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NONLOCAL PARTIAL DIFFERENTIAL EQUATIONS

by

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ABSTRACT

In this thesis linear and nonlinear partial differential equations with nonlocal terms represented by the integrals from unknown functions are studied. Since the presence of nonlocal terms makes the use of important classical methods impossible, new approaches are developed.

Nonlocal thermistor models were introduced in [ABHLR]. In order to study these models the related parametric linear problems are considered in this thesis and the existence of the solutions is established. The range of values of the parameter for which solutions are positive is found. The given estimates are independent of the right hand sides of equations. The above results are applied to the nonlinear nonlocal thermistor problem in order to obtain the existence of positive solutions.

A different nonlocal thermistor problem is considered and the conjecture of W. Allegretto and H. Xie [AX1] about the blow-up of solutions is answered.

Semilinear nonlocal elliptic equations are also studied. The existence results are established for sublinear equations using Leray-Schauder Degree methods and for the superlinear case by means of perturbation theory. The uniqueness for the sublinear case is obtained using the upper-lower solutions procedure. The above results are applied to problems of mathematical biology.

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CHAPTER I

INTRODUCTION

1.1. Nonlocal problems

Differential equations sharing a common feature - the presence of nonlocal terms - have recently appeared in various mathematical models formulated from physical phenomena (superconductivity, plasma reaction, thermal processes). In many cases the presence of nonlocal terms arises from a more adequate description of physical processes. On the other hand, nonlocal problems are often interesting from a purely mathematical point of view, since certain important classical arguments may not apply. In this thesis we concentrate on equations where a nonlocal term is represented by an integral of the unknown function.

It has already been emphasized in several articles that the presence of nonlocal terms changes significantly the behaviour of solutions as compared to the local case. As an example, let us mention the work of Chipot and Rodrigues, [CHR], where elliptic problems with coefficients depending on nonlocal terms were studied. In particular they showed that the presence of integral terms in the Dirichlet problem for even quite simple equations may imply the existence of an uncountable number of solutions. Other examples are furnished in articles where the dynamical behaviour of the solutions of reaction-diffusion equations with nonlocal terms was studied, e.g. [FEI], [CPY], [FR1], [FR2],and where it was shown that the asymptotic behaviour can be more interesting than in the corresponding local case. It is well known that if one considers homogeneous Neumann boundary conditions the only stable solutions in the local case are constants. In the nonlocal case, however, stable patterns may bifurcate from a stable constant solution.

In this thesis we consider linear and nonlinear nonlocal problems which arise from microsensors applications and mathematical biology. The nonlocal terms in these models correspond to the simulations of physical and biological phenomena which are essential for the given processes. It will be shown later that the presence of nonlocal terms influences the mathematical behaviour of these problems: it can spoil the positivity of operators and provoke the blow-up of solutions.

The reader who is interested in nonlocal problems can find more examples and applications in the following works: [AXY], [BCR], [BRO], [CAT], [CAP], [CHA], [CHR], [FIP], [FR1], [FR2], [FRG], [FUG], [HY1], [LIA], [LAL], [LZH], [RST], [ZIQ].

1.2. Description of thermistor models

One class of nonlocal equations discussed in the thesis arises in the thermistor microsensor model. Thermistors are semi-conductor type devices in which the thermal and electrical processes are closely related. In particular the current going through the thermistor leads to internal heating of the device (Joule heating) and the change of temperature due to this heating causes a change in electrical resistance. These processes are strongly affected by the external conditions such as the form of the device, the temperature and type of surrounding gas etc.

In the classical model for thermistors a local nonlinear system of partial differential equations governs the distribution of potential φ and temperature u in the device, which after scaling all mathematically irrelevant constants to unity, takes the form

$$-\vec{\nabla} \cdot (\sigma(u)\vec{\nabla}\varphi) = 0 \tag{1.2.1}$$

$$u_t - \vec{\nabla} \cdot (k(u)\vec{\nabla}u) = \sigma(u)|\vec{\nabla}\varphi|^2. \tag{1.2.2}$$

Here $\sigma(u)$ and $k(u)$ represent electrical and thermal conductivities respectively. Equation (1.2.1) is the charge conservation law and equation (1.2.2) describes heat flow with the Joule heating term given by $\sigma(u)|\vec{\nabla}\varphi|^2$. Note that since electrical processes are much faster than the thermal ones it is common practice to neglect the term φ_t. Thus (1.2.1) is an elliptic equation while (1.2.2) is parabolic. Usually the electrical and thermal conductivities are nonlinear functions of the temperature and one of the most interesting and realistic cases is when they degenerate as the temperature tends to infinity. This system is studied under suitable boundary and initial conditions. There is a great deal of literature devoted to the investigation of the steady state and time-dependent solutions of nonlinear system (1.2.1)-(1.2.2) (see for example [CIM], [CIP], [XAL], [YI2], [ACH], [AX2] and references therein).

In particular, Allegretto and Xie study in [AX1] the steady state solutions of system (1.2.1), (1.2.2) with mixed boundary conditions and the given current source I which can be expressed mathematically as a nonlocal boundary condition

$$I = \int_{S_1} \frac{\partial \varphi}{\partial n} ds$$

where S_1 is the part of the boundary. It was shown in [AX1] that for small values of the current there exists a positive steady state solution of (1.2.1), (1.2.2). Moreover if k and σ are degenerate then the system does not have steady state solutions for large values of I. It was conjectured that in this case the time dependent solutions of (1.2.1), (1.2.2) will blow-up. We address this question in Chapter IV of this thesis and show that if $\int^{\infty} \sigma(s)ds < \infty$ then the temperature will blow-up in a finite time and if $\int^{\infty} k(s)ds < \infty$ then the temperature will be unbounded for large values of the current. Another blow-up result was obtained by Antontsev and Chipot [ACH] for the Dirichlet problem (1.2.1), (1.2.2) under the condition that $\sigma(s) \to \infty$ as $s \to \infty$ and under certain restrictions on boundary and initial data. Note that in our case the blow-up occurs for all initial values of temperature. For more results on blow-up of solutions of nonlinear parabolic equations see [CPY], [HY2], [YI1].

Another part of Chapter IV is devoted to the study of a different nonlocal thermistor model which was suggested in [ABHLR] in order to study thermal conductivity gas pressure sensors. It is supposed in this model that the device is created by etching the silicon substrate and consequently there is a trench with gas under the microsensor. The loss of heat from the thermistor occurs to the substrate through the support arms and through the gas to the trench sides and bottom as well as top. Since the device is very thin compared to the depth of the trench, it is possible to modify the three-dimensional problem (1.2.1), (1.2.2) into a two-dimensional one without neglecting the loss of heat through the gas and other important effects. The time independent version of the thermal equation (1.2.2) was applied to a simple model where the thermistor was represented as a thin rectangular plate suspended over a deep trench filled with gas. Writing equation (1.2.2) in the weak form and supposing that $u \cong u(x, y)$ due to the thinness of the plate, it was found that in the region of the plate the temperature satisfies

$$-\vec{\nabla} \cdot (k(u)\vec{\nabla} u) + \frac{k(u)}{\tau}\left(\frac{-\partial u}{\partial n}\right)_{\text{top+bottom}} = \sigma(u)|\vec{\nabla}\varphi|^2$$

where $\vec{\nabla}$ operates in $x - y$ plane. Here τ is the thickness of the plate and the parameters σ and k are integral averages over this thickness. The small heat loss from the sides of the plate is neglected. The term $\frac{k(u)}{\tau}\left(\frac{-\partial u}{\partial n}\right)$ accounts for the heat loss from the surface of the plate through the gas and we apply the following argument to calculate it. We solve the equation

$$-\vec{\nabla} \cdot (k_{\text{gas}}\vec{\nabla} u) = 0$$

in a box of cross-section equal to the size of the plate and the depth equal to the depth of the trench. We assume that there is no heat flow through the side walls of the box and the temperature of the bottom is zero. This problem can be solved using separation of variables. Therefore taking the first term of the Fourier series for the solution and neglecting the heat loss from the top of the device we obtain that

$$\frac{k(u)}{\tau}\left(\frac{-\partial u}{\partial n}\right) = \eta \int_\Omega u dx dy$$

where Ω is the region occupied by the plate. Now equation (1.2.2) becomes

$$-\vec{\nabla} \cdot (k(u)\vec{\nabla} u) + \eta \int_\Omega u = \sigma(u)|\vec{\nabla}\varphi|^2 \quad \text{in} \quad \Omega. \tag{1.2.3}$$

Here the parameter η is positive and involves the geometry of the device, the gas pressure as well as other factors. Observe that only positive solutions are physically meaningful. For a more detailed derivation of this model and numerical results we refer the interested reader to [ABHLR].

We are thus led to seek positive solutions of system (1.2.1), (1.2.3) under the Dirichlet boundary conditions:

$$\varphi = \varphi_0, \qquad u = 0 \quad \text{on} \quad \partial\Omega.$$

To show the existence of solutions we apply methods of [XAL]: we first obtain a-priori bounds using Campanato spaces and then apply the Schauder Fixed Point theorem. A brief review of Campanato spaces can be found in Chapter II.

The main difficulty is to show that there are positive solutions. As we will see in the third chapter even a simpler operator L^{-1} ,where $Lu = -\Delta u + \eta \int_\Omega u, \quad u \in H_0^1(\Omega)$, does not leave the cone of positive functions invariant for large values of the parameter η, thus the positivity of the right hand side of $Lu = f$ does not imply the positivity of solutions.

1.3. Linear nonlocal problem

In Chapter III we establish several results that later find particular applications to the thermistor problems discussed above. Specifically we study the linear problem:

$$Au = -\nabla(a(x)\nabla u(x)) + \eta\psi(x)\int_\Omega \varphi u = f, \qquad u \in H_0^1(\Omega)$$

where $a(x)$, ψ, φ are sufficiently smooth functions and η is a parameter.

This problem has not receive a lot of attention. Catchpole in [CAT] studied the corresponding one-dimensional initial value problem and in particular he was interested in existence of solutions and the dimension of solution space. We briefly address the same questions for boundary value problem in n-dimensional space at the beginning of Chapter III and formulate existence and uniqueness/nonuniqueness criteria.

Another aspect of this problem which was studied earlier is the eigenvalue problem and its applications to related nonlinear equations, [FEI], [FR1]. In both works the study of spectrum location was related to the linearization of a more complicated nonlocal problem and motivated by the investigation of stability of the stationary solutions. In particular Freitas in [FR1] gives a very detailed picture of the behaviour of eigencurves when η varies. The results are obtained for the one-dimensional case but can be immediately generalized to arbitrary dimension. Let us emphasize that if $\eta > 0$ and φ and ψ are nonnegative then the nonlocal term has a stabilizing effect.

In this thesis we are interested in somewhat different question. Our main result is to show that there exist two constants $0 < \overline{\eta} = \overline{\eta}(a(\cdot))$ and $0 > \underline{\eta} = \underline{\eta}(a(\cdot))$ such that if $\underline{\eta} < \eta < \overline{\eta}$ then A^{-1} is positive, i.e. the problem

$$Au = f \tag{1.3.1}$$

has a nonnegative solution for $f \geq 0$. Since the usual methods do not work due to the presence of the nonlocal term we take the following approach. We apply first Harnack's inequality to show that for small η the solutions of (1.3.1) are positive in a small ball inside the domain Ω and then using the comparison results we establish the positivity of these solutions in the rest of the domain. We emphasize that our proof implies that $\overline{\eta}, \underline{\eta}$ are independent of the specific f. We get the even stronger result that there exists $0 < \eta_1 \leq \overline{\eta}$ which depends on φ, ψ, Ω and on the ellipticity bounds on $a(x)$, but not on $a(x)$ itself, for which the same positivity result holds for $0 \leq \eta < \eta_1$. As an important consequence of the existence of η_1 we obtain the existence of a positive solution of nonlinear system (1.2.1), (1.2.3).

It is significant for our applications to estimate η_0, the maximum possible value of $\overline{\eta}$, and we obtain an exact formula for the value of η_0 in terms of the Green's function for the problem

$$-\nabla(a(x)\nabla u(x)) = f(x) \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{on} \quad \partial\Omega.$$

While it is not evident how to find the Green's function explicitly for a general domain Ω, we were able to obtain the exact values of η_0 in the 1-dimensional case and for the case when Ω is a ball in $\mathbb{R}^n$, $n \geq 2$. We also indicate how to obtain bounds on η_0 for sufficiently smooth domains in $\mathbb{R}^2$ and some other domains in $\mathbb{R}^n$.

The existence of η_0 and the associated monotonicity properties for $\eta \leq \eta_0$ also have certain implications on the spectrum location, and we address this question in the end of Chapter III. As was mentioned above, the properties of the eigencurves of the eigenvalue problem

$$Au = \lambda u, \qquad u \in H_1^0(\Omega) \tag{1.3.2}$$

were studied in [FR1], [FEI]. Here we concentrate on properties of the eigenfunctions and in particular on the first one. We prove using Krasnosel'skii's cone theory that there exists a positive eigenfunction for (1.3.2) for $\eta < \eta_0$ and that this eigenfunction corresponds to the first eigenvalue. Furthermore we formulate analogues of the classical Barta's inequality.

1.4. Biology models and related questions

The second part of the thesis is devoted to problems arising from mathematical biology modelling. In the literature there are various interpretations of nonlocal terms from the point of view of populational dynamics. For example Feidler and Polácik in [FEI] considered the one-dimensional equation

$$u_t = u_{xx} + f(x,u) + c(x)\int_0^1 \nu(x)u(x)dx, \quad x \in (0,1)$$

and suggested that in this equation u can be viewed as population density, $f(x,u)$ a birth/death rate and the integral represents coupling by which the total population affects each individual. Another example is found in the work of Calsina and Perelló, [CAP], where the equation

$$u_t = u_{xx} + \left(x - \int_0^1 u(x)dx\right)u, \quad u \in (0,1)$$

was considered. In that paper $u(x,t)$ represents the population density at time t of the species of characteristic x which corresponds to a growth rate. Therefore here the nonlocal term appears as a competition depending on the total population.

In this thesis we adopt the interpretation suggested in the work of Furter and Grinfeld, [FUG], in order to model single-species populational dynamics with dispersal. The general approach to modelling a single species in a domain $\Omega \subset \mathbb{R}^n$, $n \geq 3$, is to suppose that the density of the population $u(x,t)$ satisfies the equation:

$$u_t = \Delta u + u \cdot \chi(x,u), \quad t > 0, \quad x \in \Omega \tag{1.4.1}$$

subject to suitable boundary and initial conditions. The function χ represents the "crowding" effect, [OKU], [FUG], and is normally supposed to be smooth and satisfy the condition that there exists $t_0 > 0$ such that $\chi(x,t_0) = 0$ for $x \in \Omega$. It was suggested in [FUG] that the "crowding" effect could depend not only on the density of the population but also on nonlocal interactions. Mathematically this situation corresponds to the case when we consider $\chi = \chi(x,u,\phi(u))$ where ϕ is a continuous functional of u.

That approach leads us to the study of semilinear differential equations of the following type:

$$u_t = \Delta u + u \cdot \chi(x,u,\phi(u)), \quad t > 0, \quad x \in \Omega \tag{1.4.2}$$

In [FUG] the model given by (1.4.2) was studied with homogeneous Neumann boundary conditions. In particular, the authors studied the steady state solutions of (1.4.2), the bifurcation from constant non-zero solutions and the behaviour of branches of positive solutions. In this thesis we consider explicitly equations of the type (1.4.2) subject to Dirichlet boundary conditions, i.e. there is a population reservoir at the boundary, in the case when $u \cdot \chi(x, u, \phi(u))$ has sublinear or superlinear growth with respect to u. The change in the boundary conditions from Neumann to Dirichlet creates differences in the solution behaviour: no longer are there positive constant solutions to the stationary problem, and the proof of the existence of a positive stationary solution occupies most of Chapter V. We also show the uniqueness of solution for the sublinear case. Stability criteria are then obtained for some cases of equation (1.4.2) and it is shown that the presence of a nonlocal term has a stabilizing effect.

Note that (1.4.2) without the nonlocal term is well studied (see e.g. [LIO] and references therein), but to the best of our knowledge the problem we consider here is new. Let us point out that many of the conventional methods of study of super- and sublinear problems fail due to the presence of nonlocal terms. Still we are able to show the existence for the sublinear case using Leray-Schauder Degree methods and for superlinear case by the perturbation theory.

1.5. Thesis outline

The thesis is structured as follows. In Chapter II we discuss briefly the general background which will be used later on in the thesis, we present chosen results from elliptic theory, degree theory, cone theory and on functional spaces. In Chapter III we address a linear nonlocal problem and study the positivity of solutions and related questions. Chapter IV is devoted to the study of nonlinear nonlocal thermistor models. We show the existence of positive solutions for one model and blow-up for another one. Finally in Chapter V we consider super- and sublinear nonlocal problems and discuss their applications to the mathematical biology. We complete the thesis with a discussion where we formulate some open questions.

CHAPTER II

BACKGROUND

In this chapter we collect and discuss briefly for the reader's convenience some results which will be used later in the thesis.

2.1. Function spaces

In this section we describe certain classes of function spaces which are useful in the study of elliptic problems. Throughout the whole section Ω is a bounded domain in $\mathbb{R}^n$, $n > 1$.

An n-tuple $\alpha = (\alpha_1, ..., \alpha_n)$ of nonnegative integers α_j is called a *multi-index* with $|\alpha| = \sum_{j=1}^{n} \alpha_j$. We denote by D^α the differential operator of order $|\alpha|$

$$D^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}.$$

We also define the *support* of a function u defined on the domain Ω as

$$\operatorname{supp} u = \overline{\{x \in \Omega : \quad u(x) \neq 0\}}.$$

Let $C^k(\Omega)$, $k \geq 0$ denote the set of all the functions u that are continuous in Ω together with all their partial derivatives D^α of orders $|\alpha| \leq k$. We put $C^\infty(\Omega) = \cap_{k=0}^{\infty} C^k(\Omega)$. The subspaces $C_0^1(\Omega)$ and $C_0^\infty(\Omega)$ of $C^1(\Omega)$ and $C^\infty(\Omega)$ respectively consist of all the functions from these spaces which have compact support in Ω.

Let $C^k(\bar{\Omega})$, $k \geq 0$ be a subspace of functions $u \in C^k(\Omega)$ for which all derivatives $D^\alpha u$ are bounded on $\bar{\Omega}$ for $0 \leq |\alpha| \leq k$. These spaces are Banach spaces under the norm

$$||u||_{C^k} = \max_{0\leq|\alpha|\leq k} \sup_{x\in\Omega} |D^\alpha u|.$$

Now we define the space $C^{k,\gamma}(\bar{\Omega})$, $k \geq 0$, $0 \leq \gamma \leq 1$ as the set of functions such that

$$u \in C^k(\bar{\Omega}) \quad \text{and} \quad \sup_{x,y\in\bar{\Omega}} \frac{|D^\alpha u(x) - D^\alpha u(y)|}{|x-y|^\gamma} < \infty \quad 0 \leq |\alpha| \leq k.$$

$C^{k,\gamma}(\bar{\Omega})$ is a Banach space under the norm

$$||u||_{C^{k,\gamma}} = ||u||_{C^k} + \max_{0\leq|\alpha|\leq k} \sup_{\substack{x,y\in\Omega \\ x\neq y}} \frac{|D^\alpha u(x) - D^\alpha u(y)|}{|x-y|^\gamma}.$$

It is easy to see that $C^{k,0}(\bar{\Omega}) = C^k(\bar{\Omega})$. Note that functions from $C^{0,1}$ are called Lipschitz continuous functions. The important property of these spaces is that for $0 \leq k \leq \infty$ and $0 \leq \gamma < \nu \leq 1$ the space $C^{k,\nu}(\bar{\Omega})$ is compactly embedded into $C^{k,\gamma}(\bar{\Omega})$. For the proof see [ADA].

Now we will give the definition and discuss properties of Sobolev Spaces.

Let $L^p(\Omega)$, $1 \leq p < \infty$, denote the set of all measurable functions which are p-integrable. This space is complete under the norm

$$||u||_{L^p} = \left(\int_\Omega |u|^p dx \right)^{1/p}.$$

Let also $L^\infty(\Omega)$ denote the set of essentially bounded functions on Ω with the norm

$$||u||_{L^\infty} = \operatorname*{ess\,sup}_{x\in\Omega} |u|.$$

We define the weak derivative for a locally integrable function u on Ω in the following way. For any multi-index α a locally integrable function v is called the αth derivative of u and is denoted as $D^\alpha u$ if

$$\int_\Omega \varphi v dx = (-1)^{|\alpha|} \int_\Omega (D^\alpha \varphi) u dx \quad \text{for all} \quad \varphi \in C_0^\infty(\Omega).$$

Finally we say that a function u belongs to the Sobolev space $W^{k,p}(\Omega)$ with integer $k \geq 0$ if $D^\alpha u \in L^p(\Omega)$ for all $|\alpha| \leq k$. $W^{k,p}(\Omega)$ is a Banach space under the following norm

$$||u||_{W^{k,p}} = \sum_{|\alpha| \leq k} ||D^\alpha u||_{L^p}.$$

We denote $W_0^{k,p}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{k,p}(\Omega)$ and note that it is also a Banach space under this norm. It is customary to denote $W^{1,2}(\Omega)$ and $W_0^{1,2}(\Omega)$ by $H^1(\Omega)$ and $H_0^1(\Omega)$ respectively. The important result is that $H_0^1(\Omega)$ can be supplied with the norm

$$||u||_{H_0^1} = \left(\int_\Omega |\nabla u|^2 dx \right)^{1/2}$$

which is equivalent to its usual norm.

Next we formulate the Sobolev Embedding Theorem.

Theorem 2.2.1. Let Ω be a bounded Lipschitzian domain in $\mathbb{R}^n$. Then we have

(1) $W^{k,p}(\Omega) \hookrightarrow L^{nk/(n-kp)}(\Omega)$ for $kp < n$;

(2) $W^{k,p}(\Omega) \hookrightarrow L^r(\Omega)$ for all $p \leq r < \infty$;

(3) $W^{k,p}(\Omega) \hookrightarrow C^{m,\lambda}(\bar{\Omega})$ where $0 \leq m + \lambda < k - \frac{n}{p}$.

The proof of this theorem can be found in [ADA].

Finally we describe Campanato Spaces which provide a powerful tool for studying elliptic problems. For more detailed exposition see [TRO].

Let Ω be a C^1 domain in $\mathbb{R}^n$, $n > 1$. For $\mu \geq 0$ define

$$L^{2,\mu}(\Omega) = \{u \in L^2(\Omega) | \sup_{\substack{x_0 \in \Omega \\ 0<\rho<\infty}} \rho^{-\mu} \int_{\Omega[x_0,\rho]} |u(x) - u_{x_0,\rho}|^2 dx < \infty\}$$

where $\Omega[x_0, \rho] = \{x \in \Omega | |x - x_0| < \rho\}$ and

$$u_{x_0,\rho} = \frac{1}{mes(\Omega[x_0, \rho])} \int_{\Omega[x_0,\rho]} u(y)dy.$$

$L^{2,\mu}$ is called a Campanato Space and is supplied with the norm

$$||u||_{L^{2,\mu}} = ||u||_{L^2} + [u]_{2,\mu}$$

where

$$[u]_{2,\mu} = \{ \sup_{\substack{x_0 \in \Omega \\ 0<\rho<\infty}} \rho^{-\mu} \int_{\Omega[x_0,\rho]} |u(x) - u_{x_0,\rho}|^2 dx\}^{1/2}.$$

The Campanato space for $0 \leq \mu \leq n + 2$ is complete under the above norm.

