Local Risk-Minimization for Change Point Models

by

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Abstract

The main aim of this thesis lies in describing, as explicitly as possible, the local-risk minimizing strategy for a change-point model. To this end, we analyze and investigate the mathematical structures of this model. The change-point model is a model that starts with a dynamic and switches to another dynamic immediately at some random time. This random time can represent the time of occurrence of an event that may affect the market and/or agents, such as the default of a firm, a catastrophic event, sudden adjustment of fiscal policies, etc. The most interesting feature of this random time lies in the fact that its behavior might not be seen through the public flow of information. This feature obliges us to enlarge the flow of information to include this random time. For this context, we develop the local-risk minimization and describe the optimal strategy using the public information. As applications of these results, we address the hedging problem for default sensitive contingent claims.

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Chapter 1

Introduction

Mathematical Finance and Modern Finance started in the Ph.D thesis of Louis Bachelier (a French mathematician) defended in 1900. During the past four decades, a tremendous development in finance theory have boosted the progress of optimal investment or optimal portfolio problems. Based on "no-arbitrage" hypothesis, these optimization problems have been extended to various building blocks. Basically speaking, optimal portfolio problems boil down into two targets: Minimization of financial risks and Maximization of utilities. The most foundational contributions to optimal portfolio problems are the works of the two Nobel Prize winners - Harry Markowitz and Robert Merton. In Markowitz's analysis, the optimal portfolio was obtained by using the variance as a measure of risk (see [29] for details), while in Merton's works, the optimal portfolio problem was addressed by using the concept of utilities of financial agents. Both Markowitz's and Merton's works have been extended to the most general market models thanks to the well-developed semimartingale theory and stochastic analysis techniques. Also, these well-developed quantitative analysis techniques led to the bloom of variety of complicated financial products and derivatives like CDO (Collateralized Debt Obligation), CDS (Credit Default Swap) and CLN (Credit Linked Note). All these financial derivatives are involved with credit risks and defaultable markets which are the key motivations for the change-point models that will be addressed in this thesis.

The performance of financial asset prices is often subject to certain random events, which suggests that the parameters (return rate, volatility and driven Brownian motions) of financial models are generally highly dependent on random times. A number of events (for example, the release of information in the press, abrupt changes in the price of raw materials or the first time a stock price hits some psychological level) can alter the asset price's dynamic abruptly. The random time when such an event occurs is called a change point and the models with (random) time -dependent parameters are called change-point models. Change-point problems have a long history, probably starting with the papers of Page [33], [34] in an a-posterior setting and of Shiryaev [43] in a quickest detection. In the context of mathematical finance, the question was investigated in [10], [21] and [45]. The modelling of change point is also related to pricing and hedging of credit risk and defaultable claims (see [7], [8], [9] and references therein).

In this thesis, we consider continuous semimartingale settings with progressive enlargement of filtrations associated to our change-point model, and we try to describe optimal portfolio that minimizes hedging risk for a given contingent claims. Precisely, we will work in the context of the Risk-Minimization Theory developed by Föllmer and Sondermann [16]. This theory provides, for a given contingent claim, the value of the portfolio and the associated hedging strategy that minimizes a particular quadratic criterion associated with accumulated trading costs. In fact, at the beginning, this theory was instituted in a martingale setting (i.e. when the stock price process is a martingale), it was extended to general semimartingale settings by Schweizer and was called as Local Risk-Minimization theory. The connection between these two theories is that when the financial asset price processes are continuous, the local risk-minimization problem can be solved as a risk-minimization problem under the so-called minimal martingale measure.

For the reader's convenience, in the remaining part of this chapter, we will provide a brief summary of the content of this thesis. A thorough summary including a motivation of the studied subjects and references to related works can be found at the start of each chapter. The rest of this thesis is constituted by four chapters (Chapters 2-5).

In Chapter 2, we will recall some stochastic tools that are important and useful throughout the thesis. These stochastic elements include semimartingale theory and basic concepts of stochastic analysis. The most important concepts in this chapter is the concept of progressive enlargement of filtrations which is essential to characterize our change-point model and the theory of local risk-minimization which is the main tool for our optimal portfolio problems.

Chapter 3 analyzes the change-point model which will be used throughout this thesis. We will demonstrate that our change-point model can be regarded as a combination of before-change dynamic and after-change dynamic. Both dynamics are semimartingales satisfying the so-called Structure Conditions with respect to the original filtration $\mathbb{F}$. This filtration can be regarded as the collection of public information related to the asset prices. We remark that the global discounted price dynamic with a change-point may not be a $\mathbb{F}$-semimartingale due to the fact that the change-point τ may not be an $\mathbb{F}$-stopping time. We assume that the change-point is an honest time which was defined in various probabilistic literature (see Millar [30], Meyer, Smythe and Walsh [30] and Barlow [6]). Under this assumption, we expand the original filtration $\mathbb{F}$ with the change-point to obtain the progressive enlarged filtration $\mathbb{G}$ which also contains the information about the change-point τ in addition to $\mathbb{F}$. We will show that the global dynamic of the discounted price process, under this assumption, is also a $\mathbb{G}$-semimartingale satisfying Structure Conditions. It is worth mentioning that the contribution of this thesis starts in Chapter 3 via the structural analysis of the model describe above.

The core of the thesis is detailed in Chapter 4. Therein, we will try to describe the local risk-minimizing hedging strategy of a given contingent claim H with respect to our change-point model under the progressive enlarged filtration $\mathbb{G}$. This description uses the flow of information $\mathbb{F}$. Since the local risk-minimization strategy is intimately related to the Galtchouk-Kunita-Watanabe decomposition of the contingent claim H under the so-called minimal martingale measure, we will also analyze the minimal martingale measure densities for before-change dynamic and after-change dynamic, and their relationships with the global $\mathbb{G}$-minimal martingale measure density. Then, we will apply the decomposition techniques related to honest time to find our $\mathbb{G}$-locally risk-minimizing strategy. Roughly speaking, we will analyze the decomposition based on three disjoint set separated by the random time (change-point) τ and then combine them to get a global picture. The last chapter (Chapter 5) presents a direct applications of our results of Chapter 4 to default sensitive contingent claims. Theses claims are popular among insurance contracts with surrender options.

Chapter 2

Elements from Stochastics and Finance

In this chapter, we will review some fundamental concepts and theories on stochastic analysis. We start with some standard notations which will be used throughout the whole thesis. Then, we will demonstrate four topics in details in the following sections: semimartingales and structure conditions in Section 2.1, projections of stochastic processes in Section 2.2, progressive enlargement of filtration in Section 2.3, local risk-minimization theory in Section 2.4. For more explanations on these topics, we refer the reader to [24], [47], [14], [36] and [37]

Consider a probability space denoted by $(\Omega, \mathcal{F}, P)$ on which we consider a family of σ-algebras $(\mathcal{F}_t)_{0\leq t\leq T}$ which is increasing ($\mathcal{F}_s \subset \mathcal{F}_t$, if $s \leq t$). We call this family a **filtration** and denote it by $\mathbb{F}$ throughout the thesis. Furthermore, we assume the filtration is right-continuous, i.e.$\mathcal{F}_t = \cap_{s>t}\mathcal{F}_s$. For each $t \in [0, T]$, $\mathcal{F}_t$ represents the aggregate information up to time t. Throughout the rest of the thesis, the filtered probability space is $(\Omega, \mathcal{F}, \mathbb{F}, P)$ and is also called the **stochastic basis** . P is the real-world probability measure. $T \in [0, \infty)$ represents a fixed time horizon. We also assume that the stochastic basis is complete, which means that the σ-algebra $\mathcal{F}$ is P-complete and that every $\mathcal{F}_t$ contains all P-null sets of $\mathcal{F}$.

Definition 2.1: A process $X = (X_t)_{t\in\mathbb{R}+}$ is called **càdlàg**, or **RCLL**, if all its paths are right-continuous ($\lim_{s\to t^+} X_s = X_t$) and admits left-hand limit($\exists \lim_{s\to t^-} X_s = X_{t-}$). The jump at time t for X is defined as $\Delta X_t := X_t - X_{t-}$.

Definition 2.2: A process $X = (X_t)_{t\in\mathbb{R}+}$ is called **adapted** if X_t is $\mathcal{F}_t$-measurable for all t.

Definition 2.3: A process $X = (X_t)_{t\in\mathbb{R}+}$ belongs to $L^p(P)$, for $p \in [1,\infty]$ if $E(|X_t|^p) < \infty$ for every $t \in \mathbb{R}+$(i.e. $|X_t|^p$ is P-**integrable** for all t).

Definition 2.4: A random time τ is a measurable mapping $\tau : \Omega \to [0,\infty]$. τ is called a **stopping time** if $\{\tau \le t\} \in \mathcal{F}_t$ for all $t \in \mathbb{R}^+$.

Hence by saying τ is a stopping time, it means that it is possible to decide whether or not $\{\tau \le t\}$ has occurred on the basis of the knowledge of $\mathcal{F}_t$. The process X^τ is called the **stopped process** at time τ, and is defined in the following way:

$$X_t^\tau := X_{t\wedge\tau} = \begin{cases} X_t, & \text{if } t \le \tau \\ X_\tau, & \text{otherwise.} \end{cases}$$

Also, the σ-algebra $\mathcal{F}_\tau$ is defined by

$$\mathcal{F}_\tau := \{A \in \mathcal{F}, A \cap \{\tau \le t\} \in \mathcal{F}_t, \text{ for all } t \ge 0\}.$$

On the product space $\Omega \times [0,T]$, we define two σ-algebra: The **optional σ-algebra** denoted by $\mathcal{O}$, which is generated by the adapted and càdlàg processes and The **predictable σ-algebra** denoted by $\mathcal{P}$, which is generated by the adapted and continuous processes.

Definition 2.5: A process is called **optional** (resp. **predictable**) if the process is $\mathcal{O}$ (resp. $\mathcal{P}$)-measurable.

The set of all real-valued processes A with A_0=0 that are RCLL, adapted and for which each path $t \mapsto A_t(\omega)$ has finite variation over each finite interval $[0, t]$ is denoted by $\mathcal{V}$. The variation of A is given by $Var(A) = \int |dA_s|$. The set of processes whose each path is non-decreasing is referred to $\mathcal{V}^+$. We denote by $\mathcal{A}$ the set of all $A \in \mathcal{V}$ that have integrable variations, that is,

$$\mathcal{A} := \{A \in \mathcal{V} : E(Var(A)_T) < \infty.\}$$

Also, we put $\mathcal{A}^+ := \mathcal{A} \cap \mathcal{V}^+$ the set of all $A \in \mathcal{V}^+$ that are integrable (i.e. $E(A_T) < \infty$). Moreover, we denote by $L(A)$ the set of all predictable processes H, satisfying

$$\int_0^T |H_s| dVar(A)_s < \infty, \quad P - a.s. \tag{2.1}$$

Then, for any $H \in L(A)$, the resulting integral of H with respect to A is denoted by $H \cdot A$, which belongs to $\mathcal{V}$. For more explanations and demonstrations on this integration and the set $L(A)$, we refer the readers to [24], page 206.

2.1 Semimartingales and Structure Conditions

In this section, we start with the most fundamental concepts in stochastic analysis: martingale, submartingale, supermartingale and local martingale. Then, we will focus on semimartingales and the structure conditions.

Definition 2.6: A **martingale** (resp. **submartingale, supermartingale**) is an adapted process X on the basis $(\Omega, \mathcal{F}, \mathbb{F}, P)$ whose P- almost all paths are RCLL, such that every X_t is integrable, and for $s \leq t$:

$$E(X_t|\mathcal{F}_s) = X_s, \quad \text{(resp. } E(X_t|\mathcal{F}_s) \geq X_s, \quad E(X_t|\mathcal{F}_s) \leq X_s \text{)}$$

Definition 2.7: A process X is a **local martingale** if there exists an increasing sequence (τ_n) of stopping times such that $\lim_{n\to\infty} \tau_n = T$ a.s. and such that each stopped process X^{τ_n} is a martingale. (τ_n) is called a localizing sequence for X.

The concept of localizing of martingales can also be extended for any class of processes. Throughout this thesis, if $\mathcal{C}$ is a class of processes, we denote by $\mathcal{C}_0$ the set of processes $X \in \mathcal{C}$ with $X_0 = 0$ and by $\mathcal{C}_{loc}$ the processes X such that there exists a localizing sequence (τ_n) for X and the stopped process $X^{\tau_n} \in \mathcal{C}$, for each τ_n. Also, we put $\mathcal{C}_{loc,0} = \mathcal{C}_0 \cap \mathcal{C}_{loc}$. Therefore, $\mathcal{M}^2_{loc}$ represents the family of locally square-integrable martingales. Now, we denote by $L^2_{loc}(X)$ the set of all predictable processes H satisfying $\int H^2 d[X,X]^* \in \mathcal{A}^+_{loc}$ such that the integration $\int H dX = H\cdot X$ is well defined and $H \cdot X \in \mathcal{M}^2_{loc}$. For more details of this integration we refer the reader to [24] (see page 204).

We denote the class of all square-integrable martingales on $(\Omega, \mathcal{F}, \mathbb{F}, P)$ by $\mathcal{M}^2(P, \mathbb{F})$ and it is the set of all $(P, \mathbb{F})$-martingales M such that $E(M_T^2) < \infty$. Also $\mathcal{M}_{loc}(P, \mathbb{F})$ represents the class of all $(P, \mathbb{F})$-local martingales. The filtration generated by a d-dimensional Brownian motion is called **Brownian filtration**. Then, from [23], it is well-known that local/square-integrable martingales adapted to Brownian filtrations admit a martingale representation as stated in the following.

Theorem 2.1: *Let $M \in \mathcal{M}_{loc}(P, \mathbb{F})$, where the filtration $\mathbb{F}$ is the Brownian filtration generated by a d-dimensional Brownian motion $W = (W^1, W^2, \ldots, W^d)$. Then there exists a predictable stochastic process*

$$\Psi = (\Psi^1, \Psi^2, \ldots, \Psi^d) : [0,T] \times \Omega \longmapsto \mathbb{R}^d$$

with

$$\int_0^T ||\Psi_t||^2 dt < \infty \quad P\text{-}a.s.$$

*$[X,X]$ is the qudratic variation of X which is defined as $[X,X]_t = [X]_t := \lim_{\|P\|\to 0} \sum_{k=1}^n (X_{t_k} - X_{t_{k-1}})^2$ where P ranges over all partititions of the inteval $[0,t]$

and

$$M_t = M_0 + \int_0^t \Psi'_s dW_s = M_0 + \sum_{i=1}^{d} \int_0^t \Psi^i_s dW^i_s$$

Furthermore, the processes Ψ *is uniquely determined with respect to* $\lambda \otimes P$*. Here,* λ *denotes the Lebesgue measure.*

Remark 2.1: We refer to [25], Lemma 2.3 and Lemma 2.4 that this martingale representation property is preserved under a new measure Q which is equivalent to P. That is for any $(Q, \mathbb{F})$-local/square martingale, there exists a Q-predictable measurable stochastic processes $\widehat{\Psi}$ such that

$$\widehat{M_t} = \widehat{M_0} + \int_0^t \widehat{\Psi}'_s d\widehat{W}_s$$

where $\widehat{W}$ is a d-dimensional $(Q, \mathbb{F})$-Brownian motion.

Definition 2.8: Let $\mathcal{T}$ be the collection of all stopping times. A measurable process X is said to be of class $(\mathbb{D})$, if $\{X_\tau I_{\{\tau<\infty\}} : \tau \in \mathcal{T}\}$ is a uniformly integrable family of random variables.

For supermartingale/submartingale in class $(\mathbb{D})$, we have the following well-known decomposition, called **Doob-Meyer decomposition** (see more details in [31]).

Theorem 2.2: *Let* X *be a right -continuous supermartingale (resp. submartingale) of class* $(\mathbb{D})$*. Then* X *can be uniquely decomposed as:*

$$X = M - A,$$

where M *is a uniformly integrable martingale, and* A *is a predictable increasing process (resp. decreasing) process with* $A_0 = 0$*.*

This decomposition also leads to define an immense class of stochastic processes, which are called semimartingales as following:

Definition 2.9: A **semimartingale** X is a process of the form

$$X = X_0 + M + A$$

with X_0 finite-valued and $\mathcal{F}_0$-measurable, $M \in \mathcal{M}_{loc,0}$ and $A \in \mathcal{V}$.

If in addition A is predictable, then X is called a **special semimartingale**.

Remark 2.2: If a semimartingale is special then the decomposition $X = X_0+M+A$ with A predictable is uniquely determined and this decomposition is called the **canonical decomposition** of X.

The integration with respect to semimartingales is also a fundamental concept in stochastic analysis and mathematical finance. Precisely, we say that a predictable processes H is integrable with respect to a semimartingale $X = X_0 + M + A$ if $H \in L^2_{loc}(M) \cap L(A) = L(X)$. In this case we define the integral of H with respect to X as

$$H \cdot X := \int H dX = \int H dM + \int H dA = H \cdot M + H \cdot A$$

Definition 2.10: For processes $X \in \mathcal{A}_{loc}$ we can define the unique (up to an evanescent set) process X^{com}, called the **compensator** under P, which is the predictable process in $\mathcal{A}_{loc}$ such that $X - X^{com}$ is a P-local martingale.

Definition 2.11: The **quadratic covariation** of two semimartingales X and Y is defined as

$$[X, Y] = XY - X_0Y_0 - X_- \cdot Y - Y_- \cdot X$$

and the **predictable quadratic covariation** of X and Y is the compensator of the quadratic covariation $[X, Y]$. It is denoted by $\langle X, Y\rangle$ and therefore also called the angle bracket of X and Y. The abbreviation $\langle X\rangle$ will be used for the angle bracket $\langle X, X\rangle$.

Remark 2.3: Since martingales/semimartingales are adapted to some reference filtration $\mathbb{F}$ under some measure P, then it is obvious that $\langle X, Y \rangle$ is measure and filtration dependent. We denote $\langle X, Y \rangle^{P,\mathbb{F}}$ in order to indicate the reference measure P and filtration $\mathbb{F}$. Changing measure or filtration may alter the angle bracket of two semimartingales.

For a semimartingale X we will denote X^c (resp. X^d) as the continuous (resp. discontinuous) local martingale part. Then we have the following theorem.

Theorem 2.3: *If X, Y are semimartingales, then*

$$[X, Y]_t = \langle X^c, Y^c \rangle_t + \sum_{s \le t} \Delta X_s \Delta Y_s. \tag{2.2}$$

Proof. see Jacod and Shiryaev in [24], page 55. □

As a result of this theorem, the predicable quadratic covariation equals the quadratic covariation, for any continuous martingales X and Y:

$$[X, Y] = \langle X, Y \rangle.$$

The following lemma will be frequently used Chapter 4.

Lemma 2.1: *Let X and Y be two semimartingales, then we have*

(1) $[X + Y, Z] = [X, Z] + [X, Z]$*, for any semimartingale Z.*

(2) If $Y \in \mathcal{V}$ and then $[X, Y] = \int \Delta X dY$ and thus if X is continuous $[X, Y] = 0$.

(3) If X and Y are local martingales, then $[X, Y] = 0$ whenever X is continuous and Y purely discontinuous.

(4) Let $H \in L(X) \cap L(Y)$, then $\left[X, \int H dY\right] = \int H d[X, Y] = H \cdot [X, Y]$

For semimartingales, the concept of orthogonality is crucial to the definition of *minimal martingale measures* in Section 2.4.

Definition 2.12: Two $(P, \mathbb{F})$-semimartingales X and Y are called **orthogonal** under a measure P if $[X, Y]$ is a local $(P, \mathbb{F})$-local martingale. That is the angle bracket $\langle X, Y\rangle^{P,\mathbb{F}} = 0$. We denote by $(X \perp Y)^{P,\mathbb{F}}$ as X and Y are mutually orthogonal under measure P and filtration $\mathbb{F}$.

For a RCLL adapted process X, an important notion in mathematical finance is the concept of an equivalent martingale measure for S, which is a probability measure Q equivalent to the original measure P such that X is a martingale under Q. If such a Q exists, then its density processes D is a strict martingale density for X under P (i.e. D is strictly positive, and both D and DX are local martingales under P). Also, the existence of such measure Q implies certain **no-arbitrage** conditions for X. (see Delbaen and Schachermayer in [13]). The assumption of no-arbitrage for X guarantees the structure conditions **(SC)**:

Definition 2.13: Suppose that a special semimartingale X admits the canonical decomposition $X = X_0 + M + A$. Then we say that X satisfies **structure conditions (SC)** if there exists a predictable process $\widehat{\lambda}$ such that

$$A_t = \int_0^t \widehat{\lambda}_u d\langle M\rangle_u \quad \textbf{and} \quad \int_0^T \widehat{\lambda}^2 d\langle M\rangle_u < \infty \quad P\text{-}a.s. \tag{2.3}$$

Under structure conditions, we can define the mean-variance tradeoff process:

Definition 2.14: If X satisfies **(SC)**, then the **mean-variance trade off process** (MVT) $\widehat{K}$ for X is defined as the increasing predictable process :

$$\widehat{K}_t = \int_0^t \widehat{\lambda}_u dA_s = \left\langle \int \widehat{\lambda} dM \right\rangle_t .$$

Remark 2.4: In the case that there exists an equivalent martingale measure for S, S automatically satisfies **(SC)** if S is continuous. We refer the reader to Theorem 1 in [40] proved by Schweizer for details. More general results in this direction can be found in Choulli/Stricker ([12]).

Then, if a semimartingale X satisfies **(SC)**, every martingale density $D \in \mathcal{M}^2_{loc}(P, \mathbb{F})$ for X can be obtained as a solution of a stochastic differential equation :

$$D_t = 1 - \int_0^t D_{u-}\lambda_u dM_u + R_t, \quad 0 \leq t \leq T, \tag{2.4}$$

for some $R \in \mathcal{M}^2_{loc,0}(P, \mathbb{F})$ orthogonal to M. The multidimensional versions of these results are formulated and proved in [40] (see Theorem 1). In this paper, the author generalizes previous results by Ansel/Stricker [2, 3] and Schweizer [39].

2.2 Projections of Stochastic Processes

In this section we review the fundamental notions of optional (resp. predictable) projections of measurable processes and dual optional (resp. predictable) projection of integrable increasing processes. Herein, the material is standard in the general theory of stochastic processes and we refer to the book [22] (Chapter V) for more details and refinements.

Theorem 2.4: *Let X be a measurable process either positive or bounded. Then, there exists a unique (up to indistinguishability) optional process, denoted by ${}^oX$, such that for every stopping time τ we have:*

$$E[X_\tau I_{\{\tau<\infty\}}|\mathcal{F}_\tau] = {}^oX_\tau I_{\{\tau<\infty\}} \quad P\text{-}a.s.$$

*The process ${}^oX$ is called the **optional projection** of X.*

Theorem 2.5: *Let X be a measurable process either positive or bounded. Then, there exists a unique predictable process, denoted by ${}^pX$, such that for every predictable stopping time τ we have:*

$$E[X_\tau I_{\{\tau<\infty\}}|\mathcal{F}_{\tau-}] = {}^pX_\tau I_{\{\tau<\infty\}} \quad P\text{-}a.s.$$

The process ${}^pX$ is called the ***predictable projection*** *of X. In particular if X is a bounded or positive martingale, then ${}^pX = X_-$*

Remark 2.5: Optional and predictable projections satisfy the following properties with the conditional expectation: if X is a measurable process and Y is an optional (resp. predictable) bounded process, then

$$^o(XY) = Y^oX, \quad (\text{resp. } ^p(XY) = Y^pX).$$

Next, we review another property of optional and predictable projections related to the integral with respect to increasing processes.

Theorem 2.6: *Let X be a non-negative measurable process and let A be an increasing process. Then we have :*

$$E\left[\int_{[0,\infty[} X_u dA_u\right] = \left[\int_{[0,\infty[} {}^oX_u dA_u\right]$$

and if in addition A is also predictable

$$E\left[\int_{[0,\infty[} X_u dA_u\right] = E\left[\int_{[0,\infty[} {}^pX_u dA_u\right]$$

Increasing processes play a central role in the general theory of stochastic processes: the main idea is to consider an increasing process as a random measure on $\mathbb{R}^+, dA_t(\omega)$, whose distribution function is $A_{\bullet}(\omega)$. Then, we shall make the convention that $A_{0-} = 0$, so that A_0 is the measure of $\{0\}$. We call an RCLL increasing process which is not adapted to $\mathbb{F} = (\mathcal{F}_t)_{0\leq t\leq T}$ as **raw increasing process**. Now, we can construct a P-measure μ on $\mathcal{B}(\mathbb{R}^+)\otimes\mathcal{F}$ in the following way: let A be a raw increasing process which is integrable, for any bounded measurable process

$$\mu(X) = E\left[\int_{[0,\infty[} X_u dA_u\right]$$

Then we can define the projections of P-measure μ:

Definition 2.15: Let μ be a P-measure on $\mathcal{B}(\mathbb{R}^+) \otimes \mathcal{F}$. We call **optional** (resp. **predictable**) **projection** of μ the P-measure μ^o (resp. μ^p) defined on $\mathcal{B}(\mathbb{R}^+) \otimes \mathcal{F}$ by:

$$\mu^o(X) = \mu({}^oX), \quad (\text{resp. } \mu^p(X) = \mu({}^pX))$$

for all measurable and bounded processes X.

Normally, μ^o (or μ^p) is not associated with ${}^oA$ (or ${}^pA$). This leads us to define the following fundamental notions of dual projections:

Definition 2.16: Let A be an integrable raw increasing process. We call **dual optional projection** of A the increasing process A^o defined by:

$$E\left[\int_{[0,\infty[} X_u dA_u^o\right] = E\left[\int_{[0,\infty[} {}^oX_u dA_u\right]$$

for any bounded measurable process X. We call **dual predictable projection** of A the predictable increasing process A^p defined by:

$$E\left[\int_{[0,\infty[} X_u dA_u^p\right] = E\left[\int_{[0,\infty[} {}^pX_u dA_u\right]$$

for any bounded measurable process X.

Remark 2.6: Since a process with finite variation (resp. integrable variation) can be written as the difference of two increasing processes (resp. integrable increasing processes), then the above definition can be extended in the same way to processes with finite and/or integrable variation.

2.3 Progressive Enlargement of Filtrations

In this section we review the theory of progressive enlargement of filtrations which was originally introduced by Millar in [32]. Then it was first independently developed by Barlow [6] and Yor [47] and further was refined by Jeulin and Yor in [28] and Jeulin in[26, 27]. For more details, we refer the reader to [48].

Here again, let $(\Omega, \mathcal{F}, \mathbb{F}, P)$ be a filtered probability space satisfying the usual conditions. We consider an $\mathcal{F}$-measurable random time τ (may not be an $\mathbb{F}$-stopping time) as a mapping $\Omega :\mapsto \mathbb{R}^+$. The progressive enlargement filtration is defined by:

Definition 2.17: The **progressive enlargement of filtration** $\mathbb{F} = (\mathcal{F}_t)$ with respect a random time τ is obtained as a collection of σ-algebras

$$\mathcal{G}_t := \bigcap_{s>t} \left(\mathcal{F}_s \vee \sigma(\tau \wedge s)\right), \text{ for all } t \in [0, T]$$

Remark that the filtration $\mathbb{G} = (\mathcal{G}_t)$ is the smallest filtration satisfying usual conditions containing $(\mathcal{F}_t)$ and making τ a stopping time.

Remark 2.7: In the case when $\mathbb{F} = \mathbb{F}^W$ (W is a Brownian motion), the filtration $\mathcal{G}_t$ is the information about the knowledge of the Brownian motion W up to time t plus the knowledge of the random time τ if the latter has occurred before time t. For other different settings of filtration $\mathbb{F}$ and $\mathbb{G}$ we refer to [18].

