University of Alberta

Sign Test for Change-Point Problem

by

Xiongsheng Jin ©

A thesis

submitted to the Faculty of Graduate Studies and Research in

partial fulfillment of the requirements for the degree of

Master of Science

in

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Department of Mathematical Sciences

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ABSTRACT

The Sign test is employed to study the change-point problem with epidemic alternative. Discussions focus on the two different cases that under the null hypothesis the population median is known or unknown. The asymptotic distributions of the test statistic under the alternative hypothesis are proved. Numerical simulation is carried out to calculate the estimated change-points, test statistic values and their P-values.

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Chapter 1

INTRODUCTION

Change-point problem originally arose in the field of quality control. When one monitors the output in a production line, one wants to keep the quality of the product within a required region and to detect the quality deviation across the threshold values as soon as possible. In Statistics, this problem can usually be modeled as follows. We have a sequence of observations of independent random variables $x_1, x_2, \cdots$ of identical distribution and want to detect whether a change at time τ could have occured in this sequence and that after time $\tau, x_\tau, x_{\tau+1}, \cdots$ have different distribution as that of $x_1, x_2, \cdots, x_{\tau-1}$. We call this problem the change-point problem and the τ the change-point.

In change-point problem, one usually needs to consider following :

[1]Testing the hypotheses:

$$
\begin{aligned}
H_0 : &\quad x_1, \cdots, x_n, \cdots \ i.i.d \sim F(x) \qquad \textit{No change.} \\
H_1 : &\quad x_1, \cdots, x_{\tau-1} \ i.i.d \sim F(x) \qquad\qquad (1.1) \\
&\quad x_\tau, \cdots, x_n, \cdots \ i.i.d \sim G(x),\ F(x) \neq G(x) for\ some\ x.\ There\ is\ a\ change.
\end{aligned}
$$

τ is the unknown change-point.

[2]Employing a suitable statistic T_n for the test problem to obtain an estimator $\hat{\tau}(n)$ for the unknown change-point τ.

[3]Discussing the properties of T_n and $\hat{\tau}(n)$ and carrying out some numerical simulations to confirm the theoretical results.

For the test hypotheses of change-point problem, usually we assume that $x_1, \cdots, x_n, \cdots$ are independent continuous random variables. Beside (1.1), there are many special forms to e the test hypotheses. For example, the test for change in the location par er can be written as

$$\begin{aligned} H_0 : &\ \ x_1, \cdots, x_n \ i.i.d \sim F(x) \\ H_1 : &\ \ x_1, \cdots, x_{\tau-1} \ i.i.d \sim F(x); \\ &\ \ x_\tau, \cdots, x_n \ i.i.d \sim F(x+\Delta), \qquad -\infty < \Delta < +\infty. \end{aligned} \tag{1.2}$$

(1.2) is equivalent to:

$$\begin{aligned} H_0 : &\ \ \Delta = 0 \\ H_1 : &\ \ \Delta \neq 0 \end{aligned} \tag{1.3}$$

Sometimes people look at

$$\begin{aligned} H_0 : &\ \ \mu_1 = \cdots = \mu_m = \mu_0, \quad \mu_i = E(x_i) \\ H_1 : &\ \ \mu_1 = \cdots = \mu_{\tau-1} \neq \mu_\tau = \cdots = \mu_n \end{aligned} \tag{1.4}$$

In these above models, one just considers the alternative hypothesis of at most one change, the so called AMOC model. A slight generalization of the AMOC model, that is often very useful, is the more than one change point model with the epidemic or square wave alternative.

$$\begin{aligned} H_0 &: \ x_1, \cdots, x_n \ i.i.d \sim F(x) \\ H_1 &: \ x_1, \cdots, x_{\tau_1 - 1}, x_{\tau_2}, \cdots, x_n \ i.i.d \sim F(x) \\ & \quad x_{\tau_1}, \cdots, x_{\tau_2 - 1} \ i.i.d \sim G(x), \ F(x) \neq G(x), \end{aligned} \tag{1.5}$$

where τ_1,τ_2 are unknown change-points.

There are also many different viewpoints used in change-point research. When one observes the output in a production line, one can use a sequential procedure where one observes the products sequentially and stops the line at a random time when one detects a change in quality, or fixed sample size procedure also called retroactive change-point detection procedure where one observes a large finite sequence of output such as the product produced in a day to determine possible change within the collection. People also use classical and Bayesian approaches, parametric and nonparametric models for change-point problem. So, there has been much research done for the change-point problem with the combination of different methods and models.

The basic AMOC problem was first considered by Page (1954, 1955) in the model (1.2). Assuming the initial value μ_0 known, Page studied testing the null hypothesis of no change ($H_0 : \Delta = 0$) against either one or two sided alternatives ($H_1 : \Delta > 0$ *or* $H_1 : \Delta \neq 0$). Let $S_0 = 0$ and $S_k = \sum_{j=1}^{k} V_j$ $k = 1, \cdots, n$

$$V_j = \begin{cases} a & \text{if } x_j \geq \mu_0, \\ b & \text{if } x_j < \mu_0 \end{cases}$$

where $a > 0$, $b > 0$ are constants, such that $E_{\mu_0}(V_j) = 0$, $j = 1, \cdots, n$. Page's decision rule rejects $H_0 : \Delta = 0$ in favour of the alternative of one change

$H_1 : \Delta > 0$, if

$$T_n = max_{0\le k\le n}\{S_k - min_{0\le i\le k} S_i\} \tag{1.6}$$

is too large.

S. Csörgő and Horváth (1983) calculated the limit distribution of T_n.

$$\begin{aligned} lim_{n\to\infty}\{T_n/(nab)^{\frac{1}{2}} < x\} &= P\{sup_{0\le t\le 1}|w(t)| \le x\} \qquad (1.7) \\ &= 1 - 4\sum_{k=1}^{\infty}(-1)^{k+1}\Phi(-(2k-1)x), \qquad x > 0. \end{aligned}$$

where w(t) is a Weiner process and Φ is the standard normal distribution function. A table for this limit distribution was also given.

G. K. Bhattacharyya and Johnson (1968) considered a general class of locally optimal rank tests for the change-point problem in the following two cases:

1 The initial distribution F_0 is known and symmetric around the origin. Testing of these hypothesis corresponds to the shift problem in model (2) with unknown $\Delta > 0$. Bhattacharyya and Johnson employed the criterion of maximizing the average local power $\overline{\beta(\Delta)} = \sum_{i=1}^{n} q_i\beta(\Delta|i)$ with respect to arbitrary set of weights q_i that satisfies $q_1 = 0,\ q_i \ge 0,\ i = 2, \cdots, n$ and $\sum_{i=1}^{n} q_i = 1$ to get a nonparametric statistic

$$T_n = \sum_{i=1}^{n} Q_i sgn(x_i) E\{-f_0'(V^{(R_i)})/f_0(V^{(R_i)})\} \tag{1.8}$$

to reject H_0 at large value of T_n. Here $V^{(1)} \le \cdots \le V^{(n)}$ is an ordered statistic of n i.i.d random variables having a distribution F_0, $Q_i = \sum_{j=1}^{i} q_i$,

$(R_1, \cdots R_n)$ is the vector of the rank of $(x_1, \cdots x_n)$ and $\beta(\Delta|i)$ is the power at Δ when the change occurs at time i. From the Bayesian viewpoint, q_i may be regarded as the prior probability of a change to occur at time i.

2 When initial level is unknown, they proposed the

$$S_n = \sum_{i=1}^{n} Q_i E\{-f'(V^{(R_i)})/f(V^{(R_i)})\}, \qquad (1.9)$$

and suggested to reject H_0 for larger value of S_n. In both cases, the tests are distribution free, they depend upon the weight function $\{q_i\}$ and are unbiased for general classes of shift alternatives. The asympototic distribution of the test statistic under the local translation alternative was also reached.

A. Sen and Srivastava (1975) proposed two nonlinear rank test for one-sided alternative with $x_i \sim N(\mu_i, \sigma^2)$, $i = 1, \cdots n$ and unknown initial μ_0 and σ^2. They suggested rejecting $H_0 : \Delta = 0$ in favor of $H_1 : \Delta > 0$ for a large value of

$$I_1 = max_{1 \le k \le n-1}\{[M_{k,n-k} - E_0(M_{k,n-k})]/[Var_0(M_{k,n-k})]^{\frac{1}{2}}\}, \qquad (1.10)$$

or

$$I_2 = max_{1 \le k \le n-1}\{[U_{k,n-k} - E_0(U_{k,n-k})]/[Var_0(U_{k,n-k})]^{\frac{1}{2}}\}, \qquad (1.11)$$

where

$$\begin{aligned} M_{k,n-k} &= \textstyle\sum_{i=k+1}^{n} \Psi\{x_i - median_{1 \le j \le n}(x_j)\}, \\ U_{k,n-k} &= \textstyle\sum_{i=k+1}^{n} \sum_{j=1}^{k} \Psi(x_i - x_j), \end{aligned}$$

and

$$\Psi(t) = \begin{cases} 1 & t > 0, \\ \\ 0 & t \leq 0, \end{cases}$$

$E_0(\cdot)$, $Var_0(\cdot)$ above denotes the mean and variance taken under null hypothesis respectively. Some Monte Carlo simulations for the estimated critical values were also provided.

A model that is similar to A.Sen and Srivastava's was studied by Hawkins(1977) for the two-sided alternative hypothesis. He provided the test statistic

$$U_n = max_{1 \leq k \leq n-1} \mid T_k \mid, \tag{1.12}$$

where

$$T_k = (\frac{n}{k(n-k)})^{\frac{1}{2}} \sum_{i=1}^{n}(x_i - \overline{x}_n), \qquad k = 1, \cdot \ \cdot, n-1.$$

The recursive formulae for the exact determination of the distribution of U_n were also proved. With the normality of $T_1, \cdots, T_n$, he got the asympototic distribution of U_n from the behavior of the maximum properties of a Gaussian process.

Pettitt (1979) proposed quite similar statistic to that of A. Sen and Srivastava for the one and two-sided tests. For the one-sided test: $H_0 : \Delta = 0$ vs $H_1 : \Delta > 0$, he suggested the statistic

$$\begin{aligned} J_1 &= min_{1 \leq k \leq n-1}\{\sum_{i=1}^{k} \sum_{j=k+1}^{n} sgn(x_i - x_j)\} \\ &= min_{1 \leq k \leq n-1}\{V_{k,n}\}. \end{aligned} \tag{1.13}$$

for the test and rejected H_0 for its large value. Here $V_{k,n} = \sum_{i=1}^{k} \sum_{j=k+1}^{n} sgn(x_i - x_j)$.

Pettitt proposed rejecting $H_0 : \Delta = 0$ in favour of the two-sided alternative $H_1 : \Delta \neq 0$ for large values of

$$J_2 = max_{1\leq k\leq n-1}|\sum_{i=1}^{k}\sum_{j=k+1}^{n} sgn(x_i - x_j)|. \tag{1.14}$$

Pettitt proved that the limit distribution of

$$y_n(x) = n^{-1}\{\frac{3}{n+1}\}^{\frac{1}{2}}V_{k,n} \tag{1.15}$$

is a Brownian bridge $y(x)$ and we know that

$$P\{sup \mid y(x) \mid\leq a\} = 1 + 2\sum_{r=1}^{\infty}(-1)^r exp(-2r^2a^2).$$

This is the limiting distribution of the Kolmogorov-Smirnov goodness of fit statistic and is extensively tabulated.

Comparing statistic I_2 and J_2, Schechtman and Wolfe (1981) proposed the following statistic

$$I_3 = max_{1\leq k\leq n-1}\{|U_{k,n-k} - E_0(U_{k,n-k})|/[Var_0(U_{k,n-k})]^{\frac{1}{2}}\} \tag{1.16}$$

for the two-sided test to reject $H_0 : \Delta = 0$ in favor of $H_1 : \Delta \neq 0$ for large value of I_3. The asymptotic properties of I_3 were also studied.

Lombard (1987) studied the smooth change model:

$H_0 : \quad \mu_1 = \cdots = \mu_n = \xi_1 \qquad$ *No change.*

$H_1 : \quad \mu_i = \theta_i \qquad i = 1, \cdots, n$

$$\theta_i = \begin{cases} \xi_1 & i \leq \tau_1 \\ \xi_1 + (i - \tau_1)(\xi_2 - \xi_1)/(\tau_2 - \tau_1) & \tau_1 < i \leq \tau_2 \\ \xi_2 & i > \tau_2. \end{cases} \tag{1.17}$$

He considered the statistic

$$q_n = \sum_{t_1=1}^{n-1} \sum_{t_2=t_1+1}^{n} \{V^*_{t_1,t_2}\}^2$$

where

$$V^*_{t_1,t_2} = \textstyle\sum_{j=t_1+1}^{t_2} \sum_{i=1}^{j} S(r_i),$$

$$S(r_i) = \{\Phi[i/(n+1)] - \overline{\Phi}\}/A,$$

$$\overline{\Phi} = n^{-1} \textstyle\sum_{i=1}^{n} \phi[i/(n+1)],$$

$$(n-1)^{-1} \textstyle\sum_{i=1}^{n} \{\phi[i/(n+1)] - \overline{\Phi}\}^2,$$

ϕ is an arbitrary score function satisfying $0 < \int_0^1 \phi^2 du < \infty$, and $r_1, \cdots, r_n$ are the ranks of $x_1, \cdots, x_n$, respectively. As $n \to \infty$, the null distribution of $n^{-5}q_n$ goes to the random variable $q = \sum_{n=1}^{\infty}(n\pi)^{-4}{Z_n}^2$ where $Z_1, \cdots, Z_n, \cdots$ are i.i.d $N(0,1)$ random variables. For the onset of trend model with $\tau_2 = n$, Lombard's statistics is ${q_n}^* = \sum_{t=1}^{n-1}\{V_{t,n}\}^2$. As $n \to \infty$,the null distribution of the $T^{-4}{q_n}^*$ approaches that of the random variable $q^* = \sum_{n=1}^{\infty} \lambda_n {Z_n}^2$ where $\lambda_1 > \lambda_2 > \cdots > 0$ is the positive real solution of the equation $tan\lambda^{-\frac{1}{4}} + tanh\lambda^{-\frac{1}{4}} = 0$. The AMOC model and multiple change point model were also discussed by Lombard.

