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Differential Operator Lie Algebras on the Ring of Laurent Polynomials

By

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research for acceptance, a thesis entitled **Differential Operator Lie Algebras on the Ring of Laurent Polynomials** submitted by **Liang Chen** here in partial fulfillment of the requirements for the degree of **Doctor of Philosophy**.

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TO MY WIFE AND MY DAUGHTER

ABSTRACT

The infinite dimensional Witt algebra and the Virasoro algebra have been extensively studied by many authors in last two decades because of their importance in both mathematics and physics. In this thesis, the differential operator Lie algebras on $\mathbb{C}[t, t^{-1}]$, the ring of Laurent polynomials, are studied. They are the natural generalizations of the Witt algebra and the Virasoro algebra and are known in the physics literature as higher spin algebras $W_{1+\infty}$ and W_∞.

In the Witt algebra, only the first order differential operators are involved. In this work, the Lie algebra consisting of all differential operators of arbitrary orders, namely, $\mathbb{C}[t, t^{-1}, \frac{d}{dt}]$ is introduced. After careful choice of a basis, it turns out that $\mathbb{C}[t, t^{-1}, \frac{d}{dt}]$ possesses a $\mathbb{Z}$-grading and a shift map that shifts the graded spaces. We call a Lie subalgebra of $\mathbb{C}[t, t^{-1}, \frac{d}{dt}]$ which is shift invariant under the shift map a shift invariant Lie subalgebra and a Lie subalgebra which is shift invariant and preserves the $\mathbb{Z}$-grading a homogeneous Lie subalgebra. We give classifications of the homogeneous Lie subalgebras of $\mathbb{C}[t, t^{-1}, \frac{d}{dt}]$ and the shift invariant Lie subalgebras of the Witt algebra and study the algebraic structure and the central extensions of these Lie algebras. Many results of the Witt algebra and the Virasoro algebra are generalized. On the representation theory side, we investigate the highest weight modules and the admissible modules. Some classes of admissible modules with 1-dimensional weight spaces are completely classified.

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INTRODUCTION

The Virasoro algebra arises as the Lie algebra of the conformal group in two dimensions and plays a fundamental role in the two dimensional conformal field theories. From a mathematical point of view, it is a central extension of the complexification of the Lie algebra $VecS^1$ of real vector fields on the unit circle S^1. Any element of $VecS^1$ is of the form $f(\theta)\frac{d}{d\theta}$, where $f(\theta)$ is a smooth real valued function on S^1, with θ a real parameter and $f(\theta+2\pi)=f(\theta)$. The Lie bracket of the vector fields is

$$\left[f(\theta)\frac{d}{d\theta}, g(\theta)\frac{d}{d\theta}\right] = (fg'-f'g)(\theta)\frac{d}{d\theta},$$

where prime stands for the derivative. A basis over $\mathbb{R}$ for $VecS^1$ is provided by the vector fields

$$\frac{d}{d\theta}, \quad cos(m\theta)\frac{d}{d\theta}, \quad sin(m\theta)\frac{d}{d\theta}, \quad (m=1,2,...).$$

To avoid convergence questions we consider this as a vector space basis, so that $f(\theta)$, $g(\theta)$ are arbitrary trigonometric polynomials, and take its linear span over $\mathbb{C}$. This permits us to introduce $e^{im\theta}$ instead of $cos(m\theta)$ and $sin(m\theta)$. We thus obtain a complex Lie algebra with basis

$$e^{im\theta}\frac{d}{d\theta} \qquad m\in\mathbb{Z},$$

and Lie bracket

$$\left[e^{im\theta}\frac{d}{d\theta}, e^{in\theta}\frac{d}{d\theta}\right] = i(n-m)e^{i(m+n)}\frac{d}{d\theta}.$$

Let us denote $e^{i\theta}$ by t. Then $e^{im\theta}\frac{d}{d\theta} = it^{m+1}\frac{d}{dt}$. The elements $t^m=e^{im\theta}$, $m\in\mathbb{Z}$ span the Laurent polynomial ring $\mathbb{C}[t,t^{-1}]$. Setting $d_m=t^{m+1}\frac{d}{dt}$, the Lie bracket becomes

$$[d_m,d_n]=(n-m)d_{m+n}, \qquad \forall m,n\in\mathbb{Z}.$$

So we see that this complex Lie algebra is exactly the well known infinite dimensional Witt algebra. We denote this Lie algebra by W. Note that each d_m can be viewed as a first order differential operator on $\mathbb{C}[t,t^{-1}]$. The Virasoro algebra Vir is the universal central extension of W and is explicitly given by

$$Vir = \sum_{m\in\mathbb{Z}}\mathbb{C}L_m + \mathbb{C}¢,$$

$$[L_m,L_n]=(n-m)L_{m+n}+\frac{m^3-m}{12}\delta_{m+n,0}¢,$$

$$[L_m,¢]=0.$$

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Vir has been extensively studied by many authors in the last two decades. The earliest mathematical reference on the Virasoro algebra that is known to us is by Gelfand and Fuchs [GF]. See also Block [Bl]. They proved that the second cohomology of W is one dimensional. The Virasoro algebra was latter realized as an algebra of operators on the representation space of a Kac-Moody algebra (see for example [FK], [KP]). In fact, an untwisted affine Kac-Moody algebra is given by

$$\tilde{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c$$

with Lie brackets

$$[x \otimes t^m, y \otimes t^n] = [x, y] \otimes t^{m+n} + m\delta_{m+n,0}c$$

and

$$[x \otimes t^m, c] = 0,$$

where $\mathfrak{g}$ is a finite dimensional simple Lie algebra. The Lie brackets

$$[L_m, y \otimes t^n] = ny \otimes t^{m+n}, \quad [L_m, c] = 0$$

join Vir and $\tilde{\mathfrak{g}}$ together as a bigger Lie algebra

$$\sum_{m \in \mathbb{Z}} \mathbb{C}L_m \oplus \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c,$$

the so called Virasoro-affine Lie algebra. A key point is that integrable highest weight representations of $\tilde{\mathfrak{g}}$ are automatically representations of the Virasoro-affine Lie algebras. This is one of the reasons that affine Lie algebras play an important part in conformal field theories. Among the significant results on the representation theory of the Virasoro algebra are Kac's formula for the determinant of the contravariant form and the determination of the characters of the irreducible highest weight modules (see for example [K2], [RW]); the completely description of the submodules of a Verma module ([FF]); the unitary conditions ([FQS], [L], [GKO]); and the proof of Kac's conjecture ([K3], [Kap], [KS], [CP], [Ma], [MP]).

In recent years, some generalizations of the Witt algebra, the so called higher spin algebras W_N, which contain generators with conformal spins k in the interval $2 \leq k \leq N$, have been introduced and studied from a variety of viewpoints (see for example [FZ], [Ba1], [Ba2], [Bi], [Mat]). However, the W_N algebras are not Lie algebras since they display an essential non-linear structure. C.N.Pope and X.Shen [PS], C.N.Pope, L.J. Romans and X.Shen [PRS1] investigated the algebras W_∞ and

$W_{1+\infty}$ obtained from W_N by letting $N \longrightarrow \infty$. They found that $W_{1+\infty}$ is a Lie algebra and W_∞ is a Lie subalgebra of $W_{1+\infty}$. Moreover, they showed in [PRS3] that $W_{1+\infty}$ is in fact the algebra of all polynomial differential operators on the unit circle, including differential operators of arbitrary order, namely, $\mathbb{C}\left[t, t^{-1}, \frac{d}{dt}\right]$. Clearly, it contains the Witt algebra as a Lie subalgebra.

$\mathbb{C}\left[t, t^{-1}, \frac{d}{dt}\right]$ and its Lie subalgebras are the objects of this thesis.

Note that two significant properties of the Witt algebra W are the following:

(i) $\mathbb{Z}$-grading:

$$[d_m, d_n] \in \mathbb{C} d_{m+n} \qquad \forall m, n \in \mathbb{Z}.$$

(ii) Shift invariance:

The linear map σ such that

$$\sigma(d_m) = d_{m+1} \qquad \forall m \in \mathbb{Z}$$

is bijective from W to itself.

After careful choice of a basis of $\mathbb{C}\left[t, t^{-1}, \frac{d}{dt}\right]$, we can extend the $\mathbb{Z}$-grading and the shift map σ to the whole of $\mathbb{C}\left[t, t^{-1}, \frac{d}{dt}\right]$ (see the Introduction of Chapter 1). We call a Lie subalgebra of $\mathbb{C}\left[t, t^{-1}, \frac{d}{dt}\right]$ with the $\mathbb{Z}$-grading property a graded Lie subalgebra; a Lie subalgebra with shift invariance property a shift-invariant Lie subalgebra; a Lie subalgebra with both properties a homogenous Lie subalgebra. For example, $\mathbb{C}\left[t, t^{-1}, \frac{d}{dt}\right]$ itself is a homogenous Lie subalgebra.

This thesis consists of two chapters. In Chapter 1, we investigate the homogenous Lie subalgebras of $\mathbb{C}\left[t, t^{-1}, \frac{d}{dt}\right]$. The first result is the classification of all the homogenous Lie subalgebras. Next, we determine the graded automorphism group of $\mathbb{C}\left[t, t^{-1}, \frac{d}{dt}\right]$, which consists of all automorphisms preserving the $\mathbb{Z}$-grading of $\mathbb{C}\left[t, t^{-1}, \frac{d}{dt}\right]$. Then we discuss the structure of the homogenous Lie subalgebras and their central extensions. On the representation theory side, we consider the highest weight modules and the bounded admissible modules of the homogenous Lie subalgebras. They are analogues of the highest weight modules and the admissible modules of the Virasoro algebra.

In Chapter 2, we investigate the shift-invariant Lie subalgebras of the Witt algebra. We find that a shift-invariant Lie subalgebra of the Witt algebra is completely determined by a one-variable polynomial. For this reason we also call a shift-invariant Lie subalgebra of W a polynomial Lie subalgebra. The corresponding central extension of a polynomial Lie subalgebra of W is called a polynomial Lie subalgebra of the Virasoro algebra. As in the case of the Virasoro algebra itself, we give the Segal-Sugawara construction for its polynomial Lie subalgebras. We then introduce a quasi-triangular decomposition for a polynomial Lie subalgebra of the Virasoro algebra and discuss the highest weight modules, the quasi-admissible and the admissible modules (see Sections 2.4 and 2.5 for the definitions) of these Lie subalgebras. In particular, we investigate a class of finite dimensional admissible modules in detail (see Sections 2.7 and 2.8).

CHAPTER 1

DIFFERENTIAL OPERATOR LIE ALGEBRAS ON $\mathbb{C}[t,t^{-1}]$

1.0 Introduction of Chapter 1

In this chapter we study the homogeneous Lie subalgebras of $\mathbb{C}\left[t,t^{-1},\frac{d}{dt}\right]$ and their representations. We have seen in the Introduction that the Witt algebra W is the complex Lie algebra of polynomial vector fields on the unit circle S^1. An element of W is a linear combination of the elements of the form $e^{in\theta}\frac{d}{d\theta}$, where θ is a real parameter, and the Lie bracket of W is given by

$$\left[e^{in\theta}\frac{d}{d\theta}, e^{im\theta}\frac{d}{d\theta}\right] = i(n-m)e^{i(m+n)\theta}\frac{d}{d\theta}.$$

If we define $t = e^{i\theta}$, then the elements $t^m = e^{im\theta}$, $m \in \mathbb{Z}$, span the Laurent polynomial ring $\mathbb{C}[t,t^{-1}]$, and $e^{im\theta}\frac{d}{d\theta} = it^{m+1}\frac{d}{dt}$ may be viewed as a first order differential operator on $\mathbb{C}[t,t^{-1}]$. Let $d_m = t^{m+1}\frac{d}{dt}$. Then $W = \sum_{m\in\mathbb{Z}}\mathbb{C}d_m$, and $[d_m,d_n] = (n-m)d_{m+n}$. The Virasoro algebra is $Vir = \sum_{m\in\mathbb{Z}}\mathbb{C}L_m + \mathbb{C}¢$ with

$$\begin{aligned}[L_m,L_n] &= (n-m)L_{m+n} + \frac{m^3-m}{12}\delta_{m+n,0}¢,\\ [L_m,¢] &= 0.\end{aligned}$$

Recently, C.N.Pope and X.Shen [PS], C.N.Pope, L.J.Roman and X.Shen [PRS1], [PRS2] studied the higher spin algebras W_∞ and $W_{1+\infty}$, the generalizations of W. The Lie algebra $W_{1+\infty}$ has basis V_m^i where $m\in\mathbb{Z}$, $i\in\mathbb{Z}_{\geq 0}$, and Lie bracket

$$\begin{aligned}\left[V_m^i,V_n^j\right] =& g_0^{ij}(m,n;\mu)V_{m+n}^{i+j} + q^2 g_2^{ij}(m,n;\mu)V_{m+n}^{i+j-2} + \dots\\ &+ q^{2r}g_{2r}^{ij}(m,n;\mu)V_{m+n}^{i+j-2r} + \dots + q^{2i}c_i(m;\mu)\delta^{ij}\delta_{m+n,0},\end{aligned}$$

where V_m^i corresponds to the m^{th} Fourier mode of a conformal spin $i+2$ field, q is a parameter, $c_i(m;\mu)$ are the central terms. The structure constants are given by

$$\begin{aligned}g_{2r}^{ij}(m,n;\mu) &= \frac{\phi_{2r}^{ij}(\mu)}{2(2r+1)!}N_{2r}^{ij}(m,n),\\ N_{2r}^{ij}(m,n) &= \sum_{k=0}^{2r+1}(-1)^k\binom{2r+1}{k}[i+1+m]_{2r+1-k}\\ &\qquad [i+1-m]_k[j+1+n]_k[j+1-n]_{2r+1-k},\\ [a]_n &= \frac{a!}{(a-n)!}.\end{aligned}$$

If we parameterize μ in terms of a variable s by $\mu = s(s+1) = -\frac{1}{4}$, then $\phi_{2r}^{ij}(\mu)$ can be expressed as

$$\phi_{2r}^{ij}(\mu) =_4 F_3\left[-\frac{1}{2}-2s, \frac{3}{2}+2s, -r-\frac{1}{2}, -r, -i-\frac{1}{2}, -j-\frac{1}{2}, i+j-2r+\frac{5}{2}; 1\right],$$

where ${}_4F_3$ is a generalized hypergeometric function (see [PRS3] for details). We note that W_∞ is a Lie subalgebra of $W_{1+\infty}$.

Later, [PRS4] proved that $W_{1+\infty}$ is nothing but the algebra of all polynomial differential operators on the unit circle, including differential operators of arbitrary order, namely

$$\mathbb{C}\left[t, t^{-1}, \frac{d}{dt}\right] = \sum_{m\in\mathbb{Z}, r\in\mathbb{Z}_{\geq 0}} \mathbb{C}t^m \left(\frac{d}{dt}\right)^r.$$

This brought a connection between the higher spin algebras and the algebra of all smooth differential operators on the unit circle.

Note that the Lie bracket of $W_{1+\infty} \cong \mathbb{C}[t, t^{-1}, \frac{d}{dt}]$ given above is very complex. We rechoose a basis for $\mathbb{C}[t, t^{-1}, \frac{d}{dt}]$ as follows: For all $m \in \mathbb{Z}$, $r \in \mathbb{Z}_{\geq 0}$, define

$$d_m^r := t^m \left(t\frac{d}{dt}\right)^r.$$

Then $\{d_m^r \mid m \in \mathbb{Z}, r \in \mathbb{Z}_{\geq 0}\}$ is a basis of $\mathbb{C}[t, t^{-1}, \frac{d}{dt}]$. The Lie bracket is

$$[d_m^r, d_n^s] = \sum_{k=0}^{r+s} \left(\binom{r}{k} n^k - \binom{s}{k} m^k\right) d_{m+n}^{r+s-k}.$$

From this we see that $\mathbb{C}\left[t, t^{-1}, \frac{d}{dt}\right]$ has $\mathbb{Z}$-grading

$$\mathbb{C}\left[t, t^{-1}, \frac{d}{dt}\right] = \sum_{m\in\mathbb{Z}} \mathfrak{g}_m, \quad [\mathfrak{g}_m, \mathfrak{g}_n] \subseteq \mathfrak{g}_{m+n},$$

where the graded subspace $\mathfrak{g}_m := \sum_{r=0}^{\infty} \mathbb{C}d_m^r$. Further, the $\mathbb{C}$-linear map σ such that

$$\sigma(d_m^r) = d_{m+1}^r \quad \forall m \in \mathbb{Z}, r \in \mathbb{Z}_{\geq 0}$$

is bijective from $\mathbb{C}\left[t, t^{-1}, \frac{d}{dt}\right]$ to itself and shifts a graded subspace to another. We call σ a shift operator of $\mathbb{C}\left[t, t^{-1}, \frac{d}{dt}\right]$ (see Section 1.1 for more details). A Lie subalgebra $\mathfrak{h}$ of $\mathbb{C}\left[t, t^{-1}, \frac{d}{dt}\right]$ is called a homogeneous Lie subalgebra if

$$\mathfrak{h} = \sum_{m\in\mathbb{Z}} \mathfrak{h}_m$$

where $\mathfrak{h}_m = \mathfrak{h} \cap \mathfrak{g}_m$, and $\sigma(\mathfrak{h}) = \mathfrak{h}$.

We organize this chapter as follows: In Section 1.1, we introduce the differential operator Lie algebra $\mathbb{C}[t,t^{-1},\frac{d}{dt}]$ and classify all of the homogeneous Lie subalgebras of it. We find that, except for a few examples, the homogeneous Lie subalgebras of $\mathbb{C}[t,t^{-1},\frac{d}{dt}]$ are determined by polynomials. In Section 1.2, we determine the automorphims of $\mathbb{C}[t,t^{-1},\frac{d}{dt}]$ which preserve the $\mathbb{Z}$-grading. These automorphisms constitute an abelian group isomorphic to $(\mathbb{C},+)\times(\mathbb{C}^*,\cdot)$. In Section 1.3, we discuss some algebraic properties of the homogeneous Lie subalgebras. In Section 1.4, we consider the central extensions of the homogeneous Lie subalgebras. In particular, we determine the universal central extension of $\mathfrak{g}\{0,1\} := \sum_{m\in\mathbb{Z}} \mathbb{C}d_m^1 + \sum_{m\in\mathbb{Z}} \mathbb{C}d_m^0$. The admissible modules of the Witt algebra with 1-dimensional weight spaces have been classified by Kaplansky and Santharoubane [Kap],[KS]. In Section 1.5, we give a classification of the admissible $\mathbb{C}[t,t^{-1},\frac{d}{dt}]$-modules with 1-dimensional weight spaces. Finally, in Section 1.6, we define the highest weight $\mathbb{C}[t,t^{-1},\frac{d}{dt}]$-modules and a contravariant hermitian form on Verma modules. A necesary condition for the hermitian form to be non-negative is also obtained.

We denote the complex number field by $\mathbb{C}$, the real number field by $\mathbb{R}$, and the integer ring by $\mathbb{Z}$. All Lie algebras considered are complex Lie Algebras.

1.1 Differential Operator Lie Algebras on $\mathbb{C}[t,t^{-1}]$

In this section, we consider the algebra of differential operators on the Laurent polynomial ring $\mathbb{C}[t,t^{-1}]$, namely $\mathbb{C}[t,t^{-1},\frac{d}{dt}]$, and give it a Lie algebra structure. We have seen in the Introduction of this chapter that by suitable choice of basis, we obtain a $\mathbb{Z}$-grading on $\mathbb{C}[t,t^{-1},\frac{d}{dt}]$ and hence we have the shift map σ on it. We call a Lie subalgebra of $\mathbb{C}[t,t^{-1},\frac{d}{dt}]$ which is $\mathbb{Z}$-graded and invariant under the shift map a homogeneous Lie subalgebra. The main result in this section is the classification of homogeneous Lie subalgebras of $\mathbb{C}[t,t^{-1},\frac{d}{dt}]$.

As a vector space over $\mathbb{C}$, $\mathbb{C}[t,t^{-1},\frac{d}{dt}]$ has a basis $\{d_m^r \mid m \in \mathbb{Z}, r \in \mathbb{Z}_{\geq 0}\}$, where $d_m^r := t^m(t\frac{d}{dt})^r$, and the action of d_m^r on $\mathbb{C}[t,t^{-1}]$ is given by

$$d_m^r \cdot t^k = k^r t^{m+k}$$

for all $k \in \mathbb{Z}$. Let $p(x) = \sum_i a_i x^i \in \mathbb{C}[x]$ be a polynomial and define

$$d_m(p(x)) = \sum_i a_i d_m^i.$$

Then

$$\begin{aligned} d_m(p(x)) \cdot t^k &= \sum_i a_i d_m^i \cdot t^k \\ &= \sum_i a_i k^i t^{m+k} \\ &= p(k) t^{m+k}. \end{aligned}$$

Moreover,

$$\begin{aligned} &\big(d_m(p(x))d_n(q(x)) - d_n(q(x))d_m(p(x))\big) \cdot t^k \\ &= q(k)d_m(p(x))t^{n+k} - p(k)d_n(q(x))t^{m+k} \\ &= \big(p(n+k)q(k) - p(k)q(k+m)\big)t^{m+n+k} \\ &= d_{m+n}\big(p(x+n)q(x) - p(x)q(x+m)\big)t^k \end{aligned}$$

for all $k \in \mathbb{Z}$.

So, if we define

$$[d_m(p(x)), d_n(q(x))] = d_{m+n}\big(p(x+n)q(x) - p(x)q(x+m)\big),$$

then $\mathbb{C}[t, t^{-1}, \frac{d}{dt}]$ is a Lie algebra. Note that if $p(x) = x^r, q(x) = x^s$, then the Lie bracket we obtained here is exactly that we have seen in the Introduction of this chapter. We note for future reference that $d_m = d_m^1 = d_m(x)$ and $d_m^s = d_m(x^s)$. We denote $\mathbb{C}[t, t^{-1}, \frac{d}{dt}]$ by $\mathfrak{g}$ in the rest of this chapter.

Definition. A Lie subalgebra of $\mathfrak{g}$ is called a differential operator Lie algebra on $\mathbb{C}[t, t^{-1}]$.

Setting

$$\mathfrak{g}_m = \sum_{r \geq 0} \mathbb{C} d_m^r$$

where $m \in \mathbb{Z}$, then $\mathfrak{g} = \sum_{m \in \mathbb{Z}} \mathfrak{g}_m$, and $[\mathfrak{g}_m, \mathfrak{g}_n] \subseteq \mathfrak{g}_{m+n}$. So $\mathfrak{g}$ is a $\mathbb{Z}$-graded Lie algebra. In particular, $[\mathfrak{g}_0, \mathfrak{g}_0] = 0$, and $[d_0^1, d_n^s] = n d_n^s$. We see that $\mathfrak{g}_0$ is an abelian Lie subalgebra of $\mathfrak{g}$ and $\mathfrak{g}_m$ is the eigenspace of $ad(d_0^1)$, the adjoint map, of eigenvalue m.

The linear map σ from $\mathfrak{g}$ to itself such that $\sigma(d_m^r) = d_{m+1}^r$ for all $m \in \mathbb{Z}, r \in \mathbb{Z}_{\geq 0}$ is called the (canonical) shift of $\mathfrak{g}$. Clearly σ is one to one and onto.

Definition. Let $\mathfrak{h} \subseteq \mathfrak{g}$ be a Lie subalgebra. $\mathfrak{h}$ is called a homogeneous Lie subalgebra of $\mathfrak{g}$ if

$$(i) \qquad \mathfrak{h} = \sum_{m \in \mathbf{Z}} \mathfrak{h}_m \qquad \textit{where} \quad \mathfrak{h}_m := \mathfrak{h} \cap \mathfrak{g}_m,$$
$$(ii) \quad \sigma(\mathfrak{h}) = \mathfrak{h}.$$

The following are examples of homogeneous Lie subalgebras of $\mathfrak{g}$:

(1) $\mathfrak{g}\{0\} := \sum_{m \in \mathbf{Z}} \mathbb{C} d_m^0$ is an abelian Lie subalgebra of $\mathfrak{g}$.

(2) For any $\alpha \in \mathbb{C}$, define $\mathfrak{g}\{x+\alpha\} := \sum_{m \in \mathbf{Z}} \mathbb{C} d_m(x+\alpha)$. Then

$$[d_m(x+\alpha), d_n(x+\alpha)] = (n-m) d_{m+n}(x+\alpha).$$

So $\mathfrak{g}\{x+\alpha\}$ is isomorphic to the infinite dimensional Witt algebra.

(3) $\mathfrak{g}\{0,1\} := \sum_{m \in \mathbf{Z}} \mathbb{C} d_m^1 + \sum_{m \in \mathbf{Z}} \mathbb{C} d_m^0$.

V.Kac and A.Raina [KR] defined the following infinite matrix Lie algebra:

$$\overline{\mathfrak{a}_\infty} := \{A = (a_{ij})_{i,j \in \mathbf{Z}} \mid a_{ij} \in \mathbb{C}, a_{ij} = 0 \quad \forall \mid i-j \mid \gg 0\},$$

where $\mid i-j \mid \gg 0$ means that $\mid i-j \mid$ is sufficiently large. The Lie bracket of $\overline{\mathfrak{a}_\infty}$ is given by $[A,B] = AB - BA$, for all $A, B \in \overline{\mathfrak{a}_\infty}$, where AB is the usual matrix multiplication. A matrix in $\overline{\mathfrak{a}_\infty}$ is a linear combination of matrices of the form

$$\sum_{i \in \mathbf{Z}} \lambda_i E_{i+m,i},$$

where $m \in \mathbb{Z}$ and E_{ij} is the matrix with 1 in i-row and j-column, with 0 elsewhere. We have the following.

Proposition 1. For any $\beta \in \mathbb{C}$, define

$$i_\beta : \mathfrak{g} \longrightarrow \overline{\mathfrak{a}_\infty}$$

to be the linear map such that

$$i_\beta(d_m^r) = \sum_{j \in \mathbf{Z}} (j+\beta)^r E_{j+m,j}.$$

Then i_β is an injective Lie algebra homomorphism.

Proof.

$$\begin{aligned}
[i_\beta(d^r_m), i_\beta(d^s_n)] &= \sum_j \sum_i (j+\beta)^r (i+\beta)^s [E_{j+m,j}, E_{i+n,i}] \\
&= \sum_j \sum_i (j+\beta)^r (i+\beta)^s \big(\delta_{j,i+n} E_{j+m,i} - \delta_{i,j+m} E_{i+n,j}\big) \\
&= \sum_i (i+n+\beta)^r (i+\beta)^s E_{i+m+n,i} \\
&\quad - \sum_j (j+\beta)^r (j+m+\beta)^s E_{j+m+n,j} \\
&= \sum_j \big((j+n+\beta)^r (j+\beta)^s - (j+\beta)^r (j+\beta+m)^s\big) E_{j+m+n,j} \\
&= \sum_j \left(\sum_k \left(\tbinom{r}{k} n^k - \tbinom{s}{k} m^k\right) (j+\beta)^{r+s-k} \right) E_{j+m+n,j} \\
&= \sum_k \left(\tbinom{r}{k} n^k - \tbinom{s}{k} m^k\right) i_\beta(d^{r+s-k}_{m+n}) \\
&= i_\beta [d^r_m, d^s_n].
\end{aligned}$$

The injectivity of i_β is clear. □

Proposition 2. $\mathfrak{h} \subseteq \mathfrak{g}$ is a homogeneous Lie subalgebra iff there exists a subspace $V \subseteq \mathbb{C}[x]$ satisfying

$$p(x+n)q(x) - p(x)q(x+m) \in V \qquad \forall p(x), q(x) \in V, \quad \forall m, n \in \mathbb{Z},$$

such that

$$\mathfrak{h} = \sum_{p(x) \in V, m \in \mathbb{Z}} \mathbb{C} d_m(p(x)).$$

Proof. Clearly, if $V \subseteq \mathbb{C}[x]$ is a subspace satisfying above condition, then $\sum_{p(x) \in V, m \in \mathbb{Z}} \mathbb{C} d_m(p(x))$ is a homogeneous Lie subalgebra of $\mathfrak{g}$. Conversely, assume that $\mathfrak{h} \subseteq \mathfrak{g}$ is a homogeneous Lie subalgebra. Then

$$\mathfrak{h} = \sum_{m \in \mathbb{Z}} \mathfrak{h}_m, \qquad where \quad \mathfrak{h}_m = \mathfrak{h} \cap \mathfrak{g}_m.$$

Note that if $d_m(p(x)) \in \mathfrak{h}_m$, then $d_n(p(x)) \in \mathfrak{h}_n$ for all $n \in \mathbb{Z}$ since $\sigma^k(d_m(p(x))) = d_{m+k}(p(x))$. Let

$$V = \{p(x) \in \mathbb{C}[x] \mid d_m(p(x)) \in \mathfrak{h} \qquad \forall m \in \mathbb{Z}\}.$$

Then for polynomials $p(x), q(x) \in V, m, n \in \mathbb{Z}$,

$$[d_m(p(x)), d_n(q(x))] = d_{m+n}\big(p(x+n)q(x) - p(x)q(x+m)\big) \in \mathfrak{h}.$$

Hence

$$p(x+n)q(x) - p(x)q(x+m) \in V, \qquad \forall m, n \in \mathbb{Z}.$$

V satisfies the required condition. □

Remark. By Proposition 2, classifying the homogeneous Lie subalgebras of $\mathfrak{g}$ is equivalent to classifying the subspaces of $\mathbb{C}[x]$ which satisfy the condition given in the proposition.

Proposition 3 (Classification of homogeneous Lie subalgebras). The following are all of the homogeneous Lie subalgebras of $\mathfrak{g}$:

(1) $\mathfrak{g}\{0\}$, $\mathfrak{g}\{0,1\}$, $\mathfrak{g}\{x+\alpha\}$ where $\alpha \in \mathbb{C}$.

(2) $\mathfrak{g}_{\langle p(x)\rangle} := \sum_{q(x)\in\langle p(x)\rangle, m\in\mathbb{Z}} \mathbb{C}d_m(q(x))$, where $\langle p(x)\rangle$ is the ideal of $\mathbb{C}[x]$ generated by $p(x)$.

Proof. We have seen that $\mathfrak{g}\{0\}$, $\mathfrak{g}\{0,1\}$, $\mathfrak{g}\{x+\alpha\}$ are all homogenous Lie subalgebras. Now for any $p(x) \in \mathbb{C}[x]$, if $q(x), r(x) \in \langle p(x)\rangle$, then clearly

$$q(x+n)r(x) - q(x)r(x+m) \in \langle p(x)\rangle,$$

for all $m, n \in \mathbb{Z}$. So $\mathfrak{g}_{\langle p(x)\rangle}$ is a homogeneous Lie subalgebra of $\mathfrak{g}$ by Proposition 2. Conversely, let $V \subseteq \mathbb{C}[x]$ be a subspace such that

$$p(x+n)q(x) - p(x)q(x+m) \in V \quad \forall p(x), q(x) \in V, \quad \forall m, n \in \mathbb{Z}.$$

Define $\mathfrak{g}_V := \sum_{m\in\mathbb{Z}, p(x)\in V} \mathbb{C}d_m(p(x))$.

(a) Suppose for all $p(x) \in V$, $deg(p(x)) = 0$. Then $V = \mathbb{C}$ and $\mathfrak{g}_V = \mathfrak{g}\{0\}$.

(b) Suppose for all $p(x) \in V, deg(p(x)) \le 1$, but there exists $p(x) \in V$ with degree 1. Assume that $p(x) = x + \alpha \in V$. If there exists $\beta \in \mathbb{C}$ such that $\beta \ne \alpha$ and $x + \beta \in V$, then $V = \mathbb{C}x + \mathbb{C}$ and $\mathfrak{g}_V = \mathfrak{g}\{0,1\}$. Otherwise, $V = \mathbb{C}(x+\alpha)$ and $\mathfrak{g}_V = \mathfrak{g}\{x+\alpha\}$.

(c) Suppose there exists $q(x) \in V$ such that $deg(q(x)) \ge 2$.

Let

$$p(x) = x^r + a_{r-1}x^{r-1} + \dots + a_0 \in V$$

be the choice with minimal degree and let

$$q(x) = x^s + b_{s-1}x^{s-1} + \dots + b_0 \in V$$

where $s \geq 2$.