The problem related to enlargement of filtration is that the local martingale structures may be altered under a larger filtration $\mathbb{G}$. In other words, a $(P, \mathbb{F})$-martingale may not remain a $(P, \mathbb{G})$-local martingale or even not a $(P, \mathbb{G})$-semimartingale. However, among random times, there is a special random time such that the pair of filtrations $(\mathbb{F}, \mathbb{G})$ satisfies the (H') hypothesis : Every $(P, \mathbb{F})$-local martingale X is $(P, \mathbb{G})$-semimartingale. This random time is called honest time which is the most studied random time after stopping times. It is defined as following:

Definition 2.18: [Jeulin,[27], Chaper V] An $\mathcal{F}$-measurable random time is called an **honest time** if for all $t > 0$ there exists an $\mathcal{F}_t$-measurable random variable ξ_t such that

$$\tau \mathbf{1}_{\{\tau \le t\}} = \xi_t \mathbf{1}_{\{\tau \le t\}}$$

The following theorem borrowed from Dellacherie and Meyer in [15] (Lemma 1) lays out a split formula for any $\mathcal{G}_t$ and $\mathcal{G}_{t-}$-measurable random variables

Theorem 2.7: *If τ is an honest time, then we have:*

For every $\mathcal{G}_t$-measurable variable Y it admits the decomposition:

$$Y = U\boldsymbol{1}_{\{t<\tau\}} + V\boldsymbol{1}_{\{t\geq\tau\}},\ \textit{where } U, V \textit{ are } \mathcal{F}_t\textit{-measurable}$$

For every $\mathcal{G}_{t-}$-measurable variable Y it admits the decomposition:

$$Y = U\boldsymbol{1}_{\{t\leq\tau\}} + V\boldsymbol{1}_{\{t>\tau\}},\ \textit{where } U, V \textit{ are } \mathcal{F}_{t-}\textit{-measurable}$$

The Azéma supermartingale associated with a random time τ is the $(P, \mathbb{F})$-optional projection of the process $\mathbf{1}_{\{\tau>t\}}$, that is

$$Z_t = E(\mathbf{1}_{\{\tau>t\}}|\mathcal{F}_t) = P(\tau > t|\mathcal{F}_t).$$

By now, we can state our main theorem of this section which is also proved by Jeulin in [27].

Theorem 2.8: *If τ is an honest time, then any $(P, \mathbb{F})$-local martingale M_t is a also a $(P, \mathbb{G})$- semimartingale and decomposes as:*

$$\begin{aligned}\widetilde{M_t} =& M_t - \int_0^{t\wedge\tau} \frac{d\langle M, Z\rangle_u^{P,\mathbb{F}}}{Z_{u-}} + \int_\tau^{\tau\vee t} \frac{d\langle M, Z\rangle_u^{P,\mathbb{F}}}{1 - Z_{u-}} \\ =& M_t - \int_0^t \boldsymbol{1}_{\{u\leq\tau\}} \frac{d\langle M, Z\rangle_u^{P,\mathbb{F}}}{Z_{u-}} + \int_0^t \boldsymbol{1}_{\{u>\tau\}} \frac{d\langle M, Z\rangle_u^{P,\mathbb{F}}}{1 - Z_{u-}}\end{aligned}$$

where $\widetilde{M}$ is a $(P, \mathbb{G})$-local martingale.

2.4 Local Risk-Minimization Theory

The Risk-minimizing hedging strategy originates from the hedging strategy described by Harrison and Kreps in [20] for complete markets. This strategy was extended by Föllmer and Sondermann [16] for incomplete markets if the underlying risky asset is still a martingale. Therein, the authors essentially suggested to determine trading strategy for a given contingent claim by minimizing the variance of future costs defined as the difference between the claim and gains made from trading on the financial market. However, this risk-minimization strategy can not be obtained in the semimartingale setting as discussed by Schweizer (1991) in [38]. The extension to semimartingales (martingale has a drift term) are developed in Schweizer [36, 37] and in Föllmer/Schweizer [17]. All the definitions and notations in this chapter can be found in those reference materials.

Consider a simple financial market consists of one risky asset X and one risk-free asset B. Also, we assume that X and B are RCLL processes adapted to the filtration $\mathbb{F}$. Furthermore, the discounted price process $S_t := \frac{X_t}{B_t}$ is assumed to belong to $L^2(P)$. In addition, we assume that there exits an equivalent martingale measure Q with square-integrable density for the discounted price process S. Hence we can exclude arbitrage opportunities in the market. This implies that S is a semimartingale under basic measure P and satisfies **(SC)** as discussed in Section 2.1. Finally we suppose that a contingent claim whose discounted value is given by a $\mathcal{F}_T$-measurable square-integrable random variable H (i.e. $H \in L^2(\mathcal{F}_T, P)$)

We see that the market $(S, \mathbb{F})$ is **complete** if any contingent claim H can be represented as a sum of a stochastic integral with respect to S and a constant (i.e. $H = c + \int_0^T \theta_u dS_u$). The integrand θ is an $\mathbb{F}$-predictable processes which provides a self-financing hedging strategy and replicates the discounted payoff H at maturity T without any risk. The hedging here means that a trader may want to cover himself against potential loss arising from a sale of a contract with discounted payoff

H at maturity T by dynamic trading strategies based on S. Since S is a $(P, \mathbb{F})$-semimartingale, we can use stochastic integral with respect to S and introduce the set $L(S)$ of all $\mathbb{F}$-predictable S-integrable processes. In an **incomplete market**, a general claim is not necessarily represented by a stochastic integral with respect to S, which means we may not replicate the discounted payoff H by only trading dynamically S without risk. Some risk results from a non-perfect hedging. Therefore, our aim is clear in this case: We are looking for an admissible strategy with minimal "risk" which replicates a given contingent claim H. We will specify this " risk" in the next subsection

2.4.1 The martingale case

For the case where S is a $(P, \mathbb{F})$-martingale, the method has been defined and developed by Föllmer and Sondermann under the name of **risk-minimization**. We adopt the market setting described above and introduce the space $L^2(S)$ of all $\mathbb{F}$-predictable process ξ such that

$$||\xi||_{L^2(S)} := \left(E\left[\int_0^T \xi_u^2 d[S]_u\right]\right)^{\frac{1}{2}} < \infty$$

Definition 2.19: For a fixed contingent claim $H \in L^2(\mathcal{F}_T, P)$, an **admissible strategy** is a pair $\varphi := (\xi, \eta)$ with $\xi \in L^2(S)$ and such that the discounted value process $V_t(\varphi) := \xi_t S_t + \eta_t, 0 \leq t \leq T$ is square-integrable and $V_T(\varphi) = H, P\text{-}a.s.$.

Definition 2.20: For any admissible strategy φ, the cost process is defined by

$$C_t(\varphi) := V_t(\varphi) - \int_0^t \xi_u dS_u, \quad 0 \leq t \leq T \tag{2.5}$$

The quantity $C_t(\varphi)$ describes the accumulated cost incurred by φ up to time t.

The risk process associated to φ is defined by

$$R_t(\varphi) := E[(C_T(\varphi) - C_t(\varphi))^2 | \mathcal{F}_t], \quad 0 \le t \le T. \tag{2.6}$$

Remark 2.8: The cost process defined above is just the value of the portfolio φ minus the accumulated gain/loss by following the trading strategy φ. Whereas, the risk process defined above can be regarded as the conditional variability of the cost up to time t compared with the terminal cost at maturity T.

Definition 2.21: An admissible strategy φ is called **risk-minimizing** if for any other admissible strategy $\widetilde{\varphi}$ such that $V_T(\widetilde{\varphi}) = V_T(\varphi), P\text{-}a.s.$, we have

$$R_t(\varphi) \le R_t(\widetilde{\varphi}) \quad P\text{-}a.s. \quad \text{for every } t \in [0, T].$$

Definition 2.22: An admissible strategy φ is called **mean-self-financing** if its cost process $C_t(\varphi)$ is a P-martingale.

Remark 2.9: A admissible strategy φ is called **self-financing** if its cost process has constant paths, that is

$$C_t \equiv C_0, \quad P\text{-}a.s. \text{ for all } t \in [0, T].$$

Any self-financing strategy is clearly mean-self-financing. Also, a self-financing strategy has the value process of the form

$$V_t = C_0 + \int_0^t \xi_u dS_u, \quad 0 \le t \le T.$$

Lemma 2.2: *If φ is a risk-minimizing strategy, then it is also mean-self-financing.*

Proof. See Lemma 2 of [16] for the proof. □

If S is a P-martingale, the risk-minimization problem is always solvable by applying the so-called Galtchouk-Kunita-Watanabe decomposition defined as follows:

Definition 2.23: The **Galtchouk-Kunita-Watanabe (GKW) decomposition** of a contingent claim H with respect to a martingale S is

$$H = H_0 + \int_0^T \xi_u^{GKW} dS_u + L_T^{GKW}, \quad P\text{-}a.s. \tag{2.7}$$

where $\xi^{GKW} \in L^2(S)$ and $L^{GKW} \in \mathcal{M}_0^2(P)$ orthogonal to S.

In fact, the set $I^2(S) = \{\int \xi dS | \xi \in L^2(S)\}$ is a stable subspace of $\mathcal{M}_0^2(P)$ (see Lemma 2.1 of [41]) and thus any $H \in L^2(\mathcal{F}_T, P)$ can be uniquely written as a decomposition:

$$H = E[H] + \int_0^T \xi_u^H dS_u + L_T^H \quad P\text{-}a.s. \tag{2.8}$$

for some $\xi^H \in L^2(S)$ and some $L^H \in \mathcal{M}_0^2(P)$ orthogonal to $I^2(S)$, which means this decomposition is already the GKW-decomposition of H. The martingale property and orthogonality of L^H and $I^2(S)$ leads us a way to compute ξ^H by using projection techniques as following:

$$\text{fair value of } H \text{ at time t } = E[H|\mathcal{F}_t] = E[H] + \int_0^t \xi_u^H dS_u + L_t^H = V_t$$

Therefore

$$dV_t = \xi_t^H dS_t + dL_t^H \Rightarrow d\langle V, S\rangle_t = \xi_t^H d\langle S\rangle_t + d\underbrace{\langle L^H, S\rangle_t}_{=0}$$

That is

$$\xi_t^H = \frac{d\langle V, S\rangle_t}{d\langle S\rangle_t} \tag{2.9}$$

The next result was obtained by Föllmer and Sondermann in [16] under the assumption that S is a square-integrable P-martingale. Schweizer also proved this result for a general local P-local martingale.

Theorem 2.9: *Suppose S is a P-martingale, then every contingent claim $H \in L^2(\mathcal{F}_T, P)$ admits a unique risk-minimizing admissible strategy φ^*. In terms of decomposition (2.8), the risk-minimizing strategy φ^* is explicitly given by*

$$\xi^* = \xi^H,$$

$$V_t(\varphi^*) = E[H|\mathcal{F}_t], \quad 0 \le t \le T,$$

$$C_t(\varphi*) = E[H] + L^H$$

Proof. See Theorem 2.4 of [41] for the proof. □

2.4.2 The semimartingale case

In the case when $S = S_0 + M + A$ is a semimartingale (i.e. A is not null), Schweizer proved that it is impossible to find an admissible strategy φ which minimizes the risk process $R_t(\varphi)$ in the sense of the Definition 2.21. Technically speaking, the reason is that the concept of risk-minimization fails in the semimartingale case because we can not control the influence of the term $\int \xi dA$ on the risk process $R(\varphi)$. More precisely, there is no analogue to the Galtchouk-Kunita-Watanabe projection techniques allowing us to decompose a contingent claim H into a stochastic integral $\int \xi dS$ and an orthogonal component. For more details and examples we refer the reader to [36] and [38].

To generalize " risk minimization " for semimartingales, Schweizer defined the new criterion which is the concept of local risk-minimization. In this subsection, we introduce the locally risk-minimizing strategy only under the continuous-time framework. The basic idea is to control hedging errors at each infinitesimal time interval by minimizing the conditional variances of instantaneous cost increments sequentially over time. First, we impose more specific assumptions on the discounted price process S on $(\Omega, \mathcal{F}, \mathbb{F}, P)$:

$$S = S_0 + M + A \text{ and satisfies } (\mathbf{SC}) \tag{2.10}$$

where $M \in \mathcal{M}^2_{loc,0}$ and $A \in \mathcal{V}$ is a predictable process null at 0. We denote by Θ_s the space of $\mathbb{F}$-predictable processes ξ such that

$$E\left[\int_0^T \xi_u^2 d[M]_u\right] + E\left[\left(\int_0^T |\xi_u dA_u|\right)^2\right] < \infty. \tag{2.11}$$

Definition 2.24: For a fixed contingent claim $H \in L^2(\mathcal{F}_T, P)$, an L^2**-admissible strategy** is a pair $\varphi = (\xi, \eta)$ such that $\xi \in \Theta_s$ and the discounted value process $V(\varphi)$ is square integrable (i.e. $V_t(\varphi) \in L^2(P)$, for each $t \in [0, T]$) and $V_T(\varphi) = H$.

Definition 2.25: An L^2-admissible strategy φ is called **mean-self-financing** if its cost process $C(\varphi)$ is a P-martingale.

Definition 2.26: A trading strategy $\Delta = (\delta, \epsilon)$ is a **small perturbation** if it satisfies the following conditions:

(1) δ is bounded

(2) The variation of $\int \delta dA$ is bounded

(3) $\delta_T = \varepsilon_T = 0$.

For any subinterval $(s, t]$ of $[0, T]$, if $\Delta = (\delta, \varepsilon)$ is a small perturbation then we define the small perturbation $\Delta|_{(s,t]}$ by

$$\Delta|_{(s,t]} = \left(\delta \mathbf{1}_{(s,t]}, \varepsilon \mathbf{1}_{[s,t)}\right).$$

Now, we introduce the main definition:

Definition 2.27: For an L^2-admissible strategy φ, a small perturbation Δ and a partition π of $[0,T]$, we define the R-quotient $r^\pi[\varphi,\Delta](t,\omega)$ as

$$r^\pi[\varphi,\Delta](t,\omega) := \sum_{t_i\in\pi} \frac{R_{t_i}(\varphi+\Delta|_{(t_i,t_{i+1}]}) - R_{t_i}(\varphi)}{E^P\left[\langle M\rangle_{t_{i+1}} - \langle M\rangle_{t_i}|\mathcal{F}_{t_i}\right]}(\omega)\mathbf{1}_{(t_i,t_{i+1}]}(t)$$

Then, we can define our new criterion for "risk-minimization ":

Definition 2.28: We say that φ is **locally risk-minimizing(LRM)** if

$$\lim_{n\to\infty} \inf r^{\pi_n}[\varphi,\Delta] \geq 0 \quad P\otimes\langle M\rangle\text{-}a.s. \text{ on } \Omega\times[0,T],$$

for every small perturbation Δ and every increasing 0-convergent sequence $(\pi_n)_{n\in\mathbb{N}}$ of partitions of $[0,T]$.

This criterion involves limit and infimum arguments which may not easy to explicitly verified. The next result characterizes the locally risk-minimizing strategy in a more convenient way.

Theorem 2.10: *Suppose S satisfies **(SC)**. Let $H \in L^2(\mathcal{F}_T, P)$ be a contingent claim and φ be an L^2-admissible strategy. If the following assumptions (1)-(3) hold, then φ is locally risk-minimizing if and only if φ is mean-self-financing and the martingale $C(\varphi)$ is $(P,\mathbb{F})$-orthogonal to M.*

(1) $\langle M\rangle$ is P-a.s. strictly increasing

(2) A is P-a.s. continuous

(3) The mean-variance tradeoff process $\widehat{K} = \int \widehat{\lambda}^2 d\langle M\rangle$ of S satisfies

$$E[\widehat{K}_T] < \infty$$

Proof. See Proposition 2.3 of [38] for the proof. □

Theorem 2.10 motivates the following definition which excludes the assumptions (1)-(3).

Definition 2.29: Let $H \in L^2(\mathcal{F}_T, P)$ be a contingent claim. An L^2-admissible strategy φ is called **pseudo-locally risk-minimizing (plrm)** for H if and only if φ is mean-self-financing and the martingale $C(\varphi)$ is $(P, \mathbb{F})$-orthogonal to M.

This definition is given for the general multi-dimensional case. If we consider a one-dimensional model and S is sufficiently well-behaved, then the plrm-strategies and LRM- strategies coincide. In general, the plrm-strategies are easier to find and to characterize, as shown in the next result.

Theorem 2.11: *A contingent claim $H \in L^2(\mathcal{F}_T, P)$ admits a pseudo-locally risk-minimizing strategy φ if and only if H can be written as*

$$H = H_0 + \int_0^T \xi_u^{FS} dS_u + L_T^{FS}, \quad P\text{-}a.s. \tag{2.12}$$

where $H_0 \in \mathbb{R}$, $\xi^{FS} \in \Theta_s$, $L^H \in \mathcal{M}_0^2(P, \mathbb{F})$ $(P, \mathbb{F})$-orthogonal to M. The plrm-strategy is given by

$$\xi = \xi^{FS}, \quad 0 \leq t \leq T$$

with cost process

$$C(\varphi) = H_0 + L^{FS}$$

and the optimal portfolio value is

$$V_t(\varphi) = C_t(\varphi) + \int_0^t \xi_u dS_u = H_0 + \int_0^t \xi_u^{FS} dS_u + L_t^{FS},$$

and

$$\eta_t = V_t(\varphi) - \xi_t^{FS} S_t.$$

Proof. It follows from the definition of pseudo-optimality and Proposition 2.3 of [17]. □

Remark 2.10: Decomposition (2.12) is well know in the literature as the **Föllmer-Schweizer (FS) decomposition**. In the martingale case it coincides with the

Galtchouk-Kunita-Watanabe -decomposition.

Due to the incompleteness of the market, the set of equivalent martingale measures is not a singleton. Next, we will see how one can obtain FS-decomposition under a specific equivalent martingale measure for S introduced in [17].

Definition 2.30: A martingale measure $\widehat{P}$ (of S)† equivalent to P with square-integrable density is called a **minimal martingale measure (MMM)** if $\widehat{P} = P$ on $\mathcal{F}_0$ and if any square-integrable P-local martingale which is orthogonal to M under P remains a local martingale under $\widehat{P}$.

We can regard the minimal martingale measure as the equivalent martingale measure that alters the martingale structure as little as possible. The following theorem demonstrates a way to find the minimal martingale measure when S is continuous.

Theorem 2.12: *Suppose S is continuous and hence satisfies **(SC)**. If the process defined by*

$$D = \mathcal{E}\left(-\int \widehat{\lambda} dM\right)^{\ddagger} \tag{2.13}$$

is a square-integrable martingale under P, then the measure $\widehat{P}$ defined by

$$\frac{d\widehat{P}}{dP} := D_T \tag{2.14}$$

is the minimal martingale measure for S.

Proof. See Theorem 3.5 of [17] for the proof. □

Remark 2.11: We recall that in Chapter 2, equation (2.4), every martingale density for S satisfying **(SC)** is a solution of the stochastic differential equation

$$D_t = 1 - \int_0^t D_{u-}\lambda_u dM_u + R_t, \quad 0 \le t \le T$$

†martingale measure means S is a martingale under the measure $\widehat{P}$

$^\ddagger\mathcal{E}(\cdot)$ denotes the Doléans-Dade exponential for some $R \in \mathcal{M}^2_{loc,0}$ orthogonal to M. Indeed, the minimal martingale measure $\widehat{P}$ is described by the martingale density corresponding to $R_t = 0, P\text{-}a.s.$ for all $t \in [0, T]$.

Theorem 2.13: *Suppose S is continuous and hence satisfies **(SC)**. Suppose also the minimal martingale measure $\widehat{P}$ for S exits. Then define the process $\widehat{V}^H$ as follows*

$$\widehat{V}^H_t := E^{\widehat{P}}[H|\mathcal{F}_t].$$

Let

$$\widehat{V}^H_t = E^{\widehat{P}}[H|\mathcal{F}_t] = \widehat{V}^H_0 + \int_0^t \widehat{\xi}^H_u dS_u + \widehat{L}^H_t \tag{2.15}$$

be the GKW-decomposition of $\widehat{V}^H_t$ with respect to S under $\widehat{P}$. If either

$$H \text{ admits a FS-decomposition} \tag{2.16}$$

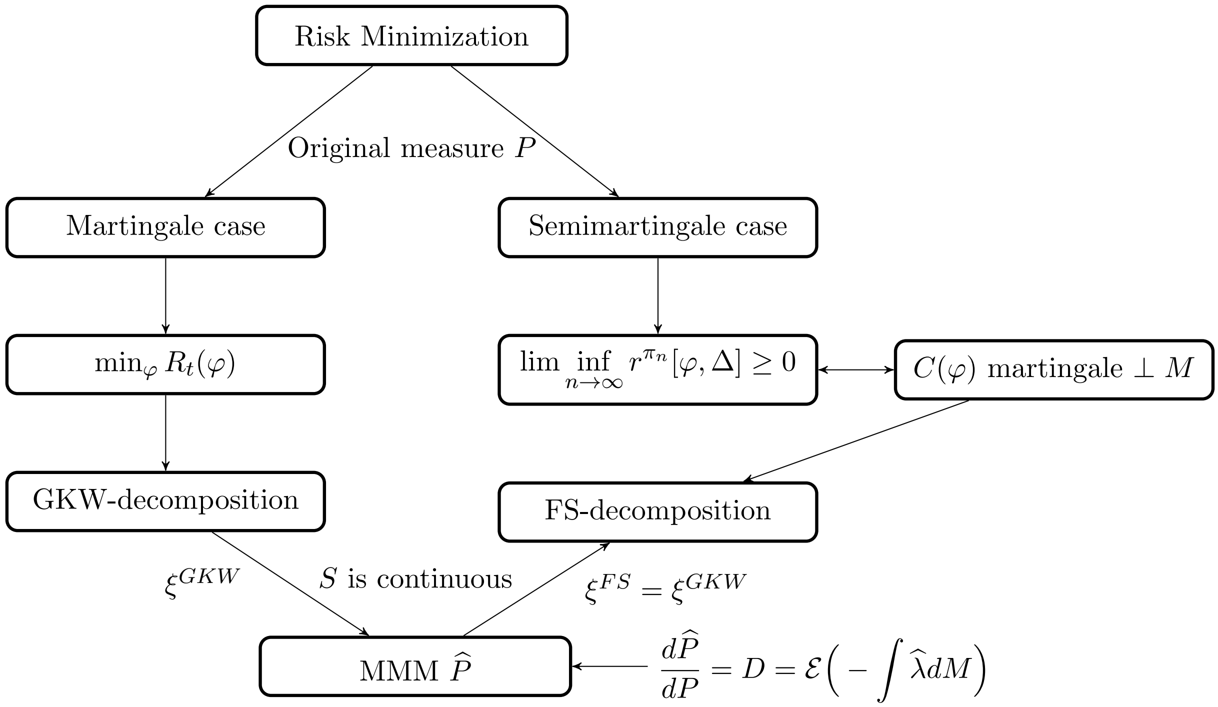
or

$$\widehat{\xi}^H \in \Theta_s \text{ and } \widehat{L}^H \in \mathcal{M}^2_0(P), \tag{2.17}$$

then (2.15) for $t = T$ gives the FS-decomposition of H and $\widehat{\xi}^H$ gives a plrm-strategy for H. A sufficient condition to guarantee the existence of $\widehat{P}$ and (2.16),(2.17) is that the mean-variance tradeoff process $\widehat{K}_t$ of S is uniformly bounded in $t \in [0, T]$ and $\omega \in \Omega$.

Proof. see Theorem 3.5 of [41] for the proof. □

Theorem 2.13 shows that for S continuous, finding a plrm-strategy for a given contingent claim $H \in L^2(\mathcal{F}_T, P)$ essentially leads us to find the GKW-decomposition of H under the minimal martingale measure $\widehat{P}$. A summary of this section is shown in the following diagram.


Risk Minimization
Original measure P
Martingale case
Semimartingale case
$\min_\varphi R_t(\varphi)$
$\lim \inf_{n\to\infty} r^{\pi_n}[\varphi, \Delta] \geq 0$
$C(\varphi)$ martingale $\perp M$
GKW-decomposition
FS-decomposition
ξ^{GKW}
S is continuous
$\xi^{FS} = \xi^{GKW}$
MMM $\widehat{P}$
$\frac{d\widehat{P}}{dP} = D = \mathcal{E}\Big(-\int \widehat{\lambda} dM\Big)$

Chapter 3

The Change Point Model

In this chapter, we introduce the change-point model which constitutes the market model considered in the rest of the thesis (Chapter 4 and 5). Herein, we will describe our change-point model as a combination of before-change dynamic and after-change dynamic, while the global dynamic constitutes a continuous semimartingale satisfying **(SC)** (see Definition 2.13) under the progressively enlarged filtration $\mathbb{G}$.

3.1 The Economical and Mathematical Model

The dynamic of financial asset is normally sensitive to certain random events, which may alter both the return rate and the volatility as a result of an abrupt change. For instance, these events include:

- Sudden adjustment of fiscal policies.
- A default of major financial institutions or counterparties.
- Natural catastrophe.
- Changes of investors' preferences.

Such events occurred intensively and particularly in financial crisis (like the 2008 financial crisis). The motivation of this change-point model is to describe the asset price dynamic under additional information or uncertainty (not just the public information or priced-in information in the asset prices) related to some random events. The occurrence time of such event is a random time, also called a **change-point**. In the following, we will describe the change-point model in detail.

To this end, we start by a given probability space $(\Omega, \mathcal{F}, P)$ and a fixed investment time horizon $T \in (0, \infty)$. On this space, consider the filtration $\mathbb{F} = (\mathcal{F}_t)_{0 \le t \le T}$ that is assumed to satisfy the usual conditions of right continuous and P-completeness. The change point is represented by the random time τ which may not be an $\mathbb{F}$-stopping time. Furthermore, by $W = (W_t)_{0 \le t \le T}$ and $W' = (W'_t)_{0 \le t \le T}$, we denote two Brownian motions defined on the filtered space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ satisfying $[W, W']_t = \rho t$ for some correlation parameter $\rho \in [-1, 1]$. Then, the financial market that we will consider throughout the thesis is defined by one risky asset and one risk-free asset. As usual, in the literature, we take the risk-free asset as the numéraire and directly work with the discounted quantities/processes. We denote by $S = (S_t)_{0 \le t \le T}$ the discounted asset price and it is assumed to satisfy the dynamic of :

$$
\begin{aligned}
dS_t &= (\mu_1(t, S_t)\mathbf{1}_{\{t \le \tau\}} + \mu_2(t, S_t)\mathbf{1}_{\{t > \tau\}})S_t dt \\
&\quad + \sigma_1(t, S_t)\mathbf{1}_{\{t \le \tau\}} S_t dW_t + \sigma_2(t, S_t)\mathbf{1}_{\{t > \tau\}} S_t dW'_t, \qquad (3.1) \\
S_0 &\text{ is positive and given .}
\end{aligned}
$$

Here the functions $\mu_i : [0, T] \times \mathbb{R} \to \mathbb{R}$ and $\sigma_i : [0, T] \times \mathbb{R} \to (0, \infty)$, for $i = 1, 2$, are Borel-measurable functions. The equation (3.1) can be interpreted as follows. The discounted asset price dynamic has a drift parameter $\mu_1(t, S_t)$ and volatility parameter $\sigma_1(t, S_t)$ up to the occurrence of the event and after this time it switches immediately to the pair of drift and volatility $(\mu_2(t, S_t), \sigma_2(t, S_t))$.

Notice that when $\rho \neq 0$, the Brownian motions W and W' are not mutually orthogonal. However, we can always construct two orthogonal $(P, \mathbb{F})$-Brownian motions (see Definition 2.12) associated to the pair (W, W')*.

$$W_t^{(1)} := W_t$$
$$W_t^{(2)} := \begin{cases} (W_t' - \rho W_t)/\sqrt{1-\rho^2} & \text{if } |\rho| \neq 1; \\ \rho W_t & \text{if } |\rho| = 1. \end{cases} \tag{3.2}$$

Indeed, it is easy to check that $[W^{(1)}, W^{(2)}] \equiv 0$ and thus $W^{(1)}, W^{(2)}$ are $(P, \mathbb{F})$-orthogonal for the case of $|\rho| \neq 1$. Now, we can replace the stochastic sources W and W' in (3.1) with $W^{(1)}$ and $W^{(2)}$ and rewrite the discounted price process as following:

$$\begin{aligned} dS_t &= \left[\mu_1(t, S_t)\mathbf{1}_{\{t\le\tau\}} + \mu_2(t, S_t)\mathbf{1}_{\{t>\tau\}}\right] S_t dt \\ &\quad + \sigma_1(t, S_t)\mathbf{1}_{\{t\le\tau\}} S_t dW_t^{(1)} + \sigma_2(t, S_t)\mathbf{1}_{\{t>\tau\}} S_t(\rho dW_t^{(1)} + \sqrt{1-\rho^2} dW_t^{(2)}), \\ S_0 &> 0. \end{aligned} \tag{3.3}$$

3.2 The Mathematical Structures of the Model

For the sake of attractive calculations, we impose some conditions on the model (3.3) and analyze its structures. The first condition is about continuity and boundedness of the parameters before and after τ ($\mu_i(t, x)$ and $\sigma_i(t, x), i = 1, 2$). This condition is crucial to the existence and uniqueness of the solution of stochastic differential equation (SDE) and also to the existence of the minimal martingale densities (see Theorem 2.12) for our change-point model.

