University of Alberta

Photodisintegration of the Deuteron

By

© Kevin Daniel Klapstein

A thesis

presented to the Faculty of Graduate Studies and Research

in partial fulfilment of the requirements for the degree

of

Master of Science

in

Theoretical Physics

Department of Physics

Edmonton, Alberta

Fall 1993

National Library of Canada

Acquisitions and Bibliographic Services Branch

395 Wellington Street
Ottawa, Ontario
K1A 0N4

Bibliothèque nationale du Canada

Direction des acquisitions et des services bibliographiques

395, rue Wellington
Ottawa (Ontario)
K1A 0N4

ISBN 0-315-88046-5

UNIVERSITY OF ALBERTA
RELEASE FORM

NAME OF AUTHOR:	Kevin Daniel Klapstein
TITLE OF THESIS:	Photodisintegration of the Deuteron
DEGREE:	Master of Science
YEAR THE DEGREE GRANTED:	1993

Kevin Daniel Klapstein
Department of Physics
University of Alberta
Edmonton, Alberta
T6G 2J1

Date:

The scientist does not study nature because it is useful;
he studies it because he delights in it,
and he delights in it because it is beautiful.
If nature were not beautiful,
it would not be worth knowing,
and if nature were not worth knowing,
life would not be worth living.
Of course I do not speak here of the beauty that strikes the senses,
the beauty of qualities and appearances;
not that I undervalue such beauty, far from it,
but it has nothing to do with science;
I mean that profounder beauty
which comes from the harmonious order of the parts,
and which a pure intelligence can grasp.

HENRI POINCARÉ

UNIVERSITY OF ALBERTA

FACULTY OF GRADUATE STUDIES AND RESEARCH

The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research for acceptance, a thesis entitled "Photodisintegration of the Deuteron " submitted by Kevin Daniel Klapstein in partial fulfilment of the requirements for the degree of Master of Science in Theoretical Physics

Professor F. C. Khanna, Supervisor

Professor N. Rodning

Professor J. Stephenson

Professor F. A. Baragar

Date:

to my parents,

who got me started,

and my wife,

who keeps me going.

Abstract

I have calculated the total and differential cross sections and photon asymmetry for the photodisintegration of the deuteron into a two body final state, i.e., into a neutron and a proton only, using realistic radial wave functions for the deuteron as well as realistic scattering state radial wave functions calculated in a coupled channel approach. Calculations were first performed non-relativistically in the long wavelength limit, using Siegert's theorem for the electric multipole matrix elements. Following this, correction terms were calculated and added to the calculation to correct for the use of both the long wavelength approximation and Siegert's thoerem. A computer program based on these calculations was written, and the results have been compared to experimental results, as well as to the results of other theoretical calculations.

While no attempt has been made to consider the important higher energy effects of the 1232 MeV excited state of the nucleons called the Δ resonance, the meson exchange currents, or relativistic corrections such as the spin-orbit force, the corrections described above make the calculation exact in so far as it goes. The calculation therefore forms an excellent starting point for an expanded calculation including these effects, and with this in mind it has been performed in such a fashion that it should be possible to add these effects into the calculation in a relatively straightforward manner. Likewise, the program readily lends itself to expansion to include the delta resonance and meson exchange current effects.

Acknowledgements

I would like to thank my supervisor, Dr. Khanna, for suggesting this problem and for his patient support and help during my work on it.

I would also like to thank the many friends among my fellow graduate students who in one way or another have aided me along the way. In particular, I would like to thank Dr. Jon Johanson, whose advice and suggestions were invaluable, and who gave me many new insights into the physics and new ideas for proceeding with the calculation.

Contents

List of Figures

Chapter 1

Introduction

1.1 Motivation

In 1934 Chadwick and Goldhaber published *"A 'Nuclear Photo-effect': Disintegration of the Diplon by γ-Rays"*[1]. They explained their motivation by saying:

> *"Heavy hydrogen was chosen as the element first to be examined, because the diplon has a small mass defect and also because it is the simplest of all nuclear systems and its properties are as important in nuclear theory as the hydrogen is in atomic theory."*

As a two-body system and the simplest nucleus, the deuteron remains the first place in which new theoretical ideas and experimental techniques are tested. Photodisintegration is of particular interest because the photon is a sensitive and well understood probe with which to study the deuteron interior at small radii. This yields information about the radial part of the deuteron wave function and the size of the D-state admixture which in turn provide information about the nucleon-nucleon force. Significant deviation from the experimental results points to flaws either in the calculation, i. e. in the approximations being used to make the calculation tractable, or in the underlying assumptions of the theory. To a degree, even an indication of error in the approximations made in the calculation provides useful information by indicating that effects which have been ignored are of greater importance than was at first assumed.

The motivation for further theoretical investigation at this time is the recent improvement in the experimental situation. Over the past decade, several machines with monochromatic or quasi-monochromatic photon beams have been developed and used to study deuteron photodisintegration. The monochromatic nature of these machines has removed the principle cause of experimental uncertainty, making it possible to subject theoretical calculations to much more stringent testing. Moreover, many of these experiments use polarized photons, and can accurately measure polarization observables, providing another rigorous test to any calculation.

1.2 History

In 1935 Bethe and Peierls[2] published the first theoretical investigation of deuteron photodisintegration. The early work used only the most dominant terms in the multipole expansion of the photon electromagnetic field and very simple deuteron wave functions. Bethe and Peierls used only the dipole moment of the deuteron dotted with the polarization of the photon for an operator while for wave functions they used the ansatz of a zero range force, giving an S-state wave function only, for the ground state and free plane waves for the final state.

This rough calculation predicts a $\sin^2\theta$ angular dependence of the differential cross section which is accurately realized at low energies, and gives the shape of the energy dependence of the total cross section quite well for energies above ~2.5 MeV up to in excess of 50 MeV. It fails, however, to predict the magnitude of the total cross section, being only about 60% of what the data shows, and also to predict either the total cross section *or* the shape of the differential cross section near the threshold of 2.2 MeV.

The error in the amplitude stems from errors in the normalization of the scattering states due to the use of a zero range force and vanishes when even simple wave functions are substituted for these[3]. The discrepancy with experiment at energies near the threshold is more interesting, and stems from the importance of the ${}^3S_1 \rightarrow {}^1S_0$ M1 transitions[1] at low energies[4], which go roughly as $k\alpha/(k^2 + \alpha^2)$. Here, $\vec{k} = (1/2)\left(\vec{k}_p - \vec{k}_n\right)$ is the wave number of the scattered proton and neutron in relative coordinates, and α is related to the deuteron radius, and is (1/2.3) inverse fermi if the radial wave function is taken to be $u_g(r) = N_g e^{-\alpha r}$.

Early improvements included the use of more reasonable deuteron wave functions[7, 8, 6] and the inclusion of higher multipoles. By 1950 it was known that:

1. Spin transitions and higher multipoles are important at higher energies[5, 6].
2. Isotropic terms which are a result of the tensor force are important to the $\theta = 0$ part of the differential cross section[9].
3. Exchange terms are important at higher energies[6, 9].
4. Near threshold the cross section is mainly the ${}^3S_1 \rightarrow {}^1S_0$ M1 transition. [3, 4].

In 1951, Feshbach and Schwinger[20] included the D-state component of the deuteron using a Yukawa-type tensor force and noted that the use of Siegert's theorem implicitly includes contributions from the exchange currents. By 1954, Berger[10] had shown that inclusion of higher multipoles has a significant effect on the *shape* of the angular distribution, and indeed it is easily shown that interference between transitions to states of different orbital angular momentum, which generally requires different multipoles, can affect the angular distribution but *not* the total cross section.

In 1958, DeSwart and Marshak[11] and DeSwart[12] began using wave functions from realistic potentials, but still with only dipole operators. They concluded

[1] The notations Mλ and Eλ denote magnetic and electric 2^λ-pole mediated transitions.

that a large D-state probability was necessary to explain the isotropic component of the differential cross section, and in the late 1950s and early 1960s various authors found that E1↔M1 interference has a large effect on polarization observables[13].

This period of work on the deuteron to some degree reached a conclusion in the mid 1960s when elaborate calculations were done using realistic deuteron wave functions and all the significant multipoles, most notably the calculation by Partovi in 1964[55], and also by Rustgi et. al. [15] and by Donnachie and O'Donnel[16, 66, 19].

1.3 Present Situation

It was clear by this time that a reasonable calculation required the inclusion of the Δ-resonance in order to be valid in the energy region in which the Δ can be excited, and also the inclusion of meson currents in order to produce reasonable results even at energies as low as 100 MeV. Even within 10 MeV of threshold, Kramer and Müller[13] had concluded that polarization observables are strongly affected by meson exchange currents due to the importance of magnetic transitions. As a result, covariant theories aimed at consistently including these effects were being developed. The approaches used have been an S-matrix approach developed by Pearlstein and Klein and a dispersion relation approach developed by Donnachie and by Sakita and Goebel. The latter of these allows the direct incorporation of the results from non-relativistic potential models into the calculation and has been extended into the π-production regime.

At present, the main thrust of the work appearing in the literature is not so much the study of the nucleon↔nucleon potential as the study of sub-nuclear degrees of freedom which have effects on meson exchange and Δ excitation. Such considerations are well beyond the scope of this thesis.

Chapter 2

Calculation

2.1 Cross Section

We consider the scattering of two particles labeled 1 and 2 having velocities $\vec{v}_1$ and $\vec{v}_2$. We choose $\vec{v}_1 \| \hat{z}$ and require that $\vec{v}_2 \| \vec{v}_1$. We follow reference [21] closely, and until the end of the section we use natural units, i.e., $\hbar$=c=1.

We let the subscript $a(b)$ indicate that a quantity refers to the two particle system before (after) the interaction, and let n indicate either a or b. If we designate the complete set of orthonormal states which form a basis for the two particle system as χ_n, then we define E_n as the eigenvalue of the hamiltonian for non-interacting particles, K, i.e.,

$$K\chi_n = E_n\chi_n. \tag{2.1}$$

We define the interaction part of the hamiltonian as

$$V \equiv H - K \tag{2.2}$$

for the interaction before the scattering, and

$$V' \equiv H - K' \tag{2.3}$$

after the interaction. With these in place we can now define the *scattering matrix*, $\mathcal{T}$, by

$$\mathcal{T} = V + V' \lim_{\eta \to 0^+} \left(\frac{1}{E_a + i\eta - H} \right) V \tag{2.4}$$

where it is to be understood that the limit is not to be taken until after the singularity has been integrated across.

Now define $\vec{P}_n$ by the equation

$$\mathcal{P}\chi_n = \vec{P}_n\chi_n \tag{2.5}$$

where $\mathcal{P}$ is the momentum operator, i.e., $\vec{P}_n$ is the total momentum of the system, and because $[\mathcal{T}, \mathcal{P}] = 0$ we can define the "reduced T-matrix" by the equation

$$\langle b|\mathcal{T}|a\rangle = \delta(\vec{P}_b - \vec{P}_a)T_{b,a} \tag{2.6}$$

With this notation, the probability for a system to make a transition from a state χ_a to a state χ_b is then

$$P_{b,a} = (2\pi)^4\delta(\vec{P}_b - \vec{P}_a)\delta(E_b - E_a)|T_{b,a}|^2\frac{1}{|\vec{v}_1 - \vec{v}_2|}F \tag{2.7}$$

where F is the total flux per unit area across the target particle due to a single beam particle.

If we now write N_B as the number of beam particles impinging on a target having N_T target particles in a time Δt and ΔN_S as the number of scattered particles, then with F_B as the flux per unit area due to the entire beam we define the scattering cross section as

$$\Delta\sigma = \frac{\Delta N_S}{N_T F_B} \tag{2.8}$$

and we get for the cross section

$$\Delta\sigma = (2\pi)^4\frac{1}{|\vec{v}_1 - \vec{v}_2|}\sum_b \delta(\vec{P}_b - \vec{P}_a)\delta(E_b - E_a)|T_{b,a}|^2 \tag{2.9}$$

Clearly, the number of scattered particles is the number of particles in all states b which are *not* the initial state. In terms of $P_{b,a}$,

$$\Delta N_S = N_T N_B \sum_b P_{b,a}, \tag{2.10}$$

where the sum over b implies an integration over the continuous quantum numbers of the particles in the final state (e.g., the momenta) as well as a sum over the discrete quantum numbers.

For a beam of N_B particles, $F_B = N_B F$ where F is the single particle flux used earlier. Thus we have

$$\begin{aligned}\Delta\sigma &= \frac{N_T N_B \sum_b P_{b,a}}{N_T N_B F} \\ &= \frac{1}{F}\sum_b P_{b,a} \end{aligned} \tag{2.11}$$

and we get for the cross section

$$\Delta\sigma = (2\pi)^4 \frac{1}{|\vec{v}_1 - \vec{v}_2|} \sum_b \delta(\vec{P}_b - \vec{P}_a)\delta(E_b - E_a)|T_{b,a}|^2 \tag{2.12}$$

Restoring the factors of $\hbar$ and writing out the momentum integrations explicitly we arrive at the expression for the cross section which will be used in our calculation,

$$\Delta\sigma = \frac{(2\pi)^4}{|\vec{v}_1 - \vec{v}_2|\,\hbar^4} \sum_{\substack{final \\ states}} \int \frac{d^3P_{1f}}{\hbar^3}\frac{d^3P_{2f}}{\hbar^3}\delta(\vec{P}_b - \vec{P}_a)\delta(E_b - E_a)|T_{b,a}|^2. \tag{2.13}$$

2.2 Interaction Hamiltonian

The hamiltonian for the system is

$$H = K + H_I \tag{2.14}$$

where

$$K = \frac{p_p^2}{2m_p} + \frac{p_n^2}{2m_n} \tag{2.15}$$

and H_I is the interaction hamiltonian. The coupling to the photon is achieved by making the minimal substitution,

$$\vec{p_j} \Rightarrow \vec{p_j} - \frac{q_j}{c}\vec{A}(\vec{r}_j). \tag{2.16}$$

where $\vec{A}(\vec{r}_j)$ is the vector potential.

With $q_p = e$ and $q_n = 0$ and letting $m_p = m_n = M$ the hamiltonian for the deuteron is

$$H = \frac{1}{2m}\left[\vec{p}_p^{\,2} - e\vec{p}_p \cdot \frac{\vec{A}_p(\vec{r}_p)}{c} - e\frac{\vec{A}_p(\vec{r}_p)}{c} \cdot \vec{p}_p + e^2\frac{\vec{A}_p(\vec{r}_p)}{c} \cdot \frac{\vec{A}_p(\vec{r}_p)}{c} + \vec{p}_n^{\,2}\right]. \qquad (2.17)$$

We are dealing with single photon processes, so we ignore the terms which are quadratic in $\vec{A}$ and the interaction hamiltonian is

$$H_I = \frac{-e}{2mc}\left(\vec{p}_p \cdot \vec{A}_p(\vec{r}_p) + \vec{A}_p(\vec{r}_p) \cdot \vec{p}_p\right). \qquad (2.18)$$

Our coordinates are defined as in the following diagram. In this geometry, $\vec{r}_1$

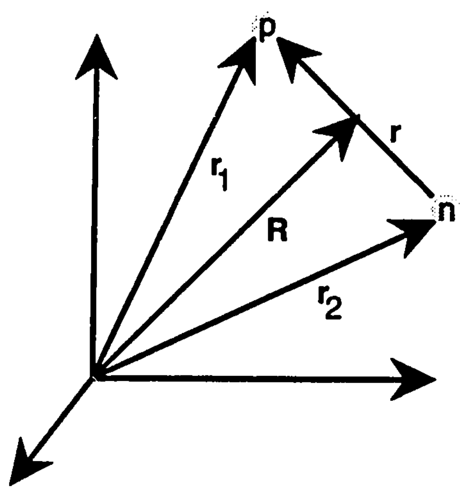

Figure 2.1: Coordinates of the deuteron system

and $\vec{r}_2$ are the positions of the proton and neutron, $\vec{R}$ is the *center of mass* coordinate, and $\vec{r}$ is the *relative* coordinate. We have

$$\begin{aligned} \vec{R} &= \tfrac{1}{2}(\vec{r}_1 + \vec{r}_2) \quad , \quad \vec{r} = \vec{r}_1 - \vec{r}_2 \\ \vec{r}_1 &= \vec{R} + \tfrac{1}{2}\vec{r} \qquad , \quad \vec{r}_2 = \vec{R} - \tfrac{1}{2}\vec{r} \end{aligned} \qquad (2.19)$$

and in a similar fashion the corresponding momenta are

$$\begin{aligned} \vec{P} &= \vec{p}_1 + \vec{p}_2 \quad , \quad \vec{p} = \tfrac{1}{2}(\vec{p}_1 - \vec{p}_2) \\ \vec{p}_1 &= \tfrac{1}{2}\vec{P} + \vec{p} \quad , \quad \vec{p}_2 = \tfrac{1}{2}\vec{P} - \vec{p}. \end{aligned} \tag{2.20}$$

Accomplishing the transition to quantum mechanics by making the substitution

$$\vec{p}_j \Rightarrow -i\hbar\vec{\nabla}_j \tag{2.21}$$

we have the interaction hamiltonian

$$H_I = \frac{e}{2mc}\left(i\hbar\vec{\nabla}_p \cdot \vec{A}_p(\vec{r}_p) + i\hbar\vec{A}_p(\vec{r}_p)\cdot\vec{\nabla}_p\right) \tag{2.22}$$

which can be rewritten in the reduced coordinates as

$$H_I = \frac{ie\hbar}{2mc}\left[\frac{1}{2}\left(\vec{\nabla}_R \cdot \vec{A}_p(\vec{r}_p) + \vec{A}_p(\vec{r}_p)\cdot\vec{\nabla}_R\right) + \left(\vec{\nabla}_r \cdot \vec{A}_p(\vec{r}_p) + \vec{A}_p(\vec{r}_p)\cdot\vec{\nabla}_r\right)\right]. \tag{2.23}$$

2.3 T Matrix

For the initial (final) state of the proton-neutron system we define the centre of mass momentum and wave number as $\vec{P}_i(\vec{P}_f)$ and $\vec{K}_i(\vec{K}_f)$. The relative momentum and wave number in the final state are $\vec{q}$ and $\vec{k}_f$, and the momentum and wave number for the photon, which is present only in the initial state, are $\vec{P}_\gamma$ and $\vec{\kappa}$.

In the Born approximation equation 2.4 becomes $\mathcal{T} \simeq H_I$, and with the above notation we have

$$\begin{aligned} \delta(\vec{P}_b - \vec{P}_a)T_{b,a} &= \int \frac{e^{-i\vec{K}_f\cdot\vec{R}}}{(2\pi)^{\frac{3}{2}}}\frac{e^{-i\vec{k}_f\cdot\vec{r}}}{(2\pi)^{\frac{3}{2}}}\left(\frac{e\hbar}{2mc}\right)\left[\left(\frac{1}{2}\right)\left(\vec{\nabla}_R \cdot \vec{A}_p(\vec{r}_p) + \vec{A}_p(\vec{r}_p)\cdot\vec{\nabla}_R\right)\right. \\ &+ \left.\left(\vec{\nabla}_r \cdot \vec{A}_p(\vec{r}_p) + \vec{A}_p(\vec{r}_p)\cdot\vec{\nabla}_r\right)\right]\frac{e^{i\vec{K}_i\cdot\vec{R}}}{(2\pi)^{\frac{3}{2}}}\phi_d(\vec{r})d^3Rd^3r \end{aligned} \tag{2.24}$$

where $\phi_d(\vec{r})$ is the wave function of the deuteron.