Case 1: $r = 0$. By the Taylor formula,

$$q(x+m) = q(x) + q'(x)m + \frac{q^{(2)}(x)}{2!}m^2 + \ldots + \frac{q^{(s)}(x)}{s!}m^s.$$

Since $q(x+m) - q(x) = q'(x)m + \frac{q^{(2)}(x)}{2!}m^2 + \ldots + \frac{q^{(s)}(x)}{s!}m^s \in V$, for all $m \in \mathbb{Z}$, and the determinant of

$$\begin{pmatrix} 1 & 1 & \ldots & 1 \\ 2 & 2^2 & \ldots & 2^s \\ \vdots & \vdots & \ldots & \vdots \\ s & s^2 & \ldots & s^s \end{pmatrix}$$

is non-zero, $q'(x), \frac{q^{(2)}(x)}{2!}, \ldots, \frac{q^{(s)}(x)}{s!} \in V$. It follows that $1, x, x^2, \ldots, x^{s-1} \in V$. But $q(x) \in V$, so also $x^s \in V$. Assume that $1, x, \ldots, x^k \in V$, $k \geq s \geq 2$. Then

$$(x+n)^2 x^k - x^2 x^k = x^k(2nx + n^2) = 2nx^{k+1} + n^2 x^k \in V$$

for all $n \in \mathbb{Z}$. Thus $x^{k+1} \in V$. By induction, $V = \mathbb{C}[x]$ and $\mathfrak{g}_V = \mathfrak{g}_{\langle 1 \rangle} = \mathfrak{g}$.

Case 2: $r = 1$, $p(x) = x + a$. Since

$$p(x)q(x) - p(x)q(x+m) = -p(x)\left(q'(x)m + \frac{q^{(2)}(x)}{2!}m^2 + \ldots + \frac{q^{(s)}(x)}{s!}m^s\right) \in V$$

for all $m \in \mathbb{Z}$, we have

$$(x+a)q'(x), \ldots, (x+a)\frac{q^{(s)}(x)}{s!} \in V.$$

Thus

$$x + a, (x+a)x, \ldots, (x+a)x^{s-1} \in V,$$

where $s \geq 2$. Assume that $(x+a), (x+a)x, \ldots, (x+a)x^k \in V$ for some $k \geq 1$. Then

$$\begin{aligned} &(x+a+n)(x+n)(x+a)x^k - (x+a)x(x+a)x^k \\ &= (x+a)x^k(2nx + n(a+n)) \in V \end{aligned}$$

for all n. Hence $(x+a)x^{k+1} \in V$. By induction, $\langle x+a \rangle \subseteq V$. Note that $\langle x+a \rangle$ has codimension 1 in $\mathbb{C}[x]$, so $\langle x+a \rangle = V$ and $\mathfrak{g}_V = \mathfrak{g}_{\langle x+a \rangle}$.

Case 3: $r \geq 2$. For any $n \in \mathbb{Z}$, since

$$\begin{aligned} &p(x+n)p(x) - p(x)p(x) \\ &= p(x)\left(p'(x)n + \frac{p^{(2)}(x)}{2!}n^2 + \ldots + \frac{p^{(r)}(x)}{r!}n^r\right) \in V, \end{aligned}$$

we have

$$p(x), xp(x), ..., x^{r-1}p(x) \in V,$$

where $r \geq 2$. Assume that

$$p(x), xp(x), ..., x^k p(x) \in V \quad for \quad some \quad k \geq r-1 \geq 1.$$

Since

$$1 \leq k-r+2 \leq k,$$

we have

$$r(x) := x^{k-r+2}p(x) \in V,$$

and

$$deg(r(x)) = k+2.$$

But

$$\begin{aligned} &r(x+n)p(x) - r(x)p(x) \\ &= p(x)\left(r'(x)n + \frac{r^{(2)}(x)}{2!}n^2 + ... + \frac{r^{(k+2)}(x)}{(k+2)!}n^{k+2}\right) \in V \end{aligned}$$

for all $n \in \mathbb{Z}$, so

$$p(x), xp(x), ..., x^{k+1}p(x) \in V.$$

By induction, $\langle p(x)\rangle \subseteq V$. If there exists $h(x) \in V \setminus \langle p(x)\rangle$, then by the choice of $p(x)$,

$$deg(h(x)) \geq deg(p(x)) \geq r.$$

Assume that $deg(h(x)) = l$ and define $h_1(x) := h(x) - x^{l-r}p(x)$. Then $h_1(x) \in V\setminus\langle p(x)\rangle$ and $deg(h_1(x)) < deg(h(x))$. Inductively, there exists a $h^*(x) \in V\setminus\langle p(x)\rangle$ such that $deg(h^*(x)) < deg(p(x))$. This is a contradiction. So we have $V = \langle p(x)\rangle$ and complete the proof. □

1.2 Graded Automorphism Group of $\mathbb{C}[t, t^{-1}, \frac{d}{dt}]$

We had seen in Section 1.1 that $\mathfrak{g} = \mathbb{C}[t, t^{-1}, \frac{d}{dt}]$ is $\mathbb{Z}$-graded as $\mathfrak{g} = \sum_{m\in\mathbb{Z}} \mathfrak{g}_m$, $\mathfrak{g}_m = \sum_{r\in\mathbb{Z}_{\geq 0}} \mathbb{C}d_m^r$. Let $\phi \in Aut(\mathfrak{g})$, the automorphism group of $\mathfrak{g}$. If $\phi(\mathfrak{g}_m) = \mathfrak{g}_m$ for all $m \in \mathbb{Z}$, we say ϕ is a graded automorphism of $\mathfrak{g}$. Let

$$Aut_G(\mathfrak{g}) = \{\phi \in Aut(\mathfrak{g}) \mid \phi \quad is \quad graded\}.$$

Then $Aut_G(\mathfrak{g})$ is a subgroup of $Aut(\mathfrak{g})$. In this section, we determine $Aut_G(\mathfrak{g})$. First we prove the following.

Proposition 1. As a Lie algebra, $\mathfrak{g}$ is generated by d^0_{-1}, d^0_1, d^2_0; that is $\mathfrak{g} = \langle d^0_{-1}, d^0_1, d^2_0 \rangle$.

Proof. Let $\mathfrak{g}' := \langle d^0_{-1}, d^0_1, d^2_0 \rangle$. Since $[d^2_0, d^0_1] = 2d^1_1 + d^0_1 \in \mathfrak{g}$, we have

$$d^1_1 \in \mathfrak{g}'.$$

Similarly, we have

$$d^1_{-1} \in \mathfrak{g}'.$$

Thus

$$d^1_0 = \frac{1}{2}[d^1_{-1}, d^1_1] \in \mathfrak{g}'.$$

Now $[d^2_0, d^1_1] = 2d^2_1 + d^1_1 \in \mathfrak{g}'$ implies that $d^2_1 \in \mathfrak{g}'$. Then $[d^1_1, d^0_1] = d^0_2 \in \mathfrak{g}'$ and $[d^2_1, d^0_1] = 2d^1_2 + d^0_2 \in \mathfrak{g}'$ imply that

$$d^1_2 \in \mathfrak{g}'.$$

Similarly,

$$d^1_{-2} \in \mathfrak{g}'.$$

Since $\mathfrak{g}\{x\}$ is generated by $\{d^1_{-2}, d^1_{-1}, d^1_0, d^1_1, d^1_2\}$,

$$\mathfrak{g}\{x\} \subseteq \mathfrak{g}'.$$

Then

$$d^0_{n+1} = [d^1_n, d^0_1] \in \mathfrak{g}'$$

for all $n \in \mathbb{Z}$. Finally from $[d^2_0, d^1_n] = 2nd^2_n + n^2 d^1_n \in \mathfrak{g}'$ for all $n \in \mathbb{Z}$, we obtain

$$d^2_n \in \mathfrak{g}'$$

for all $n \in \mathbb{Z}$. In summary, we proved that

$$d_m\left(x^k\right) \in \mathfrak{g}' \qquad \forall m \in \mathbb{Z}, k = 0, 1, 2.$$

Now assume that $d_m\left(x^k\right) \in \mathfrak{g}'$ $\forall m \in \mathbb{Z}, 0 \leq k \leq l$. Then

$$\left[d^2_m, d_n\left(x^l\right)\right] = d_{m+n}\left((x+n)^2 x^l - x^2(x+m)^l\right) \in \mathfrak{g}'$$

implies that $d_m\left(x^{l+1}\right) \in \mathfrak{g}'$ for all $m \in \mathbb{Z}$. By induction, we get $\mathfrak{g}' = \mathfrak{g}$. □

Definition. Let $\beta \in \mathbb{C}$ and define $\psi_\beta : \mathfrak{g} \longrightarrow \mathfrak{g}$ to be the unique linear map such that $\psi_\beta(d_m(p(x))) = d_m(p(x+\beta))$ for all $m \in \mathbb{Z}$ and $p(x) \in \mathbb{C}[x]$.

Let $a \in \mathbb{C}^*$ and define $\phi_a : \mathfrak{g} \longrightarrow \mathfrak{g}$ to be the unique linear map such that $\phi_a(d_m(p(x))) = a^m d_m(p(x))$ for all $m \in \mathbb{Z}$ and $p(x) \in \mathbb{C}[x]$.

Let $\tau : \mathfrak{g} \longrightarrow \mathfrak{g}$ be the linear map such that

$$\tau(d_m(p(x))) = (-1)^{m-1} d_m(p(-x-m))$$

for all $m \in \mathbb{Z}$ and $p(x) \in \mathbb{C}[x]$.

Proposition 2. Let $\Psi = \{\psi_\beta \mid \beta \in \mathbb{C}\}$ and let $\Phi = \{\phi_a \mid a \in \mathbb{C}^*\}$. Then $Aut_G(\mathfrak{g}) = (\langle\tau\rangle \ltimes \Psi) \times \Phi \cong (\mathbb{Z}_2 \ltimes (\mathbb{C},+)) \times (\mathbb{C}^*,\cdot)$, where $\tau^2 = 1$.

Proof. Since

$$\begin{aligned}
&[\psi_\beta(d_m(p(x))), \psi_\beta(d_n(p(x)))] \\
&= [d_m(p(x+\beta)), d_n(q(x+\beta))] \\
&= d_{m+n}(p(x+n+\beta)q(x+\beta) - p(x+\beta)q(x+m+\beta)) \\
&= \psi_\beta([d_m(p(x)), d_n(q(x))]),
\end{aligned}$$

$\psi_\beta \in Aut_G(\mathfrak{g})$ for all $\beta \in \mathbb{C}$. It is easy to see from the definition that $\phi_a \in Aut_G(\mathfrak{g})$ for all $a \in \mathbb{C}^*$, $\psi_\alpha\psi_\beta = \psi_{\alpha+\beta}$, $\phi_a\phi_b = \phi_{ab}$, and $\phi_a\psi_\beta = \psi_\beta\phi_a$. It is also easy to verify that $\tau \in Aut_G(\mathfrak{g})$, $\tau^2 = 1$, and $\tau\phi_a = \phi_a\tau$, $\tau\psi_\beta\tau^{-1} = \psi_{-\beta}$. Hence $(\langle\tau\rangle \ltimes \Psi) \times \Phi \cong (\mathbb{Z}_2 \ltimes (\mathbb{C},+)) \times (\mathbb{C}^*,\cdot)$ is a subgroup of $Aut_G(\mathfrak{g})$.

Now suppose $\psi \in Aut_G(\mathfrak{g})$. We show that $\psi \in (\langle\tau\rangle \ltimes \Psi) \times \Phi$. Assume that $\psi(d_m^0) = \sum_{i=0}^{l_m} a_{m,i} d_m^i$, where $m \in \mathbb{Z}$, and $a_{m,l_m} \neq 0$. Fix $m \in \mathbb{Z}$. For any $n \in \mathbb{Z}$,

$$\begin{aligned}
0 &= \psi\left[d_m^0, d_n^0\right] \\
&= \left[\psi(d_m^0), \psi(d_n^0)\right] \\
&= a_{m,l_m} a_{n,l_n} \left[d_m^{l_m}, d_n^{l_n}\right] + \ldots \\
&= a_{m,l_m} a_{n,l_n} d_{m+n}\left((x+n)^{l_m} x^{l_n} - x^{l_m}(x+m)^{l_n}\right) + \ldots \quad .
\end{aligned}$$

If $l_m > 0$, then for $n \gg 0$, $d_{m+n}\left((x+n)^{l_m} x^{l_n} - x^{l_m}(x+m)^{l_n}\right) \neq 0$. So we must have $l_m = 0$, i.e. $\psi(d_m^0) = a_m d_m^0$ for some $a_m \in \mathbb{C}^*$.

Assume that $\psi(d_m^1) = d_m(p_m(x))$. Comparing

$$\psi[d_m^1, d_n^0] = \psi(n d_{m+n}^0) = n a_{m+n} d_{m+n}^0$$

with

$$\begin{aligned}[\psi(d_m^1),\psi(d_n^0)] &= [d_m(p_m(x)), a_n d_n^0] \\ &= a_n d_{m+n}\left(p_m(x+n) - p_m(x)\right) \\ &= a_n d_{m+n}\left(p'_m(x)n + \frac{p_m^{(2)}(x)}{2!}n^2 + \dots\right),\end{aligned}$$

we obtain

$$deg\,(p'_m(x)) = 0,$$

and

$$p_m(x) = b_m x + c_m$$

for some $b_m, c_m \in \mathbb{C}$. Thus

$$\psi(d_m^1) = b_m d_m^1 + c_m d_m^0.$$

Moreover, from $\psi[d_m^1, d_n^0] = [\psi(d_m^1), \psi(d_n^0)]$, we have

(1) $$n a_{m+n} = n a_n b_m \qquad \forall m, n \in \mathbb{Z}.$$

Assume that $\psi\left(d_0^2\right) = d_0(q(x))$. Then

$$\begin{aligned}\psi\left[d_0^2, d_n^0\right] &= \psi\left(d_n\left(2nx + n^2\right)\right) \\ &= 2n\psi\left(d_n^1\right) + n^2\psi\left(d_n^0\right) \\ &= 2nb_n d_n^1 + \left(2nc_n + n^2 a_n\right)d_n^0,\end{aligned}$$

and

$$\begin{aligned}\left[\psi\left(d_0^2\right), \psi\left(d_n^0\right)\right] &= \left[d_0(q(x)), a_n d_n^0\right] \\ &= a_n d_n\left(q(x+n) - q(x)\right) \\ &= a_n d_n\left(q'(x)n + \frac{q^{(2)}(x)}{2!}n^2 + \dots\right)\end{aligned}$$

for all $n \in \mathbb{Z}$. These imply that $deg\,(q'(x)) = 1$, and $q(x) = \beta_2 x^2 + \beta_1 x + \beta_0$ for some $\beta_2, \beta_1, \beta_0 \in \mathbb{C}$, where $\beta_2 \neq 0$. From $\psi[d_0^2, d_n^0] = [\psi(d_0^2), \psi(d_n^0)]$, we get

(2) $$\begin{cases} 2nb_n = 2n\beta_2 a_n \\ 2nc_n + n^2 a_n = a_n(\beta_1 n + \beta_2 n^2) \end{cases}$$

for all $n \in \mathbb{Z}$.

With $m = 0$ and $n = 1$ in (1), and by (2), we get

$$\begin{cases} b_0 = 1 \\ b_n = \beta_2 a_n \qquad n \neq 0, \end{cases} \tag{3}$$

and

$$c_n = \frac{a_n}{2}(\beta_1 + (\beta_2 - 1)n) \qquad if \quad n \neq 0.$$

Hence

$$\begin{cases} \psi(d_n^1) = a_n \left(\beta_2 d_n^1 + \frac{\beta_1 + (\beta_2 - 1)n}{2} d_n^0\right) \qquad if \quad n \neq 0, \\ \psi(d_0^1) = d_0^1 + c_0 d_0^0. \end{cases}$$

Then from

$$\psi[d_n^1, d_{-n}^1] = -2n\psi(d_0^1) = -2n(d_0^1 + c_0 d_0^0)$$

and

$$\begin{aligned} &[\psi(d_n^1), \psi(d_{-n}^1)] \\ &= \left[a_n \left(\beta_2 d_n^1 + \frac{\beta_1 + (\beta_2 - 1)n}{2} d_n^0\right), a_{-n} \left(\beta_2 d_{-n}^1 + \frac{\beta_1 - (\beta_2 - 1)n}{2} d_{-n}^0\right)\right] \\ &= -2n\beta_2^2 a_n a_{-n} d_0^1 - n a_n a_{-n} \beta_1 \beta_2 d_0^0, \end{aligned}$$

we get $a_n a_{-n} \beta_2^2 = 1$, and $c_0 = \frac{a_n a_{-n}}{2} \beta_1 \beta_2$ for all $n \neq 0$. Hence

$$\begin{cases} c_0 = \frac{\beta_1}{2\beta_2} \\ c_n = \frac{a_n}{2}(\beta_1 + (\beta_2 - 1)n). \end{cases} \tag{4}$$

From (1) and (3), we have

$$a_{m+1} = a_1 a_m \beta_2 \qquad \forall m \in \mathbb{Z}.$$

By induction on m, we get

$$a_m = \beta_2^{m-1} a^m \qquad \forall m \in \mathbb{Z}, \tag{5}$$

where $a = a_1$. Then from (3) and (4),

$$\begin{cases} b_m = \beta_2^{m-1} a^m \beta_2 \\ c_m = \beta_2^{m-1} a^m \frac{\beta_1 + (\beta_2 - 1)m}{2} \end{cases} \tag{6}$$

for all $m \in \mathbb{Z}$.

In summary, we have proved that

$$(7)\qquad \begin{cases} \psi(d_n^0) = \beta_2^{n-1} a^n d_n^0 \\ \psi(d_n^1) = \beta_2^{n-1} a^n d_n \left(\beta_2 x + \frac{\beta_1 + (\beta_2 - 1)n}{2}\right) \end{cases}$$

for all $n \in \mathbb{Z}$. Assume that

$$\psi(d_n^r) = \beta_2^{n-1} a^n d_n \left(\left(\beta_2 x + \frac{\beta_1 + (\beta_2 - 1)n}{2}\right)^r\right)$$

for all $n \neq 0$. From $\psi[d_n^r, d_0^2] = [\psi(d_n^r), \psi(d_0^2)]$, we get

$$\psi(d_n^{r+1}) = \beta_2^{n-1} a^n d_n \left(\left(\beta_2 x + \frac{\beta_1 + (\beta_2 - 1)n}{2}\right)^{r+1}\right)$$

for all $n \neq 0$. So by induction on r, we obtain

$$\psi(d_n^r) = \beta_2^{n-1} a^n d_n \left(\left(\beta_2 x + \frac{\beta_1 + (\beta_2 - 1)n}{2}\right)^r\right)$$

for all $n \neq 0, r \in \mathbb{Z}_{\geq 0}$.

Since

$$\psi[d_n^3, d_{-n}^0] = -3n\psi(d_0^2) + 3n^2\psi(d_0^1) - n^3\psi(d_0^0),$$

and

$$\begin{aligned} &[\psi(d_n^3), \psi(d_{-n}^0)] \\ &= \left[\beta_2^{n-1} a^n d_n \left(\left(\beta_2 x + \frac{\beta_1 + (\beta_2 - 1)n}{2}\right)^3\right), \beta_2^{-n-1} a^{-n} d_{-n}(1)\right] \\ &= \beta_2^{-2} d_0 \left(\left(\beta_2 x + \frac{\beta_1 + (\beta_2 - 1)n}{2} - \beta_2 n\right)^3 - \left(\beta_2 x + \frac{\beta_1 + (\beta_2 - 1)n}{2}\right)^3\right) \\ &= \beta_2^{-2} d_0 \Big(3\left(\beta_2 x + \frac{\beta_1 + (\beta_2 - 1)n}{2}\right)^2 (-\beta_2 n) \\ &\qquad + 3\left(\beta_2 x + \frac{\beta_1 + (\beta_2 - 1)n}{2}\right)(-\beta_2 n)^2 + (-\beta_2 n)^3\Big), \end{aligned}$$

and

$$\psi\left[d_n^3, d_{-n}^0\right] = \left[\psi(d_n^3), \psi(d_{-n}^0)\right]$$

for arbitrary $n \neq 0$, using (7) and comparing the coefficients of n^3 and n, we get

$$\beta_2^2 = 1$$

and

$$\beta_0 = \beta_2^{-1}\left(\frac{\beta_1}{2}\right)^2.$$

If $\beta_2 = 1$, then (7) and $\beta_0 = \left(\frac{\beta_1}{2}\right)^2$ give us

$$\begin{cases} \psi(d_1^0) = ad_1^0 = \phi_a\psi_{\frac{\beta_1}{2}}(d_1^0) \\ \psi(d_{-1}^0) = a^{-1}d_{-1}^0 = \phi_a\psi_{\frac{\beta_1}{2}}(d_{-1}^0) \\ \psi(d_0^2) = d_0\left(\left(x+\frac{\beta_1}{2}\right)^2\right) = \phi_a\psi_{\frac{\beta_1}{2}}(d_0^2). \end{cases}$$

If $\beta_2 = -1$, then (7) and $\beta_0 = -\left(\frac{\beta_1}{2}\right)^2$ give us

$$\begin{cases} \psi(d_1^0) = ad_1^0 = \phi_a\tau\psi_{\frac{\beta_1}{2}}(d_1^0) \\ \psi(d_{-1}^0) = a^{-1}d_{-1}^0 = \phi_a\tau\psi_{\frac{\beta_1}{2}}(d_{-1}^0) \\ \psi(d_0^2) = -d_0\left(\left(-x+\frac{\beta_1}{2}\right)^2\right) = \phi_a\tau\psi_{\frac{\beta_1}{2}}(d_0^2). \end{cases}$$

Since by Proposition 1, d_0^2, d_1^0 and d_{-1}^0 generate $\mathfrak{g}$, so in both cases above

$$\psi \in \langle \tau, \psi_\beta, \phi_a \mid \beta \in \mathbb{C}, a \in \mathbb{C}^* \rangle.$$

Hence

$$Aut_G(\mathfrak{g}) = (\langle\tau\rangle \ltimes \Psi) \times \Phi.$$

□

1.3 Structure of Homogeneous Lie Subalgebras of $\mathbb{C}[t, t^{-1}, \frac{d}{dt}]$

It is obvious that $\mathfrak{g}\{0\}$ is abelian, $\mathfrak{g}\{x+\alpha\}$ is simple, and $\mathfrak{g}\{0,1\} = \mathfrak{g}\{x\} \ltimes \mathfrak{g}\{0\}$. In this section, we discuss the structure of $\mathfrak{g}_{\langle p(x)\rangle}$ further.

Proposition 1. Let $p(x), q(x) \in \mathbb{C}[x]$. Then

(i) $\mathfrak{g}_{\langle q(x)\rangle}$ is a Lie subalgebra of $\mathfrak{g}_{\langle p(x)\rangle}$ iff $p(x) \mid q(x)$.

(ii) $\mathfrak{g}_{\langle q(x)\rangle}$ is an ideal of $\mathfrak{g}_{\langle p(x)\rangle}$ iff $\langle p(x)\rangle = \langle q(x)\rangle$.

Proof. (i) is clear.

(ii) Suppose that $\mathfrak{g}_{\langle q(x)\rangle}$ is an ideal of $\mathfrak{g}_{\langle p(x)\rangle}$. Then

$$[d_m(p(x)), d_n(q(x))] = d_{m+n}\big(p(x+n)q(x) - p(x)q(x+m)\big) \in \mathfrak{g}_{\langle q(x)\rangle}.$$

Hence

$$q(x) \mid p(x)q(x+m) \quad \forall m \in \mathbb{Z}.$$

Assume that $q(x) = p(x)r(x)$. Then

$$r(x) \mid q(x+m) \quad \forall m \in \mathbb{Z}.$$

So $r(x)$ must be a constant and $\langle p(x)\rangle = \langle q(x)\rangle$. □

Proposition 2. Let

$$p(x), q(x) \in \mathbb{C}[x], \quad where \quad g.c.d(p(x), q(x)) = 1,$$

and let

$$r(x) = p(x)q(x).$$

Then

$$\mathfrak{g}_{\langle r(x)\rangle} = \mathfrak{g}_{\langle p(x)\rangle} \cap \mathfrak{g}_{\langle q(x)\rangle}.$$

In particular, if

$$r(x) = \prod_{i=1}^{r}(x+\alpha_i)^{k_i},$$

where $\alpha_1, ..., \alpha_r$ are distinct, then

$$\mathfrak{g}_{\langle r(x)\rangle} = \bigcap_{i=1}^{r} \mathfrak{g}_{\langle (x+\alpha_i)^{k_i}\rangle}.$$

Proof. Let $X \in \mathfrak{g}_{\langle p(x)\rangle} \cap \mathfrak{g}_{\langle q(x)\rangle}$. Then

$$X = \sum_m d_m\big(p(x)h_m(x)\big) = \sum_m d_m\big(q(x)k_m(x)\big),$$

where $h_m(x), k_m(x) \in \mathbb{C}[x]$. So $p(x)h_m(x) = q(x)k_m(x)$ for all $m \in \mathbb{Z}$. But $g.c.d(p(x), q(x)) = 1$, so $p(x) \mid k_m(x)$ for all $m \in \mathbb{Z}$. Assume that $k_m(x) = p(x)k_m^*(x)$, for some $k_m^*(x) \in \mathbb{C}[x]$. Then $X = \sum_m d_m(q(x)p(x)k_m^*(x)) \in \mathfrak{g}_{\langle r(x)\rangle}$.
□

Proposition 3. $\mathbb{C}d_0^0$ is the centre of $\mathfrak{g}$, which is the only proper ideal of $\mathfrak{g}$, and $\mathfrak{g}/\mathbb{C}d_0^0$ is simple.

Proof. Clearly $\mathbb{C}d_0^0 \subseteq$ centre of $\mathfrak{g}$. Now if $u = \sum_j d_j(p_j(x)) \in$ centre of $\mathfrak{g}$, since $[d_0^1, u] = \sum_j j d_j(p_j(x)) = 0$, we see that $u = d_0(p_0(x)) \in \mathfrak{g}_0$. From

$$[d_m(x), d_0(p_0(x))] = d_m\big(xp_0(x) - xp_0(x+m)\big) = 0$$

for all $m \in \mathbb{Z}$, we get $p_0(x) \in \mathbb{C}$ and $u \in \mathbb{C}d_0^0$. Hence $\mathbb{C}d_0^0$ =centre of $\mathfrak{g}$.

Let $0 \neq I \triangleleft \mathfrak{g}$ be an ideal. From $[d_0^1, I] \subseteq I$, we see that $I = \sum_{m\in\mathbb{Z}} I_m$, where $I_m = I \cap \mathfrak{g}_m$. Suppose that $I \setminus \mathbb{C}d_0^0$ is not empty and $0 \neq d_m(p(x)) \in I \setminus \mathbb{C}d_0^0$ such that $deg(p(x))$ is minimal. If $deg(p(x)) > 0$, then

$$[d_m(p(x)), d_n^0] = d_{m+n}\big(p(x+n) - p(x)\big) \in I$$

and

$$deg\big(p(x+n) - p(x)\big) < deg(p(x)).$$

This is a contradiction. So $p(x) \in \mathbb{C}$, and $d_m^0 \in I \setminus \mathbb{C}d_0^0$. This implies that

$$[d_n^1, d_m^0] = md_{m+n}^0 \in I \qquad \forall n \in \mathbb{Z}.$$

Hence $\mathfrak{g}\{0\} \subseteq I$. Note that $[d_m(x^r), d_n^0] = d_{m+n}((x+n)^r - x^r) \in I$ for all $m, n \in \mathbb{Z}, r \in \mathbb{Z}_{\geq 0}$. By induction on r, we see that $d_m(x^r) \in I, \quad \forall m \in \mathbb{Z}, r \in \mathbb{Z}_{\geq 0}$. So $I = \mathfrak{g}$. □

Proposition 4. For any $\alpha \in \mathbb{C}$, $\mathfrak{g}_{\langle x+\alpha\rangle}$ is simple.

Proof. By Section 1.2, Proposition 2, we need only to show that $\mathfrak{g}_{\langle x\rangle}$ is simple. Let $0 \neq I \triangleleft \mathfrak{g}_{\langle x\rangle}$ be an ideal. Then $I = \sum_{m\in\mathbb{Z}} I_m$, where $I_m = I \cap \mathfrak{g}_m$. Let $d_n(xp(x)) \in I$ and $deg(p(x))$ be minimal. Since

$$[d_m(x), d_n(xp(x))] = d_{m+n}((x+n)xp(x) - x(x+m)p(x+m)) \in I$$

for all $m \in \mathbb{Z}$, we have

$$deg((x+n)p(x) - (x+m)p(x+m)) \geq deg(p(x))$$

for all $m \in \mathbb{Z}$. On the other hand,

$$deg((x+n)p(x) - (x+m)p(x+m)) \leq deg(p(x)).$$

So they must be equal. Thus for any $m \in \mathbb{Z}$, there exists $p_m(x)$ with $deg(p_m(x)) = deg(p(x))$ and $d_m(xp_m(x)) \in I$. We may assume that the coefficient of the highest term of $p_m(x)$ is 1. Then

$$[d_m(x), d_n(xp_n(x))] = d_{m+n}((x+n)xp_n(x) - x(x+m)p_n(x+m)) \in I$$

for all $m,n \in \mathbb{Z}$. Fix $m \neq 0$ and choose $n = m(1 + deg(p(x)))$. Then

$$\begin{aligned}
&(x+n)p_n(x) - (x+m)p_n(x+m) \\
&= (x+n)p_n(x) - (x+m)(p_n(x) + p_n'(x)m + \dots) \\
&= (n-m)p_n(x) - xp_n(x)'m + lower \quad terms.
\end{aligned}$$

So

$$deg((x+n)p_n(x) - (x+m)p_n(x+m)) < deg(p(x))$$

unless $deg(p(x)) = 0$. So $d_m(x) \in I$ for all $m \in \mathbb{Z}$.

Now from $[d_m(x), d_n(x^{r+1})] \in I \quad \forall r \in \mathbb{Z}_{\geq 0}$, and by induction on r, we get $I = \mathfrak{g}_{\langle x \rangle}$. □

Proposition 5. For any $p(x) \in \mathbb{C}[x]$, $\mathfrak{g}_{\langle p(x) \rangle}$ is indecomposable.

Proof. If $deg(p(x)) \leq 1$, this follows from Proposition 3 and 4. Now assume that $deg(p(x)) \geq 2$. Suppose $\mathfrak{g}_{\langle p(x) \rangle} = U_1 \oplus U_2$, where U_1, U_2 are ideals of $\mathfrak{g}_{\langle p(x) \rangle}$ and $U_1 \cap U_2 = 0$ and both U_1 and U_2 are non-zero. Let

$$0 \neq y = d_{j_1}(q_1(x)) + d_{j_2}(q_2(x)) + \dots + d_{j_k}(q_k(x)) \in U_1$$

where $j_1 < j_2 < \dots < j_k$ and $q_1(x), q_2(x), \dots, q_k(x)$ are non-zero. Then for all $m \in \mathbb{Z}$,

$$\begin{aligned}
y_m := &[d_m(p(x)), y] \\
&= [d_m(p(x)), d_{j_1}(q_1(x))] + \dots \\
&= d_{m+j_1}(p(x+j_1)q_1(x) - p(x)q_1(x+m)) + \dots \in U_1.
\end{aligned}$$

Note that

$$q_1^{(m)} := p(x+j_1)q_1(x) - p(x)q_1(x+m) \neq 0$$

if $m \gg 0$, and

$$deg\left(q_1^{(m)}(x)\right) \leq deg(p(x)) + deg(q_1(x))$$

for all $m \in \mathbb{Z}$.

Let

$$0 \neq z = d_{i_1}(r_1(x)) + \dots + d_{i_s}(r_s(x)) \in U_2,$$

where $i_1 < \dots < i_2$, and $r_1(x), \dots, r_s(x)$ are non-zero. Then $[y_m, z] = 0$ for all $m \in \mathbb{Z}$. This implies

$$\begin{aligned}
&\left[d_{m+j_1}\left(q_1^{(m)}(x)\right), d_{i_1}(r_1(x))\right] \\
&= d_{m+j_1+i_1}\left(q_1^{(m)}(x+i_1)r_1(x) - q_1^{(m)}(x)r_1(x+m)\right) \\
&= 0
\end{aligned}$$

for all $m \in \mathbb{Z}$. But

$$deg\left(q_1^{(m)}(x)\right) \leq deg(p(x)) + deg(q_1(x))$$

for all $m \in \mathbb{Z}$, so

$$q_1^{(m)}(x+i_1)r_1(x) - q_1^{(m)}(x)r_1(x+m) \neq 0$$

if $m \gg 0$. Hence

$$d_{m+j_1+i_1}\left(q_1^{(m)}(x+i_1)r_1(x) - q_1^{(m)}(x)r_1(x+m)\right) \neq 0$$

if $m \gg 0$. This is a contradiction. □

1.4 Central Extensions

In this section we consider the 1-dimensional central extensions of homogeneous differential operator Lie algebras and determine the universal central extension of $\mathfrak{g}\{0,1\}$.