In the rest of the thesis the functions $\mu_i : [0, T] \times \mathbb{R} \to \mathbb{R}$ and $\sigma_i : [0, T] \times \mathbb{R} \to (0, \infty), i = 1, 2$ are assumed to satisfy:

*It can be verified that $\langle W^{(2)}\rangle_t = t$ for all $t \in [0, T]$ and $W^{(2)}$ is a martingale and thus by Lévy's characterization of Brownian motion, $W^{(2)}$ is a $(P, \mathbb{F})$-Brownian motion.

(1) There is a constant $K > 0$ such that

$$\sup_{t\in[0,T]} \max_{i=1,2} \{|\mu_i(t,x)x - \mu_i(t,y)y| + |\sigma_i(t,x)x - \sigma_i(t,y)y|\} \le K|x-y|, \quad \forall x, y \in \mathbb{R} \tag{3.4}$$

(2) There is a positive constant $C \in (0, \infty)$

$$\sup_{t\in[0,T],x\in\mathbb{R}} \max_{i=1,2} \left\{ \frac{\mu_1(t,x)}{\sigma_1(t,x)} + \frac{\mu_2(t,x)}{\sigma_2(t,x)} \right\} \le C. \tag{3.5}$$

Remark 3.1: It is well know from (3.4) that there exists a constant K' such that

$$\sup_{t\in[0,T],x\in\mathbb{R}} \max_{i=1,2} \left\{ |\sigma_i(t,x)x|^2 + |\mu_i(t,x)x|^2 \right\} \le K'(1+x^2).$$

Indeed, for any Lipschitz continuous function $f(x)$, it satisfies:

$$\begin{aligned} 2|f(x)| \le& |f(x) - f(-x)| + |f(-x) - f(x)| \\ \le& 2K|x| + 2K|x| = 4K|x|, \end{aligned}$$

which implies $|f(x)|^2 \le K'|x|^2 \le K'(1+x^2)$.

As mentioned before, the random time τ may not be an $\mathbb{F}$-stopping time and thus the processes $\mathbf{1}_{\{t<\tau\}}$ and $\mathbf{1}_{\{t\ge\tau\}}$ are not adapted to the filtration $\mathbb{F}$. To handle this undesired case, we need to enlarge the filtration $\mathbb{F}$ with the random time τ. Now, we consider the progressive enlargement of filtration with respect to τ defined in Section 2.3. To preserve the semimartingale property under the enlarged filtration $\mathbb{G}$, we impose the next condition on the random time τ:

$$\tau \text{ is an honest time} \quad \text{and} \quad Z_\tau < 1, P\text{-}a.s., \tag{3.6}$$

where Z is the Azéma supermartingale (the conditional survival probability) of τ

defined by

$$Z_t := P(\tau > t|\mathcal{F}_t), \quad 0 \le t \le T$$

It is worthy to remark that (3.6) is satisfied by many examples developed in [1] and there could be some arbitrage opportunities if $Z_\tau = 1, P\text{-}a.s.$ (see [19] for more details). Let us denote the Doob-Meyer decomposition of Z as

$$Z = X - A,$$

where X is a $(P, \mathbb{F})$-square integrable martingale and A is a $(P, \mathbb{F})$-predictable increasing process.

By applying Theorem 2.8, we get the $\mathbb{G}$-martingales $\widetilde{W}^{(i)}$ from the $\mathbb{F}$-martingales $W^i, i = 1, 2$ as follows:

$$\widetilde{W}_t^{(i)} = W_t^{(i)} - \int_0^t \theta_u^{(i)} du, \quad \text{for all } t \in [0, T] \text{ and for } i = 1, 2. \tag{3.7}$$

Here the $(P, \mathbb{G})$-predictable processes $\theta^{(i)}$ are defined as:

$$\theta_t^{(i)} = \mathbf{1}_{\{t \le \tau\}} \frac{\frac{d}{dt}\langle X, W^{(i)}\rangle_t^{P,\mathbb{F}}}{Z_{t-}} - \mathbf{1}_{\{t > \tau\}} \frac{\frac{d}{dt}\langle X, W^{(i)}\rangle_t^{P,\mathbb{F}}}{1 - Z_{t-}}. \tag{3.8}$$

Remark 3.2: It is easy to see that under (3.6) the processes $\theta^{(i)}, i = 1, 2$ are well defined. On the set $\{t \le \tau\}$, $Z_{t-} > 0$, P-$a.s.$, therefore $\mathbf{1}_{\{t \le \tau\}} \frac{1}{Z_{t-}}$ is well defined. Also the condition $Z_\tau < 1\, P\text{-}a.s.$ guarantees $Z_{t-} < 1$ on $\{t > \tau\}$ and thus $\mathbf{1}_{\{t > \tau\}} \frac{1}{1 - Z_{t-}}$ is well -defined. The processes $\theta^{(i)}$ are indeed processes with finite variation. Then, by Theorem 2.2, Lemma 2.1 and the continuity of Brownian motions $W^{(i)}$, we have $\langle \widetilde{W}^{(i)} \rangle_t^{P,\mathbb{G}} = [\widetilde{W}^{(i)}]_t = [W^{(i)}]_t = t, \forall t \in [0, T]$ and thus $\widetilde{W}^{(i)}$ are indeed $(P, \mathbb{G})$-Brownian motions by Lévy's characterization of Brownian motions. Also, for $|\rho| \ne 1$, we have $\langle \widetilde{W}^{(1)}, \widetilde{W}^{(2)} \rangle_t^{P,\mathbb{G}} = [\widetilde{W}^{(1)}, \widetilde{W}^{(2)}]_t =$

$[W^{(1)}, W^{(2)}]_t = 0, \forall t \in [0,T]$. Therefore, $\widetilde{W}^{(1)}$ and $\widetilde{W}^{(2)}$ are $(P, \mathbb{G})$-orthogonal Brownian motions.

Under (3.6), the equation (3.3) is a semimartingale-driven SDE on $(\Omega, \mathcal{F}, \mathbb{G}, P)$. Then we can apply the existence and uniqueness theorem for the strong solution of SDE (3.3) from Chapter V of Protter [35] (see also Jacod [23], Chapter XIV)

Theorem 3.1: *Under (3.4) and (3.6), there exists a unique continuous* $(P, \mathbb{G})$*-semimartingale* S *which is a solution to the SDE (3.3) on* $(\Omega, \mathcal{F}, \mathbb{G}, P)$*.*

Proof. By progressive enlargement of filtration $\mathbb{F}$ with τ, we know τ is a $\mathbb{G}$-stopping time. Then the process $\mathbf{1}_{\{t<\tau\}}$ and $\mathbf{1}_{\{t\geq\tau\}}$ are $\mathbb{G}$-adapted. Consider the following random functions: For $\omega \in \Omega, t \in [0,T], x \in \mathbb{R}$:

$$g(\omega, t, x) = \mathbf{1}_{\{t\leq\tau(\omega)\}}\mu_1(t,x)x + \mathbf{1}_{\{t>\tau(\omega)\}}\mu_2(t,x)x;$$

$$f(\omega, t, x) = \mathbf{1}_{\{t\leq\tau(\omega)\}}\sigma_1(t,x)x + \mathbf{1}_{\{t>\tau(\omega)\}}\sigma_2(t,x)x.$$

Then (3.4) implies that the f and g are random Lipschitz, in the sense of Protter [35] (see page 256). Then the existence of a unique solution S to the SDE (3.3) on $(\Omega, \mathcal{F}, \mathbb{G}, P)$ is a direct consequence of Theorem V.6 in Protter [35]. □

Finally, we will end this chapter by proving some structures under filtrations $\mathbb{G}$ for the discounted price process S as follows:

Theorem 3.2: *Under (3.4), (3.5) and (3.6), the discounted price process* S *is a* $(P, \mathbb{G})$*-special semimartingale satisfying* ***(SC)****(see Definition 2.13) and admits the canonical decomposition :*

$$S = S_0 + M^{\mathbb{G}} + A^{\mathbb{G}}. \tag{3.9}$$

Here $M_t^{\mathbb{G}} = \int_0^t (V_u S_u) d\widetilde{W}_u \in \mathcal{M}^2_{loc}(P, \mathbb{G})$ *and* $A_t^{\mathbb{G}} = \int_0^t (\mu_u S_u) du$ *is predictable with finite variation. The processes* $\mu = (\mu_t)_{0\leq t\leq T}, V = (V_t)_{0\leq t\leq T}$ *and* $\widetilde{W} = (\widetilde{W}_t)_{0\leq t\leq T}$

are given by

$$\begin{aligned}\mu_t := \boldsymbol{1}_{\{t\le\tau\}}(\mu_1(t,S_t) + \sigma_1(t,S_t)\theta_t^{(1)}) \\ + \boldsymbol{1}_{\{t>\tau\}}(\mu_2(t,S_t) + \sigma_2(t,S_t)(\rho\theta_t^{(1)} + \sqrt{1-\rho^2}\theta_t^{(2)})),\end{aligned} \tag{3.10}$$

$$V_t := \boldsymbol{1}_{\{t\le\tau\}}\sigma_1(t,S_t) + \boldsymbol{1}_{\{t>\tau\}}\sigma_2(t,S_t), \tag{3.11}$$

$$\begin{aligned}\widetilde{W}_t := \int_0^t \alpha_u d\widetilde{W}_u^{(1)} + \int_0^t \beta_u d\widetilde{W}_u^{(2)}, \\ \text{where } \alpha_t = \mathbf{1}_{\{t\le\tau\}} + \rho\mathbf{1}_{\{t>\tau\}},\ \beta_t = \sqrt{1-\rho^2}\mathbf{1}_{\{t>\tau\}}.\end{aligned} \tag{3.12}$$

Proof. To derive (3.9), we simply substitute (3.7) into the SDE (3.3) and take integration on both side afterwards. Now, we first show $M_t^{\mathbb{G}} = \int_0^t V_u S_u \widetilde{W}_u$ is a $(P,\mathbb{G})$-locally square integrable martingale. Since $\widetilde{W}_t = \int_0^t \alpha_u \widetilde{W}_u^{(1)} + \int_0^t \beta_u \widetilde{W}_u^{(2)}$ and α_t, β_t are non-negative and satisfy $\alpha_t^2 + \beta_t^2 = 1$ for all $t \in [0,T]$, then $\widetilde{W}$ is also a $\mathbb{G}$-Brownian motion. Now, we consider the sequence of $\mathbb{G}$-stopping times given by:

$$T_n = \inf\{t \ge 0 : |S_t| > n\} \wedge T, \quad n \ge 0$$

It is easy to see that T_n increases stationarily to T. Due to Remark 3.1 and the continuity of S from Theorem 3.1, we get

$$\begin{aligned}(V_{t\wedge T_n} S_{t\wedge T_n})^2 &= [\mathbf{1}_{\{t\wedge T_n<\tau\}}(\sigma_1(t\wedge T_n, S_{t\wedge T_n}))^2 + \mathbf{1}_{\{t\wedge T_n\ge\tau\}}(\sigma_2(t\wedge T_n, S_{t\wedge T_n}))^2)]S_{t\wedge T_n}^2 \\ &\le K(1+n^2)n^2 < \infty.\end{aligned}$$

Then, thanks to the Doob's maximal inequality, we obtain

$$E\left(\sup_{t\in[0,T]}|M^{\mathbb{G}}_{t\wedge T_n}|^2\right)\leq 4E(|M_{T\wedge T_n}|^2)=4KT(1+n^2)n^2<\infty.$$

This proves $M^{\mathbb{G}}$ is a $(P,\mathbb{G})$ locally square-integrable martingale.

Secondly, we notice that the process μS is $\mathbb{G}$-adapted and apply this similar method above we get $E[\int_0^{T_n}|\mu_u S_u|du]\leq TK'n(1+n)<\infty$. Therefore, the process $A^{\mathbb{G}}_t=\int_0^t\mu_u S_u dt$ is a continuous process with finite variation . Moreover $A^{\mathbb{G}}$ takes the form of :

$$A^{\mathbb{G}}_t=\int_0^t\mu_u S_u du=\int_0^t\underbrace{\frac{\mu_u}{V_u^2 S_u}}_{\widehat{\lambda}^{\mathbb{G}}_u}d\langle M^{\mathbb{G}}\rangle_u.$$

Furthermore, by denoting $\theta^{(3)}=\rho\theta^{(1)}+\sqrt{1-\rho^2}\theta^{(2)}$ and applying $(a+b)^2\leq 2a^2+2b^2$, on one hand we obtain:

$$\begin{aligned}
\int_0^T(\widehat{\lambda}^{\mathbb{G}}_u)^2 d\langle M^{\mathbb{G}}\rangle_u &= \int_0^T\left(\frac{\mu_u}{V_u}\right)^2 du\\
&=\int_0^T\left[\mathbf{1}_{\{u\leq\tau\}}\left(\frac{\mu_1(u,S_u)}{\sigma_1(u,S_u)}+\theta^{(1)}_u\right)+\mathbf{1}_{\{u>\tau\}}\left(\frac{\mu_2(u,S_u)}{\sigma_2(u,S_u)}+\theta^{(3)}_u\right)\right]^2 du\\
&\leq 4\int_0^T\mathbf{1}_{\{u\leq\tau\}}\left(\frac{\mu_1(u,S_u)}{\sigma_1(u,S_u)}\right)^2 du+4\int_0^T\mathbf{1}_{\{u\leq\tau\}}\left(\frac{\frac{d}{du}\langle X,W^{(1)}\rangle_u}{Z_{u-}}\right)^2 du\\
&+4\int_0^T\mathbf{1}_{\{u>\tau\}}\left(\frac{\mu_1(u,S_u)}{\sigma_1(u,S_u)}\right)^2 du\\
&+4\int_0^T\mathbf{1}_{\{u>\tau\}}\left(\frac{\rho\frac{d}{du}\langle X,W^{(1)}\rangle_u+\sqrt{1-\rho^2}\frac{d}{du}\langle X,W^{(2)}\rangle_u}{1-Z_{u-}}\right)^2 du
\end{aligned}$$

On the other hand, under (3.6), we have (see more details in Lemma 3.2 of [11])

$$K_1 := \sup_{u \geq 0} \left(\frac{1}{Z_{t-}} \mathbf{1}_{\{u \leq \tau\}} \right)^2 \langle X \rangle_T^2 < \infty$$

and

$$K_2 := 4 \sup_{u \geq 0} \left(\frac{1}{1 - Z_{t-}} \mathbf{1}_{\{u > \tau\}} \right)^2 \langle X \rangle_T^2 < \infty$$

Therefore we have

$$\int_0^T (\widehat{\lambda}_u^{\mathbb{G}})^2 d\langle M^{\mathbb{G}} \rangle_u \leq 4CT + 4K_1 + 4K_2 < \infty$$

This proves that S satisfies **(SC)** under $\mathbb{G}$ and the proof of the theorem is achieved.

□

Remark 3.3: We can also regard the dynamic (3.3) as :

$$dS_t = \mathbf{1}_{\{t \leq \tau\}} dS_t^{(1)} + \mathbf{1}_{\{t > \tau\}} dS_t^{(2)}, \tag{3.13}$$

where the processes $S^{(1)}, S^{(2)}$ have the following dynamics:

$$\begin{aligned} dS_t^{(1)} &= \sigma_1(t, S_t) S_t dW_t^{(1)} + \mu_1(t, S_t) S_t dt, \\ dS_t^{(2)} &= \sigma_2(t, S_t) S_u dW_t^{(3)} + \mu_2(t, S_t) S_t dt. \end{aligned} \tag{3.14}$$

Here the process $W^{(3)} := \rho W^{(1)} + \sqrt{1-\rho^2} W^{(2)}$ is a $(P, \mathbb{F})$-Brownian motion. Furthermore, the canonical decomposition of $S^{(1)}$ is given by:

$$S^{(1)} = S_0^{(1)} + M^{(1)} + A^{(1)},$$

where

$$M_t^{(1)} = \int_0^t \sigma_1(u,S_u)S_u dW_u^{(1)} \quad \text{is a } (P,\mathbb{F})\text{-local martingale}$$

and

$$A_t^{(1)} = \int_0^t \mu_1(u,S_u)S_u du = \int_0^t \underbrace{\frac{\mu_1(u,S_u)}{\sigma_1(u,S_u)^2 S_u}}_{\widehat{\lambda}_u^{(1)}} d\langle M^{(1)}\rangle_u$$

is a continuous process with finite variation. Then, due to (3.5)

$$\int_0^T (\widehat{\lambda}_u^{(1)})^2 d\langle M^{(1)}\rangle_u = \int_0^T \left(\frac{\mu_1(u,S_u)}{\sigma_1(u,S_u)}\right)^2 du \le C^2 T.$$

This implies that $S^{(1)}$ is a $(P,\mathbb{F})$-semimartingale and satisfies **(SC)** condition. Similarly, we can easily prove that

$$S^{(2)} = M^{(2)} + A^{(2)} \text{ satisfies } (\textbf{SC}),$$

where

$$M_t^{(2)} = \int_0^t \sigma_2(u,S_u)S_u dW_u^{(3)} \quad \text{is a } (P,\mathbb{F})\text{-local martingale}$$

and

$$A_t^{(2)} = \int_0^t \mu_1(u,S_u)S_u du := \int_0^t \underbrace{\frac{\mu_2(u,S_u)}{\sigma_2(u,S_u)^2 S_u}}_{\widehat{\lambda}_u^{(2)}} d\langle M^{(2)}\rangle_u.$$

is a continuous process with finite variation.

Here, also, we can easily prove that $S^{(2)}$ satisfies **(SC)** under $\mathbb{F}$. In fact,

$$\int_0^T (\widehat{\lambda}_u^{(2)})^2 d\langle M^{(2)}\rangle_u = \int_0^T \left(\frac{\mu_2(u,S_u)}{\sigma_2(u,S_u)}\right)^2 du \le C^2 T.$$

We call $S^{(1)}$ the before-change dynamic and $S^{(2)}$ the after-change dynamic. This observation is important for describing the $\mathbb{G}$-LRM strategy explicitly via using the information of $\mathbb{F}$ in the next Chapter.

Chapter 4

Locally risk-minimization for the change-point model

In this chapter we will describe the Local Risk-Minimization (LRM) strategy for the change-point model under the progressively enlarged filtration $\mathbb{G}$, as explicitly as possible, in terms of $\mathbb{F}$-LRM strategies. The filtration $\mathbb{F}$ can be regarded as the collection of pubic information reflecting the discounted risky asset price process S, whereas the filtration $\mathbb{G}$ presents all the public information and the knowledge of the change-point τ. As mentioned in Chapter 3 (see Remark 3.3), the discounted price process S can be separated into the before-change dynamic $S^{(1)}$ and the after-dynamic $S^{(2)}$. Both processes $S^{(1)}$ and $S^{(2)}$ are $(P, \mathbb{F})$-semimartingales satisfying **(SC)** under (3.4) and (3.5). Therefore, we could assume that the traders are equipped with the best knowledge of the filtration $\mathbb{F}$ so that for any contingent claim $Y \in L^2(\mathcal{F}_T, P)$ the corresponding LRM- strategy based on trading $S^{(1)}$ and/or $S^{(2)}$ can be found. Our goal in this chapter is to find the global $\mathbb{G}$-locally risk-minimizing (LRM) strategy for a contingent claim $H \in L^2(\mathcal{F}_T, P)$. Precisely, we will show that this $\mathbb{G}$-LRM strategy can be obtained as a function of $\mathbb{F}$-LRM strategies associated to five $\mathcal{F}_T$-measurable contingent claims with respect to $S^{(1)}$ and/or $S^{(2)}$. This generalizes the result obtained by Barbarin in [5]. In this thesis, we consider more general contingent claims and characterize the LRM-strategy of hedging them both on $[0, \tau]$ and $(\tau, T]$.

Throughout this chapter we assume that (3.4), (3.5) and (3.6) introduced in Chapter 3 are satisfied. For some technical reasons, we also impose the following condition on the filtration in the sequel:
The filtration $\mathbb{F}$ is generated by the two $(P, \mathbb{F})$ Brownian motions $W^{(1)}, W^{(2)}$ defined in (3.2) and $|\rho| \neq 1$ so that $W^{(1)}, W^{(2)}$ are orthogonal. That is

$$|\rho| \neq 1 \quad \text{and} \quad \mathbb{F} = \mathbb{F}^{W^{(1)}} \vee \mathbb{F}^{W^{(2)}}. \tag{4.1}$$

Under this condition, we could apply Theorem 2.1 for any square-integrable martingale adapted to filtration $\mathbb{F}$. In particular, we recall that the Azéma supermartingale (conditional survival probability) Z defined in (3.6) admits the Doob-Meyer decomposition as $Z = X - A$ where $X \in \mathcal{M}^2(P, \mathbb{F})$ and A is $\mathbb{F}$-predictable increasing process. Since all martingales adapted to Brownian filtrations are continuous, we conclude that X is continuous. Thanks to the martingale representation theorem (see Theorem 2.1), we have

$$X_t = X_0 + \int_0^t h_u^{(1)} dW_u^{(1)} + \int_0^t h_u^{(2)} dW_u^{(2)}, \tag{4.2}$$

for some $(P, \mathbb{F})$-predictable processes $h^{(i)}, i = 1, 2$ satisfying

$$E\left[\int_0^T (h_u^{(i)})^2 du\right] < \infty, \quad i = 1, 2. \tag{4.3}$$

In this case, the $\mathbb{G}$-predictable processes $\theta^{(i)}$ defined in (3.8) take the form of

$$\theta_t^{(i)} = \mathbf{1}_{\{t \le \tau\}} \frac{h_t^{(i)}}{Z_{t-}} - \mathbf{1}_{\{t > \tau\}} \frac{h_t^{(i)}}{1 - Z_{t-}}, \quad i = 1, 2. \tag{4.4}$$

Thanks to Theorem 2.8, the $\mathbb{G}$-martingale $\widetilde{X}$ obtained from the $\mathbb{F}$-martingale X

takes the form of :

$$
\begin{aligned}
\widetilde{X}_t &= X_t - \int_0^{t\wedge\tau} \frac{d\langle X\rangle_u^{P,\mathbb{F}}}{Z_{u-}} + \int_\tau^{t\vee\tau} \frac{d\langle X\rangle_u^{P,\mathbb{F}}}{1-Z_{u-}} \\
&= X_t - \int_0^{t\wedge\tau} \frac{(h_u^{(1)})^2 + (h_u^{(2)})^2}{Z_{u-}} + \int_\tau^{t\vee\tau} \frac{(h_u^{(1)})^2 + (h_u^{(2)})^2}{1-Z_{u-}} \qquad (4.5) \\
&= X_0 + \int_0^t h_u^{(1)} d\widetilde{W}_u^{(1)} + \int_0^t h_u^{(2)} \widetilde{W}_u^{(2)}.
\end{aligned}
$$

4.1 Minimal Martingale Densities associated to the Change-Point Model

The key point in finding the LRM-strategy is to find the GKW-decomposition (see Definition 2.23) for the value process of the contingent claim under the minimal martingale measure. As mentioned in Remark 3.14, the before-change dynamic $S^{(1)}$ and after-change dynamic $S^{(2)}$ are all $(P, \mathbb{F})$- semimartingales satisfying **(SC)**. Therefore, it is possible for us to find the minimal martingale measures $\widehat{P}^{1,\mathbb{F}}$ for $S^{(1)}$ and $\widehat{P}^{2,\mathbb{F}}$ for $S^{(2)}$. Also, Theorem 3.2 shows that S is also a $(P, \mathbb{G})$-semimartingale satisfying **(SC)**. In the following, we will characterize the minimal martingale density for S under $\mathbb{G}$ and establish its relationships to the minimal martingale densities of $S^{(1)}$ and $S^{(2)}$.

Lemma 4.1: *The minimal martingale measures $\widehat{P}^{1,\mathbb{F}}$ for $S^{(1)}$ and $\widehat{P}^{2,\mathbb{F}}$ for $S^{(2)}$ exist and are given by*

$$
\frac{d\widehat{P}^{1,\mathbb{F}}}{dP} = D_T^{1,\mathbb{F}} := \exp\left(-\int_0^T \frac{\mu_1(u,S_u)}{\sigma_1(u,S_u)} dW_u^{(1)} - \frac{1}{2}\int_0^T \left(\frac{\mu_1(u,S_u)}{\sigma_1(u,S_u)}\right)^2 du \right)
$$

and

$$
\frac{d\widehat{P}^{2,\mathbb{F}}}{dP} = D_T^{2,\mathbb{F}} := \exp\left(-\int_0^T \frac{\mu_2(u,S_u)}{\sigma_2(u,S_u)} dW_u^{(3)} - \frac{1}{2}\int_0^T \left(\frac{\mu_2(u,S_u)}{\sigma_2(u,S_u)}\right)^2 du \right).
$$

Here we recall that $W^{(3)} = \rho W^{(1)} + \sqrt{1-\rho^2} W^{(2)}$ is a $(P, \mathbb{F})$-Brownian motion.

Proof. From (3.14), we have for $S^{(1)}$

$$S_t^{(1)} = S_0^{(1)} + \underbrace{\int_0^t \sigma_1(u, S_u) S_u dW_u^{(1)}}_{M_t^{(1)}} + \underbrace{\int_0^t \mu_1(u, S_u) S_u du}_{A_t^{(1)}},$$

and

$$A_t^{(1)} = \int_0^t \widehat{\lambda}_u^{(1)} d\langle M^{(1)} \rangle_u \quad \text{with } \widehat{\lambda}_t^{(1)} = \frac{\mu_1(t, S_t)}{(\sigma_1(t, S_t))^2 S_t}$$

Furthermore, the mean-variance trade off process for $S^{(1)}$, $\widehat{K}_t^{(1)} = \int_0^t (\widehat{\lambda}_u^{(1)})^2 d\langle M^{(1)} \rangle_u$ is uniformly bounded under (3.4) and (3.5). In fact, we have

$$\widehat{K}_t^{(1)} = \int_0^t \left(\frac{\mu_1(u, S_u)}{\sigma_1(u, S_u)} \right)^2 du \leq C^2 T.$$

Therefore, the Novikov's condition for $D^{1,\mathbb{F}} = \mathcal{E}\left(-\int \widehat{\lambda}^{(1)} dM^{(1)} \right)$ holds, i.e.

$$E\left[\exp\left(\frac{1}{2} \widehat{K}_T^{(1)} \right) \right] < \infty.$$

Thus, $D^{1,\mathbb{F}}$ is a square-integrable martingale. Thanks to Theorem 2.13, the probability measure $\widehat{P}^{1,\mathbb{F}}$, defined by

$$\frac{d\widehat{P}^{1,\mathbb{F}}}{dP} = D_T^{1,\mathbb{F}} = \exp\left(-\int_0^T \frac{\mu_1(u, S_u)}{\sigma_1(u, S_u)} dW_u^{(1)} - \frac{1}{2} \int_0^T \left(\frac{\mu_1(u, S_u)}{\sigma_1(u, S_u)} \right)^2 du \right),$$

is the minimal martingale measure for $S^{(1)}$. For the case of $S^{(2)}$, the proof is similar by noticing that $\langle W^{(3)} \rangle_t = t$, for all $t \in [0, T]$, and thus it is also a $(P, \mathbb{F})$- Brownian motion. This completes the proof of the lemma. □

We recall that S satisfies **(SC)** under filtration $\mathbb{G}$ with

$$\widehat{\lambda}^{\mathbb{G}} = \frac{\mu}{V^2 S}$$

Then, the minimal martingale density process $D^{\mathbb{G}}$ for S under $\mathbb{G}$ is given by

$$D_t^{\mathbb{G}} = \mathcal{E}\left(-\int_0^t \widehat{\lambda}_u^{\mathbb{G}} dM_u^{\mathbb{G}}\right)_t = \exp\left(-\int_0^t \frac{\mu_u}{V_u} d\widetilde{W}_u - \frac{1}{2}\int_0^t \left(\frac{\mu_u}{V_u}\right)^2 du\right). \tag{4.6}$$

In the following, we specify the relationships between $D^{\mathbb{G}}$ and $D^{i,\mathbb{F}}$, $i = 1, 2$. For the readers' convenience, we recall some notations used frequently in the sequel

- $W^{(3)} := \rho W^{(1)} + \sqrt{1-\rho^2} W^{(2)}$ is a $(P, \mathbb{F})$-Brownian motion.
- $\widetilde{W}^{(3)} := \rho \widetilde{W}^{(1)} + \sqrt{1-\rho^2} \widetilde{W}^{(2)}$ is a $(P, \mathbb{G})$-Brownian motion.
- $\theta^{(3)} := \rho \theta^{(1)} + \sqrt{1-\rho^2} \theta^{(2)}$ is a $\mathbb{G}$-predictable process.
- $h^{(3)} := \rho h^{(1)} + \sqrt{1-\rho^2} h^{(2)}$ is a $\mathbb{F}$-predictable process.