For photons such as will be in the beam in an experiment, the vector potential will be a vector plane wave. In relative and center of mass coordinates we have

$$\begin{aligned}\vec{A}(\vec{r}_p) &= a\hat{\epsilon}e^{i\vec{\kappa}\cdot\vec{r}_p} \\ &= a\hat{\epsilon}e^{i\vec{\kappa}\cdot\vec{R}}e^{i\frac{\vec{\kappa}}{2}\cdot\vec{r}} \end{aligned} \tag{2.25}$$

so that

$$\begin{aligned}\delta(\vec{P}_b - \vec{P}_a)T_{b,a} &= a\int \frac{e^{-i\vec{K}_f\cdot\vec{R}}}{(2\pi)^{\frac{3}{2}}}\left(\frac{e\hbar}{4mc}\right)\left[\vec{\nabla}_R\cdot\hat{\epsilon}e^{i\vec{\kappa}} + \hat{\epsilon}e^{i\vec{\kappa}}\cdot\vec{\nabla}_R\right]\frac{e^{i\vec{K}_i\cdot\vec{R}}}{(2\pi)^{\frac{3}{2}}}d^3R \\ &\times \int \frac{e^{-i\vec{k}_f\cdot\vec{r}}}{(2\pi)^{\frac{3}{2}}}e^{i\frac{\vec{\kappa}}{2}\cdot\vec{r}}\phi_d(\vec{r})d^3r \\ &+ a\int \frac{e^{-i\vec{K}_f\cdot\vec{R}}}{(2\pi)^{\frac{3}{2}}}\left(\frac{e\hbar}{2mc}\right)e^{i\vec{\kappa}\cdot\vec{R}}\frac{e^{i\vec{K}_i\cdot\vec{R}}}{(2\pi)^{\frac{3}{2}}}d^3R \\ &\times \int \frac{e^{-i\vec{k}_f\cdot\vec{r}}}{(2\pi)^{\frac{3}{2}}}\left[\vec{\nabla}_r\cdot\hat{\epsilon}e^{i\frac{\vec{\kappa}}{2}\cdot\vec{r}} + \hat{\epsilon}e^{i\frac{\vec{\kappa}}{2}\cdot\vec{r}}\cdot\vec{\nabla}_r\right]\phi_d(\vec{r})d^3r \end{aligned} \tag{2.26}$$

The first half of this equation can be eliminated by evaluating the integrand on the first line. Specifically, it can easily be shown that

$$\left[\vec{\nabla}_R\cdot\hat{\epsilon}e^{i\vec{\kappa}} + \hat{\epsilon}e^{i\vec{\kappa}}\cdot\vec{\nabla}_R\right]e^{i\vec{K}_i\cdot\vec{R}} = \hat{\epsilon}\cdot\left[\left(\vec{\kappa}+2\vec{K}_i\right)e^{i\left(\vec{\kappa}+\vec{K}_i\right)\cdot\vec{R}}\right]. \tag{2.27}$$

Because we have insisted that $\vec{v}_1 \| \vec{v}_2$, i.e.,$\vec{\kappa}\|\vec{K}_i$, and because $\hat{\epsilon}\perp\vec{\kappa}$ we have

$$\hat{\epsilon}\cdot\vec{\kappa} = \hat{\epsilon}\cdot\vec{K}_i = 0 \tag{2.28}$$

Equation 2.26 now reduces to

$$\begin{aligned}\delta(\vec{P}_b - \vec{P}_a)T_{b,a} &= a\left(\frac{e\hbar}{2mc}\right)\left[\left(\frac{1}{2\pi}\right)^3\int e^{i\left(\vec{K}_i+\vec{\kappa}-\vec{K}_f\right)\cdot\vec{R}}d^3R\right] \\ &\times \frac{1}{(2\pi)^{\frac{3}{2}}}\int\frac{e^{-i\vec{k}_f\cdot\vec{r}}}{(2\pi)^{\frac{3}{2}}}\left(\vec{\nabla}_r\cdot\hat{\epsilon}e^{i\frac{\vec{\kappa}}{2}\cdot\vec{r}} + \hat{\epsilon}e^{i\frac{\vec{\kappa}}{2}\cdot\vec{r}}\cdot\vec{\nabla}_r\right)\phi_d(\vec{r})d^3r, \end{aligned} \tag{2.29}$$

where we recognize that the term inside the square brackets is just a delta function for momentum conservation,

$$\hbar^3\delta\left(\vec{P}_b - \vec{P}_a\right) = \left(\frac{1}{2\pi}\right)^3\int e^{i\left(\vec{K}_i+\vec{\kappa}-\vec{K}_f\right)\cdot\vec{R}}d^3R. \tag{2.30}$$

where $\vec{P}_a = \vec{P}_i + \vec{p}_\gamma$ so that the T-matrix is

$$T_{b,a} = \left(\frac{e\hbar^4}{2mc}\right) \int e^{-i\vec{k}_f\cdot\vec{r}} \left[\vec{\nabla}_r \cdot \vec{A}\left(\frac{\vec{\kappa}}{2}, \vec{r}\right) + \vec{A}\left(\frac{\vec{\kappa}}{2}, \vec{r}\right) \cdot \vec{\nabla}_r\right] \phi_d(\vec{r}) d^3r, \tag{2.31}$$

where the κ dependence is shown explicitly.

We write the convection current density as

$$\vec{J}_c(\vec{r}) = \left(\frac{e\hbar}{2im}\right) \left(\Psi_f^* \vec{\nabla}_r \Psi_i - \Psi_i \vec{\nabla}_r \Psi_f^*\right), \tag{2.32}$$

where the factor of 1/2 appears because only the proton has charge. This gives for the interaction hamiltonian

$$\frac{\vec{J}_c \cdot \vec{A}}{c} = \left(\frac{e\hbar}{2imc}\right) \left(\Psi_f^* \vec{A} \cdot \vec{\nabla}_r \Psi_i - \Psi_i \vec{A} \cdot \vec{\nabla}_r \Psi_f^*\right). \tag{2.33}$$

Now consider the second term. The vector identity

$$\vec{\nabla} \cdot (\Psi \vec{a}) = \vec{a} \cdot \vec{\nabla} \Psi + \Psi \vec{\nabla} \cdot \vec{a} \tag{2.34}$$

allows us to write

$$-\int \Psi_i \vec{A} \cdot \vec{\nabla}_r \Psi_f^* d^3r = -\int \left[\vec{\nabla}_r \cdot \Psi_f^* \vec{A} \Psi_i + \Psi_f^* \vec{\nabla}_r \cdot \vec{A} \Psi_i\right] d^3r \tag{2.35}$$

and applying the divergence theorem to the first term gives

$$\begin{aligned} -\int \Psi_i \vec{A} \cdot \vec{\nabla}_r \Psi_f^* d^3r &= -\int \Psi_f^* \vec{A} \Psi_i \cdot \hat{n} d\Omega_r + \int \Psi_f^* \vec{\nabla}_r \cdot \vec{A} \Psi_i d^3r \\ &= \int \Psi_f^* \vec{\nabla}_r \cdot \vec{A} \Psi_i d^3r \end{aligned} \tag{2.36}$$

and we get

$$\left\langle \Psi_f \left| \frac{(\vec{J}_c \cdot \vec{A})}{c} \right| \Psi_i \right\rangle = \left(\frac{e\hbar}{2imc}\right) \int \Psi_f^* \vec{\nabla} \cdot \vec{A} \Psi_i + \Psi_f^* \vec{A} \cdot \vec{\nabla} \Psi_i d^3r \tag{2.37}$$

so that

$$T_{b,a} = -i\hbar^3 \left(\frac{1}{2\pi}\right)^{\frac{3}{2}} \left\langle \Psi_f \left| \frac{(\vec{J}_c \cdot \vec{A})}{c} \right| \Psi_i \right\rangle. \tag{2.38}$$

So far we have not considered the coupling of the photon to the spin of the nucleons or to the mesons which we know exist in the deuteron. Unfortunately, since we are doing a non-relativistic calculation, and these phenomena are intrinsically relativistic, they are not derivable *per se*. Here we put them into the calculation by hand. We write simply

$$T_{b,a} = -i\hbar^3 \left(\frac{1}{2\pi}\right)^{\frac{3}{2}} \left\langle \Psi_f \left| \frac{\left(\vec{J} \cdot \vec{A}\right)}{c} \right| \Psi_i \right\rangle \tag{2.39}$$

where

$$\vec{J} = \vec{J}_c + \vec{J}_M + \vec{J}_{exch}. \tag{2.40}$$

The three terms on the right are the convection current density, the "magnetization" or spin dependant current density, and the "exchange" current density due to the exchange of mesons in the nucleus respectively.

The magnetization current density is given by

$$\vec{J}_M = \left(\frac{e\hbar}{2m}\right) \sum_j \delta(\vec{r} - \vec{r}_j) \vec{\nabla}_j \times \left(\Psi_f^* \vec{\sigma}_j \Psi_i\right) \tag{2.41}$$

or in reduced coordinates (which correspond to the deuteron center of mass before interaction)

$$\vec{J}_M = \left(\frac{e\hbar}{2m}\right) \vec{\nabla}_r \times \left[\Psi_f^* \left(\mu_p \vec{\sigma}_p - \mu_n \vec{\sigma}_n\right) \Psi_i\right]. \tag{2.42}$$

The exchange current is beyond the scope of this work, and we will ignore it in our calculations. It is important, however, to know that it is there and we may on occasion comment on it.

2.4 Phase Space

Combining equation 2.13 with equation 2.39 the differential cross section of the deuteron is given as

$$\Delta\sigma = \frac{\left(2\pi/\hbar^4\right)}{|v_1 - v_2|} \sum_{\substack{final \\ states}} \int d^3p_p d^3p_n \delta(\vec{P}_b - \vec{P}_a)\delta(E_b - E_a) \left| \left\langle \Psi_f \left| \frac{\left(\vec{J}\cdot\vec{A}\right)}{c} \right| \Psi_i \right\rangle \right|^2 \quad (2.43)$$

where $\vec{p}_p(\vec{p}_n)$ is the momentum of the proton (neutron) after scattering. We denote the phase space part as Γ,

$$\Gamma = \int d^3p_p d^3p_n \delta\left(\vec{p}_p + \vec{p}_n - \vec{p}_\gamma - \vec{p}_d\right) \delta\left(E_p + E_n - E_\gamma - E_d\right). \quad (2.44)$$

which in relative and center of mass coordinates is

$$\Gamma = \int d^3P d^3q \delta\left(\vec{p}_p + \vec{p}_n - (\vec{p}_\gamma + \vec{p}_d)\right) \delta\left(E_p + E_n - (E_\gamma + E_d)\right). \quad (2.45)$$

In the center of mass frame $\vec{p}_\gamma + \vec{p}_d = 0 = \vec{p}_p + \vec{p}_n$, and since we have taken $m_p \approx m_n \approx M$, $E_p = E_n$. Performing the integration over center of mass momentum we have

$$\Delta\sigma = \frac{\left(2\pi/\hbar^4\right)}{|v_1 - v_2|} \sum_{\substack{final \\ states}} \int \delta\left(2E_p - (E_\gamma + E_d)\right) \left| \left\langle \Psi_f \left| \frac{\left(\vec{J}\cdot\vec{A}\right)}{c} \right| \Psi_i \right\rangle \right|^2 q^2 dq d\Omega_q. \quad (2.46)$$

Now $\vec{p}_p = \frac{1}{2}\vec{P}_f + \vec{q}$, but for the center of mass frame, $\vec{P}_f = 0$ so $\vec{p}_p = \vec{q}$, hence $E_p^2 = c^2q^2 + m^2c^4$ which implies $2E_p dE_p = 2c^2 q dq$ so that

$$\frac{d\sigma}{d\Omega_q} = \frac{2\pi}{|v_1 - v_2|\,\hbar^4 c^2} \sum_{\substack{final \\ states}} \int q E_p \delta\left(2E_p - (E_\gamma + E_d)\right) \left| \left\langle \Psi_f \left| \frac{\left(\vec{J}\cdot\vec{A}\right)}{c} \right| \Psi_i \right\rangle \right|^2 dE_p. \quad (2.47)$$

The integration over the remaining one dimensional delta function can now be performed giving

$$\frac{d\sigma}{d\Omega_q} = \frac{\pi}{|v_1 - v_2|\,\hbar^4 c^2} \sum_{\substack{final \\ states}} \sqrt{\left(\frac{Ep}{c}\right)^2 - m^2c^2}\, E_p \left| \left\langle \Psi_f \left| \frac{\left(\vec{J}\cdot\vec{A}\right)}{c} \right| \Psi_i \right\rangle \right|^2 \quad (2.48)$$

where now $E_p = \left(\frac{E_\gamma + E_d}{2}\right)$.

Finally, from the kinematics $\sqrt{\left(\frac{E_P}{c}\right)^2 - m^2c^2} = \hbar k$ and $E_p = \sqrt{\hbar^2 c^2 k^2 + m^2 c^4}$ where $\vec{q} = \vec{p}_p$ so that

$$\frac{d\sigma}{d\Omega_p} = \frac{\pi}{\left(\frac{|v_1 - v_2|}{c}\right)(\hbar c)^3} k E_p \sum_{\substack{final\\states}} \left| \left\langle \Psi_f \left| \frac{\left(\vec{J}\cdot\vec{A}\right)}{c} \right| \Psi_i \right\rangle \right|^2 \tag{2.49}$$

which forms the basis of the rest of our calculation.

2.5 Multipole Expansion

The vector multipole expansions are well described in reference [24]. We consider a field with a definite frequency, which we can do without loss of generality since a field with an arbitrary time dependence can be constructed from this by Fourier techniques. For such a field, in the Lorentz gauge, Maxwell's equations in four-vector notation are

$$\nabla^2 A_\nu + \kappa^2 A_\nu = 0 \tag{2.50}$$

for which the solutions are

$$\vec{A}\left(\vec{\kappa},\vec{r}\right) = a\hat{\epsilon}e^{i\vec{\kappa}\cdot\vec{r}} \tag{2.51}$$

for the three vector components, and

$$\phi\left(\vec{\kappa},\vec{r}\right) = ae^{i\vec{\kappa}\cdot\vec{r}} \tag{2.52}$$

as the fourth component, $\phi\left(\vec{\kappa},\vec{r}\right) = A_0\left(\vec{\kappa},\vec{r}\right)$,where

$$a = \left(\frac{1}{2\pi}\right)\left(\frac{\hbar c}{\kappa}\right)^{\frac{1}{2}} \tag{2.53}$$

gives the correct normalization.

These can be written in a vector multipole expansion as

$$\vec{A} = a\left[\sum_{\lambda=1}^{\infty}\vec{\mathcal{L}}_{\lambda,0} + \sum_{\lambda=1}^{\infty}\sum_{\mu=\pm1} b_\mu\left(\vec{\mathcal{E}}_{\lambda,\mu} + \mu\vec{\mathcal{M}}_{\lambda,\mu}\right)\right] \tag{2.54}$$

and

$$\phi = a\sum_{\lambda=1}^{\infty}\phi_{\lambda,0}. \tag{2.55}$$

Henceforth we suppress the limits on the sums over the multipolarities λ and polarizations μ.

Taking $\vec{\kappa}\|\hat{z}$ gives $Y_l^m(\hat{\kappa}) = \delta(m_l, 0)\sqrt{(2l+1)/4\pi}$ for the Legendre polynomial Writing $j_\lambda(\kappa r)$ for the spherical Bessel function of order λ this in turn gives us

$$\phi_\lambda^\mu = i^\lambda\sqrt{4\pi(2\lambda+1)}\, j_\lambda(\kappa r)\, Y_\lambda^\mu(\hat{r}). \tag{2.56}$$

We define the vector spherical harmonics $\vec{Y}^\mu_{\lambda,\lambda,1}$ as

$$\vec{Y}^\mu_{\lambda,l,1} = \sum_{m=-l}^{l}\sum_{\nu=-1}^{1} C(l,m,1,\nu;\lambda,\mu)\, Y_l^m\hat{\mathcal{X}}_\nu, \tag{2.57}$$

where we have made the limits on the sum over projections of the spins explicit. Henceforth, it is to be assumed that sums over the projections are to be over all possible projections of the associated angular momentum unless otherwise noted with the exception of the photon polarization μ which as already noted is summed over only ±1. With the vector spherical harmonics thus defined we have also

$$\begin{aligned}\vec{\mathcal{L}}_{\lambda,\mu} &= \left(\frac{1}{i\kappa}\right)\vec{\nabla}\phi_\lambda^\mu \\ &= i^{\lambda-1}\sqrt{4\pi}\left(\sqrt{\lambda}\, j_{\lambda-1}(\kappa r)\,\vec{Y}^\mu_{\lambda,\lambda-1,1}(\hat{r}) + \sqrt{\lambda+1}\, j_{\lambda+1}(\kappa r)\,\vec{Y}^\mu_{\lambda,\lambda+1,1}(\hat{r})\right)\end{aligned} \tag{2.58}$$

$$\begin{aligned}\vec{\mathcal{E}}_\lambda^\mu &= \left(\frac{1}{\sqrt{\kappa^2\lambda(\lambda+1)}}\right)\vec{\nabla}\times\vec{L}\phi_\lambda^\mu \\ &= i^{\lambda+1}\sqrt{4\pi}\left(\sqrt{\lambda+1}\, j_{\lambda-1}(\kappa r)\,\vec{Y}^\mu_{\lambda,\lambda-1,1}(\hat{r}) + \sqrt{\lambda}\, j_{\lambda+1}(\kappa r)\,\vec{Y}^\mu_{\lambda,\lambda+1,1}(\hat{r})\right)\end{aligned} \tag{2.59}$$

and

$$\vec{\mathcal{M}}^{\mu}_{\lambda} = \left(\frac{1}{\sqrt{\lambda(\lambda+1)}}\right)\vec{L}\phi^{\mu}_{\lambda}$$
$$= i^{\lambda}\sqrt{4\pi(2\lambda+1)}\,j_{\lambda}(\kappa r)\,\vec{Y}^{\mu}_{\lambda,\lambda,1}(\hat{r}). \qquad (2.60)$$

The freedom to perform a gauge transformation on $\vec{A}$ makes it possible to make the transformed ϕ^{μ}_{λ} and $\vec{\mathcal{L}}_{\lambda,\mu}$ zero. Alternately, consider the interaction with a current density $\vec{J}$. Considering *only* these two parts of the vector potential and assuming a current with a definite frequency ω, we have

$$\int \frac{\vec{A}\cdot\vec{J}}{c}d^3r = \frac{1}{c}\int \vec{\mathcal{L}}\cdot\vec{J}d^3r - \int \phi\rho d^3r$$
$$= \frac{1}{ikc}\int \vec{\nabla}\phi\cdot\vec{J}d^3r - \int \phi\rho d^3r$$
$$= -\frac{1}{i\omega}\int \phi\left[\vec{\nabla}\cdot\vec{J} + i\omega\rho\right]d^3r \qquad (2.61)$$

where we have used the vector identity of equation 2.34 and the divergence theorem 2.36, i.e.,

$$\int \vec{\nabla}\phi\cdot\vec{J}d^3r = \int\left[\vec{\nabla}\cdot\left(\phi\vec{J}\right) - \phi\vec{\nabla}\cdot\vec{J}\right]d^3r$$
$$= \int \phi\vec{J}\cdot\hat{n}d\Omega - \int \phi\vec{\nabla}\cdot\vec{J}d^3r$$
$$= -\int \phi\vec{\nabla}\cdot\vec{J}d^3r. \qquad (2.62)$$

Finally, we note that since we have assumed a current with a definite frequency ω the equation of continuity gives

$$\vec{\nabla}\cdot\vec{J} = -\frac{\partial\rho}{\partial t}$$
$$= -i\omega\rho. \qquad (2.63)$$

The quantity in the square brackets in equation 2.61 is therefore zero, so that the effects of $\vec{\mathcal{L}}^{\mu}_{\lambda}$ and ϕ^{μ}_{λ} cancel each other out and we need concern ourselves only with the electric and magnetic multipoles, $\vec{\mathcal{E}}^{\mu}_{\lambda}$ and $\vec{\mathcal{M}}^{\mu}_{\lambda}$, giving

$$\vec{A} = a\sum_{\lambda,\mu} b_{\mu}\left(\vec{\mathcal{E}}_{\lambda,\mu} + \mu\vec{\mathcal{M}}_{\lambda,\mu}\right) \qquad (2.64)$$

2.6 The Long Wavelength Approximation

We begin by writing a small argument ($\kappa r \ll \ell$) approximation for the spherical Bessel functions,

$$j_\ell(\kappa r) \simeq \frac{(\kappa r)^\ell}{(2\ell+1)!!}. \qquad (2.65)$$

From this expression, it is clear that for small arguments we can ignore $j_{\ell+1}(\kappa r)$ by comparison with $j_{\ell-1}(\kappa r)$. Since κ is the inverse of the wavelength, this condition is called the *long wavelength approximation.*

Now consider the case of the deuteron. The argument of the Bessel function in centre of mass coordinates is $\kappa r/2$, so that the long wavelength approximation is reasonable for sufficiently small photon energies. Moreover, if we calculate the first ℓ terms exactly, the approximation will be reasonable to higher energies for the following terms, since the condition for accuracy of the expression we stated at the start of this section was that the argument of the Bessel function be small relative to its order.

