Discrete Stopping Times in Vector Lattices

by

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Abstract

The interplay between order structure and probability theory has long been studied. In recent years this has led a generalization of many of the concepts from probability theory to arbitrary vector lattices. In this thesis, we study the generalization of discrete stopping times in vector lattices. To do so, we first study the sup-completion of a vector lattice using the Maeda-Ogasawara representation of its universal completion. Using this, we show that the Daniell functional calculus for continuous functions is exactly the pointwise composition of functions in $C^\infty(K)$, where K is the Stone space of the vector lattice. These tools allow us to study unbounded stopping times and obtain a "nice" representation of them in vector lattices.

Preface

The research in this thesis is a combination of the papers *A representation of sup-completion* and *Discrete stopping times in the lattice of continuous functions*. The first paper was done in collaboration with Vladimir Troitsky, and it is accepted for publication in the *Proceedings of the American Mathematical Society*. The second paper has been published in *Positivity*. Additionally, there are some results in this thesis that do not appear in those papers. Specifically, section 2.5 contains unpublished original work.

Contents

1. Introduction and Preliminaries

Martingales and stopping times are probabilistic concepts and have proven to be extremely useful in the study of the geometry of a Banach space. Traditionally martingales and stopping times are defined in terms of measurable functions where the underlying measure space is a probability space, i.e. the measure of the whole space is 1. There is evidence in the literature that the underlying order structure of the space of measurable functions plays a central role in the study of stochastic processes and conditional expectations, for example, in [Rao76] and [GdP02]. Hence, there was hope that studying these interactions could help in understanding the geometry of a vector lattice. This connection was noted in [KLW04a, KLW05] where the authors reformulated these definitions using only order theoretic nomenclature. The idea of extending concepts from probability theory to vector lattices led to a flurry of research in recent years. A non-exhaustive list includes [Tro05, Gro14b, Amo22, GLM14, AT17, AR18, Gro14a, KLW04b]. This theory was also later used in studying the geometry of Lebesgue-Bochner spaces in [CL07, Cul07].

The aim of this work is to use the Maeda-Ogasawara representation of the universal completion of a vector lattice to study various concepts from measure free probability. In particular, using properties of almost everywhere continuous functions, we efficiently formulate and prove fundamental results in stochastic processes and stopping times in vector lattices. Representing stopping times and stopped processes as continuous functions allows us to deduce some interesting results and draw analogues with the results from classical probability theory.

The thesis is structured as follows. In Chapter 1, we begin with an overview of the thesis and state some relevant definitions and information from vector lattice theory. In Chapter 2, we provide a representation of the sup-completion using continuous functions. The sup-completion is an important tool in the theory of measure free probability [Gro21b] and having this representation allows us to simplify results for the future chapters. In Chapter 3, we recall the classical definitions of a conditional expectation operator, martingale and a stopping time. After reviewing some basic properties of these definitions, a generalized definition of these concepts in a vector lattice is provided. Several important results in probability theory use a functional calculus for their results, such as Jensen's inequality. In order to state this theorem in vector lattices, Grobler in [Gro14b] created a functional calculus using the Daniell integral. In Chapter 4, we prove that this functional calculus corresponds to the pointwise composition of continuous functions when using the Maeda-Ogasawara representation, thereby simplifying many of its properties. In Chapter 5, we use the various tools built in the preceding chapters to provide a representation of discrete stopping times and stopped processes as almost everywhere continuous functions. This allows us to study the unbounded stopped process and extend several results from [KLW04a, Kuo06, CL07, Cul07] about stopped processes.

1.1. **Preliminaries.**

We refer the reader to [AB03, AB06] for the background on vector lattices. Throughout this thesis, all vector lattices are assumed to be Archimedean. Recall that a net (x_α) in a vector lattice converges in order to x if there exists a net (u_γ), such that $u_\gamma \downarrow 0$, and for every γ there exists α_0 such that $|x_\alpha - x| \leq u_\gamma$ for all $\alpha \geq \alpha_0$. In this case, we write $x_\alpha \xrightarrow{\mathrm{o}} x$. We say that (x_α) unbounded order converges (uo-converges) to x, if $|x_\alpha - x| \wedge w \xrightarrow{\mathrm{o}} 0$ for every $w \geq 0$. Then we denote this as $x_\alpha \xrightarrow{\mathrm{uo}} x$. A sublattice is said to be regular if the inclusion map is order continuous. Given a net (x_α) in a regular sublattice Y of X, $x_\alpha \xrightarrow{\mathrm{uo}} 0$ in X if and only if $x_\alpha \xrightarrow{\mathrm{uo}} 0$ in Y. We also have that order convergence and unbounded order convergence agree for order bounded nets. Moreover, in a universally complete vector lattice, order convergence and uo-convergence agree for sequences. Another result [Pap64, Azo19] that we shall use later in this thesis is that a vector lattice is universally complete if and only if it is uo-complete. We refer the reader to [GTX17, BT22] for more information on uo-convergence. Given an operator $T : X \to Y$, we shall denote the range of T by $\mathcal{R}(T)$.

1.2. $C^\infty(K)$ **representation.**

Given a compact Hausdorff topological space K, we will write $C(K)$ for the set of all continuous functions from K to $\mathbb{R}$. The space $C(K)$ is a vector lattice under pointwise operations, and $C(K)$ is order (or Dedekind) complete if and only if K is a extremally disconnected. A subset A of a Hausdorff topological space is nowhere dense if $(\overline{A})^\circ = \emptyset$. This also implies that ∂A is a nowhere dense set for every closed set A. A set A is meagre if it is a union of a sequence of nowhere dense sets, and conversely, a set is co-meagre if its complement is meagre. K is a Baire space if countable unions of closed sets with empty interior also have empty interior. Every compact Hausdorff space is a Baire space.

Given that K is an extremally disconnected compact Hausdorff topological space, we write $C^\infty(K)$ for the vector lattice of all continuous functions from K to $[-\infty, \infty]$ that are finite almost everywhere (a.e.), that is, except on a nowhere dense set.

Two functions f and g in $C^\infty(K)$ are equal, provided that the set of points where their values differ is a nowhere dense set. Scalar multiplication, addition and lattice operations on $C^\infty(K)$ are defined pointwise a.e. It should be noted that $C^\infty(K)$ is an f-algebra with product defined pointwise a.e. Maeda-Ogasawara Theorem states that every Archimedean vector lattice $\mathcal{E}$ may be represented as an order dense sublattice of $C^\infty(K)$, where K is an extremally disconnected compact Hausdorff topological space. For $A \subseteq \overline{\mathbb{R}}$ and $\tau : K \to \overline{\mathbb{R}}$, we shall use the notation $\{\tau \in A\}$ to denote the set $\{\omega \in K : \tau(\omega) \in A\}$.

2. Sup-Completion

The concept of a sup-completion of a vector lattice was introduced in [Don82] and then further investigated in [Azo19, AN22]; this concept was utilized in a series of papers on measure-free probability by J. Grobler and C. Labuschagne [Gro14b, GL17c, GL17a, GL17b, GL19]. The intuitive idea behind sup-completion is rather simple: one wants to enlarge a function space by allowing functions that take value $+\infty$ on non-negligible sets. In particular, the sup-completion of $\mathbb{R}$ is $(-\infty, +\infty]$. However, the definition of the sup-completion and the proof of its existence (for order complete vector lattices) in [Don82] is quite technical. We shall recall this definition in Section 2.1. In Section 2.2, we prove the following theorem.

Theorem 2.0.1. *Let $\mathcal{E}$ be an order complete vector lattice represented in $C^\infty(K)$ as above. Then the sup-completion of $\mathcal{E}$ is $\{u \in C(K, \overline{\mathbb{R}}) \,:\, u \geqslant f \text{ for some } f \text{ in } \mathcal{E}\}$.*

This theorem may be viewed as an alternative (and, hopefully, more intuitive) definition of a sup-completion. Throughout this Chapter, $\mathcal{E}$ is an order complete vector lattice. In Section 2.3 we prove the sup-completion is unique using Theorem 2.0.1. In Section 2.4, we shall use our representation of the sup-completion to provide simple proofs of many results from [Azo19, AN22]. Furthermore, we prove an analogue of Yudin's theorem for sup-completions.

In [BT22, vdW18], the authors characterize order convergence in $C(K)$ and $C^\infty(K)$ spaces using topological terms. Since $\mathcal{E}^s$ is lattice, we can still define order convergence of nets in $\mathcal{E}^s$. In Section 2.5, we provide a characterisation of order convergence in the sup-completion using topological terms that is analogous to the results in [BT22]. The results in this Chapter are original work obtained in collaboration with V. Troitsky and all work except that of Section 2.5 can be found in [PT].

2.1. **Definition of Sup-Completion.**

In this section, we recall the definition of a sup-completion from [Don82]. By a ***cone*** we mean a commutative semigroup with zero $(C, +, 0)$ equipped with a non-negative scalar multiplication operation $(\lambda, a) \in \mathbb{R}_+ \times C \mapsto \lambda a \in C$, which satisfies the following conditions: $\lambda(a+b) = \lambda a + \lambda b$, $(\lambda+\mu)a = \lambda a + \mu a$, $\lambda(\mu a) = (\lambda\mu)a$, $1a = a$ and $0a = 0$ for every $a, b \in C$ and $\lambda, \mu \in \mathbb{R}$. It is easy to see that the set C_0 of all invertible elements in C is a group. For example, if $C = (-\infty, \infty]$ then C is a cone and $C_0 = \mathbb{R}$ (we take $0 \cdot \infty = 0$). The scalar multiplication on $\mathbb{R}_+ \times C_0$ may be extended to $\mathbb{R} \times C_0$ via $(-r)x = -(rx)$ when $r > 0$; it is straightforward that C_0 is a vector space over $\mathbb{R}$.

We now impose several additional conditions that describe an order on C and the way C_0 "sits" in C:

(i) C is equipped with a partial order, such that $a \leqslant b$ implies $a + c \leqslant b + c$ and $\lambda a \leqslant \lambda b$ for all $a, b, c \in C$ and $\lambda \in \mathbb{R}_+$;
(ii) C is a lattice under this order;
(iii) C has a greatest element;
(iv) C_0 has an ideal property in C in the sense that if $x \in C_0$ and $a \in C$ such that $a \leqslant x$ then $a \in C_0$
(v) C is order complete in the sense that every subset A of C has supremum; if A is bounded below then $\inf A$ exists;
(vi) C_0 is order dense in C in the following sense: $a = \sup\{x \in C_0 \,:\, x \leqslant a\}$ for every $a \in C$;
(vii) $a + (x \wedge b) = (a + x) \wedge (a + b)$ whenever $a, b \in C$ and $x \in C_0$;
(viii) for any two non-empty subsets A and B of C, if $\sup A = \sup B$ and $x \in C_0$ then $\sup\{a \wedge x \,:\, a \in A\} = \sup\{b \wedge x \,:\, b \in B\}$ in C.

It is easy to see that C_0 is an order complete vector lattice. We say that C is a sup-completion of C_0. More precisely, if $\mathcal{E}$ is an order complete vector lattice and $\mathcal{E} = C_0$ for a cone C satisfying the properties listed above, we say that C is a ***sup-completion*** of $\mathcal{E}$. It was proven in [Don82] that every order complete vector lattice admits a sup-completion; Theorem 2.0.1 provides an alternative proof of this.

2.2. **Representation of Sup-Completion.**

The proof of 2.0.1 is tedious but straightforward. Let C be the set in the theorem:

$$C = \big\{u \in C(K, \overline{\mathbb{R}}) \,:\, u \geq f \text{ for some } f \text{ in } \mathcal{E}\big\}. \tag{1}$$

By definition, $\mathcal{E}$ is a subset of C. It is easy to see that the set $\{u > -\infty\}$ is open and dense for every $u \in C$.

We will now define operations on C. Non-negative scalar multiplication on C is defined pointwise; it clearly satisfies $(\lambda + \mu)u = \lambda u + \mu u$, $\lambda(\mu u) = (\lambda\mu)u$, $1u = u$, and $0u = 0$ for every $u \in C$ and $\lambda, \mu \in \mathbb{R}$. On $\mathcal{E}$, it agrees with the non-negative scalar multiplication of $C^\infty(K)$. Defining addition on C requires some care. We do it similarly to $C^\infty(K)$. Recall that K is extremally disconnected.

Lemma 2.2.1. *Suppose that $u\colon U \to \overline{\mathbb{R}}$ is a continuous function on an open dense subset U of K. Then u extends uniquely to a function in $C(K,\overline{\mathbb{R}})$.*

Proof. Since $\overline{\mathbb{R}}$ is topologically and order isomorphic to $[-1,1]$ via, say, $\tan\frac{\pi t}{2}$, we may replace $\overline{\mathbb{R}}$ in the statement with $[-1,1]$. So suppose that $u\colon U \to [-1,1]$. Let

$$G = \Big\{ v \in C\big(K,[-1,1]\big) \;:\; \forall t \in U \quad v(t) \geqslant u(t) \Big\}.$$

Since K is extremally disconnected, $C(K)$ is order complete. It follows from $G \geqslant -\mathbb{1}$ that $w := \inf G$ exists in $C(K)$. Clearly, $w \in C\big(K,[-1,1]\big)$.

We will now show that w extends u. Fix $t \in U$. Since K is extremally disconnected, we can find a clopen set V such that $t \in V \subseteq U$. Put $v = u \cdot \mathbb{1}_V + \mathbb{1}_{K\setminus V}$; then $v \in G$ and, therefore, $w \leqslant v$; it follows that $w(t) \leqslant u(t)$. On the other hand, for every $v \in G$ we have $v \geqslant u \cdot \mathbb{1}_V - \mathbb{1}_{K\setminus V}$, so that $w \geqslant u \cdot \mathbb{1}_V - \mathbb{1}_{K\setminus V}$ and, therefore, $w(t) \geqslant u(t)$. This proves that w extends u. Since U is dense, the extension is unique. $\square$

Corollary 2.2.2. *Let $u_1, u_2 \in C(K,\overline{\mathbb{R}})$ and let $U = \{u_1 > -\infty\} \cap \{u_2 > -\infty\}$. If U is dense then there exists a unique $u \in C(K,\overline{\mathbb{R}})$ such that $u(t) = u_1(t) + u_2(t)$ for all $t \in U$.*

Proof. Define $v\colon U \to \mathbb{R} \cup \{+\infty\}$ via $v(t) = u_1(t) + u_2(t)$; v is well defined and continuous. By Lemma 2.2.1, v extends to a function $u \in C(K,\overline{\mathbb{R}})$. Uniqueness follows from the density of U. $\square$

We are now ready to define addition on C. Suppose that $u_1, u_2 \in C$. Let $U = \{u_1 > -\infty\} \cap \{u_2 > -\infty\}$. There exist $f_1, f_2 \in \mathcal{E}$ such that $f_1 \leqslant u_1$ and $f_2 \leqslant u_2$. Let V be the set where both f_1 and f_2 are finite. Then V is dense. It follows from $V \subseteq U$ that U is dense. Let u be as in Corollary 2.2.2. For every $t \in V$, we have

$$(f_1 + f_2)(t) = f_1(t) + f_2(t) \leqslant u_1(t) + u_2(t) = u(t).$$

Since V is dense, continuity of $f_1 + f_2$ and u implies that $f_1 + f_2 \leqslant u$ on K. Therefore, $u \in C$. Naturally, we define $u = u_1 + u_2$. Clearly, this definition does not depend on the choice of f_1 and f_2. It is straightforward that $u_1 + u_2 = u_2 + u_1$ and that $\lambda(u_1 + u_2) = \lambda u_1 + \lambda u_2$ when $\lambda \in \mathbb{R}_+$ and $u_1, u_2 \in C$. Furthermore, if $u_1, u_2, u_3 \in C$, we have $(u_1 + u_2) + u_3 = u_1 + (u_2 + u_3)$ because the two continuous functions agree with $u_1(t) + u_2(t) + u_3(t)$ for every t is the dense set where all the three functions are different from $-\infty$. This shows that C is a cone.

We claim that $C_0 = \mathcal{E}$. Indeed, if $u \in C_0$ then $u, -u \in C$, hence, there exist $f, g \in \mathcal{E}$ such that $f \leqslant u \leqslant g$. It follows that u is finite on an open dense set, hence $u \in C^\infty(K)$. Since $\mathcal{E}$ is order complete, it is an ideal in $C^\infty(K)$ and, therefore, $u \in \mathcal{E}$. Conversely, if $f \in \mathcal{E}$ then, clearly, f and $-f$ are both in C, hence $f \in C_0$. Note that the operations of addition and non-negative scalar multiplication that we defined on C agree with those of $C^\infty(K)$ on $C \cap C^\infty(K)$. It follows that the vector space operations induced on C_0 by C agree with the "native" operations on $C^\infty(K)$.

We define order on C pointwise. It follows from the definition of C that if $u \in C$ and $v \in C(K, \overline{\mathbb{R}})$ with $u \leqslant v$ then $v \in C$. We will now verify conditions (i)–(viii).

(i), (ii), and (iii) are straightforward. It is easy to see that lattice operations on C are pointwise.

(iv) Suppose that $v \leqslant h$ for some $v \in C$ and $h \in \mathcal{E}$. There exists $f \in \mathcal{E}$ such that $f \leqslant v \leqslant h$. It follows that $v \in C^\infty(K)$ and, furthermore, $v \in \mathcal{E}$.

Observe that if $f \leqslant u \leqslant g$ for some $f, g \in \mathcal{E}$ and $u \in C(K, \overline{\mathbb{R}})$ then $u \in \mathcal{E}$. Indeed, it follows from $f \leqslant u$ that $u \in C$; it now follows from (iv) that $u \in \mathcal{E}$.

(v) As in the proof of Lemma 2.2.1, we observe that $C(K, \overline{\mathbb{R}})$ is order complete. Let $A \subseteq C$ with $A \neq \varnothing$. It follows that $v := \sup A$ exists in $C(K, \overline{\mathbb{R}})$. Take any $w \in A$, then $v \geqslant w \in C$ implies $v \in C$, hence v is the supremum of A in C. Now suppose that $u \leqslant A$ for some $u \in C$. Then $v := \inf A$ exists in $C(K, \overline{\mathbb{R}})$; it follows from $u \leqslant v$ that $v \in C$ and, therefore, v is the infimum of A in C.

