UNIVERSITY OF ALBERTA

UNCONDITIONAL STRUCTURE OF TWISTED SUMS

BY

ADI TCACIUC ©

A thesis
submitted to the Faculty of Graduate Studies and Research
in partial fulfillment of the requirements for the degree
of **Master of Science**
in
Mathematics

DEPARTMENT OF MATHEMATICAL SCIENCES

EDMONTON, ALBERTA

FALL, 2000

0-612-59886-1

UNIVERSITY OF ALBERTA

LIBRARY RELEASE FORM

Name of Author: Adi Tcaciuc

Title of Thesis: Unconditional Structure of Twisted Sums

Degree: Master of Science

Year this Degree Granted: 2000

A Tcaciuc.
....

Adi Tcaciuc
632 Central Academic Building
Department of Mathematical Sciences
University of Alberta
Edmonton, AB
T6G 2G1

Date: September 25, 2000

ABSTRACT

We examine in this thesis the unconditional structure of twisted sums of Hilbert spaces. Johnson, Lindenstrauss and Schechtman proved that Z_2 does not have unconditional basis. In Chapter 4 we prove the same result for an arbitrary nontrivial $Z_2(\varphi)$. Further we can look at general twisted sum of l_2, $X \oplus_F Y$ for F nontrivial quasi-linear function (as described in Chapter 4), and ask the same question! We give some general conditions on $X \oplus_F Y$, which, if satisfied, would enssure that $X \oplus_F Y$ does not have unconditional basis. Later, we prove a general result that allows us to pass from a space having FDD and an unconditional basis to a subspace having UFDD. This result will be used to prove some statements about the unconditional structure of subspaces of twisted sums of l_2 with itself. Finally, we look at subspaces of twisted sums, in particular at subspaces of twisted sums of l_2 with itself, and examine when such subspaces are isomorphic to the original space.

UNIVERSITY OF ALBERTA

Faculty of Graduate Studies and Research

The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research for acceptance, a thesis entitled **Unconditional Structure of Twisted Sums** submitted by Adi Tcaciuc in partial fulfillment of the requirements for the degree of **Master of Science in Mathematics.**

Dr. R. Poliquin

Dr. N. Tomczak-Jaegermann (Supervisor)

Dr. Z. Gortel

Date: August 8, 2000

ACKNOWLEDGMENTS

I would like to express my deep gratitude to my supervisor, **Dr. Nicole Tomczak-Jaegermann**, for the help, support, advice she has given me during my studies here, and in particular for her advice and criticism in the preparation of this thesis.

My sincere appreciation is also extended to the **University of Alberta** for providing financial assistance and a great research enviroment.

Contents

Chapter 1

Introduction

Let X be a Banach space. It is usual to identify a collection of subsets of X called *closed subspaces* of X. When endowed with the structure inherited from X, these subspaces are themselves Banach spaces. For each closed subspace Y of X there is a general construction of an object called *quotient space* which is denoted by X/Y, and again has the natural structure of a Banach space coming from X. In studying the structure of Banach spaces a natural type of questions are the so-called three space problems: if both Y and X/Y have some Banach space property, does X has it as well? A property which yields a positive answer to the previous question is called a three-space property. For example, reflexivity is a three-space property. The notion of *twisted sums* gives a general setting for studying some of the three-space properties.

When X and Y are quasi-Banach spaces, a *twisted sum* of X and Y is a quasi-Banach space Z that contains a subspace X_1 isomorphic to X such that the quotient Z/X_1 is isomorphic to Y. The twisted sum Z is *trivial* if X_1 is complemented in Z, otherwise Z is *nontrivial.*

Even though questions related to twisted sums were studied earlier, the systematic study of this notion was motivated by the problem posed by Palais (the three space problem), whether there are nontrivial twisted sums of l_2 with itself. In a paper from 1973 Enflo, Lindenstrauss and Pisier gave the first example of such a space.

Their approach was "local" in nature. The notion of a quasi-linear function was introduced by Ribe in 1979; he used it to construct a nontrivial twisted sum of $\mathbb{R}$ and l_1 (example presented here). The first detailed study of quasi-linear functions and their connection to twisted sums was done by Kalton and Peck in the 1979 paper [KP] that initiated considerably further research.

Twisted sums of a Hilbert space with itself are probably the most interesting examples of twisted sums and they were used to illustrate several important points in Banach space theory. There is a particularly intersting class of twisted sums of l_2 with itself, introduced in the above mentioned paper of Kalton and Peck. If φ is a Lipschitz function on $(0, \infty)$ the space $Z_2(\varphi)$ is the completion of the space of pairs of finitely supported sequences of reals $x = (x_n)$ and $y = (y_n)$ endowed with the quasi-norm

$$\|(x,y)\| = \left(\sum_{n=1}^{\infty}(x_n - y_n\varphi(\ln(\|y\|_2/|y_n|)))^2\right)^{1/2} + \|y\|_2$$

where $\|y\|_2$ denotes the ℓ_2-norm of a sequence $y \in \ell_2$. For $\varphi(t) = t$ the space $Z_2(\varphi)$ is denoted by Z_2 and is called the **Kalton-Peck space**.

Johnson, Lindenstrauss and Schechtman proved that Z_2 does not have unconditional basis. In Chapter 4 we prove the same result for an arbitrary nontrivial $Z_2(\varphi)$. Further we can look at general twisted sum of l_2, $X \oplus_F Y$ for F nontrivial quasi-linear function (as described in Chapter 4), and ask the same question! We give some general conditions on $X \oplus_F Y$, which, if satisfied, would ensure that $X \oplus_F Y$ does not have unconditional basis. Later, we prove a general result that allows us to pass from a space having FDD and an unconditional basis to a subspace having UFDD. This result will be used to prove some statements about the unconditional structure of subspaces of twisted sums of l_2 with itself. Finally, we look at subspaces of twisted sums, in particular at subspaces of twisted sums of l_2 with itself, and examine when such subspaces are isomorphic to the original space.

There are many important questions regarding twisted sums of Hilbert spaces and perhaps the most important one is whether there exist nontrivial twisted sums of Hilbert spaces that have unconditional basis. Casazza and Kalton conjectured a negative answer to this questions.

Chapter 2

Preliminaries

In this Chapter we present basic concepts in Functional Analysis which the Banach space theory rests upon. They can be found in any textbook in Functional Analysis, see e.g., [HHZ].

2.1 Normed and Banach spaces

Definition 2.1 *Let X be a vector space. A* **norm** *on X is a real-valued function $\|.\|$ on X such that the following conditions are satisfied by all vectors x and y of X and each scalar α:*

(i) $\|x\| \geq 0$, *and* $\|x\| = 0$ *if and only of* $x = 0$;

(ii) $\|\alpha x\| = |\alpha|\|x\|$;

(iii) $\|x + y\| \leq \|x\| + \|y\|$ *(the triangle inequality);*

If we have a norm on X we can naturally define a metric by $\rho(x, y) = \|x-y\|$ which further defines the topology on X. A topological vector space X with the topology given by a norm is called a **normed space** or **normed vector space** or **normed linear space** and denoted by $(X, \|.\|)$.

Definition 2.2 *A* **Banach space** *is a complete normed linear space* $(X,\|.\|)$.

Example A Let X be the n-dimensional vector space of n-tuples of real or complex numbers ($\mathbb{R}^n$ or $\mathbb{C}^n$). The supremum norm $\|.\|_\infty$ on X is defined as follows:

$$\|x\|_\infty = \max_i |x_i|, \text{ where } x = (x_1, ..., x_n) \in X.$$

The function $\|.\|_\infty$ is easily found to be a norm on X. The space $(X, \|.\|_\infty)$ is denoted by l_∞^n.

Example B Let X be as in Example A and let $p \in [1, \infty)$. Then the function $\|.\|_p$ on X defined by:

$$\|x\|_p = (\sum_{i=1}^{n} |x_i|^p)^{1/p}, \text{ where } x = (x_1, ..., x_n) \in X.$$

is a norm on X. The space X together with the norm $\|.\|_p$ is denoted by l_p^n.

Example C Similarly, for $p \in [1, \infty)$ we introduce the normed space $l_p = l_p(\mathbb{N})$ of all scalar valued sequences $(x_i)_{i=1}^\infty$ satisfying $\sum_{i=1}^\infty |x_i|^p < \infty$ together with the norm:

$$\|x\|_p = (\sum_{i=1}^{\infty} |x_i|^p)^{1/p} \text{ for } x = (x_i)_{i=1}^\infty.$$

Definition 2.3 *An* **inner product** *(or a* **scalar product** *or a* **dot product***) is a scalar valued function* $(.,.)$ *on the product* $X \times X$ *such that:*

(i) $x \longrightarrow (x, y)$ *is a linear function for every* $y \in X$*;*

(ii) $\overline{(x, y)} = (y, x)$*, where the bar denotes the complex conjugation;*

(iii) $(x, x) \geq 0$ *for every* $x \in X$ *, and* $(x, x) = 0$ *if and only if* $x = 0$*;*

Definition 2.4 *A Banach space* X *is called a* **Hilbert space** *if there is an inner product* $(.,.)$ *on* $X \times X$ *such that* $\|x\| = \sqrt{(x, x)}$ *for every* $x \in X$*.*

It is straightforward to check that the norm $\|.\|$ on a Hilbert space H satisfies the *parallelogram identity*, namely, for every $x, y \in H$ we have:

$$\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2).$$

On the other hand , if a norm on a Banach space satisfies this equality, then, as it is directly checked, $\|.\|$ is Hilbertian norm with the inner product (x, y) defined by

$$(x, y) = \frac{1}{4}(\|x + y\|^2 - \|x - y\|^2)$$

in the real case, and by

$$(x, y) = \frac{1}{4}(\|x + y\|^2 - \|x - y\|^2 + i(\|x + iy\|^2 - \|x - iy\|^2))$$

in the complex case.

Therefore a Banach space is a Hilbert space if and only if every two-dimensional subspace of X is a Hilbert space. The parallelogram equality also gives, by inspection, that out of the spaces introduced before only l_2^n and l_2 are Hilbert spaces. It is easy to see that the inner product in l_2 is given by:

$$(x, y) = \sum_{i=1}^{\infty} x_i \cdot \overline{y_i}$$

where $x = (x_i)_{i=1}^{\infty}$ and $y = (y_i)_{i=1}^{\infty}$ are in l_2.

2.2 Linear operators between normed spaces

Definition 2.5 *Let X and Y be vector spaces. A* **linear operator** *or* **linear transformation** *from X into Y is a function $T : X \longrightarrow Y$ such that the following two conditions are satisfied whenever $x, y \in X$ and α is a scalar:*

(i) $T(x + y) = T(x) + T(y)$;

(ii) $T(\alpha x) = \alpha T(x)$.

If the scalar field ($\mathbb{R}$ or $\mathbb{C}$) is viewed as a one-dimensional vector space, then a linear operator from X into $\mathbb{R}$ or $\mathbb{C}$ is called a **linear functional**.

We say that an operator T between normed spaces X and Y is **continuous** if it is continuous in the topology defined on X and Y by the coresponding norms, and is **bounded** if $T(B)$ is a bounded subset of Y whenever B is a bounded subset of X. The collection of all bounded linear operators from X into Y is denoted by $\mathcal{B}(\mathbf{X}, \mathbf{Y})$ and has the natural structure of a vector space.

Next we'll recall a classical result about operators between normed spaces, that gives the equivalence between the notions of continuity and boundness for linear operators.

Proposition 2.6 *Let X, Y be normed spaces and T a linear operator between X and Y. Then the following are equivalent:*

(i) T is continuous;

(ii) T is continuous at some $x_0 \in X$;

(iii) T is bounded;

If X and Y are normed spaces we can define a norm on $\mathcal{B}(X, Y)$ by:

$$\|T\| = \sup\{\|Tx\|_Y; x \in X, \|x\| = 1\}$$

It is easy to see that this formula indeed defines a norm on $\mathcal{B}(X, Y)$ and if Y is a Banach space then $\mathcal{B}(X, Y)$ is a Banach space as well.

Definition 2.7 *An operator $T \in \mathcal{B}(X, Y)$ is called a* **linear isomorphism** *(or just* **isomorphism***) if it is one-to-one, onto Y, and $T^{-1} \in \mathcal{B}(X, Y)$. Two normed spaces are called* **isomorphic** *if there is a linear isomorphism T of X onto Y.*

It is easy to see that an isomorphism T carries Cauchy (convergent) sequences onto Cauchy (convergent) sequences, respectively. Therefore, if X, Y are isomorphic normed spaces and X is a Banach space, then Y is a Banach space as well. An operator $T \in \mathcal{B}(X, Y)$ is called a **(linear) isomorphism of X into Y** if it is an isomorphism of X onto a closed subspace $T(X)$ of Y.

Definition 2.8 *An operator T is called a* **linear isometry** *if it is a linear isomorphism and $\|Tx\|_Y = \|x\|_X$ for every $x \in X$. Spaces X, Y are called* **isometric** *if there exist a linear isometry T of X onto Y.*

Definition 2.9 *Consider the space X endowed with two norms, say , $X_1 = (X, \|.\|_1)$ and $X_2 = (X, \|.\|_2)$. Norms $\|.\|_1, \|.\|_2$ are called* **equivalent** *if the formal identity mapping $Id : X \longrightarrow X$ is an isomorphism between the spaces X_1 and X_2 , i.e. if there exist constants $c, C > 0$ such that:*

$$c\|x\|_2 \leq \|x\|_1 \leq C\|x\|_2,$$

for every $x \in X$.

Note that two norms $\|.\|_1$ and $\|.\|_2$ on a vector space X *need not* be equivalent just because there is *some* one-to-one bounded linear operator that maps $\|.\|_1$ onto $\|.\|_2$, even if the map is an isometric isomorphism.
It is a classical fact that if X is a finite-dimensional vector space then any two norms on X are equivalent. In particular, all finite dimensional normed spaces are Banach spaces and every normed space of dimension n is isomorphic to l_2^n.

2.3 Direct Sums and Quotient Spaces

For a pair X, Y of normed spaces we introduce a normed space $X \oplus Y$ called a **direct (topological) sum** of X and Y that consist of all ordered pairs $(x, y), x \in X, y \in Y$ together with the norm

$$\|(x, y)\| = \|x\|_X + \|y\|_Y.$$

X and Y are isometric to subspaces $\{(x, 0); x \in X\}$ and $\{(0, y); y \in Y\}$ of $X \oplus Y$, respectively. Also, $X \oplus Y$ is a Banach space if and only if both X and Y are Banach spaces.