We will need the following properties of $L^{2,\mu}$ spaces.

Theorem 2.1.2. (i) If $0 \leq \mu \leq n$ then $L^\infty(\Omega) \subset L^{2,\mu}(\Omega)$ and $L^\infty(\Omega)$ is a space of multipliers for $L^{2,\mu}(\Omega)$.

(ii) If $n < \mu \leq n + 2$ then $L^{2,\mu}(\Omega)$ is isomorphic to $C^{0,\gamma}(\bar{\Omega})$ with $\gamma = (\mu - n)/2$.

Theorem 2.1.3. Let $u \in H^1(\Omega)$ and $\nabla u \in L^{2,\mu}(\Omega)$ with $0 \leq \mu < n$. Then $u \in L^{2,\mu+2}(\Omega)$ with

$$||u||_{L^{2,\mu+2}} \leq C(||u||_{L^2} + ||\nabla u||_{L^{2,\mu}})$$

where C is independent of u.

Theorem 2.1.4. Let $u \in H_0^1(\Omega)$ satisfy

$$\int_\Omega \sum_{i,j=1}^n a^{ij} D_i u D_j v = \int_\Omega f^0 v + \sum_{i=1}^n f^i D_i v \quad \forall v \in H_0^1(\Omega),$$

with $\lambda||\xi||^2 \leq a^{ij}\xi_i\xi_j \leq \Lambda||\xi||^2$, $\Lambda \geq \lambda > 0$, $\xi \in \mathbb{R}^n \setminus \{0\}$, $f^0 \in L^{2,(\mu-2)^+}$ and $f^i \in L^{2,\mu}(\Omega)$, $i = 1, ..., n$. Then $\nabla u \in L^{2,\mu}(\Omega)$ and, moreover, we have

$$||\nabla u||_{L^{2,\mu}} \leq C\Big(||f^0||_{L^{2,(\mu-2)+}} + \sum_{i=1}^n ||f^i||_{L^{2,\mu}} + ||u||_{H^1}\Big),$$

where $0 < \mu < \mu_0 = n - 2 + 2\delta_0$, $0 < \delta_0 < 1$, and δ_0, C only depend on λ, Λ, and Ω.

All proofs can be found in [TRO].

2.2. Elliptic theory: solvability and maximum principles

The detailed presentation and proofs of the results of this paragraph can be found in [GIT].

Consider a linear differential operator of the form

$$Lu = -\sum_{i,j=1}^n a^{ij}(x) D_{ij} u + \sum_{i=1}^n b_i(x) D_i u + c(x) u, \quad a^{ij} = a^{ji} \tag{2.2.1}$$

in a bounded domain $\Omega \in \mathbb{R}^n, n \geq 2$, $i, j = 1, ..., n$. First we consider conditions for the existence of classical solutions of (2.2.1), i.e. u should at least belong to $C^2(\Omega)$, as well as several of their properties.

Definition 2.2.1. We call L *elliptic* in Ω if for all $\xi = (\xi_1, ..., \xi_n) \in \mathbb{R}^n \setminus \{0\}$ and $x \in \Omega$ it follows that

$$0 < \lambda(x)|\xi|^2 \leq a^{ij}(x)\xi_i\xi_j \leq \Lambda(x)|\xi|^2.$$

If $\lambda(x) \geq \lambda_0 > 0$ for some constant λ_0 then L is called *strictly elliptic*, and if moreover Λ/λ is bounded in Ω then L is *uniformly elliptic*.

Definition 2.2.2. The domain Ω is said to satisfy an interior (resp. exterior) sphere condition at $x_0 \in \partial\Omega$, if there exists a ball $B \subset \Omega$ (resp. $\mathbb{R}^n \setminus \Omega$) with $x_0 \in \partial B$.

The first theorem we state gives a solvability result for equation (2.2.1).

Theorem 2.2.3. Let L be uniformly elliptic in a bounded domain Ω with $c(x) \geq 0$ and let f and the coefficients of L be bounded and belong to $C^{0,\alpha}(\bar{\Omega})$. Suppose that Ω satisfies an exterior sphere condition at every boundary point. Then, if φ is continuous on $\partial\Omega$, the Dirichlet problem

$$Lu = f \quad \text{in} \quad \Omega, \qquad u = \varphi \quad \text{on} \quad \partial\Omega$$

has a unique solution $u \in C^0(\bar{\Omega}) \cap C^{2,\alpha}(\Omega)$.

The next theorem gives important information about the behaviour of the solution of the Dirichlet problem near the boundary $\partial\Omega$.

Theorem 2.2.4 (*Strong Maximum Principle*). Suppose that L is uniformly elliptic, $b_i(x)$ are bounded, $c(x) \equiv 0$ and $Lu \geq 0$ in Ω. Let $x_0 \in \partial\Omega$ be such that

(i) u is continuous at x_0;

(ii) $u(x_0) < u(x)$ for all $x \in \Omega$;

(iii) Ω satisfies an interior sphere condition at x_0.

Then the outer normal derivative of u at x_0, if it exists, satisfies the strict inequality

$$\frac{\partial u}{\partial n}(x_0) < 0.$$

If $c(x) \geq 0$ and c/λ is bounded in Ω, the same conclusion holds provided $u(x_0) \geq 0$, and if $u(x_0) = 0$ the same conclusion holds irrespective of the sign of $c(x)$.

Next we consider a linear differential operator in divergence form

$$Lu = -\sum_{i,j=1}^{n} D_i(a^{ij}(x)D_j u) + c(x)u \tag{2.2.2}$$

in a smooth bounded domain $\Omega \subset \mathbb{R}^n, n \geq 2$, $i, j = 1, ..., n$. We do not require that the coefficients a^{ij}, c be smooth and assume only that they are measurable functions on Ω. Note, that if a^{ij} are smooth then (2.2.2) is equivalent to (2.2.1). We look for solutions of the equation $Lu = 0$ in the class of generalized functions.

Definition 2.2.5. Let u be weakly differentiable and let the functions $a^{ij}(x)D_j u$ and $c(x)u$, $i = 1, ...n$ be locally integrable. Then u is said to satisfy in *weak* or *generalized* sense $Lu = 0(\leq 0, \geq 0$ respectively) in Ω if

$$L(u,v) = \int_\Omega \sum_{i,j=1}^{n} a^{ij}(x)D_j u D_i v + c(x)uv dx = 0(\leq 0, \geq 0) \tag{2.2.3}$$

for any non-negative functions $v \in C_0^1(\Omega)$. In particular (2.2.3) should hold for any $v \in H_0^1(\Omega)$. The function u is called *a solution, a subsolution* or *a supersolution* respectively.

Definition 2.2.6. Let f be locally integrable function in Ω. Then a weakly differentiable function u is called a *weak* or *generalized* solution of the inhomogeneous equation

$$Lu = f \tag{2.2.4}$$

in Ω if

$$L(u,v) = \int_\Omega f v dx, \quad \forall v \in C_0^1(\Omega). \tag{2.2.5}$$

We assume for the rest of Chapter II that L is strictly elliptic and has bounded coefficients, that is for some constants Λ and $\nu \geq 0$ we have for all $x \in \Omega$

$$\sum |a^{ij}(x)|^2 \leq \Lambda^2, \quad \lambda^{-1}|c(x)| \leq \nu^2.$$

Definition 2.2.7. A function $u \in H^1(\Omega)$ is called a solution of a generalized Dirichlet problem

$$Lu = f \quad \text{in } \Omega, \quad u = \varphi \quad \text{on } \partial\Omega \tag{2.2.6}$$

if u is the generalized solution of equation (2.2.4), $\varphi \in H^1(\Omega)$ and $u - \varphi \in H_0^1(\Omega)$.

The next theorem states the maximum principle for weak solutions.

Theorem 2.2.8. (*Weak Maximum Principle*). Let $c \geq 0$. Let $u \in H_0^1(\Omega)$ satisfy $Lu \geq 0 (\leq 0)$ in Ω. Then

$$\inf_{\Omega} u \geq \inf_{\partial\Omega} u^- \qquad (\sup_{\Omega} u \leq \sup_{\partial\Omega} u^+)$$

where $u^+ = \max\{u, 0\}$ and $u^- = \min\{u, 0\}$.

Theorem 2.2.9. (*Solvability*). Let $c(x) \geq 0$. Then for $\varphi \in H^1(\Omega)$ and $f \in L^2(\Omega)$ the generalized Dirichlet problem (2.2.6) is uniquely solvable. Moreover there exists a positive constant C depending only on L and Ω such that

$$||u||_{H^1} \leq C(||f||_{L^2} + ||\varphi||_{H^1}). \tag{2.2.7}$$

2.3. Regularity of weak solutions

In this section we give a-priori estimates and regularity theorems for weak solutions of problem (2.2.6). All proofs can be found in [GIT]. We start by stating results which suffice for the global boundedness of weak solutions.

Theorem 2.3.1. Suppose that $f \in L^{q/2}(\Omega)$ for some $q > n$. Then if u is a $H_0^1(\Omega)$ subsolution (supersolution) of (2.2.4) satisfying $u \le 0 \quad (\ge 0)$ on $\partial\Omega$ we have

$$\sup_{\Omega} u(-u) \le ||u^+(u^-)||_{L^2} + Ck$$

where $k = \lambda^{-1}||f||_{L^{q/2}}$ and $C = C(n, \nu, q, |\Omega|)$.

Theorem 2.3.2. Let $c \ge 0$ and suppose that $f \in L^{q/2}(\Omega)$ for some $q > n$. Then if u is a $H_0^1(\Omega)$ subsolution (supersolution) of (2.2.4) we have

$$\sup_{\Omega} u(-u) \le \sup_{\partial\Omega} u^+(u^-) + Ck$$

where $k = \lambda^{-1}||f||_{L^{q/2}}$ and $C = C(n, \nu, q, |\Omega|)$.

The next theorem gives us a local boundedness result.

Theorem 2.3.3 (*weak Harnack inequality*). Suppose that $f \in L^{q/2}(\Omega)$ for some $q > n$. Then if u is a $H_0^1(\Omega)$ supersolution of equation (2.2.4) in Ω, non-negative in a ball $B_{4R}(y) \subset \Omega$ and $1 \le p < n/(n-2)$, we have

$$R^{-n/p}||u||_{L^p(B_{2R}(y))} \le C(\inf_{B_R(y)} u + k(R))$$

where $k(R) = \lambda^{-1}R^{2\delta}||f||_{L^{q/2}}$, $\delta = 1 - n/q$ and $C = C(n, \Lambda/\lambda, \nu R, q, p)$.

Let us assume for the next two results that the coefficients $a^{ij} \in C^{0,\alpha}(\bar{\Omega})$

and $c, f \in L^\infty(\Omega)$. Suppose also that

$$\max_{i,j=1,\dots,n}\{||a^{ij}||_{C^{0,\alpha}}, ||c||_{L^\infty}\} \le K.$$

Then Theorems 2.3.4 and 2.3.5 give Hölder estimates for the first derivatives of weak solutions.

Theorem 2.3.4. Let $u \in C^{1,\alpha}(\bar{\Omega})$ be a weak solution of (2.2.4) in a $C^{1,\alpha}$ domain Ω, satisfying $u = \varphi$ on $\partial\Omega$, where $\varphi \in C^{1,\alpha}(\bar{\Omega})$. Then we have

$$||u||_{C^{1,\alpha}} \le C(||u||_{L^\infty} + ||\varphi||_{C^{1,\alpha}} + ||f||_{L^\infty})$$

for $C = C(n, \lambda, K, \partial\Omega)$, where λ and K are as above.

Theorem 2.3.5. Let the hypotheses of Theorem 2.3.3 hold and assume that T is a (possibly empty) $C^{1,\alpha}$ boundary portion of a domain Ω. Suppose $u \in H_0^1(\Omega)$ is a weak solution of (2.2.3) such that $u = 0$ on T (in the sense of $H_0^1(\Omega)$). Then $u \in C^{1,\alpha}(\Omega \cup T)$, and for any $\Omega' \subset\subset \Omega \cup T$ we have

$$||u||_{C^{1,\alpha}(\Omega')} \le C(||u||_{L^\infty(\Omega)} + ||f||_{L^\infty(\Omega)})$$

with $C = C(n, \lambda, K, d', T)$ where λ and K are as above and $d' = dist(\Omega', \partial\Omega - T)$.

Finally we state an existence and uniqueness theorem for continuous boundary values and a theorem concerning higher-order regularity. In these theorems L is given by (2.2.1).

Theorem 2.3.6. Let Ω be a $C^{1,1}$ domain in $\mathbb{R}^n$ and $a^{ij} \in C^0(\bar{\Omega})$, $c \in L^\infty(\Omega)$, $c \ge 0$. Then if $f \in L^p(\Omega)$, $p > n/2$, $\varphi \in C^0(\partial\Omega)$, the Dirichlet problem $Lu = f$ in Ω, $u = \varphi$ on $\partial\Omega$ has a unique solution $u \in H_{loc}^{2,p}(\Omega) \cap C^0(\bar{\Omega})$.

Theorem 2.3.7. Let u be a $H^{2,p}_{loc}(\Omega)$ solution of the equation $Lu = f$ in a domain Ω where coefficients of L and f belong to $C^{0,\alpha}(\bar{\Omega})$ with $0 < \alpha < 1$. Then $u \in C^{2,\alpha}(\bar{\Omega})$.

2.4. Eigenvalue problem

In this section we consider the eigenvalue problem

$$Au = \lambda u \tag{2.4.1}$$

where A is an operator acting from a Banach space into itself. We call the operator A *compact* if it is continuous and transformes every bounded set into a precompact set. We start by formulating the Courant Min-Max Principle:

Theorem 2.4.1. Let X be a real Hilbert space, A be a linear compact self-adjoint operator and $(Ax, x) \geq 0$ on X. Let $S = \{x \in X : ||x|| = 1\}$ and $\mathcal{F}_n$ the family of all the n-dimensional subspaces of X. Then the eigenvalues $\lambda_1 \geq \lambda_2 \geq ...$ of T are given by

$$\lambda_n = \max_{F \in \mathcal{F}_n} \min_{x \in S \cap F} (Ax, x) = \min_{F \in \mathcal{F}_{n-1}} \max_{x \in S \cap F^{\perp}} (Ax, x).$$

The proof of this classical theorem can be found for example in [DEI].

The rest of this paragraph is devoted to the Krasnosel'skii theory of positive operators, i.e. operators which map the cone of nonnegative functions into itself. More precisely we need the following results concerning the existence of a positive eigenfunction of (2.4.1) and the characterization of the corresponding eigenvalue. For more general formulations and proofs see [KRA].

Theorem 2.4.2. Let a linear positive operator $A : L^2(\Omega) \to L^2(\Omega)$ be compact. Let the relation $Af \geq \alpha f$ hold for at least one nonnegative nontrivial function $f \in L^2(\Omega)$ and some positive number α. Then the operator A has at least one nonnegative eigenvector u_0.

Definition 2.4.3. The linear operator $A : C^0(\bar{\Omega}) \to C^0(\bar{\Omega})$ is called v_0-*positive* if there exist two positive numbers α, β such that for every nontrivial nonnegative function $f \in C^0(\bar{\Omega})$

$$\alpha v_0 \leq A^n f \leq \beta v_0$$

for some n. Here $v_0 \in C^0(\bar{\Omega})$ and is nonnegative.

We call the eigenvalue with the smallest absolute value *the first eigenvalue*, the eigenvalue with the second smallest absolute value *the second* one and so on. Moreover if the first and second eigenvalues are simple we call the corresponding eigenfunctions *the first* and *the second* one respectively.

Theorem 2.4.4. Let u_0 be a nonnegative eigenfunction of a u_0-positive operator A: $Au_0 = \lambda_0 u_0$. Then λ_0 is the first eigenvalue of the operator A. Moreover λ_0 is simple.

Theorem 2.4.5. Let a linear operator $A : C^0(\bar{\Omega}) \to C^0(\bar{\Omega})$ be u_0-positive where $Au_0 = \lambda_0 u_0$. Then for arbitrary nontrivial positive $x \in C^0(\bar{\Omega})$, $x \neq ku_0$, the elements $\lambda_0 x$ and Ax are incomparable, i.e. there exist $x_1,\ x_2 \in \Omega$ such that

$$(Au_0)(x_1) < \lambda_0 u_0(x_1) \quad \text{and} \quad (Au_0)(x_2) > \lambda_0 u_0(x_2).$$

Finally we state Barta's inequality for the Laplace operator which can be treated as a specific case of Theorem 2.4.5. For another proof see [PRW].

Theorem 2.4.6. (*Barta's inequality*). Let λ_0 be the first eigenvalue of the homogeneous Dirichlet problem for the $(-\Delta)$. Then

$$\lambda_0 \geq \inf_{x \in \Omega} \left[\frac{-\Delta v}{v} \right]$$

for any positive $v \in C^2(\bar{\Omega})$.

2.5. Leray-Schauder degree

In this paragraph we will formulate the main properties of the Leray-Schauder Degree. For a detailed discussion see [DEI].

Let X be a real Banach space, $G \subset X$ an open bounded domain, F a compact operator from G to X, I an identity operator and $y \notin (I - F)(\partial G)$. On these triplets $(I - F, G, y)$ an integer-valued function *deg*, which is called Leray-Schauder Degree, is defined and it satisfies the following properties:

(L1) $deg(I, G, y) = 1$ for $y \in \Omega$;

(L2) (additivity) $deg(I-F, G, y) = deg(I-F, G_1, y) + deg(I-F, G_2, y)$ whenever G_1 and G_2 are disjoint open subsets of G such that $G = G_1 \cup G_2$ and $y \notin (I - F)(\partial G \cup \partial G_1 \cup \partial G_2)$;

(L3) (homotopy) $deg(I - H(t, \cdot), G, y(t))$ is independent of $t \in [0, a]$ whenever $H : [0, a] \times \bar{G} \to X$ is compact, $y : [0, a] \to X$ is continuous and $y(t) \notin (I - H(t, \cdot))(\partial G)$ on $[0, a]$;

(L4) $deg(I - F, G, y) \neq 0$ implies $(I - F)^{-1}(y) \neq \emptyset$.

One of the important consequences of the Leray-Schauder Degree Theory is the Schauder Fixed Point Theorem.

Theorem 2.5.1 (*The Schauder Fixed Point Theorem*). Let G be a closed convex set in a Banach space X and let F be a continuous mapping of G into itself such that the image FG is precompact. Then F has a fixed point.

The proof of this theorem can be found in [GIT].

CHAPTER III

POSITIVITY OF SOLUTIONS OF NONLOCAL LINEAR PROBLEMS

3.1. Introduction

In this chapter we consider the nonlocal problem

$$-\nabla(a(x)\nabla u(x)) + \eta\psi(x)\int_\Omega \varphi u = f(x) \tag{3.1.1}$$

in a smooth bounded domain $\Omega \subset \mathbb{R}^n$, subject to regularity and boundary conditions specified below, and involving the constant η. We show in this chapter that there exist constants $0 < \overline{\eta} = \overline{\eta}(a(\cdot))$ and $0 > \underline{\eta} = \underline{\eta}(a(\cdot))$ independent of specific f, such that if $\underline{\eta} < \eta < \overline{\eta}$ then (3.1.1) has a nonnegative solution for $f \geq 0$. Moreover we show the existence of $0 < \eta_1 \leq \overline{\eta}$ which depends only on the ellipticity bounds on $a(x)$, on φ, ψ and the shape of Ω and for which the same positivity result holds for $0 \leq \eta < \eta_1$. We also address the important questions of the estimation of η_0, the maximal possible value of $\bar{\eta}$, from below, and of the eigenfuctions properties of the corresponding eigenvalue problem.

The chapter is structured as follows. In section 3.2 we formulate our assumptions on the problem and consider the existence and uniqueness/nonuniqueness of solutions for (3.1.1). In sections 3.3 and 3.4 we prove the existence of $\overline{\eta}$, $\underline{\eta}$ and η_1. Sections 3.5 and 3.6 are devoted to the finding of bounds on η_0. Finally in section 3.7 we consider the corresponding eigenvalue problem. The tools we employ are briefly as follows: to prove the existence of $\overline{\eta}, \underline{\eta}$ and η_1 we use general elliptic theory and, in particular, Harnack's inequality. In sections 3.5 and 3.6, we make use of Green's functions. To study the eigenvalue problem, we apply variational methods and results of Krasnosel'skii's cone theory.

In conclusion, we remark that at first sight it may appear to the reader that the presence of a nonlocal term as in equation (3.1.1) has little effect upon the properties of the solution. This is not the case, as the following one-dimensional problem shows. Consider

$$-u'' + \eta \int_{\Omega} u = \sin \pi x \quad \text{in} \quad (0,1), \quad u(0) = u(1) = 0, \quad \eta > 0.$$

Solving this problem explicitly we obtain

$$u(x) = 12\eta(x^2 - x)/(\pi^3(12+\eta)) + \sin(\pi x)/\pi^2.$$

and observe that for small η, the solution $u(x)$ is positive, but as η grows larger, $u(x)$ becomes negative near the end points: $x = 0$ and $x = 1$, even though the right hand side is nonnegative and fixed.

3.2. Existence and uniqueness of solutions

To minimize technical difficulties while still presenting our ideas we impose the following regularity assumptions on $a(x)$, φ, ψ:

(A1) $a(x) \in C^{1,\alpha}(\bar{\Omega})$, $0 < \alpha < 1$, $a(x) > 0$ on $\bar{\Omega}$;

(A2) $\varphi(x), \psi(x) \in C^{\alpha}(\bar{\Omega}), \varphi, \psi \not\equiv 0$.

Moreover we always suppose that $f \not\equiv 0$. We also assume that (3.1.1) is subject to homogeneous Dirichlet boundary conditions, i.e. $u = 0$ on $\partial\Omega$.

Definition 3.2.1. We set $L(p) = q$ if and only if $-\nabla(a(x)\nabla p(x)) = q(x)$ weakly in Ω, $p = 0$ on $\partial\Omega$.

We give now an existence and uniqueness/nonuniqueness result for problem (3.1.1). The following proposition generalizes a result of Catchpole [CAT] for the initial value problem in the one-dimensional case with $\varphi \equiv \psi$.

Proposition 3.2.2. Let $L(g) = \varphi$. Then (3.1.1) has a unique solution for every $f \in L^2(\Omega)$ if $1 + \eta \int_\Omega \psi g \neq 0$. If $1 + \eta \int_\Omega \psi g = 0$ then (3.1.1) is solvable if and only if $\int_\Omega fg = 0$. Moreover in the latter case the solution is not unique.

Proof. Suppose first that $1 + \eta \int_\Omega \psi g \neq 0$. Taking the scalar product of g with both sides of (3.1.1) we obtain

$$\int_\Omega \varphi u + \eta\Big(\int_\Omega \psi g\Big)\Big(\int_\Omega \varphi u\Big) = \int_\Omega fg$$

and thus

$$\int_\Omega \varphi u = \int_\Omega fg \Big/ \Big(1 + \eta \int_\Omega \psi g\Big).$$

The last expression is well defined if $\eta \neq -1/\int_\Omega \psi g$. A direct calculation shows that, in this case, (3.1.1) is equivalent to

$$-\nabla(a(x)\nabla u) = f - \eta\psi\Big(\int_\Omega fg\Big)\Big/\Big(1 + \eta\int_\Omega \psi g\Big) \quad \text{in } \Omega, \quad u = 0 \text{ on } \partial\Omega$$

and since this problem is uniquely solvable, so is (3.1.1).