(vi) Suppose that $u \in C$ and let $A = \{f \in \mathcal{E} \ : \ f \leqslant u\}$; we need to show that $u = \sup A$. By the definition of C, A is non-empty; fix some $h \in A$. We clearly have $A \leqslant u$. Suppose $A \leqslant v$ for some $v \in C$; it suffices to show that $u \leqslant v$. Suppose not. Then there exists $t_0 \in K$ with $v(t_0) < u(t_0)$. Find a clopen neighbourhood U of t_0 such that $v(t) < u(t)$ for all $t \in U$. Since $\mathbb{1}_U$ is in $C^\infty(K)$ and $\mathcal{E}$ is order dense in $C^\infty(K)$, we can find $g \in \mathcal{E}$ such that $0 < g \leqslant \mathbb{1}_U$. Then $g(t_1) > 0$ for some $t_1 \in U$. Let f be a scalar multiple of g such that $v(t_1) < f(t_1) < u(t_1)$. It follows from $f \wedge h \leqslant f \wedge u \leqslant f$ that $f \wedge u \in \mathcal{E}$ and, therefore, $f \wedge u \in A$. However, $(f \wedge u)(t_1) > v(t_1)$, which contradicts $A \leqslant v$.

(vii) Let $u, v \in C$ and $f \in \mathcal{E}$; we need to prove that $u + (f \wedge v) = (u + f) \wedge (u + v)$. Let U be the set on which u, v, and f are all different from $-\infty$. Then U is open and dense, and it is straightforward that the functions $u + (f \wedge v)$ and $(u + f) \wedge (u + v)$ agree on U. Since they are continuous, they are equal on K.

(viii) Suppose that $\sup A = \sup B$ for two non-empty subsets A and B of C, and let $f \in \mathcal{E}$. It suffices to show that $\sup(A \wedge f) \leqslant \sup(B \wedge f)$. Suppose not. Then there exists $u \in A$ such that $u \wedge f \nleqslant h$, where $h = \sup(B \wedge f)$. There exists $t \in K$ such that $u(t) \wedge f(t) > h(t)$. Fix $\lambda \in \mathbb{R}$ such that $u(t) \wedge f(t) > \lambda > h(t)$. By continuity, we can find a clopen neighbourhood U of t such that for all $s \in U$ we have $u(s) \wedge f(s) > \lambda > h(s) \geqslant v(s) \wedge f(s)$ for $v \in B$. It follows from $u(s) \wedge f(s) > \lambda > v(s) \wedge f(s)$ that $u(s) > \lambda > v(s)$ for all $s \in U$ and $v \in B$. Consider the function $w \in C(K, \overline{\mathbb{R}})$ that is equal to λ on U and $+\infty$ on $K \setminus U$. Then $w \geqslant v$ for all $v \in B$ and, therefore, $w \geqslant b$, where $b = \sup B$. It follows that $u(s) > w(s) \geqslant b(s)$ for all $s \in U$. This contradicts $\sup A = \sup B$.

This completes the proof of the theorem.

We will now establish some useful properties of C. Here C is as in (1). Let U be a clopen subset of K. We will see that many properties in C split "nicely" to U and its complement $K \setminus U$. Let $u, v \in C$. We say that u and v are disjoint and write $u \perp v$ if their supports are disjoint. It is easy to see that if u and v are in $\mathcal{E}^u$ then this concept agrees with the disjointness there. Furthermore, if $u \perp v$ then $u \vee v = u + v$. We define the product $v\mathbb{1}_U$ pointwise; clearly, $v\mathbb{1}_U \in C$. We have $v\mathbb{1}_U \perp v\mathbb{1}_{K\setminus U}$ and, therefore, $v = (v\mathbb{1}_U) \vee (v\mathbb{1}_{K\setminus U}) = (v\mathbb{1}_U) + (v\mathbb{1}_{K\setminus U})$. From the way we defined operations on C, we easily see that $(u+v)\mathbb{1}_U = u\mathbb{1}_U + v\mathbb{1}_U$. We write $\infty\mathbb{1}_U$ for the function that is constant infinity on U and constant zero on $K \setminus U$; clearly, it is in C. For $A \subseteq C$, we write $A \cdot \mathbb{1}_U = \{v\mathbb{1}_U : v \in A\}$. It is clear that the sets $A \cdot \mathbb{1}_U$ and $A \cdot \mathbb{1}_{K\setminus U}$ are disjoint; it follows from $A \cdot \mathbb{1}_U \leqslant \infty\mathbb{1}_U$ and $A \cdot \mathbb{1}_{K\setminus U} \leqslant \infty\mathbb{1}_{K\setminus U}$ that their suprema are disjoint as well. It follows that

$$\sup A = \sup(A \cdot \mathbb{1}_U) \vee \sup(A \cdot \mathbb{1}_{K\setminus U}) = \sup(A \cdot \mathbb{1}_U) + \sup(A \cdot \mathbb{1}_{K\setminus U}).$$

Lemma 2.2.3. *For every $u \in C$ and $A \subseteq \mathcal{E}$, $\sup(u + A) = u + \sup A$.*

Proof. For every $f \in A$ we have $f \leqslant \sup A$, so that $u + f \leqslant u + \sup A$ and, therefore, $\sup(u + A) \leqslant u + \sup A$. For the converse inequality, consider first the special case when $u \in C^\infty(K)$. Then $-u \in C^\infty(K)$ and we have $\sup A = \sup(-u + u + A) \leqslant -u + \sup(u + A)$, so that $u + \sup A \leqslant \sup(u + A)$.

For the general case, let U be the closure of $\{u < \infty\}$; observe that U is clopen. By the preceding discussion, we have

$$\sup(u + A) = \sup\big((u + A) \cdot \mathbb{1}_U\big) + \sup\big((u + A) \cdot \mathbb{1}_{K\setminus U}\big) \text{ and}$$

$$u + \sup A = \big(u\mathbb{1}_U + \sup(A \cdot \mathbb{1}_U)\big) + \big(u\mathbb{1}_{K\setminus U} + \sup(A \cdot \mathbb{1}_{K\setminus U})\big)$$

It suffices to prove that

$$\sup\bigl((u+A)\cdot\mathbb{1}_U\bigr) \geqslant u\mathbb{1}_U+\sup(A\cdot\mathbb{1}_U) \text{ and } \sup\bigl((u+A)\cdot\mathbb{1}_{K\setminus U}\bigr) \geqslant u\mathbb{1}_{K\setminus U}+\sup(A\cdot\mathbb{1}_{K\setminus U}).$$

The latter inequality is satisfied trivially because u is identically infinity on $K \setminus U$. In the former inequality, we essentially reduced everything to $C^\infty(U)$. Since the restriction of u to U belongs to $C^\infty(U)$, we now use the special case to complete the proof. $\square$

2.3. **Uniqueness.**

Donner in [Don82] proved that the sup-completion is unique. His proof relies on his construction of a sup-completion. We now present an alternative proof using our Theorem 2.0.1 instead (but our proof is built on the same ideas as that in [Don82]). We need the following variant of Riesz Decomposition Property:

Lemma 2.3.1. *Let C be a sup-completion of $\mathcal{E}$. Suppose that $x \leqslant u+v$ for some $x \in \mathcal{E}$ and $u, v \in C$ with $v \geqslant 0$. Then there exist $y, z \in \mathcal{E}$ such that $x = y+z$, $y \leqslant u$, and $z \leqslant v$.*

Proof. Since $x \wedge u \in \mathcal{E}$ by (iv), we can define $y = x \wedge u$ and $z = x - x \wedge u$ in $\mathcal{E}$. We clearly have $x = y + z$ and $y \leqslant u$. It is left to verify that $z \leqslant v$. Since $z \in \mathcal{E}$, $v - z = v + (-z)$ is defined. Using (vii), we get

$$v - z = (v - x) + x \wedge u = (v - x + x) \wedge (v - x + u) = v \wedge (u + v - x) \geqslant 0.$$

$\square$

Theorem 2.3.2. *Sup-completion of an order complete vector lattice is unique. That is, if C and D are two sup-completions of $\mathcal{E}$ then there exists a bijection $J\colon D \to C$ such that $u \leqslant v$ iff $Ju \leqslant Jv$, $J(\alpha u) = \alpha Ju$, and $J(u+v) = Ju + Jv$ when $u, v \in D$ and $\alpha \geqslant 0$, and J agrees with the identity on $\mathcal{E}$.*

Proof. WLOG, C is the sup-completion that we constructed in Theorem 2.0.1, i.e., C is as in (1). For a non-empty set $A \subseteq \mathcal{E}$, we define $J\bigl(\sup_D A\bigr) = \sup_C A$. Let's verify that J is well-defined. Suppose that A and B are two non-empty subsets of $\mathcal{E}$ with $\sup_D A = \sup_D B$. For every $a \in A$, (viii) yields

$$a = \sup\nolimits_D(A \wedge a) = \sup\nolimits_D(B \wedge a) = \sup\nolimits_X(B \wedge a) \leqslant \sup\nolimits_C B.$$

It follows that $\sup_C A \leqslant \sup_C B$. The opposite inequality is similar. It is left to verify that J is defined on all of D: if $u \in D$ then it follows from (vi) that $u = \sup_D A$ where $A = \{x \in \mathcal{E} \,:\, x \leqslant u\}$. Hence, $J(u) = \sup_C A$, that is,

$$J(u) = \sup\nolimits_C\{x \in \mathcal{E} \,:\, x \leqslant u\} \tag{2}$$

Since the definition of J is symmetric, it is easy to see that J is a bijection, with the inverse given by $J^{-1}\big(\sup_C A\big) = \sup_D A$ for $A \subseteq \mathcal{E}$. It is straightforward that $J(\alpha u) = \alpha J(u)$ when $u \in D$ and $\alpha \geqslant 0$. It follows from (2) that $u \leqslant v$ implies $J(u) \leqslant J(v)$. Since the definition of J is symmetric, the converse is also satisfied, hence $u \leqslant v$ iff $J(u) \leqslant J(v)$.

It is left to show that J is additive. If $u \in D$ and $y \in \mathcal{E}$, it follows from (2) and Lemma 2.2.3 that

$$J(y+u) = \sup_C\big\{x \in \mathcal{E} \,:\, x \leqslant y+u\big\} = \sup_C\big(y+A\big) = y + \sup_C A = y + J(u),$$

where $A = \{x \in \mathcal{E} \,:\, x \leqslant u\}$. Now fix $u, v \in D$. Put $A = \{x \in \mathcal{E} \,:\, x \leqslant u\}$ and $B = \{y \in \mathcal{E} \,:\, y \leqslant v\}$. Then $u = \sup_D A$ and $v = \sup_D B$ by (vi), and $J(u) = \sup_C A$ and $J(v) = \sup_C B$ by (2). Fix $x \in A$ and $y \in B$. We have $x + y \leqslant u + v$. It follows from (2) that $x + y \leqslant J(u+v)$, so that $x \leqslant J(u+v) - y$. Taking supremum in C over $x \in A$, we get $J(u) \leqslant J(u+v) - y$ and, therefore, $J(u) + y \leqslant J(u+v)$. By Lemma 2.2.3, we have

$$J(u)+J(v) = J(u)+\sup_C B = \sup_C(J(u)+B) = \sup_C\{J(u)+y \,:\, y \in B\} \leqslant J(u+v).$$

To prove the other inequality, we first assume that $v \geqslant 0$. Note that $J(u+v) = \sup_C B$, where $B = \{x \in \mathcal{E} \,:\, x \leqslant u+v\}$. For every $x \in B$, find y and z as in Lemma 2.3.1. It follows that

$$B \subseteq \{y \in \mathcal{E} \,:\, y \leqslant u\} + \{z \in \mathcal{E} \,:\, z \leqslant v\} \leqslant J(u) + J(v),$$

so that $J(u+v) \leqslant J(u) + J(v)$.

Finally, if u and v are two arbitrary elements of D, it follows from $v^- \in \mathcal{E}$ that $J(u+v) = J(u+v^+-v^-) = J(u+v^+)-v^- \leqslant J(u)+J(v^+)-v^- = J(u)+J(v^+-v^-) = J(u) + J(v)$. □

2.4. **Applications.**

We now use our representation of sup-completion to provide simple proofs of several results of [Azo19, AN22]. We write $\mathcal{E}^u$ and $\mathcal{E}^s$ for the universal completion and the sup-completion of $\mathcal{E}$, respectively. As before, we represent $\mathcal{E}$ as an order dense sublattice of $C^\infty(K)$ for some extremally disconnected compact K; we represent $\mathcal{E}^s$ as in Theorem 2.0.1.

We start by revisiting Corollary 7 in [Azo19]. It follows immediately from Theorem 2.0.1 that every non-negative function in $C(K, \overline{\mathbb{R}})$ belongs to $\mathcal{E}^s$. In particular, we have $\mathcal{E}^u_+ \subseteq \mathcal{E}^s$. Suppose now that Y is an order dense order complete sublattice of $\mathcal{E}$. Then clearly Y is still order dense in $C^\infty(K)$, hence $C^\infty(K) = Y^u$, and

$$Y^s = \big\{u \in C(K, \overline{\mathbb{R}}) \,:\, u \geqslant f \text{ for some } f \text{ in } Y\big\}.$$

It follows that both $(\mathcal{E}^s)_+$ and $(Y^s)_+$ consist of all non-negative function in $C(K, \overline{\mathbb{R}})$ and, therefore, $(\mathcal{E}^s)_+ = (Y^s)_+$. Note that if $Y^s = \mathcal{E}^s$ then $Y = \mathcal{E}$ because every negative $f \in \mathcal{E}$ belongs to Y by (iv).

Recall that if $\mathcal{E}$ has a weak unit e, one can choose the representation so that $e = \mathbb{1}$.

Proposition 2.4.1 ([Azo19])**.** *If e is a weak unit in $\mathcal{E}_+$ and $0 \leqslant u \in \mathcal{E}^s$ then $u = \sup_n(ne \wedge u)$.*

Proof. WLOG, $e = \mathbb{1}$. Let $v = \sup_n(n\mathbb{1} \wedge u)$. Clearly, $v \leqslant u$. Fix $t \in K$. Then $v(t) \geqslant n \wedge u(t)$ for all n. Considering separately the cases when $u(t) = \infty$ and when $u(t) < \infty$, we see that $v(t) \geqslant u(t)$ and, therefore, $v \geqslant u$. $\square$

By Maeda-Ogasawara theory (see, e.g., Chapter 7 in [AB03]), there is a one-to-one correspondence between clopen subsets of K and bands in $\mathcal{E}$: if U be a clopen set in K then the set $\{x \in \mathcal{E} \,:\, \operatorname{supp} x \subseteq U\}$ is a band in $\mathcal{E}$, and every band in $\mathcal{E}$ is of this form; we denote it by B_U. The corresponding band projection P_U is given by $P_U x = x \cdot \mathbb{1}_U$. It is clear that the universal completion of B_U is $C^\infty(U)$ and, hence the sup-completion of B_U can be computed as in Theorem 2.0.1.

In particular, for $a \in \mathcal{E}$, the principal band projection P_a is given by the following: for $x \in \mathcal{E}$ and $t \in K$, we have

$$(P_a x)(t) = \begin{cases} x(t) & \text{if } t \in U, \text{ and} \\ 0 & \text{otherwise ,} \end{cases} \qquad \text{where } U = \overline{\{a \neq 0\}}.$$

The preceding formula clearly extends to the case when $a \in \mathcal{E}^s$, yielding a band projection on $\mathcal{E}$.

In Theorem 15 of [AN22], the authors prove that every element u in $\mathcal{E}^s$ can be split into its finite and infinite parts. Using our representation, this is now easy: let U be the closure of $\{u < \infty\}$, then the finite part $\mathcal{E}$ of u is defined as $x = P_U u$ and is the function that agrees with u on U and vanishes on U^C, while the infinite part w of u is defined as $P_{U^C} u$ and is the function that vanishes on U and is identically ∞ on U^C. Clearly, $u = x + w$ and $x \perp w$. It follows from $x \in C^\infty(K)$ that $x \in \mathcal{E}^u$. It follows from $u \in \mathcal{E}^s$ that $u \geqslant f$ for some $f \in \mathcal{E}$, hence $P_U u \geqslant P_U f$ and, therefore, $x \in \mathcal{E}^s$. Clearly, $u \in \mathcal{E}^u$ iff its infinite part w equals zero and u equals x, its finite part. Note also that x may be viewed as the restriction of u to U; so we may view x as an element of $C^\infty(U)$ and of U^s.

We will now present a simple proof of Riesz Decomposition Theorem in $\mathcal{E}^s$; cf. Lemma 2.3.1, [Azo19, Lemma 1], and [AN22, Lemma 8].

Lemma 2.4.2. *Suppose that $x \leqslant u + v$ for some $x, u, v \in \mathcal{E}^s$. Then there exist $y, z \in \mathcal{E}^s$ such that $x = y + z$, $y \leqslant u$, and $z \leqslant v$.*

Proof. We will decompose K into a disjoint union of clopen sets, define y and z on those sets, and then concatenate the functions to form the final decomposition. Let $A = \{u = \infty\}^\circ$, $B = \{v = \infty\}^\circ$, and $C = \{x = \infty\}^\circ$. It is clear that these three sets are clopen, the functions u, v, and x are constant infinity on the corresponding sets, and $C \subseteq A \cup B$. The restrictions of the functions to $U := K \setminus (A \cup B)$ are in $C^\infty(U)$, so we use the classical Riesz Decomposition Theorem for vector lattices to define y and z on U. On $A \cap B$, we define $y = x$ and $z = 0$. On $(A \cap C) \setminus B$, we put $y = x = \infty$ and $z = v$. On $A \setminus (B \cup C)$, we put $z = v$ and $y = x - v$ (note that $-v$ exists on this set). It is now easy to verify that y and z satisfy the requirements of the lemma on these sets. We handle $(B \cap C) \setminus A$ and $B \setminus (A \cup C)$ similarly. $\square$

Theorem 2.4.3 ([Azo19])**.** *Let $e \in \mathcal{E}_+$ be a weak unit and $u \in \mathcal{E}^s$. Then*

$$u \notin \mathcal{E}^u \qquad \textit{iff} \qquad \inf_{\lambda \in (0,\infty)} P_{(u-\lambda e)^+} e > 0.$$

Proof. WLOG, we may chose the representation so that $e = \mathbb{1}$. Let $v = \inf_{\lambda \in (0,\infty)} v_\lambda$, where $v_\lambda = P_{(u-\lambda\mathbb{1})^+}\mathbb{1}$. Suppose that $u \notin \mathcal{E}^u$. Then $\operatorname{Int}\{u = \infty\}$ in non-empty; denote this set by V. For every $t \in V$ and every $\lambda \in (0,\infty)$ we have $(u - \lambda\mathbb{1})^+(t) = \infty$, so that $v_\lambda(t) = 1$. It follows that $v_\lambda \geqslant \mathbb{1}_{\overline{V}}$ and, therefore, $v \geqslant \mathbb{1}_{\overline{V}} > 0$.

