$$K^{\lambda}{}_{\alpha} \equiv -\sigma^{-1\lambda}{}_{\beta}\sigma^{\beta}{}_{|\alpha} \quad , \quad H^{\lambda}{}_{\alpha} \equiv \sigma^{-1\lambda}{}_{\alpha} \quad . \tag{6.40}$$

Substitution of (6.40) into (6.7) gives

$$\psi(t,\sigma) = g_{\lambda\mu}(K^{\lambda}{}_{\alpha}V^{\alpha} + H^{\lambda}{}_{\alpha}\dot{\sigma}^{\alpha})(K^{\mu}{}_{\beta}V^{\beta} + H^{\mu}{}_{\beta}\dot{\sigma}^{\beta}) \quad . \tag{6.41}$$

Since $K^{\lambda}{}_{\alpha}(Z,z)$ and $H^{\lambda}{}_{\alpha}(Z,z)$ are determined solely by the gravitational field and the points Z and z we have from (6.41)

$$\partial\psi/\partial V^{\alpha} = 2g_{\lambda\mu}v^{\mu}K^{\lambda}{}_{\alpha} \quad , \qquad \partial\psi/\partial\dot{\sigma}^{\alpha} = 2g_{\lambda\mu}v^{\mu}H^{\lambda}{}_{\alpha} \quad ,$$

so that

$$\partial(-m(-\psi)^{\frac{1}{2}})/\partial V^{\alpha} = m(-\psi)^{-\frac{1}{2}}g_{\lambda\mu}v^{\mu}K^{\lambda}{}_{\alpha} = K^{\lambda}{}_{\alpha}p_{\lambda} \tag{6.42}$$

and

$$\partial(-m(-\psi)^{\frac{1}{2}})/\partial\dot{\sigma}^{\alpha} = m(-\psi)^{-\frac{1}{2}}g_{\lambda\mu}v^{\mu}H^{\lambda}{}_{\alpha} = H^{\lambda}{}_{\alpha}p_{\lambda} \quad . \tag{6.43}$$

We now find expressions for $\partial\phi/\partial V^{\alpha}$ and $\partial\phi/\partial\dot{\sigma}^{\alpha}$ in terms of $K^{\lambda}{}_{\alpha}$ and $H^{\lambda}{}_{\alpha}$. From (6.16) for n = 1: $-d\Phi/d\sigma = \int F_{\lambda\mu}v^{\lambda}l^{\mu}dt$, and from (6.15) we have

$$\Phi(\sigma) = \Phi(0) + \int_{\nu=0}^{\nu=\sigma} \frac{d\Phi}{d\nu}\, d\nu =$$

$$\int A_{,\alpha}V^{\alpha}\, dt - \int dt \int_{\nu=0}^{\nu=\sigma} F_{\lambda\mu}(z(t,\nu))v^{\lambda}(t,\nu)l^{\mu}(t,\nu)\, d\nu$$

where ν is an affine parameter along the geodesics Zz. Inserting (6.39) into the above yields

$$\Phi(\sigma) = \int \phi(t,\sigma)\, dt = \int (V^{\alpha}A_{\alpha} + V^{\alpha}\Theta_{\alpha} + \dot{\sigma}^{\alpha}\Pi_{\alpha})\, dt \tag{6.44}$$

with Θ_{α} and Π_{α} defined as

$$\Theta_{\alpha}(t) \equiv - \int_{\nu=0}^{\nu=\sigma} F_{\lambda\mu}(z(t,\nu))K^{\lambda}{}_{\alpha}(t,\nu)\dot{l}^{\mu}(t,\nu)\, d\nu \ ,$$
$$\Pi_{\alpha}(t) \equiv - \int_{\nu=0}^{\nu=\sigma} F_{\lambda\mu}(z(t,\nu))H^{\lambda}{}_{\alpha}(t,\nu)\dot{l}^{\mu}(t,\nu)\, d\nu \ . \tag{6.45}$$

(6.44) immediately gives

$$e\partial\phi/\partial V^{\alpha} = eA_{\alpha} + e\Theta_{\alpha} \quad , \qquad e\partial\phi/\partial\dot{\sigma}^{\alpha} = e\Pi_{\alpha} \quad . \tag{6.46}$$

From $L(t) = -m(-\psi)^{\frac{1}{2}}+e\phi$ and (6.42), (6.43), (6.46), we obtain p_{α} and $S^{\alpha\beta}$ in terms of $K^{\lambda}{}_{\alpha}$ and $H^{\lambda}{}_{\alpha}$:

$$p_{\alpha} = \partial L/\partial V^{\alpha} - eA_{\alpha} = K^{\lambda}{}_{\alpha}p_{\lambda} + e\Theta_{\alpha} \quad , \tag{6.47}$$

$$S^{\alpha\beta} = 2\sigma^{[\alpha}\partial L/\partial\dot{\sigma}_{\beta]} = 2\sigma^{[\alpha}(H_{\lambda}{}^{\beta]}p^{\lambda} + e\Pi^{\beta]}) \ . \tag{6.48}$$

(6.47) and (6.48) are the definitions of four-momentum and spin proposed by Dixon ((5.1) and (5.2) of [11c]).

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Appendix 1

For a relative tensor $\Phi_{\underset{\sim}{A}} = (\phi_a)_{\alpha_1 \ldots \alpha_m}{}^{\alpha_{m+1} \ldots \alpha_n}$ we have

$$(I_{\underset{\sim}{A}}{}^{\underset{\sim}{B}})_\alpha{}^\beta =$$

$$\sum_{i=1}^{m} \delta_a^b \delta_{\alpha_1}^{\beta_1} \cdots \delta_{\alpha_{i-1}}^{\beta_{i-1}} (-\delta_\alpha^{\beta_i} \delta_{\alpha_i}^\beta) \delta_{\alpha_{i+1}}^{\beta_{i+1}} \cdots \delta_{\alpha_m}^{\beta_m} \delta_{\beta_{m+1}}^{\alpha_{m+1}} \cdots \delta_{\beta_n}^{\alpha_n}$$

$$+ \sum_{i=m+1}^{n} \delta_a^b \delta_{\alpha_1}^{\beta_1} \cdots \delta_{\alpha_m}^{\beta_m} \delta_{\beta_{m+1}}^{\alpha_{m+1}} \cdots \delta_{\beta_{i-1}}^{\alpha_{i-1}} (\delta_\alpha^{\alpha_i} \delta_{\beta_i}^\beta) \delta_{\beta_{i+1}}^{\alpha_{i+1}} \cdots \delta_{\beta_n}^{\alpha_n}$$

$$- w \delta_a^b \delta_{\alpha_1}^{\beta_1} \cdots \delta_{\alpha_m}^{\beta_m} \delta_{\beta_{m+1}}^{\alpha_{m+1}} \cdots \delta_{\beta_n}^{\alpha_n} \delta_\alpha^\beta \quad .$$

When $\Phi_{\underset{\sim}{A}} = (A_\alpha, F_{\beta\gamma|\underset{\sim}{\lambda}(n)}, R^\alpha{}_{\beta\gamma\delta|\underset{\sim}{\lambda}(n)} \; ; \; n = 0, 1, \ldots)$ we have

$$M^{\underset{\sim}{A}} (I_{\underset{\sim}{A}}{}^{\underset{\sim}{B}})_\alpha{}^\beta \Phi_{\underset{\sim}{B}} = \frac{\partial L}{\partial A_{\alpha_1}} (-\delta_\alpha^{\beta_1} \delta_{\alpha_1}^\beta) A_{\beta_1} +$$

$$\sum_{n=2}^{\infty} \sum_{i=1}^{n} \frac{1}{2} m^{\alpha_1 \ldots \alpha_n} \delta_{\alpha_1}^{\beta_1} \ldots \delta_{\alpha_{i-1}}^{\beta_{i-1}} (-\delta_\alpha^{\beta_i} \delta_{\alpha_i}^\beta) \delta_{\alpha_{i+1}}^{\beta_{i+1}} \ldots \delta_{\alpha_n}^{\beta_n} F_{\beta_1 \beta_2 | \beta_3 \ldots \beta_n}$$

$$+ \sum_{n=4}^{\infty} q_{\alpha_n}{}^{\alpha_1 \ldots \alpha_{n-1}} \left(\sum_{i=1}^{n-1} \delta_{\alpha_1}^{\beta_1} \ldots \delta_{\alpha_{i-1}}^{\beta_{i-1}} (-\delta_\alpha^{\beta_i} \delta_{\alpha_i}^\beta) \delta_{\alpha_{i+1}}^{\beta_{i+1}} \cdots \delta_{\alpha_{n-1}}^{\beta_{n-1}} \delta_{\beta_n}^{\alpha_n} \right.$$

$$\left. + \delta_{\alpha_1}^{\beta_1} \ldots \delta_{\alpha_{n-1}}^{\beta_{n-1}} (\delta_\alpha^{\alpha_n} \delta_{\beta_n}^\beta) \right) R^{\beta_n}{}_{\beta_1 \beta_2 \beta_3 | \beta_4 \ldots \beta_{n-1}}$$

$$= - eu^{\beta}A_{\alpha} - \frac{1}{2}\sum_{n=2}^{\infty}\sum_{i=1}^{n} m^{\alpha_1 \cdot\cdot \alpha_{i-1}\beta\alpha_{i+1}\cdot\cdot\alpha_n} F_{\alpha_1 \cdot | \cdot\cdot \alpha_{i-1}\alpha\alpha_{i+1}\cdot\cdot\alpha_n}$$

$$- \sum_{n=4}^{\infty}\sum_{i=1}^{n-1} q_{\alpha_n}^{\alpha_1 \cdot\cdot \alpha_{i-1}\beta\alpha_{i+1}\cdot\cdot\cdot\alpha_{n-1}} R^{\alpha_n}{}_{\alpha_1 \cdot\cdot | \cdot \alpha_{i-1}\alpha\alpha_{i+1}\cdot\cdot\cdot\alpha_{n-1}}$$

$$+ \sum_{n=4}^{\infty} q_{\alpha}^{\alpha_1 \cdot\cdot\cdot \alpha_{n-1}} R^{\beta}{}_{\alpha_1\alpha_2\alpha_3 | \alpha_4 \cdot\cdot\cdot \alpha_{n-1}}$$

The symmetries of q and m together with $p_{\alpha} = P_{\alpha} - eA_{\alpha}$ give

$$M^{\underset{\sim}{A}}(I_{\underset{\sim}{A}}{}^{\underset{\sim}{B}})_{\alpha}{}^{\beta}\Phi_{\underset{\sim}{B}} = (p_{\alpha} - P_{\alpha})u^{\beta}$$

$$- \sum_{n=0}^{\infty} m^{\beta\gamma\underset{\sim}{\lambda}(n)} F_{\alpha\gamma|\underset{\sim}{\lambda}(n)} - \frac{1}{2}\sum_{n=0}^{\infty} (n+1)\, m^{\gamma\mu\underset{\sim}{\lambda}(n)\beta} F_{\gamma\mu|(\underset{\sim}{\lambda}(n)\alpha)}$$

$$+ \sum_{n=0}^{\infty} q_{\alpha}{}^{\gamma\mu\nu\underset{\sim}{\lambda}(n)} R^{\beta}{}_{\gamma\mu\nu|\underset{\sim}{\lambda}(n)} - 3\sum_{n=0}^{\infty} q_{\delta}{}^{\beta\gamma\mu\underset{\sim}{\lambda}(n)} R^{\delta}{}_{\alpha\gamma\mu|\underset{\sim}{\lambda}(n)}$$

$$- \sum_{n=0}^{\infty} (n+1)\, q_{\delta}{}^{\gamma\mu\nu\underset{\sim}{\lambda}(n)\beta} R^{\delta}{}_{\gamma\mu\nu|(\underset{\sim}{\lambda}(n)\alpha)}$$

which inserted into (2.22) gives (2.30).

Appendix 2

Consider a class of Lagrangians

$$L(u^\alpha,\ \partial_\alpha a^m,\ \theta_{\underset{\sim}{A}}) = \bar{L}(u^\alpha,\ \partial_\alpha a^m,\ \theta_{\underset{\sim}{A}},\ -u_\alpha u^\alpha,\ u^\alpha \partial_\alpha a^m)$$

where $\bar{L}$ is an arbitrary function of its last two arguments, subject to $\bar{L}(u^\alpha, \partial_\alpha a^m, \theta_{\underset{\sim}{A}}, 1, 0)$ a fixed function of u^α, $\theta_{\underset{\sim}{A}}$ and $\partial_\alpha a^m$. ($\theta_{\underset{\sim}{A}}$ is shorthand for $(a^m, e_\alpha^{(a)}, e_{\alpha|\beta}^{(a)}, \phi_A, \phi_{A|\alpha}, R^\alpha{}_{\beta\gamma\delta})$.) Let $\bar{\Delta}_\alpha^\lambda = \delta_\alpha^\lambda + u_\alpha u^\lambda (-u_\gamma u^\gamma)^{-1}$. From (any) L we can form

$$L'(u^\alpha,\ \partial_\alpha a^m,\ \theta_{\underset{\sim}{A}}) = L(u^\alpha/(-u_\lambda u^\lambda)^{1/2},\ \bar{\Delta}_\alpha^\lambda \partial_\lambda a^m,\ \theta_{\underset{\sim}{A}})$$

$$= \bar{L}(u^\alpha/(-u_\lambda u^\lambda)^{1/2},\ \bar{\Delta}_\alpha^\lambda \partial_\lambda a^m,\ \theta_{\underset{\sim}{A}},\ 1,\ 0)$$

so that each L gives the same L′.