Consider now the case when $1 + \eta \int_\Omega \psi g = 0$. The first equation in this proof gives $\int_\Omega fg = 0$, and to see that this condition is also sufficient we proceed as follows. Let us define functions v and z by $v = L^{-1}(\psi)$, $z = L^{-1}(f)$. Then (3.1.1) can be rewritten in operator form as $(I + K)u = z$, where I is the identity operator and $K(u) = \eta v \int_\Omega \varphi u$. Since K is a compact operator from $L^2(\Omega)$ to itself, by Fredholm Theory, (3.1.1) has a solution if and only if $z \perp Ker(I + K^*)$, where $K^*(\xi) = \eta\varphi \int_\Omega v\xi$. Therefore (3.1.1) has a solution if and only if $\int_\Omega z\xi = 0$ where ξ satisfies $\xi + \eta\varphi \int_\Omega v\xi = 0$, i.e. $\xi = k\varphi,\ k \in \mathbb{R}$. Consequently (3.1.1) has a solution if and only if $\int_\Omega \varphi z = 0$ or equivalently $\int_\Omega fg = 0$.

To see that the solution is not unique, observe that $kv,\ k \in \mathbb{R}$, is a solution of the homogeneous problem

$$-\nabla(a(x)\nabla u) + \eta\psi \int_\Omega \varphi u = 0 \ \text{ in } \ \Omega, \quad u = 0 \ \text{ on } \ \partial\Omega$$

if $1 + \eta \int_\Omega \psi g = 0$, and the proposition is proved.

3.3. Existence of $\underline{\eta}$ and $\overline{\eta}$

Now we turn to the proof of the existence of $\underline{\eta}$ and $\overline{\eta}$ for (3.1.1). To get our main result we need first the following lemma.

Lemma 3.3.1. Let $0 \le f \in L^2(\Omega)$. Let u be a solution of (3.1.1). Then:

1. There exists η^* independent of f such that if φ is sign-indefinite then for all $|\eta| < \eta^*$

$$\int_\Omega \varphi^+ u > 0, \quad \int_\Omega \varphi^- u > 0,$$

where $\varphi = \varphi^+ - \varphi^-$, $\varphi^+ = \max\{\varphi, 0\}$, $\varphi^- = -\min\{\varphi, 0\}$.

2. If $\varphi \ge 0$ then $\int_\Omega \varphi u > 0$ for $|\eta| < 1/|\int_\Omega \psi g|$, where $g = L^{-1}(\varphi)$.

3. If both φ, ψ are nonnegative, then for all $\eta \ge 0$ we have $\int_\Omega \varphi u > 0$ and

$$|\int_\Omega u| \Big/ \int_\Omega \varphi u \le C,$$

where C depends on the ellipticity bounds on $a(x)$, $\Omega, \varphi, \psi, \eta$ but not on f.

Proof. Let $\psi = \psi^+ - \psi^-$. Let us choose $g = L^{-1}(\varphi^+)$ and $v = L^{-1}(\varphi^-)$ and observe that g and v are positive in Ω. Let $\eta \geq 0$, and note that the case $\eta < 0$ may be treated in the same way as below. After taking the scalar product of g and v with both sides of equation (3.1.1) we get

$$\int_\Omega \varphi^+ u + \eta\Big(\int_\Omega \psi g\Big)\Big(\int_\Omega \varphi^+ u - \int_\Omega \varphi^- u\Big) = \int_\Omega fg$$

$$\int_\Omega \varphi^- u + \eta\Big(\int_\Omega \psi v\Big)\Big(\int_\Omega \varphi^+ u - \int_\Omega \varphi^- u\Big) = \int_\Omega fv.$$

From the first equation it follows that

$$\int_\Omega \varphi^+ u = \Big[\int_\Omega fg + \eta\Big(\int_\Omega \psi g\Big)\Big(\int_\Omega \varphi^- u\Big)\Big] \Big/ \Big(1 + \eta \int_\Omega \psi g\Big) \tag{3.3.1}$$

and substituting (3.3.1) into the second equation we obtain

$$\int_\Omega \varphi^- u = \frac{(\int_\Omega fv)(1 + \eta \int_\Omega \psi g) - \eta(\int_\Omega fg)(\int_\Omega \psi v)}{1 + \eta \int_\Omega \psi g - \eta \int_\Omega \psi v}. \tag{3.3.2}$$

Since the functions g and v are fixed and depend only on Ω and φ, are bounded, are zero at the boundary and by the Strong Maximum Principle have nonzero normal derivatives it follows that there exist two positive constants C_1 and C_2 such that $C_1 \leq g(x)/v(x) \leq C_2$ for all $x \in \bar{\Omega}$. Therefore (3.3.2) implies that

$$\int_\Omega \varphi^- u \geq \left(\frac{1 + \eta \int_\Omega \psi g + \eta C_1 \int_\Omega \psi^- v - \eta C_2 \int_\Omega \psi^+ v}{1 + \eta \int_\Omega \psi g - \eta \int_\Omega \psi v}\right) \int_\Omega fv. \tag{3.3.3}$$

Repeating the same procedure for $\int_\Omega \varphi^+ u$ we get

$$\int_\Omega \varphi^+ u \geq \left(\frac{1 - \eta \int_\Omega \psi v + \eta(1/C_2) \int_\Omega \psi^+ g - \eta(1/C_1) \int_\Omega \psi^- g}{1 + \eta \int_\Omega \psi g - \eta \int_\Omega \psi v}\right) \int_\Omega fg. \tag{3.3.4}$$

Since $\int_\Omega fv > 0$, $\int_\Omega fg > 0$ and since the other terms on the right hand sides of (3.3.3) and (3.3.4) do not depend on f, we can find η^* also independent of f such that for $0 \le \eta < \eta^*$ right hand sides of (3.3.3) and (3.3.4) are strictly positive what implies the first statement of the lemma.

Note, that as before

$$\int_\Omega \varphi u = \int_\Omega fg \Big/ \Big(1 + \eta \int_\Omega \psi g\Big).$$

So, for nonnegative φ, ψ we have that $\int_\Omega \varphi u > 0$ for all $\eta \ge 0$. If ψ changes sign then $\int_\Omega \varphi u > 0$ for $|\eta| < 1/|\int_\Omega \psi g|$.

Finally, (3.1.1) implies that

$$\int_\Omega u = \eta\Big(\int_\Omega \varphi u\Big)\Big(-\int_\Omega L^{-1}(\psi)\Big) + \int_\Omega L^{-1}(f).$$

Hence for nonnegative η, φ and ψ we have that

$$|\int_\Omega u| \Big/ \int_\Omega \varphi u \le \eta \int_\Omega L^{-1}(\psi) + \Big(1 + \eta \int_\Omega \psi g\Big)\Big(\int_\Omega f L^{-1}(1)\Big) \Big/ \Big(\int_\Omega fg\Big)$$

since $\int_\Omega \varphi u > 0$ by the previous remark.

Applying the previous argument we conclude that there exists a positive constant C_3 such that $L^{-1}(1)/g \le C_3$. Therefore

$$|\int_\Omega u| \Big/ \int_\Omega \varphi u \le \eta \int_\Omega L^{-1}(\psi) + C_3\Big(1 + \eta \int_\Omega \psi g\Big) \le C$$

and the lemma is proved.

The main result of this chapter is

Theorem 3.3.2. There exist $\overline{\eta} > 0$ and $\underline{\eta} < 0$ such that for every $\underline{\eta} \leq \eta \leq \overline{\eta}$ and for every $0 \leq f \in L^2(\Omega)$, the solution of (3.1.1) is nonnegative. If $f \not\equiv 0$, the solution is positive.

Proof. We prove first the theorem for the case when φ and ψ are nonnegative functions and $\eta \geq 0$, and since we can approximate a function from $L^2(\Omega)$ using smooth functions then it suffices to show the result for $f \not\equiv 0$ smooth. Set $g = L^{-1}(\varphi)$, $\tau = L^{-1}(g)$ and $v = L^{-1}(\psi)$. Then multiplying equation (3.1.1) by g yields

$$\int_\Omega \varphi u + \eta\Big(\int_\Omega \psi g\Big)\Big(\int_\Omega \varphi u\Big) = \int_\Omega fg$$

and

$$\int_\Omega \varphi u = \int_\Omega fg \Big/ \Big(1 + \eta \int_\Omega \psi g\Big). \tag{3.3.5}$$

Let us set $w = u / \int_\Omega \varphi u$. Then (3.1.1) and (3.3.5) imply that

$$-\nabla(a(x)\nabla(w + \eta v)) = f\Big(1 + \eta \int_\Omega \psi g\Big) \Big/ \int_\Omega fg.$$

Taking the scalar product with τ of both sides of the last equation yields

$$\int_\Omega g(w + \eta v) = \Big(\int_\Omega f\tau \Big/ \int_\Omega fg\Big)\Big(1 + \eta \int_\Omega \psi g\Big).$$

By the same argument as in Lemma 3.3.1 there exists a positive constant K such that $\inf_{x\in\Omega}[\tau(x)/g(x)] \geq K$. Therefore

$$\int_\Omega g(w + \eta v) \geq K(1 + \eta \int_\Omega \psi g) \geq K.$$

Note that K does not depend on f. Next we choose $\tilde{\eta}$ small enough and a subdomain $\Omega_1 \subset \Omega$ such that for all $\eta \leq \tilde{\eta}$

$$\int_{\Omega\setminus\Omega_1} g(w + \eta v) \leq K/2.$$

We can always do this, because g and v are fixed bounded nonnegative functions on Ω, $w + \eta v \geq 0$ and the absolute value of $\int_\Omega w$ is bounded by Lemma 3.3.1. Note that, consequently, the choice of the domain Ω_1 does not depend on f and w. Hence $\int_{\Omega_1} g(w + \eta v) \geq K/2$, and $\int_{\Omega_1} (w + \eta v) \geq K/(2||g||_\infty)$.

We observe that $\bar{\Omega}_1$ is compact and we cover it with a finite number of balls $B^1, ..., B^N \subset \Omega$ each of radius r. Then we can choose from this set a ball B^{j_0} such that

$$\int_{B^{j_0}} (w + \eta v) \geq K/(2N||g||_\infty).$$

Let us denote by $B^j_{r/2}$, $j = 1, ..., N$ balls of radius $r/2$ such that $B^j_{r/2} \subset B^j$ and have the same centers for all j. Now we can apply the weak Harnack's inequality to $w + \eta v$ and conclude that for the ball $B^{j_0}_{r/2} \subset B^{j_0}$ we have that $\min_{x \in B^{j_0}_{r/2}} (w + \eta v) \geq 2K_1$. It follows, that if η is small enough then $\min_{x \in B^{j_0}_{r/2}} (w) \geq K_1$.

We show next that $w \geq 0$ in $\Omega \backslash B^{j_0}_{r/2}$ and the estimate on η does not depend on j_0. To do this, we consider $N + 1$ equations:

$$-\nabla(a(x)\nabla(w + \eta v)) \geq 0 \text{ in } \Omega \backslash B^{j_0}_{r/2}, \quad w + \eta v \geq K_1 \text{ on } \partial B^{j_0}_{r/2}, \quad w + \eta v = 0 \text{ on } \partial\Omega$$

$$-\nabla(a(x)\nabla z_i) = 0 \text{ in } \Omega \backslash B^i_{r/2}, \quad z_i = K_1 \text{ on } \partial B^i_{r/2}, \quad z_i = 0 \text{ on } \partial\Omega, \ i = 1...N.$$

Then $z_i \geq 0$, $i = 1, ..., N$ and we can choose η so small that $z_i - \eta v > 0$ on $\Omega \backslash B^i_{r/2}$ for all $i = 1, ..., N$, since $\partial z_i / \partial \nu > 0$ on $\partial\Omega$, where $\partial/\partial\nu$ denotes the inward normal derivative. We get that $w + \eta v - z_{j_0} \geq 0$ in $\Omega \backslash B^{j_0}_{r/2}$. Hence for small η, it follows that $w \geq z_{j_0} - \eta v > 0$ where the estimate on η does not depend on j_0 or f.

Now we can consider the case with φ and ψ both sign indefinite in Ω and $\eta \geq 0$. Let $\varphi = \varphi^+ - \varphi^-$, $\psi = \psi^+ - \psi^-$, as before. Then from Lemma 3.3.1, it follows that for small η, $\int_\Omega \varphi^+ u \geq 0$ and $\int_\Omega \varphi^- u \geq 0$. Therefore we can rewrite equation (3.1.1) in the following form

$$-\nabla(a(x)\nabla u) + \eta(\psi^+ - \psi^-)\Big(\int_\Omega \varphi^+ u - \int_\Omega \varphi^- u\Big) = f.$$

It follows that

$$-\nabla(a(x)\nabla u) + \eta\psi^+ \int_\Omega \varphi^+ u + \eta\psi^- \int_\Omega \varphi^- u = f + \eta\psi^+ \int_\Omega \varphi^- u + \eta\psi^- \int_\Omega \varphi^+ u,$$

and we obtain

$$-\nabla(a(x)\nabla u) + \eta|\psi| \int_\Omega |\varphi| u = f + 2\eta\psi^+ \int_\Omega \varphi^- u + 2\eta\psi^- \int_\Omega \varphi^+ u.$$

So we have reduced this case to the previous one. The case when φ and ψ are both sign indefinite and $\eta < 0$ may be treated similarly.

Observe now that if $\eta < 0$ and both φ and ψ are nonnegative putting $\mu = -\eta$ we obtain from (3.1.1)

$$-\nabla(a(x)\nabla u) = f + \mu\psi \int_\Omega \varphi u \quad \text{in } \Omega, \quad u = 0 \text{ on } \partial\Omega.$$

Then if $g = L^{-1}(\varphi)$, we get

$$\int_\Omega \varphi u = \int_\Omega fg \Big/ \Big(1 - \mu \int_\Omega \psi g\Big).$$

Therefore we can conclude that if $\mu < 1/\int_\Omega \psi g$ our problem has a unique positive solution. Moreover if $\mu > 1/\int_\Omega \psi g$ then the solution changes sign. Finally, if $\mu = 1/\int_\Omega \psi g$ the solution is no longer unique.

Finally we may consider the case $\varphi, \psi \leq 0$ and $\eta < 0$ in a similar way and the theorem is proved.

Definition 3.3.3. We denote by η_0 the supremum of values of η for which solutions of (3.1.1) are positive for all nontrivial nonnegative right hand sides.

Corollary 3.3.4. Let $\eta \geq 0$ and $\varphi, \psi \geq 0$. If u solves (3.1.1) with $\eta < \eta_0$ and $0 \leq f \in L^2(\Omega)$ then the inward normal derivative of u at $x_0 \in \partial\Omega$, if it exists, satisfies $(\partial u/\partial\nu)(x_0) > 0$.

The statement of the corollary for small positive η follows immediately from the proof of Theorem 3.3.2. Specifically, since $w \geq z_{j_0} - \eta v > 0$ for small η, we easily obtain that $\partial w/\partial\nu > 0$ on $\partial\Omega$.

Proof. For simplicity we can assume that f, φ, ψ are smooth. In the notation of Theorem 3.3.2,

$$u_\eta = L^{-1}(f) - \left(\eta \int_\Omega fg \Big/ \left(1 + \eta \int_\Omega fg\right)\right) v > 0 \quad \text{in} \quad \Omega$$

if $\eta < \eta_0$ and $f \geq 0$. The normal derivative of v exists for all $x_0 \in \partial\Omega$, therefore $(\partial u_\eta/\partial\nu)(x_0)$ exists if and only if $(\partial L^{-1}(f)/\partial\nu)(x_0)$ exists. Since $u_{\eta_0} \geq 0$ in Ω and $u_{\eta_0} = 0$ on $\partial\Omega$ it follows that

$$(\partial u_{\eta_0}/\partial\nu)(x_0)$$

$$= (\partial L^{-1}(f)/\partial\nu)(x_0) - \left(\eta_0 \int_\Omega fg/(1 + \eta_0 \int_\Omega fg)\right)(\partial v/\partial\nu)(x_0) \geq 0.$$

Note, that $(\partial L^{-1}(f)/\partial\nu)(x_0)$, $(\partial v/\partial\nu)(x_0) > 0$. Thus it is easy to see that for $\eta < \eta_0$, $(\partial u_\eta/\partial\nu)(x_0) > 0$. which implies the statement of the corollary.

3.4. Existence of η_1

Next we turn to the proof of the existence of $\eta_1 > 0$ which depends only on the bounds on $a(\cdot)$, but not on $a(\cdot)$ itself and for which a positivity result similar to Theorem 3.3.2 holds. In order to do this, we prove the following three lemmas. Motivated by the physical example to which we apply our results later, we consider for the rest of this section only nonnegative φ, ψ satisfying (A2) and $\eta \geq 0$.

Lemma 3.4.1. Let $a(x)$ satisfy (A1) and (i) $0 < A \leq a(x)$, (ii) $||a(x)||_{C^{1,\alpha}} \leq D$. Let $g(x)$ and $\tau(x)$ satisfy: $L(g) = \varphi$, $L(\tau) = g$. Then

$$\inf_{x \in \Omega} [\tau(x)/g(x)] \geq K, \qquad (3.4.1)$$

where K is a positive constant which depends on A, D, Ω and φ but not on $a(x)$.

Proof. If (3.4.1) fails then there exist sequences $\{a_n(x)\}, \{g_n(x)\}$ and $\{\tau_n(x)\}$ such that the following conditions hold:

1. $0 < A \leq a_n(x), \quad ||a_n(x)||_{C^{1,\alpha}} \leq D$
2. $-\nabla(a_n(x)\nabla g_n(x)) = \varphi(x)$ in Ω, $g_n = 0$ on $\partial\Omega$
3. $-\nabla(a_n(x)\nabla \tau_n(x)) = g_n(x)$ in Ω, $\tau_n = 0$ on $\partial\Omega$
4. $\inf_{x\in\Omega} [\tau_n(x)/g_n(x)] \to 0$ as $n \to \infty$

Note that Theorem 2.2.9 implies that $g_n, \tau_n \in H_0^1(\Omega)$. Since the embedding of $C^{1,\alpha}(\bar{\Omega})$ into $C^{1,\alpha'}(\bar{\Omega})$, $\alpha' < \alpha$, is compact and all $a_n(x)$ are bounded in $C^{1,\alpha}(\bar{\Omega})$ we can find a subsequence which we will denote again as $a_n(x)$ such that $a_n(x) \to a(x)$ in $C^{1,\alpha'}(\bar{\Omega})$ as $n \to \infty$, $\alpha' < \alpha$. Obviously $a(x) \in C^{1,\alpha'}(\bar{\Omega})$,

$0 < A \leq a(x)$ and $||a(x)||_{C^{1,\alpha'}} \leq D_1 < \infty$. On the other hand, Theorem 2.3.2 implies that

$$\sup_{x \in \Omega} |g_n(x)| \leq C||\varphi||_{L^{q/2}}$$

where $q > n$ and C depends on D, and Ω. Thus by Theorem 2.3.5 we get that $g_n(x) \in C^{1,\alpha}(\bar{\Omega})$ and

$$||g_n||_{C^{1,\alpha}} \leq C_1(||g_n||_{L^\infty} + ||\varphi||_{L^\infty}) \leq C_2$$

where $C_2 = C_2(n, A, D, \partial\Omega)$. Therefore we can find a subsequence of $\{g_n\}$ which we will still denote by $\{g_n\}$ such that $g_n(x) \to g(x)$ in $C^{1,\alpha'}(\bar{\Omega})$ as $n \to \infty$. Now we show that $L(g) = \varphi$. Indeed,

$$||a_n(x)\nabla g_n(x) - a(x)\nabla g(x)||_{C^{\alpha'}} =$$

$$||(a_n(x) - a(x))\nabla g_n(x) + a(x)\nabla(g_n(x) - g(x))||_{C^{\alpha'}} \leq$$

$$C_2||a_n(x) - a(x)||_{C^{\alpha'}} + D||\nabla(g_n(x) - g(x))||_{C^{\alpha'}}$$

where C_2 and D are as above. So $a_n(x)\nabla g_n(x) \to a(x)\nabla g(x)$ in $C^{\alpha'}(\bar{\Omega})$ as $n \to \infty$.

Then for every $\xi \in C_0^\infty(\Omega)$ we have

$$\int_\Omega a(x)\nabla g(x)\nabla\xi dx - \int_\Omega \varphi\xi dx = \int_\Omega (a(x)\nabla g(x) - a_n(x)\nabla g_n(x))\nabla\xi dx \to 0.$$

So $L(g) = \varphi$. The same argument shows that $L(\tau) = g$. But then condition 4 implies that $\inf_{x\in\Omega}[\tau(x)/g(x)] = 0$. Since $\partial\tau/\partial\nu > 0$ by the Strong Maximum Principle this is a contradiction.

Lemma 3.4.2. Let $a(x)$ and $g(x)$ be as in Lemma 3.4.1. Then for every sufficiently small $\epsilon > 0$ there exists Ω_1, a neighbourhood of $\partial\Omega$, $\Omega_1 \subset \Omega$, such that $\max_{x\in\Omega_1}[g(x)] \leq \epsilon$ and the choice of Ω_1 depends on $A, D, \epsilon, \Omega, \varphi$ but not on $a(x)$.

Proof. If the statement of the lemma is not true then there exist subsequences $\{a_n(x)\}, \{g_n(x)\}$ and $\{x_n\}$ such that they satisfy Conditions 1 and 2 of Lemma 3.4.1, $x_n \to x_0 \in \partial\Omega$ as $n \to \infty$ and $g_n(x_n) \geq \epsilon$. As in the previous lemma we construct $a(x) \in C^{1,\alpha'}(\bar{\Omega})$ and $g(x) \in C^{1,\alpha'}(\bar{\Omega})$, $\alpha' < \alpha$, such that $a_n(x) \to a(x)$ in $C^{1,\alpha'}(\bar{\Omega})$ as $n \to \infty$, $a(x)$ satisfies (i), (ii), $g_n(x) \to g(x)$ in $C^{1,\alpha'}(\bar{\Omega})$ as $n \to \infty$ and $g(x) = L^{-1}(\varphi)$. But then $|(\partial g/\partial\nu)(x_0)| = \infty$, contradicting the fact that $g \in C^{1,\alpha'}(\bar{\Omega})$ and the lemma is proved.

Lemma 3.4.3. Let $a(x)$ be as in Lemma 3.4.1, $v = L^{-1}(\psi)$ and let $B \subset \Omega$ be a ball. Let $z(x)$ satisfy

$$-\nabla(a(x)\nabla z(x)) = 0 \text{ in } \Omega\backslash B, \quad z = 0 \text{ on } \partial\Omega, \quad z = K_0 \text{ on } \partial B$$

where K_0 is a positive constant. Then there exists η_* such that for all $\eta < \eta_*$

$$z - \eta v > 0 \text{ in } \Omega\backslash B$$

and η_* depends on $A, D, \Omega, B, K_0, \psi$ but not on $a(x)$.

