UNIVERSITY OF ALBERTA

AN ELASTIC INHOMOGENEITY WITH PERFECTLY BONDED INTERFACE SUBJECTED TO A GENERAL CLASS OF NONUNIFORM REMOTE LOADING

BY

CHUN IL KIM ©

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Department of Mechanical Engineering

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This dissertation is dedicated to my father, **Jo In Kim**

ABSTRACT

Problems involving elastic inhomogeneities have received considerable attention in the literature. In many of their case, complex-variable methods are used extensively and successfully to produce exact solutions and significant practical results in problems of linear plane and anti-plane elastostatics. However, such exact analyses have been more or less absent in the analogous problems from *finite* elasticity. Recent studies have provided relatively simple complex-variable formulations for problems involving elastic inhomogeneities for the class of *harmonic materials*. The results, however, are limited exclusively to uniform remote loading, whereas, in nature, remote loadings are mostly nonuniform.

In present study, we develop a general methodology for the problems of plane *finite* deformations of harmonic composite materials containing an elastic inhomogeneity subjected to nonuniform remote loading. In addition, we also derive important results of mechanics of materials concerning the subject of maintaining uniform stress distribution inside the inhomogeneity subjected to nonuniform remote loading.

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Chun Il Kim

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INTRODUCTION

1.1 GENERAL BACKGROUND

1.1.1 INTRODUCTION

Materials are widely used in modern industry for various purposes. Many of their successful applications can be found in, for example, construction fields, aerospace, biomedical applications, automobile industries and space structures. However, not every application of materials is successful at the initial stage. For example, a beverage can was first made from galvanized iron. Yet, a galvanized iron can is heavy and inadequately strong for a beverage can and hence, later, was replaced by an aluminum can. The process of finding "proper materials" for each specific purpose is not an easy task and often requires trial and error. Therefore, it is of fundamental importance that one really needs to understand the mechanical behavior of materials before one uses them.

Among various modern materials, composite materials have gained particular interest, due to the fact that technological developments in the use of materials inevitably induce severe and highly demanding working conditions so that conventional materials such as ceramics and polymeric materials may no longer meet such intensive service requirements.

1.1.2 COMPOSITE MATERIAL

Composite materials are generally referred to as multiphase materials consisting of two or more distinct constituents, which are combined to achieve desirable mechanical, thermal and/or electronic properties depending on specific service requirements. Based on the structural shape of the materials, composite materials are classified into three different classes namely; particle-reinforced, fiber-reinforced and laminar composites (see Figure 1).

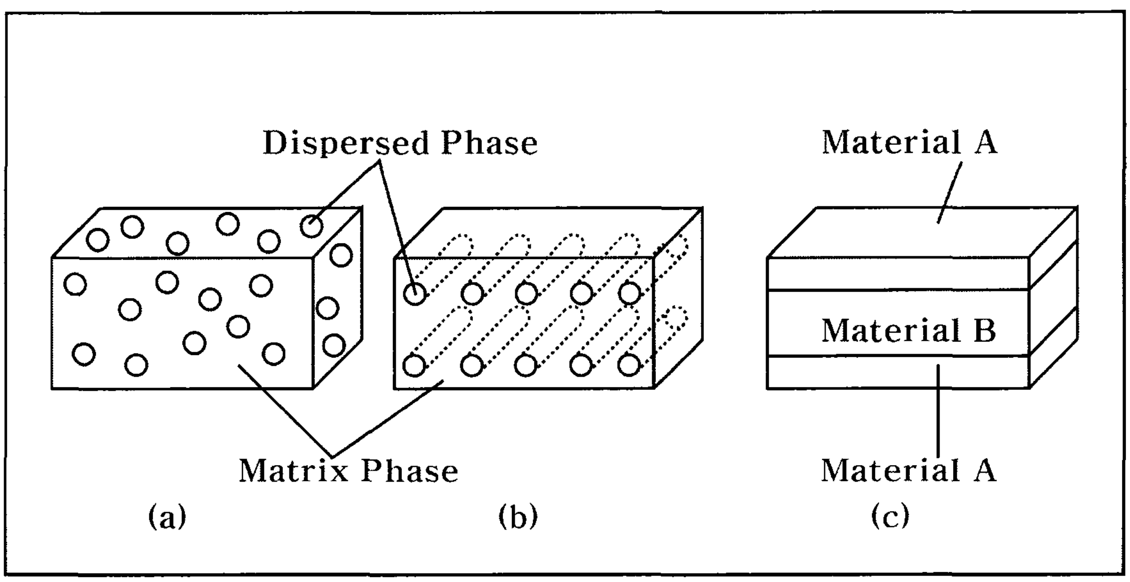

Figure 1: *Showing Three Types of Composite Materials:* (a) Particulate Composite, (b) Fiber-Reinforced Composite, and (c) Laminar Composite (From Sudak, 2000)

Particle-reinforced composites consist of a matrix material into which particles are embedded. These composites are generally subdivided into two classes referred to as large-particle composites and dispersed-strengthened composites. The large particle composites containing large amounts of coarse particles which are harder and stiffer than the matrix is manufactured to obtain specific combinations of properties rather than to improve strength. The application of the large particle composite can be found where high corrosion and fatigue-resistance are required (Tjong et al, 2005; Jian-Zhong et al, 2006). In the case of the dispersion-strength composite, the particles are extremely small in that they prevent the matrix from dislocating inducing plastic deformation and thereby increase the strength of the material (Feng et al, 2007). More practical applications of particulate composites can be found in Donald (1990).

Laminate composites consist of two or more different thin, planar and unidirectional layers called laminas. The laminas are stacked and subsequently bonded in such a way that the material properties and reinforcing directions vary with each successive layer. One particular application of laminate composites can be found in shape memory alloys (see Irie, 1989; Tupper, 2001; Zhang et al, 2007; Zhang. Y et al, 2007). Details of designing laminate composites can be found in John (1999).

Fiber-reinforced composites consist of cylindrical fibers (i.e. each cross section of the cylindrical fibers has the same planar geometry) and a surrounding bulk matrix. The reinforcement (fiber) can be embedded into the matrix material in a variety of orientations so that the composite meets each specific service requirement (see Figure 2).

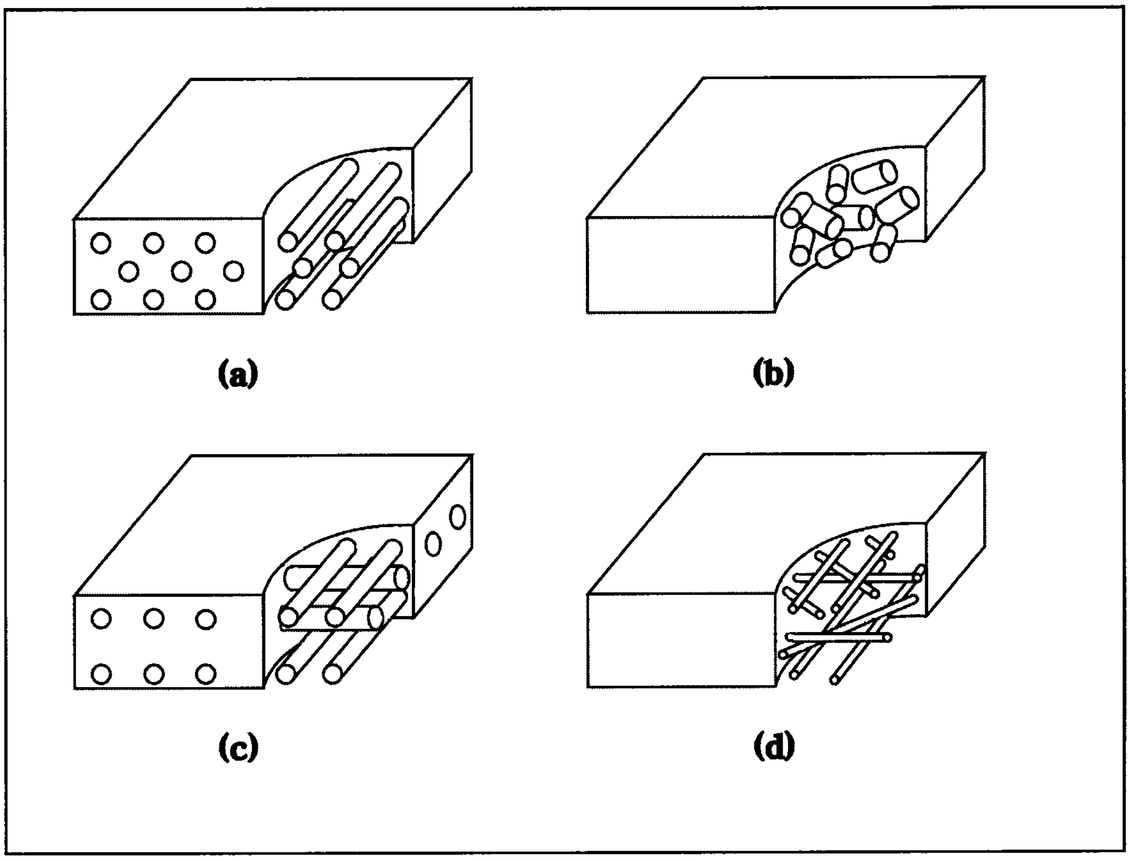

Figure 2: *Several Orientations of Fiber-Reinforced Composites:* (a) Continuous Unidirectional Fibers, (b) Randomly Oriented Short Discontinuous Fibers, (c) Orthogonal Fibers, and (d) Multiple-Ply Fibers (From Askeland, 1990)

Applications of fiber-reinforced composites are found in many different areas including aerospace, construction, automobile industries, spacecraft structures, electronics and petrochemical (see Chung, 1984; Chen et al, 2007; Karthikeyan et al, 2007). The elastic fibers acting as reinforcements absorb the external load throughout the surrounding matrix and hence, increase the strength of the composite. The reinforcement should be a material that is stiff, whereas the surrounding matrix should be tough and sufficiently ductile to support and allow for external loads to be transmitted into each reinforcement. In addition, the surrounding matrix, serving as a barrier, protects the composite from the propagation of brittle cracks induced at the vicinity of the fibers.

Due to its wide range of application in modern industry, the fiber-reinforced composite has gained particular attention and become the most predominant composite in use today. For this reason, the fiber-reinforced composite will form the fundamental framework of this dissertation.

As mentioned above, the use of reinforcements certainly improves desirable mechanical properties such as stiffness, strength, and toughness. It is well-known, however, that the reinforcement also (referred to as an inhomogeneity) disturbs the original stress field leading to local stress intensification that may further induce the total failure of the composite system. Therefore, it is of great importance to study the state of stress within the composite material, in particular, how the distribution of stress varies with the introduction of inhomogeneities.

1.2 SINGLE INHOMOGENEITY MODEL

Since, in nature, a composite body contains numerous fibers (inhomogeneities), the actual analysis of such a composite is computationally expensive and often analytically impossible. Instead, we may imagine a small section sampled from a composite and then analyze it as if the sampled section is the original composite. The central vehicle, which makes this possible, is the existence of the representative volume element (RVE) (see Pindera et al,

1995) in which a sample of the composite material is taken to be structurally the same as the whole composite with the average properties indistinguishable from those of the statistically homogeneous composite. Therefore, the entire three-dimensional fiber-reinforced composite can be analyzed by considering a single inhomogeneity embedded in infinite medium called the "matrix" followed by an *averaging procedure,* for example, the Reuss and Voigt hypotheses, the Dilute Approximation, the Self-Consistent Scheme, the Mori-Tanaka Method, and the Eshelby Equivalent Inclusion method all of which lead to *effective properties* of the composite (see Hashin, 1983; Hill, 1965; Christenson & Lo, 1979; Christenson, 1990; Jacob, 1991; Nemat & Hori, 1993; Huang & Hu, 1995).

It is widely understood that the single-inhomogeneity problem is of fundamental importance in a composite (Eshelby, 1957, 1959; Hashin, 1991). More importantly, the single-inhomogeneity model is a much simpler and more practical model for analytical approaches than other available multi-inhomogeneity composite models (Achenbach & Zhu, 1990). Therefore, it is suggested that the RVE of the composite material can be treated as a single inhomogeneity-matrix system.

In our present study, we further consider the single inhomogeneity-matrix system undergoing plane-strain deformations in that the cross section of the three-dimensional composite is subjected to plane deformations (Timoshenko & Goodier, 1970; Truesell & Toupin, 1960; Malvern, 1969; Fung, 1977; Gurtin, 1981) for simplification. The plane-strain deformation is possible when the dimension of the body in the z-direction is very large and the conditions are the same at all cross sections, so that it is sufficient to consider only a slice between two sections a unit distance apart. There are many important problems of this kind, for example, a fiber-reinforced composite (see Figure 1(b)) is stretched in the uniaxial or biaxial direction, a cylindrical tube with internal pressure and a retaining wall (e.g. dam) with lateral pressure.

For single-inhomogeneity problems most works so far, have been concerned with *linear plane elasticity* (Materials undergoing relatively small deformations see, for example, Timoshenko & Goodier, 1970; Ru & Schiavone, 1996, 1997; Ru, 1998a, 1998b; Schiavone, 2003; Van Vliet et al, 2003; Ru, et al, 2005). However, very little researches have been conducted in *nonlinear elasticity* which involves large or finite deformations (see C. Truesdell, 1965). The actual analysis of relevant problems has eluded researchers due to the existing nonlinearity in mathematical models. Yet, there is one special class of materials which mathematically behaves as a linear material. This class is the class of *harmonic* materials, first introduced by John (1960) and later explored by a number of authors see, for example, Knowles & Sternberg (1975), Varley & Cumberbatch (1980), Abeyaratne & Horgan (1984), Horgan (1989) and Li & Steigmann (1993). Therefore, in this dissertation, we will study the single inhomogeneity problem for finite deformations of a fiber-reinforced composite system in nonlinear plane elasticity using harmonic materials.

We also mention here that we consider only a well-spaced fiber-reinforced composite with fiber volume fractions up to 40% (Schmauder et al, 1992). In that way, the interaction among neighboring fibers and its influence on the stress fields on the entire composite system can be neglected so that we can isolated one fiber and study this in an infinite matrix. Finally, the effective property of the composite can be obtained by using one of the averaging procedures mentioned above.

1.3 INTERPHASE LAYER

The fiber-reinforced composite material consists of two distinctive phases, namely; a fiber (inhomogeneity) and a matrix phase. However, in nature, there is an additional phase existing between the fiber and the matrix called an " interphase or interphase layer". The interphase layer may be considered as the by-product of chemical reactions during the fabrication process or as a thin layer introduced in the design stage to improve the performance of the composite material (Horgan & Chan, 1999). Consequently, the mechanical properties of certain fiber-reinforced composite materials depend not only on the properties of the two constituent phases (fiber and matrix) but also on the nature of the bond between the fiber and the surrounding matrix (Jayaraman et al, 1993).

The existence of an interphase layer implies that the composite material is required to be considered as a three-phase constituent system (matrix phase, interphase layer and fiber). Nevertheless, due to its intrinsic complexity in forming, it is extremely difficult to provide an exact mathematical description of the behavior of the interphase layer. Extensive research has been done in the fields of micromechanics, chemistry and material sciences on the interphase layer (Brennan, 1988; Kim & Bau, 1992) and various approximation models have been proposed in an attempt to characterize the complex behavior of the interphase layer. For example, the imperfect interface model (Achenbach & Zhu, 1990; Ghahremani, 1980; Hashin, 1991) assumes that a very thin interfacial zone exists between the fiber and the matrix. In the physical sense, an 'interface' can be considered as a limiting case of the interphase layer as its thickness tends to zero. This interface, defined as a twodimensional boundary (a curved plane), distinguishes the bulk materials and, hence, the material properties are changed abruptly across the interface.

1.3.1 PERFECT BONDING

A state of perfect bonding assumes that the interface is considered as a surface across which both tractions and displacements are continuous. Such an interface is generally referred to as a "perfectly bonded interface" (see Ru &Schiavone, 1996; Ru et al, 2005). The perfect bonding assumption is an idealization for it is known that the bonding, in nature, is never perfect due to the existing cracks, voids and cavities in the interphase layer between a fiber and its surrounding matrix. In general, perfect bonding is rather conventional, yet comprehensive studies reveal that (Knowles & Sternberg, 1975; Ru, 2002) the model shows satisfactory physical correspondence in many cases. In addition, the existence of a prefect bond enables an idealization of very complex behavior such as finite plane deformations accompanying large deformations in nonlinear elasticity.

1.3.2 IMPERFECT BONDING

An imperfect bond, as the name suggests, was introduced to take into account the imperfections of the interface such as voids, cavities, cracks, and dislocations. The fundamental premise for the imperfect bonding model is that the interphase, in general, is found to be softer and weaker than both fiber and matrix (Jones & Whitter, 1967). Under this condition, the imperfect bonding can be modeled by introducing the interface across which tractions are continuous but displacements are discontinuous. A considerable amount of studies have been conducted to providing mathematical models effectively describing the behavior of the imperfectly bonded interphase. The simplest imperfect interface model is the elastic spring-type interface model in which the tractions are continuous yet, displacement jumps across the interface are directly proportional to their respective interface traction components (see, for example, Bigoni et al, 1998; Lipton & Serkove et al, 1998; steif & Hoysan, 1987; Aboudi, 1987; Gosz et al, 1992). We do not discuss this more complicated scenario in this thesis but instead note that this is an interesting area for future study especially in the area of finite deformations.

1.4 PURPOSE OF STUDY AND DISSERTATION OVERVIEW

The objectives of the present study are to consider the finite plane deformations of a harmonic composite material containing an elastic inhomogeneity embedded in a surrounding matrix of similar elastic material. The bonding at the inhomogeneity-matrix interface is considered to be perfect so that both tractions and displacements are continuous across the surface. For this purpose, complex variable techniques (Muskhelishvili, 1953; England, 1971; Brown & Churchill, 1996) are extensively incorporated in the current work.

The results derived in this dissertation will enable one to have a clear understanding of the state of stress throughout the entire fiber-reinforced composite system subjected to varying applied loadings. In particular, the complete solutions of the corresponding problems of these kinds (see Chapters. 3-5) provide physical relevance to those corresponding "real-world problems" and hence will enhance the future design of composite materials.

The thesis proceeds as follows. Chapter 2 outlines the complex-variable formulations for a special class of harmonic materials. In addition, general preliminaries such as notations, interface formulations and the state of the undeformed stage are extensively discussed for further purpose.

Chapter 3 provides the complete solution for a perfectly bonded elliptic inhomogeneity when the system is subjected to classes of nonuniform remote stress distributions. We note, in particular, the generality of our results and that existing results in the literature are obtained as special cases of the solutions derived here.

Chapter 4 examines, in detail, the analogy of Eshelby's conjecture for the particular class of materials of harmonic-type undergoing finite plane deformations. In particular, we show that, under the analogous constraints of perfect bonding and uniform remote (Piola) stress, if an inhomogeneity matrix-system is subjected to finite plane deformations, the Piola stress distribution within an elliptic inhomogeneity is necessarily uniform.

Chapter 5 discusses the idea that by adjusting the remote (Piola) stress we can design the shape of the inhomogeneity in such a way that the interior (Piola) stress distribution remains uniform.

Finally, Chapter 6 provides a summary of results obtained and some concluding remarks with some suggestions for future research.

Chapter 2
GENERAL FORMULATION AND PRELIMINARIES

2.1 INTRODUCTION

As discussed in Chapter 1, we consider the single inhomogeneity problems in which the inhomogeneity is perfectly bonded to the surrounding material referred to henceforth as 'the matrix'. Assume that all materials are 'harmonic' (see below) and that the corresponding inhomogeneity-matrix system is subjected to large plane-strain deformations. In addition, we further assume that the inhomogeneity-matrix system undergoes plane-strain deformation. Consequently, the problem of the entire system in $\mathbb{R}^3$ can be reduced the equivalent problem concerning the cross section of the system with unit thickness in $\mathbb{R}^2$.

The class of compressible *hyperelastic harmonic materials* or just *harmonic materials* was proposed by John (1960) and has attracted considerable attention in the literature (see for example, Knowles & Sternberg, 1975; Varley & Cumberbatch, 1980; Abeyaratne, 1983; Abeyaratne & Horgan, 1984; Jafari, Abeyaratne & Horgan 1984; Li & Steigmann, 1993; Horgan, 1989, 1995, 2001). For example, experimental results by Varley & Cumberbatch (1980) indicate close agreement between this class of harmonic materials and the behavior of certain rubber-like materials. In addition, Li & Steigmann (1993) and Abeyaratne (1983) have used these harmonic materials to study, respectively, the finite deformations of an annular membrane induced by the rotation of a rigid hub and the finite deformations of a crack. Recently Wang et al (2005) have extended and analyzed the notion of 'harmonic shapes' (holes and inclusions) to this class of materials.

In the following section, we present the complex-variable formulation of a model of the behavior of harmonic materials.

2.2 HARMONIC MATERIALS

2.2.1 COMPLEX-VARIABLE FORMULATION

We consider a plane cross-section of the single inhomogeneity-matrix system in which it is assumed that the matrix is unbounded. Consequently, let the inhomogeneity-matrix system occupy $\mathbb{R}^2$(described here by the generic coordinates $(x_{1,}\ x_2)$) in its undeformed (reference) configuration. The deformation describing the deformed (or current) configuration is given by:

$$w(z) = y_1(x_1,\ x_2) + iy_2(x_1,\ x_2)\,, \quad \text{for all } (x_1,\ x_2) \in \mathbb{R}^2. \tag{2.1}$$

The components of the deformation-gradient tensor are given by (see Knowles & Sternberg, 1975):

$$F_{ij} = \frac{\partial y_i}{\partial x_j} = y_{i,j}, \quad J = \det \mathbf{F} > 0 \quad , i,j = 1,2 \tag{2.2}$$

and let $\mathbf{G}$ be the left Cauchy –Green strain tensor defined by

$$\mathbf{G} = \mathbf{F}\mathbf{F}^T \quad \text{on } \mathbb{R}^2.$$

Then, from (2.1-2), we can calculate the two fundamental scalar invariants of $\mathbf{G}$:

$$\begin{aligned} R &= \lambda_1^2+\lambda_2^2 = tr\left(\mathbf{G}\right) = tr[F_{ps}F_{qs}\left(e_p\otimes e_q\right)] = F_{ps}F_{ps} = y_{p,s}y_{p,s},\; p,s=1,2, \\ J &= \lambda_1\lambda_2 = \sqrt{\det\mathbf{G}} = F_{11}F_{22}-F_{12}F_{21} = y_{1,1}y_{2,2}-y_{1,2}y_{2,1}, \end{aligned} \quad (2.3)$$

where λ_1 and λ_2 are the principal stretches of $\mathbf{G}$. From Knowles & Sternberg (1975), the governing constitutive relations and the equilibrium equation for the harmonic material are given by

$$\tau_{ij} = \frac{2}{J}\frac{\partial\Theta}{\partial R}F_{i\rho}F_{j\rho}+\frac{\partial\Theta}{\partial J}\delta_{ij},$$

$$\sigma_{ij} = 2\frac{\partial\Theta}{\partial R}F_{ij}+\frac{\partial\Theta}{\partial J}\epsilon_{i\rho}\epsilon_{j\gamma}F_{\rho\gamma},$$

$$\frac{\partial\sigma_{ij}}{\partial x_j} = 0, \quad \text{on } \mathbb{R}^2, \quad i,j,\rho,\gamma=1,2, \quad (2.4)$$

$$W\left(x_1,x_2\right) = \Theta\left(R\left(x_1,x_2\right),J\left(x_1,x_2\right)\right), \quad \text{for all } \left(x_1,\ x_2\right)\in\mathbb{R}^2,$$

where W is the strain energy density of a homogeneous and isotropic (hyper-elastic) solid. It is obvious from (2.4) that the mechanical response of a homogeneous and isotropic solid to a plane deformation is dominated entirely by the material response defined by the function Θ.

The strain energy density W for this class of harmonic materials introduced by John (1960) is characterized per unit volume of the deformed configuration by

$$W = \Theta\left(I,J\right) = 2\mu[F\left(I\right)-J], \quad I=\sqrt{R+2J}. \quad (2.5)$$

where μ is a given positive material constant and $F(I)$ a material function of I (note that $F(I)$ is different from $\mathbf{F}$). Then (2.1-3) and (2.5) lead to the representations

$$\begin{aligned} I &= \sqrt{\lambda_1^2+\lambda_2^2+2\lambda_1\lambda_2} = \lambda_1+\lambda_2 = \left|w_{,2}+iw_{,1}\right|, \\ J &= \lambda_1\lambda_2 = \det\mathbf{F} = -\operatorname{Im}\left[w_{,1}\overline{w}_{,2}\right] \\ R &= tr\left(\mathbf{F}\mathbf{F}^T\right) = \left|w_{,1}\right|^2+\left|w_{,2}\right|^2. \end{aligned} \tag{2.6}$$

From (2.5), the constitutive relations (2.4) become

$$\tau_{ij} = 2\mu\left[\frac{F'(I)}{IJ}F_{i\rho}F_{j\rho}+\left(\frac{F'(I)}{I}-1\right)\delta_{ij}\right], \tag{2.7A}$$

$$\sigma_{ij} = 2\mu\left[\frac{F'(I)}{I}F_{ij}+\left(\frac{F'(I)}{I}-1\right)\epsilon_{i\rho}\epsilon_{j\gamma}F_{\rho\gamma}\right], \tag{2.7B}$$

where

$$\frac{\partial\Theta}{\partial R} = \frac{\partial 2\mu[F(\sqrt{R+2J})-J]}{\partial R} = 2\mu F'\left(\sqrt{R+2J}\right)\frac{\partial\sqrt{R+2J}}{\partial R} = \mu\frac{F'(I)}{I},$$

$$\frac{\partial\Theta}{\partial J} = \frac{\partial 2\mu[F(\sqrt{R+2J})-J]}{\partial J} = 2\mu\left[F'\left(\sqrt{R+2J}\right)\frac{\partial\sqrt{R+2J}}{\partial J}-1\right] = 2\mu\left[\frac{F'(I)}{I}-1\right].$$

Equating (2.7B) for

$$i,\ j,\ \rho,\ \gamma = 1,2,\ \ \epsilon_{11}=\epsilon_{22}=0,\ \ \epsilon_{12}=1,\ \ \epsilon_{21}=-1,$$

we obtain that

$$\sigma_{11} = 2\mu\left(\frac{F'(I)}{I}(F_{11}+F_{22})-F_{22}\right),\quad \sigma_{22} = 2\mu\left(\frac{F'(I)}{I}(F_{11}+F_{22})-F_{11}\right),$$

$$\sigma_{12}=2\mu\left(\frac{F'(I)}{I}(F_{12}-F_{21})+F_{21}\right),\quad \sigma_{21}=2\mu\left(-\frac{F'(I)}{I}(F_{12}-F_{21})+F_{12}\right), \tag{2.8}$$

From (2.1) in conjunction with (2.2), we derive the following relation

$$\frac{F'(I)}{I}(F_{12}-F_{21})+i\frac{F'(I)}{I}(F_{11}+F_{22})=\frac{F'(I)}{I}(w_{,2}+iw_{,1}), \tag{2.9}$$

$$(\because w_{,2}=F_{12}+iF_{22},\quad w_{,1}=F_{11}+iF_{21}).$$

and from (2.7A) we have that for, $i,\ j,\ \rho=1,2,$

$$\tau_{11}=2\mu\left[\frac{F'(I)}{IJ}\left(F_{11}^2+F_{12}^2\right)+\left(\frac{F'(I)}{I}-1\right)\right],$$

$$\tau_{22}=2\mu\left[\frac{F'(I)}{IJ}\left(F_{22}^2+F_{21}^2\right)+\left(\frac{F'(I)}{I}-1\right)\right], \tag{2.10}$$

$$\tau_{12}=2\mu\frac{F'(I)}{IJ}(F_{12}F_{22}+F_{11}F_{21}),\quad (\because \delta_{11}=\delta_{22}=1,\ \delta_{12}=0),$$

Therefore, from (2.1-2) and (2.9), Equation (2.8) can be written the form

$$\sigma_{12}+i\sigma_{22}=2\mu\left[\frac{F'(I)}{I}(w_{,2}+iw_{,1})-iw_{,1}\right],$$

$$\sigma_{11}+i\sigma_{21}=2\mu i\left[w_{,2}-\frac{F'(I)}{I}(w_{,2}+iw_{,1})\right], \tag{2.11}$$

Also, from(2.1-3), Equation (2.10) leads to the representations

$$\tau_{11}+\tau_{22}=2\mu\left[\frac{IF'(I)}{J}-2\right],$$

$$\tau_{11}-\tau_{22}+2i\tau_{12}=2\mu\frac{F'(I)}{IJ}\left(w_{,1}^2+w_{,2}^2\right), \tag{2.12}$$

where

$$\begin{aligned} R &= F_{ij}F_{ij} = F_{11}^2 + F_{12}^2 + F_{22}^2 + F_{21}^2, \ \ i,\ j = 1,2, \\ w_{,1}^2 + w_{,2}^2 &= F_{11}^2 - F_{22}^2 + F_{12}^2 - F_{21}^2 + 2i\left(F_{12}F_{22} + F_{11}F_{21}\right). \end{aligned}$$

Here, σ and τ in (2.11-12) are referred to as the Piola stresses and the Cauchy stresses, respectively. The Piola stress is the measurement of stress per unit area of in the undeformed (reference) configuration and the Cauchy stress is measured per unit area of the deformed (current) configuration (see Chadwick, 1999; Fung, 1977; Gurtin, 1981).

Finally, in view of (2.2), (2.8) and the plane deformation (2.1), the equilibrium equation from the third equation of (2.4) now takes the form

$$\left[\frac{F'(I)}{I}\left(w_{,2} + iw_{,1}\right)\right]_{,1} + i\left[\frac{F'(I)}{I}\left(w_{,2} + iw_{,1}\right)\right]_{,2} = 0 \tag{2.13}$$

To deal with Equations (2.11-13), we introduce a 'stress function' (or potential) ϕ analytic with respect to the complex variable z (see Ru, 2002):

$$\begin{aligned} \frac{F'(I)}{I}\left(w_{,2} + iw_{,1}\right) &= \phi'(z), \ \ F'(I) = \left|\phi'(z)\right|, \ \ w'(z) = \left[\phi'(z)\right]^{\delta+1}, \quad (2.14) \\ I &= P\left(\left|\phi'\right|\right), \ \ P(^*) \equiv \left[F'(^*)\right]^{-1}, \\ z &= x_1 + ix_2, \ \ \text{for all } (x_1,\ x_2) \in \mathbb{R}^2, \end{aligned}$$

where $P(^*)$ denotes the inverse function of $F'(^*)$. For harmonic materials, the uniaxial Piola stress is given by (see Ru, 2002)

$$\begin{aligned} T &= 2\mu\left[F'(I) - \lambda_2\right], \quad (2.15) \\ F'(I) &= \lambda_1, \ \ \lambda_2 = P(\lambda_1) - \lambda_1, \end{aligned}$$

where λ_2 is the transverse stretch of the material. Experimental results concerning Equation (2.15) reveal that the uniaxial relations can be written in the form (see Varley & Cumberbatch, 1980)

$$T = 4\mu \frac{b\lambda_1^k + c}{\lambda_1}, \quad k = 2(1+\delta). \tag{2.16}$$

We note from Varley & Cumberbatch (1980) that this uniaxial relation (2.16) is designed to accommodate the undeformed state $\lambda_1 = \lambda_2 = 1$ at which $T = 0$. In addition, the results based on the model (2.16) were found to be in good agreement with experimental data, for example, those obtained for the large deformations of a rubber sheet containing an elliptic hole (Varley & Cumberbatch, 1980).