We will show that variation of the metric for any L gives a total energy tensor $T_\rho{}^\sigma$ that is expressible in terms of L′. Since each L determines the same L′ it then follows that $T_\rho{}^\sigma$ is independent of the particular choice of L.

To calculate $T_\rho{}^\sigma$ from L we follow the same steps as in sections 3 and 4, modifying (3.11) to include u^α.

$$I = \int L(\psi_{\underset{\sim}{A}}, \psi_{\underset{\sim}{A}|\alpha})\, d^4x + (16\pi)^{-1} \int \sqrt{-g}\, R\, d^4x \quad ,$$

$$\psi_{\underset{\sim}{A}} = (u^\alpha,\ a^m,\ e_\alpha^{(a)},\ \phi_A,\ R^\alpha{}_{\beta\gamma\delta}) \quad .$$

The variation in u^α resulting from variation of the metric may be computed from (3.1) and (3.3). Since $\eta^{\mu\alpha\beta\gamma} = (-g)^{-\frac{1}{2}}\epsilon^{\mu\alpha\beta\gamma}$ where $\epsilon^{\mu\alpha\beta\gamma}$ is the Levi-Civita permutation symbol, (3.3) implies that δN^μ is parallel to N^μ, implying δu^μ parallel to u^μ, $\delta u^\mu = \epsilon u^\mu$ say. $\delta(g_{\alpha\beta}u^\alpha u^\beta) = 0$ then implies $\delta g_{\alpha\beta}u^\alpha u^\beta + 2\epsilon g_{\alpha\beta}u^\alpha u^\beta = 0$, so that $2\epsilon = \delta g_{\alpha\beta}u^\alpha u^\beta$ and $\delta u^\mu = \frac{1}{2}\delta g_{\alpha\beta}u^\alpha u^\beta u^\mu$. This gives an additional term $(\partial L/\partial u^\mu)u^\mu u^\rho u^\sigma$ in the r.h.s. of (3.20). With ψ_A containing u^α the fundamental identity gives an additional $(\partial L/\partial u^\rho)u^\sigma$ on the r.h.s. of (3.22). The above two remarks imply that the field equations (3.24) contain an additional $(\partial L/\partial u^\lambda)\Delta^\lambda_\rho u^\sigma$ on their r.h.s. so that (3.26) and (3.27) are replaced by

$$\mathbf{t}_\rho{}^\sigma + (\partial L/\partial u^\lambda)\Delta^\lambda_\rho u^\sigma = \mathbf{t}_{\rho(\mathrm{mat})}{}^\sigma + \mathbf{t}_{\rho(\phi)}{}^\sigma$$

$$\mathbf{t}_{\rho(\mathrm{mat})}{}^\sigma \equiv L_1\delta^\sigma_\rho + \frac{\partial L}{\partial u^\lambda}\Delta^\lambda_\rho u^\sigma - (\partial_\rho a^m)\frac{\partial L}{\partial(\partial_\sigma a^m)} - e^{(a)}_{\alpha|\rho}\frac{\partial L}{\partial e^{(a)}_{\alpha|\sigma}} \tag{A2.1}$$

and $\mathbf{t}_{\rho(\phi)}{}^\sigma$ given by (3.28).

To express $\mathbf{t}_{\rho(\mathrm{mat})}{}^\sigma$ in terms of L' we note[1]

$$\frac{\partial L'}{\partial u^\alpha} = \frac{\partial L}{\partial u^\beta}\frac{\partial}{\partial u^\alpha}\left(\frac{u^\beta}{(-u_\lambda u^\lambda)^{\frac{1}{2}}}\right) + \frac{\partial L}{\partial(\partial_\beta a^m)}\frac{\partial}{\partial u^\alpha}(\overline{\Delta}^\lambda_\beta\partial_\lambda a^m)$$

$$= \frac{\partial L}{\partial u^\beta}(\delta^\beta_\alpha + u_\alpha u^\beta) + \frac{\partial L}{\partial(\partial_\beta a^m)}\frac{\partial}{\partial u^\alpha}\left(\frac{u^\lambda u_\beta}{-u_\gamma u^\gamma}\right)\partial_\lambda a^m$$

----

[1]We are not concerned about the arbitrariness of $t_\rho{}^\sigma$ or $T_\rho{}^\sigma$ as functions of u^α and $\partial_\alpha a^m$ due to the constraints. $(-u^\alpha u^\rho_\alpha)$ and $(u^\alpha\partial_\alpha a^m)$ are therefore set equal to 1 and 0 in the following formulae *after* differentiation.

$$= \frac{\partial L}{\partial u^{\beta}} \Delta^{\beta}_{\alpha} + \frac{\partial L}{\partial(\partial_{\beta} a^{m})} (\delta^{\lambda}_{\alpha} u_{\beta} + u^{\lambda} g_{\beta\alpha} + 2 u_{\alpha} u^{\lambda} u_{\beta}) \partial_{\lambda} a^{m} \quad ,$$

i.e.

$$\frac{\partial L'}{\partial u^{\alpha}} = \frac{\partial L}{\partial u^{\beta}} \Delta^{\beta}_{\alpha} + \frac{\partial L}{\partial(\partial_{\beta} a^{m})} u_{\beta} (\partial_{\alpha} a^{m}) \quad . \qquad \text{(A2.2)}$$

Similarly

$$\frac{\partial L'}{\partial(\partial_{\sigma} a^{m})} = \frac{\partial L}{\partial(\partial_{\gamma} a^{n})} \frac{\partial}{\partial(\partial_{\sigma} a^{m})} (\bar{\Delta}^{\lambda}_{\gamma} \partial_{\lambda} a^{n}) = \frac{\partial L}{\partial(\partial_{\gamma} a^{m})} \Delta^{\sigma}_{\gamma} \quad . \qquad \text{(A2.3)}$$

According to (A2.2), (A2.3), L′ satisfies

$$\frac{\partial L'}{\partial u^{\alpha}} u^{\alpha} = 0 \quad , \quad \frac{\partial L'}{\partial(\partial_{\sigma} a^{m})} u_{\sigma} = 0 \quad . \qquad \text{(A2.4)}$$

(A2.2) and (A2.3) imply

$$\frac{\partial L}{\partial u^{\lambda}} \Delta^{\lambda}_{\rho} u^{\sigma} - (\partial_{\rho} a^{m}) \frac{\partial L}{\partial(\partial_{\sigma} a^{m})} = \left[\frac{\partial L}{\partial u^{\lambda}} \Delta^{\lambda}_{\rho} + \frac{\partial L}{\partial(\partial_{\beta} a^{m})} u_{\beta} (\partial_{\rho} a^{m})\right] u^{\sigma} \quad ,$$

$$- (\partial_{\rho} a^{m}) \frac{\partial L}{\partial(\partial_{\beta} a^{m})} \Delta^{\sigma}_{\beta} = \frac{\partial L'}{\partial u^{\rho}} u^{\sigma} - (\partial_{\rho} a^{m}) \frac{\partial L'}{\partial(\partial_{\sigma} a^{m})}$$

(cf. (3.32)). These, together with (3.27) and (3.32), imply that $t^{\sigma}_{\rho(mat)}$ of (3.27) and (A2.1) are the same.

Appendix 3

The contribution to P_ρ from $L_{(A)} = enA_\alpha u^\alpha/(-u_\lambda u^\lambda)^{1/2}$ is $P_{\rho(A)} = \partial L_{(A)}/\partial u^\rho - L_{(A)}u_\rho = enA_\rho + enA_\alpha u^\alpha u_\rho - L_{(A)}u_\rho = enA_\rho$.

The contribution to $P_\rho{}^\sigma$ from $L_{(A)}$ is

$$P_{\rho(A)}^{\ \sigma} = L_{(A)}\Delta_\rho^\sigma - L_{(A)}n^{-1}(\partial_\rho a^m)(\partial n/\partial(\partial_\sigma a^m)).$$

Formulae $\gamma^{mn} = g^{\mu\nu}(\partial_\mu a^m)(\partial_\nu a^n)$, $\gamma^{mn}\gamma_{np} = \delta_p^m$, $\Delta_{\mu\nu} = \gamma_{mn}(\partial_\mu a^m)(\partial_\nu a^n)$, $\gamma = \det(\gamma^{mn})$ and $n = N\gamma^{1/2}$ (cf. section 2) imply

$$(\partial_\rho a^m)(\partial n/\partial(\partial_\sigma a^m)) = N(\partial_\rho a^m)(\tfrac{1}{2}\gamma^{-1/2})(\gamma\gamma_{pn})\partial(\gamma^{pn})/\partial(\partial_\sigma a^m)$$
$$= N\gamma^{1/2}\gamma_{mn}(\partial_\rho a^m)g^{\sigma\nu}\partial_\nu a^n = n\Delta_\rho^\sigma.$$

Therefore $P_{\rho(A)}^{\ \sigma} = 0$.

Appendix 4

$$\delta_{(e)}\mathbf{L} = \sum_{n=0}^{\infty} \mathbf{L}^{A\underset{\sim}{\alpha}(n)}\delta_{(e)}(\psi_{A|\underset{\sim}{\alpha}(n)})$$

$$= \mathbf{L}^{A}\delta_{(e)}\psi_{A} + \sum_{n=0}^{\infty} \mathbf{L}^{A\underset{\sim}{\alpha}(n)\tau}\delta_{(e)}(\psi_{A|\underset{\sim}{\alpha}(n)\tau}) \quad . \qquad (A4.1)$$

Equation (2.8) implies

$$\mathbf{L}^{A\underset{\sim}{\alpha}(n)\tau}\delta_{(e)}(\psi_{A|\underset{\sim}{\alpha}(n)\tau})$$

$$= \mathbf{L}^{A\underset{\sim}{\alpha}(n)\tau}\left(\{\delta_{(e)}(\psi_{A|\underset{\sim}{\alpha}(n)})\}_{|\tau} + (I_{A\underset{\sim}{\alpha}(n)}{}^{B\underset{\sim}{\beta}(n)})_{\rho}{}^{\sigma}\psi_{B|\underset{\sim}{\beta}(n)}\delta\Gamma^{\rho}_{\sigma\tau}\right)$$

$$= -\mathbf{L}^{A\underset{\sim}{\alpha}(n)\tau}{}_{|\tau}\delta_{(e)}(\psi_{A|\underset{\sim}{\alpha}(n)}) + (\mathrm{div})$$

$$+ \left(\mathbf{L}^{A\underset{\sim}{\alpha}(n)\tau}(I_{A}{}^{B})_{\rho}{}^{\sigma}\psi_{B|\underset{\sim}{\alpha}(n)} - n\mathbf{L}^{A\sigma\underset{\sim}{\alpha}(n-1)\tau}\psi_{A|(\rho\underset{\sim}{\alpha}(n-1))}\right)\delta\Gamma^{\rho}_{\sigma\tau}$$

according to repeated application of (2.5). If we repeat the above procedure n times we obtain

$$\mathbf{L}^{A\underset{\sim}{\alpha}(n)\tau}\delta_{(e)}(\psi_{A|\underset{\sim}{\alpha}(n)\tau})$$

$$= (-1)^{n+1}\mathbf{L}^{A\underset{\sim}{\alpha}(n)\tau}{}_{|\tau\underset{\sim}{\alpha}(n)}\delta_{(e)}\psi_{A} + (\mathrm{div})$$

$$+ \sum_{m=0}^{n}(-1)^{m}\Big(\mathbf{L}^{A\underset{\sim}{\alpha}(n-m)\underset{\sim}{\beta}(m)\tau}{}_{|\underset{\sim}{\beta}(m)}(I_{A}{}^{B})_{\rho}{}^{\sigma}\psi_{B|\underset{\sim}{\alpha}(n-m)}$$

$$- (n-m)\mathbf{L}^{A\sigma\underset{\sim}{\alpha}(n-m-1)\underset{\sim}{\beta}(m)\tau}{}_{|\underset{\sim}{\beta}(m)}\psi_{A|(\rho\underset{\sim}{\alpha}(n-m-1))}\Big)\delta\Gamma^{\rho}_{\sigma\tau} \quad .$$

Inserting this into (A4.1) and noting that

$$\sum_{n=0}^{\infty}\sum_{m=0}^{n} = \sum_{m=0}^{\infty}\sum_{n=m}^{\infty} = \sum_{m=0}^{\infty}\sum_{r=0}^{\infty} (r=n-m)$$

gives

$$\delta_{(e)}\mathbf{L} =$$

$$\sum_{n=0}^{\infty}(-1)^n \mathbf{L}^{A\underset{\sim}{\alpha}(n)}{}_{|\underset{\sim}{\alpha}(n)}\delta_{(e)}\psi_A + (\mathrm{div})$$

$$+ \sum_{m=0}^{\infty}\sum_{r=0}^{\infty}(-1)^m\Big(\mathbf{L}^{A\underset{\sim}{\alpha}(r)\underset{\sim}{\beta}(m)\tau}{}_{|\underset{\sim}{\beta}(m)}(I_A{}^B)_\rho{}^\sigma\psi_{B|\underset{\sim}{\alpha}(r)}$$

$$- r\mathbf{L}^{A\sigma\underset{\sim}{\alpha}(r-1)\underset{\sim}{\beta}(m)\tau}{}_{|\underset{\sim}{\beta}(m)}\psi_{A|(\rho\underset{\sim}{\alpha}(r-1))}\Big)\delta\Gamma^\rho_{\sigma\tau}$$

$$= \mathbf{L}^A_*\delta_{(e)}\psi_A + (\mathrm{div})$$

$$+ \sum_{r=0}^{\infty}\Big(\mathbf{L}_*^{A\underset{\sim}{\alpha}(r)\tau}(I_A{}^B)_\rho{}^\sigma\psi_{B|\underset{\sim}{\alpha}(r)} - r\mathbf{L}_*^{A\sigma\underset{\sim}{\alpha}(r-1)\tau}\psi_{A|(\rho\underset{\sim}{\alpha}(r-1))}\Big)\delta\Gamma^\rho_{\sigma\tau}$$

$$= \mathbf{L}^A_*\delta_{(e)}\psi_A + \mathbf{U}^{\tau\sigma}{}_\rho\delta\Gamma^\rho_{\sigma\tau} + (\mathrm{div})$$

according to (4.3), (4.11).