There are other ways to define a norm on the sum of the vector spaces X and Y. For example:

$$\|(x, y)\|_2 = (\|x\|_X^2 + \|y\|_Y^2)^{1/2}$$

or:

$$\|(x, y)\|_\infty = \max\{\|x\|_X, \|y\|_Y\}$$

Fortunately, all these norms are trivially equivalent.

Let X be a normed space and Y be a closed subspace of X. For $x \in X$ we consider the coset $\widehat{x}$:

$$\widehat{x} = \{z \in X; (x - z) \in Y\} = \{x + y; y \in Y\}$$

We can give the set $X/Y = \{\widehat{x}; x \in X\}$ of all cosets a vector space structure by $\widehat{x} + \widehat{y} = \widehat{x + y}$ and $\alpha\widehat{x} = \widehat{\alpha x}$, where α is a scalar. It is easy to check that

$$\|\widehat{x}\| = \inf\{\|y\|; y \in \widehat{x}\}$$

makes X/Y into a normed space.

Definition 2.10 *Let X be a Banach space, and Y a closed subspace of X. The space X/Y together with the canonical norm $\|\widehat{x}\| = \inf\{\|y\|; y \in \widehat{x}\}$ is called a* **quotient space** *of X modulo Y.*

If X is a Banach space and Y is a closed subspace of X then X/Y is also a Banach space. It is easy to check that $(X \oplus Y)/X$ is isomorphic to Y and $(X \oplus Y)/Y$ is isomorphic to X. However, X may not be isomorphic to $Y \oplus (X/Y)$!

Definition 2.11 *Let X be a Banach space and Y be a closed subspace of X. An operator $P \in \mathcal{B}(X)$ is called a* **projection** *onto Y if $P(y) = y$ for every $y \in Y$ and $P(X) = Y$.*

Equivalently, we may say that P is a projection onto Y if and only if $P^2 = P$ and P maps X onto Y. Indeed, if P is a projection then $Px \in Y$ for $x \in X$ and thus $P^2x = P(Px) = Px$. Conversely, if $P \in \mathcal{B}(X)$ satisfies $P^2 = P$ and P is a map onto Y, then given $y \in Y$, there is $x \in X$ such that $Px = y$ and then $Py = P^2x = Px = y$.

Definition 2.12 *Let X be a Banach space and Y a closed subspace of X. Then Y is said to be* **complemented** *in X if there is a bounded linear projection of X onto Y.*

It can be easily proved that when Y is complemented in X there exist a closed subspace Z of X such that $X = Y \oplus Z$. For example, we may put $Z = \ker P$. Every such Z is called a **complement** of Y in X. Also X/Y is isomorphic to Z.

We may ask ourselves whether every closed subspace of a Banach space is complemented. This is not true in general, although deciding whether a specific subspace of a given Banach space is complemented or not is often a very difficult problem.

Let us just mention the case of c_0 and l_∞. Clearly c_0 is a closed subspace of l_∞ but it can be proved that every complemented subspace of ℓ_∞ is very large (non-separable) (see e.g., [LT]), and c_0 does not satisfy this condition, hence is not complemented.

Lindenstrauss and Tzafriri showed that

Theorem 2.13 (Lindenstrauss, Tzafriri) *A Banach space X has the property that every closed subspace of X is complemented in X if and only if X is isomorphic to a Hilbert space.*

Chapter 3

Schauder Bases

In this chapter we present basic structural notions in the Banach space theory. The definitions and results can be found in many books in the Banach space theory, for example, [LT] and [HHZ].

3.1 Definitions and Properties

Definition 3.1 *A sequence* $\{x_n\}_{n=1}^{\infty}$ *in a Banach space* X *is said to be a* **Schauder basis** *of* X *if for every* $x \in X$ *there is a unique sequence of scalars* $\{a_n\}_{n=1}^{\infty}$ *so that* x *is the sum of the norm convergent series* $x = \sum_{n=1}^{\infty} a_n x_n$. *A sequence* $\{x_n\}_{n=1}^{\infty}$ *which is a Schauder basis of its closed linear span is called a* **basic sequence**. *A basis (or a basic sequence)* $\{x_n\}_{n=1}^{\infty}$ *is called* **normalized** *if* $\|x_n\| = 1$ *for all* n.

It is easy to verify that whenever $\{x_n\}_{n=1}^{\infty}$ is a Schauder basis (basic sequence) of (in) X, the sequence $\{x_n/\|x_n\|\}_{n=1}^{\infty}$ is a normalized basis (basic sequence).

If X is finite dimensional it is trivial to verify that the notion of Schauder basis coincides with that of a vector-space basis.
Also note that if a Banach space X has a Schauder basis then X is separable. Indeed, all rational linear combinations of the basis vectors form a countable dense set.

Example A If X is c_0 or l_p for $1 \leq p < \infty$, then it is easy to check that the sequence $\{e_n\}_{n=1}^{\infty}$ of standard unit vectors of X is a basis for X and that $\{\alpha_n\}_{n=1}^{\infty} = \sum_{n=1}^{\infty} \alpha_n e_n$ whenever $\{\alpha_n\}_{n=1}^{\infty} \in X$ However the sequence $\{e_n\}_{n=1}^{\infty}$ is not a basis for l_∞. Actually l_∞ does not have a Schauder basis since it is not separable, see e.g.[HHZ]

Example B Any orthonormal basis of a Hilbert space H is a Schauder basis of H. Indeed, if $\{h_n\}_{n=1}^{\infty}$ is an orthonormal basis of H then for any $x \in H$ we can write

$x = \sum_{n=1}^{\infty}(x, h_n)h_n$ where $(.,.)$ denotes the inner product in H.

From now on we shall not consider any other type of basis in infinite-dimensional Banach spaces besides Schauder basis. We shall therefore often refer to it as basis instead of Schauder basis.

If $\{x_n\}_{n=1}^{\infty}$ is a basis of a Banach space X, then the canonical projections $P_n : X \longrightarrow X$ are defined for $n \in \mathbb{N}$ by $P_n(x) = \sum_{i=1}^{n} a_i x_i$ for $x = \sum_{i=1}^{\infty} a_i x_i$.

Let $(X, \|.\|)$ be a Banach space with a basis $\{x_n\}_{n=1}^{\infty}$. For every $x = \sum_{n=1}^{\infty} a_n x_n$ in X the expression $|||x||| = \sup_n \|\sum_{i=1}^{n} a_i x_i\|$ is finite (because $\|\sum_{i=1}^{n} a_i x_i\| \to \|x\|$ as $n \to \infty$). $|||.|||$ is in fact a norm on X and also $\|x\| \leq |||x|||$ for every $x \in X$. It can be proved that X is complete also with respect to $|||.|||$ and thus, by the open mapping theorem, the norms $\|.\|$ and $|||.|||$ are equivalent. Using these remarks it immediately follows that

Proposition 3.2 *Let X be a Banach space with a basis $\{x_n\}_{n=1}^{\infty}$. Then the canonical projections P_n are bounded linear operators and* $\sup_n \|P_n\| < \infty$.

Definition 3.3 *The number* $\sup_n \|P_n\|$ *is called the* **basis constant** *of the basis $\{x_n\}_{n=1}^{\infty}$. A basis whose basis constant is* 1 *is called a* **monotone basis.**

There is a simple and useful criterion to check whether a given sequence is a Schauder basis.

Proposition 3.4 *Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of vectors in X. Then $\{x_n\}_{n=1}^{\infty}$ ia a Schauder basis of X if and only if the following three conditions hold:*

(i) $x_n \neq 0$ *for all* n.

(ii) There is a constant K so that, for every choice of scalars $\{a_i\}_{i=1}^{\infty}$ and all integers $n < m$, we have that

$$\|\sum_{i=1}^{n} a_i x_i\| \leq K \|\sum_{i=1}^{m} a_i x_i\|.$$

(iii) the closed linear span of $\{x_n\}_{n=1}^{\infty}$ is all of X.

Clearly, conditions (i) and (ii) of the previous proposition give a necessary and sufficient condition for a sequence $\{x_n\}_{n=1}^{\infty}$ to be a basic sequence. Also it is easy to see that the smallest number K for which (ii) holds is the basis constant of $\{x_n\}_{n=1}^{\infty}$.

It is natural to ask whether every infinite-dimensional separable Banach space has a basis.

This question goes back to the early 30's and appeared in an equivalent form in the *Scottish Book*. It remained open for forty years, but was finally settled in the negative in a 1973 paper by **Per Enflo**. It is true however (and classical) that every infinite-dimensional Banach space has a basic sequence.

Definition 3.5 *Two bases, $\{x_n\}_{n=1}^{\infty}$ of X and $\{y_n\}_{n=1}^{\infty}$ of Y, are said to be* **equivalent** *provided a series $\sum_{n=1}^{\infty} a_n x_n$ converges if and only if $\sum_{n=1}^{\infty} a_n y_n$ converges.*

In other words, bases are equivalent if the sequence space associated to X by $\{x_n\}_{n=1}^{\infty}$ is identical to the sequence space associated to Y by $\{y_n\}_{n=1}^{\infty}$. There is an equivalent way of saying that two bases are equivalent

Proposition 3.6 *Let $\{x_n\}_{n=1}^{\infty}$ be a basis in a Banach space X and $\{y_n\}_{n=1}^{\infty}$ be a basis in a Banach space Y. Then the following are equivalent:*

(i) $\{x_n\}_{n=1}^{\infty}$ *is equivalent to* $\{y_n\}_{n=1}^{\infty}$.

(ii) There is an isomorphism T of X onto Y such that $Tx_i = y_i$ for all i.

(iii) There are $K_1, K_2 > 0$ such that for any n and for all scalars $a_1, a_2, ..., a_n$ we have

$$\frac{1}{K_1}\|\sum_{i=1}^{n} a_i x_i\|_X \leq \|\sum_{i=1}^{n} a_i y_i\|_Y \leq K_2\|\sum_{i=1}^{n} a_i x_i\|_X.$$

Definition 3.7 *Let $\{x_n\}_{n=1}^{\infty}$ be a basic sequence in a Banach space X. A sequence of non-zero vectors $\{u_j\}_{j=1}^{\infty}$ in X of the form*

$$u_j = \sum_{n=p_j+1}^{p_{j+1}} a_n x_n$$

with $\{a_n\}_{n=1}^{\infty}$ scalars and $p_1 < p_2 < p_3 < ....$ an increasing sequence of natural numbers, is called a **block basic sequence** *or briefly a* **block basis** *of $\{x_n\}_{n=1}^{\infty}$.*

Note that a block basis of $\{x_n\}_{n=1}^{\infty}$ is a basic sequence with the basis constant less than or equal to the basis constant of $\{x_n\}_{n=1}^{\infty}$.

3.2 Duality

Let X be a Banach space with a Schauder basis $\{x_n\}_{n=1}^{\infty}$.

Definition 3.8 *The functionals $x_n^* \in X^*$ defined for each n by*

$$x_n^*(\sum_{i=1}^{\infty} a_i x_i) = a_n, \text{ for any } x = \sum_{i=1}^{\infty} a_i x_i \in X$$

are called the **biorthogonal functionals** *associated to the basis $\{x_n\}_{n=1}^{\infty}$.*

The biorthogonal functionals are bounded linear operators. Indeed, for each n we have that

$$\begin{aligned} \|x_n^*\| &= \sup_{\|x\|=1} \|x_n^*(\sum_{i=1}^{\infty} a_i x_i)\| \\ &= \sup_{\|x\|=1} |a_n| \\ &= \sup_{\|x\|=1} \frac{\|P_n x - P_{n-1} x\|}{\|x_n\|} \\ &\leq \frac{2K}{\|x_n\|} \end{aligned}$$

where K is the basis constant of $\{x_n\}_{n=1}^{\infty}$.

It can be easily shown that the sequence $\{x_n^*\}_{n=1}^{\infty}$ is a basic sequence in X^* and it is a basis for X^* if and only if the span of $\{x_n^*\}_{n=1}^{\infty}$ is all of X^*. Note that for this to happen, in particular we must have that X is separable. Hence, for example, for $X = l_1$ or $X = \mathcal{C}(0,1)$ this cannot happen for any basis. On the other hand this is always the case for X reflexive.

Definition 3.9 *A basis $\{x_n\}_{n=1}^{\infty}$ of a Banach space X is called* **shrinking** *if*

$$\|x^*|_{\{x_i\}_{i=n}^{\infty}}\| \longrightarrow 0 \text{ as } n \longrightarrow \infty$$

for every $x^* \in X^*$.

There is a simple characterization in terms of shrinking bases of the spaces for which the biorthogonal functionals form a basis of the dual:

Proposition 3.10 *Let $\{x_n\}_{n=1}^{\infty}$ be a basis for a Banach space X. Then the biorthogonal functionals $\{x_n^*\}_{n=1}^{\infty}$ form a basis of X^* if and only if $\{x_n\}_{n=1}^{\infty}$ is shrinking.*

Another important notion concerning bases is that of boundedly complete basis.

Definition 3.11 *A basis $\{x_n\}_{n=1}^{\infty}$ of a Banach space is called* **boundedly complete** *if, for every sequence of scalars $\{a_n\}_{n=1}^{\infty}$ with $\sup_n \|\sum_{i=1}^{n} a_i x_i\| < \infty$, the series $\sum_{n=1}^{\infty} a_n x_n$ converges as well.*

Example A The unit vector basis is boundedly complete for any l_p space , $1 \leq p < \infty$. An example of a basis which is not boundedly complete is the unit vector basis of c_0. Indeed, if we take for example $\{a_n\}_{n=1}^{\infty}$ to be the sequence $(1, 1, 1, ...)$ the it is easy to see that $sup_n \|\sum_{i=1}^{n} a_i x_i\| = 1 < \infty$ but the series $\sum_{n=1}^{\infty} a_n x_n$ does not converge in c_0.
By combining the notions of shrinking and boundedly complete **James** gave an elegant characterization of reflexivity in terms of bases:

Theorem 3.12 (James) *Let X be Banach space and $\{x_n\}_{n=1}^{\infty}$ a Schauder basis of X. Then X is reflexive if and only if $\{x_n\}_{n=1}^{\infty}$ is both shrinking and boundedly complete.*

3.3 Unconditional bases. Symmetric bases

A very important notion in the study of Banach spaces is that of unconditional bases. Before giving the definition of this notion we present some general facts concerning unconditional convergence.

