$, $\tilde{F}_i^{\mathbf{f}}$, and $\tilde{F}_i^{\mathbf{g}}$ are specified for each operating scenario to define the subset of the nodes of the proposed topology that are unavailable. In addition, the algorithm begins with utilizing an arbitrary controller system, as an optimal design of the network system is independent of the controller system, as discussed in Section 2.3.

Remark 3.2. In the presented design procedure, when the matrices $\tilde{F}_i^{\mathbf{u}}$, $\tilde{F}_i^{\mathbf{y}}$, $\tilde{F}_i^{\mathbf{x}_N}$, $\tilde{F}_i^{\mathbf{f}}$, and $\tilde{F}_i^{\mathbf{g}}$ are all set to identity matrices for a given i, the same design procedure as that presented in Section 2.4 is obtained (namely, the ideal operating scenario). Thus, the design procedure presented in this section is more generic, compared to the design procedure counterpart of Section 2.4.

3.2 Design of the Topology Under Unreliable Communication Links

The transfer of information between the sensor nodes $\mathcal{S}$ and actuator nodes $\mathcal{A}$ of the plant system G in (2.1), the input nodes Γ and output nodes Θ of the controller system K in (2.2), and the nodes $\mathcal{V}$ of the network system N in (2.3) can become unavailable. The unavailability may result from several factors, including packet dropouts, cyber attacks, and disconnectivity. In this section, the design of the proposed topology is addressed when the transfer of information between the nodes becomes unavailable during operation time.

3.2.1 Modelling Framework of the Topology

The design of the proposed topology under unreliable communication links is achieved by adopting the approach presented for handling erasure channels [42], [15], [29]. Namely, the status of the transfer of information is modelled as a Bernoulli random variable for each communication link $e_i \in \mathcal{E}$. The random variable is denoted by ϑ and switches between 1 and 0 to capture the availability and unavailability of the associated transfer of information (namely, successful and unsuccessful receipt of information), respectively. Given the probability of an unavailable communication link denoted by p, the probability of successful receipt of information and the probability of unsuccessful receipt of information are $\mathbb{P}(\vartheta(k) = 1) = 1-p$ and $\mathbb{P}(\vartheta(k) = 0) = p$, respectively. The random variable ϑ has a generating distribution mean and a finite variance denoted and defined as $\mu \triangleq \mathbb{E}\{\vartheta\} = 1-p$ and $\hat{\sigma}^2 = \mathbb{E}\{(\vartheta(k)-\mu)^2\} = p(1-p)$, respectively. Thus, the transfer of information between the nodes is modelled with a deterministic time-invariant mean μ and a stochastic time-varying $\hat{\Delta}$ with zero mean and variance $\hat{\sigma}^2$ (namely, $\vartheta(k) \triangleq \mu + \hat{\Delta}(k)$). The random variable ϑ is considered independent and identically distributed for $k \geq 0$, and the random variables for multiple communication links are considered independent but not necessarily identically distributed. Further, for simplicity, the random variables associated with all communication links of the proposed topology are considered to be the same (namely, having the same value of p).

Next, consider the network system N in (2.3). It is extended to capture the status of the transfer of information between the nodes of the proposed topology. This is achieved by applying a modelling approach similar to that presented in [42]. Namely, each communication link in the proposed topology is associated with an identifier label denoted by $l \in \{1, \ldots, \mathcal{L}\}$, where $\mathcal{L} = |\mathcal{E}|$ is the total number of communication links in the proposed topology. This allows for each communication link to be mapped to an identifier label using a bijective mapping operator denoted by $\mathcal{M}$. For a communication link of the proposed topology $e_l \in \mathcal{E}$, the transfer of information is defined as

$$r_l(k) = \begin{cases} \lambda_l y_j(k) & \text{if } l = \mathcal{M}(s_j, v_i), \\ \xi_l y_j(k) & \text{if } l = \mathcal{M}(s_j, a_i), \\ \sigma_l y_j(k) & \text{if } l = \mathcal{M}(s_j, \gamma_i), \\ \omega_l x_{N_j}(k) & \text{if } l = \mathcal{M}(v_j, v_i), \\ \upsilon_l x_{N_j}(k) & \text{if } l = \mathcal{M}(v_j, a_i), \\ \pi_l x_{N_j}(k) & \text{if } l = \mathcal{M}(v_j, \gamma_i), \\ \psi_l g_j(k) & \text{if } l = \mathcal{M}(\theta_j, v_i), \\ \delta_l g_j(k) & \text{if } l = \mathcal{M}(\theta_j, a_i), \\ \phi_l g_j(k) & \text{if } l = \mathcal{M}(\theta_j, \gamma_i). \end{cases}$$

For the set of communication links $\mathcal{E}$, the transfer of information is defined as $\mathbf{r}(k) \triangleq [\mathscr{G}_{\mathbf{y}}\ \mathscr{G}_{\mathbf{x}_N}\ \mathscr{G}_{\mathbf{g}}][\mathbf{y}(k)^T\ \mathbf{x}_N(k)^T\ \mathbf{g}(k)^T]^T$, where the matrices are defined as

$$\mathscr{G}_{\mathbf{y}[l,j]} = \begin{cases} \lambda_l & \text{if } l = \mathcal{M}(s_j, v_i), \\ \xi_l & \text{if } l = \mathcal{M}(s_j, a_i), \\ \sigma_l & \text{if } l = \mathcal{M}(s_j, \gamma_i), \\ 0 & \text{otherwise}, \end{cases}$$

$$\mathscr{G}_{\mathbf{x}_N[l,j]} = \begin{cases} \omega_l & \text{if } l = \mathcal{M}(v_j, v_i), \\ \upsilon_l & \text{if } l = \mathcal{M}(v_j, a_i), \\ \pi_l & \text{if } l = \mathcal{M}(v_j, \gamma_i), \\ 0 & \text{otherwise}, \end{cases}$$

$$\mathscr{G}_{\mathbf{g}[l,j]} = \begin{cases} \psi_l & \text{if } l = \mathcal{M}(\theta_j, v_i), \\ \delta_l & \text{if } l = \mathcal{M}(\theta_j, a_i), \\ \phi_l & \text{if } l = \mathcal{M}(\theta_j, \gamma_i), \\ 0 & \text{otherwise}. \end{cases}$$

Next, consider the update procedure of Section 2.1 for each node of the network system, actuator node of the plant system, and input node of the controller system. It is extended to incorporate the deterministic and stochastic components of the status of the transfer of information, in order to model the unavailability of the communication links. This leads to the extended model of the network system defined as

$$\begin{aligned} \mathbf{x}_N(k+1) &= \Omega_\mu \mathbf{x}_N(k) + \Lambda_\mu \mathbf{y}(k) + \Psi_\mu \mathbf{g}(k) + \mathscr{F}_{\mathbf{x}_N} \boldsymbol{\Delta}(k)\mathbf{r}(k), \\ \mathbf{u}(k) &= \Upsilon_\mu \mathbf{x}_N(k) + \Xi_\mu \mathbf{y}(k) + \Delta_\mu \mathbf{g}(k) + \mathscr{F}_{\mathbf{u}} \boldsymbol{\Delta}(k)\mathbf{r}(k), \\ \mathbf{f}(k) &= \Pi_\mu \mathbf{x}_N(k) + \Sigma_\mu \mathbf{y}(k) + \Phi_\mu \mathbf{g}(k) + \mathscr{F}_{\mathbf{f}} \boldsymbol{\Delta}(k)\mathbf{r}(k), \end{aligned} \tag{3.3}$$

where the system matrices are similar to those of the original network system in (2.3), $\boldsymbol{\Delta} \in \mathbb{R}^{\mathcal{L}}$ is a diagonal matrix with elements $\hat{\Delta}_l$ along its diagonal for $l \in \{1, \ldots, \mathcal{L}\}$, and

$$\mathscr{F}_{\mathbf{x}_N [i,l]} = \begin{cases} 1 & \text{if } l = l^v, \\ 0 & \text{otherwise,} \end{cases}$$
$$\mathscr{F}_{\mathbf{u}[i,l]} = \begin{cases} 1 & \text{if } l = l^a, \\ 0 & \text{otherwise,} \end{cases}$$
$$\mathscr{F}_{\mathbf{f}\,[i,l]} = \begin{cases} 1 & \text{if } l = l^\gamma, \\ 0 & \text{otherwise,} \end{cases}$$

where $l^v = \mathcal{M}(v_j, v_i) \vee \mathcal{M}(s_j, v_i) \vee \mathcal{M}(\theta_j, v_i), l^a = \mathcal{M}(v_j, a_i) \vee \mathcal{M}(s_j, a_i) \vee \mathcal{M}(\theta_j, a_i)$, and $l^\gamma = \mathcal{M}(v_j, \gamma_i) \vee \mathcal{M}(s_j, \gamma_i) \vee \mathcal{M}(\theta_j, \gamma_i)$. In the extended model of the network system in (3.3), the terms associated with vectors $\mathbf{x}_N$, $\mathbf{y}$, and $\mathbf{g}$ capture the deterministic components, while the terms associated with vector $\mathbf{r}$ capture the stochastic components, of the status of the transfer of information. The matrices $\mathscr{F}_{\mathbf{x}_N}$, $\mathscr{F}_{\mathbf{u}}$, $\mathscr{F}_{\mathbf{f}}$, and $\boldsymbol{\Delta}$ specify the stochastic components to be added to the deterministic components for the communication links.

Then, the extended model of the network system in (3.3) is decomposed into two subsystems. The first subsystem is a deterministic network system denoted by N_μ (namely, with inputs $\mathbf{y}$, $\mathbf{g}$, and $\mathbf{s}$ as well as outputs $\mathbf{u}$, $\mathbf{f}$, and $\mathbf{r}$). The second subsystem is a stochastic system denoted by $\boldsymbol{\Delta}$ (namely, with input $\mathbf{r}$ and output $\mathbf{s}$, such that $\mathbf{s}(k) = \boldsymbol{\Delta}(k)\mathbf{r}(k)$). In the presented modelling framework, when the stochastic components associated with the status of the transfer of information are all set to zero, the extended model of the network system reduces to that in (2.3).

Further, let the matrices associated with the vectors $\mathbf{w}$ and $\mathbf{z}$ of the plant system in (2.1) as well as the matrix D_K of the controller system in (2.2) to be zero matrices to simplify the derivation. The closed-loop control system (namely, the augmented plant system in (2.1), deterministic network system N_μ, and controller system in (2.2), with input vector $\mathbf{s}$ and output vector $\mathbf{r}$), denoted by M_μ, is modelled as a LTI system in discrete time as

$$\begin{aligned} \mathbf{x}(k+1) &= \mathscr{A}_\mu \mathbf{x}(k) + \mathscr{B}_\mu \mathbf{s}(k), \\ \mathbf{r}(k) &= \mathscr{C}_\mu \mathbf{x}(k), \end{aligned} \tag{3.4}$$

where the system matrices are defined as

$$\mathscr{A}_\mu = \begin{bmatrix} A + B_2\Xi_\mu C_2 & B_2\Upsilon_\mu & B_2\Delta_\mu C_K \\ \Lambda_\mu C_2 & \Omega_\mu & \Psi_\mu C_K \\ B_K\Sigma_\mu C_2 & B_K\Pi_\mu & A_K + B_K\Phi_\mu C_K \end{bmatrix},$$
$$\mathscr{B}_\mu = \begin{bmatrix} B_2\mathscr{F}_{\mathbf{u}} \\ \mathscr{F}_{\mathbf{x}_N} \\ B_K\mathscr{F}_{\mathbf{f}} \end{bmatrix}, \mathscr{C}_\mu = \begin{bmatrix} \mathscr{G}_{\mathbf{y}}C_2 & \mathscr{G}_{\mathbf{x}_N} & \mathscr{G}_{\mathbf{g}}C_K \end{bmatrix}.$$

As can be observed from the closed-loop control system in (3.4), only the system matrix $\mathscr{A}_\mu$ contains system matrices similar to those of the network system in (2.3) (namely, with coefficients denoting the weight assignments). Thus, it is expressed in terms of the system matrices of the network system in (2.3) (namely, similar to the approach presented in Section 2.2). First, consider the matrix $\mathbf{I}_{\mu_l}$ defined as a diagonal matrix with μ_l along its diagonal. An extended network parameter is defined as $\mathcal{N}_\mu \triangleq \mathbf{I}_{\mu_l}\mathcal{N}_1 + \mathcal{N}_2$, where the matrices $\mathcal{N}_1$ and $\mathcal{N}_2$ denote the network parameter $\mathcal{N}$ in (2.7) without and with the weights of the self-connectivity links of the nodes (namely, ω_{ii}), respectively, such that $\mathcal{N} = \mathcal{N}_1 + \mathcal{N}_2$. Next, the system matrix $\mathscr{A}_\mu$ is expressed in terms of the extended network parameter $\mathcal{N}_\mu$ as

$$\mathscr{A}_\mu(\mathcal{N}_\mu) = \mathcal{A} + \mathcal{B}\mathcal{N}_\mu\mathcal{C},$$

and further, the closed-loop control system is modelled as

$$\mathbf{x}(k+1) \ = (\mathscr{A}_\mu + \mathscr{B}_\mu \boldsymbol{\Delta}(k)\mathscr{C}_\mu)\mathbf{x}(k). \tag{3.5}$$

3.2.2 Condition Requirements of the Topology

First, consider the conditions required to characterize the existence of the design of the controller system. The existence of such a design is characterized in the following theorem.

Theorem 3.4. Suppose the pairs (A, B_2) and

$$\left(\begin{bmatrix} A + B_2\Xi_\mu C_2 & B_2\Upsilon_\mu \\ \Lambda_\mu C_2 & \Omega_\mu \end{bmatrix}, \begin{bmatrix} \mathbf{0} \\ \Psi_\mu \end{bmatrix} \right)$$

are stabilizable, and the pairs (C_2, A) and

$$\left(\begin{bmatrix} \mathbf{0} & \Pi_\mu \end{bmatrix}, \begin{bmatrix} A + B_2\Xi_\mu C_2 & B_2\Upsilon_\mu \\ \Lambda_\mu C_2 & \Omega_\mu \end{bmatrix} \right)$$

are detectable. Then, there exists a controller system K in (2.2) that can provide a stable closed-loop control system in (3.5).

The proof of Theorem 3.4 follows the same argument and derivation as those presented for Theorem 3.1. Then, consider the conditions required to characterize the existence of the design of the network system. The plant system in (2.1) can be stabilized using a network system if and only if the pair (A, B_2) is stabilizable and the pair (C_2, A) is detectable.

3.2.3 Design Procedure of the Topology

First, consider the mean square stability of the closed-loop control system based on the the following definition from [15].

Definition 3.1. Consider the closed-loop control system in (3.5) with state vector $\mathbf{x}$, it is mean square stable if $\mathbb{E}\{\mathbf{x}(k)\mathbf{x}(k)^T\}$ is well defined for $k \geq 0$ and $\lim_{k\to\infty} \mathbb{E}\{\mathbf{x}(k)\mathbf{x}(k)^T\} = 0$.

Next, the network system of the proposed topology under unreliable communication links is characterized based on the following theorem from [42] and [15].

Theorem 3.5. The closed-loop control system in (3.5) is mean square stable if and only if there exists a matrix $X > \mathbf{0}$ and a vector $\boldsymbol{\alpha} \in \mathbb{R}^{\mathcal{L}}$ with positive elements for all $l = 1, \ldots, \mathcal{L}$, such that the following matrix inequalities are satisfied:

$$\begin{bmatrix} X - \mathscr{B}_\mu \text{diag}\{\boldsymbol{\alpha}\}\mathscr{B}_\mu^T & \mathscr{A}_\mu \\ * & X^{-1} \end{bmatrix} > \mathbf{0}, \tag{3.6}$$

$$\begin{bmatrix} \alpha_l & \hat{\sigma}_l(\mathscr{C}_\mu)_l \\ * & X^{-1} \end{bmatrix} > \mathbf{0}. \tag{3.7}$$

In Theorem 3.5, $\hat{\sigma}_l$ relates to the probability of an unavailable communication link p, such that $\hat{\sigma}_l^2 = p(1-p)$. Further, $\hat{\sigma}_l$ can take a number of values while the resulting closed-loop control system is stable. Then, considering Theorem 3.5 and Remark 3 from [42], and observing that the set of all possible values of the probability of unavailable communication link p relates to the incrementally ordered set $\hat{\Sigma}_{\mathcal{L}} = \{\hat{\sigma}_{\min}, \ldots, \hat{\sigma}_{\max}\}$, a network system that results in a closed-loop control system in (3.5) that is mean square stable and tolerates a maximum probability of unavailable communication link p is designed by finding a solution to the optimization problem,

$$\begin{array}{ll} \text{maximize} & \hat{\sigma}_l \\ \text{subject to} & \text{the matrix inequalities} \\ & \quad \text{in (3.6) and (3.7)} \end{array}$$

and the maximum value of $\hat{\sigma}_l$ corresponds to the largest value of p and is also the largest value of the set $\hat{\Sigma}_{\mathcal{L}}$. Thus, the design procedure of the network system of the proposed topology under unreliable communication links is achieved using the following algorithm.

Algorithm 3.2 Algorithm for the Design of the Network System Under Unreliable Communication Links

Step 1. Specify A_K, B_K, C_K as arbitrary nonzero matrices, $D_K = \mathbf{0}$, $p = 0$, $\eta > 0$, and $k = 0$.

Step 2. Compute $\mathcal{N}$ and α for $l = 1, \ldots, \mathcal{L}$ by solving the convex optimization problem,

$$[\mathcal{N}, \alpha, \hat{\sigma}_l] = \arg \min_{X, \mathcal{N}, \alpha} \eta$$
$$\text{subject to}$$
$$\begin{bmatrix} X - \mathscr{B}_\mu \text{diag}\{\boldsymbol{\alpha}\} \mathscr{B}_\mu^T & \mathscr{A}_\mu(\mathcal{N}_\mu) \\ * & X_k^{-1}(2\mathbf{I} - X X_k^{-1}) \end{bmatrix} > \mathbf{0},$$
$$\begin{bmatrix} \alpha_l & \hat{\sigma}_l(\mathscr{C}_\mu)_l \\ * & X_k^{-1}(2\mathbf{I} - X X_k^{-1}) \end{bmatrix} > \mathbf{0}, \text{ and}$$
$$\begin{bmatrix} \text{diag}\{\boldsymbol{\alpha}\} & \mathbf{0} \\ * & X \end{bmatrix} > \mathbf{0}.$$

Step 3. If $\eta < 0$, go to Step 4; else, set $k = k + 1$ and return to Step 2.

Step 4. If p corresponds to $\hat{\sigma}_{\max}$, exit; else, set $p = p + 0.01$, specify $\eta > 0$ and $k = 0$, and return to Step 2.

In Algorithm 3.2, the optimization problem of Step 2 delivers a network system such that the closed-loop control system is mean square stable and tolerates a given value of p. The algorithm begins by finding a network system for $p = 0$ (namely, all communication links are available), and then, it finds network systems by incrementally increasing the value of p to its maximum value while the convergence of the algorithm remains feasible. It should be noted that the value of p is increased by 0.01; however, a different value can be specified or an alternative approach for the increase can be implemented (e.g., bisection).

Thus far, the modelling framework and the design procedure address the design of the network system of the proposed topology while the controller system is given *a-priori*. However, the use of different controller systems can result in different values of p. In order to obtain a higher maximum value of p, the optimization problem is solved based on the result from [15],

$$\mu^*_{\mathrm{MS}}(M_\mu, \boldsymbol{\Delta}) = \inf_{\tilde{\theta}} \inf_{\hat{K}} ||\tilde{\theta}^{-1} M_\mu \tilde{\theta}||^2_{\mathrm{MS}},$$

where the notation $\| G \|_{\mathrm{MS}} \triangleq \max_{i=1,\ldots,\mathcal{L}} \sqrt{\sum_{j=1}^{\mathcal{L}} \| G_{[i,j]} \|_2^2}$ defines the mean square norm of a system G, $\hat{K}$ and the inverse of $\mu^*_{\mathrm{MS}}(M_\mu, \boldsymbol{\Delta})$ denote the set of all stabilizing controller systems and the largest mean square stability margin, respectively, and $\tilde{\theta}$ is a diagonal matrix and is positive definite. The largest margin corresponds to the highest value of the maximum probability of unavailable communication links. Further, the solution to the optimization problem and the construction of the controller system can be achieved using Theorem 6.6 of [15]. More specifically, it is achieved by using the augmented plant system in (2.1) with $\mathbf{w} = \mathbf{0}$ and $\mathbf{z} = \mathbf{0}$ and the deterministic network system N_μ of Section 3.2.1, denoted by P_μ, and defined as

$$\begin{aligned}
\mathbf{x}_P(k+1) &= \tilde{\mathscr{A}}_P \mathbf{x}_P(k) + \tilde{\mathscr{B}}_{P_1} \mathbf{s}(k) + \tilde{\mathscr{B}}_{P_2} \mathbf{g}(k), \\
\mathbf{r}(k) &= \tilde{\mathscr{C}}_{P_1} \mathbf{x}_P(k) + \tilde{\mathscr{D}}_{P_{11}} \mathbf{s}(k) + \tilde{\mathscr{D}}_{P_{12}} \mathbf{g}(k), \\
\mathbf{f}(k) &= \tilde{\mathscr{C}}_{P_2} \mathbf{x}_P(k) + \tilde{\mathscr{D}}_{P_{21}} \mathbf{s}(k) + \tilde{\mathscr{D}}_{P_{22}} \mathbf{g}(k),
\end{aligned}$$

where the system matrices are defined as

$$\begin{aligned}
\tilde{\mathscr{A}}_P &= \begin{bmatrix} A + B_2 \Xi_\mu C_2 & B_2 \Upsilon_\mu \\ \Lambda_\mu C_2 & \Omega_\mu \end{bmatrix}, \tilde{\mathscr{B}}_{P_1} = \begin{bmatrix} B_2 \mathscr{F}_{\mathbf{u}} \\ \mathscr{F}_{\mathbf{x}_N} \end{bmatrix}, \\
\tilde{\mathscr{B}}_{P_2} &= \begin{bmatrix} B_2 \Delta_\mu \\ \Psi_\mu \end{bmatrix}, \tilde{\mathscr{C}}_{P_1} = \begin{bmatrix} \mathscr{G}_{\mathbf{y}} C_2 & \mathscr{G}_{\mathbf{x}_N} \end{bmatrix}, \tilde{\mathscr{D}}_{P_{11}} = \mathbf{0}, \\
\tilde{\mathscr{D}}_{P_{12}} &= \mathscr{G}_{\mathbf{g}}, \tilde{\mathscr{C}}_{P_2} = \begin{bmatrix} \Sigma_\mu C_2 & \Pi_\mu \end{bmatrix}, \tilde{\mathscr{D}}_{P_{21}} = \mathscr{F}_{\mathbf{f}}, \tilde{\mathscr{D}}_{P_{22}} = \Phi_\mu.
\end{aligned}$$

In this approach, the controller system has the same size as that of the augmented system (namely, $n + N$). To deliver a controller system of a reduced size, approaches that utilize reduction methods can be investigated. For example, model reduction and iterative computational approaches can be applied to compute the network and controller systems, while providing a higher maximum probability of unavailable communication links.

3.3 Design of the Topology Under Cyber Attacks

The transfer of information between the sensor nodes $\mathcal{S}$ and actuator nodes $\mathcal{A}$ of the plant system G in (2.1), the input nodes Γ and output nodes Θ of the controller system K in (2.2), and the nodes $\mathcal{V}$ of the network system N in (2.3) can become subject to cyber attacks. The attacks may result from an intruder, which can access as well as manipulate the transfer of information between the nodes. In this section, the design of the proposed topology is addressed when the transfer of information between the nodes is subject to cyber attacks during operation time.

3.3.1 Modelling Framework of Topology

Cyber attacks are classified into several categories [55], including denial of service (DoS), replay, and bias injection attacks. In a DoS attack, the intruder blocks the transfer of information between the nodes that are communicating. In a replay attack, the intruder eavesdrops the communication links, records the transferred information, and re-transfers the information at a subsequent time. Finally, in a bias injection attack, the intruder manipulates the transfer of information by injecting bias values.