Appendix 5

Expand $f(z)dz^{\beta}/d\tau$ in powers of σ^{α}:

$$f(Z+\sigma)dz^{\beta}/d\tau = (U^{\beta} + \dot{\sigma}^{\beta}) \sum_{n=0}^{\infty} \frac{1}{n!} \sigma^{\underset{\sim}{\lambda}(n)} \partial_{\underset{\sim}{\lambda}(n)} f(Z) \qquad (A5.1)$$

$$= U^{\beta} f(Z) + \dot{\sigma}^{\beta} f(Z) + \sum_{n=0}^{\infty} \frac{1}{(n+1)!} (U^{\beta}\sigma^{\gamma} + \dot{\sigma}^{\beta}\sigma^{\gamma}) \sigma^{\underset{\sim}{\lambda}(n)} \partial_{\gamma\underset{\sim}{\lambda}(n)} f(Z) \ .$$

To form the antisymmetric part of the summation in the above equation, consider

$$\frac{d}{d\tau}\left[\sum_{n=0}^{\infty} \frac{1}{(n+1)!} \sigma^{\beta}\sigma^{\underset{\sim}{\lambda}(n)} \partial_{\underset{\sim}{\lambda}(n)} f(Z) \right]$$

$$= \sum_{n=0}^{\infty} \frac{1}{(n+1)!} \left(\dot{\sigma}^{\beta}\sigma^{\underset{\sim}{\lambda}(n)} \partial_{\underset{\sim}{\lambda}(n)} f + n\sigma^{\beta}\dot{\sigma}^{\gamma}\sigma^{\underset{\sim}{\lambda}(n-1)} \partial_{\gamma\underset{\sim}{\lambda}(n-1)} f + \sigma^{\beta}\sigma^{\underset{\sim}{\lambda}(n)} U^{\gamma} \partial_{\gamma\underset{\sim}{\lambda}(n)} f \right)$$

$$= \dot{\sigma}^{\beta} f(Z) + \sum_{n=0}^{\infty} \left[\frac{1}{(n+2)!} \dot{\sigma}^{\beta}\sigma^{\gamma} + \frac{(n+1)}{(n+2)!} \sigma^{\beta}\dot{\sigma}^{\gamma} + \frac{1}{(n+1)!} \sigma^{\beta} U^{\gamma} \right] \sigma^{\underset{\sim}{\lambda}(n)} \partial_{\gamma\underset{\sim}{\lambda}(n)} f(Z) \ . \qquad (A5.2)$$

Using (5.24) and subtracting (A5.2) from (A5.1) gives

$$f(z)dz^{\beta}/d\tau - \frac{d}{d\tau}\left[\sum_{n=0}^{\infty} \frac{1}{(n+1)!} \sigma^{\beta}\sigma^{\underset{\sim}{\lambda}(n)} \partial_{\underset{\sim}{\lambda}(n)} f(Z) \right] \qquad (A5.3)$$

$$= f(Z) U^{\beta} - e^{-1} \sum_{n=0}^{\infty} m^{\beta\gamma\underset{\sim}{\lambda}(n)}(\tau) \partial_{\gamma\underset{\sim}{\lambda}(n)} f(Z) \ .$$

This identity is given in [5, page 242]. When used in [5] to obtain the translational equations of motion (equation 168, page 201) the total proper-time derivative is not combined with mu^α, as we do in (5.30), but is placed on the r.h.s. of the equations as part of the total four-force.

A short derivation of the polarization equations (5.17), (5.18), is available from (A5.3). Setting f(z) = $e\delta^4(x-z)$ then integrating w.r.to τ immediately gives

$$j^\alpha(x) = \int e\,dz^\alpha/d\tau\ \delta^4(x-z(\tau))\,d\tau$$

$$= \int eU^\alpha \delta^4(x-Z(\tau))\,d\tau - \sum_{n=0}^{\infty} (-1)^{n+1} \partial_{x^{\beta}\underset{\sim}{\lambda}(n)} \int m^{\alpha\beta\underset{\sim}{\lambda}(n)}(\tau)\,\delta^4(x-Z(\tau))\,d\tau$$

$$= J^\alpha(x) + \partial_\beta M^{\alpha\beta}(x)$$

according to (5.25) and (5.26).

If we set f(z) equal to $\delta^4(x-z)$ in (A5.3), then multiply the resultant identity by mu^α and integrate w.r.to τ, we obtain

$$t^{\alpha\beta}(x) = \int mu^\alpha dz^\beta/d\tau\ \delta^4(x-z(\tau))\,d\tau$$

$$= \int mu^\alpha U^\beta \delta^4(x-Z(\tau))\,d\tau - e^{-1}\sum_{n=0}^{\infty} (-1)^{n+1} \partial_{x^{\gamma}\underset{\sim}{\lambda}(n)} \int mu^\alpha m^{\beta\gamma\underset{\sim}{\lambda}(n)}(\tau)\,\delta^4(x-Z(\tau))\,d\tau$$

$$- \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)!}\, \partial_{x\underset{\sim}{\lambda}(n)} \int \frac{d(mu^\alpha)}{d\tau}\, \sigma^\beta \sigma^{\underset{\sim}{\lambda}(n)}\, \delta^4(x-Z(\tau))\,d\tau \quad .$$

From (5.20), (5.24), (5.22) and (5.23) this is

$$t^{\alpha\beta} = T^{\alpha\beta} + \partial_\gamma N^{\alpha\beta\gamma} + A^{\alpha\beta} \quad .$$

Appendix 6

Expanding $\sigma^{\alpha} f(z) dz^{\gamma}/d\tau$ gives

$$\sigma^{\alpha} f(z)\, dz^{\gamma}/d\tau = \sigma^{\alpha} f(Z+\sigma)(U^{\gamma} + \dot{\sigma}^{\gamma})$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} f(Z)_{,\underset{\sim}{\lambda}(n)} \sigma^{\alpha} \sigma^{\underset{\sim}{\lambda}(n)} (U^{\gamma} + \dot{\sigma}^{\gamma}) \quad . \qquad \text{(A6.1)}$$

Let

$$X^{\alpha\gamma} = \sigma^{\alpha} f(z)\, dz^{\gamma}/d\tau - \sum_{n=0}^{\infty} e^{-1} f(Z)_{,\underset{\sim}{\lambda}(n)} m^{\alpha\gamma\underset{\sim}{\lambda}(n)}(\tau) \quad . \qquad \text{(A6.2)}$$

From (A6.1) and (5.24) we obtain $X^{\alpha\gamma}$ =

$$\sum_{n=0}^{\infty} f(Z)_{,\underset{\sim}{\lambda}(n)} \sigma^{\underset{\sim}{\lambda}(n)} \Big[\big(\tfrac{1}{n!} - \tfrac{1}{(n+1)!}\big) \sigma^{\alpha} U^{\gamma}$$

$$+ \big(\tfrac{1}{n!} - \tfrac{(n+1)}{(n+2)!}\big) \sigma^{\alpha} \dot{\sigma}^{\gamma} + \frac{1}{(n+1)!} \sigma^{\gamma} U^{\alpha} + \frac{(n+1)}{(n+2)!} \sigma^{\gamma} \dot{\sigma}^{\alpha} \Big]$$

$$= \sum_{n=0}^{\infty} f(Z)_{,\underset{\sim}{\lambda}(n)} \sigma^{\underset{\sim}{\lambda}(n)} \Big[\frac{n}{(n+1)!} \sigma^{\alpha} U^{\gamma} + \frac{1}{(n+1)!} \sigma^{\gamma} U^{\alpha} + \frac{(n+1)^2}{(n+2)!} \sigma^{\alpha} \dot{\sigma}^{\gamma} + \frac{(n+1)}{(n+2)!} \sigma^{\gamma} \dot{\sigma}^{\alpha} \Big] \; . \qquad \text{(A6.3)}$$

When $f(z)$ is set equal to $f^{\beta}{}_{\gamma}(z)$ we find from (5.30) that the second term in the above summation is just $e^{-1}(mu^{\beta} - p^{\beta})U^{\alpha}$. This term will combine with $mu^{[\alpha}U^{\beta]}$ of (5.31) to form $p^{[\alpha}U^{\beta]}$. We wish to express the remainder of (A6.3) as a combination of derivatives of $f(Z)$ coupled to multipole moments $m^{\alpha\beta\gamma(n)}$, together with a derivative w.r.to τ. It is appropriate to consider the derivative w.r.to τ of a term having the form $\sum_{n=0}^{\infty} a_n f_{,\underset{\sim}{\lambda}(n)}(Z) \sigma^{\alpha} \sigma^{\gamma} \sigma^{\underset{\sim}{\lambda}(n)}$ where a_n are numerical coefficients. From (A6.3) the most simple choices for a_n are $(n+1)^2/(n+2)!$ or $(n+1)/(n+2)!$. Making the latter choice we find from (A6.3) and (5.24) that

$$\chi^{\alpha\gamma} - \sum_{n=0}^{\infty} \frac{1}{(n+1)!} f(Z)_{,\underset{\sim}{\lambda}(n)} \sigma^{\underset{\sim}{\lambda}(n)} \sigma^{\gamma} U^{\alpha} - \frac{d}{d\tau}\left[\sum_{n=0}^{\infty} \frac{(n+1)}{(n+2)!} f(Z)_{,\underset{\sim}{\lambda}(n)} \sigma^{\alpha}\sigma^{\gamma}\sigma^{\underset{\sim}{\lambda}(n)}\right]$$

$$= \sum_{n=0}^{\infty} f(Z)_{,\underset{\sim}{\lambda}(n)} \sigma^{\underset{\sim}{\lambda}(n)} \sigma^{\alpha}\left[\frac{n}{(n+1)!} U^{\gamma} + \left(\frac{(n+1)^2}{(n+2)!} - \frac{(n+1)}{(n+2)!}\right)\dot{\sigma}^{\gamma}\right]$$

$$- \sum_{n=0}^{\infty} \frac{(n+1)}{(n+2)!} \sigma^{\alpha}\sigma^{\gamma}\left(f(Z)_{,\underset{\sim}{\lambda}(n)\delta} U^{\delta} \sigma^{\underset{\sim}{\lambda}(n)} + n f(Z)_{,\underset{\sim}{\lambda}(n)} \sigma^{\underset{\sim}{\lambda}(n-1)} \dot{\sigma}^{\lambda_n}\right)$$

$$= \sum_{n=0}^{\infty} f(Z)_{,\underset{\sim}{\lambda}(n)\delta} \sigma^{\alpha} \sigma^{\underset{\sim}{\lambda}(n)} \sigma^{\delta}\left[\frac{(n+1)}{(n+2)!} U^{\gamma} + \frac{(n+1)(n+2)}{(n+3)!} \dot{\sigma}^{\gamma}\right]$$

$$- \sum_{n=0}^{\infty} f(Z)_{,\underset{\sim}{\lambda}(n)\delta} \sigma^{\alpha} \sigma^{\underset{\sim}{\lambda}(n)} \sigma^{\gamma}\left[\frac{(n+1)}{(n+2)!} U^{\delta} + \frac{(n+1)(n+2)}{(n+3)!} \dot{\sigma}^{\delta}\right]$$

$$= \sum_{n=0}^{\infty} e^{-1}(n+1) f(Z)_{,\underset{\sim}{\lambda}(n)\delta} m^{\delta\gamma\alpha\underset{\sim}{\lambda}(n)} \quad . \qquad \text{(A6.4)}$$

(A6.2) and (A6.4) give the required expansion:

$$\sigma^{\alpha} f(z)\, dz^{\gamma}/d\tau$$

$$= \sum_{n=0}^{\infty} e^{-1} f(Z)_{,\underset{\sim}{\lambda}(n)} m^{\alpha\gamma\underset{\sim}{\lambda}(n)} + \sum_{n=0}^{\infty} e^{-1}(n+1) f(Z)_{,\underset{\sim}{\lambda}(n)\delta} m^{\delta\gamma\alpha\underset{\sim}{\lambda}(n)}$$

$$+ \sum_{n=0}^{\infty} \frac{1}{(n+1)!} f(Z)_{,\underset{\sim}{\lambda}(n)} \sigma^{\underset{\sim}{\lambda}(n)} \sigma^{\gamma} U^{\alpha} + \frac{d}{d\tau}\left[\sum_{n=0}^{\infty} \frac{(n+1)}{(n+2)!} f(Z)_{,\underset{\sim}{\lambda}(n)} \sigma^{\alpha}\sigma^{\gamma}\sigma^{\underset{\sim}{\lambda}(n)}\right] .$$

(cf. [5, page 242]).