Recall that $\mathfrak{gl}_\infty(\mathbb{C}) = \sum_{i,j\in\mathbb{Z}} \mathbb{C}E_{i,j}$ with Lie bracket

$$[E_{i,j}, E_{k,l}] = \delta_{j,k}E_{i,l} - \delta_{l,i}E_{k,j}$$

is a Lie algebra. Define

$$\alpha : \mathfrak{gl}_\infty(\mathbb{C}) \times \mathfrak{gl}_\infty(\mathbb{C}) \longrightarrow \mathbb{C}$$

to be bilinear and

$$\begin{cases} \alpha(E_{ij}, E_{ji}) = -\alpha(E_{ji}, E_{ij}) = 1 & i \leq 0, j \geq 1 \\ \alpha(E_{ij}, E_{kl}) = 0 & otherwise. \end{cases}$$

Then α is a 2-cocycle of the Lie algebra $\mathfrak{gl}_\infty(\mathbb{C})$. Following Kac and Raina [KR], we extend α to be a 2-cocycle of the Lie algebra $\overline{\mathfrak{a}_\infty}$ as follows:

$$\alpha\left(\sum_i \lambda_i E_{i+m,i}, \sum_j \mu_j E_{j+n,j}\right) := \sum_i \sum_j \lambda_i \mu_j \alpha(E_{i+m,i}, E_{j+n,j}).$$

It is easy to check that this is a well defined Lie algebra 2-cocycle on $\overline{\mathfrak{a}_\infty}$.

Define $\mathfrak{a}_\infty := \overline{\mathfrak{a}_\infty} + \mathbb{C}\cent$ and

$$[x,y] := xy - yx + \alpha(x,y)\cent, \qquad [x,\cent] := 0, \qquad \forall x,y \in \overline{\mathfrak{a}_\infty}.$$

Then $\mathfrak{a}_\infty$ is a Lie algebra which is a 1-dimensional central extension of $\overline{\mathfrak{a}_\infty}$. Let

$$\pi : \mathfrak{a}_\infty \longrightarrow \overline{\mathfrak{a}_\infty}$$

be the canonical homomorphism given by $\pi(x) = x$, for all $x \in \overline{\mathfrak{a}_\infty}$, and $\pi(\cent) = 0$. For any Lie subalgebra $\mathfrak{h}$ of $\overline{\mathfrak{a}_\infty}$, $\pi^{-1}(\mathfrak{h})$ is a Lie subalgebra of $\mathfrak{a}_\infty$. In general, it is a 1-dimensional central extension of $\mathfrak{h}$. By Section 1.1, Proposition 1, $\mathfrak{g}$ is a Lie subalgebra of $\overline{\mathfrak{a}_\infty}$. Thus if $\mathfrak{h} \subseteq \mathfrak{g}$ is a homogeneous Lie subalgebra, then $\tilde{\mathfrak{h}} := \pi^{-1}(\mathfrak{h})$ is a 1-dimensional central extension of $\mathfrak{h}$. Precisely, if we identify d^r_m with $\sum_j j^r E_{j+m,j}$ (see Section 1.1, Proposition 1), then for polynomials $p(x), q(x)$,

$$\begin{aligned}
&\alpha\left(\sum_j p(j)E_{j+m,j}, \sum_i q(i)E_{i+n,i}\right) \\
&= \sum_j \sum_i p(j)q(i)\alpha(E_{j+m,j}, E_{i+n,i}) \\
&= \left(\sum_{1\le j\le n} p(j)q(j-n) - \sum_{1\le j\le m} p(j-m)q(j)\right)\delta_{m+n,0}.
\end{aligned}$$

So in $\tilde{\mathfrak{g}}$, the Lie brackets are

$$\begin{aligned}
[d_m(p(x)), d_n(q(x))] &= d_{m+n}(p(x+n)q(x) - p(x)q(x+m)) \\
&\quad + \left(\sum_{1\le j\le n} p(j)q(j-n) - \sum_{1\le j\le m} p(j-m)q(j)\right)\delta_{m+n,0}\cent. \\
[d_m(p(x)), \cent] &= 0.
\end{aligned}$$

Example 1:

$$\begin{aligned}
\widetilde{\mathfrak{g}\{0\}} &= \sum_{m\in\mathbb{Z}} \mathbb{C}d^0_m + \mathbb{C}\cent, \\
[d^0_m, d^0_n] &= -2m\delta_{m+n,0}\cent, \\
[d^0_m, \cent] &= 0.
\end{aligned}$$

Example 2:

$$\widetilde{\mathfrak{g}\{x\}} = \sum_{m\in\mathbb{Z}} \mathbb{C}d_m^1 + \mathbb{C}\not{c},$$

$$[d_m^1, d_n^1] = (n-m)d_{m+n}^1 + \frac{m^3-m}{6}\delta_{m+n,0}\not{c},$$

$$[d_m^1, \not{c}] = 0.$$

We see that $\widetilde{\mathfrak{g}\{x\}}$ is the Virasoro algebra, which is the universal central extension of $\mathfrak{g}\{x\}$.

Wanglai Li [Li] proved the following result: The 1-dimensional central extension of $\mathfrak{g}$ is unique up to a scalar multiple. In the rest of this section we determine the universal central extension of $\mathfrak{g}\{0,1\}$. Since $\mathfrak{g}\{0,1\}$ is perfect, its universal central extension exists.

Generally, if a Lie algebra $\mathfrak{h}$ is perfect, its universal central extension can be obtained as follows:

Consider $\wedge^2\mathfrak{h}$, the second exterior power of $\mathfrak{h}$. Let $I \subseteq \wedge^2\mathfrak{h}$ be the subspace spanned by all of the elements

$$x \wedge [y,z] + y \wedge [z,x] + z \wedge [x,y]$$

where $x, y, z \in \mathfrak{h}$. Let

$$\overline{\wedge} : \mathfrak{h} \times \mathfrak{h} \longrightarrow \frac{\wedge^2\mathfrak{h}}{I}$$

be the canonical map

$$(x,y) \longmapsto x\overline{\wedge}y := x \wedge y + I.$$

Then $\overline{\wedge}$ is the "universal" 2-cocycle for $\mathfrak{h}$, and $\mathfrak{h} \oplus (\wedge^2\mathfrak{h}/I)$ acquires a Lie algebra structure through

$$[x+u, y+v] = [x,y] + x\overline{\wedge}y \qquad where \quad x, y \in \mathfrak{h}, \quad u, v \in \frac{\wedge^2\mathfrak{h}}{I}.$$

Now

$$\tilde{\mathfrak{h}} := \left[\mathfrak{h} \oplus \frac{\wedge^2\mathfrak{h}}{I}, \mathfrak{h} \oplus \frac{\wedge^2\mathfrak{h}}{I}\right]$$

together with the restriction to $\tilde{\mathfrak{h}}$ of the natural projection of $\mathfrak{h} \oplus (\wedge^2\mathfrak{h}/I)$ onto $\tilde{\mathfrak{h}}$ is the required universal central extension (see [G] or [MoPi]).

In our case,

$$\mathfrak{g}\{0,1\} = \sum_{m\in\mathbb{Z}} \mathbb{C}d_m^1 + \sum_{m\in\mathbb{Z}} \mathbb{C}d_m^0.$$

$$\frac{\wedge^2 \mathfrak{g}\{0,1\}}{I}$$

is spanned by

$$\overline{\wedge}\left(d^1_m, d^1_n\right), \quad \overline{\wedge}\left(d^1_m, d^0_n\right), \quad \overline{\wedge}\left(d^0_m, d^0_n\right) \qquad \forall m, n \in \mathbb{Z}.$$

The 2-cocycle condition gives us

$$\begin{aligned} (1) \qquad & \overline{\wedge}\left(d^1_m, \left[d^0_n, d^0_k\right]\right) + \overline{\wedge}\left(d^0_n, \left[d^0_k, d^1_m\right]\right) + \overline{\wedge}\left(d^0_k, \left[d^1_m, d^0_n\right]\right) \\ & = -k\overline{\wedge}\left(d^0_n, d^0_{m+k}\right) + n\overline{\wedge}\left(d^0_k, d^0_{m+n}\right) \\ & = 0 \end{aligned}$$

for all $m, n, k \in \mathbb{Z}$. Let $k = -m - n$ in (1). Then

$$n\overline{\wedge}\left(d^0_{m+n}, d^0_{-m-n}\right) = (m+n)\overline{\wedge}\left(d^0_n, d^0_{-n}\right).$$

Thus

$$\mathfrak{c}_3 := \frac{\overline{\wedge}\left(d^0_n, d^0_{-n}\right)}{n}$$

is independent of n, $n \neq 0$. Let $k = 0$ in (1). Then

$$\overline{\wedge}\left(d^0_0, d^0_n\right) = 0 \qquad \forall n \in \mathbb{Z}.$$

From this, setting $m + n = 0$ in (1), we have

$$\overline{\wedge}\left(d^0_n, d^0_{k-n}\right) = 0 \qquad \forall k \neq 0.$$

Combining these results, we get

$$(2) \qquad \overline{\wedge}\left(d^0_m, d^0_n\right) = m\delta_{m+n,0}\mathfrak{c}_3.$$

Again by 2-cocycle condition,

$$\begin{aligned} (3) \qquad & \overline{\wedge}\left(d^1_m, \left[d^1_n, d^0_k\right]\right) + \overline{\wedge}\left(d^1_n, \left[d^0_k, d^1_m\right]\right) + \overline{\wedge}\left(d^0_k, \left[d^1_m, d^1_n\right]\right) \\ & = k\overline{\wedge}\left(d^1_m, d^0_{n+k}\right) - k\overline{\wedge}\left(d^1_n, d^0_{k+m}\right) + (n-m)\overline{\wedge}\left(d^0_k, d^1_{m+n}\right) \\ & = 0. \end{aligned}$$

If $m = 0$ in (3), then

$$k\overline{\wedge}\left(d^1_0, d^0_{n+k}\right) = (k+n)\overline{\wedge}\left(d^1_n, d^0_k\right).$$

With $k = -m - n$ in (3), we get

$$(4) \qquad (m-n)\overline{\wedge}\left(d^1_{m+n}, d^0_{-m-n}\right) = (m+n)\left(\overline{\wedge}\left(d^1_m, d^0_{-m}\right) - \overline{\wedge}\left(d^1_n, d^0_{-n}\right)\right).$$

Replacing m by $2m$ in (4), we obtain

$$(5) \qquad \begin{aligned} &(2m-n)\overline{\wedge}\left(d^1_{2m+n}, d^0_{-2m-n}\right) \\ &= (2m+n)\left(\overline{\wedge}\left(d^1_{2m}, d^0_{-2m}\right) - \overline{\wedge}\left(d^1_n, d^0_{-n}\right)\right). \end{aligned}$$

Replacing n by $m+n$ in (4), we obtain

$$\begin{aligned} &-n\overline{\wedge}\left(d^1_{2m+n}, d^0_{-2m-n}\right) \\ &= (2m+n)\left(\overline{\wedge}\left(d^1_m, d^0_{-m}\right) - \overline{\wedge}\left(d^1_{m+n}, d^0_{-m-n}\right)\right). \end{aligned}$$

Multiplying this by $m-n$ and replacing the $\overline{\wedge}\left(d^1_{m+n}, d^0_{-m-n}\right)$ term using (4), we obtain

$$(6) \qquad \begin{aligned} &-n(m-n)\overline{\wedge}\left(d^1_{2m+n}, d^0_{-2m-n}\right) \\ &= -2n(2m+n)\overline{\wedge}\left(d^1_m, d^0_{-m}\right) + (2m+n)(m+n)\overline{\wedge}\left(d^1_n, d^0_{-n}\right). \end{aligned}$$

From (5) and (6), we get

$$\begin{aligned} &-n(m-n)(2m+n)\overline{\wedge}\left(d^1_{2m}, d^0_{-2m}\right) + 2n(2m+n)(2m-n)\overline{\wedge}\left(d^1_m, d^0_{-m}\right) \\ &= (2m+n)2m^2\overline{\wedge}\left(d^1_n, d^0_{-n}\right). \end{aligned}$$

If $2m+n \neq 0, m-n \neq 0, m \neq 0, n \neq 0$, then

$$\frac{\overline{\wedge}\left(d^1_{2m}, d^0_{-2m}\right) - 2\overline{\wedge}\left(d^1_m, d^0_{-m}\right)}{m^2} = \frac{2\overline{\wedge}\left(d^1_m, d^0_{-m}\right)}{m(m-n)} + \frac{2\overline{\wedge}\left(d^1_n, d^0_{-n}\right)}{n(n-m)}.$$

Thus

$$\mathfrak{c}_2 := \frac{\overline{\wedge}\left(d^1_{2m}, d^0_{-2m}\right) - 2\overline{\wedge}\left(d^1_m, d^0_{-m}\right)}{m^2}$$

is independent of m, $m \neq 0$.

Finally, since

$$\overline{\wedge}\left(d^1_m, \left[d^1_n, d^1_k\right]\right) + \overline{\wedge}\left(d^1_n, \left[d^1_k, d^1_m\right]\right) + \overline{\wedge}\left(d^1_k, \left[d^1_m, d^1_n\right]\right) = 0,$$

similar calculation as above shows that

$$(*) \qquad (k+n)\overline{\wedge}\left(d^1_n, d^1_k\right) = (k-n)\overline{\wedge}\left(d^1_0, d^1_{n+k}\right)$$

for all $n, k \in \mathbb{Z}$, and

$$(**) \qquad \frac{2\left(\overline{\wedge}\left(d^1_{2m}, d^1_{-2m}\right) - 2\overline{\wedge}\left(d^1_m, d^1_{-m}\right)\right)}{m^3} = \frac{12\overline{\wedge}\left(d^1_m, d^1_{-m}\right)}{m(m^2-n^2)} + \frac{12\overline{\wedge}\left(d^1_n, d^1_{-n}\right)}{n(n^2-m^2)}$$

for all $m, n \neq 0, m^2 \neq n^2$. Thus

$$\not{c}_1 := \frac{2\left(\overline{\wedge}\left(d^1_{2m}, d^1_{-2m}\right) - 2\overline{\wedge}\left(d^1_m, d^1_{-m}\right)\right)}{m^3}$$

is independent of m, $m \neq 0$.

Now we define

$$\widetilde{d^1_m} := d^1_m + \frac{1}{m}\overline{\wedge}\left(d^1_0, d^1_m\right) \qquad where \quad m \neq 0,$$

$$\widetilde{d^1_0} := d^1_0 - \frac{1}{2}\overline{\wedge}\left(d^1_1, d^1_{-1}\right),$$

$$\widetilde{d^0_m} := d^0_m + \frac{1}{m}\overline{\wedge}\left(d^1_0, d^0_m\right) \qquad where \quad m \neq 0,$$

$$\widetilde{d^0_0} := d^0_0 - \frac{1}{2}\overline{\wedge}\left(d^0_1, d^0_{-1}\right),$$

and

$$\widetilde{\mathfrak{g}\{0,1\}} := \sum_{m\in\mathbb{Z}} \mathbb{C}\widetilde{d^1_m} + \sum_{m\in\mathbb{Z}} \mathbb{C}\widetilde{d^0_m} + \mathbb{C}\not{c}_1 + \mathbb{C}\not{c}_2 + \mathbb{C}\not{c}_3.$$

Proposition 1. $\widetilde{\mathfrak{g}\{0,1\}}$ is the universal central extension of $\mathfrak{g}\{0,1\}$ and the Lie brackets of $\widetilde{\mathfrak{g}\{0,1\}}$ are

$$\left[\widetilde{d^1_m}, \widetilde{d^1_n}\right] = (n-m)\widetilde{d^1_{m+n}} + \frac{m^3-m}{12}\delta_{m+n,0}\not{c}_1,$$

$$\left[\widetilde{d^1_m}, \widetilde{d^0_n}\right] = n\widetilde{d^0_{m+n}} + \frac{m(m-1)}{2}\delta_{m+n,0}\not{c}_2,$$

$$\left[\widetilde{d^0_m}, \widetilde{d^0_n}\right] = m\delta_{m+n,0}\not{c}_3,$$

where $\not{c}_1, \not{c}_2, \not{c}_3$ are in the centre.

Proof. Clearly,

$$\widetilde{\mathfrak{g}\{0,1\}} \subseteq \mathfrak{g}\{0,1\} \oplus \frac{\overline{\wedge}\mathfrak{g}\{0,1\}}{I}$$

and we have

$$\begin{aligned}
&\left[\widetilde{d}^1_m, \widetilde{d}^1_n\right] \\
&= \left[d^1_m, d^1_n\right] + \overline{\wedge}\left(d^1_m, d^1_n\right) \\
&= (n-m)d^1_{m+n} + \frac{n-m}{n+m}\overline{\wedge}\left(d^1_0, d^1_{m+n}\right) \qquad (using \quad (*)) \\
&= (n-m)\left(\widetilde{d^1_{m+n}} - \frac{1}{m+n}\overline{\wedge}\left(d^1_0, d^1_m\right)\right) + \frac{n-m}{n+m}\overline{\wedge}\left(d^1_0, d^1_{m+n}\right) \\
&= (n-m)\overline{\wedge}d^1_{m+n}, \qquad (provided \quad that \quad m+n \neq 0)
\end{aligned}$$

$$\left[\widetilde{d^1_m}, \widetilde{d^1_{-m}}\right]$$
$$= -2md^1_0 + \overline{\wedge}\left(d^1_m, d^1_{-m}\right)$$
$$= -2m\left(\widetilde{d^1_0} + \frac{1}{2}\overline{\wedge}\left(d^1_1, d^1_{-1}\right)\right) + \overline{\wedge}\left(d^1_m, d^1_{-m}\right)$$
$$= -2m\widetilde{d^1_0} - m\overline{\wedge}\left(d^1_1, d^1_{-1}\right) + m(m^2-1)\frac{1}{12}¢_1 + m\overline{\wedge}\left(d^1_1, d^1_{-1}\right)$$
$$(using \quad (**) \quad with \quad n=1)$$
$$= -2m\widetilde{d^1_0} + \frac{m^3-m}{12}\delta_{m+n,0}¢_1.$$

We can verify the other two commutators similarly. So $\widetilde{\mathfrak{g}\{0,1\}}$ is a Lie subalgebra of $\mathfrak{g}\{0,1\} \oplus \left(\wedge^2\mathfrak{g}\{0,1\}/I\right)$. Moreover,

$$\left[\mathfrak{g}\{0,1\} \oplus \frac{\wedge^2\mathfrak{g}\{0,1\}}{I}, \quad \mathfrak{g}\{0,1\} \oplus \frac{\wedge^2\mathfrak{g}\{0,1\}}{I}\right]$$
$$= \left[\widetilde{\mathfrak{g}\{0,1\}}, \widetilde{\mathfrak{g}\{0,1\}}\right]$$
$$= \widetilde{\mathfrak{g}\{0,1\}},$$

so $\widetilde{\mathfrak{g}\{0,1\}}$ is the universal central extension of $\mathfrak{g}\{0,1\}$. □

1.5 Admissible Modules

Let $\mathfrak{h} \subseteq \mathfrak{g}$ be a homogeneous Lie subalgebra and $\mathfrak{h} = \sum_{k\in\mathbb{Z}} \mathfrak{h}_k$. Then $\mathfrak{h}_0$ is an abelian Lie subalgebra of $\mathfrak{h}$. A $\mathfrak{h}$-module V is called admissible if

(i) $V = \sum_{\lambda\in\mathfrak{h}_0^*} V_\lambda$, where $\mathfrak{h}_0^*$ is the dual space of $\mathfrak{h}_0$, and

$$V_\lambda = \{v \in V \mid X\cdot v = \lambda(X)v \quad \forall X \in \mathfrak{h}_0\}.$$

(ii) $dim(V_\lambda) < \infty$ for all $\lambda \in \mathfrak{h}_0^*$.

For $\mathfrak{g}\{x\}$, I.Kaplansky [Kap] and I.Kaplansky and L.J.Santharoubane [KS] proved the following result:

If $V = \sum_{k\in\mathbb{Z}} \mathbb{C}v_k$ is a $\mathfrak{g}\{x\}$-module such that

$$d^1_m \cdot v_k \in \mathbb{C}v_{m+k} \quad \forall m,k \in \mathbb{Z},$$

$$d^1_1 \cdot v_k \neq 0, \quad d^1_{-1}\cdot v_k \neq 0 \quad \forall k \in \mathbb{Z},$$

then there exists $\alpha, \beta \in \mathbb{C}$ such that

$$d^1_m \cdot v_k = (k + \alpha m + \beta)v_{m+k} \quad \forall m,k \in \mathbb{Z}.$$

In this section we prove similar results for $\mathfrak{g}$ and $\mathfrak{g}\{0,1\}$.

Proposition 1. Let $V = \sum_{k\in\mathbb{Z}} \mathbb{C}v_k$ be a $\mathfrak{g}\{0,1\}$-module such that

$$d^1_m \cdot v_k \in \mathbb{C}v_{m+k}, \quad d^0_m \cdot v_k \in \mathbb{C}v_{m+k},$$

and

$$d^1_1 \cdot v_k \neq 0, \quad d^1_{-1} \cdot v_k \neq 0$$

for all $k \in \mathbb{Z}$. Then there exists $\alpha, \beta, \gamma \in \mathbb{C}$ such that

(i) $d^1_m \cdot v_k = (k + \alpha m + \beta)v_{m+k}$.

(ii) If $\alpha \neq 0, 1$, then $d^0_m \cdot v_k = \gamma v_{m+k}$ for all $m, k \in \mathbb{Z}$.

If $\alpha = 0$, then

$$d^0_m \cdot v_k = \gamma v_{m+k} \quad \forall m, k \in \mathbb{Z},$$

or

$$d^0_m \cdot v_k = \frac{(k+\beta)\gamma}{k+m+\beta} v_{m+k} \quad \forall m, k \in \mathbb{Z}.$$

If $\alpha = 1$, then

$$d^0_m \cdot v_k = \gamma v_{m+k} \quad \forall m, k \in \mathbb{Z},$$

or

$$d^0_m \cdot v_k = \frac{(k+m+\beta)\gamma}{k+\beta} v_{m+k} \quad \forall m, k \in \mathbb{Z}.$$

Proof. (i) follows from the result of Kaplansky and Santharoubane's quoted above.

(ii). First we note that by assumption,

$$d^1_1 v_k = (k + \alpha + \beta)v_{k+1} \neq 0, \quad d^1_{-1} v_k = (k - \alpha + \beta)v_{k-1} \neq 0,$$

for all $k \in \mathbb{Z}$. So $\beta \pm \alpha \notin \mathbb{Z}$. Assume that $d^0_m \cdot v_k = f(m,k)v_{m+k}$. Then from

$$d^0_m d^0_n v_k = d^0_n d^0_m v_k$$

and

$$\left[d^1_m, d^0_n\right] v_k = \left(d^1_m d^0_n - d^0_n d^1_m\right) v_k,$$

we obtain

$$(1) \qquad \begin{cases} f(n,k)f(m,n+k) = f(m,k)f(n,m+k) \\ nf(m+n,k) = f(n,k)(n+k+\alpha m+\beta) - f(n,m+k)(k+\alpha m+\beta) \end{cases}$$

for all $m, n, k \in \mathbb{Z}$.

Let $k = 0$ in (1). Then

$$\begin{cases} f(n,0)f(m,n) = f(m,0)f(n,m) \\ nf(m+n,0) = f(n,0)(n+\alpha m+\beta) - f(n,m)(\alpha m+\beta), \end{cases} \tag{2}$$

or

$$\begin{cases} f(n,0)f(m,n) = f(m,0)f(n,m) \\ f(n,m)(\alpha m+\beta) = f(n,0)(n+\alpha m+\beta) - nf(m+n,0). \end{cases} \tag{3}$$

Multiplying both sides of the second identity of (1) by $(\alpha k+\beta)(\alpha(m+k)+\beta)$ and using the second identity of (3), we obtain

(4)

$$\begin{aligned} &n\big(\alpha(m+k)+\beta\big)\big(f(m+n,0)(m+n+\alpha k+\beta) - (m+n)f(m+n+k,0)\big) \\ &= (\alpha(m+k)+\beta)(n+k+\alpha m+\beta)\big(f(n,0)(n+k\alpha+\beta) - nf(n+k,0)\big) \\ &\quad - (\alpha k+\beta)(k+\alpha m+\beta)\big(f(n,0)(n+\alpha(m+k)+\beta) - nf(m+n+k,0)\big). \end{aligned}$$

Let $m = -1, \quad n = 1$ in (4). Then

$$\begin{aligned} &(\alpha(k-1)+\beta)(k+1-\alpha+\beta)f(k+1,0) \\ &= (\alpha k+\beta)(k-\alpha+\beta)f(k,0) \\ &\quad + \big((\alpha(k-1)+\beta)(\alpha k+\beta) + (1-\alpha)(\beta-\alpha)\big)f(1,0) \\ &\quad - (\alpha(k-1)+\beta)(\alpha k+\beta)f(0,0). \end{aligned} \tag{5}$$

With $k = -1$ in (5), we get

$$(\beta-\alpha-1)f(-1,0) - 2(\beta-2\alpha)f(0,0) + (\beta-3\alpha+1)f(1,0) = 0. \tag{6}$$

Let $m = k = 1, \quad n = -1$ in (4). Then

$$(\beta+3\alpha-1)f(-1,0) - 2(\beta+2\alpha)f(0,0) + (\beta+\alpha+1)f(1,0) = 0. \tag{7}$$

Combining (6) and (7) we get

$$\alpha(1-\alpha)f(0,0) = \alpha(1-\alpha)f(1,0).$$

We consider the following three different cases:

(a). $\alpha \neq 0, 1$.

In this case, $f(0,0) = f(1,0)$ and (5) becomes

$$(8) \qquad \begin{aligned} &(\alpha(k-1)+\beta)(k+1-\alpha+\beta)f(k+1,0) \\ &= (\alpha k+\beta)(k-\alpha+\beta)f(k,0)+(1-\alpha)(\beta-\alpha)f(0,0). \end{aligned}$$

If $0 \notin \alpha\mathbb{Z}+\beta$, then by induction on k and the fact $\beta \pm \alpha \notin \mathbb{Z}$, we get $f(k,0) = f(0,0)$ for all $k \in \mathbb{Z}$. Then by (3), we get

$$f(m,n) = f(0,0) \quad \forall m,n \in \mathbb{Z}.$$

Now suppose $\alpha l + \beta = 0$, for some $l \in \mathbb{Z}$. Since $\beta \pm \alpha \notin \mathbb{Z}$, $l = 0$, or $l \geq 2$, or $l \leq -2$.

If $l = 0$, then $\beta = 0$ and (8) becomes

$$(9) \qquad (k-1)(k+1-\alpha)f(k+1,0) = k(k-\alpha)f(k,0)-(1-\alpha)f(0,0).$$

With $k = -1$ in (9), we get

$$2\alpha f(0,0) = (1+\alpha)f(-1,0)-(1-\alpha)f(0,0).$$

Hence

$$f(-1,0) = f(0,0).$$

By induction on k we have

$$f(k,0) = f(0,0)$$

for all $k \leq 1$. With $k = 2$ in (9) we get

$$(3-\alpha)f(3,0) = 2(2-\alpha)f(2,0)-(1-\alpha)f(0,0).$$

By induction on k we have

$$(10) \qquad (k-\alpha)f(k,0) = (k-1)(2-\alpha)f(2,0)-(k-2)(1-\alpha)f(0,0)$$

for all $k \geq 2$. With $n = -1, m = 2$ in the second identity of (3), we obtain

$$\begin{aligned} 2\alpha f(-1,2) &= (2\alpha-1)f(-1,0)+f(1,0) \\ &= 2\alpha f(0,0). \end{aligned}$$

or

$$f(-1,2) = f(0,0).$$

With $m = -1, n = 2$ in the second identity of (3), we obtain

$$-\alpha f(2,-1) = (-\alpha + 2)f(2,0) - 2f(1,0).$$

With these and by the first identity of (3), we get

$$-\alpha f(2,0)f(0,0) = (2-\alpha)f(2,0)f(0,0) - 2f(0,0)f(0,0),$$

or

$$f(0,0)(f(2,0) - f(0,0)).$$

If $f(0,0) \neq 0$, then $f(2,0) = f(0.0)$. Then by (10), $f(k,0) = f(0,0)$ for all $k \in \mathbb{Z}$.

If $f(0,0) = 0$, then (10) becomes

(11) $$(k-\alpha)f(k,0) = (k-1)(2-\alpha)f(2,0)$$

for all $k \geq 2$. Using this and letting $n = 1, m = 2$ in the second identity of (3), we get

$$2\alpha f(1,2) = -f(3,0).$$

With this and the fact $f(1,0) = 0$, and with $m = 1, n = 2$ in the first identity of (3), we get $f(2,0)f(3,0) = 0$. Hence by (11), $f(2,0) = f(3,0) = 0$. These imply $f(k,0) = 0 = f(0,0)$ for all $k \in \mathbb{Z}$. Now by the second identity of (3),

$$f(n,m) = f(0,0)$$

for all $m, n \in \mathbb{Z}$.

If $l \geq 2$, then

$$\alpha k + \beta \neq 0$$

for all $k < l$, and by (8),

$$f(l,0) = ... = f(1,0) = f(0,0) = ... \quad .$$

From this and with $m = l$ and $n = 1$ in the second identity of (3), we obtain $f(l+1,0) = f(l,0)$. Then by (8),

$$f(l,0) = f(l+1,0) = ... \quad .$$

So $f(k,0) = f(0,0) \quad \forall k \in \mathbb{Z}$. Now by the second identity of (3) we obtain $f(n,m) = f(0,0)$ for all $m \neq l$.

If $f(0,0) \neq 0$, multiplying both side of the first identity of (3) by $f(0,0)^{-1} = f(n,0)^{-1} = f(m,0)^{-1}$, we get $f(m,n) = f(n,m) \quad \forall m,n \in \mathbb{Z}$. Hence

$$f(m,n) = f(0,0) \quad \forall m,n \in \mathbb{Z}.$$

If $f(0,0) = 0$, then $f(n,m) = 0$ for all $m \neq l$. From this and with $m \neq 0, k = l$ in the second identity of (1), we get

$$nf(m+n,l) = f(n,l)(n+l+\alpha m+\beta). \tag{12}$$

With $n = 0$ and $l + \alpha m + \beta \neq 0$ in (12), we obtain $f(0,l) = 0$. With $m + n = 0$ in (12), we obtain $f(n,l) = 0$ for all $n \gg 0$. Now (12) gives us $nf(m+n,l) = 0$ for all $n \gg 0$ and all $m \in \mathbb{Z}$. So $f(n,l) = 0 = f(0,0)$ for all $n \in \mathbb{Z}$. We have proved that

$$f(m,n) = 0 = f(0,0) \qquad \forall m \in \mathbb{Z}.$$

The case of $l \leq -2$ can be proved similarly. So for $\alpha \neq 0, 1$, we have

$$f(m,n) = f(0,0) \qquad \forall m,n \in \mathbb{Z}.$$

(b). $\alpha = 0$.
In this case $\beta \notin \mathbb{Z}$, and (3) becomes

$$\begin{cases} f(n,0)f(m,n) = f(m,0)f(n,m) \\ \beta f(n,m) = (n+\beta)f(n,0) - nf(m+n,0), \end{cases} \tag{13}$$

and (5) becomes

$$(k+1+\beta)f(k+1,0) = (k+\beta)f(k,0) + (1+\beta)f(1,0) - \beta f(0,0). \tag{14}$$

By induction on k, we obtain from (14) that

$$f(k,0) = \frac{k(1+\beta)}{k+\beta}f(1,0) - \frac{(k-1)\beta}{k+\beta}f(0,0). \tag{15}$$

Thus

$$f(-1,0) = -\frac{1+\beta}{\beta-1}f(1,0) + \frac{2\beta}{\beta-1}f(0,0).$$

With $n = 1, m = -1$ in (13), we get

$$f(1,0)f(-1,1) = f(-1,0)f(1,-1)$$

and

$$f(1,-1) = \frac{1+\beta}{\beta} f(1,0) - \frac{1}{\beta} f(0,0).$$

With $n = -1, m = 1$ in the second identity of (13), we get

$$f(-1,1) = -\frac{1+\beta}{\beta} f(1,0) + \frac{1+2\beta}{\beta} f(0,0).$$

Combining above results, we get

$$(1+\beta) f^2(1,0) - (1+2\beta) f(1,0) f(0,0) + \beta f^2(0,0) = 0,$$

or

$$\big(f(1,0) - f(0,0)\big)\big((1+\beta) f(1,0) - \beta f(0,0)\big) = 0.$$

Hence

$$f(1,0) = f(0,0)$$

or

$$f(1,0) = \frac{\beta}{1+\beta} f(0,0).$$

If $f(1,0) = f(0,0)$, by (15), $f(k,0) = f(0,0) \quad \forall k \in \mathbb{Z}$. Then by (13),

$$f(n,m) = f(0,0) \quad \forall m,n \in \mathbb{Z}.$$

If $f(1,0) = \frac{\beta}{1+\beta} f(0,0)$, by (15), $f(k,0) = \frac{\beta}{k+\beta} f(0,0) \quad \forall k \in \mathbb{Z}$. Then by (13), we obtain

$$f(n,m) = \frac{m+\beta}{n+m+\beta} f(0,0) \quad \forall m,n \in \mathbb{Z}.$$

(c). $\alpha = 1$.