The design of the proposed topology under cyber attacks is achieved by adopting the following approach (namely, for the three mentioned categories). Each node of the proposed topology, $a_i \in \mathcal{A}$ and $s_i \in \mathcal{S}$ of the plant system, $\gamma_i \in \Gamma$ and $\theta_i \in \Theta$ of the controller system, and $v_i \in \mathcal{V}$ of the distributed and inter-connected nodes of the network system has a state that is composed of attack-free and attack-dependent components. For example, consider a node $v_i \in \mathcal{V}$ of the network system, its state under cyber attacks, denoted by $\tilde{x}_{N_i}$, is defined as $\tilde{x}_{N_i}(k) \triangleq x_{N_i}(k) + \nabla_{N_i} a_{N_i}(k)$, where $\nabla_{N_i} \in \mathbb{B}$ is a time-varying coefficient that switches between 1 and 0 to capture the existence and non-existence of a cyber attack, respectively, and $a_{N_i}(k) = x^a_{N_i}(k) - x_{N_i}(k)$ is the associated attack signal containing the type of the cyber attack defined in $x^a_{N_i}$. The type of the cyber attack is defined such that $x^a_{N_i}(k) = x_{N_i}(k_{\text{DoS}})$ under a DoS attack, where the last received information at time $k = k_{\text{DoS}}$ is used; $x^a_{N_i}(k) = x_{N_i}(k - \tau)$ under a replay attack, where the delayed information by time τ is used; and $x^a_{N_i}(k) = \varrho$ under a bias injection attack, where an injected bias value ϱ is used. Thus, the actual state x_{N_i} of node v_i is omitted and a modified state that incorporates the cyber attack (namely, defined by $x_{N_i}(k_{\text{DoS}})$, $x_{N_i}(k - \tau)$, or ϱ) is used as the received information. For the set of nodes $\mathcal{V}$, the coefficients of the nodes are combined in a coefficient matrix denoted by ∇_N, such that the coefficient of each node $v_i \in \mathcal{V}$ is placed along its diagonal, and the attack signals are combined in a vector denoted by $\mathbf{a}_N$. This leads to the modified vector $\tilde{\mathbf{x}}_N$ that denotes the state of the nodes under cyber attacks. Similarly, the sets of nodes $\mathcal{S}$ and Θ that transfer information, respectively, have modified state vectors denoted by $\tilde{\mathbf{y}}$ and $\tilde{\mathbf{g}}$, which contain the coefficient matrices denoted by $\nabla_{\mathbf{y}}$ and $\nabla_{\mathbf{g}}$, and the attack signals denoted by $\mathbf{a}_{\mathbf{y}}$ and $\mathbf{a}_{\mathbf{g}}$. Next, consider the following two assumptions.

Assumption 3.1. Suppose the transfer of information from a given node of the proposed topology is under a cyber attack. All of the transferred information from the node (namely, transferred to different nodes) is attacked similarly according to the category of the cyber attack.

Assumption 3.2. Suppose there is no direct communication between the plant and controller systems, and that the plant and controller systems are connected only through the distributed and inter-connected nodes of the network system. Let $D_K = \mathbf{0}$, and suppose the sensor nodes of the plant system and the output nodes of the controller system only transfer information to the nodes of the network system. The system matrices Δ, Σ, Ξ, and Φ are therefore omitted.

Then, consider the plant system G in (2.1), the network system N in (2.3), and the controller system K in (2.2). The vectors $\mathbf{y}$, $\mathbf{x}_N$, and $\mathbf{g}$ are replaced by the vectors $\tilde{\mathbf{y}}$, $\tilde{\mathbf{x}}_N$, and $\tilde{\mathbf{g}}$, respectively, to capture the cyber attacks on the transfer of the information from the nodes. Thus, the closed-loop control system (namely, the augmented plant system, network system, and controller system) is modelled as a LTI system in discrete time as

$$\begin{aligned}\mathbf{x}(k+1) &= \begin{bmatrix} A & B_2\Upsilon & \mathbf{0} \\ \Lambda C_2 & \Omega & \Psi C_K \\ \mathbf{0} & B_K\Pi & A_K \end{bmatrix}\mathbf{x}(k) + \begin{bmatrix} B_1 \\ \Lambda D_{21} \\ \mathbf{0} \end{bmatrix}\mathbf{w}(k) \\ &+ \begin{bmatrix} \mathbf{0} & B_2\Upsilon\nabla_N & \mathbf{0} \\ \Lambda\nabla_{\mathbf{y}} & \Omega_O\nabla_N & \Psi\nabla_{\mathbf{g}} \\ \mathbf{0} & B_K\Pi\nabla_N & \mathbf{0} \end{bmatrix}\mathbf{a}(k), \\ \mathbf{z}(k) &= \begin{bmatrix} C_1 & D_{12}\Upsilon & \mathbf{0} \end{bmatrix}\mathbf{x}(k) + D_{11}\mathbf{w}(k) \\ &+ \begin{bmatrix} \mathbf{0} & D_{12}\Upsilon\nabla_N & \mathbf{0} \end{bmatrix}\mathbf{a}(k), \end{aligned} \tag{3.8}$$

where $\mathbf{a} = [\mathbf{a}_{\mathbf{y}}^T\ \mathbf{a}_N^T\ \mathbf{a}_{\mathbf{g}}^T]^T$ and the matrix Ω_O has zeros on its diagonal and the same off-diagonal entries as the matrix Ω in (2.3).

3.3.2 Modelling Framework of the Intrusion Detection System

For the detection of cyber attacks in the proposed topology, an intrusion detection system (IDS) is designed. This is achieved by adopting an approach similar to those in the fault detection and isolation literature (for example, see [23] and [12]). Namely, each actuator node $a_i \in \mathcal{A}$ of the plant system, input node $\gamma_i \in \Gamma$ of the controller system, and node $v_i \in \mathcal{V}$ of the network system is associated with a detector system that monitors the transferred information from the neighbouring nodes. This allows for the distribution of the intrusion detection scheme amongst the nodes of the proposed topology, such that each node is capable of detecting cyber attacks on the transfer of information from its neighbouring nodes. For example, consider a node v_i of the network system, its detector system is modelled as a LTI system in discrete time as

$$\begin{aligned} x_{N_i}^D(k+1) &= A_{N_i}^D x_{N_i}^D(k) + B_{N_i}^D x_{N_i}(k), \\ r_{N_i}^D(k) &= C_{N_i}^D x_{N_i}^D(k) + D_{N_i}^D x_{N_i}(k), \end{aligned}$$

where $x_{N_i}^D$ and $r_{N_i}^D$ denote its state and residue, respectively. The states of the detector systems of all nodes of the network system of the proposed topology in an augmented manner provide the intrusion detection system at the nodes of the network system, denoted by D_N, which is modelled as a LTI system in discrete time as

$$\begin{aligned} \mathbf{x}_N^D(k+1) &= A_N^D \mathbf{x}_N^D(k) + B_N^D \mathbf{x}_N(k), \\ \mathbf{r}_N^D(k) &= C_N^D \mathbf{x}_N^D(k) + D_N^D \mathbf{x}_N(k), \end{aligned} \tag{3.9}$$

where the vectors $\mathbf{x}_N^D \in \mathbb{R}^N$ and $\mathbf{r}_N^D \in \mathbb{R}^N$, and all of its system matrices are diagonal and have suitable dimensions. Similarly, the intrusion detection systems of the actuator nodes of the plant system and the input nodes of the controller system, denoted by $D_{\mathcal{A}}$ and D_Γ, are modelled as LTI systems in discrete time as

$$\begin{aligned}\mathbf{x}_{\mathcal{A}}^D(k+1) &= A_{\mathcal{A}}^D\mathbf{x}_{\mathcal{A}}^D(k) + B_{\mathcal{A}}^D\Upsilon\mathbf{x}_N(k) + B_{\mathcal{A}}^D\Upsilon\nabla_N\mathbf{a}_N(k),\\ \mathbf{r}_{\mathcal{A}}^D(k) &= C_{\mathcal{A}}^D\mathbf{x}_{\mathcal{A}}^D(k) + D_{\mathcal{A}}^D\Upsilon\mathbf{x}_N(k) + D_{\mathcal{A}}^D\Upsilon\nabla_N\mathbf{a}_N(k),\end{aligned} \tag{3.10}$$

and

$$\begin{aligned}\mathbf{x}_{\Gamma}^D(k+1) &= A_{\Gamma}^D\mathbf{x}_{\Gamma}^D(k) + B_{\Gamma}^D\Pi\mathbf{x}_N(k) + B_{\Gamma}^D\Pi\nabla_N\mathbf{a}_N(k),\\ \mathbf{r}_{\Gamma}^D(k) &= C_{\Gamma}^D\mathbf{x}_{\Gamma}^D(k) + D_{\Gamma}^D\Pi\mathbf{x}_N(k) + D_{\Gamma}^D\Pi\nabla_N\mathbf{a}_N(k),\end{aligned} \tag{3.11}$$

respectively, where the vectors $\mathbf{x}_{\mathcal{A}}^D \in \mathbb{R}^m, \mathbf{r}_{\mathcal{A}}^D \in \mathbb{R}^m, \mathbf{x}_{\Gamma}^D \in \mathbb{R}^d$ and $\mathbf{r}_{\Gamma}^D \in \mathbb{R}^d$, and all of their system matrices are diagonal and have suitable dimensions. The closed-loop control system (namely, the plant system G in (2.1), controller system K in (2.2), network system N in (2.3) as well as the intrusion detection systems D_N in (3.9), $D_{\mathcal{A}}$ in (3.10), and D_Γ in (3.11)) is modelled as a LTI system in discrete time as

$$\begin{aligned}\bar{\mathbf{x}}(k) &= \mathscr{A}\bar{\mathbf{x}}(k) + \mathscr{B}\mathbf{v}(k),\\ \mathbf{q}(k) &= \mathscr{C}\bar{\mathbf{x}}(k) + \mathscr{D}\mathbf{v}(k),\end{aligned} \tag{3.12}$$

where the vectors $\bar{\mathbf{x}} = [\mathbf{x}_G^T\ \mathbf{x}_N^T\ \mathbf{x}_K^T\ \mathbf{x}_{\mathcal{A}}^{D^T}\ \mathbf{x}_N^{D^T}\ \mathbf{x}_\Gamma^{D^T}]^T$, $\mathbf{v} = [\mathbf{w}^T\ \mathbf{a}_{\mathbf{y}}^T\ \mathbf{a}_N^T\ \mathbf{a}_{\mathbf{g}}^T]^T$, and $\mathbf{q} = [\mathbf{z}^T\ \mathbf{r}_{\mathcal{A}}^{D^T}\ \mathbf{r}_N^{D^T}\ \mathbf{r}_\Gamma^{D^T}]^T$ denote its state, input, and output, respectively, and its system matrices are defined as

$$\mathscr{A} = \begin{bmatrix} A & B_2\Upsilon & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \Lambda C_2 & \Omega & \Psi C_K & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & B_K\Pi & A_K & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & B_{\mathcal{A}}^D\Upsilon & \mathbf{0} & A_{\mathcal{A}}^D & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & B_N^D & \mathbf{0} & \mathbf{0} & A_N^D & \mathbf{0}\\ \mathbf{0} & B_\Gamma^D\Pi & \mathbf{0} & \mathbf{0} & \mathbf{0} & A_\Gamma^D \end{bmatrix},$$

$$\mathscr{B} = \begin{bmatrix} B_1 & \mathbf{0} & B_2\Upsilon\nabla_N & \mathbf{0}\\ \Lambda D_{21} & \Lambda\nabla_{\mathbf{y}} & \Omega_O\nabla_N & \Psi\nabla_{\mathbf{g}}\\ \mathbf{0} & \mathbf{0} & B_K\Pi\nabla_N & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & B_{\mathcal{A}}^D\Upsilon\nabla_N & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & B_\Gamma^D\Pi\nabla_N & \mathbf{0} \end{bmatrix},$$

$$\mathscr{C} = \begin{bmatrix} C_1 & D_{12}\Upsilon & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & D_{\mathcal{A}}^D\Upsilon & \mathbf{0} & C_{\mathcal{A}}^D & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & D_N^D & \mathbf{0} & \mathbf{0} & C_N^D & \mathbf{0} \\ \mathbf{0} & D_\Gamma^D\Pi & \mathbf{0} & \mathbf{0} & \mathbf{0} & C_\Gamma^D \end{bmatrix},$$

$$\mathscr{D} = \begin{bmatrix} D_{11} & \mathbf{0} & D_{12}\Upsilon\nabla_N & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & D_{\mathcal{A}}^D\Upsilon\nabla_N & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & D_\Gamma^D\Pi\nabla_N & \mathbf{0} \end{bmatrix}.$$

Then, the stability of the closed-loop control system in (3.12) is characterized in the following theorem.

Theorem 3.6. Suppose the augmented system consisting of the plant, network, and controller systems defined in (3.8) is stable. Then, the closed-loop control system in (3.12) is stable if and only if each of the systems $D_{\mathcal{A}}$, D_N, and D_Γ is stable.

Proof. The matrix $\mathscr{A}$ in (3.12) has a lower triangular form with the system matrix of the augmented system in (3.8) in the upper left block and the system matrices of systems $D_{\mathcal{A}}$, D_N, and D_Γ in the lower right block along the diagonal. Thus, given that the augmented system is stable, the stability of each of the systems $D_{\mathcal{A}}$, D_N, and D_Γ is necessary and sufficient for the stability of the closed-loop control system. □

Further, the system matrices of the closed-loop control system in (3.12) are expressed in terms of matrices affine on the system matrices of the intrusion detection systems D_N in (3.9), $D_{\mathcal{A}}$ in (3.10), and D_Γ in (3.11) (namely, similar to the approach presented in Section 2.2). First, suppose the detector parameter is denoted by $\mathcal{D}$ and defined as

$$\mathcal{D} = \begin{bmatrix} D_{\mathcal{A}}^D & \mathbf{0} & \mathbf{0} & C_{\mathcal{A}}^D & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & D_N^D & \mathbf{0} & \mathbf{0} & C_N^D & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & D_\Gamma^D & \mathbf{0} & \mathbf{0} & C_\Gamma^D \\ B_{\mathcal{A}}^D & \mathbf{0} & \mathbf{0} & A_{\mathcal{A}}^D & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & B_N^D & \mathbf{0} & \mathbf{0} & A_N^D & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & B_\Gamma^D & \mathbf{0} & \mathbf{0} & A_\Gamma^D \end{bmatrix}. \tag{3.13}$$

Next, the system matrices of the closed-loop control system in (3.12) are expressed in terms of the detector parameter as

$$\mathscr{A}(\mathcal{D}) = \mathscr{A}_{11} + \mathscr{A}_{12}\mathcal{D}\mathscr{A}_{13},$$
$$\mathscr{B}(\mathcal{D}) = \mathscr{B}_{11} + \mathscr{B}_{12}\mathcal{D}\mathscr{B}_{13},$$
$$\mathscr{C}(\mathcal{D}) = \mathscr{C}_{11} + \mathscr{C}_{12}\mathcal{D}\mathscr{A}_{13},$$
$$\mathscr{D}(\mathcal{D}) = \mathscr{D}_{11} + \mathscr{D}_{12}\mathcal{D}\mathscr{B}_{13},$$

where the matrices are defined as

$$\mathscr{A}_{11} = \begin{bmatrix} A & B_2\Upsilon & \mathbf{0} & \mathbf{0} \\ \Lambda C_2 & \Omega & \Psi C_K & \mathbf{0} \\ \mathbf{0} & B_K\Pi & A_K & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}, \mathscr{A}_{12} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}, \mathscr{A}_{13} = \begin{bmatrix} \mathbf{0} & \Upsilon & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \Pi & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix},$$

$$\mathscr{B}_{11} = \begin{bmatrix} B_1 & \mathbf{0} & B_2\Upsilon\nabla_N & \mathbf{0} \\ \Lambda D_{21} & \Lambda\nabla_{\mathbf{y}} & \Omega_O\nabla_N & \Psi\nabla_{\mathbf{g}} \\ \mathbf{0} & \mathbf{0} & B_K\Pi\nabla_N & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}, \mathscr{B}_{12} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix},$$

$$\mathscr{B}_{13} = \begin{bmatrix} \mathbf{0} & \Upsilon\nabla_N & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \Pi\nabla_N & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}, \mathscr{C}_{11} = \begin{bmatrix} C_1 & D_{12}\Upsilon & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}, \mathscr{C}_{12} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{I} & \mathbf{0} \end{bmatrix},$$

$$\mathscr{D}_{11} = \begin{bmatrix} D_{11} & \mathbf{0} & D_{12}\Upsilon\nabla_N & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}, \mathscr{D}_{12} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} \end{bmatrix}.$$

3.3.3 Design Procedure of the Detection Scheme

The detection of cyber attacks in the proposed topology during operation time is achieved by monitoring the residue of each of the intrusion detection systems D_N in (3.9), $D_{\mathcal{A}}$ in (3.10), and D_Γ in (3.11). The residues are sensitive to cyber attacks, such that when $\tilde{\mathbf{x}}_N \neq \mathbf{x}_N$, $\tilde{\mathbf{y}} \neq \mathbf{y}$, and $\tilde{\mathbf{g}} \neq \mathbf{g}$, there is an associated and relatively significant change in their values. Next, the maximum number of attacked nodes for which the IDSs are computed is characterized in the following theorem.

Theorem 3.7. Suppose for some nonzero matrix $\mathrm{diag}(\nabla_{\mathbf{y}}\nabla_N\nabla_{\mathbf{g}})$, the optimization problem is solved,

$$\begin{array}{ll} \text{maximize} & \operatorname{trace}(\operatorname{diag}(\nabla_{\mathbf{y}}\nabla_N\nabla_{\mathbf{g}})) \\ \text{subject to} & \begin{bmatrix} X & Z & \mathscr{A}(\mathcal{D}) & \mathscr{B}(\mathcal{D}) \\ * & Y & \mathscr{C}(\mathcal{D}) & \mathscr{D}(\mathcal{D}) \\ * & * & X^{-1} & \mathbf{0} \\ * & * & * & \mathbf{I} \end{bmatrix} > \mathbf{0}. \end{array}$$

Then, the maximum number of attacked nodes for which the parameter $\mathcal{D}$ can be computed such that the closed-loop control system in (3.12) is asymptotically stable is equal to the number of nonzero entries along the diagonal of the matrix $\operatorname{diag}(\nabla_{\mathbf{y}}\nabla_N\nabla_{\mathbf{g}})$.

The proof of Theorem 3.7 can be easily obtained by considering Lemma 2.1 of Section 2.4 (namely, based on the results from [21, 51]) and replacing the parameter Q with the detector parameter $\mathcal{D}$ as well as the structure of the matrix $\operatorname{diag}(\nabla_{\mathbf{y}}\nabla_N\nabla_{\mathbf{g}})$.

Next, the design procedure of the intrusion detection scheme of the proposed topology is achieved using the following algorithm.

Algorithm 3.3 Algorithm for the Design of the Intrusion Detection Systems

---

Step 1. Set the outer-layer iteration index $i = 0$, and the diagonal entries of the matrix $\operatorname{diag}(\nabla_{\mathbf{y}}\nabla_N\nabla_{\mathbf{g}})$ with 1s and 0s elsewhere.
Step 2. Specify the feasibility parameter $\eta > 0$, the inner-layer iteration index $k = 0$, and X_0 as an arbitrary symmetric matrix.
Step 3. Compute $\mathcal{D}$ by solving the convex optimization problem,

$$[\mathcal{D}] = \arg \min_{X,Z,\mathcal{D},Y} \eta$$
$$\text{subject to}$$
$$\begin{bmatrix} X & Z & \mathscr{A}(\mathcal{D}) & \mathscr{B}(\mathcal{D}) \\ * & Y & \mathscr{C}(\mathcal{D}) & \mathscr{D}(\mathcal{D}) \\ * & * & X_k^{-1}(2\mathbf{I} - XX_k^{-1}) & \mathbf{0} \\ * & * & * & \mathbf{I} \end{bmatrix} > \mathbf{0}.$$

Step 4. If $\eta < 0$, exit; else, set $k = k + 1$ and return to Step 3. If no solution exists, go to Step 5.
Step 5. In the matrix $\operatorname{diag}(\nabla_{\mathbf{y}}\nabla_N\nabla_{\mathbf{g}})$, sequentially replace a 1 with a 0 in each entry of the diagonal, increment i by 1, and return to Step 2. If all attempts are made, use the last feasible solution.

In Algorithm 3.3, there are two layers. The first layer precisely specifies the number and location of cyber attacks and the second layer computes the detector parameter $\mathcal{D}$ in (3.13). The algorithm begins with assuming that all transferred information is under cyber attacks (namely, letting $\text{diag}(\nabla_{\mathbf{y}}\nabla_{N}\nabla_{\mathbf{g}}) = \mathbf{I}$), and then in a sequential manner, it removes an attack until the algorithm converges. Thus, the algorithm delivers the design of the IDSs that are sensitive to a maximum number of cyber attacks on the transfer of information between the nodes of the proposed topology.

3.4 Simulations

The design of the proposed topology under abnormal operating scenarios is demonstrated by applying the design procedures discussed in Sections 3.1, 3.2, and 3.3. More specifically, the design of the proposed topology under unreliable nodes and communication links is demonstrated using a four-tank process system, whereas the design of the intrusion detection scheme is demonstrated using a numerical example.

3.4.1 The Quadruple Tank Process System

The quadruple tank process (QTP) system from [25] is depicted in Figure 3.1. It consists of four inter-connected tanks, where tanks 1 and 2 are placed below tanks 3 and 4, and the liquid flowing to tanks 3 and 4 is regulated using valves V1 and V2 as well as pumps P1 and P2. The positions of the valves are fixed during operation time, the levels of tanks 1 and 2 are measured using sensor nodes that provide voltage measurements (namely, the outputs of the system) and the pumps use voltages applied to operate them as desired (namely, the controlled inputs of the system). The QTP system is modelled with a state vector $\mathbf{x}_G = [x_{G_1}\ x_{G_2}\ x_{G_3}\ x_{G_4}]^T$, input vector $\mathbf{u} = [u_1\ u_2]^T$, and output vector $\mathbf{y} = [y_1\ y_2]^T$. Its linearized model, discretized with a discretization time of 0.1 while under a nonminimum phase characteristic, has the system matrices defined as

$$A = \begin{bmatrix} 0.9984 & 0 & 0.0026 & 0 \\ 0 & 0.9989 & 0 & 0.0018 \\ 0 & 0 & 0.9974 & 0 \\ 0 & 0 & 0 & 0.9982 \end{bmatrix}, B_2 = \begin{bmatrix} 0.0048 & 0 \\ 0 & 0.0035 \\ 0 & 0.0077 \\ 0.0056 & 0 \end{bmatrix},$$
$$C_1 = C_2 = \begin{bmatrix} 0.5 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 \end{bmatrix}, B_1 = \mathbf{1}, D_{11} = D_{12} = D_{21} = D_{22} = \mathbf{0}.$$

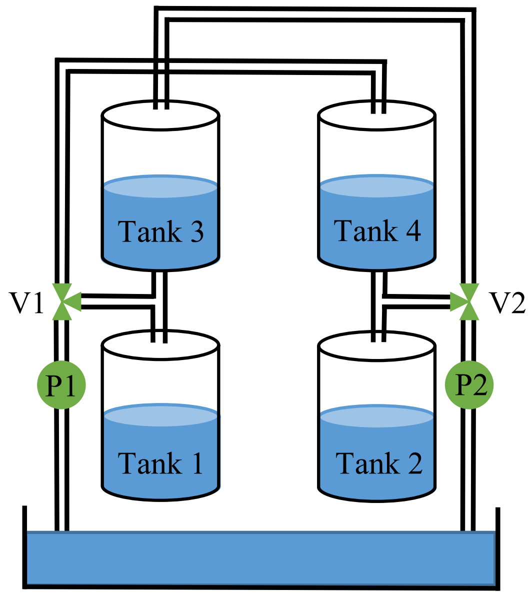

Figure 3.1: Quadruple tank process system [25] with four tanks, two valves, and two pumps.

In the two subsequent sections, the proposed topology consists of the QTP system (namely, as a plant system), a second-order controller system with a single input node γ_1 and a single output node θ_1, and a fourth-order network system with four distributed and inter-connected nodes v_1, v_2, v_3, and v_4.