When making the long wavelength approximation we drop Bessel functions of two orders higher than those we keep, without reducing the Bessel function we keep to the polynomial form. This will make it simpler to calculate correction terms later on.

2.7 Siegert's Theorem

For electric multipoles in the long wavelength approximation we can use Siegert's theorem [22], allowing us to calculate the electric multipole transition matrix elements using only the charge distribution of the nucleus, i.e., without the necessity of knowing the current densities.

The electric multipole is,

$$\vec{\mathcal{E}}_{\lambda}^{\mu} = i^{\lambda+1}\sqrt{4\pi}\left(\sqrt{\lambda+1}\,j_{\lambda-1}\left(kr\right)\vec{Y}_{\lambda,\lambda-1,1}^{\mu}\left(\hat{r}\right) - \sqrt{\lambda}\,j_{\lambda+1}\left(kr\right)\vec{Y}_{\lambda,\lambda+1,1}^{\mu}\left(\hat{r}\right)\right), \qquad (2.66)$$

which in the long wavelength approximation is

$$\vec{\mathcal{E}}_{\lambda}^{\mu} \simeq i^{\lambda+1}\sqrt{4\pi}\sqrt{\lambda+1}\,j_{\lambda-1}\left(kr\right)\vec{Y}_{\lambda,\lambda-1,1}^{\mu}\left(\hat{r}\right). \qquad (2.67)$$

The longitudinal component of the multipole expansion,

$$\vec{\mathcal{L}}_{\lambda,\mu} = i^{\lambda-1}\sqrt{4\pi}\left(\sqrt{\lambda}\,j_{\lambda-1}(kr)\,\vec{Y}_{\lambda,\lambda-1,1}^{\mu}\left(\hat{r}\right) + \sqrt{\lambda+1}\,j_{\lambda+1}\left(kr\right)\vec{Y}_{\lambda,\lambda+1,1}^{\mu}\left(\hat{r}\right)\right) \qquad (2.68)$$

becomes in the long wavelength approximation

$$\vec{\mathcal{L}}_{\lambda,\mu} \simeq i^{\lambda-1}\sqrt{4\pi}\sqrt{\lambda}\,j_{\lambda-1}(kr)\,\vec{Y}_{\lambda,\lambda-1,1}^{\mu}\left(\hat{r}\right) \qquad (2.69)$$

so that we have

$$\vec{\mathcal{E}}_{\lambda}^{\mu} \simeq -\sqrt{\frac{\lambda+1}{\lambda}}\vec{\mathcal{L}}_{\lambda,\mu} \qquad (2.70)$$

which means that the $\vec{\mathcal{E}}_{\lambda}^{\mu}$ term in $\vec{J}\cdot\vec{A}$ becomes

$$\begin{aligned}
\int \vec{\mathcal{E}}_{\lambda}^{\mu}\cdot\vec{J}d^3r &= -\sqrt{\frac{\lambda+1}{\lambda}}\int\vec{\mathcal{L}}_{\lambda}^{\mu}\cdot\vec{J}d^3r \\
&= \left(\frac{i}{k}\right)\sqrt{\frac{\lambda+1}{\lambda}}\int\vec{\nabla}\phi_{\lambda}^{\mu}\cdot\vec{J}d^3r \\
&= -\left(\frac{\omega}{k}\right)\sqrt{\frac{\lambda+1}{\lambda}}\int\phi_{\lambda}^{\mu}\rho d^3r \qquad (2.71)
\end{aligned}$$

where the continuity equation has been used. This last line, valid only in the long wavelength approximation, is Siegert's theorem.

2.8 Correction Terms

Eventually we will want to correct for our use of the long wavelength approximation and Siegert's theorem. For the magnetic multipole terms we will simply review our original calculation of the matrix elements and calculate those terms which were dropped in the long wavelength approximation. These terms will be easily calculated by analogy with the long wavelength terms.

The correction of the electric multipole terms will be more difficult. We will have to correct for the part which we did not include when we ignored the second term in the expression for the electric multipole *and* subtract off the extra contribution from the second term of the longitudinal multipole which we implicitly included when substituting $\vec{\mathcal{L}}_\lambda^\mu$ for $\vec{\mathcal{E}}_\lambda^\mu$.

We call the *electric correction multipole* $\vec{\mathcal{K}}_\lambda^\mu$. This will be the difference between the exact electric multipole and the expression used after applying Siegert's theorem. Using expressions from sections 2.5 and 2.7 we have

$$\begin{aligned}
\vec{\mathcal{K}}_\lambda^\mu &= \vec{\mathcal{E}}_\lambda^\mu + \sqrt{\frac{\lambda+1}{\lambda}}\vec{\mathcal{L}}_{\lambda,\mu} \\
&= -i^{1+\lambda}\sqrt{4\pi\lambda}\, j_{\lambda+1}(kr)\,\vec{Y}^\mu_{\lambda,\lambda+1,1}(\hat{r}) \\
&+ i^{1+\lambda}\sqrt{\frac{\lambda+1}{\lambda}}\sqrt{4\pi(\lambda+1)}\, j_{\lambda+1}(kr)\,\vec{Y}^\mu_{\lambda,\lambda+1,1}(\hat{r}) \\
&= i^{1+\lambda}(-\lambda+(\lambda+1))\sqrt{\frac{4\pi}{\lambda}}\, j_{\lambda+1}(kr)\,\vec{Y}^\mu_{\lambda,\lambda+1,1}(\hat{r}) \\
&= i^{1+\lambda}\sqrt{\frac{4\pi}{\lambda}}\, j_{\lambda+1}(kr)\,\vec{Y}^\mu_{\lambda,\lambda+1,1}(\hat{r}) \qquad (2.72)
\end{aligned}$$

This done, we must calculate matrix elements for the electric correction multipole with both the convection *and* magnetization currents. This means that there will be electric correction spin flip terms, and singlet final states will be accessible to these correction terms, unlike the case for the Siegert terms.

Finally, we will not be able to algebraically manipulate the expressions so that the action of the $\vec{\nabla}$ operator is entirely on the angular part of the deuteron ground state as we will for the magnetic multipole terms. It will be necessary to take the derivatives of the radial parts of the deuteron ground state wave functions.

Chapter 3

Matrix Elements

3.1 Wave Functions

The differential cross section is given by equation 2.49 as

$$\frac{d\sigma}{d\Omega_p} = \frac{\pi k\sqrt{(\hbar ck)^2 + (mc^2)^2}}{\left(\frac{|\vec{v}_1 - \vec{v}_2|}{c}\right)(\hbar c)^3} \left(\frac{1}{(2)(2J+1)}\right) \sum_{m_{S'}, m_J, \mu} \left| \left\langle \Psi_f \left| \frac{\left(\vec{J}\cdot\vec{A}\right)}{c} \right| \Psi_i \right\rangle \right|^2 \tag{3.1}$$

with

$$\vec{A} = a \sum_{\lambda,\mu} b_\mu \left(\vec{\mathcal{E}}_{\lambda,\mu} + \mu\vec{\mathcal{M}}_{\lambda,\mu}\right) \tag{3.2}$$

where for unpolarized photons, $b_\mu = 1/\sqrt{2}$.

We have chosen to label the initial state by the photon polarization μ and the projection of the deuteron's total angular momentum m_j, and the final state by the projection of the spin of the combined neutron-proton system, $m_{s'}$. The factor of $2\,(2J+1)$ is due to the averaging over the projections of the initial states.

With this choice for the projections the final scattering state is

$$\begin{aligned} |\Psi_f\rangle &= \sum_{S'} e^{i\vec{k}\cdot\vec{r}} \chi_{S'}^{m_{S'}} \\ &= \sum_{\substack{L',S' \\ m_{L'}}} i^{L'} 4\pi j_{L'}(kr)\, Y_{L'}^{m_{L'}*}\left(\hat{k}\right) |L', m_{l'}\rangle\, |S', m_{s'}\rangle \end{aligned} \tag{3.3}$$

for plane wave final states. In L,S,J coupling the bra-space final states is

$$\langle\Psi_f| = \sum_{\substack{L',S' \\ m_{L'}}} (-i)^{L'}\, 4\pi Y_{L'}^{m_{L'}}\left(\hat{k}\right)$$

$$\times \sum_{J',m_{J'}} j_{L'}(kr)\, C\left(L', m_{L'}, S', m_{S'}, J', m_{J'}\right) \langle L', S'; J', m_{J'}| . \qquad (3.4)$$

which for distorted waves becomes

$$\begin{aligned} \langle\Psi_f| &= \sum_{\substack{L',S' \\ m_{L'}}} (-i)^{L'}\, 4\pi Y_{L'}^{m_{L'}}\left(\hat{k}\right) \\ &\times \sum_{J',m_{J'}} f_{L',S',J'}(kr)\, C\left(L', m_{L'}, S', m_{S'}, J', m_{J'}\right) \langle L', S'; J', m_{J'}| . \end{aligned} \qquad (3.5)$$

A slight discrepancy between the deuteron magnetic moment and the sum of the proton and neutron magnetic moments indicates that there is an admixture of a higher L component in the deuteron ground state. This is reasonable if we have a tensor component in the N↔N potential, but J is still a good quantum number, so that the admixture can only consist of $L = 1$ and $L = 2$ components. The $L = 1$ part can be ruled out as not having the same parity as the obviously dominant $L = 0$ part, leaving us with a deuteron ground state wave function

$$|\Psi_i\rangle = \sum_L \left(\frac{U_L(r)}{r}\right) |L, S, J, m_J\rangle \qquad (3.6)$$

where $U_L(r) = 0$ for $L \neq 0, 2$, and $S = J = 1$.

3.2 Electric Multipole Siegert Terms

For the electric multipoles,

$$\left\langle \Psi_f \left| \frac{\vec{J} \cdot \vec{\mathcal{E}}_\lambda^\mu}{c} \right| \Psi_i \right\rangle = -\sqrt{\frac{\lambda+1}{\lambda}} \langle \Psi_f | \phi_\lambda^\mu \rho | \Psi_i \rangle \qquad (3.7)$$

which has the explicit form

$$\left\langle \Psi_f \left| \frac{\vec{J} \cdot \vec{\mathcal{E}}_\lambda^\mu}{c} \right| \Psi_i \right\rangle = -\left(i^\lambda\right) e \sqrt{\frac{(\lambda+1)(2\lambda+1)}{\lambda}} \sqrt{4\pi} \left\langle \Psi_f \left| j_\lambda\left(\frac{\kappa}{2} r\right) Y_\lambda^\mu(\hat{r}) \right| \Psi_i \right\rangle . \qquad (3.8)$$

We adopt a convention for writing the radial integrals,

$$R(L', S', J', \lambda, L, S, J) = \int f_{L',S',J'}(kr)\, j_\lambda\left(\frac{\kappa}{2} r\right) U_L(r)\, dr \qquad (3.9)$$

and the matrix element is

$$\left\langle \Psi_f \left| j_\lambda\left(\frac{\kappa}{2} r\right) Y_\lambda^\mu(\hat{r}) \right| \Psi_i \right\rangle = 4\pi \sum_{\substack{L',S',J' \\ m_{L'},m_{J'}}} \sum_L (-i)^{L'} Y_{L'}^{m_{L'}}\left(\hat{k_f}\right) R\left(L',S',J',\lambda,L,S,J\right)$$
$$\times\ C\left(L',m_{L'},S',m_{S'};J',m_{J'}\right)$$
$$\times\ \left\langle L',S',J',m_{J'} \left| Y_\lambda^\mu(\hat{r}) \right| L,S,J,m_J \right\rangle . \qquad (3.10)$$

The Wigner-Ekhart theorem gives us

$$\left\langle L',S',J',m_{J'} \left| Y_\lambda^\mu(\hat{r}) \right| L,S,J,m_J \right\rangle = \frac{C\left(J,m_J,\lambda,\mu;J',m_{J'}\right)}{\sqrt{2J+1}}$$
$$\times\ \left\langle L',S',J' \| Y_\lambda(\hat{r}) \| L,S,J \right\rangle \qquad (3.11)$$

and applying standard techniques of tensor algebra gives us

$$\delta\left(S',S\right)\left\langle L',S',J' \| Y_\lambda(\hat{r}) \| L,S,J \right\rangle = \delta\left(S',S\right)(-1)^{L-S'+J'-\lambda} \frac{1}{\sqrt{4\pi}}$$
$$\times\ \sqrt{(2J+1)(2J'+1)(2\lambda+1)(2L+1)}$$
$$\times\ C\left(L,0,\lambda,0;L',0\right)$$
$$\times\ W\left(L',J',L,J;S',\lambda\right) \qquad (3.12)$$

where W is the Racah coefficient, so the electric multipole matrix element becomes

$$\left\langle \Psi_f \left| \frac{\vec{J}\cdot\vec{\mathcal{E}}_\lambda^\mu}{c} \right| \Psi_i \right\rangle = -4\pi i^\lambda e(2\lambda+1)\sqrt{\frac{(\lambda+1)}{\lambda}}$$
$$\times \sum_{\substack{L',J',L \\ m_{L'},m_{J'}}} (-i)^{L'} R\left(L',S,J',\lambda,L,S,J\right) C\left(L,0,\lambda,0;L',0\right)$$
$$\times\ \sqrt{(2L+1)(2J+1)}\,(-1)^{L-S+J'-\lambda}$$
$$\times\ W\left(L',J',L,J;S,\lambda\right) C\left(J,m_J,\lambda,\mu;J',m_{J'}\right)$$
$$\times\ C\left(L',m_{L'},S,m_{S'};J',m_{J'}\right) Y_{L'}^{m_{L'}}\left(\hat{k}\right) . \qquad (3.13)$$

3.3 Magnetic Multipole Convection Terms

The magnetic multipole matrix element is

$$\left\langle \Psi_f \left| \frac{\vec{J}_c \cdot \vec{\mathcal{M}}^{\mu}_{\lambda}}{c} \right| \Psi_i \right\rangle = -\left(\frac{ie\hbar}{2mc}\right) \int \left[\Psi_f^* \left(\vec{\nabla}_r \Psi_i\right) \cdot \vec{\mathcal{M}}^{\mu}_{\lambda} - \Psi_i \left(\vec{\nabla}_r \Psi_f^*\right) \cdot \vec{\mathcal{M}}^{\mu}_{\lambda}\right] d^3r$$

$$= -\left(\frac{ie\hbar}{2mc}\right) \int \left[\Psi_f^* \left(\vec{\nabla}_r \Psi_i\right) \cdot \vec{\mathcal{M}}^{\mu}_{\lambda} - \Psi_i \vec{\mathcal{M}}^{\mu}_{\lambda} \cdot \vec{\nabla}_r \Psi_f^*\right] d^3r \quad (3.14)$$

The juggling of the $\vec{\nabla}$ operator using vector identities and the divergence theorem is by now familiar, and we use it to write

$$\int \Psi_i \vec{\mathcal{M}}^{\mu}_{\lambda} \cdot \vec{\nabla}_r \Psi_f^* d^3r = -\int \Psi_f^* \left(\vec{\nabla}_r \cdot \Psi_i \vec{\mathcal{M}}^{\mu}_{\lambda}\right) d^3r \quad (3.15)$$

which we can easily rewrite as

$$-\int \Psi_f^* \left(\vec{\nabla}_r \cdot \Psi_i \vec{\mathcal{M}}^{\mu}_{\lambda}\right) d^3r = -\int \Psi_f^* \left(\vec{\mathcal{M}}^{\mu}_{\lambda} \cdot \vec{\nabla}_r \Psi_i\right) d^3r - \int \Psi_f^* \left(\vec{\nabla}_r \cdot \vec{\mathcal{M}}^{\mu}_{\lambda}\right) \Psi_i d^3r. \quad (3.16)$$

If we now choose the Coulomb gauge we have $\vec{\nabla}_r \cdot \vec{\mathcal{M}}^{\mu}_{\lambda} = 0$,so that we have

$$\left\langle \Psi_f \left| \frac{\vec{J}_c \cdot \vec{\mathcal{M}}^{\mu}_{\lambda}}{c} \right| \Psi_i \right\rangle = -\left(\frac{ie\hbar}{2mc}\right) \left[\int \Psi_f^* \left(\vec{\nabla}_r \Psi_i\right) \cdot \vec{\mathcal{M}}^{\mu}_{\lambda} d^3r + \int \Psi_f^* \vec{\mathcal{M}}^{\mu}_{\lambda} \cdot \left(\vec{\nabla}_r \Psi_i\right) d^3r\right]$$

$$= -\left(\frac{ie\hbar}{mc}\right) \left[\int \Psi_f^* \vec{\mathcal{M}}^{\mu}_{\lambda} \cdot \left(\vec{\nabla}_r \Psi_i\right) d^3r\right]. \quad (3.17)$$