Theorem 4.1: *The relationship between the $\mathbb{F}$-minimal martingale density $D^{1,\mathbb{F}}$ and the $\mathbb{G}$-minimal martingale density $D^{\mathbb{G}}$ is given by:*

$$D_{t\wedge\tau}^{\mathbb{G}} = D_{t\wedge\tau}^{1,\mathbb{F}} \cdot \exp(-\Upsilon_t^{(1)}), \tag{4.7}$$

where the $\mathbb{G}$-adapted process $\Upsilon^{(1)}$ is defined by:

$$\Upsilon_t^{(1)} = \int_0^t \mathbf{1}_{\{u\leq\tau\}} \frac{h_u^{(1)}}{Z_{u-}} d\widetilde{W}_u^{(1)} + \frac{1}{2}\int_0^t \mathbf{1}_{\{u\leq\tau\}} \left(\frac{h_u^{(1)}}{Z_{u-}}\right)^2 du. \tag{4.8}$$

Proof. The proof of the theorem follows from :

$$\begin{aligned}
D_{t\wedge\tau}^{1,\mathbb{F}} &= \exp\left\{-\int_0^{t\wedge\tau} \frac{\mu_1(u,S_u)}{\sigma_1(u,S_u)} dW_u^{(1)} - \frac{1}{2}\int_0^{t\wedge\tau} \left(\frac{\mu_1(u,S_u)}{\sigma_1(u,S_u)}\right)^2 du\right\} \\
&= \exp\left\{-\left(\int_0^{t\wedge\tau} \frac{\mu_1(u,S_u)}{\sigma_1(u,S_u)} d\widetilde{W}_u^{(1)} + \int_0^{t\wedge\tau} \frac{\mu_1(u,S_u)}{\sigma_1(u,S_u)} \frac{h_u^{(1)}}{Z_{u-}} du\right)\right. \\
&\quad \left. -\frac{1}{2}\int_0^{t\wedge\tau} \left(\frac{\mu_1(u,S_u)}{\sigma_1(u,S_u)}\right)^2 du\right\}
\end{aligned}$$

$$= \exp\left\{-\int_0^{t\wedge\tau}\left(\frac{\mu_1(u,S_u)}{\sigma_1(u,S_u)}+\frac{h_u^{(1)}}{Z_{u-}}\right)d\widetilde{W}_u^{(1)}-\frac{1}{2}\int_0^{t\wedge\tau}\left(\frac{\mu_1(u,S_u)}{\sigma_1(u,S_u)}+\frac{h_u^{(1)}}{Z_{u-}}\right)^2\right\}du$$

$$\times\exp\left\{\int_0^{t\wedge\tau}\frac{h_u^{(1)}}{Z_{u-}}d\widetilde{W}_u^{(1)}+\frac{1}{2}\int_0^{t\wedge\tau}\left(\frac{h_u^{(1)}}{Z_{u-}}\right)^2du\right\}$$

$$= D^{\mathbb{G}}_{t\wedge\tau}e^{\Upsilon_t^{(1)}}.$$

This completes the proof of the theorem. □

Theorem 4.2: *The relationship between the* $\mathbb{F}$*-minimal martingale density* $D^{2,\mathbb{F}}$ *and the* $\mathbb{G}$*-minimal martingale density* $D^{\mathbb{G}}$ *is given by*

$$\frac{D_t^{\mathbb{G}}}{D_{t\wedge\tau}^{\mathbb{G}}}=\frac{D_t^{2,\mathbb{F}}}{D_{t\wedge\tau}^{2,\mathbb{F}}}\cdot\exp\left(-\Upsilon_t^{(2)}\right), \tag{4.9}$$

where the $\mathbb{G}$*-adapted process* $\Upsilon^{(2)}$ *is defined as:*

$$\Upsilon_t^{(2)}=-\int_0^t \mathbf{1}_{\{u>\tau\}}\frac{h_u^{(3)}}{1-Z_{u-}}d\widetilde{W}_u^{(3)}+\frac{1}{2}\int_0^t \mathbf{1}_{\{u>\tau\}}\left(\frac{h_u^{(3)}}{1-Z_{u-}}\right)^2du. \tag{4.10}$$

Proof. First, we notice that

$$\mathbf{1}_{\{u>\tau\}}d\widetilde{W}_u^{(3)}=\mathbf{1}_{\{u>\tau\}}(\rho d\widetilde{W}_u^{(1)}+\sqrt{1-\rho^2}d\widetilde{W}_u^{(2)})=\mathbf{1}_{\{u>\tau\}}\left(dW_t^{(3)}+\frac{h_u^{(3)}}{1-Z_{u-}}du\right)$$

and

$$\mathbf{1}_{\{u>\tau\}}\theta_u^{(3)}=\mathbf{1}_{\{u>\tau\}}\left(-\frac{h_u^{(3)}}{1-Z_{u-}}\right).$$

Then, we derive:

$$\frac{D_t^{\mathbb{G}}}{D_{t\wedge\tau}^{\mathbb{G}}}=\exp\left\{-\int_0^t \mathbf{1}_{\{u>\tau\}}\left(\frac{\mu_2(u,S_u)}{\sigma_2(u,S_u)}-\frac{h_u^{(3)}}{1-Z_{u-}}\right)d\widetilde{W}_u^{(3)}\right.$$

$$- \frac{1}{2}\int_0^t \mathbf{1}_{\{u>\tau\}} \left(\frac{\mu_2(u,S_u)}{\sigma_2(u,S_u)} - \frac{h_u^{(3)}}{1-Z_{u-}} \right)^2 du \Bigg\}$$

$$= \exp\Bigg\{ - \int_0^t \mathbf{1}_{\{u>\tau\}} \frac{\mu_2(u,S_u)}{\sigma_2(u,S_u)} dW_u^{(3)} + \int_0^t \mathbf{1}_{\{u>\tau\}} \frac{h_u^{(3)}}{1-Z_{u-}} dW_u^{(3)}$$

$$- \int_0^t \mathbf{1}_{\{u>\tau\}} \frac{\mu_2(u,S_u)}{\sigma_2(u,S_u)} \frac{h_u^{(3)}}{1-Z_{u-}} du + \int_0^t \mathbf{1}_{\{u>\tau\}} \left(\frac{h_u^{(3)}}{1-Z_{u-}} \right)^2 du$$

$$- \frac{1}{2}\int_0^t \mathbf{1}_{\{u>\tau\}} \left(\frac{\mu_2(u,S_u)}{\sigma_2(u,S_u)} - \frac{h_u^{(3)}}{1-Z_{u-}} \right)^2 du \Bigg\}$$

$$= \exp\left\{ - \int_0^t \mathbf{1}_{\{u>\tau\}} \frac{\mu_2(u,S_u)}{\sigma_2(u,S_u)} dW_u^{(3)} - \frac{1}{2}\int_0^t \mathbf{1}_{\{u>\tau\}} \left(\frac{\mu_2(u,S_u)}{\sigma_2(u,S_u)} \right)^2 du \right\}$$

$$\times \exp\left\{ \int_0^t \mathbf{1}_{\{u>\tau\}} \frac{h_u^{(3)}}{1-Z_{u-}} d\widetilde{W}_u^{(3)} - \frac{1}{2}\int_0^t \mathbf{1}_{\{u>\tau\}} \left(\frac{h_u^{(3)}}{1-Z_{u-}} \right)^2 du \right\}$$

$$= \frac{D_t^{2,\mathbb{F}}}{D_{t\wedge\tau}^{2,\mathbb{F}}} \cdot e^{-\Upsilon_t^{(2)}}.$$

This ends the proof of the theorem. □

4.2 The $\mathbb{G}$-LRM Strategy for Change-Point Model

In this section, we will determine the GKW-decomposition of the value process of a given contingent claim H that belongs to $L^2(\mathcal{F}_T, P)$ under the minimal martingale measure $\widehat{P}^{\mathbb{G}}$. Thus, throughout this section, we assume that the density process defined (4.6) is a martingale. As a result, the minimal martingale measure $\widehat{P}^{\mathbb{G}}$ is defined by

$$\frac{d\widehat{P}^{\mathbb{G}}}{dP} = D_T^{\mathbb{G}}.$$

The value process for the claim H is given by the conditional expectation under $\widehat{P}^{\mathbb{G}}$ of H with respect to the filtration $\mathbb{G}$, i.e.

$$\widehat{V}_t = E^{\widehat{P}^{\mathbb{G}}}[H|\mathcal{G}_t], \quad t \in [0, T].$$

One common method for finding such decomposition lies in assuming that the value process $\widehat{V}$, by risk-neutral valuation, is given by the arbitrage-free price $F(t, S_t) \in C^{1,2}([0,T] \times [0,\infty])$ (see [46] for example). However, this assumption may fail due to the additional information associated to the change-point τ. The reason is that the value process $\widehat{V}_t = E^{\widehat{P}^{\mathbb{G}}}[H|\mathcal{G}_t]$ may not be a Markov process and it may not depend on the S_t only. Our approach is based on decomposing the value process $\widehat{V}$ into three processes that are associated to the three disjoint sets $\{\tau > T\}$, $\{t < \tau \leq T\}$ and $\{\tau \leq t\}$ for all $t \in [0, T]$. That is

$$\widehat{V}_t = E^{\widehat{P}^{\mathbb{G}}}[H|\mathcal{G}_t] = \underbrace{E^{\widehat{P}^{\mathbb{G}}}[H\mathbf{1}_{\{T<\tau\}}|\mathcal{G}_t]}_{\widehat{V}_t^{(1)}} + \underbrace{E^{\widehat{P}^{\mathbb{G}}}[H\mathbf{1}_{\{t<\tau\leq T\}}|\mathcal{G}_t]}_{\widehat{V}_t^{(2)}} + \underbrace{E^{\widehat{P}^{\mathbb{G}}}[H\mathbf{1}_{\{t\geq\tau\}}|\mathcal{G}_t]}_{\widehat{V}_t^{(3)}}. \tag{4.11}$$

Hence, we start by analyzing each of the above processes separately.

4.2.1 Decomposition of $\widehat{V}^{(1)}$ and the corresponding $\mathcal{F}_T$-claim

This subsection analyzes deeply the value process $\widehat{V}^{(1)}$. This process is related to the case when the"change-point" event never occur before T. More importantly, we single out the corresponding $\mathcal{F}_T$-claims H_1 that can be hedged using the model $S^{(1)}$.

The following lemma derived from Theorem 2.7 is essential for this computation.

Lemma 4.2: *Suppose that τ is an* honest time*. Then, for any P-integrable random* *variable ξ, we have:*

$$E\left[\xi \mathbf{1}_{\{t<\tau\}}|\mathcal{G}_t\right] = \frac{\mathbf{1}_{\{t<\tau\}}}{Z_t} E\left[\xi \mathbf{1}_{\{t<\tau\}}|\mathcal{F}_t\right], \quad t \in [0,T].$$

Proof. A direct application of Theorem 2.7 for the $\mathcal{G}_t$-measurable random variable $Y = E[\xi|\mathcal{G}_t]$ ($t \in [0,T]$) leads to the existence of two $\mathcal{F}_t$-measurable random variables U and V such that

$$Y = U\mathbf{1}_{\{t<\tau\}} + V\mathbf{1}_{\{t\geq\tau\}}.$$

This implies that

$$Y\mathbf{1}_{\{t<\tau\}} = U\mathbf{1}_{\{t<\tau\}},$$

and

$$E[Y\mathbf{1}_{\{t<\tau\}}|\mathcal{F}_t] = E[\xi\mathbf{1}_{\{t<\tau\}}|\mathcal{F}_t] = UE[\mathbf{1}_{\{t<\tau\}}|\mathcal{F}_t] = UP(\tau > t|\mathcal{F}_t) = UZ_t.$$

Therefore, we get

$$U = \frac{1}{Z_t}E[\xi\mathbf{1}_{\{t<\tau\}}|\mathcal{F}_t] \quad \text{ on } \{Z_t > 0\},$$

and the proof follows from $\mathbf{1}_{\{t<\tau\}}E[\xi|\mathcal{G}_t] = E[\xi\mathbf{1}_{\{t<\tau\}}|\mathcal{G}_t]$, which is due to the fact that τ is a $\mathbb{G}$-stopping time.

□

The decomposition of the process $\widehat{V}^{(1)}$ is based essentially on the following four lemmas that we start with. The first lemma connects the process $\widehat{V}^{(1)}$ with the value process of an $\mathcal{F}_T$-measurable contingent claim under the model $S^{(1)}$.

Lemma 4.3: *The process $\widehat{V}^{(1)}$ defined in (4.11) can be written as follows:*

$$\widehat{V}_t^{(1)} = \frac{\mathbf{1}_{\{t<\tau\}}}{Z_t} e^{\Upsilon_t^{(1)}} E^{\widehat{P}^{1,\mathbb{F}}}[H_1|\mathcal{F}_t], \quad t \in [0,T], \tag{4.12}$$

where $H_1 = E^{\widehat{P}^{1,\mathbb{F}}}[He^{-\Upsilon_T^{(1)}}\mathbf{1}_{\{T<\tau\}}|\mathcal{F}_T] \in L^2(\mathcal{F}_T, \widehat{P}^{1,\mathbb{F}})$.

Proof. In the virtue of Theorem 4.1, we have that $D_t^{\mathbb{G}} = D_t^{1,\mathbb{F}} e^{-\Upsilon_t^{(1)}}$ on $\{t < \tau\}$ and a direct combination of this fact together with Bayes' rule and Lemma 4.2 leads to the following

$$\begin{aligned}
\widehat{V}_t^{(1)} = E^{\widehat{P}^{\mathbb{G}}}[H\mathbf{1}_{\{T<\tau\}}|\mathcal{G}_t] =& \frac{1}{D_t^{\mathbb{G}}} E^P[HD_T^{\mathbb{G}}\mathbf{1}_{\{T<\tau\}}|\mathcal{G}_t] = \frac{1}{D_t^{\mathbb{G}}} E^P[HD_T^{\mathbb{G}}\mathbf{1}_{\{T<\tau\}}\mathbf{1}_{\{t<\tau\}}|\mathcal{G}_t] \\
=& \frac{\mathbf{1}_{\{t<\tau\}}}{D_t^{\mathbb{G}} Z_t} E^P[HD_T^{\mathbb{G}}\mathbf{1}_{\{T<\tau\}}\mathbf{1}_{\{t<\tau\}}|\mathcal{F}_t] \\
=& \frac{\mathbf{1}_{\{t<\tau\}}}{D_t^{1,\mathbb{F}} Z_t} e^{\Upsilon_t^{(1)}} E^P[HD_T^{1,\mathbb{F}} e^{-\Upsilon_T^{(1)}}\mathbf{1}_{\{T<\tau\}}\mathbf{1}_{\{t<\tau\}}|\mathcal{F}_t] \\
=& \frac{\mathbf{1}_{\{t<\tau\}}}{Z_t} e^{\Upsilon_t^{(1)}} E^{\widehat{P}^{1,\mathbb{F}}}[He^{-\Upsilon_T^{(1)}}\mathbf{1}_{\{T<\tau\}}|\mathcal{F}_t] \\
=& \frac{\mathbf{1}_{\{t<\tau\}}}{Z_t} e^{\Upsilon_t^{(1)}} E^{\widehat{P}^{1,\mathbb{F}}}\left[E^{\widehat{P}^{1,\mathbb{F}}}(He^{-\Upsilon_T^{(1)}}\mathbf{1}_{\{T<\tau\}})|\mathcal{F}_T)|\mathcal{F}_t\right] \\
=& \frac{\mathbf{1}_{\{t<\tau\}}}{Z_t} e^{\Upsilon_t^{(1)}} E^{\widehat{P}^{1,\mathbb{F}}}[H_1|\mathcal{F}_t]
\end{aligned}$$

This completes the proof of the lemma. □

Remark that $M_t^{1,\mathbb{F}} = E^{\widehat{P}^{1,\mathbb{F}}}[H_1|\mathcal{F}_t]$ is a square integrable $(\widehat{P}^{1,\mathbb{F}}, \mathbb{F})$-martingale. Then, by Girsanov's theorem, the process

$$\widehat{W}_t^{(1)} = W_t^{(1)} + \int_0^t \frac{\mu_1(u, S_u)}{\sigma_1(u, S_u)} du, \tag{4.13}$$

is a $(\widehat{P}^{1,\mathbb{F}}, \mathbb{F})$-Brownian motion. Recall that $W^{(2)}$ is orthogonal to $W^{(1)}$ and thus it is also orthogonal to $M^{(1)}$ (the martingale part of $S^{(1)}$). Then, $W^{(2)}$ is still a $(\widehat{P}^{1,\mathbb{F}}, \mathbb{F})$-Brownian motion by definition of minimal martingale measure (see Definition 2.30). Under (4.1) and by the invariance of the martingale representation property under equivalent martingale measures (see Remark 2.1), we have

$$M_t^{1,\mathbb{F}} := E^{\widehat{P}^{1,\mathbb{F}}}[H_1|\mathcal{F}_t] = M_0^{1,\mathbb{F}} + \int_0^t \xi_u^{(1)} d\widehat{W}_u^{(1)} + \int_0^t \eta_u^{(1)} W_u^{(2)}, \tag{4.14}$$

for some predictable processes $\xi^{(1)} \in L^2(\widehat{W}^{(1)})$ and $\eta^{(1)} \in L^2(W^{(2)})$. The next lemma is known in the literature and will be useful in our analysis.

Lemma 4.4: *Consider the Doob-Meyer decomposition of $Z_t = P(\tau > t|\mathcal{F}_t) = X_t - A_t$. Let h be a bounded $\mathbb{G}$-predictable process. Then, the process:*

$$h_\tau \mathbf{1}_{\{\tau \leq t\}} - \int_0^{t\wedge\tau} \frac{h_u}{Z_{u-}} dA_u, t \geq 0$$

is a $(P, \mathbb{G})$-martingale.

Proof. The proof can be found in Jeulin and Yor [28] (Proposition 2). □

As a consequence of the above lemma, the process

$$L_t = \mathbf{1}_{\{\tau \leq t\}} - \int_0^t \mathbf{1}_{\{u \leq \tau\}} \frac{dA_u}{Z_{u-}} \tag{4.15}$$

is a $(P, \mathbb{G})$-martingale. Furthermore, it is easy to see L is $\mathbb{G}$-martingale with finite variations and thus it is orthogonal to any continuous $\mathbb{G}$-martingales.

Lemma 4.5: *The process $J_t = \frac{\mathbf{1}_{\{t<\tau\}}}{Z_t}$ can be decomposed as follows*

$$J_t = J_0 - \int_0^t \mathbf{1}_{\{u \leq \tau\}} \frac{1}{(Z_{u-})^2} d\widetilde{X}_u - \int_0^t \frac{1}{Z_u} dL_u + B_t^{(1)}, \quad 0 \leq t \leq T,$$

where

$$B_t^{(1)} := \sum_{u \leq t} \mathbf{1}_{\{u \leq \tau\}} \left[\Delta \left(\frac{1}{Z} \right)_u + \frac{\Delta Z_u}{(Z_{u-})^2} \right] - \int_0^t \mathbf{1}_{\{u \leq \tau\}} \frac{1}{Z_{u-}} \Delta \left(\frac{1}{Z} \right)_u dA_u \tag{4.16}$$

is a $\mathbb{G}$- predictable process with finite variation.

Proof. Let $H_t = \mathbf{1}_{\{\tau \leq t\}}$, then $\mathbf{1}_{\{t<\tau\}} = 1 - H_t$, which is a right-continuous process with finite variation. Then the quadratic variation process of $1 - H$ and J is given by

$$\left[1 - H, \frac{1}{Z} \right]_t = - \int_0^t \Delta \left(\frac{1}{Z} \right)_u dH_u.$$

Also, it is easy to see that $[Z]^c = \langle X\rangle^{P,\mathbb{F}}$ since X is continuous. An application of the integration-by-parts formula and Itô's formula afterwards, we obtain

$$
\begin{aligned}
J_t = \frac{1-H_t}{Z_t} =& J_0 + \int_0^t 1 - H_{u-} d\left(\frac{1}{Z_u}\right) + \int_0^t \frac{1}{Z_{u-}} d(1-H_u) + \left[1-H, \frac{1}{Z}\right]_t \\
=& J_0 - \int_0^t \mathbf{1}_{\{u\le\tau\}} \frac{1}{(Z_{u-})^2} dZ_u + \int_0^t \mathbf{1}_{\{u\le\tau\}} \frac{1}{(Z_{u-})^3} d[Z]^c_u \\
&+ \sum_{u\le t} \mathbf{1}_{\{u\le\tau\}} \left[\Delta\left(\frac{1}{Z}\right)_u + \frac{\Delta Z_u}{(Z_{u-})^2}\right] - \int_0^t \frac{1}{Z_{u-}} dH_u - \int_0^t \Delta\left(\frac{1}{Z}\right)_u dH_u \\
=& J_0 - \int_0^t \mathbf{1}_{\{u\le\tau\}} \frac{1}{(Z_{u-})^2} dX_u + \int_0^t \mathbf{1}_{\{u\le\tau\}} \frac{1}{(Z_{u-})^2} dA_u \\
&+ \int_0^t \mathbf{1}_{\{u\le\tau\}} \frac{1}{(Z_{u-})^3} d\langle X\rangle^{P,\mathbb{F}}_u - \int_0^t \frac{1}{Z_u} dH_u + \sum_{u\le t} \mathbf{1}_{\{u\le\tau\}} \left[\Delta\left(\frac{1}{Z}\right)_u + \frac{\Delta Z_u}{(Z_{u-})^2}\right] \\
=& J_0 - \int_0^t \mathbf{1}_{\{u\le\tau\}} \frac{1}{(Z_{u-})^2} \left(dX_u - \frac{1}{Z_{u-}} d\langle X\rangle^{P,\mathbb{F}}_u\right) \\
&+ \int_0^t \mathbf{1}_{\{u\le\tau\}} \frac{1}{(Z_{u-})^2} dA_u - \int_0^t \frac{1}{Z_u} dH_u + \sum_{u\le t} \mathbf{1}_{\{u\le\tau\}} \left[\Delta\left(\frac{1}{Z}\right)_u + \frac{\Delta Z_u}{(Z_{u-})^2}\right] \\
=& J_0 - \int_0^t \mathbf{1}_{\{u\le\tau\}} \frac{1}{(Z_{u-})^2} d\widetilde{X}_u - \int_0^t \frac{1}{Z_u} dL_u \\
&\underbrace{- \int_0^t \mathbf{1}_{\{u\le\tau\}} \frac{1}{Z_{u-}} \Delta\left(\frac{1}{Z}\right)_u dA_u + \sum_{u\le t} \mathbf{1}_{\{u\le\tau\}} \left[\Delta\left(\frac{1}{Z}\right)_u + \frac{\Delta Z_u}{(Z_{u-})^2}\right]}_{B^{(1)}_t}
\end{aligned}
$$

Since Z is $\mathbb{F}$-predictable and due to the continuity of X, the process $B^{(1)}$ is $\mathbb{G}$-predictable process with finite variation. This ends the proof of the lemma. □

Lemma 4.6: *The process* $\frac{\mathbf{1}_{\{t<\tau\}}}{Z_t} e^{\Upsilon^{(1)}_t} = J_t e^{\Upsilon^{(1)}_t}$ *can be written as:*

$$
\frac{\mathbf{1}_{\{t<\tau\}}}{Z_t} e^{\Upsilon^{(1)}_t} = J_0 - \int_0^t \mathbf{1}_{\{u\le\tau\}} \frac{h^{(2)}_u e^{\Upsilon^{(1)}_u}}{(Z_{u-})^2} d\widetilde{W}^{(2)}_u - \int_0^t \frac{e^{\Upsilon^{(1)}_u}}{Z_u} dL_u + \int_0^t e^{\Upsilon^{(1)}_u} dB^{(1)}_u,
$$

where the process $B^{(1)}$ is defined in (4.16).

Proof. Firstly, Recall that the continuous process $e^{\Upsilon_t^{(1)}}$ is defined as

$$e^{\Upsilon_t^{(1)}} = \exp\left(\int_0^t \mathbf{1}_{\{u\le\tau\}}\frac{h_u^{(1)}}{Z_{u-}}d\widetilde{W}_u^{(1)} + \frac{1}{2}\int_0^t \mathbf{1}_{\{u\le\tau\}}\left(\frac{h_u^{(1)}}{Z_{u-}}\right)^2 du\right), \quad t\in[0,T].$$

Then, thanks to Itô's formula, we have

$$de^{\Upsilon_t^{(1)}} = \mathbf{1}_{\{t\le\tau\}}\frac{h_t^{(1)}}{Z_{t-}}e^{\Upsilon_t^{(1)}}d\widetilde{W}_t^{(1)} + \mathbf{1}_{\{t\le\tau\}}\left(\frac{h_t^{(1)}}{Z_{t-}}\right)^2 e^{\Upsilon_t^{(1)}}dt, \quad \Upsilon_0^{(1)} = 0$$

Notice that $L_t = \mathbf{1}_{\{\tau\le t\}} - \int_0^t \mathbf{1}_{\{u\le\tau\}}\frac{dA_u}{Z_{u-}}$ is a finite-variation process, X is continuous and $\big[\widetilde{X}, \widetilde{W}^{(1)}\big]_t = \int_0^t h_u^{(1)}du$. Therefore, we obtain:

$$\big[J, e^{\Upsilon^{(1)}}\big]_t = -\int_0^t \mathbf{1}_{\{u\le\tau\}}\frac{h_u^{(1)}}{(Z_{u-})^3}e^{\Upsilon_u^{(1)}}d\big[\widetilde{X}, \widetilde{W}^{(1)}\big]_u = -\int_0^t \mathbf{1}_{\{u\le\tau\}}\frac{(h_u^{(1)})^2}{(Z_{u-})^3}e^{\Upsilon_u^{(1)}}du.$$

By using the integration-by-parts formula to $\frac{\mathbf{1}_{\{t<\tau\}}}{Z_t}e^{\Upsilon_t^{(1)}}$ and above equations, we derive:

$$\begin{aligned}
\frac{\mathbf{1}_{\{t<\tau\}}}{Z_t}e^{\Upsilon_t^{(1)}} =& J_0 + \int_0^t J_{u-}de^{\Upsilon_u^{(1)}} + \int_0^t e^{\Upsilon_u^{(1)}}dJ_u + \big[J, e^{\Upsilon^{(1)}}\big]_t \\
=& J_0 + \int_0^t \mathbf{1}_{\{u\le\tau\}}\frac{h_u^{(1)}e^{\Upsilon_u^{(1)}}}{(Z_{u-})^2}d\widetilde{W}_u^{(1)} + \int_0^t \mathbf{1}_{\{u\le\tau\}}\frac{h_u^{(1)}}{(Z_{u-})^2}\frac{h_u^{(1)}}{Z_{u-}}e^{\Upsilon_u^{(1)}}du \\
&- \int_0^t \mathbf{1}_{\{u\le\tau\}}\frac{e^{\Upsilon_u^{(1)}}}{(Z_{u-})^2}d\widetilde{X}_u - \int_0^t \frac{e^{\Upsilon_u^{(1)}}}{Z_u}dL_u + \int_0^t e^{\Upsilon_u^{(1)}}dB_u^{(1)} \\
&- \int_0^t \mathbf{1}_{\{u\le\tau\}}\frac{h_u^{(1)}}{(Z_{u-})^2}\frac{h_u^{(1)}}{Z_{u-}}e^{\Upsilon_u^{(1)}}du \\
=& J_0 + \int_0^t \mathbf{1}_{\{u\le\tau\}}\frac{h_u^{(1)}}{(Z_{u-})^2}e^{\Upsilon_u^{(1)}}d\widetilde{W}_u^{(1)} - \int_0^t \mathbf{1}_{\{u\le\tau\}}\frac{e^{\Upsilon_u^{(1)}}}{(Z_{u-})^2}d\widetilde{X}_u - \int_0^t \frac{e^{\Upsilon_u^{(1)}}}{Z_{u-}}dL_u
\end{aligned}$$

$$+\int_0^t e^{\Upsilon_u^{(1)}} dB_u^{(1)}$$

$$=J_0 - \int_0^t \mathbf{1}_{\{u\le\tau\}} \frac{h_u^{(2)} e^{\Upsilon_u^{(1)}}}{(Z_{u-})^2} d\widetilde{W}_u^{(2)} - \int_0^t \frac{e^{\Upsilon_u^{(1)}}}{Z_u} dL_u + \int_0^t e^{\Upsilon_u^{(1)}} dB_u^{(1)}.$$

In the last equality, we used the fact from (4.5) that

$$d\widetilde{X}_t = h_t^{(1)} d\widetilde{W}_t^{(1)} + h_t^{(2)} \widetilde{W}_t^2.$$

This ends the proof of the lemma □

Now, we can apply the martingale representation (4.14) of $M^{1,\mathbb{F}}$ to find the decomposition of $\widehat{V}^{(1)}$. The following theorem states the main result regarding $\widehat{V}^{(1)}$.