It is shown in Ru (2002) that we can write the deformation $w(z)$ in terms of two complex potentials $\phi(z)$, $\psi(z)$ as

$$w(z) = -i\phi(z) + i\frac{bw(z)\overline{w'(z)} + cz}{\overline{\phi'(z)}} + \overline{\psi(z)}, \tag{2.17}$$

and from (2.11), (2.14) and (2.18) the Piola stresses in the form:

$$-\sigma_{21} + i\sigma_{11} = \chi_{,2}, \qquad \sigma_{22} - i\sigma_{12} = \chi_{,1}, \tag{2.18}$$

where the stress function χ is defined by

$$\chi(z) \equiv \chi_1(z) + i\chi_2(z) \equiv -2\mu\left[i\phi(z) + w(z)\right]. \tag{2.19}$$

From (2.17) and (2.19), χ can be written as:

$$\chi(z) = -2\mu\left[i\frac{bw(z)\overline{w'(z)} + cz}{\overline{\phi'(z)}} + \overline{\psi(z)}\right]. \tag{2.20}$$

The stress function $\chi(z)$ in Equation (2.20) is referred to as the Piola stress function.

2.2.2 A PARTICULAR CLASS OF HARMONIC MATERIAL

According to Varley & Cumberbatch (1980), the value $k = 2.28$ (in Equation (2.16)) provides the best agreement with experimental results obtained from some rubber-like materials. In addition, due to the mathematical simplicity, the special class of harmonic materials (2.16) defined by $k = 2$ have gained particular attention (Horgan, 1989, 1995; Li & Steigmann, 1993). From (2.14) and (2.16), this special class of harmonic materials is characterized by

$$F'(I) = \frac{1}{4\alpha}\left[I + \sqrt{I^2 - 16\alpha\beta}\right], \quad P(\lambda) = 2\left(\alpha\lambda + \frac{\beta}{\lambda}\right) \geq 4\sqrt{\alpha\beta},$$

$$T(\lambda_1) = 4\mu\left[(1-\alpha)\lambda_1 - \frac{\beta}{\lambda_1}\right], \quad 1 > \alpha \geq \frac{1}{2}, \quad \beta > 0, \tag{2.21}$$

where α and β are two material constants, related to b and c by $\alpha = 1-b$ and $\beta = -c$. Here, to get a positive Piola stress and transverse stretch at very large stretching, we have that $1 > \alpha \geq \frac{1}{2}$. Similarly, to get a negative Piola stress at very large compression, we have $\beta > 0$ (Varley & Cumberbatch, 1980). To incorporate the undeformed stage, from (2.21) we require that, for $\lambda_1 = 1, T = 0$. That is,

$$T(1) = 4\mu\left[(1-\alpha)1 - \frac{\beta}{1}\right] = 0,$$

which gives (Ru, 2002)

$$\alpha + \beta = 1. \tag{2.22}$$

For example the values, $\alpha = \beta = \frac{1}{2}$ have been adopted by Li & Steigmann (1993) to study the finite deformation of an annular membrane induced by the rotation of a rigid body, and by Abeyaratne (1983) to study the finite deformation of a crack. In addition, when $k = 2$, it is noted from (2.14) that

$$w'(z) = \phi'(z), \quad \delta = 0$$
$$(\because k = 2(1+\delta) = 2),$$

and (2.17) and (2.20) become

$$iw(z) = \alpha\phi(z) + i\overline{\psi(z)} + \frac{\beta z}{\overline{\phi'(z)}} \tag{2.23}$$

$$\chi(z) = 2\mu i[(\alpha - 1)\phi(z) + i\overline{\psi(z)} + \frac{\beta z}{\overline{\phi'(z)}}], \tag{2.24}$$

which gives the two fundamental (to this study) quantities w and χ entirely in terms of two analytic functions ϕ and ψ.

2.3 PREREQUISITES

2.3.1 NOTATION AND INTERFACE FORMULATION

Consider a domain in $\mathbb{R}^2$, infinite in extent, containing a single internal elastic inhomogeneity with elastic properties different from those of the surrounding matrix. We represent the matrix by the domain S_1 and assume that the inhomogeneity occupies the region S_2. The curve Γ will denote the inhomogeneity-matrix interface. In what follows, the subscripts 1 and 2 will refer to the regions S_1 and S_2, respectively. The elastic materials occupying the matrix and the inhomogeneity belong to the class of harmonic materials characterized by Equation (2.5) and (2.21) above with associated elastic constants μ_1, α_1, β_1 and μ_2, α_2, β_2, respectively. From the previous section, the (plane) deformation function w in the matrix and in the inhomogeneity can be written, respectively, in terms of two analytic functions ϕ and ψ as:

$$iw_{\gamma}(z) = \alpha_{\gamma}\phi_{\gamma}(z) + i\overline{\psi_{\gamma}(z)} + \frac{\beta_{\gamma}z}{\overline{\phi'_{\gamma}(z)}} \quad \gamma = 1,2 \text{ (no sum over repeated indices).} \tag{2.25}$$

Similarly, the complex Piola stress function χ can be written in terms of φ and ψ in the matrix and in the inhomogeneity as:

$$\chi_{\gamma}(z) = 2\mu_{\gamma}i[(\alpha_{\gamma}-1)\phi_{\gamma}(z) + i\overline{\psi_{\gamma}(z)} + \frac{\beta_{\gamma}z}{\overline{\phi'_{\gamma}(z)}}] \quad \gamma = 1,2 \text{ (no sum over repeated indices).} \tag{2.26}$$

Since, in the present study, we adopt the perfectly bonded interface model that both tractions and displacements are continuous across the interface, the continuity conditions for the displacement and Piola stresses across the (perfectly) bonded interface Γ lead to

$$\alpha_1\phi_1(z) + i\overline{\psi_1(z)} + \frac{\beta_1 z}{\overline{\phi'_1(z)}} = \alpha_2\phi_2(z) + i\overline{\psi_2(z)} + \frac{\beta_2 z}{\overline{\phi'_2(z)}}, \tag{2.27}$$

$$\mu_1[(\alpha_1-1)\phi_1(z) + i\overline{\psi_1(z)} + \frac{\beta_1 z}{\overline{\phi'_1(z)}}] = \mu_2[(\alpha_2-1)\phi_2(z) + i\overline{\psi_2(z)} + \frac{\beta_2 z}{\overline{\phi'_2(z)}}], \tag{2.28}$$

respectively, where $z \in \Gamma$.

2.3.2 CHARACTERIZING REMOTE LOADING AND THE UNDEFORMED STAGE

Earlier, we assumed that the matrix was unbounded. Consequently, it is necessary to impose some asymptotic conditions on the inhomogeneity-matrix system. These conditions will, in reality, represent the remote loading applied to the system when subjected to plane-strain deformations. For example, in Ru et al (2005), the inhomogeneity-matrix system has been studied exclusively in the case of uniform remote loading. This scenario is not sufficiently general to accommodate the many different forms of remote loading to which such a system can be subjected. Consequently, we attempt to analyze the inhomogeneity-matrix system with as general as class of remote loadings as possible.

To consider an elliptic inhomogeneity-matrix system subjected to a class of remote stresses, it is necessary to first identify what possible forms of remote stresses can be imposed. Also, as mentioned before, it is required to ensure that the complex-variable formulations (2.21) and (2.23-24) do in fact admit the undeformed stage when all external and internal forces are removed.

From (2.18) together with (2.26), the Piola stresses in the matrix and in the inhomogeneity can be written, respectively, in terms of the two analytic functions ϕ and ψ as:

$$\begin{aligned}
-\sigma_{21}+i\sigma_{11} &= 2\mu_2 i\left[(\alpha_2-1)\left(\phi_2(z)\right)_{,2}+i\overline{\left(\psi_2(z)\right)_{,2}}+\left(\frac{\beta_2 z}{\overline{\phi_2'(z)}}\right)_{,2}\right], \\
\sigma_{22}-i\sigma_{12} &= 2\mu_2 i\left[(\alpha_2-1)\left(\phi_2(z)\right)_{,1}+i\overline{\left(\psi_2(z)\right)_{,1}}+\left(\frac{\beta_2 z}{\overline{\phi_2'(z)}}\right)_{,1}\right], \\
-\sigma_{21}^{\infty}+i\sigma_{11}^{\infty} &= 2\mu_1 i\left[(\alpha_1-1)\left(\phi_1(z)\right)_{,2}+i\overline{\left(\psi_1(z)\right)_{,2}}+\left(\frac{\beta_1 z}{\overline{\phi_1'(z)}}\right)_{,2}\right], \quad (2.29)\\
\sigma_{22}^{\infty}-i\sigma_{12}^{\infty} &= 2\mu_1 i\left[(\alpha_1-1)\left(\phi_1(z)\right)_{,1}+i\overline{\left(\psi_1(z)\right)_{,1}}+\left(\frac{\beta_1 z}{\overline{\phi_1'(z)}}\right)_{,1}\right].
\end{aligned}$$

Since the Piola stresses from (2.29) are functions of $\phi_1(z)$ and $\psi_1(z)$ and in the view of (2.25), $\psi_1(z)$ is related to $\phi_1(z)$ through $w_1(z)$, it is obvious that the complex function $\phi_1(z)$ dominates the remote stresses. It follows from (2.14) that the equilibrium equation

(2.13) can be rewritten as:

$$\phi_1'(z)_{,1} + i\phi_1'(z)_{,2} = 0. \tag{2.30}$$

Therefore, the complex function $\phi(z)$ must be chosen in such a way to satisfy the equilibrium equation (2.30). One of the successful candidates for the above equation is a monomial representation of z (e. g. z^n). In fact,

Remark 1. *Using the principal of superposition, we find that any series representation of* $\phi(z)$ *(i.e.* $\phi(z) = A_1 z + A_2 z^2 + A_3 z^3 + \cdots,\ A_\gamma \in \mathbb{C},\ (\gamma = 1, 2, 3, \cdots)$*) satisfies the equilibrium equation,*

$$\phi'(z)_{,1} + i\phi'(z)_{,2} = 0,$$

and from (2.14-15) and Knowles & Sternberg (1975),

$$\lambda_1 = |\phi'(z)| = F'(I) \neq 0 \quad \text{throughout the entire plane.}$$

The result in Remark 1 is extremely important in that it allows us to accommodate a large class of possible remote stresses, This statement is based on the well-known result from the theory of functions (Weierstrass) which states that any continuous function in a bounded domain can be uniformly approximated by a polynomial.

From Remark 1 (or (2.14-15)), we must have that, for the undeformed stage characterized by $\lambda_1 = \lambda_2 = 1$,

$$1 = \lambda_1 = F'(I) = |\phi'(z)|. \tag{2.31}$$

Also, from the deformation function (2.25), we require that, for the undeformed stage,

$$w(z) = -i\alpha\phi(z) + \overline{\psi(z)} - \frac{i\beta z}{\overline{\phi'(z)}} = z. \tag{2.32}$$

For the class of harmonic materials considered here, to ensure the desired mechanical behavior (i.e. to ensure that the undeformed stage is included), it is required from (2.21-22) that (Ru, 2002)

$$\alpha + \beta = 1. \tag{2.33}$$

Thus, it follows from (2.31-33) that the stress functions characterizing the undeformed stage are identified as:

$$\phi(z) = iz, \quad \psi(z) = 0. \tag{2.34}$$

Remark 2. *We remark that, for the class of harmonic materials considered here, a state of zero displacement or equivalently, the undeformed stage is characterized by*

$$1 = \left|\phi_\gamma'(z)\right| = F'(I),$$

$$iw_\gamma(z) = \alpha_\gamma \phi_\gamma(z) + i\overline{\psi_\gamma(z)} + \frac{\beta_\gamma z}{\overline{\phi_\gamma'(z)}} = iz, \quad \gamma = 1, 2 \text{ (no sum over repeated indices)},$$

which, in turn leads to

$$\begin{aligned} \phi_1(z) &= iz, \quad \psi_1(z) = 0, \\ \phi_2(z) &= iz, \quad \psi_2(z) = 0. \end{aligned}$$

2.3.3 SUBSIDIARY RESULTS

As stated before, it is of fundamental importance to fully understand the undeformed stage in an attempt to analyze an inhomogeneity-matrix system subjected to nonuniform remote stresses. In the present section, we provide physical interpretations of the undeformed stage in terms of mathematical representations derived in the previous section. We first note from Remark 2 that, for the undeformed stage,

$$\begin{aligned} \phi_1(z) &= iz, \quad \psi_1(z) = 0, \\ \phi_2(z) &= iz, \quad \psi_2(z) = 0, \end{aligned} \tag{2.35}$$

$$\lambda_1 = \lambda_2 = 1.$$

From (2.14) and (2.21) we now have that

$$I = P(|\phi'(z)|) = 2\left(\alpha|\phi'(z)| + \frac{\beta}{|\phi'(z)|}\right). \tag{2.36}$$

Then, the undeformed stage condition yields only that

$$I = P(|i|) = 2(\alpha + \beta) = 2,$$

where

$$1 = \lambda_1 = F'(I) = |\phi'(z)| = |i|, \quad \alpha + \beta = 1, \quad \text{(from 2.33)}.$$

This result can be easily verified directly from (2.14-15) that

$$I = P(|i|) = \lambda_1 + \lambda_2 = 2.$$

The Piola stresses inside the inhomogeneity and at the remote boundary for the undeformed stage can then be evaluated from (2.29), (2.33) and (2.35). That is

$$-\sigma_{21}+i\sigma_{11} = 2\mu_2 i\left[(\alpha_2-1)\, i z_{,2}+\beta_2\left(\frac{1}{-i}\right) z_{,2}\right] = 2\mu_2 i\left[1-(\alpha_2+\beta_2)\right]=0,$$

$$\sigma_{22}-i\sigma_{12} = 2\mu_2 i\left[(\alpha_2-1)\, i z_{,1}+\beta_2\left(\frac{1}{-i}\right) z_{,1}\right] = 2\mu_2 i\left[i\left(\alpha_2+\beta_2-1\right)\right]=0,$$

$$-\sigma_{21}^{\infty}+i\sigma_{11}^{\infty} = 2\mu_1 i\left[(\alpha_1-1)\, i z_{,2}+\beta_1\left(\frac{1}{-i}\right) z_{,2}\right] = 2\mu_1 i\left[1-(\alpha_1+\beta_1)\right]=0,$$

$$\sigma_{22}^{\infty}-i\sigma_{12}^{\infty} = 2\mu_1 i\left[(\alpha_1-1)\, i z_{,1}+\beta_1\left(\frac{1}{-i}\right) z_{,1}\right] = 2\mu_1 i\left[i\left(\alpha_1+\beta_1-1\right)\right]=0,$$

where, $\alpha_\gamma+\beta_\gamma = 1 \quad \gamma=1,2$ (no sum over repeated indices),

(i.e. $\sigma_{11}=\sigma_{22}=\sigma_{12}=\sigma_{21}=\sigma_{11}^{\infty}=\sigma_{22}^{\infty}=\sigma_{21}^{\infty}=\sigma_{12}^{\infty}=0$). For the deformation function, again, from (2.25), we have that

$$w_\gamma(z)=-i\frac{1}{2}iz-\frac{1}{2}\frac{iz}{-i}=z, \quad \gamma=1,2 \text{ (no sum over repeated indices),}$$

which implies that the material points in the reference configuration remain intact. Since we assume that there is no residual stress, the results obtained above are found to be physically correct in that if there is no load applied to the system, then, the corresponding Piola stresses and displacements are equal to zero.

Finally, (2.30) (or (Remark 1)) and (2.35) yield for the undeformed stage:

$$\phi_\gamma'(z)_{,1}+i\phi_\gamma'(z)_{,2}=0, \quad \gamma=1,2 \text{ (no sum over repeated indices),}$$

which clearly, through (2.14), shows that the equilibrium equation (2.13) is satisfied.

Chapter 3
COMPLETE SOLUTION OF AN ELLIPTIC INHOMOGENEITY SUBJECTED TO A GENERAL CLASS OF NONUNIFORM REMOTE LOADINGS IN FINITE ELASTICITY

3.1 INTRODUCTION

The mathematical analysis of problems involving plane-strain deformations of harmonic materials has recently been facilitated by the relatively simple complex-variable formulation presented in Ru (2002). For example,Wang et al (2005) have extended and analyzed the notion of 'harmonic shapes' (holes and inclusions) to this class of materials. In addition, in Ru et al (2005), the authors have used this formulation to study an important class of introductory problems involving elastic inhomogeneities embedded in harmonic materials. More precisely, the authors have obtained complete solutions of problems concerning the plane deformations of an harmonic composite material containing an elastic elliptic inhomogeneity perfectly bonded to a surrounding matrix of similar elastic material. The results presented in Ru et al (2005), however, are limited to the case where the inhomogeneity-matrix system is subjected exclusively to *uniform* remote loading. In many practical problems, however, composite materials, in particular those subjected to finite deformations, are subjected to many different forms of loading most of which are nonuniform in nature. The analysis required to study this more general class of problems, however, is extremely challenging, mainly as a result of the complications arising from the use of conformal mapping techniques with the ensuing nonlinearities in the mapped plane.

In the present Chapter, we overcome the above-mentioned difficulties and address the issues relating to nonuniform remote loading of harmonic composite materials. In particular, we extend the results in Ru et al (2005) to the case of plane finite deformations of a composite material in which an elliptic elastic inhomogeneity is embedded in the same class of harmonic materials under the assumption of nonuniform remote loading (Kim & Schiavone, 2007a). Using complex-variable methods (Muskhelishvili, 1953; England, 1971; Brown & Churchill, 1996), we obtain the complete solution for a perfectly bonded elliptic inhomogeneity when the system is subjected to classes of nonuniform remote stress distributions characterized by stress functions described by general polynomials of degree $n \geq 1$ in the corresponding complex variable z describing the matrix. The results obtained are extremely important in that, essentially, they lead to the solution of a class of problems in which the remote loading is characterized by a much wider and more practically realistic class of functions.

3.2 NOTATION AND PREREQUISITES

Consider a domain in $\mathbb{R}^2$, infinite in extent, containing a single internal elliptic elastic inhomogeneity with elastic properties different from those of the surrounding matrix. We represent the matrix by the domain S_1 and assume that the inhomogeneity occupies the region S_2. The ellipse Γ will denote the inhomogeneity-matrix interface. In what follows, the subscripts 1 and 2 will refer to the regions S_1 and S_2, respectively. The elastic materials occupying the matrix and the inhomogeneity belong to the class of harmonic materials characterized by Equation (2.5) and (2.21) above with associated elastic constants μ_1, α_1, β_1 and μ_2, α_2, β_2, respectively.

In the view of Remarks. 1 and 2, we consider problems corresponding to cases in which the inhomogeneity-matrix system is subjected to a general class of remote loadings described by analytic functions ϕ_1 and ψ_1 of the (polynomial) form:

$$\phi_1(z) = A_1 z + \sum_{n=a}^{b} A_n z^n, \psi_1(z) = B_1 z + \sum_{n=a}^{b} B_n z^n, \quad |z| \to \infty, a \geq 2, b \geq a, \quad (3.1)$$

where A_1, A_n, B_1 and B_n are known complex constants.

3.3 COMPLETE SOLUTION FOR THE ELLIPTIC INHOMOGENEITY

We proceed by first using the mapping function $z = \omega(\xi)$ to map (conformally) the infinite region outside an ellipse in the $z - plane$ to the infinite region outside the unit circle in the

$\xi - plane$. In fact (see Muskhelishvili, 1953)

$$z = \omega(\xi) = R\left(\xi + \frac{m}{\xi}\right), \quad R > 0,\ 1 \geq m \geq 0, \tag{3.2}$$

with $\xi = \rho e^{i\theta}$ and $\rho = 1$ will map the ellipse

$$x_1 = R\left(\rho + \frac{m}{\rho}\right)\cos\theta, \quad x_2 = R\left(\rho - \frac{m}{\rho}\right)\sin\theta, \tag{3.3}$$

(where $R \in \mathbb{R}$ and $\theta \in [0, 2\pi]$) and its exterior region in the complex $z - plane$, onto and outside, respectively, the unit circle in the complex $\xi - plane$.

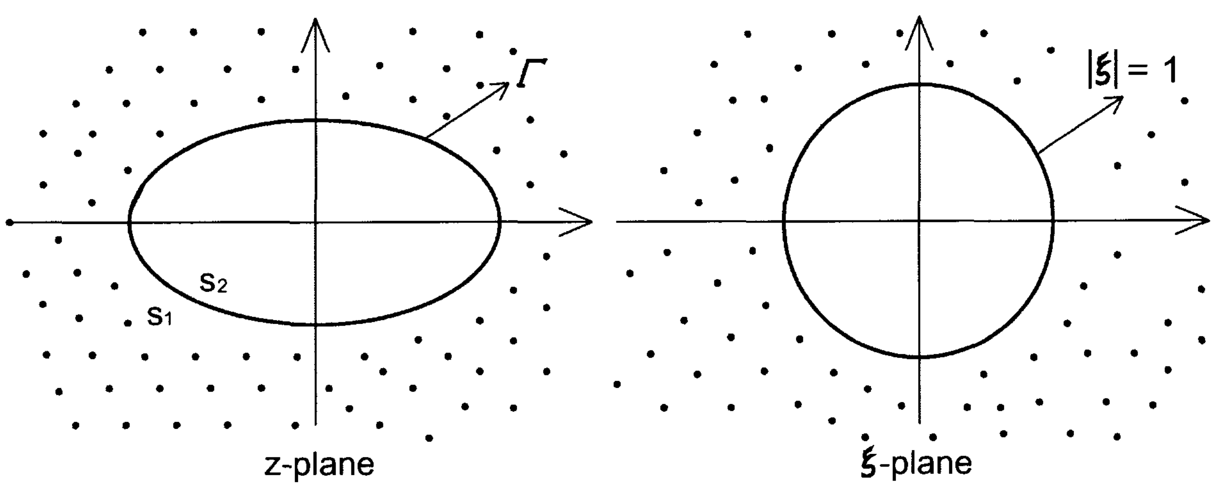

Figure 3: *The Conformal Mapping from $z - plane$ to $\xi - plane$*

In our present case of an elliptic inhomogeneity, we assume that $1 > m \geq 0$ (for $m = 1$, the ellipse becomes a straight slit; for $m = 0$, a circle). Using this mapping function and the binomial theorem, we write z^n in the $\xi - plane$ as

$$z^n = (\omega(\xi))^n = \left(R\left(\xi + \frac{m}{\xi}\right)\right)^n = R^n \sum_{r=0}^{n} \binom{n}{r} \xi^{n-2r} m^r. \tag{3.4}$$

It is clear from (3.4) that each ocurrence of z^n in the governing equations will correspond to $n+1$ terms in the $\xi - plane$. Consequently, we suggest the following representation of the corresponding functions ϕ_2 and ψ_2 inside the inhomogeneity:

$$\begin{aligned} \phi_2(z) &= C_1 z + \sum_{n=a}^{b} \left[C_n z^n + m \sum_{\alpha=0}^{\frac{n-(2+\delta_o)}{2}} C_{n,\alpha} z^{2\alpha+\delta_o} \right] \\ \psi_2(z) &= D_1 z + \sum_{n=a}^{b} \left[D_n z^n + m \sum_{\alpha=0}^{\frac{n-(2+\delta_o)}{2}} D_{n,\alpha} z^{2\alpha+\delta_o} \right], \quad z \in S_2,\ a \geq 2, b \geq a, \end{aligned} \tag{3.5}$$

where $C_1, C_n, C_{n,\alpha}, D_1, D_n$ and $D_{n,\alpha}$ are complex constants (the 'comma' notation used to denote the constants $C_{n,\alpha}$ and $D_{n,\alpha}$ is chosen only for convenience and does not denote differentiation) to be determined and δ_o is defined as:

$$\delta_o = \begin{cases} 1, & n \text{ odd}, \\ 0, & n \text{ even}. \end{cases}$$

From (2.27-28), the interface conditions can now be written in the form

$$\begin{aligned} \phi_1(z) &= K\phi_2(z) + iS\overline{\psi_2(z)} + \frac{S\beta_2 z}{\phi_2'(z)}, \\ \psi_1(z) &= i\left(-\alpha_1\overline{\phi_1(z)} - \frac{\beta_1\overline{z}}{\phi_1'(z)} + \alpha_2\overline{\phi_2(z)} + \frac{\beta_2\overline{z}}{\phi_2'(z)}\right) + \psi_2(z), \quad |z| = \Gamma. \end{aligned} \tag{3.6}$$

For convenience, we write $\phi_\gamma(z) = \phi_\gamma(w(\xi)) = \phi_\gamma(\xi)$, $\gamma = 1,2$ and similarly for the functions ψ_γ. Then, the interface condition (3.6) becomes in the $\xi - plane$:

$$\phi_1(\xi) = K\phi_2(\xi) + iS\overline{\psi_2(\xi)} + \frac{S\beta_2\omega(\xi)\overline{\omega'(\xi)}}{\overline{\phi_2'(\xi)}}, \tag{3.7}$$

$$\psi_1(\xi) = i\left(-\alpha_1\overline{\phi_1(\xi)} - \frac{\beta_1\overline{\omega(\xi)}\omega'(\xi)}{\phi_1'(\xi)} + \alpha_2\overline{\phi_2(\xi)} + \frac{\beta_2\overline{\omega(\xi)}\omega'(\xi)}{\phi_2'(\xi)}\right) + \psi_2(\xi), \quad |\xi| = 1,$$

$$\because \quad \phi_\gamma'(z) = \frac{d\phi(z)}{dz} = \frac{d\phi(\xi)}{d\xi}\frac{d\xi}{dz} = \frac{\phi'(\xi)}{\omega'(\xi)}, \quad \overline{\phi_\gamma'(z)} = \overline{\left(\frac{\phi'(\xi)}{\omega'(\xi)}\right)} = \frac{\overline{\phi'(\xi)}}{\overline{\omega'(\xi)}},$$

where the constants K and S are defined by $K = \frac{\mu_2}{\mu_1} + \alpha_2\left(1 - \frac{\mu_2}{\mu_1}\right)$, $S = \left(1 - \frac{\mu_2}{\mu_1}\right)$.

Remark 3. *Since, from Remark 1*

$$\left|\phi_\gamma'(z)\right| = F'(I) \neq 0, \quad A_1,\ C_1 \neq 0,$$
$$\gamma = 1,2 \text{ (no sum over repeated indices)},$$

and

$$\omega'(\xi) = R\left(1 - \frac{m}{\xi^2}\right) \neq 0, \quad |\xi| \geq 1, \ (\because m \neq 1),$$

it is clear that we must have

$$\phi_2'(\xi) \neq 0$$
$$\iff \overline{\phi_2'(\xi)} \neq 0, \quad |\xi| = 1.$$

In other words, from (3.4-5), we must have that

$$C_1 + \sum_{n=a}^{b}\left[nC_nR^{n-1}\left(\xi + \frac{m}{\xi}\right)^{n-1} + m\sum_{\alpha=0}^{\frac{n-(2+\delta_o)}{2}}\left[(2\alpha+\delta_0)C_{n,\alpha}R^{2\alpha+\delta_o-1}\left(\xi + \frac{m}{\xi}\right)^{2\alpha+\delta_o-1}\right]\right] \neq 0$$

$$\Longleftrightarrow$$

$$\overline{C_1}+\sum_{n=a}^{b}\left[n\overline{C_n}R^{n-1}\left(\overline{\xi}+\frac{m}{\overline{\xi}}\right)^{n-1}+m\sum_{\alpha=0}^{\frac{n-(2+\delta_o)}{2}}\left[(2\alpha+\delta_0)\,\overline{C_{n,\alpha}}R^{2\alpha+\delta_o-1}\left(\overline{\xi}+\frac{m}{\overline{\xi}}\right)^{2\alpha+\delta_o-1}\right]\right]\neq 0.$$

for $|\xi|=1$.

Since $\xi\overline{\xi}=|\xi|^2=1$, on the circular boundary in the $\xi-plane$, it follows from (3.2), (3.5) and (3.7) that the interface condition in the $\xi-plane$ becomes:

$$\phi_1(\xi)=K\left[C_1R\left(\xi+\frac{m}{\xi}\right)+\sum_{n=a}^{b}\left[C_nR^n\left(\xi+\frac{m}{\xi}\right)^n+m\sum_{\alpha=0}^{\frac{n-(2+\delta_o)}{2}}\left[C_{n,\alpha}R^{2\alpha+\delta_o}\left(\xi+\frac{m}{\xi}\right)^{2\alpha+\delta_o}\right]\right]\right]$$

$$+iS\left[\overline{D_1}R\left(\frac{1}{\xi}+\xi m\right)+\sum_{n=a}^{b}\left[\overline{D_n}R^n\left(\frac{1}{\xi}+m\xi\right)^n+m\sum_{\alpha=0}^{\frac{n-(2+\delta_o)}{2}}\left[\overline{C_{n,\alpha}}R^{2\alpha+\delta_o}\left(\frac{1}{\xi}+m\xi\right)^{2\alpha+\delta_o}\right]\right]\right]$$

$$+\frac{S\beta_2R\left(\xi+\frac{m}{\xi}\right)}{\overline{C_1}+\sum_{n=a}^{b}\left[n\overline{C_n}R^{n-1}\left(\frac{1}{\xi}+\xi m\right)^{n-1}+m\sum_{\alpha=0}^{\frac{n-(2+\delta_o)}{2}}\left[(2\alpha+\delta_0)\,\overline{C_{n,\alpha}}R^{2\alpha+\delta_o-1}\left(\frac{1}{\xi}+\xi m\right)^{2\alpha+\delta_o-1}\right]\right]},\tag{3.8}$$

for $|\xi|=1$.