Proposition 3.13 *Let $\{x_n\}_{n=1}^{\infty}$ be a sequence of vectors in a Banach space X. Then the following conditions are equivalent:*

(i) The series $\sum_{n=1}^{\infty} x_{\pi}(n)$ converges for every permutation π of natural numbers.

(ii) The series $\sum_{n=1}^{\infty} x_{n_i}$ converges for every of $n_1 < n_2 < n_3.....$

(iii) The series $\sum_{n=1}^{\infty} \varepsilon_n x_n$ for every choice of signs ε_n.

(iv) For every $\varepsilon > 0$ there exists an integer n so that $\|\sum_{i \in \sigma} x_i\| < \varepsilon$ for every finite set of integers σ which satisfies $\min\{i \in \sigma\} > n$.

A series $\sum_{n=1}^{\infty} x_n$ which satisfies one, and thus all of the above conditions. is said to be unconditionally convergent.

In finite-dimensional spaces a series $\sum_{n=1}^{\infty} x_n$ converges unconditionally if and only if converges absolutely (i.e. $\sum_{n=1}^{\infty} \|x_n\| < \infty$. In infinite dimensional spaces however we can always find a series $sum_{n=1}^{\infty} x_n$ that converges unconditionally but not absolutely. This result is due to **Dvoretzky** and **Rogers**:

Theorem 3.14 *Let X be an infinite-dimensional Banach space. Let $\{\lambda_n\}_{n=1}^{\infty}$ be a sequence of positive numbers such that $\sum_{n=1}^{\infty} \lambda_n^2 < \infty$. Then there is an unconditionally convergent series $\sum_{n=1}^{\infty} x_n$ in X such that $\|x_n\| = \lambda_n$, for every n.*

Definition 3.15 *A basis $\{x_n\}_{n=1}^{\infty}$ of a Banach space X is called* **unconditional** *if for every $x \in X$, its expansion in terms of the basis $\sum_{n=1}^{\infty} a_n x_n$ converges unconditionally.*

In view of Proposition 3.13, there are several equivalent conditions that ensure a basis is unconditional:

Proposition 3.16 *A basic sequence $\{x_n\}_{n=1}^{\infty}$ is unconditional if and only if any of the following conditions hold:*

(i) There is a constant K such that for all n and all scalars $a_1, a_2, ..., a_n$ and every subset σ of $\{1, ..., n\}$ we have

$$\|\sum_{i\in\sigma} a_i x_i\| \le K \|\sum_{i=1}^{n} a_i x_i\|.$$

(ii) There is a constant K such that for all n and all scalars $a_1, a_2, ..., a_n$ and signs $\varepsilon_i = \pm 1$ we have

$$\|\sum_{i=1}^{n} \varepsilon_i a_i x_i\| \le K \|\sum_{i=1}^{n} a_i x_i\|.$$

The best possible constant K from the condition (*iii*) in the Proposition 3.16 is called the **unconditional basis constant** of $\{x_n\}_{n=1}^{\infty}$. Of course, not every separable Banach space has an unconditional basis, and one classical example of such a space is $L_1(0,1)$ (see [LT]).
There was a natural and very important question in the structural theory of Banach spaces whether every Banach space contains an unconditional basic sequence. This long-standing problem was answered in the negative when in 1991 **Gowers** and **Maurey** constructed a Banach space that does not contain an unconditional basic sequence.

Note that a Schauder basis decomposes a Banach space into sum of 1-dimensional subspaces. It is useful in applications to consider decompositions where the components into which we decompose a given Banach space are subspaces of dimension larger than one.

Definition 3.17 *Let X be a Banach space. A sequence $\{X_n\}_{n=1}^{\infty}$ of closed subspaces of X is called a* **Schauder decomposition** *of X if every $x \in X$ has a unique representation of the form $x = \sum_{n=1}^{\infty} x_n$ with $x_n \in X_n$ for every n.*

The decomposition is called **shrinking** if for every $x^* \in X^*$ we have that

$$\|x^*|_{\{X_i\}_{i=n}^{\infty}}\| \longrightarrow 0 \text{ as } n \longrightarrow \infty$$

The decomposition is called **unconditional** if, for every $x \in X$, the series that represents x converges unconditionally.

In applications the decompositions for which $\dim X_n < \infty$ for all n play a particularly important role (note that in the general definition above $\sup_n \dim X_n$ need not be necessarily finite). Such decompositions are called **finite dimensional decompositions**, in short **FDD**. If additionally the decomposition is unconditional then it is called **unconditional finite dimensional decompositions**, in short **UFDD**.

We conclude this chapter by presenting another important concept in the study of Banach spaces. To begin, note that the unit vector basis of l_p, for $1 \leq p < \infty$, besides being unconditional, has the property that is equivalent to any of its permutations.

Definition 3.18 *A basis $\{x_n\}_{n=1}^{\infty}$ of a Banach space X is said to be* **symmetric** *if, for any permutation π of the integers, $\{x_\pi(n)\}_{n=1}^{\infty}$ is equivalent to $\{x_n\}_{n=1}^{\infty}$.*

Comparing this definition to the one for unconditional basis, it is a trivial conclusion that *every symmetric basis is unconditional.* There is a notion which is weaker than that of a symmetric basis, but of no less importance.

Definition 3.19 *A basis $\{x_n\}_{n=1}^{\infty}$ of a Banach space X is called* **subsymmetric** *if it is unconditional and, for every increasing sequence of integers $\{n_k\}_{k=1}^{\infty}$, $\{x_{n_k}\}_{k=1}^{\infty}$ is equivalent to $\{x_n\}_{n=1}^{\infty}$.*

It is not hard to prove that *every symmetric basis is subsymmetric.*

Chapter 4

Twisted Sums

In the first three section of this chapter we discuss classical results about twisted sums as they have been developed in the original paper of **Kalton** and **Peck** [KP], and described in specialized monographs (see e.g. [BL], Chapter 16). In the remaining sections we present new results regarding twisted sums of Hilbert spaces.

4.1 Introduction

Definition 4.1 *A* **quasi-norm** *on a real vector space X is a real valued function $\|.\|$ satisfying:*

(i) $\|x\| > 0$ *for* $x \in X, x \neq 0$,

(ii) $\|\alpha x\| = |\alpha|\|x\|$, *for* $x, y \in X$ *and* $\alpha \in \mathbb{R}$,

(iii) $\|x + y\| \leq k(\|x\| + \|y\|)$ *for* $x, y \in X$ *and some* $k \geq 1$.

The smallest possible k in (iii) is called the **modulus of concavity** *of* $\|.\|$

First note that in contrast with the case of normed spaces, in quasi-normed spaces which are not normed, balls with respect $\|.\|$ are not convex sets. Nevertheless, they define the topology on X and it is possible to define a metric d on X invariant under translations which determines the same topology. A quasi-normed space X is called a **quasi-Banach space** if X is complete with respect to the invariant metric d, i.e., every $\|.\|$-Cauchy sequence in X converges.

Example A The most important class of quasi-Banach spaces which are not Banach spaces is the class of $L_p(\mu)$ spaces for $0 < p < 1$ with the usual quasi-norm $\|g\|_p =$

$(\int |g|^p)^{1/p}$. In this case:

$$\|g+h\|_p \leq 2^{\frac{1}{p}-1}(\|g\|_p + \|h\|_p),$$

i.e., the modulus of concavity of of $L_p(\mu)$ is $2^{\frac{1}{p}-1}$.

A linear operator T between quasi-Banach spaces X and Y is continuous if and only if it is bounded, and we put $\|T\| = \sup\{\|Tx\|; \|x\| \leq 1\}$ (as in the convex case). The standard results depending on Baire category, like open mapping theorem and closed graph theorem, are valid for quasi-Banach spaces. On the other hand, quasi-normed spaces are not necessarily locally convex, and the Hahn-Banach theorem and other results depending on it are in general false in this context. If X is a Banach space and Y is a closed subspace of X, then the quotient space X/Y is a quasi-Banach space with the usual definition of the quotient quasi-norm. Equivalence of quasi-norms is defined the same way as for norms.

The question that motivates the study of twisted sums is the following: Given X and Y quasi-Banach spaces , what are the quasi-Banach spaces Z which contain a subspace X_1 isomorphic to X so that Z/X_1 is isomorphic to Y. Clearly the direct sum $X \oplus Y$ satisfies this condition.

Definition 4.2 *Let X and Y be quasi-Banach spaces. A* **twisted sum** *of X and Y is a quasi-Banach space Z which contains a subspace X_1 isomorphic to X so that Z/X_1 is isomorphic to Y. The twisted sum Z is* **trivial** *if X_1 is complemented in Z. Otherwise Z is* **nontrivial**.

Another way of saying that Z is a twisted sum of X and Y is to say that there exists a short exact sequence with bounded linear operators:

$$0 \longrightarrow X \xrightarrow{\;j\;} Z \xrightarrow{\;q\;} Y \longrightarrow 0$$

If X, Y and Z are such that

$$\|jx\| = \|x\|, x \in X$$

and

$$\|y\| = \inf\{\|z\|; qz = y\}, y \in Y$$

then we say that Z is an **isometric twisted sum** of X and Y. In this case Z has a subspace $j(X)$ isometric to X and $Z/j(X)$ is isomorphic to Y.

Definition 4.3 *Two twisted sums Z_1 and Z_2 of X and Y are called* **equivalent** *if there is an isomorphism T from Z_1 onto Z_2 so that the diagram:*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X & \xrightarrow{j_1} & Z_1 & \xrightarrow{q_1} & Y & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle I_X} & & \downarrow{\scriptstyle T} & & \downarrow{\scriptstyle I_Y} & & \\
0 & \longrightarrow & X & \xrightarrow{j_2} & Z_2 & \xrightarrow{q_2} & Y & \longrightarrow & 0
\end{array}$$

commutes, where I_X and I_Y are the identity operators.

4.2 Quasi-linear functions

Definition 4.4 *Let X and Y be quasi-Banach spaces. A function $F : Y \longrightarrow X$ is called* **quasi-linear** *if there is a constant M so that:*

(i) $F(tx) = tF(x)$ *for all* $t \in \mathbb{R}$ *and* $x \in Y$.

(ii) $\|F(x+y) - F(x) - F(y)\| \leq M(\|x\| + \|y\|)$ *for all* $x, y \in Y$

Definition 4.5 *Two quasi-linear functions F_1 and F_2 from Y to X are called* **equivalent** *if there is a linear map $T : Y \longrightarrow X$ and a constant C such that:*

$$\|F_1(x) - F_2(x) - Tx\| \leq C\|x\|$$

for all $x \in Y$

We shall define, for each quasi-linear function F, a twisted sum of X and Y, which will be denoted by $X \oplus_F Y$. Then we show that each twisted sum of X and Y is isomorphic to $X \oplus_F Y$, for some F.

Proposition 4.6 *Let X and Y be two quasi-Banach spaces and $F : Y \longrightarrow X$ a quasi-linear function. Then the formula:*

$$\|(x, y)\| = \|x - F(y)\| + \|y\| \tag{4.1}$$

defines a quasi-norm on the space of pairs (x, y), $x \in X$ and $y \in Y$ and this space is a twisted sum of X and Y.

Proof First we check that formula (4.1) defines a quasi-norm. For that, let C be the quasi-linearity constant of F and k_1 and k_2 be the concavity constants for X and Y respectively. We have:

$$\begin{aligned}
\|(x_1+x_2, y_1+y_2)\| = \;& \|x_1+x_2-F(y_1+y_2)\| + \|y_1+y_2\| \\
\leq \;& k_1\|x_1-F(y_1)+x_2-F(y_2)\| \\
& +k_1\|F(y_1)+F(y_2)-F(y_1+y_2)\| \\
& +k_2(\|y_1\|+\|y_2\|) \\
\leq \;& k_1^2(\|x_1-F(y_1)\|+\|x_2-F(y_2)\|) \\
& +k_1C(\|y_1\|+\|y_2\|)+k_2(\|y_1\|+\|y_2\|) \\
\leq \;& M(\|(x_1,y_1)\|+\|(x_2,y_2)\|)
\end{aligned}$$

where $M = \max(k_1^2, k_2+k_1C)$. We denote the space of pairs (x, y) endowed with this quasi-norm by $X \oplus_F Y$. Next we'll prove that $X \oplus_F Y$ is a twisted sum of X and Y.

Let $\{z_n\}_{n=1}^{\infty}$ be a Cauchy sequence in $X \oplus_F Y$ and let $q : X \oplus_F Y \longrightarrow Y$ be the map $q(x, y) = y$. q is clearly linear and we have that:

$$\begin{aligned}
\|q\| &= \sup\{\|q(x,y)\|; \|(x,y)\| = 1\} \\
&= \sup\{\|y\|; \|x-F(y)\|+\|y\| = 1\} \\
&\leq \sup\{\|x-F(y)\|+\|y\|; \|x-F(y)\|+\|y\| = 1\} \\
&= 1
\end{aligned}$$

Hence q is bounded. Moreover, we have that:

$$\|y\| \leq \inf\{\|z\|; qz = y\} \leq \|(F(y), y)\| = \|y\|$$

therefore, $\|y\| = \inf\{\|z\|; qz = y\}$.
Now consider the map $j : X \longrightarrow X \oplus_F Y$ defined by $j(x) = (x, 0)$. We have that i is linear and it is trivial to check that $\|j(x)\| = \|x\|$. We can see now that the subspace $X_0 = \{(x, 0); x \in X\}$ of $X \oplus_F Y$ is isometric to X and the quotient $(X \oplus_F Y)/X_0$ is isometric to Y. Hence we have a exact short sequence with bounded operators:

$$0 \longrightarrow X \xrightarrow{\;j\;} X \oplus_F Y \xrightarrow{\;q\;} Y \longrightarrow 0$$

It remains to check that $X \oplus_F Y$ is complete. Let $\{z_n\}_{n=1}^{\infty}$ be a Cauchy sequence in $X \oplus_F Y$, and let q be the map defined before. Since $\|q(z)\| \leq \|z\|$ for any $z \in X \oplus_F Y$ we have that $\{qz_n\}_{n=1}^{\infty}$is Cauchy. Therefore, since Y is complete and q is onto, we conclude that there is a $z \in X \oplus_F Y$ so that $qz_n \longrightarrow qz$ as $n \longrightarrow \infty$. If we let $z = (x, y)$ and $z_n = (x_n, y_n)$ then we have $\|z - z_n - (u_n, 0)\| \longrightarrow 0$ where $u_n = x - x_n - F(y - y_n) \in X$. Furthermore, we have:

$$\begin{aligned} \|u_n - u_m\| &= \|(u_n, 0) - (u_m, 0)\| \\ &= \|z - z_m - (u_m, 0) - (z - z_n - (u_n, 0)) + (z_m - z_n)\| \\ &\leq \|z - z_m - (u_m, 0)\| + \|z - z_n - (u_n, 0)\| + \|z_m - z_n\| \end{aligned}$$

therefore $\{u_n\}_{n=1}^{\infty}$ is a Cauchy sequence in X, hence converges to some $u \in X$. So we obtain in the end that $\lim_{n \to \infty} z_n = z - (u, 0)$, hence $X \oplus_F Y$ is complete.