3.4.2 The Topology Under Unreliable Nodes

In this section, the design of the proposed topology under unreliable nodes is demonstrated. Algorithm 3.1 is implemented using MATLAB's Robust Control Toolbox, where it is assumed that the input and output nodes of the controller system as well as the nodes v_1, v_2, and v_3 of the network system are unreliable and can become unavailable during operation time (namely, given a mixture of possibilities of unavailable nodes). In the worst case, the plant system is solely regulated using node v_4 of the network system. Algorithm 3.1 begins with an arbitrary controller system (namely, with eigenvalues within the unit disk) defined as

$$\mathbf{x}_K(k+1) = \begin{bmatrix} 0.5 & 0.4 \\ 0.1 & 0.2 \end{bmatrix} \mathbf{x}_K(k) + \begin{bmatrix} 1.2 \\ 0.9 \end{bmatrix} \mathbf{f}(k),$$
$$\mathbf{g}(k) = \begin{bmatrix} 2 & 1.3 \end{bmatrix} \mathbf{x}_K(k).$$

Then, the computed network and controller parameters result in the system matrices of the network and controller systems given as[†]

$$\Xi = \begin{bmatrix} 72.3425 & 15.9935 \\ 73.6337 & 17.0340 \end{bmatrix}, \Upsilon = \begin{bmatrix} 0 & 0 & 0 & 53.1574 \\ 0 & 0 & 0 & 52.5635 \end{bmatrix}, \Delta = \begin{bmatrix} 0.0013 \\ 0.0012 \end{bmatrix},$$
$$\Lambda = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ -1.9950 & -0.5769 \end{bmatrix}, \Omega = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.2562 \end{bmatrix}, \Psi = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$
$$\Sigma = \begin{bmatrix} 0.0001 & 0 \end{bmatrix}, \Pi = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}, \Phi = 0,$$

$$A_K = \begin{bmatrix} 0.9960 & 0.0211 \\ 0.0412 & 0 \end{bmatrix}, B_K = \begin{bmatrix} 0.0083 \\ 335.4874 \end{bmatrix},$$
$$C_K = \begin{bmatrix} 1.9538 & 0.0321 \end{bmatrix}, D_K = 0.$$

Next, a simulation example is implemented, as presented in Figures 3.2 and 3.3. In the figures, the nodes of the proposed topology are initially all available and operate perfectly (namely, ideal operating scenario). At time 3901, the nodes γ_1 and θ_1 of the controller system become unavailable (namely, the QTP system is regulated solely using the network system), and thus, the proposed topology reduces to the control system topology that utilizes the WCN in Figure 1.2. At time 5901, the nodes of the controller system become available, but node v_4 of the network system becomes unavailable, up until time 5951. At time 7001, the nodes γ_1 and θ_1 of the controller system and nodes v_1, v_2 and v_3 of the network system become unavailable (namely, the QTP system is regulated solely using node v_4 of the network system), up until time 8000. It can be observed that the trajectories of the states and the voltages are not significantly affected when γ_1 and θ_1, and when nodes γ_1, θ_1, v_1, v_2, and v_3 are unavailable. This is because the design of the proposed topology allows it to tolerate such abnormal operating scenarios of unavailable nodes. However, this is not the case when node v_4 becomes unavailable. It can be observed that there are spikes in the values of the trajectories of the states and the voltages. Thus, impractical results are obtained. This demonstrates the importance of achieving a design of the proposed topology that allows for tolerating unreliable nodes, and more precisely, those that are prone to become unavailable during operation time.

†Since the objective is not the optimality of the designs of the network and controller systems, the respective convergence thresholds were not lowered to their minimum values.

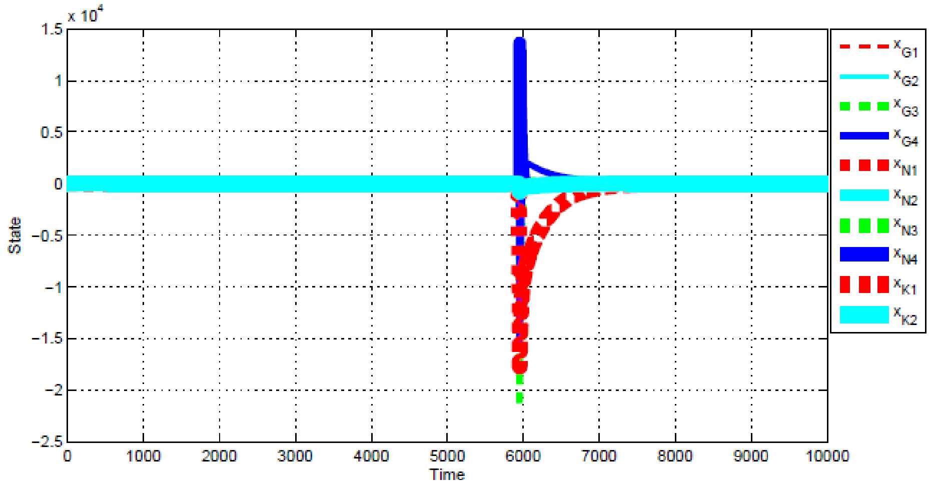

Figure 3.2: Simulation of the proposed topology under unreliable nodes - the states of the plant, network, and controller systems.

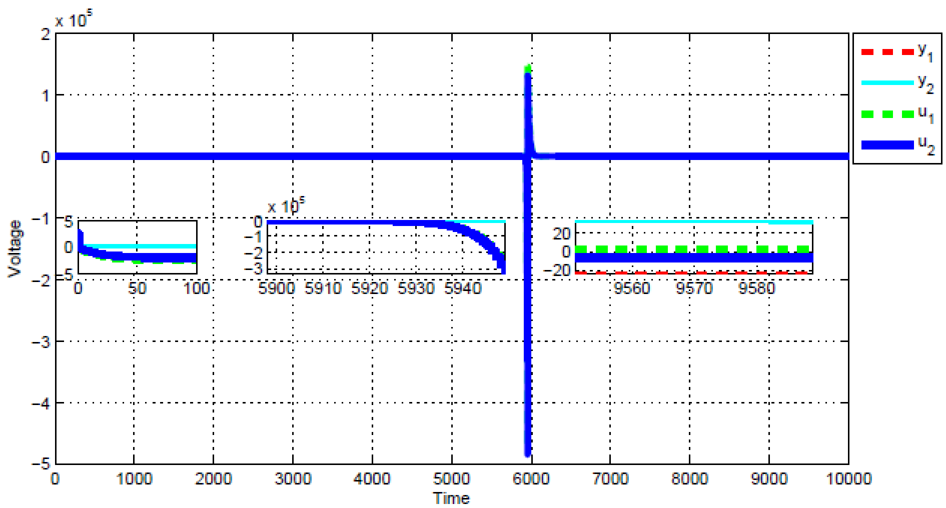

Figure 3.3: Simulation of the proposed topology under unreliable nodes - the controlled inputs and measured outputs of the plant system.

3.4.3 The Topology Under Unreliable Communication Links

In this section, the design of the proposed topology under unreliable communication links is demonstrated. Algorithm 3.2 is implemented using MATLAB's Robust Control Toolbox for a scenario of connectivity between the nodes of the proposed topology under unreliable communication links, as depicted in Figure 3.4.

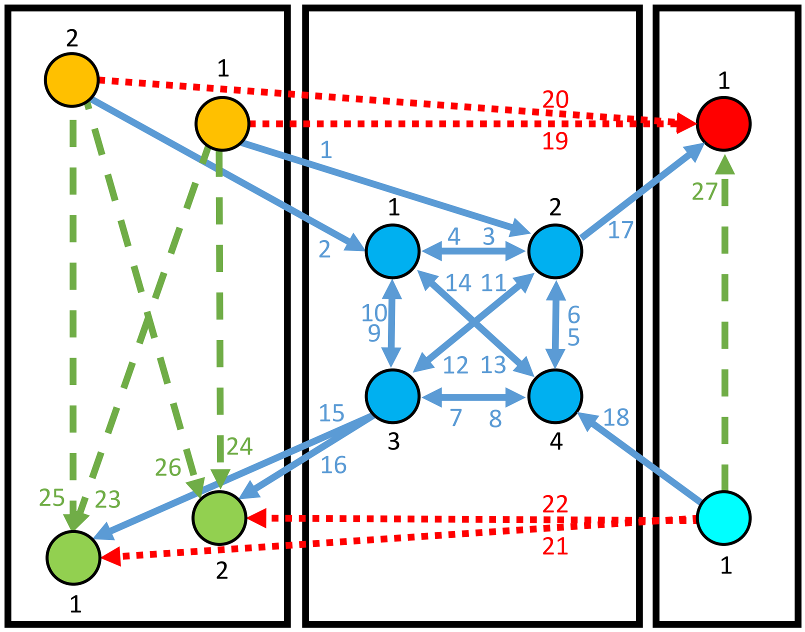

Figure 3.4: Connectivity of the nodes of the proposed topology under unreliable communicaiton links, consisting of the QTP system (left block), the network system (middle block), and the controller system (right block).

In Figure 3.4, the QTP system has its two sensor nodes (yellow circles) and two actuator nodes (green circles), the network system has its four distributed and inter-connected nodes (blue circles), and the controller system has its input node (red circle) and output node (cyan circle). Further, there are three cases of connectivity; namely, (i) the QTP and controller systems are connected indirectly through the nodes of the network system (blue solid arrows), (ii) the QTP and controller systems are directly and indirectly connected (blue solid and red dotted arrows), and (iii) all possible connections between the nodes exist (blue solid, red dotted, and green dashed arrows). Next, recall the modelling framework of Section 3.2.1. In each case, the connectivity between the nodes is defined using a set of communication links. In case (i), the set $\mathcal{E}_1 = \{e_1, \ldots, e_{18}\}$ connects the nodes of the network system with those of the QTP and controller systems. In case (ii), in addition to the set $\mathcal{E}_1$, the set $\mathcal{E}_2 = \{e_{19}, \ldots, e_{22}\}$ connects the nodes of the QTP and controller systems. Finally, in case (iii), in addition to the sets $\mathcal{E}_1$ and $\mathcal{E}_2$, the set $\mathcal{E}_3 = \{e_{23}, \ldots, e_{27}\}$ connects the nodes of the plant system and also connects the nodes of the controller system. Each communication link is assigned an identifier label (namely, according to the previously discussed bijective mapping operator). For example, consider the communication link between nodes s_1 and v_2, its assigned weight is denoted by λ_1; that of the communication link between nodes s_1 and γ_1 is denoted by σ_{19}; and that of the communication link between nodes θ_1 and a_2 is denoted by δ_{22}.

As the discretized QTP system is inherently Schur stable, the use of a controller and network system are not necessary. Thus, a design can be achieved for a maximum probability of unsuccessful receipt of information being equal to 1 (namely, $p = 1$), and the resulting weights assigned to the transfer of information between the nodes can all be set to zero. In order to demonstrate a more meaningful design case, $p = 0.5$ is specified for connectivity case (iii). In the design of the network system, Algorithm 3.2 begins with the arbitrary controller system presented in Section 3.4.2. The computed extended network parameter results in the system matrices of the network system given as

$$\Xi = \begin{bmatrix} -0.0921 & -0.0759 \\ -0.0529 & -0.0694 \end{bmatrix}, \Upsilon = \begin{bmatrix} 0 & 0 - 0.0714 & 0 \\ 0 & 0 - 0.0504 & 0 \end{bmatrix}, \Delta = \begin{bmatrix} -0.0335 \\ -0.0263 \end{bmatrix},$$

$$\Lambda = \begin{bmatrix} 0 & 0.1433 \\ 0.0288 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \Omega = \begin{bmatrix} 0.6798 & 0.1385 & 0.1180 & -0.0321 \\ 0.2378 & 0.8790 & 0.1089 & -0.0380 \\ 0.1115 & 0.1101 & 0.8617 & 0.1013 \\ -0.1151 & -0.0237 & 0.1296 & 0.9585 \end{bmatrix},$$

$$\Psi = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0.0051 \end{bmatrix}, \Sigma = \begin{bmatrix} 0.1580 & 0.1510 \end{bmatrix}, \Pi = \begin{bmatrix} 0 & 0.1535 & 0 & 0 \end{bmatrix},$$

$$\Phi = 0.0596,$$

The designed network system provides a closed-loop control system that is stable and tolerates a loss of transfer of information that is up to 50% between the nodes of the proposed topology.

3.4.4 The Topology Under Cyber Attacks

In this section, the design of the proposed topology under cyber attacks is demonstrated. Algorithm 3.3 is implemented using MATLAB's Robust Control Toolbox, where the plant system of Section 2.5 is utilized along with controller and network systems that are computed jointly using the approach discussed in Section 2.4. The algorithm delivers the design of the intrusion detection systems of all the nodes of the proposed topology, to detect cyber attacks on the information transferred from the neighbouring nodes. Algorithm 3.3 is initially implemented with setting the matrix $\mathrm{diag}(\nabla_{\mathbf{y}}\nabla_N\nabla_{\mathbf{g}}) = \mathbf{I}$ to capture the scenario of cyber attacks on the transfer of information between all the nodes of the proposed topology. In addition, it was specified that each of the nonzero entries of the system matrices of the intrusion detection systems D_N in (3.9), $D_{\mathcal{A}}$ in (3.10), and D_Γ in (3.11) to be greater than 0.1 to ensure that they are computed as nonzero, and to be less than 1 to ensure that the systems are stable (namely, satisfying Theorem 3.6). The system coefficients of the intrusion detection systems are computed for $i = 1, \ldots, 4$, and are given as

$$
\begin{aligned}
&A_{a_1}^D = 0.8155, B_{a_1}^D = 0.1660, C_{a_1}^D = 0.5102, D_{a_1}^D = 0.5437,\\
&A_{\gamma_1}^D = 0.4932, B_{\gamma_1}^D = 0.6419, C_{\gamma_1}^D = 0.5105, D_{\gamma_1}^D = 0.5448,\\
&A_{N_i}^D = 0.3985, B_{N_i}^D = 0.2154, C_{N_i}^D = 0.5077, D_{N_i}^D = 0.5105.
\end{aligned}
$$

Next, a simulation example is implemented, as presented in Figure 3.5. In Figure 3.5, nonzero initial conditions are specified for the state vectors of the systems of the proposed topology as well as $w_1 = w_2 = 0$ are specified for the external inputs of the plant system. Further, during times $90 \leq k < 100$, a cyber attack on the information transferred from node v_1 of the network system is simulated; during times $140 \leq k < 170$, a cyber attack on the information transferred from node θ_1 of the controller system is simulated; and during times $200 \leq k < 220$, a cyber attack on the information transferred from node v_4 of the network system is simulated. For the three simulated cyber attacks, the category of the cyber attack is a bias injection attack, and with injected values of 0.3 for the information transferred from v_1 and 0.1 for the information transferred from nodes v_4 and θ_1. It can be observed that at least one of the residues of the intrusion detection systems of the nodes (namely, $r_{N_1}^D, r_{N_2}^D, r_{N_3}^D, r_{N_4}^D, r_{a_1}^D$ and $r_{\gamma_1}^D$) is sensitive to one of the three simulated cyber attacks.

It should be noted that although the intrusion detection systems were successful in determining the existence of the simulated cyber attacks, their design and performance can be further investigated. For example, the effects of applying different types of cyber attacks can be studied, the use of delay-system formulation for the detection of replay attacks as well as other modelling and formulations can be investigated, and the comparison with other detection schemes of cyber attacks in the literature can be performed.

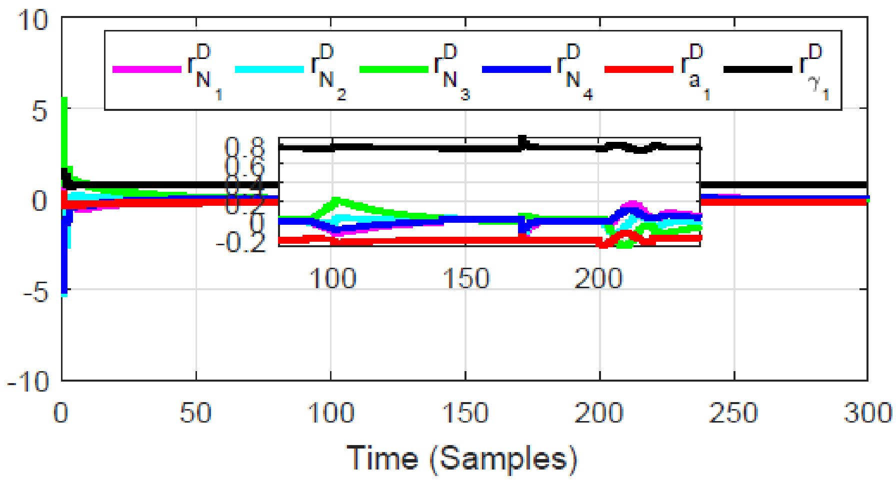

Figure 3.5: Simulation of the proposed topology under cyber attacks - the residues of the intrusion detection systems.

3.5 Summary

In this chapter, the design of the proposed topology under abnormal operating scenarios was considered. More specifically, the proposed topology was studied under unreliable nodes that can become unavailable, unreliable transfer of information such that information is lost and is not received by the nodes, and cyber attacks that block, re-transfer, and manipulate the information transferred between the nodes. Similar to Chapter 2, modelling frameworks, condition requirements, and design procedures were provided for the proposed topology under abnormal operating scenarios.

Chapter 4

Design of the Topology Under Induced Connectivity Constraints

In Chapters 2 and 3, the provided results are for the proposed topology with full connectivity. Namely, the the sensor nodes $\mathcal{S}$ and actuator nodes $\mathcal{A}$ of the plant system G in (2.1), the input nodes Γ and output nodes Θ of the controller system K in (2.2), and the nodes $\mathcal{V}$ of the network system in (2.3) as well as the communication links $\mathcal{E}$ are all fully utilized. However, a limitation on their utilization introduces savings in resources, and can lead to a simpler design as well as construction of the proposed topology. In this chapter, the design of the proposed topology under induced connectivity constraints is considered. The induced connectivity constraints define the utilization of only a subset of the nodes and the communication links between the nodes of the proposed topology.

4.1 Modelling Framework of the Topology

The design of the proposed topology under induced connectivity constraints is achieved by adopting the approach presented in the decentralized control system literature [14, 28, 37, 41, 44, 47, 56, 58]. Namely, in a DCS setup for a plant system with m actuator nodes $\mathcal{A}$ and p sensor nodes $\mathcal{S}$, each actuator node $a_i \in \mathcal{A}$ has a local controller that receives measurements from only a subset of the sensor nodes. The local controllers at the actuator nodes are non-interacting. A feedback matrix denoted by $\mathscr{L} \in \mathbb{R}^{m \times p}$ can be constructed to capture and define the feedback connections from the sensor nodes to the actuator nodes of the plant system. An entry of the feedback matrix $\mathscr{L}_{[i,j]}$ is defined as either 1 or 0 to capture whether or not sensor j is connected to actuator i, respectively. Given the DCS setup, a plant system that is controllable and observable may not be stabilized. This limitation is addressed by investigating the eigenvalues that cannot be adjusted when an output feedback controller system is utilized for regulating the open-loop system. Each such eigenvalue is referred to as a fixed mode of the system. Consider the plant system in (1.1) with $D = \mathbf{0}$. The set of eigenvalues that correspond to the fixed modes of the system, denoted by $\tilde{\Lambda}$, are defined as

$$\tilde{\Lambda} = \bigcap_{\mathscr{L} \in \mathbf{L}} \tilde{\lambda}(A + B\mathscr{L}C),$$

where $\mathbf{L}$ denotes the set of all feedback matrices. Further, if the entries of the system matrices of a given system are either zero or nonzero free parameters, the system is considered structured. If the system matrices of other systems also have the same sizes and structure, the systems are considered structurally equivalent. Then, suppose that the same feedback matrix is used for all the structurally equivalent systems and that the same fixed modes arise. Those fixed modes are considered structural fixed modes.

Next, consider the plant system G in (2.1), controller system K in (2.2), and network system N in (2.3). Each actuator node $a_i \in \mathcal{A}$ of the plant system has a local controller, whose state is denoted by x_{a_i}, modelled as a LTI system in discrete time as

$$x_{a_i}(k+1) = a_{a_i} x_{a_i}(k) + \sum_{v_j \in \mathscr{N}_{a_i}} v_{ij} x_{N_j}(k) + \sum_{s_j \in \mathscr{N}_{a_i}} \xi_{ij} y_j(k) + \sum_{\theta_j \in \mathscr{N}_{a_i}} \delta_{ij} g_j(k),$$
$$u_i(k) = x_{a_i}(k).$$

Further, each actuator node receives information from a subset of nodes from the nodes of the plant, controller, and network systems. Consider an actuator node $a_i \in \mathcal{A}$ of the plant system. The matrices $\hat{E}_{\mathbf{x}_{N_i}}$, $\hat{E}_{\mathbf{y}_i}$, and $\hat{E}_{\mathbf{g}_i}$ are defined as diagonal matrices with either 1 or 0 along their diagonal to capture the nodes from the sets $\mathcal{V}$, $\mathcal{S}$, and Θ, respectively, which transfer information to each actuator node a_i of the plant system. For the set of actuator nodes $\mathcal{A}$, the matrices in an augmented manner provide the matrices $\hat{E}_{\mathbf{x}_N}$, $\hat{E}_{\mathbf{y}}$, and $\hat{E}_{\mathbf{g}}$. Next, consider the vector defined as

$$\hat{\mathbf{q}}(k) = \begin{bmatrix} \hat{E}_{\mathbf{x}_N} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \hat{E}_{\mathbf{y}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \hat{E}_{\mathbf{g}} \end{bmatrix} \begin{bmatrix} \mathbf{x}_N(k) \\ \mathbf{y}(k) \\ \mathbf{g}(k) \end{bmatrix}.$$

The local controllers at the actuator nodes of the plant system are modelled as a LTI system in discrete time as

$$\begin{aligned} \mathbf{x}_{\mathcal{A}}(k+1) &= A_{\mathcal{A}}\mathbf{x}_{\mathcal{A}}(k) + [\Upsilon_B \ \Xi_B \ \Delta_B]\hat{\mathbf{q}}(k), \\ \mathbf{u}(k) &= \mathbf{I}\mathbf{x}_{\mathcal{A}}(k), \end{aligned} \tag{4.1}$$

where the vector $\mathbf{x}_{\mathcal{A}} \in \mathbb{R}^m$ denotes its state, the matrix $A_{\mathcal{A}}$ is a diagonal matrix that contains entries a_{a_i} along its diagonal, and the matrices Υ_B, Ξ_B, and Δ_B are block matrices, such that the rows of matrices Υ, Ξ, and Δ of the network system in (2.3) corresponding to actuator node $a_i \in \mathcal{A}$ are respectively the matrices (namely, blocks) at location $[i, i]$, and the remaining matrices are zero matrices. Similarly, each input node $\gamma_i \in \Gamma$ of the controller system has a local controller, whose state is denoted by x_{γ_i}, modelled as a LTI system in discrete time as

$$\begin{aligned} x_{\gamma_i}(k+1) &= a_{\gamma_i}x_{\gamma_i}(k) + \sum_{v_j \in \mathcal{N}_{\gamma_i}} \pi_{ij}x_{N_j}(k) + \sum_{s_j \in \mathcal{N}_{\gamma_i}} \sigma_{ij}y_j(k) + \sum_{\theta_j \in \mathcal{N}_{\gamma_i}} \phi_{ij}g_j(k), \\ f_i(k) &= x_{\gamma_i}(k). \end{aligned}$$

The matrices $\tilde{E}_{\mathbf{x}_N}$, $\tilde{E}_{\mathbf{y}}$, and $\tilde{E}_{\mathbf{g}}$ are defined to capture the nodes from the sets $\mathcal{V}, \mathcal{S}$, and Θ, respectively, which transfer information to each input node γ_i of the controller system. Next, consider the vector defined as

$$\tilde{\mathbf{q}}(k) = \begin{bmatrix} \tilde{E}_{\mathbf{x}_N} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \tilde{E}_{\mathbf{y}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \tilde{E}_{\mathbf{g}} \end{bmatrix} \begin{bmatrix} \mathbf{x}_N(k) \\ \mathbf{y}(k) \\ \mathbf{g}(k) \end{bmatrix}.$$

The local controllers at the input nodes of the controller system are modelled as a LTI system in discrete time as

$$\begin{aligned} \mathbf{x}_{\Gamma}(k+1) &= A_{\Gamma}\mathbf{x}_{\Gamma}(k) + [\Pi_B \ \Sigma_B \ \Phi_B]\tilde{\mathbf{q}}(k), \\ \mathbf{f}(k) &= \mathbf{I}\mathbf{x}_{\Gamma}(k), \end{aligned} \tag{4.2}$$

where the vector $\mathbf{x}_{\Gamma} \in \mathbb{R}^d$ denotes its state, the matrix A_{Γ} is a diagonal matrix that contains entries a_{γ_i} along its diagonal, and the matrices Π_B, Σ_B, and Φ_B are block matrices, such that the rows of matrices Π, Σ, and Φ of the network system in (2.3) corresponding to input node $\gamma_i \in \Gamma$ are respectively the matrices at location $[i, i]$, and the remaining matrices are zero matrices.

Then, consider the network system N in (2.3) whose output vectors are incorporated with the local controllers in (4.1) and (4.2). The matrices $\bar{E}_{\mathbf{x}_N}$,

$\bar{E}_{\mathbf{y}}$, and $\bar{E}_{\mathbf{g}}$ are defined to capture the nodes from the sets $\mathcal{V}, \mathcal{S}$, and Θ, respectively, which transfer information to each node v_i of the network system. The network system is then modelled as a LTI system in discrete time as

$$\mathbf{x}_N(k+1) = \begin{bmatrix} \Omega_B & \Lambda_B & \Psi_B \end{bmatrix} \begin{bmatrix} \bar{E}_{\mathbf{x}_N} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \bar{E}_{\mathbf{y}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \bar{E}_{\mathbf{g}} \end{bmatrix} \begin{bmatrix} \mathbf{x}_N(k) \\ \mathbf{y}(k) \\ \mathbf{g}(k) \end{bmatrix}, \tag{4.3}$$

where the matrices Ω_B, Λ_B, and Ψ_B are block matrices, such that the rows of matrices Ω, Λ, and Ψ of the network system in (2.3) corresponding to node $v_i \in \mathcal{V}$ are respectively the matrices at location $[i, i]$, and the remaining matrices are zero matrices.

Remark 4.1. The use of the local controllers at the actuator nodes of the plant system as well as at the input nodes of the controller system offers additional control decision capabilities. Namely, the controller system in (2.2) provides a centralized control system paradigm, and the local controllers as well as the nodes of the network system provide a decentralized control system paradigm.