In this case $\beta \notin \mathbb{Z}$ and (3) becomes

$$\text{(16)} \qquad \begin{cases} f(n,0) f(m,n) = f(m,0) f(n,m) \\ f(n,m)(m+\beta) = f(n,0)(n+m+\beta) - n f(m+n,0), \end{cases}$$

and (5) becomes

$$\text{(17)} \qquad f(k+1,0) = f(k,0) + f(1,0) - f(0,0).$$

With $k = -1$ in (17), we get

$$\text{(18)} \qquad f(-1,0) = 2f(0,0) - f(1,0).$$

With $n=-1, m=1$ in (16), we get

$$f(-1,0)f(1,-1) = f(1,0)f(-1,1),$$

and

$$f(-1,1) = \frac{\beta}{1+\beta}f(-1,0) + \frac{1}{1+\beta}f(0,0).$$

Again, with $n=1, m=-1$ in the second identity of (16), we get

$$f(1,-1) = \frac{\beta}{\beta-1}f(1,0) - \frac{1}{\beta-1}f(0,0).$$

Combining above results, we have

$$(\beta-1)(f(1,0))^2 - 2\beta f(1,0)f(-1,0) + (1+\beta)(f(-1,0))^2 = 0,$$

or

$$\big((\beta-1)f(1,0) - (\beta+1)f(-1,0)\big)\big(f(1,0) - f(-1,0)\big) = 0.$$

Hence

$$f(-1,0) = f(1,0)$$

or

$$f(-1,0) = \frac{\beta-1}{\beta+1}f(1,0).$$

Then from (18),

$$f(1,0) = f(0,0)$$

or

$$f(1,0) = \frac{1+\beta}{\beta}f(0,0).$$

If $f(1,0) = f(0,0)$, by (17), $f(k,0) = f(0,0) \quad \forall k \in \mathbb{Z}$. Then by the second identity of (16),

$$f(n,m) = f(0,0) \quad \forall m,n \in \mathbb{Z}.$$

If $f(1,0) = \frac{1+\beta}{\beta}f(0,0)$, by (17), $f(k,0) = \frac{k+\beta}{\beta}f(0,0)$. Then by the second identity of (16),

$$f(n,m) = \frac{m+n+\beta}{m+\beta}f(0,0) \quad \forall m,n \in \mathbb{Z}.$$

□

Remark. It is easy to check that the action of $\mathfrak{g}\{0,1\}$ on V given by $(i),(ii)$ in Proposition 1 indeed make V into a $\mathfrak{g}\{0,1\}$-module. Moreover, if $\gamma \neq 0$, then V is an irreducible $\mathfrak{g}\{0,1\}$-module.

Now we suppose that $V = \sum_{k\in\mathbb{Z}} \mathbb{C}v_k$ is a $\mathfrak{g}$-module and satisfies the following conditions:

$d^r_m \cdot v_k \in \mathbb{C}v_{m+k}$ for all $m,k \in \mathbb{Z}, r \in \mathbb{Z}_{\geq 0}$, and $d^1_1 \cdot v_k \neq 0$, $d^1_{-1} \cdot v_k \neq 0$ for all $k \in \mathbb{Z}$.

Since V is a $\mathfrak{g}\{0,1\}$-module by restriction, by Proposition 1, there exists $\alpha, \beta, \gamma \in \mathbb{C}$ such that one of the following occurs:

$$(a)\quad d^1_m \cdot v_k = (k+\alpha m+\beta)v_{m+k}, \quad d^0_m \cdot v_k = \gamma v_{m+k}, \quad \textit{where} \quad \beta \pm \alpha \notin \mathbb{Z}.$$

$$(b)\quad d^1_m \cdot v_k = (k+\beta)v_{m+k}, \quad d^0_m \cdot v_k = \frac{(k+\beta)\gamma}{m+k+\beta}v_{m+k}, \quad \textit{where} \quad \alpha = 0, \quad \beta \notin \mathbb{Z}.$$

$$(c)\quad d^1_m \cdot v_k = (k+m+\beta)v_{m+k}, \quad d^0_m \cdot v_k = \frac{(k+m+\beta)\gamma}{k+\beta}v_{m+k}, \quad \textit{where} \quad \alpha = 1, \quad \beta \notin \mathbb{Z}.$$

Claim 1. $\gamma \neq 0$.

Proof. If $\gamma = 0$, then $d^0_m v_k = 0 \quad \forall m,k \in \mathbb{Z}$. Comparing

$$[d^2_m, d^0_n] \cdot v_k = 2nd^1_{m+n} \cdot v_k = 2n(k+\alpha(m+n)+\beta)v_{m+n+k}$$

with

$$(d^2_m d^0_n - d^0_n d^2_m) \cdot v_k = -d^0_n d^2_m \cdot v_k = 0,$$

we get

$$2n(k+\alpha(m+n)+\beta) \cdot v_{m+n+k} = 0 \quad \forall m,n,k \in \mathbb{Z}.$$

This is a contradiction. □

Assume that $d^r_m v_k = f_r(m,k)v_{m+k}, \quad \forall m,k \in \mathbb{Z}, \quad \forall r \in \mathbb{Z}_{\geq 0}$. Since

$$[d^r_m, d^2_0] = -2md^{r+1}_m - m^2 d^r_m,$$

$$[d^r_m, d^2_0]v_k = \big(-2mf_{r+1}(m,k) - m^2 f_r(m,k)\big)v_{m+k}.$$

On the other hand

$$\left(d^r_m d^2_0 - d^2_0 d^r_m\right) v_k = \big(f_2(0,k)f_r(m,k) - f_r(m,k)f_2(0,m+k)\big)v_{m+k}.$$

So we have

$$(19) \qquad -2mf_{r+1}(m,k) = f_r(m,k)\big(m^2 + f_2(0,k) - f_2(0,m+k)\big).$$

Since

$$[d^r_m, d^0_n] = \tbinom{r}{1} n d^{r-1}_{m+n} + \tbinom{r}{2} n^2 d^{r-2}_{m+n} + \dots + \tbinom{r}{r} n^r d^0_{m+n},$$

$$[d^r_m, d^0_n]v_k$$
$$= \Big(\tbinom{r}{1} n f_{r-1}(m+n,k) + \tbinom{r}{2} n^2 f_{r-2}(m+n,k) + \dots + \tbinom{r}{r} n^r f_0(m+n,k)\Big)v_{m+n+k}.$$

On the other hand,

$$(d^r_m d^0_n - d^0_n d^r_m)v_k = \big(f_0(n,k)f_r(m,n+k) - f_r(m,k)f_0(n,m+k)\big)v_{m+n+k}.$$

So we have

(20)
$$\tbinom{r}{1} n f_{r-1}(m+n,k) + \tbinom{r}{2} n^2 f_{r-2}(m+n,k) + \dots + \tbinom{r}{r} n^r f_0(m+n,k)$$
$$= f_0(n,k)f_r(m,n+k) - f_r(m,k)f_0(n,m+k).$$

In particular, if $r=2$ and $m=0$, we get

$$f_0(n,k)\big(f_2(0,n+k) - f_2(0,k)\big) = 2nf_1(n,k) + n^2 f_0(n,k).$$

By Claim 1, $f_0(n,k) \neq 0$. So

$$(21) \qquad f_2(0,n+k) - f_2(0,k) = 2n\frac{f_1(n,k)}{f_0(n,k)} + n^2 \qquad \forall n,k \in \mathbb{Z}.$$

Then from (19),

$$(22) \qquad f_{r+1}(m,k) = f_r(m,k)\frac{f_1(m,k)}{f_0(m,k)} \qquad if \quad m \neq 0.$$

(a). $f_0(n,k) = \gamma, \quad f_1(n,k) = k + \alpha n + \beta.$

Claim 2. $\gamma = \pm 1$.
If $\gamma = 1$, then $f_r(m,k) = (k+\beta)^r$ for all $m,k \in \mathbb{Z}, r \in \mathbb{Z}_{\geq 0}$.
If $\gamma = -1$, then $f_r(m,k) = (-1)^{r-1}(k+m+\beta)^r$ for all $m,k \in \mathbb{Z}, r \in \mathbb{Z}_{\geq 0}$.

Proof. In this case, (21) becomes

$$(23) \qquad f_2(0,n+k) - f_2(0,k) = \frac{2n(k+\alpha n+\beta)}{\gamma} + n^2.$$

With $k=0$ in (23), we get

$$f_2(0,n)-f_2(0,0)=\frac{2n(\alpha n+\beta)}{\gamma}+n^2.$$

With $n+k=0$ in (23) and replacing n by $-n$, we get

$$f_2(0,n)-f_2(0,0)=\frac{2n(n-\alpha n+\beta)}{\gamma}-n^2.$$

These imply that $\gamma=1-2\alpha$.

Now (22) becomes

$$f_{r+1}(m,k)=f_r(m,k)\frac{k+\alpha m+\beta}{1-2\alpha} \qquad if \quad m\neq 0.$$

By induction on r, we get

$$f_r(m,k)=(1-2\alpha)\left(\frac{k+\alpha m+\beta}{1-2\alpha}\right)^r \qquad if \quad m\neq 0.$$

With $m+n=0$ in (20), we have

(24)
$$\begin{aligned}&\tbinom{r}{1}nf_{r-1}(0,k)+\tbinom{r}{2}n^2f_{r-2}(0,k)+\dots+\tbinom{r}{r}n^rf_0(0,k)\\&=(1-2\alpha)\big(f_r(-n,n+k)-f_r(-n,k)\big)\\&=(1-2\alpha)^{2-r}\big((k+\beta+(1-\alpha)n)^r-(k+\beta-\alpha n)^r\big).\end{aligned}$$

In particular, if $r=3$ and $k=0$, then

$$\begin{aligned}&3nf_2(0,0)+3n^2f_1(0,0)+n^3f_0(0,0)\\&=\frac{1}{1-2\alpha}\big(3\beta^2n+3\beta(1-2\alpha)n^2+(1-3\alpha+3\alpha^2)n^3\big).\end{aligned}$$

Since n is arbitrary,

$$1-2\alpha=f_0(0,0)=\frac{1}{1-2\alpha}(1-3\alpha+3\alpha^2).$$

So $\alpha=0$ or 1, and $\gamma=1-2\alpha=1$ or -1.

If $\gamma=1$, then $\alpha=0$. With $n=1$ in (24), we have

$$\begin{aligned}&\tbinom{r}{1}f_{r-1}(0,k)+\tbinom{r}{2}f_{r-1}(0,k)+\dots+\tbinom{r}{r}f_0(0,k)\\&=\tbinom{r}{1}(k+\beta)^{r-1}+\tbinom{r}{2}(k+\beta)^{r-2}+\dots+\tbinom{r}{r}.\end{aligned}$$

By induction on r, we get

$$f_r(0,k) = (k+\beta)^r.$$

Hence

$$f_r(m,k) = (k+\beta)^r \qquad \forall m,k \in \mathbb{Z}, \quad r \in \mathbb{Z}_{\geq 0}.$$

If $\gamma = -1$, then $\alpha = 1$. With $n = 1$ in (24), we have

$$\begin{aligned}
&\tbinom{r}{1} f_{r-1}(0,k) + \tbinom{r}{2} f_{r-1}(0,k) + \dots + \tbinom{r}{r} f_0(0,k) \\
&= (-1)^{2-r}\big((k+\beta)^r - (k+\beta-1)^r\big) \\
&= \tbinom{r}{1}(-1)^r (k+\beta)^{r-1} + \tbinom{r}{2}(-1)^{r-1}(k+\beta)^{r-2} + \dots + \tbinom{r}{r}(-1).
\end{aligned}$$

By induction on r, we get

$$f_r(0,k) = (-1)^{r+1}(k+\beta)^r.$$

Hence

$$f_r(m,k) = (-1)^{r-1}(k+m+\beta)^r \qquad \forall m,k \in \mathbb{Z}, \quad r \in \mathbb{Z}_{\geq 0}.$$

We complete the proof of Claim 2. □

(b). $f_0(m,k) = \frac{(k+\beta)\gamma}{m+k+\beta}$, $\quad f_1(m,k) = k+\beta$.

Claim 3. $f_r(m,k) = (-1)^{r-1}(k+m+\beta)^{r-1}(k+\beta)$.

Proof. In this case, (21) becomes

$$f_2(0,n+k) - f_2(0,k) = 2n\frac{n+k+\beta}{\gamma} + n^2. \tag{25}$$

With $k = 0$ in (25), we get

$$f_2(0,n) - f_2(0,0) = \frac{2n(n+\beta)}{\gamma} + n^2.$$

With $n + k = 0$ in (25) and replacing n by $-n$, we get

$$f_2(0,n) - f_2(0,0) = \frac{2n\beta}{\gamma} - n^2.$$

These imply that $\gamma = -1$. Hence (22) becomes

$$f_{r+1}(m,k) = -(m+k+\beta) f_r(m,k) \qquad \textit{where} \quad m \neq 0.$$

By induction on r, we get

$$f_r(m,k) = (-1)^{r-1}(k+m+\beta)^{r-1}(k+\beta)$$

for all $k, m \in \mathbb{Z}$, $m \neq 0$, and all $r \in \mathbb{Z}_{\geq 0}$. Using this and with $m = -1, n = 1$ in (20), we get

$$\begin{aligned}
&\tbinom{r}{1}f_{r-1}(0,k) + \tbinom{r}{2}f_{r-2}(0,k) + \ldots + \tbinom{r}{1}f_0(0,k) \\
&= f_0(1,k)f_r(-1,k+1) - f_r(-1,k)f_0(1,k-1) \\
&= (-1)^r\left((k+\beta)^r - (k+\beta-1)^r\right) \\
&= -\left(\tbinom{r}{1}(-1)^{r-1}(k+\beta)^{r-1} + \tbinom{r}{2}(-1)^{r-2}(k+\beta)^{r-2} + \ldots + \tbinom{r}{r}\right).
\end{aligned}$$

By induction on r, we obtain

$$f_r(0,k) = (-1)^{r-1}(k+\beta)^r.$$

Hence we complete the proof of the Claim 3. □

(c). $f_0(m,k) = \frac{(m+k+\beta)\gamma}{k+\beta}$, $f_1(m,k) = m+k+\beta$.

An argument similar to the one in (b) shows us that $\gamma = 1$ and gives us the following claim.

Claim 4. $f_r(m,k) = (k+m+\beta)(k+\beta)^{r-1}$.

Using the fact that $\mathfrak{g}$ is generated by d_0^2, d_1^0, d_{-1}^0, it is easy to see that if $f_r(m,k)$ is defined as in Claim 2, Claim 3, and Claim 4, $d_m^r \cdot v_k = f_r(m,k)v_{m+k}$ indeed gives V a $\mathfrak{g}$-module structure. In summary, we proved the following proposition.

Proposition 2. Let $V = \sum_{k\in\mathbb{Z}} \mathbb{C}v_k$. Define $d_m^r v_k := f_r(m,k)v_{m+k}$, for all $m, k \in \mathbb{Z}, r \in \mathbb{Z}_{\geq 0}$, where $f_r(m,k)$ are given by the following:

(*i*) $f_r(m,k) = (k+\beta)^r$,

(*ii*) $f_r(m,k) = (-1)^{r-1}(k+m+\beta)^r$,

(*iii*) $f_r(m,k) = (-1)^{r-1}(k+m+\beta)^{r-1}(k+\beta)$,

(*iv*) $f_r(m,k) = (k+\beta)^{r-1}(k+m+\beta)$, where $\beta \notin \mathbb{Z}$.

Then V is a $\mathfrak{g}$-module and $d_{-1}^1 v_k \neq 0, d_1^1 v_k \neq 0$. Conversely, if $V = \sum_k \mathbb{C}v_k$ is a $\mathfrak{g}$-module such that $d_m^r v_k \in \mathbb{C}v_{m+k}$ and $d_{-1}^1 v_k \neq 0, d_1^1 v_k \neq 0$, then the $\mathfrak{g}$-module structure of V is given by one of $(i), (ii)$, (iii) and (iv).

Remark. Let $p(x) = \sum_i a_i x^i \in \mathbb{C}[x]$. Then any $\mathfrak{g}$-module becomes, by restriction to $\mathfrak{g}_{\langle p(x)\rangle}$, a $\mathfrak{g}_{\langle p(x)\rangle}$-module. In particular, the $\mathfrak{g}$-module V of Proposition 2 gives rise, by restriction, to a $\mathfrak{g}_{\langle p(x)\rangle}$-module. Precisely, we have $\mathfrak{g}_{\langle p(x)\rangle}$-modules

$$V_{1,\beta} := \sum_{k\in\mathbb{Z}} \mathbb{C}v_k,$$
$$d_m(q(x))v_k = q(k+\beta)v_{m+k},$$

$$V_{2,\beta} := \sum_{k\in\mathbb{Z}} \mathbb{C}v_k,$$
$$d_m(q(x))v_k = -q(-k-m-\beta)v_{m+k},$$

$$V_{3,\beta} := \sum_{k\in\mathbb{Z}} \mathbb{C}v_k,$$
$$d_m(q(x))v_k := -q(-k-m-\beta)\frac{k+\beta}{m+k+\beta}v_{m+k},$$

and

$$V_{4,\beta} = \sum_{k\in\mathbb{Z}} \mathbb{C}v_k,$$
$$d_m(q(x))v_k = q(k+\beta)\frac{m+k+\beta}{k+\beta}v_{m+k},$$

where $q(x) \in \langle p(x)\rangle$ $m, k \in \mathbb{Z}$, and $\beta \notin \mathbb{Z}$.

Proposition 3.

Let $\beta \in \mathbb{C}\setminus\mathbb{Z}$. Assume that $k_1+\beta, ..., k_r+\beta$ are all the distinct roots of $p(x)$ which lie in $\mathbb{Z}+\beta$. Then

$$U := \mathbb{C}v_{k_1} + ... + \mathbb{C}v_{k_r}$$

is the unique maximal proper $\mathfrak{g}_{\langle p(x)\rangle}$-submodule in $V_{i,\beta}$, $i = 1,2,3,4$. And $V_{i,\beta}/U$, $i = 1,2,3,4$ are all irreducible.

Proof. We prove only the case of $V_{1,\beta}$. The proof of the cases of $V_{i,\beta}$ $i = 2,3,4$ are similar. First note that $\mathfrak{g}_{\langle p(x)\rangle}U = 0$, so U is a trivial $\mathfrak{g}_{\langle p(x)\rangle}$-submodule of $V_{1,\beta}$. Now let $K \neq 0$ be a proper $\mathfrak{g}_{\langle p(x)\rangle}$-submodule in $V_{1,\beta}$ and let $0 \neq X = \sum_{i=i_1}^{i_s} a_i v_i \in K$, where $a_i \neq 0, i = i_1, ..., i_s$. If there exists k such that $p(i_k+\beta) \neq 0$, then

$$0 \neq d_0\left(x^j p(x)\right)X = \sum_{i=i_1}^{i_s} a_i(i+\beta)^j p(i+\beta)v_i \in K$$

for all $j \in \mathbb{Z}_{\geq 0}$. Since

$$det \begin{pmatrix} 1 & 1 & \ldots & 1 \\ i_1+\beta & i_2+\beta & \ldots & i_s+\beta \\ \vdots & \vdots & \ldots & \vdots \\ (i_1+\beta)^{s-1} & (i_2+\beta)^{s-1} & \ldots & (i_s+\beta)^{s-1} \end{pmatrix} \neq 0,$$

$v_{i_k} \in K$. Then

$$d_m(p(x))v_{i_k} = p(i_k+\beta)v_{m+i_k} \in K \qquad \forall m \in \mathbb{Z},$$

and $K = V_{1,\beta}$. This is a contradiction. So we must have $p(i+\beta) = 0$ for $i = i_1, \ldots, i_s$. Hence $X \in U$. □

1.6 Highest Weight Modules

In this section, we discuss the highest weight modules of $\tilde{\mathfrak{g}}$ and $\widetilde{\mathfrak{g}\{0,1\}}$. We also define contravariant forms on Verma modules and give some necessary conditions for these forms to be non-negative.

Recall that we have an imbedding of $\tilde{\mathfrak{g}}$ into $\mathfrak{a}_\infty$ by identifying d^r_m with $\sum_j j^r E_{j+m,j}$. If we define $L^r_m = \sum_{i=0}^r \binom{r}{i}(\frac{m}{2})^{r-i} d^i_m$, then L^r_m is identified as $\sum_j (j+\frac{m}{2})^r E_{j+m,j}$ in $\mathfrak{a}_\infty$. Clearly, $\tilde{\mathfrak{g}} = \sum_{m \in \mathbb{Z}, r \in \mathbb{Z}_{\geq 0}} \mathbb{C}L^r_m + \mathbb{C}¢$. By straight calculation, we see that the commutators of $\tilde{\mathfrak{g}}$ are

$$[L^r_m, L^s_n] = \sum_{l=0}^{r+s}\left(\sum_{i+k=l}\binom{r}{i}\binom{s}{k}\left(\frac{n}{2}\right)^i\left(\frac{m}{2}\right)^k\left((-1)^k-(-1)^i\right)\right)L^{r+s-l}_{m+n}$$
$$+\left(\sum_{1\leq j\leq n}\left(j-\frac{n}{2}\right)^{r+s}-\sum_{1\leq j\leq m}\left(j-\frac{m}{2}\right)^{r+s}\right)\delta_{m+n,0}¢,$$
$$[L^r_m, ¢] = 0.$$

Define anti-linear map

$$\omega : \mathfrak{a}_\infty \longrightarrow \mathfrak{a}_\infty$$

such that

$$\omega\left(\sum_i \lambda_i E_{i+m,i}\right) = \sum_i \overline{\lambda_i} E_{i,i+m}, \quad \omega(¢) = ¢.$$

It is easy to check that ω is an anti-involution of $\mathfrak{a}_\infty$. Particularly, $\omega\mid_{\tilde{\mathfrak{g}}}$ is an anti-involution of $\tilde{\mathfrak{g}}$ and

$$\omega(L_m^r) = \sum_j \left(j + \frac{m}{2}\right)^r E_{j,j+m} = \sum_j \left(j - \frac{m}{2}\right)^r E_{j-m,j} = L_{-m}^r.$$

Let

$$\tilde{\mathfrak{g}_k} = \mathfrak{g}_k \qquad for \quad k \neq 0,$$
$$\tilde{\mathfrak{g}_0} = \mathfrak{g}_0 + \mathbb{C}\not{c},$$

and

$$\tilde{\mathfrak{g}_+} = \sum_{k>0} \tilde{\mathfrak{g}_k}, \quad \tilde{\mathfrak{g}_-} = \sum_{k<0} \tilde{\mathfrak{g}_k}.$$

Then

$$\tilde{\mathfrak{g}} = \tilde{\mathfrak{g}_-} \oplus \tilde{\mathfrak{g}_0} \oplus \tilde{\mathfrak{g}_+}.$$

Definition. Let $U(\tilde{\mathfrak{g}})$ be the universal enveloping algebra of $\tilde{\mathfrak{g}}$ and $\Lambda \in \tilde{\mathfrak{g}_0}^*$, the dual space of $\tilde{\mathfrak{g}_0}$. Let $J(\Lambda)$ be the left ideal of $U(\tilde{\mathfrak{g}})$ generated by $\tilde{\mathfrak{g}_+}$ and

$$\{X - \Lambda(X) \mid X \in \tilde{\mathfrak{g}_0}\},$$

where we identify 1 with the identity of $U(\tilde{\mathfrak{g}})$. $M(\Lambda) := U(\tilde{\mathfrak{g}})/J(\Lambda)$ is called a Verma module. Any quotient of $M(\Lambda)$ is called a highest weight module of $\tilde{\mathfrak{g}}$ of highest weight Λ.

By the Poincare-Birkhoff-Witt theorem, $M(\Lambda)$ has basis

$$L_{-n}^{k_{n,1}} \ldots L_{-n}^{k_{n,s_n}} \ldots . L_{-1}^{k_{1,1}} \ldots L_{-1}^{k_{1,s_1}} v_+$$

where

$$k_{j,1} \geq k_{j,2} \geq \ldots \geq k_{j,s_j}, \quad 1 \leq j \leq n, \quad n \in \mathbb{Z}_{\geq 0},$$

and v_+ is the image of 1 in $M(\Lambda)$.

Note that $[L_0^1, L_n^s] = nL_n^s$, for all $n \in \mathbb{Z}$ and $s \in \mathbb{Z}_{\geq 0}$. We decompose $M(\Lambda)$ as a sum of eigenspaces of L_0^1 in the following.

For any $\mu \in \mathbb{C}$, define

$$M^\mu := \{u \in M(\Lambda) \mid L_0^1 u = \mu u\}.$$

i.e. M^μ is the eigenspace of L_0^1 of eigenvalue μ.

Since:

$$L_0^1 \cdot L_{-n}^{k_{n,1}} \dots L_{-n}^{k_{n,s_n}} \dots . L_{-1}^{k_{1,1}} \dots L_{-1}^{k_{1,s_1}} v_+$$
$$= (\lambda - (ns_n + (n-1)s_{(n-1)} + \dots + s_1)) L_{-n}^{k_{n,1}} \dots L_{-n}^{k_{n,s_n}} \dots . L_{-1}^{k_{1,1}} \dots L_{-1}^{k_{1,s_1}} v_+,$$

for any basis element of $M(\Lambda)$, where $\lambda = \Lambda(L_0^1)$, it is easy to see that $M(\Lambda) = \sum_{k \geq 0} M^{\lambda - k}$ and $M^\lambda = \mathbb{C}v_+$. If $N \subseteq M(\Lambda)$ is a proper $\tilde{\mathfrak{g}}$-module, then $N \cap M^\lambda = 0$. Hence $M(\Lambda)$ is indecomposable and contains a unique maximal proper submodule $N(\Lambda)$. Let $V(\Lambda) := M(\Lambda)/N(\Lambda)$. Then $V(\Lambda)$ is the unique irreducible highest weight module of highest weight Λ. Hence we proved the following proposition.

Proposition 1. (i) $M(\Lambda) = \sum_{k \in \mathbb{Z}_{\geq 0}} M^{\lambda-k}$, $M^\lambda = \mathbb{C}v_+$, and $M(\Lambda)$ is indecomposable.

(ii) $M(\Lambda)$ has a unique proper maximal submodule $N(\Lambda)$ and $V(\Lambda) = M(\Lambda)/N(\Lambda)$ is the unique irreducible highest weight module of highest weight Λ.

Define a total order $<$ on $\{L_m^r \mid m \in \mathbb{Z}, r \in \mathbb{Z}_{\geq 0}\} \bigcup \{\cent\}$ as follows:

$$L_m^r < L_n^s$$

iff

$$m < n$$

or

$$m = n, \quad r < s.$$

And

$$L_0^k < \cent < L_1^0 \quad \forall k \in \mathbb{Z}.$$

Then an element of $U(\tilde{\mathfrak{g}})$ is a linear combination of elements of the form

$$R = L_{-j_1}^{k_1} L_{-j_2}^{k_2} \dots L_{-j_s}^{k_s} (L_0^0)^{r_0} \dots (L_0^k)^{r_n} \cent^l L_{i_1}^{l_1} \dots L_{i_t}^{l_t}, \tag{1}$$

where $L_{-j_1}^{k_1} \leq \dots \leq L_{-j_s}^{k_s} < L_0^0$, $\cent < L_{i_1}^{l_1} \leq \dots \leq L_{i_t}^{l_t}$. For $u \in M(\Lambda)$, define $\langle u \rangle$ to be the coefficient of the highest weight vector v_+ in the expansion of u with respect to $M(\Lambda) = \sum_{k \in \mathbb{Z}_{\geq 0}} M^{\lambda-k}$. If we extend the anti-involution ω of $\tilde{\mathfrak{g}}$ to $U(\tilde{\mathfrak{g}})$ by $\omega(XY) = \omega(Y)\omega(X)$, then we have $\langle \omega(R)v_+ \rangle = \overline{\langle Rv_+ \rangle}$ provided that $\Lambda(L_0^k) \in \mathbb{R}$ for $k = 0, 1, 2 \dots$ and $\Lambda(\cent) \in \mathbb{R}$.

Proposition 2.

(i) Assume that $\Lambda(L_0^k) \in \mathbb{R}$ for all $k \in \mathbb{Z}_{\geq 0}$ and $\Lambda(\not{c}) \in \mathbb{R}$. Then $M(\Lambda)$ carries a unique contravariant hermitian form $\langle .,. \rangle$ such that $\langle v_+, v_+ \rangle = 1$;

(ii) $\langle M^{\lambda-k}, M^{\lambda-l} \rangle = 0 \quad$ if $k \neq l$;

(iii) $ker\langle .,. \rangle = N(\Lambda)$. Hence $V(\Lambda)$ carries a unique contravariant hermitian form such that $\langle v, v \rangle = 1$ and this form is non-degenerate, where $v = v_+ + N(\Lambda)$.

Proof. (i),(ii). For any monomials $P(v_+) = L_{-j_1}^{k_1} ... L_{-j_s}^{k_s} v_+$ and $Q(v_+) = L_{-i_1}^{l_1} ... L_{-i_t}^{l_t} v_+$, define $\langle P(v_+), Q(v_+) \rangle := \langle \omega(P) Q(v_+) \rangle$. This is a well defined contravariant hermitian form on $M(\Lambda)$. (see [S], [KR] or [MoPi]). Moreover, $\langle P(v_+), Q(v_+) \rangle = 0$, if $j_1 + j_2 + ... + j_s \neq i_1 + i_2 + ... + i_t$.

(iii). By definition,

$$ker\langle .,. \rangle = \{u \in M(\Lambda) \mid \langle u, w \rangle = 0 \quad \forall w \in M(\Lambda)\}.$$

Clearly $ker\langle .,. \rangle$ is a proper submodule of $M(\lambda)$. Moreover, if $V \subseteq M(\Lambda)$ is a submodule and $P(v_+) \in V, \quad Q(v_+) \in M(\Lambda)$, then $\omega(Q)P(v_+) \in V$. So if V is a proper submodule of $M(\Lambda)$, then $\langle \omega(Q) P(v_+) \rangle = 0$, i.e. $\langle P(v_+), Q(v_+) \rangle = 0$. Hence $P(v_+) \in ker\langle .,. \rangle$ and $V \subseteq ker\langle .,. \rangle$. So we proved that

$$ker\langle .,. \rangle = N(\Lambda).$$

□

An important question is when $\langle .,. \rangle$ is non-negative on $M(\Lambda)$, hence positive definite on $V(\Lambda)$. For Virasoro algebra $\widetilde{\mathfrak{g}\{x\}}$, D.Friedan, Z.Qiu, and S.Shenker [FQS], and R.Langlands [L] gave a necessary condition for the corresponding form $\langle .,. \rangle$ on Verma module to be non-negative. P.Goddard,A.Kent and D.Olive [GKO] proved that the condition is also sufficient. For $\tilde{\mathfrak{g}}$, even though we still don't know examples for which $\langle .,. \rangle$ are non-negative on $M(\Lambda)$, we can prove the following necessary condition. First, for $c \in \mathbb{R}, \quad \underline{h} = (h_0, h_1, h_2, ...) \in \mathbb{R}^\infty$, we define an infinite real matrix $A(\underline{h}, c)$ as follows:

$$A(\underline{h}, c) = (A_{ij}(\underline{h}, c))_{i,j=1,2,...} \quad ,$$

where

$$A_{ij}(\underline{h}, c) := \sum_{l=0}^{i+j} \binom{i+j}{l} \left(\frac{1}{2}\right)^l ((-1)^l - 1)\, h_{i+j-l} - \left(\frac{1}{2}\right)^{i+j} c.$$

Clearly,

$$A_{ij}(\underline{h}, c) = A_{ji}(\underline{h}, c).$$

Proposition 3. Let $\Lambda \in \tilde{\mathfrak{g}_0}^*$ be such that $h_i = \Lambda(L_0^i) \in \mathbb{R} \quad \forall i \in \mathbb{Z}_{\geq 0}$ and $c = \Lambda(¢) \in \mathbb{R}$. Then a necessary condition for $\langle .,. \rangle$ to be non-negative on $M(\Lambda)$ is that $A(\underline{h}, c)$ is positive semidefinite.