For model (1.3) or (1.4), we have a general statistic to reject H_0 in favour of H_1 for large values of

$$max_{1 \leq k \leq n}\{\mid S_k - kS_n \mid /(k(1-k/n))^{\frac{1}{2}}\}. \tag{1.18}$$

Csörgö and Horváth (1986) studied the above statistic by considering

$$Z_n(t) = \begin{cases} (S_{[(n+1)t]} - [(n+1)t]S_n/n)/(n^{\frac{1}{2}}\sigma) & 0 \leq t < 1, \\ 0 & t = 1, \end{cases} \tag{1.19}$$

where $\sigma^2 = E(x_1 - E(x_1))^2$.

They proved that the process $Z_n(t)$, $(0 \leq t \leq 1)$ has the same asymptotic behaviour as the uniform quantile and empirical processes. Many asympototic properties of nonparametric statistics were also given in their paper.

Gombay (1994) considered rank and sign stastistic for the epidemic alternative model of (1.5). She suggested the statistic

$$T_n = max_{k<l}|n^{\frac{1}{2}}\sum_{i=k}^{l-1} S(R_i)|.$$

for the rank test and proved the asymptotic distribution of T_n:

$$lim_{n\to\infty}P\{T_n \leq c\} = 1 - \sum_{j=1}^{\infty} 2(4j^2c^2 - 1)e^{-2j^2c^2}. \qquad (1.20)$$

The asympototic consistency of T_n was proved under some regularity conditions.

For the sign statistic, Gombay proposed the statisitc

$$U_n = max_{1\leq k<l\leq n}\sum_{i=k}^{l-1} sgn(x_i - \xi_0). \qquad (1.21)$$

for the test:

$$H_O: \quad x_i,\ i = 1,\cdots,n\ have\ known\ median\ \xi_0$$

$$H_1: \quad x_i,\ i = 1,\cdots,\tau_1 - 1,\tau_2,\cdots,n\ have\ median\ \xi_0$$

$$x_i,\ i = \tau_1 - 1,\cdots,\tau_2\ have\ median\ \xi_1,\ \xi_0 \neq \xi_1.$$

Gombay suggested that H_0 be rejected for large values of U_n when $\xi_1 > \xi_2$, and similarly for the $\xi_1 < \xi_0$ case. She also got the asymptotic distribution under the null hypothesis of the U_n

$$lim_{n\to\infty}P\{n^{-\frac{1}{2}}U_n \geq c\} = 1 - \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}exp\{-\pi^2\frac{2k+1^2}{8c^2}\}, \quad c > 0. \quad (1.22)$$

Based on the research of Gombay, I consider the sign statistic for the epidemic alternative hypothesis. When ξ_0 is known, the asymptotic distribution under the alternative hypothesis is proved.

When the initial value ξ_0 is unknown, I suggest the use of the statistic

$$M(n) = max_{1 \leq k < l < n} n^{-\frac{1}{2}} | \sum_{k \leq i < l} sgn(x_i - \hat{\xi}_n)|.$$

where $\hat{\xi}_n = median(x_1, \cdots, x_n)$. The exact distribution of $n^{\frac{1}{2}} M(n)$ under H_0 is same as that of the maximum deviation in a simple symmetric random walk and it has been calculated exactly for each sample size.

To estimate the change-points, I use

$$(\hat{\tau}_1(n), \hat{\tau}_2(n)) = argmax_{k<l} |n^{-\frac{1}{2}} \sum_{i=k}^{l-1} sgn(x_i - \hat{\xi}_n)|.$$

as the estimatiors of τ_1,τ_2.

Finally, under H_1, I proved the asymptotic normality of $M(n)$ and

$$|\hat{\tau}_1(n) - \tau_1| + |\hat{\tau}_2(n) - \tau_2| = O_p(1)$$

In the third part, I calculated the power of sign test and do some numerical simulations.

Chapter 2

SIGN TEST FOR THE CHANGE-POINT PROBLEM WITH KNOWN INITIAL MEDIAN

Let $x_1, \cdots, x_n$ be a sequence of independent continuous random variables. Consider the following hypothesis test with two change-points :

$$\begin{aligned} H_O: &\quad x_i,\ i = 1, \cdots, n \ \textit{have known median}\ \xi_0, \\ H_1: &\quad x_i,\ i = 1, \cdots, \tau_1 - 1, \tau_2, \cdots, n \ \textit{have median}\ \xi_0, \qquad (2.1) \\ &\quad x_i,\ i = \tau_1, \cdots, \tau_2 - 1 \ \textit{have median}\ \xi_1, \quad \xi_1 \neq \xi_0, \end{aligned}$$

where ξ_0 is known. The unknown integers τ_1, τ_2 are the change-points. We assume $\tau_1 = [n\lambda_1]$, $\tau_2 = [n\lambda_2]$ for some $0 < \lambda_1 < \lambda_2 < 1$. By [a], we denote the integer part of a.

We employ the sign statistic for our test problem. Let

$$\begin{aligned} U_n &= max_{k<l} \sum_{i=k}^{l-1} sgn(x_i - \xi_0) \\ &= max_{k<l}(S_{l-1} - S_{k-1}). \qquad (2.2) \end{aligned}$$

where $S_k = \sum_{i=1}^{k} sgn(x_i - \xi_0)$.

Under H_0, Gombay (1994) proved that the exact distribution of U_n is

$$P\{U_n \geq N\} = 1 - \frac{2}{2N+1}\sum_{j=1}^{2N}(c(j))^n s(j(N+1))\frac{1+c(j)}{s(j)}\frac{1-(-1)^j}{2}, \qquad (2.3)$$

where N is a positive integer and

$$c(j) = cos(\frac{j\pi}{2N+1}), \qquad s(j) = sin(\frac{j\pi}{2N+1}).$$

Also, the asymptotic distribution of the test statistic under the null hypothesis H_0 was shown to be

$$\lim_{n\to\infty} P\{n^{-\frac{1}{2}}U_n \geq c\} = 1 - \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}exp(-\pi^2\frac{(2k+1)^2}{8c^2}), \qquad (2.4)$$

for all $c > 0$.

Denote $\delta_n = P\{x_{\tau_1} > \xi_0\}$. We consider the following two cases of alternatives.

$$\begin{aligned} &(i): \quad \delta_n = \delta \qquad for\ all\ n \\ &(ii): \quad \delta_n \to \tfrac{1}{2} \ \ and \ \ \sqrt{n}|\delta_n - \tfrac{1}{2}| \to \infty. \end{aligned} \qquad (2.5)$$

Case (i) is the fixed alternative, while case (ii) is the local but not contiguous alternative.

Assume $\delta_n > \frac{1}{2}$, as the other case is similar by symmetry.

Lemma 2.1 *Under the hypothesis H_1:*

$$P\{max_{1\leq l<\tau_2}(S_l - S_{\tau_2-1}) \geq max_{\tau_2\leq l}(S_l - S_{\tau_2-1})\} \to 0, \qquad as\ n\to\infty. \quad (2.6)$$

Proof: We want to prove the lemma by considering l in different areas for the first term and keeping the second term unchanged.

(a) $l < \tau_1$

$$max_{l<\tau_1}(S_l - S_{\tau_2-1}) = max_{l<\tau_1}[(S_l - S_{\tau_1-1}) + (S_{\tau_1-1} - S_{\tau_2-1})].$$

Note that $S_l - S_{\tau_1-1}$, $l < \tau_1$ is a simple symmetic random walk.

In the sum $S_{\tau_1-1} - S_{\tau_2-1}$, the terms are

$$sgn(x_i - \xi_0) = \begin{cases} 1 & \text{w.p} \quad \delta_n \qquad \tau_1 \leq i < \tau_2 \\ -1 & \text{w.p} \quad 1-\delta_n \end{cases} \tag{2.7}$$

so

$$\begin{aligned} E(sgn(x_i - \xi_0)) &= \delta_n + (-1)(1-\delta_n) \\ &= 2\delta_n - 1 > 0 \qquad \tau_1 \leq i < \tau_2 \end{aligned} \tag{2.8}$$

$$\begin{aligned} Var(sgn(x_i - \xi_0)) &= E[(sgn(x_i - \xi_0))^2] - (Esgn(x_i - \xi_0))^2 \\ &= \delta_n + (1-\delta_n) - (1-2\delta_n)^2 \\ &= 4\delta_n(1-\delta_n) \qquad \tau_1 \leq i < \tau_2 \end{aligned} \tag{2.9}$$

from (2.8), (2.9) we have

$$\begin{aligned} E(S_{\tau_1-1} - S_{\tau_2-1}) &= -(\tau_2 - \tau_1)(2\delta_n - 1) \\ &< 0 \end{aligned} \tag{2.10}$$

$$Var(S_{\tau_1-1} - S_{\tau_2-1}) = (\tau_2 - \tau_1)4\delta_n(1-\delta_n) \tag{2.11}$$

Employ $C.L.T$ to $S_{\tau_1-1} - S_{\tau_2-1}$

$$\frac{1}{\sqrt{(\tau_2-\tau_1)4\delta_n(1-\delta_n)}}[S_{\tau_1-1} - S_{\tau_2-1} + (\tau_2-\tau_1)(2\delta_n - 1)] \xrightarrow{D} N(0,1) \qquad n \to \infty \tag{2.12}$$

Note that $\tau_2 - \tau_1 = n(\lambda_2 - \lambda_1)$.

Under the condition of (i) or (ii) of (2.5)

$$-\frac{(\tau_2-\tau_1)(2\delta_n-1)}{\sqrt{(\tau_2-\tau_1)4\delta_n(1-\delta_n)}} \to -\infty, \qquad n \to \infty, \tag{2.13}$$

so

$$\frac{S_{\tau_1-1} - S_{\tau_2-1}}{\sqrt{(\tau_2-\tau_1)4\delta_n(1-\delta_n)}} \tag{2.14}$$

has mean that converges to $-\infty$ and it has asymptotic variance 1.

According to Billingsley (1968), the properties of the simple random walk $S_l - S_{\tau_1-1}$ are

$$\frac{1}{\sqrt{\tau_1-1}} max_{l\le\tau_1}(S_l - S_{\tau_1-1}) \sim |N(0,1)|, \tag{2.15}$$

hence

$$\begin{aligned} &\frac{1}{\sqrt{(\tau_2-\tau_1)4\delta_n(1-\delta_n)}} max_{l\le\tau_1}(S_l - S_{\tau_1-1}) \\ &= \frac{\sqrt{\tau_1-1}}{\sqrt{(\tau_2-\tau_1)4\delta_n(1-\delta_n)}}\frac{1}{\sqrt{\tau_1-1}} max_{l\le\tau_1}(S_l - S_{\tau_1-1}) \\ &= O_p(1). \end{aligned} \tag{2.16}$$

Similarly, for the simple random variable $S_l - S_{\tau_2-1}, \tau_2 \le l \le n$,

$$\begin{aligned} &\frac{1}{2\sqrt{(\tau_2-\tau_1)\delta_n(1-\delta_n)}} max_{\tau_2\le l}(S_l - S_{\tau_2-1}) \\ &= \frac{\sqrt{n-\tau_2+1}}{2\sqrt{(\tau_2-\tau_1)\delta_n(1-\delta_n)}}\frac{1}{\sqrt{n-\tau_2+1}} max_{\tau_2\le l}(S_l - S_{\tau_2-1}) \\ &= O_p(1). \end{aligned} \tag{2.17}$$

Combine (2.14) (2.16) (2.17), we get

$$P\{max_{l<\tau_1}(S_l - S_{\tau_2-1}) \geq max_{\tau_2\leq l}(S_l - S_{\tau_2-1})\} \to 0 \qquad as\ n \to \infty \quad (2.18)$$

(*b*) $\tau_1 \leq l < \tau_2$.

First consider the case (i) of (2.5), that is when $\delta_n = \delta$ for all n. It is a well known fact (see, page 72 in Billingsley (1968)) that

$$\frac{1}{b_n} max_{\tau_2\leq l\leq n}(S_l - S_{\tau_2-1}) \xrightarrow{D} |N(0,1)|, \qquad n \to \infty, \quad (2.19)$$

where $b_n = \sqrt{n - \tau_2 + 1}$.

Let $\{\epsilon_n\}$ such that $\epsilon_n b_n \to \infty$ and $\epsilon_n \to 0$.