The closed-loop control system with state vector $\mathbf{x} = [\mathbf{x}_G^T \ \mathbf{x}_{\mathcal{A}}^T \ \mathbf{x}_N^T \ \mathbf{x}_K^T \ \mathbf{x}_\Gamma^T \ ]^T$ is modelled as a LTI system in discrete time as

$$\begin{aligned} \mathbf{x}(k+1) &= \mathscr{A}\mathbf{x}(k) + \mathscr{B}\mathbf{w}(k), \\ \mathbf{z}(k) &= \mathscr{C}\mathbf{x}(k) + \mathscr{D}\mathbf{w}(k), \end{aligned} \tag{4.4}$$

where the system matrices are defined as

$$\mathscr{A} = \begin{bmatrix} A & B_2 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \Xi_B \hat{E}_{\mathbf{y}} C_2 & A_{\mathcal{A}} & \Upsilon_B \hat{E}_{\mathbf{x}_N} & \Delta_B \hat{E}_{\mathbf{g}} C_K & \Delta_B \hat{E}_{\mathbf{g}} D_K \\ \Lambda_B \bar{E}_{\mathbf{y}} C_2 & \mathbf{0} & \Omega_B \bar{E}_{\mathbf{x}_N} & \Psi_B \bar{E}_{\mathbf{g}} C_K & \Psi_B \bar{E}_{\mathbf{y}} D_K \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & A_K & B_K \\ \Sigma_B \tilde{E}_{\mathbf{y}} C_2 & \mathbf{0} & \Pi_B \tilde{E}_{\mathbf{x}_N} & \Phi_B \tilde{E}_{\mathbf{g}} C_K & A_\Gamma + \Phi_B \tilde{E}_{\mathbf{g}} D_K \end{bmatrix},$$

$$\mathscr{B} = \begin{bmatrix} B_1 \\ \Xi_B \hat{E}_{\mathbf{y}} D_{21} \\ \Lambda_B \bar{E}_{\mathbf{y}} D_{21} \\ \mathbf{0} \\ \Sigma_B \tilde{E}_{\mathbf{y}} D_{21} \end{bmatrix}, \mathscr{C} = \begin{bmatrix} C_1 & D_{12} & \mathbf{0} \end{bmatrix}, \mathscr{D} = D_{11}.$$

Then, the system matrices of the closed-loop control system in (4.4) are expressed in terms of matrices affine on the system matrices of the local controllers in (4.1) and (4.2), network system in (4.3), and controller system in

(2.2) (namely, similar to the approach presented in Section 2.2). First, suppose the decomposed network parameters and controller parameter are defined as

$$\begin{aligned} \mathcal{N}_1 &= \left[\begin{array}{ccc} \Xi_B & \Upsilon_B & \Delta_B \end{array}\right], \mathcal{N}_2 = \left[\begin{array}{ccc} \Lambda_B & \Omega_B & \Psi_B \end{array}\right], \\ \mathcal{N}_3 &= \left[\begin{array}{ccc} \Sigma_B & \Pi_B & \Phi_B \end{array}\right], \mathcal{K} = \left[\begin{array}{cc} D_K & C_K \\ B_K & A_K \end{array}\right]. \end{aligned} \tag{4.5}$$

The system matrices $\mathscr{A}$ and $\mathscr{B}$ are expressed in terms of the decomposed network parameters $\mathcal{N} \triangleq \{\mathcal{N}_1, \mathcal{N}_2, \mathcal{N}_3\}$ along with the system matrices $A_{\mathcal{A}}$ and A_Γ as

$$\begin{aligned} \mathscr{A}(A_{\mathcal{A}}, A_\Gamma, \mathcal{N}) &= \mathscr{A}_{A_{\mathcal{A}}}^T A_{\mathcal{A}} \mathscr{A}_{A_{\mathcal{A}}} + \mathscr{A}_{A_\Gamma}^T A_\Gamma \mathscr{A}_{A_\Gamma} + \mathscr{A}_{\mathcal{N}_3} \\ &\quad + \mathscr{A}_{\mathcal{N}_4} \mathcal{N}_1 \mathscr{A}_{\mathcal{N}_7} + \mathscr{A}_{\mathcal{N}_5} \mathcal{N}_2 \mathscr{A}_{\mathcal{N}_8} + \mathscr{A}_{\mathcal{N}_6} \mathcal{N}_3 \mathscr{A}_{\mathcal{N}_9}, \\ \mathscr{B}(A_{\mathcal{A}}, A_\Gamma, \mathcal{N}) &= \mathscr{B}_{\mathcal{N}_1} + \mathscr{A}_{\mathcal{N}_4} \mathcal{N}_1 \mathscr{B}_{\mathcal{N}_2} + \mathscr{A}_{\mathcal{N}_5} \mathcal{N}_2 \mathscr{B}_{\mathcal{N}_3} \\ &\quad + \mathscr{A}_{\mathcal{N}_6} \mathcal{N}_3 \mathscr{B}_{\mathcal{N}_4}, \end{aligned}$$

where the matrices are defined as

$$\begin{aligned} &\mathscr{A}_{A_{\mathcal{A}}} = \left[\begin{array}{ccc} \mathbf{0} & \mathbf{I} & \mathbf{0} \end{array}\right], \mathscr{A}_{A_\Gamma} = \left[\begin{array}{cc} \mathbf{0} & \mathbf{I} \end{array}\right], \\ &\mathscr{A}_{\mathcal{N}_3} = \left[\begin{array}{ccccc} A & B_2 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & A_K & B_K \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array}\right], \mathscr{A}_{\mathcal{N}_4} = \left[\begin{array}{c} \mathbf{0} \\ \mathbf{I} \\ \mathbf{0} \end{array}\right], \mathscr{A}_{\mathcal{N}_5} = \left[\begin{array}{c} \mathbf{0} \\ \mathbf{I} \\ \mathbf{0} \end{array}\right], \\ &\mathscr{A}_{\mathcal{N}_6} = \left[\begin{array}{c} \mathbf{0} \\ \mathbf{I} \end{array}\right], \mathscr{A}_{\mathcal{N}_7} = \left[\begin{array}{ccccc} \hat{E}_{\mathbf{y}} C_2 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \hat{E}_{\mathbf{x}_N} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \hat{E}_{\mathbf{g}} C_K & \hat{E}_{\mathbf{g}} D_K \end{array}\right], \\ &\mathscr{A}_{\mathcal{N}_8} = \left[\begin{array}{ccccc} \bar{E}_{\mathbf{y}} C_2 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \bar{E}_{\mathbf{x}_N} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \bar{E}_{\mathbf{g}} C_K & \bar{E}_{\mathbf{g}} D_K \end{array}\right], \\ &\mathscr{A}_{\mathcal{N}_9} = \left[\begin{array}{ccccc} \tilde{E}_{\mathbf{y}} C_2 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \tilde{E}_{\mathbf{x}_N} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \tilde{E}_{\mathbf{g}} C_K & \tilde{E}_{\mathbf{g}} D_K \end{array}\right], \mathscr{B}_{\mathcal{N}_1} = \left[\begin{array}{c} B_1 \\ \mathbf{0} \end{array}\right], \\ &\mathscr{B}_{\mathcal{N}_2} = \left[\begin{array}{c} \hat{E}_{\mathbf{y}} D_{21} \\ \mathbf{0} \end{array}\right], \mathscr{B}_{\mathcal{N}_3} = \left[\begin{array}{c} \bar{E}_{\mathbf{y}} D_{21} \\ \mathbf{0} \end{array}\right], \mathscr{B}_{\mathcal{N}_4} = \left[\begin{array}{c} \tilde{E}_{\mathbf{y}} D_{21} \\ \mathbf{0} \end{array}\right]. \end{aligned}$$

Then, the system matrix $\mathscr{A}$ is expressed in terms of the controller parameter $\mathcal{K}$ as

$$\mathscr{A}(\mathcal{K}) = \mathscr{A}_{\mathcal{K}_1} + \mathscr{A}_{\mathcal{K}_2} \mathcal{K} \mathscr{A}_{\mathcal{K}_5} + \mathscr{A}_{\mathcal{K}_3} \mathcal{K} \mathscr{A}_{\mathcal{K}_5} + \mathscr{A}_{\mathcal{K}_4} \mathcal{K} \mathscr{A}_{\mathcal{K}_5},$$

where the matrices are defined as

$$\mathscr{A}_{\mathcal{K}_1} = \begin{bmatrix} A & B_2 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \Xi_B \hat{E}_{\mathbf{y}} C_2 & A_{\mathcal{A}} & \Upsilon_B \hat{E}_{\mathbf{x}_N} & \mathbf{0} & \mathbf{0} \\ \Lambda_B \bar{E}_{\mathbf{y}} C_2 & \mathbf{0} & \Omega_B \bar{E}_{\mathbf{x}_N} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \Sigma_B \tilde{E}_{\mathbf{y}} C_2 & \mathbf{0} & \Pi_B \tilde{E}_{\mathbf{x}_N} & \mathbf{0} & A_\Gamma \end{bmatrix}, \mathscr{A}_{\mathcal{K}_2} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \Delta_B \hat{E}_{\mathbf{g}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \\ \mathbf{0} & \mathbf{0} \end{bmatrix},$$

$$\mathscr{A}_{\mathcal{K}_3} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \Psi_B \bar{E}_{\mathbf{g}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \mathscr{A}_{\mathcal{K}_4} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \Phi_B \tilde{E}_{\mathbf{g}} & \mathbf{0} \end{bmatrix}, \mathscr{A}_{\mathcal{K}_5} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{I} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} \end{bmatrix}.$$

Remark 4.2. The setup of the proposed topology with local controllers at the actuator nodes of the plant system as well as at the input nodes of the controller system does not lead to a more complex structure of the proposed topology. On the contrary, the setup leads to a simplified proposed topology that requires the use of a reduced number of nodes and communication links between the nodes. More specifically, the sizes of the local controllers are the same as the vectors $\mathbf{u}$ and $\mathbf{f}$ (namely, m and d, respectively), as can be observed from the systems in (4.1) and (4.2).

Remark 4.3. The use of local controllers at the actuator nodes of the plant system in the control system topology that utilizes the WCN in [37, 41] only allows for receiving information from the nodes of the WCN, and not from the sensor nodes of the plant system. In the presented setup of this chapter, the use of local controllers at the actuator nodes of the plant system as well as at the input nodes of the controller system allows for receiving information from any sensor node $s_i \in \mathcal{S}$ of the plant system, input node $\gamma_i \in \Gamma$ of the controller system, and node $v_i \in \mathcal{V}$ of the network system. In addition, the modelling frameworks and design procedures presented in Chapters 2 and 3 assume some constraints on the system matrices of the controller and network systems (e.g. $D_K = \mathbf{0}$ and $\Phi = \mathbf{0}$). In the presented setup of this chapter, such constraints are eliminated. Thus, the design of the proposed topology presented in this chapter is more generic as well as cost-efficient.

4.2 Condition Requirements of the Topology

First, consider the local controllers in (4.1) and (4.2), and the network system in (4.3). The local controllers at the actuator nodes of the plant system in (4.1) use the information transferred from neighbouring nodes, defined by the feedback matrix denoted by $\hat{\mathscr{L}}$ (namely, specified by the 1s in the matrices $\hat{E}_{\mathbf{x}_N}$, $\hat{E}_{\mathbf{y}}$, and $\hat{E}_{\mathbf{g}}$ to capture the nodes that transfer information to the local controllers). The local controllers at the input nodes of the controller system in (4.2) use the information transferred from neighbouring nodes, defined by the feedback matrix denoted by $\tilde{\mathscr{L}}$ (namely, specified by the 1s in the matrices $\tilde{E}_{\mathbf{x}_N}$, $\tilde{E}_{\mathbf{y}}$, and $\tilde{E}_{\mathbf{g}}$). Similarly, the nodes of the network system in (4.3) use the information transferred from neighbouring nodes, defined by the feedback matrix denoted by $\bar{\mathscr{L}}$ (namely, specified by the 1s in the matrices $\bar{E}_{\mathbf{x}_N}$, $\bar{E}_{\mathbf{y}}$, and $\bar{E}_{\mathbf{g}}$). Next, suppose the plant system in (2.1) is stabilizable and detectable, and consider the specified DCS setup and feedback connections. The closed-loop control system in (4.4) can be stabilized using the proposed topology if and only if all of its fixed modes are stable [56], [37]. Thus, the existence of fixed modes can introduce problems when they are not stable, and therefore, the elimination of fixed modes is desirable. This motivates the design of the proposed topology such that all structural fixed modes are eliminated.

Then, consider the modelling of the plant system in (2.1) and the controller system in (2.2) using a graph-theoretic approach. The plant system is modelled with a set of nodes $\mathcal{V}_G = \{v_{G_1}, \ldots, v_{G_n}\}$ to represent its states as well as a set of sensor nodes $\mathcal{S}$ and a set of actuator nodes $\mathcal{A}$. The nodes $\mathcal{V}_G$ are connected amongst each other using the set of links (namely, edges) denoted by $\mathcal{E}_{\mathcal{V}_G \leftrightarrow \mathcal{V}_G}$; they are connected with the sensor nodes $\mathcal{S}$ using the set of links denoted by $\mathcal{E}_{\mathcal{V}_G \to \mathcal{S}}$; and they are also connected with the actuator nodes $\mathcal{A}$ using the set of links denoted by $\mathcal{E}_{\mathcal{A} \to \mathcal{V}_G}$. Similarly, the controller system is modelled with a set of nodes $\mathcal{V}_K = \{v_{K_1}, \ldots, v_{K_r}\}$ to represent its states as well as a set of input nodes Γ and a set of output nodes Θ. The nodes $\mathcal{V}_K$ are connected amongst each other using the set of links denoted by $\mathcal{E}_{\mathcal{V}_K \leftrightarrow \mathcal{V}_K}$; they are connected with the input nodes Γ using the set of links denoted by $\mathcal{E}_{\Gamma \to \mathcal{V}_K}$; and they are also connected with the output nodes using the set of links denoted by $\mathcal{E}_{\mathcal{V}_K \to \Theta}$.

The design of the proposed topology such that the closed-loop control system in (4.4) has no structural fixed modes is achieved by extending the results from [37, 44]. Consider the following assumption, lemma, and theorem.

Assumption 4.1. For the proposed topology, suppose the local controllers in (4.1) are used at the actuator nodes of a stabilizable and detectable plant system in (2.1) and that they use feedback connections defined in a feedback matrix $\hat{\mathscr{L}}$; the local controllers in (4.2) are used at the input nodes of the controller system in (2.2) and that they use feedback connections defined in a feedback matrix $\tilde{\mathscr{L}}$; and the nodes of the network system in (4.3) use feedback connections defined in a feedback matrix $\bar{\mathscr{L}}$.

Lemma 4.1. For the proposed topology satisfying Assumption 4.1, the closed-loop control system has no structural fixed modes outside of the origin (namely, other than at the origin) if and only if every state node of the plant system $v_{G_i} \in \mathcal{V}_G$ belongs to a strong component that: (i) includes an edge from the set $\mathcal{E}_{\mathcal{S}\to\mathcal{A}}$ or (ii) formed by a path that includes edges from the sets $\mathcal{E}_{\mathcal{S}\to\mathcal{V}}$, $\mathcal{E}_{\mathcal{V}\leftrightarrow\mathcal{V}}$, $\mathcal{E}_{\mathcal{V}\to\mathcal{A}}$, $\mathcal{E}_{\mathcal{S}\to\Gamma}$, $\mathcal{E}_{\Theta\to\mathcal{A}}$, $\mathcal{E}_{\mathcal{V}\to\Gamma}$, $\mathcal{E}_{\Theta\to\mathcal{V}}$, $\mathcal{E}_{\Theta\to\Gamma}$, $\mathcal{E}_{\Gamma\to\mathcal{V}_K}$, $\mathcal{E}_{\mathcal{V}_K\leftrightarrow\mathcal{V}_K}$, and $\mathcal{E}_{\mathcal{V}_K\to\Theta}$ which connects nodes from the sets $\mathcal{V}$ and $\mathcal{V}_K$.

Proof. The proof follows from the results presented in [37, 44]. For condition (i), the existence of a strong component for each state node $v_{G_i} \in \mathcal{V}_G$ of the plant system that includes an edge from the set $\mathcal{E}_{\mathcal{S}\to\mathcal{A}}$ allows for having a cycle for node v_{G_i}, such that it is reachable from an actuator node (namely, one local controller) and a sensor node is reachable from the state node v_{G_i}. For condition (ii), a similar observation can be made for the existence of a strong component for each state node v_{G_i} formed by nodes and communication links from the listed sets. Thus, in both conditions (i) and (ii), any state node of the plant system belongs to a cycle, and therefore, no structural fixed modes will exist outside of the origin. □

Similar to the discussion in [37], for discrete-time systems, structural fixed modes at the origin will not introduce issues, as they are within the unit disk. Thus, the plant system can be stabilized. Further, if it is necessary to eliminate all structural fixed modes, a condition for the existence of a set of disjoint cycles covering all state nodes needs to be added to the result presented in Lemma 4.1 [37, 44].

Theorem 4.1. For the proposed topology satisfying Assumption 4.1, almost all plant systems that are structurally equivalent to that in (2.1) can be stabilized if each state node $v_{G_i} \in \mathcal{V}_G$ belongs to a cycle that consists of any of the following:

i. An edge from the set $\mathcal{E}_{\mathcal{S}\to\mathcal{A}}$;

ii. An edge from each of the sets $\mathcal{E}_{\mathcal{S}\to\mathcal{V}}$, $\mathcal{E}_{\mathcal{V}\leftrightarrow\mathcal{V}}$, and $\mathcal{E}_{\mathcal{V}\to\mathcal{A}}$ to connect to at least one node of the network system;

iii. An edge from each of the sets $\mathcal{E}_{\mathcal{S}\to\Gamma}$, $\mathcal{E}_{\Gamma\to\mathcal{V}_K}$, $\mathcal{E}_{\mathcal{V}_K\leftrightarrow\mathcal{V}_K}$, $\mathcal{E}_{\mathcal{V}_K\to\Theta}$, and $\mathcal{E}_{\Theta\to\mathcal{S}}$ to connect to at least one state node of the controller system; and

iv. An edge from each of the sets $\mathcal{E}_{\mathcal{S}\to\mathcal{V}}$, $\mathcal{E}_{\mathcal{V}\leftrightarrow\mathcal{V}}$, $\mathcal{E}_{\mathcal{V}\to\Gamma}$, $\mathcal{E}_{\Gamma\to\mathcal{V}_K}$, $\mathcal{E}_{\mathcal{V}_K\leftrightarrow\mathcal{V}_K}$, $\mathcal{E}_{\mathcal{V}_K\to\Theta}$, $\mathcal{E}_{\Theta\to\mathcal{V}}$, and $\mathcal{E}_{\mathcal{V}\to\mathcal{A}}$ to connect to at least one node of the network system as well as the controller system.

Proof. First, consider condition (i). Each state node $v_{G_i} \in \mathcal{V}_G$ belongs to a cycle that is formed using a communication link from the set $\mathcal{E}_{\mathcal{S}\to\mathcal{A}}$. Therefore, condition (i) of Lemma 4.1 holds. In condition (ii), each state node belongs to a cycle formed using communication links from the specified sets, by only using a set of nodes of the network system (i.e., resulting in the control system topology that utilizes the WCN as well as the result presented in Theorem 4 of [37]). In conditions (iii) and (iv), cycles are formed by only using a set of nodes of the controller system and nodes of both the controller and network systems, respectively. Therefore, condition (ii) of Lemma 4.1 holds. Thus, under conditions (i)-(iv), all state nodes of the plant system are strong components. Therefore, no structural fixed modes exist due to the plant system; and hence it can be stabilized. Similar to the argument in [37], any nodes of the network or controller systems that are not strong components will introduce structural fixed modes, and as a solution, their connections can be eliminated by setting to zero the weight corresponding to the respective communication links. Then, the closed-loop control system will have no structural fixed modes outside of the origin, and hence it can be stabilized. □

Further, the connectivity between the nodes of the proposed topology to result in the closed-loop control system without structural fixed modes is established according to the formation procedure outlined in the following algorithm.

In Algorithm 4.1, different cycles are formed to include all state nodes of the plant system. Namely, the algorithm initially attempts to form such cycles by establishing communication links between the sensor and actuator nodes of the plant system (Step 3). For nodes that cannot be incorporated in such cycles, the algorithm establishes communication links between the sensor nodes of the plant system and the input nodes of the controller system as well

Algorithm 4.1 Algorithm for the Connectivity Between the Nodes of the Proposed Topology Without Structural Fixed Modes

Step 1. Set $i = 1$ and n as the number of states of the plant system, and specify A_K, B_K, C_K, and D_K as arbitrary nonzero matrices.
Step 2. For state node $v_{G_i} \in \mathcal{V}_G$, determine sensor nodes and actuator nodes from the plant system that can connect to the state node.
Step 3. For an actuator node that is within a communication range of a sensor node, establish a communication link; if possible and if $i = n$, exit; or if $i \neq n$, set $i = i+1$ and return to Step 2; if no such cycle can be formed, go to Step 4.
Step 4. For an input node and an output node from the controller system that are within a communication range of a sensor node and an actuator node of the plant system, respectively, establish two communication links to form a cycle; if possible and if $i = n$, exit; or if $i \neq n$, set $i = i + 1$ and return to Step 2; if no such cycle can be formed, go to Step 5.
Step 5. For intermediate nodes from the network system that are within a communication range of a sensor node and an actuator node of the plant system, and possibly, an input and an output node of the controller system, establish communication links to form a cycle; if $i = n$, exit; or if $i \neq n$, set $i = i + 1$ and return to Step 2.

as between the output nodes of the controller system and the input nodes of the plant system (Step 4). Finally, for nodes that cannot be incorporated in such cycles as well, the algorithm establishes communication links between the sensor nodes of the plant system and the nodes of the network system as well as the nodes of the network system and the actuator nodes of the plant system (Step 5). Then, the number of communication links established between the nodes of the proposed topology is characterized in the following corollary.

Corollary 4.1. Suppose Algorithm 4.1 is capable of forming a topology that satisfies the conditions given in Theorem 4.1. Then, for the set of essential sensor nodes that are reachable from the state nodes denoted by $\mathcal{S}_G \subseteq \mathcal{S}$ and the set of essential actuator nodes that can reach the state nodes denoted by $\mathcal{A}_G \subseteq \mathcal{A}^*$, the minimum number of communication links that are established is given as

$$\min(|\mathcal{S}_{\mathrm{G}}|, |\mathcal{A}_{\mathrm{G}}|), \tag{4.6}$$

*The notions of essential sensor nodes and essential actuator nodes refer to the smallest set of sensor nodes that are reachable from all state nodes and the smallest set of actuator nodes that can reach all state nodes, respectively, similar to the notion in [37].

and the maximum number of communication links is given as

$$\sum_{i=1}^{|\mathcal{S}_G|} d(s_i \in \mathcal{S}_G, \gamma_i \in \Gamma) + \sum_{i=1}^{|\mathcal{A}_G|} d(\theta_i \in \Theta, a_i \in \mathcal{A}_G). \tag{4.7}$$

Proof. First, consider the possibility of forming cycles for each state node $v_{G_i} \in \mathcal{V}_G$ by only establishing communication links between sensor and actuator nodes, and assume that the cycles are formed by only establishing communication links equal to at most the minimum number of sensor nodes $\mathcal{S}_G$ or actuator nodes $\mathcal{A}_G$. This leads to the result in (4.6). Next, consider the case of cycles that only can be formed such that for each sensor node from the set $\mathcal{S}_G$ and each actuator node from the set $\mathcal{A}_G$, an independent path must be used to connect to the input and output nodes of the controller system, and possibly through nodes of the network system. This leads to the result in (4.7). □

Remark 4.4. The proposed topology without the controller system in (2.2) reduces to the control system topology that utilizes the WCN in Figure 1.2. In this case, the nodes of the network system are used in forming the cycles to contain the state nodes of the plant system. However, unlike the approach in [37], the actuator nodes of the plant system of the proposed topology can receive information directly from the sensor nodes of the plant system. Thus, cycles can be formed with and without the nodes of the network system. Therefore, the modelling framework and design procedure presented in this section are more generic.

4.3 Strategic Formation of the Topology

The connectivity between the nodes of the proposed topology such that the closed-loop control system has no structural fixed modes was discussed. However, the discussed approach does not account for any cost and benefit associated with establishing the communication links between the nodes. In this section, the formation of the proposed topology such that the utilized nodes and the established communication links are strategically specified is delivered, by adopting the approaches presented for handling social networks and networked systems [24, 49, 50]. Namely, in a social network, a utility function is constructed for each individual, denoted by v_i, from the graph of connected individuals, denoted by $\mathscr{G}$, such that it is defined as

$$\varphi_i = w_{ii} + \sum_{j \neq i} \varsigma^{t_{ij}} w_{ij} - \sum_{(v_i, v_j) \in \mathscr{G}} c_{ij}, \tag{4.8}$$

where w_{ij} denotes the intrinsic value of individual v_j to v_i, t_{ij} denotes the number of links in the shortest path, c_{ij} denotes the cost of establishing a link between the individuals v_j and v_i, and $0 < \varsigma < 1$ allows for capturing the value that individual v_i derives from the connection with v_j in proportion to the distance between the individuals [24]. The approach results in a trade-off between the distance between the individuals and the cost of the connectivity. Further, this approach was extended and applied to inter-connection between multi-layer networks, by developing utility functions based on that in (4.8) for establishing links between nodes that are t hops away [49] and [50].