Using the definition

$$\vec{\mathcal{M}}^{\mu}_{\lambda} \cdot \vec{\nabla}_r = \left(\frac{-\kappa}{2\sqrt{\lambda(\lambda+1)}}\right) \vec{r} \times \vec{\mathcal{L}}_{\lambda,\mu} \cdot \vec{\nabla}_r, \quad (3.18)$$

and using the triple product rule we have for the matrix element

$$\left\langle \Psi_f \left| \frac{\vec{J}_c \cdot \vec{\mathcal{M}}^{\mu}_{\lambda}}{c} \right| \Psi_i \right\rangle = -\left(\frac{e\hbar\kappa}{2mc\sqrt{\lambda(\lambda+1)}}\right) \left\langle \Psi_f \left| \vec{\mathcal{L}}_{\lambda,\mu} \cdot \vec{L} \right| \Psi_i \right\rangle. \quad (3.19)$$

In the long wavelength approximation,

$$\vec{\mathcal{L}}_{\lambda,\mu} \simeq i^{\lambda-1}\sqrt{4\pi}\sqrt{\lambda}\, j_{\lambda-1}\left(\frac{\kappa}{2}r\right) \vec{Y}^{\mu}_{\lambda,\lambda-1,1}(\hat{r}) \quad (3.20)$$

so that

$$\left\langle \Psi_f \left| \frac{\vec{J}_c \cdot \vec{\mathcal{M}}_\lambda^\mu}{c} \right| \Psi_i \right\rangle = i^{\lambda+1} \left(\frac{e\hbar\kappa}{mc} \right) \sqrt{\frac{\pi}{\lambda+1}} \left\langle \Psi_f \left| j_{\lambda-1}\left(\frac{\kappa}{2}r\right) \vec{Y}^\mu_{\lambda,\lambda-1,1}(\hat{r}) \cdot \vec{L} \right| \Psi_i \right\rangle . \tag{3.21}$$

Writing the wave functions explicitly, this is

$$\begin{aligned} \left\langle \Psi_f \left| j_{\lambda-1} \vec{Y}^\mu_{\lambda,\lambda-1,1} \cdot \vec{L} \right| \Psi_i \right\rangle &= \sum_{L',S',m_{L'}} (-i)^{L'} 4\pi Y_{L'}^{m_{L'}} \left(\hat{k}\right) \\ &\times \sum_{J',m_{J'}} \sum_{L,S} R(L',S',J',\lambda-1,L,S,J) \\ &\times C(L',m_{L'},S',m_{S'};J',m_{J'}) \\ &\times \sum_\sigma (-1)^\sigma C(\lambda-1,\mu+\sigma,1,-\sigma;\lambda,\mu) \\ &\times \left\langle L',S',J',m_{J'} \left| Y^{\mu+\sigma}_{\lambda-1} L_{-\sigma} \right| L,S,J,m_J \right\rangle . \end{aligned} \tag{3.22}$$

To evaluate the matrix element $\left\langle L',S',J',m_{J'} \left| Y^{\mu+\sigma}_{\lambda-1} L_{-\sigma} \right| L,S,J,m_J \right\rangle$ we expand the wave functions $|L,S,J,m_J\rangle$ as $|L,m_L\rangle\,|S,m_S\rangle$. For an operator $\mathcal{O}_\lambda^\mu$ which affects only L and not S we have

$$\begin{aligned} \langle L',S',J',m_{J'} |\mathcal{O}^\mu_\lambda| L,S,J,m_J\rangle &= \sum_{\alpha,\beta}\sum_{\alpha',\beta'} C(L',\alpha',S',\beta';J',m_{J'},) \\ &\times C(L,\alpha,S,\beta;J,m_J,) \\ &\times \langle L',\alpha' |\mathcal{O}^\mu_\lambda| L,\alpha\rangle \langle S',\beta' \mid S,\beta\rangle \\ &= \sum_{\alpha,\beta}\sum_{\alpha',\beta'} C(L',\alpha',S',\beta';J',m_{J'},) \\ &\times C(L,\alpha,S,\beta;J,m_J,) \\ &\times \langle L',\alpha' |\mathcal{O}^\mu_\lambda| L,\alpha\rangle \,\delta(S',S)\,\delta(\beta',\beta) . \end{aligned} \tag{3.23}$$

where α, β, and so on are summed over all possible projections of the angular momenta to which they correspond in the Clebsch-Gordon coefficients.

Using

$$L_{-\sigma}|L,\alpha\rangle = \sqrt{L(L+1)}C(L,\alpha,1,-\sigma;L,\alpha-\sigma)\,|L,\alpha-\sigma\rangle \tag{3.24}$$

and evaluating the matrix element of the spherical harmonics we get

$$\left\langle L', S', J', m_{J'} \left| Y_{\lambda-1}^{\mu+\sigma} L_{-\sigma} \right| L, S, J, m_J \right\rangle = \sqrt{\frac{2\lambda-1}{4\pi}} \sqrt{\frac{L(L+1)(2L+1)}{(2L'+1)}}$$
$$\times \; \delta(S', S)\, C(L, 0, \lambda-1, 0; L', 0)$$
$$\times \; \sum_{\beta} C(L, m_J - \beta, S, \beta; J, m_J)$$
$$\times \; C(L, m_J - \beta - \sigma, \lambda - 1, \mu + \sigma; L', m_J + \mu - \beta)$$
$$\times \; C(L, m_J - \beta, 1, -\sigma; L', m_J - \beta - \sigma)$$
$$\times \; C(L', m_J - \beta + \mu, S', \beta; J', m_{J'}) . \qquad (3.25)$$

so the transition matrix element is

$$\left\langle \Psi_f \left| \frac{\vec{J}_c \cdot \vec{\mathcal{M}}_{\lambda}^{\mu}}{c} \right| \Psi_i \right\rangle = i^{\lambda+1} \left(\frac{2\pi\kappa e\hbar}{mc} \right) \sqrt{\frac{2\lambda-1}{\lambda+1}} \sum_{L',L} (-i)^{L'} \sqrt{\frac{L(L+1)(2L+1)}{(2L'+1)}}$$
$$\times \; \sum_{J',S} R(L', S, J', \lambda-1, L, S, J)\, C(L, 0, \lambda-1, 0; L', 0)$$
$$\times \; C(L', m_J + \mu - m_{S'}, S, m_{S'}; J', m_J + \mu)\, Y_{L'}^{*m_J+\mu-m_{S'}}(\hat{k})$$
$$\times \; \sum_{\sigma,\beta} (-1)^{\sigma} C(\lambda-1, \mu+\sigma, 1, -\sigma; \lambda, \mu)$$
$$\times \; C(L, m_J - \beta, S, \beta; J, m_J)$$
$$\times \; C(L, m_J - \beta - \sigma, \lambda - 1, \mu + \sigma; L', m_J + \mu - \beta)$$
$$\times \; C(L, m_J - \beta, 1, -\sigma; L', m_J - \beta - \sigma)$$
$$\times \; C(L', m_J - \beta + \mu, S, \beta; J', m_{J'}) \qquad (3.26)$$

where we have used the orthogonality of the Clebsch-Gordon coefficients to perform the sums over as many of the projections as possible.

3.4 Magnetic Multipole Spin Terms

Recalling equation 2.42

$$\vec{J}_M = \left(\frac{e\hbar}{2m} \right) \vec{\nabla} \times \left[\Psi_f^* \left(\mu_p \vec{\sigma}_p - \mu_p \vec{\sigma}_n \right) \Psi_i \right] . \qquad (3.27)$$

the terms we have to calculate are of the form

$$\int \vec{\nabla} \times \left(\Psi_f^* \vec{\sigma}_N \Psi_i\right) \cdot \vec{\mathcal{M}}_\lambda^\mu d^3r. \tag{3.28}$$

With the vector identity $\left(\vec{\nabla} \times \vec{B}\right) \cdot \vec{A} = \vec{B} \cdot \left(\vec{\nabla} \times \vec{A}\right) - \vec{\nabla} \cdot \left(\vec{A} \times \vec{B}\right)$ the magnetic multipole spin term is

$$\begin{aligned} \int \vec{\nabla} \times \left(\Psi_f^* \vec{\sigma}_N \Psi_i\right) \cdot \vec{\mathcal{M}}_\lambda^\mu d^3r &= \int \left(\Psi_f^* \vec{\sigma}_N \Psi_i\right) \cdot \left(\vec{\nabla}_r \times \vec{\mathcal{M}}_\lambda^\mu\right) d^3r \\ &- \int \vec{\nabla} \cdot \left(\vec{\mathcal{M}}_\lambda^\mu \times \left(\Psi_f^* \vec{\sigma}_N \Psi_i\right)\right) d^3r \\ &= \int \left(\Psi_f^* \vec{\sigma}_N \Psi_i\right) \cdot \left(\vec{\nabla} \times \vec{\mathcal{M}}_\lambda^\mu\right) d^3r \end{aligned} \tag{3.29}$$

where the divergence theorem has been used to eliminate one of the terms.

The next obvious task is to calculate the curl of $\vec{\mathcal{M}}_\lambda^\mu$. We have

$$\begin{aligned} \vec{\nabla} \times \vec{\mathcal{M}}_\lambda^\mu &= i^\lambda \sqrt{4\pi (2\lambda+1)} \vec{\nabla} \times \left(j_\lambda\left(\frac{\kappa}{2}r\right) \vec{Y}_{\lambda,\lambda,1}^\mu (\hat{r})\right) \\ &= i^\lambda \sqrt{4\pi (2\lambda+1)} \sum_q (-1)^q \hat{\mathcal{X}}_{-q} \left(\vec{\nabla} \times \left(j_\lambda\left(\frac{\kappa}{2}r\right) \vec{Y}_{\lambda,\lambda,1}^\mu (\hat{r})\right)\right)_q \\ &= i^{\lambda+1} \sqrt{6\pi\lambda (2\lambda+1)} W(\lambda-1,1,\lambda,1;\lambda,1)\kappa \\ &\times j_{\lambda-1}\left(\frac{\kappa}{2}r\right) \sum_q C(\lambda-1,-(\mu+q),1,q;\lambda,-\mu) Y_{\lambda-1}^{\mu+q}(\hat{r}) \hat{\mathcal{X}}_{-q}. \end{aligned} \tag{3.30}$$

In evaluating $\vec{\nabla} \times \left(j_\lambda\left(\frac{\kappa}{2}r\right) \vec{Y}_{\lambda,\lambda,1}^\mu (\hat{r})\right)$ we have used the long wavelength approximation, specifically,

$$\begin{aligned} \left(\vec{\nabla} \times \left(j_\lambda\left(\frac{\kappa}{2}r\right) \vec{Y}_{\lambda,\lambda,1}^\mu (\hat{r})\right)\right)_q &= -i\sqrt{2} \sum_\sigma C(1,\sigma,1,q-\sigma;1,q) \\ &\times C(\lambda,\mu+q-\sigma,1,-(q-\sigma);\lambda,\mu) \\ &\times (-1)^{q-\sigma} \vec{\nabla}_\sigma j_\lambda\left(\frac{\kappa}{2}r\right) Y_\lambda^{\mu+(q-\sigma)}(\hat{r}) \end{aligned} \tag{3.31}$$

and

$$\begin{aligned} \vec{\nabla}_\sigma j_\lambda\left(\frac{\kappa}{2}r\right) Y_\lambda^{\mu+(q-\sigma)}(\hat{r}) &= \sum_{K=\lambda\pm1} \sqrt{\frac{2\lambda+1}{2K+1}} C(\lambda,0,1,0;K,0) \\ &\times C(\lambda,\mu+q-\sigma,1,\sigma;K,\mu+q) \\ &\times Y_K^{\mu+q}(\hat{r}) D^K \left(j_\lambda\left(\frac{\kappa}{2}r\right)\right). \end{aligned} \tag{3.32}$$

where,

$$D^{\lambda+1} = \frac{d}{dr} - \frac{\lambda}{r} \tag{3.33}$$

and

$$D^{\lambda-1} = \frac{d}{dr} + \frac{\lambda+1}{r}. \tag{3.34}$$

When operating on the Bessel functions these give

$$D^{\lambda+1}\left(j_\lambda\left(\frac{\kappa}{2}r\right)\right) = -\left(\frac{\kappa}{2}\right) j_{\lambda+1}\left(\frac{\kappa}{2}r\right) \tag{3.35}$$

and

$$D^{\lambda-1}\left(j_\lambda\left(\frac{\kappa}{2}r\right)\right) = \left(\frac{\kappa}{2}\right) j_{\lambda-1}\left(\frac{\kappa}{2}r\right) \tag{3.36}$$

so that ignoring the term for which $K = \lambda + 1$ meets our definition of the long wavelength approximation.

Returning our attention to equation 3.30 we consider more carefully the process of taking the curl of the vector potential $\vec{A}$. Our calculation has produced a multiplication by $\kappa/2$, but if we consider $\vec{\nabla}_{\vec{r}_N} \times \vec{A}(\vec{r}_N)$ and note that $\hat{\kappa} \perp \hat{\imath}, \hat{\jmath}$ and $\| \hat{k}$ then only the $\partial/\partial z$ term in the curl need be considered. Next, because $\hat{\epsilon} = \hat{\epsilon}^+$ or $\hat{\epsilon}^-$, both $\perp \hat{k}$, the curl reduces to $\hat{\jmath}\epsilon_x\partial/\partial z - \hat{\imath}\epsilon_y\partial/\partial z$ which gives

$$\begin{aligned}
\vec{\nabla}_{\vec{r}_N} \times \vec{A}(\vec{r}_N) &\propto \vec{\nabla}_{\vec{r}_N} \times \hat{\epsilon}^{\pm} e^{i\vec{\kappa}\cdot\vec{r}_N} \\
&\propto \hat{\epsilon}^{\mp} \frac{\partial}{\partial z_N} e^{i\kappa z_N} \\
&\propto \hat{\epsilon}^{\mp} i\kappa e^{i\kappa z_N} \\
&\propto \hat{\epsilon}^{\mp} i\kappa e^{i\vec{\kappa}\cdot\vec{r}_N}.
\end{aligned} \tag{3.37}$$

Now consider $\vec{\nabla}_{\vec{r}} \times \vec{A}(\vec{r})$. We still have the transversality conditions as before, but now

$$\begin{aligned}
\vec{\nabla}_{\vec{r}} \times \vec{A}(\vec{r}) &\propto \vec{\nabla}_{\vec{r}} \times \hat{\epsilon}^{\pm} e^{\frac{i\vec{\kappa}}{2}\cdot\vec{r}_N} \\
&\propto \hat{\epsilon}^{\mp} \frac{i\kappa}{2} e^{\frac{i\vec{\kappa}}{2}\cdot\vec{r}_N}.
\end{aligned} \tag{3.38}$$

so that where our calculation has produced a multiplication by $\kappa/2$ we must multiply by κ instead, giving

$$\vec{\nabla}\times\vec{\mathcal{M}}^{\mu}_{\lambda}=i^{\lambda+1}2\sqrt{6\pi\lambda\left(2\lambda+1\right)}W\left(\lambda-1,1,\lambda,1;\lambda,1\right)\kappa j_{\lambda-1}\left(\frac{\kappa}{2}r\right)\vec{Y}^{\mu}_{\lambda,\lambda-1,1}\left(\hat{r}\right). \tag{3.39}$$

where we have recoupled the vector spherical harmonics.

We now have matrix elements of the form

$$\left\langle\Psi_f\left|\frac{e\hbar}{2m}\vec{\nabla}\times\left(\mu_p\vec{\sigma}_p\cdot\vec{\mathcal{M}}^{\mu}_{\lambda}\left(\vec{r}\right)-\mu_n\vec{\sigma}_n\cdot\vec{\mathcal{M}}^{\mu}_{\lambda}\left(-\vec{r}\right)\right)\right|\Psi_i\right\rangle. \tag{3.40}$$

From the definition of $\vec{\mathcal{M}}^{\mu}_{\lambda}\left(\vec{r}\right)$ we can write

$$\vec{\mathcal{M}}^{\mu}_{\lambda}\left(-\vec{r}\right)\propto j_{\lambda}\left(-\frac{\kappa}{2}r\right)\vec{Y}^{\mu}_{\lambda,\lambda,1}\left(\hat{r}\right)=\left(-1\right)^{\lambda}j_{\lambda}\left(\frac{\kappa}{2}r\right)\vec{Y}^{\mu}_{\lambda,\lambda,1}\left(\hat{r}\right) \tag{3.41}$$

so that

$$\vec{\mathcal{M}}^{\mu}_{\lambda}\left(-\vec{r}\right)=\left(-1\right)^{\lambda}\vec{\mathcal{M}}^{\mu}_{\lambda}\left(\vec{r}\right). \tag{3.42}$$

This gives us

$$\begin{aligned}\left\langle\Psi_f\left|\frac{\vec{J_M}\cdot\vec{\mathcal{M}}^{\mu}_{\lambda}}{c}\right|\Psi_i\right\rangle &= \frac{e\hbar\kappa}{mc}\left(i\right)^{\lambda+1}\sqrt{6\pi\lambda\left(2\lambda+1\right)}W\left(\lambda-1,1,\lambda,1;\lambda,1\right)\\ &\times\left\langle\Psi_f\left|\left(\mu_p\vec{\sigma}_p-\left(-1\right)^{\lambda}\mu_n\vec{\sigma}_n\right)\cdot j_{\lambda-1}\left(\frac{\kappa}{2}r\right)\vec{Y}^{\mu}_{\lambda,\lambda-1,1}(\hat{r})\right|\Psi_i\right\rangle\end{aligned} \tag{3.43}$$

and if we now define $\mu^{\pm}=\mu_p\pm\mu_n$ and $\vec{S}^{\pm}=\vec{S}_p\pm\vec{S}_n=(1/2)(\vec{\sigma}_p\pm\vec{\sigma}_n)$ we can write

$$\begin{aligned}\left\langle\Psi_f\left|\frac{\vec{J_M}\cdot\vec{\mathcal{M}}^{\mu}_{\lambda}}{c}\right|\Psi_i\right\rangle &= \frac{e\hbar\kappa}{mc}\left(i\right)^{\lambda+1}\sqrt{6\pi\lambda\left(2\lambda+1\right)}W\left(\lambda-1,1,\lambda,1;\lambda,1\right)\\ &\times\left\langle\Psi_f\left|\left(\mu^{+}\vec{S}^{\pm}+\mu^{-}\vec{S}^{\mp}\right)\cdot j_{\lambda-1}\left(\frac{\kappa}{2}r\right)\vec{Y}^{\mu}_{\lambda,\lambda-1,1}\left(\hat{r}\right)\right|\Psi_i\right\rangle\end{aligned} \tag{3.44}$$

where for $\lambda=1,3,5\ldots$ we use the top sign and for $\lambda=2,4,6\ldots$ we use the bottom sign. This leaves us with the task of finding matrix elements of the form

$$\begin{aligned}\left\langle\Psi_f\left|\vec{S}^{\pm}\cdot\vec{Y}^{\mu}_{\lambda,\lambda,1}\left(\hat{r}\right)\right|\Psi_i\right\rangle &= \sum_{L',S',J'}\sum_{L,S}\sum_{m_{L'},m_{J'}}4\pi\left(-i\right)^{L'}Y^{m_{L'}}_{L'}\left(\hat{k}\right)\\ &\times C\left(L',m_{L'},S',m_{S'};J',m_{J'}\right)\\ &\times\left\langle L',S',J',m_{J'}\left|\vec{S}^{\pm}\cdot\vec{Y}^{\mu}_{\lambda,\lambda-1,1}\left(\hat{r}\right)\right|L,S,J,m_{J'}\right\rangle\end{aligned} \tag{3.45}$$