Denote $A = \{max_{\tau_1\leq l<\tau_2}(S_l - S_{\tau_2-1}) \geq max_{\tau_2\leq l\leq n}(S_l - S_{\tau_2-1})\}$.

Then we have

$$P\{\frac{1}{b_n} max_{\tau_2\leq l\leq n}(S_l - S_{\tau_2-1}) < \epsilon_n\} = O(\epsilon_n). \quad (2.20)$$

and

$$\begin{aligned} &P\{max_{\tau_1\leq l<\tau_2}(S_l - S_{\tau_2-1}) \geq max_{\tau_2\leq l\leq n}(S_l - S_{\tau_2-1})\} \\ &= P\{A \cap (\tfrac{1}{b_n} max_{\tau_2\leq l\leq n}(S_l - S_{\tau_2-1}) \geq \epsilon_n)\} \\ &\quad + P\{A \cap (\tfrac{1}{b_n} max_{\tau_2\leq l\leq n}(S_l - S_{\tau_2-1}) < \epsilon_n)\} \qquad (2.21) \\ &\leq P\{max_{\tau_1\leq l<\tau_2}(S_l - S_{\tau_2-1}) \geq b_n\epsilon_n\} \\ &\quad + P\{\tfrac{1}{b_n} max_{\tau_2\leq l\leq n}(S_l - S_{\tau_2-1}) < \epsilon_n\}. \end{aligned}$$

The first term in (2.21) can be written as

$$P\{\max_{1\leq j\leq(\tau_2-\tau_1)} \sum_{i=1}^{j} \eta_i \geq b_n\epsilon_n\}, \quad (2.22)$$

where

$$\eta_i = -sgn(x_{i+\tau_1-1} - \xi_0) \qquad 1 \leq i \leq (\tau_2 - \tau_1),$$

Denote $\tilde{S}_j(\omega) = \sum_{i=1}^{j} \eta_i(\omega)$.

By the Strong Law of Large Numbers

$$P\{\omega : \lim_{j\to\infty} \frac{\tilde{S}_j(\omega)}{j} = a\} = 1 \tag{2.23}$$

where $a = E\eta_i = 1 - 2\delta < 0$.

i.e. $\exists \Omega_0 \subset \Omega$, $P(\Omega_0) = 0$ s.t. for all $\omega \notin \Omega_0$

$$\frac{\tilde{S}_j(\omega)}{j} \to a < 0, \qquad j \to \infty, \tag{2.24}$$

i.e. for all $\omega \notin \Omega_0$, $\exists j_0 = j_0(\omega)$ s.t.

$$\tilde{S}_j(\omega) < 0, \qquad for\ all\ j \geq j_0. \tag{2.25}$$

Combine (2.25) and $b_n\epsilon_n \to \infty$, for every $\omega \notin \Omega_0$

$$max_{1\leq j\leq j_0}\tilde{S}_j(\omega) < b_n\epsilon_n, \qquad when\ n\ is\ large. \tag{2.26}$$

so

$$P\{\omega : max_{1\leq j\leq(\tau_2-\tau_1-1)}\tilde{S}_j > b_n\epsilon_n\} \to 0, \qquad n \to \infty. \tag{2.27}$$

Using (2.20) and (2.27), we get

$$P\{max_{\tau_1\leq l<\tau_2}(S_l - S_{\tau_2-1}) \geq max_{\tau_2\leq l}(S_l - S_{\tau_2-1})\} \to 0, \qquad as\ n \to \infty. \tag{2.28}$$

The last case is $\tau_1 \leq l < \tau_2$ and $\sqrt{n}|\delta_n - \frac{1}{2}| \to 0$ as $n \to \infty$.

Let r satisfy $\lambda_1 < r < \lambda_2$ and rn is an integer.

$$max_{\tau_1 \le l \le rn}(S_l - S_{\tau_2 - 1}) = max_{\tau_1 \le l \le rn}(S_l - S_{rn}) + (S_{rn} - S_{\tau_2 - 1}). \qquad (2.29)$$

$$\begin{aligned} E(S_{rn} - S_{\tau_2-1}) &= (\tau_2 - 1 - rn)(1 - 2\delta_n) \\ &\approx (\lambda_2 - r)(1 - 2\delta_n)n. \end{aligned}$$

$$\begin{aligned} Var(S_{rn} - S_{\tau_2-1}) &= (\tau_2 - 1 - rn)4\delta_n(1 - \delta_n) \\ &\approx (\lambda_2 - r)4\delta_n(1 - \delta_n)n \\ &= dn. \end{aligned}$$

where $d = (\lambda_2 - r)4\delta_n(1 - \delta_n)$.

The same way as in (a) by C.L.T.

$$\frac{(S_{rn} - S_{\tau_2-1}) - (\tau_2 - 1 - rn)(1 - 2\delta_n)}{\sqrt{4\delta_n(1 - \delta_n)(\tau_2 - 1 - rn)}} \xrightarrow{D} N(0,1), \qquad n \to \infty. \qquad (2.30)$$

On the other hand, we have

$$\begin{aligned} &\tfrac{1}{\sqrt{dn}} max_{\tau_1 \le l \le rn}(S_l - S_{rn}) \\ &= \tfrac{\sqrt{r-\lambda_1}}{\sqrt{d}} \tfrac{1}{\sqrt{n(r-\lambda_1)}} max_{\tau_1 \le l \le rn}(S_l - S_{rn}) \qquad (2.31) \\ &= O_p(1), \end{aligned}$$

and

$$\begin{aligned} &\tfrac{1}{\sqrt{dn}} max_{\tau_2 \le l \le n}(S_l - S_{\tau_2-1}) \\ &= \tfrac{\sqrt{1-\lambda_2}}{\sqrt{d}} \tfrac{1}{\sqrt{n-\tau_2}} max_{\tau_2 \le l \le n}(S_l - S_{\tau_2-1}) \qquad (2.32) \\ &= O_p(1). \end{aligned}$$

The mean of the dominating term in (2.29) is

$$E\left(\frac{S_{rn} - S_{\tau_2 - 1}}{\sqrt{dn}}\right) = \frac{(\tau_2 - 1 - rn)(1 - 2\delta_n)}{\sqrt{dn}} \tag{2.33}$$
$$\rightarrow -\infty, \qquad as\ n \rightarrow \infty.$$

Join (2.30), (2.31), (2.32), (2.33), we have

$$P\{max_{\tau_1 \leq l \leq rn}(S_l - S_{\tau_2 - 1}) \geq max_{\tau_2 \leq l \leq n}(S_l - S_{\tau_2 - 1})\} \rightarrow 0, \qquad n \rightarrow \infty. \tag{2.34}$$

Now consider a sequence $r_{(n)} \nearrow \lambda_2$, $r_{(n)}n$ is an integer. We want to show that

$$P\{max_{r_{(n)}n \leq l < \tau_2}(S_l - S_{\tau_2 - 1}) \geq max_{\tau_2 \leq l}(S_l - S_{\tau_2 - 1})\} \rightarrow 0, \qquad n \rightarrow \infty. \tag{2.35}$$

Dividing by $\sqrt{n}$ on both sides of the inequality of (2.35) and employing (2.32), we can write (2.35) equivalently as

$$P\{\frac{1}{\sqrt{n}} max_{1 \leq j \leq \tilde{r}_{(n)}n} \tilde{S}_j \geq O_p(1)\} \rightarrow 0, \qquad n \rightarrow \infty. \tag{2.36}$$

where $\tilde{S}_j$ has independent, identically distributed terms with mean $(1 - 2\delta_n) < 0$ and finite variance $4\delta_n(1 - \delta_n)$ and $\tilde{r}_{(n)} = \frac{[\lambda_2 n]}{n} - r_{(n)} \searrow 0$. But

$$\begin{aligned} max_{1 \leq j \leq \tilde{r}_{(n)}n} \tilde{S}_j &\leq max_{1 \leq j \leq \tilde{r}_{(n)}n}(\tilde{S}_j + j(2\delta_n - 1)) \\ &= max_{1 \leq j \leq \tilde{r}_{(n)}n} \tilde{S}_j^{\,*}, \end{aligned} \tag{2.37}$$

where $\tilde{S}_j^{\,*}$ has terms with mean zero and

$$\begin{aligned} \frac{1}{\sqrt{n}} max_{1 \leq j \leq \tilde{r}_{(n)}n} \tilde{S}_j^{\,*} &= \frac{\sqrt{\tilde{r}_{(n)}n}}{\sqrt{n}} \frac{1}{\sqrt{\tilde{r}_{(n)}n}} max_{1 \leq j \leq \tilde{r}_{(n)}n} S_j^{\,*} \\ &= O_p(\sqrt{\tilde{r}_{(n)}}). \end{aligned} \tag{2.38}$$

From (2.36), (2.38), using that $\tilde{r}_{(n)} \rightarrow 0$, we can conclude (2.35) is true.

It remains to prove that

$$P\{\frac{1}{\sqrt{n}} max_{rn<l<r_{(n)}n}(S_l - S_{\tau_2-1}) \geq O_p(1)\} \rightarrow 0, \qquad n \rightarrow \infty. \tag{2.39}$$

We may write

$$\begin{aligned} & max_{rn<l<r_{(n)}n}\frac{1}{\sqrt{n}}(S_l - S_{\tau_2-1}) \\ & = max_{rn<l<r_{(n)}n}\frac{1}{\sqrt{n}}\{(S_l - S_{r_{(n)}n}) + (S_{r_{(n)}n} - S_{\tau_2-1})\} \qquad (2.40) \\ & \leq max_{rn<l<r_{(n)}n}\frac{1}{\sqrt{n}}\{(S_l - S_{r_{(n)}n} + (r_{(n)}n - l)(2\delta_n - 1)) + (S_{r_{(n)}n} - S_{[\lambda_2 n]-1})\}, \end{aligned}$$

Similarly as before

$$\begin{aligned} & max_{rn<l<r_{(n)}n}\tfrac{1}{\sqrt{n}}\{(S_l - S_{r_{(n)}n}) + (r_{(n)}n - l)(2\delta_n - 1)\} \\ & \quad = O_p(1), \qquad\qquad n \rightarrow \infty. \end{aligned} \tag{2.41}$$

we have

$$\begin{aligned} & E\{\tfrac{1}{\sqrt{n}}(S_{r_{(n)}n} - \quad {}_{\lambda_2 n]-1})\} \\ & \quad \approx \sqrt{n}(\lambda_2 - r_{(n)})(1 - 2\delta_n) < 0, \end{aligned} \tag{2.42}$$

and

$$\begin{aligned} & S.D.\{\tfrac{1}{\sqrt{n}}(S_{r_{(n)}n} - S_{[\lambda_2 n]-1})\} \\ & \quad \approx \sqrt{4\delta_n(1 - \delta_n)(\lambda_2 - r_{(n)})} \end{aligned} \tag{2.43}$$

Hence, if we choose a seguence $r_{(n)} \rightarrow \lambda_2$ such that we also have $\sqrt{n(\lambda_2 - r_{(n)})}(1 - 2\delta_n) \rightarrow -\infty$. We get that (2.39) holds.

Now consider

$$\begin{aligned} & max_{\tau_1 \leq l \leq \tau_2}(S_l - S_{\tau_2-1}) \\ & \quad = max\{max_{\tau_1 \leq l < rn}(S_l - S_{\tau_2-1}), \\ & \quad\ max_{rn \leq l < r_{(n)}n}(S_l - S_{\tau_2-1}), max_{r_{(n)}n \leq l < \tau_2}(S_l - S_{\tau_2-1})\} \\ & \quad = max\{A_1,\ A_2,\ A_3\}, \end{aligned} \tag{2.44}$$

where

$$\begin{aligned} A_1 &= max_{\tau_1 \leq l < \tau n}(S_l - S_{\tau_2 - 1}), \\ A_2 &= max_{\tau n \leq l < \tau_{(n)} n}(S_l - S_{\tau_2 - 1}), \\ A_3 &= max_{\tau_{(n)} n \leq l < \tau_2}(S_l - S_{\tau_2 - 1}). \end{aligned}$$

Also, denote $B = max_{\tau_2 \leq l}(S_l - S_{\tau_2 - 1})$ and we have

$$\begin{aligned} &P\{max(A_1,\ A_2,\ A_3) > B\} \\ &\quad = P\{A_1 > B \ or\ A_2 > B \ or\ A_3 > B\} \\ &\quad \leq P\{A_1 > B\} + P\{A_2 > B\} + P\{A_3 > B\} \\ &\quad \to 0, \qquad\qquad n \to \infty. \end{aligned} \tag{2.45}$$

Combining the above (2.34), (2.35) and (2.39), we get

$$P\{max_{\tau_1 \leq l < \tau_2}(S_l - S_{\tau_2 - 1}) \geq max_{\tau_2 \leq l}(S_l - S_{\tau_2 - 1})\} \to 0, \qquad n \to \infty, \tag{2.46}$$

as claimed, and the Lemma 2.1 is proved. □

Lemma 2.2 *Under hypothesis H_1:*

$$P\{max_{k \geq \tau_1}(S_{\tau_1} - S_k) \geq max_{k < \tau_1}(S_{\tau_1} - S_k)\} \to 0 \qquad as\ n \to \infty \tag{2.47}$$