Choosing $f(z) = f^{\beta}{}_{\gamma}(z)$ gives an expression for

$\sigma^{\alpha} f^{\beta}{}_{\gamma}(z) dz^{\gamma}/d\tau$ which on substitution into (5.31) gives the spin equation of motion (5.33).

Appendix 7

Let $A^{\alpha\beta}$ denote

$$(4\pi)^{-1}(f^{\alpha}{}_{\lambda}H^{\beta\lambda} - \tfrac{1}{4}f_{\mu\nu}f^{\mu\nu}g^{\alpha\beta}) \quad . \tag{A7.1}$$

The field equations $f^{\alpha\beta}{}_{,\beta} = 4\pi(J^{\alpha}+M^{\alpha\beta}{}_{,\beta})$, $f_{[\alpha\beta,\gamma]} = 0$ imply

$$A_{\alpha}{}^{\beta}{}_{,\beta} = - f_{\alpha\beta}J^{\beta} - \tfrac{1}{2}M^{\lambda\mu}f_{\lambda\mu,\alpha} \quad . \tag{A7.2}$$

Define

$$M^{\alpha\beta\underset{\sim}{\gamma}(m)}(x) \equiv \sum_{n=0}^{\infty}(-1)^{n}m^{\alpha\beta\underset{\sim}{\gamma}(m)\underset{\sim}{\lambda}(n)}{}_{,\underset{\sim}{\lambda}(n)}. \tag{A7.3}$$

then

$$m^{\alpha\beta\underset{\sim}{\gamma}(m)} = M^{\alpha\beta\underset{\sim}{\gamma}(m)} + M^{\alpha\beta\lambda\underset{\sim}{\gamma}(m)}{}_{,\lambda} \quad . \tag{A7.4}$$

From (A7.4) we have

$$M^{\lambda\mu}f_{\lambda\mu,\alpha} = m^{\lambda\mu}f_{\lambda\mu,\alpha} - (M^{\lambda\mu\tau}f_{\lambda\mu,\alpha})_{,\tau} + M^{\lambda\mu\tau}f_{\lambda\mu,\alpha\tau} \;. \tag{A7.5}$$

Similarly

$$M^{\lambda\mu\tau}f_{\lambda\mu,\alpha\tau} = m^{\lambda\mu\tau}f_{\lambda\mu,\alpha\tau} - (M^{\lambda\mu\tau\gamma}f_{\lambda\mu,\alpha\tau})_{,\gamma} + M^{\lambda\mu\tau\gamma}f_{\lambda\mu,\alpha\tau\gamma} \;.$$

Repeated application of (A7.4) gives

$$M^{\lambda\mu} f_{\lambda\mu,\alpha} \tag{A7.6}$$

$$= \sum_{n=0}^{\infty} m^{\lambda\mu\underset{\sim}{\tau}(n)} f_{\lambda\mu,\alpha\underset{\sim}{\tau}(n)} - \Big(\sum_{n=0}^{\infty} M^{\lambda\mu\tau\underset{\sim}{\gamma}(n)} f_{\lambda\mu,\alpha\underset{\sim}{\gamma}(n)} \Big)_{,\tau} \quad .$$

From (5.35), (A7.2) and (A7.6) we obtain

$$\partial_\beta T^{\alpha\beta}_{(mat)} = - A_\alpha{}^\beta{}_{,\beta} + \tfrac{1}{2}\partial_\tau \Big(\sum_{n=0}^{\infty} M^{\lambda\mu\tau\underset{\sim}{\gamma}(n)} f_{\lambda\mu,\alpha\underset{\sim}{\gamma}(n)} \Big) \quad .$$

According to (5.37) and (A7.1) this is just (5.36).

Appendix 8

In the equation following (5.38) we wish to express the two summation terms on the r.h.s. as $T^{[\alpha\beta]}_{(em)}$ (cf. (5.37)) plus a divergence. From (A7.4) we have

$$\sum_{n=0}^{\infty} f^{\alpha}{}_{\gamma,\underset{\sim}{\lambda}(n)} m^{\beta\gamma\underset{\sim}{\lambda}(n)}$$

$$= \sum_{n=0}^{\infty} f^{\alpha}{}_{\gamma,\underset{\sim}{\lambda}(n)} M^{\beta\gamma\underset{\sim}{\lambda}(n)} + \partial_{\tau}\Big(\sum_{n=0}^{\infty} f^{\alpha}{}_{\gamma,\underset{\sim}{\lambda}(n)} M^{\beta\gamma\tau\underset{\sim}{\lambda}(n)}\Big) - \sum_{n=0}^{\infty} f^{\alpha}{}_{\gamma,\underset{\sim}{\lambda}(n)\tau} M^{\beta\gamma\tau\underset{\sim}{\lambda}(n)}$$

$$= f^{\alpha}{}_{\gamma} M^{\beta\gamma} + \partial_{\gamma}\Big(\sum_{n=0}^{\infty} f^{\alpha}{}_{\kappa,\underset{\sim}{\lambda}(n)} M^{\beta\kappa\gamma\underset{\sim}{\lambda}(n)}\Big) \quad . \qquad \text{(A8.1)}$$

and

$$\sum_{n=0}^{\infty} (n+1) f^{\alpha}{}_{\gamma,\delta\underset{\sim}{\lambda}(n)} m^{\gamma\delta\beta\underset{\sim}{\lambda}(n)}$$

$$= \sum_{n=0}^{\infty} (n+1) f^{\alpha}{}_{\gamma,\delta\underset{\sim}{\lambda}(n)} M^{\gamma\delta\beta\underset{\sim}{\lambda}(n)} + \partial_{\tau}\Big(\sum_{n=0}^{\infty} (n+1) f^{\alpha}{}_{\gamma,\delta\underset{\sim}{\lambda}(n)} M^{\gamma\delta\beta\tau\underset{\sim}{\lambda}(n)}\Big)$$

$$- \sum_{n=0}^{\infty} (n+1) f^{\alpha}{}_{\gamma,\delta\underset{\sim}{\lambda}(n)\tau} M^{\gamma\delta\beta\tau\underset{\sim}{\lambda}(n)} \quad .$$

The last term is $-\sum_{n=0}^{\infty} n f^{\alpha}{}_{\gamma,\delta\underset{\sim}{\lambda}(n)} M^{\gamma\delta\beta\underset{\sim}{\lambda}(n)}$ which combines with the first term on the r.h.s to give

$$\sum_{n=0}^{\infty} (n+1) f^{\alpha}{}_{\gamma,\delta\underset{\sim}{\lambda}(n)} m^{\gamma\delta\beta\underset{\sim}{\lambda}(n)} \qquad \text{(A8.2)}$$

$$= \sum_{n=0}^{\infty} f^{\alpha}{}_{\gamma,\delta\underset{\sim}{\lambda}(n)} M^{\gamma\delta\beta\underset{\sim}{\lambda}(n)} + \partial_{\gamma}\Big(\sum_{n=0}^{\infty} (n+1) f^{\alpha}{}_{\kappa,\delta\underset{\sim}{\lambda}(n)} M^{\kappa\delta\beta\gamma\underset{\sim}{\lambda}(n)}\Big) \quad .$$

Substitution of (A8.1) and (A8.2) into the equation following (5.38) and use of (5.37), (5.39) gives (5.40).

Appendix 9

From section 6.2 we have

$$\Phi(\sigma) = \int A_\lambda(z(t,\sigma))v^\lambda(t,\sigma)\ dt$$

and

$$-\ d\Phi/d\sigma = \int F_{\mu\nu}v^\mu l^\nu\ dt \quad .$$

Continued differentiation and use of (6.4), (6.10) gives (neglecting derivatives of curvature and squares of curvature):

$$-d^2\Phi/d\sigma^2 = \int \left(F_{\mu\nu|\lambda}v^\mu l^\nu l^\lambda + F_{\mu\nu}l^\nu \frac{\delta l^\mu}{\delta t}\right) dt \quad ,$$

$$-d^3\Phi/d\sigma^3 = \int \left(F_{\mu\nu|\lambda_1\lambda_2}v^\mu l^\nu l^{\lambda_1} l^{\lambda_2} + 2F_{\mu\nu|\lambda}l^\nu l^\lambda \frac{\delta l^\mu}{\delta t} + F_{\mu\nu}R_\lambda{}^\mu{}_{\rho\sigma}v^\rho l^\sigma l^\nu l^\lambda\right) dt \quad ,$$

$$-d^4\Phi/d\sigma^4$$

$$= \int \Big(F_{\mu\nu|\lambda_1\lambda_2\lambda_3}v^\mu l^\nu l^{\lambda_1} l^{\lambda_2} l^{\lambda_3} + 3F_{\mu\nu|\lambda_1\lambda_2}l^\nu l^{\lambda_1} l^{\lambda_2}\frac{\delta l^\mu}{\delta t}$$

$$+ 3F_{\mu\nu|\lambda_1}R_\lambda{}^\mu{}_{\rho\sigma}v^\rho l^\sigma l^\nu l^\lambda l^{\lambda_1} + F_{\mu\nu}R_\lambda{}^\mu{}_{\rho\sigma}l^\sigma l^\nu l^\lambda \frac{\delta l^\rho}{\delta t}\Big)\ dt \quad .$$

The above expressions are all of the form (6.16). We may verify (6.16) by induction: differentiating (6.16) gives

$$-d^{n+1}\Phi/d\sigma^{n+1}$$

$$= \int \Big(F_{\mu\nu|\underset{\sim}{\lambda}(n)} v^{\mu} l^{\nu} l^{\underset{\sim}{\lambda}(n)} + (1+(n-1)) F_{\mu\nu|\underset{\sim}{\lambda}(n-1)} l^{\nu} l^{\underset{\sim}{\lambda}(n-1)} \frac{\delta l^{\mu}}{\delta t}$$

$$+ \left((n-1)+\frac{1}{2}(n-1)(n-2)\right) F_{\mu\nu|\underset{\sim}{\lambda}(n-2)} R_{\lambda}{}^{\mu}{}_{\rho\sigma} v^{\rho} l^{\sigma} l^{\nu} l^{\lambda} l^{\underset{\sim}{\lambda}(n-2)}$$

$$+ \left(\frac{1}{2}(n-1)(n-2)+\frac{1}{6}(n-1)(n-2)(n-3)\right) F_{\mu\nu|\underset{\sim}{\lambda}(n-3)} R_{\lambda}{}^{\mu}{}_{\rho\sigma} l^{\sigma} l^{\nu} l^{\lambda} l^{\underset{\sim}{\lambda}(n-3)} \frac{\delta l^{\rho}}{\delta t} \Big) dt.$$

The coefficients in the above expression are n, $n(n-1)$ and $n(n-1)(n-2)$ so that the above is just (6.16) with n replaced by $n+1$.

Appendix 10

Consider a composite particle consisting of n point particles with world-lines $z^{\lambda}_{(i)}(t)$ $(i = 1,2,\ldots n)$. With a common central world-line $Z^{\alpha}(t)$, the Lagrangian $L_{(i)}$ of each particle i may be expanded in powers of connecting vectors $\sigma^{\alpha}_{(i)}$. The total action for the composite particle in an external field ϕ_A is then

$$I = \int L\, dt \quad \text{with} \quad L = \sum_{i=1}^{n} L_{(i)} \tag{A10.1}$$

and

$$L_{(i)} = L_{(i)}(V^{\alpha}, g_{\alpha\beta}, \sigma^{\alpha}_{(i)}, \dot{\sigma}^{\alpha}_{(i)}, \Psi_{\underset{\sim}{A}}) \tag{A10.2}$$

I is a functional of

$$X_A(t) \equiv (z^{\lambda}_{(i)}(t), \ i = 1,\ldots n) \tag{A10.3}$$

The 4n translational equations of motion are obtained from $dI/d\varepsilon = 0$ for infinitesimal displacements of each $z^{\lambda}_{(i)}(t)$ with endpoints held fixed. In other words, from demanding $dI/d\varepsilon = 0$ for arbitrary variations $\partial X_A(t,\varepsilon)/\partial\varepsilon$ subject to $X_A(t_i,\varepsilon) = X_A(t_i,0)$ $(i = 1,2)$. This, however, gives too much information when one is interested in the overall motion and spin of the composite particle and not its detailed internal dynamics. Each particle i is governed by equations (6.27), (6.30) and the equations for the composite particle are obviously given by summing over i in these. The equations for the total four-momentum and total spin

$$P_\alpha = \sum_{i=1}^{n} \frac{\partial L_{(i)}}{\partial V^\alpha} \,, \quad S^\alpha{}_\beta = \sum_{i=1}^{n} \sigma^{[\alpha}_{(i)} \frac{\partial L_{(i)}}{\partial \dot{\sigma}^{\beta]}_{(i)}} \,, \quad M^A_{\sim} = \sum_{i=1}^{n} \frac{\partial L_{(i)}}{\partial \Psi_{\underset{\sim}{A}}} \,, \qquad (A10.4)$$

are then (6.27) and (6.30).

To compare with Chapter 2, Section 3, let $e^{(a)}_\alpha(t)$ denote any orthonormal tetrad defined along $Z^\alpha(t)$. Two types of (constrained) variation in $X_A(t)$ will be discussed below that are essentially the variations considered in Ch. 2, Sec. 3. The action principle appearing there is then a consequence of the more general principle $dI/d\varepsilon = 0$ for *arbitrary* $\partial X_A/\partial\varepsilon$ of this chapter. Instead of independent variation of each $z^\lambda_{(i)}(t)$, the action principle of Ch. 2, Sec. 3, selects just the variations needed to deduce the *composite* equations.