Conversely, suppose that $v > 0$. Then the set $W := \{v > 0\}$ is open. Fix $t \in W$. For every $\lambda \in (0,\infty)$ it follows from $v_\lambda \geqslant v$ that $v_\lambda(t) > 0$, so that $(u - \lambda\mathbb{1})^+(t) > 0$ and, therefore, $u(t) \geqslant \lambda$. It follows that $u(t) = \infty$ for all $t \in W$ and therefore, $u \notin C^\infty(K)$. $\square$

It is proved in Theorem 19 in [AN22] that if $0 \leqslant u \in \mathcal{E}^s$ and e is a weak unit in $\mathcal{E}$ then $P_w e = \inf_{\lambda \in (0,\infty)} P_{(u-\lambda e)^+} e$, where w is the infinite part of u. This fact can now be easily proved analogously to Theorem 2.4.3. Other properties of decompositions of u into the finite and the infinite part in [AN22] can be proved in a similar way.

Fix a weak unit e in $\mathcal{E}_+$. WLOG, we may assume that e corresponds to $\mathbb{1}$ in the $C^\infty(K)$ representation of $\mathcal{E}^u$. Similarly to how we defined addition on $C^\infty(K)$, one can define multiplication, making $\mathcal{E}^u$ into an f-algebra with e being a multiplicative unit. Recall that $(\mathcal{E}^s)_+$ consists of all continuous positive functions from K to $[0,\infty]$. Similarly to how we defined addition on $\mathcal{E}^s$, we can define product on $(\mathcal{E}^s)_+$. That is, $uv = w$ iff $u(t)v(t) = w(t)$ for all t in an open dense set; we again follow the convention that $0 \cdot \infty = 0$. It is easy to see that the resulting product agrees with that defined in Section 3.2 of [AN22].

Recall the following well-known fact of vector lattice theory: given any identity or inequality that involves finitely many variables, and linear and lattice operations, if it is valid in $\mathbb{R}$ then it remains valid in every vector lattice. This fact follows easily from Krein-Kakutani Representation Theorem, see, e.g., [AB06, Theorem 4.29]. For example, the identity $(x-y)^+ \wedge (y-z)^+ \wedge (z-x)^+ = 0$ is valid in $\mathbb{R}$ (for any $x, y, z \in \mathbb{R}$) and, therefore, it is valid in every vector lattice. We will now extend this idea to sup-completions.

Proposition 2.4.4. *Consider a formula*

$$\Phi(r_1, \dots, r_n) \leqslant \Psi(r_1, \dots, r_n), \tag{3}$$

where $r_1, \dots, r_n$ are formal variables and Φ and Ψ are expressions only involving addition, positive scalar multiplication, $\vee$, and $\wedge$. Suppose that (3) *is valid when $r_1, \dots, r_n$ are interpreted as (arbitrary) elements of $\mathbb{R}$. Then* (3) *remains valid if $r_1, \dots, r_n$ are interpreted as elements of $\mathcal{E}^s$.*

Proof. Both $\Phi(r_1, \dots, r_n)$ and $\Psi(r_1, \dots, r_n)$ are continuous functions on $\mathbb{R}^n$, increasing in every variable. It follows that they may be extended continuously to functions from $\big(\mathbb{R} \cup \{\infty\}\big)^n$ to $\mathbb{R} \cup \{\infty\}$, and inequality (3) remains valid for all $r_1, \dots, r_n \in \mathbb{R} \cup \{\infty\}$. Now let $x_1, \dots, x_n \in \mathcal{E}^s$; view them as functions in $C(K, \overline{\mathbb{R}})$. Put $A = \bigcup_{i=1}^n \{x_i = -\infty\}$; then A is nowhere dense. By the preceding argument, we have

$$\Phi\big(x_1(t), \dots, x_n(t)\big) \leqslant \Psi\big(x_1(t), \dots, x_n(t)\big)$$

for all $t \notin A$. It follows from the way we defined operations in $\mathcal{E}^s$ that $\Phi(x_1, \dots, x_n) \leqslant \Psi(x_1, \dots, x_n)$ where the expressions are interpreted in $\mathcal{E}^s$. ☐

Remark 2.4.5. If we restrict $x_1, \dots, x_n$ to $(\mathcal{E}^s)_+$ then we may, in a similar way, allow the formulae Φ and Ψ to involve products; in this case, it suffices to verify (3) for all $r_1, \dots, r_n \in \mathbb{R}_+$.

The method described above allows us to immediately deduce various identities and inequalities in $\mathcal{E}^s$, e.g., $x + (y \vee z) = (x+y) \vee (x+z)$, $x + y = x \vee y + x \wedge y$, $x(y+z) = xy + xz$, etc; cf. Lemmas 11, 12, and 24 in [AN22].

2.5. **Convergence in Sup-Completion.**

We now use the representation from the above sections to obtain a criterion for order convergence in a sup-completion. The results in this section are motivated by the results from Section 3 of [BT22] and use many of the results from that paper. The general idea is to split elements of the sup-completion into its finite and infinite parts, study the convergence on them separately and, combine the results in the end to obtain convergence over the sup-completion. While $\mathcal{E}^s$ is not a vector lattice, it is still a lattice and we can consider order convergence in it. We state the following definition of order convergence in a lattice below:

Definition 2.5.1. Given a net $(x_\alpha)_{\alpha\in\Delta}$ in $\mathcal{E}^s$ and $x \in \mathcal{E}^s$, we say that $x_\alpha \xrightarrow{\text{o}} x$ if there exists two nets $(a_\gamma)_{\gamma\in\Gamma}$ and $(b_\gamma)_{\gamma\in\Gamma}$ such that $a_\gamma \uparrow x, b_\gamma \downarrow x$, and for every $\gamma \in \Gamma$ there exists $\alpha_0 \in \Delta$ such that for every $\alpha \geq \alpha_0$ we have $a_\gamma \leq x_\alpha \leq b_\gamma$.

This definition is equivalent to saying that $x_\alpha \xrightarrow{\text{o}} x$ if there exists two non-empty sets A and B such that $\sup A = x = \inf B$ and for every $a \in A$ and $b \in B$ there exists α_0 such that $a \leq x_\alpha \leq b$ for all $\alpha \geq \alpha_0$.

Lemma 2.5.2. *For $G \subseteq \mathcal{E}^s_+, \sup G = \infty\mathbb{1}$ iff for every non-empty open set U and every $n \in \mathbb{R}_+$ there exists a non-empty open set $V \subseteq U$ and $g \in G$ with $g(t) > n$ for all $t \in V$.*

Proof. Suppose that $\sup G \neq \infty\mathbb{1}$. Then there exists $f \in \mathcal{E}^s_+$ with $G \leq f < \infty\mathbb{1}$. Hence, there exists a point $t \in K$ such that $f(t) < \infty$. Since K is a totally disconnected space, we can find a clopen subset $U \subset K$ containing t such that $f(U) < \infty$. Hence, by compactness there exists $n \in \mathbb{N}$ such that f is less than n on U. It follows that every $g \in G$ is less than n on every open subset V of U.

Suppose now that there exists an open non-empty set U and $n \in \mathbb{R}_+$ such that for every non-empty open subset $V \subseteq U$ and $g \in G$ there exists $t \in V$ such that $g(t) \leq n$. So for a given $g \in G$, we have that $\{g \leq n\}$ is a closed set and it intersects every open subset of U. This implies that $\{g \leq n\}$ contains U. Thus, we have that $\cap_{g\in G}\{g \leq n\} \supseteq U$ and clearly, $\sup G \leq \infty\mathbb{1}_{\overline{U}^c} + n\mathbb{1}_{\overline{U}} < \infty\mathbb{1}$. ☐

Lemma 2.5.3. *For $G \subseteq \mathcal{E}^s$, TFAE:*

(i) $\sup G = \infty\mathbb{1}$

(ii) There exists a dense set D such that $\sup_{g\in G} g(t) = \infty$ for every $t \in D$;

(iii) There exists a co-meagre set D such that $\sup_{g\in G} g(t) = \infty$ for every $t \in D$.

Proof. By Lemma 2.5.2, we have (ii) implies (i). Also, (iii) implies (ii) since in a Baire space, a co-meagre set is dense. Thus, it remains to prove (i) implies (iii). Let us assume that $\sup G = \infty\mathbb{1}$. Then for $n \in \mathbb{N}$ let us denote $V_n := \bigcup_{g\in G}\{g > n\}$ which is clearly an open set. By Lemma 2.5.2, for every open set U, we have that there exists a $t \in U$ and $g \in G$ such that $g(t) > n$ and thereby, $t \in V_n$. Hence, V_n is an open dense subset of K. Let $D := \bigcap_{n\in\mathbb{N}} V_n$. Then D is a co-meagre set such that for every $t \in D$ and $n \in \mathbb{N}$, we have $\sup_{g\in G} g(t) \geq n$. This results in $\sup_{g\in G} g(t) = \infty$ for every $t \in D$. $\square$

Lemma 2.5.4. *Let $X \in \mathcal{E}_+^s$ and $(a_\gamma), (b_\gamma) \subset \mathcal{E}_+^s$ such that $a_\gamma \uparrow X$ and $b_\gamma \downarrow X$. Then there exists a co-meagre set $M \subseteq K$ such that $a_\gamma(t) \uparrow X(t)$ and $b_\gamma(t) \downarrow X(t)$ for every $t \in M$.*

Proof. Let us denote $U = \overline{\{X < \infty\}}$, which gives that U is a clopen set and $X\mathbb{1}_U \in C^\infty(K)$. WLOG, we can assume that every element of the net b_γ is finite on the set where X is finite. If not we can replace b_γ with $b_\gamma \wedge (X + \mathbb{1})$ since $b_\gamma \downarrow X$ if and only if $b_\gamma \wedge (X + \mathbb{1}) \downarrow X$. So now let us split all the elements as follows: $a_\gamma = a_\gamma\mathbb{1}_U + a_\gamma\mathbb{1}_{U^c}$ and $b_\gamma = b_\gamma\mathbb{1}_U + b_\gamma\mathbb{1}_{U^c}$. This results in $a_\gamma\mathbb{1}_U \uparrow X\mathbb{1}_U, a_\gamma\mathbb{1}_{U^c} \uparrow X\mathbb{1}_{U^c}, b_\gamma\mathbb{1}_U \downarrow X\mathbb{1}_U$ and $b_\gamma\mathbb{1}_{U^c} \downarrow X\mathbb{1}_{U^c}$. Clearly, $a_\gamma\mathbb{1}_U \in C^\infty(K)_+$ since $a_\gamma\mathbb{1}_U \leq X\mathbb{1}_U \in C^\infty(K)_+$. Similarly, $b_\gamma\mathbb{1}_U \in C^\infty(K)_+$ since $b_\gamma\mathbb{1}_U \leq X\mathbb{1}_U + \mathbb{1}_U \in C^\infty(K)_+$. Hence, by [Remark 4.1, [BT22]], there exists a co-meagre subset of U such that convergence on it is pointwise.

Thus, it remains to show that upward convergence on U^c is pointwise on a co-meagre set. Note that the value for X on U^c is ∞ everywhere. When $b_\gamma\mathbb{1}_{U^c} \downarrow X\mathbb{1}_{U^c}$, we clearly have that $b_\gamma\mathbb{1}_{U^c}(t) = \infty = X\mathbb{1}_{U^c}(t)$ for every $t \in U^c$. Let us denote $G = (a_\gamma\mathbb{1}_{U^c})_{n\in\mathbb{N}} \subset \mathcal{E}_+^s$. Then $\sup G = \infty\mathbb{1}_{U^c}$ and by applying Lemma 2.5.3 we get that there exists a co-meagre subset V of U^c such that $a_\gamma(t) \uparrow \infty$ for every $t \in V$. Therefore, upon combining the co-meagre subsets of U and U^c, we obtain the desired co-meagre subset M of K where $a_\gamma(t) \uparrow X(t)$ and $b_\gamma(t) \downarrow X(t)$ for every $t \in M$. $\square$

Theorem 2.5.5. *Let $X \in \mathcal{E}_+^s$ and $(X_\alpha)_{\alpha\in\Delta} \subset \mathcal{E}_+^s$. Then $X_\alpha \xrightarrow{o} X$ iff for every non-empty open set U and every $n \in \mathbb{N}$ there exists a non-empty clopen $V \subseteq U$ and an index α_0 such that at least one of the following is true:*

- *X_α, X are finite on V and $|X_\alpha - X|$ is less than $\frac{1}{n}$ on V whenever $\alpha \geq \alpha_0$.*
- *$X_\alpha, X \geq n$ on V whenever $\alpha \geq \alpha_0$.*

Proof. Let us suppose that $X_\alpha \xrightarrow{o} X$. Fix a non-empty set U and $n \in \mathbb{N}$. Let us denote $W := \{X = \infty\}^\circ$ and consider the case when $W \cap U$ is non-empty. Then there exists two nets $(a_\gamma)_{\gamma\in\Gamma}$ and $(b_\gamma)_{\gamma\in\Gamma}$ such that $a_\gamma \uparrow X, b_\gamma \downarrow X$, and for every $\gamma \in \Gamma$ there exists $\alpha_0 \in \Delta$ such that for every $\alpha \geq \alpha_0$ we have $a_\gamma \leq X_\alpha \leq b_\gamma$. Clearly, $a_\gamma \uparrow X$ implies that $a_\gamma \cdot \mathbb{1}_W \uparrow X \cdot \mathbb{1}_W$. Since, W is a clopen set, we can consider the functions $a_\gamma \cdot \mathbb{1}_W, X \cdot \mathbb{1}_W$ to belong to $[C^\infty(W)]^s$. Then upon applying Lemma 2.5.2 we get that there exists a clopen set $V \subseteq W \cap U$ and $\gamma \in \Gamma$ with $a_\gamma(t) > n$ for all

$t \in V$. Therefore, we have $X_\alpha(t), X(t) \geq n$ for all $\alpha \geq \alpha_0$ and $t \in V$.

Instead suppose that $W \cap U$ is empty. By multiplying the net by $\mathbb{1}_{W^c}$, we have that $X_\alpha \cdot \mathbb{1}_{W^c} \xrightarrow{o} X \cdot \mathbb{1}_{W^c}$. This is true because, $a_\gamma \cdot \mathbb{1}_{W^c}$ and $b_\gamma \cdot \mathbb{1}_{W^c}$ remain an eventual lower and upper bound respectively. However, we also have that $(X_\alpha \cdot \mathbb{1}_{W^c}) \wedge (X \cdot \mathbb{1}_{W^c} + \mathbb{1}_{W^c}) \xrightarrow{o} X \cdot \mathbb{1}_{W^c}$. But since all the functions belong to $C^\infty(K)_+$, we have that $(X_\alpha \cdot \mathbb{1}_{W^c}) \wedge (X \cdot \mathbb{1}_{W^c} + \mathbb{1}_{W^c}) \xrightarrow{uo} X \cdot \mathbb{1}_{W^c}$ and using [Remark 4.1, [BT22]] we get that there exists a non-empty open set $V \subseteq U$ and α_0 such that

$$|(X_\alpha \cdot \mathbb{1}_{W^c}) \wedge (X \cdot \mathbb{1}_{W^c} + \mathbb{1}_{W^c}) - X \cdot \mathbb{1}_{W^c}|(t) \leq \frac{1}{n}$$

for all $\alpha \geq \alpha_0$ and $t \in V$. But this clearly implies that X_α, X are finite on V and $|X_\alpha - X|$ is less than $\frac{1}{n}$ on V whenever $\alpha \geq \alpha_0$.

We shall now prove the converse. Fix an open non-empty set U and $n \in \mathbb{N}$. Let V and α_0 be as in the assumption. Let us suppose that the first condition is true, then $|X_\alpha - X|$ is less than $\frac{1}{n}$ on V whenever $\alpha \geq \alpha_0$. This implies that X_α and X are finite on the clopen set V and thus, $X_\alpha \cdot \mathbb{1}_V, X \cdot \mathbb{1}_V \in C(K)_+$ for all $\alpha \geq \alpha_0$. Then set $h = X\mathbb{1}_V - \frac{1}{n}\mathbb{1}_V$ and $g = X + \frac{1}{n}\mathbb{1}_V + \infty\mathbb{1}_{V^c}$. Thus, we have that $h \leq X, X_\alpha \leq g$ for every $\alpha \geq \alpha_0$. Now if the second condition is true, set $g = \infty\mathbb{1}$ and $h = n\mathbb{1}_V$. Since $X_\alpha, X \geq n$ on V whenever $\alpha \geq \alpha_0$ we get that $h \leq X, X_\alpha \leq g$.

Upon repeating this process for every pair (U, n) where U is an open set in K, we get a collection of functions g and h which we shall denote by G and H respectively. Clearly $\sup H \leq X$ and $\inf G \geq X$. Using Lemma 2.5.2, we can deduce that $(\sup H) \cdot \mathbb{1}_{\{X=\infty\}^\circ} = \infty\mathbb{1}_{\{X=\infty\}^\circ}$. From the above step, we have that $\{X = \infty\}^\circ \subseteq \{g = \infty\}, \forall g \in G$ and thus, $(\inf G) \cdot \mathbb{1}_{\{X=\infty\}^\circ} = \infty\mathbb{1}_{\{X=\infty\}^\circ}$. On the other hand, if U is an open subset of $\overline{\{X < \infty\}}$ then by compactness of the clopen subsets and by passing to a sequence of clopen sets, we can find a point $t \in U$ such that $(\sup H)(t) = X(t)$ and $(\inf G)(t) = X(t)$. Hence, $\sup H$ and $\inf G$ are equal to X on a dense set and therefore, $\inf G = X$ and $\sup H = X$. Pick any $g \in G$ and $h \in H$. Then there exists $\alpha_1, \alpha_2 \in \Delta$ such that for every $\alpha \geq \alpha_1$ we have $X_\alpha \leq g$ and for every $\alpha \geq \alpha_2$ we have $h \leq X_\alpha$. Finally denoting $\alpha_0 = \alpha_1 \vee \alpha_2$ proves that $X_\alpha \xrightarrow{o} X$. □

The above topological property allows us to deduce properties regarding the order convergence in the sup-completion.