Proof. For $(a_0, a_1, ... a_r) \in \mathbb{R}^{r+1}$, if $\langle .,. \rangle$ is non-negative, then

$$\left\langle \left(\sum_{i=1}^{r} a_i L_{-1}^i \right) v_+, \left(\sum_{i=1}^{r} a_i L_{-1}^i \right) v_+ \right\rangle \geq 0,$$

i.e.

$$\left\langle v_+, \sum_{i=1}^{r} \sum_{j=1}^{r} a_i a_j [L_1^i, L_{-1}^j] v_+ \right\rangle \geq 0.$$

Since

$$[L_1^i, L_{-1}^j] = \sum_{l=0}^{i+j} \binom{i+j}{l} \left(\frac{1}{2} \right)^l \left((-1)^l - 1 \right) L_0^{i+j-l} - \left(\frac{1}{2} \right)^{i+j} ¢,$$

$$\sum_i \sum_j a_i a_j \left(\sum_{l=0}^{i+j} \binom{i+j}{l} \left(\frac{1}{2} \right)^l \left((-1)^l - 1 \right) h_{i+j-l} - \left(\frac{1}{2} \right)^{i+j} c \right) \geq 0.$$

This is

$$\sum_i \sum_j a_i a_j A_{ij}(\underline{h}, c) \geq 0,$$

or

$$\begin{pmatrix} a_0 & a_1 & \dots & a_r & 0 & \dots \end{pmatrix} A(\underline{h}, c) \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_r \\ 0 \\ \vdots \end{pmatrix} \geq 0.$$

So $A(\underline{h}, c)$ is positive semidefinite. □

Finally, we consider the highest weight modules of $\widetilde{\mathfrak{g}\{0,1\}}$. Take

$$\{L_m^1, L_m^0 \mid \forall m \in \mathbb{Z}\} \cup \{¢\}$$

as a basis. Then

$$\widetilde{\mathfrak{g}\{0,1\}} = \sum_{m \in \mathbb{Z}} \mathbb{C} L_m^1 + \sum_{m \in \mathbb{Z}} \mathbb{C} L_m^0 + \mathbb{C}¢,$$

and

$$[L^1_m, L^1_n] = (n-m)L^1_{m+n} - \frac{1}{12}(m^3 + 2m)\delta_{m+n,0}\cent,$$
$$[L^1_m, L^0_n] = nL^0_{m+n} - \frac{m}{2}\delta_{m+n,0}\cent,$$
$$[L^0_m, L^0_n] = -m\delta_{m+n,0}\cent.$$

Let

$$L_m = -L^1_m - \frac{1}{8}\delta_{m,0}\cent,$$
$$A_m = -L^0_m - \frac{1}{2}\delta_{m,0}\cent,$$
$$\cent_1 = -\cent.$$

Then

$$[L_m, L_n] = (n-m)L_{m+n} + \frac{m^3 - m}{12}\delta_{m+n,0}\cent_1,$$
$$[L_m, A_n] = -nA_{m+n},$$
$$[A_m, A_n] = m\delta_{m+n,0}\cent_1,$$

and

$$\omega(L_m) = L_{-m}, \quad \omega(A_m) = A_{-m}, \quad \omega(\cent_1) = \cent_1.$$

Let

$$\widetilde{\mathfrak{g}\{0,1\}}_+ = \sum_{m>0} \mathbb{C}L_m + \sum_{m>0} \mathbb{C}A_m,$$
$$\widetilde{\mathfrak{g}\{0,1\}}_- = \sum_{m<0} \mathbb{C}L_m + \sum_{m<0} \mathbb{C}A_m,$$

and

$$\widetilde{\mathfrak{g}\{0,1\}}_0 = \mathbb{C}L_0 + \mathbb{C}A_0 + \mathbb{C}\cent_1.$$

Then

$$\widetilde{\mathfrak{g}\{0,1\}} = \widetilde{\mathfrak{g}\{0,1\}}_+ \oplus \widetilde{\mathfrak{g}\{0,1\}}_0 \oplus \widetilde{\mathfrak{g}\{0,1\}}_-.$$

For $\Lambda \in \widetilde{\mathfrak{g}\{0,1\}}_0^*$, as in the case of $\tilde{\mathfrak{g}}$, we have Verma module $M(\Lambda)$, the unique maximal proper submodule $N(\Lambda)$ and $V(\Lambda) := M(\Lambda)/N(\Lambda)$. Moreover, if

$$h := \Lambda(L_0), \quad a := \Lambda(A_0), \quad c := \Lambda(\cent_1) \in \mathbb{R},$$

there exists a contravariant hermitian form $\langle .,. \rangle$ on $M(\Lambda)$ such that $ker\langle .,. \rangle = N(\Lambda)$.

Proposition 4. Let

$$\widetilde{\mathfrak{g}\{0,1\}} = \sum_{m\in\mathbb{Z}} \mathbb{C}L_m + \sum_{m\in\mathbb{Z}} \mathbb{C}A_m + \mathbb{C}\cent_1,$$
$$\Lambda \in (\mathbb{C}L_0 + \mathbb{C}A_0 + \mathbb{C}\cent_1)^*,$$

be such that

$$h = \Lambda(L_0), \quad a = \Lambda(A_0), \quad c = \Lambda(\cent_1) \in \mathbb{R}.$$

Then a necessary condition for $\langle .,. \rangle$ to be non-negative on $M(\Lambda)$ is

(i)
$$(h, c) \geq (0, 1),$$

or

(ii)
$$(h, c) = (h_m^{r,s}, c_m) \qquad m \geq 0,$$

where

$$h_m^{r,s} = \frac{((m+3)r - (m+2)s)^2 - 1}{4(m+2)(m+3)},$$
$$c_m = 1 - \frac{6}{(m+2)(m+3)} \qquad 1 \leq s \leq r \leq m+1.$$

And for any given (h, c) satisfying (i) or (ii),

$$-\sqrt{2hc} \leq a \leq \sqrt{2hc}.$$

Proof. The conditions (i) and (ii) on (h, c) are given by [FQS] and [L]. For any given pair (h, c) satisfying (i) and (ii), since $\forall \alpha \in \mathbb{R}$,

$$\langle (L_{-1} + \alpha A_{-1})v_+ \mid (L_{-1} + \alpha A_{-1})v_+ \rangle \geq 0,$$

we have

$$2h + 2\alpha a + \alpha^2 c \geq 0,$$

or

$$2\alpha a \geq -2h - \alpha^2 c.$$

Thus

$$\begin{cases} a \geq \frac{-h}{\alpha} - \frac{\alpha c}{2} & \forall \alpha \in \mathbb{R}_{>0} \\ a \leq -\frac{h}{\alpha} - \frac{\alpha c}{2} & \forall \alpha \in \mathbb{R}_{<0} \end{cases}.$$

Define $f(\alpha) := -\frac{h}{\alpha} - \frac{\alpha c}{2}$ and let

$$f'(\alpha) = \frac{h}{\alpha^2} - \frac{c}{2} = 0.$$

We get $\alpha = \pm\sqrt{\frac{2h}{c}}$ and

$$-\sqrt{2hc} = f\left(\sqrt{\frac{2h}{c}}\right) \leq a \leq f\left(-\sqrt{\frac{2h}{c}}\right) = \sqrt{2hc}.$$

□

CHAPTER 2

POLYNOMIAL LIE SUBALGEBRAS OF THE VIRASORO ALGEBRA

2.0 Introduction of Chapter 2

In this chapter we study the polynomial Lie subalgebras of the Witt algebra and the Virasoro algebra, and their representations. We have seen in the Introduction that the Witt algebra is an infinite dimensional simple Lie algebra given by $W = \sum_{m\in\mathbb{Z}} \mathbb{C}d_m$ with Lie bracket $[d_m, d_n] = (n-m)d_{m+n}$. The Virasoro algebra is the universal central extension of W, and given by $Vir = \sum_{m\in\mathbb{Z}} \mathbb{C}L_m + \mathbb{C}\not{c}$ with Lie brackets

$$[L_m, L_n] = (n-m)L_{m+n} + \frac{m^3-m}{12}\delta_{m+n,0}\not{c},$$
$$[L_m, \not{c}] = 0.$$

Both of them have been extensively studied by many authors. However, the subalgebras of W and Vir have still not been considered seriously. Some obvious subalgebras of W are the following:

(a) Finite dimensional subalgebras

$$\mathbb{C}x, \qquad x \in W.$$
$$\mathbb{C}d_0 + \mathbb{C}d_m, \qquad m \in \mathbb{Z}.$$
$$\mathbb{C}d_{-m} + \mathbb{C}d_0 + \mathbb{C}d_m, \qquad m \in \mathbb{Z}.$$

(b)

$$\sum_{m\in\mathbb{Z}} \mathbb{C}d_{mk} \qquad k \in \mathbb{Z}_{>0}.$$

(c)

$$\sum_{m\geq k} \mathbb{C}d_m \qquad k \in \mathbb{Z}_{\geq -1},$$
$$\sum_{m\leq k} \mathbb{C}d_m \qquad k \in \mathbb{Z}_{\leq 1}.$$

(d) Intersections of subalgebras of type (b) and (c) above.

Before we introduce the polynomial Lie subalgebras of W and Vir, let us recall the shift map from W to itself such that

$$\sigma(d_m) = d_{m+1} \qquad \forall m \in \mathbb{Z}.$$

Even though σ is not a Lie algebra homomorphism, it is invertible and possesses the following two properties:

$$[\sigma(x), \sigma(y)] = \sigma^2[x, y]$$

and

$$[\sigma(x), y] + [x, \sigma(y)] = 2\sigma[x, y]$$

for all $x, y \in W$.

We call a linear map from a Lie algebra to itself with these two properties a shift operator. We'll see in Section 2.1 that the shift operators of W constitute a commutative algebra $\mathbb{C}[\sigma, \sigma^{-1}]$ which is isomorphic to the Laurent polynomial ring $\mathbb{C}\left[t, t^{-1}\right]$. We say a Lie subalgebra V of W is shift invariant if $\sigma(V) = V$. It turns out that $V \subseteq W$ is a shift invariant Lie subalgebra if and only if

$$V = p(\sigma)W = \{p(\sigma)x \mid x \in W\}$$

for some polynomial $p(t) \in \mathbb{C}[t]$. We call $p(\sigma)W$ the polynomial Lie subalgebra of W associated with $p(t)$.

Next, we consider the canonical Lie homomorphism

$$\pi : Vir \longrightarrow W$$

with

$$\pi(L_m) = d_m, \qquad \forall m \in \mathbb{Z},$$
$$\pi(\cent) = 0.$$

If V is a Lie subalgebra of W, then $\pi^{-1}(V)$ is a Lie subalgebra of Vir. In particular, if $V = p(\sigma)W$, then $\pi^{-1}(V) = \pi^{-1}(p(\sigma)W)$ is a Lie subalgebra of Vir. We call it the polynomial Lie subalgebra of Vir associated with $p(t)$.

This chapter is organized as follows: In Section 2.1, we introduce the shift operators for an arbitrary Lie algebra, determine all of the shift operators of the Witt algebra, and show that a Lie subalgebra of W is shift invariant if and only if it is a polynomial Lie subalgebra. In Section 2.2, we discuss the structure of the polynomial Lie subalgebras of W and in Section 2.3, following the method of the Segal-Sugawara construction for the Virasoro algebra, we realize the elements of the polynomial Lie subalgebras of the Virasoro algebra as the Segal-Sugawara operators on certain Fock spaces. In section 2.4. we define category $\mathbb{O}_p$ of $Vir(p)$-modules, where $Vir(p)$ is the polynomial subalgebra of Vir associated with the polynomial $p = p(x)$. This is an analogue of category $\mathbb{O}$ of the Virasoro algebra modules. We go on to discuss the highest weight modules, in particular the Verma modules of $Vir(p)$. In Section 2.5, quasi-admissible modules and admissible modules of $Vir(p)$ are introduced. They are the natural generalizations of admissible modules of the Virasoro algebra. The irreducible admissible modules of the Virasoro algebra have been classified by Mathieu [Ma] and Martin and Piard [MP]. In Section 2.6, we give a family of irreducible quasi-admissible modules of the Virasoro algebra. We discuss a class of admissible modules of $Vir(1-t)$ in detail in Section 2.7, where in particular, a class of finite dimensional modules is determined and the necessary and sufficient conditions for their indecomposibility is obtained. Finally, we give a brief comment on the admissible modules of the general polynomial Lie subalgebras in Section 2.8.

We always assume that the base field is the complex field $\mathbb{C}$ in our discussion.

2.1 Shift Operators of the Witt Algebra

In this section, we introduce shift operators for an arbitrary Lie algebra and determine all of the shift operators of the Witt algebra. The general concept of shift operators is motivated by the following observation. The Witt algebra is a $\mathbb{Z}$-graded Lie algebra, $W = \sum_{m\in\mathbb{Z}} \mathbb{C}d_m, \quad [d_m, d_n] = (n-m)d_{m+n}$. Consider the linear map

$$\sigma : W \longrightarrow W$$

defined by $\sigma(d_i) = d_{i+1}$. Straight calculation shows that

$$(1) \qquad \begin{cases} [\sigma(d_i), \sigma(d_j)] = \sigma^2([d_i, d_j]), \\ [\sigma(d_i), d_j] + [d_i, \sigma(d_j)] = 2\sigma([d_i, d_j]). \end{cases}$$

We give the following definition.

Definition.

Let $\mathfrak{g}$ be a Lie algebra and let $S \subseteq End(\mathfrak{g})$ be a commutative (associative) subalgebra. S is called a shift operator algebra on $\mathfrak{g}$ if for any $\sigma, \tau \in S$, we have

$$[\sigma(x), \tau(y)] + [\tau(x), \sigma(y)] = 2\sigma\tau([x, y]) \qquad \forall x, y \in \mathfrak{g}.$$

An element of S is called a shift operator on $\mathfrak{g}$ (relative to S).

Remark. If $\sigma = \tau$, then

$$[\sigma(x), \sigma(y)] = \sigma^2[x, y].$$

If $\tau = 1$, then

$$[\sigma(x), y] + [x, \sigma(y)] = 2\sigma[x, y].$$

Proposition 1. Let $\sigma \in End(\mathfrak{g})$. If

$$[\sigma(x), \sigma(y)] = \sigma^2([x, y])$$

and

$$[\sigma(x), y] + [x, \sigma(y)] = 2\sigma([x, y])$$

for all $x, y \in \mathfrak{g}$, then $\mathbb{C}[\sigma]$ is a shift operator algebra on $\mathfrak{g}$. Hence σ is a shift operator (relative to $\mathbb{C}[\sigma]$). Moreover, if σ^{-1} exists, then $\mathbb{C}[\sigma, \sigma^{-1}]$ is a shift operator algebra on $\mathfrak{g}$.

Proof. Assume that $m \geq n \geq 0$.

If $n > 0$, then

$$\begin{aligned}
&[\sigma^m(x), \sigma^n(y)] + [\sigma^n(x), \sigma^m(y)] \\
&= \sigma^2[\sigma^{m-1}(x), \sigma^{n-1}(y)] + \sigma^2[\sigma^{n-1}(x), \sigma^{m-1}(y)] \\
&= \sigma^2\left([\sigma^{m-1}(x), \sigma^{n-1}(y)] + [\sigma^{n-1}(x), \sigma^{m-1}(y)]\right) \\
&= \sigma^2 \cdot 2\sigma^{m-1+n-1}[x, y] \qquad (by \quad induction) \\
&= 2\sigma^{m+n}[x, y] \qquad \forall x, y \in \mathfrak{g}.
\end{aligned}$$

If $n = 0, m \geq 2$, then

$$\begin{aligned}
&[\sigma^m(x), y] + [x, \sigma^m(y)] \\
&= [\sigma^m(x), y] + [\sigma^{m-1}(x), \sigma(y)] + [\sigma(x), \sigma^{m-1}(y)] + [x, \sigma^m(y)] \\
&\quad - \left([\sigma^{m-1}(x), \sigma(y)] + [\sigma(x), \sigma^{m-1}(y)]\right) \\
&= 2\sigma[\sigma^{m-1}(x), y] + 2\sigma[x, \sigma^{m-1}(y)] \qquad (by \quad hypothesis) \\
&\quad - \sigma^2\left([\sigma^{m-2}(x), y] + [x, \sigma^{m-2}(y)]\right) \\
&= 2\sigma \cdot 2\sigma^{m-1}[x, y] - 2\sigma^m[x, y] \\
&\qquad (by \quad induction) \\
&= 2\sigma^m[x, y] \qquad \forall x, y \in \mathfrak{g}.
\end{aligned}$$

So

$$[\sigma^m(x), \sigma^n(y)] + [\sigma^n(x), \sigma^m(y)] = 2\sigma^{m+n}[x, y]$$

for all $m, n \in \mathbb{Z}$, and $x, y \in \mathfrak{g}$. This implies that for all polynomials $p(t), q(t)$,

$$[p(\sigma)x, q(\sigma)y] + [q(\sigma)x, p(\sigma)y] = 2p(\sigma)q(\sigma)[x, y] \qquad \forall x, y \in \mathfrak{g}.$$

If σ^{-1} exists, then

$$[x,y] = [\sigma\sigma^{-1}(x), \sigma\sigma^{-1}(y)] = \sigma^2[\sigma^{-1}(x), \sigma^{-1}(y)].$$

So

$$[\sigma^{-1}(x), \sigma^{-1}(y)] = \sigma^{-2}[x,y],$$

and

$$[\sigma^{-1}(x), y] + [x, \sigma^{-1}(y)] = \sigma^{-2}[x, \sigma(y)] + \sigma^{-2}[\sigma(x), y]$$
$$= \sigma^{-2} \cdot 2\sigma[x,y] = 2\sigma^{-1}[x,y].$$

Then we obtain

$$[\sigma^{-m}(x), \sigma^{-n}(y)] = \sigma^{-m-n}[x,y]$$

and

$$[\sigma^{-m}(x), \sigma^{-n}(y)] + [\sigma^{-n}(x), \sigma^{-m}(y)] = \sigma^{-m-n}[x,y],$$

for all $m,n \geq 0$. Finally,

$$\begin{aligned}
&[\sigma^{m}(x), \sigma^{-n}(y)] + [\sigma^{-n}(x), \sigma^{m}(y)] \\
&= \sigma^{-2n}\left([\sigma^{m+n}(x), y] + [x, \sigma^{m+n}(y)]\right) \\
&= \sigma^{-2n}\left(2\sigma^{m+n}[x,y]\right) \\
&= 2\sigma^{m-n}[x,y]
\end{aligned}$$

for all $m,n \in \mathbb{Z}$. Hence for any Laurent polynomials $p(t), q(t)$,

$$[p(\sigma)(x), q(\sigma)(y)] + [q(\sigma)(x), p(\sigma)(y)] = 2p(\sigma)q(\sigma)[x,y].$$

We have proved the proposition. □

More generally, we have the following:

Proposition 2. Let $S \subseteq End(\mathfrak{g})$ be a commutative subalgebra with generators $\{\sigma_i \mid i \in I\}$. Then S is a shift operator algebra on $\mathfrak{g}$ iff

$$[\sigma_{i_1}...\sigma_{i_m}x, \sigma_{j_1}...\sigma_{j_n}y] + [\sigma_{j_1}...\sigma_{j_n}x, \sigma_{i_1}...\sigma_{i_m}y] = 2\sigma_{i_1}...\sigma_{i_m}\sigma_{j_1}...\sigma_{j_n}[x,y]$$

for all $\sigma_{i_k}, \sigma_{j_l}$ and all $x, y \in \mathfrak{g}$.

Proof. Obvious. □

Example 1 Let $\mathfrak{g} = \mathbb{C}x + \mathbb{C}y$ be the 2-dimensional Lie algebra with Lie bracket $[x, y] = y$. If $\sigma \in End(\mathfrak{g})$ is a shift operator, then

$$\sigma \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

for some $a, b, c, d \in \mathbb{C}$. Since

$$[ax + by, y] + [x, cx + dy] = 2(cx + dy),$$

we have $c = 0, a = d$. i.e.

$$\sigma \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Let

$$\tau = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Then $\sigma = aI + b\tau$. It is thus clear that $\mathbb{C}I + \mathbb{C}\tau$ is the only shift operator algebra on $\mathfrak{g}$ of dimension greater than 1.

Example 2 For $sl_2(\mathbb{C}) = \mathbb{C}e + \mathbb{C}h + \mathbb{C}f$, where $[e, f] = h, [h, e] = 2e, [h, f] = -2f$, if σ is a shift operator, then

$$\sigma \begin{pmatrix} e \\ h \\ f \end{pmatrix} = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \begin{pmatrix} e \\ h \\ f \end{pmatrix}$$

for some $a_i, b_i, c_i \in \mathbb{C},\ i = 1, 2, 3$.

From

$$[\sigma(h), e] + [h, \sigma(e)] = 2\sigma[h, e] = 4\sigma(e),$$

we obtain

$$a_3 = 0, \quad b_2 = a_1, \quad b_3 = -4a_2.$$

From

$$[\sigma(h), f] + [h, \sigma(f)] = 2\sigma[h, f] = -4\sigma f,$$

we obtain

$$c_1 = 0, \quad c_3 = b_2, \quad b_1 = -4c_2.$$

Finally, from

$$[\sigma(e), f] + [e, \sigma(f)] = 2\sigma[e, f] = 2\sigma(h),$$

we obtain

$$a_2 = c_2 = 0.$$

So we have

$$a_1 = b_2 = c_3$$

and

$$a_2 = a_3 = b_1 = b_3 = c_1 = c_2 = 0.$$

Thus $\{aI \mid a \in \mathbb{C}\}$ are the only shift operators of $sl_2(\mathbb{C})$. Example 3 Let $\mathfrak{g}$ be a Lie algebra and $\overline{\mathfrak{g}} := \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$ be the loop algebra. Define a linear map

$$\sigma : \overline{\mathfrak{g}} \longrightarrow \overline{\mathfrak{g}}$$

by $\sigma(x \otimes t^m) = x \otimes t^{m+1}$ for all $x \in \mathfrak{g}$ and $m \in \mathbb{Z}$. Then $\mathbb{C}[\sigma, \sigma^{-1}]$ is a shift operator algebra on $\overline{\mathfrak{g}}$.

We now consider the shift operators of the Witt algebra.

Proposition 3. Let

$$\sigma : W \longrightarrow W$$

be the linear map defined by $\sigma(d_i) = d_{i+1}$. Then $\mathbb{C}[\sigma, \sigma^{-1}]$ is a shift operator algebra of W. Moreover, if $\tau \in End(W)$ is a shift operator on W, then $\tau \in \mathbb{C}[\sigma, \sigma^{-1}]$.

Proof. Clearly σ is invertible. Using (1) and Proposition 1, $\mathbb{C}[\sigma, \sigma^{-1}]$ is a shift operator algebra on W.

Conversely, suppose that $\tau \in End(W)$ is a shift operator, and

$$\tau(d_i) = \sum_j a_{ij} d_j = \sum_j a_{ij} \sigma^{j-i} d_i = p_i(\sigma) d_i \qquad \forall i \in \mathbb{Z},$$

where $p_i(t) := \sum_j a_{ij} t^{j-i}$ is a Laurent polynomial. Comparing

$$2\tau[d_0, d_j] = 2j\tau(d_j) = 2jp_j(\sigma)d_j$$

with

$$\begin{aligned}
&[\tau(d_0), d_j] + [d_0, \tau(d_j)] \\
&= [p_0(\sigma)d_0, d_j] + [d_0, p_j(\sigma)d_j] \\
&= [p_0(\sigma)d_0, d_j] + [d_0, p_0(\sigma)d_j] + [p_j(\sigma)d_0, d_j] + [d_0, p_j(\sigma)d_j] \\
&\quad - ([d_0, p_0(\sigma)d_j] + [p_j(\sigma)d_0, d_j]) \\
&= 2jp_0(\sigma)d_j + 2jp_j(\sigma)d_j - ([d_0, p_0(\sigma)d_j] + [p_j(\sigma)d_0, d_j]),
\end{aligned}$$

we get

$$(*) \qquad [d_0, p_0(\sigma)d_j] + [p_j(\sigma)d_0, d_j] = 2jp_0(\sigma)d_j.$$

Hence

$$\begin{aligned}
&[d_0, (p_0(\sigma) - p_j(\sigma))d_j] \\
&= [d_0, p_0(\sigma)d_j] - [d_0, p_j(\sigma)d_j] \\
&= [d_0, p_0(\sigma)d_j] + [p_j(\sigma)d_0, d_j] - \left([p_j(\sigma)d_0, d_j] + [d_0, p_j(\sigma)d_j]\right) \\
&= 2jp_0(\sigma)d_j - 2jp_j(\sigma)d_j \qquad (by \quad (*)) \\
&= 2j\left(p_0(\sigma) - p_j(\sigma)\right)d_j.
\end{aligned}$$

i.e. $(p_0(\sigma) - p_j(\sigma))d_j$ is an eigenvector of d_0 of eigenvalue $2j$. So

$$(p_0(\sigma) - p_j(\sigma))d_j = \alpha_j\sigma^j d_j,$$

and

$$p_j(\sigma) = p_0(\sigma) - \alpha_j\sigma^j$$

for some $\alpha_j \in \mathbb{C}$. Now from

$$\begin{aligned}
&[\tau(d_j), d_{-j}] + [d_j, \tau(d_{-j})] \\
&= \left[\left(p_0(\sigma) - \alpha_j\sigma^j\right)d_j, d_{-j}\right] + \left[d_j, \left(p_0(\sigma) - \alpha_{-j}\sigma^{-j}\right)d_{-j}\right] \\
&= [p_0(\sigma)d_j, d_{-j}] + [d_j, p_0(\sigma)d_{-j}] \\
&\quad - \alpha_j[d_{2j}, d_{-j}] - \alpha_{-j}[d_j, d_{-2j}] \\
&= 2p_0(\sigma)[d_j, d_{-j}] + 3j\alpha_j d_j + 3j\alpha_{-j}d_{-j} \\
&= -4jp_0(\sigma)d_0 + 3j\alpha_j d_j + 3j\alpha_{-j}d_{-j},
\end{aligned}$$

and

$$2\tau[d_j, d_{-j}] = -4j\tau(d_0) = -4jp_0(\sigma)d_0,$$

we obtain $\alpha_j = 0$. Hence

$$p_j(\sigma) = p_0(\sigma), \quad \forall j \in \mathbb{Z},$$

and $\tau = p_0(\sigma) \in \mathbb{C}[\sigma, \sigma^{-1}]$. □

Remark.

(1) $\mathbb{C}[\sigma, \sigma^{-1}] \cong \mathbb{C}[t, t^{-1}]$ as commutative algebras over $\mathbb{C}$, where t is an indeterminate.

(2) For any $p(\sigma) \in \mathbb{C}[\sigma, \sigma^{-1}]$, $p(\sigma)W$ is a Lie subalgebra of W, and $\sigma^m p(\sigma)W = p(\sigma)W$ for all $m \in \mathbb{Z}$.

Proposition 4. Let σ be defined as in Proposition 1 and let $\mathfrak{g} \leq W$ be a Lie subalgebra. Then $\mathfrak{g}$ is σ-invariant iff $\mathfrak{g} = p(\sigma)W$ for some $p(\sigma) \in \mathbb{C}[\sigma]$.

Proof. If $\mathfrak{g} = p(\sigma)W$ for some $p(\sigma) \in \mathbb{C}[\sigma]$, then clearly $\mathfrak{g}$ is a σ-invariant Lie subalgebra of W. Conversely, suppose that $0 \neq \mathfrak{g} \leq W$ and $\sigma(\mathfrak{g}) = \mathfrak{g}$. Then there exists

$$0 \neq x = \sum_{i=0}^{r} a_i d_i = p_x(\sigma) d_0 \in \mathfrak{g}.$$

Assume that x is an element of $\mathfrak{g}$ such that $p_x(t)$ has minimal degree. For any $y \in \mathfrak{g}$, $y \neq 0$, we may write

$$y = \sigma^{m_y} p_y(\sigma) d_0$$

for some $p_y(t) \in \mathbb{C}[t], p_y(0) \neq 0$, and $m_y \in \mathbb{Z}$. Then

$$\sigma^{-m_y} y = p_y(\sigma) d_0.$$

Moreover, if we write

$$p_y(t) = q(t)p_x(t) + r(t),$$

with some $q(t), r(t) \in \mathbb{C}[t]$ and $deg(r(t)) < deg(p_x(t))$, then $r(t) = 0$ by the minimality of $p(t)$. Thus

$$\sigma^{-m_y} y = p_y(\sigma) d_0 = p_x(\sigma) q(\sigma) d_0 \in p_x(\sigma)W.$$

So $y = \sigma^{m_y} q(\sigma) p_x(\sigma) d_0 \in p_x(\sigma)W$, and hence $\mathfrak{g} \subseteq \sum_{m \in \mathbb{Z}} \sigma^m p_x(\sigma)W = p_x(\sigma)W$. On the other hand, an element of $p_x(\sigma)W$ can be written as $\sigma^m p_x(\sigma) q(\sigma) d_0$ for some $m \in \mathbb{Z}$ and some $q(t) \in \mathbb{C}[t]$. Since $p_x(\sigma) d_0 \in \mathfrak{g}$ and $\sigma(\mathfrak{g}) = \mathfrak{g}$, we have $p_x(\sigma)W = \mathfrak{g}$, and hence $\mathfrak{g} = p_x(\sigma)W$. □

2.2 Polynomial Lie Subalgebras of the Witt Algebra

In Section 2.1, we proved that if $\sigma \in End(W)$ is defined by $\sigma(d_i) = d_{i+1}$ for all i, then a Lie subalgebra $\mathfrak{g}$ of W is σ-invariant iff $\mathfrak{g} = p(\sigma)W$ for some $p(t) \in \mathbb{C}[t]$. We give the following definition.

Definition. A Lie subalgebra V of W is called a polynomial Lie subalgebra if $V = p(\sigma)W$ for some polynomial $p(t) \in \mathbb{C}[t]$. We use $W(p(t))$ to denote $p(\sigma)W$ and use $d_m^{p(t)}$ to denote $p(\sigma)d_m$.

In this section, we discuss the structure of such algebras and give a condition for $p(t)$ such that $W(p(t))$ admits an anti-linear anti-involution.

Proposition 1. Let $p(t), q(t) \in \mathbb{C}[t]$, $p(0) \neq 0, q(0) \neq 0$. Then

(i) $W(q(t)) \leq W(p(t))$ iff $p(t) \mid q(t)$;

(ii) If $g.c.d(p(t), q(t)) = 1$, then

$$W(p(t)q(t)) = W(p(t)) \cap W(q(t)).$$

Consequently, if

$$p(t) = \prod_{i=1}^{r}(1+\alpha_i t)^{k_i},$$

where $\alpha_1, \ldots \alpha_r$ are distinct , then

$$W(p(t)) = \bigcap_{i=1}^{r} W\left((1+\alpha_i t)^{k_i}\right).$$

Proof. (i) If $p(t) \mid q(t)$, then

$$q(t) = p(t)r(t)$$

for some $r(t) \in \mathbb{C}[t]$ and

$$W(q(t)) = q(\sigma)W = p(\sigma)r(\sigma)W \subseteq p(\sigma)W = W(p(t)).$$

So

$$W(q(t)) \leq W(p(t)).$$

Conversely, if $W(q(t)) \leq W(p(t))$, then

$$d_0^{q(t)} = d_0^{p(t)r(t)}$$

for some $r(t) \in \mathbb{C}[t]$. Hence

$$p(t) \mid q(t).$$

(ii) By (i),

$$W(p(t)q(t)) \leq W(p(t)) \cap W(q(t)).$$

Conversely, if

$$0 \neq x \in W(p(t)) \cap W(q(t)),$$

we may write

$$x = d_m^{x(t)}$$

for some $m \in \mathbb{Z}$, $x(t) \in \mathbb{C}[t]$ and $x(0) \neq 0$. Since $x \in W(p(t))$, we have

$$x = d_m^{x(t)} = d_m^{p(t)r(t)}$$

for some $r(t) \in \mathbb{C}[t]$ and hence

$$p(t) \mid x(t).$$

Similarly, we have

$$q(t) \mid x(t).$$

But $g.c.d(p(t), q(t)) = 1$. So

$$p(t)q(t) \mid x(t)$$

and

$$x \in W(p(t)q(t)).$$

Hence $W(p(t)q(t)) = W(p(t)) \cap W(q(t))$. □

Lemma 1. If $q(t) \in \mathbb{C}[t]$, and $q(0) \neq 0$, then

$$[d_m, q(\sigma)d_n] = q(\sigma)[d_m, d_n] + q'(\sigma)\sigma d_{m+n}$$

for all $m, n \in \mathbb{Z}$.