Let each $\sigma^\alpha_{(i)}$ have scalar components $\sigma_{(i)a}$ w.r.to $e^{(a)}_\alpha$:

$$\sigma^\alpha_{(i)} = \sigma_{(i)a} e^{(a)\alpha} \,, \quad \dot{\sigma}^\alpha_{(i)} = \dot{\sigma}_{(i)a} e^{(a)\alpha} + \sigma_{(i)a} \dot{e}^{(a)\alpha} . \qquad (A10.5)$$

It follows from (A10.1), (A10.2), (A10.5) that L may be written as[1]

$$L = L(V^\alpha,\, e^{(a)}_\alpha,\, \dot{e}^{(a)}_\alpha,\, \Psi_{\underset{\sim}{A}},\, \sigma_{(i)a},\, \dot{\sigma}_{(i)a}) \quad . \qquad (A10.6)$$

---

[1] L denotes the Lagrangian both as a function of $(g_{\alpha\beta}, \sigma^\alpha_{(i)}, \dot{\sigma}^\alpha_{(i)})$ and as a function of $(e^{(a)}_\alpha, \dot{e}^{(a)}_\alpha, \sigma_{(i)a}, \dot{\sigma}_{(i)a})$ since the particular meaning will always be clear from the arguments differentiating L.

From (A10.5) it follows that

$$\frac{\partial L}{\partial \dot{e}^{(a)\beta}} = \sum_{i=1}^{n} \frac{\partial L_{(i)}}{\partial \dot{\sigma}^{\gamma}_{(i)}} \frac{\partial \dot{\sigma}^{\gamma}_{(i)}}{\partial \dot{e}^{(a)\beta}} = \sum_{i=1}^{n} \frac{\partial L_{(i)}}{\partial \dot{\sigma}^{\gamma}_{(i)}} \delta^{\gamma}_{\beta} \sigma_{(i)a}$$

giving

$$e^{(a)\alpha} \frac{\partial L}{\partial \dot{e}^{(a)\beta}} = \sum_{i=1}^{n} \sigma^{\alpha}_{(i)} \frac{\partial L_{(i)}}{\partial \dot{\sigma}^{\beta}_{(i)}} .$$

Definitions (2.19) and (A10.4) for spin are therefore the same. Furthermore, we have P_{α} and M^{A} given by (A10.4) when L is written either in terms of the connecting vectors or in terms of the tetrad.

Consider the equations resulting from $dI/d\varepsilon = 0$ for arbitrary $\partial Z^{\alpha}/\partial\varepsilon$ with $e^{(a)}_{\alpha}$ and $\sigma_{(i)a}$ held fixed by parallel propagation. This is exactly the action principle generating (2.23)[1]. $dI/d\varepsilon = 0$ is given by summing (5.24) over i, and $\delta e^{(a)}_{\alpha}/\delta\varepsilon = 0$, $\delta\sigma_{(i)a}/\delta\varepsilon = 0$, imply $\delta\sigma^{\alpha}_{(i)}/\delta\varepsilon = 0$[2]. The resulting equations are therefore (6.27) (i.e. (2.23)) for the composite particle.

Equation (2.22) was obtained by variation of $e^{(a)}_{\alpha}$ subject to $\delta g_{\alpha\beta}/\delta\varepsilon = 0$, with $\partial Z^{\alpha}/\partial\varepsilon = 0$, $\partial\sigma_{(i)a}/\partial\varepsilon = 0$. This

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[1]The scalar components $\sigma_{(i)a}$ play no part in the derivation of the equations of motion since they are held fixed here and in the next type of variation considered. As scalars their infinitesimal generators are zero so they do not contribute to the invariance identities. No mention of the dependence of L on scalars such as $\sigma_{(i)a}$ and electric charge e need therefore be made.

[2]The resulting variation in $z^{\lambda}_{(i)}$ is given by
$0 = \delta\sigma^{\alpha}_{(i)}/\delta\varepsilon = \sigma^{\alpha}_{(i)}|_{\beta}\partial Z^{\beta}/\partial\varepsilon + \sigma^{\alpha}_{(i)}|_{\lambda}\partial z^{\lambda}_{(i)}/\partial\varepsilon$.
For zero curvature (where $\sigma^{\alpha}_{(i)} = z^{\alpha}_{(i)} - Z^{\alpha}$, $\sigma^{\alpha}_{(i)}|_{\beta} = -\delta^{\alpha}_{\beta}$ and $\sigma^{\alpha}_{(i)}|_{\lambda} = \delta^{\alpha}_{\lambda}$) this is a "rigid" translation $\partial z^{\lambda}_{(i)}(t,\varepsilon)/\partial\varepsilon = \partial Z^{\lambda}(t,\varepsilon)/\partial\varepsilon$.

induces a variation in $X_A(t)$ given by $\delta\sigma^{\alpha}_{(i)}/\delta\varepsilon = \sigma_{(i)a}\delta e^{(a)\alpha}/\delta\varepsilon = \sigma^{\alpha}_{(i)|\lambda}\partial z^{\lambda}_{(i)}/\partial\varepsilon$[1], and from (6.24) gives $dI/d\varepsilon = \sum_i\int(\partial L(t)/\partial\sigma^{\alpha}_{(i)})\delta\sigma^{\alpha}_{(i)}/\delta\varepsilon dt = 0$. Let $\delta e^{(a)}_{\alpha}/\delta\varepsilon = \Omega_{\alpha}{}^{\beta}e^{(a)}_{\beta}$ where $\delta g_{\alpha\beta}/\delta\varepsilon = 0$ implies $\Omega^{(\alpha\beta)} = 0$. Then $\delta\sigma^{\alpha}_{(i)}/\delta\varepsilon = \sigma_{(i)a}\Omega^{\alpha\beta}e^{(a)}_{\beta} = \Omega^{\alpha\beta}\sigma_{(i)\beta}$. Inserting this into $dI/d\varepsilon = 0$ gives the spin equation of motion as $\sum_i \delta L(t)/\delta\sigma^{[\alpha}_{(i)}\sigma^{\beta]}_{(i)} = 0$. Equation (6.29) and the equation following it then give (6.30) (i.e. (2.22)) for the total spin.

To summarize, extremizing I for *arbitrary* variations $\partial z^{\lambda}_{(i)}/\partial\varepsilon$ leads to equations of motion (6.27), (6.30) for each i, while the constrained variations induced in $z^{\lambda}_{(i)}$ from those of Chap. 2, Sec. 3, give the same equations summed over i for the total four-momentum and spin.

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[1]For zero curvature $\sigma^{\alpha}_{(i)|\lambda} = (z^{\alpha}_{(i)} - Z^{\alpha})_{,\lambda} = \delta^{\alpha}_{\lambda}$ so that $\partial z^{\alpha}_{(i)}(t,\varepsilon)/\partial\varepsilon = \delta\sigma^{\alpha}_{(i)}(t,\varepsilon)/\delta\varepsilon = \Omega^{\alpha\beta}(t)\sigma_{(i)\beta}$ which is a "rigid" rotation.

described by "stream lines", each labelled by a particular set of values for a^m. For both fluids and solids the number of particles in an infinitesimal flux tube d^3a is assumed to be a constant of the motion given by $N(a^m)d^3a$. In terms of the gradients $\partial_\mu a^m$ and $N(a^m)$ the numerical flux may be written as

$$N^\mu(x) = N(a^m)\eta^{\mu\alpha\beta\gamma}(\partial_\alpha a^1)(\partial_\beta a^2)(\partial_\gamma a^3) \qquad (3.3)$$

where $\eta^{\mu\alpha\beta\gamma}$ is the completely antisymmetric permutation tensor[1]. Conservation of particle number implies that

$$N^\mu{}_{|\mu} = 0 \quad . \qquad (3.4)$$

Defining γ_{mn} by

$$\gamma^{mn}\gamma_{np} = \delta^m_p \quad , \qquad \gamma^{mn} \equiv g^{\mu\nu}(\partial_\mu a^m)(\partial_\nu a^n) \quad , \qquad (3.5)$$

distances between neighbouring points a^m and a^m+da^m of the material in the local rest frame are given by

$$(ds^2)_\perp = \gamma_{mn} da^m da^n \quad . \qquad (3.6)$$

A useful form for the number density n is its expression in terms of $N(a^m)$ and γ^{mn},

---

1. $\varepsilon^{\mu\alpha\beta\gamma} = \sqrt{-g}\eta^{\mu\alpha\beta\gamma}$ where $\varepsilon^{\mu\alpha\beta\gamma}$ are the Levi-Civita permutation symbols.

$$n(x^{\mu}) = N(a^{m})(\det\gamma^{mn})^{\frac{1}{2}} \quad . \tag{3.7}$$

The material will be assumed to be in *adiabatic* motion with the entropy per particle S constant along each world line, $S = S(a^{m})$. The entropy current $S^{\mu} = S(a^{m})N^{\mu}$ according to (3.4) then satisfies $S^{\mu}{}_{|\mu} = 0$.

3.3 Gravitational Field Equations Derived from Variation of Tetrad

The gravitational field equations are derived from variation of an orthonormal tetrad field $e^{(a)}_{\alpha}(x)$ that satisfies

$$\eta_{ab}e^{(a)}_{\alpha}\, e^{(b)}_{\beta} = g_{\alpha\beta} \quad , \qquad g^{\alpha\beta}e^{(a)}_{\alpha}\, e^{(b)}_{\beta} = \eta^{ab} \quad , \tag{3.8}$$

$$\eta_{ab} = \eta^{ab} = \text{diag}(1,1,1,-1) \quad (a,b = 1 \text{ to } 4) \quad . \tag{3.9}$$

The sixteen component tetrad field plays a dual role: the ten symmetrized products $\eta_{ab}e^{(a)}_{\alpha}\, e^{(b)}_{\beta} = g_{\alpha\beta}$ define the metric and determine the gravitational field, while the dependence of the Lagrangian density upon the six angular velocities $w^{ab} = e^{(a)\alpha}\, e^{(b)}_{\alpha|\beta}u^{\beta}$ determines the internal spin of the medium. Because of this second role the Lagrangian density will depend explicitly on the tetrad field in addition to its dependence via the metric tensor $g_{\alpha\beta}$. One could have considered a Lagrangian density depending upon both $g_{\alpha\beta}$ and $e^{(a)}_{\alpha}$ (and then varied $g_{\alpha\beta}$ to obtain the ten gravitational equations and varied six independent components of the tetrad field to obtain spin equations). However, since the orthonormality conditions $g_{\alpha\beta} = \eta_{ab}\, e^{(a)}_{\alpha}\, e^{(b)}_{\beta}$ are ten relations between the twenty-six variables $(g_{\alpha\beta},\ e^{(a)}_{\alpha})$, we have sixteen independent variables and it is much simpler to choose the $e^{(a)}_{\alpha}$ as the sixteen. We take, therefore, arbitrary *independent* variations in the $e^{(a)}_{\alpha}$ to obtain ten gravitational equations and six spin equations.

Helpful guidance for the following calculations comes from Rosenfeld and Belinfante [7]. They consider spin flux due to fields only, while a total spin flux due to both material and field is considered here.

The spin and translational equations of motion for the medium and the system of field equations for the applied fields are all obtained from the four-dimensional action integral

$$I = \int L(\psi_{\underset{\sim}{A}},\ \psi_{\underset{\sim}{A}|\alpha})\, d^4x + (16\pi)^{-1} \int \sqrt{-g}\, R\, d^4x \qquad (3.10)$$

in which we have made the specific choice of the curvature scalar R for the gravitational free-field Lagrangian[1]. The Lagrangian density L is taken to be an unspecified function of the fields

---

[1]According to Lovelock's theorem [27a], *any* choice $L(g_{\alpha\beta},\ \partial_\varepsilon g_{\alpha\beta}, \ldots,\ \partial_{\varepsilon_1}\ldots\partial_{\varepsilon_n} g_{\alpha\beta}, \ldots)$ which leads to second order field equations for minimally coupled sources is variationally equivalent to the curvature scalar $R = g^{\alpha\beta} R_{\alpha\beta}$ $(R_{\alpha\beta} = R^{\mu}{}_{\alpha\mu\beta})$.

$$\psi_{\underset{\sim}{A}} = (a^m,\ e^{(a)}_\alpha,\ \phi_A,\ R^\alpha{}_{\beta\gamma\delta}) \qquad (3.11)$$

and first covariant derivatives $\psi_{A|\alpha}$. ϕ_A is an arbitrary set of fields interacting with the gravitational field and the medium. We are restricting the dependence of L to only *first* derivatives in order to simplify the discussion. To simplify further we will also assume that no derivatives of $R^\alpha{}_{\beta\gamma\delta}$ enter. This is still sufficiently general to cover most cases of practical interest. (For the analysis of the general case see Chapter 4.) The Lagrangian density L is assumed to be constructed from a^m and $a^m{}_{|\mu} = \partial_\mu a^m$ so as to be independent of the particular choice of "material co--ordinates" a^m. ($N^\mu(x)$ and $n(x)$, given by equations (3.3) and (3.7), are both invariant under $a^m \to \bar{a}^m = f^m(a^n)$.)