Theorem 2.5.6. *Let $X \in \mathcal{E}_+^s$ and $(X_\alpha)_{\alpha \in \Delta} \subset \mathcal{E}_+^s$. If $X_\alpha \xrightarrow{o} X$ then X_α converges to X pointwise on a co-meagre set. The converse is true for countable nets.*

Proof. Suppose that $X_\alpha \xrightarrow{o} X$. Then there exists two nets $(a_\gamma)_{\gamma\in\Gamma}$ and $(b_\gamma)_{\gamma\in\Gamma}$ such that $a_\gamma \uparrow X, b_\gamma \downarrow X$, and for every $\gamma \in \Gamma$ there exists $\alpha_0 \in \Delta$ such that for every $\alpha \geq \alpha_0$ we have $a_\gamma \leq X_\alpha \leq b_\gamma$. By Lemma 2.5.4, we have that there exists a co-meagre set M such that $a_\gamma(t) \uparrow X(t)$ and $b_\gamma(t) \downarrow X(t)$ for every $t \in M$ and, therefore, $\lim_\alpha X_\alpha(t) = X(t)$.

WLOG, we prove the converse for sequences. Let X_n be a sequence that converges to X on a co-meagre set D. Fix an non-empty open set U and $n \in \mathbb{N}$. Let us first assume that there exists a non-empty clopen subset V of U such that X is equal to infinity on V. Then due to pointwise convergence on a co-meagre subset of V and by Lemma 2.5.3, we know that $\sup(X_n \cdot \mathbb{1}_V) = \infty\mathbb{1}_V = X \cdot \mathbb{1}_V$. On the other hand, let us assume that there exists no clopen subset of U such that X is equal to infinity on them. i.e., $X\mathbb{1}_U \in C^\infty(K)$. So consider $Y_n = X_n \wedge (X + \mathbb{1})$. Then $Y_n\mathbb{1}_U \in C^\infty(K)$ and therefore $Y_n\mathbb{1}_U$ converges to $X\mathbb{1}_U$ pointwise on a co-meagre subset of U. Then using [Remark 4.1, [BT22]], we have that $Y_n\mathbb{1}_U \xrightarrow{uo} X\mathbb{1}_U$ in $C^\infty(K)$ and therefore, $Y_n\mathbb{1}_U \xrightarrow{o} X\mathbb{1}_U$ in $C^\infty(K)$. Thus, there exists an open subset V of U and an index n_0 such that for all $n \geq n_0$, we have $|Y_n - X| \leq \frac{1}{n}$ on V. Then applying Lemma 2.5.5 finishes the proof. $\square$

3. Basics of Measure-Free probability

Conditional expectations have been studied in an operator theoretic setting in [Rao76, GdP02], as positive operators acting on L_p-spaces. In [KLW04a, KLW05, Kuo06], it is shown that many of the fundamental results about conditional expectation operators and stochastic processes can be formulated and proved in the measure-free framework of vector lattices. In this Chapter, the generalized vector lattice definitions of many of the concepts from classical probability theory are provided as well as the motivation behind these definitions. This chapter serves as a preliminary to the Chapters 4 and 5 and does not contain any original work by the author.

In Section 3.1, we recall the classical definitions of a conditional expectation operator. After reviewing some basic properties of this definition, a generalized definition of a conditional expectation operator in a vector lattice is produced. As in the classical probability theory, we then show that this operator can be extended to its natural domain using the sup-completion. Until recently, there was uncertainty about whether this generalized conditional expectation operator could be constructed in vector lattices other than L_p spaces. In [Amo22], the author constructed the example of a conditional expectation operator on $C(K)$, which we will detail in the end of the Section.

We then turn our attention to martingales and filtrations. In Section 3.2, after reviewing some properties, the generalized definition of a sub (super) martingale and filtration in a vector lattice is inferred. It is then shown that many of the properties of martingales from classical probability theory can be stated in the vector lattice setting, such as the Doob-Meyer decomposition theorem for sub (super) martingales. This finally allows us to study discrete stopping times in vector lattices in Section 3.3. Stopping times on vector lattices are defined as increasing families of projections. We also define stopped processes and the stopped conditional expectation here, and show the compatibility of the stopped process and stopped conditional expectation. We conclude with stating the vector lattice versions of the optional stopping theorems.

3.1. **Conditional expectation operator.**

Throughout this chapter, we fix $(\Omega, \mathcal{F}, \mu)$ to be a probability space and denote χ to be the constant one function in $L_1(\mu) := L_1(\Omega, \mathcal{F}, \mu)$. A random variable is a real valued element of $L_1(\mu)$ and the expectation of the random variable f is

$$\mathbb{E}(f) = \int_\Omega f dP$$

For Σ a sub$-\sigma-$algebra of $\mathcal{F}$, an element $g \in L_1(\Omega, \Sigma, \mu)$ is called the conditional expectation of f relative to Σ if g is $\Sigma-$measurable and

$$\int_A g d\mu = \int_A f d\mu$$

for all $A \in \Sigma$. In this case, we denote g by $\mathbb{E}[f|\Sigma]$. Conditional expectations play an important role in probability theory and satisfy some nice properties. Namely, $\mathbb{E}(\cdot|\Sigma) : L_1(\mu) \to L_1(\Omega, \Sigma, \mu)$ is a projection on $L_1(\mu)$ since $L_1(\Omega, \Sigma, \mu) \subseteq L_1(\mu)$. Moreover, $\mathbb{E}(\cdot|\Sigma)$ is a positive linear map. As μ is a probability measure, the constant function χ is a weak unit in $L_1(\mu)$. Also $f \in L_1(\mu)$ is a weak unit of $L_1(\mu)$ if and only if $f > 0$ a.e., and in this case $\mathbb{E}(f|\Sigma)$ is a weak unit of $L_1(\Omega, \Sigma, \mu)$. We refer the reader to [Wil91] for more information on conditional expectations on probability spaces.

In the fact the above mentioned properties characterize conditional expectations. This is due to the result by [Rao76], which relates contractive projections to conditional expectations.

Theorem 3.1.1. *If $T : L_1(\mu) \to L_1(\mu)$ is a positive contractive projection with $T\chi = \chi$, then $Tf = \mathbb{E}(f|\Sigma)$ where $f \in L_1(\Omega, \Sigma, \mu)$, for a unique $\sigma-$algebra $\Sigma \subset \mathcal{F}$.*

Conditional expectations also are also order continuous. That is, if f_n is an almost everywhere (a.e.) increasing sequence in $L_1(\mu)$ with a.e. pointwise limit $f \in L_1(\mu)$, then $\mathbb{E}(f_n|\Sigma)$ is an increasing sequence in $L_1(\Omega, \Sigma, \mu)$ with a.e. pointwise limit $\mathbb{E}(f|\Sigma)$. We summarize the above properties of conditional expectation in the following:

(i) $f \to \mathbb{E}(f|\Sigma)$ is linear;
(ii) if $f \geq 0$ then $\mathbb{E}(f|\Sigma) \geq 0$;
(iii) $\mathbb{E}(\chi|\Sigma) = \chi$;
(iv) $\mathbb{E}[\mathbb{E}(f|\Sigma)|\Sigma] = \mathbb{E}(f|\Sigma)$, i.e. $\mathbb{E}(\cdot|\Sigma)$ is idempotent;
(v) If $f_n \uparrow f$ in $L_1(\Omega, \mathcal{F}, P)$ then $\mathbb{E}(f_n|\Sigma) \uparrow \mathbb{E}(f|\Sigma)$ in $L_1(\Omega, \Sigma, P)$, i.e. $\mathbb{E}(\cdot|\Sigma)$ is order continuous.

Therefore, it is evident from properties i-v, that a conditional expectation is a linear positive order continuous projection on $L_1(\mu)$, that maps weak units to weak units and has range which is an order complete sublattice of $L_1(\mu)$. Now it is easy to see that the properties above are stated using using lattice terminology can be used to define a conditional expectation operator on arbitrary vector lattices. Now in order to generalize the property $\mathbb{E}(\chi|\mathcal{F}) = \chi$, we assume that our vector lattice has a weak unit and require that the conditional expectation of a weak unit is again a weak unit. These properties along with the following Theorem 3.1.2, motivate Definition 3.1.3 for conditional expectation operators on a vector lattice.

Theorem 3.1.2 (Theorem 2.2, [KLW05])**.** *Let $\mathcal{E}$ be a vector lattice with weak unit and T be a positive order continuous projection on $\mathcal{E}$. There is a weak unit e of $\mathcal{E}$ with $T(e) = e$ if and only if $T(w)$ is a weak unit of $\mathcal{E}$ for each weak unit w in $\mathcal{E}$.*

Definition 3.1.3 (Definition 2.3, [KLW05])**.** Let $\mathcal{E}$ be a vector lattice with weak unit. A positive order continuous projection $\mathbb{F}$ on $\mathcal{E}$ with range, $\mathcal{R}(\mathbb{F})$, an order complete vector sublattice of $\mathcal{E}$, is called a conditional expectation operator if $\mathbb{F}(e)$ is a weak unit of $\mathcal{E}$ for each weak unit e in $\mathcal{E}$.

Of course, if $\mathcal{E} = L_1(\mu)$ is a probability space and Σ is a sub$-\sigma-$algebra of $\mathcal{F}$, then $\mathcal{E}$ has the weak unit $e = \chi$ and

$$\mathbb{F}f = \mathbb{E}(f|\Sigma)$$

is a vector lattice conditional expectation operator on $\mathcal{E}$ with $\mathbb{F}e = e$.

Conditional expectation operators satisfy an averaging property [[KLW05], Theorem 5.3]. That is, if $f \in \mathcal{R}(\mathbb{F})$ and $g \in \mathcal{E}$ with $fg \in \mathcal{E}$ then $\mathbb{F}(fg) = f\mathbb{F}(g)$, where the product of two elements is done in the universal completion of $\mathcal{E}$. We also note that the range of a strictly positive conditional expectation operator is a regular sublattice of $\mathcal{E}$.

Remark 3.1.4. Let X be a vector lattice and let Y be an order complete vector lattice. Then Veksler's Theorem [AB06] states that every positive order continuous operator $T : X \to Y$ can be uniquely extended to a positive order continuous operator $\tilde{T} : X^\delta \to Y$ where X^δ is the order completion of X. This extension is defined as $\tilde{T}x = \sup T([0, x] \cap X)$ for $x \in X^\delta_+$. Thus, every conditional expectation operator $\mathbb{F}$ on $\mathcal{E}$ can be uniquely extended to define a conditional expectation operator on $\mathcal{E}^\delta$. Hence, WLOG we can consider our conditional expectation operator to be defined on an order complete vector lattice. In fact, a conditional expectation operator can be extended to a domain larger than the order completion of $\mathcal{E}$. We delve into this extension below.

In probability theory every conditional expectation operator $\mathbb{F}$ can be extended to a conditional expectation $\tilde{\mathbb{F}}$ on the, so called, natural domain of $\mathbb{F}$, denoted by dom($\mathbb{F}$), see [GdP02]. In [KLW05], the authors studied the analogue of this extension for conditional expectation operators on vector lattices. They showed that for any conditional expectation $\mathbb{F}$ on $\mathcal{E}$, there exists a largest vector sublattice of $\mathcal{E}^u$ called the natural domain of $\mathbb{F}$, to which $\mathbb{F}$ extends uniquely to as a conditional expectation operator. The natural domain of $\mathbb{F}$ is denoted by $L_1(\mathbb{F})$. Below, we briefly detail the construction of this domain for strictly positive conditional expectation operators on vector lattices. This natural domain satisfies some nice properties. In particular in [GLM14], it was proven that if the conditional expectation $\mathbb{F}$ is extended to a conditional expectation $\tilde{\mathbb{F}}$ on dom($\mathbb{F}$), then $\mathcal{R}(\tilde{\mathbb{F}}) = \mathcal{R}(\mathbb{F})^u$. i.e., the range of the extension is actually an f-algebra.

The following proof extends the domain of a strictly positive conditional expectation operator. We refer the reader to [Kuo06] for a proof not requiring the conditional expectation operator to be strictly positive. Let D be the set of elements x in $\mathcal{E}^u_+$ for which there is a net (x_α) in $\mathcal{E}$ such that $x_\alpha \uparrow x$ and $(\mathbb{F}x_\alpha)$ is bounded in $\mathcal{E}^u$. Then set $\mathbb{F}x = \sup \mathbb{F}x_\alpha \in \mathcal{E}^u$ for $x \in D$ and $L_1(\mathbb{F}) = D - D$. It is shown that $\mathbb{F}$ is well defined on D and that $\mathbb{F}$ remains a conditional expectation operator on $D - D$. In [GLM14], the authors suggested an alternate (and equivalent) approach to define the natural domain of $\mathbb{F}$ using the sup-completion $\mathcal{E}^s$. We detail this construction here as proven in [Azo19]. Let A denote the collection of elements $0 \leq x \in \mathcal{E}^s$ such that $\mathbb{F}x := \sup \mathbb{F}x_\alpha \in \mathcal{E}^u$ where (x_α) is any increasing net in $\mathcal{E}$ such that $x_\alpha \uparrow x$. Then one can prove that such elements are contained in $\mathcal{E}^u$ and then set dom $\mathbb{F} = A - A$. It will be enough to prove that the set A is in fact the same as the set denoted above by D. Assume by contradiction that $x \in A \setminus \mathcal{E}^u$. Then there exists $u > 0$ such that $x \geq nu$ for all $n \in \mathbb{N}$, so then $\mathbb{F}x \geq n\mathbb{F}u$. So $\mathbb{F}u = 0$ which contradicts the fact that $\mathbb{F}$ is strictly positive. Since $\mathcal{E}^u_+ \subseteq \mathcal{E}^s_+$, it follows that $D \subset A$ which completes the proof. In view of this, given a conditional expectation operator $\mathbb{F}$ on $\mathcal{E}$, we shall say that $\mathcal{E}$ is $\mathbb{F}-$universally complete if dom $\mathbb{F} = \mathcal{E}$.

To conclude this section, we construct an example of a conditional expectation operator on a $C(K)$ space. The following example was constructed by Ben Amor in [Amo22]. The definitions below can be found in [BKM21].

Definition 3.1.5. Let K be a Tychonoff space. We call a point $x \in K$ perfectly disconnected in K if it is not simultaneously an accumulation point of two disjoint subsets. If every point of K is perfectly disconnected, then K called perfectly disconnected.

Definition 3.1.6. Let K be a topological space. The Alexandroff duplicate of K is the space created by taking two (disjoint) copies of K, say $A(K) = K \cup K'$.

From the definition, we have that every perfectly disconnected space is extremally disconnected. In [Theorem 2.4, [BKM21]] the authors prove that if K is perfectly disconnected and the set of isolated points of K is clopen then $A(K)$ is extremally disconnected. We now, construct an example of a conditional expectation operator on a $C(K)$ space.

Theorem 3.1.7. *Let K be a perfectly disconnected, compact Hausdorff space such that the set of isolated points of K is clopen. Then there exists a non trivial strictly positive conditional expectation operator on $C(A(K))$.*

Proof. Consider the map $T : C(A(K)) \to C(A(K))$ that maps $f \to g$ where $g(x) = \frac{f(x)+f(x')}{2}$ for every x in $A(K)$. We will prove that T is a conditional expectation on $C(A(K))$. With the pointwise order, it is clear that T is strictly positive, order continuous and satisfies $T\mathbb{1} = \mathbb{1}$.

Moreover, T is a projection since for all $f \in C(A(K))$ and for all x in $A(K)$, we have

$$\begin{aligned} T \circ T(f(x)) &= T\left(\frac{f(x)+f(x')}{2}\right) \\ &= \frac{1}{2}(T(f(x)) + T(f(x'))) \\ &= \frac{1}{2}\left(\frac{f(x)+f(x')}{2} + \frac{f(x')+f(x)}{2}\right) \\ &= T(f(x)). \end{aligned}$$

Observe that

$$\mathcal{R}(T) = \{g \in C(A(K)) \text{ such that } g(x) = g(x') \text{ for all } x \in A(K)\}.$$

It remains to show that $\mathcal{R}(T)$ is an order complete vector sublattice of $C(A(K))$. However to show this we recall [Theorem 3.1, [AAP94]] which states that the range of a positive projection on an order complete vector lattice is order complete. This completes the proof. $\square$

3.2. **Martingales and Filtrations.**

For the rest of this Chapter, we shall assume $\mathcal{E}$ to be an order complete vector lattice with a weak unit e.

On the probability space $(\Omega, \mathcal{F}, \mu)$ a filtration is an increasing family of sub$-\sigma-$algebras of $\mathcal{F}$. The law of total expectation gives that

$$\mathbb{E}[\mathbb{E}[f|\Sigma_i]|\Sigma_j] = \mathbb{E}[f|\Sigma_i] = \mathbb{E}[\mathbb{E}[f|\Sigma_j]|\Sigma_i]$$

for $i \leq j$ and $f \in L_1(\Omega, \mathcal{F}, \mu)$. However, as $\mathbb{E}[\cdot|\Sigma_i]$ is a projection, we can rewrite this as

$$\mathcal{R}(\mathbb{E}[\cdot|\Sigma_i]) \subset \mathcal{R}(\mathbb{E}[\cdot|\Sigma_{i+1}])$$

for all $i \in \mathbb{N}$, and requiring that the family of conditional expectations commute. This allows to formulate the following definition for the filtration on a vector lattice.

Definition 3.2.1. A filtration on $\mathcal{E}$ is a family of conditional expectations, $(\mathbb{F}_i)_{i\in\mathbb{N}}$, on $\mathcal{E}$ with $\mathbb{F}_i\mathbb{F}_j = \mathbb{F}_j\mathbb{F}_i = \mathbb{F}_i$ for all $j \geq i$.

Filtrations play a critical role in probability theory and are a foundation for martingale theory. So having the above definition in hand allows for defining submartingales and supermartingales in vector lattices. However, we shall first need to define stochastic processes first. A stochastic process is simply a collection of random variables indexed by a subset of $\mathbb{R}_+$. That is, $(X_t)_{t\in T}$ is said to be stochastic process if $X_t \in L_1(\mu)$ for all $t \in T$ where $T \subset \mathbb{R}_+$. The stochastic process $(X_t)_{t\in T}$ is said to be adapted to a filtration, $(\Sigma_t)_{t\in T}$, if X_t is an Σ_t-measurable function for every $t \in T$.