Proof. We prove the lemma by induction on $deg(q(t))$. Assume that the lemma is true for $q(t)$. Then

$$\begin{aligned}
&[d_m, q(\sigma)(1+\alpha\sigma)d_n] \\
&= [d_m, q(\sigma)d_n] + \alpha[d_m, q(\sigma)d_{n+1}] \\
&= q(\sigma)(n-m)d_{m+n} + \alpha q(\sigma)(n+1-m)d_{m+n+1} \\
&\quad + q'(\sigma)\sigma d_{m+n} + \alpha q'(\sigma)\sigma d_{m+n+1} \\
&= (n-m)q(\sigma)(d_{m+n} + \alpha\sigma d_{m+n}) + \alpha q(\sigma)\sigma d_{m+n} + q'(\sigma)(1+\alpha\sigma)d_{m+n+1} \\
&= q(\sigma)(1+\alpha\sigma)[d_m, d_n] + (q(\sigma)(1+\alpha\sigma))'\sigma d_{m+n}.
\end{aligned}$$

The lemma follows. □

Proposition 2. Let $p(t), q(t) \in \mathbb{C}[t], p(0) \neq 0, q(0) \neq 0$. Then $W(q(t)) \triangleleft W(p(t))$ iff

$$p(t) \mid q(t)$$

and

$$(*) \qquad \frac{p(t)}{g.c.d(p(t), p'(t))} = \frac{q(t)}{g.c.d(q(t), q'(t))}.$$

Proof. Note that the condition (*) is equivalent to the following condition:

For any $1 + \alpha t$,

$$(1 + \alpha t) \mid q(t)$$

iff

$$(1 + \alpha t) \mid p(t).$$

Suppose that $p(t) \mid q(t)$ and (*) holds. Write $q(t) = p(t)r(t)$. By Lemma 1,

$$(**) \qquad \begin{aligned} [p(\sigma)d_m, q(\sigma)d_n] &= [p(\sigma)d_m, p(\sigma)r(\sigma)d_n] \\ &= (p(\sigma))^2[d_m, r(\sigma)d_n] \\ &= (p(\sigma))^2\big(r(\sigma)[d_m, d_n] + r'(\sigma)\sigma d_{m+n}\big) \\ &= p(\sigma)q(\sigma)[d_m, d_n] + (p(\sigma))^2 r'(\sigma)\sigma d_{m+n}. \end{aligned}$$

Since by the condition, $q(t) \mid (p(t))^2 r'(t)$, we have

$$[p(\sigma)d_m, q(\sigma)d_n] \in W(q(t)).$$

So

$$W(q(t)) \triangleleft W(p(t)).$$

Conversely, suppose that $W(q(t)) \triangleleft W(p(t))$ and there exists $1 + \alpha t$ such that $q(t)$ is divisible by $1 + \alpha t$ but $p(t)$ is not divisible by $1 + \alpha t$. Assume that $s \in \mathbb{Z}_{\geq 1}$ is the maximal integer such that $(1 + \alpha t)^s \mid q(t)$ and $q(t) = p(t)r(t)$ for some $r(t) \in \mathbb{C}[t]$. Using (**),

$$p(\sigma)q(\sigma)[d_m, d_n] + (p(\sigma))^2 r'(\sigma)\sigma d_{m+n} = [p(\sigma)d_m, q(\sigma)d_n] \in W(q(t)).$$

So

$$(p(\sigma))^2 r'(\sigma)\sigma d_{m+n} \in W(q(t)),$$

and hence

$$q(t) \mid (p(t))^2 r'(t).$$

But $p(t)$ is not divisible by $1 + \alpha t$, so $(1 + \alpha t)^s \mid r'(t)$ and then $(1 + \alpha t)^{s+1} \mid q(t)$. This is a contradiction. □

Proposition 3. Let $p(t) \in \mathbb{C}[t]$. Then

(i) $W(p(t))$ has a trivial centre.

(ii) $W(p(t))$ is indecomposable.

(iii) $W(p(t))$ is simple iff $W(p(t)) = W$.

Proof. (i) is clear.

(ii) Suppose that

$$W(p(t)) = W_1 \oplus W_2,$$

where $0 \neq W_1 \triangleleft W(p(t)), \quad 0 \neq W_2 \triangleleft W(p(t))$. Let

$$0 \neq x = d_m^{p(t)x(t)} = p(\sigma)x(\sigma)d_m \in W_1,$$

where $x(t) \in \mathbb{C}[t]$. Then

$$\begin{aligned}\left[x, d_{n-m}^{p(t)x(t)}\right] &= (p(\sigma)x(\sigma))^2 [d_m, d_{n-m}] \\ &= (n-2m)(p(\sigma)x(\sigma))^2 d_n \in W_1\end{aligned}$$

for all $n \in \mathbb{Z}$. So

$$\left(p(\sigma)x(\sigma)\right)^2 d_n \in W_1$$

for all $n \neq 2m$. Similar argument as above implies that

$$W\left((p(t)x(t))^4\right) \subseteq W_1.$$

Similarly, there is $y(t) \in \mathbb{C}[t]$ such that

$$W\left((p(t)y(t))^4\right) \subseteq W_2.$$

Hence

$$0 \neq W\left((p(t)x(t)y(t))^4\right) \subseteq W_1 \cap W_2.$$

This is a contradiction.

(iii) If $deg(p(t)) > 0$, then $W\left(p(t)^2\right) \neq 0$ is a proper ideal of $W(p(t))$. If $deg(p(t)) = 0$, then $W(p(t)) = W$. □

Proposition 4. Let $p(t), q(t) \in \mathbb{C}[t]$. If there exists $a \neq 0$ such that $q(t) = p(at)$, then $W(p(t)) \cong W(q(t))$.

Proof. Let

$$\phi : W(p(t)) \longrightarrow W(q(t))$$

be the linear map such that

$$\phi(p(\sigma)d_m) = a^m q(\sigma)d_m.$$

Assume that $p(t) = \sum_i a_i t^i$. Then

$$\begin{aligned}
\phi([p(\sigma)d_m, p(\sigma)d_n] &= \phi\left((p(\sigma))^2(n-m)d_{m+n}\right) \\
&= (n-m)\phi\left(\sum_i a_i p(\sigma)d_{m+n+i}\right) \\
&= (n-m)\sum_i a_i q(\sigma)a^{m+n+i}d_{m+n+i} \\
&= (n-m)q(\sigma)a^{m+n}q(\sigma)d_{m+n} \\
&= (q(\sigma))^2 a^{m+n}[d_m, d_n] \\
&= [a^m q(\sigma)d_m, a^n q(\sigma)d_n] \\
&= [\phi(p(\sigma)d_m), \phi(p(\sigma)d_n)].
\end{aligned}$$

The proposition follows. □

Recall that the anti-linear map

$$\omega : W \longrightarrow W$$

such that

$$\omega(d_m) = d_{-m}$$

is an anti-involution of W. For a polynomial $p(t) \in \mathbb{C}[t]$, $\omega\mid_{W(p(t))}$ is in general not an anti-involution of $W(p(t))$. However, we have the following proposition.

Proposition 5. Let $p(t) = \sum_{i=0}^r a_i t^i$. Then $\omega\mid_{W(p(t))}$ is an anti-involution of $W(p(t))$ iff $p(t)$ satisfies

$$t^r \overline{p}\left(\frac{1}{t}\right) = bp(t)$$

for some $b \in S^1$, the unit circle in $\mathbb{C}$. Here $\overline{p}(t) = \sum_{i=0}^r \overline{a_i} t^i$, and $\overline{a_i}$ is the complex conjugate of a_i. In this case $\omega\left(d_m^{p(t)}\right) = bd_{-m-r}^{p(t)}$.

Proof. If ω is an anti-involution of $W(p(t))$, then for any $m \in \mathbb{Z}$,

$$\omega(d_m^{p(t)}) = \omega\left(\sum_{i=0}^r a_i d_{m+i}\right) = \sum_{i=0}^r \overline{a_i} d_{-m-i} \in W(p(t)).$$

So

$$\sum_{i=0}^{r} \overline{a_i} d_{-m-i} = b_m d^{p(t)}_{-m-r}$$

for some $b_m \in \mathbb{C}$. This implies that

$$b_m a_i = \overline{a_{r-i}} \quad i = 0, 1, ..., r \quad \forall m \in \mathbb{Z}.$$

Hence b_m is independent of m. Assume that $b_m = b$. Then from $\omega^2 = 1$ we get $b\bar{b} = 1$, i.e. $b \in S^1$. So $p(t)$ satisfies

$$t^r \overline{p}(\frac{1}{t}) = bp(t).$$

Conversely, if $t^r \overline{p}(\frac{1}{t}) = bp(t)$ for some $b \in S^1$, it is easy to see that

$$\omega\left(d^{p(t)}_m\right) = b d^{p(t)}_{-m-r}$$

and

$$\omega\left[d^{p(t)}_m, d^{p(t)}_n\right] = \left[\omega\left(d^{p(t)}_n\right), \omega\left(d^{p(t)}_m\right)\right].$$

i.e. ω is an anti-involution of $W(p(t))$. □

2.3 Segal-Sugawara Construction

In this section we define the polynomial subalgebras of the Virasoro algebra and realize the elements of these polynomial subalgebras as Segal-Sugawara operators on certain Fock spaces. Recall that the Virasoro algebra is the universal central extension of the Witt algebra.

$$\begin{aligned} Vir &= \sum_{m \in \mathbb{Z}} \mathbb{C}L_m + \mathbb{C}¢, \\ [L_m, L_n] &= (n-m)L_{m+n} + \frac{m^3 - m}{12}\delta_{m+n,0}¢, \\ [L_m, ¢] &= 0. \end{aligned}$$

Let

$$\pi : Vir \longrightarrow W$$

be the canonical homomorphism. i.e.

$$\pi(L_m) = d_m, \quad \pi(¢) = 0.$$

For any $p = p(t) = \sum_{i=0}^{r} a_i t^i \in \mathbb{C}[t]$, we denote $\pi^{-1}(W(p(t)))$ by $Vir(p)$. Clearly $Vir(p)$ is a Lie subalgebra of Vir and

$$\{L_m^p = \sum_{i=0}^{r} a_i L_{m+i} \mid \forall m \in \mathbb{Z}\} \cup \{\cent\}$$

is a basis of $Vir(p)$. The Lie brackets of $Vir(p)$ are given by

$$[L_m^p, L_n^p] = (n-m)\sum_{i=0}^{r} a_i L_{m+n+i}^p + \sum_{i,k} a_i a_k \frac{(m+i)^3-(m+i)}{12}\delta_{m+n+i+k,0}\cent,$$
$$[L_m^p, \cent] = 0.$$

Definition. $Vir(p)$ is called the polynomial Lie subalgebra of Vir associated with $p(t)$.

Now we give the Segal-Sugawara construction of $Vir(p)$. In the case of $p(t) = 1$, this is the usual Segal-Sugawara construction of the Virasoro algebra.

Fix a polynomial

$$p = p(t) = \sum_{i=0}^{r} a_i t^i \in \mathbb{C}[t].$$

Let $\{a_m^p \mid m \in \mathbb{Z}\}$ be a set of symbols and define

$$\mathbb{A}_p := \sum_{m\in\mathbb{Z}} \mathbb{C}a_m^p + \mathbb{C}$$

to be the abstract vector space with basis $\{a_m^p \mid m \in \mathbb{Z}\} \cup \{1\}$. $\mathbb{A}_p$ may be made into a Lie algebra with commutators

$$[a_m^p, a_n^p] = \sum_{i=0}^{r} a_i \left(m + \frac{i}{2}\right)\delta_{m+n+i,0},$$
$$[a_m^p, 1] = 0.$$

Clearly,

$$[[\mathbb{A}_p, \mathbb{A}_p], \mathbb{A}_p] = 0.$$

We call $\mathbb{A}_p$ the Heisenberg algebra associated with $p(t)$. If we define

$$\mathbb{A}_p(\pm) := \sum_{m>0} \mathbb{C}a_{\pm m}^p$$

and

$$\mathfrak{h}_p := \mathbb{C}a_0^p + \mathbb{C},$$

then $\mathbb{A}_p(+)$ and $\mathfrak{h}_p$ are abelian Lie subalgebras of $\mathbb{A}_p$, and

$$\mathbb{A}_p = \mathbb{A}_p(-) \oplus \mathfrak{h}_p \oplus \mathbb{A}_p(+).$$

Let $U(\mathbb{A}_p)$ be the universal enveloping algebra of $\mathbb{A}_p$, let $h \in \mathbb{C}$, and let $J_p(h)$ be the left ideal of $U(\mathbb{A}_p)$ generated by $\mathbb{A}_p(+)$ and $a_0^p - h$. Define

$$M_p(h) := U(\mathbb{A}_p)/J_p(h).$$

Then $M_p(h)$ is a left $\mathbb{A}_p$-module. Moreover, if $v \in M_p(h)$, then $a_m^p \cdot v = 0$ for $m \gg 0$. We call $M_p(h)$ a Fock space. The Segal-Sugawara operators on $M_p(h)$ are defined to be

$$d_m^p := -\frac{1}{2}\sum_{j\in\mathbb{Z}} : a_{-j}^p a_{m+j}^p :,$$

where

$$: a_m^p a_n^p := \begin{cases} a_m^p a_n^p & m \leq n \\ a_n^p a_m^p & m > n. \end{cases}$$

By similar calculation as that in the Segal-Sugawara construction of the Virasoro algebra (see for example [KR]), we get

$$[a_k^p, d_m^p] = \sum_{i=0}^{r} a_i \left(k + \frac{i}{2}\right) a_{m+i+k}^p,$$

$$[d_m^p, d_n^p] = (n-m)\sum_{i=0}^{r} a_i d_{m+n+i}^p + \sum_{i,k} a_i a_k \left(\frac{(m+i)^3 - (m+i)}{12} + \frac{1}{16}(k-i)(m+i)^2\right) \delta_{m+n+i+k,0}.$$

Define

$$\mathfrak{g}_p := \sum_{m\in\mathbb{Z}} \mathbb{C} d_m^p + \mathbb{C}.$$

Then $\mathfrak{g}_p$ with Lie brackets as above is a Lie algebra.

Proposition 1. For any $p = p(t) \in \mathbb{C}[t]$, with $p(0) \neq 0$, $Vir(p) \cong \mathfrak{g}_p$.

Proof. We are going to show that there exist $\{b_m \in \mathbb{C} \mid m \in \mathbb{Z}\}$ such that

$$\phi : Vir(p) \longrightarrow \mathfrak{g}_p,$$

$$\phi(L_m^p) = d_m^p + b_m, \quad \phi(\mathfrak{c}) = 1,$$

is a Lie algebra isomorphism. Comparing the commutators of $Vir(p)$ with those of $\mathfrak{g}_p$, we see that $\phi[L_m^p, L_n^p] = [\phi L_m^p, \phi L_n^p]$ iff

$$(*) \qquad (n-m)\sum_{i=0}^{r} a_i b_{m+n+i} = \sum_{i,k} a_i a_k \frac{(k-i)(m+i)^2}{16}\delta_{m+n+i+k,0}$$
$$= \sum_{l=0}^{2r}\left(\sum_{i+k=l} a_i a_k \frac{(k-i)(2mi+i^2)}{16}\right)\delta_{m+n+l,0}.$$

Since

$$\sum_{i+k=l} a_i a_k \frac{(k-i)(2mi+i^2)}{16}$$
$$= \sum_{i<k, i+k=l} a_i a_k \frac{(k-i)(2mi+i^2)}{16} + \sum_{i>k, i+k=l} a_i a_k \frac{(k-i)(2mi+i^2)}{16}$$
$$= (2m+l) \sum_{i<k, i+k=l} \frac{(i-k)(l-2i)}{16} a_i a_k,$$
$$= -(2m+l) \sum_{i<k, i+k=l} \frac{(i-k)^2}{16} a_i a_k,$$

(*) becomes

$$(n-m)\sum_{i=0}^{r} a_i b_{m+n+i} = -\sum_{l=0}^{2r}\left((2m+l) \sum_{i<k, i+k=l} \frac{(i-k)^2}{16} a_i a_k\right)\delta_{m+n+l,0}.$$

If $\delta_{m+n+l,0} = 1$, we have

$$(2m+l)\sum_{i=0}^{r} a_i b_{m+n+i} = (2m+l) \sum_{i<k, i+k=l} \frac{(i-k)^2}{16} a_i a_k,$$

or

$$\sum_{i=0}^{r} a_i b_{m+n+i} = \sum_{i<k, i+k=-m-n} \frac{(i-k)^2}{16} a_i a_k.$$

If $\delta_{m+n+l,0}$ is 0, for all $l = 0, 1, \ldots, r$, we have

$$\sum_{i=0}^{r} a_n b_{m+n+i} = 0.$$

Consider the linear system

$$\sum_{i=0}^{r} a_i x_{i-l} = c_l, \qquad where \quad c_l \in \mathbb{C}, \quad l \in \mathbb{Z}.$$

Since $a_0 \neq 0$ and $a_r \neq 0$, this linear system has infinitely many solutions. Hence there exist a set $\{b_m \mid m \in \mathbb{Z}\}$ which satisfies (*). This completes the proof. □

Let $p = p(t) \in \mathbb{C}[t]$, $Vir(p)$ be the polynomial subalgebra of Vir associated with $p(t)$. Let

$$Vir(p)_+ = \sum_{m>0} \mathbb{C}L_m^p,$$

$$\mathfrak{h} = \mathbb{C}L_0^p + \mathbb{C}\mathfrak{c},$$

$$Vir(p)_- = \sum_{m<0} \mathbb{C}L_m^p.$$

Then

$$Vir(p) = Vir(p)_- \oplus \mathfrak{h} \oplus Vir(p)_+.$$

Further, $Vir(p)_+$ is a Lie subalgebra of $Vir(p)$, $\mathfrak{h}$ is an abelian Lie subalgebra, and

$$[\mathfrak{h}, Vir(p)_+] \subseteq Vir(p)_+.$$

We call

$$Vir(p) = Vir(p)_- \oplus \mathfrak{h} \oplus Vir(p)_+$$

the quasi-triangular decomposition of $Vir(p)$.

2.4 Highest Weight Modules of $Vir(p)$

We have seen in last section that for $p = p(t) \in \mathbb{C}[t]$ with $p(0) \neq 0$, $Vir(p)$ has quasi-triangular decomposition

$$Vir(p) = Vir(p)_- \oplus \mathfrak{h} \oplus Vir(p)_+$$

In this section, we define the category $\mathbb{O}_p$ of $Vir(p)$-modules, which is an analogue of the category $\mathbb{O}$ of the Virasoro algebra modules, and discuss highest weight $Vir(p)$-modules.

First, we introduce a partial order $\leq$ on $\mathfrak{h}^*$, the dual space of $\mathfrak{h}$, as follows: For any $\lambda, \mu \in \mathfrak{h}^*$,

$$\lambda \geq \mu$$

iff

$$\lambda(L_0^p) - \mu(L_0^p) \in \mathbb{Z}_{\geq 0}.$$

Now we give the following definition.

Definition. Let V be a $Vir(p)$-module. It is a $\mathfrak{h}$-module if we restrict to $\mathfrak{h}$.

V is called $\mathfrak{h}$-triangular if

$$V = \sum_{\lambda \in \mathfrak{h}^*} V_\lambda,$$

where $\{V_\lambda \mid \lambda \in \mathfrak{h}^*\}$ are $\mathfrak{h}$-submodules and satisfy the following conditions:

(i) If $\lambda \leq \mu$, then V_μ is a $\mathfrak{h}$-submodule of V_λ.

(ii) $V_\lambda = \left\{v \in V \mid h \cdot v = \lambda(h)v \left(mod \quad \sum_{\mu>\lambda} V_\mu\right) \quad \forall h \in \mathfrak{h}\right\}$.

$V_\lambda / \sum_{\mu>\lambda} V_\mu$ is called a quasi-weight space. λ is called a quasi-weight if

$$V_\lambda / \sum_{\mu>\lambda} V_\mu \neq 0.$$

A non-zero element $v \in V_\lambda / \sum_{\mu>\lambda} V_\mu$ is called a quasi-weight vector. The multiplicity of λ is defined to be

$$dim \left(V_\lambda / \sum_{\mu>\lambda} V_\mu \right)$$

and denoted by $mult(\lambda)$. For a $\mathfrak{h}$-triangular module V, we define

$$P(V) := \{\lambda \in \mathfrak{h}^* \mid \lambda \quad is \quad a \quad quasi-weight\}$$

and for $\lambda \in \mathfrak{h}^*$, we define

$$D(\lambda) := \{\mu \in h^* \mid \mu \leq \lambda\}.$$

V is called $\mathfrak{h}$-diagonalizable if

$$V = \sum_{\lambda \in \mathfrak{h}^*} V'_\lambda,$$

where

$$V'_\lambda := \{v \in V \mid hv = \lambda(h)v \quad \forall h \in \mathfrak{h}\}.$$

V'_λ is called a weight space. λ is called a weight if $V'_\lambda \neq 0$. A non-zero element of V'_λ is called a weight vector.

Note that if $V = \sum_{\lambda \in \mathfrak{h}^*} V'_\lambda$ is $\mathfrak{h}$-diagonalizable, then V is $\mathfrak{h}$-triangular. In fact, for any $\lambda \in \mathfrak{h}^*$, define

$$V_\lambda := \sum_{\mu \geq \lambda} V'_\mu.$$

Then

$$V = \sum_{\lambda \in \mathfrak{h}^*} V_\lambda$$

and

$$V'_\lambda \cong V_\lambda / \sum_{\mu > \lambda} V_\mu$$

is a quasi-weight space. Moreover, a weight is a quasi-weight, and a weight vector is quasi-weight vector.

The category $\mathbb{O}_p$ of $Vir(p)$-modules is defined as follows:

Objects: $Vir(p)$-modules

$$V = \sum_{\lambda \in h^*} V_\lambda,$$

where V is $\mathfrak{h}$-triangular and $mult(\lambda) < \infty$ for all $\lambda \in \mathfrak{h}^*$, and there exist $\lambda_1, ..., \lambda_n \in \mathfrak{h}^*$ such that

$$P(V) \subseteq \bigcup_{i=1}^{n} D(\lambda_i).$$

Morphisms: $Vir(p)$-module homomorphisms.

Let

$$\mathbb{O}'_p := \{V \in \mathbb{O}_p \mid V \quad is \quad \mathfrak{h} - diagonalizable\}.$$

Then $\mathbb{O}'_p$ is a subcategory of $\mathbb{O}_p$.

The following facts are clear:

If $V_1, V_2 \in \mathbb{O}_p \left(or \quad \mathbb{O}'_p\right)$, then $V_1 \oplus V_2, \quad V_1 \otimes V_2 \in \mathbb{O}_p \left(or \quad \mathbb{O}'_p\right)$.

If $V_1 \in \mathbb{O}_p \left(or \quad \mathbb{O}'_p\right), \quad V_2 \leq V_1$, then $V_2 \in \mathbb{O}_p \left(or \quad \mathbb{O}'_p\right)$ and $V_1/V_2 \in \mathbb{O}_p \left(or \quad \mathbb{O}'_p\right)$.

Important examples of modules from $\mathbb{O}_p$ are the highest weight modules. Let V be a $Vir(p)$-module. Let $U(Vir(p))$ be the universal enveloping algebra of $Vir(p)$. If there exists $\Lambda \in \mathfrak{h}^*$ and $v \in V$ such that

$$Vir(p)_+ \cdot v = 0,$$

$$h \cdot v = \Lambda(h)v, \quad \forall h \in \mathfrak{h},$$

and

$$V = U(Vir(p)) \cdot v,$$

then V is called a highest weight module. Λ is called the highest weight and v is called a highest weight vector. If V is a highest weight module of $Vir(p)$ with highest weight Λ, then

$$V = \sum_{\lambda \leq \Lambda} V_\lambda,$$

where

$$V_\Lambda = \mathbb{C}v,$$

and

$$V_\lambda := \left\{ u \in V \mid h \cdot u = \lambda(h) u \quad \left(mod \quad \sum_{\mu > \lambda} V_\mu \right) \quad \forall h \in \mathfrak{h} \right\}.$$

Clearly, $mult(\lambda) < \infty$ for all weight λ. So $V \in \mathbb{O}_p$.

Definition. A highest weight $Vir(p)$-module $M_p(\Lambda)$ with highest weight Λ is called a Verma module if every highest weight $Vir(p)$- module with highest weight Λ is a quotient of $M_p(\Lambda)$.

Proposition 1. For any $\Lambda \in \mathfrak{h}^*$, there exists a unique Verma module $M_p(\Lambda)$ with highest weight Λ up to isomorphism.

Proof. Let $U(Vir(p))$ be the universal enveloping algebra of $Vir(p)$, $J_p(\Lambda)$ be the left ideal of $U(Vir(p))$ generated by $Vir(p)_+$ and $\{h - \Lambda(h) \mid \forall h \in \mathfrak{h}\}$, and set

$$M_p(\Lambda) = U(Vir(p))/J_p(\Lambda).$$

The left multiplication on $U(Vir(p))$ induces a structure of $U(Vir(p))$-module on $M_p(\Lambda)$. Hence $M_p(\Lambda)$ is a $Vir(p)$- module. Moreover, $M_p(\Lambda)$ is a highest weight $Vir(p)$-module with highest weight Λ and highest weight vector $1 + J_p(\Lambda)$. Let V be a $Vir(p)$-module with highest weight Λ and highest weight vector v_Λ. The set of the annihilators of v_Λ is a left ideal J_1 of $U(Vir(p))$ and $J_p(\Lambda) \subseteq J_1$. So we have an epimorphism of $Vir(p)$-module from $M_p(\Lambda)$ to V. The uniqueness of $M_p(\Lambda)$ follows from the universal property of $M_p(\Lambda)$. □

Remark. By the Poincare-Birkhoff-Witt theorem, $M_p(\Lambda)$ has a basis

$$L^p_{-i_n} \ldots L^p_{-i_1} v_p,$$

where $0 < i_1 \leq i_2 \leq \ldots \leq i_n$ and $v_p = 1 + J_p(\Lambda)$. Note that if $p = 1$, then $\mathbb{O}_1$ is the usual category $\mathbb{O}$ of Vir-modules. If $V \in \mathbb{O}_1$, then $V \in \mathbb{O}_p$ for any $p(t) \in \mathbb{C}[t]$. In particular, if $\Lambda_1 \in (\mathbb{C}L_0 + \mathbb{C}c\!\!/)^*$, then $M_1(\Lambda_1)$ is the Verma module of Vir of highest weight Λ_1, and $M_1(\Lambda_1) \in \mathbb{O}_p$.

Proposition 2. Let
$$p(t) = \sum_{i=0}^{r} a_i t^i \in \mathbb{C}[t], \quad a_0 = 1.$$
Let
$$\Lambda_p \in (\mathbb{C}L_0^p + \mathbb{C}¢)^*$$
and
$$\Lambda_1 \in (\mathbb{C}L_0 + \mathbb{C}¢)^*$$
be such that $\Lambda_p(L_0^p) = \Lambda_1(L_0)$ and $\Lambda_p(¢) = \Lambda_1(¢)$. Then
$$M(\Lambda) \cong M_p(\Lambda_p)$$
as $Vir(p)$-modules.

Proof. Let
$$\psi : M_p(\Lambda_p) \longrightarrow M_1(\Lambda_1)$$
be the linear map such that
$$L^p_{-i_n} ... L^p_{-i_1} v_p \mapsto L^p_{-i_n} ... L^p_{-i_1} v_1,$$
where v_p is the image of 1 in $M_p(\Lambda_p)$, and v_1 is the image of 1 in $M_1(\Lambda_1)$. Clearly,
$$U(Vir(p)) \xhookrightarrow{i} U(Vir) \xrightarrow{\pi} M_1(\Lambda_1) = \frac{U(Vir)}{J_1(\Lambda_1)}$$
is a $Vir(p)$-module homomorphism.

Next we show ψ is onto. Since
$$L_{-n}v_1 = 0 \quad \forall -n > 0$$
and
$$L_{-n-1}v_1 = L^p_{-n-1}v_1 - (a_1 L_{-n} + ... + a_r L_{-n-1+r})v_1,$$
so for any $n \in \mathbb{Z}$, $L_{-n}v_1$ lies in the linear span of
$$\{L^p_m v_1 \mid m \in \mathbb{Z}\}.$$

Assume that all $L_{-i_n} ... L_{-i_1} v_1$ lie in the linear span of
$$\{L^p_{-k_j} ... L^p_{-k_1} v_1 \mid k_1, ..., k_j \in \mathbb{Z}, j = 1, 2, ..., n\}$$

for some $n > 0$. For $-m \gg 0$, $L_{-m}L_{-i_n}...L_{-i_1}v_1 = 0$. Also

$$\begin{aligned} &L_{-m-1}L_{-i_n}...L_{-i_1}v_1 \\ &= L^p_{-m-1}L_{-i_n}...L_{-i_1}v_1 - (a_1L_{-m} + ... + a_rL_{-m-1+r})L_{-i_n}...L_{-i_1}v_1. \end{aligned}$$

We may assume, by induction, that all $L_{-m+j}L_{-i_n}...L_{-i_1}v_1$, $j = 0, ..., r$ lie in the linear span of

$$\left\{L^p_{-k_j}...L^p_{-k_1}v_1 \mid k_1, ..., k_j \in \mathbb{Z}, j = 1, ..., n+1\right\}.$$

Then

$$L_{-m-1}L_{-i_n}...L_{-i_1}v_1 \in \left\{L^p_{-k_j}...L^p_{-k_1}v_1 \mid k_1, ..., k_j \in \mathbb{Z}, j = 1, ..., n+1\right\}.$$

Hence ψ is onto.

Finally, we show ψ is 1-1. Define a linear order on monomials

$$L_{-i_n}...L_{-i_1}v_1, \quad 0 < i_1 \le ... \le i_n$$

as follows:

$$L_{-i_n}...L_{-i_1}v_1 < L_{-j_m}...L_{-j_1}v_1$$

iff

$$n < m$$

or

$$n = m, \quad i_n = j_n, \quad ... \quad , i_k = j_k, \quad i_{k-1} < j_{k-1}.$$

For any $x \in M_1(\Lambda_1)$, we can write $x = x_1 + ... + x_s$ where $\{x_i \mid i = 1, ..., s\}$ are monomials, and $x_1 > ... > x_s$. We call x_1 the leading term of x. For example, the leading term of

$$L^p_{-k_j}...L^p_{-k_1}v_1$$

is

$$L_{-k_j}...L_{-k_1}v_1.$$

Now suppose that

$$\sum_{i_1, ..., i_n} a_{i_n...i_1}L^p_{-i_n}...L^p_{-i_1}v_1 = 0,$$

where $a_{i_n...i_1} \neq 0$. If we write the left hand side of above as $u_1 + ... + u_s$ with $u_1 > u_2 > ... > u_s$, then

$$u_1 = u_2 = ... = u_s = 0.$$

On the other hand, the leading term u_1 has the form

$$a_{i_n \dots i_1} L_{-i_n} \dots L_{-i_1} v_1$$

which is non-zero. This is a contradiction. So

$$\{L^p_{-i_n} \dots L^p_{-i_1} v_1 \mid 0 < i_1 \leq \dots \leq i_n, n \in \mathbb{Z}_{\geq 0}\}$$

is linearly independent and hence ψ is 1-1. □

Proposition 3. Let $M(\Lambda)$ be the Verma module of Vir of highest weight Λ and $N \subseteq M(\Lambda)$. Then N is a $Vir(p)$-submodule iff N is a Vir-submodule.

Proof. If N is a $Vir(p)$-submodule and $x \in N$, then $L_m x = 0$, for $m \gg 0$. Assume that $p(t) = \sum_{i=0}^r a_i t^i$ with $a_0 = 1$. Since for all $n \in \mathbb{Z}$,

$$L_{n-1} x = L^p_{n-1} x - (a_1 L_n + \dots a_r L_{n+r-1}) x,$$

we see by induction on n that $L_n x \in N$ for all $n \in \mathbb{Z}$. Hence N is a Vir-submodule. □

The following results follow from Proposition 2 and 3 above, Section 2.2 Proposition 5 and the results of the highest weight modules of Vir (see for example [FQS], [L], [GKO] or [KR]).