Theorem 4.3: *Consider the representation of the continuous $(\widehat{P}^{1,\mathbb{F}}, \mathbb{F})$-martingale $M_t^{1,\mathbb{F}} = E^{\widehat{P}^{1,\mathbb{F}}}(H_1|\mathcal{F}_t)$:*

$$M_t^{1,\mathbb{F}} = M_0^{1,\mathbb{F}} + \int_0^t \xi_u^{(1)} \widehat{W}_u^{(1)} + \int_0^t \eta_u^{(1)} dW_u^{(2)}.$$

Then, $\widehat{V}_t^{(1)} = \frac{\mathbf{1}_{\{t<\tau\}}}{Z_t} e^{\Upsilon_t^{(1)}} E^{\widehat{P}^{1,\mathbb{F}}}(H_1|\mathcal{F}_t)$ can be written as:

$$\widehat{V}_t^{(1)} = J_0 M_0^{1,\mathbb{F}} + \int_0^t \mathbf{1}_{\{u\le\tau\}} \frac{e^{\Upsilon_u^{(1)}}}{Z_{u-}} \xi_u^{(1)} d\widehat{W}_u + L_t^{(1)}, \tag{4.17}$$

where

$$\widehat{W}_t = \widetilde{W}_t + \int_0^t \frac{\mu_u}{V_u} du \text{ is a } (\widehat{P}^{\mathbb{G}}, \mathbb{G})\text{-Brownian motion by Girsanov's theorem}$$

and

$$L_t^{(1)} := \int_0^t \mathbf{1}_{\{u\le\tau\}} \frac{e^{\Upsilon_u^{(1)}}}{Z_{u-}} \left(\eta_u^{(1)} - \frac{h_u^{(2)}}{Z_{u-}} M_u^{1,\mathbb{F}} \right) d\widetilde{W}_u^{(2)} - \int_0^t M_u^{1,\mathbb{F}} \frac{e^{\Upsilon_u^{(1)}}}{Z_u} dL_u \tag{4.18}$$

is a $(P, \mathbb{G})$-martingale orthogonal to $M^{\mathbb{G}}$ and thus is still a $(\widehat{P}^{\mathbb{G}}, \mathbb{G})$-martingale.

Proof. Firstly, we compute $\left[M^{1,\mathbb{F}}, Je^{\Upsilon^{(1)}}\right]_t$:

$$\begin{aligned}
\left[M^{1,\mathbb{F}}, Je^{\Upsilon^{(1)}}\right]_t &= -\int_0^t \mathbf{1}_{\{u\leq\tau\}} \frac{h_u^{(2)} e^{\Upsilon_u^{(1)}}}{Z_{u-}} \xi_u^{(1)} \underbrace{d[\widehat{W}^{(1)}, \widetilde{W}^{(2)}]_u}_{d[W^{(1)},W^{(2)}]_u=0} \\
&\quad - \int_0^t \mathbf{1}_{\{u\leq\tau\}} \frac{h_u^{(2)} e^{\Upsilon_u^{(1)}}}{(Z_u)^2} \eta_u^{(1)} \underbrace{d[W^{(2)}, \widetilde{W}^{(2)}]_u}_{d[W^{(2)},W^{(2)}]_u=du} \\
&= -\int_0^t \mathbf{1}_{\{u\leq\tau\}} \frac{h_u^{(2)}}{(Z_{u-})^2} e^{\Upsilon_u^{(1)}} \eta_u^{(1)} du.
\end{aligned}$$

Then, by using the integration-by-parts formula, we have

$$\begin{aligned}
\widehat{V}_t^{(1)} &= J_t e^{\Upsilon_t^{(1)}} M_t^{1,\mathbb{F}} \\
&= J_0 M_0^{1,\mathbb{F}} + \int_0^t J_{u-} e^{\Upsilon_u^{(1)}} dM_u^{1,\mathbb{F}} + \int_0^t M_u^{1,\mathbb{F}} d(J_u e^{\Upsilon_u^{(1)}}) + \left[M^{1,\mathbb{F}}, Je^{\Upsilon^{(1)}}\right]_t \\
&= J_0 M_0^{1,\mathbb{F}} + \int_0^t \mathbf{1}_{\{u\leq\tau\}} \frac{e^{\Upsilon_u^{(1)}}}{Z_{u-}} \xi_u^{(1)} d\widehat{W}_u^{(1)} + \int_0^t \mathbf{1}_{\{u\leq\tau\}} \frac{e^{\Upsilon_u^{(1)}}}{Z_{u-}} \eta_u^{(1)} dW_u^{(2)} \\
&\quad - \int_0^t \mathbf{1}_{\{u\leq\tau\}} \frac{h_u^{(2)} e^{\Upsilon_u^{(1)}}}{(Z_{u-})^2} M_u^{1,\mathbb{F}} d\widetilde{W}_u^{(2)} - \int_0^t \frac{e^{\Upsilon_u^{(1)}}}{Z_u} M_u^{1,\mathbb{F}} dL_u + \int_0^t M_u^{1,\mathbb{F}} e^{\Upsilon_u^{(2)}} dB_u^{(1)} \\
&\quad - \int_0^t \mathbf{1}_{\{u\leq\tau\}} \frac{h_u^{(2)} e^{\Upsilon_u^{(1)}}}{(Z_{u-})^2} \eta_u^{(1)} du \\
&= J_0 M_0^{1,\mathbb{F}} + \int_0^t \mathbf{1}_{\{u\leq\tau\}} \frac{e^{\Upsilon_u^{(1)}}}{Z_{u-}} \xi_u^{(1)} d\widehat{W}_u^{(1)} + L_t^{(1)} + \int_0^t M_u^{1,\mathbb{F}} e^{\Upsilon_u^{(1)}} dB_u^{(1)} \\
&= J_0 M_0^{1,\mathbb{F}} + \int_0^t \mathbf{1}_{\{u\leq\tau\}} \frac{e^{\Upsilon_u^{(1)}}}{Z_{u-}} \xi_u^{(1)} d\widehat{W}_u + L_t^{(1)} + \int_0^t M_u^{1,\mathbb{F}} e^{\Upsilon_u^{(1)}} dB_u^{(1)}.
\end{aligned}$$

In the last two equalities, we used the facts that

$$\mathbf{1}_{\{u\le\tau\}}d\widetilde{W}^{(2)}_u = \mathbf{1}_{\{u\le\tau\}}(dW^{(2)}_u - \frac{h^{(2)}_u}{Z_{u-}}du)$$

and

$$\begin{aligned}
\mathbf{1}_{\{u\le\tau\}}d\widehat{W}_u =& \mathbf{1}_{\{u\le\tau\}}d\widetilde{W}_u + \mathbf{1}_{\{u\le\tau\}}\frac{\mu_u}{V_u}du \\
=& \mathbf{1}_{\{u\le\tau\}}d\widetilde{W}^{(1)}_u + \mathbf{1}_{\{u\le\tau\}}\left(\frac{\mu_1(u,S_u)}{\sigma_1(u,S_u)} + \theta^{(1)}_u\right)du \\
=& \mathbf{1}_{\{u\le\tau\}}[dW^{(1)}_u - \theta^{(1)}_u du] + \mathbf{1}_{\{u\le\tau\}}\left(\frac{\mu_1(u,S_u)}{\sigma_1(u,S_u)} + \theta^{(1)}_u\right)du \qquad (4.19) \\
=& \mathbf{1}_{\{u\le\tau\}}dW^{(1)}_u + \mathbf{1}_{\{u\le\tau\}}\frac{\mu_1(u,S_u)}{\sigma_1(u,S_u)}du \\
=& \mathbf{1}_{\{u\le\tau\}}d\widehat{W}^{(1)}_u.
\end{aligned}$$

Secondly, we will prove that $L^{(1)}$ is $(P,\mathbb{G})$-strongly orthogonal to $M^{\mathbb{G}}$. To this end, we need to verify that $\langle L^{(1)}, M^{\mathbb{G}}\rangle^{P,\mathbb{G}} = 0$. Since $M^{\mathbb{G}}$ is continuous and $[L, M^{\mathbb{G}}] = 0$, then we obtain

$$\begin{aligned}
\langle L^{(1)}, M^{\mathbb{G}}\rangle^{P,\mathbb{G}}_t =& \int_0^t \mathbf{1}_{\{u\le\tau\}}\frac{e^{\Upsilon^{(1)}_u}}{Z_{u-}}\left(\eta^{(1)}_u - \frac{h^{(2)}_u}{Z_{u-}}M^{1,\mathbb{F}}_u\right)V_uS_u d[\widetilde{W}^{(2)}, \widetilde{W}]_u \\
=& \int_0^t \mathbf{1}_{\{u\le\tau\}}\frac{e^{\Upsilon^{(1)}_u}}{Z_{u-}}\left(\eta^{(1)}_u - \frac{h^{(2)}_u}{Z_{u-}}M^{1,\mathbb{F}}_u\right)V_uS_u(\mathbf{1}_{\{u\le\tau\}} + \rho\mathbf{1}_{\{u>\tau\}})\underbrace{d[\widetilde{W}^{(2)}, \widetilde{W}^{(1)}]_u}_{=0} \\
&+ \int_0^t \mathbf{1}_{\{u\le\tau\}}\frac{e^{\Upsilon^{(1)}_u}}{Z_{u-}}\left(\eta^{(1)}_u - \frac{h^{(2)}_u}{Z_{u-}}M^{1,\mathbb{F}}_u\right)V_uS_u\sqrt{1-\rho^2}\mathbf{1}_{\{u>\tau\}}d[\widetilde{W}^{(2)}, \widetilde{W}^{(2)}]_u \\
=& 0.
\end{aligned}$$

Therefore, $L^{(1)}$ and $M^{\mathbb{G}}$ are $(P,\mathbb{G})$-mutually orthogonal. By definition of the minimal martingale measure $\widehat{P}^{\mathbb{G}}$, we know that $L^{(1)}$ is still a $(\widehat{P}^{\mathbb{G}},\mathbb{G})$-martingale. The proof of the theorem will be achieved if we prove that the process $\int_0^t e^{\Upsilon_u^{(1)}} dB_u^{(1)}$ is null. To this end, we remark that $\int_0^t M_u^{1,\mathbb{F}} e^{\Upsilon_u^{(2)}} dB_u^{(1)}$ is a predictable process with finite variation and $\widehat{V}_t^{(1)}$, $\int_0^t \mathbf{1}_{\{u\le\tau\}} \frac{e^{\Upsilon_u^{(1)}}}{Z_{u-}} \xi_u^{(1)} d\widehat{W}_u$ and $L_t^{(1)}$ are all $(\widehat{P}^{\mathbb{G}},\mathbb{G})$-martingales and thus $\int_0^t M_u^{1,\mathbb{F}} e^{\Upsilon_u^{(2)}} dB_u^{(1)}$ is a predictable $(\widehat{P}^{\mathbb{G}},\mathbb{G})$-martingale which equals to the constant 0. This ends the proof of the theorem. □

4.2.2 Decomposition of $\widehat{V}^{(2)}$ and the corresponding $\mathcal{F}_T$-claims

In this subsection, we will analyze the value process $\widehat{V}_t^{(2)} = E^{\widehat{P}^{\mathbb{G}}}[H\mathbf{1}_{\{t<\tau\le T\}}|\mathcal{G}_t]$. More importantly, we dig out the corresponding $\mathcal{F}_T$-claims H_2, H_3 and H_4 that can be hedged using the model $S^{(1)}$.

To obtain the decomposition of $\widehat{V}^{(2)}$, we first need the following two lemmas.

Lemma 4.7: *Consider the process $H_t = \mathbf{1}_{\{\tau\le t\}}$. The process $\widehat{V}_t^{(2)} = E^{\widehat{P}^{\mathbb{G}}}[H\mathbf{1}_{\{t<\tau\le T\}}|\mathcal{G}_t]$ can be written as*

$$\widehat{V}_t^{(2)} := E^{\widehat{P}^{\mathbb{G}}}[H\mathbf{1}_{\{t<\tau\le T\}}|\mathcal{G}_t] = \frac{\mathbf{1}_{\{t<\tau\}}}{Z_t} e^{\Upsilon_t^{(1)}} \left(\frac{1}{D_t^{1,\mathbb{F}}}\right) E\left[M_T^{\mathbb{F}} \int_t^T h_u dH_u |\mathcal{F}_t\right].$$

Here the processes $M^{\mathbb{F}}$ and h are defined as:

$$M_t^{\mathbb{F}} = E(D_T^{2,\mathbb{F}} H|\mathcal{F}_t) \text{ is a continuous } (P,\mathbb{F})\text{-martingale,}$$

$$h_t = \frac{D_t^{1,\mathbb{F}}}{D_t^{2,\mathbb{F}}} e^{-\Upsilon_t^{(1)}} \text{ is a } \mathbb{G}\text{-predictable process.}$$

Proof. Firstly, on the set $\{\tau\le T\}$, we derive that :

$$D_T^{\mathbb{G}} = D_{T\wedge\tau}^{\mathbb{G}} \frac{D_T^{\mathbb{G}}}{D_{T\wedge\tau}^{\mathbb{G}}} = D_\tau^{1,\mathbb{F}} e^{-\Upsilon_\tau^{(1)}} \frac{D_T^{2,\mathbb{F}}}{D_\tau^{2,\mathbb{F}}} = D_T^{2,\mathbb{F}} h_\tau.$$

Secondly, Lemma 4.2 together with the application of Baye's rule give us:

$$
\begin{aligned}
\widehat{V}_t^{(2)} =& E^{\widehat{P}^{\mathbb{G}}}[H\mathbf{1}_{\{t<\tau\leq T\}}|\mathcal{G}_t] \\
=& \frac{1}{D_t^{\mathbb{G}}} E[D_T^{\mathbb{G}} H\mathbf{1}_{\{t<\tau\leq T\}}|\mathcal{G}_t] \\
=& \frac{\mathbf{1}_{\{t<\tau\}}}{D_t^{1,\mathbb{F}} Z_t} e^{\Upsilon_t^{(1)}} E[D_T^{2,\mathbb{F}} H h_\tau \mathbf{1}_{\{t<\tau\leq T\}}|\mathcal{F}_t] \\
=& \frac{\mathbf{1}_{\{t<\tau\}}}{D_t^{1,\mathbb{F}} Z_t} e^{\Upsilon_t^{(1)}} E\left[E(D_T^{2,\mathbb{F}} H|\mathcal{F}_T)\int_t^T h_u dH_u|\mathcal{F}_t\right] \\
=& \frac{\mathbf{1}_{\{t<\tau\}}}{D_t^{1,\mathbb{F}} Z_t} e^{\Upsilon_t^{(1)}} E\left[M_T^{\mathbb{F}}\int_t^T h_u dH_u|\mathcal{F}_t\right].
\end{aligned}
$$

This ends the proof of the lemma. □

The next lemma connects the process $\widehat{V}^{(2)}$ to the value process of an $\mathcal{F}_T$-claim under the model $S^{(1)}$.

Lemma 4.8: *The process* $E\left[M_T \int_t^T h_u dH_u|\mathcal{F}_t\right]$ *can be written as:*

$$
\begin{aligned}
E\left[M_T \int_t^T h_u dH_u|\mathcal{F}_t\right] =& E\left[D_T^{1,\mathbb{F}} H_2|\mathcal{F}_t\right] - D_t^{1,\mathbb{F}}\int_0^t \frac{M_u^{\mathbb{F}}}{D_u^{2,\mathbb{F}}}\widetilde{h}_u dA_u \\
&+ M_t^{\mathbb{F}} E\left[D_T^{1,\mathbb{F}}\int_t^T \frac{\widetilde{h}_u}{D_u^{2,\mathbb{F}}} dA_u|\mathcal{F}_t\right],
\end{aligned}
$$

where the process $\widetilde{h}_t$ *is the* $\mathbb{F}$*-predictable projection of the process* $\mathbf{1}_{\{t\leq\tau\}}\frac{1}{Z_{t-}e^{\Upsilon_t^{(1)}}}$*, i.e.*

$$
\widetilde{h} = {}^{p,\mathbb{F}}\left(\mathbf{1}_{\{\cdot\leq\tau\}}\frac{1}{Z_- e^{\Upsilon^{(1)}}}\right) \tag{4.20}
$$

and H_2 *is an* $\mathbb{F}_T$*-measurable random variable defined by:*

$$
H_2 := \int_0^T \frac{M_u^{\mathbb{F}}\widetilde{h}_u}{D_u^{2,\mathbb{F}}} dA_u.
$$

Proof. Firstly, we notice that

$$E\left[M_T^{\mathbb{F}}\int_t^T h_u dH_u|\mathcal{F}_t\right] = \underbrace{E\left[(M_T^{\mathbb{F}}-M_t^{\mathbb{F}})\int_t^T h_u dH_u|\mathcal{F}_t\right]}_{(a)} + \underbrace{M_t^{\mathbb{F}} E\left[\int_t^T h_u dH_u|\mathcal{F}_t\right]}_{(b)}. \tag{4.21}$$

Furthermore, for any $\mathbb{G}$-predictable process η, due to $\mathbb{F}\subset\mathbb{G}$, we deduce that

$$\begin{aligned} E\left[\int_t^T h_u dH_u|\mathcal{F}_t\right] =& E\left[E\left(\int_t^T \eta_u dH_u|\mathcal{G}_t\right)|\mathcal{F}_t\right] \\ =& E\left[\underbrace{E\left(\int_t^T \eta_u dL_u|\mathcal{G}_t\right)}_{=0} + E\left(\int_t^T \mathbf{1}_{\{u\leq\tau\}}\frac{\eta_u}{Z_{u-}}dA_u|\mathcal{G}_t\right)|\mathcal{F}_t\right] \\ =& E\left[\int_t^T \mathbf{1}_{\{u\leq\tau\}}\frac{\eta_u}{Z_{u-}}dA_u|\mathcal{F}_t\right] \end{aligned}$$

Secondly, due to above observation and $[M^{\mathbb{F}},\int hdH]\equiv 0$, the term (a) in (4.21) can be written as follows:

$$\begin{aligned} (a) =& E\left[\int_t^T\left(\int_t^{u-} h_s dH_s\right)dM_u^{\mathbb{F}}|\mathcal{F}_t\right] + E\left[\int_t^T M_u^{\mathbb{F}} h_u dH_u|\mathcal{F}_t\right] \\ =& \underbrace{E\left[\int_t^T\left(\int_t^{u-} h_s dH_s\right)d\widetilde{M}_u^{\mathbb{F}}|\mathcal{F}_t\right]}_{=0} + E\left[\int_t^T \underbrace{\mathbf{1}_{\{u\leq\tau\}}\frac{\left(\int_t^{u-} h_s dH_s\right)}{Z_{u-}}}_{=0} d\langle X, M^{\mathbb{F}}\rangle_u|\mathcal{F}_t\right] \\ &+ E\left[\int_t^T M_u^{\mathbb{F}} h_u dH_u|\mathcal{F}_t\right] \\ =& E\left[\int_t^T \mathbf{1}_{\{u\leq\tau\}}\frac{M_u^{\mathbb{F}} h_u}{Z_{u-}}dA_u|\mathcal{F}_t\right] = E\left[\int_t^T D_u^{1,\mathbb{F}}\frac{M^{\mathbb{F}_u}\widetilde{h}_u}{D_u^{2,\mathbb{F}}}dA_u|\mathcal{F}_t\right] \\ =& E\left[D_T^{1,\mathbb{F}}\int_t^T \frac{M_u^{\mathbb{F}}}{D_u^{2,\mathbb{F}}}\widetilde{h}_u dA_u|\mathcal{F}_t\right] \end{aligned}$$

$$=E\left[D_T^{1,\mathbb{F}}\int_0^T \frac{M_u^{\mathbb{F}}\widetilde{h}_u}{D_u^{2,\mathbb{F}}}dA_u|\mathcal{F}_t\right]-D_t^{1,\mathbb{F}}\int_0^t \frac{M_u^{\mathbb{F}}}{D_u^{2,\mathbb{F}}}\widetilde{h}_u dA_u.$$

Similarly, the term (b) takes the form of :

$$\begin{aligned}(b)=&M_t^{\mathbb{F}}E\left[\int_t^T h_u dH_u|\mathcal{F}_t\right]=M_t^{\mathbb{F}}E\left[\int_t^T \mathbf{1}_{\{u\leq\tau\}}\frac{h_u}{Z_{u-}}dA_u|\mathcal{F}_t\right]\\ =&M_t^{\mathbb{F}}E\left[\int_t^T \frac{D_u^{1,\mathbb{F}}}{D_u^{2,\mathbb{F}}}\widetilde{h}_u dA_u|\mathcal{F}_u\right]=M_t^{\mathbb{F}}E\left[D_T^{1,\mathbb{F}}\int_t^T \frac{\widetilde{h}_u}{D_u^{2,\mathbb{F}}}dA_u|\mathcal{F}_t\right].\end{aligned}$$

This completes the proof of the lemma. □

Now, we can combine the above two lemmas and obtain a further decomposition of the process $\widehat{V}^{(2)}$:

$$\begin{aligned}\widehat{V}_t^{(2)}=&\frac{\mathbf{1}_{\{t<\tau\}}}{Z_t}e^{\Upsilon_t^{(1)}}\frac{1}{D_t^{1,\mathbb{F}}}E\left[M_T^{\mathbb{F}}\int_t^T h_u dH_u|\mathcal{F}_t\right]\\ =&J_t e^{\Upsilon_t^{(1)}}\frac{1}{D_t^{1,\mathbb{F}}}E\left[D_T^{1,\mathbb{F}}H_2|\mathcal{F}_t\right]-J_t e^{\Upsilon_t^{(1)}}\frac{D_t^{1,\mathbb{F}}}{D_t^{1,\mathbb{F}}}\int_0^t \frac{M_u^{\mathbb{F}}\widetilde{h}_u}{D_u^{2,\mathbb{F}}}dA_u\\ &+J_t e^{\Upsilon_t^{(1)}}M_t^{\mathbb{F}}\frac{1}{D_t^{1,\mathbb{F}}}E\left[D_T^{1,\mathbb{F}}\int_t^T \frac{\widetilde{h}_u}{D_u^{2,\mathbb{F}}}dA_u|\mathcal{F}_t\right]\\ =&\underbrace{J_t e^{\Upsilon_t^{(1)}}E^{\widehat{P}^{1,\mathbb{F}}}[H_2|\mathcal{F}_t]-J_t e^{\Upsilon_t^{(1)}}\int_0^t \frac{M_u^{\mathbb{F}}\widetilde{h}_u}{D_u^{2,\mathbb{F}}}dA_u}_{(A)}\\ &+\underbrace{J_t e^{\Upsilon_t^{(1)}}M_t^{\mathbb{F}}E^{\widehat{P}^{1,\mathbb{F}}}\left[\int_t^T \frac{\widetilde{h}_u}{D_u^{2,\mathbb{F}}}dA_u|\mathcal{F}_t\right]}_{(B)}.\end{aligned}\tag{4.22}$$

We first consider the term (A), which is relatively easy to compute. Here, we will apply the martingale representation of the $(\widehat{P}^{1,\mathbb{F}},\mathbb{F})$-martingale $M_t^{2,\mathbb{F}}:=E^{\widehat{P}^{1,\mathbb{F}}}[H_2|\mathcal{F}_t]$.

Theorem 4.4: *Consider the representation of the continuous* $(\widehat{P}^{1,\mathbb{F}}, \mathbb{F})$*-martingale* $M_t^{2,\mathbb{F}} = E^{\widehat{P}^{1,\mathbb{F}}}(H_2|\mathcal{F}_t)$*:*

$$M_t^{2,\mathbb{F}} = M_0^{2,\mathbb{F}} + \int_0^t \xi_u^{(2)} d\widehat{W}_u^{(1)} + \int_0^t \eta_u^{(2)} dW_u^{(2)}. \tag{4.23}$$

Then, the process $(A) = J_t e^{\Upsilon_t^{(1)}} E^{\widehat{P}^{1,\mathbb{F}}}[H_2|\mathcal{F}_t] - J_t e^{\Upsilon_t^{(1)}} \int_0^t \frac{M_u^{\mathbb{F}} \widetilde{h}_u}{D_u^{2,\mathbb{F}}} dA_u$ *can be written as:*

$$(A) = J_0 M_0^{2,\mathbb{F}} + \int_0^t \mathbf{1}_{\{u \leq \tau\}} \frac{e^{\Upsilon_u^{(1)}}}{Z_{u-}} \xi_u^{(2)} d\widehat{W}_u + L_t^{(2)} + B_t^{(2)}, \tag{4.24}$$

where

$$\widehat{W}_t = \widetilde{W}_t + \int_0^t \frac{\mu_u}{V_u} du \text{ is a } (\widehat{P}^{\mathbb{G}}, \mathbb{G})\text{-Brownian motion by Girsanov's theorem.}$$

and

$$\begin{aligned} L_t^{(2)} := &\int_0^t \mathbf{1}_{\{u \leq \tau\}} \frac{e^{\Upsilon_u^{(1)}}}{Z_{u-}} \left(\eta_u^{(2)} - \frac{h_u^{(2)}}{Z_{u-}} \left(M_u^{2,\mathbb{F}} - \int_0^{u-} \frac{M_s^{\mathbb{F}} \widetilde{h}_s}{D_s^{2,\mathbb{F}}} dA_s \right) \right) d\widetilde{W}_u^{(2)} \\ &- \int_0^t \frac{e^{\Upsilon_u^{(1)}}}{Z_u} \left(M_u^{2,\mathbb{F}} - \frac{M_u^{\mathbb{F}} \widetilde{h}_u}{D_u^{2,\mathbb{F}}} \Delta A_u - \int_0^{u-} \frac{M_s^{\mathbb{F}} \widetilde{h}_s}{D_s^{2,\mathbb{F}}} dA_s \right) dL_u \end{aligned} \tag{4.25}$$

is a $(P, \mathbb{G})$*-martingale orthogonal to* $M^{\mathbb{G}}$ *and thus is still a* $(\widehat{P}^{\mathbb{G}}, \mathbb{G})$*-martingale, and the* $\mathbb{G}$*-predictable finite variation process* $B^{(2)}$ *is defined as:*

$$\begin{aligned} B_t^{(2)} := &\int_0^t e^{\Upsilon_u^{(1)}} \left(M_u^{2,\mathbb{F}} - \frac{M_u^{\mathbb{F}} \widetilde{h}_u}{D_u^{2,\mathbb{F}}} \Delta A_u - \int_0^{u-} \frac{M_s^{\mathbb{F}} \widetilde{h}_s}{D_s^{2,\mathbb{F}}} dA_s \right) dB_u^{(1)} \\ &+ \int_0^t \mathbf{1}_{\{u \leq \tau\}} \frac{e^{\Upsilon_u^{(1)}} M_u^{\mathbb{F}} \widetilde{h}_u}{Z_{u-} D_u^{2,\mathbb{F}}} dA_u \end{aligned} \tag{4.26}$$

Proof. Firstly, the decomposition of $J_t e^{\Upsilon_t^{(1)}} M_t^{2,\mathbb{F}}$ follows the proof of Theorem 4.17. However, in this case, we can not conclude that the $\mathbb{G}$-predictable process with finite variation $\int_0^t M_u^{2,\mathbb{F}} e^{\Upsilon_u^{(1)}} dB_u^{(1)}$ is equal to 0 since the process $J_t e^{\Upsilon_t^{(1)}} M_t^{2,\mathbb{F}}$ is not guaranteed to be a $\mathbb{G}$-martingale.