On the probability space $(\Omega, \mathcal{F}, \mu)$, the family of pairs $(f_i, \mathcal{F}_i)_{i\in\mathbb{N}}$ is called a submartingale if $(\mathcal{F}_i)_{i\in\mathbb{N}}$ is a filtration, $f_i \in L_1(\Omega, \mathcal{F}_i, \mu)$ for each $i \in \mathbb{N}$ and $\mathbb{E}[f_j|\mathcal{F}_i] \geq f_i$, for all $j \geq i$. Similarly, the family of pairs $(f_i, \mathcal{F}_i)_{i\in\mathbb{N}}$ is called a supermartingale if $(\mathcal{F}_i)_{i\in\mathbb{N}}$ is a filtration, $f_i \in L_1(\Omega, \mathcal{F}_i, \mu)$ for each $i \in \mathbb{N}$ and $\mathbb{E}[f_j|\mathcal{F}_i] \leq f_i$, for all $j \geq i$. The stochastic process is called a martingale if it is simultaneously both a submartingale and a supermartingale.

This allows us to formulate the vector lattice definitions of the above concepts. In the setting of vector lattices, a stochastic process is a collection of elements in the vector lattice $\mathcal{E}$ indexed by a subset of $\mathbb{R}_+$. The process $(X_t)_{t\in T}$ is adapted to the filtration $(\mathbb{F}_t)_{t\in T}$ if $X_t \in \mathcal{R}(\mathbb{F}_t)$ for every $t \in T$. The stochastic process is said to be right continuous if T is an interval and $\text{o-}\lim_{s\downarrow t} X_s = X_t$. The stochastic process is said to be discrete if $T = \mathbb{N}$. In this thesis, we shall only concern ourselves with discrete time stochastic processes. The following is the definition of (sub, super) martingale in vector lattices.

Definition 3.2.2. If (X_t) is a stochastic process adapted to $(\mathbb{F}_t)$, we call $(X_t, \mathbb{F}_t)$ a super-martingale (respectively sub-martingale) if $\mathbb{F}_t(X_s) \leq X_t$ (respectively $\mathbb{F}_t(X_s) \geq$

X_t) for all $t \leq s$. The process is called a martingale if it is a sub-martingale and super-martingale.

Some the classical results from probability regarding martingales have been extended to the vector lattice setting in [KLW04a]. We state the vector lattice Doob-Meyer decomposition proven in [KLW04a] below.

Theorem 3.2.3. *Let $(f_i, \mathbb{F}_i)$ be a discrete sub (super) martingale and let*

$$A_j = \sum_{i=1}^{j-1} \mathbb{F}_i(f_{i+1} - f_i)$$
$$M_j = f_j - A_j$$

for all $j \in \mathbb{N}$. Then we have the unique decomposition $f_i = M_i + A_i, i \in \mathbb{N}$, of $(f_i, \mathbb{F}_i)$ with $(M_j, \mathbb{F}_j)$ a martingale, (A_j) positive and increasing (negative and decreasing), $A_1 = 0$ and $A_{j+1} \in \mathcal{R}(\mathbb{F}_j)$ for all $j \in \mathbb{N}$.

3.3. **Stopping times.**
In this section, we shall first recall some basic definitions on stopping times. Then similar to the rest of the chapter, we motivate and state the vector lattice analogues of the stopping times.

Definition 3.3.1. Let (Σ_i) denote a filtration on the measure space $(\Omega, \mathcal{F}, \mu)$ and $(f_i) \subset L_1(\mu)$ be a sequence adapted to this filtration.

(a) A stopping time adapted to (Σ_i) is a map $\tau : \Omega \to \mathbb{N} \cup \{\infty\}$ such that $\tau^{-1}(\{1, \ldots, i\}) \in \Sigma_i$ for each $i \in \mathbb{N}$. A stopping time τ is said to be bounded if there exists $n \in \mathbb{N}$ such that $\tau(\omega) \leq n$ almost everywhere on Ω.

(b) If τ is a bounded stopping time adapted to the filtration, then the stopped element corresponding to τ is the pair (f_τ, Σ_τ) where

$$f_\tau = \sum_i f_i \cdot \chi_{\tau^{-1}(i)} \text{ and } \Sigma_\tau = \{A \subset \Omega : A \cap \tau^{-1}(\{i\}) \in \Sigma_i, \forall i \in \mathbb{N}\}.$$

(c) If τ is a stopping time adapted to the filtration, then the stopped process is defined to be the sequence of pairs $(f_{\tau \wedge n\chi}, \Sigma_{\tau \wedge n\chi})_{n \in \mathbb{N}}$.

Let us denote $\mathbb{T}$ to be the set of all stopping times adapted to the filtration (Σ_i). Then we can define a partial ordering on $\mathbb{T}$ as follows. If $\sigma, \tau \in \mathbb{T}$, define $\sigma \leq \tau$ if and only if $\sigma(\omega) \leq \tau(\omega)$ for almost all $\omega \in \Omega$. With this partial ordering, it is easy to verify that $\sigma \vee \tau$ and $\sigma \wedge \tau$ are again stopping times. Some properties of stopping times and stopped processes are that Σ_τ is a sub-sigma algebra of $\mathcal{F}$ and $(\Sigma_{\tau \wedge n\chi})_{n \in \mathbb{N}}$ forms a filtration over $L_1(\mu)$. Moreover, if $(f_i) \subset L_1(\mu)$ is adapted to (Σ_i), then

$(f_{\tau\wedge n\chi})_{n\in\mathbb{N}}$ is adapted to the filtration $(\Sigma_{\tau\wedge n\chi})_{n\in\mathbb{N}}$. More information about stopping times in classical probability theory can be found in [Wil91].

There is a correspondence between a bounded stopping time τ adapted to a filtration (Σ_i) and a commuting sequence (P_i) of linear band projections on $L_1(\mu)$, as was established in [KLW04a]. Indeed, for $f \in L_1(\mu)$, define the projection $P_i f = f \cdot \chi_{\tau^{-1}(\{1,\ldots,i\})}$ for each $i \in \mathbb{N}$. Then, for $i \leq j$ and $0 \leq f \in L_1(\mu)$, the inequality $0 \leq P_i f \leq P_j f \leq f$ gives $0 \leq P_i \leq P_j \leq I$, where I denotes the identity function, which implies (P_i) is an increasing sequence of band projections on $L_1(\mu)$. Moreover, (P_i) satisfies $P_i P_j = P_{i\wedge j}$. This follows directly from identity

$$\chi_{\tau^{-1}(\{1,\ldots,i\})} \cdot \chi_{\tau^{-1}(\{1,\ldots,j\})} = \chi_{\tau^{-1}(\{1,\ldots,i\})\cap\tau^{-1}(\{1,\ldots,j\})} = \chi_{\tau^{-1}(\{1,\ldots,i\wedge j\})}$$

Lastly, since $\tau^{-1}(\{1,\ldots,i\}) \in \Sigma_i$ for each $i \in \mathbb{N}$, it follows that

$$P_i\mathbb{E}(f|\Sigma_j) = \chi_{\tau^{-1}(\{1,\ldots,i\})}\mathbb{E}(f|\Sigma_j) = \mathbb{E}(\chi_{\tau^{-1}(\{1,\ldots,i\})}f|\Sigma_j) = \mathbb{E}(P_i f|\Sigma_j)$$

for all $i \leq j$ and $f \in L_1(\mu)$.

Thus, a stopping time τ adapted to a filtration (Σ_i) is an increasing sequence (P_i) of commuting band projections on $L_1(\mu)$ for which $\mathbb{F}_j P_i = P_i \mathbb{F}_j$ for all $i \leq j$, where each $\mathbb{F}_j := \mathbb{E}(\cdot|\Sigma_j)$. Furthermore, if τ is bounded, then there exists n_0 such that $P_i = I$ for all $i \geq n_0$. Motivated by the above observations, we formulate a definition for a stopping time on a vector lattice:

Definition 3.3.2. Let $(\mathbb{F}_i)_{i\in\mathbb{N}}$ be a filtration on $\mathcal{E}$. A stopping time $P := (P_i)_{i\in\mathbb{N}}$ is defined to be an increasing sequence of band projections such that $P_0 = 0$ and $\mathbb{F}_j P_i = P_i \mathbb{F}_j$ whenever $i \leq j$.

In particular, each P_i is order continuous, $0 \leq P_i \leq I$ and $\mathcal{R}(P_i)$ is a band in $\mathcal{E}$, hence an order complete sublattice. Note that $P_n e \in \mathcal{F}_n$, because $P_n e = P_n \mathbb{F}_n e = \mathbb{F}_n P_n e$. The stopping time $P := (P_i)$ is said to be bounded if there exists n_0 so that $P_i = I$ for all $i \geq n_0$.

It should be noted that if $\tau \leq \sigma$ are stopping times in $L_1(\mu)$ then

$$\sigma^{-1}(\{1,\ldots,i\}) = \{\omega|\sigma(\omega) \leq i\} \subset \{\omega|\tau(\omega) \leq j\} = \tau^{-1}(\{1,\ldots,j\})$$

for all $i \leq j$. As before, if we set $P_i(f) = f \cdot \chi_{\tau^{-1}(\{1,\ldots,i\})}$ and $S_i(f) = f \cdot \chi_{\sigma^{-1}(\{1,\ldots,i\})}$ then

$$\chi_{\tau^{-1}(\{1,\ldots,i\})} \cdot \chi_{\sigma^{-1}(\{1,\ldots,i\})} = \chi_{\sigma^{-1}(\{1,\ldots,i\})} = \chi_{\sigma^{-1}(\{1,\ldots,i\})} \cdot \chi_{\tau^{-1}(\{1,\ldots,i\})}$$

for all $i \le j$ and thus,

$$S_i P_j = S_i = P_j S_i \text{ for all } i \le j.$$

For order continuous positive projections P_i and S_i bounded above by the identity on $L_1(\mu)$, the above equality is equivalent to $S_i \le P_i$ for all $i \in \mathbb{N}$. Moreover, the collection of stopping times are a lattice. Given two stopping times, τ and σ in $L_1(\mu)$, then we denote the element $\tau \wedge \sigma$ as the map from $\Omega \to \mathbb{N}$ where $(\tau \wedge \sigma)(x) = \tau(x) \wedge \sigma(x)$. Hence, if we denote by (P_i) and (S_i) respectively the projections on $L_1(\mu)$ associated with the stopping times τ and σ then $P_i f = f \cdot \chi_{\tau^{-1}(\{1,\ldots,i\})}$ and $S_i f = f \cdot \chi_{\sigma^{-1}(\{1,\ldots,i\})}$ for $f \in L_1(\mu)$ and

$$(P \wedge S)_i(f) = f \cdot \chi_{(\tau\wedge\sigma)^{-1}(\{1,\ldots,i\})} = f \cdot \chi_{(\tau)^{-1}(\{1,\ldots,i\})} \wedge f \cdot \chi_{(\sigma)^{-1}(\{1,\ldots,i\})} = (P)_i(f) \wedge (S)_i(f)$$

Combining the above remarks allows us to define a lattice structure on stopping times in arbitrary vector lattice. Given two stopping times $P = (P_i)$ and $S = (S_i)$ on $\mathcal{E}$, we denote $P \le S$ if $P_i \le S_i$ for all $i \in \mathbb{N}$. Then it follows that $(P \wedge S)_i := (P_i \wedge S_i)$ and $(P \vee S)_i = (P_i \vee S_i)$. It is routine to verify that the above gives us a partial ordering on the set of all stopping times on $\mathcal{E}$ and that the collection of stopping times on $\mathcal{E}$ is a lattice.

We now consider stopped processes. As introduced earlier, we denote $P_i f = f \cdot \chi_{\tau^{-1}(\{1,\ldots,i\})}$ for $f \in L_1(\mu)$ then the above definition of f_τ becomes $f_\tau = \sum_{i\in\mathbb{N}}(P_i - P_{i-1})f_i$ where $P_0 = 0$ and the above sum is finite as the stopping time τ is bounded. This allows us to state the required generalization for vector lattices.

To generalize the definition of Σ_τ requires us to consider the conditional expectation $\mathbb{E}[\cdot|\Sigma_\tau]$ instead of the $\sigma-$algebra Σ_τ. Here we observe that if $f \in L_1(\mu)$ then

$$\mathbb{E}[f \cdot \chi_{\tau^{-1}(\{i\})}|\Sigma_\tau] = \mathbb{E}[f \cdot \chi_{\tau^{-1}(\{i\})}|\Sigma_i] = \chi_{\tau^{-1}(\{i\})} \cdot \mathbb{E}[f|\Sigma_i]$$

and thus

$$\mathbb{E}[f|\Sigma_\tau] = \sum \mathbb{E}[f \cdot \chi_{\tau^{-1}(\{i\})}|\Sigma_i] = \sum \chi_{\tau^{-1}(\{i\})} \cdot \mathbb{E}[f|\Sigma_i]$$

If we denote $\mathbb{F}_i f = \mathbb{E}[f|\Sigma_i]$ then

$$\mathbb{E}[f|\Sigma_\tau] = \sum \mathbb{F}_i(P_i - P_{i-1})f = \sum (P_i - P_{i-1})\mathbb{F}_i f$$

Hence the definition of a stopped process in a vector lattice is as follows:

Definition 3.3.3. Let $P = (P_i)$ be a bounded stopping time adapted to the filtration $(\mathbb{F}_i)$ and (X_i) be an adapted stochastic process. Then we denote the stopped element as $(X_P, \mathbb{F}_P)$ where

$$X_P := \sum_{i=1}^{\infty}(P_i - P_{i-1})X_i.$$

and the conditional expectation operator $\mathbb{F}_P : \mathcal{E} \to \mathcal{E}$ is defined for $X \in \mathcal{E}$ as

$$\mathbb{F}_P X = \sum_{i=1}^{\infty}(P_i - P_{i-1})\mathbb{F}_i X$$

Thus from the definition, it is clear that the operator $\mathbb{F}_P$ returns the stopped element for the stochastic process $(\mathbb{F}_i X)$ when the input is X. Before moving on, we will need to establish that $\mathbb{F}_P$ is indeed a conditional expectation operator on $\mathcal{E}$. We shall denote $\mathcal{R}(\mathbb{F}_P) := \mathcal{F}_P$. The following theorems were proven in [KLW04a].

Theorem 3.3.4. *Let P be a bounded stopping time, then $\mathbb{F}_P$ is a positive linear order continuous projection with*

$$\mathcal{F}_P = \{f \in \mathcal{E} | P_i f \in \mathcal{F}_i \textit{ for all } i \in \mathbb{N}\}.$$

Theorem 3.3.5. *Let P be a bounded stopping time adapted to the filtration $(\mathbb{F}_i)$. The operator $\mathbb{F}_P : \mathcal{E} \to \mathcal{E}$ defined for $X \in \mathcal{E}$ as $\mathbb{F}_P X = \sum_{i=1}^{\infty}(P_i - P_{i-1})\mathbb{F}_i X$ is a conditional expectation on $\mathcal{E}$. Moreover, if $(X_i) \subset \mathcal{E}$ with $X_i \in \mathbb{F}_i$ for all $i \in \mathbb{N}$, then $\mathbb{F}_P X_P = X_P$.*

Given a stopping time $P = (P_i)$ and $n \in \mathbb{N}$, we denote $P \wedge ne$ to be the stopping time with

$$(P \wedge ne)_i = \begin{cases} P_i, & i < n \\ I, & i \geq n \end{cases}$$

Then given an adapted stochastic process (X_i), we define the stopped process to be the family of stopped elements $(X_{P\wedge ne})_{n\in\mathbb{N}}$. By the above theorems, this stopped process is adapted to filtration $(\mathbb{F}_{P\wedge ne})_{n\in\mathbb{N}}$.

To conclude this chapter, we recall some optional stopping time theorems that were proved in [KLW04a, CL07].

Lemma 3.3.6 (Lemma 4.9, [KLW04a])**.** *Let $S \leq P$ be bounded stopping times adapted to the filtration $(\mathbb{F}_i)$ on a $\mathcal{E}$. Let (X_i) be an increasing (decreasing) stochastic process adapted to the filtration $(\mathbb{F}_i)$. Then the stopped elements X_S and X_P satisfy the inequality $X_S \leq (\geq) X_P$.*

Theorem 3.3.7 (Theorem 4.11, [KLW04a]). *Let $(X_i, \mathbb{F}_i)$ be a (sub, super) martingale and $S \leq P$ bounded stopping times adapted to the filtration $(\mathbb{F}_i)$, then $\mathbb{F}_S X_P (\geq, \leq) = X_S$.*

4. Functional Calculus

A major result in probability theory is Jensen's inequality and in order to state it in the setting of vector lattices, we need a functional calculus on our vector lattice $\mathcal{E}$. Grobler created a functional calculus on order complete vector lattices using the Daniell integral [Gro14b] and this has since played a major role in the theory of measure-free probability in vector lattices [AT17, AR18, Gro14a, AN]. In the Section 4.1, we briefly recall the construction of the functional calculus using the Daniell integral, as shown in [Gro14b].

It of interest to explore the characterization of the functional calculus when the elements of the vector lattice are considered using their representation in $C^\infty(K)$. There was sufficient reason to believe that for continuous functions the functional calculus developed by Grobler corresponds to the pointwise composition of functions. There have been partial results suggesting this hypothesis in [AT17] where the authors developed ideas to study the functional calculus using convex functions. In Section 4.2, we prove this result and illustrate the advantages of studying the functional calculus through the $C^\infty(K)$ by improving several results of [Gro14b, AT17] via simple proofs.

In Section 4.3, we define the multivariate functional calculus for vector lattices as the composition of continuous functions. We conclude this chapter by proving the multivariate extension of Jensen's inequality in vector lattices. The results in this Chapter are original work and have appeared in [Pol24]. Throughout this Chapter, we fix an order complete vector lattice $\mathcal{E}$ with a weak unit e and a Maeda-Ogasawara representation of $\mathcal{E}$ as an order dense ideal in $C^\infty(K)$, where K is the Stone space of $\mathcal{E}$, with e corresponding to $\mathbb{1}$.