Proof. If the statement of the lemma fails then there exist sequences $\{a_n(x)\}$, $\{z_n(x)\}$, $\{v_n(x)\}$ and $\{\eta_n\}$ such that the following statements are true:

1. $0 < A \leq a_n(x)$, $||a_n(x)||_{C^{1,\alpha}} \leq D$
2. $-\nabla(a_n(x)\nabla v_n(x)) = \psi(x)$ in Ω, $v_n = 0$ on $\partial\Omega$

3. $-\nabla(a_n(x)\nabla z_n(x)) = 0$ in $\Omega\backslash B$, $z_n = 0$ on $\partial\Omega$, $z_n = K_0$ on ∂B

4. $\eta_n \to 0$ as $n \to \infty$

5. There exists $x_n \in \Omega\backslash B$ such that $(z_n - \eta_n v_n)(x_n) \leq 0$.

Then in the same way as before we get $a_n(x) \to a(x)$ in $C^{1,\alpha'}(\bar{\Omega})$, $v_n(x) \to v(x)$ in $C^{1,\alpha'}(\bar{\Omega})$, $z_n(x) \to z(x)$ in $C^{1,\alpha'}(\overline{\Omega\backslash \bar{B}})$ as $n \to \infty$, $\alpha' < \alpha$, and $a(x), v(x)$ and $z(x)$ satisfy

$$-\nabla(a(x)\nabla v(x)) = \psi(x) \text{ in } \Omega, \quad v = 0 \text{ on } \partial\Omega$$

$$-\nabla(a(x)\nabla z(x)) = 0 \text{ in } \Omega\backslash B, \quad z = 0 \text{ on } \partial\Omega, \quad z = K_0 \text{ on } \partial B.$$

Since $\Omega\backslash B$ is bounded, without loss of generality $x_n \to x_0 \in \overline{\Omega\backslash B}$ as $n \to \infty$. We consider three possibilities: 1) $x_0 \in \Omega\backslash\bar{B}$; 2) $x_0 \in \partial B$; 3) $x_0 \in \partial\Omega$. In the first case we get $(z_n - \eta_n v_n)(x_n) \to z(x_0)$ as $n \to \infty$ and consequently $z(x_0) \leq 0$ which contradicts the maximum principle. Since $K_0 > 0$ we get a contradiction in the second case too. We show now that the third case is also impossible. Consider a normal $[x_n, t_n]$ through x_n to the boundary of Ω, $t_n \in \partial\Omega$, and the corresponding inner normal vector ν_n. Then there exists $y_n \in [x_n, t_n]$ such that $y_n \to x_0$ as $n \to \infty$ and $(\partial(z_n - \eta_n v_n)/\partial\nu_n)(y_n) \leq 0$. Thus passing to the limit we obtain that $(\partial z_n/\partial\nu)(x_0) \leq 0$ which contradicts the Strong Maximum Principle, and the lemma is proved.

Theorem 3.4.4. Let $a(x)$ satisfy (A1) and conditions (i),(ii) from Lemma 3.4.1 and let $\varphi, \psi \geq 0$. Then there exists $\eta_1 > 0$ such that for every $\eta \leq \eta_1$ and every nonnegative nontrivial $f(x) \in L^2(\Omega)$ there exists a positive solution of (3.1.1). Moreover η_1 depends on $A, D, \Omega, \psi, \varphi$ but not on $a(x)$.

Proof. We need only combine Lemmas 3.4.1, 3.4.2 and 3.4.3 and the proof of Theorem 3.3.2.

Obviously an exact analogue of Corollary 3.3.4 may also be obtained by the above procedure.

3.5. General formula for η_0

Since the methods of Section 3.3 are non-constructive we now pass to the problem of estimating η_0. In general this is very difficult, and we can give an explicit estimate only in some special cases. In this section we obtain an explicit formula for η_0 and in the next one we give the resulting values as applications. We conjecture that this approach could also be used to obtain the existence of η_0, however we feel that the earlier procedure is simpler and also provides the existence of η_1.

Consider (3.1.1) with $\eta \geq 0$, $a(x), \psi(x) \geq 0, \varphi(x) \geq 0$ satisfying hypotheses (A1) and (A2) and homogeneous Dirichlet boundary conditions. Let $G(x, y)$ be the Green's function of the operator L defined in section 3.2, $g = L^{-1}(\varphi)$, $v = L^{-1}(\psi)$ and $K = \sup_{y\in\Omega} \sup_{x\in\Omega} [g(y)v(x)/G(x,y)]$. Note, that $G(x, y)$ exists even for the case $a(x) \in L^{\infty}(\Omega)$, [STA]. Then we have

Theorem 3.5.1. Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain. Then if

$$\int_{\Omega} v(x)\varphi(x)dx < K < \infty,$$

it follows that

$$\eta_0 = (K - \int_{\Omega} \varphi(x)v(x)dx)^{-1}. \tag{3.5.1}$$

Note that from the results of Section 3.3 and from the last theorem it follows that K is always finite. Since the previous examples show that η_0 cannot be infinite, then we must always have $\int_\Omega \varphi(x)v(x)dx < K$. Observe, that since G also exists for $a(x) \in L^\infty(\Omega)$, then η_0 will also exist whenever the conditions of Theorem 3.5.1 hold.

Proof. Without loss of generality set $\int_\Omega \varphi(x)u(x)dx = 1$ in equation (3.1.1). Then (3.1.1) becomes

$$-\nabla(a(x)\nabla u(x)) = -\eta\psi(x) + f(x) \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \quad \int_\Omega \varphi u = 1. \quad (3.5.2)$$

It is enough to consider only $f(x) \in C^\infty(\bar{\Omega})$. We look for a solution $u(x)$ of (3.5.2) of the form

$$u = -\eta v + h, \quad h = L^{-1}(f).$$

and observe that h can be written in integral form as $h(x) = \int_\Omega G(x,y)f(y)dy$, where f satisfies

$$\int_\Omega \int_\Omega G(x,y)f(y)\varphi(x)dydx = 1 + \eta \int_\Omega \varphi(x)v(x)dx. \quad (3.5.3)$$

Note that $u(x) > 0$ if and only if $h(x) > \eta v(x)$ for all $x \in \Omega$, and furthermore if $u(x)$ is negative at some point then there exists $x_0 \in \Omega$ such that $u(x_0) = 0$, since $\int_\Omega \varphi u > 0$ and thus $u(x)$ cannot be negative everywhere.

We show that for $\eta \leq (K - \int_\Omega \varphi(x)v(x)dx)^{-1}$ condition (3.5.3) and the condition that there exists a point $x_0 \in \Omega$ such that $h(x_0) = \eta v(x_0)$ are contradictory. Indeed, if $h(x_0) = \eta v(x_0)$ for some $x_0 \in \Omega$ then

$$h(x) = \eta v(x_0) \int_\Omega G(x,z)f(z)dz \Big/ \int_\Omega G(x_0,z)f(z)dz.$$

Note that here we can take arbitrary f since we can always find a constant C such that Cf satisfies (3.5.3) and because the last expression is invariant with respect to multiplication of f by a scalar. Therefore

$$\int_\Omega \varphi(x)h(x)dx \leq \eta \sup_{x_0\in\Omega}\sup_{f\geq 0}\left[\int_\Omega v(x_0)g(z)f(z)dz \Big/ \int_\Omega G(x_0,z)f(z)dz\right].$$

We claim that

$$\sup_{x_0\in\Omega}\sup_{f\geq 0}\left[\int_\Omega v(x_0)g(z)f(z)dz \Big/ \int_\Omega G(x_0,z)f(z)dz\right] =$$

$$\sup_{x_0\in\Omega}\sup_{y\in\Omega}[v(x_0)g(y)/G(x_0,y)] = K.$$

Indeed, first it is easy to see from the definition of K that

$$\sup_{x_0\in\Omega}\sup_{f\geq 0}\left[\int_\Omega v(x_0)g(z)f(z)dz \Big/ \int_\Omega G(x_0,z)f(z)dz\right] \leq K.$$

On the other hand , if we take a sequence $\{f_n(z)\}$ which tends to $\delta(y-z)$, $y\in\Omega$ when $n\to\infty$ then

$$K = \sup_{y\in\Omega}\sup_{x_0\in\Omega}\lim_{n\to\infty}\frac{\int_\Omega v(x_0)g(z)f_n(z)dz}{\int_\Omega G(x_0,z)f_n(z)dz}.$$

Therefore since $K > \int_\Omega \varphi(x)v(x)dx$ then for $\eta < (K - \int_\Omega \varphi(x)v(x)dx)^{-1}$ it follows that

$$\int_\Omega \varphi(x)h(x)dx < 1 + \eta\int_\Omega \varphi(x)v(x)dx.$$

But this contradicts (3.5.3), so we can conclude that $u(x)$ is positive for the given range of η.

We show now that η_0 is actually equal to $(K - \int_\Omega \varphi(x)v(x)dx)^{-1}$, i.e. that if $\eta > (K - \int_\Omega \varphi(x)v(x)dx)^{-1}$, then there exists f such that the solution of (3.1.1) changes sign. Indeed, if $\eta > (K - \int_\Omega \varphi(x)v(x)dx)^{-1}$ then we can find two points $x_0, y_0 \in \Omega$ such that

$$\eta > \left(v(x_0)g(y_0)/G(x_0,y_0) - \int_\Omega \varphi(z)v(z)dz\right)^{-1}. \qquad (3.5.4)$$

We construct a sequence $0 \le f_n(x) \to \delta(x-y_0)$ as $n \to \infty$ and solve the problems

$$-\nabla(a(x)\nabla w_n) + \eta\psi \int_\Omega \varphi w_n = f_n \quad \text{in} \quad \Omega, \quad w_n = 0 \quad \text{on} \quad \partial\Omega.$$

Using expression (3.3.5) for $\int_\Omega \varphi(z)w_n(z)dz$ and the fact that $\int_\Omega \psi(z)g(z)dz = \int_\Omega \varphi(z)v(z)dz$ we rewrite the last problem as

$$-\nabla(a(x)\nabla w_n)(x) = f_n(x) - \eta\psi(x)\int_\Omega f_n(z)g(z)dz \Big/ \left(1 + \eta\int_\Omega \varphi(z)v(z)dz\right)$$

and therefore

$$w_n(x) = L^{-1}(f_n)(x) - \eta v(x)\int_\Omega f_n(z)g(z)dz \Big/ \left(1 + \eta\int_\Omega \varphi(z)v(z)dz\right).$$

Then (3.5.4) implies that when $n \to \infty$

$$w_n(x_0) \to -\eta v(x_0)g(y_0) \Big/ \left(1 + \eta\int_\Omega \varphi(z)v(z)dz\right) + G(x_0,y_0) < 0$$

and thus we conclude that $w_n(x_0) < 0$ for large n. The theorem is proved.

3.6. Applications of the general formula for η_0

This section is devoted to the application of the general formula (3.5.1) to the special case of equation (3.1.1). Specifically we consider (3.1.1) with $a(x), \psi(x), \varphi(x) \equiv 1$. Then we have the following

Theorem 3.6.1. Let Ω be a unit ball B_1 in $\mathbb{R}^n$, $n \geq 1$. Then

$$\eta_0 = \frac{n^2(n+2)}{\sigma_n(2^{n-1}(n+2)-1)},$$

where σ_n is the surface area of the unit ball in $\mathbb{R}^n$ with $\sigma_1 = 2$.

Proof. To prove the theorem we consider three separate cases.

Case 1. If $n = 1$ and $\Omega = (-1, 1)$ then

$$G(x,y) = \frac{1}{2}\begin{cases} (x+1)(1-y) & \text{if} \quad -1 \leq x \leq y \leq 1 \\ (1-x)(y+1) & \text{if} \quad -1 \leq y \leq x \leq 1 \end{cases}$$

$$g(x) = v(x) = (x+1)(1-x)/2.$$

Direct calculations yield that

$$K = \frac{1}{2}\sup_{y\in\Omega}\sup_{x\in\Omega}\left\{ \begin{array}{lll} (y+1)(1-x) & \text{if} & -1 \leq x \leq y \leq 1 \\ (1-y)(x+1) & \text{if} & -1 \leq y \leq x \leq 1 \end{array} \right\} = 2$$

and

$$\int_{-1}^{1} v(x)dx = \frac{1}{2}\int_{-1}^{1}(1-x^2)dx = 2/3.$$

Substituting these values into (3.5.1) we obtain that $\eta_0 = 3/4$ and therefore the statement of the theorem is verified for $n = 1$.

Case 2. When $n = 2$ the Green's function and $v(x) \equiv g(x)$ are

$$G(x,y) = \frac{1}{4\pi}\ln\left|\frac{1-\bar{x}y}{x-y}\right|^2, \quad v(x) = (1-|x|^2)/4,$$

where $x = (x_1, x_2)$, $y = (y_1, y_2)$ are identified with complex numbers $x = x_1 + ix_2$, $y = y_1 + iy_2$. To apply the formula for η_0 given in Theorem 3.5.1 we first have to find

$$K = \frac{\pi}{4} \sup_{y \in B_1} \sup_{x \in B_1} \left[(1 - |x|^2)(1 - |y|^2) \Big/ \ln \left| \frac{1 - \bar{x}y}{x - y} \right|^2 \right].$$

Since $(-\Delta)$ is invariant with respect to the rotations of a ball then if T denotes any rotation, we have $G(Tx, Ty) = G(x, y)$. Therefore without loss of generality we may put $x = (x_1, 0)$. We can see that

$$\left| \frac{1 - \bar{x}y}{x - y} \right|^2 = \frac{(1 - x_1 y_1)^2 + x_1^2 y_1^2}{(x_1 - y_1)^2 + y_2^2} = 1 + \frac{(1 - |x|^2)(1 - |y|^2)}{|x - y|^2}.$$

For convenience we denote

$$F(x, y) = (1 - |x|^2)(1 - |y|^2) \Big/ \ln \left(1 + \frac{(1 - |x|^2)(1 - |y|^2)}{|x - y|^2} \right).$$

Next we observe that

$$\sup_{y \in B_1} \sup_{x \in B_1} F(x, y) = \sup_{\substack{0 \le r_1 \le 1 \\ 0 \le r_2 \le 1}} \sup_{\substack{|x| = r_1 \\ |y| = r_2}} F(x, y)$$

and

$$\sup_{\substack{|x| = r_1 \\ |y| = r_2}} F(x, y) = (1 - r_1^2)(1 - r_2^2) \Big/ \ln \left(1 + \frac{(1 - r_1^2)(1 - r_2^2)}{(r_1 + r_2)^2} \right) =$$

$$(1 - r_1^2)(1 - r_2^2) \Big/ \ln \left(\frac{1 + r_1 r_2}{r_1 + r_2} \right)^2.$$

Our claim is that

$$\sup_{y \in B_1} \sup_{x \in B_1} F(x, y) = \sup_{\substack{0 \le r_1 \le 1 \\ 0 \le r_2 \le 1}} (1 - r_1^2)(1 - r_2^2) \Big/ \ln \left(1 + \frac{(1 - r_1^2)(1 - r_2^2)}{(r_1 + r_2)^2} \right) = 4. \tag{3.6.1}$$

Indeed, first note that

$$\lim_{\substack{r_1\to 1\\ r_2\to 1}} (1-r_1^2)(1-r_2^2)\Big/\ln\left(1+\frac{(1-r_1^2)(1-r_2^2)}{(r_1+r_2)^2}\right) =$$

$$\lim_{\substack{r_1\to 1\\ r_2\to 1}} (1-r_1^2)(1-r_2^2)\Big/\left(\frac{(1-r_1^2)(1-r_2^2)}{(r_1+r_2)^2}\right) = \lim_{\substack{r_1\to 1\\ r_2\to 1}} (r_1+r_2)^2 = 4.$$

Therefore to prove (3.6.1) it is enough to show that

$$(1-r_1^2)(1-r_2^2)\Big/\ln\left(1+\frac{(1-r_1^2)(1-r_2^2)}{(r_1+r_2)^2}\right) \le 4 \qquad (3.6.2)$$

or equivalently

$$(1-r_1^2)(1-r_2^2) \le 8\ln\left(\frac{1+r_1r_2}{(r_1+r_2)}\right).$$

After the change of variables $z = 1-r_1$, $w = 1-r_2$, we get

$$wz(2-z)(2-w) \le 8\ln\left(\frac{(2-w)(2-z)}{2-z-w} - 1\right).$$

The left and right hand sides of the previous inequality are equal when $z = w = 0$, so it suffices to show that the partial derivatives with respect to z and w of the left hand side are less than or equal to the corresponding derivatives of the right hand side. Since both sides are symmetric it is enough to check only the partial derivative with respect to w, i.e.

$$2z(2-z)(1-w) \le \frac{8(2-z)z}{(2-w-z)((2-z)(2-w)-2+z+w)}.$$

Simplifying the last inequality we get that (3.6.2) is equivalent to

$$r_2(r_1+r_2)(1+r_1r_2) \le 4.$$

But this inequality is obvious since $0 \leq r_1 \leq 1, \quad 0 \leq r_2 \leq 1$. Finally we obtain from the definition of K and $F(x,y)$ that

$$K = \frac{\pi}{4} \sup_{y \in B_1} \sup_{x \in B_1} F(x,y) = \pi.$$

Taking in consideration that $\int_{B_1} (1 - |x|^2)/4 = \pi/8$ we get from (3.5.1) that $\eta_0 = 8/(7\pi)$ and the theorem is proved for $n = 2$.

Case 3. In the case when $n \geq 3$ we recall from [MIKH] that

$$G(x,y) = (\sigma_n(n-2))^{-1} \begin{cases} |x-y|^{2-n} - |x/|x| - y|x||^{2-n}, & \text{if } x \neq 0 \\ |x-y|^{2-n} - 1, & \text{if } x = 0 \end{cases}$$

$$v(x) = (1 - |x|^2)/(2n), \quad x \in B_1,$$

and therefore we should find

$$K = \sup_{y \in B_1} \sup_{x \in B_1} [(1 - |x|^2)(1 - |y|^2)/(4n^2 G(x,y))].$$

As before we set

$$F(x,y) = \frac{(1 - |x|^2)(1 - |y|^2)}{\sigma_n(n-2)G(x,y)}.$$

and, as in Case 2, suppose without loss of generality that $x = (x_1, 0, ..., 0)$, $y = (y_1, y_2, ..., y_n)$ and $x_1 \geq 0$. Then

$$F(x,y) = \frac{(1 - |x_1|^2)(1 - |y|^2)}{|x-y|^{2-n} - |x/x_1 - x_1 y|^{2-n}}$$

$$= \frac{(1 - |x_1|^2)(1 - |y|^2)|x-y|^{n-2}|x/x_1 - x_1 y|^{n-2}}{|x/x_1 - x_1 y|^{n-2} - |x-y|^{n-2}}$$

$$= \frac{(1-|x_1|^2)(1-|y|^2)|x-y|^{n-2}|x/x_1 - x_1 y|^{n-2}}{(|x/x_1 - x_1 y| - |x-y|)\sum_{k=0}^{n-3}|x-y|^k|x/x_1 - x_1 y|^{n-3-k}}$$

$$= \frac{(1-|x_1|^2)(1-|y|^2)|x-y|^{n-2}|x/x_1 - x_1 y|^{n-2}(|x-y| + |x/x_1 - x_1 y|)}{(|x/x_1 - x_1 y|^2 - |x-y|^2)\sum_{k=0}^{n-3}|x-y|^k|x/x_1 - x_1 y|^{n-3-k}}.$$

Observing that

$$|x/x_1 - x_1 y|^2 - |x-y|^2 = (1-x_1y_1)^2 + x_1^2(y_2^2+\ldots+y_n^2) - (x_1-y_1)^2 - (y_2^2+\ldots+y_n^2)$$

$$= (1-|x_1|^2)(1-|y|^2)$$

we can rewrite $F(x,y)$ as

$$F(x,y) = \frac{|x-y|^{n-2}|x/x_1 - x_1 y|^{n-2}(|x-y| + |x/x_1 - x_1 y|)}{\sum_{k=0}^{n-3}|x-y|^k|x/x_1 - x_1 y|^{n-3-k}}.$$

Dividing the numerator and the denominator of $F(x,y)$ by $|x-y|^{n-2}|x/x_1 - x_1 y|^{n-2}$ yields

$$F(x,y) = \frac{(|x-y| + |x/x_1 - x_1 y|)}{\sum_{k=0}^{n-3}|x-y|^{k-n+2}|x/x_1 - x_1 y|^{-k-1}} =$$

$$\left(\sum_{k=0}^{n-3}|x-y|^{k-n+1}|x/x_1 - x_1 y|^{-k-1}\right)^{-1} + \left(\sum_{k=0}^{n-3}|x-y|^{k-n+2}|x/x_1 - x_1 y|^{-k-2}\right)^{-1}$$

The supremum of $F(x,y)$ is thus reached when $x = (1, 0, \ldots, 0)$ and $y = (-1, 0, \ldots, 0)$ and is equal to $2^{n+1}/(n-2)$. Therefore $K = \sigma_n 2^{n-1} n^{-2}$.

With some elementary calculations we get

$$\int_{B_1} (1-|x|^2)/(2n)dx = \sigma_n n^{-2}/(n+2).$$

and thus from (3.5.1) we get that

$$\eta_0 = \frac{n^2(n+2)}{\sigma_n(2^{n-1}(n+2)-1)},$$

The theorem is proved.

Remark 3.6.2. If we can map our domain Ω into the ball in such a way that the differential operator under consideration transforms into the Laplace operator, then we can apply Theorem 3.6.1 in order to estimate η_0 from below. Conformal mappings of smooth domains in $\mathbb{R}^2$ can serve as an example for such a situation for a variety of domains Ω. In general the difficulty in applying Theorem 3.5.1 are in the estimation of the resulting v, g and $G(x,y)$.

3.7. Some remarks about the eigenvalue problem

In this section we are interested in the eigenvalue problem associated with (3.1.1) and first consider the self-adjoint problem

$$-\Delta u + \eta\varphi \int_\Omega \varphi u = \lambda u \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{on} \quad \partial\Omega, \quad \eta \geq 0. \tag{3.7.1}$$

The properties of eigenvalues of (3.7.1) and of the nonself-adjoint version of this problem were studied in [FR1], [FEI]. Here we mainly want to look at the behaviour of the first eigenvalue and the first eigenfunction of (3.7.1). A more general case obtained by replacing Δu by $\nabla(a(x)\nabla u)$ can be treated in the same way.

Proposition 3.7.1. The first eigenvalue of (3.7.1), $\lambda_0(\eta)$, satisfies the bounds $0 < \Lambda_1 \leq \lambda_0(\eta) \leq \Lambda_2 < \infty$, with

$$\Lambda_1 = \lambda_0(0), \quad \Lambda_2 = (\alpha^2/(\alpha^2+\beta^2))\lambda_0(0) + (\beta^2/(\alpha^2+\beta^2))\lambda_1(0)$$

where $\lambda_0(0), \lambda_1(0)$ are respectively the first and second eigenvalues of the unperturbed problem, and α, β are chosen in such way that

$$\int_\Omega \varphi[\alpha u_0(0) + \beta u_1(0)] = 0.$$

Here $u_0(0)$ and $u_1(0)$ are the first and the second eigenfunctions of the unperturbed problem, such that $||u_0||_{L^2(\Omega)} = ||u_1||_{L^2(\Omega)} = 1$.