From (3.4) together with (3.8), we finally obtain , for $|\xi|=1$,

$$\phi_1(\xi)=KC_1R\left(\xi+\frac{m}{\xi}\right)+iS\overline{D_1}R\left(\frac{1}{\xi}+\xi m\right)+\sum_{n=a}^{b}\left[R^n\sum_{r=0}^{n}\binom{n}{r}\xi^{n-2r}\left(KC_nm^r+iS\overline{D_n}m^{n-r}\right)\right]$$

$$+\sum_{n=a}^{b}\left[\sum_{\alpha=0}^{\frac{n-(2+\delta_o)}{2}}\left[R^{2\alpha+\delta_o}\sum_{r=0}^{2\alpha+\delta_o}\binom{2\alpha+\delta_o}{r}\xi^{2(\alpha-r)+\delta_o}\left(KC_{n,\alpha}m^{r+1}+iS\overline{D_{n,\alpha}}m^{2\alpha+\delta_o-r+1}\right)\right]\right]+$$

$$\frac{S\beta_2 R\left(\xi+\frac{m}{\xi}\right)}{\overline{C_1}+\sum_{n=a}^{b}\left[n\overline{C_n}R^{n-1}\left(\frac{1}{\xi}+\xi m\right)^{n-1}+m\sum_{\alpha=0}^{\frac{n-(2+\delta_o)}{2}}\left[\left(2\alpha+\delta_0\right)\overline{C_{n,\alpha}}R^{2\alpha+\delta_o-1}\left(\frac{1}{\xi}+\xi m\right)^{2\alpha+\delta_o-1}\right]\right]}, \tag{3.9}$$

where

$$R^n\left(\xi+\frac{m}{\xi}\right)^n = R^n\sum_{r=0}^{n}\binom{n}{r}\xi^{n-2r}m^r,$$

$$R^{2\alpha+\delta_o}\left(\xi+\frac{m}{\xi}\right)^{2\alpha+\delta_o} = R^{2\alpha+\delta_o}\sum_{r=0}^{2\alpha+\delta_o}\binom{2\alpha+\delta_o}{r}\xi^{2(\alpha-r)+\delta_o}m^r$$

$$R^n\left(\frac{1}{\xi}+m\xi\right)^n = R^n\left(m\xi+\frac{1}{\xi}\right)^n = R^n\sum_{r=0}^{n}\binom{n}{r}\xi^{n-2r}m^{n-r}$$

$$R^{2\alpha+\delta_o}\left(\frac{1}{\xi}+m\xi\right)^{2\alpha+\delta_o} = R^{2\alpha+\delta_o}\left(m\xi+\frac{1}{\xi}\right)^{2\alpha+\delta_o} = R^{2\alpha+\delta_o}\sum_{r=0}^{2\alpha+\delta_o}\binom{2\alpha+\delta_o}{r}\xi^{2(\alpha-r)+\delta_o}m^{n-r}.$$

If we adopt the form $\phi_1$from (3.9) for the exterior region $|\xi|>1$, we have that

$$\phi_1(\xi)=\xi\left(KC_1R+iS\overline{D_1}Rm\right)+\frac{1}{\xi}\left(KC_1Rm+iS\overline{D_1}R\right)+SR_{(n,\xi)}$$

$$+\sum_{n=a}^{b}\left[R^n\sum_{r=0}^{n}\left[\binom{n}{r}\xi^{n-2r}X_{n,(n,r)}\right]+\sum_{\alpha=0}^{\frac{n-(2+\delta_o)}{2}}\left[R^{2\alpha+\delta_o}\sum_{r=0}^{2\alpha+\delta_o}\binom{2\alpha+\delta_o}{r}\xi^{2(\alpha-r)+\delta_o}X_{n,(\alpha,r)}\right]\right]$$

$$,|\xi|>1, \tag{3.10}$$

where the constants,

$$X_{n,(n,r)}=KC_nm^r+iS\overline{D_n}m^{n-r},\quad X_{n,(\alpha,r)}=KC_{n,\alpha}m^{r+1}+iS\overline{D_{n,\alpha}}m^{2\alpha+\delta_o-r+1},$$

and

$$R_{(n,\xi)} = \frac{\beta_2 R\left(\xi + \frac{m}{\xi}\right)}{\overline{C_1} + \sum_{n=a}^{b}\left[n\overline{C_n}R^{n-1}\left(\frac{1}{\xi}+\xi m\right)^{n-1} + m\sum_{\alpha=0}^{\frac{n-(2+\delta_0)}{2}}\left[(2\alpha+\delta_0)\overline{C_{n,\alpha}}R^{2\alpha+\delta_o-1}\left(\frac{1}{\xi}+\xi m\right)^{2\alpha+\delta_o-1}\right]\right]}.$$

From (3.1) and (3.4), the remote loading condition in the $\xi - plane$ is taken to be

$$\phi_1(\xi) = A_1 R\xi + \sum_{n=a}^{b}\left[A_n R^n \sum_{r=0}^{\frac{n-\delta_0}{2}}\binom{n}{r}\xi^{n-2r}m^r\right], \quad |\xi| \to \infty, \tag{3.11}$$

Consequently, from (3.10), we have that

$$\phi_1(\xi) = \xi\left(KC_1R + iS\overline{D_1}Rm\right) + \lim_{|\xi|\to\infty}\left(SR_{(n,\xi)}\right)$$

$$+\sum_{n=a}^{b}\left[R^n\sum_{r=0}^{\frac{n-\delta_0}{2}}\left[\binom{n}{r}\xi^{n-2r}X_{n,(n,r)}\right] + \sum_{\alpha=0}^{\frac{n-(2+\delta_o)}{2}}\left[R^{2\alpha+\delta_o}\sum_{r=0}^{\alpha}\binom{2\alpha+\delta_o}{r}\xi^{2(\alpha-r)+\delta_o}X_{n,(\alpha,r)}\right]\right]$$

$$, |\xi| \to \infty, \tag{3.12}$$

$$\because n - 2r \geq \delta_o \implies r \leq \frac{n-\delta_o}{2},$$

$$2(\alpha - r) + \delta_o \geq \delta_o \implies r \leq \alpha, \quad |\xi| \to \infty.$$

Clearly, this must also satisfy the remote loading condition (3.11), from which we derive the equations defining the unknown complex constants:

$$A_1R\xi + \sum_{n=a}^{b}\left[A_nR^n\sum_{r=1}^{\frac{n-\delta_0}{2}}\binom{n}{r}\xi^{n-2r}m^r\right] = \xi\left(KC_1R + iS\overline{D_1}Rm\right) + \lim_{|\xi|\to\infty}\left(SR_{(n,\xi)}\right)$$

$$+\sum_{n=a}^{b}\left[R^n\sum_{r=1}^{\frac{n-\delta_0}{2}}\left[\binom{n}{r}\xi^{n-2r}X_{n,(n,r)}\right] + \sum_{\alpha=0}^{\frac{n-(2+\delta_o)}{2}}\left[R^{2\alpha+\delta_o}\sum_{r=0}^{\alpha}\binom{2\alpha+\delta_o}{r}\xi^{2(\alpha-r)+\delta_o}X_{n,(\alpha,r)}\right]\right], \tag{3.13A}$$

$$\sum_{n=a}^{b} [A_n] \xi^n = \sum_{n=a}^{b} \left[X_{n,(n,r=0)}\right] \xi^n = \sum_{n=a}^{b} \left[KC_n + iS\overline{D_n}m^n\right] \xi^n. \tag{3.13B}$$

Remark 4. *As a direct consequence of Remark 3, we have that*

$$KC_1R + iS\overline{D_1}Rm - \frac{1}{\xi^2}\left(KC_1Rm + iS\overline{D_1}R\right) + SR'(n,\xi)$$
$$+\sum_{n=a}^{b}\left[R^n\sum_{r=0}^{n}(n-2r)\binom{n}{r}\xi^{n-2r-1}X_{n,(n,r)}\right]$$
$$+\sum_{n=a}^{b}\left[\sum_{\alpha=0}^{\frac{n-(2+\delta_o)}{2}}\left[R^{2\alpha+\delta_o}\sum_{r=0}^{2\alpha+\delta_o}(2(\alpha-r)+\delta_o)\binom{2\alpha+\delta_o}{r}\xi^{2(\alpha-r)+\delta_o-1}X_{n,(\alpha,r)}\right]\right] \neq 0,$$
$$\forall\xi : |\xi| \geq 1.$$

This implies, together with Remark 3 and Equations (3.13A-B), that there are restrictions on the constants $A_1, A_n, B_1 and\ B_n$ characterizing the remote stresses.

It remains to determine the analytic function $\psi_1(\xi)$. From the interface conditions (3.7), (3.4-5) and (3.10) it follows that, for $|\xi| = 1$,

$$\psi_1(\xi) = -i\alpha_1\left[\frac{1}{\xi}\left(K\overline{C_1}R - iSD_1Rm\right) + \xi\left(K\overline{C_1}Rm - iSD_1R\right)\right]$$
$$+i\alpha_2\overline{C_1}R\left(\frac{1}{\xi} + m\xi\right) + D_1R\left(\xi + \frac{m}{\xi}\right)$$
$$-i\alpha_1\sum_{n=a}^{b}\left[R^n\sum_{r=0}^{n}\left[\binom{n}{r}\xi^{2r-n}\overline{X_{n,(n,r)}}\right] + \sum_{\alpha=0}^{\frac{n-(2+\delta_o)}{2}}\left[R^{2\alpha+\delta_o}\sum_{r=0}^{2\alpha+\delta_o}\binom{2\alpha+\delta_o}{r}\xi^{2(r-\alpha)-\delta_o}\overline{X_{n,(\alpha,r)}}\right]\right]$$
$$+i\alpha_2\sum_{n=a}^{b}\left[\overline{C_n}R^n\sum_{r=0}^{n}\left[\binom{n}{r}\xi^{2r-n}m^r\right] + \sum_{\alpha=0}^{\frac{n-(2+\delta_o)}{2}}\left[\overline{C_{n,\alpha}}R^{2\alpha+\delta_o}\sum_{r=0}^{2\alpha+\delta_o}\binom{2\alpha+\delta_o}{r}\xi^{2(r-\alpha)-\delta_o}m^{r+1}\right]\right]$$
$$+\sum_{n=a}^{b}[D_nR^n\sum_{r=0}^{n}[\binom{n}{r}\xi^{2r-n}m^{n-r}]$$
$$+\sum_{\alpha=0}^{\frac{n-(2+\delta_o)}{2}}[D_{n,\alpha}R^{2\alpha+\delta_o}\sum_{r=0}^{2\alpha+\delta_o}\binom{2\alpha+\delta_o}{r}\xi^{2(r-\alpha)-\delta_o}m^{2\alpha+\delta_o-r+1}]]$$

$$+\overline{R_{(n,\xi)}}i\left(1-\alpha_1 S\right)-\frac{i\beta_1 R^2\left(\frac{1}{\xi}+m\xi\right)\left(1-\frac{m}{\xi^2}\right)}{\phi_1'\left(\xi\right)}, \tag{3.14}$$

where

$$\overline{R_{(n,\xi)}}=$$

$$\frac{\beta_2 R\left(\frac{1}{\xi}+m\xi\right)}{C_1+\sum_{n=a}^{b}\left[nC_nR^{n-1}\left(\xi+\frac{m}{\xi}\right)^{n-1}+m\sum_{\alpha=0}^{\frac{n-(2+\delta_o)}{2}}\left[\left(2\alpha+\delta_0\right)C_{n,\alpha}R^{2\alpha+\delta_o-1}\left(\xi+\frac{m}{\xi}\right)^{2\alpha+\delta_o-1}\right]\right]},$$

and

$$\phi_1'\left(\xi\right)=KC_1R+iS\overline{D_1}Rm-\frac{1}{\xi^2}\left(KC_1Rm+iS\overline{D_1}R\right)+SR'\left(n,\xi\right)$$
$$+\sum_{n=a}^{b}\left[R^n\sum_{r=0}^{n}\left[\left(n-2r\right)\binom{n}{r}\xi^{n-2r-1}X_{n,(n,r)}\right]\right]$$
$$+\sum_{n=a}^{b}\left[\sum_{\alpha=0}^{\frac{n-(2+\delta_o)}{2}}\left[R^{2\alpha+\delta_o}\sum_{r=0}^{2\alpha+\delta_o}\left(2\left(\alpha-r\right)+\delta_o\right)\binom{2\alpha+\delta_o}{r}\xi^{2(\alpha-r)+\delta_o-1}X_{n,(\alpha,r)}\right]\right],$$

and

$$R^n\left(\bar{\xi}+\frac{m}{\bar{\xi}}\right)^n = R^n\left(\frac{1}{\xi}+m\xi\right)^n=R^n\sum_{r=0}^{n}\binom{n}{r}\xi^{2r-n}m^r,$$

$$R^{2\alpha+\delta_o}\left(\bar{\xi}+\frac{m}{\bar{\xi}}\right)^{2\alpha+\delta_o} = R^{2\alpha+\delta_o}\left(\frac{1}{\xi}+m\xi\right)^{2\alpha+\delta_o}=R^{2\alpha+\delta_o}\sum_{r=0}^{2\alpha+\delta_o}\binom{2\alpha+\delta_o}{r}\xi^{\delta_o-2(\alpha-r)}m^r$$

$$R^n\left(m\bar{\xi}+\frac{1}{\bar{\xi}}\right)^n = R^n\left(\frac{m}{\xi}+\xi\right)^n=R^n\sum_{r=0}^{n}\binom{n}{r}\xi^{2r-n}m^{n-r}$$

$$R^{2\alpha+\delta_o}\left(m\bar{\xi}+\frac{1}{\bar{\xi}}\right)^{2\alpha+\delta_o} = R^{2\alpha+\delta_o}\sum_{r=0}^{2\alpha+\delta_o}\binom{2\alpha+\delta_o}{r}\xi^{\delta_o-2(\alpha-r)}m^{n-r},\quad \because \xi\bar{\xi}=\left|\xi\right|^2=1.$$

From (3.1) and (3.4), ψ_1 must also satisfy the asymptotic condition

$$\psi_1(z) = B_1 R\xi + \sum_{n=a}^{b}\left[B_n R^n \sum_{r=\frac{n+\delta_o}{2}}^{n} \binom{n}{r} \xi^{2r-n} m^{n-r}\right], \quad |\xi| \to \infty.$$

Hence, If we adopt ψ_1 from (3.14) for $|\xi| > 1$, we require that

$$B_1 R\xi + \sum_{n=a}^{b}\left[B_n R^n \sum_{r=\frac{n+\delta_o}{2}}^{n-1} \binom{n}{r} \xi^{2r-n} m^{n-r}\right] = \xi\left[-i\alpha_1\left(K\overline{C_1}Rm - iSD_1R\right) + i\alpha_2\overline{C_1}Rm + D_1R\right]$$

$$+\sum_{n=a}^{b}\left[R^n \sum_{r=\frac{n+\delta_o}{2}}^{n-1} \binom{n}{r} \xi^{2r-n}\left(-i\alpha_1\overline{X_{n,(n,r)}} + i\alpha_2\overline{C_n}m^r + D_n m^{n-r}\right)\right] +$$

$$\sum_{n=a}^{b}\left[\sum_{\alpha=0}^{\frac{n-(2+\delta_o)}{2}}\left[R^{2\alpha+\delta_o}\sum_{r=\alpha+\delta_o}^{2\alpha+\delta_o}\binom{2\alpha+\delta_o}{r}\xi^{2(r-\alpha)-\delta_o}\left(-i\alpha_1\overline{X_{n,(\alpha,r)}} + i\alpha_2\overline{C_{n,\alpha}}m^{r+1} + D_{n,\alpha}m^{2\alpha+\delta_o-r+1}\right)\right]\right]$$

$$+\lim_{|\xi|\to\infty}\left(\overline{R_{(n,\xi)}}i\left(1-\alpha_1 S\right)\right) - \lim_{|\xi|\to\infty}\left(\frac{i\beta_1 R^2\left(\frac{1}{\xi}+m\xi\right)\left(1-\frac{m}{\xi^2}\right)}{\phi_1'(\xi)}\right), \qquad (3.15\text{ A})$$

$$\sum_{n=a}^{b}\left[B_n\right]\xi^n = \sum_{n=a}^{b}\left[-i\alpha_1\overline{X_{n,(n,r=n)}} + i\alpha_2\overline{C_n}m^n + D_n\right]\xi^n, \qquad (3.15\text{B})$$

$$\because 2r - n \geq \delta_o \implies r \geq \frac{n+\delta_o}{2},$$

$$2(r-\alpha) - \delta_o \geq \delta_o \implies r \geq \alpha + \delta_o, \quad |\xi| \to \infty,$$

where $X_{n,(n,r)}$ and $X_{n,(\alpha,r)}$ are defined in (3.10).

SUMMARY

The complete solution of the inhomogeneity-matrix system subjected to the general class of remote loadings described by (3.1) is given by

$$\phi_2(z) = C_1 z + \sum_{n=a}^{b}\left[C_n z^n + m\sum_{\alpha=0}^{\frac{n-(2+\delta_o)}{2}} C_{n,\alpha} z^{2\alpha+\delta_o}\right] \tag{3.16}$$

$$\psi_2(z) = D_1 z + \sum_{n=a}^{b}\left[D_n z^n + m\sum_{\alpha=0}^{\frac{n-(2+\delta_o)}{2}} D_{n,\alpha} z^{2\alpha+\delta_o}\right],\ z\in S_2,\ a\geq 2, b\geq a,$$

$$\phi_1(\xi) = \xi\left(KC_1R + iS\overline{D_1}Rm\right) + \frac{1}{\xi}\left(KC_1Rm + iS\overline{D_1}R\right) + SR_{(n,\xi)} \tag{3.17}$$

$$+\sum_{n=a}^{b}\left[R^n\sum_{r=0}^{n}\left[\binom{n}{r}\xi^{n-2r}X_{n,(n,r)}\right] + \sum_{\alpha=0}^{\frac{n-(2+\delta_o)}{2}}\left[R^{2\alpha+\delta_o}\sum_{r=0}^{2\alpha+\delta_o}\binom{2\alpha+\delta_o}{r}\xi^{2(\alpha-r)+\delta_o}X_{n,(\alpha,r)}\right]\right]$$
$$,|\xi|>1,$$

and

$$\psi_1(\xi) = -i\alpha_1\left[\frac{1}{\xi}\left(K\overline{C_1}R - iSD_1Rm\right) + \xi\left(K\overline{C_1}Rm - iSD_1R\right)\right]$$

$$+i\alpha_2\overline{C_1}R\left(\frac{1}{\xi}+m\xi\right) + D_1R\left(\xi+\frac{m}{\xi}\right)$$

$$-i\alpha_1\sum_{n=a}^{b}\left[R^n\sum_{r=0}^{n}\left[\binom{n}{r}\xi^{2r-n}\overline{X_{n,(n,r)}}\right] + \sum_{\alpha=0}^{\frac{n-(2+\delta_o)}{2}}\left[R^{2\alpha+\delta_o}\sum_{r=0}^{2\alpha+\delta_o}\binom{2\alpha+\delta_o}{r}\xi^{2(r-\alpha)-\delta_o}\overline{X_{n,(\alpha,r)}}\right]\right]$$

$$+i\alpha_2\sum_{n=a}^{b}\left[\overline{C_n}R^n\sum_{r=0}^{n}\left[\binom{n}{r}\xi^{2r-n}m^r\right] + \sum_{\alpha=0}^{\frac{n-(2+\delta_o)}{2}}\left[\overline{C_{n,\alpha}}R^{2\alpha+\delta_o}\sum_{r=0}^{2\alpha+\delta_o}\binom{2\alpha+\delta_o}{r}\xi^{2(r-\alpha)-\delta_o}m^{r+1}\right]\right]$$

$$+\sum_{n=a}^{b}[D_nR^n\sum_{r=0}^{n}[\binom{n}{r}\xi^{2r-n}m^{n-r}]$$

$$+\sum_{\alpha=0}^{\frac{n-(2+\delta_o)}{2}}\left[D_{n,\alpha}R^{2\alpha+\delta_o}\sum_{r=0}^{2\alpha+\delta_o}\binom{2\alpha+\delta_o}{r}\xi^{2(r-\alpha)-\delta_o}m^{2\alpha+\delta_o-r+1}\right]]$$

$$+\overline{R_{(n,\xi)}}i\left(1-\alpha_1 S\right)-\frac{i\beta_1 R^2\left(\frac{1}{\xi}+m\xi\right)\left(1-\frac{m}{\xi^2}\right)}{\phi_1'(\xi)}, \tag{3.18}$$

where, again, $|\xi|>1$. Here, the unknown complex constants $C_1, C_n, C_{n,\alpha}, D_1, D_n$ and $D_{n,\alpha}$ are determined by the equations

$$A_1 R\xi+\sum_{n=a}^{b}\left[A_n R^n\sum_{r=1}^{\frac{n-\delta_0}{2}}\binom{n}{r}\xi^{n-2r}m^r\right]=\xi\left(KC_1R+iS\overline{D_1}Rm\right)+\lim_{|\xi|\to\infty}\left(SR_{(n,\xi)}\right)$$

$$+\sum_{n=a}^{b}\left[R^n\sum_{r=1}^{\frac{n-\delta_0}{2}}\left[\binom{n}{r}\xi^{n-2r}X_{n,(n,r)}\right]+\sum_{\alpha=0}^{\frac{n-(2+\delta_o)}{2}}\left[R^{2\alpha+\delta_o}\sum_{r=0}^{\alpha}\binom{2\alpha+\delta_o}{r}\xi^{2(\alpha-r)+\delta_o}X_{n,(\alpha,r)}\right]\right], \tag{3.19A}$$

$$\sum_{n=a}^{b}\left[A_n\right]\xi^n=\sum_{n=a}^{b}\left[X_{n,(n,r=0)}\right]\xi^n=\sum_{n=a}^{b}\left[KC_n+iS\overline{D_n}m^n\right]\xi^n, \tag{3.19B}$$

$$B_1R\xi+\sum_{n=a}^{b}\left[B_nR^n\sum_{r=\frac{n+\delta_o}{2}}^{n-1}\binom{n}{r}\xi^{2r-n}m^{n-r}\right]=\xi\left[-i\alpha_1\left(K\overline{C_1}Rm-iSD_1R\right)+i\alpha_2\overline{C_1}Rm+D_1R\right]$$

$$+\sum_{n=a}^{b}\left[R^n\sum_{r=\frac{n+\delta_o}{2}}^{n-1}\binom{n}{r}\xi^{2r-n}\left(-i\alpha_1\overline{X_{n,(n,r)}}+i\alpha_2\overline{C_n}m^r+D_nm^{n-r}\right)\right]+$$

$$\sum_{n=a}^{b}\left[\sum_{\alpha=0}^{\frac{n-(2+\delta_o)}{2}}\left[R^{2\alpha+\delta_o}\sum_{r=\alpha+\delta_o}^{2\alpha+\delta_o}\binom{2\alpha+\delta_o}{r}\xi^{2(r-\alpha)-\delta_o}\left(-i\alpha_1\overline{X_{n,(\alpha,r)}}+i\alpha_2\overline{C_{n,\alpha}}m^{r+1}+D_{n,\alpha}m^{2\alpha+\delta_o-r+1}\right)\right]\right]$$

$$+\lim_{|\xi|\to\infty}\left(\overline{R_{(n,\xi)}}i\left(1-\alpha_1S\right)\right)-\lim_{|\xi|\to\infty}\left(\frac{i\beta_1R^2\left(\frac{1}{\xi}+m\xi\right)\left(1-\frac{m}{\xi^2}\right)}{\phi_1'(\xi)}\right), \tag{3.19C}$$

$$\sum_{n=a}^{b}\left[B_n\right]\xi^n=\sum_{n=a}^{b}\left[-i\alpha_1\overline{X_{n,(n,r=n)}}+i\alpha_2\overline{C_n}m^n+D_n\right]\xi^n, \tag{3.19D}$$

where, from Remarks. 3 and 4, we must have that

$$A_1,\ C_1 \neq 0,$$

$$C_1+\sum_{n=a}^{b}\left[nC_nR^{n-1}\left(\xi+\frac{m}{\xi}\right)^{n-1}+m\sum_{\alpha=0}^{\frac{n-(2+\delta_o)}{2}}\left[(2\alpha+\delta_0)\,C_{n,\alpha}R^{2\alpha+\delta_o-1}\left(\xi+\frac{m}{\xi}\right)^{2\alpha+\delta_o-1}\right]\right]\neq 0$$

$$\Longleftrightarrow$$

$$\overline{C_1}+\sum_{n=a}^{b}\left[n\overline{C_n}R^{n-1}\left(\overline{\xi}+\frac{m}{\overline{\xi}}\right)^{n-1}+m\sum_{\alpha=0}^{\frac{n-(2+\delta_o)}{2}}\left[(2\alpha+\delta_0)\,\overline{C_{n,\alpha}}R^{2\alpha+\delta_o-1}\left(\overline{\xi}+\frac{m}{\overline{\xi}}\right)^{2\alpha+\delta_o-1}\right]\right]\neq 0,$$

$$\text{for } |\xi|=1, \text{ and}$$

$$KC_1R+iS\overline{D_1}Rm-\frac{1}{\xi^2}\left(KC_1Rm+iS\overline{D_1}R\right)+SR'(n,\xi)$$
$$+\sum_{n=a}^{b}\left[R^n\sum_{r=0}^{n}(n-2r)\binom{n}{r}\xi^{n-2r-1}X_{n,(n,r)}\right]$$
$$+\sum_{n=a}^{b}\left[\sum_{\alpha=0}^{\frac{n-(2+\delta_o)}{2}}\left[R^{2\alpha+\delta_o}\sum_{r=0}^{2\alpha+\delta_o}(2(\alpha-r)+\delta_o)\binom{2\alpha+\delta_o}{r}\xi^{2(\alpha-r)+\delta_o-1}X_{n,(\alpha,r)}\right]\right]\neq 0,$$
$$\forall\xi:|\xi|\geq 1.$$

This solution is rather complicated since it accommodates an extremely general class of remote loadings. However, the computation of the complete solution from this more general framework is not difficult when considering particular cases. For example, in Section 3.5, we demonstrate the case in which the system is subjected to a 'simple' nonuniform remote loading characterized by stress functions $\phi_1(z) = A_1z + A_3z^3$, $\psi_1(z) = B_1z + B_3z^3$, $|z| \to \infty$, in the matrix. Before, we do this, however, we use the above solution to derive general expressions for the important mechanical quantities describing the system.

3.4 FURTHER DISCUSSION

As a consequence of the results obtained above, we can draw the following interesting conclusions.

3.4.1 Expressions for I and J Inside the Elliptic Inhomogeneity

From (2.1) and (2.6), R and J can be calculated as follow:

$$J = \det \mathbf{F} = -\operatorname{Im}\left[w_{,1}\overline{w}_{,2}\right] = y_{1,1}y_{2,2} - y_{1,2}y_{2,1}, \tag{3.20}$$

$$R = tr(\mathbf{F}\mathbf{F}^T) = \left|w_{,1}\right|^2 + \left|w_{,2}\right|^2 = y_{p,s}y_{p,s}\ , ...p, s = 1, 2, \tag{3.21}$$

Hence, in the view of (2.1), Equation (2.5) yields

$$\begin{aligned} I &= \sqrt{R + 2J} = \sqrt{tr(\mathbf{F}\mathbf{F}^T) + 2J}, \\ &= \sqrt{\left(y_{1,1} + y_{2,2}\right)^2 + \left(y_{1,2} - y_{2,1}\right)^2}, \end{aligned} \tag{3.22}$$

where

$$\left(y_{1,1} + y_{2,2}\right)^2 + \left(y_{1,2} - y_{2,1}\right)^2 = 2\left(y_{1,1}y_{2,2} - y_{1,2}y_{2,1},\right) + y_{11}^2 + y_{22}^2 + y_{12}^2 + y_{21}^2.$$

Equation (2.25) and (3.16) give us that

$$\begin{aligned} w_2(z) &= -i\alpha_2\left[C_1 z + \sum_{n=a}^{b}\left(C_n z^n + m\sum_{\alpha=0}^{\frac{n-(2+\delta_o)}{2}} C_{n,\alpha} z^{2\alpha+\delta_o}\right)\right] \\ &+\overline{D}_1\overline{z} + \sum_{n=a}^{b}\left(\overline{D}_n\overline{z}^n + m\sum_{\alpha=0}^{\frac{n-(2+\delta_o)}{2}} \overline{D_{n,\alpha}}\overline{z}^{2\alpha+\delta_o}\right) \\ &- \frac{i\beta_2 z}{\overline{C_1} + \sum_{n=a}^{b}\left(n\overline{C_n}\overline{z}^{n-1} + m\sum_{\alpha=0}^{\frac{n-(2+\delta_o)}{2}}\left(2\alpha+\delta_o\right)\overline{C_{n,\alpha}}\overline{z}^{2\alpha+\delta_o-1}\right)}. \end{aligned} \tag{3.23}$$

Consequently, from (2.14) and (2.21), or directly from (2.36) I has the following expression within the inhomogeneity

$$I = P\left(\left|\phi_2'(z)\right|\right) = 2\left(\alpha\left|\phi_2'(z)\right| + \frac{\beta}{\left|\phi_2'(z)\right|}\right).$$

Therefore, from(3.16), we have that

$$\begin{aligned} I &= 2[\alpha_2 \left| C_1 + \sum_{n=a}^{b} [nC_n z^{n-1} + m \sum_{\alpha=0}^{\frac{n-(2+\delta_o)}{2}} (2\alpha+\delta_o)\, C_{n,\alpha} z^{2\alpha+\delta_o-1}] \right| \\ &+ \frac{\beta_2}{\left| C_1 + \sum_{n=a}^{b} [nC_n z^{n-1} + m \sum_{\alpha=0}^{\frac{n-(2+\delta_o)}{2}} (2\alpha+\delta_o)\, C_{n,\alpha} z^{2\alpha+\delta_o-1}] \right|}], \end{aligned} \tag{3.24}$$

and from Remark 3.

$$\left|\phi_2'(z)\right| = \left| C_1 + \sum_{n=a}^{b} \left[nC_n z^{n-1} + m \sum_{\alpha=0}^{\frac{n-(2+\delta_o)}{2}} (2\alpha+\delta_o)\, C_{n,\alpha} z^{2\alpha+\delta_o-1} \right] \right| \neq 0,$$

$$I^2 - 16\alpha_2\beta_2 = 4\left[\alpha_2\left|\phi_2'(z)\right| - \frac{\beta_2}{\left|\phi_2'(z)\right|}\right]^2 > 0. \tag{3.25}$$

which guarantees that $F'(I)$, from (2.21) is well-defined.

3.4.2 Stress Distribution

From (2.18), the Piola stresses inside the inhomogeneity and at the remote boundary are given by

$$\begin{aligned} -\sigma_{21} + i\sigma_{11} &= (\chi_2)_{,2}, \quad \sigma_{22} - i\sigma_{12} = (\chi_2)_{,1}, \\ -\sigma_{21}^{\infty} + i\sigma_{11}^{\infty} &= (\chi_1)_{,2}, \quad \sigma_{22}^{\infty} - i\sigma_{12}^{\infty} = (\chi_1)_{,1}. \end{aligned} \tag{3.26}$$

From Equation (2.26), (3.1) and (3.16)

$$\chi_1(z) = 2\mu_1 i[(\alpha_1 - 1)(A_1 z + \sum_{n=a}^{b} A_n z^n) + i(\overline{B_1}\overline{z} + \sum_{n=a}^{b} \overline{B_n}\overline{z}^n) + \frac{\beta_1 z}{\overline{A_1} + \sum_{n=a}^{b} n\overline{A_n}\overline{z}^{n-1}}],$$

$$\begin{aligned} \chi_2(z) &= 2\mu_2 i[(\alpha_2 - 1)(C_1 z + \sum_{n=a}^{b}(C_n z^n + m \sum_{\alpha=0}^{\frac{n-(2+\delta_o)}{2}} C_{n,\alpha} z^{2\alpha+\delta_o})) \\ &+ i(\overline{D}_1\overline{z} + \sum_{n=a}^{b}(\overline{D}_n\overline{z}^n + m \sum_{\alpha=0}^{\frac{n-(2+\delta_o)}{2}} \overline{D_{n,\alpha}}\overline{z}^{2\alpha+\delta_o})) \\ &+ \frac{\beta_2 z}{\overline{C_1} + \sum_{n=a}^{b}(n\overline{C_n}\overline{z}^{n-1} + m\sum_{\alpha=0}^{\frac{n-(2+\delta_o)}{2}} (2\alpha+\delta_o)\overline{C_{n,\alpha}}\overline{z}^{2\alpha+\delta_o-1})}]. \end{aligned} \tag{3.27}$$

so that the Piola stresses both inside the inhomogeneity and at the remote boundary can be calculated from (3.19A-D), (3.26-27).