Writing out the dot product as a tensor product

$$\vec{S}^{\pm} \cdot \vec{Y}^{\mu}_{\lambda,\lambda-1,1} = \sum_q C(\lambda-1,\mu+q,1,-q;\lambda,\mu) \times \sum_{\Lambda} C(\lambda-1,\mu+q,1,-q;\Lambda,\mu)\left[Y^{\mu+q}_{\lambda-1}\otimes S^{\pm}{}^{-q}_{1}\right]^{\mu}_{\Lambda} \quad (3.46)$$

and applying the Wigner-Ekhart theorem

$$\langle L',S',J',m_{J'}|\left[Y^{\mu+q}_{\lambda-1}\otimes S^{\pm}{}^{-q}_{1}\right]^{\mu}_{\Lambda}|L,S,J,m_J\rangle = \frac{C(J,m_J,\Lambda,\mu;J',m_{J'})}{\sqrt{2J'+1}} \times \langle L',S',J'\|\left[Y^{\mu+q}_{\lambda-1}\otimes S^{\pm}{}^{-q}_{1}\right]_{\Lambda}\|L,S,J\rangle \quad (3.47)$$

The reduced matrix element can be decomposed into its orbital and spin components

$$\langle L',S',J'\|\left[Y^{\mu+q}_{\lambda-1}\otimes S^{\pm}{}^{-q}_{1}\right]_{\Lambda}\|L,S,J\rangle = \sqrt{(2J+1)(2J'+1)(2\Lambda+1)} \times X(L',S',J';L,S,J;\lambda-1,1,\Lambda) \times \langle S'\|S^{\pm}\|S\rangle\langle L'\|Y_{\lambda-1}\|L\rangle \quad (3.48)$$

where X is the 9-j symbol. $\langle L'\|Y_{\lambda-1}\|L\rangle$ is easily evaluated, but $\langle S'\|S^{\pm}\|S\rangle$ must be written out more fully as

$$\langle S'\|S^{\pm}\|S\rangle = \left\langle\frac{1}{2},\frac{1}{2},S'\right\| S_p \pm S_n \left\|\frac{1}{2},\frac{1}{2},S\right\rangle \quad (3.49)$$

where the factors of 1/2 are the spins of the proton and neutron. We decompose this to separate the proton and neutron spin functions thus,

$$\left\langle\frac{1}{2},\frac{1}{2},S'\right\| S_p \pm S_n \left\|\frac{1}{2},\frac{1}{2},S\right\rangle = -\sqrt{(2S+1)(2S'+1)}W\left(\frac{1}{2},S',\frac{1}{2},S;\frac{1}{2},1\right) \times \left[(-1)^{S'}\left\langle\frac{1}{2}\right\|S_p\left\|\frac{1}{2}\right\rangle \pm (-1)^{S}\left\langle\frac{1}{2}\right\|S_n\left\|\frac{1}{2}\right\rangle\right] \quad (3.50)$$

leaving only reduced matrix elements of the operators between their eigenstates.

The transition matrix element is then

$$\begin{aligned}
\left\langle \Psi_f \left| \frac{\vec{J}_M \cdot \vec{\mathcal{M}}_\lambda^\mu}{c} \right| \Psi_i \right\rangle &= -(i)^{\lambda+1} \left[\left(\frac{12\pi e\hbar\kappa}{mc} \right) \sqrt{\lambda(2\lambda-1)}(2\lambda+1) \right] \\
&\times W(\lambda-1,1,\lambda,1;\lambda,1) \\
&\times \sum_{L',S',J'} \sum_{L,S} (-i)^{L'} Y_{L'}^{m_J+\mu-m_{S'}}\left(\hat{k}\right) \\
&\times C(\lambda,0,\lambda-1,0;L',0) \\
&\times R(L',S',J',\lambda-1,L,S,J) \\
&\times \sqrt{(2S+1)(2S'+1)(2J+1)(2L+1)} \\
&\times C(J,m_J,\lambda,\mu;J',m_J+\mu) \\
&\times C(L',m_J+\mu-m_{S'},S',m_{S'};J',m_J+\mu) \\
&\times W\left(\frac{1}{2},S',\frac{1}{2},S;\frac{1}{2},1\right) \\
&\times X(L',S',J';L,S,J;\lambda-1,1,\lambda) \\
&\times \left((-1)^{S'}\mu_p \pm (-1)^S \mu_n\right) \qquad (3.51)
\end{aligned}$$

3.5 Corrections to Siegert Terms

3.5.1 Convection Terms

Recall equation 2.72

$$\vec{\mathcal{K}}_\lambda^\mu = i^{1+\lambda}\sqrt{\frac{4\pi}{\lambda}}\, j_{\lambda+1}\left(\frac{\kappa}{2}r\right) \vec{Y}_{\lambda,\lambda+1,1}^\mu(\hat{r}). \qquad (3.52)$$

By proceeding in exactly the same way as we did in section 3.3 we get

$$\begin{aligned}
\left\langle \Psi_f \left| \frac{\vec{J}_c \cdot \vec{\mathcal{K}}_\lambda^\mu}{c} \right| \Psi_i \right\rangle &= -\left(\frac{ie\hbar}{2mc}\right) \left\langle \Psi_f \left| \vec{\mathcal{K}}_\lambda^\mu \cdot \vec{\nabla} \right| \Psi_i \right\rangle \\
&= -\left(\frac{e\hbar}{2mc}\right) \left[\left\langle \Psi_f \left| \frac{1}{r^2} \vec{\mathcal{K}}_\lambda^\mu \cdot \vec{r} \times \vec{L} \right| \Psi_i \right\rangle \right. \\
&+ \left. i \left\langle \Psi_f \left| \frac{1}{r} \vec{\mathcal{K}}_\lambda^\mu \cdot \vec{r} \frac{\partial}{\partial r} \right| \Psi_i \right\rangle \right] \qquad (3.53)
\end{aligned}$$

where we have used the identity

$$\vec{\nabla} = \frac{\vec{r}}{|r|}\frac{\partial}{\partial r} - \frac{i}{r^2}\vec{r}\times\vec{L}. \tag{3.54}$$

Beginning with the first term

$$\frac{1}{r^2}\vec{\mathcal{K}}^{\mu}_{\lambda}\cdot\left(\vec{r}\times\vec{L}\right) = i^{\lambda+1}\sqrt{\frac{4\pi}{\lambda}}\frac{1}{r^2}j_{\lambda+1}\left(\frac{\kappa}{2}r\right)\left[\vec{Y}^{\mu}_{\lambda,\lambda+1,1}(\hat{r})\cdot\left(\vec{r}\times\vec{L}\right)\right] \tag{3.55}$$

and with

$$\vec{r} = \sum_{\alpha}(-1)^{\alpha}\hat{\mathcal{X}}_{-\alpha}\sqrt{\frac{4\pi}{3}}\,rY_1^{\alpha}(\hat{r}) \tag{3.56}$$

we have

$$\vec{r}\times\vec{L} = -i\sqrt{2}\sum_{q}(-1)^{q}\hat{\mathcal{X}}_{-q}\sum_{\alpha}\sqrt{\frac{4\pi}{3}}\,rY_1^{\alpha}(\hat{r})\,L_{q-\alpha}C(1,\alpha,1,q-\alpha;1,q) \tag{3.57}$$

In terms of the angular functions $C^{\mu}_{\lambda} = \sqrt{4\pi/(2\lambda+1)}Y^{\mu}_{\lambda}$ this gives us

$$\begin{aligned}\vec{Y}^{\mu}_{\lambda,\lambda+1,1}(\hat{r})\cdot\left(\vec{r}\times\vec{L}\right) &= -i\sqrt{\frac{2\lambda+1}{2\pi}}\,r\sum_{\Lambda=\lambda,\lambda+2}\sum_{\alpha,\beta}(-1)^{-\beta}\\ &\times\ C(1,\alpha,1,-(\alpha+\beta);1,-\beta)\\ &\times\ C(\lambda+1,\mu+\beta,1,-\beta;\lambda,\mu)\\ &\times\ C(\lambda+1,\mu+\beta,1,\alpha;\Lambda,\mu+\alpha+\beta)\\ &\times\ C(\lambda+1,0,1,0;\Lambda,0)\\ &\times\ C^{\mu+\beta+\alpha}_{\Lambda}L_{-(\alpha+\beta)}\end{aligned} \tag{3.58}$$

so

$$\begin{aligned}\frac{1}{r^2}\vec{\mathcal{K}}^{\mu}_{\lambda}\cdot\left(\vec{r}\times\vec{L}\right) &= i^{\lambda}\sqrt{\frac{2(2\lambda+1)}{\lambda}}\,\frac{1}{r}j_{\lambda+1}\left(\frac{\kappa}{2}r\right)\sum_{\Lambda=\lambda,\lambda+2}\sum_{\alpha,\beta}(-1)^{-\beta}\\ &\times\ C(1,\alpha,1,-(\alpha+\beta);1,-\beta)\\ &\times\ C(\lambda+1,\mu+\beta,1,-\beta;\lambda,\mu)\\ &\times\ C(\lambda+1,\mu+\beta,1,\alpha;\Lambda,\mu+\alpha+\beta)\\ &\times\ C(\lambda+1,0,1,0;\Lambda,0)\\ &\times\ C^{\mu+\beta+\alpha}_{\Lambda}L_{-(\alpha+\beta)}.\end{aligned} \tag{3.59}$$

The matrix element of $C_{\Lambda}^{\mu+\beta+\alpha} L_{-(\alpha+\beta)}$ is easily evaluated in direct analogy with the evaluation of equation 3.25 giving at last

$$
\begin{aligned}
\left\langle \Psi_f \left| \frac{1}{r^2} \vec{\mathcal{K}}_{\lambda}^{\mu} \cdot \vec{r} \times \vec{L} \right| \Psi_i \right\rangle &= \sum_{\substack{L',J' \\ L}} \sum_{\Lambda=\lambda,\lambda+2} \sum_{\substack{\alpha,\beta \\ \sigma}} (i)^{L'+\lambda} (-1)^{L'-\beta} 4\pi \\
&\times \sqrt{\frac{2(2\lambda+1) L(L+1)(2L+1)}{\lambda(2L'+1)}} Y_{L'}^{m_J+\mu-m_{S'}}\left(\hat{k}\right) \\
&\times C(L', m_J+\mu-m_{S'}, 1, m_{S'}; J', m_J+\mu) \\
&\times R(L', S, J', \lambda+1, L, S, J) \\
&\times C(1, \alpha, 1, -(\alpha+\beta); 1, -\beta) \\
&\times C(\lambda+1, \mu+\beta, 1, -\beta; \lambda, \mu) \\
&\times C(\lambda+1, \mu+\beta, 1, \alpha; \Lambda, \mu+\alpha+\beta) \\
&\times C(\lambda+1, 0, 1, 0; \Lambda, 0) \\
&\times C(L, 0, \Lambda, 0; L', 0) \\
&\times C(L', m_J+\mu-\sigma, 1, \sigma; J', m_J+\mu) \\
&\times C(L, m_J-\sigma, 1, \sigma; 1, m_J) \\
&\times C(L, m_J-\sigma-\alpha-\beta, \Lambda, \mu+\alpha+\beta; L', m_J+\mu-\sigma) \\
&\times C(L, m_J-\sigma, 1, -(\alpha+\beta); L, m_J-\sigma-(\alpha+\beta)).
\end{aligned}
\tag{3.60}
$$

For the second term we write the operator as

$$\frac{1}{r} \vec{\mathcal{K}}_{\lambda}^{\mu} \cdot \vec{r} \frac{\partial}{\partial r} = i^{\lambda+1} \sqrt{\frac{4\pi}{\lambda}} \frac{1}{r} j_{\lambda+1}\left(\frac{\kappa}{2} r\right) \left[\vec{Y}_{\lambda,\lambda+1,1}^{\mu}(\hat{r}) \cdot \vec{r}\right] \frac{\partial}{\partial r}. \tag{3.61}$$

The dot product

$$\vec{Y}_{\lambda,\lambda+1,1}^{\mu}(\hat{r}) \cdot \vec{r} = \sum_{\beta} (-1)^{-\beta} r \sqrt{\frac{4\pi}{3}} C(\lambda+1, \mu+\beta, 1, -\beta; \lambda, \mu) Y_{\lambda+1}^{\mu+\beta}(\hat{r}) Y_1^{-\beta}(\hat{r}) \tag{3.62}$$

becomes

$$\vec{Y}_{\lambda,\lambda+1,1}^{\mu} \cdot \vec{r} = r \sqrt{\frac{2\lambda+1}{4\pi}} \sum_{\Lambda=\lambda,\lambda+2} \sum_{\beta} (-1)^{-\beta}$$

$$\times \quad C(\lambda+1,\mu+\beta,1,-\beta;\lambda,\mu)$$
$$\times \quad C(\lambda+1,\mu+\beta,1,-\beta;\Lambda,\mu)$$
$$\times \quad C(\lambda+1,0,1,0;\Lambda,0)\,C^{\mu}_{\Lambda} \tag{3.63}$$

so that the operator is

$$\begin{aligned}\frac{1}{r}\vec{\mathcal{K}}^{\mu}_{\lambda}\cdot\vec{r}\frac{\partial}{\partial r} &= (i)^{\lambda+1}\sqrt{\frac{2\lambda+1}{\lambda}}j_{\lambda+1}\sum_{\Lambda=\lambda,\lambda+2}\sum_{\beta}(-1)^{-\beta}\\ &\times \quad C(\lambda+1,\mu+\beta,1,-\beta;\lambda,\mu)\\ &\times \quad C(\lambda+1,\mu+\beta,1,-\beta;\Lambda,\mu)\\ &\times \quad C(\lambda+1,0,1,0;\Lambda,0)\,C^{\mu}_{\Lambda}\frac{\partial}{\partial r}.\end{aligned} \tag{3.64}$$

The matrix element is now easy to evaluate. We define

$$Q(L',S',J',\lambda,L,S,J) = \int f_{L',S',J'}(kr)\,j_{\lambda}\!\left(\frac{\kappa}{2}r\right)\left(\left(\frac{\partial}{\partial r}\frac{U_L(r)}{r}\right)\right)r\,dr \tag{3.65}$$

and the matrix element of the second term is

$$\begin{aligned}i\left\langle\Psi_f\left|\frac{1}{r}\vec{\mathcal{K}}^{\mu}_{\lambda}\cdot\vec{r}\frac{\partial}{\partial r}\right|\Psi_i\right\rangle &= \sum_{L',J',L}(i)^{L'+\lambda+1}(-1)^{J'+L'-\lambda-1}(12\pi)\sqrt{\frac{2\lambda+1}{\lambda}}\\ &\times \quad Q(L',1,J',\lambda+1,L,1,1)\,Y^{m_J+\mu-m_{S'}}_{L'}\left(\hat{k}\right)\\ &\times \quad C(L',m_J+\mu-m_{S'},1,m_{S'};J',m_J+\mu)\\ &\times \quad \sum_{\Lambda=\lambda,\lambda+2}C(\lambda+1,0,1,0;\Lambda,0)\\ &\times \quad C(1,m_J,\Lambda,\mu;J',m_J+\mu)\\ &\times \quad C(1,0,\Lambda,0;J',0)\\ &\times \quad W(L',J',L,1;1,\Lambda)\\ &\times \quad \sum_{\beta}(-1)^{-\beta}C(\lambda+1,\mu+\beta,1,-\beta;\lambda,\mu)\\ &\times \quad C(\lambda+1,\mu+\beta,1,-\beta;\Lambda,\mu).\end{aligned} \tag{3.66}$$

3.5.2 Magnetization Terms

This entire matrix element is calculated by an almost complete analogy with the calculation of the magnetic multipole interaction with the magnetization current calculated in section 3.4.

Recall equation 2.42,

$$\vec{J}_M = \left(\frac{e\hbar}{2m}\right) \vec{\nabla}_r \times \left[\Psi_f^* \left(\mu_p \vec{\sigma}_p - \mu_n \vec{\sigma}_n\right) \Psi_i\right]. \tag{3.67}$$

This will give us matrix elements of the form

$$\int \vec{\nabla} \times \left(\Psi_f^* \vec{\sigma}_N \Psi_i\right) \cdot \vec{\mathcal{K}}_\lambda^\mu d^3r = \int \left(\Psi_f^* \vec{\sigma}_N \Psi_i\right) \cdot \left(\vec{\nabla} \times \vec{\mathcal{K}}_\lambda^\mu\right) d^3r \tag{3.68}$$

giving us terms like

$$\left\langle \Psi_f \middle| \mu \vec{\sigma} \cdot \left(\vec{\nabla} \times j_{\lambda+1}\left(\frac{\kappa}{2}r\right) \vec{Y}_{\lambda,\lambda+1,1}^\mu(\hat{r})\right) \middle| \Psi_i \right\rangle. \tag{3.69}$$

We find $\vec{\nabla} \times \left(j_{\lambda+1}\left(\frac{\kappa}{2}r\right) \vec{Y}_{\lambda,\lambda+1,1}^\mu(\hat{r})\right)$ by the method used in equation 3.31 giving

$$\begin{aligned} \left\langle \Psi_f \middle| \frac{\vec{J}_M \cdot \vec{\mathcal{K}}_\lambda^\mu}{c} \middle| \Psi_i \right\rangle &\simeq -(i)^\lambda \frac{e\hbar\kappa}{mc} \sqrt{\frac{3\pi(\lambda+1)}{2\lambda}} \\ &\times W(\lambda, 1, \lambda+1, 1; \lambda+1, 1) \\ &\times \left[\left\langle \Psi_f \middle| \mu_p \vec{\sigma}_p \cdot \left(j_\lambda\left(\frac{\kappa}{2}r\right) \vec{Y}_{\lambda+1,\lambda,1}^\mu(\hat{r})\right) \middle| \Psi_i \right\rangle\right. \\ &- \left.(-1)^\lambda \left\langle \Psi_f \middle| \mu_n \vec{\sigma}_n \cdot \left(j_\lambda\left(\frac{\kappa}{2}r\right) \vec{Y}_{\lambda+1,\lambda,1}^\mu(\hat{r})\right) \middle| \Psi_i \right\rangle\right]. \end{aligned} \tag{3.70}$$