Proof: It can be proved by using the method in the proof of Lemma 2.1 and observing the symmetry. □

To estimate τ_1, τ_2, it is customary to use

$$\begin{aligned} &(\hat{\tau}_1(n), \hat{\tau}_2(n)) \\ &\quad = argmax_{k<l}(S_{l-1} - S_{k-1}) \\ &\quad = \{(min(k), max(l)) :\ \sum_{i=k}^{l-1} sgn(x_i - \xi_0) = max_{1 \leq u < v \leq n} \sum_{i=u}^{v-1} sgn(x_i - \xi_0)\}. \end{aligned} \tag{2.48}$$

Combining Lemma 2.1 and Lemma 2.2 together, when $n \to \infty$ we get that

$$\begin{aligned} &P\{\hat{\tau}_1 < \tau_1\} \to 1, \\ &P\{\hat{\tau}_2 > \tau_2\} \to 1. \end{aligned} \tag{2.49}$$

We can write

$$\begin{aligned} &max_{k<l}(S_{l-1} - S_{k-1}) \\ &\quad = max_{k<l}\{(S_{l-1} - S_{\tau_2-1}) + (S_{\tau_2-1} - S_{\tau_1}) + (S_{\tau_1} - S_{k-1})\} \\ &\quad = max_{k<l}\{max_l(S_{l-1} - S_{\tau_2-1}) + max_k(S_{\tau_1} - S_{k-1})\} + (S_{\tau_2-1} - S_{\tau_1}), \end{aligned} \tag{2.50}$$

and

$$\begin{aligned} max_l(S_{l-1} - S_{\tau_2-1}) &= max\{max_{l<\tau_2-1}(S_{l-1} - S_{\tau_2-1}), max_{\tau_2\leq l}(S_{l-1} - S_{\tau_2-1})\} \\ max_k(S_{\tau_1} - S_{k-1}) &= max\{max_{k\geq\tau_1}(S_{\tau_1} - S_{k-1}), max_{k<\tau_1}(S_{\tau_1} - S_{k-1})\} \end{aligned}$$

To get the asymptotic distribution of $\sqrt{n}max_{k<l}(S_{l-1} - S_{k-1})$ under the alternative hypothesis, we have by the properties of simple symmetric random walk that

$$\frac{1}{\sqrt{n\lambda_1}} max_{1\leq i<\tau_1} \sum_{j=1}^{i} sgn(x_j - \xi_0) \xrightarrow{D} |N_1(0,1)|, \qquad n \to \infty, \tag{2.51}$$

and

$$\frac{1}{\sqrt{n(1-\lambda_1)}} max_{\tau_2\leq i\leq n} \sum_{j=1}^{i} sgn(x_j - \xi_0) \xrightarrow{D} |N_2(0,1)|, \qquad n \to \infty, \tag{2.52}$$

where N_1 and N_2 are indepedent standard normal random variables.

Furthermore (2.12) can also be written as

$$\frac{1}{\sqrt{4n\delta_n(1-\delta_n)(\lambda_2 - \lambda_1)}}\{\sum_{\tau_1\leq i<\tau_2} sgn(x_i - \xi_0) - \mu_n\} \to N_3(0,1), \tag{2.53}$$

where N_3 is a standard normal random variable, independent of N_1 and N_2, and $\mu_n \approx n(\lambda_2 - \lambda_1)(2\delta_n - 1)$. So using Lemmas 2.1 and 2.2 and (2.49), we get that the asymptotic distribution of $n^{-\frac{1}{2}} max_{k<l}(S_{l-1} - S_{k-1})$ under the alternative hypothesis is that of

$$\sqrt{\lambda_1}|N_1| + \sqrt{1-\lambda_2}|N_2| + 2\sqrt{\delta_n(1-\delta_n)(\lambda_2-\lambda_1)}N_3 + M_n, \tag{2.54}$$

where $M_n = (2\delta_n - 1)\sqrt{\frac{n(\lambda_2-\lambda_1)}{4\delta_n(1-\delta_n)}}$. As $M_n > 0$, from (2.55) the consistency of our test follows.

Recall that

$$P\{\hat{\tau}_1(n) \leq \tau_1(n)\} \rightarrow 1,$$

$$P\{\hat{\tau}_2(n) \geq \tau_2(n)\} \rightarrow 1.$$

In the case of symmetric simple random walk $\{S_i\}$, given $\hat{\tau}_1(n) < \tau_1(n)$, $\hat{\tau}_2(n) > \tau_2(n)$, the distribution of $\hat{\tau}_1(n)$ and $\hat{\tau}_2(n)$ can be obtained. By svmmetry, it is sufficient to consider

$$\begin{aligned} &P\{\hat{\tau}_2(n) = j \mid \hat{\tau}_2(n) \geq \tau_2(n)\} \\ &\quad = P\{argmax_{\tau_2 \leq l < n}(S_l - S_{\tau_2 - 1}) = j\} \\ &\quad = P\{S_k < 0;\ k = 1, \cdots, j\}P\{S_k - S_j \leq 0;\ k = j+1, \cdots, n\} \\ &\quad = P\{S_k < 0;\ k = 1, \cdots, j\}P\{S_l \leq 0;\ l = 1, \cdots, n-j\}. \end{aligned} \tag{2.55}$$

using time reversal and indepedent increments property, the expressions for the two factors are well known:

$$\begin{aligned} &P\{S_k < 0;\ k = 1, \cdots, j\} \\ &\quad = \begin{cases} \frac{1}{2}P\{S_j = 0\} = \binom{j}{j/2}(\frac{1}{2})^{j+1} & j\ is\ even \\ \sum_{b=-1}^{-j} \frac{b}{j}P\{S_j = b\} & j\ is\ odd \end{cases} \end{aligned} \tag{2.56}$$

and

$$P\{S_k \leq 0;\ k = 1, \cdots, L\}$$

$$= \begin{cases} P\{S_L = 0\} & j \textit{ is even.} \\ P\{S_{L+1} = 0\} & j \textit{ is odd.} \end{cases} \tag{2.57}$$

From (2.49), $P\{\hat{\tau}_1 > \tau_1\} \rightarrow 0$ and $P\{\hat{\tau}_2 < \tau_2\} \rightarrow 0$, we conclude that $(\hat{\tau}_1(n), \hat{\tau}_2(n)) = argmax_{k<l}(S_{l-1} - S_{k-1})$ is not a good estimation for the change-points $(\tau_1(n), \tau_2(n))$.

We will see in the next chapter that we can get a better estimator for the change-points (τ_1, τ_2) even though we do not know the initial value of the median ξ_0, but have to use its estimator under H_0.

Chapter 3

SIGN TEST FOR THE CHANGE-POINT PROBLEM WITH UNKNOWN MEDIAN

Let $x_1, \cdots, x_n$ be a sequence of independent continuous random variables. We want to consider the following hypothesis test for the change-point problem.

$$\begin{aligned} H_0 :\ & x_i \overset{D}{=} Y;\ i = 1, \cdots, n \\ H_1 :\ & x_i \overset{D}{=} Y;\ i = 1, \cdots, \tau_1 - 1, \tau_2, \cdots, n \\ & x_i \overset{D}{=} Z;\ i = \tau_1, \cdots, \tau_2 - 1 \quad Y \neq Z. \end{aligned} \tag{3.1}$$

H_1 is called the epidemic or square-wave alternative. We assume Y and Z have distribution functions $F(x)$ and $G(x)$ respectively where $F(x) \neq G(x)$ at least for some x and

$$F^{-1}(\frac{1}{2}) = \xi_0, \qquad G^{-1}(\frac{1}{2}) = \xi_A,$$

where ξ_0 and ξ_A are unknown. Also unknown are parameters τ_1 and τ_2 the change-points. We assume $\tau_1 = [n\lambda_1]$, $\tau_2 = [n\lambda_2]$, for some $0 < \lambda_1 < \lambda_2 < 1$.

First we consider the distribution of test statistic under H_0.

Let

$$\hat{\xi}_n = median\{x_1, \cdots, x_n\}, \tag{3.2}$$

and

$$V_n(u) = n^{-\frac{1}{2}} \sum_{1 \le i \le [nu]} sgn(x_i - \hat{\xi}_n), \qquad 0 \le u \le 1. \tag{3.3}$$

We employ the following statistic

$$\begin{aligned} M(n) &= max_{k<l} |V_n(\frac{l-1}{n}) - V_n(\frac{k-1}{n})| \\ &= max_{1 \le k < l \le n} |n^{-\frac{1}{2}} \sum_{k \le i < l} sgn(x_i - \hat{\xi}_n)|. \end{aligned} \tag{3.4}$$

Under H_0, as $n \to \infty$

$$V_n(u) = n^{-\frac{1}{2}} \sum_{1 \le i \le [nu]} sgn(x_i - \hat{\xi}_n) \xrightarrow{D} B(u), \qquad 0 \le u \le 1. \tag{3.5}$$

where $\{B(u); 0 \le u \le 1\}$ is a Brownian bridge (see Billingsley 1968).

The asymptotic distribution of $M(n)$, under H_0, when $\lambda_2 = 1$, is the same as that of the Kolmogorov-Smirnov test of the equal sample sizes, $m = \frac{n}{2}$ in case n is even and $\frac{n-1}{2}$ when n is odd, (*in this case* $sgn(0) = 0$). This case is the at-most-one-change alternative

$$\begin{aligned} H_1^* : \ & x_1, \cdots, x_{\tau-1} \sim F(x), \\ & x_\tau, \cdots, x_n \sim G(x), \quad F(x) \neq G(x). \end{aligned}$$

Tables for this distribution can be used.

When $0 < \lambda_1 < \lambda_2 < 1$ is assumed, i.e. when we test H_0 against H_1, from Gnedenko (1954, p.53), we have the exact formula:

$$\begin{aligned} & P\{M(n) < z\} \\ & = 1 + \tfrac{2}{\binom{2m}{m}}\{[\alpha \textstyle\sum_{s=1}^{[\frac{m}{\alpha+1}]}\binom{2m}{m-s(\alpha+1)} - (\alpha-1)\sum_{s=1}^{[\frac{m}{\alpha}]}\binom{2m}{m-s\alpha}] \qquad (3.6) \\ & -[\textstyle\sum_{i=1}^{\alpha}\sum_{s=1}^{[\frac{m+i}{\alpha+1}]}\binom{2m}{m+i-s(\alpha+1)} - \sum_{i=1}^{\alpha-1}\sum_{s=1}^{[\frac{m+i}{\alpha}]}\binom{2m}{m+i-s\alpha}]\}, \end{aligned}$$

where $m = [\frac{n}{2}]$, $\alpha = [z\sqrt{n}]$.

Now we consider the distribution of test statistic under the alternative hypothesis. Suppose

$$\delta = P\{x_{\tau_1} \leq \xi_0\} > \frac{1}{2} \qquad (3.7)$$

where $\xi_0 = F^{-1}(\frac{1}{2})$. We use the notation

$$\hat{\xi}_n = median\{x_1, \cdots, x_n\}, \qquad under\ H_1. \qquad (3.8)$$

$$H(x) = [1 - (\lambda_2 - \lambda_1)]F(x) + (\lambda_2 - \lambda_1)G(x) \qquad (3.9)$$

Let $\eta_0 = H^{-1}(\frac{1}{2})$. Note that η_0 is an unknown number that does not depend on n. We have

$$G^{-1}(\frac{1}{2}) < H^{-1}(\frac{1}{2}) < F^{-1}(\frac{1}{2}). \qquad (3.10)$$

Let

$$\begin{aligned} s &= P\{x_i \leq \eta_0\} < \tfrac{1}{2} \qquad i = 1, \cdots, \tau_1 - 1, \tau_2, \cdots, n, \\ s' &= P\{x_i \leq \eta_0\} > \tfrac{1}{2} \qquad i = \tau_1, \cdots, \tau_2 - 1. \end{aligned} \qquad (3.11)$$

By the well known property of quantile process (see e.g Csörgő-Révész 1981), we get

$$\hat{\xi}_n - \eta_0 = O_p(n^{-\frac{1}{2}}), \qquad for\ large\ n. \tag{3.12}$$

From (3.11), we get that

$$\begin{aligned} r_n &= P\{x_i \leq \hat{\xi}_n\} < \tfrac{1}{2}, \qquad i = 1, \cdots, \tau_1 - 1, \tau_2, \cdots, n. \\ r_n^{'} &= P\{x_i \leq \hat{\xi}_n\} > \tfrac{1}{2}, \qquad i = \tau_1, \cdots, \tau_2 - 1. \end{aligned} \tag{3.13}$$