In addition, from (2.12) the Cauchy stresses inside the inhomogeneity are given by

$$\tau_{11} + \tau_{22} = 2\mu_2[\frac{IF'(I)}{J} - 2],$$

$$\tau_{11} - \tau_{22} + 2i\tau_{12} = 2\mu_2\frac{F'(I)}{IJ}[(w_2)_{,2}^2 + (w_2)_{,1}^2].$$

In our case, from Equation (2.14) and (3.20-25)

$$F'(I) = |\phi_2'(z)| = \left| C_1 + \sum_{n=a}^{b}\left[nC_n z^{n-1} + m \sum_{\alpha=0}^{\frac{n-(2+\delta_o)}{2}} (2\alpha+\delta_o) C_{n,\alpha} z^{2\alpha+\delta_o-1}\right]\right| \neq 0.$$

from which the Cauchy stresses can be calculated.

3.5 EXAMPLE: The Case $a = b = 3,\ \delta_o = 1$: Quadratic Remote Loading

The general polynomial solutions mentioned above can accommodate many different forms of applied remote loading. Among these, we consider the particular class of remote loading, characterized by the stress functions $\phi_1(z) = A_1 z + A_3 z^3$, $\psi_1(z) = B_1 z + B_3 z^3$, $|z| \to \infty$.

3.5.1 Complete Solutions

From (3.1) and (3.16), the stress functions at the remote boundary and inside the inhomogeneity are respectively given by

$$\begin{aligned} \phi_1(z) &= A_1 z + A_3 z^3, \quad \psi_1(z) = B_1 z + B_3 z^3, \quad |z| \to \infty, \\ \phi_2(z) &= C_1 z + C_3 z^3 + mzC_{3,0}, \quad \psi_2(z) = D_1 z + D_3 z^3 + mzD_{3,0}. \end{aligned} \tag{3.28}$$

From the general solution (3.19A-D), the unknown complex constants $C_1, C_3, C_{3,0}, D_1, D_3$ and $D_{3,0}$ are completely determined by the equations

$$A_1 R\xi + 3A_3 R^3 m\xi = \xi\left(KC_1 R + iS\overline{D_1}Rm\right) + \lim_{|\xi|\to\infty}\left(S\frac{\beta_2 R\left(\xi + \frac{m}{\xi}\right)}{\overline{C_1} + 3\overline{C_3}R^2\left(\frac{1}{\xi} + m\xi\right)^2 + m\overline{C_{3,0}}}\right)$$

$$+3R^3\xi X_{n=3,(n=3,r=1)} + R\xi X_{n=3,(\alpha=0,r=0)}, \tag{3.29A}$$

$$B_1 R\xi + 3B_3 R^3 \xi m = \xi\left[-i\alpha_1\left(K\overline{C_1}Rm - iSD_1 R\right) + i\alpha_2\overline{C_1}Rm + D_1 R\right]$$
$$+ \lim_{|\xi|\to\infty}\left(i(1-\alpha_1 S)\frac{\beta_2 R\left(\frac{1}{\xi} + m\xi\right)}{C_1 + 3C_3 R^2\left(\xi + \frac{m}{\xi}\right)^2 + mC_{3,0}}\right)$$

$$-\lim_{|\xi|\to\infty}\left(\frac{i\beta_1 R^2\left(\frac{1}{\xi}+m\xi\right)\left(1-\frac{m}{\xi^2}\right)}{\phi_1'(\xi)}\right)+3R^3\xi\left(-i\alpha_1\overline{X_{n=3,(n=3,r=2)}}+i\alpha_2\overline{C_3}m^2+D_3m\right)$$

$$+R\xi\left(-i\alpha_1\overline{X_{n=3,(\alpha=0,r=1)}}+i\alpha_2\overline{C_{3,0}}m^2+D_{3,0}m\right), \qquad (3.29B)$$

$$A_3 = KC_3 + iS\overline{D_3}m^3, \quad B_3 = -i\alpha_1\overline{X_{3,(n=3,r=3)}} + i\alpha_2\overline{C_3}m^3 + D_3, \qquad (3.29C)$$

where

$$\phi_1'(\xi) = KC_1R + iS\overline{D_1}Rm - \frac{1}{\xi^2}\left(KC_1Rm + iS\overline{D_1}R\right)$$

$$+S\left(\frac{\beta_2 R\left(\xi+\frac{m}{\xi}\right)}{\overline{C_1}+3\overline{C_3}R^2\left(\frac{1}{\xi}+m\xi\right)^2+m\overline{C_{3,0}}}\right)'$$

$$+R^3\sum_{r=0}^{3}\left[\binom{3}{r}(3-2r)X_{n=3,(n=3,r)}\xi^{2(1-r)}\right]+R\sum_{r=0}^{1}\left[\binom{1}{r}(1-2r)\xi^{-2r}X_{n=3,(\alpha=0,r)}\right],$$

and $A_1, A_2, B_1$and B_2 are prescribed complex constants. Assume $C_3,\ C_{3,0} \neq 0$, then from (3.10), we finally have that

$$A_\gamma = KC_\gamma + iS\overline{D_\gamma}m^\gamma, \quad B_\gamma = -i\alpha_1\left[K\overline{C_\gamma}m^\gamma - iSD_\gamma\right] + i\alpha_2\overline{C_\gamma}m^\gamma + D_\gamma,$$

$$\gamma = 1, 3 \ \text{(no sum over repeated indices)},$$

$$3A_3R^2m = 3R^2\left[KC_3m + iS\overline{D_3}m^2\right] + KC_{3,0}m + iS\overline{D_{3,0}}m^2,$$

$$\begin{aligned}3B_3R^2m &= 3R^2\left[-i\alpha_1\left(K\overline{C_3}m^2 - iSD_3m\right) + i\alpha_2\overline{C_3}m^2 + D_3m\right]\\ &\quad -i\alpha_1\left(K\overline{C_{3,0}}m^2 - iSD_{3,0}m\right) + i\alpha_2\overline{C_{3,0}}m^2 + D_{3,0}m.\end{aligned}$$

From Remarks. 3 and 4, we must have

$$C_1 + 3C_3R^2\left(\xi + \frac{m}{\xi}\right)^2 + mC_{3,0} \neq 0 \tag{3.30}$$

$$\Longleftrightarrow \overline{C_1} + 3\overline{C_3}R^2\left(\overline{\xi} + \frac{m}{\overline{\xi}}\right)^2 + m\overline{C_{3,0}} \neq 0, \quad |\xi| = 1,$$

and, in the surrounding matrix

$$KC_1R + iS\overline{D_1}Rm - \frac{1}{\xi^2}\left(KC_1Rm + iS\overline{D_1}R\right) + S\left(\frac{\beta_2R\left(\xi + \frac{m}{\xi}\right)}{\overline{C_1} + 3\overline{C_3}R^2\left(\frac{1}{\xi} + m\xi\right)^2 + m\overline{C_{3,0}}}\right)'$$

$$+R^3\sum_{r=0}^{3}\left[\binom{3}{r}(3-2r)\,X_{n=3,(n=3,r)}\xi^{2(1-r)}\right]$$

$$+R\sum_{r=0}^{1}\left[\binom{1}{r}(1-2r)\,\xi^{-2r}X_{n=3,(\alpha=0,r)}\right] \neq 0,\ \forall\xi : |\xi| \geq 1. \tag{3.31}$$

3.5.2 Determination of I and J Inside the Elliptic Inhomogeneity

From (3.23), the deformation functions now take the form

$$w_2 = -i\alpha_2[(C_1 + mC_{3,0})\,z + C_3z^3] + \left(\overline{D_1} + m\overline{D_{3,0}}\right)\overline{z} + \overline{D_3}\overline{z}^3 - \frac{i\beta_2 z}{\overline{C_1} + m\overline{C_{3,0}} + 3\overline{C_3}\overline{z}^2}, \tag{3.32}$$

and from (3.24)

$$I = 2\left(\alpha_2\left|C_1 + mC_{3,0} + 3C_3z^2\right| + \frac{\beta_2}{|C_1 + mC_{3,0} + 3C_3z^2|}\right),$$

$$|\phi_2'(z)| = \left|C_1 + mC_{3,0} + 3C_3z^2\right| \neq 0, \text{(from Remark 3)},$$

$$I^2 - 16\alpha_2\beta_2 = 4\left(\alpha_2\left|C_1 + mC_{3,0} + 3C_3z^2\right| - \frac{\beta_2}{|C_1 + mC_{3,0} + 3C_3z^2|}\right)^2 \geqslant 0, \tag{3.33}$$

from which I can be calculated. Henceforth, $F'(I)$, from (2.21) is well defined. Also J can now be determined from (3.20) together with (3.32).

$$J = -\operatorname{Im}\left[(w_2)_{,1}(\overline{w}_2)_{,2}\right],$$

where

$$w_2 = -i\alpha_2[(C_1 + mC_{3,0})\,z + C_3 z^3] + \left(\overline{D_1} + m\overline{D_{3,0}}\right)\bar{z} + \overline{D_3}\bar{z}^3 - \frac{i\beta_2 z}{\overline{C_1} + m\overline{C_{3,0}} + 3\overline{C_3}\bar{z}^2}.$$

3.5.3 Deformed Contour

From (3.32), we can easily plot an example of the deformed contour (see Figure 4). The fact that the corresponding complex constants $C_1, C_3, C_{3,0}, D_1, D_2$ and $D_{3,0}$ satisfy (3.30-31) guarantee that there is no overlapping.

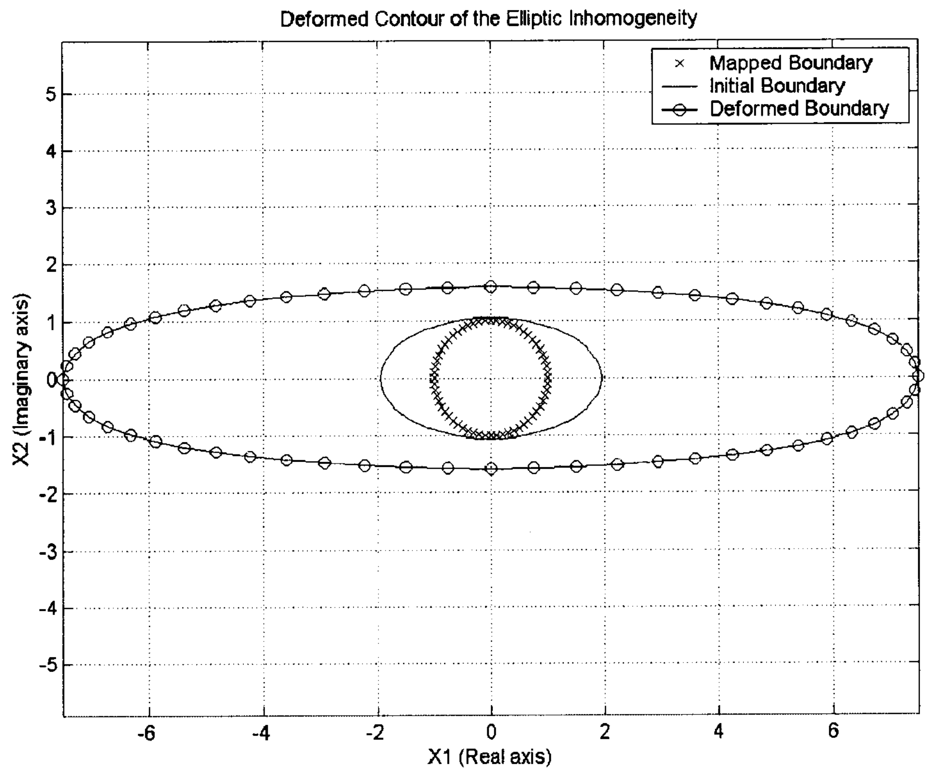

Figure 4: *Example of a Deformed Elliptic Inhomogeneity Subjected to a Remote Loading Characterized by (3.28).* Here, we have taken $(A_1 = 0.75i,\ A_3 = -0.02i,\ B_1 = 2.4,\ B_3 = -0.01,\ \mu_1 = 0.27,\ \mu_2 = 0.3,\ \alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 0.5)$

3.5.4 Admissibility Conditions

From (2.1), (3.20) and (3.32-33), inside the inhomogeneity,

$$I^2 > 16\alpha_2\beta_2, \quad J > 0. \tag{3.34}$$

As in Ru (2002), (3.34) again ensure that there is no overlapping of the deformation field (interior to the elliptic inhomogeneity).

3.5.5 The Piola Stress Distribution

From (3.27), the Piola stresses in the matrix and in the inhomogeneity are described, respectively, by:

$$\chi_1(z) = 2\mu_1 i[(\alpha_1 - 1)(A_1 z + A_3 z^3) + i(\overline{B_1}\bar{z} + \overline{B_3}\bar{z}^3) + \frac{\beta_1 z}{\overline{A_1} + 3\overline{A_3}\bar{z}^2}],$$

$$\begin{aligned}\chi_2(z) = {} & 2\mu_2 i[(\alpha_2 - 1)((C_1 + mC_{3,0})\, z + C_3 z^3) + \\ & i((\overline{D_1} + m\overline{D_{3,0}})\, \bar{z} + \overline{D_3}\bar{z}^3) + \frac{\beta_2 z}{\overline{C_1} + m\overline{C_{3,0}} + 3\overline{C_3}\bar{z}^2}].\end{aligned}$$

Thus, the corresponding Piola stresses can be found from (3.26) as follows:

$$-\sigma_{21} + i\sigma_{11} = 2\mu_2 i[i(\alpha_2 - 1)(C_1 + mC_{3,0} + 3C_3 z^2) + \overline{D_1} + m\overline{D_{3,0}} + 3\overline{D_3}\overline{z}^2 + i\beta_2 \left(\frac{\overline{C_1} + m\overline{C_{3,0}} + 3\overline{C_3}\overline{z}(\overline{z} + 2z)}{(\overline{C_1} + m\overline{C_{3,0}} + 3\overline{C_3}\overline{z}^2)^2} \right)],$$

$$\sigma_{22} - i\sigma_{12} = 2\mu_2 i[(\alpha_2 - 1)(C_1 + mC_{3,0} + 3C_3 z^2) + i(\overline{D_1} + m\overline{D_{3,0}} + 3\overline{D_3}\overline{z}^2) + \beta_2 \left(\frac{\overline{C_1} + m\overline{C_{3,0}} + 3\overline{C_3}\overline{z}(\overline{z} - 2z)}{(\overline{C_1} + m\overline{C_{3,0}} + 3\overline{C_3}\overline{z}^2)^2} \right)],$$

$$-\sigma_{21}^{\infty} + i\sigma_{11}^{\infty} = 2\mu_1 i[i(\alpha_1 - 1)(A_1 + 3A_3 z^2) + \overline{B_1} + 3\overline{B_3}\overline{z}^2 + i\beta_1 \lim_{|z|\to\infty} \left(\frac{\overline{A_1} + 3\overline{A_3}\overline{z}(\overline{z} + 2z)}{(\overline{A_1} + 3\overline{A_3}\overline{z}^2)^2} \right)],$$

$$\sigma_{22}^{\infty} - i\sigma_{12}^{\infty} = 2\mu_1 i[(\alpha_1 - 1)(A_1 + 3A_3 z^2) + i(\overline{B_1} + 3\overline{B_3}\overline{z}^2) + \beta_1 \lim_{|z|\to\infty} \left(\frac{\overline{A_1} + 3\overline{A_3}\overline{z}(\overline{z} - 2z)}{(\overline{A_1} + 3\overline{A_3}\overline{z}^2)^2} \right)].$$

For example, the Piola stresses at the origin ($z = x_1 + ix_2,\ x_1 = x_2 = 0$, center of elliptic inhomogeneity) can be calculated as

$$-\sigma_{21} + i\sigma_{11} = 2\mu_2 i[i(\alpha_2 - 1)\left(C_1 + mC_{3,0}\right) + \overline{D_1} + m\overline{D_{3,0}} + i\beta_2 \frac{1}{\overline{C_1} + m\overline{C_{3,0}}}],$$

$$\sigma_{22} - i\sigma_{12} = 2\mu_2 i[(\alpha_2 - 1)\left(C_1 + mC_{3,0}\right) + i\left(\overline{D_1} + m\overline{D_{3,0}}\right) + \beta_2 \frac{1}{\overline{C_1} + m\overline{C_{3,0}}}].$$

and, clearly, the corresponding deformation is , from (3.32), $w_2\left(z\right) = 0$.

Remark 5. *We note from (3.28) and (3.29A-C) that if we set* $A_3 = B_3 = C_3 = C_{3,0} = D_3 = D_{3,0} = 0$*, we obtain exactly the corresponding solutions for the simpler case* *presented in Ru et al (2005) when the elliptic inhomogeneity-matrix system is subjected to uniform remote loading.*

3.6 THE CASE OF THE CIRCULAR INHOMOGENEITY

As we previously mentioned, the solution derived above is sufficiently general to accommodate a wide class of inhomogeneity problems. In this section, from the general solution presented in the previous sections, we obtain the complete set of solutions for a perfectly bonded *circular* inhomogeneity-matrix system (a particular class of an elliptic inhomogeneity) subjected to the same class of remote stresses (Kim & Schiavone, 2007b).

3.6.1 Complete Solution in the Case of the Circular Inhomogeneity

By setting $m = 0$, the mapping function (3.2) becomes:

$$z = \omega(\xi) = R\xi, \quad R > 0, \;\; \xi = \rho e^{i\theta}, \;\; \rho = 1, \tag{3.35}$$

which implies that we have a circle whose radius is R in the $z-plane$ as a special case of an ellipse. In other words, we can actually derive the complete solution for the circular inhomogeneity problems from the general solution presented in the previous section, by applying the the following condition:

$$z = \omega(\xi) = R\left(\xi + \frac{m}{\xi}\right), \quad R > 0, \;\; m = 0,$$

(a circular mapping function: a particular case of the ellipse).

From (3.16) together with (3.2), we obtain, for $m = 0$,

$$\begin{aligned}\phi_2(z) &= C_1 R\xi + \sum_{n=a}^{b} C_n R\xi^n \\ \psi_2(z) &= D_1 R\xi + \sum_{n=a}^{b} D_n R\xi^n, \quad R\xi \in S_2,\ a \geq 2, b \geq a,\end{aligned} \tag{3.36}$$

and from (3.17-18), we have that, for $m = 0$,

$$\phi_1(\xi) = KC_1 R\xi + \frac{R}{\xi} iS\overline{D_1} + \sum_{n=a}^{b} (R^n \xi^n KC_n) + \sum_{n=a}^{b} (\xi^{-n} iS\overline{D_n}) + SR_{(n,\xi)},$$

$$, |\xi| > 1, \quad \left(\because \binom{n}{0} = \binom{n}{n} = 1 \right), \tag{3.37}$$

$$\psi_1(\xi) = -i\alpha_1 \left[K\overline{C_1}\frac{R}{\xi} - iSD_1 R\xi \right] + i\alpha_2 \overline{C_1}\frac{R}{\xi} + D_1 R\xi$$

$$-i\alpha_1 \sum_{n=a}^{b} \left[R^n \xi^{-n} \overline{KC_n} + R^n \xi^n \overline{\overline{iSD_n}} \right] + i\alpha_2 \sum_{n=a}^{b} (\overline{C_n} R^n \xi^{-n}) + \sum_{n=a}^{b} (D_n R^n \xi^n) \tag{3.38}$$

$$+\overline{R_{(n,\xi)}} i (1 - \alpha_1 S) - \frac{i\beta_1 R^2 \frac{1}{\xi}}{\phi_1'(\xi)},$$

where

$$R_{(n,\xi)} = \frac{\beta_2 R\xi}{\overline{C_1} + \sum_{n=a}^{b} \left[n\overline{C_n} \left(\frac{R}{\xi}\right)^{n-1} \right]},$$

and

$$\begin{aligned} X_{n,(n,r)} &= KC_n m^r + iS\overline{D_n} m^{n-r} = \begin{cases} KC_n, & r = 0, \\ iS\overline{D_n}, & r = n, \end{cases} \\ , X_{n,(\alpha,r)} &= KC_{n,\alpha} m^{r+1} + iS\overline{D_{n,\alpha}} m^{2\alpha+\delta_o - r+1} = 0. \end{aligned}$$

Finally, from (3.19A-D), the unknown complex constants $C_1,\ C_n,\ D_1$, and D_n are determined, for $m = 0$

$$A_1 R\xi \;=\; KC_1 R\xi + \frac{S\beta_2 R\xi}{\overline{C_1}},\quad \sum_{n=a}^{b} [A_n]\, R\xi^n = \sum_{n=a}^{b} [KC_n]\, R\xi^n, \qquad (3.39A)$$
$$\because\ X_{n,(n,r=0)} = KC_n + iS\overline{D_n}m^n = KC_n,$$

$$B_1 R\xi \;=\; R\xi D_1 (1-\alpha_1 S),\quad \sum_{n=a}^{b} [B_n]\, R\xi^n = \sum_{n=a}^{b} [D_n (1-\alpha_1 S)]\, R\xi^n, \ (3.39B)$$
$$\because\ \overline{X_{n,(n,r=n)}} = K\overline{C}_n m^n - iSD_n = -iSD_n.$$

where

$$K = \frac{\mu_2}{\mu_1} + \alpha_2\left(1 - \frac{\mu_2}{\mu_1}\right),\quad S = \left(1 - \frac{\mu_2}{\mu_1}\right).$$

Hence, from (3.36-3.39(A, B)), we can derive the complete set of solutions of the inhomogeneity-matrix system subjected to remote loading described by (3.1). In fact, by letting $R \equiv R_c$ and in the view of (3.35), Equations (3.36-3.39(A, B)) immediately reduced to the corresponding set of solutions in Kim & Schiavone (2007b). We also mention here that the solution for the circular inhomogeneity-matrix system obtained in this section further reduces to the results presented in Ru et al (2005) and Ogden & Isherwood (1978) provided that the same conditions are considered.

SUMMARY

The complete solution for the circular inhomogeneity cases is given by

$$\phi_2(z) = C_1 z + \sum_{n=a}^{b} C_n z^n,\quad \psi_2(z) = D_1 z + \sum_{n=a}^{b} D_n z^n,\quad |z| < R_c,\quad a \geq 2, b \geq a,\ (3.40)$$

$$\phi_1(z) = X_1 z + \sum_{n=a}^{b}(X_n z^n) + \frac{X_{-1}}{z} + \sum_{n=a}^{b}(\frac{X_{-n}}{z^n}) + R(z), \quad |z| > R_c, \tag{3.41}$$

and

$$\psi_1(z) = -i\alpha_1[\frac{\overline{X_1}R_c^2}{z} + \sum_{n=a}^{b}(\frac{\overline{X_n}R_c^{2n}}{z^n}) + \frac{\overline{X_{-1}}}{R_c^2}z + \sum_{n=a}^{b}(\frac{\overline{X_{-n}}}{R_c^{2n}}z^n) + \frac{S\beta_2 R_c^2}{z[\sum_{n=a}^{b}(nC_n z^{n-1}) + C_1]}]$$

$$-\frac{i\beta_1 R_c^2}{z[X_1 + \sum_{n=a}^{b}(nX_n z^{n-1}) - \frac{X_{-1}}{z^2} - \sum_{n=a}^{b}(\frac{nX_{-n}}{z^{n+1}}) + R'(z)]}$$

$$+i\alpha_2[\frac{\overline{C_1}R_c^2}{z} + \sum_{n=a}^{b}(\frac{\overline{C_n}R_c^{2n}}{z^n})] + \frac{i\beta_2 R_c^2}{z(C_1 + \sum_{n=a}^{b} nC_n z^{n-1})} + D_1 z + \sum_{n=a}^{b} D_n z^n, \tag{3.42}$$

where , $|z| > R_c$. Here, the unknown complex constants C_1, C_n, D_1 and D_n are determined by the equations

$$A_1 = [\frac{\mu_2}{\mu_1} + \alpha_2(1 - \frac{\mu_2}{\mu_1})]C_1 + (1 - \frac{\mu_2}{\mu_1})\frac{\beta_2}{\overline{C_1}}, \quad B_1 = D_1[1 - \alpha_1(1 - \frac{\mu_2}{\mu_1})],$$

$$\sum_{n=a}^{b} A_n z^n = \sum_{n=a}^{b}[(\frac{\mu_2}{\mu_1} + \alpha_2(1 - \frac{\mu_2}{\mu_1}))C_n]z^n, \quad \sum_{n=a}^{b} B_n z^n = \sum_{n=a}^{b}[D_n(1 - \alpha_1(1 - \frac{\mu_2}{\mu_1}))]z^n, \tag{3.43}$$

where, from Remarks. 3 and 4, we must have that for $R \equiv R_c,\ z = R\xi$,

$$A_1,\ C_1 \neq 0,$$

$$C_1 + \sum_{n=a}^{b} nC_n z^{n-1} \neq 0 \Longleftrightarrow \overline{C_1} + \sum_{n=a}^{b} n\overline{C_n z^{n-1}} \neq 0, \quad \forall z : |z| \leq R_c,$$

$$X_1 + \sum_{n=a}^{b}(nX_n z^{n-1}) - \frac{X_{-1}}{z^2} - \sum_{n=a}^{b}(\frac{nX_{-n}}{z^{n+1}}) + R'(z) \neq 0, \quad \forall z : |z| \geq R_c.$$

3.6.2 Expressions for I and J Inside the Circular Inhomogeneity

From (3.23) we have for $m = 0$

$$w_2 = -i\alpha_2[C_1 z + \sum_{n=a}^{b} C_n z^n] + \overline{D_1}\bar{z} + \sum_{n=a}^{b} \overline{D_n}\bar{z}^n - \frac{i\beta_2 z}{\overline{C_1} + \sum_{n=a}^{b} n\overline{C_n} \overline{z^{n-1}}}, \tag{3.44}$$

and from (3.24)

$$I = 2\left(\alpha_2 \left|C_1 + \sum_{n=a}^{b} nC_n z^{n-1}\right| + \frac{\beta_2}{\left|C_1 + \sum_{n=a}^{b} nC_n z^{n-1}\right|}\right). \tag{3.45}$$

Finally, from Remark 3

$$|\phi_2'(z)| = \left|C_1 + \sum_{n=a}^{b} nC_n z^{n-1}\right| \neq 0.$$

$$I^2 - 16\alpha_2\beta_2 = 4\left(\alpha_2 \left|C_1 + \sum_{n=a}^{b} nC_n z^{n-1}\right| - \frac{\beta_2}{\left|C_1 + \sum_{n=a}^{b} nC_n z^{n-1}\right|}\right)^2 > 0, \tag{3.46}$$

which guarantees that $F'(I)$, from (2.21) is well-defined.

3.6.3 Stress Distribution

From (3.27), we have that, for $m = 0$

$$\chi_1(z) = 2\mu_1 i[(\alpha_1 - 1)(A_1 z + \sum_{n=a}^{b} A_n z^n) + i(\overline{B_1}\bar{z} + \sum_{n=a}^{b} \overline{B_n}\bar{z}^n) + \frac{\beta_1 z}{\overline{A_1} + \sum_{n=a}^{b} n\overline{A_n} \overline{z^{n-1}}}],$$

$$\chi_2(z) = 2\mu_2 i[(\alpha_2 - 1)(C_1 z + \sum_{n=a}^{b} C_n z^n) + i(\overline{D_1}\bar{z} + \sum_{n=a}^{b} \overline{D_n}\bar{z}^n) + \frac{\beta_2 z}{\overline{C_1} + \sum_{n=a}^{b} n\overline{C_n} \overline{z^{n-1}}}], \tag{3.47}$$

so that the Piola stresses both inside the inhomogeneity and at the remote boundary can be calculated from (3.43), (3.26) and (3.47).

In addition, from Equation (2.14), (3.21-22) and (3.44-46), we have that, for $m = 0$

$$F'(I) = |\phi_2'(z)| = \left| C_1 + \sum_{n=a}^{b} nC_n z^{n-1} \right| \neq 0,$$

from with the Cauchy stresses inside the inhomogeneity can be evaluated by (2.12)

3.6.4 Example: The Case $a = b = 2$.

Similar to the previous section, here, we consider the particular class of remote stresses, characterized by the stress functions $\phi_1(z) = A_1 z + A_2 z^2$, $\psi_1(z) = B_1 z + B_2 z^2$,where $A_1, A_2, B_1$and B_2 are prescribed complex constants.

Complete Solutions

From (3.1) and (3.40), the stress functions at the remote boundary and inside the inhomogeneity, respectively, are given by

$$\begin{aligned} \phi_1(z) &= A_1 z + A_2 z^2, \quad \psi_1(z) = B_1 z + B_2 z^2, \quad |z| \to \infty, \\ \phi_2(z) &= C_1 z + C_2 z^2, \quad \psi_2(z) = D_1 z + D_2 z^2, \quad |z| < R_c \end{aligned} \tag{3.48}$$

From the general solution (3.43), the unknown complex constants C_1, C_2, D_1, and D_2 are completely determined by the equations

$$\begin{aligned} A_1 &= [\frac{\mu_2}{\mu_1} + \alpha_2(1 - \frac{\mu_2}{\mu_1})]C_1 + (1 - \frac{\mu_2}{\mu_1})\frac{\beta_2}{\overline{C_1}}, \quad B_1 = D_1[1 - \alpha_1(1 - \frac{\mu_2}{\mu_1})], \\ A_2 &= [\frac{\mu_2}{\mu_1} + \alpha_2(1 - \frac{\mu_2}{\mu_1})]C_2, \quad B_2 = D_2[1 - \alpha_1(1 - \frac{\mu_2}{\mu_1})]. \end{aligned} \tag{3.49}$$

From Remarks. 3 and 4, we must have

$$C_1 + 2C_2 z \neq 0 \Longleftrightarrow \overline{C_1} + 2\overline{C_2}\overline{z} \neq 0, \quad \forall z : |z| \leq R_c, \tag{3.50}$$

and in the surrounding matrix

$$KC_1 + 2KC_2z - \frac{iS\overline{D_1}R_c^2}{z^2} - \frac{2iS\overline{D_2}R_c^4}{z^3} + S\beta_2\left(\frac{\frac{4\overline{C_2}R_c^2}{z} + \overline{C_1}}{(\frac{2\overline{C_2}R_c^2}{z} + \overline{C_1})^2}\right) \neq 0, \quad \forall z : |z| \geq R_c. \tag{3.51}$$

Determination of I and J Inside the inhomogeneity

From (3.44), the deformation function inside the inhomogeneity now takes the form

$$w_2 = -i\alpha_2[C_1z + C_2z^2] + \overline{D_1}\bar{z} + \overline{D_2}\bar{z}^2 - \frac{i\beta_2 z}{\overline{C_1} + 2\overline{C_2}\bar{z}}, \tag{3.52}$$

and from (3.45)

$$I = 2\left(\alpha_2 |C_1 + 2C_2z| + \frac{\beta_2}{|C_1 + 2C_2z|}\right),$$
$$|\phi_2'(z)| = |C_1 + 2C_2z| \neq 0, \text{(from Remark. 2)},$$

$$I^2 - 16\alpha_2\beta_2 = 4\left(\alpha_2 |C_1 + 2C_2z| - \frac{\beta_2}{|C_1 + 2C_2z^1|}\right)^2 \geqslant 0, \tag{3.53}$$

from which I can be calculated. Henceforth, $F'(I)$, from (2.21) again is well defined. Also J can now be determined from (3.20) together with (3.52).

$$J = -\operatorname{Im}\left[(w_2)_{,1}(\overline{w}_2)_{,2}\right],$$

where

$$w_2 = -i\alpha_2[C_1z + C_2z^2] + \overline{D_1}\bar{z} + \overline{D_2}\bar{z}^2 - \frac{i\beta_2 z}{\overline{C_1} + 2\overline{C_2}\bar{z}}.$$

Deformed contour

From (41), we can easily plot the deformed contour (see Figure 5). The fact that the corresponding complex constants C_1, C_2, D_1 and D_2 satisfy (3.50) and (3.51) guarantee that there is no overlapping.