Proof. In order to prove the proposition we use the Courant Min-Max Principle. Specifically

$$\lambda_0(\eta) = \inf_{u\in H_0^1(\Omega)} \left(||\nabla u||_{L^2}^2 + \eta \left(\int_\Omega \varphi u \right)^2 \right) \Big/ ||u||_{L^2}^2$$

$$\geq \inf_{u\in H_0^1(\Omega)} ||\nabla u||_{L^2}^2 \Big/ ||u||_{L^2}^2 = \lambda_0(0),$$

$$\lambda_0(\eta) = \inf_{u\in H_0^1(\Omega)} \left(||\nabla u||_{L^2}^2 + \eta \left(\int_\Omega \varphi u \right)^2 \right) \Big/ ||u||_{L^2}^2 \leq \inf_{\substack{u\in H_0^1(\Omega) \\ \int_\Omega \varphi u = 0}} ||\nabla u||_{L^2}^2 \Big/ ||u||_{L^2}^2.$$

Since $\int_\Omega \varphi[\alpha u_0(0) + \beta u_1(0)] = 0$, and $u_0(0)$ and $u_1(0)$ are orthogonal we obtain

$$\lambda_0(\eta) \leq (\alpha^2 ||\nabla u_0(0)||_{L^2}^2 + \beta^2 ||\nabla u_1(0)||_{L^2}^2)/(\alpha^2 + \beta^2)$$

$$= (\alpha^2/(\alpha^2 + \beta^2))\lambda_0(0) + (\beta^2/(\alpha^2 + \beta^2))\lambda_1(0).$$

The proposition is proved.

Remark 3.7.2. Let $u_0(\eta)$ be the first eigenfuction of $-\nabla(a(x)\nabla u) + \eta\varphi \int_\Omega \varphi u$ with norm equal to 1 in $H_0^1(\Omega)$. Note that $\lambda_0(\eta)$ is monotone increasing in η by the Courant Min-Max Principle and is bounded by Proposition 3.7.1. We want to study the asymptotic behaviour of this family of functions when $\eta \to \infty$. Since all $u_0(\eta)$ are bounded in $H_0^1(\Omega)$ we can find a subsequence which converges to some function $\bar{u}$ strongly in $L^2(\Omega)$ and weakly in $H_0^1(\Omega)$. Taking into consideration that

$$\int_\Omega \varphi u = \lambda \int_\Omega L^{-1}(\varphi) u \Big/ \left(1 + \eta \int_\Omega L^{-1}(\varphi)\varphi\right),$$

where L is the operator defined in Section 3.2, and that $\lambda_0(\eta) \to \lambda_\infty$ when $\eta \to \infty$ we obtain that $\bar{u}$ satisfies the following nonself-adjoint problem

$$-\nabla(a(x)\nabla \bar{u}) + \lambda_\infty \varphi \int_\Omega L^{-1}(\varphi)\bar{u} \Big/ \int_\Omega L^{-1}(\varphi)\varphi = \lambda_\infty \bar{u}.$$

The same is true for other eigenfunctions and eigenvalues. In particular it means that we can determine the limit values of eigenvalues of our self-adjoint problem by considering the intersections of eigencurves of the latter nonself-adjoint problem with the line $\eta = \lambda$.

Consider now the nonself-adjoint problem

$$-\nabla(a(x)\nabla u) + \eta\psi \int_\Omega \varphi u = \lambda u \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{on} \quad \partial\Omega, \quad \eta \geq 0. \tag{3.7.2}$$

We are interested in the positivity of the first eigenfunction of (3.7.2). In general for large η, the first eigenfunction u_0 changes sign as the following example shows.

Example 3.7.3. Consider (3.7.2) with $a(x) \equiv 1$ and $\psi(x) \equiv \varphi(x) \equiv 1$. Assume $u_0(\eta) \geq 0$ and thus observe that the integral of u_0 can not be equal to zero. Since equation (3.7.2) is linear we can put the value of the integral of u_0 equal to one. Then integrating (3.7.2) and using Green's formula we obtain

$$\int_{\partial\Omega} \partial u_0(\eta)/\partial\nu = \lambda_0(\eta) - \eta|\Omega|. \tag{3.7.3}$$

Since $\lambda_0(\eta)$ is bounded above by Proposition 3.7.1, we have that if η is large then $\partial u_0(\eta)/\partial\nu < 0$ at some part of $\partial\Omega$ and consequently that $u_0(\eta)$ changes sign.

Example 3.7.4. In some cases u_0 remains positive for all $\eta > 0$. Indeed, if we choose $\varphi \equiv \psi$ in such way that $\int_\Omega \varphi v_0 = 0$, where $v_0 > 0$ is the first eigenfunction of operator L, then $u_0(x) \equiv v_0(x)$ for all $\eta > 0$.

The next theorem concerns the existence of a positive eigenfunction of (3.7.2) for $\eta < \eta_0$.

Theorem 3.7.5. Let $\varphi, \psi \geq 0$. Then problem (3.7.2) has a positive eigenfunction u_0 for $\eta < \eta_0$. Moreover u_0 is in fact the first eigenfunction of (3.7.2) and the corresponding eigenvalue λ_0 is simple.

Proof. Let us denote by A_η the following operator

$$A_\eta u = -\nabla(a(x)\nabla u) + \eta\psi\int_\Omega \varphi u, \quad u \in H_0^1(\Omega).$$

Then by Proposition 3.2.1 and Theorem 3.3.2 for $\eta < \eta_0$ the operator A_η^{-1} : $L^2(\Omega) \to L^2(\Omega)$ is well defined and positive. Moreover, since $f \in L^2(\Omega)$ and φ, ψ are nonnegative we have

$$||u||^2_{H_0^1} \leq C\Big(\int_\Omega a(x)|\nabla u|^2 + \Big(\int_\Omega \psi u\Big)\Big(\int_\Omega \varphi u\Big)\Big) \leq C||f||_{L^2}||u||_{H_0^1}$$

and so $u \in H_0^1\Omega$. This implies that the operator A_η^{-1} is completely continuous since the inclusion of $H_0^1(\Omega)$ into $L^2(\Omega)$ is compact. Therefore by Theorem 2.4.2 in order to prove the first statement of the theorem, it is enough to show that there exists a nonnegative nontrivial function $f_\eta \in L^2(\Omega)$ such that $A_\eta^{-1} f_\eta \geq$

$\alpha_\eta f_\eta$ for some positive α_η. Let $f_\eta \in C_0^\infty(\Omega)$, $0 \le f_\eta < 1$. Then $A_\eta^{-1} f_\eta > 0$ in Ω and $A_\eta^{-1} f_\eta \ge \alpha_\eta > 0$ on $supp(f_\eta)$ for some α_η. Consequently, $A_\eta^{-1} f_\eta \ge \alpha_\eta f_\eta$ in Ω, and we obtain that there exists a nonnegative eigenfunction u_0 of (3.7.2). Furthermore Theorem 3.3.2 implies that u_0 is positive and the first statement of the theorem is proved.

Let us show now that the positive eigenfunction u_0 corresponds to the first eigenvalue λ_0 and this eigenvalue is simple. Indeed, since we can rewrite (3.7.2) as

$$-\nabla(a(x)\nabla u) + \eta\psi \int_\Omega \varphi u - \lambda u = 0 \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{on} \quad \partial\Omega, \quad \eta \ge 0$$

then Theorem 2.3.1 implies that $u_0 \in C(\bar{\Omega})$ and therefore by Theorem 2.3.5 $u_0 \in C^{1,\alpha}(\bar{\Omega})$. It means that the normal derivative of u_0 is well defined and positive (Corollary 3.3.4). Moreover if we consider the operator $A_\eta^{-1} : C(\bar{\Omega}) \to C^{1,\alpha}(\bar{\Omega})$ we obtain with the help of Theorem 3.3.2 and Corollary 3.3.4 that $u = A_\eta^{-1} f$ is also positive and have positive normal derivative on $\partial\Omega$ for $\eta < \eta_0$. Therefore we can find positive numbers γ and β such that

$$\gamma u_0 \le A_\eta^{-1} f \le \beta u_0$$

and thus the operator A_η^{-1} is u_0-positive for $\eta < \eta_0$. So Theorem 2.4.4 implies that u_0 is the first eigenfunction and the corresponding eigenvalue λ_0 is simple.

We also obtain from the fact that A_η^{-1} is u_0-positive for $\eta < \eta_0$ and as a consequence of Theorem 2.4.5 that for every $u \in C^2(\bar{\Omega})$, $u > 0$ in Ω, $u = 0$ on

$\partial\Omega$

$$\lambda_0(\eta) \geq \inf_{x\in\Omega}\left[\left(-\nabla(a(x)\nabla u) + \eta\psi\int_\Omega \varphi u\right)\Big/ u\right].$$

This is an analogue of the classical Barta's inequality for our problem.

Remark 3.7.6. Equation (3.7.3) provides an upper estimate on η_0 for the particular cases of equation (3.7.2) with $\varphi \equiv \psi \equiv 1$ but for a general domain. We can easily see that $\eta_0 \leq \lambda_1(0)/\int_\Omega \psi$, where $\lambda_1(0)$ is the second eigenvalue of L. If we embed the smaller domain Ω' for which we know the second eigenvalue $\lambda_1'(0)$ of L, into Ω, then the monotonicity properties of eigenvalues [MIKH] imply that $\lambda_1'(0) > \lambda_1(0)$, and therefore $\eta_0 < \lambda_1'(0)/\int_\Omega \psi$.

CHAPTER IV

NONLOCAL THERMISTOR PROBLEMS

4.1. Introduction

In this chapter we consider two nonlocal problems which arise in the modelling of thermal and electrical processes in thermistors. We recall that thermistors are electrical devices whose resistivity depends on temperature. Usually the processes in thermistors are described by the local system (1.2.1), (1,2,2).

In this chapter we shall first study the nonlocal version of problem (1.2.1), (1.2.2) which arise in microsensor devices where heat loss to the surrounding gas is of great importance. In particular we are interested in the case when the thermal conductivity $k(s)$ and the electrical conductivity $\sigma(s)$ tend to zero as s tends to infinity. This behaviour of k and σ is typical of that encountered in thermistor structures that arise in microsensor applications. Sections 4.2 and 4.3 are devoted to the investigation of existence and positivity of steady state solution of the thermistor problem with a nonlocal term appearing in the temperature equation.

In sections 4.4 and 4.5 we discuss system (1.2.1), (1.2.2) with nonlocal boundary conditions and we show that under suitable assumptions on k, σ and the shape of the thermistor blow-up occurs. We recall that Lacey in [LAC] studied one-dimensional thermistor model with constant thermal conductivity and showed the blow-up of solutions under the condition that the electrical conductivity decreases rapidly enough. In our work we impose weaker restrictions on the shape of thermistor and consider non-constant thermal conductivity.

4.2. Nonlocal thermistor problem with heat loss to the surrounding gas

We consider the following nonlinear nonlocal system:

$$-\vec{\nabla} \cdot (\sigma(u)\vec{\nabla}\varphi) = 0 \quad \text{in} \quad \Omega, \quad \varphi = \varphi_0 \text{ on } \partial\Omega \tag{4.2.1}$$

$$-\vec{\nabla} \cdot (k(u)\vec{\nabla}u) + \eta \int_\Omega u = \sigma(u)|\vec{\nabla}\varphi|^2 \quad \text{in} \quad \Omega, \quad u = 0 \text{ on } \partial\Omega. \tag{4.2.2}$$

We suppose here that $\eta \geq 0$.

The existence of positive solutions of (4.2.1),(4.2.2) when $\eta = 0$ was studied under a variety of assumptions (see [XAL], [CIM], [CIP] and the references therein). In the next section we shall prove the existence of positive solutions of (4.2.1), (4.2.2) for small η under the following hypotheses:

(H1) $\Omega \subset \mathbb{R}^n$ is a smooth bounded domain;

(H2) $\varphi_0(x) \in C^{2,\beta}(\bar{\Omega})(\beta > 0)$ and there exists a positive constant φ_M such that $0 \leq \varphi_0(x) \leq \varphi_M$ on $\bar{\Omega}$;

(H3) $\sigma(t)$ and $k(t)$ are positive and smooth for all $t \in \mathbb{R}$, are bounded above, $k(t)/\sigma(t)$ is a decreasing function of t and $\int_0^\infty k(t)/\sigma(t)dt > \varphi_M^2/2$.

Let us emphasize that under these hypotheses problem (4.2.1), (4.2.2) can be degenerate, i.e. $\sigma(s), k(s) \to 0$ as $s \to \infty$.

4.3. Existence of positive solutions

The proof of the existence of solutions of (4.3.1) and (4.3.2) follows the same general scheme, as introduced in [XAL] with the differences due entirely to the presence of the nonlocal term. Thus we will discuss the main steps of the proof concentrating only on these differences, and basing ourselves on the results given earlier in Chapter III.

We rewrite system (4.2.1), (4.2.2) in weak form as

$$\varphi - \varphi_0 \in H_0^1(\Omega), \quad \int_\Omega \sigma(u)\nabla\varphi\nabla\psi = 0 \quad \forall\psi \in H_0^1(\Omega) \tag{4.3.1}$$

$$u \in H_0^1(\Omega),$$

$$\int_\Omega k(u)\nabla u\nabla v + \eta\Big(\int_\Omega u\Big)\Big(\int_\Omega v\Big) = -\int_\Omega \sigma(u)\varphi\nabla\varphi\nabla v \quad \forall v \in H_0^1(\Omega). \tag{4.3.2}$$

We start by proving the L^∞-boundedness of solutions of (4.3.1), (4.3.2).

Lemma 4.3.1. Let (u, φ) be a $C^\alpha(\bar\Omega)$ solution of (4.3.1), (4.3.2). Then

$$0 \le \varphi(x) \le \varphi_M \quad \text{on } \bar\Omega, \quad u(x) \le M \quad \text{on } \bar\Omega, \tag{4.3.3}$$

where M is a positive constant such that $\int_0^M k(t)/\sigma(t)dt = \varphi_M^2/2$. Note that M is independent of u and φ but depends on k and σ.

Proof. Note first that since $u \in C^\alpha(\bar\Omega)$ then u is bounded above and below on $\bar\Omega$ and so are $k(u)$ and $\sigma(u)$. In particular this implies that equation (4.3.1) is uniformly elliptic and by the Weak Maximum Principle $0 \le \varphi(x) \le \varphi_M$ in $\bar\Omega$. Moreover by Theorem 2.2.4 we have $\varphi \in C^1(\bar\Omega)$.

Let ξ satisfy

$$\xi \in H_0^1(\Omega), \quad \int_\Omega k(u)\nabla\xi\nabla\psi = \int_\Omega 1\cdot\psi \quad \forall\psi \in H_0^1(\Omega). \tag{4.3.4}$$

Then the boundedness of $k(u)$ yields that (4.3.4) is uniformly elliptic and once again the Weak Maximum Principle implies that $\xi > 0$ in Ω. Thus since for $\eta \geq 0$ we have

$$\int_\Omega u = \int_\Omega \sigma(u)|\nabla\varphi|^2\xi \Big/ \left(1 + \eta\int_\Omega \xi\right) > 0,$$

and by (4.3.2)

$$\int_\Omega k(u)\nabla u \nabla v \leq -\int_\Omega \sigma(u)\varphi\nabla\varphi\nabla v \quad \forall\ 0 \leq v \in H_0^1(\Omega).$$

Let $\zeta(x) = \varphi^2(x)/2 + \int_0^{u(x)}[k(t)/\sigma(t)]dt$ and $\zeta_0(x) = \varphi_0^2(x)/2$. Then $\zeta(x), \zeta_0(x) \in H^1(\Omega)$ and furthemore $\zeta(x)$ satifies

$$\zeta - \zeta_0 \in H_0^1(\Omega), \quad \int_\Omega \sigma(u)\nabla\zeta\nabla v \leq 0 \quad \forall\ 0 \leq v \in H_0^1(\Omega).$$

Thus the Weak Maximum Principle applied to ζ yields that

$$\int_0^{u(x)} k(t)/\sigma(t)dt \leq \varphi^2(x)/2 + \int_0^{u(x)} k(t)/\sigma(t)dt \leq \varphi_M^2/2$$

and we conclude by (H3) that $u(x) \leq M$ on $\bar{\Omega}$ where M satisfies the conditions of the lemma.

We choose now a smooth monotone increasing function $\chi(t)$ satisfying the following properties: 1) $\chi(t) = t$ for $0 \leq t \leq M$; 2) $-1 \leq \chi(t) \leq M + 1$ for all $t \in \mathbb{R}$. We consider the truncated problem

$$\varphi - \varphi_0 \in H_0^1(\Omega), \quad \int_\Omega \bar{\sigma}(u)\nabla\varphi\nabla\psi = 0 \quad \forall\psi \in H_0^1(\Omega) \tag{4.3.5}$$

$$u \in H_0^1(\Omega),$$

$$\int_\Omega \bar{k}(u)\nabla u\nabla v + \eta\Big(\int_\Omega u\Big)\Big(\int_\Omega v\Big) = -\int_\Omega \bar{\sigma}(u)\varphi\nabla\varphi\nabla v \quad \forall v \in H_0^1(\Omega), \quad (4.3.6)$$

where $\bar{\sigma}(t) = \sigma(\chi(t))$, $\bar{k}(t) = k(\chi(t))$, and note that all $C^\alpha(\bar{\Omega})$ solutions of (4.3.5), (4.3.6) satisfy bounds (4.3.3). Indeed, the bounds for φ follow immediately from the maximum principle. Then repeating the proof of Lemma 4.3.1 we obtain that

$\int_0^{\bar{u}} \bar{k}(t)/\bar{\sigma}(t)dt \le \varphi_M^2/2$, where $\bar{u}$ is the maximum of the solution of (4.3.5), (4.3.6). If $\bar{u} > M$ then from the definition of M we get that $\int_M^{\bar{u}} \bar{k}(t)/\bar{\sigma}(t)dt \le 0$ which is impossible.

In next three lemmas we establish $H^1(\Omega)$ and $L^{2,\mu}(\Omega)$ bounds for the solutions of the truncated system.

Lemma 4.3.2. Let (u, φ) be a $C^\alpha(\bar{\Omega})$ solution of (4.3.5), (4.3.6). Then

$$||\varphi||_{H^1} \le C||\varphi_0||_{H^1}, \qquad ||u||_{H^1} \le C||\varphi_0||_{H^1}.$$

Proof. Note first that there exist two positive constants λ_m and λ_M such that

$$\lambda_m \le \bar{k}(u), \bar{\sigma}(u) \le \lambda_M \quad (4.3.7)$$

for any u solving (4.3.5), (4.3.6). Putting $\psi = \varphi - \varphi_0$ in (4.3.5) we obtain immediately that

$$\int_\Omega \bar{\sigma}(u)\nabla\varphi\nabla(\varphi - \varphi_0) = 0$$

and therefore

$$\int_\Omega \bar{\sigma}(u)|\nabla(\varphi - \varphi_0)|^2 = -\int_\Omega \bar{\sigma}(u)\nabla\varphi_0\nabla(\varphi - \varphi_0).$$

So (4.3.7) and Hölder's inequality imply that

$$||\nabla(\varphi - \varphi_0)||_{L^2} \le C||\nabla\varphi_0||_{L^2} \le C||\varphi_0||_{H^1}.$$

Since $\varphi - \varphi_0 \in H_0^1(\Omega)$ we have that $||\varphi - \varphi_0||_{H^1} \le C||\nabla(\varphi - \varphi_0)||_{L^2}$ and thus

$$||\varphi||_{H^1} \le ||\varphi - \varphi_0||_{H^1} + ||\varphi_0||_{H^1} \le C||\varphi_0||_{H^1}.$$

Finally since $u \in H_0^1(\Omega)$ putting $v = u$ in (4.3.6) yields

$$\int_\Omega \bar{k}(u)|\nabla u|^2 \le \int_\Omega \bar{k}(u)|\nabla u|^2 + \eta\Big(\int_\Omega u\Big)^2 = -\int_\Omega \bar{\sigma}(u)\varphi\nabla\varphi\nabla u$$

and since $\varphi \in L^\infty(\Omega)$ the same argument as before implies the second statement of the lemma.

To establish $L^{2,\mu}$-estimates we need the following lemma the proof of which can be found in [St, Theorem 8.5].

Lemma 4.3.3. Let $G(x,y)$ and $\bar{G}(x,y)$ be Green's functions of uniformly elliptic operators L and $\bar{L}$ respectively. Then for every compact subset $\Omega' \subset \Omega$ there exist positive constants C_1, C_2 such that

$$C_1 \le \frac{G(x,y)}{\bar{G}(x,y)} \le C_2 \quad \text{for all} \quad x, y \in \Omega'.$$

Here C_1, C_2 depend on Ω, Ω' and on bounds on coefficients of L and $\bar{L}$.

Lemma 4.3.4. Let (u,φ) be a $C^\alpha(\bar{\Omega})$ solution of (4.3.5), (4.3.6). Then

$$||\nabla\varphi||_{L^{2,\mu}}, ||\nabla u||_{L^{2,\mu}} \le C(||\varphi_0||_{H^1} + ||\nabla\varphi_0||_{L^{2,\mu}})$$

where C depends only on $\lambda_m, \lambda_M, M, \varphi_M$ and Ω and $0 < \mu < \mu_0 = N - 2 + 2\delta_0$, $0 < \delta_0 < 1$ depends only on λ_m and λ_M.