Lemma 3.1 *Under the alternative hypothesis H_1, for $0 < u < 1$, we have*

$$E\{V_n(u)\} \tag{3.14}$$

$$= \begin{cases} n^{\frac{1}{2}} u C_1 + O_p(n^{-\frac{1}{2}}), & u < \lambda_1, \\ n^{\frac{1}{2}}[\lambda_1 C_1 + (u - \lambda_1) C_2] + O_p(n^{-\frac{1}{2}}), & \lambda_1 \leq u < \lambda_2, \\ n^{\frac{1}{2}}[(u - \lambda_2 + \lambda_1) C_1 + (\lambda_2 - \lambda_1) C_2] + O_p(n^{-\frac{1}{2}}), & \lambda_2 \leq u. \end{cases}$$

$$Var\{n^{\frac{1}{2}} V_n(u)\} \tag{3.15}$$

$$= \begin{cases} nuD_1 + \frac{nu(nu-1)}{2} O_p(n^{-\frac{1}{2}}), & u < \lambda_1, \\ n\lambda_1 D_1 + (u - \lambda_1) n D_2 + \frac{nu(nu-1)}{2} O_p(n^{-\frac{1}{2}}), & \lambda_1 \leq u < \lambda_2, \\ (u + \lambda_1 - \lambda_2) n D_1 + (\lambda_2 - \lambda_1) n D_2 \frac{nu(nu-1)}{2} O_p(n^{-\frac{1}{2}}), & \lambda_2 \leq u. \end{cases}$$

where $C_1 = 1 - 2r_n$, $C_2 = 1 - 2r_n^{'}$, $D_1 = 4r_n(1 - r_n)$, $D_2 = 4r_n^{'}(1 - r_n^{'})$

Proof: For the sake of brevity, we give the proof for $u \geq \lambda_2$. The proof for other cases are quite similar. Hence they will be omitted. Note that $x_i \overset{D}{=} Y$, for $i < [n\lambda_1]$, or $i \geq [n\lambda_2]$, and $x_i \overset{D}{=} Z$ for $[n\lambda_1] \leq i < [n\lambda_2]$. To calculate the mean, we have

$$
\begin{aligned}
&E\{V_n(u)\} \\
&= E\{n^{-\frac{1}{2}} \sum_{1\le i\le [nu]} sgn(x_i - \hat{\xi}_n\} \\
&= n^{-\frac{1}{2}}[\sum_{1\le i<[n\lambda_1]} E\{sgn(x_i - \hat{\xi}_n)\} + \sum_{[n\lambda_1]\le i<[n\lambda_2]} E\{sgn(x_i - \hat{\xi}_n)\} \\
&\quad + \sum_{[n\lambda_2]\le i<[nu]} E\{sgn(x_i - \hat{\xi}_n)\}] \\
&= n^{-\frac{1}{2}}[(n\lambda_1 - 1)E\{sgn(Y - \hat{\xi}_n)\} + n(\lambda_2 - \lambda_1)E\{sgn(Z - \hat{\xi}_n)\} \\
&\quad +(nu - n\lambda_2 + 1)E\{sgn(Y - \hat{\xi}_n)\}] + O_p(n^{-\frac{1}{2}}) \qquad (3.16) \\
&= n^{-\frac{1}{2}}[n(\lambda_1 - 1)C_1 + n(\lambda_2 - \lambda_1)C_2 + (nu - n\lambda_2 + 1)C_1] + O_p(n^{-\frac{1}{2}}) \\
&= n^{\frac{1}{2}}[\lambda_1 C_1 + (\lambda_2 - \lambda_1)C_2 + (u - \lambda_2)C_1] + n^{-\frac{1}{2}}([nu] - nu + 1) + O_p(n^{-\frac{1}{2}}) \\
&= n^{\frac{1}{2}}[(n - \lambda_2 + \lambda_1)C_1 + (\lambda_2 - \lambda_1)C_2] + O_p(n^{-\frac{1}{2}}).
\end{aligned}
$$

with

$$
\begin{aligned}
E\{sgn(Y - \hat{\xi}_n)\} &= P\{Y > \hat{\xi}_n\} + (-1)P\{Y \le \hat{\xi}_n\} \\
&= 1 - 2P\{Y \le \hat{\xi}_n\} \qquad (3.17) \\
&= 1 - 2r_n.
\end{aligned}
$$

and $E\{sgn(Z - \hat{\xi}_n)\} = 1 - 2r_n'$, we get (3.14)

For the variance calculation, we have

$$
\begin{aligned}
Var\{sgn(Y - \hat{\xi}_n)\} &= E\{[sgn(Y > \hat{\xi}_n)]^2\} - [E\{sgn(Y - \hat{\xi}_n\}]^2 \\
&= 1 - (1 - 2r_n)^2 \qquad (3.18) \\
&= 4r_n(1 - r_n).
\end{aligned}
$$

$$Var\{sgn(Z-\hat{\xi}_n)\} = 4r_n'(1-r_n')$$

and

$$\begin{aligned} Var\{n^{\frac{1}{2}}V_n(u)\} &= Var\{\sum_{1\le i\le [nu]} sgn(x_i-\hat{\xi}_n)\} \\ &= \sum_{1\le i\le [nu]} Var\{sgn(x_i-\hat{\xi}_n)\} \\ &\quad + \sum_{1\le i<j\le [nu]} Cov\{sgn(x_i-\hat{\xi}_n), sgn(x_j-\hat{\xi}_n)\} \end{aligned} \tag{3.19}$$

(1) We consider first $i \; and \; j < [n\lambda_1] \; or \ge [n\lambda_2]$

$$\begin{aligned} &Cov\{sgn(x_i-\hat{\xi}_n), sgn(x_j-\hat{\xi}_n)\} \\ &= E\{sgn(x_i-\hat{\xi}_n)sgn(x_j-\hat{\xi}_n)\} - E\{sgn(x_i-\hat{\xi}_n)\}E\{sgn(x_j-\hat{\xi}_n)\} \\ &= E\{sgn(x_i-\eta_0)sgn(x_j-\eta_0)\} - (1-2r_n)^2 \\ &\quad + E\{sgn(x_i-\hat{\xi}_n)sgn(x_j-\hat{\xi}_n) - sgn(x_i-\eta_0)sgn(x_j-\eta_0)\} \\ &= E\{sgn(x_i-\eta_0)\}E\{sgn(x_j-\eta_0)\} - (1-2r_n)^2 \\ &\quad + E\{sgn(x_i-\hat{\xi}_n)sgn(x_j-\hat{\xi}_n) - sgn(x_i-\eta_0)sgn(x_j-\eta_0)\} \\ &= (1-2s)^2 - (1-2r_n)^2 \\ &\quad + E\{sgn(x_i-\hat{\xi}_n)sgn(x_j-\hat{\xi}_n) - sgn(x_i-\eta_0)sgn(x_j-\eta_0)\}. \end{aligned} \tag{3.20}$$

Because when $x_i \; and \; x_j \notin (\hat{\xi}_n \wedge \eta_0, \; \hat{\xi}_n \vee \eta_0)$

$$sgn(x_i-\hat{\xi}_n)sgn(x_j-\hat{\xi}_n) - sgn(x_i-\eta_0)sgn(x_j-\eta_0) = 0$$

we have

$$\begin{aligned} &|E\{sgn(x_i-\hat{\xi}_n)sgn(x_j-\hat{\xi}_n) - sgn(x_i-\eta_0)sgn(x_j-\eta_0)\}| \\ &\le \int_{(\hat{\xi}_n\wedge\eta_0,\hat{\xi}_n\vee\eta_0)\times(\hat{\xi}_n\wedge\eta_0,\hat{\xi}_n\vee\eta_0)} 2dF_{i,j} \\ &= O_p(n^{-\frac{1}{2}}) \end{aligned} \tag{3.21}$$

where $F_{i,j}$ is the joint d.f of x_i and x_j.

Because $F_{i,j}$ is continuous and by (3.12)

$$(1-2s)^2-(1-2r_n)^2=4(1-r_n-s)(r_n-s) \tag{3.22}$$

As in (3.21), we get

$$\begin{aligned}|r_n-s| &= |P\{x_i\le\hat{\xi_n}\}-P\{x_i\le\eta_0\}| \\ &\le P\{\hat{\xi_n}\wedge\eta_0\le x_i<\hat{\xi_n}\vee\eta_0\} \\ &= \int_{(\hat{\xi_n}\wedge\eta_0,\hat{\xi_n}\vee\eta_0)} dF_i \\ &= O_p(n^{-\frac{1}{2}})\end{aligned} \tag{3.23}$$

Combining (3.21), (3.22), (3.23), we have

$$Cov\{sgn(x_i-\hat{\xi_n}),sgn(x_j-\hat{\xi_n})\}=O_p(n^{-\frac{1}{2}}), \tag{3.24}$$

for i *and* j $<n\lambda_1$ *or* $\ge n\lambda_2$

(*ii*) For $i<n\lambda_1$ *or* $\ge n\lambda_2$ and $n\lambda_1\le j<n\lambda_2$

$$\begin{aligned}&Cov\{sgn(x_i-\hat{\xi_n}),sgn(x_j-\hat{\xi_n})\} \\ &= E\{sgn(x_i-\hat{\xi_n})sgn(x_j-\hat{\xi_n})\}-E\{sgn(x_i-\hat{\xi_n})\}E\{sgn(x_j-\hat{\xi_n})\} \\ &= E\{sgn(x_i-\eta_0)sgn(x_j-\eta_0)\}-(1-2r_n)(1-2r_n') \\ &\quad +E\{sgn(x_i-\hat{\xi_n})sgn(x_j-\hat{\xi_n})-sgn(x_i-\eta_0)sgn(x_j-\eta_0)\} \\ &= (1-2s)(1-2s')-(1-2r_n)(1-2r_n') \\ &\quad +E\{sgn(x_i-\hat{\xi_n})sgn(x_j-\hat{\xi_n})-sgn(x_i-\eta_0)sgn(x_j-\eta_0)\}\end{aligned} \tag{3.25}$$

Because

$$|ab-cd|\le|b||a-c|+|c||b-d|,$$

. have

$$|(1-2s)(1-2s')-(1-2r_n)(1-2r'_n)| \\ \leq 2|1-2s||r_n-s|+2|1-2r_n||r'_n-s'| \tag{3.26}$$

As in (3.23), we have

$$|r'_n-s'|=O_p(n^{-\frac{1}{2}}). \tag{3.27}$$

And

$$(1-2s)(1-2s')-(1-2r_n)(1-2r'_n)=O_p(n^{-\frac{1}{2}}) \tag{3.28}$$

Using (3.21), (3.25), (3.28), we have proved

$$Cov\{sgn(x_i-\hat{\xi}_n),sgn(x_j-\hat{\xi}_n)\}=O_p(n^{-\frac{1}{2}}), \tag{3.29}$$

for $i<[n\lambda_1]$ or $\leq[n\lambda_2]$ and $[n\lambda_1]\leq j<[n\lambda_2]$. So, we have

$$\begin{aligned} &Var\{n^{\frac{1}{2}}V_n(u)\} \\ &=(n\lambda_1-1)D_1+n(\lambda_2-\lambda_2)D_2+([nu]-n\lambda_2+1)D_1+\frac{[nu]([nu]-1)}{2}O_p(n^{-\frac{1}{2}}) \\ &=n(\lambda_1-\lambda_2+u)D_1+n(\lambda_2-\lambda_2)D_2+([nu]-nu)D_1+\frac{nu(nu-1)}{2}O_p(n^{-\frac{1}{2}}). \\ &=n(\lambda_1-\lambda_2+u)D_1+n(\lambda_2-\lambda_2)D_2+\frac{nu(nu-1)}{2}O_p(n^{-\frac{1}{2}})\}. \end{aligned} \tag{3.30}$$

where

$$\frac{[nu]([nu]-1)}{2}O_p(n^{-\frac{1}{2}})=\frac{nu(nu-1)}{2}O_p(n^{-\frac{1}{2}}) \tag{3.31}$$

Let $\hat{\tau}_1(n)$, $\hat{\tau}_2(n)$ be the estimators of unknown change-points τ_1, τ_2. Similarly as in Chapter 2, we use the following estimators $\hat{\tau}_1(n)$, $\hat{\tau}_2(n)$ for τ_1 and τ_2, respectively:

$$
\begin{aligned}
& (\hat{\tau}_1(n),\ \hat{\tau}_2(n)) \\
= \quad & argmax_{k<l}|S_{l-1} - S_{k-1}| && (3.32) \\
= \quad & \{(min\{k\}, max\{l\} : |\sum_{i=k}^{l-1} sgn(x_i - \hat{\xi}_n)| = max_{1\leq u<v\leq n}|\sum_{i=u}^{v-1} sgn(x_i - \hat{\xi}_n)|\}.
\end{aligned}
$$

Theorem 3.1 *Under the alternative hypothesis H_1, if* (3.7) *is true, then*

$$|\hat{\tau}_1(n) - \tau_1(n)| + |\hat{\tau}_2(n) - \tau_2(n)| = O_p(1), \qquad (3.33)$$

where $\hat{\tau}_1(n)$, $\hat{\tau}_2(n)$ are the estimators of unknown change-points of τ_1, τ_2. Furthermore

$$\frac{1}{\sigma\sqrt{(\lambda_2 - \lambda_1)n}}\{\sum_{j=\hat{\tau}_1}^{\hat{\tau}_2} sgn(x_j - \hat{\xi}_n) - (\lambda_2 - \lambda_1)nC_2\} \xrightarrow{D} N(0,1). \qquad (3.34)$$

where C_2 is defined in (3.15) and $\sigma = 2\sqrt{s'(1-s')}$.

Proof: For the sake of brevity, we give the proof for the alternative hypothesis $H_2 \subset H_1$, where $\lambda_2 = 1$. The more general claim will easily follow from this case. From now on we will drop the index of λ and τ.