■

Next we'll prove a theorem, due to Kalton, which gives the correspondence between equivalence classes of quasi-linear functions and twisted sums.

Theorem 4.7 *Let X and Y be two quasi-Banach spaces. Then the correspondence $F \leftrightarrow X \oplus_F Y$ is a one-to-one correspondence between equivalence classes of quasi-linear functions from Y to X and equivalence classes of twisted sums of X and Y. In particular, $X \oplus_F Y$ is a trivial twisted sum if and only if F is equivalent to a linear function.*

Proof First, we'll prove that if Z is a twisted sum of X and Y, then there is a quasi-linear function $F : X \longrightarrow Y$ so that Z is equivalent to $X \oplus_F Y$. Consider the short exact sequence giving the twisted sum Z:

$$0 \longrightarrow X \xrightarrow{\ j\ } Z \xrightarrow{\ q\ } Y \longrightarrow 0$$

Let $\varphi : Y \longrightarrow Z$ be a linear mapping (not necessarily continuous) such that $q\varphi(y) = y$ for all $y \in Y$ (i.e. φ is a linear right inverse of q). To construct such a φ we define it first on a Hamel (algebraic) basis of Y and then extend it by linearity. Since q is onto and bounded q is a open mapping, therefore, by the open mapping theorem, there is a constant $M > 0$ such that for any $y \in Y$ there exist $x_y \in Z$ with $\|x_y\| \leq M\|y\|$ satisfying $q(x_y) = y$. Define $\psi : Y \longrightarrow Z$ by $\psi(y) = x_y$. We can define this ψ such that it is homogeneous, by defining it first on the unit ball of Y and then extend it by homogeneity on whole Y.
Consider now the map $F : Y \longrightarrow X$ defined by:

$$F(y) = j^{-1}(\psi(y) - \varphi(y))$$

Note that F is well defined since Ker$q$ =Imj and $\psi(y)-\varphi(y)\in$Ker$q$ for any $y\in Y$. Clearly F is homogeneous and next we'll check that F is quasi-linear:

$$\begin{aligned}\|F(x+y)-F(x)-F(y)\| = & \|j^{-1}(\psi(x+y)-\psi(x)-\psi(y))\| \\ \leq & \|j^{-1}\|\|\psi(x+y)-\psi(x)-\psi(y)\| \\ \leq & \|j^{-1}\|k_1^2(\|\psi(x+y)\|-\|\psi(x)\|-\|\psi(y)\|) \\ \leq & \|j^{-1}\|k_1^2 M(\|x+y\|+\|x\|+\|y\|) \\ \leq & \|j^{-1}\|k_1^2 M(k_2+1)(\|x\|+\|y\|)\end{aligned}$$

where k_1, k_2 are the moduli of concavity for Z and Y respectively. Hence F is quasi-linear. We'll prove next that Z is equivalent with $X\oplus_F Y$.

Define $T : Z \longrightarrow X\oplus_F Y$ by:

$$Tz = (j^{-1}(z-\varphi(qz)), qz)$$

Again, note that T is well defined since $z-\varphi(qz)\in$Ker$q$. Also T is clearly linear. We have:

$$\begin{aligned}\|Tz\| = & \|qz\|+\|j^{-1}(z-\varphi(qz))-F(qz)\| \\ = & \|qz\|+\|j^{-1}(z-\varphi(qz))-j^{-1}(\psi(qz)-\varphi(qz))\| \\ = & \|qz\|+\|j^{-1}(z-\psi(qz))\| \\ \leq & \|j^{-1}\|\|z-\psi(qz)\|+\|q\|\|z\| \\ \leq & \|j^{-1}\|(\|z\|+M\|q\|\|z\|)+\|q\|\|z\| \\ \leq & (\|j^{-1}\|(1+M\|q\|)+\|q\|)\|z\|\end{aligned}$$

therefore T is bounded. Next we'll prove that T is one-to-one and onto.

$$\begin{aligned}T(z_1)=T(z_2) \Longleftrightarrow & \left\{\begin{array}{c} j^{-1}(z_1-\varphi(qz_1)) = j^{-1}(z_2-\varphi(qz_2)) \\ qz_1 = qz_2\end{array}\right. \\ \Longleftrightarrow & \left\{\begin{array}{c} z_1-\varphi(qz_1) = z_2-\varphi(qz_2) \\ qz_1 = qz_2\end{array}\right. \\ \Longleftrightarrow & \; z_1 = z_2\end{aligned}$$

therefore T is one-to-one. Given $(x,y)\in X\oplus_F Y$ ir is easy to check that for $z=j(x)+\varphi(y)$ we have $T(z)=(x,y)$, hence T is onto. Moreover the required diagrams commute, therefore T is an equivalence between Z and $X\oplus_F Y$.

Assume now that $X\oplus_{F_1}Y$ and $X\oplus_{F_2}Y$ are equivalent twisted sums, and let $T : X\oplus_{F_1}Y \longrightarrow X\oplus_{F_2}Y$ be the isomorphism which gives the equivalence. Then, since the required diagrams commute, there exist a linear operator $L : Y \longrightarrow X$ such that

$T(x,y) = (x + Ly, y)$. Since T is bounded we have that $\|T(x,y)\| \leq \|T\|\|(x,y)\|$ for any $(x,y) \in X \oplus_{F_1} Y$. So, for $(F_1(y), y) \in X \oplus_{F_1} Y$, for any $y \in Y$ we obtain that

$$\|T(F_1(y), y)\| = \|(F_1(y) + Ly, y)\| = \|y\| + \|F_1(y) - F_2(y) + Ly\| \tag{4.2}$$

and

$$\|(F_1(y), y)\| = \|y\| + \|F_1(y) - F_1(y)\| = \|y\| \tag{4.3}$$

The relations (4.2), (4.3) the boundness of T yield:

$$\|F_1(y) - F_2(y) + Ly\| \leq (\|T\| - 1)\|y\|$$

and thus F_1 is equivalent to F_2.

Also if F_1 and F_2 are equivalent then $X \oplus_{F_1} Y$ and $X \oplus_{F_2} Y$ are equivalent twisted sums as well. Indeed, the map $T(x,y) = (x + Ly, y)$ where L is such that $\|F_1(y) - F_2(y) + Ly\| \leq K\|y\|$ is an equivalence map between $X \oplus_{F_1} Y$ and $X \oplus_{F_2} Y$.

■

Quasi-linear maps are in general discontinuous so it may come as a surprise thet they actually can be extended from any dense subspace of Y to the whole space in an essentially unique way.

Proposition 4.8 *Suppose that X and Y are quasi-Banach spaces and Y_0 is a dense subspace of Y. Suppose that $F_0 : Y_0 \longrightarrow X$ is a quasi-linear function. Then F_0 can be extended to a quasi-linear function $F : Y \longrightarrow X$ and F is unique up to equivalence.*

Proof Consider the space $X \oplus_{F_0} Y_0$ with a quasi-norm given by :

$$\|(x,y)\| = \|x - F_0(y)\| + \|y\|$$

Let Z be the completion of this space. We'll verify that Z is a twisted sum of X and Y. Define $j : X \longrightarrow X \oplus_{F_0} Y_0$ by $j(x) = (x, 0)$ and $q' : X \oplus_{F_0} Y_0 \longrightarrow Y$ by $q'(x,y) = y$. Then q' extends to a quotient map q of Z onto Y. To prove that the sequence:

$$0 \longrightarrow X \xrightarrow{\ j\ } Z \xrightarrow{\ q\ } Y \longrightarrow 0$$

is a short exact sequence, remains to prve that Ker$q$=$j(X)$. We clearly have $j(x)$ $\subseteq$Ker$q$. Now, if $z \in Z$ such that $q(z) = 0$, then $z = \lim_n (x_n, y_n)$, where $(x_n, y_n) \in$

$X \oplus_{F_0} Y_0$ for all n, and $q(x_n, y_n) \longrightarrow q(z) = 0$, hence $y_n \longrightarrow 0$. But $\|(F_0(y_n), y_n)\| = \|y_n\| \longrightarrow 0$, so we have that:

$$\begin{aligned} \|(x_n - F_0(y_n), 0) - z\| &\leq \|(x_n - F_0(y_n), 0) - (x_n, y_n)\| + \|(x_n, y_n) - z\| \\ &= \|(F_0(y_n), y_n)\| + \|(x_n, y_n) - z\| \\ &= \|y_n\| + \|(x_n, y_n) - z\| \longrightarrow 0 \text{ as } n \longrightarrow \infty \end{aligned}$$

Hence $z = \lim_n (x_n - F_0(y_n), 0)$; thus $(x_n - F_0(y_n))$ converges to some x_0 in X and $z = j(x_0)$. Therefore $j(X) = \text{Ker}q$.
Now by Theorem 4.7, Z is equivalent to $X \oplus_H Y$ for some quasi-linear map H, therefore there exists an isomorphism $T : Z \longrightarrow X \oplus_H Y$ such that the following diagram commutes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \xrightarrow{j} & Z & \xrightarrow{q} & Y & \longrightarrow & 0 \\ & & \downarrow I_X & & \downarrow T & & \downarrow I_Y & & \\ 0 & \longrightarrow & X & \longrightarrow & X \oplus_H Y & \longrightarrow & Y & \longrightarrow & 0 \end{array}$$

Then the restriction of T to $X \oplus_{F_0} Y_0$ has the form $T(x, y) = (x + Ly, y)$ where $L : Y_0 \longrightarrow X$ is linear. From this we obtain that

$$\|F_0(y) - H(y) + L(y)\| \leq \|T\| \|y\|$$

for any $y \in Y_0$. Now define the function $F : Y \longrightarrow X$ by

$$F(y) = \begin{cases} H(y) - L(y) & \text{if } y \notin Y_0 \\ F_0(y) & \text{if } y \in Y_0 \end{cases}$$

It is trivial to verify that F is quasi-linear, and the uniqueness up to equivalence follows from the uniqueness of the completion.

■

Example We'll construct a nontrivial twisted sum of $\mathbb{R}$ and l_1. First we define a non-trivial quasi-linear function $F : l_1 \longrightarrow \mathbb{R}$. From the previous theorem, is enough to define F on on the subspace $\mathbb{R}^\infty$ of finitely supported sequences on l_1, with the norm from l_1. Let:

$$F(x) = \sum x_n \ln |x_n| - \left(\sum x_n\right) \cdot \ln \left|\sum x_n\right|, \; x = (x_1, x_2, ...) \in \mathbb{R}^\infty$$

where $0 \ln 0$ is taken as 0. It is easy to see that F is quasi-linear . We'll prove that F is not equivalent to 0, i.e. there isn't any linear function $\varphi : l_1 \longrightarrow \mathbb{R}$ such that:

$$|F(x) - \varphi(x)| \leq M\|x\|. \tag{4.4}$$

Note that $F(e_n) = 0$, where $\{e_n\}_{n=1}^{\infty}$ denotes the unit vector basis in l_1. Assuming there exist a linear function φ satisfying (4.4) then we have that $|\varphi(e_n)| \leq M$ for all n, hence φ is bounded on the unit sphere of $\mathbb{R}^\infty$. On the other hand:

$$x_n = \frac{1}{n}\sum_{i=1}^{n} e_i \in S(\mathbb{R}^\infty) \text{ for any } n.$$

and $F(x_n) = -\ln n$. So:

$$\begin{aligned} |\varphi(x_n)| = & \ |F(x_n) - \varphi(x_n) - F(x_n)| \\ \geq & \ ||F(x_n)| - |F(x_n) - \varphi(x_n)|| \\ = & \ |F(x_n)| - |F(x_n) - \varphi(x_n)| \text{ for } n \text{ large enough} \\ \geq & \ \ln n - M \longrightarrow \infty \text{ as } n \longrightarrow \infty \end{aligned}$$

contradiction with φ bounded on $S(\mathbb{R}^\infty)$.
Therefore $\mathbb{R} \oplus_F l_1$ is a non-trivial twisted sum of $\mathbb{R}$ and l_1. In particular, $\mathbb{R} \oplus_F l_1$ is **not** isomorphic to a Banach space, since in a Banach space any 1-dimensional subspace is complemented.

This example makes it clear that the natural setting for discussion of twisted sums is that of quasi-Banach spaces. In general, even when X and Y are Banach spaces, the quasi-norm in $X \oplus_F Y$ may fail to be equivalent to a norm (as in the example above). However, given some *regularity conditions* on X and Y, we may conclude that $X \oplus_F Y$ is isomorphic to a Banach space. The *regularity conditions* involve notions as *type, cotype, superreflexivity.* Without entering into the definitions of these terms, we'll list the following theorem, due to **Kalton** and **Peck**

Theorem 4.9 (Kalton,Peck)

(i) If X and Y have nontrivial type then any twisted sum Z of X and Y is isomorphic to a Banach space.

(ii) If X and Y are superreflexive then so is any twisted sum of X and Y.

Let us note that l_1 is a prime example of a Banach space without nontrivial type. As it will be needed in the next chapters, we also note that l_2 has a nontrivial type and is superreflexive.