Proof. The proof of this lemma is based on Campanato Space theory, [Chapter II], and follows basically the proof of [Theorem 2.3, XAL]. We rewrite (4.3.5) in the following form

$$\Phi \in H_1^0(\Omega) \quad \int_\Omega \bar{\sigma}(u)\nabla\Phi\nabla\psi = -\int_\Omega \bar{\sigma}(u)\nabla\varphi_0\nabla\psi \quad \forall\psi \in H_1^0(\Omega) \tag{4.3.8}$$

where $\varphi(x) = \Phi(x) + \varphi_0(x)$ for $x \in \bar{\Omega}$. Thus by Theorem 2.3.3

$$||\nabla\Phi||_{L^{2,\mu}} \leq C(||\bar{\sigma}(u)\nabla\varphi_0||_{L^{2,\mu}} + ||\Phi||_{H^1}) \tag{4.3.9}$$

where $0 < \mu < \mu_0 = N - 2 + 2\delta_0$ and $0 < \delta_0 < 1$ depends only on λ_m and λ_M. Furthermore since $\int_\Omega \bar{\sigma}(u)|\nabla\varphi|^2 v = -\int_\Omega \bar{\sigma}(u)\varphi\nabla\varphi\nabla v$ for all $v \in H_1^0(\Omega)$ then applying Theorem 2.3.3 to (4.3.6) we obtain

$$||\nabla u||_{L^{2,\mu}} \leq C(||\eta\int_\Omega u||_{L^{2,(\mu-2)+}} + ||\bar{\sigma}(u)\varphi\nabla\varphi||_{L^{2,\mu}} + ||u||_{H^1}). \tag{4.3.10}$$

Since by Theorem 2.3.1 $L^\infty(\Omega)$ is a space of multipliers for $L^{2,\mu}(\Omega)$ for $0 \leq \mu \leq N$ we have

$$||\bar{\sigma}(u)\nabla\varphi_0||_{L^{2,\mu}} \leq ||\bar{\sigma}(u)||_{L^\infty}||\nabla\varphi_0||_{L^{2,\mu}} \leq \lambda_M||\nabla\varphi_0||_{L^{2,\mu}} \tag{4.3.11}$$

$$||\bar{\sigma}(u)\varphi\nabla\varphi||_{L^{2,\mu}} \leq ||\bar{\sigma}(u)||_{L^\infty}||\varphi||_{L^\infty}||\nabla\varphi||_{L^{2,\mu}} \leq \lambda_M\varphi_M||\nabla\varphi||_{L^{2,\mu}}. \tag{4.3.12}$$

To estimate the term $\eta\int_\Omega u$ from above independently of η and u we replace $\int_\Omega u$ by its expression

$$\int_\Omega u = \int_\Omega \bar{\sigma}(u)|\nabla\varphi|^2\xi \Big/ \left(1 + \eta\int_\Omega \xi\right),$$

where ξ satisfies

$$\xi \in H_0^1(\Omega), \quad \int_\Omega \bar{k}(u)\nabla\xi\nabla\psi = \int_\Omega \psi \quad \forall\psi \in H_0^1(\Omega) \tag{4.3.13}$$

and conclude that

$$\eta\int_\Omega u \leq \int_\Omega \bar{\sigma}(u)|\nabla\varphi|^2\xi \Big/ \int_\Omega \xi.$$

The numerator of the last expression can be bounded using the same technique as before:

$$||\int_\Omega \bar{\sigma}(u)|\nabla\varphi|^2\xi||_{L^{2,(\mu-2)+}} = ||\int_\Omega \bar{\sigma}(u)\varphi\nabla\varphi\nabla\xi||_{L^{2,(\mu-2)+}}$$

$$\leq C\lambda_M\varphi_M||\nabla\varphi||_{L^2}||\nabla\xi||_{L^2} \leq C\frac{\lambda_M}{\lambda_m}\varphi_M||\varphi||_{H^1}.$$

We claim, that $\int_\Omega \xi \geq C_1 > 0$, where C_1 is a constant independent of u. Indeed, let $\bar{G}(x,y)$ and $G(x,y)$ be the Green's functions of equation (4.3.13) and the Laplace operator with homogeneous Dirichlet boundary conditions respectively, and let $\Omega' \subset \Omega$ be a fixed compact set. Then, since $\bar{k}(u)$ is bounded above and below independently of u, Lemma 4.3.3 implies that $\bar{G}(x,y)/G(x,y) \geq C_2 > 0$ for all $x, y \in \Omega'$, where C_2 depends on Ω, Ω' and on the L^∞ bounds on $\bar{k}$. Thus, since ξ is nonnegative,

$$\int_\Omega \xi(x)dx \geq \int_{\Omega'} \xi(x)dx \geq C_2\int_{\Omega'}\int_{\Omega'} G(x,y)dxdy$$

and

$$\eta\int_\Omega u \leq C||\varphi||_{H^1}. \tag{4.3.14}$$

Finally combining Lemma 4.3.2 and estimates (4.3.11), (4.3.12), (4.3.14) we obtain from (4.3.9) and (4.3.10) the statements of Lemma 4.3.4.

Lemma 4.3.5. There exist positive constants K, δ_0 such that for any $C^\alpha(\bar{\Omega})$ solution (u, φ) of (4.3.5), (4.3.6) we have

$$||u||_{C^\gamma} \leq K \quad \text{and} \quad ||\varphi||_{C^\gamma} \leq K$$

for all $0 < \gamma < \delta_0$.

This result follows immediately from Theorems 2.3.1, 2.3.2 and Lemma 4.3.4.

Theorem 4.3.6. The problem (4.3.5), (4.3.6) has at least one solution for every $\eta \geq 0$. Moreover there exists $\eta_1 > 0$ independent of u and φ such that for all $\eta \leq \eta_1$ there exists a positive solution of (4.3.5), (4.3.6) and consequently of the initial problem (4.2.1), (4.2.2).

Remark 4.3.7. Computer simulations show that for η large the solution of (4.2.1), (4.2.2) may become negative at some points.

Proof. The proof of existence is based on the application of the Schauder Fixed Point Theorem and follows exactly the proof of [Theorem 3.1, AX]. Therefore there exists at least one $C^\gamma(\bar{\Omega})$ solution of (4.3.5), (4.3.6) satisfying $||u||_{C^\gamma(\bar{\Omega})} \leq K$, $||\varphi||_{C^\gamma(\bar{\Omega})} \leq K$ for $0 < \gamma < \delta_0 < 1$, where δ_0, K are positive constants depending on the problem data, but not on u and φ. Moreover applying consequently Theorems 2.2.5, 2.2.6 and 2.2.7 along the same lines as in Remark 4.3.3 we obtain that (u, φ) is actually a classical solution of (4.3.5), (4.3.6) and $||u||_{C^{2,\alpha_0}(\bar{\Omega})} \leq K_1$, $||\varphi||_{C^{2,\alpha_0}(\bar{\Omega})} \leq K_2$ where $0 < \alpha_0 < 1$ and K_1, K_2 are positive constants independent of u, φ. Therefore $\bar{k}(u) \geq D$ and $||\bar{k}(u)||_{C^{1,\alpha_0}(\bar{\Omega})} \leq C$ for all u solutions of (4.3.5), (4.3.6). Here D depends on the upper bound M of u

obtained in Lemma 4.3.1 and both D and C are independent of the partucular u and φ. Theorem 3.4.4 yields the existence of η_1 independent of u, φ such that for all $\eta \leq \eta_1$ there exists a positive solution of (4.3.5), (4.3.6) and, consequently, of (4.2.1), (4.2.2). Thus the theorem is proved.

4.4. Nonlocal thermistor problem with a given current

The next two sections are devoted to a thermistor problem with nonlocal boundary conditions corresponding to the case of a current source. We thus consider the following system of nonlinear equations in the cylinder $Q = \Omega \times \{t > 0\}$ where Ω is a smooth bounded domain that may be considered to be in $\mathbb{R}^3 = \{(x, y, z)\}$ without loss of generality:

$$u_t - \nabla(k(u)\nabla u) = \sigma(u)|\nabla\varphi|^2 \quad \text{in} \quad Q \tag{4.4.1}$$

$$-\nabla(\sigma(u)\nabla\varphi) = 0 \quad \text{in} \quad Q \tag{4.4.2}$$

and we associate with (4.4.1) and (4.4.2) the following boundary and initial conditions: we decompose the boundary of Ω into two ways: $\partial\Omega = S_0 \cup S_1 \cup S_2$ and $\partial\Omega = \Gamma_D \cup \Gamma_N$ and we impose the following mixed boundary conditions

$$\varphi = 0 \quad \text{on} \quad S_0 \times \{t > 0\}, \quad \varphi = \lambda(t) \quad \text{on} \quad S_1 \times \{t > 0\}, \quad \frac{\partial\varphi}{\partial n} = 0 \quad \text{on} \quad S_2 \times \{t > 0\} \tag{4.4.3}$$

$$u = 0 \quad \text{on} \quad \Gamma_D \times \{t > 0\}, \quad \frac{\partial u}{\partial n} = 0 \quad \text{on} \quad \Gamma_N \times \{t > 0\} \tag{4.4.4}$$

and initial conditions

$$u(\cdot, 0) = u_0. \tag{4.4.5}$$

Finally we suppose that the device is driven by a current source and the total current through the part of the boundary S_1 is represented by a nonlocal condition

$$I(t) = \int_{S_1} \sigma(u) \frac{\partial \varphi}{\partial n}. \tag{4.4.6}$$

Here $I(t), \lambda(t)$ are positive smooth functions, I is given and λ is to be determined.

Setting $\varphi = \lambda \psi$ we rewite problem (4.4.1)-(4.4.6) in more convenient form:

$$\left.\begin{array}{c} u_t - \nabla(k(u)\nabla u) = \lambda(t)^2 \sigma(u) |\nabla \psi|^2 \ \text{ in } \ Q \\ -\nabla(\sigma(u)\nabla\psi) = 0 \ \text{ in } \ Q \\ \psi = 0 \text{ on } S_0 \times \{t > 0\}, \psi = 1 \text{ on } S_1 \times \{t > 0\}, \frac{\partial \psi}{\partial n} = 0 \text{ on } S_2 \times \{t > 0\} \\ u = 0 \ \text{ on } \ \Gamma_D \times \{t > 0\}, \ \ \frac{\partial u}{\partial n} = 0 \ \text{ on } \ \Gamma_N \times \{t > 0\} \\ u(\cdot, 0) = u_0 \\ I(t) = \lambda(t) \int_{S_1} \sigma(u) \frac{\partial \psi}{\partial n} \end{array}\right\} (P)$$

Note that if $I(t) \not\equiv 0$ then $\lambda(t) \not\equiv 0$ and solutions (u, ψ, λ) of (P) generate solutions (u, φ, λ) of the original problem (4.4.1)-(4.4.6). The converse is also true.

Definition 4.4.1. The triplet (u, ψ, λ) is called an *almost classical* solution of (P) if u, ψ, λ exist for $t > T$ for some positive constant T and u, ψ are classical everywhere except at $\overline{S}_0 \cap \overline{S}_2$, $\overline{S}_1 \cap \overline{S}_2$, $\overline{\Gamma}_D \cap \overline{\Gamma}_N$ for $t > 0$.

We impose the following conditions on u_0, k, σ, I and the boundary of Ω:

(C1) u_0 is non-negative and smooth in Ω;

(C2) $\sigma(s), k(s) \in C^1(\mathbb{R}^+) \cap L^\infty(\mathbb{R}^+)$, where $\mathbb{R}^+ = \{t \geq 0\}$ and $\sigma(s), k(s) > 0$ for $s > 0$;

(C3) there exist positive constants I_0 and I_1 such that $\int_0^t I(s)^2 ds \geq tI_0^2 - I_1$;

(C4) (1) $S_0, S_1, S_2, \Gamma_D, \Gamma_N$ are smooth; (2) Γ_D, S_0, S_1 are nonempty closed subsets of $\partial\Omega$.

Note that (C3) holds if for example $I(t) \equiv constant$ or $I(t)$ is a periodic function with period T. Moreover we suppose that the following essential hypothesis holds:

(C5) There exist two planes, which we may choose without the loss of generality to be $z = a$ and $z = b$ for some $0 < a < b$, such that $\Omega \cap \{(x, y, z) | a \leq z \leq b\} = A \times [a, b]$ where A denotes a smooth domain in $\mathbb{R}^2$. Furthermore: $S_0 \subset \bar{\Omega} \cap \{z < a\}, S_1 \subset \bar{\Omega} \cap \{z > b\}$ and $\frac{\partial u}{\partial n} = 0$ on $\partial\Omega \cap \{a \leq z \leq b\}$.

The last hypothesis means that we require that our thermistor have a small cylindrical part but we do not impose any restrictions on the rest of the thermistor. In electrical engineering, this situation corresponds to the so-called narrowing process, i.e. the case when, due to possible process malfunction, the thermistor has a part which is much narrower than the rest of the device. Note that the model considered in [LAC] requires that the whole thermistor have cylindrical shape.

In was shown in [AX1] that if $\int^\infty k(s)/\sigma(s) ds < \infty$ then the problem (P) under hypothesis (C2), (C4) and (C5) has no steady state solutions for large value of constant current I. We show in the next section using methods developed in [AX1] and [ACH] that for the problem (P) there exists a current value, determined by the narrow region, such that blow-up occurs if the input exceeds this value under the assumption of the convergence of $\int^{\infty} k(s)ds$ or $\int^{\infty} \sigma(s)ds$.

4.5. Nonexistence of solutions for problem (P)

Let first $\sigma(s)$ satisfy the following condition:

(R1) $\int^{\infty} \sigma(s)ds < \infty, \quad \sigma'(s) \leq 0, \quad \forall s \geq 0.$

In particular (R1) implies that $\sigma(s) \to 0$ as $s \to \infty$. Then we have

Theorem 4.5.1. Let hypotheses (C1)-(C5) and (R1) hold. There exists a value $\tilde{I} > 0$ such that if $I_0 > \tilde{I}$ then all almost classical solutions (u, ψ, λ) of problem (P) will blow up in finite time.

Proof. Suppose that we can find an almost classical solution (u, ψ, λ) of problem (P) which exists for all $t > 0$. Then observe that (u, ψ) is classical in $Q_A = \{x \in A \times [a, b], t > 0\}$. The second equation of (P) and corresponding boundary conditions imply that for any $a < z < b$

$$I(t) = \lambda(t) \int_{S_1} \sigma(u) \frac{\partial \psi}{\partial n} = -\lambda(t) \int_{A(z)} \sigma(u) \frac{\partial \psi}{\partial n}, \tag{4.5.1}$$

where $A(z) = \{(x, y, z) | (x, y) \in A\}$. Using Hölder's inequality we conclude that

$$I(t)^2 \leq \lambda(t)^2 \mu(A) \int_{A(z)} \sigma^2(u) |\nabla \psi|^2, \tag{4.5.2}$$

where $\mu(A)$ denotes the usual Lebesgue measure of the domain A. Next we introduce the function

$$Y(z, t) = \int_{A(z)} \int_{u(x,y,z,t)}^{\infty} \sigma(s) ds dx dy$$

where $z \in (a, b)$. Condition (R1) and the positivity of u imply that $Y(z, t)$ is well defined and nonnegative. Taking the derivative of $Y(z, t)$ with respect to t

and using the first equation of (P) we obtain

$$Y'(z,t) = -\int_{A(z)} \sigma(u)u_t dxdy$$

$$= -\int_{A(z)} [\sigma(u)\nabla(k(u)\nabla u) + \lambda(t)^2\sigma^2(u)|\nabla\psi|^2]dxdy.$$

Since

$$\sigma(u)\nabla(k(u)\nabla u) = \nabla(\sigma(u)k(u)\nabla u) - \sigma'(u)k(u)|\nabla u|^2$$

the assumption that $\sigma'(u) < 0$ and (4.5.2) yield that

$$Y'(z,t) = -\int_{A(z)} \nabla(\sigma(u)k(u)\nabla u)dxdy$$

$$+\int_{A(z)} \sigma'(u)k(u)|\nabla u|^2 dxdy - \int_{A(z)} \lambda(t)^2\sigma^2(u)|\nabla\psi|^2 dxdy$$

$$\leq -\int_{A(z)} \nabla(\sigma(u)k(u)\nabla u)dxdy - \frac{I(t)^2}{\mu(A)}. \tag{4.5.3}$$

To deal with the integral of the right hand side of the last inequality we apply the transform $w = \int_0^u k(s)\sigma(s)ds$ and get

$$-\int_{A(z)} \nabla(\sigma(u)k(u)\nabla u)dxdy = -\int_{A(z)} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) w dxdy.$$

Observe now that since $\frac{\partial w}{\partial n} = 0$ on $\{\partial A \times [a,b], t > 0\}$ and the normal to the cylinder $A \times [a,b]$ has no z-component then

$$\int_{A(z)} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) w dxdy = 0 \tag{4.5.4}$$

and therefore

$$-\int_{A(z)} \nabla(\sigma(u)k(u)\nabla u)dxdy = -\frac{\partial^2}{\partial z^2}\left[\int_{A(z)} w dxdy\right].$$

Obviously $\xi(z,t) = \int_{A(z)} w dx dy$ is positive for $a < z < b$ and $t > 0$ and is sufficiently smooth with respect to z. Thus an application of Barta's inequality yields that

$$\lambda_1 \geq \inf_{a \leq z \leq b} \left[\frac{-\frac{\partial^2}{\partial z^2}(\xi(z,t))}{\xi(z,t)} \right]$$

where λ_1 denotes the first eigenvalue of the homogeneous Dirichlet problem $-\frac{\partial^2}{\partial z^2}$ on (a,b), i.e. $\lambda_1 = \frac{\pi^2}{(b-a)^2}$. So there exists $z_0 \in (a,b)$ such that

$$-\frac{\partial^2}{\partial z^2} \left[\int_{A(z_0)} w dx dy \right] \leq \lambda_1 \int_{A(z_0)} w dx dy. \tag{4.5.5}$$

Substituting (4.5.5) into (4.5.3) with $z = z_0$ we conclude that

$$Y'(z_0,t) \leq \lambda_1 \int_{A(z_0)} w dx dy - \frac{I(t)^2}{\mu(A)} \leq \lambda_1 \mu(A) ||k||_{L^\infty} \int_0^\infty \sigma(s) ds - \frac{I(t)^2}{\mu(A)}.$$

Now integrating the last expression with respect to t yields

$$0 \leq Y(z_0,t) \leq Y(z_0,0) - \frac{1}{\mu(A)} \int_0^t I(s)^2 ds + t\lambda_1 \mu(A) ||k||_\infty \int_0^\infty \sigma(s) ds,$$

where

$$Y(z_0,0) = \int_{A(z_0)} \int_{u_0(z_0,x,y)}^\infty \sigma(s) ds dx dy.$$

Using (C3) we obtain that

$$0 \leq Y(z_0,t) \leq Y(z_0,0) + \frac{I_1}{\mu(A)} - t\left(\frac{I_0^2}{\mu(A)} - \lambda_1 \mu(A) ||k||_\infty \int_0^\infty \sigma(s) ds \right),$$

and we conclude finally that a necessary condition for the solution (u, ψ, λ) to exist globally is

$$I_0^2 \leq \frac{\pi^2}{(b-a)^2} (\mu(A))^2 ||k||_\infty \int_0^\infty \sigma(s) ds := \tilde{I}^2.$$

The theorem is proved.

Note that the value of $\tilde{I}$ does not depend on a particular u_0.

We now replace condition (R1) with the following

(R2) $\int_0^\infty k(s)ds < \infty$.

Then we have

Theorem 4.5.2. Let hypotheses (C1)-(C5) and (R2) hold. Then there exists a value $I^* > 0$ such that if $I_0 > I^*$ and (u, ψ, λ) is an almost classical solution of (P) which exists for all $t > 0$ then u is unbounded in $\Omega \times \{t > 0\}$.

Proof. Observe first that (4.5.1) which is still valid under the conditions of Theorem 4.5.2, and Hölder's inequality imply that

$$I(t)^2 \leq \lambda(t)^2 \int_{A(z)} \sigma(u) \int_{A(z)} \sigma(u)|\nabla\psi|^2. \tag{4.5.6}$$

Setting now $w = \int_0^u k(s)ds$ we conclude that in $A \times [a, b]$ functions u and w satisfy

$$u_t - \Delta w = \lambda(t)^2 \sigma(u)|\nabla\psi|^2 \tag{4.5.7}$$

with $\frac{\partial w}{\partial n} = 0$ on $\partial A \times [a, b]$. Next for any $z \in (a, b)$ we integrate (4.5.7) over $A(z)$ and using the same observation (4.5.4) as in Theorem 4.5.1 we obtain

$$\frac{\partial}{\partial t}\left(\int_{A(z)} udxdy\right) - \frac{\partial^2}{\partial^2 z}\left(\int_{A(z)} wdxdy\right) = \lambda(t)^2 \int_{A(z)} \sigma(u)|\nabla\psi|^2.$$

Therefore, (4.5.6) and Barta's inequality yield that for some $z_0 \in (a, b)$

$$\frac{I(t)^2}{||\sigma||_{L^\infty}\mu(A)} \leq \frac{\partial}{\partial t}\left(\int_{A(z_0)} udxdy\right) + \lambda_1 \sup_{z \in (a,b)}\left(\int_{A(z_0)} wdxdy\right)$$

$$\leq \frac{\partial}{\partial t}\Big(\int_{A(z_0)} u dx dy\Big) + \lambda_1 \mu(A)\Big(\int_0^\infty k(s) ds\Big),$$

where as before $\lambda_1 = \frac{\pi^2}{(b-a)^2}$. Thus

$$\frac{\partial}{\partial t}\Big(\int_{A(z_0)} u dx dy\Big) \geq \frac{I(t)^2}{||\sigma||_{L^\infty}\mu(A)} - \lambda_1 \mu(A)\Big(\int_0^\infty k(s) ds\Big)$$

and integrating the last inequality with respect to t implies that

$$\int_{A(z_0)} u dx dy \geq \int_{A(z_0)} u_0 dx dy + \frac{1}{||\sigma||_{L^\infty}\mu(A)}\Big(\int_0^t I(s)^2 ds\Big) - t\lambda_1 \mu(A)\int_0^\infty k(s) ds$$

$$\geq \int_{A(z_0)} u_0 dx dy - \frac{I_1}{||\sigma||_{L^\infty}\mu(A)} + t\Big(\frac{I_0^2}{||\sigma||_{L^\infty}\mu(A)} - \lambda_1 \mu(A)\Big(\int_0^\infty k(s) ds\Big)\Big).$$

Therefore if

$$I_0^2 \geq \lambda_1 ||\sigma||_{L^\infty}[\mu(A)]^2\Big(\int_0^\infty k(s) ds\Big) := (I^*)^2$$

then the solution u is unbounded and the theorem is proved.

Observe that the critical value I^* coincides exactly with the critical value of the constant current obtained in [AX1]. This is characterized by the following property: if the current I is greater than I^*, then there are no stationary solutions of problem (P) and correspondently (4.4.1)-(4.4.6).

CHAPTER V

SEMILINEAR NONLOCAL PROBLEMS

5.1. Introduction

In this chapter we deal with semilinear partial differential equations involving a nonlocal term of the type:

$$u_t = \Delta u + u \cdot \chi(x, u, \phi(u)), \quad t > 0, \quad x \in \Omega \tag{5.1.1}$$

which has been proposed by Furter and Grinfeld [FUG] to describe population dynamics processes. Here Ω is a bounded smooth domain in $\mathbb{R}^n$ and $\phi(u)$ is a continuous functional representing a nonlocal term. We assume explicitly $u = 0$ on $\partial\Omega$ and $n \geq 3$. We prove the existence of positive stationary solutions and show that such a solution is unique for sublinear case. Stability criteria are then obtained for some cases of equation (5.1.1).

Our main tools are Leray-Schauder Degree Theory, upper-lower solution procedures and the Maximum Principle for the proof of existence and Picone's identity for the proof of uniqueness. We emphasize that upper-lower solution methods work only for some special cases and can not be applied in general. This is related to the form of the term $\chi(x, u, \phi(u))$ and especially to the lack of monotonicity. We recall in this regard that it was shown by Fukagai, Kusano and Yoshida, [FKY], that the general upper-lower solution procedure fails for superlinear local equations since upper and lower solutions then turn out to be actual solutions, although it is well known that this method works well for sublinear local problems. However in the nonlocal case the upper-lower solution procedure may fail even for sublinear problems as the following elementary equation shows. Consider

$$-\Delta u + \eta \int_\Omega u = h(x) \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{on} \quad \partial\Omega, \tag{5.1.2}$$

with nontrivial $0 \leq h(x) \in L^\infty(\Omega)$. Then the results from Chapter III show that for some $h(x)$ and large $\eta > 0$ the unique solution u must actually be negative somewhere in Ω. Yet equation (5.1.2) admits $\bar{u} = K$ and $\underline{u} = 0$ as an upper-lower solution pair, with K a large positive constant.

The chapter is structured as follows: we first prove the existence and uniqueness theorems for the sublinear case and then pass to the superlinear problem. We then conclude with examples which include equation (5.1.1) and applications to biological models. We discuss their stability and also briefly address the linear case.