4.1. **Daniell Integral.**

Denote by $F(\mathbb{R})$ the algebra consisting of all finite unions of disjoint left open right closed intervals $(a, b], (a, \infty)$ and $(\infty, b]$ with $a, b \in \mathbb{R}$. Let $\mathbb{L}$ be the vector lattice of real valued functions of the form:

$$f = \sum_{i=1}^{n} a_i \mathbb{1}_{S_i}, S_i \in F(\mathbb{R}) \tag{4}$$

where $(S_i)_{i=1}^n$ is a partition of $\mathbb{R}$. The order relation of $\mathbb{L}$ is defined by $f \leq g$ if $f(t) \leq g(t)$ for every $t \in \mathbb{R}$.

Definition 4.1.1. A positive linear function $I : \mathbb{L} \to \mathcal{E}$ is called an $\mathcal{E}-$valued Daniell integral on $\mathbb{L}$ whenever, for every sequence (f_n) in $\mathbb{L}$ that satisfies $f_n(t) \downarrow 0$ for every $t \in \mathbb{R}$, it follows that $I(f_n) \downarrow 0$.

We note that I need not be an order continuous operator. Consider $I : \mathbb{L} \to \mathbb{R}$ where $I(f) = f(0)$. Then I is a $\mathbb{R}-$valued Daniell integral and consider the sequence $f_n = \mathbb{1}_{(-\frac{1}{n},0]}$. Then $f_n \downarrow 0$ in $\mathbb{L}$ but $I(f_n) \not\to 0$.

We shall detail the construction of a specific Daniell integral developed by Grobler that shall be useful in defining a functional calculus on $\mathcal{E}$. For $Y \in \mathcal{E}$ we denote P_Y to be the band projection associated with the band generated by Y. Fix $X \in \mathcal{E}$, then the right continuous spectral system of X is the increasing right-continuous stochastic process $A = (A_t)_{t\in\mathbb{R}}$ where $A_t = e - P_{(X-te)^+}e$. We denote by A_∞ and $A_{-\infty}$ respectively, the supremum and infimum of the process. To define the $\mathcal{E}-$valued Daniell integral, we define a vector lattice measure μ_A with respect to $(A_t)_{t\in\mathbb{R}}$ as follows:

- $\mu_A(a, b] = A_b - A_a$ where $(a, b] \in F(\mathbb{R})$.
- For any finite disjoint union of half open intervals, we have $\mu_A(\cup_{i=1}^n I_i) = \sum_{i=1}^n \mu_A(I_i)$

Then μ_A is a countably additive $\mathcal{E}-$valued measure on $F(\mathbb{R})$ as shown in [Lemma 3.7, [Gro14b]] and the functional calculus is defined for elements of $\mathbb{L}$ as:

$$I(f) = \sum_{i=1}^{n} a_i \mu_A(S_i), \text{ where } f \in \mathbb{L} \text{ as in (4)}$$

Define $\mathbb{L}^\uparrow := \{f : \mathbb{R} \to \overline{\mathbb{R}} : \exists (f_n)_{n\in\mathbb{N}} \subset \mathbb{L}, \text{ such that } f_n(t) \uparrow f(t) \text{ for every } t \in \mathbb{R}\}$. Then the integral $I : \mathbb{L} \to \mathcal{E}$ can be extended to an integral from $\mathbb{L}^\uparrow$ to $\mathcal{E}^s$ as follows: for $f \in \mathbb{L}^\uparrow$, we define $I(f) = \sup_n I(f_n)$ where (f_n) is a sequence in $\mathbb{L}$ such that $f_n \uparrow f$. Then [Lemma 3.2, [Gro14b]] states that this extension is well-defined. The extension also satisfies the following properties.

Lemma 4.1.2 (Lemma 3.4, [Gro14b])**.** *The extension of I to $\mathbb{L}^\uparrow$ is well-defined and satisfies the following the properties.*

- *If $f, g \in \mathbb{L}^\uparrow$ and $f \leq g$, then $I(f) \leq I(g)$;*
- *If $f \in \mathbb{L}^\uparrow$ and $0 \leq c < \infty$, then $cf \in \mathbb{L}^\uparrow$ and $I(cf) = cI(f)$;*
- *If $f, g \in \mathbb{L}^\uparrow$, then $f + g \in \mathbb{L}^\uparrow$ and $I(f + g) = I(f) + I(g)$;*
- *If $(f_n)_{n\in\mathbb{N}}$ is a sequence in $\mathbb{L}^\uparrow$ and $f_n(t) \uparrow f(t)$ for every t , then $f \in \mathbb{L}^\uparrow$ and $I(f_n) \uparrow I(f)$.*

4.2. **Representation of functional calculus.**
Given a continuous function $f \in C(\mathbb{R})$ and $X \in C^{\infty}(K)$, let U be the set on which X is finite. Clearly, U is open and dense, and the composition of f and X is defined, finite, and continuous on U. Then the composition extends uniquely to a function in $C^{\infty}(K)$. We shall denote this extended function by $f \circ X$. Also note that by the Maeda-Ogasawara theory, there is a one-to-one correspondence between clopen subsets of K and bands in $\mathcal{E}$. For an element $Y \in \mathcal{E}$, the clopen set corresponding to the band generated by Y is the set $\overline{\{Y \neq 0\}}$. Hence the corresponding band projection of e is $P_Y e = \mathbb{1}_{\overline{\{Y \neq 0\}}}$.

Lemma 4.2.1. *Let $f \in \mathbb{L}$ and $\{-\infty = \gamma_0 < \gamma_1 < \cdots < \gamma_n < \infty\}$ be a partition of $\mathbb{R}$ such that $f = a_\infty \mathbb{1}_{(\gamma_n,\infty)} + \sum_{i=1}^{n} a_i \mathbb{1}_{(\gamma_{i-1},\gamma_i]}$. Then the Daniell integral of f is*

$$I(f) = a_\infty \mathbb{1}_{\overline{\{X>\gamma_n\}}} + \sum_{i=1}^{n} a_i \mathbb{1}_{\overline{\{X>\gamma_{i-1}\}} \setminus \overline{\{X>\gamma_i\}}}$$

Proof. Note that since the weak unit e is fixed, it corresponds to $\mathbb{1}$ in the stone representation. Now, we will first prove the result in the case when $f = \mathbb{1}_S$ where $S = (a, b] \in F(\mathbb{R})$. So,

$$\begin{aligned} I(f) = \mu_A(S) = A_b - A_a &= (e - P_{(X-be)^+}e) - (e - P_{(X-ae)^+}e) \\ &= P_{(X-ae)^+}e - P_{(X-be)^+}e \end{aligned}$$

For $(X - ae)^+$, we have the corresponding band projection $P_{(X-ae)^+}e = \mathbb{1}_{\overline{\{X>a\}}}$. Hence,

$$I(f) = P_{(X-ae)^+}e - P_{(X-be)^+}e = \mathbb{1}_{\overline{\{X>a\}} \setminus \overline{\{X>b\}}}$$

It is easy to see that $\overline{\{X > a\}} \setminus \overline{\{X > b\}}$ is clopen. If $S = (a, \infty) \in F(\mathbb{R})$, the above argument can be adapted to give us

$$\begin{aligned} I(f) = \mu_A(S) = A_\infty - A_a &= \sup_{b\in\mathbb{R}} (e - P_{(X-be)^+}e) - (e - P_{(X-ae)^+}e) \\ &= \sup_{b\in\mathbb{R}} (P_{(X-ae)^+}e - P_{(X-be)^+}e) \\ &= \sup_{b\in\mathbb{R}} \mathbb{1}_{\overline{\{X>a\}} \setminus \overline{\{X>b\}}} \\ &= \mathbb{1}_{\overline{\{X>a\}}} \end{aligned}$$

Similarly, if $S = (-\infty, b] \in F(\mathbb{R})$, we have $I(f) = \mathbb{1}_{K \setminus \overline{\{X>b\}}} = \mathbb{1}_{\overline{\{X>-\infty\}} \setminus \overline{\{X>b\}}}$. Therefore when $f \in \mathbb{L}$ is a piece-wise constant function of the form $f = a_\infty \mathbb{1}_{(\gamma_n,\infty)} + \sum_{i=1}^{n} a_i \mathbb{1}_{(\gamma_{i-1},\gamma_i]}$, we have:

$$I(f) = a_\infty \mathbb{1}_{\overline{\{X>\gamma_n\}}} + \sum_{i=1}^{n} a_i \mathbb{1}_{\overline{\{X>\gamma_{i-1}\}} \setminus \overline{\{X>\gamma_i\}}}$$

$\square$

Lemma 4.2.2. *Let $f \in \mathbb{L}$ such that f has bounded support. Then for an element $X \in \mathcal{E}$, there exists an open dense set $H \subseteq K$ such that $I(f)(\omega) = f(X(\omega))$ for every $\omega \in H$.*

Proof. Let us suppose that $(a, b]$ is the support of f. Then there exists a partition $\{a = \alpha_0 < \alpha_1 < \cdots < \alpha_k = b\}$ of $(a, b]$ such that $f = \sum_{j=1}^{k} a_j \mathbb{1}_{(\alpha_{j-1}, \alpha_j]}$. Then Lemma 4.2.1 gives us

$$I(f) = \sum_{j=1}^{k} a_j \mathbb{1}_{\overline{\{X>\alpha_{j-1}\}} \setminus \overline{\{X>\alpha_j\}}}$$

Now set $H_j = \overline{\{X > \alpha_j\}} \cap \{X \leq \alpha_j\}$ and $N = \{X = \infty\}$. Now, $\{X > \alpha_j\}$ is an open set and $\{X \leq \alpha_j\}$ is a closed set of K satisfying $\{X > \alpha_j\} \cap \{X \leq \alpha_j\} = \emptyset$ which implies that H_j is a closed nowhere dense set. Since each of the sets are closed and nowhere dense, upon setting $M = \left(\bigcup_{j=1}^{k} H_j\right) \cup N$ we have that M is a closed nowhere dense set and $H := K \setminus M$ is an open dense set. Let $\omega \in H$. Then we consider two separate cases:

Case 1. If $X(\omega) \notin (a, b]$: Then $f(X(\omega)) = 0 = I(f)(\omega)$.

Case 2. If $X(\omega) \in (a, b]$: Then there exists j_0 such that $X(\omega) \in (\alpha_{j_0-1}, \alpha_{j_0}]$: Then we have that $f(X(\omega)) = a_{j_0}$. However, we have that $\omega \in \overline{\{X > \alpha_{j_0-1}\}}$ and $\omega \in \{X \leq \alpha_{j_0}\}$. By definition of the set H, $\omega \notin \overline{\{X > \alpha_{j_0}\}}$ and thus, $I(f)(\omega) = a_{j_0}$. Hence, $I(f)(\omega) = f(X(\omega))$.

Therefore, we have that $I(f)(\omega) = f(X(\omega))$ for every $\omega \in H$. This implies that $f \circ X = I(f) \in \mathcal{E}$ when $f \in \mathbb{L}$. $\square$

Lemma 4.2.3. *Let $f \in C(\mathbb{R})$ be a positive continuous function with bounded support. Then for an element $X \in \mathcal{E}$, we have $I(f) = f \circ X \in \mathcal{E}_+$.*

Proof. Because f has bounded support implies $f \leq \lambda \mathbb{1}_{(a,b]}$ for some $a, b \in \mathbb{R}$. Then $I(f) \leq \lambda I(\mathbb{1}_{(a,b]}) \in \mathcal{E}$. As $\mathcal{E}$ is order complete, we have $I(f) \in \mathcal{E}_+$. Since f has bounded support, f is an uniformly continuous function and there exists a sequence (f_k) in $\mathbb{L}$ such that $f_k \uparrow\leq f$ and (f_k) converges to f uniformly. WLOG, passing to a subsequence, we have $0 \leq f - f_k \leq \frac{1}{k}\mathbb{1}_{\mathbb{R}}$. By Lemma 4.2.2, $f_k \circ X \in \mathcal{E}$ and thus, $f \circ X - f_k \circ X = (f - f_k) \circ X \leq (\frac{1}{k}\mathbb{1}_{\mathbb{R}}) \circ X = \frac{1}{k}\mathbb{1}_K$. Therefore, $f_k \circ X$ converges to $f \circ X$ relatively uniformly in $\mathcal{E}$. Similarly, $I(f) - I(f_k) = I(f - f_k) \leq I(\frac{1}{k}\mathbb{1}_{\mathbb{R}}) = \frac{1}{k}\mathbb{1}_K$. So $I(f_k)$ converges to $I(f)$ relatively uniformly in $\mathcal{E}$. Since $I(f_k) = f_k \circ X$ by Lemma 4.2.2, passing to the limit gives, $I(f) = f \circ X$. $\square$

In view of the above theorem, we will show that for continuous functions the Daniell functional calculus satisfies a nice representation. The idea for the proof of the theorem can be found in Groblers' paper. Let $\mathbb{L}_0^{\uparrow}$ be the set of all real valued positive functions in $\mathbb{L}^{\uparrow}$ and

$$\mathbb{L}_u = \{f : f = g - h \text{ such that } g, h \in \mathbb{L}_0^{\uparrow} \text{ and } I(g), I(h) \in \mathcal{E}^u\}$$

Then analogous to the proof of [Proposition 3.5, [Gro14b]] we can show that $\mathbb{L}_u$ is a vector space and I has a well-defined extension to $\mathbb{L}_u$ by defining for $f = g - h \in \mathbb{L}_u$, $I(f) = I(g) - I(h)$. The proof also shows that the extension is positive and linear on $\mathbb{L}_u$.

Theorem 4.2.4. *Let $f \in C(\mathbb{R})$. Then for an element $X \in \mathcal{E}$, we have $I(f) = f \circ X$.*

Proof. Let $f \in C(\mathbb{R})$ be a positive continuous function. Then we can find a sequence $(f_n) \subset C(\mathbb{R})$ of positive continuous functions with bounded support as follows: $f_n(t) = f(t)$ if $t \in [-n, n]$, and $f_n(t) = 0$ if $t \notin [-n-1, n+1]$. Then $f_n \uparrow f$ and combining Lemma 4.1.2 and Lemma 2.5.4 gives that $I(f_n) \uparrow I(f)$ pointwise on a co-meagre subset of K. However, we also have that $f_n \circ X(\omega) \uparrow f \circ X(\omega)$ where ω belongs to $\{X < \infty\}$, an open dense set. Since Lemma 4.2.3 gives that $I(f_n) = f_n \circ X$, we can conclude that $I(f) = f \circ X$ on a co-meagre set and thus on a dense set by the Baire category theorem. As f is a positive continuous function, implies that $f \circ X \in \mathcal{E}^u_+$ and therefore $f \circ X \in \mathcal{E}^s_+$. Since $I(f)$ and $f \circ X$ are continuous functions equal on a dense subset of K, we have $I(f) = f \circ X$ everywhere on K. However, $f \circ X \in \mathcal{E}^u$ implies that $I(f) = f \circ X \in \mathcal{E}^u$. Hence, $C(\mathbb{R})_+ \subseteq \mathbb{L}_u$ and thus, $C(\mathbb{R}) \subseteq \mathbb{L}_u$. So given $f \in C(\mathbb{R})$, applying the preceding argument to f^+ and f^-, we get

$$I(f) = I(f^+) - I(f^-) = f^+ \circ X - f^- \circ X = f \circ X$$

□

The following are some simple corollaries resulting from the above theorem which improve upon results of [Corollary 4.3, [Gro14b]], [Proposition 2.6, [AT17]] and [Proposition 2.6, [AT17]]. The results below follow from the properties of continuous functions.

Corollary 4.2.5. *Let $X \in \mathcal{E}$ and $f \in C(\mathbb{R})$. Then $I(f) = f(X) \in \mathcal{E}^u$.*

Corollary 4.2.6. *Let $I : \mathbb{L}_u \to \mathcal{E}^u$ and $f, g \in C(\mathbb{R})$. Then $I(f \vee g) = I(f) \vee I(g), I(f \wedge g) = I(f) \wedge I(g)$ and $I(|f|) = |I(f)|$. That is, the restriction of I to $C(\mathbb{R})$ is a lattice homomorphism.*

Corollary 4.2.7. *Let $X \in \mathcal{E}$ and $f, g \in C(\mathbb{R})$. Then $I(fg) = (fg)(X) = f(X)g(X) = I(f)I(g)$.*

The following proposition is a stronger statement than [Proposition 4.6, [Gro14b]] and [Proposition 2.8, [AT17]]. Recall the following convergence criterion for elements of $C^\infty(K)$ from [Theorem 3.7, [BT22]] that states that $x_n \xrightarrow{\text{uo}} x$ if and only if x_n converges to x pointwise on a co-meagre set.

Proposition 4.2.8. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. If $x_n \xrightarrow{\text{uo}} x$ in $\mathcal{E}$ then $f(x_n) \xrightarrow{\text{uo}} f(x)$ in $\mathcal{E}^u$.*

Proof. By the convergence criterion, $x_n(\omega) \to x(\omega)$ for every $\omega \in H$ where $H \subseteq K$ is co-meagre set. By continuity of f this implies that $f[x_n(\omega)] \to f[x(\omega)]$ for $\omega \in H$. Therefore, $f(x_n) \xrightarrow{\text{uo}} f(x)$. $\square$

4.3. **Multivariate functional calculus.**
Since the Daniell functional calculus for continuous functions corresponds to the pointwise composition of functions, we can extend this to the concept of multivariate continuous functions. Given $f \in C(\mathbb{R}^n, \mathbb{R})$ and $\mathbf{X} = (X_i)_{i=1}^n \subset \mathcal{E}$ where $n \in \mathbb{N}$, let U be the set on which all the X_i are finite. Then U is an open dense set and $f(X_1, \ldots, X_n)$ is a well-defined continuous function on U. Then denote $f(\mathbf{X})$ to be the unique extension of $f(X_1, \ldots, X_n)$ in $C^\infty(K)$. This enables us to prove the multivariate version of Jensen's inequality. The univariate version of the Jensen's inequality in the setting of vector lattices was proved in [Theorem 4.4, [Gro14b]]. Let $\mathbf{X} = (X_1, \ldots, X_n)$ and for a given conditional expectation operator $\mathbb{F}$ on $\mathcal{E}$ let $\mathbb{F}\mathbf{X} := (\mathbb{F}X_1, \ldots, \mathbb{F}X_n)$.