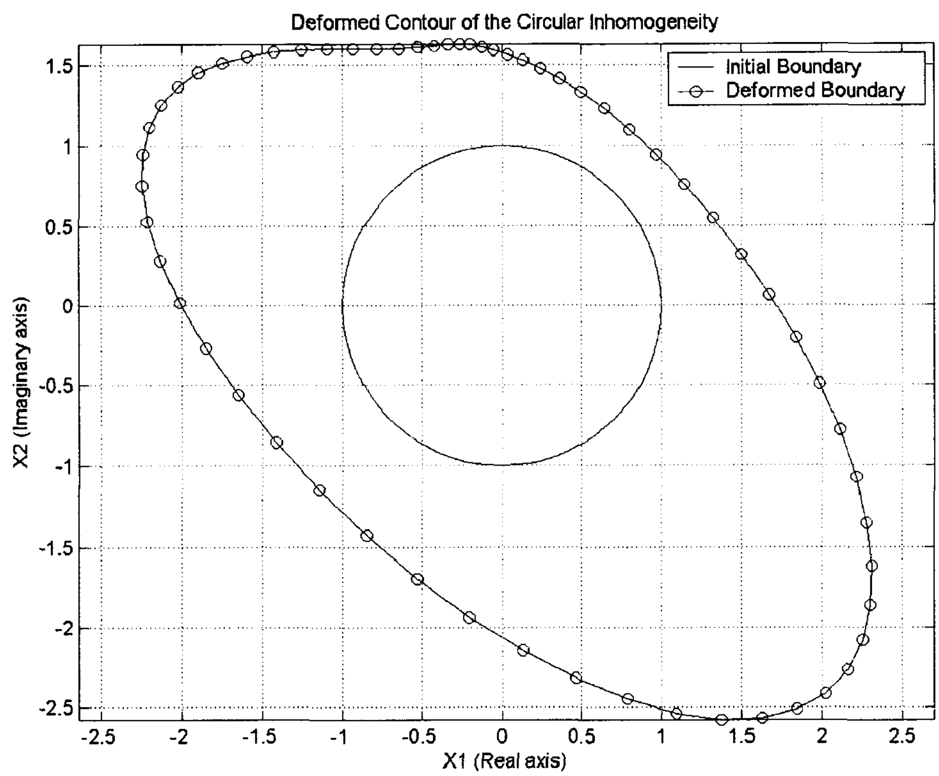

Figure 5: *Example of a Deformed Circular Inhomogeneity Subjected to a Remote Loading Characterized by (3.48).* Here, we have taken *(*$A_1 = 3.5$, $A_2 = 1$, $B_1 = 0.5$, $B_2 = 0.25$, $\mu_1 = 0.27$, $\mu_2 = 0.3$, $\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 0.5$*)*

Admissibility conditions

From (3.20) and (3.52-53), inside inhomogeneity,

$$I^2 > 16\alpha_2\beta_2, \quad J > 0. \tag{3.54}$$

Again, as in Ru (2002), (3.54) ensure that there is no overlapping of the deformation field (interior to the circular inhomogeneity).

The Piola Stress Distribution

From (3.47), the Piola stress in the inhomogeneity and the matrix is described, respectively, by:

$$\chi_1(z) = 2\mu_1 i[(\alpha_1 - 1)(A_1 z + A_2 z^2) + i(\overline{B_1}\bar{z} + \overline{B_2}\bar{z}^2) + \frac{\beta_1 z}{\overline{A_1} + 2\overline{A_2}\bar{z}}],$$

$$\chi_2(z) = 2\mu_2 i[(\alpha_2 - 1)(C_1 z + C_2 z^2) + i(\overline{D_1}\bar{z} + \overline{D_2}\bar{z}^2) + \frac{\beta_2 z}{\overline{C_1} + 2\overline{C_2}\bar{z}}],$$

Thus, the corresponding Piola stress can be found from (3.26) as follows:

$$-\sigma_{21} + i\sigma_{11} = 2\mu_2 i[i(\alpha_2 - 1)(C_1 + 2C_2 z) + \overline{D_1} + 2\overline{D_2}\bar{z} + i\beta_2\left(\frac{\overline{C_1} + 2\overline{C_2}(\bar{z} + z)}{(\overline{C_1} + 2\overline{C_2}\bar{z})^2}\right)],$$

$$\sigma_{22} - i\sigma_{12} = 2\mu_2 i[(\alpha_2 - 1)(C_1 + 2C_2 z) + i(\overline{D_1} + 2\overline{D_2}\bar{z}) + \beta_2\left(\frac{\overline{C_1} + 2\overline{C_2}(\bar{z} - z)}{(\overline{C_1} + 2\overline{C_2}\bar{z})^2}\right)],$$

$$-\sigma_{21}^{\infty} + i\sigma_{11}^{\infty} = 2\mu_1 i[i(\alpha_1 - 1)(A_1 + 2A_2 z) + \overline{B_1} + 2\overline{B_2}\bar{z} + i\beta_1 \lim_{|z|\to\infty}\left(\frac{\overline{A_1} + 2\overline{A_2}(\bar{z} + z)}{(\overline{A_1} + 2\overline{A_2}\bar{z})^2}\right)],$$

$$\sigma_{22}^{\infty} - i\sigma_{12}^{\infty} = 2\mu_1 i[(\alpha_1 - 1)(A_1 + 2A_2 z) + i(\overline{B_1} + 2\overline{B_2}\bar{z}) + \beta_1 \lim_{|z|\to\infty}\left(\frac{\overline{A_1} + 2\overline{A_2}(\bar{z} - z)}{(\overline{A_1} + 2\overline{A_2}\bar{z})^2}\right)].$$

Note that the Piola stress at the origin ($z = x_1 + ix_2,\ x_1 = x_2 = 0$, center of circular inhomogeneity) can be calculated as

$$-\sigma_{21} + i\sigma_{11} = 2\mu_2 i[i(\alpha_2 - 1)C_1 + \overline{D_1} + i\beta_2 \frac{1}{\overline{C_1}}],$$

$$\sigma_{22} - i\sigma_{12} = 2\mu_2 i[(\alpha_2 - 1)C_1 + i\overline{D_1} + \beta_2 \frac{1}{\overline{C_1}}].$$

and, clearly, the corresponding deformation is, from (3.52), $w_2 = 0$.

3.7 CONCLUSIONS

In this Chapter, we consider an inhomogeneity-matrix system from a particular class of compressible hyperelastic materials of harmonic-type undergoing finite plane deformations. We obtain the complete solution for a perfectly bonded elliptic inhomogeneity when the system is subjected to general classes of remote Piola stresses characterized by stress functions described by general polynomials of degree $n \geq 1$. In addition, from the general solution, we also derived the complete set of solutions for a perfectly bonded *circular* inhomogeneity-matrix system (a particular class of an elliptic inhomogeneity) subjected to the same class of remote stresses. The solutions presented here are extremely important in that they essentially lead to solutions of problems in which the inhomogeneity-matrix system is subjected to a wide class of remote loading conditions and so can accommodate a correspondingly large set of physically relevant problems. We note, in particular, the generality of our results and that existing results in the literature are obtained as special cases of the solutions derived here. For example, the results found in Ru et al (2005) for the elliptic inhomogeneity-matrix system subjected to uniform remote loading are obtained here as special cases (see Appendix). In addition, concerning for the case of the circular inhomogeneity problems, the general solutions derived here further reduce to the corresponding results presented in Ogden & Isherwood (1978) as far as the same conditions are applied.

Chapter 4
ESHELBY'S CONJECTURE

4.1 INTRODUCTION

It is well-known that 'peak' in the stress distribution inside an inhomogeneity are responsible for the failure of the corresponding fiber and ultimately the entire composite. It is of interest, therefore, to design an inhomogeneity in which the stress distribution remains uniform. One of the most celebrated results in solid mechanics is concerned specifically with this problem. This result is known as Eshelby's conjecture.

Eshelby's conjecture in linear plane and anti-plane elasticity states that the stress distribution inside an inhomogeneity, perfectly bonded to an infinite isotropic elastic medium, subjected at infinity to a uniform stress field, is uniform if and only if the inhomogeneity is elliptical (Sendeckyj, 1970; Ru & Schiavone, 1996). In Ru et al (2005), the authors examined one side of Eshelby's conjecture for the plane deformations of the same class of compressible hyperelastic materials of harmonic-type considered in this thesis. More specifically, it was shown that, under the analogous constraints of perfect bonding and uniform remote Piola stress, if the Piola stress within the inhomogeneity is uniform, the inhomogeneity is necessarily elliptical except for the special case when the complex potentials at the remote boundary and inside the inhomogeneity takes a particular restricted form:

$$\phi_1(z) = Az, \ \psi_1(z) = Bz, \quad |z| \to \infty,$$

$$\phi_2(z) = Cz, \ \psi_2(z) = Bz, \quad \text{inside the inhomogeneity},$$

where

$$A = C\frac{\mu_2}{\mu_1} + \left(1 - \frac{\mu_2}{\mu_1}\right)\left[\alpha_2 C + \frac{\beta_2}{\overline{C}}\right],$$
$$\left(1 - \frac{\mu_2}{\mu_1}\right) B = 0, \quad \alpha_1 \overline{A} + \frac{\beta_1}{A} = \alpha_2 \overline{C} + \frac{\beta_2}{C}),$$

in which case the shape of the inhomogeneity can be taken as arbitrary. This latter result is extremely interesting in that it does not arise in the analogous results from linear elasticity and can be attributed to the particular class of materials being considered here. It is of interest, therefore, to examine if the converse result (i.e. the 'other side' of Eshelby's conjecture) holds true for this class of materials. Namely, if the Piola stress distribution within an elliptic inhomogeneity is necessarily uniform when the same inhomogeneity-matrix system is subjected to finite plane deformations and uniform remote loading (Kim et al, 2007).

To examine Eshelby's conjecture, we begin by assuming that, at infinity, the Piola stress is uniform so that

$$\phi_1(z) = Az + O_{(1)}, \quad \psi_1(z) = Bz + O_{(2)}, \quad |z| \to \infty,$$

where A, B, $O_{(1)}$ and $O_{(2)}$ are prescribed complex constants.

4.2 COMPLEX VARIABLE FORMULATION

As a consequence of (2.27-28) and the remote loading condition, we seek analytic functions $\phi_\gamma(z)$ and $\psi_\gamma(z)$, $\gamma = 1,2$ in S_1 and S_2 ,respectively, such that

$$\phi_1(z) = K\phi_2(z) + iS\overline{\psi_2(z)} + \frac{S\beta_2 z}{\overline{\phi_2'(z)}}, \quad z \in \Gamma, \tag{4.1}$$

$$\psi_1(z) = i\left(-\alpha_1\overline{\phi_1(z)} - \frac{\beta_1\bar{z}}{\phi_1'(z)} + \alpha_2\overline{\phi_2(z)} + \frac{\beta_2\bar{z}}{\phi_2'(z)}\right) + \psi_2(z), \quad z \in \Gamma,$$

$$\phi_1(z) = Az + O_{(1)}, \quad \psi_1(z) = Bz + O_{(2)}, \quad |z| \to \infty, \tag{4.2}$$

where

$$K = \frac{\mu_2}{\mu_1} + \alpha_2\left(1 - \frac{\mu_2}{\mu_1}\right), \quad S = 1 - \frac{\mu_2}{\mu_1},$$

For the class of harmornic materials considered here, we have from (2.21-22):

$$\begin{aligned} 1 &> \alpha_\gamma \geq 1/2,\ \beta_\gamma > 0,\ \alpha_\gamma + \beta_\gamma = 1,\ \mu_\gamma > 0, \\ \gamma &= 1,2 \text{ (no sum over repeated indices).} \end{aligned} \tag{4.3}$$

It is well-known that the finite region bounded by an ellipse can be mapped conformally into a circle, but the corresponding transformation is complicated and inconvenient Muskhelishvili (1953). Instead, consider the enclosed region S_2 to be cut along the segment $D = \{(x_1, 0) : -2R \leq x_1 \leq 2R\}$ connecting the foci. This cut may be thought of as an ellipse, which is confocal with Γ but whose minor axis is zero. Hence the cut region in S_2 may be thought of as the limiting case of a region between two confocal ellipses.

In fact, Muskhelishvili (1953),

$$z = \omega(\xi) = R\left(\xi + \frac{1}{\xi}\right), \quad R > 0, \tag{4.4}$$

will map the ellipse

$$x_1 = R\left(\rho + \frac{1}{\rho}\right)\cos\theta, \quad x_2 = R\left(\rho - \frac{1}{\rho}\right)\sin\theta, \tag{4.5}$$

where R, $\rho > 0$ and $\theta \in [0, 2\pi]$ and its exterior region in the complex $z - plane$, onto and outside, respectively, the circle of radius ρ in the complex $\xi - plane$. Hence, using the transformation (4.4), the region S_1 is transformed into the exterior of the circle with radius R_* and the region $S_2 \backslash D$ into the ring $1 < |\xi| < R_*$ in the $\xi - plane$. The segment D and the boundary Γ are mapped onto the circles $|\xi| = 1$ and $|\xi| = R_*$, respectively, where,

$$R_* = \frac{a + \sqrt{a^2 - 4R^2}}{2R} > 1, \qquad a = R\left(R_* + \frac{1}{R_*}\right).$$

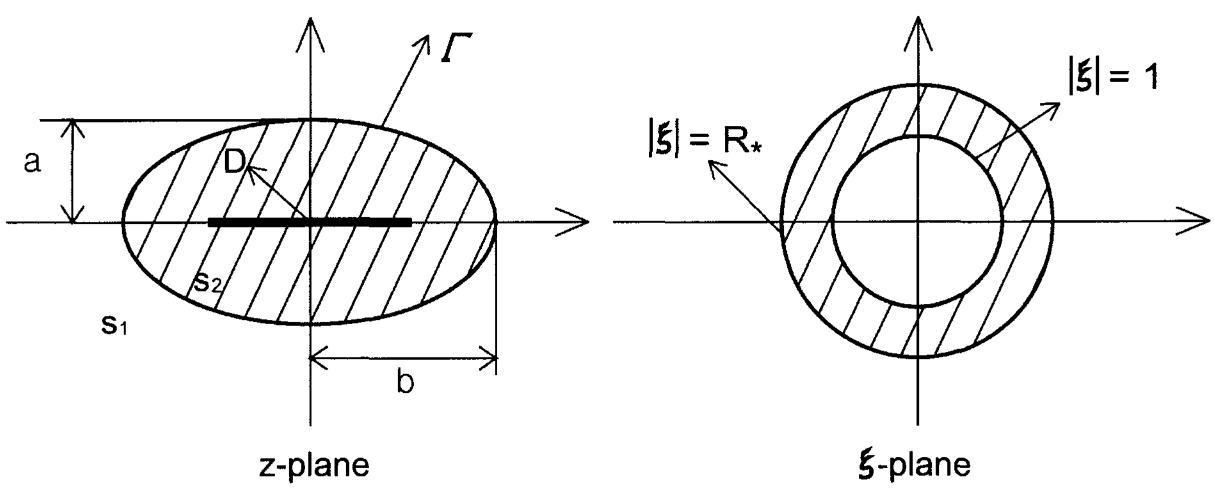

Figure 6: *The Conformal Mapping from $z - plane$ to $\xi - plane$*

Since $\phi_2(z)$ is analytic in S_2, we must have

$$\phi_2(z) = \phi_2(\overline{z}), \quad \psi_2(z) = \psi_2(\overline{z}) \quad z \in D. \tag{4.6}$$

For convenience, we write $\phi_\gamma(\omega(\xi)) = \phi_\gamma(\xi)$, $\gamma = 1,2$ and similarly for the functions ψ_γ so that, in the $\xi - plane$, condition (4.6) becomes

$$\phi_2(\xi) = \phi_2(\overline{\xi}), \quad \psi_2(\xi) = \psi_2(\overline{\xi}), \quad \forall\, \xi : |\xi| = 1.$$

Thus, in the $\xi - plane$, we seek the solution $\phi_\gamma(\xi)$, $\psi_\gamma(\xi)$, $\gamma = 1,2$ in the regions $|\xi| > R_*$ and $1 < |\xi| < R_*$, respectively, such that

$$\phi_1(\xi) = K\phi_2(\xi) + iS\overline{\psi_2(\xi)} + \frac{S\beta_2\omega(\xi)\overline{\omega'(\xi)}}{\overline{\phi_2'(\xi)}}, \quad |\xi| = R_*, \tag{4.7}$$

$$\psi_1(\xi) = -i\alpha_1\overline{\phi_1(\xi)} - \frac{i\beta_1\overline{\omega(\xi)}\omega'(\xi)}{\phi_1'(\xi)} + i\alpha_2\overline{\phi_2(\xi)} + \frac{i\beta_2\overline{\omega(\xi)}\omega'(\xi)}{\phi_2'(\xi)} + \psi_2(\xi), \quad |\xi| = R_*,$$

$$\because \quad \phi_\gamma'(z) = \frac{d\phi(z)}{dz} = \frac{d\phi(\xi)}{d\xi}\frac{d\xi}{dz} = \frac{\phi'(\xi)}{\omega'(\xi)}, \quad \overline{\phi_\gamma'(z)} = \overline{\left(\frac{\phi'(\xi)}{\omega'(\xi)}\right)} = \frac{\overline{\phi'(\xi)}}{\overline{\omega'(\xi)}},$$

$$\phi_2(\xi) = \phi_2(\overline{\xi}), \quad \psi_2(\xi) = \psi_2(\overline{\xi}), \quad \forall\, \xi : |\xi| = 1, \tag{4.8}$$

$$\phi_1(\xi) = AR\xi + O_{(1)}, \quad \psi_1(\xi) = BR\xi + O_{(2)}, \quad |\xi| \to \infty. \tag{4.9}$$

Since $\phi_2(\xi)$ and $\psi_2(\xi)$ are analytic in the ring $1 < |\xi| < R_*$, they have Laurent series representations

$$\phi_2(\xi) = \sum_{n=-\infty}^{n=\infty} C_n \xi^n, \quad \psi_2(\xi) = \sum_{n=-\infty}^{n=\infty} D_n \xi^n, \quad C_n, D_n \in \mathbb{C}.$$

Letting $\xi = \gamma e^{i\theta}$, from (3.8), we obtain Muskhelishvili (1953)

$$C_n = C_{-n}, \quad D_n = D_{-n}, \tag{4.10}$$

$$\phi_2(\xi) = C_o + \sum_{n=1}^{n=\infty} C_n \left(\xi^n + \frac{1}{\xi^n}\right), \quad \psi_2(\xi) = D_o + \sum_{n=1}^{n=\infty} D_n \left(\xi^n + \frac{1}{\xi^n}\right), \quad 1 < |\xi| < R_*.$$

Remark 6. *Since, from Remark 1*

$$\left|\phi'_\gamma(z)\right| = F'(I) \neq 0, \quad A_1,\ C_1 \neq 0,$$

and

$$\omega'(\xi) \neq 0, \quad |\xi| > 1,$$

it is clear that we must have

$$\phi'_2(\xi) \neq 0 \iff \overline{\phi'_2(\xi)} \neq 0, \quad |\xi| > 1.$$

In other words, from (4.10), we must have that

$$\begin{aligned} \sum_{n=1}^{n=\infty} nC_n \left(\xi^{n-1} - \frac{1}{\xi^{n+1}}\right) &\neq 0 \\ \iff \overline{\sum_{n=1}^{n=\infty} nC_n \left(\xi^{n-1} - \frac{1}{\xi^{n+1}}\right)} &\neq 0, \quad \forall \xi : |\xi| = R_* . \end{aligned}$$

It follows from (4.10) and the interface condition (4.7) that we now have, for $|\xi| = R_*$

$$\phi_1(\xi) = K\left[C_o + \sum_{n=1}^{n=\infty} C_n\left(\xi^n + \frac{1}{\xi^n}\right)\right] + iS\left[\overline{D_o + \sum_{n=1}^{n=\infty} D_n\left(\xi^n + \frac{1}{\xi^n}\right)}\right]$$

$$+\frac{S\beta_2 R\left(\xi + \frac{1}{\xi}\right)\overline{R\left(1 - \frac{1}{\xi^2}\right)}}{\overline{\sum_{n=1}^{n=\infty} nC_n\left(\xi^{n-1} - \frac{1}{\xi^{n+1}}\right)}}.$$

Since $\xi\overline{\xi} = |\xi|^2 = R_*^2$ at the interface,

$$\phi_1(\xi) = K\left[C_o + \sum_{n=1}^{n=\infty} C_n\left(\xi^n + \frac{1}{\xi^n}\right)\right] + iS\left[\overline{D_o} + \sum_{n=1}^{n=\infty}\overline{D_n}\left(\frac{R_*^{2n}}{\xi^n} + \frac{\xi^n}{R_*^{2n}}\right)\right]$$

$$+\frac{S\beta_2 R^2\left(\xi + \frac{1}{\xi}\right)\left(1 - \frac{\xi^2}{R_*^4}\right)}{\sum_{n=1}^{n=\infty} n\overline{C_n}\left(\frac{R_*^{2(n-1)}}{\xi^{n-1}} - \frac{\xi^{n+1}}{R_*^{2(n+1)}}\right)}. \tag{4.11}$$

If we adopt ϕ_1 from (4.11) for $|\xi| > R_*$, at the remote boundary, we have

$$\phi_1(\xi) = K\left[C_o + \sum_{n=1}^{n=\infty} C_n\xi^n\right] + iS\left[\overline{D_o} + \sum_{n=1}^{n=\infty}\overline{D_n}\frac{\xi^n}{R_*^{2n}}\right] + \lim_{|\xi|\to\infty}\left(SR_{(n,\xi)}\right), \quad |\xi| \to \infty, \tag{4.12}$$

where

$$R_{(n,\xi)} = \frac{\beta_2 R^2\left(\xi + \frac{1}{\xi}\right)\left(1 - \frac{\xi^2}{R_*^4}\right)}{\sum_{n=1}^{n=\infty} n\overline{C_n}\left(\frac{R_*^{2(n-1)}}{\xi^{n-1}} - \frac{\xi^{n+1}}{R_*^{2(n+1)}}\right)}.$$

Also from (4.9),

$$\phi_1(\xi) = AR\xi + O_{(1)}, \quad |\xi| \to \infty. \tag{4.13}$$

Therefore, it follows from (4.12-13) that

$$AR\xi + O_{(1)} = K\left[C_o + \sum_{n=1}^{n=\infty} C_n\xi^n\right] + iS\left[\overline{D_o} + \sum_{n=1}^{n=\infty}\overline{D_n}\frac{\xi^n}{R_*^{2n}}\right] + \lim_{|\xi|\to\infty}\left(SR_{(n,\xi)}\right), \quad |\xi| \to \infty. \tag{4.14}$$

Equating coefficients of ξ, we require from (4.14) that

$$AR\xi + O_{(1)} = K\left[C_o + C_1\xi\right] + iS\left[\overline{D_o} + \overline{D_1}\frac{\xi}{R_*^2}\right] + \lim_{|\xi|\to\infty}\left(SR_{(n,\xi)}\right), \tag{4.15}$$

$$\sum_{n=2}^{n=\infty}\left(KC_n + \frac{iS\overline{D_n}}{R_*^{2n}}\right)\xi^n = 0, \quad |\xi| \to \infty. \tag{4.16}$$

From (4.11-12) and (4.16), for

$$\phi_1(\xi) = KC_o + iS\overline{D_o} + \left(KC_1 + \frac{iS\overline{D_1}}{R_*^2}\right)\xi + \sum_{n=1}^{\infty}\left[\left(KC_n + iS\overline{D_n}R_*^{2n}\right)\frac{1}{\xi^n}\right] + SR_{(n,\xi)}, \quad |\xi| > R_* \tag{4.17}$$

Remark 7. *As a direct consequence of Remark 6, we have that*

$$KC_1 + \frac{iS\overline{D_1}}{R_*^2} + \sum_{n=1}^{\infty}\left[-n\left(KC_n + iS\overline{D_n}R_*^{2n}\right)\frac{1}{\xi^{n+1}}\right] + SR'(n,\xi) \neq 0$$

$$\forall\xi : |\xi| > R_*.$$

Similarly, from the interface condition (4.7) together with (4.10) and (3.12), it follows for $|\xi| = R_*$,

$$\psi_1(\xi) = -i\alpha_1\left[K\overline{C_o} - iSD_o + \left(K\overline{C_1} - \frac{iSD_1}{R_*^2}\right)\frac{R_*^2}{\xi} + \sum_{n=1}^{\infty}\left[\left(K\overline{C_n} - iSD_nR_*^{2n}\right)\frac{\xi^n}{R_*^{2n}}\right] + S\overline{R_{(n,\xi)}}\right]$$

$$-\frac{i\beta_1 R^2\left(\frac{R_*^2}{\xi} + \frac{\xi}{R_*^2}\right)\left(1 - \frac{1}{\xi^2}\right)}{KC_1 + \frac{iS\overline{D_1}}{R_*^2} + \sum_{n=1}^{\infty}\left[-n\left(KC_n + iS\overline{D_n}R_*^{2n}\right)\frac{1}{\xi^{n+1}}\right] + SR'(n,\xi)} + D_o \tag{4.18}$$

$$+i\alpha_2\left[\overline{C_o} + \sum_{n=1}^{\infty}\overline{C_n}\left(\frac{R_*^{2n}}{\xi^n} + \frac{\xi^n}{R_*^{2n}}\right)\right] + \frac{i\beta_2 R^2\left(\frac{R_*^2}{\xi} + \frac{\xi}{R_*^2}\right)\left(1 - \frac{1}{\xi^2}\right)}{\sum_{n=1}^{\infty} nC_n\left(\xi^{n-1} - \frac{1}{\xi^{n+1}}\right)} + \sum_{n=1}^{\infty} D_n\left(\xi^n + \frac{1}{\xi^n}\right),$$

where from (4.12)

$$\overline{R_{(n,\xi)}} = \frac{\beta_2 R^2 \left(\frac{R_*^2}{\xi} + \frac{\xi}{R_*^2}\right)\left(1 - \frac{1}{\xi^2}\right)}{\sum_{n=1}^{\infty} nC_n \left(\xi^{n-1} - \frac{1}{\xi^{n+1}}\right)}.$$

Thus, at the remote boundary, we have from (4.18)

$$\psi_1(\xi) = -i\alpha_1 \left[K\overline{C_o} - iSD_o + \sum_{n=1}^{\infty}\left[\left(\frac{K\overline{C_n}}{R_*^{2n}} - iSD_n\right)\xi^n\right] + \lim_{|\xi|\to\infty}\left(S\overline{R_{(n,\xi)}}\right)\right] - \lim_{|\xi|\to\infty}\left(Q_{(n,\xi)}\right)$$

$$+i\alpha_2\left[\overline{C_o} + \sum_{n=1}^{\infty}\frac{\overline{C_n}}{R_*^{2n}}\xi^n\right] + \lim_{|\xi|\to\infty}\left(i\overline{R_{(n,\xi)}}\right) + D_o + \sum_{n=1}^{\infty} D_n\xi^n, \quad |\xi| \to \infty, \tag{4.19}$$

where

$$Q_{(n,\xi)} = \frac{i\beta_1 R^2\left(\frac{R_*^2}{\xi} + \frac{\xi}{R_*^2}\right)\left(1 - \frac{1}{\xi^2}\right)}{KC_1 + \frac{iS\overline{D_1}}{R_*^2} - \sum_{n=1}^{\infty}\left[n\left(KC_n + iS\overline{D_n}R_*^{2n}\right)\frac{1}{\xi^{n+1}}\right] + SR'(n,\xi)}.$$

From (4.9), ψ_1 must also satisfy the asymptotic condition

$$\psi_1(\xi) = BR\xi + O_{(2)}, \quad |\xi| \to \infty.$$

As in (4.14), we require from (4.19) that

$$BR\xi + O_{(2)} = -i\alpha_1\left[K\overline{C_o} - iSD_o + \sum_{n=1}^{\infty}\left[\left(\frac{K\overline{C_n}}{R_*^{2n}} - iSD_n\right)\xi^n\right] + \lim_{|\xi|\to\infty}\left(S\overline{R_{(n,\xi)}}\right)\right]$$

$$-\lim_{|\xi|\to\infty}\left(Q_{(n,\xi)}\right) + i\alpha_2\left[\overline{C_o} + \sum_{n=1}^{\infty}\frac{\overline{C_n}}{R_*^{2n}}\xi^n\right] + \lim_{|\xi|\to\infty}\left(i\overline{R_{(n,\xi)}}\right) + D_o + \sum_{n=1}^{\infty} D_n\xi^n, \quad |\xi| \to \infty.$$

Therefore, equating coefficients of ξ, we have

$$BR\xi + O_{(2)} = -i\alpha_1\left[K\overline{C_o} - iSD_o + \left(\frac{K\overline{C_1}}{R_*^2} - iSD_1\right)\xi\right]$$

$$+\lim_{|\xi|\to\infty}\left[i\overline{R_{(n,\xi)}}\left(1-\alpha_1 S\right)\right]+-\lim_{|\xi|\to\infty}\left(Q_{(n,\xi)}\right)+i\alpha_2\left[\overline{C_o}+\frac{\overline{C_1}}{R_*^2}\xi\right]+D_o+D_1\xi, \quad (4.20)$$

$$\sum_{n=2}^{\infty}\left[\left\{\frac{i\overline{C_n}}{R_*^{2n}}\left(-\alpha_1 K+\alpha_2\right)+D_n\left(1-\alpha_1 S\right)\right\}\xi^n\right]=0. \quad (4.21)$$

Recall (4.16). Together with (4.21), since $\xi = \rho e^{i\theta}, \quad \rho > 0$, we derive the following equations for $C_n, D_n, \quad n = 2, 3\ldots$

$$KC_n+\frac{iS\overline{D_n}}{R_*^{2n}}=0, \quad (4.22)$$

$$\frac{i\left(\alpha_1 K-\alpha_2\right)C_n}{R_*^{2n}}+\overline{D_n}\left(1-\alpha_1 S\right)=0. \quad (4.23)$$

Also from (4.3), for the class of harmornic materials discussed here

$$K, \ (1-\alpha_1 S)\neq 0. \quad (4.24)$$

We consider the following cases:

4.2.1 Case 1 : When $S\overline{D_n} = 0$

From (4.22) that we must have $KC_n = 0, \ n \geq 2$, and hence from (4.24)

$$C_n = 0, \quad n \geq 2,$$

Then, it follows from (4.23-4.24)

$$D_n = 0, \quad n \geq 2.$$

4.2.2 Case 2 : When $(\alpha_1 K - \alpha_2) C_n = 0$

From (4.23) together with (4.24)

$$D_n = 0, \quad n \geq 2,$$

Then, (4.22) and again (4.24) yield that

$$C_n = 0, \quad n \geq 2.$$

4.2.3 Case 3 : When $S\overline{D_n} = (\alpha_1 K - \alpha_2) C_n = 0$

In this case Equations (4.22-24) yield only that

$$C_n = D_n = 0, \quad n \geq 2.$$

4.2.4 Case 4 : When $S\overline{D_n}$, $(\alpha_1 K - \alpha_2) C_n \neq 0$

For the case when both $S\overline{D_n}$ and $(\alpha_1 K - \alpha_2) C_n$ are not equal to zero, we start with the assumption that

$$C_n, \ D_n \neq 0.$$

By letting $C_n = \mathrm{Re}(C_n) + i\,\mathrm{Im}(C_n)$, $D_n = \mathrm{Re}(D) + i\,\mathrm{Im}(D_n)$, we solve Equations (4.22-23) and obtain

$$\left[K(1 - \alpha_1 S) + \frac{(\alpha_1 K - \alpha_2) S}{R_*^{4n}}\right] C_n = 0, \tag{4.25}$$

$$\left[K(1 - \alpha_1 S) + \frac{(\alpha_1 K - \alpha_2) S}{R_*^{4n}}\right] D_n = 0.$$

Suppose that $C_n,\ D_n \neq 0$ then we must have

$$\left[K\left(1-\alpha_1 S\right)+\frac{\left(\alpha_1 K-\alpha_2\right) S}{R_*^{4n}}\right]=0, \tag{4.26}$$

otherwise, we have the same result as in Cases. 1-3, that is, $C_n = D_n = 0, \quad n \geq 2$. From (4.2), rewriting (4.26) in terms of α_γ and $\mu_\gamma > 0$ $\gamma = 1, 2$, we find that (4.26) reduces to

$$\left(\frac{\mu_2}{\mu_1}\right)^2 \alpha_1\left(\alpha_2-1\right)\left(\frac{1}{R_*^{4n}}-1\right)+\left(\frac{\mu_2}{\mu_1}\right)\left[1+\left(\alpha_1+\alpha_2-2\alpha_1\alpha_2\right)\left(\frac{1}{R_*^{4n}}-1\right)\right]$$

$$+\alpha_2\left(\alpha_1-1\right)\left(\frac{1}{R_*^{4n}}-1\right)=0$$

The two roots $\left(\frac{\mu_2}{\mu_1}\right)_1$ and $\left(\frac{\mu_2}{\mu_1}\right)_2$ are easily found to be:

$$\left(\frac{\mu_2}{\mu_1}\right)_1 = \frac{-\mathcal{B}+\sqrt{\mathcal{B}^2-4\mathcal{AC}}}{2\mathcal{A}}, \tag{4.27}$$

$$\left(\frac{\mu_2}{\mu_1}\right)_2 = \frac{-\mathcal{B}-\sqrt{\mathcal{B}^2-4\mathcal{AC}}}{2\mathcal{A}},$$

where

$$\mathcal{A} = \alpha_1\left(\alpha_2-1\right)\left(\frac{1}{R_*^{4n}}-1\right),\ \mathcal{B}=1+\left(\alpha_1+\alpha_2-2\alpha_1\alpha_2\right)\left(\frac{1}{R_*^{4n}}-1\right),$$

$$\mathcal{C} = \alpha_2\left(\alpha_1-1\right)\left(\frac{1}{R_*^{4n}}-1\right).$$

It is not difficult to show that (4.27) means that, (see Figures. 7 and 8)

$$\left(\frac{\mu_2}{\mu_1}\right)_1 < 0, \quad \left(\frac{\mu_2}{\mu_1}\right)_2 < 0.$$

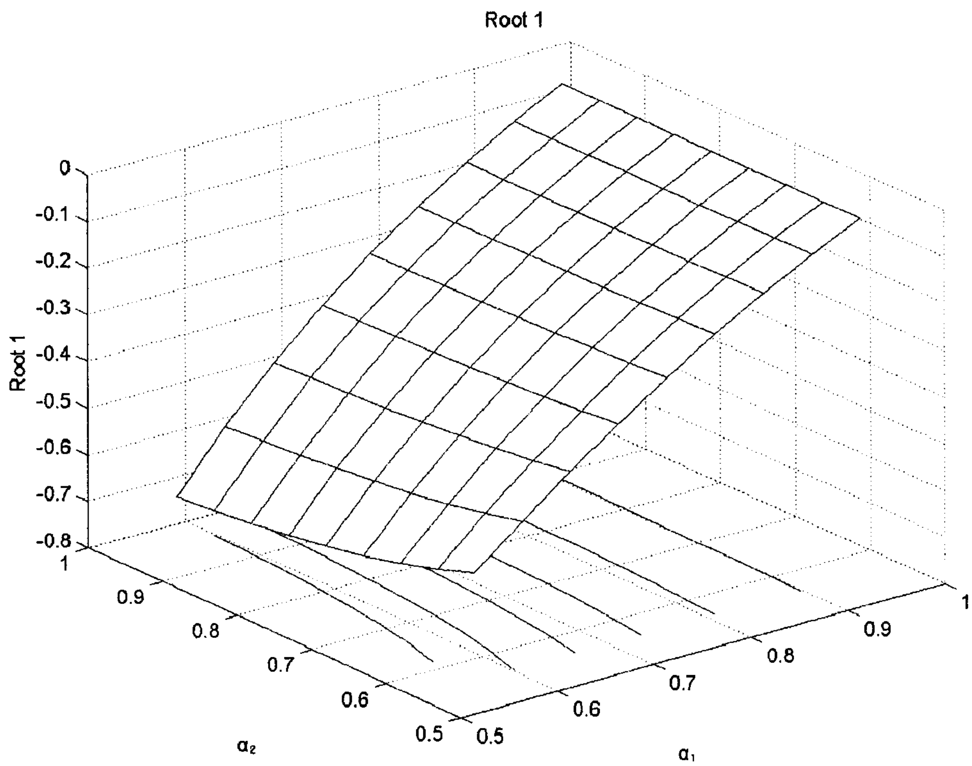

Figure 7: *First Root (equation 4.27):* $(\frac{\mu_2}{\mu_1})_1$, where$(1 > \alpha_\gamma \geq 1/2, \;\; \gamma = 1, 2)$

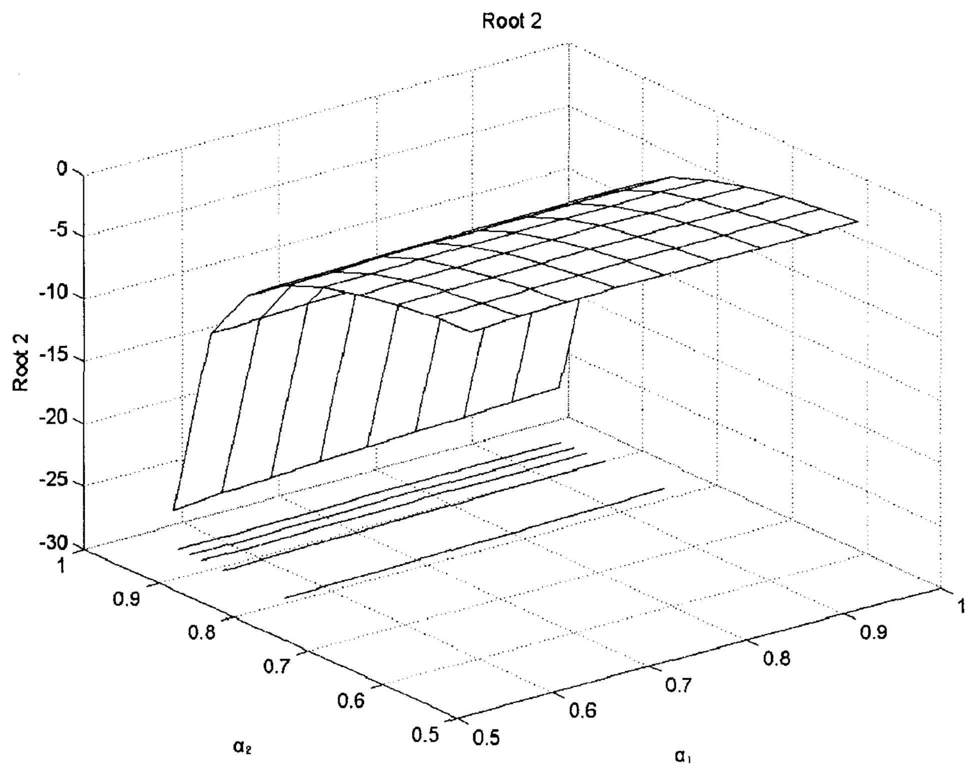

Figure 8: *Second Root (equation 4.27):* $(\frac{\mu_2}{\mu_1})_2$, where$(1 > \alpha_\gamma \geq 1/2, \ \ \gamma = 1, 2)$

This contradicts the assumptions made on the elastic constants for this class of materials, namely Ru (2002):

$$\begin{aligned} 1 &> \alpha_\gamma \geq 1/2, \ \beta_\gamma > 0, \ \alpha_\gamma + \beta_\gamma = 1, \ \mu_\gamma > 0, \\ \gamma &= 1, 2 \text{ (no sum over repeated indices).} \end{aligned}$$

Consequently, (4.26) cannot be true and

$$\left[K\left(1 - \alpha_1 S\right) + \frac{\left(\alpha_1 K - \alpha_2\right) S}{R_*^{2n}} \right] \neq 0. \tag{4.28}$$

From (4.25), (4.28) now implies that $C_n = D_n = 0, \quad n \geq 2$.

In conclusion, from (4.10), in the $\xi - plane$

$$\phi_2(\xi) = C_o + C_1\left(\xi + \frac{1}{\xi}\right), \quad \psi_2(\xi) = D_o + D_1\left(\xi + \frac{1}{\xi}\right), \quad 1 < |\xi| < R_*,$$

$$\because \sum_{n=1}^{n=\infty} C_n = C_1, \ \sum_{n=1}^{n=\infty} D_n = D_1, \quad (\because C_n = D_n = 0, \quad n \geq 2),$$

and through the mapping function (4.4), we have that in the $z - plane$

$$\phi_2(z) = C_o + \frac{C_1}{R}z, \quad \psi_2(z) = D_o + \frac{D_1}{R}z, \quad z \in S_2,$$

$$\because \ \frac{z}{R} = \left(\xi + \frac{1}{\xi}\right),$$

which implies from (2.18) and (2.28) that the state of (Piola) stress inside the inhomogeneity is necessarily uniform. In fact, from (2.26), we derive the Piola stress function χ inside the inhomogeneity:

$$\chi_2(z) = 2\mu_2 i[(\alpha_2 - 1)\left(C_o + \frac{C_1}{R}z\right) + i\overline{\left(D_o + \frac{D_1}{R}z\right)} + \frac{\beta_2 Rz}{\overline{C_1}}],$$

and from (2.18), the Piola stress inside inhomogeneity can be calculated as

$$-\sigma_{21} + i\sigma_{11} = (\chi_2)_{,2} = 2\mu_2 i[(\alpha_2 - 1)\frac{iC_1}{R} + \frac{\overline{D_1}}{R} + \frac{i\beta_2 R}{\overline{C_1}}],$$

$$\sigma_{22} - i\sigma_{12} = (\chi_2)_{,1} = 2\mu_2 i[(\alpha_2 - 1)\frac{C_1}{R} + \frac{i\overline{D_1}}{R} + \frac{\beta_2 R}{\overline{C_1}}],$$

which, clearly, is uniform.

We have just proved the following result concerning necessity in Eshelby's conjecture for this class of harmonic materials:

Theorem 1. *For this class of harmonic materials, if an inhomogeneity-matrix system is subjected to finite plane deformations and uniform remote loading, the Piola stress distribution within an elliptic inhomogeneity is necessarily uniform.*

4.3 SOLUTION FOR THE UNIFORM INTERIOR STRESS CASE

We can, in fact, obtain the complete solution corresponding to the elliptic inhomogeneity-matrix system under the assumption of uniform remote loading. In fact, from the results obtained in the previous section, the analytic functions ϕ_2 and ψ_2 take the form

$$\phi_2(z) = C_o + \frac{C_1}{R}z, \quad \psi_2(z) = D_o + \frac{D_1}{R}z, \tag{4.29}$$

where $C_o, C_1, D_o, D_1 \in \mathbb{C}$. From (4.4) and (4.7), on $|\xi| = R_*$,

$$\phi_1(\xi) = K\left[C_o + C_1\left(\xi + \frac{1}{\xi}\right)\right] + iS\left[\overline{D_o} + \overline{D_1}\left(\frac{R_*^2}{\xi} + \frac{\xi}{R_*^2}\right)\right] + \frac{S\beta_2 R^2\left(\xi + \frac{1}{\xi}\right)}{\overline{C_1}}, \tag{4.30}$$

where, from Remark 1, $C_1, \overline{C_1} \neq 0$. Also from (4.9)

$$\phi_1(\xi) = AR\xi + O_{(1)}, \quad |\xi| \to \infty. \tag{4.31}$$

Then, without loss of generality, the Laurent series expansion for ϕ_1 takes the form:

$$\phi_1(\xi) = \sum_{n=-\infty}^{n=1} X_n \xi^n, \quad |\xi| > R_*. \tag{4.32}$$

It follows from (4.30-32) that

$$\phi_1(\xi) = X_o + X_{-1}\frac{1}{\xi} + X_1\xi, \quad X_{-2} = X_{-3} = X_{-4} = \cdots = 0, \quad |\xi| > R_*, \tag{4.33}$$

where

$$X_o = O_{(1)} = KC_o + iS\overline{D_o},$$

$$X_1 = AR = KC_1 + \frac{iS\overline{D_1}}{R_*^2} + \frac{S\beta_2 R^2}{\overline{C_1}},$$

$$X_{-1} = KC_1 + iS\overline{D_1}R_*^2 + \frac{S\beta_2 R^2}{\overline{C_1}}.$$

In the view of Remark 7, we must have from (4.33) that

$$X_1 \neq \frac{X_{-1}}{\xi^2}, \quad \forall \xi : |\xi| > R_*.$$

From the interface condition (4.7), (4.29) and (4.33), it follows that for $|\xi| = R_*$,

$$\psi_1(\xi) = -i\alpha_1 \left[\overline{X_o} + \overline{X_{-1}}\frac{\xi}{R_*^2} + \overline{X_1}\frac{R_*^2}{\xi}\right] - \frac{i\beta_1 R^2 \left(\frac{R_*^2}{\xi} + \frac{\xi}{R_*^2}\right)\left(1 - \frac{1}{\xi^2}\right)}{X_1 - \frac{X_{-1}}{\xi^2}} \tag{4.34}$$

$$+ i\alpha_2 \left[\overline{C_o} + \overline{C_1}\left(\frac{R_*^2}{\xi} + \frac{\xi}{R_*^2}\right)\right] + \frac{i\beta_2 R^2 \left(\frac{R_*^2}{\xi} + \frac{\xi}{R_*^2}\right)}{C_1} + D_o + D_1\left(\xi + \frac{1}{\xi}\right).$$

From (4.9), ψ_1 must also satisfy the asymptotic condition

$$\psi_1(\xi) = BR\xi + O_{(2)}, \quad |\xi| \to \infty.$$

Hence, if we adopt (4.34) for the region $|\xi| > R_*$, we require that

$$-i\alpha_1 \left[\overline{X_o} + \overline{X_{-1}}\frac{\xi}{R_*^2}\right] - \frac{i\beta_1 R^2}{X_1 R_*^2}\xi + i\alpha_2 \left[\overline{C_o} + \frac{\overline{C_1}}{R_*^2}\xi\right] + \frac{i\beta_2 R^2}{C_1 R_*^2}\xi + D_o + D_1\xi =$$

$$BR\xi + O_{(2)}, \quad |\xi| \to \infty.$$

Thus, since $X_1 = AR$, we require

$$O_{(2)} = -i\alpha_1\overline{X_o} + i\alpha_2\overline{C_o} + D_o,$$

$$-\frac{i\alpha_1\overline{X_{-1}}}{R_*^2}-\frac{i\beta_1 R}{AR_*^2}+\frac{i\alpha_2\overline{C_1}}{R_*^2}+\frac{i\beta_2 R^2}{C_1R_*^2}+D_1=BR.$$

SUMMARY

The complete solution corresponding to the elliptic inhomogeneity-matrix system subjected to uniform remote loading is given by

$$\phi_2(z)=C_o+\frac{C_1}{R}z,\quad \psi_2(z)=D_o+\frac{D_1}{R}z,\quad |z|\in S_2, \tag{4.35}$$

$$\phi_1(\xi)=X_o+X_{-1}\frac{1}{\xi}+X_1\xi,\ X_o=O_{(1)},\ X_1=AR,\quad |\xi|>R_* \tag{4.36}$$

and

$$\begin{aligned}\psi_1(\xi) &= -i\alpha_1\left[\overline{X_o}+\overline{X_{-1}}\frac{\xi}{R_*^2}+\overline{X_1}\frac{R_*^2}{\xi}\right]-\frac{i\beta_1R^2\left(\frac{R_*^2}{\xi}+\frac{\xi}{R_*^2}\right)\left(1-\frac{1}{\xi^2}\right)}{X_1-\frac{X_{-1}}{\xi^2}} \\ &\quad +i\alpha_2\left[\overline{C_o}+\overline{C_1}\left(\frac{R_*^2}{\xi}+\frac{\xi}{R_*^2}\right)\right]+\frac{i\beta_2R^2\left(\frac{R_*^2}{\xi}+\frac{\xi}{R_*^2}\right)}{C_1}+D_o+D_1\left(\xi+\frac{1}{\xi}\right).\end{aligned} \tag{4.37}$$

where , $|\xi|>R_*$. Here, the complex constants C_o, C_1, D_o and D_1 are determined by the equations

$$\begin{aligned}KC_o+iS\overline{D_o}, &= X_o=O_{(1)}, \\ -i\alpha_1\overline{X_o}+i\alpha_2\overline{C_o}+D_o &= O_{(2)},\end{aligned}$$

$$
\begin{aligned}
AR &= KC_1 + \frac{iS\overline{D_1}}{R_*^2} + \frac{S\beta_2 R^2}{\overline{C_1}}, \qquad (4.38) \\
X_{-1} &= KC_1 + iS\overline{D_1}R_*^2 + \frac{S\beta_2 R^2}{\overline{C_1}}, \\
BR &= -\frac{i\alpha_1 \overline{X_{-1}}}{R_*^2} - \frac{i\beta_1 R}{AR_*^2} + \frac{i\alpha_2 \overline{C_1}}{R_*^2} + \frac{i\beta_2 R^2}{C_1 R_*^2} + D_1.
\end{aligned}
$$

where from Remarks. 6 and 7

$$
\begin{aligned}
C_1 &\neq 0 \Longleftrightarrow \overline{C_1} \neq 0, \\
X_1 &\neq \frac{X_{-1}}{\xi^2}, \quad \forall \xi : |\xi| > R_*.
\end{aligned}
$$

We note that the results obtained in Ru et al (2005) arise as a special case of those obtained here (see Appendix).

4.4 FURTHER DISCUSSION

From the results obtained in Section 4.3, we easily obtain the following important properties of this inhomogeneity-matrix system. Since, these properties will provide the actual response of the elliptic inhomogeneity-matrix system under the prescribed remote loading.

4.4.1 Expressions for I and J Inside the Inhomogeneity

From (2.1) and (2.6), R and J can be calculated as follow:

$$
J = \det \mathbf{F} = -\operatorname{Im}\left[w_{,1}\overline{w}_{,2}\right] = y_{1,1}y_{2,2} - y_{1,2}y_{2,1}, \qquad (4.39)
$$

$$R = tr(\mathbf{F}\mathbf{F}^T) = |w_{,1}|^2 + |w_{,2}|^2 = y_{p,s}y_{p,s}\ , ...p, s = 1, 2, \tag{4.40}$$

Hence, in the view of (2.1), Equation (2.5) yields

$$\begin{aligned} I &= \sqrt{R + 2J} = \sqrt{tr(\mathbf{F}\mathbf{F}^T) + 2J}, \\ &= \sqrt{(y_{1,1} + y_{2,2})^2 + (y_{1,2} - y_{2,1})^2}, \end{aligned} \tag{4.41}$$

where

$$(y_{1,1} + y_{2,2})^2 + (y_{1,2} - y_{2,1})^2 = 2\left(y_{1,1}y_{2,2} - y_{1,2}y_{2,1},\right) + y_{11}^2 + y_{22}^2 + y_{12}^2 + y_{21}^2.$$

Equation (2.25) and (4.35) give us that

$$w_2 = -i\alpha_2[C_o + \frac{C_1}{R}z] + \overline{D_o} + \frac{\overline{D_1}}{R}\overline{z} - \frac{i\beta_2 Rz}{\overline{C_1}}. \tag{4.42}$$

Consequently, I takes the following form within the inhomogeneity

$$I = 2\left(\alpha_2\left|\frac{C_1}{R}\right| + \frac{\beta_2}{\left|\frac{C_1}{R}\right|}\right), \tag{4.43}$$

and from Remark 6.

$$|\phi_2'(z)| = \left|\frac{C_1}{R}\right| \neq 0, \quad \forall\xi : |\xi| > 1,$$

$$I^2 - 16\alpha_2\beta_2 = 4\left(\alpha_2\left|\frac{C_1}{R}\right| - \frac{\beta_2}{\left|\frac{C_1}{R}\right|}\right)^2 > 0, \tag{4.44}$$

which guarantees that $F'(I)$, from (2.21) is well-defined.

4.4.2 Stress Distribution Inside Inhomogeneity

From (2.18), the Piola stresses inside the inhomogeneity and at the remote boundary are given, respectively, by

$$\begin{aligned} -\sigma_{21} + i\sigma_{11} &= (\chi_2)_{,2}\,, \quad \sigma_{22} - i\sigma_{12} = (\chi_2)_{,1}\,, \\ -\sigma_{21}^{\infty} + i\sigma_{11}^{\infty} &= (\chi_1)_{,2}\,, \quad \sigma_{22}^{\infty} - i\sigma_{12}^{\infty} = (\chi_1)_{,1}\,. \end{aligned} \tag{4.45}$$

From Equation (2.26), (4.2) and (4.29) (or (4.35))

$$\chi_1(z) = 2\mu_1 i[(\alpha_1 - 1)(Az + O_{(1)}) + i(\overline{B}\bar{z} + \overline{O_{(2)}}) + \frac{\beta_1 z}{\overline{A}}],$$

$$\chi_2(z) = 2\mu_2 i[(\alpha_2 - 1)(\frac{C_1}{R}z + C_o) + i(\frac{\overline{D_1}}{R}\bar{z} + \overline{D_o}) + \frac{\beta_2 z R}{\overline{C_1}}].$$

Thus, the corresponding uniform Piola stress can be found from (4.45) as follows:

$$-\sigma_{21} + i\sigma_{11} = 2\mu_2 i[i(\alpha_2 - 1)\frac{C_1}{R} + \frac{\overline{D_1}}{R} + \frac{i\beta_2 R}{\overline{C_1}}],$$

$$\sigma_{22} - i\sigma_{12} = 2\mu_2 i[(\alpha_2 - 1)\frac{C_1}{R} + \frac{i\overline{D_1}}{R} + \frac{\beta_2 R}{\overline{C_1}}],$$

$$-\sigma_{21}^{\infty} + i\sigma_{11}^{\infty} = 2\mu_1 i[i(\alpha_1 - 1)A + \overline{B} + \frac{i\beta_1}{\overline{A}}],$$

$$\sigma_{22}^{\infty} - i\sigma_{12}^{\infty} = 2\mu_1 i[(\alpha_1 - 1)A + i\overline{B} + \frac{\beta_1}{\overline{A}}].$$

4.4.3 Deformed Contour

From the expression (4.42) of the deformation inside the inhomogeneity, it is clear that the curve Γ remains elliptical after deformation. Hence the deformed curve does not admit overlapping (see Figure 9).

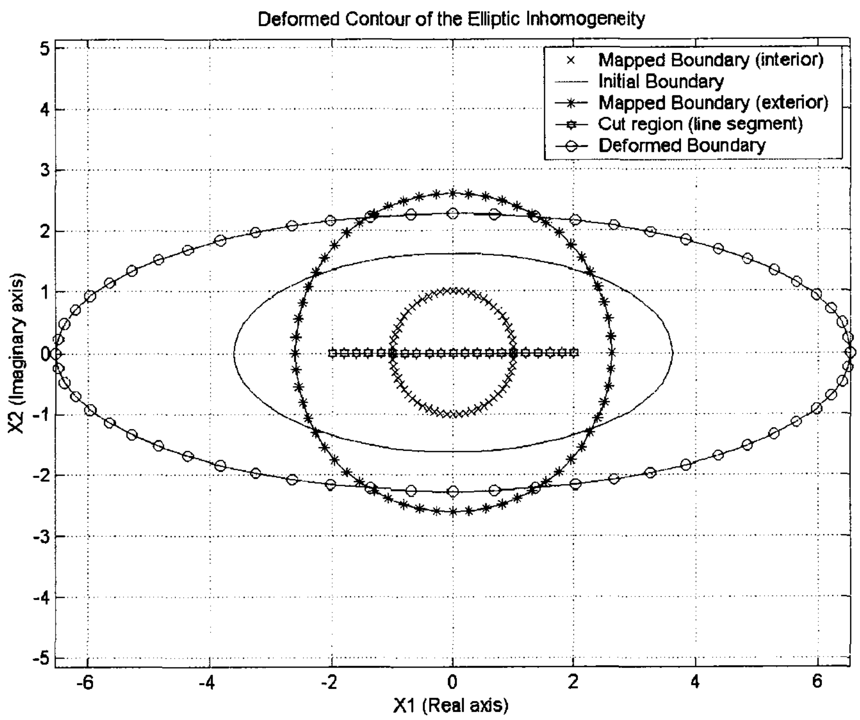

Figure 9: *Deformed Contour of the Elliptic Inhomogeneity.* Here, we have taken $(A = 0.21i, B = -0.33, O_{(1)} = O_{(2)} = 0, \mu_1 = 0.27, \mu_2 = 0.3, \alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 0.5)$

4.4.4 Admissibility Conditions

From (4.40) and (4.42-44), inside the inhomogeneity,

$$I^2 > 16\alpha_2\beta_2, \quad J > 0. \tag{4.46}$$

As in Ru (2002), (4.46) again ensures that there is no overlapping of the deformation field (interior to the inhomogeneity). More precisely, (4.46) guaranties that the strain energy density function $F'(I)$ from (2.21) remains a positive real valued function. In addition, $J > 0$ implies that relative orientation of the line elements is preserved under deformation (see, Gurtin, 1981; Chadwick, 1999; Fung, 1977). Henceforth, there is no overlapping after deformation.

4.5 CONCLUSIONS

We have show that, under the analogous constraints of perfect bonding and uniform remote (Piola) stress if the inhomogeneity-matrix system is subjected to finite plane deformations, the Piola stress distribution within the elliptic inhomogeneity is necessarily uniform. In addition, we have obtained the complete solution for this system for any uniform remote stress distribution.

Chapter 5
DESIGNING AN INHOMOGENEITY WITH UNIFORM INTERIOR STRESS

5.1 INTRODUCTION

Following our investigation of Eshelby's conjecture, it is of interest to examine whether we can design the remote loading and the shape of the inhomogeneity (fiber) to achieve a state of uniform stress inside the inhomogeneity.

In this Chapter, we address this issue and consider an inhomogeneity-matrix system in which the inhomogeneity is perfectly bonded to the surrounding matrix and subjected to some arbitrary nonuniform remote loading. In fact, we show that if the Piola stress distribution inside the inhomogeneity is uniform and the system is subjected to linear remote loading, the inhomogeneity is necessarily hypotrochoidal, with no exceptions (Kim & Schiavone, 2007c). We also obtain the complete solution of the inhomogeneity-matrix system in this case. In addition, we consider the more general case when the system is subjected to nonuniform remote loading characterized by stress functions of the form of n^{th} degree polynomials in the complex variable z describing the matrix. In this case, by finding the complete solution of the system, we show that we can again obtain uniform stress inside a hypotrochoidal inhomogeneity with $n+1$ cusps. Finally, we illustrate our results with an example.

In the view of Remarks. 1 and 2, we first consider the simpler problem corresponding to the case in which the inhomogeneity-matrix system is subjected to linear remote loading characterized by analytic functions ϕ_1 and ψ_1 of the form:

$$\phi_1(z) = A_1 z + A_2 z^2, \quad \psi_1(z) = B_1 z + B_2 z^2, \quad |z| \to \infty, \tag{5.1}$$

where A_1, A_2, B_1 and B_2 are prescribed complex constants.

5.2 UNIFORM STRESS DISTRIBUTION INSIDE THE INHOMOGENEITY

Suppose, that the state of Piola stress inside the inhomogeneity is uniform, characterized by

$$\phi_2(z) = Cz, \quad \psi_2(z) = Dz, \quad z \in S_2, \tag{5.2}$$

where C and D are two complex constants to be determined.

First, we note that the interface conditions (2.27-28) can be written in the form

$$\phi_1(z) = K\phi_2(z) + i\mathcal{S}\overline{\psi_2(z)} + \frac{\mathcal{S}\beta_2 z}{\phi_2'(z)},$$

$$\psi_1(z) = i\left(-\alpha_1\overline{\phi_1(z)} - \frac{\beta_1\overline{z}}{\phi_1'(z)} + \alpha_2\overline{\phi_2(z)} + \frac{\beta_2\overline{z}}{\phi_2'(z)}\right) + \psi_2(z), \quad z \in \Gamma, \tag{5.3}$$

where the constants K and $\mathcal{S}$ are defined by $K = \frac{\mu_2}{\mu_1} + \alpha_2\left(1 - \frac{\mu_2}{\mu_1}\right)$ and $\mathcal{S} = \left(1 - \frac{\mu_2}{\mu_1}\right)$.