Next, consider the sensor nodes $\mathcal{S}$ and actuator nodes $\mathcal{A}$ of the plant system in (2.1), input nodes Γ and output nodes Θ of the controller system in (2.2), and nodes $\mathcal{V}$ of the network system in (4.3). The establishing of communication links between the nodes is associated with a cost as well as a benefit. The cost and benefit result from several factors, including their relative location, their power levels and transmission capabilities, and their vulnerability to unavailability. Then, consider a state node $v_{G_i} \in \mathcal{V}_G$ of the plant system, and suppose $b(t)$ denotes the benefit function for a cycle formed with the least number of hops to contain the state node v_{G_i} (namely, from a sensor node to an actuator node which connect to v_{G_i}) such that it is monotonically decreasing with the number of hops t. Further, suppose $c(l_j^x)$ denotes the cost function for establishing a communication link $l_j \in \mathcal{E}$ and superscript x denotes the category of the established communication link. The set of communication links used in forming the cycles to contain the state nodes of the plant system is characterized using the following assumption and lemma.

Assumption 4.2. For the proposed topology, suppose it is possible to establish communication links between any nodes to form a cycle that contains any state node $v_{G_i} \in \mathcal{V}_G$, and that only one such cycle is formed with no repeated nodes in paths of communication links of a single direction.

Lemma 4.2. For the proposed topology satisfying Assumptions 4.1 and 4.2, the number of hops from a sensor node to an actuator node of the plant system is defined as $t \in \{1, \ldots, 2(|\mathcal{V}| - 1) + 4\}$, and the cycle to contain any state node $v_{G_i} \in \mathcal{V}_G$ should be formed by strictly utilizing communication link(s) from:

i. The set of communication links $\mathcal{E}_{\mathcal{S}\to\mathcal{A}}$ if

$$b(1) - b(t^b) > c(l^a) - \sum_{j=1}^{t^b} c(l_j^b), \tag{4.9}$$

where $t^b \in \{2, \ldots, 2(|\mathcal{V}| - 1) + 4\}$, $l^a \in \mathcal{E}_{\mathcal{S}\to\mathcal{A}}$, and $l_j^b \in \mathcal{E} \setminus \mathcal{E}_{\mathcal{S}\to\mathcal{A}}$.

ii. The sets of communication links $\mathcal{E}_{\mathcal{S}\to\mathcal{V}}$, $\mathcal{E}_{\mathcal{V}\leftrightarrow\mathcal{V}}$, and $\mathcal{E}_{\mathcal{V}\to\mathcal{A}}$ if

$$b(t^c) - b(1) > \sum_{j=1}^{t^c} c(l_j^c) - c(l^a),$$

$$b(t^c) - b(t^b) > \sum_{j=1}^{t^c} c(l_j^c) - \sum_{j=1}^{t^b} c(l_j^b),$$

where $t^c \in \{2, \ldots, 2(|\mathcal{V}| - 1) + 2\}$ and $l_j^c \in \mathcal{E}_{\mathcal{S}\to\mathcal{V}} \bigcup \mathcal{E}_{\mathcal{V}\leftrightarrow\mathcal{V}} \bigcup \mathcal{E}_{\mathcal{V}\to\mathcal{A}}$.

iii. The sets of communication links $\mathcal{E}_{\mathcal{S}\to\Gamma}$ and $\mathcal{E}_{\Theta\to\mathcal{A}}$ if

$$b(2) - b(t^b) > \sum_{j=1}^{2} c(l_j^d) - \sum_{j=1}^{t^b} c(l_j^b),$$

$$b(2) - b(1) > \sum_{j=1}^{2} c(l_j^d) - c(l^a),$$

where $l_j^d \in \mathcal{E}_{\mathcal{S}\to\Gamma} \bigcup \mathcal{E}_{\Theta\to\mathcal{A}}$.

iv. The sets of communication links $\mathcal{E} \setminus \mathcal{E}_{\mathcal{S}\to\mathcal{A}}$ if

$$b(t^b) - b(1) > \sum_{j=1}^{t^b} c(l_j^b) - c(l^a),$$

$$b(t^b) - b(t^c) > \sum_{j=1}^{t^b} c(l_j^b) - \sum_{j=1}^{t^c} c(l_j^c),$$

$$b(t^b) - b(2) > \sum_{j=1}^{t^b} c(l_j^b) - \sum_{j=1}^{2} c(l_j^d).$$

Proof. Consider a cycle formed by establishing one or multiple communication links between a sensor node and an actuator node to contain a state node v_{G_i}.

The minimum number of hops is obtained when the sensor node is directly connected to the actuator node, and thus, the actuator node is 1 hop away. The maximum number of hops is obtained when the sensor node is indirectly connected to the actuator node through all nodes of the network system and the controller system (e.g., from a sensor node by hopping over all nodes of the network system in one direction to reach the controller system, and then from the controller system by hopping over all nodes of the network system in the opposite direction to reach the actuator node), and thus, the actuator node is $2(|\mathcal{V}|-1)+4$ hops away. Then, consider condition (i), where a cycle a is formed by connecting a sensor node directly to an actuator node to contain a state node, its utility function is given as $\varphi^a = b(1) - c(l^a)$. Alternatively, suppose a cycle b is formed by establishing communication links to indirectly connect the sensor node to the actuator node to contain the state node (namely, the cycle uses t^b hops with the specified communication links), its utility function is given as $\varphi^b = b(t^b) - \sum_{j=1}^{t^b} c(l_j^b)$. For cycle a to be more optimal than cycle b, its utility function must be larger for any number of hops t^b. This leads to the condition in (4.9). By using the specified communication links, the minimum and maximum number of hops, and comparing the utility functions for the cycles in each of the conditions (ii), (iii), and (iv), the respective listed results are obtained. □

Remark 4.5. In Lemma 4.2, the formation of the proposed topology is characterized under different possible scenarios of interest; namely, when it is only desirable to establish communication links to the actuator nodes of the plant system (condition (i)), such as in the decentralized control system setup discussed in Section 4.1; when it is only desirable to establish communication links to the nodes of the network system (condition (ii)), such as in the control system topology that utilizes the WCN in Figure 1.2; when it is only desirable to establish communication links to the nodes of the controller system (condition (iii)), such as in the standard control system topology in Figure 1.1 (namely, resulting in a centralized control system paradigm with a decentralized control system setup); and when it is desirable to establish communication links to the controller system by hopping over nodes of the network system (condition (iv)), such as in the proposed control system topology.

Next, consider the following function defined in terms of the nodes of the system:

$$\begin{aligned}\mathcal{P} = &\min(|\mathcal{V}_G|, |\mathcal{A}|) + \min(|\mathcal{V}_G|, |\mathcal{S}|) \\ &+ \min(|\mathcal{V}_G|, |\Gamma|) + \min(|\mathcal{V}_G|, |\Theta|).\end{aligned} \tag{4.10}$$

The lower and upper bounds of the number of established communication links are characterized in the following theorem.

Theorem 4.2. For the proposed topology satisfying Assumptions 4.1 and 4.2, the minimum and maximum numbers of communication links established in the proposed topology are 1 and $2(|\mathcal{V}| - 1) + \mathcal{P}$, respectively.

Proof. Suppose that a sensor node $s_i \in \mathcal{S}$ can be reached by all state nodes and an actuator node $a_i \in \mathcal{A}$ can reach all state nodes. Then, it suffices to establish a single communication link between the sensor node s_i and the actuator node a_i (namely, from the set $\mathcal{E}_{\mathcal{S}\to\mathcal{A}}$) to form cycles that contain all state nodes, and thus, the minimum number of communication links established in the proposed topology is 1. Then, suppose that: (i) different sensor nodes and actuator nodes of the plant system are used for every state node $v_{G_i} \in \mathcal{V}_G$, and similarly, different input nodes and output nodes of the controller system are used for every state node; and (ii) the nodes of the network system are connected as a bus topology, and thus, resulting in the longest path with no repeated nodes in a single direction and the highest utility function (namely, no alternative connections are more optimal). Based on condition (i), the maximum number of communication links that connect the nodes of the network system with the nodes of the plant and controller systems, from the sets $\mathcal{E}_{\mathcal{S}\to\mathcal{V}}$, $\mathcal{E}_{\mathcal{V}\to\Gamma}$, $\mathcal{E}_{\Theta\to\mathcal{V}}$, and $\mathcal{E}_{\mathcal{V}\to\mathcal{A}}$, is $\mathcal{P}$ in (4.10). Further, based on condition (ii), the maximum number of communication links between the nodes of the network system is $2(|\mathcal{V}| - 1)$ (i.e., the same bus topology can be shared between the different nodes of the plant and controller systems). Thus, the maximum number of communication links established in the proposed topology is as listed. □

Next, suppose the proposed topology strictly utilizes a network system, whose nodes are connected using a standard connectivity configuration (namely, such as a ring, star, and full network configuration). The lower and upper bounds of the number of established communication links are characterized in the following theorem.

Theorem 4.3. For the proposed topology satisfying Assumptions 4.1 and 4.2, suppose it is required to connect the plant and controller systems through a network system, and that the nodes of the network system are connected as either a ring, star, or full network configuration. Then, in order to form cycles to include all state nodes $\mathcal{V}_G$, where for each sensor node, the actuator node is t hops away, the minimum number of communication links is 4, and

i. For a ring network configuration, the maximum number of communication links is $2|\mathcal{V}| + \mathcal{P}$, where $|\mathcal{V}| \geq 2$ and $t \in \{4, \ldots, 2(|\mathcal{V}| - 1) + 4\}$.

ii. For a star network configuration, the maximum number of communication links is $2(|\mathcal{V}| - 1) + \mathcal{P}$, where $|\mathcal{V}| \geq 4$ and $t \in \{4, \ldots, 8\}$.

iii. For a full network configuration, the maximum number of communication links is $|\mathcal{V}|(|\mathcal{V}|-1)+\mathcal{P}$, where $|\mathcal{V}| \geq 3$ and $t \in \{4, \ldots, 2(|\mathcal{V}|-1)+4\}$.

Proof. Suppose that a sensor node $s_i \in \mathcal{S}$ can be reached by all state nodes and an actuator node $a_i \in \mathcal{A}$ can reach all state nodes. Then, consider the use of only one node of the network system, whether connected as a ring, star, or full network configuration, to connect the nodes of the plant system with the nodes of the controller system. Then, it suffices to establish 4 communication links, from the sets $\mathcal{E}_{\mathcal{S}\to\mathcal{V}}$, $\mathcal{E}_{\mathcal{V}\to\Gamma}$, $\mathcal{E}_{\Theta\to\mathcal{V}}$, and $\mathcal{E}_{\mathcal{V}\to\mathcal{A}}$, to form cycles that contain all state nodes, and thus, the minimum number of communication links established in the proposed topology is 4. Next, suppose that different sensor nodes and actuator nodes of the plant system are used for every state node $v_{G_i} \in \mathcal{V}_G$, and similarly, different input nodes and output nodes of the controller system are used for every state node. Then, for a ring network configuration, the maximum number of communication links that connect the nodes of the network system with the nodes of the plant and controller systems, from the sets $\mathcal{E}_{\mathcal{S}\to\mathcal{V}}$, $\mathcal{E}_{\mathcal{V}\to\Gamma}$, $\mathcal{E}_{\Theta\to\mathcal{V}}$, and $\mathcal{E}_{\mathcal{V}\to\mathcal{A}}$, is $\mathcal{P}$ in (4.10), and the maximum number of communication links between the nodes of the network system is $2|\mathcal{V}|$. Following the same approach for the star and full network configurations in each of the conditions (ii) and (iii), the respective listed results are obtained. □

Remark 4.6. For a bus network configuration, the maximum number of communication links was addressed in Theorem 4.2. However, the minimum number is 4, similar to the ring, star, and full network configurations.

4.4 Design Procedure of the Topology

The formation of the proposed topology such that cycles are formed to contain each state node v_{G_i} of the plant system was discussed. However, the discussed approach is for the cycles to be formed separately, and not collectively to account for establishing and sharing common communication links. In this section, the formation of cycles that have common communication links to utilize a lower reduced number of nodes and communication links is addressed. First, consider the following utility function for the state nodes $\mathcal{V}_G$ of the plant system defined as

$$\varphi = \sum_{i=1}^{n} b(t_i) - \sum_{i=1}^{n} \sum_{j=1}^{|\mathcal{E}_i \setminus \mathcal{E}_c|} c(l_j) - \sum_{s=1}^{|\mathcal{E}_c|} c(l_s), \tag{4.11}$$

where $\mathcal{E}_c$ denotes the set of common communication links that are shared by at least two state nodes. In the second and third terms in the utility function in (4.11), the communication links l_j and l_s belong to the set of communication links that are either unique to each state node or shared by at least two state nodes, respectively. Then, a more optimal design of the proposed topology that results in the maximum utility function for all possible scenarios of connectivity between the nodes is achieved by solving the optimization problem,

$$\begin{aligned} [\mathcal{E}] = \arg \max_{\mathcal{E} \in \mathbf{E}} \varphi \\ \text{subject to} \\ \text{Assumptions 4.1 and 4.2,} \end{aligned} \tag{4.12}$$

where $\mathbf{E}$ denotes the set of all possible scenarios of connectivity between the nodes of the proposed topology. Then, consider the modelling framework presented in Section 4.1 and suppose $\mathcal{S} = \{\{A_{\mathcal{A}}, A_{\Gamma}, \mathcal{N}\}, \mathcal{K}\}$. The strategic formation of the proposed topology is achieved by computing $A_{\mathcal{A}}, A_{\Gamma}, \mathcal{N}$, and $\mathcal{K}$ (namely, the design variables) using the following algorithm.

In Algorithm 4.2, the design of the proposed topology is delivered by considering the benefit and cost associated with establishing the communication links between the nodes of the proposed topology as well as utilizing shared communication links when possible. Namely, the algorithm begins with forming the proposed topology by maximizing the utility function of the optimization problem in (4.12) (Step 2). This is achieved according to given specifica-

Algorithm 4.2 Algorithm for the Design of the Proposed Topology Under Induced Connectivity Constraints and with a Strategic Formation Approach

Step 1. Specify A_K, B_K, C_K, and D_K as arbitrary nonzero matrices.
Step 2. Obtain $\hat{E}_{\mathbf{x}_N}$, $\hat{E}_{\mathbf{y}}$, $\hat{E}_{\mathbf{g}}$, $\tilde{E}_{\mathbf{x}_N}$, $\tilde{E}_{\mathbf{y}}$, $\tilde{E}_{\mathbf{g}}$, $\bar{E}_{\mathbf{x}_N}$, $\bar{E}_{\mathbf{y}}$, and $\bar{E}_{\mathbf{g}}$ by solving the optimization problem in (4.12).
Step 3. Iteratively compute $A_{\mathcal{A}}$, A_Γ, $\mathcal{N}$, and $\mathcal{K}$ by solving the convex optimization problem,

$$\begin{gathered}[\mathcal{S}, Y, \kappa] = \arg \min_{X,Z,\mathcal{S},Y,\kappa} \kappa \\ \text{subject to} \\ \kappa\mathbf{I} - Y > \mathbf{0}, \text{ and} \\ \begin{bmatrix} X & Z & \mathscr{A}(\mathcal{S}) & \mathscr{B}(\mathcal{S}) \\ * & Y & \mathscr{C}(\mathcal{S}) & \mathscr{D}(\mathcal{S}) \\ * & * & X_k^{-1}(2\mathbf{I} - XX_k^{-1}) & \mathbf{0} \\ * & * & * & \mathbf{I} \end{bmatrix} > \mathbf{0}. \end{gathered} \tag{4.13}$$

tions for the required application. For example, the communication links are established when the following are specified:

i. The minimum number of sensor nodes and actuator nodes of the plant system in (2.1) that connect to all its state nodes $\mathcal{V}_G$ (namely, essential sensor and actuator nodes);

ii. The incorporation of the controller system in (2.2) and the network system in (4.3) in the closed-loop control system;

iii. The configuration of the nodes of the network system (e.g., bus, ring, star, and full network configurations); and

iv. The cost and benefit associated with establishing the communication links between the nodes in relation to the respective number of hops.

Then, the algorithm computes the design variables to deliver the proposed topology such that the optimization problem in (4.13) has a minimum performance measure (Step 3). This is achieved using the iterative approach of Chapter 2, such that the design variables are iteratively computed until there is no significant reduction in all the performance measures.

4.5 Simulations

The design of the proposed topology under induced connectivity constraints is demonstrated using the QTP system presented in Section 3.4.1. This is achieved using the design procedure presented in Section 4.4. First, consider the system matrices of the QTP system and the structure of its state space model. It can be observed that different cycles can be formed to contain the state nodes v_{G_1}, v_{G_2}, v_{G_3}, and v_{G_4} with using the sensor nodes s_1 and s_2 as well as the actuator nodes a_1 and a_2, such that the closed-loop control system does not have structural fixed modes. Further, it can be observed that state nodes v_{G_3} and v_{G_4} are reachable from the state nodes v_{G_1} and v_{G_2}, respectively. Therefore, connecting sensor node s_1 with actuator node a_2 as well as sensor node s_2 with actuator node a_1 form cycles that contain all state nodes in a more efficient manner (namely, $a_2 \to v_{G_3} \to v_{G_1} \to s_1$ and $a_1 \to v_{G_4} \to v_{G_2} \to s_2$). Thus, the design objective is to form the two cycles with and without incorporating nodes of the network and controller systems.

The strategic formation of the proposed topology to establish communication links between the nodes of the plant, network, and controller systems is investigated for two scenarios of connectivity between the nodes of the proposed topology, as depicted in Figure 4.1.

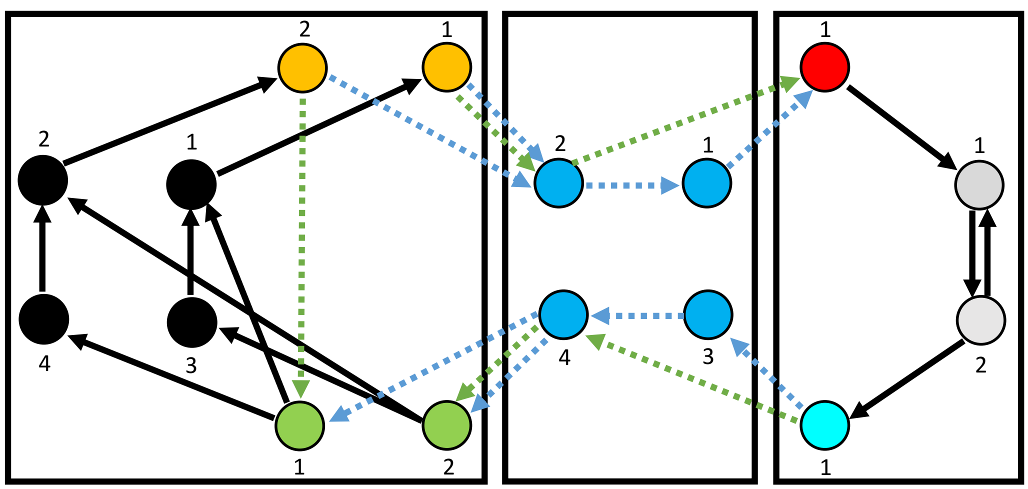

Figure 4.1: Connectivity scenarios of the nodes of the proposed topology under induced connectivity constraints, consisting of the QTP system (left block), the network system (middle block), and the controller system (right block).

More specifically, the closed-loop control system consists of the QTP system (namely, as a plant system), a second-order controller system with a single input node and a single output node, and a fourth-order network system with four distributed and inter-connected nodes. In Figure 4.1, the QTP system with its four nodes (black circles), two sensor nodes (yellow circles), and two actuator nodes (green circles) is controlled using a network system with its four nodes (blue circles) and a controller system with its two state nodes (grey circles), input node (red circle), and output node (cyan circle). The two different scenarios of connectivity are captured using the connections between the nodes (green and blue dotted arrows), such that communication links are established to satisfy Theorem 4.1, where:

i. In the first scenario, the path $s_1 \to v_2 \to \gamma_1 \to \theta_1 \to v_4 \to a_2$ as well as the communication link $s_2 \to a_1$ are formed; and

ii. In the second scenario, the paths $s_1 \to v_2 \to v_1 \to \gamma_1 \to \theta_1 \to v_3 \to v_4 \to a_2$ as well as $s_2 \to v_2 \to v_1 \to \gamma_1 \to \theta_1 \to v_3 \to v_4 \to a_1$ are formed.

It should be noted that the connections within the controller system (namely, $\gamma_1 \to v_{K_1} \to v_{K_2} \to \theta_1$) are omitted, and that in the second scenario, the path $v_2 \to v_1 \to \gamma_1 \to \theta_1 \to v_3 \to v_4$ offers shared communication links to connect the sensor and actuator nodes of the plant system.

Then, Algorithm 4.2 is implemented to deliver the design variables $A_{\mathcal{A}}$, A_{Γ}, $\mathcal{N}$, and $\mathcal{K}$. Namely, the algorithm begins with an arbitrary controller system (Step 1); then, the matrices $\hat{E}_{\mathbf{x}_N}$, $\hat{E}_{\mathbf{y}}$, $\hat{E}_{\mathbf{g}}$, $\tilde{E}_{\mathbf{x}_N}$, $\tilde{E}_{\mathbf{y}}$, $\tilde{E}_{\mathbf{g}}$, $\bar{E}_{\mathbf{x}_N}$, $\bar{E}_{\mathbf{y}}$, and $\bar{E}_{\mathbf{g}}$ are specified for the listed communication links that are established between the nodes (Step 2); and finally, the optimization problem is solved using MATLAB's Robust Control Toolbox (Step 3), by iteratively computing the design variables to deliver a stable and optimal closed-loop control system while reaching a lower threshold associated with the performance measure†. Further, for the first scenario, the nonzero entries of the computed system matrices of the network system are given as

$$\upsilon_{24} = -249.1982, \xi_{12} = -1.0906, \pi_{12} = -0.0001426,$$
$$\omega_{22} = 0.0119, \omega_{44} = 0.3491, \lambda_{21} = -0.6590, \psi_{41} = -7.3533,$$

†Since the objective is not the optimality of the designs of the systems, the respective convergence thresholds were not lowered to their minimum values.

and the system matrices of the controller system in (2.2), the local controllers at the actuator nodes of the plant system in (4.1), and the local controller at the input nodes of the controller system in (4.2) are given as

$$A_K = \begin{bmatrix} -0.0331 & -0.6127 \\ 0.0163 & 0.3031 \end{bmatrix}, B_K = \begin{bmatrix} 15.3253 \\ -19.4123 \end{bmatrix},$$
$$C_K = \begin{bmatrix} 0.0032 & 0.0323 \end{bmatrix}, D_K = 1.0590,$$
$$A_{\mathcal{A}} = \begin{bmatrix} 0.0271 & 0 \\ 0 & 0.0271 \end{bmatrix}, A_\Gamma = -0.0372.$$

For the second scenario, the nonzero entries of the computed system matrices of the network system are given as

$$\upsilon_{14} = -0.0532, \upsilon_{24} = -0.0546, \pi_{11} = -0.0281,$$
$$\omega_{11} = -0.9641, \omega_{22} = 0.0496, \omega_{33} = -0.0640,$$
$$\omega_{44} = 0.5749, \omega_{12} = 0.0232, \omega_{43} = -0.0182,$$
$$\lambda_{21} = 1.7994, \lambda_{22} = -1.0667, \psi_{31} = -4.5122,$$

and the system matrices of the controller system, the local controllers at the actuator nodes of the plant system, and the local controller at the input nodes of the controller system are given as

$$A_K = \begin{bmatrix} 0.0633 & -0.0562 \\ -0.1116 & 0.1029 \end{bmatrix}, B_K = \begin{bmatrix} -0.2381 \\ 0.2991 \end{bmatrix},$$
$$C_K = \begin{bmatrix} -0.0161 & 0.0149 \end{bmatrix}, D_K = 0.0253,$$
$$A_{\mathcal{A}} = \begin{bmatrix} 0.9935 & 0 \\ 0 & 0.9935 \end{bmatrix}, A_\Gamma = 0.9945.$$

Remark 4.7. It should be noted that the matrix D_K was computed to be nonzero, unlike the constraint given in Chapters 2 and 3. In addition, the weight assigned to the self-connectivity links of the nodes of the network system (namely, ω_{ii}) represents the state of node v_i of the network system and is computed to be nonzero when the node is part of the proposed topology (namely, when it is active), and zero otherwise (e.g., in the first scenario, nodes v_1 and v_3 are not used, and therefore $\omega_{11} = \omega_{33} = 0$). In addition, in both scenarios, the communication links to be established are specified with the assumption that the corresponding cycles are the optimal cycles, which generate the maximum value for the utility function in (4.11). If no such specifications are given, the optimization problem in (4.12) needs to be solved.