Proposition 4. Let $p = p(t) = \sum_{i=0}^r a_i t^i$, $\Lambda \in (\mathbb{C} L^p_0 + \mathbb{C} \not{c})^*$. Then

(i) The Verma module $M_p(\Lambda)$ of $Vir(p)$ of highest weight Λ is indecomposable. There exists a unique maximal proper submodule N_p. And

$$L_p(\Lambda) := M_p(\Lambda)/N_p$$

is the unique irreducible highest weight module of highest weight Λ. Moreover, if

$$\Lambda(\not{c}) > 1, \quad \Lambda(L^p_0) > 0,$$

then

$$M_p(\Lambda) = L_p(\Lambda).$$

i.e. $M_p(\Lambda)$ is irreducible.

(ii) If $p(t)$ satisfies $t^r \overline{p}(\frac{1}{t}) = b p(t)$ for some $b \in S^1$, then $L_p(\Lambda)$ is unitarizable iff $\Lambda(\not{c}) \geq 1, \Lambda(L^p_0) \geq 0$, or

$$\Lambda(\not{c}) = 1 - \frac{6}{(m+2)(m+3)}, \qquad m = 0, 1, 2, \dots$$

$$\Lambda(L^p_0) = \frac{((m+3)r - (m+2))^2 - 1}{4(m+2)(m+3)}, \qquad r, s \in N, \quad 1 \leq s \leq r \leq m+1.$$

2.5 Quasi-admissible Modules

A Vir-module V is called admissible if $\mathbb{C}L_0 + \mathbb{C}\not{c}$ acts semisimply on V and all weight spaces are finite dimensional [MP]. In this section we give the definition of a quasi-admissible module of $Vir(p)$ and discuss a class of such modules which are called polynomial modules.

Definition. Let V be a $Vir(p)$-module. If $V = \sum_{\lambda \in h^*} V_\lambda$ is $\mathfrak{h}$-triangular and each quasi-weight space is finite dimensional, then V is called a quasi-admissible module of $Vir(p(t))$. If in addition, the dimensions of the quasi-weight spaces are uniformly bounded, then V is called a bounded quasi-admissible module. A quasi-admissible module V of $Vir(p)$ is called admissible if it is $\mathfrak{h}$-diagonalizable.

The highest weight modules of $Vir(p)$ we discussed in Section 2.4 are examples of quasi-admissible modules. But they are not bounded in general. We introduce a class of bounded quasi-admissible modules as follows:

Let

$$p = p(t) = \sum_{i=0}^{r} a_i t^i \in \mathbb{C}[t],$$

and let $Vir(p)$ be the polynomial Lie subalgebra of Vir associated with $p(t)$. Fix $\alpha \in \mathbb{C}$ and fix a polynomial

$$q = q(t) = \sum_{j=0}^{s} b_j t^j \in \mathbb{C}[t].$$

Let

$$V = V_{q,\alpha} = \sum_{k \in \mathbb{Z}} \mathbb{C} v_k$$

be the $\mathbb{C}$-space with basis $\{v_k \mid k \in \mathbb{Z}\}$. We define the actions of $Vir(p)$ on V as follows:

$$L_m^p \cdot v_k = (k + \alpha m) \sum_{i=0}^{r} a_i v_{m+k+i} + \sum_{j=0}^{s} b_j v_{m+k+j},$$

$$\not{c} \cdot v_k = 0.$$

Proposition 1. Let $p = p(t)$, $q = q(t)$, α be as above. Then $V_{q,\alpha}$ is a $Vir(p)$-module.

Proof. Since

$$
\begin{aligned}
&L_n^p L_m^p v_k \\
&= L_n^p \left((k+\alpha m) \sum_i a_i v_{m+k+i} + \sum_j b_j v_{m+k+j} \right) \\
&= (k+\alpha m) \sum_i a_i \left((m+k+i+\alpha n) \sum_l a_l v_{m+k+i+n+l} + \sum_j b_j v_{m+k+j+n+i} \right) \\
&\quad + \sum_j b_j \left((m+k+j+\alpha n) \sum_i a_i v_{m+k+j+n+i} + \sum_s b_s v_{m+k+j+n+s} \right) \\
&= \sum_{i,l} a_i a_l (k+\alpha m)(m+i+k+\alpha n) v_{m+n+k+i+l} \\
&\quad + \sum_{i,j} a_i b_j (k+\alpha m) v_{m+n+k+i+j} \\
&\quad + \sum_{i,j} a_i b_j (m+k+j+\alpha n) v_{m+n+k+i+j} \\
&\quad + \sum_{j,s} b_j b_s v_{m+n+k+j+s},
\end{aligned}
$$

and similarly

$$
\begin{aligned}
L_m^p L_n^p v_k = &\sum_{i,l} a_i a_l (k+\alpha n)(n+i+k+\alpha m) v_{m+n+k+i+l} \\
&+ \sum_{i,j} a_i b_j (k+\alpha n) v_{m+n+k+i+j} \\
&+ \sum_{i,j} a_i b_j (n+k+j+\alpha m) v_{m+n+k+i+j} \\
&+ \sum_{j,s} b_j b_s v_{m+n+k+j+s},
\end{aligned}
$$

we have

$$
\begin{aligned}
&(L_m^p L_n^p - L_n^p L_m^p) v_k \\
&= \sum_{i,l} a_i a_l (n-m)(k+\alpha(m+n+i)) v_{m+n+k+i+l} \\
&\quad + \sum_{i,j} (n-m) a_i b_j v_{m+n+k+i+j} \\
&= (n-m) \sum_i a_i \left(\sum_l a_l (k+\alpha(m+n+i)) v_{m+n+k+i+l} + \sum_j b_j v_{m+n+k+i+j} \right) \\
&= (n-m) \left(\sum_i a_i L_{m+n+i}^p v_k \right) \\
&= [L_m^p, L_n^p] v_k.
\end{aligned}
$$

So $V_{q,\alpha}$ is a $Vir(p)$-module. □

Remark. From

$$
\begin{aligned}
L_0^p v_k &= k \sum_{i=0}^{r} a_i v_{k+i} + \sum_{j=0}^{s} b_j v_{k+j} \\
&= (ka_0 + b_0) v_k + k \sum_{i=1}^{r} a_i v_{k+i} + \sum_{j=1}^{s} b_j v_{k+j} \\
&\in (ka_0 + b_0) v_k + \sum_{l>k} \mathbb{C} v_l,
\end{aligned}
$$

we see that

$$\frac{\sum_{l \geq k} \mathbb{C} v_l}{\sum_{l \geq k+1} \mathbb{C} v_l}$$

is the quasi-weight space of quasi-weight λ_k, where

$$\lambda_k(L_0^p) = ka_0 + b_0$$

and

$$\lambda_k(¢) = 0.$$

Hence $V_{q,\alpha}$ is a bounded quasi-admissible $Vir(p)$-module.

Definition. $V_{q,\alpha} = \sum_{k \in \mathbb{Z}} \mathbb{C} v_k$ is called the polynomial module of $Vir(p)$ associated with the polynomial $q(t)$ and α.

Remark. Since $\not c$ always acts as 0 on a polynomial module, we will ignore $\not c$ when we consider the polynomial modules.

Proposition 2. If $q_1 = q_1(t) \in \mathbb{C}[t]$, $q_2 = q_2(t) = q_1(t) + mp(t)$, and $\alpha \in \mathbb{C}$, then $V_{q_1,\alpha} \cong V_{q_2,\alpha}$ as $Vir(p)$-modules.

Proof. Assume that

$$V_{q_1,\alpha} = \sum_{k\in\mathbb{Z}} \mathbb{C}v_k, \quad V_{q_2,\alpha} = \sum_{k\in\mathbb{Z}} \mathbb{C}v'_k,$$

and

$$q_1(t) = \sum_{j=0}^{s} b_j t^j.$$

Then

$$q_2 = q_2(t) = \sum_{j=0}^{s} b_j t^j + m\sum_{i=0}^{r} a_i t^i.$$

Define

$$\phi : \quad V_{q_1,\alpha} \longrightarrow V_{q_2,\alpha}$$

to be linear and

$$\phi(v_k) = v'_{k-m}.$$

Then

$$\begin{aligned}
\phi\left(L^p_m v_k\right) &= \phi\left((k+\alpha m)\sum_i a_i v_{m+k+i} + \sum_j b_j v_{m+k+j}\right) \\
&= (k+\alpha m)\sum_i a_i v'_{k+i} + \sum_j b_j v'_{k+j} \\
&= (k-m+\alpha m)\sum_i a_i v'_{k+i} + m\sum_i a_i v'_{k+i} + \sum_j b_j v'_{k+j} \\
&= L^p_m \cdot v'_{k-m} \\
&= L^p_m \phi(v_k).
\end{aligned}$$

Clearly, ϕ is 1-1 and onto. So ϕ is a module isomorphism. □

Proposition 3. Let

$$p = p(t), q = q(t), r(t) \in \mathbb{C}[t],$$

with $p(0) \neq 0, r(0) \neq 0$, and

$$p(t) \mid r(t).$$

If $V = V_{q,\alpha}$ is a polynomial module of $Vir(p(t))$, then V is a polynomial module of $Vir(r(t))$ as well.

Proof. We need only prove the case $r(t) = p(t)(1+at)$. Assume that

$$q(t) = \sum_{j=0}^{s} b_j t^j, \quad p(t) = \sum_{i=0}^{r} a_i t^i.$$

Then

$$\begin{aligned}
L_m^{r(t)} v_k &= (L_m^p + aL_{m+1}^p) v_k \\
&= (k+\alpha m) \sum_i a_i v_{m+k+i} + \sum_j b_j v_{m+k+j} \\
&\quad + a(k+\alpha m+\alpha) \sum_i a_i v_{m+k+1+i} + a \sum_j b_j v_{m+k+j+1} \\
&= (k+\alpha m) \left(a_0 v_{m+k} + \sum_{i=1}^{r} (a_i + a a_{i-1}) v_{m+k+i} + a a_r v_{m+k+r+1} \right) \\
&\quad + a\alpha \sum_{i=0}^{r} a_i v_{m+k+1+i} + \sum_{j=0}^{s} b_j v_{m+k+j} + a \sum_{j+0}^{s} b_j v_{m+k+j+1} \\
&= (k+\alpha m) \sum_{i=0}^{r+1} c_i v_{m+k+i} \\
&\quad + a\alpha \sum_{i=0}^{r} a_i v_{m+k+1+i} + \sum_{j=0}^{s} b_j v_{m+k+j} + a \sum_{j+0}^{s} b_j v_{m+k+j+1},
\end{aligned}$$

where

$$\sum_{j=0}^{r+1} c_j t^j = (1+at)p(t) = r(t).$$

So V ia a polynomial module of $Vir(r(t))$. □

Remark. In general, if $p(t) \mid r(t)$, a polynomial module of $Vir(r(t))$ may not be obtained from a polynomial module of $Vir(p(t))$ by restriction.

Proposition 4. Let $p(t) = \sum_{i=0}^{r} a_i t^i$ be such that $t^r \overline{p}(\frac{1}{t}) = bp(t)$ for some $b \in S^1$. Let $\alpha \in \mathbb{C}$ and $q(t) = \sum_{i=0}^{r} b_j t^j$ satisfy

$$\alpha + \overline{\alpha} = 1, \quad b_j = a_j(j - \overline{\alpha} r) + \overline{b b_{r-j}}, \quad j = 0, 1, \ldots, r.$$

If we define a hermitian form $\langle .,. \rangle$ on $V_{q(t),\alpha}$ such that $\langle v_k, v_l \rangle = \delta_{kl}$, then $V_{q,\alpha}$ is a unitary module of $Vir(p)$.

Proof. By Section 2.2, Proposition 5, $\omega(L^p_m) = \bar{b}L^p_{-m-r}$ is an anti-involution of $Vir(p)$. To prove the above proposition, we need only show

$$\langle L^p_m v_k, v_l\rangle = \langle v_k, \omega(L^p_m)v_l\rangle \qquad \forall k, l \in \mathbb{Z}.$$

Since

$$\langle L^p v_k, v_l\rangle = \sum_{i=0}^{r}(a_i(k+\alpha m)+b_i)\delta_{m+k+i,l},$$

on the other hand

$$\begin{aligned}\langle v_k, \omega(L^p_m)v_l\rangle &= \langle v_k, \bar{b}L^p_{-m-r}v_l\rangle \\ &= \bar{b}\sum_{i=0}^{r}\bar{a}_i(l-\bar{\alpha}(m+r))\delta_{k,l-m-r+i} + \bar{b}\sum_{i=0}^{r}\bar{b}_i\delta_{k,l-m-r+i} \\ &= \sum_{i=0}^{r} a_{r-i}(l-\bar{\alpha}(m+r))\delta_{k,l-m-r+i} + \bar{b}\sum_{i=0}^{r}\bar{b}_i\delta_{k,l-m-r+i} \\ &= \sum_{i=0}^{r} a_i(l-\bar{\alpha}(m+r))\delta_{k,l-m-i} + \bar{b}\sum_{i=0}^{r}\bar{b}_{r-i}\delta_{k,l-m-i} \\ &= \sum_{i=0}^{r} a_i(k+(1-\bar{\alpha})m)\delta_{k+m+i,l} + \sum_{i=0}^{r}\left(a_i(i-\bar{\alpha}r)+\bar{b}\bar{b}_{r-i}\right)\delta_{k+m+i,l} \\ &= \sum_{i=0}^{r}(a_i(k+\alpha m)+b_i)\delta_{m+k+i,l}, \qquad (by \quad conditions)\end{aligned}$$

we get $\langle L^p_m v_k, v_l\rangle = \langle v_k, \omega(L^p_m)v_l\rangle$. □

2.6 Quasi-admissible Modules of the Virasoro Algebra

Irreducible admissible modules of the Virasoro algebra have been classified by O.Mathieu [M], C.Martin and A.Piard [MP]. In this section we discuss a class of irreducible quasi-admissible modules of the Virasoro algebra.

Proposition 1. Let $a \neq 0$ and let

$$V_{at,0} = \sum_{k\in\mathbb{Z}}\mathbb{C}v_k$$

be the polynomial Vir-module associated with at and 0. Then $V_{at,0}$ is an irreducible Vir-module.

Proof. Let $0 \neq U \subseteq V_{at,0}$ be a Vir-submodule, and let

$$0 \neq x = \sum_{j=r}^{s} c_j v_j \in U.$$

If $r \neq 0$, then

$$L_{-r}x = rc_r v_0 + v' \in U$$

for some $v' \in \sum_{i>0} \mathbb{C}v_i$. So without loss of generality, we may assume that

$$x = v_0 + c_1 v_1 + ... + c_n v_n \in U,$$

and x is chosen with minimal n. Since $L_m v_k = kv_{m+k} + av_{m+k+1}$, we have

(1)

$$\begin{aligned} L_0 x &= \sum_i c_i L_0 v_i \\ &= (a+c_1)v_1 + (ac_1 + 2c_2)v_2 + ... + (ac_{n-1} + nc_n)v_n + ac_n v_{n+1} \in U, \end{aligned}$$

(2)
$$\begin{aligned} L_0 L_0 x = (a+c_1)v_1 &+ \big(a(a+c_1) + 2(ac_1 + 2c_2)\big)v_2 + ... \\ &+ \big(a(ac_{n-2} + (n-1)c_{n-1}) + n(ac_{n-1} + nc_n)\big)v_n \\ &+ \big(a(ac_{n-1} + nc_n) + (n+1)ac_n\big)v_{n+1} + a^2 c_n v_{n+2} \in U, \end{aligned}$$

and

(3)
$$\begin{aligned} L_1 x = (a+c_1)v_2 &+ (ac_1 + 2c_2)v_3 + ... \\ &+ (ac_{n-2} + (n-1)c_{n-1})v_n + (ac_{n-1} + nc_n)v_{n+1} + ac_n v_{n+2} \in U. \end{aligned}$$

From (2) and (3) we get

(4)
$$\begin{aligned} L_0 L_0 x - aL_1 x = (a+c_1)v_1 &+ 2(ac_1 + 2c_2)v_2 + ... \\ &+ n(ac_{n-1} + nc_n)v_n + a(n+1)c_n v_{n+1} \in U. \end{aligned}$$

Suppose $n > 0$.
In the case of $a + c_1 \neq 0$, by (1) and (4), we obtain

(5)
$$\begin{aligned} 0 \neq & L_0 L_0 x - aL_1 x - (n+1)L_0 x \\ &= -n(a+c_1)v_1 + ... + (-2)(ac_{n-1} + nc_n)v_n \in U. \end{aligned}$$

Note that by the minimality of n,

$$ac_{n-1} + nc_n \neq 0.$$

In the case of $a + c_1 = 0$, again by (1) and (4), we obtain

$$\begin{aligned} 0 \neq & L_0L_0x - aL_1x - 2L_0x \\ = & (ac_2 + 3c_3)v_3 + \ldots + a(n-1)c_nv_{n+1} \in U. \end{aligned}$$

Applying L_{-2}, we get

$$\begin{aligned} 0 \neq & L_{-2}(L_0L_0x - aL_1x - 2L_0x) \\ = & 3(ac_2 + 3c_3)v_1 + \ldots + a(n-1)c_nv_n \in U, \end{aligned} \tag{6}$$

where

$$a(n-1)c_n \neq 0.$$

Now we see that in any case, there exists some

$$0 \neq x_1 = b_1v_1 + .. + b_{n-1}v_{n-1} + b_nv_n \in U,$$

with $b_n \neq 0$. Assume that $b_n = 1$. Then

$$x - c_nx_1 = v_0 + d_1v_1 + \ldots + d_{n-1}v_{n-1} \in U.$$

This is a contradiction. So $n = 0$ and $v_0 \in U$. Then from

$$L_mv_0 = av_{m+1} \in U \qquad \forall m \in \mathbb{Z},$$

we see that all $v_m \in U$. So $U = V_{at,0}$. □

Proposition 2. $V_{at,0} \cong V_{a't,0}$ only if $a = a'$.

Proof. Let

$$\phi : V_{at,0} \longrightarrow V_{a't,0}$$

be a Vir-module isomorphism and

$$V_{at,0} = \sum_{k \in \mathbb{Z}} \mathbb{C}v_k, \quad V_{a't,0} = \sum_{k \in \mathbb{Z}} \mathbb{C}v'_k.$$

Assume that

$$\phi(v_0) = \sum_{j=k}^{n} c_jv'_j,$$

where c_k, c_n are non-zero.

Since $L_{-1}v_0 = av_0$, we have

$$\phi(L_{-1}v_0) = a\phi(v_0) = \sum_{j=k}^{n} ac_j v'_j.$$

On the other hand,

$$L_{-1}\phi(v_0) = L_{-1}\sum_{j=k}^{n} c_j v'_j = \sum_{j=k}^{n} c_j(jv'_{j-1} + a'v'_j).$$

These imply that $a'c_n v'_n = ac_n v'_n$. But $c_n \neq 0$, So $a = a'$. □

2.7 Admissible Modules of $Vir(1-t)$

Let $p(t) = \sum_{i=0}^{r} a_i t^i$. From now on we will concern ourselves mainly with the polynomial modules $V_{q(t),\alpha}$, where $q(t) = \sum_{j=0}^{r} a_j \beta_j t^j$.

In this section, we consider the case $p = p(t) = 1 + at$. By Section 2.2, Proposition 4, it suffices to consider the case $p = p(t) = 1 - t$.

Let $q(t) = \beta_0 - \beta_1 t, \alpha \in \mathbb{C}$. For any $r \in \mathbb{Z}$, we have the polynomial $Vir(1-t)$-module associated with $\beta_0 - (\beta_1 + r)t$ and α which is given by

$$V_{q,\alpha}^{(r)} := V_{\beta_0-(r+\beta_1)t,\alpha} = \sum_{k\in\mathbb{Z}} \mathbb{C}v_k^{(r)}$$

with

$$L_m^p v_k^{(r)} = (k + \alpha m + \beta_0)v_{m+k}^{(r)} - (k + \alpha m + r + \beta_1)v_{m+k+1}^{(r)}.$$

Proposition 1. Let $u_k^{(r+1)} := v_k^{(r)} - v_{k+1}^{(r)}$. Then $U := \sum_{k\in\mathbb{Z}} \mathbb{C}u_k^{(r+1)} \subset V_{q,\alpha}^{(r)}$ is a $Vir(1-t)$-submodule and $U \cong V_{q,\alpha}^{(r+1)}$.

Proof. Since

$$\begin{aligned}
L_m^p v_k^{(r+1)} &= L_m^p v_k^{(r)} - L_m^p v_{k+1}^{(r)} \\
&= (k + \alpha m + \beta_0)v_{k+m}^{(r)} - (k + \alpha m + r + \beta_1)v_{k+m+1}^{(r)} \\
&\quad - (k + 1 + \alpha m + \beta_0)v_{k+m+1}^{(r)} + (k + 1 + \alpha m + r + \beta_1)v_{k+m+2}^{(r)} \\
&= (k + \alpha m + \beta_0)\left(v_{k+m}^{(r)} - v_{k+m+1}^{(r)}\right) \\
&\quad - (k + 1 + \alpha m + r + \beta_1)\left(v_{m+k+1}^{(r)} - v_{k+m+2}^{(r)}\right) \\
&= (k + \alpha m + \beta_0)v_{k+m}^{(r+1)} - (k + \alpha m + r + 1 + \beta_1)v_{m+k+1}^{(r+1)}.
\end{aligned}$$

The proposition follows. □

Remark. By Proposition 1, If we identify $v_k^{(r+1)}$ with $v_k^{(r)} - v_{k+1}^{(r)}$, then we identify $V_{q,\alpha}^{(r+1)}$ as a submodule of $V_{q,\alpha}^{(r)}$. Under this identification, we have a sequence of $Vir(1-t)$-modules

$$... > V_{q,\alpha}^{(-1)} > V_{q,\alpha}^{(0)} > V_{q,\alpha}^{(1)} > V_{q,\alpha}^{(2)} > ... \quad .$$

Definition. We define the following $Vir(1-t)$-modules.

$$V_{q,\alpha}(-\infty,+\infty) := \bigcup_{r\in\mathbb{Z}} V_{q,\alpha}^{(r)},$$
$$V_{q,\alpha}(l,+\infty) := \bigcup_{r\geq l} V_{q,\alpha}^{(r)} = V_{q,\alpha}^{(l)},$$
$$V_{q,\alpha}(-\infty,n) := \frac{V_{q,\alpha}(-\infty,+\infty)}{V_{q,\alpha}(n,+\infty)},$$
$$V_{q,\alpha}(l,n) := \frac{V_{q,\alpha}(l,\infty)}{V_{q,\alpha}(n,+\infty)}, \qquad (l < n).$$

All of them are bounded quasi-admissible $Vir(1-t)$-modules and each quasi-weight space is 1-dimensional. In particular, $V_{q,\alpha}(l,n)$ is finite dimensional. In the rest of this section, we discuss $V_{q,\alpha}(l,n)$ and $V_{q,\alpha}(-\infty,n)$ in detail. We'll see that $V_{q,\alpha}(l,n)$ and $V_{q,\alpha}(-\infty,n)$ are admissible $V_{q,\alpha}(1-t)$-modules.

1. Finite dimensional modules $V_{q,\alpha}(l,n)$, where $l < n$.

Since

$$V_{q,\alpha}^{(l)} = \mathbb{C}v_0^{(l)} + V_{q,\alpha}^{(l+1)}$$
$$= \mathbb{C}v_0^{(l)} + \mathbb{C}v_0^{(l+1)} + ... + \mathbb{C}v_0^{(n-1)} + V_{q,\alpha}^{(n)},$$

we have

$$V_{q,\alpha}(l,n) = \mathbb{C}\overline{v}_0{}^{(l)} + ... + \mathbb{C}\overline{v}_0{}^{(n-1)},$$

where $\overline{v}_0{}^{(i)} = v_0^{(i)} + V_{q,\alpha}^{(n)}$. Moreover, $\overline{v}_0{}^{(l)}, ..., \overline{v}_0{}^{(n-1)}$ are linear independent. Hence

$$dim V_{q,\alpha}(l,n) = n - l.$$

For further discussion, we need the following functions.

Definition.

$$\left\langle {\beta \atop k} \right\rangle := \begin{cases} 0 & k < 0 \\ 1 & k = 0 \\ \frac{\beta(\beta+1)...(\beta+k-1)}{k!} & k > 0, \end{cases}$$

where $\beta \in \mathbb{C}, k \in \mathbb{Z}$.

Remark. If $\beta \in \mathbb{Z}_{\geq 0}$, then $\left\langle {\beta \atop k} \right\rangle = \binom{\beta+k-1}{k}$.

Lemma 1. For all $\beta \in \mathbb{C}$ and $k \in \mathbb{Z}$,

$$\left\langle {\beta \atop k} \right\rangle - \left\langle {\beta \atop k-1} \right\rangle = \left\langle {\beta-1 \atop k} \right\rangle.$$

Proof. This is clear for $k \leq 1$. Assume that $k > 1$. Then

$$\begin{aligned}
\left\langle {\beta \atop k} \right\rangle - \left\langle {\beta \atop k-1} \right\rangle &= \frac{\beta(\beta+1)...(\beta+k-1)}{k!} - \frac{\beta(\beta+1)...(\beta+k-2)}{(k-1)!} \\
&= \left\langle {\beta \atop k-1} \right\rangle \left(\frac{\beta+k-1}{k} - 1 \right) \\
&= \left\langle {\beta \atop k-1} \right\rangle \frac{\beta-1}{k} \\
&= \left\langle {\beta-1 \atop k} \right\rangle.
\end{aligned}$$

□

Lemma 2. For any $m \in \mathbb{Z}$, $j = 1, 2, ..., n-l$,

$$\overline{v}_m{}^{(n-j)} = \sum_{i=0}^{j-1} \left\langle {-m \atop i} \right\rangle \overline{v}_0{}^{(n-j+i)},$$

where $\overline{v}_m{}^{(k)} = v_m^{(k)} + V_{q,\alpha}^{(n)}$, $k = l, ..., n-1$.

Proof. First suppose that $m \geq 0$ and the lemma is true for m. Since

$$\overline{v}_m{}^{(n-j)} - \overline{v}_{m+1}{}^{(n-j)} = \overline{v}_m{}^{(n-j+1)},$$

we have

$$\begin{aligned}
\overline{v}_{m+1}{}^{(n-j)} &= \overline{v}_m{}^{(n-j)} - \overline{v}_m{}^{(n-j+1)} \\
&= \sum_{i=0}^{j-1} \left\langle {-m \atop i} \right\rangle \overline{v}_0{}^{(n-j+i)} - \sum_{i=0}^{j-2} \left\langle {-m \atop i} \right\rangle \overline{v}_0{}^{(n-j+i+1)} \\
&= \overline{v}_0{}^{(n-j)} + \sum_{i=1}^{j-1} \left(\left\langle {-m \atop i} \right\rangle - \left\langle {-m \atop i-1} \right\rangle \right) \overline{v}_0{}^{(n-j+i)} \\
&= \overline{v}_0{}^{(n-j)} + \sum_{i=1}^{j-1} \left\langle {-m-1 \atop i} \right\rangle \overline{v}_0{}^{(n-j+i)} \\
&= \sum_{i=0}^{j-1} \left\langle {-m-1 \atop i} \right\rangle \overline{v}_0{}^{(n-j+i)}.
\end{aligned}$$

By induction on m we see that the lemma is true for all $m \geq 0$.

Next we have

$$\overline{v}_{-m}{}^{(n-1)} = \overline{v}_0{}^{(n-1)}$$

for all $m \geq 0$. Assume that

$$\overline{v}_{-m}{}^{(n-j)} = \sum_{i=0}^{j-1} \left\langle {m \atop i} \right\rangle \overline{v}_0{}^{(n-j+i)}$$

for all $m \geq 0$. Since

$$\overline{v}_{-1}{}^{(n-j-1)} - \overline{v}_0{}^{(n-j-1)} = \overline{v}_{-1}{}^{(n-j)},$$

$$\begin{aligned}\overline{v}_{-1}{}^{(n-j-1)} &= \overline{v}_0{}^{(n-j-1)} + \overline{v}_{-1}{}^{(n-j)} \\ &= \overline{v}_0{}^{(n-j-1)} + \sum_{i=0}^{j-1} \left\langle {1 \atop i} \right\rangle \overline{v}_0{}^{(n-j+i)} \\ &= \sum_{i=0}^{j} \left\langle {1 \atop i} \right\rangle \overline{v}_0{}^{(n-j-1+i)}.\end{aligned}$$

Assume that

$$\overline{v}_{-m}{}^{(n-j-1)} = \sum_{i=0}^{j} \left\langle {m \atop i} \right\rangle \overline{v}_0{}^{(n-j-1+i)}.$$

Then

$$\begin{aligned}\overline{v}_{-m-1}{}^{(n-j-1)} &= \overline{v}_{-m}{}^{(n-j-1)} + \overline{v}_{-m-1}{}^{(n-j)} \\ &= \sum_{i=0}^{j} \left\langle {m \atop i} \right\rangle \overline{v}_0{}^{(n-j-1+i)} + \sum_{i=0}^{j-1} \left\langle {m+1 \atop i} \right\rangle \overline{v}_0{}^{(n-j+i)} \\ &= \overline{v}_0{}^{(n-j-1)} + \sum_{i=1}^{j} \left(\left\langle {m \atop i} \right\rangle + \left\langle {m+1 \atop i-1} \right\rangle \right) \overline{v}_0{}^{(n-j+i-1)} \\ &= \sum_{i=0}^{j} \left\langle {m+1 \atop i} \right\rangle \overline{v}_0{}^{(n-j+i-1)}.\end{aligned}$$

By induction on m we get

$$\overline{v}_{-m}{}^{(n-j-1)} = \sum_{i=0}^{j-1} \left\langle {m \atop i} \right\rangle \overline{v}_0{}^{(n-j+i-1)},$$

for all $m \in \mathbb{Z}_{>0}$. Then by induction on j, the lemma is true for all $m < 0$. We have completed the proof. □

Proposition 2. Let

$$\phi_{q,\alpha}(l,n) : Vir(1-t) \longrightarrow \mathfrak{gl}\,(V_{q,\alpha}(l,n))$$

be the representation given by

$$\phi_{q,\alpha}(l,n)\,(L_m^p)\,\overline{v}_k{}^{(n-j)} = L_m^p \overline{v}_k{}^{(n-j)}, \qquad j = 1, 2, ..., n-l.$$

Then

$$\frac{Vir(1-t)}{ker\phi_{q,\alpha}(l,n)}$$

is a $n-l$ dimensional solvable Lie algebra. Moreover, if we take $\left\{\overline{v}_0{}^{(l)}, ..., \overline{v}_0{}^{(n-1)}\right\}$ as a basis of $V_{q,\alpha}(l,n)$, then

$$\begin{aligned}
&\phi_{q,\alpha}(l,n)(L_m^p)\overline{v}_0{}^{(n-j)} \\
&= \sum_{i=0}^{j-1} \left(-\langle{}_i^{-m}\rangle(n-j+\beta_1-\beta_0) + \langle{}_{i-1}^{-m}\rangle(\alpha m + n - j + \beta_1)\right) \overline{v}_0{}^{(n-j+i)}.
\end{aligned}$$

Proof.

$$\begin{aligned}
&L_m^p \cdot \overline{v}_0{}^{(n-j)} \\
&= (\alpha m + \beta_0)\overline{v}_m{}^{(n-j)} - (\alpha m + (n-j) + \beta_1)\overline{v}_{m+1}{}^{(n-j)} \\
&= -(n-j+\beta_1-\beta_0)\overline{v}_m{}^{(n-j)} \\
&\quad + (\alpha m + (n-j) + \beta_1)\left(\overline{v}_m{}^{(n-j)} - \overline{v}_{m+1}{}^{(n-j)}\right) \\
&= -(n-j+\beta_1-\beta_0)\overline{v}_m{}^{(n-j)} + (\alpha m + (n-j) + \beta_1)\overline{v}_m{}^{(n-j+1)} \\
&= -(n-j+\beta_1-\beta_0)\sum_{i=0}^{j-1}\langle{}_i^{-m}\rangle\overline{v}_0{}^{(n-j+i)} \\
&\quad + (\alpha m + (n-j) + \beta_1)\sum_{i=0}^{j-2}\langle{}_i^{-m}\rangle\overline{v}_0{}^{(n-j+i+1)} \\
&= -(n-j+\beta_1-\beta_0)\overline{v}_0{}^{(n-j)} \\
&\quad + \sum_{i=1}^{j-1}\left(-\langle{}_i^{-m}\rangle(n-j+\beta_1-\beta_0) + \langle{}_{i-1}^{-m}\rangle(\alpha m + n - j + \beta_1)\right)\overline{v}_0{}^{(n-j+i)}.
\end{aligned}$$

□

Lemma 3. For any $\beta \in \mathbb{C}$, $\quad k \in \mathbb{Z}_{\geq 0}$,

$$\sum_{i=0}^{k} (-1)^{k-i} \left\langle {\beta \atop i} \right\rangle \left\langle {\beta+i \atop k-i} \right\rangle = 0.$$