From (5.2) and the first of (5.3), we now have that

$$\phi_1(z) = KCz + i\$\overline{D}\overline{z} + \frac{\$\beta_2 z}{\overline{C}}, \quad z \in \Gamma. \tag{5.4}$$

Without loss of generality, we assume that there is a mapping function (Muskhelishvili, 1953)

$$z = \omega(\xi) = R\left(\xi + \frac{m_1}{\xi} + \frac{m_2}{\xi^2} + \cdots\right), \quad R \neq 0, \tag{5.5}$$

which maps the region S_1 onto the exterior of the unit circle in the $\xi - plane$. Here, for simplicity, we have chosen the origin of the coordinate system such that no constant term appears on the right-hand side of (5.5). For convenience, we wirte $\phi_\gamma(z) = \phi_\gamma(\omega(\xi)) = \phi_\gamma(\xi)$, $\gamma = 1, 2$ and similarly for the function ψ_γ. From (5.1) and (5.5), the remote loading condition in the $\xi - plane$ is taken to be

$$\phi_1(\xi) = A_1 R\xi + A_2 R^2\left(\xi^2 + 2m_1\right), \quad |\xi| \to \infty. \tag{5.6}$$

From (5.4-5), in the $\xi - plane$, the interface condition becomes:

$$\phi_1(\xi) = \left(KC + \frac{\$\beta_2}{\overline{C}}\right)\omega(\xi) + i\$\overline{D}\overline{\omega(\xi)}, \quad |\xi| = 1. \tag{5.7}$$

In addition, expanding $\phi_1(\xi)$ in Laurent series outside the unit circle $|\xi| = 1$, and using (5.6), we obtain

$$\phi_1(\xi) = X_2\xi^2 + X_1\xi + X_o + X_{-1}\frac{1}{\xi} + X_{-2}\frac{1}{\xi^2} + \cdots, \tag{5.8}$$

$$X_1 = A_1 R \neq 0, \; X_2 = A_2 R^2 \neq 0, \; X_o = 2A_2 R^2 m_1 \quad |\xi| > 1,$$

$$(\because A_1 R\xi + A_2 R^2\left(\xi^2 + 2m_1\right) = X_2\xi^2 + X_1\xi + X_o, \quad |\xi| \to \infty).$$

Since $\xi\overline{\xi} = |\xi|^2 = 1$ at the interface, to satisfy the boundary condition (5.7), it is clear that, from (5.8), we must have, for $|\xi| = 1$,

$$\begin{aligned}\phi_1(\xi) &= X_2\xi^2 + X_1\xi + X_o + X_{-1}\frac{1}{\xi} + X_{-2}\frac{1}{\xi^2} + \cdots \qquad (5.9)\\ &= \left(KC + \frac{\$\beta_2}{\overline{C}}\right) R\left(\xi + \frac{m_1}{\xi} + \frac{m_2}{\xi^2} + \cdots\right) + i\$\overline{D}R\left(\frac{1}{\xi} + m_1\xi + m_2\xi^2 + \cdots\right).\end{aligned}$$

We consider the following cases:

5.2.1 Case 1: When $\$D = 0, KC + \frac{\$\beta_2}{\overline{C}} \neq 0$

In this case (5.9) yields:

$$\begin{aligned}\phi_1(\xi) &= X_2\xi^2 + X_1\xi + X_o + X_{-1}\frac{1}{\xi} + X_{-2}\frac{1}{\xi^2} + \cdots\\ &= \left(KC + \frac{\$\beta_2}{\overline{C}}\right) R\left(\xi + \frac{m_1}{\xi} + \frac{m_2}{\xi^2} + \cdots\right), \quad |\xi| = 1.\end{aligned}$$

It is clear that this equation requires $X_2 = 0$. However, from (5.8), $X_2 \neq 0$, so that we cannot have solutions for this case (Note that the case $X_2 = 0$ corresponds to the case when the inhomogeneity- matrix system is subjected to uniform remote loading (see, for example, Ru et al, 2005).

5.2.2 Case 2: When $KC + \frac{\$\beta_2}{\overline{C}} = 0, \$D \neq 0$

In this case (5.9) yields

$$\begin{aligned}\phi_1(\xi) &= X_2\xi^2 + X_1\xi + X_o + X_{-1}\frac{1}{\xi} + X_{-2}\frac{1}{\xi^2} + \cdots \\ &= i\$\overline{D}R\left(\frac{1}{\xi} + m_1\xi + m_2\xi^2 + \cdots\right), \quad |\xi| = 1.\end{aligned}$$

However, from (5.8), we must have

$$\begin{aligned}X_1 &= A_1R = m_1 \neq 0, \\ X_o &= 2A_2R^2m_1 \neq 0, \quad A_1, A_2, R \neq 0,\end{aligned}$$

which again implies that there can be no solutions in this case.

5.2.3 Case 3: When $\$D \neq 0$ and $KC + \frac{\$\beta_2}{\overline{C}} \neq 0$

From (5.9), it is clear that we must have

$$m_1 = m_3 = m_4 = \cdots = 0.$$

Then, the mapping function (5.5) must take the form:

$$z = \omega(\xi) = R\left(\xi + \frac{m_2}{\xi^2}\right), \quad R, m_2 \neq 0. \tag{5.10}$$

In other words, we have just proved the following result.

Theorem 2. *If the inhomogeneity-matrix system is subjected to linear remote loading characterized by the quadratic stress functions* $\phi_1(z) = A_1z + A_2z^2, \psi_1(z) = B_1z + B_2z^2$ *and the state of (Piola) stress inside the inhomogeneity is uniform, then the* *inhomogeneity is a hypotrochoid with three cusps (Muskhelishvili, 1953) with mapping function*

$$z = \omega(\xi) = R\left(\xi + \frac{m}{\xi^n}\right), \quad R > 0, \ 0 < m \leq \frac{1}{n}, \quad n = 2,$$

provided that

$$m, \ \$D, \ KC + \frac{\$\beta_2}{\overline{C}} \neq 0.$$

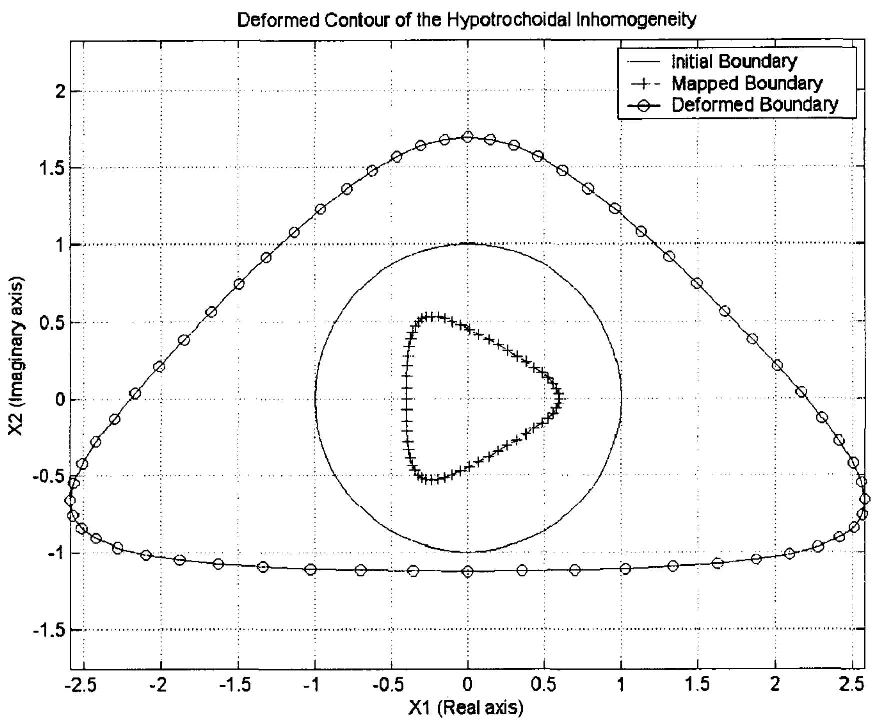

Figure 10: *Deformed Contour of the Hypotrochoidal Inhomogeneity with* 3 *Cusps* ($n = 2$). Here, we have taken ($A_1 = 1, A_3 = 0.17, \mu_1 = 0.27, \mu_2 = 0.3, \alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 0.5$)

5.3 COMPLETE SOLUTION OF A HYPOTROCHOIDAL INHOMOGENEITY

We now proceed to find the complete solution in the case corresponding to Theorem 2. To this end, assume that Γ is a hypotrochoid with three cusps and, without loss of generality, assume that the coordinate system is chosen to be symmetric about the principal axes of the hypotrochoid so that the corresponding mapping function is given by:

$$z = \omega(\xi) = R\left(\xi + \frac{m}{\xi^2}\right), \quad R > 0, \;\; 0 < m \leq \frac{1}{2}. \tag{5.11}$$

We seek the solution corresponding to the following uniform Piola stress distribution inside the inhomogeneity:

$$\phi_2(z) = Cz, \quad \psi_2(z) = Dz, \quad z \in S_2, \tag{5.12}$$

where C and D are two complex constants to be determined. From (5.7) and (5.11), on $|\xi| = 1$,

$$\phi_1(\xi) = \left(KC + \frac{\$\beta_2}{\overline{C}}\right) R\left(\xi + \frac{m}{\xi^2}\right) + i\$\overline{D}R\left(\frac{1}{\xi} + m\xi^2\right). \tag{5.13}$$

Also, from (5.1) and (5.11),

$$\phi_1(\xi) = A_1 R\xi + A_2 R^2 \xi^2, \quad |\xi| \to \infty. \tag{5.14}$$

It follows from (5.8) and (5.13-14) that

$$\begin{aligned} \phi_1(\xi) &= X_2\xi^2 + X_1\xi + \frac{X_{-1}}{\xi} + \frac{X_{-2}}{\xi^2}, \quad X_o = X_{-3} = X_{-4} = \cdots = 0, \quad (5.15) \\ |\xi| &> 1, \end{aligned}$$

where

$$X_2 = A_2 R^2 = i\mathcal{S}\overline{D}Rm, \quad X_1 = A_1 R = \left(KC + \frac{\mathcal{S}\beta_2}{\overline{C}}\right) R,$$

$$X_{-1} = \frac{A_2 R^2}{m} = i\mathcal{S}\overline{D}R, \quad X_{-2} = A_1 Rm = \left(KC + \frac{\mathcal{S}\beta_2}{\overline{C}}\right) Rm,$$

and, from Theorem 2,

$$m, \ \ \mathcal{S}D, \ \ KC + \frac{\mathcal{S}\beta_2}{\overline{C}} \neq 0.$$

Remark 8. *Since, throughout the entire plane (see Remark 1)*

$$|\phi'(z)| = F'(I) \neq 0$$

and

$$\omega'(\xi) \neq 0, \quad |\xi| > 1,$$

it is clear that we must have

$$\phi_1'(\xi) \neq 0, \quad |\xi| > 1.$$

In other words, from (5.8), we must have that

$$2X_2\xi + X_1 - \frac{X_{-1}}{\xi^2} - \frac{2X_{-2}}{\xi^3} \neq 0, \quad \forall \xi : |\xi| > 1.$$

This implies, together with (5.15), that there are restrictions on the constants A_1, A_2, B_1*and* B_2 *characterizing the remote stresses.*

It remains to determine the analytic function $\psi_1(z)$. From the interface conditions the (5.3)

$$\psi_1(\xi) = i\left(-\alpha_1\overline{\phi_1(\xi)} - \frac{\beta_1\overline{\omega(\xi)}\omega'(\xi)}{\phi_1'(\xi)} + \alpha_2\overline{\phi_2(\xi)} + \frac{\beta_2\overline{\omega(\xi)}\omega'(\xi)}{\phi_2'(\xi)}\right) + \psi_2(\xi), \quad |\xi| = 1,$$

$$\because \quad \phi_\gamma'(z) = \frac{d\phi(z)}{dz} = \frac{d\phi(\xi)}{d\xi}\frac{d\xi}{dz} = \frac{\phi'(\xi)}{\omega'(\xi)}, \quad \overline{\phi_\gamma'(z)} = \overline{\left(\frac{\phi'(\xi)}{\omega'(\xi)}\right)} = \frac{\overline{\phi'(\xi)}}{\overline{\omega'(\xi)}},$$

and form (5.11-12) and (5.15), it follows that, for $|\xi| = 1$,

$$\psi_1(\xi) = -i\alpha_1\left(\frac{\overline{X_2}}{\xi^2} + \frac{\overline{X_1}}{\xi} + \overline{X_{-1}}\xi + \overline{X_{-2}}\xi^2\right) - \frac{i\beta_1 R^2\left(\frac{1}{\xi} + m\xi^2\right)\left(1 - \frac{2m}{\xi^3}\right)}{2X_2\xi + X_1 - \frac{X_{-1}}{\xi^2} - \frac{2X_{-2}}{\xi^3}}$$

$$+ i\alpha_2\overline{C}R\left(\frac{1}{\xi} + m\xi^2\right) + \frac{i\beta_2 R\left(\frac{1}{\xi} + m\xi^2\right)}{C} + DR\left(\xi + \frac{m}{\xi^2}\right). \tag{5.16}$$

From (5.1) and (5.11), ψ_1 must also satisfy the asymptotic condition

$$\psi_1(\xi) = B_1 R\xi + B_2 R^2\xi^2, \quad |\xi| \to \infty.$$

Hence, If we adopt ψ_1 from (5.16) for $|\xi| > 1$, we require that

$$\begin{aligned} &-i\alpha_1\left(\overline{X_{-1}}\xi + \overline{X_{-2}}\xi^2\right) - \frac{i\beta_1 R^2 m}{2X_2}\xi + i\alpha_2\overline{C}Rm\xi^2 + \frac{i\beta_2 Rm}{C}\xi^2 + DR\xi \\ = \;& B_1 R\xi + B_2 R^2\xi^2, \quad |\xi| \to \infty. \end{aligned}$$

Thus, since $X_{-1} = \dfrac{A_2 R^2}{m}$, $X_{-2} = A_1 Rm$, we require that

$$B_1 = -\frac{i\alpha_1\overline{A_2}R}{m} - \frac{i\beta_1 m}{2A_2 R} + D,$$

$$B_2 = -\frac{i\alpha_1\overline{A_1}m}{R} + \frac{i\alpha_2\overline{C}m}{R} + \frac{i\beta_2 m}{RC}.$$

From (5.15), by letting $A_1 = \operatorname{Re} A_1 + i \operatorname{Im} A_1,\ C = \operatorname{Re} C + i \operatorname{Im} C$, we can solve for C and D, in terms of A_1 and A_2. In fact,

$$A_1 \overline{C} = K |C|^2 + \mathcal{S}\beta_2,$$

which yields

$$K \left[(\operatorname{Re} C)^2 + (\operatorname{Im} C)^2 \right] - \operatorname{Re} A_1 \operatorname{Re} C - \operatorname{Im} A_1 \operatorname{Im} C$$

$$+ i \left(\operatorname{Re} A_1 \operatorname{Im} C - \operatorname{Im} A_1 \operatorname{Re} C \right) + \mathcal{S}\beta_2 = 0,$$

Then, we can derive two separate equations from the above equation by equating real and imaginary parts, respectively:

$$\operatorname{Re} A_1 \operatorname{Im} C - \operatorname{Im} A_1 \operatorname{Re} C = 0, \quad \text{(imaginary part)} \tag{5.18}$$

$$K \left[(\operatorname{Re} C)^2 + (\operatorname{Im} C)^2 \right] - \operatorname{Re} A_1 \operatorname{Re} C - \operatorname{Im} A_1 \operatorname{Im} C + \mathcal{S}\beta_2 = 0, \quad \text{(real part)}. \tag{5.19}$$

From (5.18)

$$\operatorname{Im} C = \frac{\operatorname{Im} A_1}{\operatorname{Re} A_1} \operatorname{Re} C, \tag{5.20}$$

and from (5.21), Equation (5.19) can now be written in the form

$$K (\operatorname{Re} C)^2 \left[\frac{|A_1|^2}{(\operatorname{Re} A_1)^2} \right] - \operatorname{Re} C \left[\frac{|A_1|^2}{\operatorname{Re} A_1} \right] + \mathcal{S}\beta_2 = 0, \tag{5.21}$$

where

$$|A_1|^2 = A_1 \overline{A_1} = (\operatorname{Re} A_1)^2 + (\operatorname{Im} A_1)^2 .$$

Solving Equation (5.21) in terms of $\operatorname{Re} C$:

$$\operatorname{Re} C = \frac{\operatorname{Re} A_1 \left[|A_1| \pm \sqrt{|A|^2 - 4\beta_2 \mathcal{S} K} \right]}{2K\,|A|}, \tag{5.22}$$

$$\operatorname{Im} C = \frac{\operatorname{Im} A_1 \left[|A_1| \pm \sqrt{|A|^2 - 4\beta_2 \mathcal{S} K} \right]}{2K\,|A|}, \text{ (from (5.20)).}$$

Recalling that $C = \operatorname{Re} C + i \operatorname{Im} C$, we finally have the expression of the complex constant C in terms of A_1:

$$C = \mathcal{J} A_1, \tag{5.23}$$

where

$$\mathcal{J} = \left(\frac{|A_1| \pm \sqrt{|A|^2 - 4\beta_2 \mathcal{S} K}}{2K\,|A|} \right).$$

In addition, from (5.15), the unknown complex constant D can be determined by the relation:

$$D = \frac{iR}{\mathcal{S} m} \overline{A_2}. \tag{5.24}$$

Consequently, the constants C and D are therefore completely determined from (5.23) and (5.24). Also, from (5.17), B_1 and B_2 can be written in terms of A_1 and A_2 :

$$B_1 = \overline{A_2} \frac{iR}{m} \left(\frac{1}{\mathcal{S}} - \alpha_1 \right) - \frac{1}{A_2} \left(\frac{i\beta_1 m}{2R} \right), \tag{5.25}$$

$$B_2 = \overline{A_1} \frac{im}{R} \left(\alpha_2 \overline{\mathcal{J}} - \alpha_1 \right) + \frac{1}{A_1} \left(\frac{i\beta_2 m}{R \mathcal{J}} \right).$$

SUMMARY

The complete solution of the inhomogeneity-matrix system subjected to linear remote loading described by (5.1) is given by

$$\phi_2(z) = Cz, \quad \psi_2(z) = Dz, \quad |z| \in S_2, \tag{5.26}$$

$$\phi_1(\xi) = X_2\xi^2 + X_1\xi + \frac{X_{-1}}{\xi} + \frac{X_{-2}}{\xi^2}, \tag{5.27}$$

where

$$X_2 = A_2R^2 = i\$\overline{D}Rm, \quad X_1 = A_1R = \left(KC + \frac{\$\beta_2}{\overline{C}}\right)R,$$

$$X_{-1} = \frac{A_2R^2}{m} = i\$\overline{D}R, \quad X_{-2} = A_1Rm = \left(KC + \frac{\$\beta_2}{\overline{C}}\right)Rm,$$

and

$$\psi_1(\xi) = -i\alpha_1\left(\frac{\overline{X_2}}{\xi^2} + \frac{\overline{X_1}}{\xi} + \overline{X_{-1}}\xi + \overline{X_{-2}}\xi^2\right) - \frac{i\beta_1R^2\left(\frac{1}{\xi} + m\xi^2\right)\left(1 - \frac{2m}{\xi^3}\right)}{2X_2\xi + X_1 - \frac{X_{-1}}{\xi^2} - \frac{2X_{-2}}{\xi^3}}$$

$$+i\alpha_2\overline{C}R\left(\frac{1}{\xi} + m\xi^2\right) + \frac{i\beta_2R\left(\frac{1}{\xi} + m\xi^2\right)}{C} + DR\left(\xi + \frac{m}{\xi^2}\right), \tag{5.28}$$

where , $|\xi| > 1$. Here, the complex constants C and D are determined by the equations

$$C = \mathcal{J}A_1, \quad D = \frac{iR}{\$m}\overline{A_2},$$

$$\mathcal{J} = \left(\frac{|A_1| \pm \sqrt{|A_1|^2 - 4\beta_2\$K}}{2K|A_1|}\right),$$

where, from Remark 8 and Theorem 2, we must have that

$$2X_2\xi + X_1 - \frac{X_{-1}}{\xi^2} - \frac{2X_{-2}}{\xi^3} \neq 0, \quad \forall\xi : |\xi| > 1,$$

$$C, \; A_1, \; A_2 \neq 0 \text{ (to guarantee non-uniform remote stresses)},$$

$$m, \; \mathbb{S}D, \; KC + \frac{\mathbb{S}\beta_2}{\overline{C}} \neq 0.$$

Finally, B_1 and B_2 are given by:

$$B_1 = \overline{A_2}\frac{iR}{m}\left(\frac{1}{\mathbb{S}} - \alpha_1\right) - \frac{1}{A_2}\left(\frac{i\beta_1 m}{2R}\right),$$

$$B_2 = \overline{A_1}\frac{im}{R}\left(\alpha_2\overline{\mathbb{J}} - \alpha_1\right) + \frac{1}{A_1}\left(\frac{i\beta_2 m}{R\mathbb{J}}\right).$$

5.4 SOLUTION OF A HYPOTROCHOIDAL INHOMOGENEITY (GENERAL)

We can extend the results obtained above, to the case when the inhomogeneity-matrix system is subjected to non-uniform (not necessarily linear) remote loading characterized by the analytic functions ϕ_1 and ψ_1 in the form:

$$\phi_1(z) = A_1 z + A_n z^n, \quad \psi_1(z) = B_1 z + B_n z^n, \quad |z| \to \infty, \quad n \geq 2. \tag{5.29}$$

Supppose that the inhomogeneity is hypotrochoidal with $n+1$ cusps described by the conformal mapping function (Muskhelishvili, 1953),

$$z = \omega(\xi) = R\left(\xi + \frac{m_n}{\xi^n}\right), \quad R > 0, \; 0 < m_n \leq \frac{1}{n}. \tag{5.30}$$

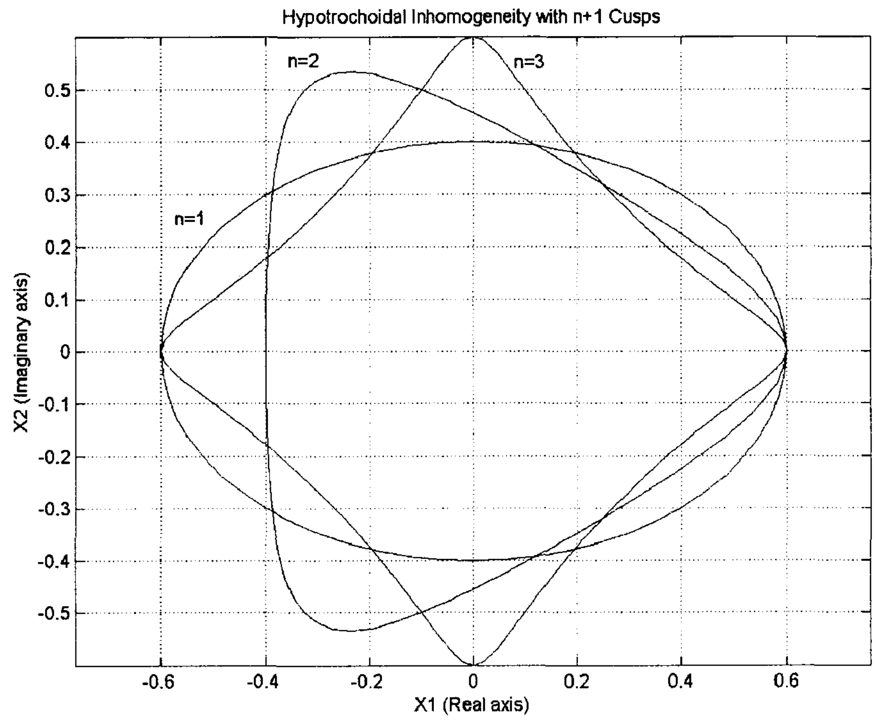

Figure 11: *Example of Hypotrochoidal Inhomogeneity with* $n+1$ *Cusps.*

We assune that the state of Piola stress inside the inhomogeneity is uniform and described by the stress functions:

$$\phi_2(z) = Cz, \quad \psi_2(z) = Dz, \quad z \in S_2. \tag{5.31}$$

From (5.1) and (5.30)

$$\phi_1(\xi) = A_1 R\xi + A_n R^n \xi^n, \quad \psi_1(\xi) = B_1 R\xi + B_n R^n \xi^n, \quad |\xi| \to \infty. \tag{5.32}$$

Applying the same procedure used in Section 5.4 above, we find that the analytic functions ϕ_1 and ψ_1 for $|\xi| > 1$ are given by

$$\phi_1(\xi) = X_n\xi^n + X_1\xi + \frac{X_{-1}}{\xi} + \frac{X_{-n}}{\xi^n}, \tag{5.33}$$

where

$$X_n = A_nR^n = i\mathcal{S}\overline{D}Rm_n, \quad X_1 = A_1R = \left(KC + \frac{\mathcal{S}\beta_2}{\overline{C}}\right)R,$$

$$X_{-1} = \frac{A_nR^n}{m_n} = i\mathcal{S}\overline{D}R, \quad X_{-n} = A_1Rm_n = \left(KC + \frac{\mathcal{S}\beta_2}{\overline{C}}\right)Rm_n,$$

$$K = \frac{\mu_2}{\mu_1} + \alpha_2\left(1 - \frac{\mu_2}{\mu_1}\right), \quad \mathcal{S} = \left(1 - \frac{\mu_2}{\mu_1}\right),$$

and

$$\psi_1(\xi) = -i\alpha_1\left(\frac{\overline{X_n}}{\xi^n} + \frac{\overline{X_1}}{\xi} + \overline{X_{-1}}\xi + \overline{X_{-n}}\xi^n\right) - \frac{i\beta_1R^2\left(\frac{1}{\xi} + m_n\xi^n\right)\left(1 - \frac{nm_n}{\xi^{n+1}}\right)}{nX_n\xi^{n-1} + X_1 - \frac{X_{-1}}{\xi^2} - \frac{nX_{-n}}{\xi^{n+1}}}$$

$$+i\alpha_2\overline{C}R\left(\frac{1}{\xi} + m_n\xi^n\right) + \frac{i\beta_2R\left(\frac{1}{\xi} + m_n\xi^n\right)}{C} + DR\left(\xi + \frac{m_n}{\xi^n}\right), \tag{5.34}$$

The complex constants C and D are determined by the following general equations

$$C = \mathcal{J}A_1, \quad D = \frac{iR^{n-1}}{\mathcal{S}m_n}\overline{A_n}, \tag{5.35}$$

$$\mathcal{J} = \left(\frac{|A_1| \pm \sqrt{|A_1|^2 - 4\beta_2\mathcal{S}K}}{2K|A_1|}\right),$$

Also, from Remark 2 and Theorem 1, we must have that

$$nX_n\xi^{n-1} + X_1 - \frac{X_{-1}}{\xi^2} - \frac{nX_{-n}}{\xi^{n+1}} \neq 0, \quad \forall\xi : |\xi| > 1,$$

$$C, \ A_1, \ A_n \neq 0 \ \text{(to guarantee non-uniform remote stresses)},$$

$$m_n, \ \$D, \ KC + \frac{\$\beta_2}{\overline{C}} \neq 0.$$

Consequently, from (5.33)and (5.35), the complex constants B_1 and B_2 in (5.29) are specifically choosen in such a way that

$$B_1 = \overline{A_n}\frac{-iR^{n-1}}{m_n}\left(\frac{1}{\$} - \alpha_1\right) - \frac{1}{A_n}\left(\frac{i\beta_1 m_n}{nR^{n-1}}\right),$$

$$Bn = \overline{A_1}\frac{im_n}{R^{n-1}}\left(\alpha_2\overline{J} - \alpha_1\right) + \frac{1}{A_1}\left(\frac{i\beta_2 m_n}{R^{n-1}J}\right).$$

For the case $n = 2$, it is clear from (5.29-35) that the general solution reduces to the corresponding results discussed in Section 5.4.

5.5 FURTHER DISCUSSION

As a consequence of the results obtained above, we can draw the following interesting conclusions.

5.5.1 Expressions for I and J Inside the Inhomogeneity

From (2.1) and (2.6), R and J can be calculated as follow:

$$J = \det \mathbf{F} = -\operatorname{Im}\left[w_{,1}\overline{w}_{,2}\right] = y_{1,1}y_{2,2} - y_{1,2}y_{2,1}, \tag{5.36}$$

$$R = tr(\mathbf{F}\mathbf{F}^T) = |w_{,1}|^2 + |w_{,2}|^2 = y_{p,s}y_{p,s} \ , ...p, s = 1, 2, \tag{5.37}$$

Hence, in the view of (2.1), Equation (2.5) yeilds

$$\begin{aligned} I &= \sqrt{R + 2J} = \sqrt{tr(\mathbf{F}\mathbf{F}^T) + 2J}, \\ &= \sqrt{(y_{1,1} + y_{2,2})^2 + (y_{1,2} - y_{2,1})^2}, \end{aligned} \tag{5.38}$$

where

$$(y_{1,1} + y_{2,2})^2 + (y_{1,2} - y_{2,1})^2 = 2\left(y_{1,1}y_{2,2} - y_{1,2}y_{2,1},\right) + y_{11}^2 + y_{22}^2 + y_{12}^2 + y_{21}^2.$$

Equations (2.25-26) give us that

$$w_2(z) = -i\alpha_2 Cz + \overline{D}z - \frac{i\beta_2 z}{\overline{C}}. \tag{5.39}$$

Consequently, I takes the following form within the inhomogeneity

$$I = 2\left(\alpha_2 |C| + \frac{\beta_2}{|C|}\right), \tag{5.40}$$

and from Remark 2,

$$|\phi_2'(z)| = |C| \neq 0.$$

$$I^2 - 16\alpha_2\beta_2 = 4\left(\alpha_2 |C| - \frac{\beta_2}{|C|}\right)^2 > 0, \tag{5.41}$$

which guarantees that $F'(I)$, from (1) is well-defined.

5.5.2 Stress Distribution

From (2.18), the Piola stresses inside the inhomogeneity and at the remote boundary are given by

$$-\sigma_{21} + i\sigma_{11} = (\chi_2)_{,2}, \quad \sigma_{22} - i\sigma_{12} = (\chi_2)_{,1}, \tag{5.42}$$

$$-\sigma_{21}^{\infty} + i\sigma_{11}^{\infty} = (\chi_1)_{,2}, \quad \sigma_{22}^{\infty} - i\sigma_{12}^{\infty} = (\chi_1)_{,1}.$$

From Equation (2.26), (5.29) and (5.31)

$$\chi_1(z) = 2\mu_1 i[(\alpha_1 - 1)(A_1 z + A_n z^n) + i(\overline{B_1}\overline{z} + \overline{B_n}\overline{z^n}) + \frac{\beta_1 z}{\overline{A_1} + n\overline{A_n z^{n-1}}}],$$

$$\chi_2(z) = 2\mu_2 i[(\alpha_2 - 1)Cz + i\overline{D}\overline{z} + \frac{\beta_2 z}{\overline{C}}], \tag{5.43}$$

Thus, the corresponding Piola stress can be found from (41) as follows:

$$-\sigma_{21} + i\sigma_{11} = 2\mu_2 i[i(\alpha_2 - 1)C + \overline{D} + \frac{i\beta_2}{\overline{C}}], \tag{5.44}$$

$$\sigma_{22} - i\sigma_{12} = 2\mu_2 i[(\alpha_2 - 1)C + i\overline{D} + \frac{\beta_2}{\overline{C}}] \text{ (uniform)}$$

$$-\sigma_{21}^{\infty} + i\sigma_{11}^{\infty} = 2\mu_1 i[i(\alpha_1 - 1)(A_1 + nA_n z^{n-1}) + \overline{B_1} + n\overline{B_n z^{n-1}}], \tag{5.45}$$

$$\sigma_{22}^{\infty} - i\sigma_{12}^{\infty} = 2\mu_1 i[(\alpha_1 - 1)(A_1 + nA_n z^{n-1}) + i(\overline{B_1} + n\overline{B_n z^{n-1}})] \text{ (non-uniform)}.$$

Hence, the Piola stresses inside the inhomogeneity (uniform) and at the remote boundary (non-uniform) can be calculated from (5.44-45).