Under an arbitrary variation $\delta e^{(a)}_\alpha$ of the tetrad field, and the resulting variation

$$\delta g_{\rho\sigma} = 2\eta_{ab}\, e^{(b)}_{(\rho}\, \delta e^{(a)}_{\sigma)} \qquad (3.12)$$

of the metric, variation $\delta_{(e)}L$ has contributions from three sources:

1. the explicit dependence of L on $e^{(a)}_\alpha$, $e^{(a)}_{\alpha|\beta}$;
2. the variation of the affine connexion hidden in the covariant derivatives $e^{(a)}_{\alpha|\beta}$, $\phi_{A|\alpha}$;
3. the variation of $R^\alpha{}_{\beta\gamma\delta}$.

Accordingly, from (2.8) we have $\delta_{(e)}L =$

$$(\delta \mathbf{L}/\delta e_\sigma^{(a)})\delta e_\sigma^{(a)} + \mathbf{U}^{\tau\sigma}{}_\rho \delta\Gamma^\rho_{\sigma\tau} + \mathbf{Q}_\alpha{}^{\beta\gamma\delta}\delta R^\alpha{}_{\beta\gamma\delta} + (\mathrm{div}) \qquad (3.13)$$

where $\delta \mathbf{L}/\delta\psi_A$ and $\mathbf{U}^{\tau\sigma}{}_\rho$ were defined in (2.12) and (2.14),

$$\mathbf{Q}_\alpha{}^{\beta\gamma\delta} = \sqrt{-g}\; Q_\alpha{}^{\beta\gamma\delta} \equiv \partial \mathbf{L}/\partial R^\alpha{}_{\beta\gamma\delta} \qquad (3.14)$$

and (div) represents a divergence $\partial_\alpha(\ldots)$.

To re-express the last two terms of (3.13) in terms of $\delta e_\sigma^{(a)}$, we note that

$$\delta R^\alpha{}_{\beta\gamma\delta} = 2(\delta\Gamma^\alpha_{\beta[\delta})_{|\gamma]} \;, \quad \delta\Gamma^\rho_{\sigma\tau} = (\delta g)^\rho_{(\sigma|\tau)} - \tfrac{1}{2}(\delta g)_{\sigma\tau}{}^{|\rho} \qquad (3.15)$$

enable us to write the identities

$$\mathbf{Q}_\alpha{}^{\beta\gamma\delta}\delta R^\alpha{}_{\beta\gamma\delta} = -2\mathbf{Q}_\rho{}^{\sigma[\mu\tau]}{}_{|\mu}\delta\Gamma^\rho_{\sigma\tau} + (\mathrm{div}) \qquad (3.16)$$

$$\mathbf{U}^{\tau\sigma}{}_\rho\delta\Gamma^\rho_{\sigma\tau} = \tfrac{1}{2}(\tfrac{1}{2}(\mathbf{S}^{\sigma\tau\rho} + \mathbf{S}^{\rho\tau\sigma}) - \mathbf{U}^{\tau(\rho\sigma)})_{|\tau}\delta g_{\rho\sigma} + (\mathrm{div}) \qquad (3.17)$$

valid for *arbitrary* tensor densities $\mathbf{Q}_\alpha{}^{\beta\gamma\delta}$, $\mathbf{U}^{\tau\sigma}{}_\rho$. We have defined the "spin flux"

$$\mathbf{S}^{\sigma\tau\rho} \equiv 2\mathbf{U}^{\rho[\sigma\tau]} \;. \qquad (3.18)$$

Using the variation in R

$$(16\pi)^{-1}\delta(\sqrt{-g}R) = -(8\pi)^{-1}\sqrt{-g}G^{\alpha\beta}e_{(a)\alpha}\delta e_\beta^{(a)} \qquad (3.19)$$

where $G^{\alpha\beta}$ is the Einstein tensor, $G^{\alpha\beta} = R^{\alpha\beta} - \frac{1}{2}g^{\alpha\beta}R$, the action principle $\delta_{(e)}I = 0$ gives, with the aid of (3.13), (3.16), (3.17) and (3.12)

$$(8\pi)^{-1}\sqrt{-g}G^{\rho\sigma} = (\delta \mathbf{L}/\delta e^{(a)}_{\sigma})e^{(a)\rho}$$
$$+ (\tfrac{1}{2}(\mathbf{S}^{\sigma\tau\rho} + \mathbf{S}^{\rho\tau\sigma}) - \mathbf{U}^{\tau(\rho\sigma)})_{|\tau} + 4\mathbf{Q}^{\lambda(\rho\sigma)\mu}{}_{|\mu\lambda} . \qquad (3.20)$$

The symmetry of $G^{\rho\sigma}$ and the second and third terms of the right hand side in ρ,σ gives at once the six *equations of motion for spin*:

$$e^{(a)}_{[\rho}(\delta \mathbf{L}/\delta e^{(a)}_{\sigma]}) = 0 \quad . \qquad (3.21)$$

The gravitational field equations are the remaining ten equations of (3.20). As a first step towards reducing these to a familiar form, the identity (2.16) is used to replace $(\delta \mathbf{L}/\delta e^{(a)}_{\sigma})e^{(a)\rho}$ with terms having a direct physical interpretation. Let R_A and $\mathbf{Q}^A$ denote $R^{\alpha}{}_{\beta\gamma\delta}$ and $\mathbf{Q}_{\alpha}{}^{\beta\gamma\delta}$. In the present context of (3.10) and (3.11), the identity (2.16) reads[1]

$$(\delta \mathbf{L}/\delta e^{(a)}_{\sigma})e^{(a)}_{\rho} = \mathbf{U}^{\tau\sigma}{}_{\rho|\tau} + \mathbf{t}_{\rho}{}^{\sigma} + \mathbf{Q}^A(I_A{}^B)_{\rho}{}^{\sigma}R_B \qquad (3.22)$$

modulo the *nongravitational field equations*:

----

[1]The last term of (3.22) has the explicit form $\mathbf{Q}^A(I_A{}^B)_{\rho}{}^{\sigma}R_B = \mathbf{Q}_{\rho}{}^{\beta\gamma\delta}R^{\sigma}{}_{\beta\gamma\delta} - 3\mathbf{Q}_{\alpha}{}^{\sigma\gamma\delta}R^{\alpha}{}_{\rho\gamma\delta}$.

$$\delta L/\delta \phi_A = 0 \quad . \tag{3.23}$$

Substitution of (3.22) into (3.20) gives the following form of the *gravitational field equations*:

$$(8\pi)^{-1}\sqrt{-g}\; G^{\rho\sigma} = \sqrt{-g}\; T^{\rho\sigma} \equiv \tag{3.24}$$

$$t^{\rho\sigma} + \tfrac{1}{2}(S^{\sigma\tau\rho} + S^{\rho\tau\sigma} + S^{\sigma\rho\tau})_{|\tau} + 4Q^{\lambda(\rho\sigma)\mu}{}_{|\mu\lambda} + Q^A(\mathcal{T}_A{}^B)^{\rho\sigma}R_B \;.$$

3.4 Matter and Field Decomposition of Energy-Momentum and Spin

Equation (3.24) identifies $T^{\rho\sigma}$ as the "correct" (symmetric, covariantly constant) total energy-momentum tensor. It is expressed in terms of a canonical energy tensor density $t_\rho{}^\sigma$, a spin flux $S^{\rho\sigma\tau}$ and gravitational quadrupole terms. The presence of interaction terms in L implies that $t_\rho{}^\sigma$ and $S^{\rho\sigma\tau}$ will not in general be simply a sum of free material and field parts, but will contain (respectively) interaction momenta and spin. As Israel [6a] points out for the electromagnetic case:

> "This expression for the *total* energy tensor is the fundamental result, and questions about which part should be called the 'electromagnetic energy tensor' are mere semantics and to a large extent, superfluous. However, if a prescription is desired, even though it be arbitrary, the least that one should

demand is that it be simple, natural, general, and unambiguous."

In the context of Lagrangian dynamics one immediately has a simple, natural, general and unambiguous split into matter and field parts from the following considerations.

Consider

$$t_\rho^{\ \sigma} = \mathbf{L}\delta_\rho^\sigma - \psi_{\underset{\sim}{A}|\rho}\mathbf{L}^{\underset{\sim}{A}\sigma}$$

$$= \mathbf{L}\delta_\rho^\sigma - (\partial_\rho a^m)\frac{\partial \mathbf{L}}{\partial(\partial_\sigma a^m)} - e_{\alpha|\rho}^{(a)}\frac{\partial \mathbf{L}}{\partial(e_{\alpha|\sigma}^{(a)})} - \phi_{A|\rho}\frac{\partial \mathbf{L}}{\partial\phi_{A|\sigma}} .$$

If a convention must be adopted for a split then the obvious one is to designate the second and third terms as belonging to the material since their form involves differentiating and multiplying by *material* derivatives $\partial_\alpha a^m$ and $e_{\alpha|\beta}^{(a)}$. Similarly the last term, obtained by differentiation and multiplication by $\phi_{A|\alpha}$, is regarded as field energy-momentum.[1] The splitting of $t_\rho^{\ \sigma}$ is completed by decomposing $\mathbf{L}$ into a sum

$$\mathbf{L} = \mathbf{L}_1(\psi_{\underset{\sim}{A}},\ \psi_{\underset{\sim}{A}|\alpha}) + \mathbf{L}_2(\psi_{\underset{\sim}{A}},\ \psi_{\underset{\sim}{A}|\alpha}) \qquad (3.25)$$

in which $\mathbf{L}_1 = \sqrt{-g}L_1$ and $\mathbf{L}_2 = \sqrt{-g}L_2$ represent matter and field parts respectively. We thus have

----

[1] This is a very general procedure. If a set of variables $\psi_{\underset{\sim}{A}}$ on which a Lagrangian density $\mathbf{L}$ depends are of two types, $\psi_{\underset{\sim}{A}} = (\theta_A,\ \eta_A)$, then our convention immediately splits $\psi_{\underset{\sim}{A}|\rho}\mathbf{L}^{\underset{\sim}{A}\sigma}$ into energy-momentum $\theta_{A|\rho}\partial\mathbf{L}/\partial\theta_{A|\sigma}$ of type one and energy-momentum $\eta_{A|\rho}\partial\mathbf{L}/\partial\eta_{A|\sigma}$ of type two.

$$t_\rho^{\ \sigma} = t_{\rho(\phi)}^{\ \sigma} + t_{\rho(\mathrm{mat})}^{\ \sigma} \tag{3.26}$$

where

$$t_{\rho(\mathrm{mat})}^{\ \sigma} = L_1 \delta_\rho^\sigma - (\partial_\rho a^m) \frac{\partial L}{\partial(\partial_\sigma a^m)} - e_{\alpha|\rho}^{(a)} \frac{\partial L}{\partial(e_{\alpha|\sigma}^{(a)})} \tag{3.27}$$

defines the material energy-momentum tensor density and

$$t_{\rho(\phi)}^{\ \sigma} = \sqrt{-g}\ t_{\rho(\phi)}^{\ \sigma} = L_2 \delta_\rho^\sigma - \phi_{A|\rho} \partial L / \partial \phi_{A|\sigma} \tag{3.28}$$

defines the canonical energy tensor density for the fields ϕ_A. The particular split of L into L_1 and L_2 is of minor importance. Equations (3.27) and (3.28) involve the *total* L except for the "diagonal" terms $L_1 \delta_\rho^\sigma$ and $L_2 \delta_\rho^\sigma$. Changing the split of L will merely redistribute energy-momentum between these diagonal terms.

The conditions $u^\alpha \partial_\alpha a^m = 0$, $u_\alpha u^\alpha = -1$ determine the four-velocity u^α as a function of $\partial_\alpha a^m$ ($m = 1,2,3$) (cf. Sec. 2). If u^α appears in L it is to be regarded as such. Noting that the second term on the right hand side of (3.27) is orthogonal to u^α only in its first index, we write it as a sum of a convective four-momentum flux and a stress term by projection (on the second index) parallel and orthogonal to u^α. This can be expressed neatly by introducing a new Lagrangian that includes u^α among its variables. In terms of the projection operator

$$\Delta_\alpha^\beta = \delta_\alpha^\beta + u_\alpha u^\beta \tag{3.29}$$

and $L(a^m, \partial_\alpha a^m, e_\alpha^{(a)}, e_{\alpha|\beta}^{(a)}, \phi_A, \phi_{A|\beta}, R^\alpha{}_{\beta\gamma\delta})$ we define

$$L'(u^\alpha, a^m, \partial_\alpha a^m, \ldots) \equiv L(a^m, \Delta_\alpha^\beta \partial_\beta a^m, \ldots) \tag{3.30}$$

It then follows that

$$\frac{\partial L'}{\partial u^\rho} = \frac{\partial L}{\partial(\partial_\alpha a^m)} u_\alpha (\partial_\rho a^m) \quad , \quad \frac{\partial L'}{\partial(\partial_\sigma a^m)} = \frac{\partial L}{\partial(\partial_\alpha a^m)} \Delta_\alpha^\sigma \quad , \tag{3.31}$$

giving

$$-(\partial_\rho a^m)\frac{\partial L}{\partial(\partial_\sigma a^m)} = \frac{\partial L'}{\partial u^\rho} u^\sigma - (\partial_\rho a^m)\frac{\partial L'}{\partial(\partial_\sigma a^m)} \quad . \tag{3.32}$$

From (3.31) L' satisfies

$$\frac{\partial L'}{\partial u^\alpha} u^\alpha = 0 \quad , \quad u_\alpha \frac{\partial L'}{\partial(\partial_\alpha a^m)} = 0 \quad . \tag{3.33}$$

It seems most simple to introduce u^α in (3.30), although we could have considered a Lagrangian dependent on both $\partial_\alpha a^m$ and u^α at the outset in (3.10) and (3.11). Such a Lagrangian is largely arbitrary as a function of $\partial_\alpha a^m$ and u^α due to the constraints $u_\alpha u^\alpha = -1$, $u^\alpha \partial_\alpha a^m = 0$, so that a class of Lagrangians is associated with any given Lagrangian. One can show that all members of a given class of Lagrangians give the same energy-momentum on variation of the metric (Appendix 2). The neatest presentation of the results is in terms of L' whose dependence on u^α and $\partial_\alpha a^m$ has been delimited to a unique form by conditions (3.33).