Hence

$$\begin{pmatrix} 1 & \left\langle {l+\beta_1 \atop 1} \right\rangle & \left\langle {l+\beta_1 \atop 2} \right\rangle & \dots & \left\langle {l+\beta_1 \atop n-l-1} \right\rangle \\ 0 & 1 & \left\langle {l+1+\beta_1 \atop 1} \right\rangle & \dots & \left\langle {l+\beta_1+1 \atop n-l-2} \right\rangle \\ \vdots & \dots & \dots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \left\langle {n-1+\beta_1 \atop 1} \right\rangle \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} 1 & -\left\langle {l+\beta_1 \atop 1} \right\rangle & \left\langle {l+\beta_1 \atop 2} \right\rangle & \dots & (-1)^{n-l-1} \left\langle {l+\beta_1 \atop n-l-1} \right\rangle \\ 0 & 1 & -\left\langle {l+1+\beta_1 \atop 1} \right\rangle & \dots & \dots \\ \vdots & \dots & \dots & \ddots & \vdots \\ 0 & 0 & \dots & \dots & -\left\langle {n-1+\beta_1 \atop 1} \right\rangle \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}.$$

Proof.

$$\begin{aligned} &\sum_{i=0}^{k} (-1)^{k-i} \left\langle {\beta \atop i} \right\rangle \left\langle {\beta+i \atop k-i} \right\rangle \\ &= \sum_{i=0}^{k} (-1)^{k-i} \frac{\beta(\beta+1)\dots(\beta+i-1)\cdot(\beta+i)\dots(\beta+k-1)}{i!\cdot(k-i)!} \\ &= \frac{\beta(\beta+1)\dots(\beta+k-1)}{k!} \sum_{i=0}^{k} (-1)^{k-i} \frac{k!}{i!(k-i)!} \\ &= \left\langle {\beta \atop k} \right\rangle \sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} \\ &= \left\langle {\beta \atop k} \right\rangle (1-1)^k \\ &= 0. \end{aligned}$$

$\square$

Lemma 4. For any $a \in \mathbb{C}$, $r, s \in \mathbb{Z}_{\geq 0}$,

$$(*) \qquad \sum_{i+j=r} (-1)^j \left\langle {a \atop i} \right\rangle \left\langle {s+a-j \atop j} \right\rangle = (-1)^r \left\langle {s-r \atop r} \right\rangle .$$

Proof. Clearly, the lemma is true if $a = 0$. Assume that the lemma is true for $a = -k$, $k \in \mathbb{Z}_{\geq 0}$. Then

$$\begin{aligned}
&\sum_{i+j=r} (-1)^j \binom{-k-1}{i} \left\langle {s-k-1-j \atop j} \right\rangle \\
&= \sum_{i+j=r} (-1)^{j+i} \binom{k+1}{i} \left\langle {s-k-1-j \atop j} \right\rangle \\
&= (-1)^r \sum_{i+j=r} \binom{k+1}{i} \left\langle {s-k-1-j \atop j} \right\rangle \\
&= (-1)^r \sum_{i+j=r} \binom{k}{i} \left(\left\langle {s-k-1-j \atop j} \right\rangle + \left\langle {s-k-j \atop j-1} \right\rangle \right) \\
&= (-1)^r \sum_{i+j=r} \binom{k}{i} \left\langle {s-k-j \atop j} \right\rangle \qquad (by \quad Lemma \quad 1) \\
&= \sum_{i+j=r} (-1)^j \left\langle {k \atop i} \right\rangle \left\langle {s-k-j \atop j} \right\rangle \\
&= (-1)^r \left\langle {s-r \atop r} \right\rangle . \qquad (by \quad induction)
\end{aligned}$$

So we proved that the lemma is true for all $a = -k \in \mathbb{Z}_{\leq 0}$. Note that if we view a as a variable, then the left hand side of (*) is a polynomial of degree r. So it is a constant and must be $(-1)^r \left\langle {s-r \atop r} \right\rangle$. □

Proposition 3. $V_{q,\alpha}(l, n)$ is an admissible $Vir(1-t)$-module and each weight space is 1-dimensional. Precisely, let

$$\begin{pmatrix} u_l \\ u_{l+1} \\ \vdots \\ u_{n-1} \\ u_{n-1} \end{pmatrix} = \begin{pmatrix} 1 & \left\langle {l+\beta_1 \atop 1} \right\rangle & \cdots & \cdots & \left\langle {l+\beta_1 \atop n+l-1} \right\rangle \\ 0 & 1 & \left\langle {l+1+\beta_1 \atop 1} \right\rangle & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \left\langle {n-2+\beta_1 \atop 1} \right\rangle \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \begin{pmatrix} \overline{v}_0{}^{(l)} \\ \overline{v}_0{}^{(l+1)} \\ \vdots \\ \overline{v}_0{}^{(n-2)} \\ \overline{v}_0{}^{(n-1)} \end{pmatrix} .$$

Then

$$V_{q,\alpha}(l, n) = \sum_{i=l}^{n-1} \mathbb{C} u_i .$$

Moreover,

(**)

$$
\begin{aligned}
&L^p_m u_k \\
&= \sum_{s=0}^{n-k-1} \left(\sum_{r=0}^{s} (-1)^{s-r-1} \left\langle {-m \atop r} \right\rangle \left((k+\beta_1-\beta_0) \left\langle {r \atop s-r} \right\rangle + \alpha m \left\langle {r+1 \atop s-r-1} \right\rangle \right) \right) u_{k+s} \\
&= -(k+\beta_1-\beta_0) \left\langle {-m \atop 0} \right\rangle u_k \\
&\quad + \left(-(k+\beta_1-\beta_0) \left\langle {-m \atop 1} \right\rangle + \alpha m \left\langle {-m \atop 0} \right\rangle \right) u_{k+1} \\
&\quad + \left(-(k+\beta_1-\beta_0) \left(\left\langle {-m \atop 2} \right\rangle - \left\langle {-m \atop 1} \right\rangle \right) + \alpha m \left(\left\langle {-m \atop 1} \right\rangle - \left\langle {-m \atop 0} \right\rangle \right) \right) u_{k+2} \\
&\quad + \Bigg(-(k+\beta_1-\beta_0) \left(\left\langle {-m \atop 3} \right\rangle - 2 \left\langle {-m \atop 2} \right\rangle + \left\langle {-m \atop 1} \right\rangle \right) \\
&\qquad\quad + \alpha m \left(\left\langle {-m \atop 2} \right\rangle - 2 \left\langle {-m \atop 1} \right\rangle + \left\langle {-m \atop 0} \right\rangle \right) \Bigg) u_{k+3} \\
&\quad + \ldots
\end{aligned}
$$

In particular,

$$L^p_0 u_k = -(k+\beta_1-\beta_0) u_k, \quad k = l, \ldots, n-1.$$

i.e. the u_k are weight vectors.

Proof. By definition

$$u_k = \sum_{i=0}^{n-k-1} \left\langle {k+\beta_1 \atop i} \right\rangle \overline{v}_0{}^{(k+i)},$$

where $k = l, \ldots, n-1$. By Lemma 3,

$$\overline{v}_0{}^{k} = \sum_{j=0}^{n-k-1} (-1)^j \left\langle {k+\beta_1 \atop j} \right\rangle u_{k+j},$$

where $k = l, ..., n-1$. Then by Proposition 2,

$$
\begin{aligned}
L_m u_k &= \sum_{i=0}^{n-k-1} \left\langle {k+\beta_1 \atop i} \right\rangle L_m^p \overline{v}_0{}^{(k+i)} \\
&= \sum_{i=0}^{n-k-1} \left\langle {k+\beta_1 \atop i} \right\rangle \sum_{r=0}^{n-k-1-i} \Big(- \left\langle {-m \atop r} \right\rangle (k+i+\beta_1-\beta_0) \\
&\qquad\qquad + \left\langle {-m \atop r-1} \right\rangle (\alpha m + k + i + \beta_1) \Big) \overline{v}_0{}^{(k+i+r)} \\
&= \sum_{i=0}^{n-k-1} \left\langle {k+\beta_1 \atop i} \right\rangle \sum_{r=0}^{n-k-1-i} \Big(- \left\langle {-m \atop r} \right\rangle (k+i+\beta_1-\beta_0) \\
&\qquad + \left\langle {-m \atop r-1} \right\rangle (\alpha m + k + i + \beta_1) \Big) \sum_{j=0}^{n-k-i-r-1} (-1)^j \left\langle {k+i+r+\beta_1 \atop j} \right\rangle u_{k+i+r+j} \\
&= \sum_{s=0}^{n-k-1} \Big(\sum_{r+i+j=s} (-1)^j \left\langle {k+\beta_1 \atop i} \right\rangle \left\langle {k+\beta_1+s-j \atop j} \right\rangle \\
&\qquad \Big(- \left\langle {-m \atop r} \right\rangle (k+i+\beta_1-\beta_0) + \left\langle {-m \atop r-1} \right\rangle (\alpha m + k + i + \beta_1) \Big) \Big) u_{k+s}.
\end{aligned}
$$

Let us denote the coefficient of u_{k+s} by a_{k+s}. Then

$$
\begin{aligned}
a_{k+s} &= \sum_{r=0}^{s} \left(\sum_{i+j=s-r} (-1)^{j+1} \left\langle {k+\beta_1 \atop i} \right\rangle \left\langle {k+\beta_1+s-j \atop j} \right\rangle \left\langle {-m \atop r} \right\rangle (k+\beta_1-\beta_0+i) \right) \\
&\quad + \sum_{r=1}^{s} \left(\sum_{i+j=s-r} (-1)^{j} \left\langle {k+\beta_1 \atop i} \right\rangle \left\langle {k+\beta_1+s-j \atop j} \right\rangle \left\langle {-m \atop r-1} \right\rangle (\alpha m + k + \beta_1 + i) \right) \\
&= \sum_{r=0}^{s} \left((k+\beta_1-\beta_0) \left\langle {-m \atop r} \right\rangle \sum_{i+j=s-r} (-1)^{j+1} \left\langle {k+\beta_1 \atop i} \right\rangle \left\langle {k+\beta_1+s-j \atop j} \right\rangle \right) \\
&\quad + \sum_{r=1}^{s} \left(\alpha m \left\langle {-m \atop r-1} \right\rangle \sum_{i+j=s-r} (-1)^{j} \left\langle {k+\beta_1 \atop i} \right\rangle \left\langle {k+\beta_1+s-j \atop j} \right\rangle \right) \\
&\quad + \sum_{r=0}^{s} \left(\sum_{i+j=s-r} (-1)^{j+1} \left\langle {k+\beta_1 \atop i} \right\rangle \left\langle {k+\beta_1+s-j \atop j} \right\rangle \left\langle {-m \atop r} \right\rangle i \right) \\
&\quad + \sum_{r=1}^{s} \left(\sum_{i+j=s-r} (-1)^{j} \left\langle {k+\beta_1 \atop i} \right\rangle \left\langle {k+\beta_1+s-j \atop j} \right\rangle \left\langle {-m \atop r-1} \right\rangle (k+\beta_1+i) \right).
\end{aligned}
$$

Note that, replacing r by $r+1$, the last summand can be written as

$$\sum_{r=0}^{s-1}\left(\sum_{i+j=s-r-1}(-1)^j\langle{}^{k+\beta_1}_{i+1}\rangle\langle{}^{k+\beta_1+s-j}_{j}\rangle\langle{}^{-m}_{r}\rangle(i+1)\right).$$

Replacing $i+1$ by i, we see that the last two summand cancel. The first two summand are simplified using Lemma 4. Then we have

$$\begin{aligned} a_{k+s} &= (k+\beta_1-\beta_0)\sum_{r=0}^{s}(-1)^{s-r-1}\langle{}^{r}_{s-r}\rangle\langle{}^{-m}_{r}\rangle+\alpha m\sum_{r=1}^{s}(-1)^{s-r}\langle{}^{r}_{s-r}\rangle\langle{}^{-m}_{r-1}\rangle \\ &= \sum_{s=0}^{n-k-1}\left(\sum_{r=0}^{s}(-1)^{s-r-1}\langle{}^{-m}_{r}\rangle\left((k+\beta_1-\beta_0)\langle{}^{r}_{s-r}\rangle+\alpha m\langle{}^{r+1}_{s-r-1}\rangle\right)\right)u_{k+s}. \end{aligned}$$

□

Proposition 4. Suppose $n-l\geq 2$. $V_{q,\alpha}(l,n)$ is decomposable iff

(a) $n-l=2$ and $n-2+\beta_1-\beta_0+\alpha=0$. In this case,

$$V_{q,\alpha}(l,n)=\mathbb{C}u_l\oplus\mathbb{C}u_{n-1}.$$

(b) $n-l\geq 3$, $\alpha=1$ and $n-1+\beta_1-\beta_0=0$. In this case,

$$V_{q,\alpha}(l,n)=\sum_{i=l}^{n-2}\mathbb{C}u_i\oplus\mathbb{C}u_{n-1}.$$

(c) $n-l\geq 3$, $\alpha=0$ and $l+\beta_1-\beta_0=0$. In this case,

$$V_{q,\alpha}(l,n)=\mathbb{C}u_l\oplus\sum_{j=l+1}^{n-1}\mathbb{C}u_j.$$

Proof. Suppose

$$V_{q,\alpha}(l,n)=U_1\oplus U_2,$$

where U_1,U_2 are two non-trivial submodules. We see from Proposition 3 that

$$u_k,\quad k=l,...,n-1$$

are weight vectors of L_0^p of different weights. So if $\sum_k a_k u_k\in U_i$, $i=1,2$, then $u_k\in U_i$ whenever $a_k\neq 0$. Assume that

$$u_l,...,u_{n-j_0}\in U_1,$$

and

$$u_{n-j_0+1} \in U_2,$$

where $j_0 \geq 2$.

With $k = n - j_0$ in Proposition 3 (**), we get

$$\begin{aligned} L_m^p u_{n-j_0} = &-(n - j_0 + \beta_1 - \beta_0)u_{n-j_0} \\ &+ m(n - j_0 + \beta_1 - \beta_0 + \alpha)u_{n-j_0+1} \\ &+ u' \in U_1 \end{aligned}$$

for all $m \in \mathbb{Z}$, where $u' \in \sum_{i \geq 2} \mathbb{C}u_{n-j_0+i}$. So we have

$$n - j_0 + \beta_1 - \beta_0 + \alpha = 0$$

for some $j_0 \in \{2, ..., n - j\}$. And then

$$n - j_0 + i + \beta_1 - \beta_0 + \alpha \neq 0$$

for all $i \geq 1$. Since $u_{n-j_0+1} \in U_2$, and again by Proposition 3 (**),

$$\begin{aligned} L_m^p u_{n-j_0+i} = &-(n - j_0 + i + \beta_1 - \beta_0)u_{n-j_0+i} \\ &+ m(n - j_0 + i + \beta_1 - \beta_0 + \alpha)u_{n-j_0+i+1} \\ &+ u \end{aligned}$$

for all $m \in \mathbb{Z}$, where $u \in \sum_{s \geq 2} \mathbb{C}u_{n-j_0+i+s}$. We see that

$$U_1 = \sum_{i=l}^{n-j_0} \mathbb{C}u_i,$$

and

$$U_2 = \sum_{i=n-j_0+1}^{n-1} \mathbb{C}u_i.$$

(a) If $n - l = 2$, then $j_0 = 2$ and $n - 2 + \beta_1 - \beta_0 + \alpha = 0$. By Proposition 3,

$$L_m^p u_l = \alpha u_l$$

and

$$L_m^p u_{n-1} = (1 - \alpha)u_{n-1}.$$

So

$$V_{q,\alpha}(l,n) = \mathbb{C}u_l \oplus \mathbb{C}u_{n-1}.$$

(b) If $n - l \geq 3$ and $l \leq n - j_0 - 1$, With $k = n - j_0 - 1$ in Proposition 3 (**) and using

$$n - j_0 + \beta_1 - \beta_0 + \alpha = 0,$$

we get

$$\begin{aligned} L_m^p u_{n-j_0-1} = &(\alpha+1)u_{n-j_0-1} \\ &+ m u_{n-j_0} \\ &+ \frac{(1-\alpha)}{2}\left(m^2+m\right)u_{n-j_0+1} \\ &+ (1-2\alpha)\left(\left\langle {-m \atop 3} \right\rangle - m^2\right)u_{n-j_0+2} \\ &+ u \in U_1 \end{aligned}$$

for all $m \in \mathbb{Z}$, where $u \in \sum_{i\geq 3} \mathbb{C}u_{n-j_0+i}$. From this we get $\alpha = 1$ and $j_0 = 2$. Hence

$$U_1 = \sum_{i=l}^{n-2} \mathbb{C}u_i,$$

and

$$U_2 = \mathbb{C}u_{n-1}.$$

For $k \geq 0$, using Proposition 3, we get

$$\begin{aligned} &L_m^p u_{n-2-k} \\ &= \sum_{s=0}^{k+1}\left(\sum_{r=0}^{s}(-1)^{s-r-1}\left\langle {-m \atop r} \right\rangle\left((-k-1)\left\langle {r \atop s-r} \right\rangle + m\left\langle {r+1 \atop s-r-1} \right\rangle\right)\right)u_{n-2-k+s}. \end{aligned}$$

The coefficient of u_{n-1} is

$$
\begin{aligned}
&\sum_{r=0}^{k+1}(-1)^{k-r}\left\langle {-m \atop r}\right\rangle\left(-(k+1)\left\langle {r \atop k+1-r}\right\rangle + m\left\langle {r+1 \atop k-r}\right\rangle\right)\\
&=\sum_{r=1}^{k+1}(-1)^{k-r+1}\left\langle {-m \atop r}\right\rangle(k+1)\left\langle {r \atop k+1-r}\right\rangle + \sum_{r=0}^{k}(-1)^{k-r}\left\langle {-m \atop r}\right\rangle m\left\langle {r+1 \atop k-r}\right\rangle\\
&=\sum_{r=0}^{k}(-1)^{k-r}\left\langle {-m \atop r+1}\right\rangle(k+1)\left\langle {r+1 \atop k-r}\right\rangle + \sum_{r=0}^{k}(-1)^{k-r}\left\langle {-m \atop r}\right\rangle m\left\langle {r+1 \atop k-r}\right\rangle\\
&=\sum_{r=0}^{k}(-1)^{k-r}\left\langle {r+1 \atop k-r}\right\rangle\left((k+1)\left\langle {-m \atop r+1}\right\rangle + m\left\langle {-m \atop r}\right\rangle\right)\\
&=\sum_{r=0}^{k}(-1)^{k-r}\left\langle {r+1 \atop k-r}\right\rangle\left((k-r)\left\langle {-m \atop r+1}\right\rangle + r\left\langle {-m \atop r}\right\rangle\right)\\
&=\sum_{r=0}^{k-1}(-1)^{k-r}\left\langle {r+1 \atop k-r}\right\rangle(k-r)\left\langle {-m \atop r+1}\right\rangle + \sum_{r=1}^{k}(-1)^{k-r}\left\langle {r+1 \atop k-r}\right\rangle r\left\langle {-m \atop r}\right\rangle\\
&=\sum_{r=0}^{k-1}(-1)^{k-r}\left\langle {-m \atop r+1}\right\rangle\left((k-r)\left\langle {r+1 \atop k-r}\right\rangle - (r+1)\left\langle {r+2 \atop k-r-1}\right\rangle\right)\\
&=0.
\end{aligned}
$$

So $L^p_m u_{n-2-k} \in U_1$ for all $k \geq 0$. U_1 is indeed a submodule and

$$V_{q,\alpha}(l,n) = U_1 \oplus U_2.$$

(c) $n - l \geq 3$ and $l = n - j_0$.

In this case

$$U_1 = \mathbb{C}u_l = \mathbb{C}u_{n-j_0},$$

and

$$U_2 = \sum_{i=l+1}^{n-1} \mathbb{C}u_i.$$

Since $j_0 \geq 3$,

$$0 \neq u_{n-j_0+1}, \quad 0 \neq u_{n-j_0+2} \in U_2.$$

By Proposition 3,

$$L^p_m u_{n-j_0} = -\alpha u_{n-j_0} + \alpha\left(\left\langle {-m \atop 2}\right\rangle - m^2\right) u_{n-j_0+2} + u$$

for all $m \in \mathbb{Z}$, where $u \in \sum_{i\geq 3} \mathbb{C}u_{n-j_0+i}$. This forces that

$$\alpha = 0,$$

and hence

$$l + \beta_1 - \beta_0 = 0.$$

Then

$$L^p_m u_{n-j_0} = L^p_m u_l = 0$$

for all $m \in \mathbb{Z}$. Again we proved that

$$V_{q,\alpha} = U_1 \oplus \bar{U}_2.$$

We have completed the proof. □

Proposition 5. Let

$$q(t) = \beta_0 - \beta_1 t,$$
$$q'(t) = \beta'_0 - \beta'_1 t, \qquad \alpha, \alpha' \in \mathbb{C}, l, n, l', n' \in \mathbb{Z},$$

be such that $n - l = n' - l' > 0$. We have the following:

(i) If $n - l = n' - l' = 1$, then

$$V_{q,\alpha}(l,n) \cong V_{q',\alpha'}(l',n')$$

iff

$$l + \beta_1 - \beta_0 = l' + \beta'_1 - \beta'_0.$$

(ii) If $n - l = n' - l' \geq 2$, then

$$V_{q,\alpha}(l,n) \cong V_{q',\alpha'}(l',n')$$

iff

$$\alpha = \alpha',$$

and

$$l + \beta_1 - \beta_0 = l' + \beta'_1 - \beta'_0.$$

Proof.

(i) By Proposition 3,

$$L^p_m u_{n-1} = -(n - 1 + \beta_1 - \beta_0) u_{n-1},$$

and

$$L^p_m u'_{n'-1} = -(n'-1+\beta'_1-\beta'_0)u'_{n'-1}$$

for all $m \in \mathbb{Z}$. So (i) is clear.

(*ii*) Again by Proposition 3,

$$L^p_m \begin{pmatrix} u_{n-2} \\ u_{n-1} \end{pmatrix} = \begin{pmatrix} -(n-2+\beta_1-\beta_0) & m(n-2+\alpha+\beta_1-\beta_0) \\ 0 & -(n-1+\beta_1-\beta_0) \end{pmatrix} \begin{pmatrix} u_{n-2} \\ u_{n-1} \end{pmatrix},$$

and

$$L^p_m \begin{pmatrix} u'_{n'-2} \\ u'_{n'-1} \end{pmatrix} = \begin{pmatrix} -(n'-2+\beta'_1-\beta'_0) & m(n'-2+\alpha'+\beta'_1-\beta'_0) \\ 0 & -(n'-1+\beta'_1-\beta'_0) \end{pmatrix} \begin{pmatrix} u'_{n'-2} \\ u'_{n'-1} \end{pmatrix}.$$

So if

$$V_{q,\alpha}(l,n) \cong V_{q',\alpha'}(l',n')$$

then

$$\alpha = \alpha'$$

and

$$l+\beta_1-\beta_0 = l'+\beta'_1-\beta'_0.$$

Conversely, if

$$\alpha = \alpha'$$

and

$$l+\beta_1-\beta_0 = l'+\beta'_1-\beta'_0,$$

we see from Proposition 3 (**) that

$$V_{q,\alpha}(l,n) \cong V_{q',\alpha'}(l',n').$$

□

2. $V_{q,\alpha}(-\infty,n) = V_{q,\alpha}(-\infty,+\infty)/V_{q,\alpha}(n,+\infty)$.

For $j = 1, 2, \ldots$, let

$$\overline{v}_0{}^{(n-j)} = v_0^{(n-j)} + V_{q,\alpha}(n,+\infty),$$

and let

$$u_{n-j} = \sum_{i=0}^{j-1} \left\langle {n-j+\beta_1 \atop i} \right\rangle \overline{v}_0{}^{(n-j+i)}.$$

Then

$$V_{q,\alpha}(-\infty,n)=\sum_{j=1}^{\infty}\mathbb{C}\overline{v}_0{}^{(n-j)}$$
$$=\sum_{j=1}^{\infty}\mathbb{C}u_{n-j},$$

and L_0^p acts on u_{n-j}, $j=1,2,\ldots$ diagonally.

Since for any $l<n$, $V_{q,\alpha}(l,+\infty)$ is a submodule of $V_{q,\alpha}(-\infty,+\infty)$, $V_{q,\alpha}(l,n)$ is a submodule of $V_{q,\alpha}(-\infty,n)$. Moreover,

$$V_{q,\alpha}(l,n)=\sum_{j=1}^{n-l}\mathbb{C}\overline{v}_0{}^{(n-j)}=\sum_{j=1}^{(n-l)}\mathbb{C}u_{n-j},$$

and

$$V_{q,\alpha}(-\infty,n)=\bigcup_{l>n}V_{q,\alpha}(l,n).$$

Proposition 6. $V_{q,\alpha}(-\infty,n)$ is an admissible $Vir(1-t)$-module, and each weight space is 1-dimensional. Moreover, $V_{q,\alpha}(-\infty,n)$ is decomposable only if

$$\alpha=1$$

and

$$n-1+\beta_1-\beta_0=0.$$

In this case,

$$V_{q,\alpha}(-\infty,n)=V_{q,\alpha}(-\infty,n-1)\oplus\mathbb{C}u_{n-1}.$$

Proof. Follows from Proposition 4 by letting $l\longrightarrow-\infty$. □

2.8 Admissible Modules of $Vir(p(t))$

In this section, we assume that $p=p(t)=\sum_{i=0}^{r}a_it^i$, where $a_0=a_r=1$, and consider the admissible $Vir(p)$-modules. Let $q=q(t)=\sum_{i=0}^{r}a_i\beta_it^i$, $\alpha\in\mathbb{C}$. For each $j\in\mathbb{Z}$, we have the polynomial $Vir(p)$-modules associated with $\sum_{i=0}^{r}a_i(\beta_i+ji)t^i$ and α, which is given by

$$V_{q,\alpha}^{(j)}:=\sum_{k\in\mathbb{Z}}\mathbb{C}v_k^{(j)},$$

with

$$L_m^pv_k^{(j)}:=\sum_{i=0}^{r}a_i(k+\alpha m+\beta_i+ji)v_{m+k+i}^{(j)}.$$

Analogously to Section 2.7 Proposition 1, we have

Proposition 1. Let

$$u_k^{j+1} := \sum_{i=0}^{r} a_i v_{k+i}^{(j)}.$$

Then

$$U := \sum_{k\in\mathbb{Z}} \mathbb{C} u_k^{(j+1)} \subset V_{q,\alpha}^{(j)}$$

is a $Vir(p)$-module and $U \cong V_{q,\alpha}^{(j+1)}$.

By Proposition 1, if we identify $v_k^{(j+1)}$ with $\sum_{i=0}^{r} a_i v_{k+i}^{(j)}$ in $V_{q,\alpha}^{(j)}$, then we identify $V_{q,\alpha}^{(j+1)}$ as a submodule of $V_{q,\alpha}^{(j)}$ and obtain the following $Vir(p)$-module sequence

$$\dots > V_{q,\alpha}^{(-1)} > V_{q,\alpha}^{(0)} > V_{q,\alpha}^{(1)} > V_{q,\alpha}^{(2)} > \dots \quad .$$

Define

$$V_{q,\alpha}(-\infty,+\infty) := \bigcup_{j\in\mathbb{Z}} V_{q,\alpha}^{(j)},$$

$$V_{q,\alpha}(l,+\infty) := \bigcup_{j\geq l} V_{q,\alpha}^{(j)} = U_{q,\alpha}^{(l)},$$

$$V_{q,\alpha}(-\infty,n) := \frac{V(-\infty,+\infty)}{V_{q,\alpha}(n,+\infty)},$$

and

$$V_{q,\alpha}(l,n) := \frac{V_{q,\alpha}(l,+\infty)}{V_{q,\alpha}(n,\infty)} \qquad l < n.$$

Then all of these are bounded quasi-admissible modules of $Vir(p)$. Moreover, $V_{q,\alpha}(-\infty,n)$ and $V_{q,\alpha}(l,n)$ are admissible modules. The proof of these statements are similar to that in the case of $Vir(1-t)$ in last section but tedious. We will only prove the following:

Proposition 2.

(*i*) $V_{q,\alpha}(l,n)$ is an admissible $Vir(p)$- module and $dim V_{q,\alpha}(l,n) = (n-l)r$.

(*ii*) Let

$$\phi_{q,\alpha}(l,n) : Vir(p) \longrightarrow \mathfrak{gl}(V_{q,\alpha}(l,n))$$

be the representation given by

$$\phi_{q,\alpha}(l,n)(L_m^p)\bar{v}_k{}^{(j)} = L_m^p \bar{v}_k{}^{(j)},$$

where $\bar{v}_k{}^{(j)} = v_k^{(j)} + V_{q,\alpha}^{(n)}, j = l, \dots, n-1$. Then

$$\frac{Vir(p)}{ker\phi_{q,\alpha}(l,n)}$$

is a solvable Lie algebra.

Proof. Since

$$V_{q,\alpha}^{(j)} = \mathbb{C}v_0^{(j)} + ... + \mathbb{C}v_{r-1}^{(j)} + V_{q,\alpha}^{(j+1)},$$

for all $j \in \mathbb{Z}$, we have

$$V_{q,\alpha}(l,n) = \mathbb{C}\overline{v}_0{}^{(l)} + ... + \mathbb{C}\overline{v}_{r-1}{}^{(l)} + ... + \mathbb{C}\overline{v}_0{}^{(n-1)} + ... + \mathbb{C}\overline{v}_{r-1}{}^{(n-1)},$$

where

$$\overline{v}_k^{(j)} = v_k^{(j)} + V_{q,\alpha}(n,+\infty),$$

and they are linear independent. So

$$dim V_{q,\alpha}(l,n) = (n-l)r.$$

Now for $j = l, ..., n-1$,

$$\begin{aligned}
L_m^p v_k^{(j)} &= \sum_{i=0}^{r} a_i(k + \alpha m + \beta_i + ji) v_{m+k+i}^{(j)} \\
&= \sum_{i=0}^{r} a_i(\beta_i + ji) v_{m+k+i}^{(j)} + (k + \alpha m) \sum_{i=0}^{r} a_i v_{m+k+i}^{(j)} \\
&= \beta_0 v_{m+k}^{(j)} + \sum_{i=1}^{r} a_i(\beta_i + ji) v_{m+k+i}^{(j)} + (k + \alpha m) v_{m+k}^{(j+1)} \\
&= -\beta_0 \sum_{i=1}^{r} a_i v_{m+k+i}^{(j)} + \sum_{i=1}^{r} a_i(\beta_i + ji) v_{m+k+i}^{(j)} + (\beta_0 + k + \alpha m) v_{m+k}^{(j+1)} \\
&= \sum_{i=1}^{r} a_i(ji + \beta_i - \beta_0) v_{m+k+i}^{(j)} \quad \left(mod \quad V_{q,\alpha}^{(j+1)}\right).
\end{aligned}$$

So

$$L_n^p L_m^p v_k^{(j)} = L_m^p L_n^p v_k^{(j)} \quad \left(mod \quad V_{q,\alpha}^{(j+1)}\right)$$

for all $m, n \in \mathbb{Z}$. There exist $w_0^{(j)}, ..., w_{r-1}^{(j)} \in V_{q,\alpha}^{(j)}$ and $\lambda_0^{(j)}, ..., \lambda_{r-1}^{(j)} \in Vir(p)^*$, the restricted dual space of $Vir(p)$, such that

$$L_m^p w_k^{(j)} = \lambda_k^{(j)}(L_m^p) w_k^{(j)} \quad \left(mod \quad V_{q,\alpha}^{(j+1)}\right),$$

where

$$k = 0, 1, ..., r-1,$$

$$j = l, ..., n - 1.$$

$$m, k \in \mathbb{Z}.$$

Let

$$\overline{w}_k{}^{(j)} = w_k^{(j)} + V_{q,\alpha}^{(n)}, \quad k = 0, ..., r - 1, \quad j = l, ..., n - 1.$$

Then

$$\left\{\overline{w}_k{}^{(j)} \mid k = 0, ..., r - 1, \quad j = l, ..., n - 1\right\}$$

is a basis of $V_{q,\alpha}(l, n)$. Under this basis, $\phi_{q,\alpha}(l, n)(L_m^p)$ is an upper triangular matrix for every $m \in \mathbb{Z}$. Moreover, we may choose a basis

$$\left\{u_k^{(j)} \mid k = 0, ..., r - 1, j = l, ..., n - 1\right\}$$

for $V_{q,\alpha}(l, n)$ such that under this basis L_0^p acts diagonally. We have completed the proof. □

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