First we prove

$$|\hat{\tau(n)} - \tau(n)| = O_p(1) \qquad (3.35)$$

This is equivalent to

$$
\begin{aligned}
& \lim_{k\to\infty} limsup_{n\to\infty} P\{max_{i\leq\tau-k} V_n(\tfrac{i}{n}) \geq max_{\tau-k<i<\tau+k} V_n(\tfrac{i}{n})\} \\
& + \lim_{k\to\infty} limsup_{n\to\infty} P\{max_{\tau+k\leq i} V_n(\tfrac{i}{n}) \geq max_{\tau-k<i<\tau+k} V_n(\tfrac{i}{n})\} = 0 \qquad (3.36)
\end{aligned}
$$

We show that the first term on the left hand side of (3.36) is zero. The claim for the second term can be proved the same way by symmetry. Hence it will be omitted.

Since $x_i,\ i < \tau$ are identically distributed,

$$\begin{aligned}
&P\{max_{i\leq\tau-k}V_n(\frac{i}{n}) \geq max_{\tau-k<l<\tau+k}V_n(\frac{l}{n})\} \\
&\quad\leq P\{max_{i\leq\tau-k}\sum_{j=1}^{i} sgn(x_j-\hat{\xi_n}) \geq max_{\tau-k<l<\tau}\sum_{j=1}^{l} sgn(x_j-\hat{\xi_n})\} \\
&\quad= P\{\exists i, i\leq\tau-k : \sum_{j=1}^{i} sgn(x_j-\hat{\xi_n}) \geq max_{\tau-k<l<\tau}\sum_{j=1}^{l} sgn(x_j-\hat{\xi_n})\} \\
&\quad= P\{\exists i, i\leq\tau-k : 0 \geq max_{\tau-k<l<\tau}\sum_{j=i+1}^{l} sgn(x_j-\hat{\xi_n})\} \qquad (3.37) \\
&\quad= P\{\exists i, i\leq\tau-k : 0 \geq \sum_{j=i+1}^{\tau-k} sgn(x_j-\hat{\xi_n}) \\
&\quad\ \ +max_{\tau-k<l<\tau}\sum_{j=\tau-k+1}^{l} sgn(x_j-\hat{\xi_n})\} \\
&\quad= 1-P\{\forall i,\ i\leq\tau-k : 0 < \sum_{j=i+1}^{\tau-k} sgn(x_j-\hat{\xi_n}) \\
&\quad\ \ +max_{\tau-k<l<\tau}\sum_{j=\tau-k+1}^{l} sgn(x_j-\hat{\xi_n})\} \\
&\quad= 1-P\{0 < min_{1\leq i\leq\tau-k}\sum_{j=i+1}^{\tau-k} sgn(x_j-\hat{\xi_n}) \\
&\quad\ \ +max_{\tau-k<l<\tau}\sum_{j=\tau-k+1}^{l} sgn(x_j-\hat{\xi_n})\} \\
&\quad= P\{0 \geq \frac{1}{\sqrt{k}}min_{1\leq i\leq\tau-k}\sum_{j=i+1}^{\tau-k} sgn(x_j-\hat{\xi_n}) \\
&\quad\ \ +\frac{1}{\sqrt{k}}max_{\tau-k<l<\tau}\sum_{j=\tau-k+1}^{l} sgn(x_j-\hat{\xi_n})\}
\end{aligned}$$

Let $k = 1, 2, \cdots$, be a sequence and $m(k)$ another sequence, such that

$$\frac{m(k)}{\sqrt{k}} \to \infty, \qquad \frac{m(k)}{k} \to 0. \tag{3.38}$$

$$\begin{aligned}
&\tfrac{1}{\sqrt{k}}\textstyle\sum_{j=1}^{m(k)} sgn(x_j - \hat{\xi}_n) \\
&\quad \geq \tfrac{1}{\sqrt{k}}\textstyle\sum_{j=1}^{m(k)} sgn(x_j - \eta_0) - \tfrac{1}{\sqrt{k}}\textstyle\sum_{j=1}^{m(k)} |sgn(x_j - \hat{\xi}_n) - sgn(x_j - \eta_0)| \quad (3.39) \\
&\overset{D}{=} cW(\tfrac{m(k)}{k}) + \delta\tfrac{m(k)}{\sqrt{k}} - \tfrac{2}{\sqrt{k}}\textstyle\sum_{j=1}^{m(k)} I_j + o_p(1),
\end{aligned}$$

where $c,\ \delta > 0$ are constants, $W(\cdot)$ is a Wiener process, I_j is the indicator of the event $x_j \in \{\hat{\xi}_n \wedge \eta_0, \hat{\xi}_n \vee \eta_0\}$. As $|\hat{\xi}_n - \eta_0| = O_p(n^{-\frac{1}{2}})$, we get that $I_j = O_p(n^{-\frac{1}{2}})$.

As $\frac{m(k)}{k} \to 0$, by the continuity of the Wiener process

$$min_{1\leq j\leq m(k)}W(\frac{j}{k}) \xrightarrow{P} 0 \tag{3.40}$$

and we also have $\delta = Esgn(x_j - \eta_0) = 1 - 2s > 0$ (see (3.11)).

Now,

$$\begin{aligned}
\tfrac{1}{\sqrt{k}}\textstyle\sum_{j=1}^{m(k)} I_j &= \tfrac{m(k)}{\sqrt{k}}O_p(n^{-\frac{1}{2}}) \\
&= o_p(1),
\end{aligned}$$

So we get

$$-[min_{1\leq l\leq m(k)}\frac{1}{\sqrt{k}}\sum_{j=1}^{l} sgn(x_j - \hat{\xi}_n)]^- = o_p(1). \tag{3.41}$$

where $[u]^- = min(u, 0)$. It is known that

$$min_{0<t<1}W(t) = O_p(1). \tag{3.42}$$

Also

$$-[\frac{1}{\sqrt{k}}min_{m(k)<l\leq k}sgn(x_j - \hat{\xi}_n)]^-$$

$$= -[min_{m(k)<l\leq k}\{\frac{1}{\sqrt{k}}\sum_{j=1}^{l} sgn(x_j - \eta_0) + \frac{1}{\sqrt{k}}\sum_{j=1}^{l}[sgn(x_j - \hat{\xi}_n) - sgn(x_j - \eta_0)]\}]^-$$

$$\leq -[min_{0<t<1}W(t) + o_p(1) - \frac{2}{\sqrt{k}}\sum_{j=1}^{k} I_j + \delta\frac{m(k)}{\sqrt{k}}]^-. \tag{3.43}$$

As $I_j = O_p(n^{-\frac{1}{2}})$, and $\frac{m(k)}{\sqrt{k}} \to \infty$, we get that

$$-[min_{m(k)<l\leq k}\frac{1}{\sqrt{k}}sgn(x_j - \hat{\xi}_n)]^- = o_p(1). \tag{3.44}$$

To show that

$$-[min_{k<l<\tau-k}\frac{1}{\sqrt{k}}\sum_{j=1}^{l} sgn(x_j - \hat{\xi}_n)]^- = o_p(1). \tag{3.45}$$

we consider

$$\begin{aligned} &\tfrac{1}{\sqrt{k}}\textstyle\sum_{j=1}^{l} sgn(x_j - \hat{\xi}_n) \\ &\geq \tfrac{\sqrt{l}}{\sqrt{k}}\tfrac{1}{\sqrt{l}}\textstyle\sum_{j=1}^{l} sgn(x_j - \hat{\xi}_n) - \tfrac{2}{k}\textstyle\sum_{j=1}^{l} I_j \qquad (3.46) \\ &= \sqrt{\tfrac{l}{k}}O_p((loglogl)^{\frac{1}{2}}) + \tfrac{1}{\sqrt{k}}l\delta_n + \tfrac{1}{\sqrt{k}}lO_p(n^{-\frac{1}{2}}), \quad l \leq \tau. \end{aligned}$$

Here we used that by the law of iterated logarithm

$$liminf_{n\to\infty}\frac{S_n}{\sqrt{n}} \overset{a.s.}{=} O((loglogn)^{\frac{1}{2}}),$$

if S_n is the sum of mean zero independent identically distributed random variables that have finite variance. From (3.46) we got that (3.45) is true. Putting (3.41), (3.44) and (3.45) we have

$$\begin{aligned} &-[\frac{1}{\sqrt{k}}min_{1\leq i\leq\tau-k}\sum_{j=i+1}^{\tau-k} sgn(x_j - \hat{\xi}_n)]^- \\ &\overset{D}{=} -[\frac{1}{\sqrt{k}}min_{1\leq l\leq\tau-k}\sum_{j=1}^{l} sgn(x_j - \hat{\xi}_n)]^- \qquad (3.47) \end{aligned}$$

$$
\begin{aligned}
&= max\{-[\ min_{1\leq l\leq m(k)}\frac{1}{\sqrt{k}}\sum_{j=1}^{l} sgn(x_j - \hat{\xi}_n)]^-, \\
&\quad -[min_{m(k)<l\leq k}\frac{1}{\sqrt{k}}\sum_{j=1}^{l} sgn(x_j - \hat{\xi}_n)]^-, \\
&\quad -[min_{k<l<\tau-k}\frac{1}{\sqrt{k}}\sum_{j=1}^{l} sgn(x_j - \hat{\xi}_n)]^-\} \\
&= o_p(1). \qquad (3.48)
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
&\tfrac{1}{\sqrt{k}}max_{\tau-k<l<\tau}\textstyle\sum_{\tau-k}^{l} sgn(x_j - \hat{\xi}_n) \\
&\overset{D}{=} \tfrac{1}{\sqrt{k}}max_{1<l<k}\textstyle\sum_{j=1}^{l} sgn(x_{j+\tau-k-1} - \hat{\xi}_n) \qquad (3.49) \\
&\geq \tfrac{1}{\sqrt{k}}max_{1<l<k}\textstyle\sum_{j=1}^{l} sgn(x_{j+\tau-k-1} - \eta_0) - \tfrac{2}{\sqrt{k}}\textstyle\sum_{j=1}^{l} I_j.
\end{aligned}
$$

The error of the approximation is

$$
\begin{aligned}
\tfrac{2}{\sqrt{k}}\textstyle\sum_{j=1}^{l} I_j \ &= \tfrac{l}{\sqrt{k}}O_p(n^{-\frac{1}{2}}) \\
&= \sqrt{k}O_p(n^{-\frac{1}{2}}).
\end{aligned}
$$

Then we get by the strong law of large number

$$
\begin{aligned}
&\tfrac{1}{\sqrt{k}}max_{1\leq l\leq k}\textstyle\sum_{j=1}^{l} sgn(x_{j+\tau-k-1} - \eta_0) \\
&\geq \sqrt{k}\tfrac{1}{k}\textstyle\sum_{j=1}^{k} sgn(x_{j+\tau-k-1} - \eta_0) \qquad (3.50) \\
&\overset{a.s.}{\rightarrow} \infty, \qquad k \rightarrow \infty.
\end{aligned}
$$

As $Esgn(x_j - \eta_0) > 0$, combining (3.47) and (3.49) we get

$$
\lim_{k\rightarrow\infty} limsup_{n\rightarrow\infty}P\{max_{i\leq\tau-k}V(\frac{i}{n}) \geq max_{\tau-k<i<\tau+k}V(\frac{i}{n})\} = 0
$$

As the sign statistic can be looked as a rank statistic with score function

$$
\phi(u) = \begin{cases} -1, & 0 \leq u < \frac{1}{2} \\ 1, & \frac{1}{2} < u \leq 1 \\ 0 & otherwise. \end{cases} \qquad (3.51)
$$

we can use the Hajek (1968) result for the two sample sign statistic to get

$$\frac{V_n(\lambda) - \mu_n}{\sigma_n} \xrightarrow{D} N(0,1), \tag{3.52}$$

where

$$\mu_n = E\{V_n(\lambda)\}, \quad \sigma_n = Var V_n(\lambda)$$

In the one change-point case, the test statistic

$$\begin{aligned} M(n) &= max_{1 \leq k \leq n} V_n(\tfrac{k}{n}) \\ &= \tfrac{1}{\sqrt{n}} \textstyle\sum_{j=1}^{\hat{\tau}(n)} sgn(x_j - \hat{\xi}_n) \end{aligned} \tag{3.53}$$

If $\hat{\tau} < \tau$

$$\begin{aligned} M(n) - V_n(\lambda) &= -\tfrac{1}{\sqrt{n}} \textstyle\sum_{j=\hat{\tau}(n)+1}^{\tau(n)} sgn(x_j - \hat{\xi}_n) \\ &\overset{D}{=} -\tfrac{1}{\sqrt{n}} \textstyle\sum_{j=1}^{k} sgn(x_{j+\hat{\tau}} - \hat{\xi}_n) \\ &= o_p(1) \end{aligned} \tag{3.54}$$

and similar statememt is true if $\hat{\tau} > \tau$.

It is easy to see that for $\sigma_n = 2\sqrt{r_n(1 - r_n)}$ and $\sigma = 2\sqrt{s(1 - s)}$

$$\frac{\sigma_n}{\sigma} \xrightarrow{P} 1 \tag{3.55}$$

so by Slutsky's theorem, we can replace σ_n by σ in (3.51), and the proof of the theorem is concluded. □

Besides showing the asymptotic distribution of the test statistic, the above Theorem implies the consistency of our test. Furthermore, the Theorem allows a comparision between statistic used for two-sample problems and those for change-point problems. It shows that asymptotically they have same limit. The important implication of this is, that when we compare different change-point detection procedures, the results of asymptotic relation efficiencies of two-sample test are valid for the change-point tests as well. This statememnt is of course true only for at-most-one-change and for epidemic alternative cases.