Theorem 4.3.1. *Let $f \in C(\mathbb{R}^n, \mathbb{R})$ be a convex function and $\mathbf{X} = (X_i)_{i=1}^n \subset \mathcal{E}$. Let $\mathbb{F}$ be a conditional expectation defined on $\mathcal{E}$. If $f(\mathbf{X}) \in \mathcal{E}$, then $\mathbb{F}(f(\mathbf{X})) \geq f(\mathbb{F}\mathbf{X})$.*

Proof. Since f is a convex function, it is a fact from analysis that there exists a sequence of affine functions $L_m : \mathbb{R}^n \to \mathbb{R}$ of the form $L_m(t) = \langle a_m, t\rangle + b_m$ for some $a_m \in \mathbb{R}^n, b_m \in \mathbb{R}$ such that for every $t \in \mathbb{R}^n$, we have

$$f(t) = \sup_{m \in \mathbb{N}} L_m(t)$$

Since $f \geq L_m$, we have $f(\mathbf{X}) \geq L_m(\mathbf{X})$ for every m. Since $\mathbb{F}$ is a positive linear projection and $f(\mathbf{X}) \in \mathcal{E}$, we have

$$\mathbb{F}(f(\mathbf{X})) \geq \mathbb{F}(L_m(\mathbf{X})) = L_m(\mathbb{F}\mathbf{X}), \forall m \in \mathbb{N} \tag{5}$$

For $m \in \mathbb{N}$, let $L'_m = L_1 \vee \cdots \vee L_m$. Then L'_m are an increasing sequence of continuous functions such that $f(t) = \sup_{m \in \mathbb{N}} L'_m(t)$ and thus $L'_m(\mathbb{F}\mathbf{X}) \uparrow f(\mathbb{F}\mathbf{X})$ by Theorem

4.1.2. By Corollary 4.2.6 and 5, we have

$$L'_m(\mathbb{F}\mathbf{X}) = \bigvee_{i=1}^{m} L_i(\mathbb{F}\mathbf{X}) \leq \mathbb{F}(f(\mathbf{X}))$$

Thus, it follows that $f(\mathbb{F}\mathbf{X}) \leq \mathbb{F}(f(\mathbf{X}))$. $\square$

Corollary 4.3.2. *Let $(X_t^{(i)}, \mathbb{F}_t, \mathcal{F}_t)_{i=1}^n$ be a finite collection of martingales with the filtration $(\mathbb{F}_t)_{t\in\mathbb{N}}$ and let $g \in C(\mathbb{R}^n, \mathbb{R})$ be a convex function. If $g(\mathbf{X}_t) = g(X_t^{(1)}, \ldots, X_t^{(n)}) \in \mathcal{E}$ for all t, then $(g(\mathbf{X}_t), \mathbb{F}_t, \mathcal{F}_t)$ is a sub-martingale.*

Proof. It follows from Jensen's inequality that for $t < s$, we have

$$\mathbb{F}_t[g(\mathbf{X}_s)] \geq g[\mathbb{F}_t(\mathbf{X}_s)] = g(\mathbf{X}_t)$$

and thus $(g(\mathbf{X}_t), \mathbb{F}_t, \mathcal{F}_t)$ is a sub-martingale. $\square$

5. Discrete stopping times

Stopping times play an integral part in probability theory [Wil91]. This notion has been extended to vector lattices in [KLW04a] as detailed in Chapter 3. In [CL07], the authors extended this theory to unbounded stopping times in a Banach lattice. In this Chapter, we shall further extend this theory by studying unbounded stopping times in vector lattices. To achieve this, we shall use the representation of the sup-completion from Chapter 2.

In Section 5.1, we show that every stopping time corresponds to an element of $\mathcal{E}^s$ that satisfies some additional constraints. This representation resembles the definition of a stopping time in probability theory with the noted difference that we are now in $C(K,\overline{\mathbb{R}})$ instead. Considering our stopping times to be elements of a vector lattice imparts a natural order on them that agrees with the ordering of stopping times considered in Chapter 3. Moreover, this allows us to easily deduce properties about stopping times as well as proving the vector lattice version of the Début theorem. We then consider the representation of a stochastic process stopped by a bounded stopping time in a vector lattice. It is then shown that if we have an unbounded stopping time τ, then the sequence of stopped elements corresponding to $\tau \wedge n\mathbb{1}$ is an uo$-$Cauchy sequence. Thus, we have a natural definition for the unbounded stopped element. This definition is the shown to be the vector lattice extension to that considered in [CL07]. The results in this chapter are original and have appeared in [Pol24]. For the rest of this thesis, we fix an order complete vector lattice $\mathcal{E}$ with a weak unit e and a Maeda-Ogasawara representation of $\mathcal{E}$ as an order dense ideal in $C^\infty(K)$, where K is the Stone space of $\mathcal{E}$, with e corresponding to $\mathbb{1}$. Furthermore, we fix the filtration $(\mathbb{F}_i)_{i\in\mathbb{N}}$ on $\mathcal{E}$ and we shall denote $\mathcal{F}_i := \mathcal{R}(\mathbb{F}_i)$ for $i \in \mathbb{N}$.

5.1. Representation of Stopping times.

We begin with the following lemma:

Lemma 5.1.1. *Let $(U_n)_{n\in\mathbb{N}}$ be a sequence of pairwise disjoint clopen sets in K, and set $\tau = \sup_{n\in\mathbb{N}} n\mathbb{1}_{U_n}$ where the supremum is taken in $C^\infty(K)_+$. Then $\{\tau = n\} = U_n, \forall n \in \mathbb{N}$ and $\mathcal{R}(\tau) \subseteq \mathbb{N}\cup\{0\}\cup\{\infty\}$.*

Proof. Since $(n\mathbb{1}_{U_n})_{n\in\mathbb{N}}$ are pairwise disjoint functions in $C^\infty(K)_+$, we have that $\tau \in C^\infty(K)$. Since band projections are order continuous,

$$\mathbb{1}_{U_k}\tau = \mathbb{1}_{U_k}(\sup_n n\mathbb{1}_{U_n}) = \sup_n n\mathbb{1}_{U_n\cap U_k} = k\mathbb{1}_{U_k}.$$

Hence, $U_k \subseteq \{\tau = k\}^\circ$ for all $k \in \mathbb{N}$. But let us suppose that $\exists n \in \mathbb{N}, \exists \alpha \in V := \{\tau = n\}^\circ$ such that $\alpha \notin U_n$. Then consider the function

$$T = \tau - \mathbb{1}_{V \setminus U_n}$$

Clearly, $\tau = T$ on $(V \setminus U_n)^c$ and $T(V \setminus U_n) = n - 1$ and thus $T < \tau$. However, $\mathbb{1}_{U_k} T = \mathbb{1}_{U_k}\tau - \mathbb{1}_{(V\setminus U_n)\cap U_k} = k\mathbb{1}_{U_k} - \mathbb{1}_{(V\setminus U_n)\cap U_k}$. Since $V \cap U_k = \emptyset$, we have $\mathbb{1}_{(V\setminus U_n)\cap U_k} = 0$ and thus $T \geq k\mathbb{1}_{U_k}, \forall k \in \mathbb{N}$. But this is a contradiction, and thus, we have $\{\tau = n\}^\circ = U_n$.

The support of τ is $\overline{\bigcup_{i=1}^\infty U_i}$ and $\tau(U_i) \in \mathbb{N}, \forall i \in \mathbb{N}$. Let $\omega \in \partial(\bigcup_{i=1}^\infty U_i)$. Then there exists a sequence $(\omega_n) \subset \bigcup_{i=1}^\infty U_i$ such that $\omega_n \to \omega$ and the tail of the sequence does not belong to any U_i. Therefore, WLOG, we can assume that $\omega_n \in \bigcup_{i=n}^\infty U_i$ and thus $\tau(\omega_n) \geq n$. Therefore, $\tau(\omega) = \infty$ and hence $\mathcal{R}(\tau) \subseteq \mathbb{N} \cup \{0\} \cup \{\infty\}$. However, since the range of τ is discrete, for $n \in \mathbb{N}$ we have $\{\tau = n\} = \{\tau < n - \frac{1}{2}\} \cup \{\tau > n + \frac{1}{2}\}$. Hence, $\{\tau = n\}$ is an open set and we have $\{\tau = n\} = U_n$. $\square$

Maeda-Ogasawara theorem allows to represent the stopping time in terms of continuous functions on the Stone space of $\mathcal{E}$. Since the (P_n) are band projections, each of the P_n correspond to multiplication by a function of the form $\mathbb{1}_{W_n}$ where W_n is a clopen set, with the (W_n) being an increasing sequence of clopen sets. Let $P'_n = P_n - P_{n-1}$ for $n \geq 1$ and $P'_0 = 0$. Then each of the P'_n remains a band projection and corresponds to multiplication by a function of the form $\mathbb{1}_{U_n}$ where $U_n := W_n \setminus W_{n-1}$ are pairwise disjoint clopen sets. Let $V = K \setminus (\overline{\bigcup_{n=1}^\infty U_n})$ and set $\tau = \infty\mathbb{1}_V + \sup_n n\mathbb{1}_{U_n}$. Lemma 5.1.1 gives that $\sup_{n\in\mathbb{N}} n\mathbb{1}_{U_n} \in C^\infty(K)$ and thus $\tau \in \mathcal{E}^s$ with $\mathcal{R}(\tau) \subseteq \mathbb{N} \cup \{\infty\}$. Furthermore, for $n \in \mathbb{N}$, we have $\{\tau = n\} = U_n$ is a clopen set and hence

$$\mathbb{1}_{\{\tau=n\}} = P'_n E = P_n E - P_{n-1}E \in \mathcal{F}_n$$

Since $\mathbb{1}_{\{\tau=n\}} \in \mathcal{F}_n$ we also have $\mathbb{1}_{\{\tau\leq n\}} \in \mathcal{F}_n$.

In fact, the converse is true as well. That is, if $\tau \in \mathcal{E}^s$ satisfying $\mathcal{R}(\tau) \subseteq \mathbb{N} \cup \infty$ and $\mathbb{1}_{\{\tau=n\}} \in \mathcal{F}_n$ for all $n \in \mathbb{N}$, then τ corresponds to a stopping time. To see this, define the operator $P_n : \mathcal{E} \to \mathcal{E}$ via $P_n(f) = f \cdot \mathbb{1}_{\{\tau\leq n\}}$ for $f \in \mathcal{E}$. Clearly, $(P_n)_{n\in\mathbb{N}}$ is an increasing sequence of band projections such that $P_0 = 0$. Then upon using the averaging property of conditional expectation operators, we obtain that

$$\mathbb{F}_n P_n(f) = \mathbb{F}_n f \cdot \mathbb{1}_{\{\tau\leq n\}} = \mathbb{1}_{\{\tau\leq n\}} \cdot \mathbb{F}_n(f) = P_n\mathbb{F}_n(f)$$

Hence, τ corresponds to a stopping time. We summarize the above in the theorem below.

Theorem 5.1.2. *Every stopping time corresponds to an element $\tau \in \mathcal{E}^s$ that satisfies $\mathcal{R}(\tau) \subseteq \mathbb{N} \cup \{\infty\}$ and $\mathbb{1}_{\{\tau=n\}} \in \mathcal{F}_n$ for all $n \in \mathbb{N}$. The converse is also true.*

Representing stopping times as above is useful in proving their various properties. A variant of Lemma 5.1.3 given below has been proven in [Gro11] by Grobler in the case of continuous processes (i.e., when the index is $[0, \infty)$). However, the properties of discrete stopping times can not be deduced directly from there. Hence, we explicitly derive the result using the representation of stopping times and stopped processes from Theorem 5.1.2.

Lemma 5.1.3. *The set of stopping times is closed under the following operations.*

- *If σ, τ are stopping times then so are $\sigma \vee \tau, \sigma \wedge \tau$ and $\sigma + \tau$.*
- *If $(\tau_n)_{n\in\mathbb{N}}$ is a sequence of stopping times, then $\inf \tau_n$ and $\sup \tau_n$ are stopping times. This includes the case where τ_n is increasing (or decreasing) to the limit τ.*

Proof. If σ, τ are stopping times then $\{\sigma \vee \tau \leq n\} = \{\sigma \leq n\} \cap \{\tau \leq n\}$ and $\{\sigma \wedge \tau \leq n\} = \{\sigma \leq n\} \cup \{\tau \leq n\}$ and thus we have

$$\mathbb{1}_{\{\sigma\vee\tau\leq n\}} = \mathbb{1}_{\{\sigma\leq n\}} \wedge \mathbb{1}_{\{\tau\leq n\}} \in \mathcal{F}_n.$$

$$\mathbb{1}_{\{\sigma\wedge\tau\leq n\}} = \mathbb{1}_{\{\sigma\leq n\}} \vee \mathbb{1}_{\{\tau\leq n\}} \in \mathcal{F}_n.$$

Since, $\sigma \vee \tau, \sigma \wedge \tau \in \mathcal{E}^s$ with $\mathcal{R}(\sigma \vee \tau), \mathcal{R}(\sigma \wedge \tau) \subseteq \mathbb{N} \cup \{\infty\}$, we have that $\sigma \vee \tau$ and $\sigma \wedge \tau$ are stopping times. Also $\{\sigma + \tau \leq n\} = \bigcup_{0\leq s\leq n}\{\sigma = s\} \cap \{\tau = n - s\}$ and $\mathbb{1}_{\{\sigma=s\}}, \mathbb{1}_{\{\tau=n-s\}} \in \mathcal{F}_n$. Hence $\mathbb{1}_{\{\sigma+\tau\leq n\}} \in \mathcal{F}_n$ and, $\sigma + \tau$ is a stopping time.

Let us denote $\sup \tau_n = \tau$. WLOG, (τ_n) is increasing; otherwise, replace τ_n with $\tau_1 \vee \cdots \vee \tau_n$. We first observe that since $\tau_n \in \mathcal{E}^s_+$, τ is well defined and contained in $\mathcal{E}^s_+$. We claim that $\mathcal{R}(\tau) \subseteq \mathbb{N} \cup \{\infty\}$. Suppose not. Then there exists a point $\omega \in K$ such that $\tau(\omega) = \alpha \notin \mathbb{N} \cup \{\infty\}$ and hence there exists a clopen set $V \subseteq K$ such that $\tau(V) \subseteq (\lfloor\alpha\rfloor, \lfloor\alpha\rfloor + 1)$. Let $\sigma = \tau \cdot \mathbb{1}_{V^c} + \lfloor\alpha\rfloor\mathbb{1}_V$. Clearly, $\sigma \geq \tau_n$ for every $n \in \mathbb{N}$, which is a contradiction. Therefore, $\mathcal{R}(\tau) \subseteq \mathbb{N} \cup \{\infty\}$ and since τ is a continuous function taking discrete values, $\{\tau \leq k\}$ is a clopen set for $k \in \mathbb{N}$. By Lemma 2.5.4, there exists a co-meagre set $M \subseteq K$ such that $\tau_n(\omega) \uparrow \tau(\omega)$ for every $\omega \in M$. Then for a fixed $k \in \mathbb{N}$, we have

$$\{\tau \leq k\} \cap M = \bigcap_n \{\tau_n \leq k\} \cap M$$

which implies that $\mathbb{1}_{\{\tau\leq k\}} = \mathbb{1}_{\left(\bigcap_n\{\tau_n\leq k\}\right)^\circ}$ on a co-meagre set.

We claim that $\mathbb{1}_{\left(\bigcap_n\{\tau_n\le k\}\right)^\circ} = \inf_n \mathbb{1}_{\{\tau_n\le k\}}$ in $\mathcal{E}$. Since, $\bigcap_n\{\tau_n \le k\} \subseteq \{\tau_m \le k\}$ for every m, we have $\mathbb{1}_{\left(\bigcap_n\{\tau_n\le k\}\right)^\circ} \le \inf_n \mathbb{1}_{\{\tau_n\le k\}}$. On the other hand, let $\sigma := \inf_n \mathbb{1}_{\{\tau_n\le k\}}$. Using a similar argument as above, we can show that $\mathcal{R}(\sigma) \subseteq \{0\}\cup\{1\}$ and therefore, $V := \operatorname{supp}\sigma$ is a clopen set. Then $V \subseteq \{\tau_n \le k\}, \forall n \in \mathbb{N} \implies V \subseteq \bigcap_n\{\tau_n \le k\} \implies V \subseteq \left(\bigcap_n\{\tau_n \le k\}\right)^\circ$. Hence, $\sigma \le \mathbb{1}_{\left(\bigcap_n\{\tau_n\le k\}\right)^\circ}$ which establishes the claim. Hence, by the Baire category theorem, $\mathbb{1}_{\{\tau\le k\}} = \inf_n \mathbb{1}_{\{\tau_n\le k\}} \in \mathcal{F}_k$ and thus $\sup_n \tau_n$ is a stopping time. Similarly, we can argue for $\inf \tau_n$. $\square$

Remark 5.1.4. Let τ be a stopping time and (n_k) be an increasing sequence in $\mathbb{N}$. Then clearly there exists a positive, increasing continuous function $g : \mathbb{R}_+ \cup \{\infty\} \to \mathbb{R}_+ \cup \{\infty\}$ such that $g(k) = n_k, g(0) = 0, g(\infty) = \infty$ and $g(t) \ge t$ for all t. Since g is a continuous function, $g(\tau)$ is a well defined element of $\mathcal{E}^s_+$ with $\mathcal{R}(g(\tau)) = \mathbb{N}\cup\{\infty\}$. We claim that $g(\tau)$ is again a stopping time. Now, $\{g(\tau) = n_k\} = \{\tau = k\}$ and hence $\mathbb{1}_{\{g(\tau)=n_k\}} = \mathbb{1}_{\{\tau=k\}} \in \mathcal{F}_k \subseteq \mathcal{F}_{n_k}$. If $n \notin (n_k)_{k=1}^\infty$ then $\{g(\tau) = n\} = \emptyset$ and $\mathbb{1}_{\{g(\tau)=n\}} = 0 \in \mathcal{F}_n$. Thus, $g(\tau)$ is a stopping time. We note that $g(\tau)$ is only determined by τ and (n_k), and does not depend on the choice of g.

The following theorem is a vector lattice version of the Début theorem [Fis13] for discrete stochastic processes. For any stopping time, there exists an adapted stochastic process and a subset of $\mathbb{R}$ such that the corresponding hitting time will be precisely this stopping time. The stochastic process can be chosen intuitively and similar to the classical probability case. It will be 1 until just before the stopping time is reached, from which on, it will be 0. The increasing process, therefore, first hits the set $\{1\}$ at the stopping time.