Further, in the first and second scenarios of connectivity between the nodes of the proposed topology, it is only required to establish five and eight communication links, respectively, to form the proposed topology, and where a stable and optimal closed-loop control system is delivered. In contrast, using the design procedures presented in Chapters 2 and 3, it would be required to establish over forty communication links. Thus, the design procedure presented in Section 4.4 provides a significantly more efficient approach to the formation of the proposed topology.

4.6 Summary

In this chapter, the design of the proposed topology under induced connectivity constraints was considered. More specifically, the following were delivered:

- The definition and the modelling of the proposed topology with using a decentralized control system setup as well as the modelling framework to facilitate the design of the proposed topology;
- The conditions required to characterize the connectivity between the nodes of the proposed topology;
- The strategic formation of the connectivity between the nodes of the proposed topology; and
- The design procedure of the proposed topology by using an algorithm for computing its design variables.

Chapter 5

Design of the Topology Under Additional Specifications

In Chapters 2, 3, and 4, the provided results are for the proposed topology under ideal operating scenarios, abnormal operating scenarios, and induced connectivity constraints. In this chapter, the design of the proposed topology under two additional specifications is considered. Firstly, the design of the proposed topology by using a model reduction approach to remove nodes and associated communication links is considered (namely, as an alternative approach to that of Chapter 4 to utilize a reduced number of nodes and communication links). Secondly, the design of the proposed topology by segregating the set of distributed and inter-connected nodes of the network system into two independent sets of nodes is considered (namely, with different objectives in relation to the connectivity between the nodes of the proposed topology).

5.1 Model Reduction of the Topology

Similar to the discussion of Chapter 4, a limitation on the utilization of the nodes and the communication links between the nodes of the proposed topology can lead to a simpler design as well as construction of the proposed topology. In contrast to the approach of Chapter 4, where the design of the topology begins by specifying a number of nodes and communication links to form the topology, in this section, the design of the proposed topology is re-evaluated to determine which nodes and associated communication links can be removed to result in a reduced size of the proposed topology.

5.1.1 Modelling Framework of the Topology

The design of the proposed topology under the removal of nodes and associated communication links is achieved by adopting the approach presented in the model reduction literature [8, 10]. Namely, nodes are removed from the set of distributed and inter-connected nodes $\mathcal{V}$ of the network system and state nodes are removed from the state nodes $\mathcal{V}_K$ of the controller system. The resulting proposed topology therefore consists of the plant system, as well as the reduced network and controller systems. More specifically, consider the closed-loop control system in (2.4), denoted by M, with state vector $\mathbf{x} = [\mathbf{x}_G^T \mathbf{x}_N^T \mathbf{x}_K^T]^T$, input vector $\mathbf{w}$, and output vector $\mathbf{z}$. The objective is to realize a closed-loop control system, denoted by $\tilde{M}$, with state vector $\tilde{\mathbf{x}} = [\mathbf{x}_G^T \tilde{\mathbf{x}}_N^T \tilde{\mathbf{x}}_K^T]^T$, input vector $\mathbf{w}$, and output vector $\tilde{\mathbf{z}}$, and modelled in discrete time as

$$\begin{aligned} \tilde{\mathbf{x}}(k+1) &= \tilde{\mathscr{A}}\tilde{\mathbf{x}}(k) + \tilde{\mathscr{B}}\mathbf{w}(k), \\ \tilde{\mathbf{z}}(k) &= \tilde{\mathscr{C}}\tilde{\mathbf{x}}(k) + \tilde{\mathscr{D}}\mathbf{w}(k), \end{aligned} \tag{5.1}$$

where $\tilde{N} < N$ and $\tilde{r} < r$, and $\tilde{\mathbf{x}}_N \in \mathbb{R}^{\tilde{N}}$ and $\tilde{\mathbf{x}}_K \in \mathbb{R}^{\tilde{r}}$ denote the state vectors of the reduced network and controller systems, respectively, and all of its system matrices have suitable dimensions.

Next, consider the decomposed and rearranged vector $\mathbf{x} = [\mathbf{x}_G^T \tilde{\mathbf{x}}_N^T \hat{\mathbf{x}}_N^T \tilde{\mathbf{x}}_K^T \hat{\mathbf{x}}_K^T]^T$, such that $\mathbf{x}_G, \tilde{\mathbf{x}}_N, \tilde{\mathbf{x}}_K$ correspond to the nodes that are to remain, and $\hat{\mathbf{x}}_N$ and $\hat{\mathbf{x}}_K$ correspond to the nodes that are to be removed. Then, the resulting system matrices of the closed-loop control system in (2.4) are defined as

$$\begin{aligned} \mathscr{A} &= \begin{bmatrix} \mathscr{A}_{11} & \mathscr{A}_{12} & \mathscr{A}_{13} & \mathscr{A}_{14} & \mathscr{A}_{15} \\ \mathscr{A}_{21} & \mathscr{A}_{22} & \mathscr{A}_{23} & \mathscr{A}_{24} & \mathscr{A}_{25} \\ \mathscr{A}_{31} & \mathscr{A}_{32} & \mathscr{A}_{33} & \mathscr{A}_{34} & \mathscr{A}_{35} \\ \mathscr{A}_{41} & \mathscr{A}_{42} & \mathscr{A}_{43} & \mathscr{A}_{44} & \mathscr{A}_{45} \\ \mathscr{A}_{51} & \mathscr{A}_{52} & \mathscr{A}_{53} & \mathscr{A}_{54} & \mathscr{A}_{55} \end{bmatrix}, \mathscr{B} = \begin{bmatrix} \mathscr{B}_{11} \\ \mathscr{B}_{21} \\ \mathscr{B}_{31} \\ \mathscr{B}_{41} \\ \mathscr{B}_{51} \end{bmatrix}, \\ \mathscr{C} &= \begin{bmatrix} \mathscr{C}_{11} & \mathscr{C}_{12} & \mathscr{C}_{13} & \mathscr{C}_{14} & \mathscr{C}_{15} \end{bmatrix}, \mathscr{D} = \mathscr{D}. \end{aligned}$$

Further, suppose the states of the closed-loop control system in (2.4) are categorized as either slow or fast dynamic states to correspond to the nodes that are to remain and to be removed, respectively. In order to indicate a no change in the state for a discrete-time system, $\hat{\mathbf{x}}_N(k+1) = \hat{\mathbf{x}}_N(k)$ and $\hat{\mathbf{x}}_K(k+1) = \hat{\mathbf{x}}_K(k)$. Thus, in the closed-loop control system in (2.4), the vectors $\hat{\mathbf{x}}_N$ and $\hat{\mathbf{x}}_K$ can be expressed in terms of the vectors $\mathbf{x}_G$, $\tilde{\mathbf{x}}_N$, $\tilde{\mathbf{x}}_K$, and $\mathbf{w}$. The system matrices of the closed-loop control system in (5.1) are then defined as

$$\tilde{\mathscr{A}} = \begin{bmatrix} \tilde{\mathscr{A}_{11}} & \tilde{\mathscr{A}_{12}} & \tilde{\mathscr{A}_{13}} \\ \tilde{\mathscr{A}_{21}} & \tilde{\mathscr{A}_{22}} & \tilde{\mathscr{A}_{23}} \\ \tilde{\mathscr{A}_{31}} & \tilde{\mathscr{A}_{32}} & \tilde{\mathscr{A}_{33}}, \end{bmatrix}, \tilde{\mathscr{B}} = \begin{bmatrix} \tilde{\mathscr{B}}_{11} \\ \tilde{\mathscr{B}}_{21} \\ \tilde{\mathscr{B}}_{31} \end{bmatrix},$$

$$\tilde{\mathscr{C}} = \begin{bmatrix} \tilde{\mathscr{C}_{11}} & \tilde{\mathscr{C}_{12}} & \tilde{\mathscr{C}_{13}} \end{bmatrix}, \tilde{\mathscr{D}} = \tilde{\mathscr{D}}_{11},$$

where the matrices are defined as

$$\begin{aligned}
&\tilde{\mathscr{A}_{11}} = \mathscr{A}_{11} + \mathscr{A}_{13}\bar{\mathscr{A}_{31}} + \mathscr{A}_{15}\bar{\mathscr{A}_{51}}, \tilde{\mathscr{A}_{12}} = \mathscr{A}_{12} + \mathscr{A}_{13}\bar{\mathscr{A}_{32}} + \mathscr{A}_{15}\bar{\mathscr{A}_{52}},\\
&\tilde{\mathscr{A}_{13}} = \mathscr{A}_{14} + \mathscr{A}_{13}\bar{\mathscr{A}_{34}} + \mathscr{A}_{15}\bar{\mathscr{A}_{54}}, \tilde{\mathscr{A}_{21}} = \mathscr{A}_{21} + \mathscr{A}_{23}\bar{\mathscr{A}_{31}} + \mathscr{A}_{25}\bar{\mathscr{A}_{51}},\\
&\tilde{\mathscr{A}_{22}} = \mathscr{A}_{22} + \mathscr{A}_{23}\bar{\mathscr{A}_{32}} + \mathscr{A}_{25}\bar{\mathscr{A}_{52}}, \tilde{\mathscr{A}_{23}} = \mathscr{A}_{24} + \mathscr{A}_{23}\bar{\mathscr{A}_{34}} + \mathscr{A}_{25}\bar{\mathscr{A}_{54}},\\
&\tilde{\mathscr{A}_{31}} = \mathscr{A}_{41} + \mathscr{A}_{43}\bar{\mathscr{A}_{31}} + \mathscr{A}_{45}\bar{\mathscr{A}_{51}}, \tilde{\mathscr{A}_{32}} = \mathscr{A}_{42} + \mathscr{A}_{43}\bar{\mathscr{A}_{32}} + \mathscr{A}_{45}\bar{\mathscr{A}_{52}},\\
&\tilde{\mathscr{A}_{33}} = \mathscr{A}_{44} + \mathscr{A}_{43}\bar{\mathscr{A}_{34}} + \mathscr{A}_{45}\bar{\mathscr{A}_{54}}, \tilde{\mathscr{B}}_{11} = \mathscr{B}_{11} + \mathscr{A}_{13}\bar{\mathscr{B}}_{31} + \mathscr{A}_{15}\bar{\mathscr{B}}_{51},\\
&\tilde{\mathscr{B}}_{21} = \mathscr{B}_{21} + \mathscr{A}_{23}\bar{\mathscr{B}}_{31} + \mathscr{A}_{25}\bar{\mathscr{B}}_{51}, \tilde{\mathscr{B}}_{31} = \mathscr{B}_{41} + \mathscr{A}_{43}\bar{\mathscr{B}}_{31} + \mathscr{A}_{45}\bar{\mathscr{B}}_{51},\\
&\tilde{\mathscr{C}_{11}} = \mathscr{C}_{11} + \mathscr{C}_{13}\bar{\mathscr{A}_{31}} + \mathscr{C}_{15}\bar{\mathscr{A}_{51}}, \tilde{\mathscr{C}_{12}} = \mathscr{C}_{12} + \mathscr{C}_{13}\bar{\mathscr{A}_{32}} + \mathscr{C}_{15}\bar{\mathscr{A}_{52}},\\
&\tilde{\mathscr{C}_{13}} = \mathscr{C}_{14} + \mathscr{C}_{13}\bar{\mathscr{A}_{34}} + \mathscr{C}_{15}\bar{\mathscr{A}_{54}}, \tilde{\mathscr{D}}_{11} = \mathscr{D} + \mathscr{C}_{13}\bar{\mathscr{B}}_{31} + \mathscr{C}_{15}\bar{\mathscr{B}}_{51},
\end{aligned}$$

the remaining matrices are defined as

$$\begin{aligned}
&\bar{\mathscr{A}_{31}} = \eta(\mathscr{A}_{35}(\mathbf{I} - \mathscr{A}_{55})^{-1}\mathscr{A}_{51} + \mathscr{A}_{31}), \bar{\mathscr{A}_{32}} = \eta(\mathscr{A}_{35}(\mathbf{I} - \mathscr{A}_{55})^{-1}\mathscr{A}_{52} + \mathscr{A}_{32}),\\
&\bar{\mathscr{A}_{34}} = \eta(\mathscr{A}_{35}(\mathbf{I} - \mathscr{A}_{55})^{-1}\mathscr{A}_{54} + \mathscr{A}_{34}), \bar{\mathscr{B}}_{31} = \eta(\mathscr{A}_{35}(\mathbf{I} - \mathscr{A}_{55})^{-1}\mathscr{B}_{51} + \mathscr{B}_{31}),\\
&\bar{\mathscr{A}_{51}} = \beta(\mathscr{A}_{53}(\mathbf{I} - \mathscr{A}_{33})^{-1}\mathscr{A}_{31} + \mathscr{A}_{51}), \bar{\mathscr{A}_{52}} = \beta(\mathscr{A}_{53}(\mathbf{I} - \mathscr{A}_{33})^{-1}\mathscr{A}_{32} + \mathscr{A}_{52}),\\
&\bar{\mathscr{A}_{54}} = \beta(\mathscr{A}_{53}(\mathbf{I} - \mathscr{A}_{33})^{-1}\mathscr{A}_{34} + \mathscr{A}_{54}), \bar{\mathscr{B}}_{51} = \beta(\mathscr{A}_{53}(\mathbf{I} - \mathscr{A}_{33})^{-1}\mathscr{B}_{31} + \mathscr{B}_{51}),
\end{aligned}$$

and the variables η and β are defined as

$$\begin{aligned}
\eta &= (\mathbf{I} - (\mathbf{I} - \mathscr{A}_{33})^{-1}\mathscr{A}_{35}(\mathbf{I} - \mathscr{A}_{55})^{-1}\mathscr{A}_{53})^{-1}(\mathbf{I} - \mathscr{A}_{33})^{-1},\\
\beta &= (\mathbf{I} - (\mathbf{I} - \mathscr{A}_{55})^{-1}\mathscr{A}_{53}(\mathbf{I} - \mathscr{A}_{33})^{-1}\mathscr{A}_{35})^{-1}(\mathbf{I} - \mathscr{A}_{55})^{-1}.
\end{aligned}$$

Next, the error between the original closed-loop control system in (2.4) and the reduced closed-loop control system in (5.1) with state vector $\zeta \triangleq [\mathbf{x}^T \, \tilde{\mathbf{x}}^T]^T$, input vector $\mathbf{w}$, and output vector $\rho = \mathbf{z} - \tilde{\mathbf{z}}$, denoted by E, is modelled as a LTI system in discrete time as

$$\begin{aligned}
\zeta(k+1) &= \mathscr{A}_e\zeta(k) + \mathscr{B}_e\mathbf{w}(k),\\
\rho(k) &= \mathscr{C}_e\zeta(k) + \mathscr{D}_e\mathbf{w}(k),
\end{aligned} \tag{5.2}$$

where the system matrices are defined as

$$\mathscr{A}_e = \begin{bmatrix} \mathscr{A} & \mathbf{0} \\ \mathbf{0} & \tilde{\mathscr{A}} \end{bmatrix}, \mathscr{B}_e = \begin{bmatrix} \mathscr{B} \\ \tilde{\mathscr{B}} \end{bmatrix},$$

$$\mathscr{C}_e = \begin{bmatrix} \mathscr{C} & -\tilde{\mathscr{C}} \end{bmatrix}, \mathscr{D}_e = \begin{bmatrix} \mathscr{D} - \tilde{\mathscr{D}} \end{bmatrix}.$$

5.1.2 Design Procedure of the Topology

The characterization of the reduced closed-loop control system in (5.1) is achieved using the Hankel norm performance of the error system in (5.2). First, consider the following two propositions and theorem from [18] and references therein.

Proposition 5.1. The error system in (5.2) is asymptotically stable if

$$\lim_{k\to\infty} |\zeta(k)| = \mathbf{0}$$

when $\mathbf{w}(k) = \mathbf{0}$. Further, for an asymptotically stable error system in (5.2), $\rho \in \ell_2[0,\infty)$ when $\mathbf{w} \in \ell_2[0,\infty)$.

Proposition 5.2. The error system in (5.2) is asymptotically stable with a Hankel norm error performance defined by $\alpha > 0$ if the error system is asymptotically stable and

$$\frac{\sum\limits_{k=T}^{\infty} \rho^T(k)\rho(k)}{\sum\limits_{k=0}^{T-1} \mathbf{w}^T(k)\mathbf{w}(k)} < \alpha^2$$

under zero initial condition for all $\mathbf{w} \in \ell_2[0,\infty)$ with $\mathbf{w}(k) = \mathbf{0}$ for all $k \geq T$.

Theorem 5.1. The error system in (5.2) is asymptotically stable with a guaranteed Hankel norm error performance α if there exists matrices $\mathcal{S}_1 > \mathbf{0}$ and $\mathcal{S}_2 > \mathbf{0}$ satisfying the following LMIs:

$$\begin{bmatrix} \mathscr{A}_e^T \mathcal{S}_1 \mathscr{A}_e - \mathcal{S}_1 & \mathscr{A}_e^T \mathcal{S}_1 \mathscr{B}_e \\ * & \mathscr{B}_e^T \mathcal{S}_1 \mathscr{B}_e - \alpha^2 \mathbf{I} \end{bmatrix} < \mathbf{0}, \tag{5.3}$$

$$\begin{bmatrix} \mathscr{A}_e^T \mathcal{S}_2 \mathscr{A}_e - \mathcal{S}_2 & \mathscr{C}_e^T \\ * & -\mathbf{I} \end{bmatrix} < \mathbf{0}, \tag{5.4}$$

$$\mathcal{S}_1 - \mathcal{S}_2 \geq \mathbf{0}. \tag{5.5}$$

Next, a greedy algorithm-based approach is proposed to remove nodes from the nodes of the network system and state nodes from the state nodes of the controller system (namely a model reduction approach), such as follows:

i. Removing nodes from the network system, and then removing state nodes from the state nodes of the controller system;

ii. Removing state nodes from the state nodes of the controller system, and then removing nodes from the network system; and

iii. Removing nodes from the network system and state nodes from the state nodes of the controller system simultaneously, and when no longer possible, removing nodes from the remaining system.

More specifically, approaches (i), (ii), (iii) are implemented using the following two algorithms (namely, approach (ii) is implemented using Algorithm 5.1 with replacing the variables and conditions appropriately).

Algorithm 5.1 Algorithm for the Design of the Reduced Proposed Topology with Model Reduction Approach (i)

Step 1. Specify $\tilde{N}$ and $\tilde{r}$; set $\tilde{\mathbf{x}}_N = \mathbf{x}_N$ and $\tilde{\mathbf{x}}_K = \mathbf{x}_K$; and set $p = N$ and $s = r$.
Step 2. For $i = 1 : p$, remove $\tilde{\mathbf{x}}_N(i)$ and compute $\alpha(i)$ from the convex optimization problem (with $\eta = \alpha^2$),

$$[\alpha] = \arg \min_{\alpha, \mathcal{S}_1, \mathcal{S}_2} \eta$$
$$\text{subject to}$$
$$\text{conditions (5.3) - (5.5), and}$$
$$\eta > 0, \mathcal{S}_1 > \mathbf{0}, \text{ and } \mathcal{S}_S > \mathbf{0}.$$

Remove $\tilde{\mathbf{x}}_N(i)$ that corresponds to the lowest $\alpha(i)$ and update $\tilde{\mathbf{x}}_N$.
Step 3. If no solution exists or if $\dim(\tilde{\mathbf{x}}_N) = \tilde{N}$, go to Step 4; else, set $p = \dim(\tilde{\mathbf{x}}_N)$ and return to Step 2.
Step 4. For $j = 1 : s$, remove $\tilde{\mathbf{x}}_K(j)$ and compute $\alpha(j)$. Remove $\tilde{\mathbf{x}}_K(j)$ that corresponds to the lowest $\alpha(j)$ and update $\tilde{\mathbf{x}}_K$.
Step 5. If no solution exists or if $\dim(\tilde{\mathbf{x}}_K) = \tilde{r}$, exit; else, set $s = \dim(\tilde{\mathbf{x}}_K)$ and go to Step 4.

Algorithms 5.1 and 5.2 attempt to remove nodes from the network system and state nodes from the state nodes of the controller system until the desired reduced size of the proposed topology is realized or the reduction is not possible. Thus, the algorithms deliver either an original or a reduced closed-loop control system that consists of the same or a fewer number of nodes of the network system and state nodes of the controller system, respectively.

Remark 5.1. Suppose a solution exists at the reduction steps of Algorithm 5.1, and the number of nodes to be removed from the network system are denoted by $\hat{N}$ and the state nodes to be removed from the state nodes of the controller system by $\hat{r}$, and the number of nodes at an iteration step l associated with

Algorithm 5.2 Algorithm for the Design of the Reduced Proposed Topology with Model Reduction Approach (iii)

Step 1. Specify $\tilde{N}$ and $\tilde{r}$; set $\tilde{\mathbf{x}}_N = \mathbf{x}_N$ and $\tilde{\mathbf{x}}_K = \mathbf{x}_K$; and set $p = N$ and $s = r$.
Step 2. For $i = 1 : p$ and $j = 1 : s$, remove the pair $(\tilde{\mathbf{x}}_N(i), \tilde{\mathbf{x}}_K(j))$ and compute $\alpha(i,j)$. Remove $\tilde{\mathbf{x}}_N(i)$ and $\tilde{\mathbf{x}}_K(j)$ that correspond to the lowest $\alpha(i,j)$ and update $\tilde{\mathbf{x}}_N$ and $\tilde{\mathbf{x}}_K$. If no solution exists, go to Step 4 and then Step 6.
Step 3. If $\dim(\tilde{\mathbf{x}}_K) = \tilde{r}$ and $\dim(\tilde{\mathbf{x}}_N) = \tilde{N}$, exit; if $\dim(\tilde{\mathbf{x}}_K) = \tilde{r}$, set $p = \dim(\tilde{\mathbf{x}}_N)$ and go to Step 4; if $\dim(\tilde{\mathbf{x}}_N) = \tilde{N}$, set $s = \dim(\tilde{\mathbf{x}}_K)$ and go to Step 6; else set $p = \dim(\tilde{\mathbf{x}}_N)$ and $s = \dim(\tilde{\mathbf{x}}_K)$ and return to Step 2.
Step 4. For $i = 1 : p$, remove $\tilde{\mathbf{x}}_N(i)$ and compute $\alpha(i)$. Remove $\tilde{\mathbf{x}}_N(i)$ that corresponds to the lowest $\alpha(i)$ and update $\tilde{\mathbf{x}}_N$.
Step 5. If no solution exists or if $\dim(\tilde{\mathbf{x}}_N) = \tilde{N}$, exit; else, set $p = \dim(\tilde{\mathbf{x}}_N)$ and return to Step 4.
Step 6. For $j = 1 : s$, remove $\tilde{\mathbf{x}}_K(j)$ and compute $\alpha(j)$. Remove $\tilde{\mathbf{x}}_K(j)$ that corresponds to the lowest $\alpha(j)$ and update $\tilde{\mathbf{x}}_K$.
Step 7. If no solution exists or if $\dim(\tilde{\mathbf{x}}_K) = \tilde{r}$, exit; else, set $s = \dim(\tilde{\mathbf{x}}_K)$ and return to Step 6.

the network system by $\tilde{N}(l)$ and those with the controller system by $\tilde{r}(l)$. The total number of iteration steps is $\hat{N} + \hat{r}$, and at every iteration step, there is a total of $\tilde{N}(l)$ or $\tilde{r}(l)$ number of possibilities.

Remark 5.2. Suppose a solution exists at the reduction steps of Algorithm 5.2. There is a total of $\hat{N}$ or $\hat{r}$ number of iteration steps, if $\hat{N} > \hat{r}$ or $\hat{r} > \hat{N}$, respectively. Further, at every iteration step l where a pair of nodes is to be removed, there is a total of $\tilde{N}(l) \times \tilde{r}(l)$ number of possibilities, and where a single node is to be removed, there is a total of $\tilde{N}(l)$ or $\tilde{r}(l)$ number of possibilities.

5.2 Separation of the Network System of the Topology

The proposed control system topology of Chapters 2, 3, and 4 consists of a plant system, a controller system, and a network system, as depicted in Figure 2.1. However, the set of distributed and inter-connected nodes that form the network system can be segregated into subsets of nodes that have different objectives. For example, two subsets of nodes can be formed. The first subset of nodes handles the transfer of information from the sensor nodes

$\mathcal{S}$ of the plant system to the input nodes Γ of the controller system, and the second subset of nodes handles the transfer of information from the output nodes Θ of the controller system to the actuator nodes $\mathcal{A}$ of the plant system. The first and second subsets of nodes form a transfer network system and a receiving network system, respectively. This control system topology is similar to that of using inbound and outbound networks in industrial control systems [1]. Thus, it can offer additional practicality. In this section, the proposed topology of Chapters 2, 3, and 4 is further extended, such that its network system is decomposed into a transfer network system and a receiving network system.