Chapter 4

SIMULATION

In this Chapter we will consider the powers of our hypothesis tests in the previous chapters.

[1] The initial median is known case.

In Chapter 2, for the test (2.1),

$$\begin{aligned} H_0 :\ & x_i\ has\ median\ \xi_0\ for\ i = 1, \cdots, n. \\ H_1 :\ & x_i\ has\ median\ \xi_0\ for\ i = 1, \cdots, \tau_1 - 1\ \ \tau_2, \cdots, n. \\ & x_i\ has\ median\ \xi_1\ for\ i = \tau_1, \cdots, \tau_2 - 1, \end{aligned} \tag{4.1}$$

we consider the following test statistic

$$U_n = max_{k<l} \sum_{i=k}^{l-1} sgn(x_i - \xi_0). \tag{4.2}$$

To estimate the unknown change points τ_1 and τ_2, we use

$$(\hat{\tau_1}(n), \hat{\tau_2}(n)) = argmax_{k<l}(S_{l-1} - S_{k-1}) \tag{4.3}$$

Under H_0, Gombay proved that

$$P_{H_0}(U_n \geq N) = 1 - \frac{2}{2N+1} \sum_{j=1}^{2N} (c(j))^n s(j(N+1)) \frac{1 + c(j)}{s(j)} \frac{1 - (-1)^j}{2}. \tag{4.4}$$

where N is a positive integer and

$$c(j) = cos(\frac{j\pi}{2N+1}), \qquad s(j) = sin(\frac{j\pi}{2N+1}).$$

Under the alternative hypothesis H_1, we have the distribution of the test statistic U_n

$$\sqrt{\lambda_1}|N_1| + \sqrt{1-\lambda_2}|N_2| + 2\sqrt{\delta_n(1-\delta_n)(\lambda_2-\lambda_1)}N_3 + M_n. \tag{4.5}$$

All the notations here are the same as that in the Chapter 2.

For the hypothesis test (2.1), given significant level α, we reject H_0 in favour of H_1, if $U_n \geq N_\alpha$, where N_α is a positive number and

$$P_{H_0}(U_n \geq N_\alpha) \leq \alpha.$$

The power of the test is

$$\begin{aligned} &P_{H_1}(U_n \geq N_\alpha) \qquad (4.6)\\ &= P\{\sqrt{\lambda_1}|N_1| + \sqrt{1-\lambda_2}|N_2| + 2\sqrt{\delta_n(1-\delta_n)(\lambda_2-\lambda_1)}N_3 + M_n > N_\alpha\}. \end{aligned}$$

For the standard normal random variable $X \sim N(0,1)$, from

$$P\{|X| < x\} = \int_{-x}^{x} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx,$$

we get the density function of $|N(0,1)|$

$$f_{|X|}(x) = \sqrt{\frac{2}{\pi}} e^{-x^2/2}.$$

Also denote

$$a = \sqrt{\lambda_1},$$

$$b = \sqrt{1 - \lambda_2},$$

$$c = 2\sqrt{\delta_n(1 - \delta_n)(\lambda_2 - \lambda_1)},$$

$$d = N_\alpha - M_n.$$

we have

$$P_{H_1}(U_n \geq N_\alpha)$$

$$= \int\int\int_{ax+by+cz>d,\ x>0,\ y>0} \frac{\sqrt{2}}{\pi^{\frac{3}{2}}} exp\{-\tfrac{1}{2}(x^2 + y^2 + z^2)\}dxdydz. \tag{4.7}$$

When $n \to \infty$, $M \to \infty$, *and* $N_\alpha - M_n = d \to -\infty$, we can simply find

$$\lim_{n \to \infty} P_{H_1}(U_n \geq N_\alpha) = 1, \tag{4.8}$$

That means the power of the test converges to one as $n \to \infty$.

Unfortunately, we can not get the explicit expression for the integer (4.7). Numerical calculations are needed to carry out for different a, b, c, d.

To get some idea about the power for fixed sample size test, we do some simulation . We assume that the population distributions are normal and uniform. For hypothesis test (2.1), we consider seven different cases for test with epidemic alternatives. From (4.2), (4.3) and (4.4), we calculated the estimated change-points $\hat{\tau}_1$ and $\hat{\tau}_2$, test statistic values u_n and their P-values list in Table 4.1. In the table, rnorm(20) denotes a set of twenty observations from a standard normal population, rnorm(20, 2, 1) denotes a set of twenty observations from a normal population with the mean of 2 and standard deviation of 1, runif(20) denotes a

Table 4.1: Initial median is known case

H_1	τ_1	τ_2	$\hat{\tau}_1$	$\hat{\tau}_2$	u_n	P-value
rnorm(20), rnorm(20,2,1), rnorm(20)	21	41	13	48	23	0.004539
rnorm(20), rnorm(20,1.5,1), rnorm(20)	21	41	8	40	24	0.002933
rnorm(20), rnorm(20,1,1), rnorm(20)	21	41	6	50	20	0.016071
rnorm(20), rnorm(20,0.5,1), rnorm(20)	21	41	8	18	8	0.548926
runif(20), runif(20,0.75,1.75), runif(20)	21	41	8	59	29	0.000223
runif(20), runif(20,0.5,1.5), runif(20)	21	41	10	56	32	0.000039
runif(20), runif(20,0.25,1.25), runif(20)	21	41	3	59	18	0.033904

set of twenty observations from a uniform population in [0, 1] and runif(20, 0.5, 1.5) denotes a set of twenty observations from a uniform population in [0.5, 1.5].

We consider only the case where the variance of the distribution population does not change. From the Table (4.1), we can see when the difference between ξ_0 and ξ_1 is getting large, the statistics value u_n will likely get larger and there will be a more significant P-value. We confirm that the change-point estimators $\hat{\tau}_1$ and $\hat{\tau}_2$ are not good as our theory has predicted. They should be close to 21 and 41 but they are not. But we are able to detect the changes in all but one case, as the P-value is small.

Now we do the simulation on a real world data. We consider the sign test for Lombard's (1987) data which give the radii of circular indentations cut by a milling machine. The sample size is 100. The data are time-ordered and to be read row by row. A constant, 3.9, has been subtracted from all the data. We assume that they are independent random variables.

1.010	1.066	0.975	0.921	1.165	1.027	1.100	0.981	0.977	1.106
0.932	0.990	0.940	0.877	0.987	0.958	1.112	0.878	1.029	0.971
1.004	1.087	1.038	1.119	0.768	1.096	1.114	1.007	0.978	0.957
0.884	1.004	1.032	1.130	0.961	1.066	1.029	1.107	1.150	1.190
1.152	1.049	1.183	0.993	1.161	0.988	1.087	1.034	0.889	1.109
1.196	1.098	0.954	0.986	0.943	1.058	0.960	1.073	0.904	1.171
1.060	1.189	1.019	1.213	1.204	1.148	1.033	1.023	1.145	0.994
1.147	1.054	1.059	0.972	1.141	1.082	0.931	0.848	1.039	1.043
1.016	1.027	0.932	0.879	0.754	0.911	0.971	1.180	0.849	0.870
1.003	0.843	1.018	1.145	0.995	0.895	1.085	1.055	0.992	1.141

To do the calculation, first we get an estimator of the initial median $\xi_0 = 0.987$ based on first 15 observations. Then we do calculation just like the known initial median case, we get the test statistic value is 34 and the two estimated change-points are $\hat{\tau}_1 = 16$ and $\hat{\tau}_2 = 82$. The P-value for the test is 0.001050026 and clearly indicates that there are changes along the sequence. The corresponding test of Pettitt (1979) of H_0 against one change alternative got the P-value of 0.1324 and did not detect signal change in this data.

For test (2.1), under the null hypothesis H_0, we use the relation (2.3) to calculate the exactly critical value $N_\alpha(n)$ for given n and α listed in Table 4.2.

[2] The initial median is unknown case.

Let's consider the hypothesis test (3.1), we use

$$M(n) = max_{1 \leq k < l \leq n} n^{-\frac{1}{2}} \Big| \sum_{k \leq i < l} sgn(x_i - \hat{\xi}_n) \Big|. \tag{4.9}$$

The distribution of the test statistic $M(n)$ under H_0 is

$$P\{M(n) \leq z\} \tag{4.10}$$

Table 4.2: $N_\alpha(n)$ for test (2.1) that $P_{H_0}\{U_n \geq N_\alpha(n)\} \leq \alpha$

n \ α	0.1	0.05	0.025	0.01	0.005	0.0025	0.001
4	4						
5	4	5					
6		5	6				
7	5		6	7			
8		6	7		8		
9	6	7		8		9	
10	6	7	8		9		10
11	7		8	9		10	11
12	7	8		9	10		11
13	7	8	9	10		11	12
14	7	8	9	10	11	12	13
15	8	9	10	11		12	13
16	8	9	10	11	12		13
17	8	9	10	11	12	13	14
18	8	10	11	12		13	14
19	9	10	11	12	13	14	15
20	9	10	11	12	13	14	15
21	9	10	11	13		14	16
22	9	11	12	13	14	15	16
23	9	11	12	13	14	15	16
24	10	11	12	14		15	17
25	10	11	12	14	15	16	17
26	10	11	12	14	15	16	17
27	10	12	13	14		16	18
28	10	12	13	15	16	17	18

$$
\begin{aligned}
&= 1 + \frac{2}{\binom{2m}{m}}\{[\alpha \sum_{s=1}^{[\frac{m}{\alpha+1}]} \binom{2m}{m-s(\alpha+1)} - (\alpha-1)\sum_{s=1}^{[\frac{m}{\alpha}]}\binom{2m}{m-s\alpha}] \\
&-[\sum_{i=1}^{\alpha} \sum_{s=1}^{[\frac{m+i}{\alpha+1}]} \binom{2m}{m+i-s(\alpha+1)} - \sum_{s=1}^{[\frac{m+i}{\alpha+1}]} \binom{2m}{m+i-s(\alpha+1)} - \sum_{i=1}^{\alpha-1} \sum_{s=1}^{[\frac{n+i}{\alpha}]} \binom{2m}{m+i-s\alpha}]\}
\end{aligned}
$$

Here all the notations are same as that in the Chapter 3.

Table 4.3: Initial median is unknown case

H_1	τ_1	τ_2	$\hat{\tau}_1$	$\hat{\tau}_2$	$n^{\frac{1}{2}}m(n)$	P-value
rnorm(20), rnorm(20,2,1), rnorm(20)	21	41	20	48	22	0.0000008
rnorm(20), rnorm(20,1.5,1), rnorm(20)	21	41	12	40	20	0.0000203
rnorm(20), rnorm(20,1,1), rnorm(20)	21	41	7	50	15	0.0088002
rnorm(20), rnorm(20,0.5,1), rnorm(20)	21	41	8	40	12	0.1025916
runif(20), runif(20,0.75,1.75), runif(20)	21	41	19	40	19	0.0000846
runif(20), runif(20,0.50,1.50), runif(20)	21	41	23	46	17	0.0010595
runif(20), runif(20,0.25,1.25), runif(20)	21	41	20	33	7	0.8687585

Table 4.2 : (*continued*)

n \ α	0.1	0.05	0.025	0.01	0.005	0.0025	0.001
29	11	12	13	15	16	17	18
30	11	12	14	15	16	17	19
35	12	13	15	16	18	19	20
40	12	14	16	18	19	20	22
45	13	15	17	19	20	21	23
50	14	16	18	20	21	23	24
55	15	17	18	21	22	24	26
60	15	17	19	22	23	26	27
70	16	19	21	23	25	27	29
80	18	20	22	25	27	29	31
90	19	21	24	27	29	30	33
100	20	22	25	28	30	32	35
110	21	24	26	29	32	34	36
120	21	25	27	31	33	35	38
130	22	26	28	32	34	37	40
140	23	27	30	33	36	38	41
150	24	27	31	34	37	39	42
200	28	32	35	40	43	46	49
250	31	35	39	44	48	51	55
300	34	39	43	49	52	56	60
350	37	42	47	52	57	60	65
400	39	45	50	56	60	64	70
500	44	50	56	63	68	72	78
1000	62	71	79	89	96	102	110
2000	88	100	112	126	135	144	156

Given significance level α, we reject H_0 in favour of H_1 for large value of

$M(n)$, such that

$$P_{H_0}\{M(n) \geq M_\alpha(n)\} \leq \alpha. \tag{4.11}$$

where $M_\alpha(n)$ is a positive number.

From (3.34), we have the distribution of $M(n)$ under the alternative hypothesis H_1. Then, at the significance level α, the power of test (3.1) is

$$\begin{aligned} & P_{H_1}\{M(n) \geq M_\alpha(n)\} \\ & = P_{H_1}\{\tfrac{M(n)-|\mu|}{\sigma} \geq \tfrac{M_\alpha(n)-|\mu|}{\sigma}\} \\ & \approx 1 - \Phi(\tfrac{M_\alpha-|\mu|}{\sigma}) \end{aligned} \tag{4.12}$$

where Φ is the cumulative distribution function of the standard normal random variable.