5.2. A sublinear problem

We consider the following nonlocal nonlinear problem in a smooth bounded domain $\Omega \subset \mathbb{R}^n$, $n \geq 3$,

$$-\Delta u + g(x,u)\phi(u) = f(x,u) \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{on} \quad \partial\Omega \tag{5.2.1}$$

We are interested in positive solutions of (5.2.1). Here $f(x,u)$ and $g(x,u)$ denote functions with nontrivial dependence on u and $\phi(u)$ is a continuous functional from $H_0^1(\Omega)$ to $\mathbb{R}$, which maps bounded sets to bounded sets. We suppose here that $f(x,t)$ is sublinear with respect to t and reasonably smooth, that is:

(F1) $\lim_{t \to \infty} \frac{f(x,t)}{t} = 0$ uniformly for $x \in \Omega$;

(F2) $f(x,t)$ is locally Lipschitz continuous with respect to t on $\mathbb{R}^+$ uniformly for $x \in \Omega$, where $\mathbb{R}^+ = \{t \in \mathbb{R} \mid t \geq 0\}$ and belongs to $C^{0,\alpha}(\bar{\Omega})$, $0 < \alpha < 1$, uniformly for $t \in K$ for any compact $K \subset \mathbb{R}^+$;

(F3) $f(x,t) \geq 0$ for $t > 0$, $f(x,t) = f(x,0)$ for $t < 0$, $x \in \Omega$ and there exists $t_0 > 0$ such that $f(x,t) > 0$ for $0 < t < t_0$.

Note that condition (F1) includes the case when $f(x,t)$ vanishes for large t.

The functional ϕ represents the nonlocal term in problem (5.2.1) and we require that $\phi(u)$ satisfy

(Φ1) $\phi(0) = 0$;

(Φ2) $\phi(u) \geq \phi(v)$ for $u, v \in H_0^1(\Omega)$, $u \geq v$ in Ω with equality holding only if $u = v$.

We suppose that $g(x,t)$ is also locally Lipschitz continuous with respect to t uniformly for $x \in \Omega$, belongs to $C^{0,\alpha}(\bar{\Omega})$, $0 < \alpha < 1$, uniformly for $t \in K$ for any compact $K \subset \mathbb{R}^+$ and satisfies

(G1) $g(x,t) > 0$ for $t > 0, x \in \Omega$, $g(x,t) = g(x,0) = 0$ for $t \leq 0, x \in \Omega$ and $g(x,t)/t$ is Lipschitz continuous at $t = 0$ with respect to t uniformly for $x \in \Omega$;

(G2) there exists $0 < \beta$ such that $\lim_{t \to \infty} \frac{g(x,t)}{t^\beta} = 0$ uniformly for $x \in \Omega$.

Observe that since we seek positive solutions, the behaviour of $g(x,t)$ and $f(x,t)$ for $t < 0$ is irrelevant. If we are interested only in nonnegative nontrivial solutions of (5.2.1) then the condition on the smoothness of $g(x,t)/t$ at $t = 0$ can be dropped. We assume finally that the positive solutions of

$$-\Delta u + \phi(u)g(x,u) = f(x,u) + tw \quad \text{in } \Omega, \quad u = 0 \quad \text{on} \quad \partial\Omega$$

with $t \geq 0$ and $0 \leq w \in C_0^\infty(\bar{\Omega})$, are separated from 0. As examples of explicit conditions which suffice for this to hold, we state:

(F4) For $0 \leq u \in H_0^1(\Omega) \cap L^\infty(\Omega)$

$$\inf_{x\in\Omega} \frac{f(x,u)}{u} \geq \frac{K}{||u||_{L^\infty}^s}$$

where $K > 0$ and $0 < s \leq 1$ and both K and s are independent of the particular u;

(H1) there exist positive constants M and $l \geq 0$ such that for any $0 < u \in H_0^1(\Omega)$

$$\left|\left|\frac{\phi(u)g(x,u)}{u}\right|\right|_{L^\infty} \leq M||u||_{L^\infty}^l.$$

That these conditions are indeed sufficient will be shown in the sequel. In addition we remark that most of the conditions on f, ϕ and g were motivated by the examples that follow, and were chosen to minimize technicalities in the presentation. The given proofs almost invariably hold in more general situations with no changes.

5.3. Existence of solutions for sublinear case

First we prove the following lemma.

Lemma 5.3.1. Let $0 < u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ solve

$$-\Delta u + \lambda g(x,u) = f(x,u), \quad \lambda \geq 0 \tag{5.2.1}$$

under conditions (G1), (G2), (F1)-(F3). Then $||u||_{L^\infty} \le C$ with C independent of u.

Proof. Since $u \in H_0^1(\Omega)$ is positive then $g(x,u)$ is also positive in Ω and we have that $-\Delta u \le f(x,u)$. Condition (F1) implies that for any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that $f(x,u) \le \varepsilon u + C_\varepsilon$. We thus have $-\Delta u - \varepsilon u \le C_\varepsilon$ and applying Theorem 2.3.1 we conclude

$$||u||_{L^\infty} \le C_1(||u||_{L^2} + 1) \le C$$

for some constants C_1, C independent of u.

Other proofs may also be easily given using bootstrapping arguments and results in, e.g., [ADN].

We next establish the existence of positive solutions.

Theorem 5.3.2. Problem (5.2.1) has a positive solution $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ under assumptions (F1)-(F4), (Φ1), (Φ2), (G1), (G2) and (H1).

Proof. We consider a modified problem

$$-\Delta u + \lambda^+ g(x,u) = f(x,u) \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{on} \quad \partial\Omega \tag{5.3.2}$$

$$\lambda = \phi(u) \tag{5.3.3}$$

where λ^+ denotes as usual the positive part of λ. Multiplying first (5.3.2) by the negative part u^- of any nontrivial solution u we obtain using (G1)

$$-\int_\Omega |\nabla u^-|^2 = \int_\Omega f(x,u)u^- \ge 0$$

and thus all solutions of (5.3.2) in $H_0^1(\Omega) \cap L^\infty(\Omega)$ are nonnegative and $\lambda = \lambda^+$. Since moreover $g(x,t)/t$ is Lipschitz continuous at $t = 0$ with respect to t uniformly for $x \in \Omega$, we rewrite (5.3.2) as

$$-\Delta u + \lambda\left(\frac{g(x,u)}{u}\right)u = f(x,u) \geq 0$$

and by Theorem 2.3.5 $u \in C^{1,\alpha}(\bar{\Omega})$ and therefore the Strong Maximum Principle implies that u is positive.

We define the operator $T(\lambda, u)$ in the following way

$$T(\lambda,u) = \{\frac{1}{2}(\lambda^+ + \phi(u)), -\lambda^+(-\Delta)^{-1}g(x,u) + (-\Delta)^{-1}f(x,u)\}$$

where $(-\Delta)^{-1}$ denotes the inverse of the Laplace operator with homogeneous Dirichlet boundary conditions. It is easy to see that fixed points of T yield solutions of (5.2.1). We define also the space

$$H = \mathbb{R} \times (H_0^1(\Omega) \cap L^\infty(\Omega))$$

with norm

$$||(\lambda,u)||_H = (\lambda^2 + ||u||^2_{H_0^1})^{1/2} + ||u||_{L^\infty}.$$

We show first that T acts from H to H. Let $T(\lambda,u) = (\mu, v)$. Then it is evident that the μ belongs to $\mathbb{R}$. Next, since $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ then by Theorem 2.2.9

$$||v||_{H_0^1} \leq C(||g(x,u)||_{L^2} + ||f(x,u)||_{L^2})$$

and (F1) and (G2) imply that

$$||v||_{H_0^1} \leq C(||u^\beta||_{L^2} + ||u||_{L^2} + 1) \leq C(||u||^\beta_{L^\infty} + ||u||_{L^2} + 1) < \infty.$$

Finally by [ADN], [GIT]

$$||v||_{L^\infty} \leq C(||g(x,u)||_{L^p} + ||f(x,u)||_{L^p}) + ||u||_{H_0^1}$$

for large p. Thus since $u \in L^\infty(\Omega)$ we conclude that v belongs to $H_0^1(\Omega) \cap L^\infty(\Omega)$.

To show that T is continuous consider a sequence $\{(\lambda_n, u_n)\}$ from H such that $(\lambda_n, u_n) \to (\lambda, u)$ in the norm of H as $n \to \infty$. If we put $T(\lambda_n, u_n) = (\mu_n, v_n)$ and $T(\lambda, u) = (\mu, v)$ then the continuity of ϕ yields that

$$|\mu_n - \mu| \leq \frac{1}{2}|\lambda_n - \lambda| + |\phi(u_n) - \phi(u)| \to 0$$

as $n \to \infty$. Moreover

$$\begin{aligned} &\int_\Omega |\nabla(v_n - v)|^2 \\ &\leq \int_\Omega |\lambda_n| \, |g(x,u_n) - g(x,u)| \, |v_n - v| + |\lambda_n - \lambda| \int_\Omega |g(x,u)| \, |v_n - v| \\ &\quad + \int_\Omega |f(x,u_n) - f(x,u)| \, |v_n - v| \\ &\leq C\Big(\int_\Omega |u_n - u| \, |v_n - v| + |\lambda_n - \lambda| \int_\Omega |v_n - v| + \int_\Omega |u_n - u| \, |v_n - v| \Big) \\ &\leq C||v - v_n||_{L^2}(||u - u_n||_{L^2} + |\lambda - \lambda_n| + ||u - u_n||_{L^2}). \end{aligned}$$

Therefore $||v_n - v||_{H_0^1} \to 0$ as $n \to \infty$. Finally

$$\begin{aligned} &||v_n - v||_{L^\infty} \\ &\leq C(||\lambda g(x,u) - \lambda_n g(x,u_n)||_{L^p} + ||f(x,u) - f(x,u_n)||_{L^p}) + ||v_n - v||_{H_0^1} \\ &\leq C(|\lambda_n - \lambda| + ||u - u_n||_{L^p} + ||u - u_n||_{L^p}) + ||v_n - v||_{H_0^1} \end{aligned}$$

$$\leq C(|\lambda_n - \lambda| + ||u - u_n||^p_{L^\infty}) + ||v_n - v||_{H_0^1} \to 0$$

as $n \to \infty$, and we conclude that T is continuous from H to H.

Finally we prove that T is a compact operator. Let $\{(\lambda_n, u_n)\}$ be a bounded sequence from H. By continuity $g(x, u_n)$ and $f(x, u_n)$ are bounded in $L^\infty(\Omega)$ and by Theorem 2.3.5 $\{v_n\}$ is bounded in $C^{1,\alpha}(\bar{\Omega})$. Thus we can find a subsequence of $\{(\mu_n, v_n)\}$ which converges in H.

Next, we employ Leray-Schauder Degree Theory to show the existence of fixed points of T. Note first that all positive solutions (i.e. by (G1) all solutions) of $(\lambda, u) = \nu T(\lambda, u)$ are bounded in H for all $0 \leq \nu \leq 1$. Indeed the equation $(\lambda, u) = \nu T(\lambda, u)$ implies that

$$-\Delta u + \nu\lambda g(x, u) = \nu f(x, u) \tag{5.3.4}$$

$$\lambda - \frac{1}{2}\nu(\lambda + \phi(u)) = 0. \tag{5.3.5}$$

Then multiplying (5.3.4) by u and integrating over Ω yields

$$\int_\Omega |\nabla u|^2 \leq \nu \int_\Omega f(x, u)u$$

and therefore by (F1)

$$||u||^2_{H_0^1} \leq \int_\Omega (\varepsilon u^2 + C_\varepsilon u) \leq \varepsilon ||u||^2_{L^2} + C_\varepsilon ||u||_{L^2}.$$

Choosing ε small, we obtain that $||u||_{H_0^1} \leq C$. Moreover Lemma 5.3.1 implies that $||u||_{L^\infty} \leq C$. Finally (5.3.5) yields that

$$\lambda(1 - \frac{\nu}{2}) = \frac{\nu}{2}\phi(u) \leq C$$

and we conclude that λ is bounded. A homotopy property (L3) of degree implies that

$deg(I-T, B_R, 0) = deg(I, B_R, 0) = 1$ where I is the identity operator and B_R denotes a ball in H of large radius R. We remind the reader that here deg denotes the Leray-Schauder Degree.

Next, we consider the operator equation

$$(\lambda, u) = T(\lambda, u) + t(0, v) \tag{5.3.6}$$

with $t \geq 0$ and v chosen to be the eigenfunction corresponding to the first eigenvalue μ_1 of $(-\Delta)$ with homogeneous Dirichlet boundary conditions. Once again (G1) implies that all solutions of (5.3.6) are positive. Equation (5.3.6) is equivalent to the system

$$-\Delta u + \lambda g(x, u) = f(x, u) + t\mu_1 v \tag{5.3.7}$$

$$\lambda = \phi(u). \tag{5.3.8}$$

We want to show that there exists $r > 0$ such that any solution of (5.3.7), (5.3.8) satisfies $||(\lambda, u)||_H > r$.

By Theorem 2.4.5 we have

$$\mu_1 + \sup_{x\in\Omega} \frac{\phi(u)g(x,u)}{u} \geq \inf_{x\in\Omega}\left[\frac{-\Delta u + \phi(u)g(x,u)}{u}\right]$$
$$= \inf_{x\in\Omega}\left[\frac{f(x,u)}{u} + \frac{t\mu_1 v}{u}\right] \geq \inf_{x\in\Omega}\left[\frac{f(x,u)}{u}\right]$$

and by (F4) and (H1) there exists $K > 0$ such that

$$\frac{K}{||u||^s_{L^\infty}} \leq \mu_1 + M||u||^l_{L^\infty},$$

therefore $||u||_{L^\infty} \geq C$. Thus there exists a constant $r > 0$ such that $||u||_{H_0^1 \cap L^\infty} \geq ||u||_{L^\infty} \geq r$, and so any solution of (5.3.7), (5.3.8) with $t \geq 0$ satisfies $||(\lambda, u)||_H > r$.

Thus choosing a small r and constructing a suitable homotopy we obtain

$$deg(I - T, B_r, 0) = deg(I - T - t(0, v), B_r, 0) = 0.$$

The additivity property (L2) of the Leray-Schauder degree implies that

$$deg(I - T, B_R \backslash \overline{B}_r, 0) = 1$$

and we conclude by (L4) that the operator T has at least one fixed point in the annulus $B_R \backslash \overline{B}_r$. We must show that if (λ, u) is one of these fixed points then both λ and u are nontrivial. If $\lambda = 0$ then by (5.3.3) we have $\phi(u) = 0$ and therefore $u \equiv 0$. Thus it follows that λ, and therefore also u, is nontrivial and consequently u is a positive $H_0^1(\Omega) \cap L^\infty(\Omega)$-solution of (5.2.1).

Remark 5.3.3. It follows from results of Chapter II that u is actually mooth. Indeed, we rewrite (5.2.1) as

$$-\Delta u = -\phi(u) g(x, u) + f(x, u) \quad \text{in } \Omega, \quad u = 0 \quad \text{on} \quad \partial\Omega \qquad (5.3.9)$$

and since u is bounded $f(x, u)$ and $g(x, u)$ are also bounded by (F1) and (G2). Therefore all conditions of Theorem 2.3.5 hold and $u \in C^{1,\alpha}(\bar{\Omega})$ for some $0 < \alpha < 1$. Next, condition (F2) and conditions on $g(x, t)$ imply that $f(x, u)$ and $g(x, u)$ are smooth enough to belong to $L^{q/2}(\Omega)$ for some $q > n/2$ and we conclude by Theorem 2.3.6 that $u \in H^{2,q}_{loc}(\Omega)$. The same conditions on f and g and the smoothness of u imply that the right hand side of (5.3.9) belongs to $C^{0,\alpha}(\Omega)$ and finally we obtain using Theorem 2.3.7 that $u \in C^{2,\alpha}(\bar{\Omega})$.

5.4. Uniqueness of solution for sublinear case

We turn next to the investigation of the uniqueness of the solution of (5.2.1). Assume that the following holds:

$$\text{(G3)} \qquad \frac{g(x,t_1)}{t_1} - \frac{g(x,t_2)-g(x,t_1)}{t_2-t_1} \leq 0 \quad \text{for any} \quad 0 < t_1 < t_2.$$

We also recall that for the problem $Au = f$ in Ω, $u \in H_0^1(\Omega)$, where A is a differential operator, functions $v, w \in H^1(\Omega)$ are called sub- and supersolution respectively if they satisfy

$$Av \leq f \quad \text{in } \Omega, \quad v \leq 0 \quad \text{on } \partial\Omega$$

$$Aw \geq f \quad \text{in } \Omega, \quad w \geq 0 \quad \text{on } \partial\Omega.$$

We need the following technical lemma.

Lemma 5.4.1. Let the sequence $\{z_n\}$, $z_n \in C^2(\bar{\Omega})$, $n = 1, 2, \ldots$, converge in $C^2(\bar{\Omega})$ to $z \in C^2(\bar{\Omega})$ and let $0 \leq z_n \leq z$ in $\bar{\Omega}$. Let $\Omega_\varepsilon = \{x \in \Omega \mid z(x) > \varepsilon\}$ for $\varepsilon > 0$. Then

$$\lim_{n\to\infty}\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon} z_n^2\left[\frac{(-\Delta z_n)}{z_n} - \frac{(-\Delta z)}{z}\right] = 0.$$

Proof. We show first that

$$\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon} z_n^2\left[\frac{(-\Delta z_n)}{z_n} - \frac{(-\Delta z)}{z}\right] = \int_{\Omega} z_n^2\left[\frac{(-\Delta z_n)}{z_n} - \frac{(-\Delta z)}{z}\right]$$

uniformly in n. Indeed since $|z_n/z| \le 1$ and $(-\Delta z_n)$ are uniformly bounded we conclude that

$$\int_{\Omega\setminus\Omega_\varepsilon} z_n^2 \left| \frac{(-\Delta z_n)}{z_n} - \frac{(-\Delta z)}{z} \right| \le \int_{\Omega\setminus\Omega_\varepsilon} |z_n(-\Delta z_n)| + \int_{\Omega\setminus\Omega_\varepsilon} |z_n(-\Delta z)| \le C\varepsilon$$

and the uniform convergence follows.

Next it is easy to see that

$$\lim_{n\to\infty} \int_{\Omega_\varepsilon} z_n^2 \left[\frac{(-\Delta z_n)}{z_n} - \frac{(-\Delta z)}{z} \right] = 0.$$

Thus applying the theorem about the commuting limits we obtain the statement of the lemma.

Theorem 5.4.2. Let $f(x,u)$, $g(x,u)$ and $\phi(u)$ satisfy the conditions of Theorem 5.3.2 and (G3). Assume in addition that for any $0 < t_1 < t_2$

$$\frac{f(x,t_2) - f(x,t_1)}{t_2 - t_1} - \frac{f(x,t_1) - f(x,0)}{t_1} \le 0.$$

Then problem (5.2.1) has unique positive solution from $H_0^1(\Omega) \cap L^\infty(\Omega)$.

Proof. We proved already existence and positivity. Suppose that there exist two positive solutions u, v of problem (5.2.1), and assume without loss of generality that $\phi(v) \ge \phi(u)$. Then (5.2.1) implies

$$-\Delta v + g(x,v)\phi(u) \le f(x,v). \tag{5.4.1}$$

Consider now an auxillary problem

$$-\Delta w + g(x,w)\phi(u) = f(x,w) \quad \text{in} \quad \Omega, \qquad w = 0 \quad \text{on} \quad \partial\Omega \tag{5.4.2}$$

where u is the solution of (5.2.1). Inequality (5.4.1) shows that v is a subsolution for problem (5.4.2). Next, (G1), (G3) and (F1) imply that there exists a linear function which bounds $\phi(u)g(x,t)$ from below uniformly in x and another linear function with lesser slope which bounds $f(x,t)$ from above uniformly in x. Thus we can find a large constant N satisfying $g(x,N)\phi(u) > f(x,N)$ and $N > \max\{||v||_{L^\infty}, ||u||_{L^\infty}\}$ which will be a supersolution for (5.4.2). Since $f(x,t)$, $g(x,t)$ are locally Lipschitz continuous we can find $M \geq 0$ such that the operator $f(x,t) - \phi(u)g(x,t) + Mt$ is monotone on the order interval $< v, N >$. We define the operator S which acts from $H_0^1(\Omega)$ to $H_0^1(\Omega)$ in the following way

$$Sr = p \;\Leftrightarrow\; -\Delta p + g(x,r)\phi(u) + Mp = f(x,r) + Mr \quad \text{in } \Omega, \quad p = 0 \quad \text{on } \partial\Omega. \tag{5.4.3}$$

We show first that S maps the interval $< v, N >$ into itself. Indeed if $v \leq r \leq N$ then subtracting (5.4.3) from (5.4.1) and taking the scalar product of the result with $(v-p)^+$ we obtain

$$\int_\Omega |\nabla(v-p)^+|^2 + M\int_\Omega |(v-p)^+|^2$$

$$\leq \phi(u)\int_\Omega (f(x,v) - f(x,r) - g(x,v) + g(x,r) + M(v-r))(v-p)^+ \leq 0.$$

and therefore $p \geq v$. A similar argument shows that $p \leq N$ and moreover S is monotone on $< v, N >$.

Next we show that S is continuous. Let us consider a sequence $\{r_n\}$, $n = 1, 2, \ldots$, $r_n \in H_0^1(\Omega)$, $r_n \to r$ as $n \to \infty$ in $H_0^1(\Omega)$ and put $Sr_n = p_n$ and $Sr = p$.

Then the local Lipschitz continuity of f and g implies that

$$\int_{\Omega} |\nabla(p - p_n)|^2 + M \int_{\Omega} |p - p_n|^2 \leq C \int_{\Omega} (r - r_n)(p - p_n).$$

Thus $||p - p_n||_{H_0^1} \leq C||r - r_n||_{L^2}$ and hence $p_n \to p$ as $n \to \infty$ in $H_0^1(\Omega)$. Note that since the inclusion of $H_0^1(\Omega)$ into $L^2(\Omega)$ is compact then the previous argument shows that S is a compact operator.

We construct a sequence $\{\bar{w}_n\}$ as follows:

$$-\Delta \bar{w}_n + g(x, \bar{w}_{n-1})\phi(u) + M\bar{w}_n = f(x, \bar{w}_{n-1}) + M\bar{w}_{n-1} \quad \text{in } \Omega, \quad \bar{w}_n = 0 \quad \text{on } \partial\Omega$$

for $n = 1, 2, \ldots$ with $\bar{w}_0 = N$. Then by monotonicity we have

$$\bar{w}_0 \geq \bar{w}_1 \geq \bar{w}_2 \geq \ldots \geq v$$

and we set $w = \lim_{n \to \infty} \bar{w}_n$. Observe now that w solves (5.4.2), $w \geq v$ and by monotonicity if $v \leq p \leq N$ is the solution of (5.4.2) then $p \leq w$. Thus since $N > ||u||_{L_\infty}$ we have $u \leq w$. Furthermore the same argument as in Remark 5.3.3 implies that $w \in C^{2,\alpha}(\bar{\Omega})$.