In addition, the Cauchy stresses from (2.12) inside the inhomogeneity are given by

$$\tau_{11}+\tau_{22} = 2\mu_2[\frac{IF'(I)}{J}-2],$$

$$\tau_{11}-\tau_{22}+2i\tau_{12} = 2\mu_2\frac{F'(I)}{IJ}[(w_2)_{,2}^2+(w_2)_{,1}^2],$$

which, in our case, from Equation (2.14) and (5.36-41),

$$F'(I)=|\phi_2'(z)|=|C|\neq 0,$$

are also uniform. In fact, from (5.4), we obtain

$$I=2\left(\alpha_2|C|+\frac{\beta_2}{|C|}\right),$$

and from (5.36) and (5.39)

$$J = -\operatorname{Im}\left[(w_2(z))_{,1}\overline{(w_2(z))_{,2}}\right]$$

$$= \alpha_2|C|^2+2\alpha_2\beta_2+|D|^2+\frac{\beta_2}{|C|^2},$$

where

$$(w_2(z))_{,1}=-i\alpha_2 C+\overline{D}-\frac{i\beta_2}{\overline{C}},\ \ (w_2(z))_{,2}=\alpha_2 C+i\overline{D}+\frac{\beta_2}{\overline{C}}.$$

Consequently, from (2.21), the Cauchy stresses are can be calculated as

$$\tau_{11}+\tau_{22}=2\mu_2[\frac{2\left(\alpha_2|C|^2+\beta_2\right)}{\alpha_2|C|^2+2\alpha_2\beta_2+|D|^2+\frac{\beta_2}{|C|^2}}-2],$$

$$\tau_{11}-\tau_{22}+2i\tau_{12}=2\mu_2\frac{|C|\,[(w_2(z))_{,2}^2+(w_2(z))_{,1}^2]}{2\left(\alpha_2|C|+\frac{\beta_2}{|C|}\right)\left[\alpha_2|C|^2+2\alpha_2\beta_2+|D|^2+\frac{\beta_2}{|C|^2}\right]},$$

which, clearly, indicate that the Cauchy stresses inside the inhomogeneity are uniform.

5.6 EXAMPLE: THE CASE $n = 3$

The general solutions mentioned above can accommodate many different forms of applied remote loading. Among these, we consider the particular class of remote stresses, characterized by the stress functions $\phi_1(z) = A_1 z + A_2 z^3$, $\psi_1(z) = B_1 z + B_2 z^3$, where $A_1, A_3, B_1$and B_3 are prescribed complex constants in such a way that

$$B_1 = \overline{A_3}\frac{iR^2}{m_3}\left(\frac{1}{\$} - \alpha_1\right) - \frac{1}{A_3}\left(\frac{i\beta_1 m_3}{3R^2}\right), \tag{5.46}$$

$$B_3 = \overline{A_1}\frac{im_3}{R^2}\left(\alpha_2\overline{J} - \alpha_1\right) + \frac{1}{A_1}\left(\frac{i\beta_2 m_3}{R^2 J}\right),$$

where

$$J = \left(\frac{|A_1| \pm \sqrt{|A_1|^2 - 4\beta_2 \$ K}}{2K\,|A_1|}\right),\ K = \frac{\mu_2}{\mu_1} + \alpha_2\left(1 - \frac{\mu_2}{\mu_1}\right),\ \$ = \left(1 - \frac{\mu_2}{\mu_1}\right), \tag{5.47}$$

and the corresponding conformal mapping function (hypotrochoid with $n = 3$) (Muskhelishvili, 1953), from (5.30), is given by

$$z = \omega(\xi) = R\left(\xi + \frac{m_3}{\xi^3}\right),\quad R > 0,\ 0 < m_3 \leq \frac{1}{3}, \tag{5.48}$$

5.6.1 Complete Solutions

From (5.29) and (5.31), the stress functions at the remote boundary and inside the inhomogeneity, respectively, are given by

$$\begin{aligned} \phi_1(z) &= A_1 z + A_3 z^3, \quad \psi_1(z) = B_1 z + B_3 z^3, \quad |z| \to \infty, \\ \phi_2(z) &= Cz, \quad \psi_2(z) = Dz, \quad z \in S_2. \end{aligned} \tag{5.49}$$

From the general solution (5.35), the complex constants C and D are completely determined by the equations

$$X_3 = A_3 R^3 = i\mathscr{S}\overline{D}Rm_3, \quad X_1 = A_1 R = \left(KC + \frac{\mathscr{S}\beta_2}{\overline{C}}\right) R, \tag{5.50}$$

$$X_{-1} = \frac{A_3 R^3}{m_3} = i\mathscr{S}\overline{D}R, \quad X_{-3} = A_1 R m_3 = \left(KC + \frac{\mathscr{S}\beta_2}{\overline{C}}\right) R m_3,$$

$$C = \mathscr{J} A_1, \quad D = \frac{iR^2}{\mathscr{S} m_3}\overline{A_3}. \tag{5.51}$$

Also, from Remark 8 and Theorem 2, we must have that

$$C, \ A_1, \ A_3 \neq 0 \text{ (to guarantee non-uniform remote stresses)},$$

and in the surriounding matrix

$$3X_3\xi^2 + X_1 - \frac{X_{-1}}{\xi^2} - \frac{3X_{-3}}{\xi^4} \neq 0, \quad \forall \xi : |\xi| > 1,$$

$$m_3, \ \mathscr{S}D, \ KC + \frac{\mathscr{S}\beta_2}{\overline{C}} \neq 0. \tag{5.52}$$

5.6.2 Determination of I and J Inside the Inhomogeneity

From (5.39) together with (5.51), the deformation function inside the inhomogeneity now takes the form

$$w_2(z) = -i\alpha_2 \mathcal{J} A_1 z - \frac{iR^2}{\mathcal{S}m_3} A_3 z - \frac{i\beta_2 z}{\overline{\mathcal{J} A_1}}. \tag{5.53}$$

and from (5.40-41) and (5.51)

$$I = 2\left(\alpha_2 |\mathcal{J} A_1| + \frac{\beta_2}{|\mathcal{J} A_1|}\right),$$

$$|\phi_2'(z)| = |\mathcal{J} A_1| \neq 0,$$

$$I^2 - 16\alpha_2\beta_2 = 4\left(\alpha_2 |\mathcal{J} A_1| - \frac{\beta_2}{|\mathcal{J} A_1|}\right)^2 > 0, \tag{5.54}$$

from which I can be calculated. Consequently, $F'(I)$, from (2.21) is well-defined. Also J can now be determined from (5.36) together with (5.53),

$$J = -\operatorname{Im}\left[(w_2)_{,1} (\overline{w}_2)_{,2}\right],$$

where

$$w_2(z) = -i\alpha_2 \mathcal{J} A_1 z - \frac{iR^2}{\mathcal{S}m_3} A_3 z - \frac{i\beta_2 z}{\overline{\mathcal{J} A_1}}.$$

which implies, together with (5.33), that the Cauchy stresses are uniform inside the inhomogeneity (see Equation (5.45) below).

5.6.3 Deformed Contour

From (5.53), we can easily plot the deformed contour (see Figure 12). The fact that the corresponding complex constants C and D satisfy (5.52) guarantee that there is no overlapping.

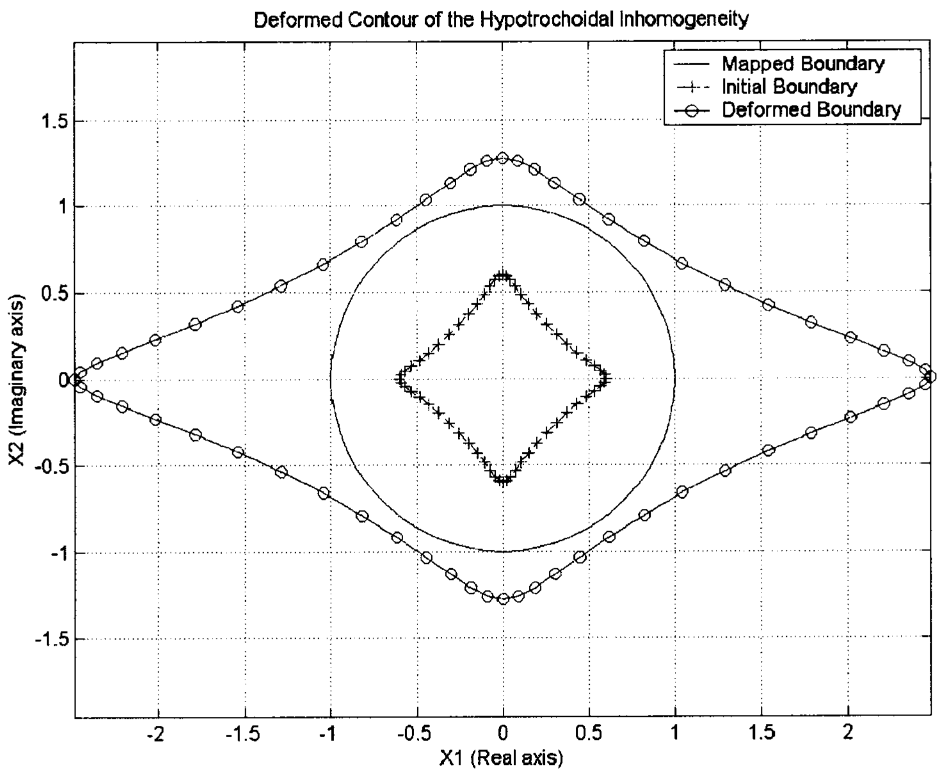

Figure 12: *Deformed Contour of the Hypotrochoidal Inhomogeneity with* 4 *Cusps* ($n = 3$).

Here, we have taken ($A_1 = i, A_3 = 0.25i, \mu_1 = 0.27, \mu_2 = 0.3, \alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 0.5$)

5.6.4 Admissibility Conditions

From (5.36) and (5.53-54), inside inhomogeneity,

$$I^2 > 16\alpha_2\beta_2, \quad J > 0. \tag{5.55}$$

As in Ru (2002), (5.55) ensures that there is no overlapping of the deformation field (interior to the inhomogeneity).

5.6.5 The Piola Stress Distribution

From (5.44-45) and (5.51), the Piola stress in the inhomogeneity and the matrix is described, respectively, by:

$$
\begin{aligned}
-\sigma_{21} + i\sigma_{11} &= 2\mu_2 i[i(\alpha_2 - 1)\Im A_1 - \frac{iR^2}{\Re m_3} A_3 + \frac{i\beta_2}{\Im A_1}], \\
\sigma_{22} - i\sigma_{12} &= 2\mu_2 i[(\alpha_2 - 1)\Im A_1 + \frac{R^2}{\Re m_3} A_3 + \frac{\beta_2}{\Im A_1}], \\
-\sigma_{21}^{\infty} + i\sigma_{11}^{\infty} &= 2\mu_1 i[i(\alpha_1 - 1)(A_1 + 3A_3 z^2) + \overline{B_1} + 3\overline{B_3 z^2}], \\
\sigma_{22}^{\infty} - i\sigma_{12}^{\infty} &= 2\mu_1 i[(\alpha_1 - 1)(A_1 + 3A_3 z^2) + i(\overline{B_1} + 3\overline{B_3 z^2})].
\end{aligned}
$$

Again, from the above, the state of (Piola) stress inside the inhomogeneity is uniform.

5.7 CONCLUSIONS

We have discussed the idea that by adjusting the remote (Piola) stress we can design the shape of the inhomogeneity in such a way that the interior (Piola) stress distribution remains uniform. In fact, using complex variable techniques, we have shown that a uniform (Piola) stress distribution can be achieved inside the inhomogeneity when the system is subjected to linear remote loading, in which case, the inhomogeneity is necessarily hyptrochoidal with three cusps. In this case, we have obtained the complete solution of the corresponding inhomogeneity-matrix system in this. In addition, we also considered the more general case when the system is subjected to nonuniform remote loading characterized by stress functions of the form of n^{th} degree polynomials in the complex variable z describing the matrix. By finding the complete solution of the corresponding system, we have shown that we can again obtain uniform stress inside a hypotrochoidal inhomogeneity with $n+1$ cusps.

Chapter 6
CONCLUSIONS AND FUTURE WORK

6.1 CONCLUSIONS

In this dissertation, certain important problems concerning a single inhomogeneity-matrix system of harmonic-type undergoing finite plane deformations are considered. For an interface condition, the state of perfect bonding generally referred to as a "perfectly bonded interface" is assumed so that both traction and displacement is continuous across the interface. By successfully incorporating complex variable techniques, we have analyzed the inhomogeneity-matrix system subjected to varying remote loading conditions (not necessarily uniform) and derived novel results of each corresponding problem discussed through Chapters. 3-5.

Chapter 3 addresses and analyzes the problems of an elliptic inhomogeneity subjected to classes of nonuniform remote stress distribution. Despite its practical importance in finite plane elastostatics, so far, there have been no closed form solutions available for such problems. This is due mainly to complications arising from the use of conformal mapping techniques with the ensuing nonlinearities in the mapped plane.

In the present study, we overcome the above-mentioned difficulties and address the issues relating to nonuniform remote loading of harmonic composite materials. In particular, we extend the results in Ru et al (2005) to the case of plane finite deformations of a composite material in which an elliptic elastic inhomogeneity is embedded in the same class of harmonic materials under the assumption of nonuniform remote loading. Using complex-variable methods, we obtain the complete solution for a perfectly bonded elliptic inhomogeneity when the system is subjected to classes of nonuniform remote stress distributions characterized by stress functions described by general polynomials of degree $n \geq 1$ in the corresponding complex variable z describing the matrix. The results obtained are extremely important in that, essentially, they lead to the solution of a class of problems in which the remote loading is characterized by a much wider and more practically realistic class of functions. We also mention here that the solutions obtained in Chapter 3 are general in that they can accommodate the existing results in the literatures (Ru et al, 2005; Ogden & Isherwood, 1978) as special cases of the solutions. In particular, the methodology proposed in Chapter 3 enables us to consider several interesting subjects involving the finite plane deformation of an inhomogeneity-matrix system.

Eshelby's conjecture in finite plane elastostatics is examined in Chapter 4. What is particularly interesting in the present study is to ask if the above statement holds true for the special class of harmonic material considered here. One side of Eshelby's conjecture for this material undergoing finite plane deformations was examined Ru et al (2005). In particular, it was shown there that under the analogous constraints of perfect bonding and uniform remote Piola stress, if the Piola stress within the inhomogeneity is uniform, the inhomogeneity is necessarily elliptical except for the special case when the remote loading takes a particular restricted form, in which case the shape of the inhomogeneity can be taken as arbitrary. The 'other side' of the conjecture, however, remained unfinished due to the existing complications arising from the use of mapping (conformal) techniques. By virtue of the analogy developed in Chapter 3, we overcome the above-mentioned difficulties and examine, in detail, the converse result. More precisely, it was shown in Chapter 4 that the Piola stress distribution within a perfectly bonded elliptic inhomogeneity is necessarily uniform when the same inhomogeneity-matrix system is subjected to finite plane deformations and uniform remote loading. This statement completes the Eshelby's conjecture for the class of harmonic materials considered here. In addition, the complete solution for this system for any uniform remote stress distribution is obtained which, in turn, reduces to the results in Ru et al (2005). Further, the issue of "uniform stress distribution inside inhomogeneity" provided motivation for the study of designing an inhomogeneity with uniform interior stress (Chapter 5).

The state of uniform interior stress distribution is a desirable state in that, by inducing uniform stress distribution in a system, one can remove unfavorable effects such as stress intensification and concentration both leading to the failure of a system. Therefore, it is of particular interest to see if we can actually achieve this state for an inhomogeneity-matrix system undergoing finite plane deformations in the same class of harmonic materials considered here. In fact, using complex variable techniques and the analogy developed in Chapter 3, it is shown in Chapter 5 that if the Piola stress distribution inside the inhomogeneity is uniform and the system is subjected to linear remote loading, the inhomogeneity is necessarily hypotrochoidal, with no exceptions. The complete solution of the inhomogeneity-matrix system is obtained in this case. In particular, we extend the above results to the more general case when the system is subjected to nonuniform remote loading characterized by stress functions of the form of n^{th} degree polynomials in the complex variable z describing the matrix. In this case, by finding the complete solution of the system, we show that we can again obtain uniform stress inside a hypotrochoidal inhomogeneity with $n+1$ cusps. In other words, by adjusting the cusps of a hypotrochoidal inhomogeneity and the remote loading, we can always achieve uniform stress distribution inside the inhomogeneity subjected to a relatively wide class of remote stresses discussed in Section 5.5.

The results derived in this dissertation will enable one to have a clear understanding of the state of the stress of an inhomogeneity-matrix system. In particular, the complete solutions of the corresponding problems in Chapters. 3-5, provide physical relevance to those corresponding "real-world problems". This will eventually enhance the future design of composite materials especially for fiber-reinforced composites.

6.2 FUTURE WORK

The most important extension of our work concerns the incorporation of a more realistic interface model. One which is 'imperfect' in the sense that it can include the effects of voids, cracks and imperfections in the glue layer between the fiber and the matrix. This has been done to a certain extent for linear elasticity (Ru & Schiavone, 1996, 1997; Ru, 1998a, 1998b; Sudak et al, 1999) but not, in any way, for finite elasticity. This is a subject of extreme interest and practical significance from nano-mechanics to traditional areas of continuum mechanics.

It is suggested that we begin with a model of a 'linear interface' embedded in a non-linear material (Aboudi, 1987; Steif & Hoysan, 1987; Gosz & Achebach, 1992; Benveniste, 1985; Zhong & Meguid, 1996). This is a good starting point for our analysis but since this works is absolutely new, we expect this to be extremely challenging yet immensely new circling area of research.

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Appendix A
DERIVATIONS

A.1 DERIVATION OF RESULTS IN CHAPTER. 3

From the general solution presented in Chapter 3, we derived the corresponding results from Ru et al (2005). In fact, by setting

$$A_n = B_n = 0, \quad n \geq 2,$$

we derive from (3.1) that the complex potentials characterizing the remote stress are taken to be

$$\phi_1(z) = A_1 z, \quad \psi_1(z) = B_1 z \quad |z| \to \infty.$$

Then, the general solution (3.19B) and (3.19D) yield only that

$$0 = \sum_{n=a}^{b} [A_n] \xi^n = \sum_{n=a}^{b} \left[X_{n,(n,r=0)}\right] \xi^n = \sum_{n=a}^{b} \left[KC_n + iS\overline{D_n} m^n\right] \xi^n,$$

$$0 = \sum_{n=a}^{b} [B_n] \xi^n = \sum_{n=a}^{b} \left[-i\alpha_1 \overline{X_{n,(n,r=n)}} + i\alpha_2 \overline{C_n} m^n + D_n\right] \xi^n, \quad a \geq 2, b \geq a,$$

where

$$\overline{X_{n,(n,r=n)}} = K\overline{C_n} m^n - iSD_n,$$

which, in turn leads to

$$KC_n + iS\overline{D_n} m^n = 0,$$

$$i(\alpha_1 K - \alpha_2) C_n m^n + \overline{D_n}(1 - \alpha_1 S) = 0.$$

For the class of harmonic materials discussed here,

$$K,\ (1-\alpha_1 S)\neq 0.$$

Therefore, to satisfy the above equations, it is shown in Chapter 4 (see Cases. 1-4) that we must have

$$C_n = D_n = 0, \quad n \geq 2,$$

which implies that the subsidiary complex constants are taken to be

$$C_{n,\alpha} = D_{n,\alpha} = 0, \quad n \geq 2.$$

Hence, Equation (3.19A) and (3.19C) become:

$$A_1 R\xi = \xi\left(KC_1R + iS\overline{D_1}Rm\right) + \lim_{|\xi|\to\infty}\left(S\frac{\beta_2 R\left(\xi+\frac{m}{\xi}\right)}{\overline{C_1}}\right)$$

$$B_1 R\xi = \xi\left[-i\alpha_1\left(K\overline{C_1}Rm - iSD_1R\right) + i\alpha_2\overline{C_1}Rm + D_1R\right]$$

$$+\lim_{|\xi|\to\infty}\left[\left(\frac{\beta_2 R\left(\xi+\frac{m}{\xi}\right)}{\overline{C_1}}\right)i\left(1-\alpha_1 S\right)\right] - \lim_{|\xi|\to\infty}\left[\frac{i\beta_1 R^2\left(\frac{1}{\xi}+m\xi\right)\left(1-\frac{m}{\xi^2}\right)}{\phi_1'(\xi)}\right],$$

where, from (3.17),

$$\phi_1(\xi) = \xi\left(KC_1R + iS\overline{D_1}Rm\right) + \frac{1}{\xi}\left(KC_1Rm + iS\overline{D_1}R\right) + S\frac{\beta_2 R\left(\xi+\frac{m}{\xi}\right)}{\overline{C_1}},$$

$$\phi_1'(\xi) = KC_1R + iS\overline{D_1}Rm + S\frac{\beta_2 R}{\overline{C_1}} - \frac{1}{\xi^2}\left(KC_1Rm + iS\overline{D_1}RS + \frac{\beta_2 Rm}{\overline{C_1}}\right).$$

Thus, at the remote boundary, we obtain,

$$A_1 R = K C_1 R + iS\overline{D_1} Rm + S\frac{\beta_2 R}{\overline{C_1}}$$

$$B_1 R = -i\alpha_1 \left(K\overline{C_1} Rm - iSD_1 R\right) + i\alpha_2 \overline{C_1} Rm + D_1 R + \frac{\beta_2 Rm}{C_1} i\left(1-\alpha_1 S\right) - \frac{i\beta_1 Rm}{A_1},$$

$$(\because A_1 = KC_1 + iS\overline{D_1}m + S\frac{\beta_2}{\overline{C_1}}).$$

Consequently, by setting $R = \lambda$, $Rm = \lambda_1$ and recall $K = \frac{\mu_2}{\mu_1} + \alpha_2\left(1 - \frac{\mu_2}{\mu_1}\right)$, $S = \left(1 - \frac{\mu_2}{\mu_1}\right)$, we obtain the solution for the corresponding elliptic inhomogeneity problem under the assumption of uniform remote stresses (Ru et al, 2005) characterized by the complex potentials:

$$\phi_1(z) = A_1 z, \quad \psi_1(z) = B_1 z \quad |z| \to \infty.$$

where

$$A_1\lambda = \left[\left(\frac{\mu_2}{\mu_1} + \alpha_2\left(1 - \frac{\mu_2}{\mu_1}\right)\right)C_1 + \left(1 - \frac{\mu_2}{\mu_1}\right)\frac{\beta_2 R}{\overline{C_1}}\right]\lambda + i\left(1 - \frac{\mu_2}{\mu_1}\right)\overline{D_1}\lambda_1$$

$$B_1\lambda = -i\alpha_1\overline{X_1} - \frac{i\beta_1\lambda_1}{A_1} + i\left(\alpha_2\overline{C_1} + \frac{\beta_2\lambda_1}{C_1}\right)\lambda_1 + D_1\lambda,$$

and

$$X_1 = \left[\left(\frac{\mu_2}{\mu_1} + \alpha_2\left(1 - \frac{\mu_2}{\mu_1}\right)\right)C_1 + \left(1 - \frac{\mu_2}{\mu_1}\right)\frac{\beta_2 R}{\overline{C_1}}\right]\lambda_1 + i\left(1 - \frac{\mu_2}{\mu_1}\right)\overline{D_1}\lambda.$$

Finally, from (3.5), we obtain the complex potentials ϕ_2 and ψ_2 inside the inhomogeneity, for $C_n = C_{n,\alpha} = D_n = D_{n,\alpha} = 0, \quad n \geq 2$,

$$\phi_2(z) = C_1 z, \quad \psi_2(z) = D_1 z, \quad z \in S_2.$$

A.2 DERIVATION OF RESULTS IN CHAPTER. 4

In this section, we show that the solution derived in Chapter 4 can be reduced to the results presented in Ru et al (2005). We first proceeds by noting the difference between the mapping functions (conformal). In Ru et al (2005), the infinite region outside an ellipse in the $z-plane$ is mapped (conformally) to the infinite region outside the unit circle in the $\xi-plane$ by the following mapping function (see Muskhelishvili, 1953):

$$z = \omega(\xi) = \lambda\xi + \frac{\lambda_1}{\xi}, \quad \lambda \neq 0,$$

which, with $\xi = \rho e^{i\theta}$ and $\rho = 1$ will map the ellipse

$$x_1 = \left(\lambda\rho + \frac{\lambda_1}{\rho}\right)\cos\theta, \quad x_2 = \left(\lambda\rho - \frac{\lambda_1}{\rho}\right)\sin\theta,$$

(where $\lambda, \lambda_1 \in \mathbb{R}$ and $\theta \in [0, 2\pi]$) and its exterior region in the complex z-plane, onto and outside, respectively, the unit circle in the complex $\xi-plane$ (note that ρ is designed to have unit value). Such a mapping function is convenient when the forms of complex potentials ϕ_2 and ψ_2 inside the inhomogeneity are prescribed, yet it is not suitable for the purpose of identifying the forms of complex potentials describing the state of stresses inside inhomogeniety.

Therefore, in Chapter 4, we adopt the following mapping function (conformal).

$$z = \omega(\xi) = R\left(\xi + \frac{1}{\xi}\right), \quad R > 0,$$

which maps the ellipse

$$x_1 = R\left(\rho + \frac{1}{\rho}\right)\cos\theta, \quad x_2 = R\left(\rho - \frac{1}{\rho}\right)\sin\theta,$$

where R, $\rho > 0$ and $\theta \in [0, 2\pi]$ and its exterior region in the complex z-plane, onto and outside, respectively, the circle of radius ρ in the complex $\xi - plane$. Consequently, the region S_1 is transformed into the exterior of the circle with radius R_* and the region $S_2 \backslash D$ into the ring $1 < |\xi| < R_*$ in the $\xi - plane$. The segment D and the boundary Γ are mapped onto the circles $|\xi| = 1$ and $|\xi| = R_*$, respectively, where (see Figure 6),

$$R_* = \frac{a + \sqrt{a^2 - 4R^2}}{2R} > 1, \quad a = R\left(R_* + \frac{1}{R_*}\right).$$

The only difference here is that ρ is not a fixed value so that the mapping function accommodates the region inside an elliptic inhomogeneity. By comparing the above mapping functions, we derive the relations:

$$\lambda = R\rho, \quad \lambda_1 = \frac{R}{\rho},$$

and, since at the interface $\rho = R_*$, the above equations are now taken to be

$$\lambda = RR_*, \quad \lambda_1 = \frac{R}{R_*}.$$

Next, the complex potentials in Chapter 4 and in Ru et al (2005) are given, respectively,

$$\phi_1(z) = Az + O_{(1)}, \quad \psi_1(z) = Bz + O_{(2)}, \quad |z| \to \infty,$$

$$\phi_2(z) = C_o + \frac{C_1}{R}z, \quad \psi_2(z) = D_o + \frac{D_1}{R}z, \quad |z| \in S_2, \text{(Chapter 4)},$$

and

$$\phi_1(z) = Az, \quad \psi_1(z) = Bz, \quad |z| \to \infty,$$

$$\phi_2(z) = Cz, \quad \psi_2(z) = Dz, \quad |z| \in S_2, \quad \text{(Ru et al,2005)}.$$

From Cases. 1-4 in Chapter 4 together with (4.38), we have that, for $O_{(1)} = O_{(2)} = 0$,

$$\phi_1(z) = Az, \quad \psi_1(z) = Bz, \quad |z| \to \infty,$$

$$\phi_2(z) = \frac{C_1}{R}z, \quad \psi_2(z) = \frac{D_1}{R}z, \quad |z| \in S_2,$$

$$(\because C_o = D_o = 0).$$

Then, we define the complex constants in such a way that

$$\frac{C_1}{R} = C, \quad \frac{D_1}{R} = D,$$

and recall

$$\lambda = RR_*, \quad \lambda_1 = \frac{R}{R_*}.$$

Therefore, from (4.38)

$$ARR_* = \left(KC_1 + \frac{iS\overline{D_1}}{R_*^2} + \frac{S\beta_2 R^2}{\overline{C_1}} \right) R_*$$

$$= KCRR_* + \frac{iS\overline{D}R}{R_*} + \frac{S\beta_2 RR_*}{\overline{C}} = KC\lambda + iS\overline{D}\lambda_1 + \frac{S\beta_2\lambda}{\overline{C}} = A\lambda,$$

$$\frac{X_{-1}}{R_*} = \left(KC_1 + iS\overline{D_1}R_*^2 + \frac{S\beta_2 R^2}{\overline{C_1}} \right) \frac{1}{R_*}$$

$$= KC\frac{R}{R_*} + iS\overline{D}RR_* + \frac{S\beta_2 R}{\overline{C}R_*} = KC\lambda_1 + iS\overline{D}\lambda + \frac{S\beta_2}{\overline{C}}\lambda_1 = X_1,$$

$$BRR_* = \left(-\frac{i\alpha_1 \overline{X_{-1}}}{R_*^2} - \frac{i\beta_1 R}{AR_*^2} + \frac{i\alpha_2 \overline{C_1}}{R_*^2} + \frac{i\beta_2 R^2}{C_1 R_*^2} + D_1 \right) R_*$$

$$= -\frac{i\alpha_1 \overline{X_1} R_*}{R_*} - \frac{i\beta_1 R}{AR_*} + \frac{i\alpha_2 \overline{C} R}{R_*} + \frac{i\beta_2 R}{CR_*} + DRR_*$$

$$-i\alpha_1 \overline{X_1} - \frac{i\beta_1}{A}\lambda_1 + i\alpha_2 \overline{C}\lambda_1 + \frac{i\beta_2}{C}\lambda_1 + D\lambda = B\lambda.$$

Consequently, replacing $K = \frac{\mu_2}{\mu_1} + \alpha_2 \left(1 - \frac{\mu_2}{\mu_1}\right), \quad S = \left(1 - \frac{\mu_2}{\mu_1}\right),$

$$A_1\lambda = \left[\left(\frac{\mu_2}{\mu_1} + \alpha_2 \left(1 - \frac{\mu_2}{\mu_1}\right)\right) C_1 + \left(1 - \frac{\mu_2}{\mu_1}\right) \frac{\beta_2 R}{\overline{C_1}} \right] \lambda + i \left(1 - \frac{\mu_2}{\mu_1}\right) \overline{D_1} \lambda_1$$

$$X_1 = \left[\left(\frac{\mu_2}{\mu_1} + \alpha_2 \left(1 - \frac{\mu_2}{\mu_1}\right)\right) C_1 + \left(1 - \frac{\mu_2}{\mu_1}\right) \frac{\beta_2 R}{\overline{C_1}} \right] \lambda_1 + i \left(1 - \frac{\mu_2}{\mu_1}\right) \overline{D_1} \lambda,$$

$$B_1\lambda = -i\alpha_1 \overline{X_1} - \frac{i\beta_1 \lambda_1}{A_1} + i \left(\alpha_2 \overline{C_1} + \frac{\beta_2 \lambda_1}{C_1} \right) \lambda_1 + D_1 \lambda,$$

we find that these results are identical to those presented in Ru et al (2005) for the solution of the elliptic inhomogeneity problem under uniform remote stresses.