For the simulation, first we consider the same data used in the simulations summarized in the Table 4.1. Assuming the intial medians of population under the null hypothesis are unknown, we calculate the medians of the observations. Using relation (3.4), we get the estimated change-points $\hat{\tau}_1$ and $\hat{\tau}_2$, test statistic values $n^{\frac{1}{2}}m(n)$ and their P-values listed in Table 4.3.

From the Table 4.1 and Table 4.3, we find that estimated change-points calculated from statistic $M(n)$ are closer to the real change-points than that from statistic U_n.

For the Lombard's (1987) data, we use the meadian of the total data $\hat{\xi}_n = 1.027$ as the intial median of the population distribution is assumed to be unknown. Then we calculated the estimated change-points $\hat{\tau}_1 = 32$, $\hat{\tau}_2 = 76$. The

Table 4.4: $n^{\frac{1}{2}} M_{\alpha}(n) \quad P_{H_0}\{M(n) \geq M_{\alpha}(n)\} \leq \alpha$

n \ α	0.1	0.05	0.025	0.01	0.005	0.0025	0.001
14		6					7
15		6					7
16	6		7				8
17	6		7				8
18		7		8			9
19		7		8			9
20	7		8			9	10
21	7		8			9	10
22		8		9			10
23		8		9			10
24		8	9			10	11
25		8	9			10	11
26	8		9		10		11
27	8		9		10		11
28		9		10		11	12
29		9		10		11	12
30	9		10		11		12
31	9		10		11		12
32	9		10	11		12	13
33	9		10	11		12	13
34	9	10		11		12	13
35	9	10		11		12	13
36		10	11		12		13
37		10	11		12		13

test statistic value $n^{\frac{1}{2}}m(n))$ is 18 and its P-value is 0.0520979, indicating the presence of changes. These are in good agreement with the Gombay's (1994) results of rank test, with the suggestion of the cusum plot and with Lombard's (1987) conclusins.

In the general case for test (3.1), under the null hypothesis H_0, we use relation (3.6) to calculate the exact critical value $n^{\frac{1}{2}}M_\alpha(n)$ for given n and α listed in Table 4.4.

Table 4.4 : (*continued*)

n \ α	0.1	0.05	0.025	0.01	0.005	0.0025	0.001
38	10		11	12		13	14
39	10		11	12		13	14
40	10	11		12	13		14
45	11		12	13		14	15
50	11	12	13	14		15	16
55	12	13	14		15	16	17
60		13	14	15	16	17	18
65	13	14	15	16		17	18
70	14	15	16		17	18	19
80	14	16	17	18		19	21
90	15	17	18	19	20	21	22
100	16	17	19	20	21	22	23
110	17	18	19	21	22	23	24
120	18	19	20	22	23	24	25
130	18	20	21	23	24	25	26
140	19	21	22	24	25	26	27
150	20	21	23	24	26	27	28
160	20	22	24	25	26	28	29
170	21	23	24	26	27	28	30
180	22	23	25	27	28	29	31
190	23	24	26	28	29	30	32
200	23	25	26	28	30	31	32

APPENDIX

[1] Program to calculate the test statistics value u_n of (1.2) and the estimated change-points $\hat{\tau}_1$ and $\hat{\tau}_2$ when the mean ξ_0 is known.

```
v < -  Obeservations of $H_1$
x < -  sign(v - $\xi_0$)
z < -  x * 0
w < -  x * 0
for(i in 1 : length(x))  {
        y < -  x * 0
        for(j in i : length(x))  {
                s < -  0
                for(k in i : j)  {
                        s < -  s + x[k] }
                y[j] < -  s
        }
        a < -  y[i]
        t < -  i
        for(l in i :  length(x))  {
                b < -  y[l]
                if(a < b)  {
                        a < -  b
                        t < -  l }
        }
        z[i] < -  a
        w[i] < -  t
}
print(z)
print(w)
a < -  z[1]
t < -  1
for(m in 2 : length(x))  {b < -  z[m]
        if(a < b)  {t < -  m
                a < -  b
        }
}
```

```
print("The Statistics value $u_n$ is")
print(a)
print("The estimated $\hat{\tau}_1$ is")
print(t - 1)
print("The estimated $\hat{\tau}_2$ is")
print(w[t])
```

[2] Program to calculate the test statistics $n^{\frac{1}{2}}m(n)$ and the estimated change-points $\hat{\tau}_1$ and $\hat{\tau}_2$ when the mean is unknown.

```
v <- Observations in $H_1$
e <- median(v)
x <- sign(v - e)
z <- x * 0
w <- x * 0
for(i in 1 : length(x)) {
        y <- x * 0
        for(j in i : length(x)) {
                s <- 0
                for(k in i : j) {
                        s <- s + x[k] }
                y[j] <- s
        }
        a <- abs(y[i])
        t <- i
        for(l in i : length(x)) {
                b <- abs(y[l])
                if(a < b) {
                        a <- b
                        t <- l
                }
        }
        z[i] <- a
        w[i] <- t
}
print(z)
print(w)
a <- z[1]
t <- 1
```

```
    for(l in 2 : length(x)) {
            b <- z[l]
            if(a < b) {
                    a <- b
                    t <- l
            }
    }
    print("The Statistics value of $n^{\frac{1}{2}}m(n)$ is")
    print(a)
    print("The estimated $\hat{\tau}_1$ is")
    print(t - 1)
    print("The estimated $\hat{\tau}_2$ is")
    print(w[t])
```

[3] Program to calculate the P-Value for the test statistics $n^{\frac{1}{2}}m(n)$ when the mean is unknown.

```
n <- Sample size
a <- $n^{\frac{1}{2}}m(n)$
m <- floor(n/2)
d <- 2 * m
sum1 <- 0
sum2 <- 0
sum3 <- 0
sum4 <- 0
% calculate $\sum_{i=1}^{\alpha}\sum_{s=1}^{\frac{m+i}{\alpha+1}}\binom{2m}{m+i-s(\alpha+1)}$ %
for(i in 1 : a) {b <- floor((m + i)/(a + 1))
        sum11 <- 0
        for(s in 1 : b) {c <- m + i - s * (a + 1)
                if(c > 0 && c <= d) {
                        e <- 1
                        f <- c - 1
                        for(k in 0 : f) {e <- e * (d - k)/(c - k) }
                }
                else if(c == 0) {e <- 1}
                else {e <- 0}
                sum11 <- sum11 + e
        }
        sum1 <- sum1 + sum11
}
```

```
% calculate $\sum_{i=1}^{\alpha-1}\sum_{s=1}^{\frac{m+i}{\alpha}}\binom{2m}{m+i-s\alpha}$ %
a1 < - a - 1
for(i in 1 : a1) {b < - floor((m + i)/a)
        sum22 < - 0
        for(s in 1 : b) {c < - m + i - s * a
                if(c > 0 && c <= d) {e < - 1
                        f < - c - 1
                        for(k in 0 : f) {e < - e * (d - k)/(c - k)}
                }
                else if(c == 0) {e < - 1 }
                else {e < - 0 }
                sum22 < - sum22 + e
        }
        sum2 < - sum2 + sum22
}
% calculate $(\alpha - 1)\sum_{s=1}^{\frac{m}{\alpha}}\binom{2m}{m-s\alpha}$ %
g < - floor(m/a)
for(i in 1 : g) {c < - m - i * a
        if(c > 0 && c <= d) {e < - 1
                f < - c - 1
                for(k in 0 : f) {e < - e * (d - k)/(c - k)}
        }
        else if(c == 0) {e < - 1 }
        else {e < - 0 }
        sum3 < - sum3 + e
}
sum3 < - (a - 1) * sum3
% calculate $\alpha\sum_{s=1}^{\frac{m}{\alpha+1}}\binom{2m}{m-s(\alpha+1)}$ %
h < - floor(m/(a + 1))
for(i in 1 : h) {c < - m - i * (a + 1)
        if(c > 0 && c <= d) {e < - 1
                f < - c - 1
                for(k in 0 : f) {e < - e * (d - k)/(c - k) }
        }
        else if(c == 0) {e < - 1 }
        else {e < - 0 }
        sum4 < - sum4 + e
}
sum4 < - a * sum4
```

```
% calculate (2m m) %
m1 <- m - 1
e <- 1
for(i in 0 : m1) { e <- e * (d - i)/(m - i) }
p <- 2 * (sum1 - sum2 + sum3 - sum4)/e
print("The P - Value for the test statistics value with unknow mean is")
print(p)
```

[4] Program to calculate the critical value for statistics U_n of (2.2) when the initial mean ξ_0 is known.

```
n <- Sample size
for(n1 in  : n) {
        print(" * * * *n = * * *")
        print(n1)
        for(k in 1 : n1) {s <- 0
                k1 <- 2 * k
                for(j in 1 : k1) {
                        a <- cos(j * pi/(2 * k + 1))
                        b <- sin(j * pi/(2 * k + 1))
                        c <- sin(j * pi * (k + 1)/(2 * k + 1))
                        s <- s + (a^n1) * c * (1 + a) * (1 - (-1)^j)/(2 * b)
                }
                p <- 1 - (2 * s)/(2 * k + 1)
                if(p > 0.05 && p <= 0.1) { print("M1 = ")
                        print(k) }
                if(p > 0.025 && p <= 0.05) { print("M2 = ")
                        print(k) }
                if(p > 0.01 && p <= 0.025) { print("M3 = ")
                        print(k) }
                if(p > 0.005 && p <= 0.01) { print("M4 = ")
                        print(k) }
                if(p > 0.0025 && p <= 0.005) { print("M5 = ")
                        print(k) }
                if(p > 0.001 && p <= 0.0025) { print("M6 = ")
                        print(k) }
                if(p <= 0.001) { print("M7 = ")
                        print(k) }
        }
}
```

[5] Program to calculate the P-Value for the test statistics value u_n of (2.2) when the initial mean ξ_0 is known.

```
n < - Sample size
N < - u_n
N1 < - 2 * N
s < - 0
for(i in 1 : N1) {
        a < - cos(i * pi/(2 * N + 1))
        b < - sin(i * pi/(2 * N + 1))
        c < - sin(i * pi * (N + 1)/(2 * N + 1))
        s < - s + (a^n) * c * (1 + a) * (1 - (-1)^i)/(2 * b)
}
p < - 1 - 2 * s/(2 * N + 1)
print("The P - Value for the test of known mean is")
print(p)
```

[6] Program to calculate the critical values for statistics $M(n)$ of (3.4) when the initial mean is unknown

```
n < - sample size
m < - floor(n/2)
d < - 2 * m
for(a in 1 : n) {
        sum1 < - 0
        sum2 < - 0
        sum3 < - 0
        sum4 < - 0
        for(i in 1 : a) {
                b < - floor((m + i)/(a + 1))
                sum11 < - 0
                for(s in 1 : b) {
                        c < - m + i - s * (a + 1)
                        if(c > 0 && c <= d) {
                                e < - 1
                                f < - c - 1
                                for(k in 0 : f) {
                                        e < -e * (d - k)/(c - k) }
                        }
                else if(c == 0) {e < - 1 }
                else {e < - 0 }
                sum11 < - sum11 + e
        }
        sum1 < - sum1 + sum11
}
a1 < - a - 1
for(i in 1 : a1) {
        b < - floor((m + i)/a)
        sum22 < - 0
        for(s in 1 : b) {
                c < - m + i - s * a
                if(c > 0 && c <= d) {
                        e < - 1
                        f < - c - 1
                        for(k in 0 : f) { e < - e * (d - k)/(c - k) }
                }
                else if(c == 0) { e < - 1 }
                else { e < - 0 }
                sum22 < -sum22 + e
        }
        sum2 < -sum2 + sum22
}
g < - floor(m/a)
for(i in 1 : g) {
        c < - m - i * a
        if(c > 0 && c <= d) {
                e < - 1
                f < - c - 1
                for(k in 0 : f) { e < - e * (d - k)/(c - k) }
        }
        else if(c == 0) { e < - 1 }
        else { e < - 0 }
        sum3 < -sum3 + e
}
sum3 < - (a - 1) * sum3
```

```
    h < - floor(m/(a + 1))
    for(i in 1 : h) {
            c < - m - i * (a + 1)
            if(c > 0 && c <= d) {
                    e < - 1
                    f < - c - 1
                    for(k in 0 : f) { e < - e * (d - k)/(c - k) }
            }
            else if(c == 0) { e < - 1 }
            else { e < - 0 }
            sum4 < - sum4 + e
    }
    sum4 < - a * sum4
    m1 < - m - 1
    e < - 1
    for(i in 0 : m1) {
            e < - e * (d - i)/(m - i)}
    p < - 2 * (sum1 - sum2 + sum3 - sum4)/e
    if(p > 0.05 && p <= 0.1) { print("M1 = ")
            print(a) }
    if(p > 0.025 && p <= 0.05) { print("M2 = ")
            print(a) }
    if(p > 0.01 && p <= 0.025) { print("M3 = ")
            print(a) }
    if(p > 0.005 && p <= 0.01) { print("M3 = ")
            print(a) }
    if(p > 0.0025 && p <= 0.005) { print("M5 = ")
            print(a) }
    if(p > 0.001 && p <= 0.0025) { print("M6 = ")
            print(a) }
    if(p <= 0.001) { print("M7 = ")
            print(a) }
}
```

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