We will show that $w \equiv u$ and to do this we will use Picone's Identity in the following form, [ALL]:

$$\int_{\Omega} \psi^2 \left| \nabla \left(\frac{\varphi}{\psi} \right) \right|^2 = \int_{\Omega} |\nabla \varphi|^2 - \int_{\Omega} \frac{\varphi^2}{\psi}(-\Delta \psi)$$

where $\varphi \in C_0^\infty(\bar{\Omega})$, $\psi > 0$ in Ω and $\psi \in C^2(\bar{\Omega})$.

Let $\{u_n\}$ and $\{w_n\}$ be two sequences of functions from $C_0^\infty(\bar{\Omega})$ such that $u_n \to u$ and $w_n \to w$ in $C^2(\bar{\Omega})$ as $n \to \infty$, with $0 \leq w_n - u_n \leq w - u$. Remark

5.3.3 allows us to apply Picone's identity to $w_n - u_n$ and u and we obtain by direct calculation:

$$\int_\Omega u^2 \left|\nabla\left(\frac{w_n - u_n}{u}\right)\right|^2 = \int_\Omega |\nabla(w_n - u_n)|^2 - \int_\Omega \frac{(w_n - u_n)^2}{u}(-\Delta u)$$
$$= \int_\Omega (w_n - u_n)^2 \left[\frac{-\Delta(w_n - u_n)}{w_n - u_n} - \frac{-\Delta u}{u}\right].$$

Define $\Omega_\varepsilon = \{x \in \Omega \mid w(x) - u(x) > \varepsilon\}$ for $\varepsilon > 0$. Then we have by the countable additivity of Lebesgue's integral

$$\int_\Omega (w_n - u_n)^2 \left[\frac{-\Delta(w_n - u_n)}{w_n - u_n} - \frac{-\Delta u}{u}\right]$$
$$= \lim_{\varepsilon \to 0} \int_{\Omega_\varepsilon} (w_n - u_n)^2 \left[\frac{-\Delta(w_n - u_n)}{w_n - u_n} - \frac{-\Delta u}{u}\right]$$

and equations (5.2.1) and (5.4.2) imply

$$\int_{\Omega_\varepsilon} (w_n - u_n)^2 \left[\frac{-\Delta(w_n - u_n)}{w_n - u_n} - \frac{-\Delta u}{u}\right]$$
$$= \int_{\Omega_\varepsilon} (w_n - u_n)^2 \left[\frac{-\Delta(w_n - u_n)}{w_n - u_n} - \frac{-\Delta(w - u)}{w - u}\right]$$
$$+ \int_{\Omega_\varepsilon} (w_n - u_n)^2 \left[\frac{-\Delta(w - u)}{w - u} - \frac{-\Delta u}{u}\right]$$
$$= \int_{\Omega_\varepsilon} (w_n - u_n)^2 \left[\frac{-\Delta(w_n - u_n)}{w_n - u_n} - \frac{-\Delta(w - u)}{w - u}\right]$$
$$+ \int_{\Omega_\varepsilon} (w_n - u_n)^2 \left[\frac{f(x,w) - f(x,u)}{w - u} - \frac{f(x,u)}{u}\right]$$
$$+ \int_{\Omega_\varepsilon} \phi(u)(w_n - u_n)^2 \left[-\frac{g(x,w) - g(x,u)}{w - u} + \frac{g(x,u)}{u}\right].$$

Since $w > u$ in Ω_ε then the conditions of the theorem and (G3) yield that for $x \in \Omega_\varepsilon$

$$\frac{f(x,w) - f(x,u)}{w - u} - \frac{f(x,u) - f(x,0)}{u} \leq 0$$

$$-\frac{g(x,w)-g(x,u)}{w-u}+\frac{g(x,u)}{u}\leq 0$$

and thus

$$\int_\Omega u^2\left|\nabla\left(\frac{w_n-u_n}{u}\right)\right|^2\leq\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon}(w_n-u_n)^2\left[\frac{-\Delta(w_n-u_n)}{w_n-u_n}-\frac{-\Delta(w-u)}{w-u}\right]. \tag{5.4.4}$$

Applying Lemma 5.4.1 and Fatou's Theorem we can pass now to the limit as $n\to\infty$ and conclude by (5.4.4) that

$$\int_\Omega u^2\left[\nabla\left(\frac{w-u}{u}\right)\right]^2\leq 0$$

and therefore $w\equiv u$. But then we also have $u\geq v$ and $\phi(u)\leq\phi(v)$ which imply that $v\equiv u$, and the uniqueness follows.

5.5. An example

In this section we consider equation (5.2.1) with $g(x,u)=u^\alpha$, $\alpha\geq 1$, if $u\geq 0$ and $g(x,u)=0$ if $u<0$, $\phi(u)=\int_\Omega u$ and $f(x,u)=(u+\tau)^\gamma$ where $\tau\geq 0$ and $0<\gamma<1$, i.e.

$$-\Delta u+u^\alpha\int_\Omega u=(u+\tau)^\gamma \quad \text{in } \Omega,\quad u>0 \text{ in } \Omega,\quad u=0 \quad \text{on} \quad \partial\Omega. \tag{5.5.1}$$

In this case $\phi(u)$ and $g(x,u)$ satisfy hypotheses ($\Phi 1$), ($\Phi 2$), (G1), (G2) and $f(x,u)=(u+\tau)^\gamma$ satisfies (F1) and (F3). Note that $(u+\tau)^\gamma$ is not in $C^{0,1}_{loc}(\mathbb{R}^+)$ but it is in $C^{0,\gamma}_{loc}(\mathbb{R}^+)$, $0<\gamma<1$ and this smoothness is sufficient for the proof of existence. We also note that hypothesis (F4) holds, since

$$\inf_{x\in\Omega}\frac{(u+\tau)^\gamma}{u}\geq\inf_{x\in\Omega}\frac{u^\gamma}{u}\geq\frac{1}{||u||^{1-\gamma}_{L^\infty}}.$$

Finally $\|\int_\Omega u\|_{L^\infty} \leq C\|u\|_{L^\infty}$ and (H1) holds. Therefore we can apply Theorem 5.3.2 to obtain the existence of a positive solution of (5.5.1). Moreover since $g(x,t)$ satisfies (G3) for $t > 0$, Remark 5.3.3 holds and since monotonicity can replace smoothness in the argument of Theorem 5.4.2 we conclude that the positive solution of (5.5.1) is unique.

5.6. A superlinear problem

In this section we study the situation when $f(x,t)$ has a superlinear growth as $t \to \infty$. In particular let $f(x,t)$ satisfy the earlier conditions (F2)-(F3) and instead of the sublinearity at infinity we assume that the following holds:

(F5) $\lim_{t\to\infty} \frac{f(x,t)}{t^s} = r(x)$ uniformly for $x \in \Omega$ for some $1 < s < (n+2)/(n-2)$, $n > 2$ and $0 < r(x) \in C^{0,\alpha}(\bar{\Omega})$.

We consider first the case when $f(x,t)$ has the following behaviour near $t = 0$:

(F6) $f(x,0) = 0$ and $\lim_{t\to+0} \frac{f(x,t)}{t} < \mu_1$, where μ_1 is the first eigenvalue of $(-\Delta)$ with homogeneous Dirichlet boundary conditions.

We begin by formulating the following local results, which show that if the nonlocal term is small, existence can be proved immediately from the local case by routine perturbation arguments.

Proposition 5.6.1. Let $\tilde{C} = \{w | w \in C^1(\bar{\Omega}),\ \ w = 0 \text{ on } \partial\Omega\}$ equipped with the C^1 norm and let $f(x,t)$ satisfy (F2), (F3), (F5), (F6). Set $F(u) = (-\Delta)^{-1}(f(x,u))$. Then $F : \tilde{C} \to \tilde{C}$, $F(u) \geq 0$ and there exist constants $r, R > 0$ such that $deg(I - F, B_R \backslash \overline{B}_r, 0) \neq 0$, where B_a denotes the open ball of radius a

in $\tilde{C}$.

Proposition 5.6.2. There exists $\delta > 0$ such that if $F_1 : \dot{C} \to \dot{C}$ is a continuous and compact map with $||F_1(v)||_{C^1} < \delta$ for $v \in B_R$, $v > 0$, and the conclusions of Proposition 5.6.1 hold for F, then there exists $0 < u \in B_R$ such that $u = F(u) + F_1(u)$.

The proofs of these propositions can be found in [ANZ].

Now we state our next theorem.

Theorem 5.6.3. Consider problem (5.2.1) under conditions (F2), (F3), (F5), (F6), (Φ1), (Φ2), (G1). Let

$$||(-\Delta)^{-1}(\phi(v)g(x,v))||_{C^1} < \delta \quad \text{for} \quad 0 < v \in B_R \tag{5.6.1}$$

where δ and R are positive constants from Propositions 5.6.1 and 5.6.2. Then there exists a positive solution of problem (5.2.1).

Proof. The result of the theorem follows immediately from Propositions 5.6.1 and 5.6.2 if we put $F_1 v = -(-\Delta)^{-1}(\phi(v)g(x,v))$.

Remark 5.6.4. Note that in many cases condition (5.6.1) can be easily checked. Indeed if we consider $\phi(v)g(x,v) = \eta v \int_\Omega v$ or $\phi(v)g(x,v) = \eta \int_\Omega v$ with $\eta > 0$ then (5.6.1) holds for small η. Unfortunately we were not able to prove the existence of solutions of the superlinear problem in the general case, i.e. for η large, but we conjecture that the statement of Theorem 5.6.3 will be still true.

Next we consider the case when $f(x,0) > 0$. We will need the following local result

Proposition 5.6.5. Let $f(x,t)$ satisfy (F2), (F3), (F5) and $f(x,0) > 0$. Then the problem

$$-\Delta u = \lambda f(x,u) \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{on} \quad \partial\Omega$$

has a positive solution in $\tilde{C}$ for small λ.

This result is well known, see for example [LIO] and references therein, but we were unable to find an explicit proof and thus we sketch one for the reader's convenience.

Proof. Note first that by Theorem 2.2.3 the problem $-\Delta v = f(x,0)$ has a unique solution in $\tilde{C}$ such that $v > 0$ in Ω. So we can find two balls B_r and B_R in $\tilde{C}$ such that $v \in B_R \backslash \overline{B_r}$ and therefore $deg(I - (-\Delta)^{-1}(f(x,0)), B_R \backslash \bar{B}_r, 0) = 1$.

We define now $F_1 w = (-\Delta)^{-1}[f(x,\lambda w) - f(x,0)]$ with $\lambda > 0$. Then the Lipschitz continuity of $f(x,t)$ and Theorem 2.3.4 imply that

$$||F_1 w||_{C^1} \leq C(||F_1 w||_{L^\infty} + ||f(x,\lambda w) - f(x,0)||_{L^\infty})$$

$$\leq C||f(x,\lambda w) - f(x,0)||_{L^\infty} \leq C\lambda ||w||_{L^\infty} \leq K\lambda$$

for $w \in B_R$ with $K > 0$. Thus if we choose λ small enough we can apply Proposition 5.6.2 and conclude that there exists a positive solution of $-\Delta w = f(x,\lambda w)$ in $\tilde{C}$. Putting now $u = \lambda w$ we obtain the statement of the proposition.

Remark 5.6.6. We recall that the result of Proposition 5.6.5 (and consequently what follows) can not hold in general for all $\lambda > 0$. To see this, choose for example $f(x,u) = u^s + \varepsilon$ with $1 < s < (n+2)/(n-2)$ and $\varepsilon > 0$ and observe that if v is the eigenfunction of $(-\Delta)$ with Dirichlet boundary conditions corresponding to the first eigenvalue then we have:

$$\mu_1 \int_\Omega vu = \int_\Omega v(-\Delta)u = \lambda \int_\Omega v(u^s + \varepsilon).$$

Choosing λ so large that $2\mu_1 t \le \lambda(t^s + \varepsilon)$ then gives $\int_\Omega vu = 0$, contradicting the assumption $u > 0$.

Consider now the parametrized version of (5.2.1)

$$-\Delta u + g(x,u)\phi(u) = \lambda f(x,u) \quad \text{in } \Omega, \quad u = 0 \quad \text{on} \quad \partial\Omega. \tag{5.6.2}$$

Then we can prove the following theorem.

Theorem 5.6.7. Let $f(x,t)$ satisfy (F2), (F3), (F5) and $f(x,0) > 0$, $\phi(u)$ satisfy (Φ1), (Φ2) and $g(x,t)$ satisfy (G1). There exist $\lambda^* > 0$ such that problem (5.6.2) with $\lambda \le \lambda^*$ has a positive solution.

Proof. The proof proceeds the same way as in Proposition 5.6.5 except that we put now

$$F_1 w = (-\Delta)^{-1}\left[f(x,\lambda w) - f(x,0) - \frac{g(x,\lambda w)}{\lambda}\phi(\lambda w)\right]$$

and note that (G1) implies that

$$\left\|\frac{g(x,\lambda w)\phi(\lambda w)}{\lambda}\right\|_C \le C\left\|\frac{g(x,\lambda w}{\lambda w}\right\|_C \|w\|_C \le \frac{\delta}{2}$$

for small λ and $w \in B_R$.

5.7. Applications to mathematical biology

As it was mentioned before the study of semilinear equations was inspired by the model which describes single-species population dynamics with dispersal.

So we suppose that the density of the population $u(x,t)$ satisfies the equation:

$$u_t = \Delta u + u \cdot \chi(x, u, \phi(u)), \quad t > 0, \quad x \in \Omega \tag{5.7.1}$$

subject to suitable boundary and initial conditions. The function χ represents the "crowding" effect and ϕ is a continuous functional of u. In what follows we investigate steady state solutions of (5.7.1) with homogeneous Dirichlet boundary conditions and various χ.

Example 5.7.1. We start by considering $\chi(u) = a(1-u) - \int_\Omega u^m$ where a, m are positive constants such that $a > \mu_1$ and $1 \le m \le 2$. In the notations of Section 5.7.2 we have that $f(x,u) = au(1-u)$ if $0 \le u \le 1$ and $f(x,u) \equiv 0$ if $u > 1$, $g(x,u) = u$ and $\phi(u) = \int_\Omega u^m$. Then $\phi(u)$ satisfies (Φ1)-(Φ2), $f(x,u)$ satisfies (F1)-(F3), but not (F4). Nevertheless we still can ensure that solutions of (5.3.6) are bounded away from zero. Indeed, repeating the argument from Theorem 5.3.2 we obtain

$$\mu_1 + M||u||^m_{L^\infty} \ge a - a||u||_{L^\infty}$$

and thus $||u||_{L^\infty} \ge C$. Therefore the arguments of Theorems 5.3.2 and 5.4.1 hold and we conclude that in this case (5.7.1) has a unique positive solution u_0 such that $u_0 \le 1$.

The natural question which arises is that of the stability of u_0, and in order to deal with this problem we will apply the spectral theory for nonlocal problems elaborated in [FR1]. Note that it was shown in [FEI], [FR1] that in general the linearized nonlocal problems can have very complicated dynamics, and to simplify matters we first guarantee that the linearized eigenvalue problem does not have complex eigenvalues and then investigate the stability.

To do this we will need the following results the proofs of which can be found in [FR1]. We consider the one-parameter family of linear operators

$$L_\eta u = -\Delta u + bu + \eta\psi \int_\Omega \varphi u, \quad u \in H_1^0(\Omega)$$

with $\psi, \varphi \in C(\bar{\Omega})$. Let $\{\nu_k(\eta)\}$, $k = 1, 2, \ldots$ denote the complete set of eigenvalues of L_η and $\{w_k\}$ be the complete system of eigenfunctions of L_0. Then we have

Proposition 5.7.2.

1. $$\nu_k'(0) = \left(\int_\Omega \varphi w_k\right)\left(\int_\Omega \psi w_k\right)\Big/\left(\int_\Omega w_k^2\right)$$

2. Let $I \subset \mathbb{R}$ be such that $\nu_k(\eta)$ is real for $\eta \in I$. Then $\nu_k'(\eta)$ does not change sign for $\eta \in I$.

Proposition 5.7.3. If the product $\left(\int_\Omega \varphi w_k\right)\left(\int_\Omega \psi w_k\right)$ does not change sign for all k then all $\nu_k(\eta)$ remain real for all real η.

We start by writing the linearized operator for problem 5.7.1 at u_0 which is of the form

$$L_m v = -\Delta v + bv + mu_0 \int_\Omega u_0^{m-1} v$$

where $b(x) = -a(1-2u_0(x)) + \int_\Omega u_0^m$. If $m = 2$ then the operator L_2 is selfadjoint and all eigenvalues of $L_2 v_k = \nu_k(2) v_k$ are real. If $1 \le m < 2$ then the situation becomes more complicated. We recall that if we let $\{w_k\}$ be the complete system of eigenfunctions of the corresponding local operator L_0, i.e. $L_0 w_k = -\Delta w_k +$

$bw_k = \lambda_k w_k$, and if u_0 is such that the following condition holds:

$$\left(\int_\Omega u_0 w_k\right)\left(\int_\Omega u_0^{m-1} w_k\right) \text{ does not change sign for all } k \tag{5.7.2}$$

then Proposition 5.7.3 implies that all eigenvalues of L_m are real. Unfortunately condition (5.7.2) is very hard to check explicitly except that for $m = 2$ it is automatically satisfied.

Assuming that (5.7.2) holds, since $w_1 > 0$ and $u_0 > 0$ in Ω we conclude that $\left(\int_\Omega u_0 w_k\right)\left(\int_\Omega u_0^{m-1} w_k\right) \geq 0$ for all k, and Proposition 5.7.3 yields that $\nu_k(m) \geq \lambda_k$ for all k, and moreover $\nu_1(m) > \lambda_1$. Thus if $\lambda_1 \geq 0$ we conclude that u_0 is an asymptotically stable solution of the nonlocal problem.

Example 5.7.4. Next we consider the problem (5.7.1) with $\chi(x,u) = p(x)u^{\gamma-1} - \eta \int_\Omega u$, i.e.

$$-\Delta u + \eta u \int_\Omega u = p(x)u^\gamma \quad \text{in } \Omega, \quad u = 0 \quad \text{on} \quad \partial\Omega \tag{5.7.3}$$

where η, γ are positive constants, $1 < \gamma < \frac{n+2}{n-2}$ and $0 < p(x) \in C^{0,\alpha}(\bar{\Omega})$. In this case $f(x,u) = p(x)u^\gamma$ is superlinear and satisfies (F5), (F6), and it follows from Theorem 5.6.3 that problem (5.7.3) has a positive solution for small η. If we now put $w = \lambda u$, then equation (5.7.3) becomes

$$-\Delta w + \frac{\eta}{\lambda} w \int_\Omega w = \lambda^{1-\gamma} p(x) w^\gamma \quad \text{in } \Omega, \quad w = 0 \quad \text{on} \quad \partial\Omega.$$

Thus for any general η_0 by choosing $\lambda = \eta/\eta_0$ we obtain that w solves

$$-\Delta w + \eta_0 w \int_\Omega w = \lambda^{1-\gamma} p(x) w^\gamma \quad \text{in } \Omega, \quad w = 0 \quad \text{on} \quad \partial\Omega.$$

Note that since η was small it follows that $\lambda^{1-\gamma}$ will be large. Therefore we conclude that for the parametrized version of (5.7.3), i.e. (5.7.3) with right hand side which is equal to $\nu p(x)u^{\gamma}$, for any η there exists ν_0 such that for $\nu \geq \nu_0$ our problem will have a positive solution.

Example 5.7.5. Our last example deals with the case when $\lambda(x, u) = a - \int_\Omega p(x)u(x)dx$. Let us consider the problem

$$-\Delta u + u \int_\Omega p(x)u(x)dx = au \quad \text{in } \Omega, \quad u = 0 \quad \text{on} \quad \partial\Omega, \tag{5.7.4}$$

with $0 \leq p(x) \in C^{0,\alpha}(\bar{\Omega})$, $p(x) \not\equiv 0$ and $a > \mu_1$ where μ_1 is the first eigenvalue of $-\Delta u = \mu u$ with homogeneous Dirichlet boundary conditions. The right hand side of problem (5.7.4) is now linear with respect to u and we can not apply directly results of Sections 5.3, 5.4 and 5.6. But it is possible to check that the first eigenfunction $0 < v_1$ of $-\Delta u = \mu u$ with $\int_\Omega p(x)v_1(x)dx = a - \mu_1$ solves (5.7.4). Moreover using the same argument as in Theorem 5.4.2 we can show that this solution is unique.

The stability analysis for problem (5.7.4) shows that the nonlocal term plays a stabilizing role. Indeed, using once again ideas from [FR1] we write the linearized operator at v_1 in the form

$$\begin{aligned} Lv &= -\Delta v + \left(\int_\Omega p(x)v_1(x)dx - a \right) v + v_1 \int_\Omega p(x)v(x)dx \\ &= -\Delta v - \mu_1 v + v_1 \int_\Omega p(x)v(x)dx. \end{aligned}$$

Let $\{v_k\}$, $k = 1, 2...$ be the complete system of eigenfunction of the operator $-\Delta v - \mu_1 v = \lambda v$ with the corresponding eigenvalues $\{\lambda_k\}$. Then since

$\left(\int_\Omega v_1^2\right)\left(\int_\Omega pv_1\right) > 0$ and $\left(\int_\Omega v_1 v_k\right)\left(\int_\Omega pv_k\right) = 0$ for all $k > 1$ we conclude again by Proposition 5.7.3 that all eigenvalues $\{\nu_k\}$ of the operator L are real. Moreover Proposition 5.7.2 implies that $\nu_1 > \lambda_1 = 0$ and $\nu_k = \lambda_k$ for $k > 1$ and once again v_1 is asymptotically stable.

CHAPTER VI

DISCUSSION

In this thesis we have studied some partial differential equations with nonlocal terms. We established results about existence, blow-up, positivity and uniqueness of the solutions of these problems. We also consider applications to the nonlocal microsensor models and mathematical biology. The previous results and models considered in [ABHLR], [AX1], [FUG] were improved and generalized in the following aspects:

[1] The results of Chapters III and IV give mathematical justification to the numerical calculations presented in [ABHLR]. Since only positive solutions of the nonlocal system describing processes in thermistors have physical sense we studied the problem of positivity of solutions. As far as we are aware this question was not addressed before. The methods which we applied differ essentially from the classical ones.

[2] We answered the conjecture formulated in [AX1] about the blow-up of problem (P) (Chapter IV). We formulated conditions under which the blow-up of temperature occurs. In electrical engineering the situation considered in this chapter corresponds to the narrowing processes and the estimation of the value of current for which the blow-up occurs is of some importance in applications.

[3] We generalized to the n-dimensional case and to Dirichlet boundary conditions the model suggested in [FUG] describing the single-species populational

dynamics with nonlocal crowding effect (Chapter V). We established the existence of positive stationary solutions and discussed their stability.

Since these nonlocal problems are just starting to attract widespread attention there are several open questions in this area. We conclude the thesis by mentioning some of them:

[1] We obtained in Chapter III the formula for the critical value of the parameter η for which (3.1.1) has positive solutions. Unfortunately this formula depends on the Green's function of the domain and thus is not usually explicitly known. Therefore it would be useful to obtain an estimate from below for this critical value of η.

[2] It would be interesting to consider nonlocal thermistor problem (4.2.1), (4.2.2) with various boundary conditions, in particular mixed or nonlocal conditions.

[3] In Chapter V we showed the existence of positive solutions of equation (5.2.1) with certain restrictions on the norm of the nonlocal term. We were not able to show existence for the general case but conjecture that this is the case.

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