4.3 Twisted sums of l_2

Definition 4.10 *A function $\varphi : X \to Y$ between two Banach spaces X and Y is called* **Lipschitz** *if there exist a constant K such that for any $x, y \in X$ we have*

$$\|\varphi(x) - \varphi(y)\|_Y \leq K \cdot \|x - y\|_X.$$

There is a general simple way to build quasi-linear functions from l_2 to l_2. Let $\mathcal{L}$ denote the class of Lipschitz functions $\varphi : \mathbb{R}^+ \longrightarrow \mathbb{R}$. For $\varphi \in \mathcal{L}$ we can define a map $F_0 : \mathbb{R}^\infty \longrightarrow l_2$, where $\mathbb{R}^\infty$ is the subspace of finitely supported sequences of l_2, given by:

$$F_0(x)[k] = \begin{cases} x_k \cdot \varphi\left(\ln \frac{\|x\|}{|x_k|}\right) & \text{, for } x_k \neq 0 \\ 0 & \text{, for } x_k = 0 \end{cases} \tag{4.5}$$

Proposition 4.11 *F_0 defined in (4.5) is a quasi-linear function.*

Proof First we'll show that for all scalars a, b we have:

$$|(a+b)\varphi(\ln|a+b|) - a\varphi(\ln|a|) - b\varphi(\ln|b|)| \leq 2L(|a| + |b|), \tag{4.6}$$

where L is the Lipschitz constant of φ, and $0\varphi(\ln 0)$ is taken to be 0. Also, without loss of generality, we may assume that $\varphi(0) = 0$. Note that for any $0 < t < 1$ we have $|t \ln|t|| \leq 1/e$. We distinguish two cases:
Case 1 $|a| + |b| = 1$. In this case we have:

$$\begin{aligned} &|(a+b)\varphi(\ln|a+b|) - a\varphi(\ln|a|) - b\varphi(\ln|b|)| \\ &\quad\leq L(|(a+b)\ln|a+b|| + |a\ln|a|| + |b\ln|b||) \\ &\quad\leq 3L/e \\ &\quad\leq 2L. \end{aligned}$$

Case 2 $|a| + |b| \neq 1$ In this case put $s = |a| + |b|$ and let $c = a/s$, $d = b/s$ and $\psi(t) = \varphi(t + \ln s) - \varphi(\ln t)$. We have now that $|c| + |d| = 1$ and therefore we have the inequality (4.6) for c, d and ψ, according to *Case 1*. Doing the calculations, we obtain the required inequality for a, b and φ.

Define now $G : \mathbb{R}^\infty \longrightarrow l_2$ by:

$$G(x)[k] = \begin{cases} x_k \cdot \varphi\left(\ln \frac{1}{|x_k|}\right) & \text{, for } x_k \neq 0 \\ 0 & \text{, for } x_k = 0 \end{cases}$$

From (4.6) we obtain that:

$$\|G(x+y)-G(x)-G(y)\| \leq 2L(\|x\|+\|y\|), \text{for } x,y \in \mathbb{R}^\infty$$

Fix now $u \in \mathbb{R}^\infty$ with $\|u\| \leq 1$. From the definition of G and from the fact that φ is Lipschitz it is easy to see that the kth coordinate of $\|u\|G(u/\|u\|) - G(u)$ is bounded by $L|u_k \ln \|u\||$. Therefore:

$$\| \|u\|G(u/\|u\|) - G(u)\| \leq L\|u\| |\ln \|u\|| \leq L/e.$$

Note that $F_0(x) = \|x\|G(x/\|x\|)$. For $\|x\| + \|y\| = 1$ we have:

$$\begin{aligned}
\|F_0(x+y) - F_0(x) - F_0(y)\| &= \|G(x+y) - \|x\|G(x/\|x\|) - \|y\|G(y/\|y\|)\| \\
&\leq \|G(x+y) - G(x) - G(y)\| \\
&+ \| \|x\|G(x/\|x\|) - G(x)\| \\
&+ \| \|y\|G(y/\|y\|) - G(y)\| \\
&\leq 2L + L/e + L/e.
\end{aligned}$$

Since F_0 is homogeneous, it follows from above that is quasi-linear.

■

Since $F_0 : \mathbb{R}^\infty \longrightarrow l_2$ is quasi-linear and $\mathbb{R}^\infty$ is dense in l_2 it follows from Proposition 4.8 that F_0 can be extended to quasi-linear function $F : l_2 \longrightarrow l_2$ and the extension is unique up to equivalence. For any $\varphi \in \mathcal{L}$ we'll denote by $Z_2(\varphi)$ the twisted sum $l_2 \oplus_F l_2$, where F is defined as above. Note that the definition is unambiguos, since the extensions F of F_0 are all equivalent. So, in other words, $Z_2(\varphi)$ is the completion of the space of pairs of finitely supported sequences $x = \sum x_n e_n$ and $y = \sum y_n h_n$ (where by $\{e_n\}_{n=1}^\infty$ and $\{h_n\}_{n=1}^\infty$ we denote the usual unit vector basis in the two l_2 spaces) endowed with the quasi-norm:

$$\|(x,y)\| = \left(\sum_{n=1}^\infty (x_n - y_n \varphi(\ln(\|y\|_2/|y_n|)))^2 \right)^{1/2} + \|y\|_2 \tag{4.7}$$

For $\varphi(t) = t$ the space $Z_2(\varphi)$ is denoted by Z_2 and is called the **Kalton-Peck space.** We mentioned in the previous section that l_2 has nontrivial type, therefore, from Theorem 4.9 we have that the quasi-norm from (4.7) is equivalent to a norm, i.e. $Z_2(\varphi)$ is a Banach space.

A result of **Kalton** and **Peck** gives sufficient conditions for $Z_2(\varphi)$ to be trivial (we also say that the twisted sum **splits**).

Theorem 4.12 (Kalton,Peck) *Let φ and ψ be in $\mathcal{L}$. Then $Z_2(\varphi)$ and $Z_2(\psi)$ are equivalent twisted sums if and only if* $\sup\limits_{t>0} |\varphi(t) - \psi(t)| < \infty$.

It follows in particular that, if $Z_2(\varphi)$ splits, so $Z_2(\varphi)$ is isomorphic to l_2 and the converse is also true!

Remark: In 1975 , **Enflo, Lindenstrauss** and **Pisier** gave a negativ answer to the **Palais problem**: If both Y and X/Y are isomorphic to a Hilbert space, is X necessarily isomorphic to a Hilbert space? The spaces $Z_2(\varphi)$ with $\sup_{t>0} |\varphi(t)| = \infty$ give an alternative answer to the problem. Indeed, if $\sup_{t>0} |\varphi(t)| = \infty$ then $Z_2(\varphi)$ does not split, hence is **not** isomorphic to l_2, but both $Y = span\{(e_n, 0), n \in \mathbb{N}\}$ and X/Y **are** isomorphic to l_2 , by the very definition of a twisted sum.

4.4 Unconditional structure of $Z_2(\varphi)$

In this section will examine the unconditional structure of $Z_2(\varphi)$ **Johnson, Lindenstrauss** and **Schechtman** proved in [JLS] that the space Z_2 does not have unconditional basis. We'll prove the same result for an arbitrary nontrivial $Z_2(\varphi)$ Let $\varphi \in \mathcal{L}$ such that $Z_2(\varphi)$ does not split. Denote by E_n the two-dimensional subspace of $Z_2(\varphi)$ spanned by the vectors $(e_n, 0)$ and $(0, h_n)$, i.e. $E_n = span\{(e_n, 0), (0, h_n)\}$.

Proposition 4.13 *The spaces E_n, $n \in \mathbb{N}$ form a 2-dimensional UFDD for $Z_2(\varphi)$, which is 1-unconditional (i.e. $\|\sum u_n\| = \|\sum \varepsilon_n u_n\|$ for every choice of finitely many vectors $u_n \in E_n$ and signs $\varepsilon_n = \pm 1$). Moreover this UFDD is symmetric in the sense that for any permutation $\pi : \mathbb{N} \longrightarrow \mathbb{N}$ we have $\|(x_\pi, y_\pi)\| = \|(x, y)\|$ where $x_\pi(n) = x(\pi(n))$.*

Proof

Fix $u_1, u_2, ..., u_n$ arbitrary n vectors with $u_k \in E_k$ for any k between 1 and n and fix arbitrary n signs $\varepsilon_1, \varepsilon_2, ..., \varepsilon_n$. Let $u_k = (a_k e_k, b_k h_k)$ for any k. Also note that the standard vector basis of l_2 is unconditional, so $\|\sum \varepsilon_n a_n e_n\| = \|\sum a_n e_n\|$ for any $a = \{a_n\}_{n=1}^{\infty} \in l_2$. Let F_0 be the function defined by 4.5. Therefore we have:

$$
\begin{aligned}
\|\sum_{k=1}^{n} \varepsilon_k u_k\| &= \|(\sum_{k=1}^{n} \varepsilon_k a_k e_k, \sum_{k=1}^{n} \varepsilon_k b_k h_k)\| \\
&= \|\sum_{k=1}^{n} \varepsilon_k b_k h_k)\| + \|\sum_{k=1}^{n} \varepsilon_k a_k e_k - F_0(\sum_{k=1}^{n} \varepsilon_k b_k h_k)\| \\
&= \|\sum_{k=1}^{n} b_k h_k)\| + \|\sum_{k=1}^{n}(\varepsilon_k a_k - \varepsilon_k b_k \varphi(\ln(\|\sum_{k=1}^{n} b_k h_k)\|/|b_k|))e_k\| \\
&= \|\sum_{k=1}^{n} b_k h_k)\| + \|\sum_{k=1}^{n} \varepsilon_k(a_k - b_k \varphi(\ln(\|\sum_{k=1}^{n} b_k h_k)\|/|b_k|))e_k\| \\
&= \|\sum_{k=1}^{n} b_k h_k)\| + \|\sum_{k=1}^{n} a_k e_k - F_0(\sum_{k=1}^{n} b_k h_k)\| \\
&= \|\sum_{k=1}^{n} u_k\|
\end{aligned}
$$

Hence the spaces E_n form a 1-unconditional 2-dimensional UFDD for $Z_2(\varphi)$. Simple calculations and the fact that the standard unit basis in l_2 is symmetric gives the fact that this UFDD is symmetric.

■

Also from the previous proposition follows easily that the sequence $e_1, h_1, e_2, h_2, e_3, h_3, \ldots..$ is a basis for $Z_2(\varphi)$

To prove that $Z_2(\varphi)$ doesn't have an unconditional basis, we'll prove a slightly stronger result, that $Z_2(\varphi)$ doesn't even have **Gordon-Lewis local unconditional structure.**

Definition 4.14 *A Banach space X is said to have* **Gordon-Lewis local unconditiona structure(in short l.u.st.)** *if for every finite dimensional subspace E of X the inclusion operator $I : E \longrightarrow X$ factors through a finite dimensional space U with an unconditional basis in an uniform manner, i.e. there exist a K such that for every finite dimensional subspace E of X there exist a finite dimensional space U and operators $T : E \longrightarrow U$ and $S : U \longrightarrow X$ such that $ST = I$ and $\|T\| \cdot \|S\| \cdot ubc(U) \leq K$, where $ubc(U)$ is the unconditionality constant of U.*

This definition provides a weak notion of unconditionality. A space having an unconditional basis has l.u.st. as well, but the converse is not true.

For our purpose, we'll need a result that gives a characterization of superreflexive Banach spaces that admit a finite-dimensional UFDD and have l.u.st. The following theorem is due to **Johnson, Lindenstrauss** and **Schechtman**, [JLS]:

Theorem 4.15 *Let E be a superreflexive Banach space with an unconditional decomposition into finite-dimensional subspaces $\{E_k\}_{k=1}^{\infty}$. Then E has local unconditional structure if and only if there is a Banach space $F \supset E$ with an unconditional basis $\{z_{n,k}\}_{n \leq m_k, k<\infty}$ so that $E_k \subset \text{span}\{z_{n,k}\}_{n=1}^{m_k}$ for all k, and so that there is a bounded linear projection P from F onto E with $Pz_{n,k} \in E_k$ for all k and $n \leq m_k$.*

Before proving the main result, we'll need the following two Lemmas:

Lemma 4.16 *Let φ be a Lipschitz function such that $(\varphi(\ln\sqrt{n}))_n$ is bounded. Then $\sup_{t>0} |\varphi(t)| < \infty$.*

Proof We have that there exist a $M > 0$ such that for any n, $|\varphi(\ln\sqrt{n})| < M$. Pick $t > 0$. Then there exists a unique n such that $\ln\sqrt{n} \leq t < \ln\sqrt{n+1}$.

Then:

$$\begin{aligned}
|\varphi(t)| &\leq |\varphi(t) - \varphi(\ln\sqrt{n+1})| + |\varphi(\ln\sqrt{n+1})| \\
&\leq |t - \ln\sqrt{n+1}| + M \\
&\leq |\ln\sqrt{n} - \ln\sqrt{n+1}| + M \\
&= \tfrac{1}{2}\ln(1+\tfrac{1}{n}) + M \\
&\leq \tfrac{1}{2}\ln 2 + M
\end{aligned}$$

Hence $\sup_{t>0} |\varphi(t)| < \infty$.

■

Lemma 4.17 *Let A be a 2 x 2 matrix $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$. Define $T : E_n \longrightarrow E_n$ by:*

$$T(x_n e_n, y_n h_n) = ((\alpha x_n + \beta y_n)e_n, (\gamma x_n + \delta y_n)h_n)$$

and define formally an operator $\tilde{T}$ on $Z_2(\varphi)$ by:

$$\tilde{T}\left(\sum_{n=1}^{\infty} x_n e_n, \sum_{n=1}^{\infty} y_n h_n\right) = \sum_{n=1}^{\infty} T(x_n e_n, y_n h_n).$$

Then: $\tilde{T}$ continuous if and only if $\alpha = \delta$ and $\gamma = 0$.

Proof For the **if** part assume that $\alpha = \delta$ and $\gamma = 0$. Let F_0 be the function defined by (4.5) and F a extension given by Theorem 4.8 such that $Z_2(\varphi) = l_2 \oplus_F l_2$.
When $\alpha = \delta$ and $\gamma = 0$, T is:

$$T(x_n e_n, y_n h_n) = ((\alpha x_n + \beta y_n)e_n, \alpha y_n h_n)$$

and

$$\tilde{T}(\sum_{n=1}^{\infty} x_n e_n, \sum_{n=1}^{\infty} y_n h_n) = (\sum_{n=1}^{\infty}(\alpha x_n + \beta y_n)e_n, \sum_{n=1}^{\infty} \alpha y_n h_n)$$

Then:

$$\begin{aligned}
\|\tilde{T}(\sum_{n=1}^{\infty} x_n e_n, \sum_{n=1}^{\infty} y_n h_n)\| = &\ \|(\sum_{n=1}^{\infty}(\alpha x_n + \beta y_n)e_n, \sum_{n=1}^{\infty} \alpha y_n h_n)\| \\
= &\ |\alpha|\|y\| + \|\sum_{n=1}^{\infty} \alpha x_n e_n + \sum_{n=1}^{\infty} \beta y_n e_n - F(\alpha \sum_{n=1}^{\infty} y_n h_n)\| \\
= &\ |\alpha|\|y\| + \|\alpha(\sum_{n=1}^{\infty} x_n e_n - F(\sum_{n=1}^{\infty} y_n h_n)) + \beta \sum_{n=1}^{\infty} y_n e_n\| \\
\leq &\ |\alpha|\|y\| + |\alpha|\|\sum_{n=1}^{\infty} x_n e_n - F(\sum_{n=1}^{\infty} y_n h_n)\| + |\beta|\|y\| \\
= &\ (|\alpha| + |\beta|)\|y\| + |\alpha|\|x - F(y)\| \\
\leq &\ (|\alpha| + |\beta|)\|(x, y)\|
\end{aligned}$$

Hence $\tilde{T}$ is bounded.