Secondly, we derive the decomposition of $J_t e^{\Upsilon^{(1)}_t} \int_0^t \frac{M^{\mathbb{F}}_u \widetilde{h}_u}{D^{2,\mathbb{F}}_u} dA_u$ as follows:

$$\left[J e^{\Upsilon^{(1)}}, \frac{M^{\mathbb{F}} \widetilde{h}}{D^{2,\mathbb{F}}} \cdot A \right]_t = - \int_0^t \frac{e^{\Upsilon^{(1)}_u} M^{\mathbb{F}}_u \widetilde{h}_u}{Z_u D^{2,\mathbb{F}}_u} \Delta A_u dL_u$$

$$+ \int_0^t \frac{e^{\Upsilon^{(1)}_u} M^{\mathbb{F}}_u \widetilde{h}_u}{D^{2,\mathbb{F}}_u} \Delta A_u dB^{(1)}_u$$

then, a direct application of the integration-by-parts formula leads to :

$$J_t e^{\Upsilon^{(1)}_t} \int_0^t \frac{M^{\mathbb{F}}_u \widetilde{h}_u}{D^{2,\mathbb{F}}_u} dA_u = \int_0^t \mathbf{1}_{\{u \le \tau\}} \frac{e^{\Upsilon^{(1)}_u} M^{\mathbb{F}}_u \widetilde{h}_u}{Z_{u-} D^{2,\mathbb{F}}_u} dA_u$$

$$- \int_0^t \mathbf{1}_{\{u \le \tau\}} \frac{h^{(2)}_u e^{\Upsilon^{(1)}_u}}{(Z_{u-})^2} \left(\int_0^{u-} \frac{M^{\mathbb{F}}_s \widetilde{h}_s}{D^{2,\mathbb{F}}_s} dA_s \right) d\widetilde{W}^{(2)}_u$$

$$- \int_0^t \frac{e^{\Upsilon^{(1)}_u}}{Z_u} \left(\int_0^{u-} \frac{M^{\mathbb{F}}_s \widetilde{h}_s}{D^{2,\mathbb{F}}_s} dA_s \right) dL_u$$

$$+ \int_0^t e^{\Upsilon^{(1)}_u} \left(\int_0^{u-} \frac{M^{\mathbb{F}}_s \widetilde{h}_s}{D^{2,\mathbb{F}}_s} dA_s \right) dB^{(1)}_u + \left[J e^{\Upsilon^{(1)}}, \frac{M^{\mathbb{F}} \widetilde{h}}{D^{2,\mathbb{F}}} \cdot A \right]_t .$$

Finally, combining the decompositions of $J_t e^{\Upsilon^{(1)}_t} M^{2,\mathbb{F}}_t$ and $J_t e^{\Upsilon^{(1)}_t} \int_0^t \frac{M^{\mathbb{F}}_u \widetilde{h}_u}{D^{2,\mathbb{F}}_u} dA_u$ will ends the proof of (4.24). To prove, the $(P, \mathbb{G})$-martingale $L^{(2)}$ is orthogonal to the martingale part of S under $\mathbb{G}$ (i.e.$M^{\mathbb{G}}$), we refer the reader to the proof of Theorem 4.17. □

To find the decomposition of $(B) = J_t e^{\Upsilon^{(1)}_t} M^{\mathbb{F}}_t E^{\widehat{P}^{1,\mathbb{F}}} \left[\int_t^T \frac{\widetilde{h}_u}{D^{2,\mathbb{F}}_u} dA_u | \mathcal{F}_t \right]$, we denote the $\mathcal{F}_T$-measurable random variables H_3 and H_4 by

$$H_3 := \frac{D^{2,\mathbb{F}}_T}{D^{1,\mathbb{F}}_T} H \quad \text{and} \quad H_4 := \int_0^T \frac{\widetilde{h}_u}{D^{2,\mathbb{F}}_u} dA_u,$$

and the corresponding continuous $(\widehat{P}^{1,\mathbb{F}},\mathbb{F})$-martingales $M^{3,\mathbb{F}}$ and $M^{4,\mathbb{F}}$by

$$M_t^{3,\mathbb{F}} := E^{\widehat{P}^{1,\mathbb{F}}}[H_3|\mathcal{F}_t] \quad \text{and} \quad M_t^{4,\mathbb{F}} := E^{\widehat{P}^{1,\mathbb{F}}}[H_4|\mathcal{F}_t].$$

Then, we observe that

$$M_t^{\mathbb{F}} := E(D_T^{2,\mathbb{F}}H|\mathcal{F}_t) = D_t^{1,\mathbb{F}}E^{\widehat{P}^{1,\mathbb{F}}}\left(\frac{D_T^{2,\mathbb{F}}}{D_T^{1,\mathbb{F}}}H|\mathcal{F}_t\right) = D_t^{1,\mathbb{F}}M_t^{3,\mathbb{F}}$$

and therefore

$$\begin{aligned} J_t e^{\Upsilon_t^{(1)}} M_t^{\mathbb{F}} E^{\widehat{P}^{1,\mathbb{F}}}\left[\int_t^T \frac{\widetilde{h}_u}{D_u^{2,\mathbb{F}}}dA_u|\mathcal{F}_t\right] =& J_t e^{\Upsilon_t^{(1)}} D_t^{1,\mathbb{F}} M_t^{3,\mathbb{F}} E^{\widehat{P}^{1,\mathbb{F}}}\left[\int_t^T \frac{\widetilde{h}_u}{D_u^{2,\mathbb{F}}}dA_u|\mathcal{F}_t\right] \\ =& J_t e^{\Upsilon_t^{(1)}} D_t^{1,\mathbb{F}} M_t^{3,\mathbb{F}} M_t^{4,\mathbb{F}} \\ & - J_t e^{\Upsilon_t^{(1)}} D_t^{1,\mathbb{F}} M_t^{3,\mathbb{F}} \int_0^t \frac{\widetilde{h}_u}{D_u^{2,\mathbb{F}}} dA_u. \end{aligned} \tag{4.27}$$

Hence, the final decomposition of (B) in (4.22) is derived from the following five lemmas.

Lemma 4.9: *The process $Je^{\Upsilon^{(1)}}D^{1,\mathbb{F}}$ admits the decomposition :*

$$J_t e^{\Upsilon_t^{(1)}} D_t^{1,\mathbb{F}} = J_0 - \int_0^t \mathbf{1}_{\{u\le\tau\}} \frac{e^{\Upsilon_u^{(1)}} D_u^{1,\mathbb{F}}}{Z_{u-}} \left(\frac{\mu_1(u,S_u)}{\sigma_1(u,S_u)}\right) d\widehat{W}_t + L_t^{(3)} + B_t^{(3)}.$$

Here

$$L^{(3)} := -\int_0^t \mathbf{1}_{\{u\le\tau\}} \frac{h_u^{(2)} e^{\Upsilon_u^{(1)}} D_u^{1,\mathbb{F}}}{(Z_{u-})^2} d\widetilde{W}_u^2 - \int_0^t \frac{e^{\Upsilon_u^{(1)}} D_u^{1,\mathbb{F}}}{Z_u} dL_u, \tag{4.28}$$

is a $(P,\mathbb{G})$-martingale orthogonal to $M^{\mathbb{G}}$ and thus is still a $(\widehat{P}^{\mathbb{G}},\mathbb{G})$-martingale, *and the $\mathbb{G}$-predictable finite variation process $B^{(3)}$ is defined by*

$$B_t^{(3)} := \int_0^t e^{\Upsilon_u^{(1)}} D_u^{1,\mathbb{F}} dB_u^{(1)} + \int_0^t \boldsymbol{1}_{\{u\le\tau\}} \frac{e^{\Upsilon_u^{(1)}} D_u^{1,\mathbb{F}}}{Z_{u-}} \left(\frac{\mu_1(u,S_u)}{\sigma_1(u,S_u)}\right)^2 du. \tag{4.29}$$

Proof. Recall that $D^{1,\mathbb{F}}$ takes the form of

$$\begin{aligned} D_t^{1,\mathbb{F}} =& 1 - \int_0^t \left(\frac{\mu_1(u,S_u)}{\sigma_1(u,S_u)}\right) D_t^{1,\mathbb{F}} dW_t^{(1)} \\ =& 1 - \int_0^t \left(\frac{\mu_1(u,S_u)}{\sigma_1(u,S_u)}\right) D_t^{1,\mathbb{F}} d\widehat{W}_t^{(1)} + \int_0^t \left(\frac{\mu_1(u,S_u)}{\sigma_1(u,S_u)}\right)^2 D_u^{1,\mathbb{F}} du. \end{aligned}$$

Then, it is easy to verify that $[Je^{\Upsilon^{(1)}}, D^{1,\mathbb{F}}] \equiv 0$ since $W^{(1)}$ and $W^{(2)}$ are mutually orthogonal. The direct application of the integration part formula together with Lemma 4.6 leads to

$$\begin{aligned} J_t e^{\Upsilon_t^{(1)}} D_t^{1,\mathbb{F}} =& J_0 - \int_0^t \mathbf{1}_{\{u\le\tau\}} \frac{e^{\Upsilon_u^{(1)}} D_u^{1,\mathbb{F}}}{Z_{u-}} \left(\frac{\mu_1(u,S_u)}{\sigma_1(u,S_u)}\right) d\widehat{W}_t^{(1)} \\ &+ \int_0^t \mathbf{1}_{\{u\le\tau\}} \frac{e^{\Upsilon_u^{(1)}} D_u^{1,\mathbb{F}}}{Z_{u-}} \left(\frac{\mu_1(u,S_u)}{\sigma_1(u,S_u)}\right)^2 du \\ &- \int_0^t \mathbf{1}_{\{u\le\tau\}} \frac{h_u^{(2)} e^{\Upsilon_u^{(1)}} D_u^{1,\mathbb{F}}}{(Z_{u-})^2} d\widetilde{W}_u^2 - \int_0^t \frac{e^{\Upsilon_u^{(1)}} D_u^{1,\mathbb{F}}}{Z_u} dL_u, \\ &+ \int_0^t e^{\Upsilon_u^{(1)}} D_u^{1,\mathbb{F}} dB_u^{(1)} \\ =& J_0 - \int_0^t \mathbf{1}_{\{u\le\tau\}} \frac{e^{\Upsilon_u^{(1)}} D_u^{1,\mathbb{F}}}{Z_{u-}} \left(\frac{\mu_1(u,S_u)}{\sigma_1(u,S_u)}\right) d\widehat{W}_t^{(1)} \\ &+ L_t^{(3)} + B_t^{(3)} \end{aligned}$$

In the last equality, we use the fact from (4.19) that $\mathbf{1}_{\{u\le\tau\}} d\widehat{W}_u^{(1)} = \mathbf{1}_{\{u\le\tau\}} d\widehat{W}_t$ □

Lemma 4.10: *Consider the martingale representation of a continuous $(\widehat{P}^{1,\mathbb{F}}, \mathbb{F})$-*

martingale $m^{\widehat{P}^{1,\mathbb{F}}}$*:*

$$m_t^{\widehat{P}^{1,\mathbb{F}}} = m_0^{\widehat{P}^{1,\mathbb{F}}} + \int_0^t \xi_u \widehat{W}_u^{(1)} + \int_0^t \eta_u dW_u^{(2)},$$

then, the process $Je^{\Upsilon^{(1)}} D^{1,\mathbb{F}} m^{\widehat{P}^{1,\mathbb{F}}}$ *admits the decomposition :*

$$\begin{aligned} J_t e^{\Upsilon_t^{(1)}} D_t^{1,\mathbb{F}} m_t^{\widehat{P}^{1,\mathbb{F}}} =& J_0 m_0^{\widehat{P}^{1,\mathbb{F}}} + \int_0^t \mathbf{1}_{\{u \leq \tau\}} \frac{e^{\Upsilon_u^{(1)}} D_u^{1,\mathbb{F}}}{Z_{u-}} \left[\xi_u - m_u^{\widehat{P}^{1,\mathbb{F}}} \left(\frac{\mu_1(u, S_u)}{\sigma_1(u, S_u)} \right) \right] d\widehat{W}_u \\ &+ \int_0^t m_u^{\widehat{P}^{1,\mathbb{F}}} dL_u^{(3)} + \int_0^t \mathbf{1}_{\{u \leq \tau\}} \frac{e^{\Upsilon_u^{(1)}} D_u^{1,\mathbb{F}}}{Z_u-} \eta_u d\widetilde{W}_u^{(2)} \\ &+ \int_0^t m_u^{\widehat{P}^{1,\mathbb{F}}} dB_u^{(3)} - \int_0^t \mathbf{1}_{\{u \leq \tau\}} \frac{e^{\Upsilon_u^{(1)}} D_u^{1,\mathbb{F}}}{Z_{u-}} \left(\frac{\mu_1(u, S_u)}{\sigma_1(u, S_u)} \right) \xi_u du. \end{aligned}$$

Proof. Firstly, we calculate $[Je^{\Upsilon^{(1)}}, m^{\widehat{P}^{1,\mathbb{F}}}]$ as:

$$\begin{aligned} [Je^{\Upsilon^{(1)}} D^{1,\mathbb{F}}, m^{\widehat{P}^{1,\mathbb{F}}}]_t =& - \int_0^t \mathbf{1}_{\{u \leq \tau\}} \frac{e^{\Upsilon_u^{(1)}} D_u^{1,\mathbb{F}}}{Z_{u-}} \left(\frac{\mu_1(u, S_u)}{\sigma_1(u, S_u)} \right) \xi_u du \\ &- \int_0^t \mathbf{1}_{\{u \leq \tau\}} \frac{h_u^{(2)} e^{\Upsilon_u^{(1)}} D_u^{1,\mathbb{F}}}{(Z_{u-})^2} \eta_u du. \end{aligned}$$

Secondly, a direct application of the integration-by-parts formula together with Lemma 4.9 will give us the desired result. □

Lemma 4.11: *For any continuous finite variation process* B*, the process* $Je^{\Upsilon^{(1)}} D^{1,\mathbb{F}} B$ *admits the decomposition:*

$$\begin{aligned} J_t e^{\Upsilon_t^{(1)}} D_t^{1,\mathbb{F}} B_t =& J_0 B_0 - \int_0^t \mathbf{1}_{\{u \leq \tau\}} \frac{e^{\Upsilon_u^{(1)}} D_u^{1,\mathbb{F}} B_u}{Z_{u-}} \left(\frac{\mu_1(u, S_u)}{\sigma_1(u, S_u)} \right) d\widehat{W}_u \\ &+ \int_0^t B_u dL_u^{(3)} + \int_0^t B_u dB_u^{(3)} + \int_0^t \mathbf{1}_{\{u \leq \tau\}} \frac{e^{\Upsilon_u^{(1)}} D_u^{1,\mathbb{F}}}{Z_{u-}} dB_u \end{aligned}$$

Proof. Since B is a continuous process with finite variation, it is easy to verify that

$[Je^{\Upsilon^{(1)}}D^{1,\mathbb{F}}, B] \equiv 0$. Then, the result is obvious via the direct application of the integration-by-parts formula together with Lemma 4.9. □

Lemma 4.12: *Consider the martingale representations of $M^{3,\mathbb{F}}$ and $M^{4,\mathbb{F}}$:*

$$M_t^{3,\mathbb{F}} = M_0^{3,\mathbb{F}} + \int_0^t \xi_u^{(3)} d\widehat{W}_u^{(1)} + \int_0^t \eta_u^{(3)} dW_u^{(2)} \tag{4.30}$$

$$, M_t^{4,\mathbb{F}} = M_0^{4,\mathbb{F}} + \int_0^t \xi_u^{(4)} d\widehat{W}_u^{(1)} + \int_0^t \eta_u^{(4)} dW_u^{(2)}. \tag{4.31}$$

Then, the process $Je^{\Upsilon^{(1)}}D^{1,\mathbb{F}}M^{3,\mathbb{F}}M^{4,\mathbb{F}}$ admits the decomposition :

$$\begin{aligned} J_t e^{\Upsilon_t^{(1)}} D_t^{1,\mathbb{F}} M_t^{3,\mathbb{F}} M_t^{4,\mathbb{F}} =& J_0 M_0^{3,\mathbb{F}} M_0^{4,\mathbb{F}} + \int_0^t \mathbf{1}_{\{u\le\tau\}} \frac{e^{\Upsilon_u^{(1)}} D_u^{1,\mathbb{F}}}{Z_{u-}} \Big[\xi_u^{(3)} M_u^{4,\mathbb{F}} + \xi_u^{(4)} M_u^{3,\mathbb{F}} \\ & - \frac{\mu_1(u,S_u)}{\sigma_1(u,S_u)} \left(M_u^{3,\mathbb{F}} + M_u^{4,\mathbb{F}} + \int_0^u (\xi_s^{(3)}\xi_s^{(4)} + \eta_s^{(3)}\eta_s^{(4)}) ds \right) \Big] d\widehat{W}_u \\ & + L_t^{(4)} + B_t^{(4)}, \end{aligned}$$

where

$$\begin{aligned} L_t^{(4)} :=& \int_0^t \left[M_u^{3,\mathbb{F}} + M_u^{4,\mathbb{F}} + \int_0^u (\xi_s^{(3)}\xi_s^{(4)} + \eta_s^{(3)}\eta_s^{(4)}) ds \right] dL_u^{(3)} \\ & + \int_0^t \mathbf{1}_{\{u\le\tau\}} \frac{e^{\Upsilon_u^{(1)}} D_u^{1,\mathbb{F}}}{Z_{u-}} \left(\eta_u^{(3)} M_u^{4,\mathbb{F}} + \eta_u^{(4)} M_u^{3,\mathbb{F}} \right) d\widetilde{W}_u^{(2)} \end{aligned} \tag{4.32}$$

is a $(P,\mathbb{G})$-martingale orthogonal to $\mathbb{G}$ and thus is still a $(\widehat{P}^{\mathbb{G}},\mathbb{G})$-martingale, and the $\mathbb{G}$-predictable finite variation process $B^{(4)}$ is defined by

$$\begin{aligned} B_t^{(4)} :=& \int_0^t \left[M_u^{3,\mathbb{F}} + M_u^{4,\mathbb{F}} + \int_0^u (\xi_s^{(3)}\xi_s^{(4)} + \eta_s^{(3)}\eta_s^{(4)}) ds \right] dB_u^{(3)} \\ & + \int_0^t \mathbf{1}_{\{u\le\tau\}} \frac{e^{\Upsilon_u^{(1)}} D_u^{1,\mathbb{F}}}{Z_{u-}} \left[\xi_s^{(3)}\xi_s^{(4)} + \eta_s^{(3)}\eta_s^{(4)} - \left(\frac{\mu_1(u,S_u)}{\sigma_1(u,S_u)} \right) \left(\xi_u^{(3)} M_u^{4,\mathbb{F}} + \xi_u^{(4)} M_u^{3,\mathbb{F}} \right) \right] du. \end{aligned} \tag{4.33}$$

Proof. Firstly, since $M^{3,\mathbb{F}}$ and $M^{4,\mathbb{F}}$ are continuous, $M^{3,\mathbb{F}}M^{4,\mathbb{F}}$ can be written as:

$$M_t^{3,\mathbb{F}}M_t^{4,\mathbb{F}} = M_0^{3,\mathbb{F}}M_0^{4,\mathbb{F}} + \int_0^t M_u^{3,\mathbb{F}}dM_u^{4,\mathbb{F}} + \int_0^t M_u^{4,\mathbb{F}}dM_u^{3,\mathbb{F}} + [M^{3,\mathbb{F}}, M^{4,\mathbb{F}}]_t$$

Furthermore, the processes $\int M^{3,\mathbb{F}}dM^{4,\mathbb{F}}$ and $\int M^{4,\mathbb{F}}dM^{3,\mathbb{F}}$ take the form:

$$\int_0^t M_u^{3,\mathbb{F}}dM_u^{4,\mathbb{F}} = \int_0^t \xi_u^{(4)}M_u^{3,\mathbb{F}}d\widehat{W}_u + \int_0^t \eta_u^{(4)}M_u^{3,\mathbb{F}}dW_u^{(2)}$$

$$\int_0^t M_u^{4,\mathbb{F}}dM_u^{3,\mathbb{F}} = \int_0^t \xi_u^{(3)}M_u^{4,\mathbb{F}}d\widehat{W}_u + \int_0^t \eta_u^{(3)}M_u^{4,\mathbb{F}}dW_u^{(2)}$$

and the continuous finite variation process $[M^{3,\mathbb{F}}, M^{4,\mathbb{F}}]$ takes the form:

$$[M^{3,\mathbb{F}}, M^{4,\mathbb{F}}]_t = \int_0^t (\xi_u^{(3)}\xi_u^{(4)} + \eta_u^{(3)}\eta_u^{(4)})du.$$

Then, the proof of the lemma follows from a combination of Lemma 4.10 and Lemma 4.11. □

The final step to derive the decomposition of part (B), due to (4.27), is to find the decomposition of the process $J_t e^{\Upsilon_t^{(1)}} D_t^{1,\mathbb{F}} M_t^{3,\mathbb{F}} \int_0^t \frac{\widetilde{h}_u}{D_u^{2,\mathbb{F}}} dA_u$. This is the aim of the following lemma.

Lemma 4.13: *Consider the representation of $M^{3,\mathbb{F}}$ as:*

$$M_t^{3,\mathbb{F}} = M_0^{3,\mathbb{F}} + \int_0^t \xi_u^{(3)}d\widehat{W}_u^{(1)} + \int_0^t \eta_u^{(3)}dW_u^{(2)},$$

and denote the $\mathbb{G}$-predictable finite variation process by $A_t^{\widetilde{h}} = \int_0^t \frac{\widetilde{h}_s}{D_s^{2,\mathbb{F}}}dA_s$. Then, the process $J_t e^{\Upsilon_t^{(1)}} D_t^{1,\mathbb{F}} M_t^{3,\mathbb{F}} A_t^{\widetilde{h}}$ admits the decomposition:

$$J_t e^{\Upsilon_t^{(1)}} D_t^{1,\mathbb{F}} M_t^{3,\mathbb{F}} A_t^{\widetilde{h}} = \int_0^t \mathbf{1}_{\{u\leq\tau\}} \frac{e^{\Upsilon_u^{(1)}} D_u^{1,\mathbb{F}} A_{u-}^{\widetilde{h}}}{Z_{u-}} \left[\xi_u^{(3)} - M_u^{3,\mathbb{F}}\left(\frac{\mu_1(u,S_u)}{\sigma_1(u,S_u)}\right)\right] d\widehat{W}_u$$

$$+L_t^{(5)} + B_t^{(5)}.$$

Here

$$L^{(5)} := \int_0^t M_u^{3,\mathbb{F}} A_u^{\widetilde{h}} dL_u^{(3)} + \int_0^t \boldsymbol{1}_{\{u\leq\tau\}} \frac{e^{\Upsilon_u^{(1)}} D_u^{1,\mathbb{F}}}{Z_{u-}} A_{u-}^{\widetilde{h}} d\widetilde{W}_u^{(2)} \tag{4.34}$$

is a $(P,\mathbb{G})$-martingale orthogonal to $\mathbb{G}$ and thus is still a $(\widehat{P}^{\mathbb{G}},\mathbb{G})$-martingale and the $\mathbb{G}$-predictable finite variation process $B^{(5)}$ is defined by:

$$\begin{aligned} B_t^{(5)} := &\int_0^t \boldsymbol{1}_{\{u\leq\tau\}} \frac{e^{\Upsilon_u^{(1)}} D_u^{1,\mathbb{F}}}{Z_{u-}} dA_u^{\widetilde{h}} + \int_0^t M_u^{3,\mathbb{F}} A_{u-}^{\widetilde{h}} dB_t^{(3)} \\ &- \int_0^t \boldsymbol{1}_{\{u\leq\tau\}} \frac{e^{\Upsilon_u^{(1)}} D_u^{1,\mathbb{F}} \xi_u^{(3)}}{Z_{u-}} \left(\frac{\mu_1(u,S_u)}{\sigma_1(u,S_u)}\right) A_{u-}^{\widetilde{h}} du \end{aligned} \tag{4.35}$$

Proof. Firstly, due to Lemma 4.10, we can calculate $[Je^{\Upsilon^{(1)}} D^{1,\mathbb{F}} M^{3,\mathbb{F}}, A^{\widetilde{h}}]$ as follows:

$$[Je^{\Upsilon^{(1)}} D^{1,\mathbb{F}} M^{3,\mathbb{F}}, A^{\widetilde{h}}]_t = -\int_0^t \mathbf{1}_{\{u\leq\tau\}} \frac{e^{\Upsilon_u^{(1)}} D_u^{1,\mathbb{F}}}{Z_{u-}} \Delta A_u^{\widetilde{h}} dL_u$$

Secondly, an application of the integration-by-parts formula leads to :

$$\begin{aligned} J_t e^{\Upsilon_t^{(1)}} D_t^{1,\mathbb{F}} M_t^{3,\mathbb{F}} A_t^{\widetilde{h}} = &\int_0^t A_{u-}^{\widetilde{h}} d(J_u e^{\Upsilon_u^{(1)}} D_u^{1,\mathbb{F}} M_u^{3,\mathbb{F}}) + \int_0^t J_{u-} e^{\Upsilon_u^{(1)}} D_u^{1,\mathbb{F}} M_u^{3,\mathbb{F}} dA_u^{\widetilde{h}} \\ &+ [Je^{\Upsilon^{(1)}} D^{1,\mathbb{F}} M^{3,\mathbb{F}}, A^{\widetilde{h}}]_t. \end{aligned}$$

This ends the proof of lemma. □

Then, by combining Lemma 4.12 and Lemma 4.13 in the following, we derive the final decomposition of term $(B) = J_t e^{\Upsilon_t^{(1)}} M_t^{\mathbb{F}} E^{\widehat{P}^{1,\mathbb{F}}} \left[\int_t^T \frac{\widetilde{h}_u}{D_u^{2,\mathbb{F}}} dA_u | \mathcal{F}_t\right]$, which is described in (4.22).

Theorem 4.5: *Consider the martingale representations of $M^{3,\mathbb{F}}$ and $M^{4,\mathbb{F}}$:*

$$M_t^{3,\mathbb{F}} = M_0^{3,\mathbb{F}} + \int_0^t \xi_u^{(3)} d\widehat{W}_u^{(1)} + \int_0^t \eta_u^{(3)} dW_u^{(2)},$$

$$M_t^{4,\mathbb{F}} = M_0^{4,\mathbb{F}} + \int_0^t \xi_u^{(4)} d\widehat{W}_u^{(1)} + \int_0^t \eta_u^{(4)} dW_u^{(2)}.$$

Then the process $(B) = J_t e^{\Upsilon_t^{(1)}} M_t^{\mathbb{F}} E^{\widehat{P}^{1,\mathbb{F}}} \left[\int_t^T \frac{\widetilde{h}_u}{D_u^{2,\mathbb{F}}} dA_u | \mathcal{F}_t \right]$ *admits the decomposition:*

$$\begin{aligned} J_t e^{\Upsilon_t^{(1)}} M_t^{\mathbb{F}} E^{\widehat{P}^{1,\mathbb{F}}} \left[\int_t^T \frac{\widetilde{h}_u}{D_u^{2,\mathbb{F}}} dA_u | \mathcal{F}_t \right] =& J_0 M_0^{3,\mathbb{F}} M_0^{4,\mathbb{F}} \\ &+ \int_0^t \mathbf{1}_{\{u \leq \tau\}} \frac{e^{\Upsilon_u^{(1)}} D_u^{1,\mathbb{F}}}{Z_{u-}} \widetilde{\xi}_u d\widehat{W}_u \\ &+ L_t^{(4)} - L_t^{(5)} + B_t^{(4)} - B_t^{(5)} \end{aligned}$$

where the $\mathbb{G}$*-predictable process* $\widetilde{\xi}$ *is defined by*

$$\begin{aligned} \widetilde{\xi}_t :=& \left[\xi_t^{(3)} M_t^{4,\mathbb{F}} + \xi_t^{(4)} M_t^{3,\mathbb{F}} - \frac{\mu_1(t, S_t)}{\sigma_1(t, S_t)} \left(M_t^{3,\mathbb{F}} + M_t^{4,\mathbb{F}} + \int_0^t (\xi_u^{(3)} \xi_u^{(4)} + \eta_u^{(3)} \eta_u^{(4)}) du \right) \right] \\ &- A_{t-}^{\widetilde{h}} \left[\xi_t^{(3)} - M_t^{3,\mathbb{F}} \left(\frac{\mu_1(t, S_t)}{\sigma_1(t, S_t)} \right) \right] \end{aligned} \tag{4.36}$$

and the processes $L^{(4)}$, $L^{(5)}$, $B^{(4)}$ *and* $B^{(5)}$ *are defined in (4.32),(4.34),(4.33) and (4.35), respectively.*

Now, we are in the stage of giving the final decomposition of the process $\widehat{V}^{(2)}$.