By the same methods as gave us 3.44 we get

$$\begin{aligned} &\left\langle \Psi_f \middle| \left(\mu_p \vec{\sigma}_p + (-1)^\lambda \mu_n \vec{\sigma}_n\right) \cdot \left(j_\lambda\left(\frac{\kappa}{2}r\right) \vec{Y}_{\lambda+1,\lambda,1}^\mu(\hat{r})\right) \middle| \Psi_i \right\rangle \\ = &\sum_{L',S',J'} \sum_{L,S} \sum_{m_{L'},m_{J'}} 4\pi(-i)^{L'} Y_{L'}^{m_{L'}}\left(\hat{k}\right) C\left(L', m_{L'}, S', m_{S'}; J', m_{J'}\right) \\ \times &\left\langle L', S', J', m_{J'} \middle| \left(\mu^+ \vec{S}^\pm + \mu^- \vec{S}^\mp\right) \cdot j_\lambda\left(\frac{\kappa}{2}r\right) \vec{Y}_{\lambda+1,\lambda,1}^\mu(\hat{r}) \middle| L, S, J, m_J \right\rangle \end{aligned} \tag{3.71}$$

where now for $\lambda = 1, 3, 5\ldots$ we use the *bottom* sign and for $\lambda = 2, 4, 6\ldots$ we use the *top* sign. The matrix element $\left\langle L', S', J', m_{J'} \left| \vec{S}^{\pm} \cdot \vec{Y}^{\mu}_{\lambda+1,\lambda,1}(\hat{r}) \right| L, S, J, m_J \right\rangle$ is evaluated in exactly the same way as equations 3.45 through 3.50 giving us the final result,

$$\begin{aligned}
\left\langle \Psi_f \left| \frac{\vec{J}_M \cdot \vec{\mathcal{K}}^{\mu}_{\lambda}}{c} \right| \Psi_i \right\rangle &= -(i)^{\lambda} \frac{12\pi e\hbar\kappa}{mc} \sqrt{\frac{(\lambda+1)(2\lambda+1)(2\lambda+3)}{\lambda}} \\
&\times W(\lambda, 1, \lambda+1, 1; \lambda+1, 1) \\
&\times \sum_{L',S',J'} \sum_{L,S} (-i)^{L'} Y_{L'}^{m_J+\mu-m_{s'}}\left(\hat{k}\right) \\
&\times C(L, 0, \Lambda, 0; L', 0) \\
&\times R(L', S', J', \lambda, L, S, J) \\
&\times \sqrt{(2S+1)(2S'+1)(2J+1)(2L+1)} \\
&\times C(J, m_J, \lambda+1, \mu; J', m_J+\mu) \\
&\times C(L', m_{L'}, S', m_{S'}; J', m_J+\mu) \\
&\times W\left(\frac{1}{2}, S', \frac{1}{2}, S; \frac{1}{2}, 1\right) \\
&\times X(L', S', J'; L, S, J; \lambda, 1, \lambda+1) \\
&\times \left((-1)^{S'} \mu_p \pm (-1)^{S} \mu_n\right). \qquad (3.72)
\end{aligned}$$

3.6 Corrections to Magnetic Convection Terms

The correction term is

$$\left\langle \Psi_f \left| \frac{\vec{J}_c \cdot \vec{\mathcal{N}}^{\mu}_{\lambda}}{c} \right| \Psi_i \right\rangle = i^{\lambda+1} \left(\frac{e\hbar\kappa}{mc}\right) \sqrt{\frac{\pi}{\lambda}} \left\langle \Psi_f \left| j_{\lambda+1}\left(\frac{\kappa}{2} r\right) \vec{Y}^{\mu}_{\lambda,\lambda+1,1}(\hat{r}) \cdot \vec{L} \right| \Psi_i \right\rangle. \qquad (3.73)$$

The matrix element becomes

$$\begin{aligned}
\left\langle \Psi_f \left| j_{\lambda+1} \vec{Y}^{\mu}_{\lambda,\lambda+1,1} \cdot \vec{L} \right| \Psi_i \right\rangle &= \sum_{L',S',m_{L'}} (-i)^{L'} 4\pi Y_{L'}^{m_{L'}}\left(\hat{k}\right) \\
&\times \sum_{J',m_{J'}} \sum_{L,S} R(L', S', J', \lambda+1, L, S, J) \\
&\times C(L', m_{L'}, S', m_{S'}; J', m_{J'})
\end{aligned}$$

$$\times \sum_{\sigma} (-1)^{\sigma} C(\lambda+1, \mu+\sigma, 1, -\sigma; \lambda, \mu)$$
$$\times \left\langle L', S', J', m_{J'} \left| Y_{\lambda+1}^{\mu+\sigma} L_{-\sigma} \right| L, S, J, m_J \right\rangle \quad (3.74)$$

and following the steps which gave us equation 3.25 we have

$$\begin{aligned}
\left\langle L', S', J', m_{J'} \left| Y_{\lambda+1}^{\mu+\sigma} L_{-\sigma} \right| L, S, J, m_J \right\rangle &= \sqrt{\frac{2\lambda+3}{4\pi}} \sqrt{\frac{L(L+1)(2L+1)}{(2L'+1)}} \\
&\times \delta(S', S)\, C(L, 0, \lambda+1, 0; L', 0) \\
&\times \sum_{\beta} C(L, m_J-\beta, S, \beta; J, m_J) \\
&\times C(L, m_J-\beta-\sigma, \lambda+1, \mu+\sigma; L', m_J+\mu-\beta) \\
&\times C(L, m_J-\beta, 1, -\sigma; L, m_J-\beta-\sigma) \\
&\times C(L', m_J-\beta+\mu, S', \beta; J', m_{J'}), \qquad (3.75)
\end{aligned}$$

so the matrix element is

$$\begin{aligned}
\left\langle \Psi_f \left| \frac{\vec{J}_c \cdot \vec{\mathcal{N}}_{\lambda}^{\mu}}{c} \right| \Psi_i \right\rangle &= i^{\lambda+1} \left(\frac{2\pi e \hbar \kappa}{mc} \right) \sqrt{\frac{2\lambda+3}{\lambda}} \sum_{L', J'} \sum_{L,S} (-i)^{L'} Y_{L'}^{m_J+\mu-m_{S'}} \left(\hat{k}\right) \\
&\times R(L', S', J', \lambda+1, L, S, J) \\
&\times C(L', m_J+\mu-m_{S'}, S', m_{S'}; J', m_J+\mu) \\
&\times \sqrt{\frac{L(L+1)(2L+1)}{(2L'+1)}} C(L, 0, \lambda+1, 0; L', 0) \\
&\times \sum_{\sigma,\beta} (-1)^{\sigma} C(\lambda+1, \mu+\sigma, 1, -\sigma; \lambda, \mu) \\
&\times C(L, m_J-\beta-\sigma, \lambda+1, \mu+\sigma; L', m_J+\mu-\beta) \\
&\times C(L, m_J-\beta, S, \beta; J, m_J) \\
&\times C(L, m_J-\beta, 1, -\sigma; L, m_J-\beta-\sigma) \\
&\times C(L', m_J-\beta+\mu, S, \beta; J', m_J+\mu). \qquad (3.76)
\end{aligned}$$

3.7 Corrections to Magnetic Spin Terms

We add to equation 3.30 a correction term $\vec{\mathcal{W}}_\lambda^\mu$ giving for an operator

$$\begin{aligned}\vec{\nabla}\times\vec{\mathcal{W}}_\lambda^\mu &= i^\lambda\sqrt{6\pi(\lambda+1)(2\lambda+1)}W(\lambda+1,1,\lambda,1;\lambda,1)\kappa \\ &\times\ j_{\lambda+1}\left(\frac{\kappa}{2}r\right)\sum_q C(\lambda+1,-(\mu+q),1,q;\lambda,-\mu)Y_{\lambda+1}^{\mu+q}(\hat{r})\hat{\mathcal{X}}_{-q}.\end{aligned} \tag{3.77}$$

This gives us

$$\begin{aligned}\left\langle\Psi_f\left|\frac{\vec{J_M}\cdot\vec{\mathcal{W}}_\lambda^\mu}{c}\right|\Psi_i\right\rangle &= \frac{e\hbar\kappa}{mc}(i)^{\lambda+1}\sqrt{6\pi(\lambda+1)(2\lambda+1)}W(\lambda+1,1,\lambda,1;\lambda,1) \\ &\times\ \left\langle\Psi_f\left|\left(\mu^+\vec{S}^\pm-\mu^-\vec{S}^\mp\right)\cdot j_{\lambda+1}\left(\frac{\kappa}{2}r\right)\vec{Y}_{\lambda,\lambda+1,1}^\mu(\hat{r})\right|\Psi_i\right\rangle\end{aligned} \tag{3.78}$$

and the matrix element is

$$\begin{aligned}\left\langle\Psi_f\left|\frac{\vec{J_M}\cdot\vec{\mathcal{W}}_\lambda^\mu}{c}\right|\Psi_i\right\rangle &= -(i)^{\lambda+1}\left[\left(\frac{12\pi e\hbar\kappa}{mc}\right)\sqrt{(\lambda+1)(2\lambda+3)(2\lambda+1)}\right] \\ &\times\ W(\lambda+1,1,\lambda,1;\lambda,1) \\ &\times\ \sum_{L',S',J'}\sum_{L,S}(-i)^{L'}Y_{L'}^{m_J+\mu-m_{S'}}\left(\hat{k}\right) \\ &\times\ C(L,0,\lambda+1,0;L',0) \\ &\times\ R(L',S',J',\lambda+1,L,S,J) \\ &\times\ \sqrt{(2S+1)(2S'+1)(2J+1)(2L+1)} \\ &\times\ C(J,m_J,\lambda,\mu;J',m_J+\mu) \\ &\times\ C(L',m_J+\mu-m_{S'},S',m_{S'};J',m_J+\mu) \\ &\times\ W\left(\frac{1}{2},S',\frac{1}{2},S;\frac{1}{2},1\right) \\ &\times\ X(L',S',J';L,S,J;\lambda+1,1,\lambda) \\ &\times\ \left((-1)^{S'}\mu_p\pm(-1)^S\mu_n\right)\end{aligned} \tag{3.79}$$

where for $\lambda=1,3,5\ldots$ we use the top sign and for $\lambda=2,4,6\ldots$ we use the bottom sign.

Chapter 4

Results

4.1 Initial and Scattering States

To evaluate the expressions derived in the previous chapters a program was written and executed on the DEC Station 3100. Radial wave functions for the scattering states and the deuteron ground states were calculated by programs based on reference [51] with additional deuteron ground states coming from reference [52] for comparison. Although the wave functions of reference [51] explicitly contain the Δ components in the ground state, only the nucleon components were used in the calculation.

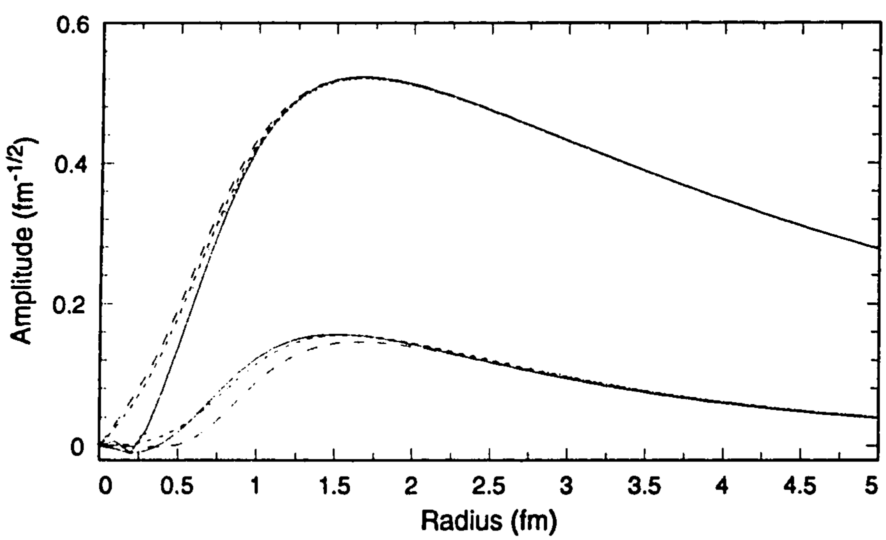

Figure 4.1: Nucleon S (upper) and D (lower) ground state radial wave functions: Solid, [51]; long dash, 4.38% D-state [52]; medium dash, 4.99% D-state [52]; short dash, 5.61% D-state [52].

Figure 4.1 shows the radial wave functions for the deuteron ground state. The top curves are the S-states components, and the bottom are the D-state components. The principle difference between the three wave functions of [52] and the wave function [51] is the behavior at small radii. For this reason we expect that any differences in our calculation will occur mainly at higher energies, where the deuteron is probed to a smaller radius.

4.2 The Total Cross Section

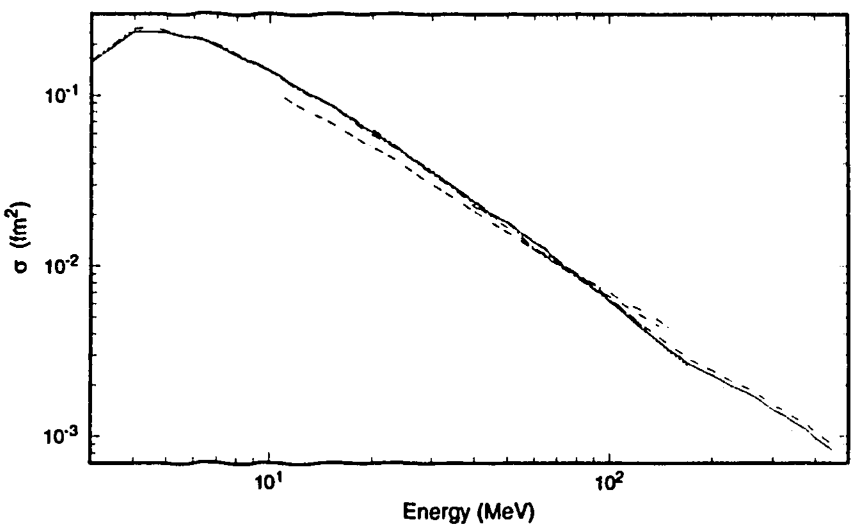

Figure 4.2: Theoretical curves for the total cross section: Solid, using [51]; dash and dot using 5.61% and 4.38% D-state [52];dot dash,[55]; double dot dash,[46]; double dash dot,[66].

Figure 4.2 shows theoretical curves for the total cross section from just above threshold to 500 MeV on a log-log plot. The calculations shown from the literature are calculations performed in the same manner as the current calculation, having in them no meson exchange currents beyond those which are included by the use of Siegert's theorem, no consideration of the Δ resonance, and no relativistic corrections, i.e., no spin orbit terms. The different calculations are in general agreement with each other, with the calculation of [66] being the only significantly deviating line below 100 MeV. By 500 MeV the effect of the different radial wave functions has become pronounced, and a 1.3% difference in the D-state probabilities between the ground states of [52] causes a 20% difference in the total cross section.

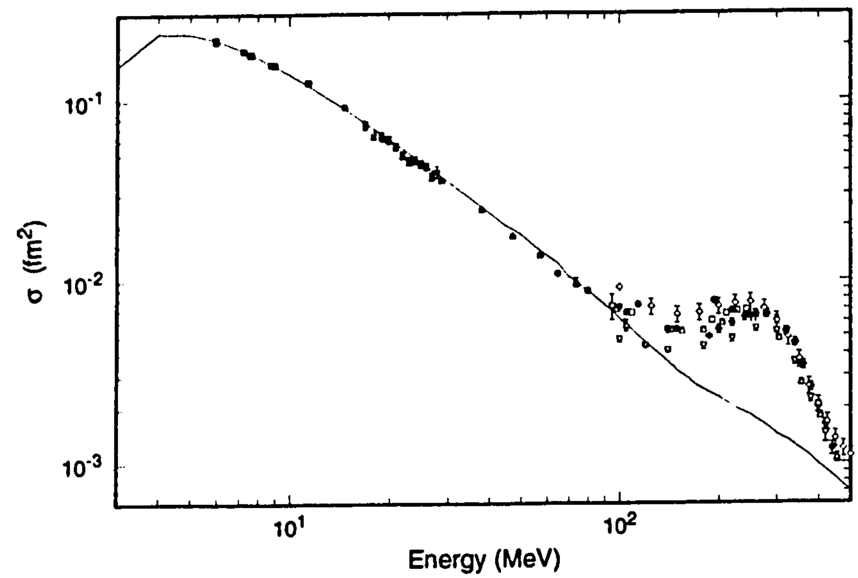

Figure 4.3: Total cross section with experimental points: Lowest energy open square, [29]; open up triangles, [32]; open diamond, [33]; open down triangle, [35]; solid circles, [37]; solid squares, [38]; solid up triangles, [39]; solid diamonds, [40]; crosses in delta region, [41]; solid down triangles, [42]; pluses, [34]; higher energy open squares, [43]; crosses at low energy, [54].

Figure 4.3 shows the total cross section calculated with the ground state of reference [51] (nucleon components only), which will be used unless otherwise noted from here on, and a selection of data points from experiments. The agreement with experiment is excellent up to about 100 MeV, in spite of the absence of relativistic corrections and meson effects beyond those included by the use of Siegert's theorem. After 100 MeV, the experimental cross section begins to rise rapidly due to the presence of the $\Delta(1236)$ resonance. The calculation does not follow the data since no attempt was made to include the Δ in the calculation. At 500 MeV, when the delta should no longer be a significant contributor to the total cross section, the data is still about 30% higher than the calculation. This is not surprising since the calculation was performed without relativistic corrections, without meson exchange currents beyond those included through the use of Siegert's theorem and using only a limited number of multipoles, and therefore should not necessarily be valid at higher energies.

4.3 Differential Cross Sections

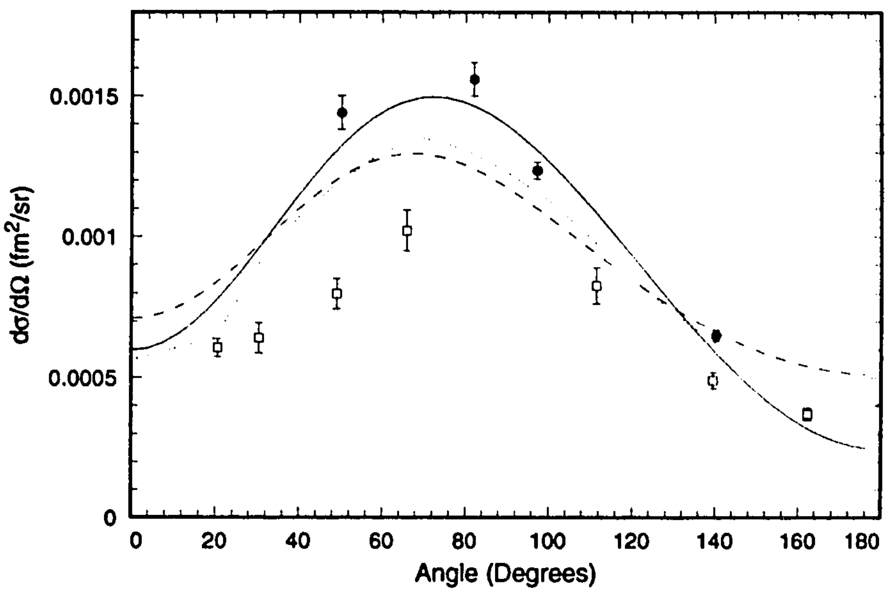

Figure 4.4: Differential cross section at 60 MeV: Solid, present work; dash, [55]; dot, [70]; open squares, [60]; solid circles, [30].