Theorem 5.1.5. *Let τ be a discrete stopping time. Then there exists an adapted process $(X_n)_{n\in\mathbb{N}}$ such that $\tau(\omega) = \inf\{t \in \mathbb{N} : X_t(\omega) = 1\}$.*

Proof. Let $S_n = \{\tau \le n\}$ and let $X_n = \mathbb{1}_{S_n}$ for all $n \in \mathbb{N}$. Since $\mathbb{1}_{\{\tau\le n\}} \in \mathcal{F}_n$, this implies that $X_n \in \mathcal{F}_n$ and thus, the stochastic process $(X_n)_{n\in\mathbb{N}}$ is increasing and adapted. Let $\sigma : K \to \overline{\mathbb{R}}$ defined via $\sigma(\omega) = \inf\{t \in \mathbb{N} : X_t(\omega) = 1\}$. Then $\{\sigma = n\} = \{X_{n-1} = 0\} \cap \{X_n = 1\}$. However, we also have $\{\tau = n\} = \{X_{n-1} = 0\} \cap \{X_n = 1\}$ for every $n \in \mathbb{N}$. Moreover, $\omega \in \{\tau = \infty\} \iff X_n(\omega) = 0, \forall n \in \mathbb{N} \iff \sigma(\omega) = \inf\{\emptyset\} = \infty$. Hence, $\tau(\omega) = \inf\{t \in \mathbb{N} : X_t(\omega) = 1\}$. $\square$

For a given stopping time $\tau \in \mathcal{E}^s$ and $n \in \mathbb{N}$, Theorem 5.1.3 shows that $\tau\wedge n\mathbb{1}$ is also a stopping time. So for an adapted process $(X_n)_{n\in\mathbb{N}}$, there exists a stopped element $X_{\tau\wedge n\mathbb{1}}$ for every $n \in \mathbb{N}$. So the stopped process corresponding to τ and $(X_n)_{n\in\mathbb{N}}$ is the sequence of stopped elements $(X_{\tau\wedge n\mathbb{1}})_{n\in\mathbb{N}}$.

Remark 5.1.6. It is evident from above that a stopping time τ is bounded precisely when $\tau \in C(K)$. The above representation of stopping times enables us to do the same for the stopped processes. Let τ be a bounded stopping time. Then there exists $N \in \mathbb{N}$ such that $\mathcal{R}(\tau) \leq N$. Since $P_n - P_{n-1} = P'_n$, given an adapted process $(X_n, \mathbb{F}_n)_{n\in\mathbb{N}}$ we have $P'_n X_n = X_n.\mathbb{1}_{\{\tau=n\}}$. Hence by Definition 3.3.3, $X_\tau = \sum_{n=1}^{\infty} X_n.\mathbb{1}_{\{\tau=n\}}$. Moreover, as τ is a continuous function with values in $\mathbb{N}$, this implies that $\{\tau = n\}$ is a clopen set for every $n \leq N$ and $\{\tau = n\} = \emptyset$ when $n > N$. Therefore, $X_\tau = \sum_{n=1}^{N} X_n \cdot \mathbb{1}_{\{\tau=n\}}$. Let $\omega \in K$, then evaluated pointwise, we have $X_\tau(\omega) = X_{\tau(\omega)}(\omega)$. Thus, the stopped process is $(X_{\tau\wedge n\mathbb{1}})_{n\in\mathbb{N}}$ evaluated pointwise.

Similarly, we can provide a pointwise evaluation of the conditional expectation operator corresponding to the stopping time. For a bounded stopping time τ, the stopping time conditional expectation operator $\mathbb{F}_\tau : \mathcal{E} \to \mathcal{E}$ is defined via $\mathbb{F}_\tau X = \sum_{i=1}^{n} \mathbb{1}_{\{\tau=i\}} \cdot \mathbb{F}_i X$. Thus using the same argument as done for stopped processes, we can evaluate for $\omega \in K$ as $(\mathbb{F}_\tau X)(\omega) = (\mathbb{F}_{\tau(\omega)} X)(\omega)$. We formalize this in the following theorem.

Theorem 5.1.7. *Let $(X_n, \mathbb{F}_n)_{n\in\mathbb{N}}$ be an adapted process on $\mathcal{E}$. Then, given a bounded discrete stopping time τ, the stopped element $X_\tau \in \mathcal{E}$ is the function evaluated pointwise. That is, at $\omega \in K$, the value of the stopped element is $X_{\tau(\omega)}(\omega)$. Moreover, for the stopping time conditional expectation operator $\mathbb{F}_\tau : \mathcal{E} \to \mathcal{E}$ we have $(\mathbb{F}_\tau X)(\omega) = (\mathbb{F}_{\tau(\omega)} X)(\omega)$ for $\omega \in K$.*

The following lemma has been proven in [Cul07] when $\mathcal{E}$ is an order continuous Banach lattice and σ and τ are unbounded stopping times. Below, we prove the lemma for when σ and τ are bounded stopping times in a vector lattice.

Lemma 5.1.8. *Let $\sigma \leq \tau$ be two bounded stopping times and let $(\mathbb{F}_i)$ be a filtration. Then $\mathbb{F}_\sigma = \mathbb{F}_\tau \mathbb{F}_\sigma = \mathbb{F}_\sigma \mathbb{F}_\tau$.*

Proof. Let $X \in \mathcal{E}$ and $\omega \in K$. Then evaluating pointwise, we have

$$(\mathbb{F}_\tau \mathbb{F}_\sigma X)(\omega) = \mathbb{F}_\tau[(\mathbb{F}_{\sigma(\omega)} X)(\omega)] = (\mathbb{F}_{\tau(\omega)} \mathbb{F}_{\sigma(\omega)} X)(\omega)$$

Since $\sigma \leq \tau$, we have that $\sigma(\omega) \leq \tau(\omega)$ and by the definition of a filtration, we have that $\mathbb{F}_{\tau(\omega)} \mathbb{F}_{\sigma(\omega)} X = \mathbb{F}_{\sigma(\omega)} X$. Thus, $(\mathbb{F}_\tau \mathbb{F}_\sigma X)(\omega) = (\mathbb{F}_{\sigma(\omega)} X)(\omega) = (\mathbb{F}_\sigma X)(\omega)$. Similarly, we can show that $(\mathbb{F}_\sigma \mathbb{F}_\tau X)(\omega) = (\mathbb{F}_\sigma X)(\omega)$. Consequently, $\mathbb{F}_\sigma = \mathbb{F}_\sigma \mathbb{F}_\tau = \mathbb{F}_\tau \mathbb{F}_\sigma$. $\square$

We shall extend the definition of a stopped element to unbounded stopping times contained in $\mathcal{E}^u$. To do this, let $\tau \in \mathcal{E}^u$ be a stopping time and let $\omega \in \cap_{i=1}^{\infty}\{X_k < \infty\} \cap \{\tau < \infty\}$. Then ω belongs to a co-meagre set. Let us suppose that $\tau(\omega) = k$. Then notice that $\lim_{n\to\infty} X_{\tau(\omega)\wedge n\mathbb{1}(\omega)}(\omega) = X_k(\omega) < \infty$. By [Corollary 3.10, [BT22]], we have that $X_{\tau\wedge n\mathbb{1}}$ is an uo-Cauchy sequence in $\mathcal{E}^u$ and thus uo-converges to an element in $\mathcal{E}^u$. Thus, we make the following definition.

Definition 5.1.9. Let τ be a stopping time contained in $\mathcal{E}^u$ and let (X_n) an adapted process. Then we define the stopped element as

$$X_\tau = \text{uo-}\lim_{n\to\infty} X_{\tau\wedge n\mathbb{1}}$$

Thus, the unbounded stopped element is in $\mathcal{E}^u$. Note that we can not define the conditional expectation operator $(\mathbb{F}_\tau)$ corresponding to the stopping element. This is because, since the sequence $(\mathbb{F}_{\tau\wedge n\mathbb{1}}X)$ is uo-Cauchy, the uo-limit of this exists in $\mathcal{E}^u$. And thus, we can get the operator $\mathbb{F}_\tau : \mathcal{E} \to \mathcal{E}^u$ defined as $\mathbb{F}_\tau X = \text{uo-}\lim_{n\to\infty} \mathbb{F}_{\tau\wedge n\mathbb{1}}X$ for $X \in \mathcal{E}$. Therefore, $\mathbb{F}_\tau$ fails the critical property of being a conditional expectation operator.

Remark 5.1.10. Similar to Theorem 5.1.6, we can provide a pointwise computation of X_τ when $\tau \in \mathcal{E}^u$. To begin with, we claim that the function $X_{\tau(\omega)}(\omega)$ computed pointwise for $\omega \in K$ is a continuous function. Let (a, b) be an open interval and let $\omega_0 \in K$ such that $X_{\tau(\omega_0)}(\omega_0) \in (a, b)$. Then there exists $k \in \mathbb{N}$ such that $X_k(\omega_0) \in (a, b)$ and $\tau(\omega_0) = k$. But we know that $X_k^{-1}(a, b)$ and $\{\tau = k\}$ are open sets. Thus we can find a neighborhood of ω_0, say U, such that $U \subseteq \{\tau = k\} \cap X_k^{-1}(a, b)$. Then for any element $\omega \in U$, we have that $X_{\tau(\omega)}(\omega) = X_k(\omega) \in (a, b)$. Therefore, $X_{\tau(\omega)}(\omega)$ computed pointwise is a continuous function and belongs to $\mathcal{E}^u$. Moreover, from the above, we notice that there exists a co-meagre set M such that for $\omega \in M$, we have $\lim_{n\to\infty} X_{\tau(\omega)\wedge n\mathbb{1}(\omega)}(\omega) = X_{\tau(\omega)}(\omega) < \infty$. Thus, for $\omega \in K$ we have that the value of X_τ at ω is $X_{\tau(\omega)}(\omega)$.

Remark 5.1.11. We should note that the idea of unbounded stopping times for Banach lattices were previously considered in [CL07, Cul07]. In [CL07], the definition of a stopped element was defined as exactly as in Definition 3.3.3 with the additional remark that the definition is also valid for unbounded stopping times if $(X_P, \mathbb{F}_P)$ exists. Definition 5.1.9 generalises the definition considered by Cullender and Labuschagne. To check this, assume that $(X_P, \mathbb{F}_P)$ exists. i.e., the partial sums $(X_P)_n = \sum_{i=1}^{n}(P_i - P_{i-1})X_i$ and $\mathbb{F}_P X = \sum_{i=1}^{n}(P_i - P_{i-1})\mathbb{F}_i X$ converge in norm. This then implies that there must exist a subsequence that uo-converges to the same limit. However, since the original sequence is uo-Cauchy implies that the norm limit and uo-limit are the same.

The following proposition improves upon [Lemma 5.3, [KLW04a]].

Proposition 5.1.12. *Let $(X_n, \mathbb{F}_n)_{n\in\mathbb{N}}$ be an increasing adapted process. Then the following statements hold.*

- *Let σ and τ be two stopping times contained in $\mathcal{E}^u$. Then $X_{\sigma\vee\tau} = X_\sigma \vee X_\tau$ and $X_{\sigma\wedge\tau} = X_\sigma \wedge X_\tau$.*
- *Let τ_n be a sequence of stopping times such that $\tau := \sup_n \tau_n$ is contained in $\mathcal{E}^u$. Then $X_\tau = \sup_n X_{\tau_n}$.*
- *Let τ_n be a sequence of stopping times contained in $\mathcal{E}^u$. Then $X_{\inf_n \tau_n} = \inf_n X_{\tau_n}$.*

Proof. By Lemma 5.1.3, we have that $\sigma \vee \tau$ is a stopping time, and therefore, $X_{\sigma\vee\tau}$ is a continuous function in $\mathcal{E}$. Let $\omega \in K$, then $X_{(\sigma\vee\tau)(\omega)}(\omega) = X_{\sigma(\omega)\vee\tau(\omega)}(\omega)$. WLOG, $\tau(\omega) \geq \sigma(\omega)$. Then $X_{(\sigma\vee\tau)(\omega)}(\omega) = X_{\tau(\omega)}(\omega)$. Since (X_n) is an increasing process, we have $X_{\tau(\omega)} \geq X_{\sigma(\omega)}$. Therefore, we have: $X_{(\sigma\vee\tau)(\omega)}(\omega) = X_{\tau(\omega)}(\omega) \vee X_{\sigma(\omega)}(\omega)$. Since the equality is valid for every point in K, we have $X_{\sigma\vee\tau} = X_\sigma \vee X_\tau$. Similarly, we can conclude for $X_{\sigma\wedge\tau} = X_\sigma \wedge X_\tau$.

WLOG, (τ_n) is increasing; otherwise, replace τ_n with $\tau_1 \vee \cdots \vee \tau_n$. Since τ_n and τ are in $\mathcal{E}^u$, X_{τ_n} and X_τ are well defined elements of $\mathcal{E}$. For an arbitrary $\omega \in K$, we have $X_{\tau(\omega)}(\omega) \geq X_{\tau_n(\omega)}(\omega)$ for every $n \in \mathbb{N}$. Therefore, $X_\tau \geq \sup X_{\tau_n}$. Let $\sigma \geq X_{\tau_n}$ for every $n \in \mathbb{N}$. By [Lemma 3.6, [BT22]], there exists a co-meagre subset $D \subseteq K$ such that $\tau_n(\omega) \uparrow \tau(\omega)$ for every $\omega \in D$. Fix $\omega \in D \setminus \{\tau = \infty\}$. Since $\mathcal{R}(\tau) \subseteq \mathbb{N} \cup \infty \cup \{0\}$, for large enough n we have $\tau_n(\omega) = \tau(\omega)$. Hence, $\sigma(\omega) \geq X_{\tau_n(\omega)}(\omega) = X_{\tau(\omega)}(\omega)$. Therefore, by the Baire Category theorem $X_\tau = \sup_n X_{\tau_n}$. Similarly, we can prove for the infimum. ☐

The next theorem is a variation of [Theorem 7.3.6, [Cul07]]. In that theorem, the author proves the desired result when $\mathcal{E}$ is an order continuous Banach lattice while we obtain the analogous result for bounded stopping but by removing the the requiring of an order continuous Banach lattice.

Theorem 5.1.13. *Let $(\mathbb{F}_i)_{i\in\mathbb{N}}$ be a filtration on $\mathcal{E}$ and let $\mathbb{D}$ be a net of bounded stopping times. Then the following statements hold:*

(i) $o\text{-}\lim_{\tau\in\mathbb{D}} \mathbb{F}_\tau f = f$ if and only if f is an order limit point of the set $\bigcup_{\tau\in\mathbb{D}} \mathcal{R}(\mathbb{F}_\tau)$.

(ii) Let (f_i) be a martingale relative to $(\mathbb{F}_i)$. Then $o-\lim_{\tau\in\mathbb{D}} f_\tau = g$ if and only if $f_\tau = \mathbb{F}_\tau g$ for each $\tau \in \mathbb{D}$ and g is an order limit point of the set $\bigcup_{\tau\in\mathbb{D}} \mathcal{R}(\mathbb{F}_\tau)$.

Proof. i) Let us suppose that $o-\lim_{\tau\in\mathbb{D}} \mathbb{F}_\tau f = f$. Since $\mathbb{F}_\tau$ is a projection, it follows that $\mathbb{F}_\tau f \in \mathcal{R}(\mathbb{F}_\tau)$ and thus f is an order limit point of $\bigcup_{\tau\in\mathbb{D}} \mathcal{R}(\mathbb{F}_\tau)$. For the converse, let us suppose that f is an order limit point of the set $\bigcup_{\tau\in\mathbb{D}} \mathcal{R}(\mathbb{F}_\tau)$. Then there exists a net $(h_\gamma) \subset \bigcup_{\tau\in\mathbb{D}} \mathcal{R}(\mathbb{F}_\tau)$ such that $h_\gamma \xrightarrow{o} f$. By the definition of order convergence, there exists a net (u_β) such that $u_\beta \downarrow 0$ and for every β there exists γ_0 such that $|h_\gamma - f| \leq u_\beta$ whenever $\gamma \geq \gamma_0$. Moreover by Theorem 5.1.8, for h_{γ_0}, we have that there exists $\tau_0 \in \mathbb{D}$ such that $h_{\gamma_0} \in \mathcal{R}(\mathbb{F}_\tau)$ for all $\tau \geq \tau_0$. Therefore, we have

$$\begin{aligned}
|\mathbb{F}_\tau f - f| &\leq |\mathbb{F}_\tau f - h_{\gamma_0}| + |h_{\gamma_0} - f| \\
&\leq |\mathbb{F}_\tau f - \mathbb{F}_\tau h_{\gamma_0}| + |h_{\gamma_0} - f| \\
&\leq \mathbb{F}_\tau |f - h_{\gamma_0}| + |h_{\gamma_0} - f| \\
&\leq \mathbb{F}_\tau u_\beta + u_\beta.
\end{aligned}$$

Since $\mathbb{F}_\tau$ is an order continuous operator for every τ we have that $\mathbb{F}_\tau u_\beta + u_\beta$ is a decreasing net such that $\mathbb{F}_\tau u_\beta + u_\beta \downarrow 0$. Thus, we have $o-\lim_{\tau\in\mathbb{D}} \mathbb{F}_\tau f = f$, as needed.

ii) Let us suppose that f_τ order converges to g. Then it is clear by Proposition 3.3.5 that g is an order limit of points in $\bigcup_{\tau\in\mathbb{D}} \mathcal{R}(\mathbb{F}_\tau)$. Moreover from Theorem 3.3.7, we have that $\mathbb{F}_\tau f_\sigma = f_\tau$ when $\tau \leq \sigma$. Thus, $\mathbb{F}_\tau g = \mathbb{F}_\tau(o-\lim_{\sigma\in\mathbb{D}} f_\sigma) = o-\lim_{\sigma\in\mathbb{D}} \mathbb{F}_\tau f_\sigma = f_\tau$. For the converse from Part i), since g is an order limit of $\bigcup_{\tau\in\mathbb{D}} \mathcal{R}(\mathbb{F}_\tau)$, we get that $o-\lim_{\tau\in\mathbb{D}} f_\tau = o-\lim_{\tau\in\mathbb{D}} \mathbb{F}_\tau g = g$.

□

Remark 5.1.14. There is also the notion of stopping times for continuous stochastic processes in vector lattices, as discussed by Grobler in [Gro10, Gro11, Gro21a]. However, obtaining the $C^\infty(K)$ representation of the continuous stopping times and stopped processes falls outside the scope of the methods used in this chapter. This is due to the fact that we make use of the Baire category theorem at various points in this paper, which does not hold for an arbitrary union of nowhere dense sets.

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