For the **only if** part assume that $\tilde{T}$ is bounded.

Let $u_n = \frac{1}{\sqrt{n}} \sum_{k=1}^{n}(e_k, 0)$. Note that $\|u_n\| = 1$ and

$$\tilde{T}(u_n) = \frac{1}{\sqrt{n}}\tilde{T}(\sum_{k=1}^{n}(e_k, 0)) = \frac{1}{\sqrt{n}}\left(\alpha \sum_{k=1}^{n} e_k, \gamma \sum_{k=1}^{n} h_k\right)$$

Then:

$$\begin{aligned}
\|\tilde{T}(u_n)\| = &\ \tfrac{1}{\sqrt{n}}\left(|\gamma|\sqrt{n} + \|\alpha \sum_{k=1}^{n} e_k - \gamma F_0(\sum_{k=1}^{n} h_k)\|\right) \\
= &\ \tfrac{1}{\sqrt{n}}\left(|\gamma|\sqrt{n} + \|\alpha \sum_{k=1}^{n} e_k - \gamma\varphi(\ln\sqrt{n}) \sum_{k=1}^{n} e_k\|\right) \\
= &\ \tfrac{1}{\sqrt{n}}(|\gamma|\sqrt{n} + \sqrt{n}|\gamma\varphi(\ln\sqrt{n}) - \alpha|) \\
= &\ |\gamma| + |\gamma\varphi(\ln\sqrt{n}) - \alpha| \longrightarrow \infty \text{ from Lemma 4.16 if } \gamma \neq 0
\end{aligned}$$

Since $\tilde{T}$ is bounded we must have $\gamma = 0$.

To prove $\alpha = \delta$, take:

$$u_n = \frac{1}{\sqrt{n}}\left(\varphi(\ln\sqrt{n}) \sum_{k=1}^{n} e_k, \sum_{k=1}^{n} h_k\right)$$

Again note that $\|u_n\| = 1$. We have

$$T(u_n) = \frac{1}{\sqrt{n}}\left((\alpha\varphi(\ln\sqrt{n}) + \beta)e_k, \delta h_k\right)$$

so

$$\tilde{T}(u_n) = \frac{1}{\sqrt{n}}\left((\alpha\varphi(\ln\sqrt{n}) + \beta)\sum_{k=1}^{n} e_k, \delta\sum_{k=1}^{n} h_k\right)$$

Hence:

$$\begin{aligned}\|\tilde{T}(u_n)\| = &\ \tfrac{1}{\sqrt{n}}\left(|\delta|\sqrt{n} + \|((\alpha-\delta)\varphi(\ln\sqrt{n}) + \beta)\sum_{k=1}^{n} e_k\|\right)\\ = &\ |\delta| + |(\alpha-\delta)\varphi(\ln\sqrt{n}) + \beta| \longrightarrow \infty \text{ from Lemma 4.16 if } \alpha \neq \delta\end{aligned}$$

Again, since $\tilde{T}$ is bounded we must have $\alpha = \delta$. This concludes the proof.

■

And now we have all the tools to prove the main result for spaces $Z_2(\varphi)$. Later on we'll prove a stronger version of this theorem.

Theorem 4.18 *The space $Z_2(\varphi)$ has unconditional basis if and only if $Z_2(\varphi)$ is trivial.*

Proof The **if** part is immediate , since $Z_2(\varphi)$ being trivial implies that $Z_2(\varphi)$ is isomorphic to l_2 and the standard unit vector basis is unconditional in l_2.

For the **only if** part we'll prove that if $Z_2(\varphi)$ is non-trivial then $Z_2(\varphi)$ doesn't have l.u.st., hence doesn't have an unconditional basis as well.
Assume by contradiction that $Z_2(\varphi)$ has l.u.st. The argument that will yield an contradiction is divided into several steps.

Step 1 We shall construct a bounded linear operator $\tilde{T} : Z_2(\varphi) \longrightarrow Z_2(\varphi)$ with the property that $\tilde{T}E_k \subset E_k$ for all k and so that the matrices A_k of the restriction of $\tilde{T}$ to E_k with respect to the natural basis satisfy:

$$\inf_k dist\left(A_k, \left\{\begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix} ; \alpha, \beta \in \mathbb{R}\right\}\right) > 0 \tag{4.8}$$

where the distance is taken with respect to any fixed norm on the space of 2x2 matrices. To build this $\tilde{T}$ assume that F, $z_{n,k}$'s and P are as in Theorem 4.15, and let $z^*_{n,k} \in F^*$ be the biorthogonal vectors to $(z_{n,k})_{n,k}$. We claim that the operator defined as:

$$T_J u = P\left(\sum_{k,n\in J} z^*_{n,k}(u) z_{n,k}\right), u \in Z_2(\varphi) \tag{4.9}$$

is bounded for any set of indices J. Indeed,

$$\begin{aligned}
\|T_J u\| &= \|P\left(\sum_{k,n\in J} z^*_{n,k}(u) z_{n,k}\right)\| \\
&\leq \|P\| \cdot \|\sum_{k,n\in J} z^*_{n,k}(u) z_{n,k}\| \\
&\leq \|P\| \cdot K \cdot \|\sum_{k,n} z^*_{n,k}(u) z_{n,k}\| \\
&= \|P\| \cdot K \cdot \|u\|
\end{aligned}$$

where K is the unconditionality constant of $(z_{n,k})_{n,k}$. Hence T_J is bounded for any set of indices J.

Next, for each k, we shall find subsets $J_k \subset \{1, 2, ..., m_k\}$ such that the 2x2 matrices A_k corresponding to the restrictions of $P\left(\sum_{n\in J_k} z^*_{n,k}(u) z_{n,k}\right)$ to E_k satisfy:

$$dist\left(A_k, \left\{\begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix}; \alpha, \beta \in \mathbb{R}\right\}\right) \geq \mu \tag{4.10}$$

for all k and some fixed $\mu > 0$.

Fix k. For each $1 \leq n \leq m_k$ consider the rank one operator $S_n : E_k \longrightarrow E_k$ defined as $S_n u = P\left(z^*_{n,k}(u) z_{n,k}\right)$. Since S_n is rank-one and E_k is two-dimensional, its matrix with respect to the natural basis is either of the form

$$\begin{pmatrix} a_n & b_n \\ \alpha_n a_n & \alpha_n b_n \end{pmatrix} \text{ or } \begin{pmatrix} 0 & 0 \\ a_n & b_n \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \alpha_n a_n & \alpha_n b_n \end{pmatrix}$$

where , in the second case we consider $\alpha_n = 1$ to unify notations. Note that

$$\sum_{n=1}^{m_k} S_n u = \sum_{n=1}^{m_k} P\left(z^*_{n,k}(u) z_{n,k}\right) = P\left(\sum_{n=1}^{m_k} z^*_{n,k}(u) z_{n,k}\right) = Pu = u,$$

for any $u \in E_k$. Hence the sum of all the m_k matrices is the identity matrix, so we have that $\sum_{n=1}^{m_k} \alpha_n b_n = 1$. Now summing the absolute values of the upper right entries of the matrices of the first form and the lower right entries of the matrices of the second form and taking into account that $(z_{n,k})_{n,k}$ is an unconditional basis for F we obtain that there is an absolute constant K_0 such that $\sum_{n=1}^{m_k} |b_n| \leq K_0$. Let $I_k := \{n; |\alpha_n| \geq \frac{1}{2K_0}\}$ We claim that $\sum_{n\in I_k} \alpha_n b_n \geq \frac{1}{2}$ Indeed if we assume the contrary, we get

$$\begin{aligned} 1 = |\sum_{n=1}^{m_k} \alpha_n b_n| \leq & \ |\sum_{n\in I_k} \alpha_n b_n| + |\sum_{n\notin I_k} \alpha_n b_n| \\ < & \ \tfrac{1}{2} + \tfrac{1}{2K_0} \cdot \sum_{n\notin I_k} |b_n| \\ \leq & \ \tfrac{1}{2} + \tfrac{1}{2K_0} \cdot K_0 \\ = & \ 1 \end{aligned}$$

and we obtain a contradiction. Hence $\sum_{n\in I_k} \alpha_n b_n \geq \frac{1}{2}$.
Now fix an $n \in I_k$ and assume that for this n the matrix is of the first form. We claim that at least one of the following possibilities must occur:

i) $\alpha_n a_n \geq \frac{1}{1+2K_0}|\alpha_n b_n|$

ii) $-\alpha_n a_n \geq \frac{1}{1+2K_0}|\alpha_n b_n|$

iii) $a_n - \alpha_n b_n \geq \frac{1}{1+2K_0}|\alpha_n b_n|$

iv) $\alpha_n b_n - a_n \geq \frac{1}{1+2K_0}|\alpha_n b_n|$

Assume that none of the above occur. Then the negations of i) and ii) imply that $|\alpha_n a_n| < \frac{1}{1+2K_0}|\alpha_n b_n|$, hence, since $n \in I_k$, we obtain

$$|a_n| < \frac{1}{1+2K_0}|b_n| \leq \frac{2K_0}{1+2K_0}|\alpha_n b_n| \tag{4.11}$$

Also the negations of iii) and iv) imply that

$$|a_n - \alpha_n b_n| < \frac{1}{1+2K_0}|\alpha_n b_n| \tag{4.12}$$

Therefore, from (4.11) and (4.12) we obtain that:

$$\begin{aligned} |\alpha_n b_n| = |a_n - (a_n - \alpha_n b_n)| \leq & \ |a_n| + |a_n - \alpha_n b_n| \\ < & \ \tfrac{2K_0}{1+2K_0}|\alpha_n b_n| + \tfrac{1}{1+2K_0}|\alpha_n b_n| \\ = & \ |\alpha_n b_n| \end{aligned}$$

and we obtain a contradiction. Hence our claim is true. If $n \in I_k$ is such that the matrix of S_n is of the second form then we obtain trivially that $|\alpha_n b_n| \geq \frac{1}{1+2K_0}|\alpha_n b_n|$ hence the analogue of iii) or iv) for this n hold.

Since $\sum_{n\in I_k} \alpha_n b_n \geq \frac{1}{2}$ we have that there exist a subset $J_k \subset I_k$ so that $\sum_{n\in J_k} \alpha_n b_n \geq \frac{1}{8}$ and so that one of the possibilities i)-iv) holds for any choice of $n \in J_k$. We'll prove next that for this choice of J_k condition (4.10) holds. For convenience, we'll consider the distance with respect to the supremum norm on the space of 2x2 matrices. For a fixed k, we can distinguish two cases:

1) The possibilities i) or ii) occur and in this case the lower left corner of A_k, $\sum_{n\in J_k} \alpha_n a_n$, satisfies:

$$|\sum_{n\in J_k} \alpha_n a_n| \geq \frac{1}{1+2K_0} \sum_{n\in J_k} |\alpha_n b_n| \geq \frac{1}{8(1+2K_0)}$$

therefore, the distance to the set of matrices of the form $\begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix}$ is at least $\frac{1}{8(1+2K_0)}$.

2) the possibilities iii) or iv) occur. In this case note that the absolute value of difference between the upper left corner and the lower right corner of A_k satisfies:

$$|\sum_{n\in J_k} a_n - \alpha_n b_n| \geq \sum_{n\in J_k} |a_n - \alpha_n b_n| \geq \frac{1}{1+2K_0} \sum_{n\in J_k} |\alpha_n b_n| \geq \frac{1}{8(1+2K_0)}$$

Therefore we have

$$\begin{aligned} \max_{\alpha\in\mathbb{R}}\{|\sum_{n\in J_k} a_n - \alpha|, |\sum_{n\in J_k} \alpha_n b_n - \alpha|\} \geq\ & \tfrac{1}{2}(|\sum_{n\in J_k} a_n - \alpha| + |\sum_{n\in J_k} \alpha_n b_n - \alpha|) \\ \geq\ & \tfrac{1}{2}|\sum_{n\in J_k} a_n - \alpha_n b_n| \\ \geq\ & \tfrac{1}{16(1+2K_0)}. \end{aligned}$$

Hence, the distance to the set of matrices of the form $\begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix}$ is at least $\frac{1}{16(1+2K_0)}$. So the relation (4.10) holds for $\mu = \frac{1}{16(1+2K_0)}$.

Now we'll take $J = \bigcup_k J_k$ and define $\tilde{T} := T_J$ where T_J is defined as in the formula (4.9).

We already checked that $\tilde{T}$ is bounded and it is easy to verify that $\tilde{T}E_k \subset E_k$. Also, from the previous considerations, the estimate (4.8) holds as well.

Step 2 Next we'll prove that the existence of such a $\tilde{T}$ as defined above contradicts Lemma 4.17.

We have that $\tilde{T}$ is a diagonal operator with respect to the 2-dimensional decomposition and the matrices of the restriction of this operator to E_k are A_k. Hence, we can find a subsequence $(k_i)_i$ of integers such that all the matrices A_{k_i} are small enough perturbations of a fixed matrix which cannot be of the type $\begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix}$ since we have that

$$\inf_{k_i} dist\left(A_{k_i}, \left\{\begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix} ; \alpha, \beta \in \mathbb{R}\right\}\right) > 0$$

Since the UFDD is symmetric (actually one only needs the subsymmetry) we have that $\overline{span}\{E_{k_i}\}_{i=1}^{\infty}$ is naturally isometric to $Z_2(\varphi)$. But it is easy to see that Lemma 4.17 clearly continues to hold when all the matrices A_{k_i} are small enough perturbations of one fixed matrix. Therefore we obtained a contradiction with the mentioned lemma, since the diagonal operator (with respect to UFDD) we built is bounded but its restrictions to E_k are not of the form (or at least small perturbations) of a matrix of the type $\begin{pmatrix} \alpha & \beta \\ 0 & \alpha \end{pmatrix}$.