Theorem 4.6: *Consider the martingale representations of* $M^{2,\mathbb{F}}$, $M^{3,\mathbb{F}}$ *and* $M^{4,\mathbb{F}}$ *as:*

$$M_t^{2,\mathbb{F}} = M_0^{2,\mathbb{F}} + \int_0^t \xi_u^{(2)} d\widehat{W}_u^{(1)} + \int_0^t \eta_u^{(2)} dW_u^{(2)}$$

$$M_t^{3,\mathbb{F}} = M_0^{3,\mathbb{F}} + \int_0^t \xi_u^{(3)} d\widehat{W}_u^{(1)} + \int_0^t \eta_u^{(3)} dW_u^{(2)}$$

$$M_t^{4,\mathbb{F}} = M_0^{4,\mathbb{F}} + \int_0^t \xi_u^{(4)} d\widehat{W}_u^{(1)} + \int_0^t \eta_u^{(4)} dW_u^{(2)}$$

Then, the process $\widehat{V}_t^{(2)} = E^{\widehat{P}^{\mathbb{G}}}[\mathbf{1}_{\{t<\tau\leq T\}} H|\mathcal{G}_t]$ *admits the decomposition:*

$$\widehat{V}_t^{(2)} = J_0(M_0^{2,\mathbb{F}} + M_0^{3,\mathbb{F}} + M_0^{4,\mathbb{F}}) + \int_0^t \mathbf{1}_{\{u\leq\tau\}} \frac{e^{\Upsilon_u^{(1)}}}{Z_{u-}} \left[\xi_u^{(2)} + D_u^{1,\mathbb{F}} \widetilde{\xi}_u\right] \widehat{W}_u$$

$$+L_t^{(2)} + L_t^{(4)} - L_t^{(5)} + B_t^{(2)} + B_t^{(4)} - B_t^{(5)},$$

where the process $\widetilde{\xi}$ *is defined in (4.36) and* $L^{(2)}$, $L^{(4)}$, $L^{(5)}$, $B^{(2)}$,$B^{(4)}$, $B^{(5)}$ *are defined in (4.25), (4.32) ,(4.34), (4.26), (4.33) and (4.35), respectively.*

Proof. Since $\widehat{V}_t^{(2)} = (A) + (B)$, the proof follows from a combination of Theorem 4.4 and Theorem 4.6. □

4.2.3 Decomposition of $\widehat{V}^{(3)}$ and the corresponding $\mathcal{F}_T$-claim

In this subsection, we will deal with the decomposition of the process $\widehat{V}^{(3)}$. This will be achieved due to the following three lemmas.

Lemma 4.14: *The process* $\widehat{V}_t^{(3)}$ *can be written as*

$$\widehat{V}_t^{(3)} = E^{\widehat{P}^{\mathbb{G}}}[H\mathbf{1}_{\{t\geq\tau\}}|\mathcal{G}_t] = \frac{\mathbf{1}_{\{t\geq\tau\}}}{1-Z_t} e^{\Upsilon_t^{(2)}} E^{\widehat{P}^{2,\mathbb{F}}}[HE(e^{-\Upsilon_T^{(2)}} \mathbf{1}_{\{t\geq\tau\}}|\mathcal{F}_T)|\mathcal{F}_t].$$

Proof. Thanks to Theorem 2.7, we have

$$E[\xi\mathbf{1}_{\{t\geq\tau\}}|\mathcal{G}_t] = \frac{\mathbf{1}_{\{t\geq\tau\}}}{1-Z_t} E[\xi\mathbf{1}_{\{t\geq\tau\}}|\mathcal{F}_t],$$

for the random variable $\xi = \frac{D_T^{2,\mathbb{F}}}{D_\tau^{2,\mathbb{F}}} e^{-\Upsilon_T^{(2)}} \frac{D_\tau^{2,\mathbb{F}}}{D_t^{2,\mathbb{F}}} e^{\Upsilon_t^{(2)}} H$. Then, we obtain:

$$\widehat{V}_t^{(3)} = E^{\widehat{P}^{\mathbb{G}}}[H\mathbf{1}_{\{t\geq\tau\}}|\mathcal{G}_t]$$

$$= E\left[\frac{D_T^{\mathbb{G}}}{D_t^{\mathbb{G}}} H\mathbf{1}_{\{t\geq\tau\}}|\mathcal{G}_t\right]$$

$$= E\left[\frac{D_T^{\mathbb{G}}}{D_\tau^{\mathbb{G}}} \cdot \frac{D_\tau^{\mathbb{G}}}{D_t^{\mathbb{G}}} H \mathbf{1}_{\{t \geq \tau\}} | \mathcal{G}_t\right]$$

$$= E\left[\frac{D_T^{2,\mathbb{F}}}{D_\tau^{2,\mathbb{F}}} e^{-\Upsilon_T^{(2)}} \frac{D_\tau^{2,\mathbb{F}}}{D_t^{2,\mathbb{F}}} e^{\Upsilon_t^{(2)}} H \mathbf{1}_{\{t \geq \tau\}} | \mathcal{G}_t\right]$$

$$= \frac{\mathbf{1}_{\{t \geq \tau\}}}{1 - Z_t} e^{\Upsilon_t^{(2)}} E\left[\frac{D_T^{2,\mathbb{F}}}{D_t^{2,\mathbb{F}}} H E(e^{-\Upsilon_T^{(2)}} \mathbf{1}_{\{t \geq \tau\}} | \mathcal{F}_T) | \mathcal{F}_t\right]$$

$$= \frac{\mathbf{1}_{\{t \geq \tau\}}}{1 - Z_t} e^{\Upsilon_t^{(2)}} E^{\widehat{P}^{2,\mathbb{F}}}[H E(e^{-\Upsilon_T^{(2)}} \mathbf{1}_{\{t \geq \tau\}} | \mathcal{F}_T) | \mathcal{F}_t].$$

This ends the proof of the lemma.

□

Unfortunately, the process $U_t = E^{\widehat{P}^{2,\mathbb{F}}}[HE(e^{-\Upsilon_t^{(2)}} \mathbf{1}_{\{t \geq \tau\}} | \mathcal{F}_T) | \mathcal{F}_t]$ is not a $(\widehat{P}^{2,\mathbb{F}}, \mathbb{F})$-martingale. It is actually a submartingale, since for each $0 < s \leq t \leq T$,

$$\begin{aligned}
E^{\widehat{P}^{2,\mathbb{F}}}[U_t | \mathcal{F}_s] =& E^{\widehat{P}^{2,\mathbb{F}}}[E^{\widehat{P}^{2,\mathbb{F}}}[HE(e^{-\Upsilon_t^{(2)}} \mathbf{1}_{\{t \geq \tau\}} | \mathcal{F}_T) | \mathcal{F}_t] | \mathcal{F}_s] \\
\geq& E^{\widehat{P}^{2,\mathbb{F}}}[E^{\widehat{P}^{2,\mathbb{F}}}[HE(e^{-\Upsilon_T^{(2)}} \mathbf{1}_{\{s \geq \tau\}} | \mathcal{F}_T) | \mathcal{F}_t] | \mathcal{F}_s] \\
=& E^{\widehat{P}^{2,\mathbb{F}}}[HE(e^{-\Upsilon_T^{(2)}} \mathbf{1}_{\{s \geq \tau\}} | \mathcal{F}_T) | \mathcal{F}_s] = U_s.
\end{aligned}$$

Thus, the best we can do is to consider the decomposition of U as a sum of a martingale part and a predictable increasing part as follows:

$$U_t = E^{\widehat{P}^{2,\mathbb{F}}}[HE(e^{-\Upsilon_T^{(2)}} \mathbf{1}_{\{t \geq \tau\}} | \mathcal{F}_T) | \mathcal{F}_t] = M_t^{5,\mathbb{F}} + A_t^{5,\mathbb{F}}, \tag{4.37}$$

where the $M^{5,\mathbb{F}}$ is a $(\widehat{P}^{2,\mathbb{F}}, \mathbb{F})$-martingale and $A^{5,\mathbb{F}}$ is a predictable increasing process. The martingale $M^{5,\mathbb{F}}$ again leads us to consider its martingale representation. We construct the following processes whose validation are left to the readers.

$$W^{(4)} = \sqrt{1 - \rho^2} W^{(1)} - \rho W^{(2)} \Rightarrow (P, \mathbb{F})\text{-Brownian motion}$$

$$\widetilde{W}^{(4)} = \sqrt{1-\rho^2}\widetilde{W}^{(1)} - \rho\widetilde{W}^{(2)} \Rightarrow (P,\mathbb{G})\text{-Brownin motion}$$

$$h^{(4)} = \sqrt{1-\rho^2}h^{(1)} - \rho h^{(2)} \Rightarrow \mathbb{F}\text{-predictable process}$$

Basically, here we construct another two Brownian motions which are orthogonal to the driven Brownian motions of the after-change dynamics $S^{(2)}$. That is $(W^{(4)} \perp W^{(3)})^{P,\mathbb{F}}$ and $(\widetilde{W}^{(4)} \perp \widetilde{W}^{(3)})^{P,\mathbb{G}}$. Still, by Girsanov's theorem, the process

$$\widehat{W}_t^{(3)} = W^{(3)} + \int_0^t \frac{\mu_2(u,S_u)}{\sigma_2(u,S_u)}du \tag{4.38}$$

is a $(\widehat{P}^{2,\mathbb{F}},\mathbb{F})$- Brownian motion. Also, notice that $\mathbb{F} = \mathbb{F}^{W^{(1)}} \vee \mathbb{F}^{W^{(2)}} = \mathbb{F}^{W^{(3)}} \vee \mathbb{F}^{W^{(4)}}$. Then the martingale $M^{5,\mathbb{F}}$ can be written as

$$M_t^{5,\mathbb{F}} = M_0^{5,\mathbb{F}} + \int_0^t \xi_u^{(5)}\widehat{W}_u^{(3)} + \int_0^t \eta_u^{(5)}dW_u^{(4)} \tag{4.39}$$

for some predictable processes $\xi^{(5)}$ and $\eta^{(5)}$. To derive the decomposition of $\widehat{V}^{(3)}$, we also need the following two lemmas.

Lemma 4.15: *The process* $K_t = \frac{\mathbf{1}_{\{t\geq\tau\}}}{1-Z_t}$ *can be decomposed as follows*

$$K_t = K_0 + \int_0^t \frac{1}{1-Z_u}dH_u + \int_0^t \mathbf{1}_{\{u>\tau\}}\frac{1}{(1-Z_{u-})^2}d\widetilde{X}_u + C_t^{(1)},$$

where

$$C_t^{(1)} = -\int_0^t \mathbf{1}_{\{u>\tau\}}\frac{1}{(1-Z_{u-})^2}dA_u + \sum_{u\leq t}\mathbf{1}_{\{u>\tau\}}\left[\Delta\left(\frac{1}{1-Z}\right)_u - \frac{\Delta Z_u}{(1-Z_{u-})^2}\right] \tag{4.40}$$

is a $\mathbb{G}$*-predictable process with finite variation.*

Proof. First of all, the condition $Z_\tau < 1$, *P-a.s.* guarantees the process K_t is well-defined. By our convention, let $H_t = \mathbf{1}_{\{\tau\leq t\}}$, we have $[H,\frac{1}{1-Z}]_t = \int\Delta(\frac{1}{1-Z})dH$ for all $t \geq 0, P$-$a.s.$ since H is a finite variation process. Also $[Z]^c = \langle X \rangle^{P,\mathbb{F}}$. Then apply the integration-by-parts formula, we obtain:

$$
\begin{aligned}
K_t =& \frac{H_t}{1 - Z_t} \\
=& K_0 + \int_0^t H_{u-} d\left(\frac{1}{1 - Z_u}\right) + \int_0^t \frac{1}{1 - Z_{u-}} dH_u + \int_0^t \Delta \left(\frac{1}{1 - Z}\right)_u dH_u \\
=& K_0 + \int_0^t H_{u-} d\left(\frac{1}{1 - Z_u}\right) + \int_0^t \frac{1}{1 - Z_u} dH_u \\
=& K_0 + \int_0^t \mathbf{1}_{\{u>\tau\}} \frac{1}{(1 - Z_{u-})^2} dZ_u + \int_0^t \mathbf{1}_{\{u>\tau\}} \frac{1}{(1 - Z_{u-})^3} d[Z]^c_u \\
&+ \sum_{u \leq t} \mathbf{1}_{\{u>\tau\}} \left[\Delta \left(\frac{1}{1 - Z}\right)_u - \frac{\Delta Z_u}{(1 - Z_{u-})^2}\right] + \int_0^t \frac{1}{1 - Z_u} dH_u \\
=& K_0 + \int_0^t \mathbf{1}_{\{u>\tau\}} \frac{1}{(1 - Z_{u-})^2} dX_u - \int_0^t \mathbf{1}_{\{u>\tau\}} \frac{1}{(1 - Z_{u-})^2} dA_u \\
&+ \int_0^t \mathbf{1}_{\{u>\tau\}} \frac{1}{(1 - Z_{u-})^3} d\langle X \rangle^{P,\mathbb{F}}_u + \sum_{u \leq t} \mathbf{1}_{\{u>\tau\}} \left[\Delta \left(\frac{1}{1 - Z}\right)_u - \frac{\Delta Z_u}{(1 - Z_{u-})^2}\right] \\
&+ \int_0^t \frac{1}{1 - Z_u} dH_u \\
=& K_0 + \int_0^t \mathbf{1}_{\{u>\tau\}} \frac{1}{(1 - Z_{u-})^2} d\widetilde{X}_u + \int_0^t \frac{1}{1 - Z_u} dH_u + C^{(1)}_t.
\end{aligned}
$$

In the last equality we used the fact that

$$
\mathbf{1}_{\{u>\tau\}} d\widetilde{X}_u = \mathbf{1}_{\{u>\tau\}} \left(dX_u - \frac{d\langle X \rangle^{P,\mathbb{F}}_u}{1 - Z_{u-}}\right).
$$

This completes the proof of the lemma. □

Lemma 4.16: *The process* $\frac{\mathbf{1}_{\{t\geq\tau\}}}{1-Z_t}e^{\Upsilon^{(2)}_t} = K_t e^{\Upsilon^{(2)}_t}$ *can be written as:*

$$\frac{\mathbf{1}_{\{t\geq\tau\}}}{1-Z_t}e^{\Upsilon^{(2)}_t} = K_t e^{\Upsilon^{(2)}_t} = K_0 + \int_0^t \frac{e^{\Upsilon^{(2)}_u}}{1-Z_u}dH_u + \int_0^t \mathbf{1}_{\{u>\tau\}}\frac{h^{(4)}_u e^{\Upsilon^{(2)}_u}}{(1-Z_{u-})^2}d\widetilde{W}^{(4)}_u + C^{(2)}_t,$$

where

$$C^{(2)}_t = \int_0^t e^{\Upsilon^{(2)}_u} dC^{(1)}_u \tag{4.41}$$

is a $\mathbb{G}$*-predictable process with finite variation.*

Proof. Recall that the process $e^{\Upsilon^{(2)}}$ is defined by

$$e^{\Upsilon^{(2)}_t} = \exp\left\{-\int_0^t \mathbf{1}_{\{u>\tau\}}\frac{h^{(3)}_u}{1-Z_{u-}}d\widetilde{W}^{(3)}_u + \frac{1}{2}\int_0^t \mathbf{1}_{\{u>\tau\}}\left(\frac{h^{(3)}_u}{1-Z_{u-}}\right)^2 du\right\}.$$

Then, Itô's formula leads to

$$de^{\Upsilon^{(2)}_t} = -\mathbf{1}_{\{t>\tau\}}\frac{h^{(3)}_t}{1-Z_{t-}}e^{\Upsilon^{(2)}_t}d\widetilde{W}^{(3)}_t + \mathbf{1}_{\{t>\tau\}}\left(\frac{h^{(3)}_t}{1-Z_{t-}}\right)e^{\Upsilon^{(2)}_t}dt.$$

As a result, we get

$$\left[K, e^{\Upsilon^{(2)}}\right] = -\int_0^t \mathbf{1}_{\{u>\tau\}}\frac{h^{(3)}_u e^{\Upsilon^{(2)}_u}}{(1-Z_{u-})^3}d[\widetilde{X},\widetilde{W}^{(3)}]_u = -\int_0^t \mathbf{1}_{\{u>\tau\}}\frac{(h^{(3)}_u)^2 e^{\Upsilon^{(2)}_u}}{(1-Z_{u-})^3}du.$$

Thanks to integration-by-parts formula, we obtain :

$$\begin{aligned}
K_t e^{\Upsilon^{(2)}_t} =& K_0 + \int_0^t K_{t-}de^{\Upsilon^{(2)}_u} + \int_0^t e^{\Upsilon^{(2)}_u}dK_u + [K, e^{\Upsilon^{(2)}}]_t \\
=& K_0 - \int_0^t \mathbf{1}_{\{u>\tau\}}\frac{h^{(3)}_u e^{\Upsilon^{(2)}_u}}{(1-Z_{u-})^2}d\widetilde{W}^{(3)}_u + \int_0^t \mathbf{1}_{\{u>\tau\}}\frac{(h^{(3)}_u)^2 e^{\Upsilon^{(2)}_u}}{(1-Z_{u-})^3}du \\
&+ \int_0^t \mathbf{1}_{\{u>\tau\}}\frac{e^{\Upsilon^{(2)}_u}}{(1-Z_{u-})^2}d\widetilde{X}_u + \int_0^t \frac{e^{\Upsilon^{(2)}_u}}{1-Z_u}dH_u + \int_0^t e^{\Upsilon^{(2)}_u}dC^{(1)}_u \\
&- \int_0^t \mathbf{1}_{\{u>\tau\}}\frac{(h^{(3)}_u)^2 e^{\Upsilon^{(2)}_u}}{(1-Z_{u-})^3}du
\end{aligned}$$

$$
\begin{aligned}
=&K_0 - \int_0^t \mathbf{1}_{\{u>\tau\}} \frac{h_u^{(3)} e^{\Upsilon_u^{(2)}}}{(1-Z_{u-})^2} d\widetilde{W}_u^{(3)} + \int_0^t \mathbf{1}_{\{u>\tau\}} \frac{e^{\Upsilon_u^{(2)}}}{(1-Z_{u-})^2} d\widetilde{X}_u \\
&+ \int_0^t e^{\Upsilon_u^{(2)}} dC_u^{(1)} + \int_0^t \frac{e^{\Upsilon_u^{(2)}}}{1-Z_u} dH_u \\
=&K_0 + \int_0^t \frac{e^{\Upsilon_u^{(2)}}}{1-Z_u} dH_u + \int_0^t \mathbf{1}_{\{u>\tau\}} \frac{h_u^{(4)} e^{\Upsilon_u^{(2)}}}{(1-Z_{u-})^2} d\widetilde{W}_u^{(4)} + C_t^{(2)}.
\end{aligned}
$$

In the last equality, we used the fact that

$$
d\widetilde{X}_u - h_u^{(3)} d\widetilde{W}_u^{(3)} = h_u^{(4)} d\widetilde{W}_u^{(4)}.
$$

This completes the proof of the lemma. □

Now, we can apply the martingale representations of $M^{5,\mathbb{F}}$ to find the decomposition of $\widehat{V}^3$.

Theorem 4.7: *Consider the martingale representation of the $(\widehat{P}^{2,\mathbb{F}}, \mathbb{F})$-martingale $M_t^{5,\mathbb{F}}$ defined in (4.37) as:*

$$
M_t^{5,\mathbb{F}} = M_0^{5,\mathbb{F}} + \int_0^t \xi_u^{(5)} \widehat{W}_u^{(3)} + \int_0^t \eta_u^{(5)} dW_u^{(4)}
$$

which means $U_t = E^{\widehat{P}^{2,\mathbb{F}}}[HE(e^{-\Upsilon_T^{(2)}} \mathbf{1}_{\{t\geq\tau\}} | \mathcal{F}_T) | \mathcal{F}_t]$ admits the decomposition:

$$
U_t = E^{\widehat{P}^{2,\mathbb{F}}}[HE(e^{-\Upsilon_T^{(2)}} \mathbf{1}_{\{t\geq\tau\}} | \mathcal{F}_T) | \mathcal{F}_t] = U_0 + \int_0^t \xi_u^{(5)} \widehat{W}_u^{(3)} + \int_0^t \eta_u^{(5)} dW_u^{(4)} + A_t^{5,\mathbb{F}}
$$

Then , $\widehat{V}_t^{(3)} = \frac{\mathbf{1}_{\{t\geq\tau\}}}{1-Z_t} e^{\Upsilon_t^{(2)}} E^{\widehat{P}^{2,\mathbb{F}}}[HE(e^{-\Upsilon_T^{(2)}} \mathbf{1}_{\{t\geq\tau\}} | \mathcal{F}_T) | \mathcal{F}_t] = K_t e^{\Upsilon_t^{(2)}} U_t$ can be written as:

$$
\begin{aligned}
K_t e^{\Upsilon_t^{(2)}} U_t =& \frac{\mathbf{1}_{\{t\geq\tau\}}}{1-Z_t} e^{\Upsilon_t^{(2)}} E^{\widehat{P}^{2,\mathbb{F}}}(He^{-\Upsilon_T^{(2)}} \mathbf{1}_{\{t\geq\tau\}} | \mathcal{F}_t) \\
=& K_0 U_0 + \int_0^t \mathbf{1}_{\{u>\tau\}} \frac{e^{\Upsilon_u^{(2)}}}{1-Z_{u-}} \xi_u^{(5)} d\widehat{W}_u + L_t^{(6)} + C_t^{(3)},
\end{aligned}
\tag{4.42}
$$

where

$$\widehat{W}_t = \widetilde{W}_t + \int_0^t \frac{\mu_u}{V_u} du \text{ is a } (\widehat{P}^{\mathbb{G}}, \mathbb{G})\text{-Brownian motion by Girsanov's theorem,}$$

and

$$L_t^{(6)} = \int_0^t \mathbf{1}_{\{u>\tau\}} \frac{e^{\Upsilon_u^{(2)}}}{1-Z_{u-}} \left(\eta_u^{(5)} + \frac{h_u^{(4)}}{1-Z_{u-}} U_{u-} \right) d\widetilde{W}_u^{(4)} + \int_0^t e^{\Upsilon_u^{(2)}} \widetilde{U}_u dL_u \tag{4.43}$$

is a $(P, \mathbb{G})$*-martingale strongly orthogonal to* $M^{\mathbb{G}}$ *and thus is still a* $(\widehat{P}^{\mathbb{G}}, \mathbb{G})$*-martingale, and the* $\mathbb{G}$*-predictable finite process* $C^{(3)}$ *is defined by:*

$$C_t^{(3)} = \int_0^t \mathbf{1}_{\{u\le\tau\}} \frac{e^{\Upsilon_u^{(2)}} \widetilde{U}_u}{Z_{u-}} dA_u + \int_0^t \mathbf{1}_{\{u>\tau\}} \frac{e^{\Upsilon_u^{(2)}}}{1-Z_{u-}} dA_u^{5,\mathbb{F}} + \int_0^t U_u dC_u^{(2)}, \tag{4.44}$$

with $\widetilde{U}_t = E^{\widehat{P}^{2,\mathbb{F}}}\left[HE\left[\frac{e^{-\Upsilon_T^2} \mathbf{1}_{\{t\le\tau\}}}{1-Z_t} | \mathcal{F}_T \right] | \mathcal{F}_t \right]$.

Proof. Firstly, we calculate $[M^{5,\mathbb{F}}, Ke^{\Upsilon^{(2)}}]$:

$$\begin{aligned}
[Ke^{\Upsilon^{(2)}}, M^{5,\mathbb{F}}]_t &= \int_0^t \mathbf{1}_{\{u>\tau\}} \frac{h_u^{(4)} e^{\Upsilon_u^{(2)}}}{(1-Z_{u-})^2} \xi_u^{(5)} \underbrace{d[\widetilde{W}^{(4)}, \widehat{W}^{(3)}]_u}_{d[W^{(4)}, W^{(3)}]_u = 0} \\
&+ \int_0^t \mathbf{1}_{\{u>\tau\}} \frac{h_u^{(4)} e^{\Upsilon_u^{(2)}}}{(1-Z_{u-})^2} \eta_u^{(5)} \underbrace{d[\widetilde{W}^{(4)}, W^{(4)}]_u}_{du} \\
&= \int_0^t \mathbf{1}_{\{u>\tau\}} \frac{h_u^{(4)} e^{\Upsilon_u^{(2)}} \eta_u^{(5)}}{(1-Z_{u-})^2} du.
\end{aligned}$$

Secondly, we calculate $[Ke^{\Upsilon^{(2)}}_, A^{5,\mathbb{F}}]$

$$[Ke^{\Upsilon^{(2)}}_, A^{5,\mathbb{F}}]_t = \int_0^t \frac{e^{\Upsilon^{(2)}_u}}{1-Z_u}\Delta A^{5,\mathbb{F}}_u dH_u + \int_0^t \Delta A^{5,\mathbb{F}}_u dC^{(2)}_u.$$

Therefore

$$\begin{aligned}
[Ke^{\Upsilon^{(2)}}, U] =& [Ke^{\Upsilon^{(2)}}, M^{5,\mathbb{F}}] + [Ke^{\Upsilon^{(2)}}, A^{5,\mathbb{F}}] \\
=& \int_0^t \mathbf{1}_{\{u>\tau\}} \frac{h^{(4)}_u e^{\Upsilon^{(2)}_u} \eta^{(5)}_u}{(1-Z_{u-})^2} du + \int_0^t \frac{e^{\Upsilon^{(2)}_u}}{1-Z_u}\Delta A^{5,\mathbb{F}}_u dH_u + \int_0^t \Delta A^{5,\mathbb{F}}_u dC^{(2)}_u
\end{aligned}$$

Then, apply the integration-by-parts formula, we obtain:

$$\begin{aligned}
K_t e^{\Upsilon^{(2)}_t} U_t =& \frac{\mathbf{1}_{\{t\geq\tau\}}}{1-Z_t} e^{\Upsilon^{(2)}_t} U_t \\
=& K_0 U_0 + \int_0^t U_{u-} d(K_u e^{\Upsilon^{(2)}_u}) + \int_0^t K_{u-} e^{\Upsilon^{(2)}_u} dU_u + [U, Ke^{\Upsilon^{(2)}}]_u \\
=& K_0 U_0 + \int_0^t \frac{e^{\Upsilon^{(2)}_u}}{1-Z_u} U_{u-} dH_u + \int_0^t \mathbf{1}_{\{u>\tau\}} \frac{h^{(4)}_u e^{\Upsilon^{(2)}_u}}{(1-Z_{u-})^2} U_{u-} d\widetilde{W}^{(4)}_u \\
&+ \int_0^t U_{u-} dC^{(2)}_u + \int_0^t \mathbf{1}_{\{u>\tau\}} \frac{e^{\Upsilon^{(2)}_u}\xi^{(5)}_u}{1-Z_{u-}} d\widehat{W}^{(3)}_u + \int_0^t \mathbf{1}_{\{u>\tau\}} \frac{e^{\Upsilon^{(2)}_u}}{1-Z_{u-}} dA^{5,\mathbb{F}}_u \\
&+ \underbrace{\int_0^t \mathbf{1}_{\{u>\tau\}} \frac{e^{\Upsilon^{(2)}_u}\eta^{(5)}_u}{1-Z_{u-}} dW^{(4)}_u + \int_0^t \mathbf{1}_{\{u>\tau\}} \frac{h^{(4)}_u e^{\Upsilon^{(2)}_u}\eta^{(5)}_u}{(1-Z_{u-})^2} du}_{\int_0^t \mathbf{1}_{\{u>\tau\}} \frac{e^{\Upsilon^{(2)}_u}\eta^{(5)}_u}{1-Z_{u-}} d\widetilde{W}^{(4)}_u} \\
&+ \int_0^t \frac{e^{\Upsilon^{(2)}_u}}{1-Z_u}\Delta A^{5,\mathbb{F}}_u dH_u + \int_0^t \Delta A^{5,\mathbb{F}}_u dC^{(2)}_u \\
=& K_0 U_0 + \int_0^t \mathbf{1}_{\{u>\tau\}} \frac{e^{\Upsilon^{(2)}_u}}{1-Z_{u-}} \xi^{(5)}_u d\widehat{W}^{(3)}_u + \int_0^t e^{\Upsilon^{(2)}_u} \widetilde{U}_u dL_u + \int_0^t \mathbf{1}_{\{u\leq\tau\}} \frac{e^{\Upsilon^{(2)}_u}\widetilde{U}_u}{Z_{u-}} dA_u \\
&+ \int_0^t \frac{e^{\Upsilon^{(2)}_u}}{1-Z_{u-}} \mathbf{1}_{\{u>\tau\}} \left(\eta^{(5)}_u + \frac{h^{(4)}_u}{1-Z_{u-}} U_{u-}\right) d\widetilde{W}^{(4)}_u + \int_0^t \mathbf{1}_{\{u>\tau\}} \frac{e^{\Upsilon^{(2)}_u}}{1-Z_{u-}} dA^{5,\mathbb{F}}_u
\end{aligned}$$

$$+\int_0^t U_u dC_u^{(2)}$$

$$=K_0U_0+\int_0^t \mathbf{1}_{\{u>\tau\}}\frac{e^{\Upsilon_u^{(2)}}}{1-Z_{u-}}\xi_u^{(5)}d\widehat{W}_u+L_t^{(6)}+C_t^{(3)}.$$