5.2.1 Definition of the Topology

The proposed extended control system topology consists of the following systems: a plant system, a controller system, and intermediate transfer and receiving network systems. The plant system has a set of sensor nodes $\mathcal{S} = \{s_1, \ldots, s_p\}$ and a set of actuator nodes $\mathcal{A} = \{a_1, \ldots, a_m\}$; the controller system has a set of input nodes $\Gamma = \{\gamma_1, \ldots, \gamma_d\}$ and a set of output nodes $\Theta = \{\theta_1, \ldots, \theta_t\}$; and the intermediate transfer and receiving network systems have sets of distributed and inter-connected nodes $\mathcal{V}_T = \{v_{T_1}, \ldots, v_{T_{n_T}}\}$ and $\mathcal{V}_R = \{v_{R_1}, \ldots, v_{R_{n_R}}\}$, respectively. In contrast to the proposed topology of Chapters 2, 3, and 4, the different types of nodes of the proposed extended topology are restricted in terms of connectivity. More specifically, the sensor nodes $\mathcal{S}$ of the plant system provide measurements to the actuator nodes $\mathcal{A}$ of the plant system, the input nodes Γ of the controller system, and only to the distributed and inter-connected nodes $\mathcal{V}_T$ of the intermediate transfer network system. Similarly, the output nodes Θ of the controller system send control commands to the sets of nodes $\mathcal{A}$, Γ, and $\mathcal{V}_R$. The connectivity between the nodes of the proposed extended topology is achieved using a set of wireless communication links $\mathcal{E}$. An example of the proposed extended control system topology is depicted in Figure 5.1, where the connectivity from the sets of nodes $\mathcal{S}$, $\mathcal{V}_R$, and Θ to the set of nodes $\mathcal{A}$ is represented using solid, dotted, and dashed red arrows, and is denoted by $\mathcal{E}_{\mathcal{S}\to\mathcal{A}}$, $\mathcal{E}_{\mathcal{V}_R\to\mathcal{A}}$, and $\mathcal{E}_{\Theta\to\mathcal{A}}$, respectively; that from $\mathcal{S}$ and $\mathcal{V}_T$ to $\mathcal{V}_T$ using dashed and solid black arrows, and denoted by $\mathcal{E}_{\mathcal{S}\to\mathcal{V}_T}$ and $\mathcal{E}_{\mathcal{V}_T\leftrightarrow\mathcal{V}_T}$, respectively; that from Θ and $\mathcal{V}_R$ to $\mathcal{V}_R$ using dashed and solid green arrows, and denoted by $\mathcal{E}_{\Theta\to\mathcal{V}_R}$ and $\mathcal{E}_{\mathcal{V}_R\leftrightarrow\mathcal{V}_R}$, respectively; and that from $\mathcal{S}$, $\mathcal{V}_T$, and Θ to Γ using dashed, dotted, and solid yellow arrows, and denoted by $\mathcal{E}_{\mathcal{S}\to\Gamma}$, $\mathcal{E}_{\mathcal{V}_T\to\Gamma}$, and $\mathcal{E}_{\Theta\to\Gamma}$, respectively.

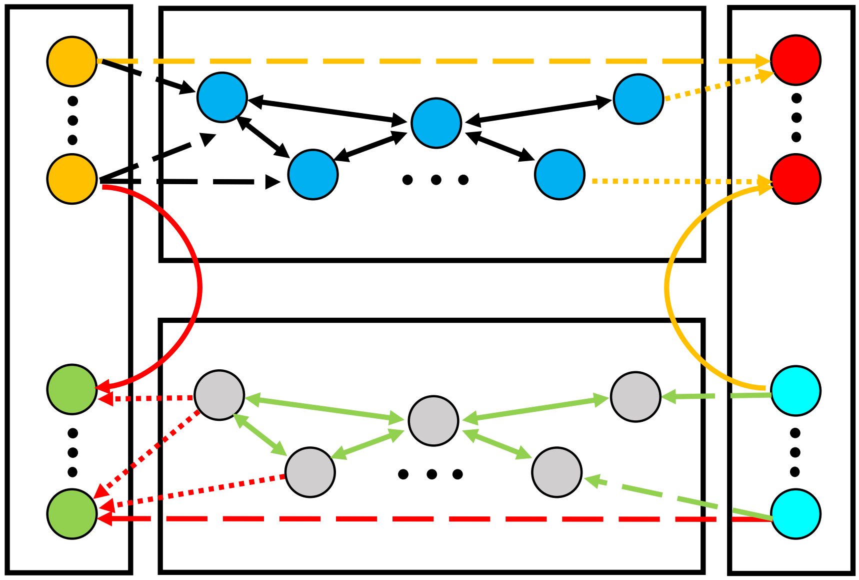

Figure 5.1: An example of the proposed extended control system topology with a plant system (left block), a transfer network system (middle top block), a controller system (right block), and a receiving network system (middle bottom block), where the sensor nodes (yellow circles) and actuator nodes (green circles) of the plant system, the distributed and inter-connected nodes (blue circles) of the transfer network system, the input nodes (red circles) and output nodes (cyan circles) of the controller system, and the distributed and inter-connected nodes (grey circles) of the receiving network system are connected using wireless communication links (different colors and shapes of arrows).

Similar to the definitions in Section 2.1, the plant system and the controller system, denoted by G and K, are modelled as in (2.1) and (2.2), respectively. Further, similar to the nodes of the network system in (2.3), each node of the intermediate transfer and receiving network systems updates its state based on its current state as well as the current states of its neighbouring nodes. Similarly, each actuator node of the plant system and input node of the controller system updates its state based on the current states of its neighbouring nodes. First, consider the transfer network system, the update procedure is modelled for each node of the transfer network system with state x_{T_i} and input node of the controller system with state f_i in discrete time as

$$x_{T_i}(k+1) = \omega_{T_{ii}} x_{T_i}(k) + \sum_{\upsilon_{T_j} \in \mathscr{N}_{\upsilon_{T_i}}} \omega_{T_{ij}} x_{T_j}(k) + \sum_{s_j \in \mathscr{N}_{\upsilon_{T_i}}} \lambda_{T_{ij}} y_j(k),$$
$$f_i(k) = \sum_{\upsilon_{T_j} \in \mathscr{N}_{\gamma_i}} \upsilon_{T_{ij}} x_{T_j}(k) + \sum_{s_j \in \mathscr{N}_{\gamma_i}} \xi_{T_{ij}} y_j(k),$$

where the coefficients $\omega_{T_{ij}}$, $\lambda_{T_{ij}}$, $\upsilon_{T_{ij}}$, and $\xi_{T_{ij}}$ denote the weights assigned to the states received by node i from node j, and $\omega_{T_{ii}}$ denotes the weight assigned to the self-connectivity link. The states of the nodes of the transfer network system and the input nodes of the controller system in an augmented manner provide the transfer network system with state vector $\mathbf{x}_T = [x_{T_1}, \ldots, x_{T_{n_T}}] \in \mathbb{R}^{n_T}$, denoted by T, which is modelled as a LTI system in discrete time as

$$\begin{aligned} \mathbf{x}_T(k+1) &= \Omega_T \mathbf{x}_T(k) + \Lambda_T \mathbf{y}(k), \\ \mathbf{f}(k) &= \Upsilon_T \mathbf{x}_T(k) + \Xi_T \mathbf{y}(k), \end{aligned} \tag{5.6}$$

where the matrices Ω_T, Λ_T, Υ_T, and Ξ_T contain the coefficients denoting the weight assignments. Next, consider the receiving network system, the update procedure is modelled for each node of the receiving network system with state x_{R_i} and actuator node of the plant system with state u_i in discrete time as

$$x_{R_i}(k+1) = \omega_{R_{ii}} x_{R_i}(k) + \sum_{\upsilon_{R_j} \in \mathscr{N}_{\upsilon_{R_i}}} \omega_{R_{ij}} x_{R_j}(k) + \sum_{\theta_j \in \mathscr{N}_{\upsilon_{R_i}}} \lambda_{R_{ij}} g_j(k),$$
$$u_i(k) = \sum_{\upsilon_{R_j} \in \mathscr{N}_{a_i}} \upsilon_{R_{ij}} x_{R_j}(k) + \sum_{\theta_j \in \mathscr{N}_{a_i}} \xi_{R_{ij}} g_j(k),$$

where the coefficients $\omega_{R_{ij}}$, $\lambda_{R_{ij}}$, $\upsilon_{R_{ij}}$, and $\xi_{R_{ij}}$ denote the weights assigned to the states received by node i from node j, and $\omega_{R_{ii}}$ denotes the weight assigned to the self-connectivity link. The states of the nodes of the receiving network system and the actuator nodes of the plant system in an augmented manner provide the receiving network system with state vector $\mathbf{x}_R = [x_{R_1}, \ldots, x_{R_{n_R}}] \in \mathbb{R}^{n_R}$, denoted by R, which is modelled as a LTI system in discrete time as

$$\begin{aligned} \mathbf{x}_R(k+1) &= \Omega_R \mathbf{x}_R(k) + \Lambda_R \mathbf{g}(k), \\ \mathbf{u}(k) &= \Upsilon_R \mathbf{x}_R(k) + \Xi_R \mathbf{g}(k), \end{aligned} \tag{5.7}$$

where the system matrices Ω_R, Λ_R, Υ_R, and Ξ_R contain the coefficients denoting the weight assignments*. Further, the values of the system matrices of

*In the modelling of the transfer and receiving network systems, the set of communication links $\mathcal{E}_{\mathcal{S} \to \mathcal{A}}$ and $\mathcal{E}_{\Theta \to \Gamma}$ are omitted.

transfer network system in (5.6) and of those of the receiving network system in (5.7), in addition to those of the controller system in (2.2), are determined to deliver a control system configuration that satisfies specific objectives (e.g., stability and performance), and they are therefore the design variables.

5.2.2 Modelling Framework of the Topology

The plant system in (2.1), the controller system in (2.2), the transfer network system in (5.6), and the receiving network system in (5.7) are connected through feedback (namely, through the communication links between the nodes). This results in the feedback setup of the closed-loop control system depicted in Figure 5.2.

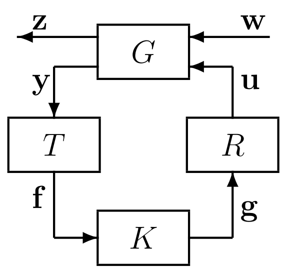

Figure 5.2: Feedback setup of the closed-loop control system consisting of the plant, controller, and transfer and receiving network systems.

Then, the closed-loop control system is modelled as a LTI system in discrete time as

$$\begin{aligned}
\mathbf{x}(k+1) &= \underbrace{\begin{bmatrix} \mathscr{A}_{11} & \dots & \mathscr{A}_{14} \\ \vdots & \ddots & \vdots \\ \mathscr{A}_{41} & \dots & \mathscr{A}_{44} \end{bmatrix}}_{\mathscr{A}} \mathbf{x}(k) + \underbrace{\begin{bmatrix} \mathscr{B}_{11} \\ \vdots \\ \mathscr{B}_{41} \end{bmatrix}}_{\mathscr{B}} \mathbf{w}(k), \\
\mathbf{z}(k) &= \underbrace{\begin{bmatrix} \mathscr{C}_{11} & \dots & \mathscr{C}_{14} \end{bmatrix}}_{\mathscr{C}} \mathbf{x}(k) + \mathscr{D}\mathbf{w}(k),
\end{aligned} \tag{5.8}$$

where the vector $\mathbf{x} \triangleq [\mathbf{x}_G^T \ \mathbf{x}_T^T \ \mathbf{x}_K^T \ \mathbf{x}_R^T]^T$ denotes its state, and the system matrices are defined as

$$
\begin{aligned}
\mathscr{A}_{11} &= A + B_2\Xi_R D_K \Xi_T C_2, \mathscr{A}_{12} = B_2\Xi_R D_K \Upsilon_T, \\
\mathscr{A}_{13} &= B_2\Xi_R C_K, \mathscr{A}_{14} = B_2\Upsilon_R, \mathscr{A}_{21} = \Lambda_T C_2, \\
\mathscr{A}_{22} &= \Omega_T, \mathscr{A}_{23} = \mathbf{0}, \mathscr{A}_{24} = \mathbf{0}, \mathscr{A}_{31} = B_K \Xi_T C_2, \\
\mathscr{A}_{32} &= B_K \Upsilon_T, \mathscr{A}_{33} = A_K, \mathscr{A}_{34} = \mathbf{0}, \mathscr{A}_{41} = \Lambda_R D_K \Xi_T C_2, \\
\mathscr{A}_{42} &= \Lambda_R D_K \Upsilon_T, \mathscr{A}_{43} = \Lambda_R C_K, \mathscr{A}_{44} = \Omega_R, \\
\mathscr{B}_{11} &= B_1 + B_2\Xi_R D_K \Xi_T D_{21}, \mathscr{B}_{21} = \Lambda_T D_{21}, \\
\mathscr{B}_{31} &= B_K \Xi_T D_{21}, \mathscr{B}_{41} = \Lambda_R D_K \Xi_T D_{21}, \\
\mathscr{C}_{11} &= C_1 + D_{12}\Xi_R D_K \Xi_T C_2, \mathscr{C}_{12} = D_{12}\Xi_R D_K \Upsilon_T, \\
\mathscr{C}_{13} &= D_{12}\Xi_R C_K, \mathscr{C}_{14} = D_{12}\Upsilon_R, \\
\mathscr{D} &= D_{11} + D_{12}\Xi_R D_K \Xi_T D_{21}.
\end{aligned}
$$

As can be observed from the system matrices of the closed-loop control system in (5.8), the system matrices of the plant, controller, and transfer and receiving network systems are coupled together in a nonlinear manner. Next, the system matrices are expressed in terms of matrices affine on the system matrices of the controller, transfer network, and receiving network systems (namely, similar to the approach presented in Section 2.2). First, consider the transfer network system in (5.6) and suppose the associated transfer network parameter is denoted by $\mathcal{T}$ and defined as

$$
\mathcal{T} = \begin{bmatrix} \Xi_T & \Upsilon_T \\ \Lambda_T & \Omega_T \end{bmatrix}. \tag{5.9}
$$

The system matrices of the closed-loop control system in (5.8) are defined in terms of the transfer network parameter as

$$
\begin{aligned}
\mathscr{A}(\mathcal{T}) &= \mathscr{A}_1^{\mathcal{T}} + \mathscr{B}_1^{\mathcal{T}} \mathcal{T} \mathscr{C}_1^{\mathcal{T}}, \\
\mathscr{B}(\mathcal{T}) &= \mathscr{B}_2^{\mathcal{T}} + \mathscr{B}_1^{\mathcal{T}} \mathcal{T} \mathscr{C}_2^{\mathcal{T}}, \\
\mathscr{C}(\mathcal{T}) &= \mathscr{C}_3^{\mathcal{T}} + \mathscr{C}_4^{\mathcal{T}} \mathcal{T} \mathscr{C}_1^{\mathcal{T}}, \\
\mathscr{D}(\mathcal{T}) &= D_{11} + \mathscr{C}_4^{\mathcal{T}} \mathcal{T} \mathscr{C}_2^{\mathcal{T}},
\end{aligned}
$$

where the matrices are defined as

$$\mathscr{A}_1^{\mathcal{T}} = \begin{bmatrix} A & \mathbf{0} & B_2\Xi_R C_K & B_2\Upsilon_R \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & A_K & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \Lambda_R C_K & \Omega_R \end{bmatrix}, \mathscr{B}_1^{\mathcal{T}} = \begin{bmatrix} B_2\Xi_R D_K & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \\ B_K & \mathbf{0} \\ \Lambda_R D_K & \mathbf{0} \end{bmatrix}, \mathscr{B}_2^{\mathcal{T}} = \begin{bmatrix} B_1 \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix},$$

$$\mathscr{C}_1^{\mathcal{T}} = \begin{bmatrix} C_2 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} \end{bmatrix}, \mathscr{C}_2^{\mathcal{T}} = \begin{bmatrix} D_{21} \\ \mathbf{0} \end{bmatrix}, \mathscr{C}_3^{\mathcal{T}} = \begin{bmatrix} C_1 & \mathbf{0} & D_{12}\Xi_R C_K & D_{12}\Upsilon_R \end{bmatrix},$$

$$\mathscr{C}_4^{\mathcal{T}} = \begin{bmatrix} D_{12}\Xi_R D_K & \mathbf{0} \end{bmatrix}.$$

Next, consider the controller system in (2.2), and suppose the associated controller parameter is denoted by $\mathcal{K}$ and defined as

$$\mathcal{K} = \begin{bmatrix} D_K & C_K \\ B_K & A_K \end{bmatrix}. \tag{5.10}$$

The system matrices of the closed-loop control system in (5.8) are defined in terms of the controller parameter as

$$\begin{aligned} \mathscr{A}(\mathcal{K}) &= \mathscr{A}_1^{\mathcal{K}} + \mathscr{B}_1^{\mathcal{K}}\mathcal{K}\mathscr{C}_1^{\mathcal{K}}, \\ \mathscr{B}(\mathcal{K}) &= \mathscr{B}_2^{\mathcal{K}} + \mathscr{B}_1^{\mathcal{K}}\mathcal{K}\mathscr{C}_2^{\mathcal{K}}, \\ \mathscr{C}(\mathcal{K}) &= \mathscr{C}_3^{\mathcal{K}} + \mathscr{C}_4^{\mathcal{K}}\mathcal{K}\mathscr{C}_1^{\mathcal{K}}, \\ \mathscr{D}(\mathcal{K}) &= D_{11} + \mathscr{C}_4^{\mathcal{K}}\mathcal{K}\mathscr{C}_2^{\mathcal{K}}, \end{aligned}$$

where the matrices are defined as

$$\mathscr{A}_1^{\mathcal{K}} = \begin{bmatrix} A & \mathbf{0} & \mathbf{0} & B_2\Upsilon_R \\ \Lambda_T C_2 & \Omega_T & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \Omega_R \end{bmatrix}, \mathscr{B}_1^{\mathcal{K}} = \begin{bmatrix} B_2\Xi_R & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \\ \Lambda_R & \mathbf{0} \end{bmatrix}, \mathscr{B}_2^{\mathcal{K}} = \begin{bmatrix} B_1 \\ \Lambda_T D_{21} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix},$$

$$\mathscr{C}_1^{\mathcal{K}} = \begin{bmatrix} \Xi_T C_2 & \Upsilon_T & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} \end{bmatrix}, \mathscr{C}_2^{\mathcal{K}} = \begin{bmatrix} \Xi_T D_{21} \\ \mathbf{0} \end{bmatrix}, \mathscr{C}_3^{\mathcal{K}} = \begin{bmatrix} C_1 & \mathbf{0} & \mathbf{0} & D_{12}\Upsilon_R \end{bmatrix},$$

$$\mathscr{C}_4^{\mathcal{K}} = \begin{bmatrix} D_{12}\Xi_R & \mathbf{0} \end{bmatrix}.$$

Finally, consider the receiving network system in (5.7), and suppose the associated receiving network parameter is denoted by $\mathcal{R}$ and defined as

$$\mathcal{R} = \begin{bmatrix} \Xi_R & \Upsilon_R \\ \Lambda_R & \Omega_R \end{bmatrix}. \tag{5.11}$$

The system matrices of the closed-loop control system in (5.8) are defined in terms of the receiving network parameter as

$$\begin{aligned} \mathscr{A}(\mathcal{R}) &= \mathscr{A}_1^{\mathcal{R}} + \mathscr{B}_1^{\mathcal{R}}\mathcal{R}\mathscr{C}_1^{\mathcal{R}}, \\ \mathscr{B}(\mathcal{R}) &= \mathscr{B}_2^{\mathcal{R}} + \mathscr{B}_1^{\mathcal{R}}\mathcal{R}\mathscr{C}_2^{\mathcal{R}}, \\ \mathscr{C}(\mathcal{R}) &= \mathscr{C}_3^{\mathcal{R}} + \mathscr{C}_4^{\mathcal{R}}\mathcal{R}\mathscr{C}_1^{\mathcal{R}}, \\ \mathscr{D}(\mathcal{R}) &= D_{11} + \mathscr{C}_4^{\mathcal{R}}\mathcal{R}\mathscr{C}_2^{\mathcal{R}}, \end{aligned}$$

where the matrices are defined as

$$\mathscr{A}_1^{\mathcal{R}} = \begin{bmatrix} A & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \Lambda_T C_2 & \Omega_T & \mathbf{0} & \mathbf{0} \\ B_K \Xi_T C_2 & B_K \Upsilon_T & A_K & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}, \mathscr{B}_1^{\mathcal{R}} = \begin{bmatrix} B_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}, \mathscr{B}_2^{\mathcal{R}} = \begin{bmatrix} B_1 \\ \Lambda_T D_{21} \\ B_K \Xi_T D_{21} \\ \mathbf{0} \end{bmatrix},$$

$$\mathscr{C}_1^{\mathcal{R}} = \begin{bmatrix} D_K \Xi_T C_2 & D_K \Upsilon_T & C_K & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix}, \mathscr{C}_2^{\mathcal{R}} = \begin{bmatrix} D_K \Xi_T D_{21} \\ \mathbf{0} \end{bmatrix}, \mathscr{C}_3^{\mathcal{R}} = \begin{bmatrix} C_1 & \mathbf{0} \end{bmatrix},$$

$$\mathscr{C}_4^{\mathcal{R}} = \begin{bmatrix} D_{12} & \mathbf{0} \end{bmatrix}.$$

5.2.3 Design Procedure of the Topology

The design procedure of the proposed extended topology is addressed by extending the joint design approach of Section 2.4 to iteratively compute the controller parameter in (5.10), the transfer network parameter in (5.9), and the receiving network parameter in (5.11). First, consider Algorithms 2.1 and 2.2 of Section 2.4, and suppose the set containing the controller, and transfer and receiving network parameters is denoted and defined as $\mathcal{M} = \{\mathcal{T}, \mathcal{K}, \mathcal{R}\}$. The design procedure of the controller, transfer network, and receiving network systems of the proposed extended topology is achieved using the following algorithm.

In Algorithm 5.3, the algorithm begins with computing initial values for the matrix X as well as for the parameter $\mathcal{M}$ to result in a closed-loop control system that is stable, but not optimal (Steps 1, 2, and 3). Then, the computed initial values are used to compute values for the parameter $\mathcal{M}$ to result in a closed-loop control system that is stable as well as optimal (namely, with respective to the energy-to-peak performacne measure) (Steps 4, 5, and 6). Further, the joint design of the controller, transfer network, and receiving network systems of the proposed extended topology is achieved using the following algorithm.

In Algorithm 5.4, the computation of the controller, transfer network, and receiving network parameters is implemented iteratively until there is no significant change in the performance measures. More specifically, each of the controller, transfer network, and receiving network systems is designed given the remaining two systems, while sequentially iterating through the design of each of the systems until no significant reduction is observed in the respective performance measure (namely, $\epsilon_{\mathcal{M}}$). Further, the algorithm starts with an arbitrary controller and receiving network systems, and a transfer network

Algorithm 5.3 Algorithm for the Separate Design of Optimal Controller, Transfer Network and Receiving Network Systems

Step 1. Specify $k = 0$, $\eta_{\mathcal{M}} > 0$, and X_0 as an arbitrary symmetric matrix.
Step 2. Compute X by solving the convex optimization problem,

$$[X] = \arg \min_{X,Z,\mathcal{M},Y} \eta_{\mathcal{M}}$$
$$\text{subject to}$$
$$\begin{bmatrix} X & Z & \mathscr{A}(\mathcal{M}) & \mathscr{B}(\mathcal{M}) \\ * & Y & \mathscr{C}(\mathcal{M}) & \mathscr{D}(\mathcal{M}) \\ * & * & X_k^{-1}(2\mathbf{I} - XX_k^{-1}) & \mathbf{0} \\ * & * & * & \mathbf{I} \end{bmatrix} > \mathbf{0}. \quad (5.12)$$

Step 3. If $\eta_{\mathcal{M}} < 0$, go to Step 4; else, set $k = k + 1$ and go to Step 2.
Step 4. Set $k = 0$ and $\epsilon_{\mathcal{M}} > 0$.
Step 5. Compute $\mathcal{M}$, Y, and κ by solving the convex optimization problem,

$$[\mathcal{M}, Y, \kappa] = \arg \min_{X,Z,\mathcal{M},Y,\kappa} \kappa$$
$$\text{subject to}$$
$$\text{condition (5.12) and}$$
$$\kappa\mathbf{I} - Y > \mathbf{0}.$$

Step 6. If $\kappa < \epsilon_{\mathcal{M}}$, exit; else, set $k = k + 1$ and go to Step 5.