Therefore $Z_2(\varphi)$ doesn't have l.u.st. and the proof is complete.

■

4.5 Arbitrary twisted sums of l_2

We begin by observing that in the Theorem 4.18 the particular form of the norm in $Z_2(\varphi)$ was not used, but rather the properties of $Z_2(\varphi)$ derived from previous results turned to be of much importance. This suggest that the theorem can be extended for arbitrary twisted sums of l_2 that satisfy certain properties. In this section we shall examine an extension of the main theorem for $l_2 \oplus_F l_2$ where F is an arbitrary quasilinear function. We also shall describe some conditions on F which ensure that a version of the main theorem holds.

Let $F : l_2 \longrightarrow l_2$ be a non-trivial quasi-linear function and consider the twisted sum $l_2 \oplus_F l_2$. Assume for the moment that F is such that $l_2 \oplus_F l_2$ has canonical UFDD which is subsymetric. Recall that the norm in $l_2 \oplus_F l_2$ is given by

$$\|(x, y)\| = \|y\| + \|x - Fy\|.$$

We'll denote by $Z_2(F)$ the space $l_2 \oplus_F l_2$. Recall from previous sections that $Z_2(F)$ is trivial if and only if there exist an linear operator $T : l_2 \longrightarrow l_2$ and a constant K such that

$$\|Fx - Tx\| \leq K\|x\|.$$

Hence, $Z_2(F)$ is **not** trivial if for any $S : l_2 \longrightarrow l_2$ linear and for any K, there exist an $x = x(K, S) \in l_2$, $\|x\| = 1$ such that

$$\|Fx - Sx\| \geq K. \tag{4.13}$$

As before, we'll denote by E_n the subspace of $l_2 \oplus_F l_2$ spanned by $(e_n, 0)$ and $(0, h_n)$. To prove that Theorem 4.18 "works" in this case as well we'll need the following result, a version of Lemma 4.17:

Lemma 4.19 *Assume that the space $Z_2(F) = l_2 \oplus_F l_2$ has canonical UFDD (i.e.$Z_2(F) = \sum_{n=1}^{\infty} E_n$), which is subsymmetric.*
Let A be a 2 x 2 matrix $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$. Define $T : E_n \longrightarrow E_n$ by:

$$T(x_n e_n, y_n h_n) = ((\alpha x_n + \beta y_n)e_n, (\gamma x_n + \delta y_n)h_n)$$

and define formally an operator $\tilde{T}$ on $Z_2(F)$ by:

$$\tilde{T}(\sum_{n=1}^{\infty} x_n e_n, \sum_{n=1}^{\infty} y_n h_n) = \sum_{n=1}^{\infty} T(x_n e_n, y_n h_n).$$

Then: $\tilde{T}$ continuous $\Leftrightarrow \alpha = \delta$ and $\gamma = 0$

Proof The argument used in Lemma 4.17 for the **if** part is valid here as well.

To prove the **only if** part assume that $\tilde{T}$ is bounded. For $u = \sum_{i=1}^{\infty} a_i e_i \in l_2$ we have that:

$$\tilde{T}(u, 0) = \sum_{i=1}^{\infty} T(a_i e_i, 0) = \sum_{i=1}^{\infty} (\alpha a_i e_i, \gamma a_i h_i)$$

Hence

$$\begin{aligned} \|\tilde{T}(u,0)\| &= |\gamma| \|\sum_{i=1}^{\infty} a_i h_i\| + \|\alpha \sum_{i=1}^{\infty} a_i e_i - \gamma F(\sum_{i=1}^{\infty} a_i h_i)\| \\ &= |\gamma| \|u\| + |\gamma| \cdot \|\tfrac{\alpha}{|\gamma|} \sum_{i=1}^{\infty} a_i e_i - F(\sum_{i=1}^{\infty} a_i h_i)\| \end{aligned}$$

Define the operator $S : l_2 \longrightarrow l_2$ on the basis elements by $S(h_i) = \frac{\alpha}{|\gamma|} e_i$ and extend it to the whole of l_2 by linearity. Since $Z_2(F)$ is nontrivial, from (4.13) we have that for any $K > 0$ there exist an $u_K \in l_2$, $\|u_K\| = 1$ such that $\|Su_K - Fu_K\| \geq K$. Then, if $\gamma \neq 0$

$$\|\tilde{T}(u_K, 0)\| = |\gamma| + |\gamma| \|Su_K - Fu_K\| \geq |\gamma|(K+1) \longrightarrow \infty \text{ as } K \longrightarrow \infty.$$

But since $\tilde{T}$ is bounded and $\|(u_K, 0)\| = \|u_K\| = 1$ the above condition cannot hold. Therefore $\gamma = 0$.

For the second part let $u = \sum_{i=1}^{\infty} a_i h_i \in l_2$ and write $F(u) = \sum_{i=1}^{\infty} b_i e_i$.

Then

$$\tilde{T}(F(\sum_{i=1}^{\infty} a_i h_i), \sum_{i=1}^{\infty} a_i h_i) = (\sum_{i=1}^{\infty}(\alpha b_i + \beta a_i)e_i, \delta \sum_{i=1}^{\infty} a_i h_i).$$

So we have

$$\begin{aligned}\|\tilde{T}(F(u), u)\| = & \ |\delta|\|u\| + \|\alpha \sum_{i=1}^{\infty} b_i e_i + \beta \sum_{i=1}^{\infty} a_i e_i - \delta F(\sum_{i=1}^{\infty} a_i h_i)\| \\ = & \ |\delta|\|u\| + \|\alpha F(u) + \beta \sum_{i=1}^{\infty} a_i e_i - \delta F(u)\| \\ = & \ |\delta|\|u\| + \|\beta \sum_{i=1}^{\infty} a_i e_i + (\alpha - \delta)F(u)\| \\ = & \ |\delta|\|u\| + |\alpha - \delta| \cdot \|\tfrac{\beta}{|\alpha-\delta|} \sum_{i=1}^{\infty} a_i e_i - F(u)\|.\end{aligned}$$

As before, define $S : l_2 \longrightarrow l_2$ by $S(h_i) = \frac{\beta}{|\alpha-\delta|} e_i$ and extend it by linearity to all of l_2. Since $Z_2(F)$ is nontrivial, from (4.13) we have that for any $K > 0$ there exist a $u_K \in l_2$, $\|u_K\| = 1$ such that $\|Su_K - Fu_K\| \geq K$.
Then, if $\alpha \neq \delta$,

$$\begin{aligned}\|\tilde{T}(F(u_K), u_K)\| = & \ |\delta| + |\alpha - \delta| \cdot \|Su_K - Fu_K\| \\ \geq & \ |\delta| + |\alpha - \delta| \cdot K \longrightarrow \infty \text{ as } K \longrightarrow \infty\end{aligned}$$

Note that $\|(F(u_K), u_K)\| = \|u_K\| + \|F(u_k) - F(u_k)\| = 1$ and since $\tilde{T}$ is bounded the above convergence cannot hold. Therefore $\alpha = \delta$ and this concludes the proof.

■

Now we can formulate our main theorem which strengthen Theorem 4.18.

Theorem 4.20 *Let $Z_2(F)$ be a twisted sum of l_2 with itself such that the canonical 2-dimensional decomposition is unconditional and subsymmetric. Then $Z_2(F)$ has an unconditional basis if and only if $Z_2(F)$ is a trivial twisted sum.*

As remarked before, now that we have Lemma 4.19, the proof of this theorem is similar to that of Theorem 4.18.

A natural question that appears is what conditions on F would ensure that $Z_2(F)$ admits a canonical subsymmetric UFDD? The following proposition gives an partial answer to this question.

Notation If $x = (x_1, x_2, ...)$ is an element in l_2 and $\varepsilon = (\varepsilon_1, \varepsilon_2, ...)$ is a vector of signs (i.e. $\varepsilon_i = \pm 1$) then by εx we understand the vector $(\varepsilon_1 x_1, \varepsilon_2 x_2, ...)$ of l_2.

Proposition 4.21 *Let $F : l_2 \longrightarrow l_2$ be a quasi linear function with the property that there exist a constant K such that for any finitely supported vector $u \in l_2$ and for any choices of signs $\varepsilon = (\varepsilon_1, \varepsilon_2, ...)$ the following inequality holds*

$$\|F(\varepsilon u) - \varepsilon F(u)\| \leq K\|u\|.$$

Then $Z_2(F)$ admits the canonical UFDD.

Proof We'll prove that there exist a constant C such that for any finite number of vectors $u_1, u_2, ..., u_n$ with $u_k \in E_k$ and for any signs $\varepsilon = (\varepsilon_1, \varepsilon_2, ...\varepsilon_n)$ we have that

$$\|\sum_{k=1}^{n} \varepsilon_k u_k\| \leq C\|\sum_{k=1}^{n} u_k\|$$

For any k let $u_k = (a_k e_k, b_k h_k)$. Again note that the standard vector basis of l_2 is 1-unconditional, so $\|\sum \varepsilon_n a_n e_n\| = \|\sum a_n e_n\|$ for any $a = \{a_n\}_{n=1}^{\infty} \in l_2$. Therefore we have:

$$\begin{aligned}
\|\sum_{k=1}^{n} \varepsilon_k u_k\| = \ & \|(\sum_{k=1}^{n} \varepsilon_k a_k e_k, \sum_{k=1}^{n} \varepsilon_k b_k h_k)\| \\
= \ & \|\sum_{k=1}^{n} \varepsilon_k b_k h_k\| + \|\sum_{k=1}^{n} \varepsilon_k a_k e_k - F(\sum_{k=1}^{n} \varepsilon_k b_k h_k)\| \\
\leq \ & \|\sum_{k=1}^{n} \varepsilon_k b_k h_k\| + \|\sum_{k=1}^{n} \varepsilon_k a_k e_k - \varepsilon F(\sum_{k=1}^{n} b_k h_k)\| \\
+ \ & \|\varepsilon F(\sum_{k=1}^{n} b_k h_k) - F(\sum_{k=1}^{n} \varepsilon_k b_k h_k)\| \\
\leq \ & \|\sum_{k=1}^{n} b_k h_k\| + \|\sum_{k=1}^{n} a_k e_k - F(\sum_{k=1}^{n} b_k h_k)\| + K\|\sum_{k=1}^{n} b_k h_k\| \\
\leq \ & (K+1)\|\sum_{k=1}^{n} u_k\|
\end{aligned}$$

Hence the spaces E_n form a 2-dimensional UFDD for $Z_2(F)$ and this concludes the proof.

■

4.6 Subspaces of twisted sums

In this section we'll explore the unconditional structure of subspaces of twisted sums of l_2. We'll begin by proving a general theorem, which will allow to pass from a space having an FDD and an unconditional basis to a subspace having UFDD.

Theorem 4.22 *Let X be a Banach space and $\{e_n\}_{n=1}^\infty$ a monotone normalized unconditional basis of X. Assume that $X = \sum_{n=1}^\infty E_n$ is a shrinking FDD. Then there exist a sequence $(n_k)_{k=1}^\infty$ of natural numbers such that $X = \sum_{k=1}^\infty E_{n_k}$ is unconditional.*

Proof We'll construct by induction the sequence $(E_{n_k})_k$ such that $X = \sum_{k=1}^\infty E_{n_k}$ is unconditional.

Let $E_{n_1} := E_1$. Since E_1 is finite dimensional, its unit sphere is compact, hence for any $\delta > 0$ we can find a finite δ-net of the unit sphere. Pick a finite $\frac{1}{8}$-net of $S_1 = \{x \in E_{n_1}; \|x\| = 1\}$, say $W_1 = \{w_1^1, w_1^2, ..., w_1^{k_1}\}$ with

$$w_1^i = \sum_{j=1}^\infty a_j^i e_j \text{ for } i = 1, 2, ..., k_1$$

For any $1 \le i \le k_1$ there exist a p_i such that $\| \sum_{j=p_i+1}^\infty a_j^i e_j\| < \frac{1}{8}$. Let $u_1^i = \sum_{j=1}^{p_i} a_j^i e_j$, for any $i = 1, 2, ...k_1$. Then we have that $\|w_1^i - u_1^i\| < \frac{1}{8}$, for any $i = 1, 2, ...k_1$. Let $q_1 := \max_{1 \le i \le k_1} p_i$.

Since $\sum_{n=1}^\infty E_n$ is shrinking we have that

$$\|x^*|_{span[E_i]_n^\infty}\| \longrightarrow 0 \text{ for any } x^* \in X^*$$

For any i, let $x_i^* : X \longrightarrow \mathbb{R}$ be defined by $x_i^*(\sum_{j=1}^\infty a_j e_j) = a_i$. Clearly $x_i^* \in X^*$. Now, for any $1 \le i \le q_1$ there exist $r_i > q_1$ such that for any $w \in S_m$ with $m > r_i$ we have that $|x_i^*(w)| < \frac{1}{4 \cdot 4^2 \cdot q_1}$.

Let $r = \max_{1 \le i \le q_1} r_i$ and pick an $n_2 > r$. In S_{n_2} consider a finite $\frac{1}{2 \cdot 4^2}$-net, say $W_2 = \{w_2^1, w_2^2, ..., w_2^{k_2}\}$, with:

$$w_2^i = \sum_{j=1}^\infty b_j^i e_j \text{ for } i = 1, 2, ..., k_2$$

For any i between 1 and k_2 there exist a $s_i > q_1$ such that $\| \sum_{j=s_i+1}^\infty b_j^i e_j\| < \frac{1}{4 \cdot 4^2}$. Let $u_2^i = \sum_{j=q_1+1}^{s_i} b_j^i e_j$, for any $i = 1, 2, ...k_2$. We have that

$$\|w_2^i - u_2^i\| \le \| \sum_{j=1}^{q_1} b_j^i e_j\| + \| \sum_{j=s_i+1}^\infty b_j^i e_j\|$$

But we've just seen that $\| \sum_{j=s_i+1}^\infty b_j^i e_j\| < \frac{1}{4 \cdot 4^2}$ and:

$$\| \sum_{j=1}^{q_1} b_j^i e_j\| \le \sum_{j=1}^{q_1} |b_j^i| = \sum_{j=1}^{q_1} |x_j^*(w_2^i)| \le \sum_{j=1}^{q_1} \frac{1}{4 \cdot 4^2 \cdot q_1} = \frac{1}{4 \cdot 4^2}$$

Hence, for any $1 \leq i \leq k_2$ we have that $\|w_2^i - u_2^i\| \leq \frac{1}{4 \cdot 4^2} + \frac{1}{4 \cdot 4^2} = \frac{1}{2 \cdot 4^2}$

Continuing in this manner we obtain a sequence $(E_{n_k})_k$ such that for each S_{n_k} we have an finite $\frac{1}{2 \cdot 4^k}$-net W_{n_k} with the property that for any $w_{n_k}^i \in W_{n_k}$ there exist an $u_{n_k}^i \in X$ such that $\|w_{n_k}^i - u_{n_k}^i\| < \frac{1}{2 \cdot 4^k}$

Also from the previous construction we see that $(u_{n_k}^{i_k})_k$ is a block basic sequence of $\{e_n\}_{n=1}^\infty$ for **any** choice of i_k in the index set of W_{n_k}. Since $\{e_n\}_{n=1}^\infty$ is unconditional, any such block is unconditional as well.