In the last two equalities, we first used the fact that $U=\Delta U+U_-=\Delta A^{5,\mathbb{F}}+U_-$ and denoted $\widetilde{U}_t=E^{\widehat{P}^{2,\mathbb{F}}}\left[HE\left[\frac{e^{-\Upsilon_T^2}\mathbf{1}_{\{t\le\tau\}}}{1-Z_t}|\mathcal{F}_T\right]|\mathcal{F}_t\right]$, which is well-defined for all $t\in[0,T]$ since $Z_\tau<1, P$-$a.s.$. Therefore, the process $\int_0^t e^{\Upsilon_u^{(2)}}\widetilde{U}_u dH_u$ is locally integrable and thus has a compensator, which leads to

$$\int_0^t e^{\Upsilon_u^{(2)}}\widetilde{U}dH_u=\int_0^t e^{\Upsilon_u^{(2)}}dL_u+\int_0^t \mathbf{1}_{\{u\le\tau\}}\frac{e^{\Upsilon_u^{(2)}}\widetilde{U}_u}{Z_{u-}}dA_u.$$

Secondly, we used the fact that

$$\begin{aligned}\mathbf{1}_{\{u>\tau\}}d\widehat{W}_u=&\mathbf{1}_{\{u>\tau\}}d\widetilde{W}_u+\mathbf{1}_{\{u>\tau\}}\frac{\mu_u}{V_u}du\\
=&\mathbf{1}_{\{u>\tau\}}d\widetilde{W}_u^{(3)}+\mathbf{1}_{\{u>\tau\}}\left(\frac{\mu_2(u,S_u)}{\sigma_2(u,S_u)}+\theta_u^{(3)}\right)du\\
=&\mathbf{1}_{\{u>\tau\}}[dW_u^{(3)}-\theta_u^{(3)}du]+\mathbf{1}_{\{u>\tau\}}\left(\frac{\mu_2(u,S_u)}{\sigma_2(u,S_u)}+\theta_u^{(3)}\right)du\\
=&\mathbf{1}_{\{u>\tau\}}dW_u^{(3)}+\mathbf{1}_{\{u>\tau\}}\frac{\mu_2(u,S_u)}{\sigma_2(u,S_u)}du\\
=&\mathbf{1}_{\{u>\tau\}}d\widehat{W}_u^{(3)}.\end{aligned}$$

This finishes the proof of (4.42). Finally, we prove that $L^{(6)}$ is $(P,\mathbb{G})$- strongly orthogonal to $M^{\mathbb{G}}$. Since $M^{\mathbb{G}}$ is continuous and recall that $\widetilde{W}^{(4)}=\sqrt{1-\rho^2}\widetilde{W}^{(1)}-\rho\widetilde{W}^{(2)}$, then we have:

$$\langle L^{(6)},M^{\mathbb{G}}\rangle_t^{P,\mathbb{G}}=[L^{(6)},M^{\mathbb{G}}]_t$$

$$= \int_0^t \mathbf{1}_{\{u>\tau\}} \frac{e^{\Upsilon_u^{(2)}}}{1 - Z_{u-}} \left(\eta_u^{(5)} + \frac{h^{(4)}}{1 - Z_{u-}} M_u^{3,\mathbb{F}} \right) V_u S_u d[\widetilde{W}^{(4)}, M^{\mathbb{G}}]_u$$

$$= \int_0^t \mathbf{1}_{\{u>\tau\}} \frac{e^{\Upsilon_u^{(2)}}}{1 - Z_{u-}} \left(\eta_u^{(5)} + \frac{h^{(4)}}{1 - Z_{u-}} M_u^{3,\mathbb{F}} \right) V_u S_u \mathbf{1}_{\{u \leq \tau\}} \sqrt{1-\rho^2} d[\widetilde{W}^{(1)}, \widetilde{W}^{(1)}]_u$$

$$+ \int_0^t \mathbf{1}_{\{u>\tau\}} \frac{e^{\Upsilon_u^{(2)}}}{1 - Z_{u-}} \left(\eta_u^{(5)} + \frac{h^{(4)}}{1 - Z_{u-}} M_u^{3,\mathbb{F}} \right) V_u S_u \mathbf{1}_{\{u>\tau\}} \rho \sqrt{1-\rho^2} d[\widetilde{W}^{(1)}, \widetilde{W}^{(1)}]_u$$

$$- \int_0^t \mathbf{1}_{\{u>\tau\}} \frac{e^{\Upsilon_u^{(2)}}}{1 - Z_{u-}} \left(\eta_u^{(5)} + \frac{h^{(4)}}{1 - Z_{u-}} M_u^{3,\mathbb{F}} \right) V_u S_u \mathbf{1}_{\{u>\tau\}} \rho \sqrt{1-\rho^2} d[\widetilde{W}^{(2)}, \widetilde{W}^{(2)}]_u$$

$$=0.$$

In the last equality we use the fact that $d[\widetilde{W}^{(4)}, \widetilde{W}]_u = \alpha_u \sqrt{1-\rho^2} - \rho\beta_u = \sqrt{1-\rho^2}\mathbf{1}_{\{u \leq \tau\}}$ and $\mathbf{1}_{\{u \leq \tau\}}\mathbf{1}_{\{u>\tau\}} = 0$ for all $t \geq 0$. Therefore, $L^{(6)}$ and $M^{\mathbb{G}}$ are $(P, \mathbb{G})$-mutually orthogonal and thus by definition of the minimal martingale measure $\widehat{P}^{\mathbb{G}}$, we know that $L^{(6)}$ is still a $(\widehat{P}^{\mathbb{G}}, \mathbb{G})$-martingale and this ends the proof of the theorem □

4.2.4 The Galtchouk-Kunita-Watanabe Decomposition of H

In this subsection, we combine the decompositions of $\widehat{V}^{(1)}$, $\widehat{V}^{(2)}$ and $\widehat{V}^{(3)}$ and get the Galtchouk-Kunita-Watanabe decomposition of H with respect to the $\mathbb{G}$-dynamic of the discounted asset price S under the minimal martingale measure $\widehat{P}^{\mathbb{G}}$.

We denote by $\widehat{W}$ the $(\widehat{P}^{\mathbb{G}}, \mathbb{G})$-version of the $(P, \mathbb{G})$-Brownian motion $\widetilde{W}$*. The $\mathbb{G}$-discounted asset price dynamic under $\widehat{P}^{\mathbb{G}}$ can be written as

$$dS_t = V_t S_t d\widehat{W}_t.$$

Theorem 4.8: *The Galtchouk-Kunita-Watanabe decomposition of H with respect*

*By Girsanov's theorem $\widehat{W}_t = \widetilde{W}_t + \int_0^t \frac{\mu_u}{V_u} du$ is a $(\widehat{P}^{\mathbb{G}}, \mathbb{G})$-Brownian motion *to S under the minimal martingale measure $\widehat{P}^{\mathbb{G}}$ is*

$$H = H_0 + \int_0^T \xi_t^H dS_t + L_T^H,$$

where

$$\xi_t^H = \frac{\mathbf{1}_{\{t \le \tau\}} \frac{e^{\Upsilon_t^{(1)}}}{Z_{t-}}(\xi_t^{(1)} + \xi_t^{(2)} + D_t^{1,\mathbb{F}} \widetilde{\xi}_t) + \mathbf{1}_{\{t > \tau\}} \frac{e^{\Upsilon_t^{(2)}}}{1 - Z_{t-}} \xi_t^{(5)}}{V_t S_t}, \tag{4.45}$$

the processes $\xi^{(1)}$, $\xi^{(2)}$ and $\widetilde{\xi}$ are defined (4.14),(4.23) and (4.36) respectively,

$$L^H := L^{(1)} + L^{(2)} + L^{(4)} - L^{(5)} + L^{(6)}, \tag{4.46}$$

and the processes $L^{(1)}, L^{(2)}, L^{(4)}$, $L^{(5)}$ and $L^{(6)}$ are defined in (4.18),(4.25),(4.32), (4.34) and (4.43) respectively.

Proof. On the one hand we know the $(\widehat{P}^{\mathbb{G}}, \mathbb{G})$-martingale $\widehat{V}_t = E^{\widehat{P}^{\mathbb{G}}}[H|\mathcal{G}_t]$ admits the unique Galtchouk-Kunita-Watanabe decomposition with respect to S, that is

$$\widehat{V}_t^{(3)} = E^{\widehat{P}^{\mathbb{G}}}[H] + \int_0^t \psi^H dS_t + L_t^H.$$

On the other hand, due to $\widehat{V}_t = \widehat{V}_t^{(1)} + \widehat{V}_t^{(2)} + \widehat{V}_t^{(3)}$, we combine Theorem 4.3, Theorem 4.6, Theorem 4.7 and the uniqueness of Galtchouk-Kunita-Watanabe decomposition of H, and we conclude that the sum of predictable terms with finite variation should vanish, that is:

$$B_t^{(2)} + B_t^{(4)} - B_t^{(5)} + C_t^{(3)} = 0.$$

Finally, the proof of the theorem is achieved since $dS = VSd\widehat{W}$. □

Remark 4.1: We remark that the martingale representations of the martingales $M^{1,\mathbb{F}}, M^{2,\mathbb{F}}, M^{3,\mathbb{F}}, M^{4,\mathbb{F}}$ and $M^{5,\mathbb{F}}$ actually give us the Galtchouk-Kunita-Watanabe decompositions of five $\mathbb{F}_T$-measurable contingent claims with respect $S^{(1)}$ and $S^{(2)}$

to under the corresponding minimal martingale measures. That is

$$\frac{\xi_t^{(1)}}{\sigma_1(t,S_t)S_t} \Rightarrow \mathbb{F}\text{-LRM strategy for } H_1 = E[He^{-\Upsilon_T^{(1)}}\mathbf{1}_{\{T<\tau\}}|\mathcal{F}_T]$$

$$\frac{\xi_t^{(2)}}{\sigma_1(t,S_t)S_t} \Rightarrow \mathbb{F}\text{-LRM strategy for } H_2 = \int_0^T \frac{M_u^{\mathbb{F}}\widetilde{h}_u}{D_u^{2,\mathbb{F}}} dA_u$$

$$\frac{\xi_t^{(3)}}{\sigma_1(t,S_t)S_t} \Rightarrow \mathbb{F}\text{-LRM strategy for } H_3 = \frac{D_T^{2,\mathbb{F}}}{D_T^{1,\mathbb{F}}} H$$

$$\frac{\xi_t^{(4)}}{\sigma_1(t,S_t)S_t} \Rightarrow \mathbb{F}\text{-LRM strategy for } H_4 = \int_0^T \frac{\widetilde{h}_u}{D_u^{2,\mathbb{F}}} dA_u$$

$$\frac{\xi_t^5}{\sigma_2(t,S_t)S_t} \Rightarrow \mathbb{F}\text{-LRM strategy for } H_5 = M_T^{5,\mathbb{F}}.$$

Since the global discounted price dynamic S is a continuous $(P,\mathbb{G})$-semimartingale, then the strategy ξ^H defined in (4.45) is exactly the $\mathbb{G}$-LRM strategy for H by trading S. This $\mathbb{G}$-strategy can be interpreted in this way:

1 Follow the strategy $\frac{e^{\Upsilon^{(1)}}(\xi^{(1)}+\xi^{(2)}+D^{1,\mathbb{F}}\widetilde{\xi})}{Z_-\sigma^1 S^{(1)}}$ before the change-point event occurs

2 Switch to $\frac{e^{\Upsilon^{(2)}}\xi^{(5)}}{(1-Z_-)\sigma^2 S^{(2)}}$ after the change-point event occurs

This tells us that an investor who is equipped with the knowledge of the markets $(S^{(1)},\mathbb{F})$, $(S^{(2)},\mathbb{F})$ and the behavior of the change point τ can hedge a given contingent claim H under the market $(S,\mathbb{G})$ (in the sense of local risk-minimization) based on the hedging strategies using information of $\mathbb{F}$ for five different contingent claims related to H.

Chapter 5

Application to Default Sensitive Contingent Claims

In this chapter, we apply the results derived in Chapter 4 to a specific class of contingent claims. These claims are called **default sensitive contingent claims** which are mostly involved with insurance contracts. The change-point (random time) τ in our model can be regarded as a default time or a surrender time which are assumed to be a stopping time in the traditional actuarial literature (see [4, 42, 44] for examples). However, here we only assume τ is a random time that satisfies (3.6) (i.e. τ is honest and $Z_\tau < 1, P$-$a.s.$). We now describe these default sensitive contingent claims as the insurer's payments in certain insurance contract. A plentiful variety of life insurance contracts can be modelled as a combination of the following three types of insurer's payments which we wish to hedge.

(1) The discounted payoff that the insurer has to pay at the end of the contract term (fixed time horizon T) is an $\mathcal{F}_T$-measurable P-square integrable random variable and is denoted by $g(T,\omega)$. At time T, the insurer has to pay

$$g(T,\omega)\mathbf{1}_{\{\tau>T\}}.$$

(2) The discounted amount that the insurer has to pay when the policyholder surrenders before T which is denoted by $\mathbf{1}_{\{0<\tau\leq T\}}R(\tau,\omega)$ with $(R(t,\omega))_{0\leq t\leq T}$ being an $\mathbb{F}$-predictable bounded process. That is

$$\mathbf{1}_{\{0<\tau\leq T\}}R(\tau,\omega)=\int_0^T R(u,\omega)dH_u,$$

where $H_t=\mathbf{1}_{\{t\leq\tau\}}$.

(3) The discounted payoff that the insurer has to pay as long as the policyholder has not surrendered. These payoffs can be modelled through their cumulative value up to time t, denoted by $C(t,\omega)$. We assume $C(t,\omega)$ is a right-continuous increasing $\mathbb{F}$-adapted process. Then, the cumulative payoff up to surrender is then given by :

$$C(T,\omega)\mathbf{1}_{\{\tau>T\}}+C(\tau_-,\omega)\mathbf{1}_{\{0<\tau\leq T\}}=\int_0^T(1-H_u)dC(u,\omega)$$

where $H_t=\mathbf{1}_{\{t\leq\tau\}}$. Also, we assume $C(0,\omega)=0$ and $C(t,\omega)=C(t_-,\omega)$.

5.1 Payment at Maturity of the Contract

For the first type of insurer's payments, the value process under the minimal martingale measure $\widehat{P}^{\mathbb{G}}$ can be obtained by replacing H in (4.11) with $g_T\mathbf{1}_{\{\tau>T\}}=g(T,\omega)\mathbf{1}_{\{\tau>T\}}$. That is

$$\widehat{V}_t^g=E^{\widehat{P}^{\mathbb{G}}}[g_T\mathbf{1}_{\{\tau>T\}}|\mathcal{G}_t].$$

In this case, the process $\widehat{V}_t^g$ is similar to the process $\widehat{V}^{(1)}$ defined in (4.11), then we can apply Theorem 4.3 to get the $\mathbb{G}$-LRM strategy for $g(T,\omega)\mathbf{1}_{\tau>T}$. Recall

$$\begin{aligned}\widehat{V}_t^g&=\frac{\mathbf{1}_{\{t<\tau\}}}{Z_t}e^{\Upsilon_t^{(1)}}E^{\widehat{P}^{1,\mathbb{F}}}[g_Te^{-\Upsilon_T^{(1)}}\mathbf{1}_{\{\tau>T\}}|\mathcal{F}_t]\\&=\frac{\mathbf{1}_{\{t<\tau\}}}{Z_t}e^{\Upsilon_t^{(1)}}E^{\widehat{P}^{1,\mathbb{F}}}[E^{\widehat{P}^{1,\mathbb{F}}}[g_Te^{-\Upsilon_T^{(1)}}\mathbf{1}_{\{\tau>T\}}|\mathcal{F}_T]|\mathcal{F}_t].\end{aligned}$$

Then, we have $U_t^g = E^{\widehat{P}^{1,\mathbb{F}}}[E^{\widehat{P}^{1,\mathbb{F}}}[g_T e^{-\Upsilon_T^{(1)}} \mathbf{1}_{\{T<\tau\}} | \mathcal{F}_T | \mathcal{F}_t] \in \mathcal{M}^2(\widehat{P}^{1,\mathbb{F}}, \mathbb{F})$. So the Föllmer-Schweizer decomposition of $g_T \mathbf{1}_{\{\tau > T\}}$ under with respect to S is given in the following theorem.

Theorem 5.1: *For $g_T = g(T, \omega)$ being a F_T-measurable and P-square integrable random variable, suppose $U_t^g = E^{\widehat{P}^{1,\mathbb{F}}}[E^{\widehat{P}^{1,\mathbb{F}}}[g_T e^{-\Upsilon_T^{(1)}} \mathbf{1}_{\{T<\tau\}} | \mathcal{F}_T] | \mathcal{F}_t]$ admits the martingale representation*

$$U_t^g = U_0^g + \int_0^t \phi_u^{(1)} d\widehat{W}_u^{(1)} + \int_0^t \phi_u^{(2)} dW_u^{(2)}. \tag{5.1}$$

Then the $(P, \mathbb{G})$-Föllmer-Schweizer decomposition of $g_T \mathbf{1}_{\{\tau > T\}}$ is given by

$$g_T \mathbf{1}_{\{\tau > T\}} = U_0^g + \int_0^T \mathbf{1}_{\{u \le \tau\}} \frac{e^{\Upsilon_u^{(1)}} \phi_u^{(1)}}{Z_{u-} \sigma_1(u, S_u) S_u^{(1)}} dS_u^{(1)} + L_T^g$$

where

$$L_t^g = \int_0^t \mathbf{1}_{\{u \le \tau\}} \frac{e^{\Upsilon_u^{(1)}}}{Z_{u-}} \left(\phi_u^{(2)} - \frac{h_u^{(2)}}{Z_{u-}} U_u^g \right) d\widetilde{W}_u^{(2)} - \int_0^t U_u^g \frac{e^{\Upsilon_u^{(1)}}}{Z_u} dL_u{}^*$$

is a $(P, \mathbb{G})$ martingale orthogonal to the martingale part of $S^{(1)}$.

Proof. see proof of Theorem 4.3 □

Remark 5.1: The martingale representation of (5.1) can also be written as

$$U_t^g = U_0^g + \int_0^t \frac{\phi_u^{(1)}}{\sigma_1(u, S_u) S_u} dS_u^{(1)} + \int_0^t \phi_u^{(2)} dW_u^{(2)}$$

where $S_t^{(1)} = \sigma_1(t, S_t) S_t d\widehat{W}_t^{(1)}$ is a $(\widehat{P}^{1,\mathbb{F}}, \mathbb{F})$-martingale via Girsanov's theorem. Because $W^{(2)}$ is orthogonal to $W^{(1)}$, this represents the $(\widehat{P}^{1,\mathbb{F}}, \mathbb{F})$-GKW decomposition of the $\mathcal{F}_T$-measurable, P-square integrable random variable $E[g_T e^{-\Upsilon_T^{(1)}} \mathbf{1}_{\{T<\tau\}} | \mathcal{F}_T]$ with respect to $S^{(1)}$ (the before-change dynamic). Therefore, the $\mathbb{F}$-LRM strategy

*Here L is defined in (4.15)

for the contingent claim $f = E[g_T e^{-\Upsilon_T^{(1)}} \mathbf{1}_{\{T<\tau\}} | \mathcal{F}_T]$ based on trading $S^{(1)}$ is given by

$$\xi_t^f = \frac{\phi_t^{(1)}}{\sigma_1(t, S_t) S_t}.$$

Whereas, the $\mathbb{G}$-LRM strategy for the original contingent claim $g_T \mathbf{1}_{\{\tau>T\}}$ is given by

$$\xi_t^g = \mathbf{1}_{\{t\leq\tau\}} \frac{e^{\Upsilon_t^{(1)}}}{Z_{t-}} \xi_t^f.$$

5.2 Payment at Surrender Time

The second type of discounted payoffs is the payments at the surrender time. We have the value process under minimal martingale measure $\widehat{P}^{\mathbb{G}}$ as : With $R_t = R(t, \omega)$

$$\begin{aligned}
\widehat{V}_t^R &= E^{\widehat{P}^{\mathbb{G}}} \left[\int_0^T R_u dH_u \mid \mathcal{G}_t \right] = E^{\widehat{P}^{\mathbb{G}}} \left[\int_0^t R_u dH_u + \int_t^T R_u dH_u | \mathcal{G}_t \right] \\
&= \int_0^t R_u dH_u + E^{\widehat{P}^{\mathbb{G}}} \left[\mathbf{1}_{\{t<\tau\leq T\}} R_\tau | \mathcal{G}_t \right] \\
&= \int_0^t R_u dL_u + \int_0^t \mathbf{1}_{\{u\leq\tau\}} \frac{R_u}{Z_{u-}} dA_u + E^{\widehat{P}^{\mathbb{G}}} \left[\mathbf{1}_{\{t<\tau\leq T\}} R_\tau | \mathcal{G}_t \right].
\end{aligned}$$

The process $E^{\widehat{P}^{\mathbb{G}}}[\mathbf{1}_{\{t<\tau\leq T\}} R_\tau | \mathcal{G}_t]$ is similar to the process $\widehat{V}^{(2)}$ defined in Chapter 4. Therefore, we can apply the result associated to $\widehat{V}^{(2)}$ to get the decomposition of $E^{\widehat{P}^{\mathbb{G}}}[\mathbf{1}_{\{t<\tau\leq T\}} R_\tau | \mathcal{G}_t]$. Due to Lemma 4.7, we first obtain:

$$E^{\widehat{P}^{\mathbb{G}}}[\mathbf{1}_{\{t<\tau\leq T\}} R_\tau | \mathcal{G}_t] = \frac{\mathbf{1}_{\{t<\tau\}}}{D_t^{1,\mathbb{F}} Z_t} e^{\Upsilon_t^{(1)}} E\left[D_T^{2,\mathbb{F}} \int_t^T R_u h_u dH_u | \mathcal{F}_t \right],^{\dagger}$$

where

$$h_t := \frac{D_t^{1,\mathbb{F}}}{D_t^{2,\mathbb{F}}} e^{-\Upsilon_t^{(1)}}.$$

†In this case, $H = 1$ and thus the process $M^{\mathbb{F}}$ coinsides with $D^{2,\mathbb{F}}$

In this case, the $\mathbb{F}_T$-measurable random variables corresponding to H_2, H_3 and H_4 are defined as:

$$\widetilde{H}_2 := \int_0^T \widetilde{h}_u R_u dA_u,$$

$$\widetilde{H}_3 := \frac{D_T^{2,\mathbb{F}}}{D_T^{1,\mathbb{F}}},$$

$$\widetilde{H}_4 := \int_0^T \frac{\widetilde{h}_u R_u}{D_u^{2,\mathbb{F}}} dA_u,$$

where $\widetilde{h}_u := {}^{p,\mathbb{F}}\left(\frac{\mathbf{1}_{\{\cdot<\tau\}}}{Z_- e^{\Upsilon^{(1)}}}\right)$ and the corresponding $(\widehat{P}^{1,\mathbb{F}}, \mathbb{F})$-martingales are defined by:

$$\widetilde{M}_t^{2,\mathbb{F}} := E^{\widehat{P}^{1,\mathbb{F}}}[\widetilde{H}_2|\mathcal{F}_t], \quad \widetilde{M}_t^{3,\mathbb{F}} := E^{\widehat{P}^{1,\mathbb{F}}}[\widetilde{H}_3|\mathcal{F}_t] \quad \text{and} \quad \widetilde{M}_t^{4,\mathbb{F}} := E^{\widehat{P}^{1,\mathbb{F}}}[\widetilde{H}_4|\mathcal{F}_t].$$

The following theorem is just the consequence of Theorem 4.6, which lays out the $\mathbb{G}$-LRM strategy for $\int_0^T R_u dH_u$.

Theorem 5.2: *Consider the martingale representations of $\widetilde{M}^{2,\mathbb{F}}$, $\widetilde{M}^{3,\mathbb{F}}$ and $\widetilde{M}^{4,\mathbb{F}}$ as:*

$$\widetilde{M}_t^{2,\mathbb{F}} = \widetilde{M}_0^{2,\mathbb{F}} + \int_0^t \phi_u^{(2)} d\widehat{W}_u^{(1)} + \int_0^t \psi_u^{(2)} dW_u^{(2)}$$

$$\widetilde{M}_t^{3,\mathbb{F}} = \widetilde{M}_0^{3,\mathbb{F}} + \int_0^t \phi_u^{(3)} d\widehat{W}_u^{(1)} + \int_0^t \psi_u^{(3)} dW_u^{(2)}$$

$$\widetilde{M}_t^{4,\mathbb{F}} = \widetilde{M}_0^{4,\mathbb{F}} + \int_0^t \phi_u^{(4)} d\widehat{W}_u^{(1)} + \int_0^t \psi_u^{(4)} dW_u^{(2)},$$

and denote the $\mathbb{G}$-predictable finite variation process by $A_t^R := \int_0^t \frac{\widetilde{h}_u R_u}{D_u^{2,\mathbb{F}}} dA_u$ Then, the $\mathbb{G}$-LRM strategy ξ^R for the claim $\int_0^T R_u dH_u$ takes the form of:

$$\xi_t^R := \mathbf{1}_{\{t\leq\tau\}} \frac{e^{\Upsilon_t^{(1)}}}{Z_{t-}\sigma_1(t, S_t) S_t} \left[\phi_t^{(2)} + D_t^{1,\mathbb{F}} \widetilde{\phi}_t\right],$$

where the $\mathbb{G}$*-predictable process* $\widetilde{\phi}$ *is given by*

$$\widetilde{\phi}_t := \left[\phi_t^{(3)}\widetilde{M}_t^{4,\mathbb{F}} + \phi_t^{(4)}\widetilde{M}_t^{3,\mathbb{F}} - \frac{\mu_1(t,S_t)}{\sigma_1(t,S_t)}\left(\widetilde{M}_t^{3,\mathbb{F}} + \widetilde{M}_t^{4,\mathbb{F}} + \int_0^t (\phi_u^{(3)}\phi_u^{(4)} + \psi_u^{(3)}\psi_u^{(4)})du\right)\right]$$
$$- A_{t-}^R\left[\phi_t^{(3)} - \widetilde{M}_t^{3,\mathbb{F}}\left(\frac{\mu_1(t,S_t)}{\sigma_1(t,S_t)}\right)\right]. \tag{5.2}$$

Proof. The form of strategy ξ^R is the consequence of Theorem 4.6. Furthermore, due to the uniqueness of the Galtchouk-Kunita-Watanabe decomposition of the $(\widehat{P}^{\mathbb{G}}, \mathbb{G})$-martingale $\widehat{V}^R = E^{\widehat{P}^{\mathbb{G}}}\left[\int_0^T R_u dH_u\right]$, the $\mathbb{G}$-predictable finite variation terms in the decomposition of the process $E^{\widehat{P}^{\mathbb{G}}}[\mathbf{1}_{\{t<\tau\leq T\}}R_\tau]$ together with the $\mathbb{G}$-predictable process $\int_0^t \mathbf{1}_{\{u\leq\tau\}}\frac{R_u}{Z_{u-}}dA_u$ should vanish to 0. Also, we notice that the process $\int_0^t R_u dL_u$ is a $(P,\mathbb{G})$-martingale orthogonal to $M^{\mathbb{G}}$ and thus it is still a $(\widehat{P}^{\mathbb{G}}, \mathbb{G})$-martingale. This means that this process coincides with the orthogonal part of the Galtchouk-Kunita-Watanabe decompositions of $\widehat{V}^R$. Finally, since S is a continuous $(P,\mathbb{G})$-semimartingale satisfying **(SC)**, the Galtchouk-Kunita-Watanabe decomposition of $\widehat{V}^R$ actually leads to the LRM-strategy for the claim $\int_0^T R_u dH_u$. □

We can find similar result for the third type of discounted payoff:

$$C(T,\omega)\mathbf{1}_{\{\tau>T\}} + C(\tau_-,\omega)\mathbf{1}_{\{0<\tau\leq T\}} = \int_0^T (1-H_u)dC(u,\omega)$$

In this case, we notice that this payoff can be written as the sum of the first two with $g(T,\omega) = C(T,\omega)$ and $R(t,\omega) = C(t_-,\omega)$.

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