Algorithm 5.4 Algorithm for the Joint Design of Optimal Controller, Transfer Network and Receiving Network Systems

Step 1. Specify the reduction parameter β, and set $i = 1$, $\epsilon_{\mathcal{T}}(i < 1) = \epsilon_{\mathcal{K}}(i < 2) = \epsilon_{\mathcal{R}}(i < 3) = -1$, and $\mathcal{K}$ and $\mathcal{R}$ as arbitrary matrices.
Step 2. Compute $\mathcal{T}$ using Algorithm 5.3.
Step 3. If $\epsilon_{\mathcal{T}}(i) > \beta\epsilon_{\mathcal{T}}(i-3)$, $\epsilon_{\mathcal{R}}(i-1) > \beta\epsilon_{\mathcal{R}}(i-4)$, and $\epsilon_{\mathcal{K}}(i-2) > \beta\epsilon_{\mathcal{K}}(i-5)$, exit; else, set $i = i + 1$.
Step 4. Compute $\mathcal{K}$ using Algorithm 5.3.
Step 5. If $\epsilon_{\mathcal{K}}(i) > \beta\epsilon_{\mathcal{K}}(i-3)$, $\epsilon_{\mathcal{T}}(i-1) > \beta\epsilon_{\mathcal{T}}(i-4)$, and $\epsilon_{\mathcal{R}}(i-2) > \beta\epsilon_{\mathcal{R}}(i-5)$, exit; else, set $i = i + 1$.
Step 6. Compute $\mathcal{R}$ using Algorithm 5.3.
Step 7. If $\epsilon_{\mathcal{R}}(i) > \beta\epsilon_{\mathcal{R}}(i-3)$, $\epsilon_{\mathcal{K}}(i-1) > \beta\epsilon_{\mathcal{K}}(i-4)$, and $\epsilon_{\mathcal{T}}(i-2) > \beta\epsilon_{\mathcal{T}}(i-5)$, exit; else, set $i = i + 1$ and go to Step 2.

system is designed; then, using the receiving network system and the designed transfer network system, a controller system is designed; then, using the designed controller and transfer network systems, a receiving network system is designed; and repeating the sequence until no significant reduction is observed in the performance measures.

Thus far, the design of the proposed extended topology as well as the proposed topology of Chapters 2, 3, and 4 considers a closed-loop control system that consists of a single plant system and a single controller system. However, the design of the topologies can also be achieved for a closed-loop control system that consists of multiple plant systems and multiple controller systems (namely, as an extension to the discussion on using the WCN to control multiple plant systems [42]). For the proposed extended topology, the nodes of the transfer network system and the nodes of the receiving network system are utilized for the transfer of information between the plant systems and the controller systems. Consider a closed-loop control system with plant systems $\mathbf{G} = \{G^i, \ldots, G^b\}$ and controller systems $\mathbf{K} = \{K^1, \ldots, K^b\}$, which can have different orders. The nodes $\mathcal{V}_T$ of the transfer network system and the nodes $\mathcal{V}_R$ of the receiving network system connect each pair of plant system and controller system (namely, G_i and K_i for $i = 1, \ldots, b$). The modelling framework of Section 5.2.2 and the design procedure of Section 5.2.3 can be applied to closed-loop control systems with multiple plant systems and multiple controller systems. Further, the design of such a control system topology can be achieved by extending Algorithms 5.3 and 5.4, such that a separate LMI condition (namely, in (5.12)) is utilized for the closed-loop control system with each pair of plant system and controller system. The resulting design variables are the system matrices of the controller systems A_K^i, B_K^i, C_K^i, and D_K^i as well as the system matrices of the transfer network system Ω_T, Λ_T^i, Υ_T^i, and Ξ_T^i, and the system matrices of the receiving network system Ω_R, Λ_R^i, Υ_R^i, and Ξ_R^i.

5.3 Simulations

The design of the proposed topology under the additional specifications is demonstrated by applying the design procedures discussed in Sections 5.1.2 and 5.2.3.

5.3.1 Model Reduction of the Topology

In this section, the design of the proposed topology under the removal of nodes and associated communication links is demonstrated. First, consider the second-order plant system, fourth-order network system, and second-order controller system of Section 2.5, with only w_2 and z_2. The objective of the design procedure is to remove nodes from the network system and state nodes from the state nodes of the controller system until they each have a single node. Algorithms 5.1 and 5.2 are implemented using MATLAB's Robust Control Toolbox for the three approaches discussed in Section 5.1.2, and the following results are obtained:

i. When applying approach (i), only three nodes of the network system were removed;

ii. When applying approach (ii), only a single state node of the controller system and two nodes of the network system were removed; and

iii. When applying approach (iii), only a single state node of the controller system and three nodes of the network system were removed.

The specific results associated with applying each of the three approaches are presented in the following table, where the set of node(s) removed at each iteration step in the algorithms is denoted by τ.

Table 5.1: Results of the removal of nodes and associated communication links by applying approaches (i), (ii), and (iii)

Iteration	Approach (i)		Approach (ii)		Approach (iii)	
Step	τ	η	τ	η	τ	η
1	$\{v_1\}$	0.0079	$\{v_{K_1}\}$	0.0027	$\{v_3, v_{K_2}\}$	0.0121
2	$\{v_3\}$	0.0209	$\{v_1\}$	0.003	$\{v_2\}$	0.0604
3	$\{v_4\}$	1.8035	$\{v_3\}$	0.0187	$\{v_4\}$	0.0047

Next, a simulation example is implemented, as presented in Figures 5.3 – 5.8. In the simulation example, a time-varying input w_2 is utilized such that $w_2 = 10$ when $10 \leq k \leq 20$; $w_2 = 20$ when $40 \leq k \leq 60$; and $w_2 = 0$ otherwise.

From Figures 5.3 – 5.8, the following can be observed:

i. When applying approach (i), the removal of the one or the two nodes from the network system results in the reduced closed-loop control sys-

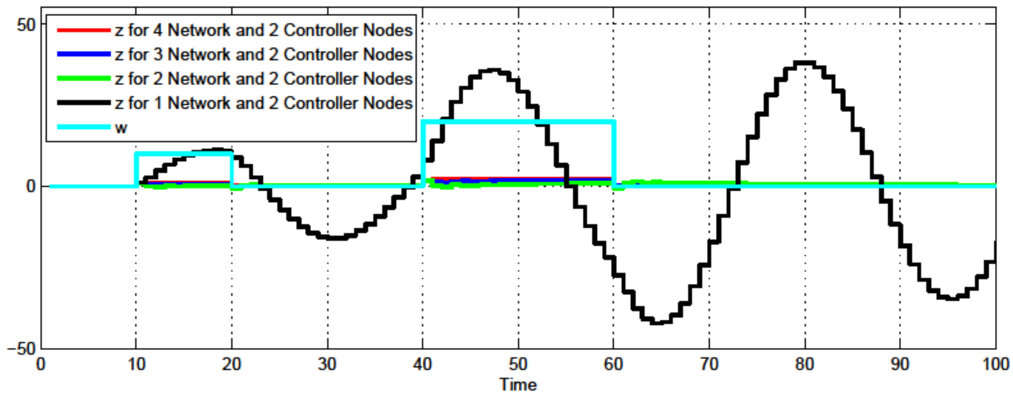

Figure 5.3: Simulation of the proposed topology under the removal of nodes and associated communication links by applying approach (i) - the inputs and outputs of the original and reduced closed-loop control systems.

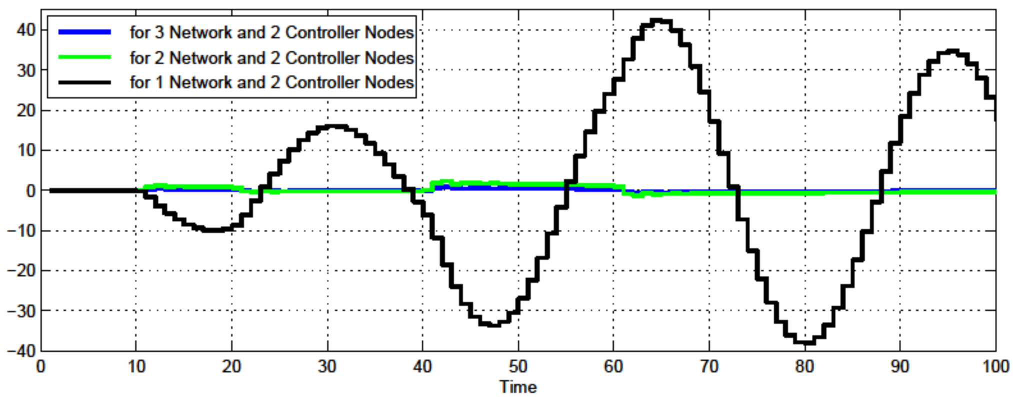

Figure 5.4: Simulation of the proposed topology under the removal of nodes and associated communication links by applying approach (i) - the outputs of the error systems.

tems having output signals that are similar to that of the original closed-loop control system, and in the error system having small output signals. However, the removal of the three nodes from the network system results in the reduced closed-loop control system and the error system having output signals that are significantly deteriorated.

ii. When applying approach (ii), the removal of the nodes results in the reduced closed-loop control systems having output signals that are similar to that of the original closed-loop control system, and in the error system having relatively small output signals.

iii. When applying approach (iii), the removal of the state node from the

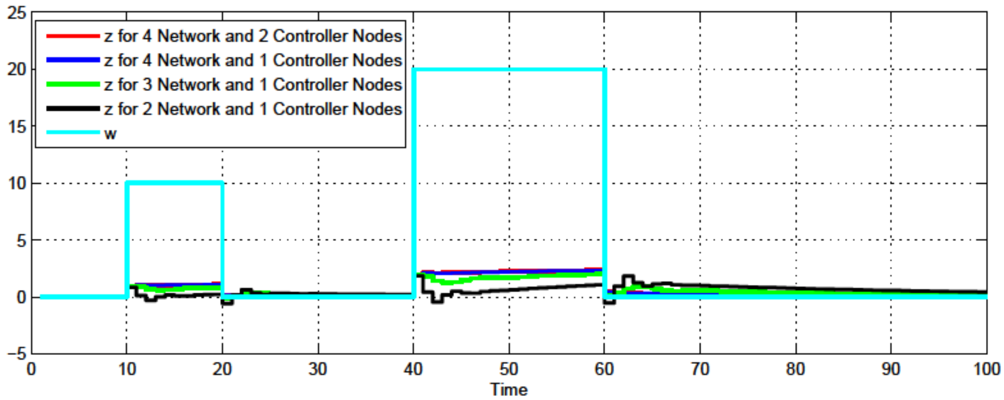

Figure 5.5: Simulation of the proposed topology under the removal of nodes and associated communication links by applying approach (ii) - the inputs and outputs of the original and reduced closed-loop control systems.

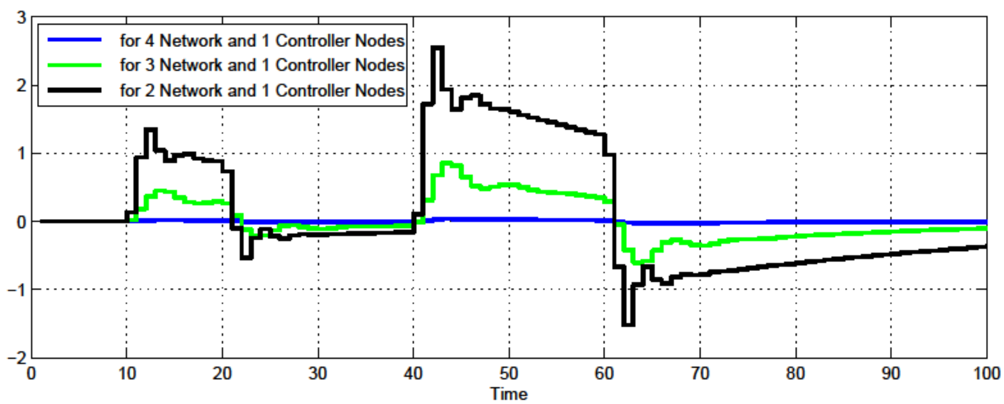

Figure 5.6: Simulation of the proposed topology under the removal of nodes and associated communication links by applying approach (ii) - the outputs of the error systems.

state nodes of the controller system and either the single node or the three nodes from the network system results in the reduced closed-loop control systems having output signals that are similar to that of the original closed-loop control system, and in the error system having relatively small output signals. However, the removal of the state node from the state nodes of the controller system and the two nodes from the network system results in the reduced closed-loop control system having a slightly different output signal than that of the original closed-loop control system, and in the error system having a slightly large output signal.

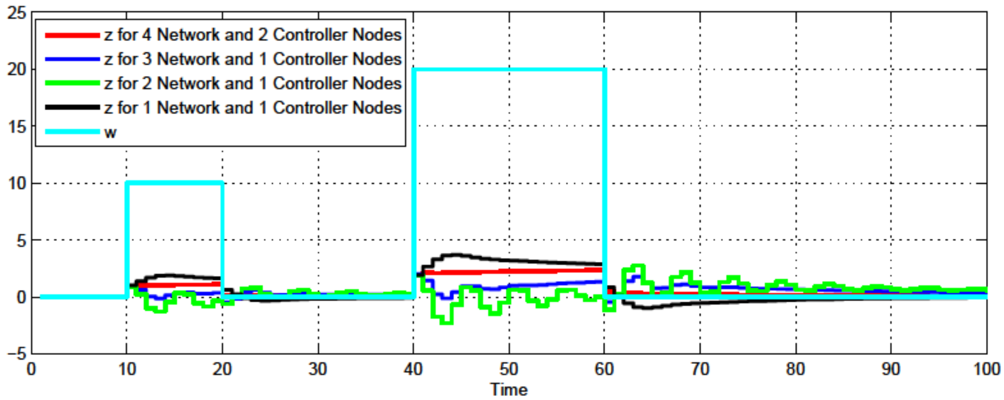

Figure 5.7: Simulation of the proposed topology under the removal of nodes and associated communication links by applying approach (iii) - the inputs and outputs of the original and reduced closed-loop control systems.

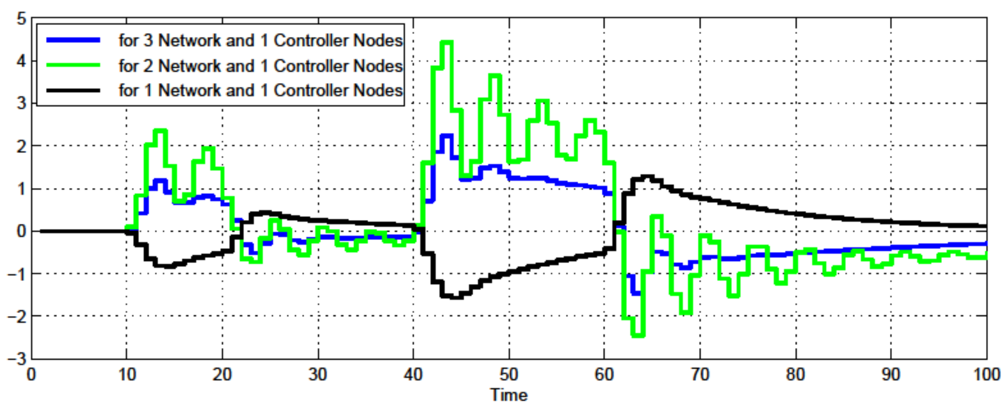

Figure 5.8: Simulation of the proposed topology under the removal of nodes and associated communication links by applying approach (iii) - the outputs of the error systems.

As can be observed, the removal of different sets of nodes has a different impact. Thus, the selection of the nodes and associated communication links to be removed from the proposed topology should receive careful consideration. In addition, it should be noted that although the approaches and the respective algorithms allowed for the removal of nodes and communication links, their improvement can be further investigated. More specifically, the selected node and pair of nodes to be removed at each iteration step provide the best solution at that given iteration step. However, this does not guarantee that the selected nodes to be removed based on all iteration steps provide the best solution. Therefore, alternative approaches can be studied to provide more optimal results.

5.3.2 Separation of the Network System of the Topology

In this section, the design of the proposed extended topology is demonstrated. First, consider a closed-loop control system that consists of the following systems:

- A second-order plant system G^1 that has a single actuator node $\mathcal{A}^1 = \{a_1^1\}$ and a single sensor node $\mathcal{S}^1 = \{s_1^1\}$ as presented in Section 2.5, and a respective second-order controller system K^1 that has a single input node $\Gamma^1 = \{\gamma_1^1\}$ and a single output node $\Theta^1 = \{\theta_1^1\}$;

- A third-order plant system G^2 that has two actuator nodes $\mathcal{A}^2 = \{a_1^2, a_2^2\}$ and two sensor nodes $\mathcal{S}^2 = \{s_1^2, s_2^2\}$ whose system matrices are given as

$$A = \begin{bmatrix} 0.2 & 0.1 & 0.6 \\ 0.1 & 1 & 0 \\ 1 & 1.1 & 0.2 \end{bmatrix}, B_1 = \begin{bmatrix} 0.3 & 0.5 \\ 0.2 & 0.5 \\ 0 & 0.3 \end{bmatrix}, B_2 = \begin{bmatrix} 0 & 0.1 \\ 0.3 & 0.1 \\ 0.2 & 0.2 \end{bmatrix},$$
$$C_1 = \begin{bmatrix} 0.1 & 0.3 & 0.3 \\ 0.6 & 0.2 & 0.4 \end{bmatrix}, C_2 = \begin{bmatrix} 0 & 0.6 & 0.1 \\ 0.2 & 0.4 & 0.4 \end{bmatrix},$$
$$D_{11} = \begin{bmatrix} 0.2 & 0.4 \\ 0.4 & 0.4 \end{bmatrix}, D_{12} = \begin{bmatrix} 0.4 & 0.2 \\ 0.4 & 0.4 \end{bmatrix}, D_{21} = \begin{bmatrix} 0 & 0.3 \\ 0.5 & 0.2 \end{bmatrix}, D_{22} = \mathbf{0},$$

  and a respective first-order controller system with a single input node $\Gamma^2 = \{\gamma_1^2\}$ and a single output node $\Theta^2 = \{\theta_1^2\}$; and

- A second-order transfer network system with the set of nodes $\mathcal{V}_T = \{v_{T_1}, v_{T_2}\}$ and a second-order receiving network system with the set of nodes $\mathcal{V}_R = \{v_{R_1}, v_{R_2}\}$.

Algorithms 5.3 and 5.4 are implemented using MATLAB's Robust Control Toolbox to compute the controller parameter $\mathcal{K}$ in (5.10), the transfer network parameter $\mathcal{T}$ in (5.9), and the receiving network parameter $\mathcal{R}$ in (5.11), while setting $\beta = 0.9$. The resulting convergence thresholds (namely, representing the approximate lowest values which were found using a bisection approach) are presented in the following table; they were obtained when no significant change occurs in a short period of time.

The weights assigned to the transfer of information between the nodes of the transfer network system and between the nodes of the receiving network

Table 5.2: Results of the joint design of the transfer network, controller, and receiving network systems

Iteration Step	ϵ_T	ϵ_K	ϵ_R
1	4.634	–	–
2	–	3.9333	–
3	–	–	3.0478
4	2.6978	–	–
5	–	2.6869	–
6	–	–	2.568
7	2.4834	–	–
8	–	2.4658	–
9	–	–	2.40771

system are computed as

$$\Omega_T = \begin{bmatrix} -0.6766 & -1.4432 \\ -0.0241 & 0.4012 \end{bmatrix}, \Omega_R = \begin{bmatrix} 0.1473 & 0.0057 \\ -0.0871 & 0.4239 \end{bmatrix},$$

and the weights assigned to the transfer of information between the nodes of the first pair of plant and controller systems (namely, G^1 and K^1) and those of the transfer and receiving network systems are computed as

$$\Lambda_T^1 = \begin{bmatrix} 0.0740 \\ -0.0083 \end{bmatrix}, \Upsilon_T^1 = \begin{bmatrix} 0.0131 & -0.8965 \end{bmatrix}, \Xi_T^1 = -0.0316,$$
$$\Lambda_R^1 = \begin{bmatrix} -0.1463 \\ -0.0184 \end{bmatrix}, \Upsilon_R^1 = \begin{bmatrix} -0.6807 & -7.4188 \end{bmatrix}, \Xi_R^1 = 4.6553,$$

and the weights assigned to the transfer of information between the nodes of the second pair of plant and controller systems (namely, G^2 and K^2) and those of the transfer and receiving network systems are computed as

$$\Lambda_T^2 = \begin{bmatrix} -0.7039 & -0.3134 \\ 0.0168 & 0.0536 \end{bmatrix} \Upsilon_T^2 = \begin{bmatrix} -47.4329 & -57.2675 \end{bmatrix},$$
$$\Xi_T^2 = \begin{bmatrix} -29.9894 & -13.6862 \end{bmatrix}, \Lambda_R^2 = \begin{bmatrix} 0.0613 \\ -0.0412 \end{bmatrix},$$
$$\Upsilon_R^2 = \begin{bmatrix} 5.6884 & 11.5379 \\ -3.3764 & -30.6617 \end{bmatrix}, \Xi_R^2 = \begin{bmatrix} 0.3416 \\ 0.1440 \end{bmatrix}.$$

Further, the system matrices of the first and second controller systems K^1 and K^2 are computed as

$$A_K^1 = \begin{bmatrix} -0.6254 & 0.2679 \\ 0.4999 & -0.0996 \end{bmatrix}, B_K^1 = \begin{bmatrix} -0.2555 \\ 0.4038 \end{bmatrix},$$
$$C_K^1 = \begin{bmatrix} -1.1518 & -1.0893 \end{bmatrix}, D_K^1 = 1.0858,$$
$$A_K^2 = 0.2647,\ B_K^2 = -0.0146,\ C_K^2 = -0.1838,\ D_K^2 = 0.1124.$$

The design of the controller, transfer network, and receiving network systems delivers a closed-loop control system that is stable as well as optimal (namely, with respect to the energy-to-peak performance measure). Further, it should be noted that the use of different values of β in Algorithm 5.4 results in different designs of the controller, transfer network, and receiving network systems.

5.4 Summary

In this chapter, the design of the proposed topology under two additional specifications was considered. More specifically, the following were delivered:

- The design of the proposed topology by using a model reduction approach to remove nodes and associated communication links (namely, an alternative approach to that of Chapter 4 to utilize a reduced number of nodes and communication links); and

- The design of the proposed topology by segregating the set of distributed and inter-connected nodes of the network system into two independent sets of nodes, such that a set of nodes is responsible for the transfer of information from the plant system to the controller system and the other set of nodes is responsible for the transfer of information from the controller system to the plant system.

Also, the modelling of the closed-loop control system and the modelling framework to facilitate the design of the proposed topology were presented; and the design procedure of the proposed topology by using algorithms for computing its design variables was addressed.

Chapter 6

Conclusion and Future Work

6.1 Conclusion

In this thesis, an alternative control system topology was proposed for the control of connected disperse systems, and it was modelled, studied, and designed. The proposed topology consists of the following systems: (i) a plant system with sensor and actuator nodes, (ii) a controller system with input and output nodes, and (iii) an intermediate network system with distributed and inter-connected nodes. The nodes of the network system can route information as well as perform computational tasks, and they allow for the connectivity between very distant nodes of the proposed topology. Further, the plant system is regulated using the controller and network systems, in an individual or a cooperative manner. This introduces both centralized and decentralized control system paradigms, and therefore, more flexibility is offered in making control decisions. More specifically, the following were delivered for the proposed topology:

- Definitions of its different components;
- Modelling of the closed-loop control system, and modelling frameworks to facilitate its design;
- Conditions required to characterize the existence of its design and its behaviour; and
- Design procedures for computing its design variables.

Further, the deliverables were provided for different topics. Namely, the proposed topology was addressed under ideal operating scenarios, failures in the operation of the nodes and the transfer of information, cyber attacks on the nodes and the transfer of information, induced connectivity constraints, model reduction, as well as segregation of the nodes of the network system into two independent sets of nodes with different objectives.

6.2 Future Work

In this thesis, the presented topics address some important directions for the implementation of the proposed topology. Further, there are several other possible directions for the improvement of the proposed topology and its design; the topics considered as promising future directions are listed as follows.

- Secure control system design: To further mitigate the effects of cyber attacks in the proposed topology, its design while accounting for potential cyber attacks is desirable. For example, this can be achieved by detecting and removing attacked nodes and information, as well as reconfiguring the connectivity between the nodes during operation time.

- Event-triggered control system design: To further introduce savings in resources in the proposed topology, its design while accounting for an event-triggered operation is desirable. For example, this can be achieved by allowing for the transfer of information as well as the inclusion of nodes and communication links only when certain events occur during operation time.

- Dynamic topological formation design: To further enhance the formation of the proposed topology and provide optimal connectivity between the nodes, its design with dynamic clustering of nodes is desirable. For example, this can be achieved by allowing for nodes to enter and leave a formation of any cluster consisting of a subset of nodes during operation time.

- Autonomous control system design: To further improve the distributed making of control decisions of the proposed topology, its design with more autonomous nodes is desirable. For example, this can be achieved by incorporating intelligent nodes that are capable of making control decisions in a more independent manner with less information received from neighbouring nodes during operation time.

- Communication induced optimal design: To further the flexibility and robustness of the proposed topology, its design while accounting for communication induced characteristics is desirable. For example, this can be achieved by accounting for any time delays in the transfer of information between any nodes which only route information, as well as the communication and mobility range of the nodes to meet specific application requirements during operation time.

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