Next we'll show that **any** sequence $(w_{n_k})_k$ with $w_{n_k} \in S_{n_k}$ is equivalent to a block basis of $\{e_n\}_{n=1}^\infty$. Fix such a sequence $(w_{n_k})_k$. Then for any $w_{n_k} \in S_{n_k}$, there exists an $w_{n_k}^{i_k} \in W_{n_k}$ such that

$$\|w_{n_k} - w_{n_k}^{i_k}\| \leq \frac{1}{2 \cdot 4^k}, \text{ since } W_{n_k} \text{ is an } \tfrac{1}{2 \cdot 4^k}\text{-net.}$$

As we've seen before, for this $w_{n_k}^{i_k}$ there exists an $u_{n_k}^{i_k} \in X$ such that

$$\|w_{n_k}^{i_k} - u_{n_k}^{i_k}\| \leq \frac{1}{2 \cdot 4^k}.$$

We claim that $(w_{n_k})_k \sim (u_{n_k}^{i_k})_k$. Indeed,

$$\|w_{n_k} - u_{n_k}^{i_k}\| \leq \|w_{n_k} - w_{n_k}^{i_k}\| + \|w_{n_k}^{i_k} - u_{n_k}^{i_k}\| \leq \frac{1}{2 \cdot 4^k} + \frac{1}{2 \cdot 4^k} = \frac{1}{4^k}$$

Therefore:

$$\sum_{k=1}^{\infty} \|w_{n_k} - u_{n_k}^{i_k}\| \leq \sum_{k=1}^{\infty} \frac{1}{4^k} = \frac{1}{3}$$

so indeed $(w_{n_k})_k \sim (u_{n_k}^{i_k})_k$, and this concludes the proof.

■

We'll prove next a small general result regarding subspaces of twisted sums:

Lemma 4.23 *Let X and Y be two quasi-Banach spaces and Z a twisted sum of X and Y. Let Z_0 be a subspace of Z. Then there exist subspaces X_0 of X and Y_0 of Y such that Z_0 is a twisted sum of X_0 and Y_0.*

Proof Indeed suppose we have the exact sequence with bounded linear operators:

$$0 \longrightarrow X \xrightarrow{\ j\ } Z \xrightarrow{\ q\ } Y \longrightarrow 0$$

Define:
$$X_0 := \{x \in X; j(x) \in Z_0\} \text{ and } Y_0 := q(Z_0).$$

Clearly X_0 and Y_0 are subspaces of X and Y respectively and it is trivial to check that we have the following short exact sequence with bounded operators:

$$0 \longrightarrow X_0 \xrightarrow{j_0} Z_0 \xrightarrow{q_0} Y_0 \longrightarrow 0$$

where $j_0 = j|_{X_0}$ and $q_0 = q|_{Z_0}$

■

In the case of $Z_2(F)$, any subspace Z_0 of $Z_2(F)$ is a twisted sum of subspaces of l_2. If these subspaces are infinite dimensional then Z_0 is again a twisted sum of l_2 with itself. Hence Z_0 is equivalent with $Z_2(G)$ for some quasi-linear function $G : l_2 \longrightarrow l_2$.

As a corollary to the Theorem 4.22 we have the following results:

Theorem 4.24 *Let $Z_2(F)$ be a twisted sum of l_2 with itself such that $Z_2(F)$ has canonical shrinking FDD which is subsymmetric. Then $Z_2(F)$ has an unconditional basis if and only if $Z_2(F)$ is a trivial twisted sum.*

Proof The **if** part is obvious, and we'll prove now the **only if** part.

From subsymmetry we have that the subspace $\sum_{k=1}^{\infty} E_{n_k}$ of $Z_2(F)$ is isomorphic to $Z_2(F)$ for any sequence $(n_k)_{k=1}^{\infty}$ of natural numbers. Assume that $Z_2(F)$ has an unconditional basis. Then, from the Theorem4.22 we have that there exist a sequence $(n_k)_{k=1}^{\infty}$ of natural numbers such that the sum $\sum_{k=1}^{\infty} E_{n_k}$ is unconditional. But, from the Theorem 4.20 we have that the twisted sum $\sum_{k=1}^{\infty} E_{n_k}$ is trivial and, since it is isomorphic to $Z_2(F)$, we obtain that $Z_2(F)$ is trivial as well.

■

Theorem 4.25 *Let $Z_2(F)$ be a twisted sum of l_2 with itself such that $Z_2(F)$ has canonical shrinking FDD. Then if $Z_2(F)$ has an unconditional basis then there exists a sequence $(n_k)_{k=1}^{\infty}$ of natural numbers such that the sum $\sum_{k=1}^{\infty} E_{n_k}$ is a trivial twisted sum of l_2 with itself.*

Proof Since $Z_2(F)$ has an unconditional basis and canonical shrinking FDD, from Theorem4.22 we have that there exist a sequence $(n_k)_{k=1}^{\infty}$ of natural numbers such that the sum $\sum_{k=1}^{\infty} E_{n_k}$ is unconditional. But then from Theorem 4.20 we have that the twisted sum $\sum_{k=1}^{\infty} E_{n_k}$ is trivial.

■

Clearly, not every twisted sum of l_2 has necessarily canonical FDD. However, if F satisfies some conditions, then we can conclude that $Z_2(F)$ has FDD. The result is given by the following proposition.

Proposition 4.26 *Let $F : l_2 \longrightarrow l_2$ be a quasi-linear function such that for every $y \in l_2$*

$$\|F(y - P_n y)\| \longrightarrow 0 \text{ for } n \longrightarrow \infty$$

where P_n are the canonical projections in l_2. Then $Z_2(F)$ admits canonical FDD.

Proof

Let $x = \sum_{k=1}^{\infty} a_k e_k$ and $y = \sum_{k=1}^{\infty} b_k h_k$ be in l_2. We have to show that

$$\|(x, y) - \sum_{k=1}^{n} (a_k e_k, b_k h_k)\| \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

We have:

$$\begin{aligned}
\|(x, y) - \sum_{k=1}^{n} (a_k e_k, b_k h_k)\| &= \|(\sum_{k=1}^{\infty} a_k e_k, \sum_{k=1}^{\infty} b_k h_k) - \sum_{k=1}^{n} (a_k e_k, b_k h_k)\| \\
&= \|(\sum_{k=n+1}^{\infty} a_k e_k, \sum_{k=n+1}^{\infty} b_k h_k)\| \\
&= \| \sum_{k=n+1}^{\infty} b_k h_k\| + \| \sum_{k=n+1}^{\infty} a_k e_k - F(\sum_{k=n+1}^{\infty} b_k h_k)\| \\
&= \|y - P_n y\| + \|x - P_n x - F(y - P_n y)\| \\
&\leq \|y - P_n y\| + \|x - P_n x\| + \|F(y - P_n y)\|
\end{aligned}$$

But $\|y - P_n y\|$ and $\|x - P_n x\|$ both tend to 0, being the norms of the tails of elements of l_2, and $\|F(y - P_n y)\|$ tends to 0 by the hypothesis. Therefore we have that

$$\|(x, y) - \sum_{k=1}^{n} (a_k e_k, b_k h_k)\| \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

Hence $(x, y) = \sum_{k=1}^{\infty} (a_k e_k, b_k h_k)$. Also, since the representations $x = \sum_{k=1}^{\infty} a_k e_k$ and $y = \sum_{k=1}^{\infty} b_k h_k$ are unique it follows easily that the above representation of (x, y) is unique as well. Therefore $Z_2(F)$ has canonical FDD and the proof is complete.

■

Note that if in particular F is continuous at 0 the hypothesis of the previous proposition are satisfied, hence $Z_2(F)$ has canonical FDD.

Next we'll give a characterization for subspaces of $Z_2(F)$. We start by presenting a lemma which follows easily from Theorem 4.7.

Lemma 4.27 *Let Z be a twisted sum of quasi-Banach spaces X and Y and let Z_0 be a subspace of Z and X_0, Y_0 be the subspaces of X and Y, respectively described before such that Z_0 is a twisted sum of X_0 and Y_0. Then there exist a quasi-linear function $F : Y \longrightarrow X$ such that Z is equivalent with $X \oplus_F Y$ and $F(Y_0) \subset X_0$*

Proof We have the following two exact sequences:

$$0 \longrightarrow X \xrightarrow{j} Z \xrightarrow{q} Y \longrightarrow 0$$

and

$$0 \longrightarrow X_0 \xrightarrow{j_0} Z_0 \xrightarrow{q_0} Y_0 \longrightarrow 0$$

where $j_0 = j|_{X_0}$ and $q_0 = q|_{Z_0}$. Recall from Theorem 4.7 that when we built a quasi linear function F such that Z is equivalent with $X \oplus_F Y$ we constructed the functions $\varphi : Y \longrightarrow Z$ a linear right inverse of q and $\psi : Y \longrightarrow Z$ a bounded right inverse of q and then set

$$F(y) = j^{-1}(\psi(y) - \varphi(y)).$$

Note from the proof of Theorem 4.7 that we can construct this functions such that $\varphi(Y_0) \subset Z_0$ and $\psi(Y_0) \subset Z_0$. Then the function F obtained in this way has the property that $F(Y_0) \subset X_0$

Now consider Z a twisted sum of l_2 with itself and Z_0 a subspace of Z. We saw that Z_0 is a twisted sum of X_0 and Y_0 for some X_0 and Y_0 subspaces of l_2. Then there exists a quasi-linear function $F : l_2 \longrightarrow l_2$ such that $F(Y_0) \subset X_0$ and Z is equivalent with $Z_2(F)$. Assume that X_0 and Y_0 are infinite dimensional, hence isometric with l_2 and let $T : l_2 \longrightarrow X_0$ and $S : l_2 \longrightarrow Y_0$ be the coresponding isometries. Define $G : l_2 \longrightarrow l_2$ by

$$G = T^{-1} \circ F \circ S$$

Note that G is well defined since $F(Y_0) \subset X_0$.

■

Theorem 4.28 *In the previous setting, we have* *(i) G is a quasi-linear function*

(ii) $Z_2(G)$ is equivalent with Z_0

Proof
(i) For any x, y in l_2 we have:

$$\begin{aligned} \|G(x+y)-G(x)-G(y)\| &= \|T^{-1}(F(S(x+y))-F(Sx)-F(Sy))\| \\ &\leq \|F(Sx+Sy)-F(Sx)-F(Sy)\| \\ &\leq k(\|Sx\|+\|Sy\|) \\ &= k(\|x\|+\|y\|) \end{aligned}$$

where k is the quasi-linearity constant of F. The homogeneity of G follows easily from the homogeneity of F, T and S. Hence G is quasi-linear.
(ii) To prove that $Z_2(G)$ and Z_0 are equivalent it is enough to show that

$$\|(x,y)\|_G = \|(Tx, Sy)\|_F \text{ for any } x, y \in l_2.$$

We have:

$$\begin{aligned} \|(Tx,Sy)\|_F &= \|Sy\|+\|Tx-F(Sy)\| \\ &= \|y\|+\|T^{-1}(Tx-F(Sy))\| \\ &= \|y\|+\|x-T^{-1}(F(Sy))\| \\ &= \|y\|+\|x-G(y)\| \\ &= \|(x,y)\|_G \end{aligned}$$

Hence $Z_2(G)$ and Z_0 are equivalent and the proof is complete.

■

Corollary 4.29 *In the previous setting, if there exist a linear (not necessarily continuous) function $Q : l_2 \longrightarrow l_2$ such that*

$$\|T^{-1} \circ F \circ S(y) - F(y) - Q(y)\| \leq C\|y\|$$

for some constant C and for any $y \in l_2$ then $Z_2(F)$ and $Z_2(G)$ are equivalent.

Proof Follows immediately from the previous theorem and the definition of equivalent twisted sums.

■

4.7 Closing Remarks

Certain results presented here hold in a more general context. All the results leading to the Theorem 4.18 can be extended if we replace l_2 by a space X having a 1-unconditional symmetric basis and making some assumption on the twisted sum of X with itself. Questions regarding this problem have been studied in [CK]. For the purpose of this thesis we considered only real quasi-Banach spaces, but similar constructions work for the complex case as well.

Twisted sums proved to be a useful tool in approaching several problems in the Banach space theory. As mentioned before, they give a general way to construct spaces which fail the three space property, i.e. spaces Z which for some $0 < p < \infty$ contain a subspace X such that both X and Z/X are isomorphic to l_p while Z itself is not isomorphic to l_p. Also using twisted sums **Kalton** produced an explicit example of a Banach space that is not isomorphic to its complex conjugate.

The main conjecture related to the material presented here is that of **Kalton** and **Casazza** who conjectured that if a twisted sum of a Hilbert space with itself has local unconditional structure, then it must be trivial, i.e. isomorphic to a Hilbert space.

Bibliography

[BL] Y. Benyamini and J. Lindenstrauss, *Geometric Nonlinear Functional Analysis*, American Mathematical Society, **2000**.

[CK] P.G. Casazza and N.J. Kalton, *Unconditional bases and unconditional finite-dimensional decompositions in Banach spaces*, Israel Journal of Mathematics **95** (1996), 349–373.

[KP] N.J. Kalton and N.T. Peck, *Twisted sums of sequence spaces and the three space problem*, Transactions of the AMS **225** (1979), 1–30.

[JLS] W.B. Johnson, J. Lindenstrauss and G. Schechtman, *On the relation between several notions of unconditional structure*, Israel Journal of Mathematics **37** (1980), 120–129.

[LT] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces I*, Springer-Verlag, **1977**.

[HHZ] P. Habala, P. Hájek and V. Zizler, *Introduction to Banach Spaces*, MATFYS Press, Charles University, Prague, **1997**.