Figure 4.4 shows the differential cross section at 60 MeV. The squares are data from 1955 while the circles are from a monochromatic 1983 experiment, and the circles should therefore be conside[illegible] the more reliable data points. The present work then appears to fit the data better than the calculation of [55]. The principle differences between this calculation and that of [55] are the radial wave functions used for the scattering states and deuteron ground states. The calculation of [70] includes meson exchange and delta effects, but the majority of the difference between it and the present calculation is due to the relativistic corrections which it also contains, and which interferes destructively with the rest of the calculation to reduce the cross section in general.

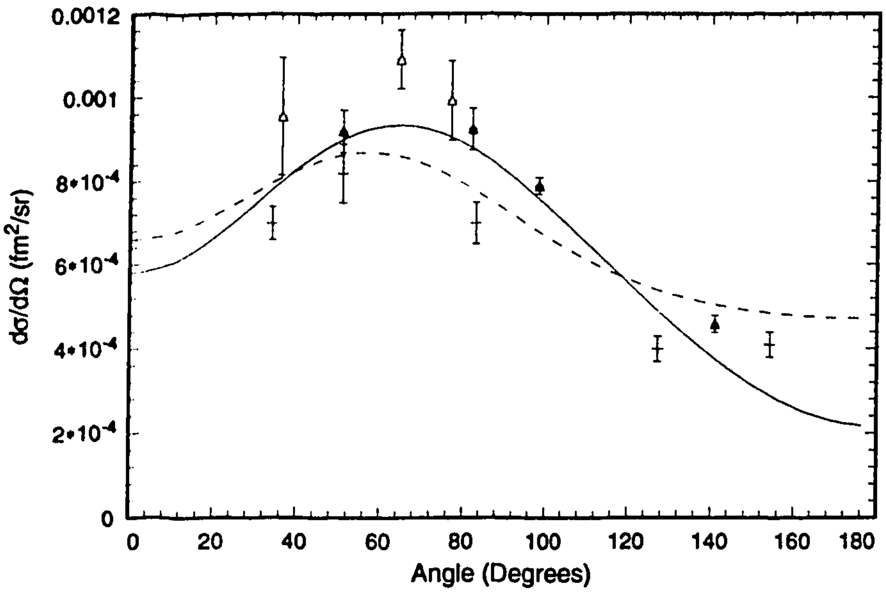

Figure 4.5: Differential cross section at 80 MeV: Solid, present work; dashed line, [55]; open triangles, [33]; plus, [37]; solid triangles, [59].

Figure 4.5 shows the differential cross section at 80 MeV. It is not possible to make a choice between the different calculations. The data are scattered and contradictory, and there is no reason to prefer the results of one of the experiments over the results of the others.

Figure 4.6 shows several theoretical calculations at 100 MeV, and one at 107 MeV which is close enough for reasonable comparison. The striking discrepancy between the calculations of [55] and [66] is not due to the 7% difference in photon energy. The discrepancy results from the neglect of higher order and correction terms in [66], and in differences between radial wave functions. The agreement of the present calculations with either of these two calculations, while not inspiring,

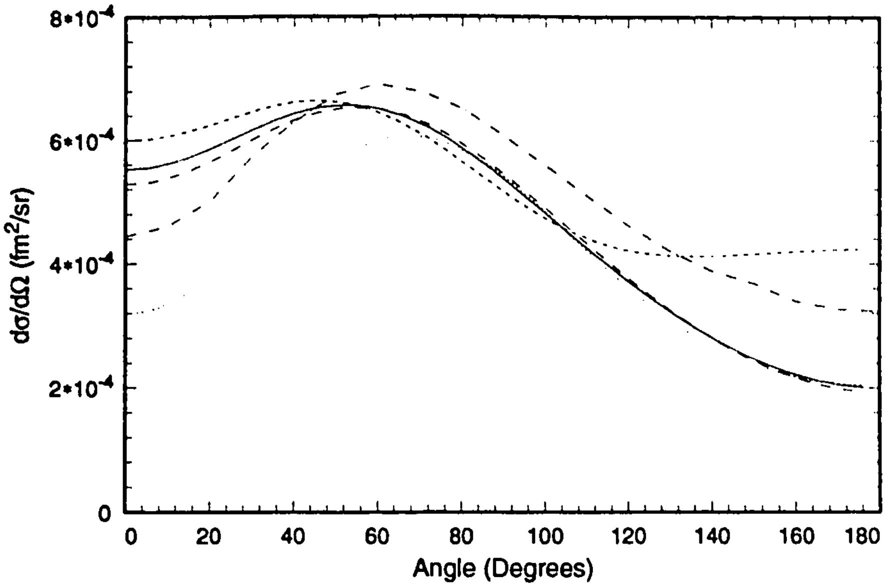

Figure 4.6: Theoretical curves for the differential cross section at 100 MeV: Solid, present work; long dash, 4.38% [52]; medium dash, [55]; short dash, 107MeV[66]; dot dash[70].

is better than their agreement with each other. The calculation of [70] including the meson exchange and Δ resonance effects and relativistic corrections is also shown. The dominant effect of these three is again the inclusion of relativistic corrections, which lowers the cross section throughout. The effect of the meson exchange current and Δ effects was to increase the cross section, especially around 90 degrees and slightly larger angles, so that the combined effect has been a lowering of the cross section at forward and backward angles.

Figure 4.7 shows the differential cross section at 100 MeV with data from various experiments. There is no convincing agreement with the data from either the present calculation or the "classical" calculation of [55] which contains essentially the same physics. The discrepancy with data at small angles is a problem common to many such deuteron photodisintegration calculations, and is related to the neglect

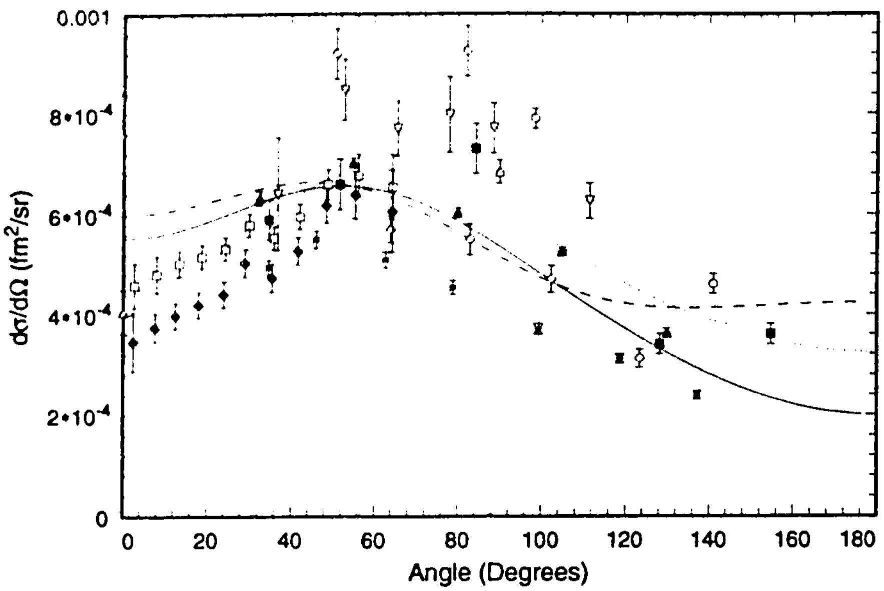

Figure 4.7: Differential cross section at 100 MeV with experimental points: Solid, present work; dash, [55]; dot[70]; open circles, [27]; open squares, [29]; open up triangles, [32]; diamonds, [33]; down triangles, [35]; crosses [37]; solid squares [44]; solid up triangles [58].

relativistic corrections and meson exchange terms beyond those in the Siegert operators, as evidenced by the improved agreement of the calculation of [70] which includes these terms.

Figure 4.8 shows the effects of the different matrix elements on the differential cross section. The addition of the magnetic correction terms makes a barely discernible difference to the cross section at this energy, so different lines are not shown for the addition of the electric and magnetic correction terms. It is clear that the correction terms should already be included at this energy, since the forward angle cross section is about 13% lower with the correction terms.

Figure 4.9 shows the differential cross section at 140 MeV, the last energy at which we can reasonably hope for any agreement before the effects of the Δ resonance completely overwhelm our calculation. While the data are scattered and somewhat

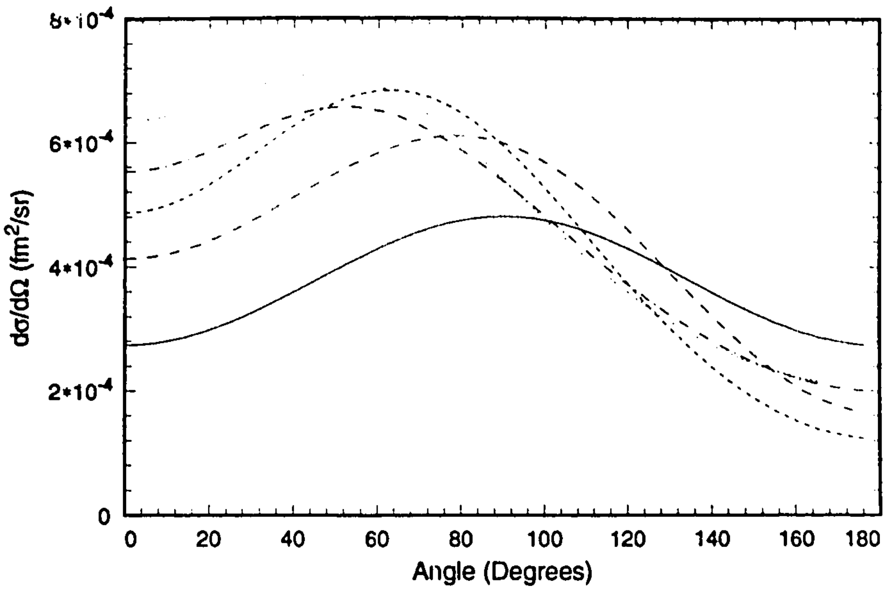

Figure 4.8: Effects of different terms on the differential cross section at 100 MeV: Solid, E1; long dash, E1&M1; med. dash, E1,E2&M1; short dash, E1,E2,E3,M1&M2; dash-dot, E1,E2,E3,M1&M2 and correction terms.

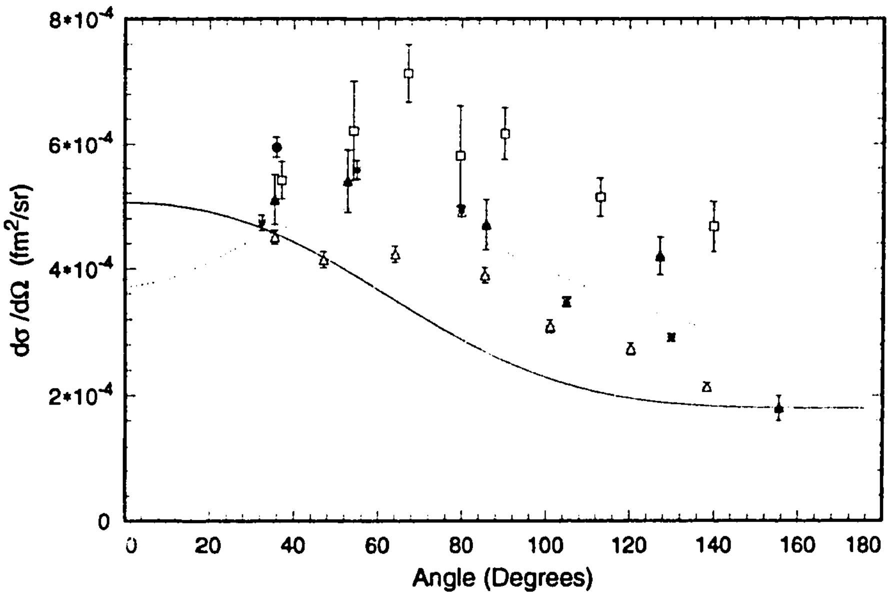

Figure 4.9: Differential cross section at 140 MeV: Solid line, present work; dot [70]; solid circle[34]; open triangles, [35]; solid triangles, [37]; crosses, [44]; squares [33].

contradictory, the results with a monochromatic photon beam experiment (crosses) are to be preferred. The amplitude is a little higher than the calculation, and the general shape appears to be better described by the calculation of [70], the most probable explanation being that we are beginning to see in earnest the effects of the Δ.

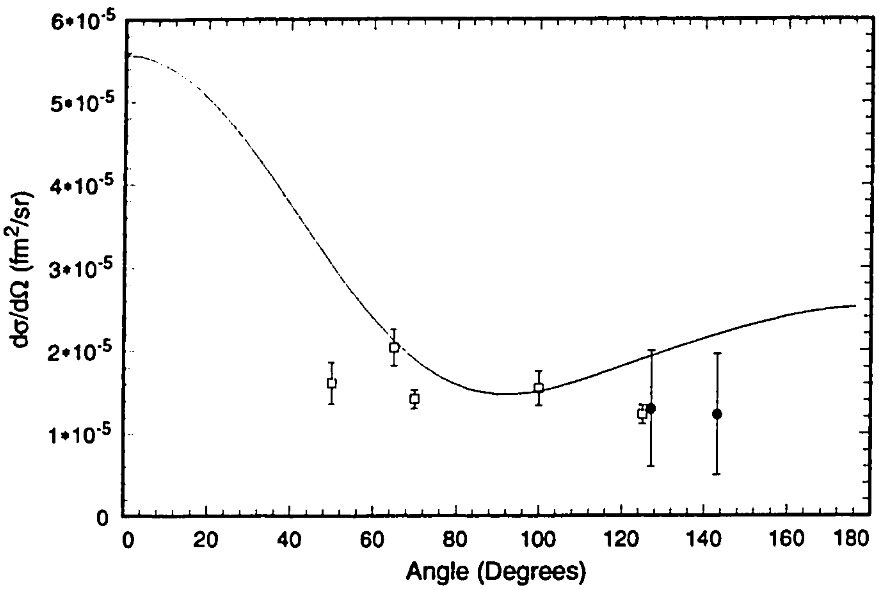

Figure 4.10: Differential cross section at 800 MeV:Solid, present work, open squares, [53]; solid circles,[34]

Figure 4.10 shows the differential cross section at 800 MeV. While we abandoned comparison with experiment after 140 MeV due to the dominance of the delta resonance, it is our hope that at energies beyond the delta resonance, our calculation will again show some agreement with experiment. Since high energy photons will probe the deuteron at small radii, the ground state wave functions we have used here may give cross sections substantially different from what would be achieved with more conventional wave functions such as those of reference [52].

4.4 Photon Asymmetry

For linearly polarized photons we can define in the notation of [55] the photon asymmetry $\Sigma(\theta)$ by the equation

$$\frac{d\sigma}{d\Omega_p} = I_o(\theta)\left[1 + \Sigma_l \Sigma(\theta) \cos(2\phi)\right]. \tag{4.1}$$

Here Σ_l is the degree of linear polarization, $\Sigma_l = \left(\Sigma_x^2 + \Sigma_y^2\right)^{1/2}$ where Σ_x and Σ_y are the Stoke parameters of the photon. It is an easy matter to construct the matrix elements for linearly polarized photons from our matrix elements for circularly polarized photons, and the asymmetry is then calculated as the difference between the cross sections with linearly polarized photons, $\Sigma_l = 1$, and unpolarized photons, $\Sigma_l = 0$, divided by the differential cross section, at $\phi = 0$.

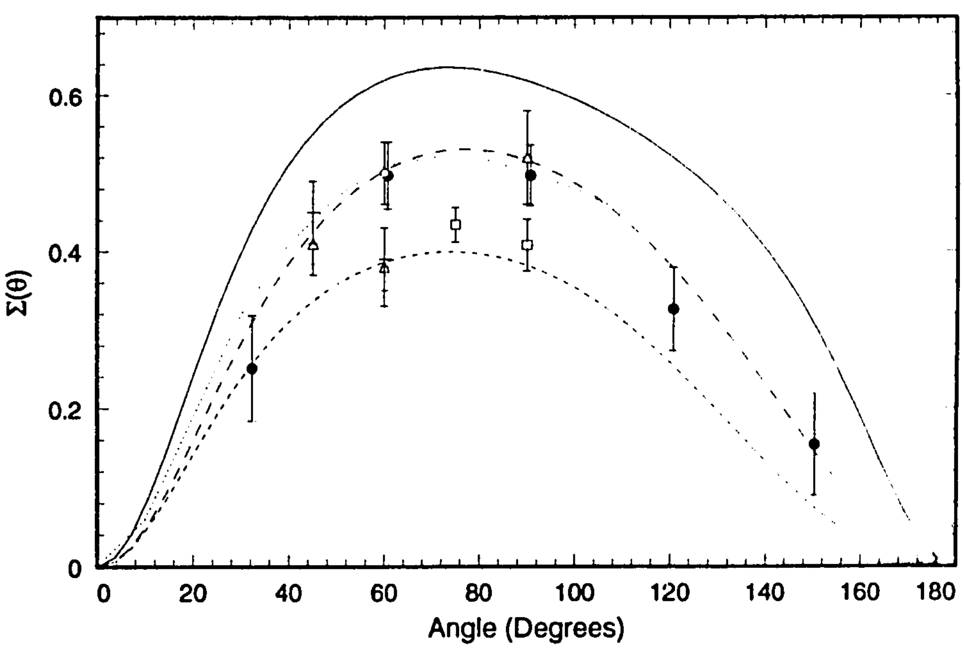

Figure 4.11: Photon asymmetry at 60 MeV: Solid, present work; long dash, [55]; med dash, [66]; dot, [70]; solid circles, [62]; open squares,[63]; open triangles, [64]; pluses, [67];open circle, [68].

Figure 4.11 shows the photon asymmetry as a function of angle for 60 MeV photons. The general shape of the theoretical curves is similar for all three theoretical curves. The calculations of [55] and [70] are very similar and appear to offer the best fit to the data at this energy. The calculation of [66] appears a little low, with the current calculation being somewhat high relative to the data.

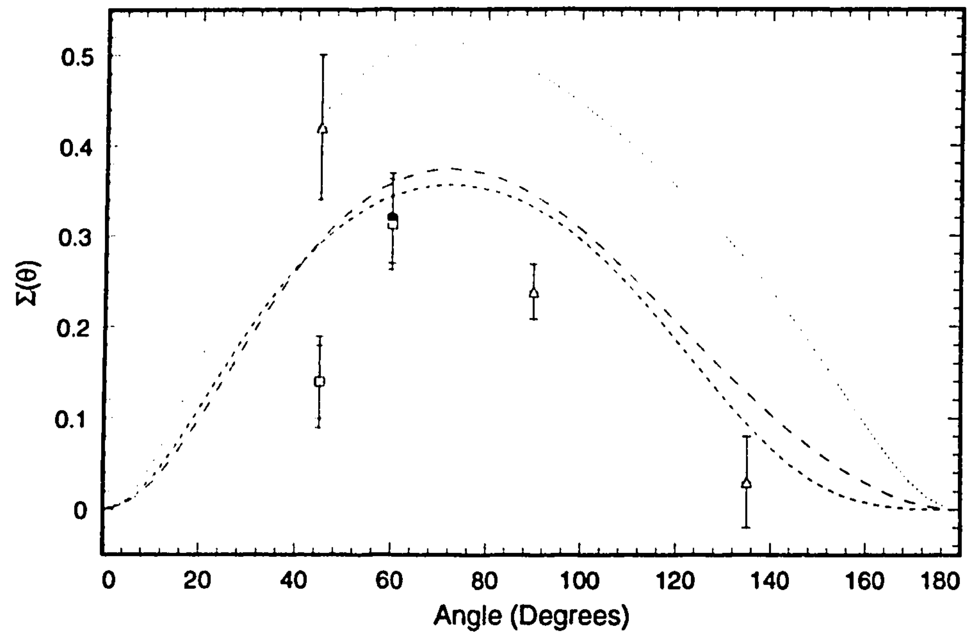

Figure 4.12: Photon asymmetry at 80 MeV: Dot, present work; long dash, [55], med. dash, [66]; solid circles, [64]; open squares,[67]; up triangles, [69].