The dependence of the Lagrangian on the angular velocities $w^{ab} = e^{(a)\alpha} e^{(b)}_{\alpha|\beta} u^\beta$ determines the internal spin. (This is demonstrated in Appendix 10 for a single particle, and hence for the material as a whole.) We therefore assume that $e^{(a)}_{\alpha|\beta}$ appears in the Lagrangian via w^{ab} or, equivalently, via $\dot{e}^{(a)}_\alpha = e^{(a)}_{\alpha|\beta} u^\beta$.

With $L'(u^\alpha, .., e^{(a)}_{\alpha|\beta}, ...) = \sqrt{-g}\, L(u^\alpha, .., \dot{e}^{(a)}_\alpha, ..)$ (3.34)

it follows that

$$- e^{(a)}_{\alpha|\rho} \frac{\partial \mathbf{L}}{\partial e^{(a)}_{\alpha|\sigma}} = - \sqrt{-g}\, e^{(a)}_{\alpha|\rho} \frac{\partial L}{\partial \dot{e}^{(a)}_\alpha} u^\sigma = \left\{ \sqrt{-g} \frac{\partial L}{\partial u^\rho} - \frac{\partial \mathbf{L'}}{\partial u^\rho} \right\} u^\sigma . \quad (3.35)$$

(3.32) and (3.35) give

$$- (\partial_\rho a^m) \frac{\partial \mathbf{L}}{\partial(\partial_\sigma a^m)} - e^{(a)}_{\alpha|\rho} \frac{\partial \mathbf{L}}{\partial \mathbf{e}^{(a)}_{\alpha|\sigma}} = \sqrt{-g} \left\{ \frac{\partial L}{\partial u^\rho} u^\sigma - (\partial_\rho a^m) \frac{\partial L}{\partial(\partial_\sigma a^m)} \right\} .$$

Inserting this into (3.27) yields

$$t_{\rho(\mathrm{mat})}^{\ \sigma} = \left\{ \frac{\partial L}{\partial u^\rho} - L_1 u_\rho \right\} u^\sigma + \left\{ L_1 \Delta^\sigma_\rho - (\partial_\rho a^m) \frac{\partial L}{\partial(\partial_\sigma a^m)} \right\} \quad (3.36)$$

where $\mathbf{t}_{\rho(\mathrm{mat})}^{\ \sigma} = \sqrt{-g}\, t_{\rho(\mathrm{mat})}^{\ \sigma}$. Accordingly we define the *canonical material four-momentum* P_ρ and the *pressure tensor* $p_\rho^{\ \sigma}$ as

$$P_\rho = \partial L/\partial u^\rho - L_1 u_\rho$$

$$p_\rho^{\ \sigma} = L_1 \Delta^\sigma_\rho - (\partial_\rho a^m) \frac{\partial L}{\partial(\partial_\sigma a^m)} \quad (3.37)$$

so that

$$t_{\rho(\text{mat})}^{\ \sigma} = P_\rho u^\sigma + P_\rho^{\ \sigma} \quad . \tag{3.38}$$

From the same considerations as the first footnote of this section the total spin flux can be decomposed into matter and field parts. (2.14) and (3.18) give

$$\mathbf{S}^{\rho\sigma\tau} = 2\mathbf{U}^{\tau[\rho\sigma]} = 2\mathbf{L}^{\underset{\sim}{A}\tau}(I_{\underset{\sim}{A}}{}^{\underset{\sim}{B}})^{[\sigma\rho]}\psi_{\underset{\sim}{B}}$$

$$= 2\frac{\partial \mathbf{L}}{\partial e_{\alpha|\tau}^{(a)}}(-\delta_\alpha^{[\rho}g^{\sigma]\beta})e_\beta^{(a)} + 2\frac{\partial \mathbf{L}}{\partial \phi_{A|\tau}}(I_A{}^B)^{[\sigma\rho]}\phi_B \quad .$$

In terms of the *material spin* $S^{\rho\sigma}$

$$S^{\rho\sigma} \equiv 2e^{(a)[\rho}\frac{\partial L}{\partial \dot{e}_{\sigma]}^{(a)}} \tag{3.39}$$

and *field spin flux* $S_{(\phi)}^{\rho\sigma\tau}$

$$S_{(\phi)}^{\rho\sigma\tau} \equiv 2\frac{\partial L}{\partial \phi_{A|\tau}}(I_A{}^B)^{[\sigma\rho]}\phi_B \tag{3.40}$$

we have $$(-g)^{-\frac{1}{2}}\mathbf{S}^{\rho\sigma\tau} = S^{\rho\sigma}u^\tau + S_{(\phi)}^{\rho\sigma\tau} \quad . \tag{3.41}$$

This decomposition is independent of the split of **L** since (3.39) and (3.40) are in terms of the total **L**.

3.5 Balance Laws

The Einstein tensor satisfies $G^{\alpha\beta}{}_{|\beta} = 0$ and $G^{[\alpha\beta]} = 0$. The Einstein equations (3.24) are therefore inconsistent unless $T^{\alpha\beta}$ satisfies the same identities. These identities for $T^{\alpha\beta}$ may be derived from the action principle by demanding that I takes an extremal value for the actual translational motion and spin evolution and then simplifying with the aid of (3.23). Setting $\delta I = 0$ for "variation of world-lines", variation of six spin co-ordinates and variation of the non-gravitational fields will thus ensure the consistency of (3.24). The resulting ten equations for the four-momentum and spin are usually referred to as balance laws, equations of motion or as "local" conservation laws for total spin and total four-momentum.

The spin equations have been given already in (3.21). According to (3.20) they immediately imply $T^{[\alpha\beta]} = 0$. To express them as a balance law, making use of (3.30), (3.34) and (3.39) gives the spin equations (3.21) in the following form:

$$\frac{1}{2}(S_\rho{}^\sigma u^\alpha)_{|\alpha} = e^{(a)}_{[\rho} \frac{\partial L}{\partial e^{(a)}_{\sigma]}} + \dot{e}^{(a)}_{[\rho} \frac{\partial L}{\partial \dot{e}^{(a)}_{\sigma]}} \quad . \qquad (3.42)$$

Identity (2.11) applied to

$$L(u^\alpha, a^m, \partial_\mu a^m, e^{(a)}_\alpha, \dot{e}^{(a)}_\alpha, \Phi_{\underset{\sim}{A}}) \quad , \qquad (3.43)$$

$(\Phi_{\underset{\sim}{A}} = (\phi_A, \phi_{A|\alpha}, R^\alpha{}_{\beta\gamma\delta}))$ gives $\quad e^{(a)}_\rho \frac{\partial L}{\partial e^{(a)}_\sigma} + \dot{e}^{(a)}_\rho \frac{\partial L}{\partial \dot{e}^{(a)}_\sigma} =$

$$\frac{\partial L}{\partial u^\rho} u^\sigma - (\partial_\rho a^m) \frac{\partial L}{\partial(\partial_\sigma a^m)} + \frac{\partial L}{\partial \Phi_{\underset{\sim}{A}}} (I_{\underset{\sim}{A}}{}^{\underset{\sim}{B}})_\rho{}^\sigma \Phi_{\underset{\sim}{B}} + \delta^\sigma_\rho L \quad . \tag{3.44}$$

Comparison of (3.42) and (3.44) gives the *balance law for spin*

$$\tfrac{1}{2}(S^{\rho\sigma}u^\alpha)_{|\alpha} = t^{[\rho\sigma]}_{(mat)} + (\partial L/\partial\Phi_{\underset{\sim}{A}})(I_{\underset{\sim}{A}}{}^{\underset{\sim}{B}})^{[\rho\sigma]}\Phi_{\underset{\sim}{B}} \quad . \tag{3.45}$$

The equations governing the translational motion are obtained from extremization of the action integral on "varying the world-lines"[1]. We consider a 1-parameter family of tetrad fields $e^{(a)}_\alpha(x,\varepsilon)$ and congruences $x^\mu(a^m,t,\varepsilon)$. We extremize the action integral $I(\varepsilon)$ of (3.10) subject to fixed end-points $x^\mu(a^m,t_i,\varepsilon) = x^\mu(a^m,t_i,0)$ $(i = 1,2)$ a $\delta e^{(a)}_\alpha/\delta\varepsilon = 0$ ($e^{(a)}_\alpha$ attached to world-lines held fixed by parallel propagation). For the infinitesimal variation $x^\mu(a^m,t,\varepsilon) = x^\mu(a^m,t,0) + \varepsilon\xi^\mu(a^m,t)$ the accompanying *absolute* variations are:

$$\delta\Phi_{\underset{\sim}{A}} = \varepsilon\Phi_{\underset{\sim}{A}|\rho}\xi^\rho \quad , \quad \delta(\sqrt{-g}\,d^4x) = \varepsilon\xi^\alpha{}_{|\alpha}\sqrt{-g}\,d^4x \quad ,$$

$$\delta a^m = 0 \quad , \quad \delta(\partial_\mu a^m) = -\varepsilon(\partial_\rho a^m)\xi^\rho{}_{|\mu} \quad ,$$

$$\delta u^\mu = \varepsilon\Delta^\mu_\rho\xi^\rho{}_{|\sigma}u^\sigma \quad , \quad \delta(d\tau) = -\varepsilon\xi_{\alpha|\beta}u^\alpha u^\beta d\tau$$

and $\delta e^{(a)}_\alpha = 0$ implying (3.46)

$$\delta(\dot{e}^{(a)}_\alpha) = \varepsilon e^{(a)}_\lambda R^\lambda{}_{\alpha\mu\nu}u^\mu\xi^\nu + \frac{\delta e^{(a)}_\alpha}{\delta t}\,\delta\left(\frac{dt}{d\tau}\right) \quad .$$

---

[1] For the application of this technique to non-spinning matter, cf. [28].

The action principle then gives for arbitrary $\xi^{\mu}(a^m, t)$

$$0 = \frac{dI}{d\varepsilon} = \frac{d}{d\varepsilon} \int \sqrt{-g}\; L(u^{\alpha}, a^m, \partial_{\mu} a^m, e_{\alpha}^{(a)}, \dot{e}_{\alpha}^{(a)}, \Phi_{\underset{\sim}{A}})\, d^4x =$$

$$\int \Bigg(L\xi^{\alpha}{}_{|\alpha} + \frac{\partial L}{\partial u^{\mu}} \Delta^{\mu}{}_{\rho} \xi^{\rho}{}_{|\sigma} u^{\sigma} - \frac{\partial L}{\partial(\partial_{\sigma} a^m)} (\partial_{\rho} a^m) \xi^{\rho}{}_{|\sigma} \tag{3.47}$$

$$+ \frac{\partial L}{\partial \dot{e}_{\alpha}^{(a)}} (e_{\lambda}^{(a)} R^{\lambda}{}_{\alpha\mu\nu} u^{\mu} \xi^{\nu} + e_{\alpha}^{(a)} \xi_{\rho|\sigma} u^{\rho} u^{\sigma}) + \frac{\partial L}{\partial \Phi_{\underset{\sim}{A}}} \Phi_{\underset{\sim}{A}|\rho} \xi^{\rho} \Bigg) \sqrt{-g}\; d^4x \quad .$$

The first of (3.33), together with (3.34), implies

$$\frac{\partial L}{\partial u^{\alpha}} u^{\alpha} + \frac{\partial L}{\partial \dot{e}_{\alpha}^{(a)}} \dot{e}_{\alpha}^{(a)} = 0 \quad . \tag{3.48}$$

Hence

$$\frac{\partial L}{\partial u^{\mu}} \Delta^{\mu}{}_{\rho} \xi^{\rho}{}_{|\sigma} u^{\sigma} + \frac{\partial L}{\partial \dot{e}_{\alpha}^{(a)}} \dot{e}_{\alpha}^{(a)} \xi_{\rho|\sigma} u^{\rho} u^{\sigma} = \frac{\partial L}{\partial u^{\rho}} \xi^{\rho}{}_{|\sigma} u^{\sigma} \quad . \tag{3.49}$$

Substitution of (3.49) into (3.47) and integration by parts gives the following *four-momentum balance law*

$$t_{\rho\,(\mathrm{mat})}{}^{\sigma}{}_{|\sigma} = -\frac{1}{2} R_{\rho\alpha\beta\gamma} S^{\beta\gamma\alpha}_{(\mathrm{mat})} + \frac{\partial L}{\partial \Phi_{\underset{\sim}{A}}} \Phi_{\underset{\sim}{A}|\rho} \quad . \tag{3.50}$$

3.6 Einstein-Lorentz Theory for Dielectrics

To illustrate the results of this chapter we now consider a charged dipolar medium and its interaction with a Maxwell-Einstein field, described by a vector potential A_{α} and the metric tensor.