Figure 4.12 shows the photon asymmetry as a function of angle for 80 MeV photons. None of the curves seem to be in good agreement with the data. It is surprising to note that the calculations of [55] and [66] are becoming more similar, since the principle difference between them is that the later lacks some of the higher order terms and correction terms in the multipole expansion which one expects should give an increasingly large contribution with increasing energy.

Figure 4.13 shows the photon asymmetry as a function of angle for 100 MeV photons. Again, none of the curves is in agreement with the data, although [55] and [70] substantially agree with each other.

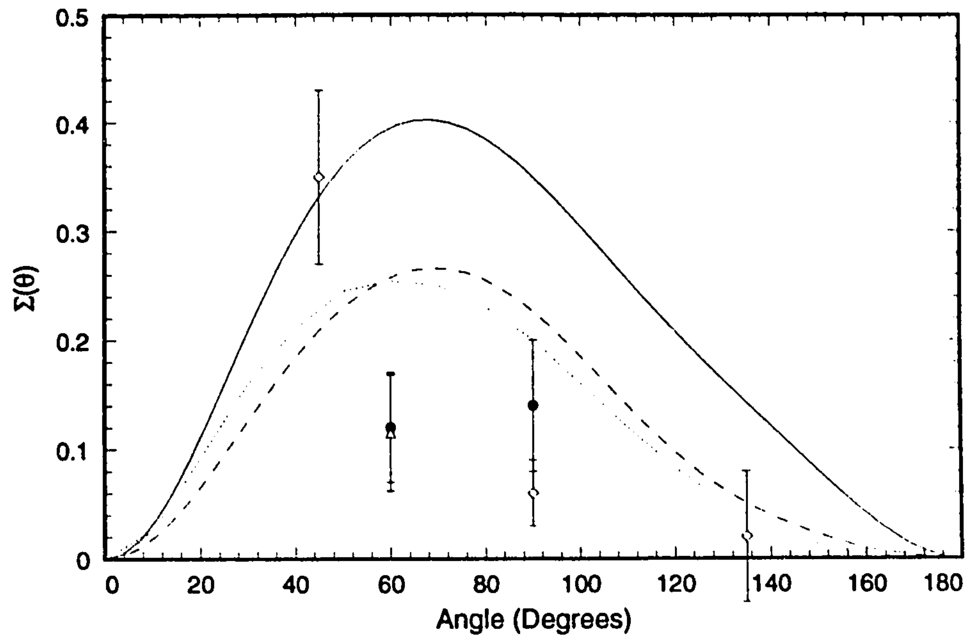

Figure 4.13: Photon asymmetry at 100 MeV: Solid, present work; dash, [55]; dot, [70]; solid circles, [64]; triangles, [67]; diamonds, [69].

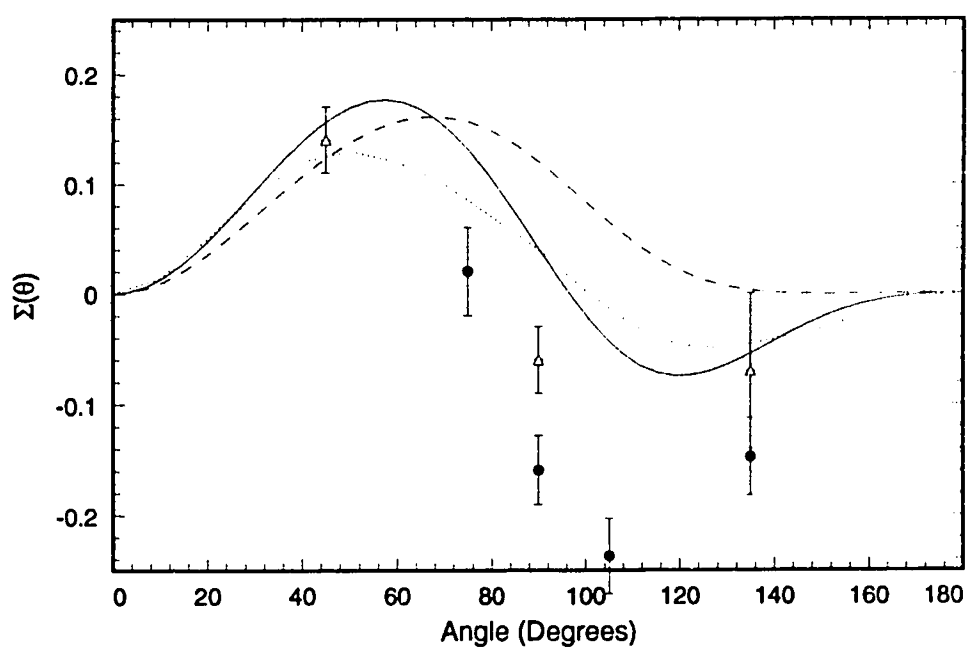

Figure 4.14: Photon asymmetry at 140 MeV: Solid, present work; dash, [55]; dot, [70]; open triangles, [69]; solid circles, [65].

Figure 4.14 shows the photon asymmetry as a function of angle for 140 MeV photons. No fit between the data and any of the calculations is apparent. The calculation of [55] is no longer at all similar to the present calculation, which is actually in reasonable agreement with the calculations of [70]. In both this and the preceding graph, it should be noted that the photon asymmetry is highly model dependent. In particular, reference [70] shows a variety of other calculations with presumably identical physics to the curves shown from that reference, but having used different potential models to calculate the ground and scattering states. The differences between these different calculations are at least as large as the differences between the calculations shown and the data.

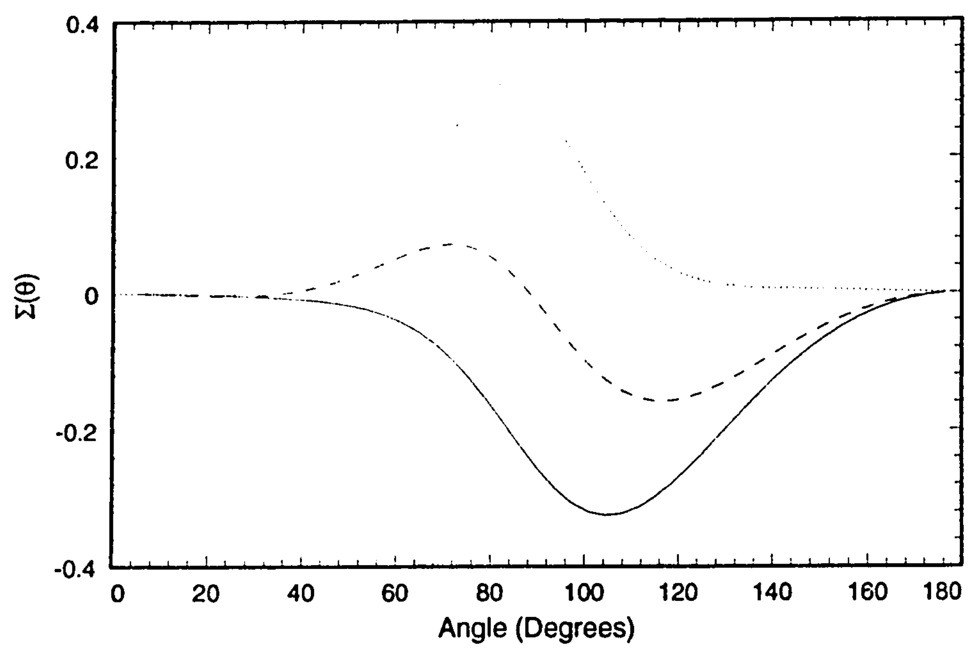

Figure 4.15: Photon asymmetry at higher energies: Solid, 500MeV; long dash, 700MeV; short dash, 900MeV.

Figure 4.15 shows the photon asymmetry for a variety of energies above the delta resonance. There is no data to compare to as yet, but data should be available from the Mainz accelerator in the near future. It is to be expected that meson exchange currents and relativistic corrections will be important at these energies, and little can be said other than that the curves of figure 4.15 show what the contributions of the nucleon components to the photon asymmetries look like.

Chapter 5

Conclusion

The agreement of the present calculation with data for the total cross section is very good up to about 100 MeV, after which point the $\Delta(1232)$ resonance, which is not included in this calculation, overwhelms the total cross section.

The agreement of the differential cross section with data is not quite so satisfying. Although there is still some similarity between the data and the present calculation, the fit achieved by the calculation of [70] is significantly better, especially at small angles. Such agreement as is achieved is mainly fortuitous, due largely to the fact that meson exchange and isobar currents tend to increase the cross section while relativistic corrections tend to decrease it, so that the two effects cancel each other out to a great extent. None the less, the shape is significantly affected by these terms by 100 MeV, and even more so by 140 MeV.

Photon asymmetry also did not exhibit good agreement between theory and experiment. Even at 60 MeV, the present calculation was significantly higher than the data, while the calculations of [70], including isobar currents, meson exchange currents, and relativistic corrections, and of [55], which has essentially the same content in terms of physics as the present calculation, both show reasonable agreement with the data. At this energy, the most likely source of the discrepancy of the present calculation with the data is the extreme sensitivity of the photon asymmetry to the potential model chosen.

By 140 MeV, the calculations of [70] and [55] are not as close to the data as the present calculation which, however, does not itself agree with the data. Such similarity to the data as exists at this energy must be attributed to coincidence, since it is not reasonable to expect that the physics included in the present calculation is sufficient to produce an accurate description of the processes involved at this energy.

What the present calculation has done is only to examine the nucleon contributions to the cross sections and photon asymmetries using radial wave functions from reference [51]. As expected, the effects of isobar and meson exchange currents and still more importantly of relativistic corrections are significant, especially at higher energies, as is obvious from a comparison of our results to calculations including these effects such as the curves shown from reference [70]. For a calculation to fully examines the influence of using the wave functions of [51] on the cross section and photon asymmetry these effects must be included.

Bibliography

[1] J. Chadwick and M. Goldhaber, Nature, *134,* 237 (1934)

[2] H. Bethe and R. Peierls, Proc. Roy. Soc., *A148,* 146 (1935)

[3] H. Bethe and C. L ys. Rev., *77,* 647 (1950)

[4] E. Fermi, Phys. Rev (1935)

[5] M Rose and G. Goertzel, Phys Rev., *72,* 749 (1947)

[6] J. Marshall and E. Guth, Phys. Rev., *78,* 738 (1950)

[7] J. Levinger, Phys. Rev., *76,* 699L (1949)

[8] L. Schiff, Phys. Rev., *78,* 733 (1950)

[9] T. Hu and H. Massey, Proc. Roy. Soc., *A196,* 135 (1949)

[10] J. Berger, Phys. Rev., *94,* 1698 (1934)

[11] J. deSwart and R. Marshak, Physica, *25,* 1001 (1959)

[12] J. deSwart, Physica, *25,* 233 (1959)

[13] G. Kramer and D. Müller, Z. Phys., *158,* 204 (1960)

[14] F. Partovi, Ann. Phys. (N. Y.), *27,* 79 (1964)

[15] M. Rustgi, W. Zernik, G. Breit, and D. Andrews, Phys. Rev., *120,* 1881 (1960)

[16] A. Donnachie, Nucl. Phys., *32,* 637 (1962)

[17] A. Donnachie, Nucl. Phys., *37,* 594 (1962)

[18] A. Donnachie and P. O'Donnell, Nucl. Phys., *53,* 128 (1964)

[19] A. Donnachie and P. O'Donnell, Nucl. Phys., *59,* 385 (1964)

[20] H. Feshbach and J. Schwinger, Phys. Rev., *84,* 194 (1951)

[21] M Goldberger and K. Watson, *Collision Theory*, John Wiley and Sons, (1964)

[22] A. J. F. Siegert, Phys. Rev., *52,* 787 (1937)

[23] J. D. Jackson, *Classical Electrodynamics*, John Wiley and Sons, (1975)

[24] H. C. Lee, *Nuclear Charge, Convection Current, and Magnetization Current Densities*, Atomic Energy of Canada Limited, (1975)

[25] H. Horie et. al., *Tables of the Racah Coefficients*, Pan Pacific Press, Tokyo, Japan (1960)

[26] E. De Sanctis, M. Anghinolfi, G.P. Capitani, P. Corvisiero, P. Di Giacomo, C. Guaraldo, V. Lucherini, E. Polli, A.R. Reolon, G. Ricco, M. Sanzone, and A. Zucchiatti, Phys. Rev. C *34,* 413 (1986)

[27] H.O. Meyer, J.R. Hall, M. Hugi, H.J. Karwowski, R.E. Pollock, and P. Schwandt, Phys. Rev. C *31,* 309 (1985)

[28] P. Levi Sandri, E. De Sanctis, M. Anghinolfi, N. Bianchi, G.P. Capitani, P. Corvisiero, C. Guaraldo, V. Lucherini, V. Muccifora, E. Polli, A.R. Reolon, G. Ricco, P. Rossi, M. Sanzone, and M. Taiuti, Phys. Rev. Lett. *59,* 2543 (1987)

[29] P. Levi Sandri, M. Anghinolfi, N. Bianchi, G.P. Capitani, P. Corvisiero, E. De Sanctis, C. Guaraldo, V. Lucherini, V. Muccifora, E. Polli, A.R. Reolon, G. Ricco, P. Rossi, M. Sanzone, and M. Taiuti, Phys. Rev. C *39,* 1701 (1989)

[30] K. Baba, I. Endo, H. Fukuma, K. Inoue, T. Kawamoto, T. Ohsugi, Y. Sumi, T. Takeshita, S. Uehara, Y. Yano, and T. Maki, Phys. Rev. C *28,* 286 (1983)

[31] P. Dougan, T. Kivikas, K. Lugnér, V. Ramsay, and W. Stiefler, Z Physik A *276,* 55 (1976)

[32] J.C. Keck and A.V. Tollestrup, Phys. Rev. *101,* 360 (1956)

[33] P. Dougan, V. Ramsey, and W. Stiefler, Z. Phys. A *280,* 341 (1977)

[34] R. Kose, W. Paul, and K. Stockhorst, Z. Phys. *202,* 364 (1967)

[35] M. Kanazawa, S. Homma, M. Koike, Y. Murata, H. Okuno, F. Soga, N. Yoshikawa, and A. Sasaki, Phys. Rev. C *35,* 1828 (1987)

[36] E.A. Whalin, B. D. Schriever, and A.O. Hanson, Phys. Rev. *101,* 377 (1956)

[37] Y. Birenbaum, S. Kahane, and R. Moreh, Phys. Rev. C *32,* 1825 (1985)

[38] R. Bernabei, A. Incicchitti, M. Mattioli, P. Picozza, and D. Prosperi, Phys. Rev. Lett. *57,* (1986)

[39] J. Arends, H.J. Gassen, A. Hegerath, B. Mecking, G. Nöldeke, P. Prenzel, T. Reichelt, and A. Voswinkel, Nucl. Phys. *A412,* 509 (1984)

[40] M. Hugi, J.M. Cameron, M. Ahmad, J. Collot, G. Gaillard, J.S. Wesick, G.W.R. Edwards, H.Fielding, D.A. Hutcheon, R. Abegg, C.A. Miller, P. Kitching, N.E. Davison, N.R. Stevenson, and I.J. Van Heerden, Nucl. Phys. *A472,* 701 (1987)

[41] J.M. Cameron, C.A. Davis, H. Fielding, P. Kitching, J. Pasos, J. Soukup, J. Uegaki, J. Wesick, and H.S. Wilson, Nucl. Phys. *A458,* 637 (1986)

[42] E. DeSanctis, G.P. Capitani, P. Ki Giacomo, C. Guaraldo, V. Lucherini, E. Polli, A.R. Reolon, R. Scrimaglio, M. Anghinolfi, P. Corvisiero, G. Ricco, M. Sanzone, and A. Zucchiatti, Phys. Rev. Lett. *54,* 1639 (1985)

[43] D.M. Skopik, Y.M. Shin, M.C. Phenneger, and J.J. Murphy, II, Phys. Rev. C *9,* 531 (1974)

[44] K.M. Schmitt and H. Arenhövel, Few Body Systems *6*, 117 (1989)

[45] S.F. Pate, C. Bloch, G. Xu, S.M. Bowyer, T.W. Bowyer, W.W. Jacobs, H.-O. Meyer, E. Pierce, J. Sowinski, S.E. Vigdor, C. Whiddon, and S.W. Wissink, Phys. Rev. Lett. *70*, 3205 (1993)

[46] J.A. Galey, Phys. Rev. *117*, 763 (1960)

[47] Lew Allen, Jr., Phys. Rev. *98*, 705 (1955)

[48] B. Weissman et al., Nucl. Phys. *A174*, 129 (1971)

[49] R. Dymarz, C.J. Morningstar, R. Gourishankar, and F.C. Khanna, Nucl. Phys. *A507*, 531 (1990)

[50] R. Machleidt, *The Meson Theory of Nuclear Forces and Nuclear Structure*, Advances in Nuclear Physics *19*, 189 (1989)

[51] T.-S. H. Lee, Photodisintegration of the Deuteron at Medium and High Energies, Fifth Workshop on Perspectives in Nuclear Physics at Intermediate Energies, ICTP, Trieste, Italy 6-10 May (1991)

[52] F. Partovi, Annals of Physics, *27*, 79-113, (1964)

[53] Schmitt and Arenhovel, Few Body Systems *6*, 117 (1989)

[54] S.F. Pate, et al. Phys Rev Lett *70*,3205 (1993)

[55] J.A. Galey, Phys Rev. *117*,763 (1960)

[56] Lew Allen, Jr., Phys Rev *98*, 705 (1955)

[57] B. Weissman et al., Nuc Phys *A174*, 129 (1971)

[58] M. P. DePascale, et. al. Phys. Rev. C *32*, 1830 (1985)

[59] V. Barannik et. al., Sov. Jour. Nuc. Fnys. *38*, 667 (1983)

[60] I. E. Vnukov et. al., Sov. Jour. Nuc. Phys. *47*, 583 (1988)

[61] V. G. Gorbenko et. al. Sov. Jour. Nuc. Phys. *35*, 627 (1982)

[62] A. Donnachie, Nuc Phys *32*, 637 (1962)

[63] V. V. Vas'kov et. al. JETP Lett. *43*, 663 (1986)

[64] M. P. DePascale, et. al. Preprint LNF 84/65

[65] F. F. Liu, Phys Rev *138*, 1443b (1965)

[66] H. Arenhövel and M. Sanzone, Few Body Systems,Supplement 3 (1991)