The phenomenological current[1] J^α and the (skew--symmetric) displacement tensor $H^{\alpha\beta}$ are defined by

$$J^\alpha \equiv \partial L/\partial A_\alpha \quad , \quad H^{\alpha\beta} \equiv -4\pi \partial L/\partial A_{\beta|\alpha} = -8\pi \partial L/\partial F_{\alpha\beta} \quad , \qquad (3.51)$$

in which it has been assumed that $A_{\alpha|\beta}$ appears in the Lagrangian only through the skew-symmetric combination $F_{\alpha\beta} = 2\partial_{[\alpha}A_{\beta]}$. The *electromagnetic field equations* are therefore (cf. eq.(3.23))

$$\partial_{[\alpha}F_{\beta\gamma]} = 0 \quad , \quad H^{\alpha\beta}{}_{|\beta} = 4\pi J^\alpha \quad , \qquad (3.52)$$

and imply conservation of free charge

$$J^\alpha{}_{|\alpha} = 0 \quad . \qquad (3.53)$$

In order that equations (3.52) reduce to those of Maxwell in the absence of matter, L must reduce to the free--field electromagnetic Lagrangian[2]

$$L_0 = -(16\pi)^{-1} F_{\mu\nu}F^{\mu\nu} \quad . \qquad (3.54)$$

Defining the electromagnetic polarization tensor

$$M^{\alpha\beta} \equiv 2\partial(L - L_0)/\partial F_{\alpha\beta} \qquad (3.55)$$

----

[1] The microscopic current is $j^\alpha \equiv \delta(L-L_0)/\delta A_\alpha = J^\alpha + M^{\alpha\beta}{}_{|\beta}$.
[2] Lovelock [27 b,c] has shown that the most general $L_0(A_\alpha, \partial_\beta A_\alpha, g_{\alpha\beta})$ for which $\delta L_0/\delta A_\alpha = F^{\alpha\beta}{}_{|\beta}$ is given by (3.54) plus a trivial divergence term.

leads at once to the usual Lorentz polarization relations

$$H^{\alpha\beta} = F^{\alpha\beta} - 4\pi M^{\alpha\beta} \quad . \qquad (3.56)$$

Since only in *non-conducting* material can dissipative processes be expected to be absent, we therefore demand that L satisfies

$$J^{\alpha} \equiv \partial L/\partial A_{\alpha} = eN^{\alpha} \quad . \qquad (3.57)$$

From (3.4) and (3.53) it follows that the charge per particle e satisfies $de/d\tau = 0$. Assumption (3.57) amounts to assuming that the (undifferentiated) potential A_{α} appears in L only through a bilinear interaction term

$$L_{(A)} = \sqrt{-g}\; eA_{\alpha}N^{\alpha} = eN(a^{m})A_{\alpha}\varepsilon^{\alpha\beta\gamma\delta}(\partial_{\beta}a^{1})(\partial_{\gamma}a^{2})(\partial_{\delta}a^{3}) \quad . \qquad (3.58)$$

From (3.58) it is apparent that $L_{(A)}$ is independent of the metric and therefore does not contribute to $T^{\rho\sigma}$. Furthermore, since $L_{(A)}$ is the only gauge-dependent part of L, (with $A_{\alpha|\beta}$ entering via $F_{\alpha\beta}$,) $T^{\rho\sigma}$ must be gauge invariant. To ensure that the decomposition of $T^{\rho\sigma}$ is into gauge invariant parts, some adjustments must be made to the definitions of matter and field energy-momentum and spin. The redefinitions are simplest when $L_{(A)}$ is assumed to be part of L_1. Then it is easily seen from (3.37) that both $P_{\rho}{}^{\sigma}$ and a "kinetic four-momentum" p_{ρ}

$$p_\rho \equiv P_\rho - enA_\rho \tag{3.59}$$

are gauge invariant (Appendix 3). From (3.28) and (3.40) we obtain

$$t_{\rho(\phi)}{}^{\sigma} = L_2\delta^\sigma_\rho - (4\pi)^{-1}A_{\alpha|\rho}H^{\alpha\sigma} \;, \quad 4\pi S^{\rho\sigma\tau}_{(\phi)} = 2A^{[\rho}H^{\sigma]\tau} \;, \tag{3.60}$$

so that

$$t_{\rho(\phi)}{}^{\sigma} + \tfrac{1}{2}(S_{\rho(\phi)}{}^{\tau\sigma} + S^{\sigma\tau}_{(\phi)\rho} + S^{\sigma}{}_{\rho(\phi)}{}^{\tau})_{|\tau} = T_{\rho(em)}{}^{\sigma} - A_\rho J^\sigma \tag{3.61}$$

where we have defined a gauge-invariant electromagnetic energy tensor by

$$T_{\rho(em)}{}^{\sigma} \equiv (4\pi)^{-1}F_{\rho\alpha}H^{\sigma\alpha} + L_2\delta^\sigma_\rho \quad . \tag{3.62}$$

From (3.26), (3.38), (3.59) and (3.61) we obtain

$$\begin{aligned}(-g)^{-\frac{1}{2}}\mathbf{t}_\rho{}^{\sigma} &+ \tfrac{1}{2}(S_{\rho(\phi)}{}^{\tau\sigma} + S^{\sigma\tau}_{(\phi)\rho} + S^{\sigma}{}_{\rho(\phi)}{}^{\tau})_{|\tau} \\ &= p_\rho u^\sigma + P_\rho{}^{\sigma} + T_{\rho(em)}{}^{\sigma} \quad .\end{aligned} \tag{3.63}$$

This, together with (3.41), gives the *gravitational field equations* (3.24) in the form

$$\begin{aligned}(8\pi)^{-1}G^{\rho\sigma} = T^{\rho\sigma} &\equiv T^{\rho\sigma}_{(mat)} + T^{\rho\sigma}_{(em)} \\ &+ \tfrac{1}{2}(S^{\rho\tau\sigma}_{(mat)} + S^{\sigma\tau\rho}_{(mat)} + S^{\sigma\rho\tau}_{(mat)})_{|\tau} + 4Q^{\lambda(\rho\sigma)\mu}{}_{|\mu\lambda} + Q^{A}(I_A{}^{B})^{\rho\sigma}R_B\end{aligned} \tag{3.64}$$

in which the tensors

$$T^{\rho\sigma}_{(mat)} = p^{\rho}u^{\sigma} + P^{\rho\sigma} \quad , \qquad S^{\rho\sigma\tau}_{(mat)} = S^{\rho\sigma}u^{\tau} \quad , \tag{3.65}$$

represent the material fluxes of four-momentum and spin angular momentum. Their divergences, according to (3.45) and (3.50), may be expressed as

$$\frac{1}{2}S^{\rho\sigma\tau}_{(mat)|\tau} = T^{[\rho\sigma]}_{(mat)} - F^{[\rho}{}_{\alpha}M^{\sigma]\alpha} - 4R^{[\rho}{}_{\alpha\beta\gamma}Q^{\sigma]\alpha\beta\gamma} \quad , \tag{3.66}$$

$$(T_{(mat)} + T_{(em)})_{\rho}{}^{\sigma}{}_{|\sigma} = -\frac{1}{2}R_{\rho\alpha\beta\gamma}S^{\beta\gamma\alpha}_{(mat)} + Q_{\alpha}{}^{\beta\gamma\delta}R^{\alpha}{}_{\beta\gamma\delta|\rho} \quad , \tag{3.67}$$

where we have used the result

$$T_{\rho}{}^{\sigma}{}_{(em)|\sigma} = - F_{\rho\alpha}J^{\alpha} - \frac{1}{2}M^{\mu\nu}F_{\mu\nu|\rho} \tag{3.68}$$

which follows from (3.62), (3.56) and (3.52). As we remarked in Section 5, these equations can be used to verify directly that the total energy tensor $T^{\rho\sigma}$ is symmetric and conserved.

From (3.62) $T^{\rho\sigma}_{(em)}$ depends on the split of L only through its second (diagonal) term. A change in the decomposition of L will merely redistribute terms between the diagonal parts of the the material and field energy tensors. (3.62) differs from proposals of Abraham, and Einstein and Laub [2] for the localization of electromagnetic energy and momentum in a dielectric medium. The split (3.64) of $T^{\rho\sigma}$ into matter and field may be called a generalized Minkowski splitting since (3.62) becomes the tensor proposed by Minkowski [2] when L_2 is chosen to equal $-(16\pi)^{-1}F_{\alpha\beta}H^{\alpha\beta}$. The choice (3.54) for L_2 seems to be of special importance. This gives a $T^{\rho\sigma}_{(em)}$ that has been recognized for some time in special relativistic dielectric theory as being of special significance [3a, 3b, 4a], while Israel and Stewart [6b] have recently given persuasive reasons for its use.

Field equations (3.64) and (3.52) (with (3.56)) both contain divergence terms, namely the "Belinfante-Rosenfeld" spin term and $4Q^{\lambda(\rho\sigma)\mu}{}_{|\mu\lambda}$ in (3.64) and the polarization current $4\pi M^{\alpha\beta}{}_{|\beta}$ in (3.52). This suggests interpreting the spin and quadrupole terms as "gravitational polarization" contributions (polarization of energy-momentum). Chapter 5 will explore the concept of gravitational polarization.

3.7 Spinning Fluids and Dust

Recalling that equations (3.7) and (3.5) determine the number density $n(x)$ as a function of a^m and $\partial_\alpha a^m$, assume that L depends on $\partial_\alpha a^m$ only via n:

$$L(u^\alpha, a^m, \partial_\alpha a^m, \ldots) = L_F(u^\alpha, a^m, n, \ldots) \quad . \qquad (3.69)$$

From $(\partial_\rho a^m)(\partial n/\partial(\partial_\sigma a^m)) = n\Delta^\sigma_\rho$ (Appendix 3) it follows that the pressure tensor (3.37) in terms of *pressure* P is

$$P_\rho{}^\sigma = P\Delta^\sigma_\rho \quad , \qquad P \equiv L_I - n\partial L_F/\partial n \quad . \qquad (3.70)$$

The general considerations of this paper are therefore appropriate for the description of ideal (dissipation-free) spinning fluids whose spin density is convective, given by $S^{\alpha\beta\gamma}_{(mat)} = S^{\alpha\beta}u^{\gamma}$. These are the "Weyssenhoff" spinning fluids of [26]. (A much more complicated (and realistic) description of spinning fluids may be obtained in [6d].)

It was noted in sections 3.4 and 3.6 how a change in the decomposition of the total Lagrangian will re-distribute "diagonal" energy-momentum between matter and field. The definition of material pressure P will therefore depend on the particular split of L, as may be seen from (3.70). Setting P equal to zero is therefore only a meaningful criterion for "dust" as long one is dealing with a fixed decomposition of L. Consider the case where L_2 is taken to be the free-field Lagrangian $L_2 = L_0(\phi_A, \phi_{A|\alpha}, g_{\alpha\beta})$ (all the interaction terms in L_F allocated to the material). From (3.70) the definition $P = 0$ for dust is $n\partial L_1/\partial n = L_1$ giving $L_1 = nL_D(u^\alpha, \ldots)$ with L_D independent of n. L_D is then the single particle Lagrangian of (2.18) (modulo $-u_\alpha u^\alpha$ factors).

Finally, when L_F is a function only of n, $L_F = -\rho(n)$, from (3.37) and (3.70) we obtain the usual expressions for P_ρ and P for a non-spinning fluid in a gravitational field.

$$P_\rho = \rho u_\rho \quad , \qquad P = n(\partial\rho/\partial n) - \rho \quad . \qquad (3.71)$$

4. HIGHER DERIVATIVE COUPLING

Chapter 3 considered Lagrangians depending on variables $\psi_{\underset{\sim}{A}}$ and first covariant derivatives $\psi_{\underset{\sim}{A}|\alpha}$. We now discuss the more general situation where L is allowed to depend on higher derivatives of the fields. The notation of Chapter 2, Sec. 4 is used with $\underset{\sim}{\alpha}(n)$ denoting a symmetrized set of indices. With the convention that $\partial L/\partial \Psi_{\underset{\sim}{A}}$ has the same symmetries as $\Psi_{\underset{\sim}{A}}$, the use of symmetrized indices implies that the multipole moments (defined as $\partial L/\partial \phi_{A|\underset{\sim}{\alpha}(n)}$) have the same symmetries as in special relativity.

4.1 Generalization of Fundamental Identity (2.16)

Consider a scalar density $L(\Psi_{\underset{\sim}{A}})$ where $\Psi_{\underset{\sim}{A}} = (\psi_{A|\underset{\sim}{\alpha}(n)}, n = 0, 1, \ldots)$ and extend definitions (2.13), (2.12), (2.14) as follows:

$$L^{A\alpha_1\ldots\alpha_n} = L^{A\underset{\sim}{\alpha}(n)} \equiv \partial L/\partial \psi_{A|\underset{\sim}{\alpha}(n)} \quad , \tag{4.1}$$

$$L_*^{A\underset{\sim}{\alpha}(n)} \equiv \sum_{m=0}^{\infty} (-1)^m \; L^{A\underset{\sim}{\alpha}(n)\underset{\sim}{\beta}(m)}{}_{|\underset{\sim}{\beta}(m)} \quad , \tag{4.2}$$

$$\overline{U}^{\tau\sigma}{}_{\rho} \equiv (I_A{}^B)_{\rho}{}^{\sigma} \sum_{n=0}^{\infty} L_*^{A\tau\underset{\sim}{\alpha}(n)} \psi_{B|\underset{\sim}{\alpha}(n)} \quad , \tag{4.3}$$

$$\overline{t}_{\rho}{}^{\sigma} \equiv \delta^{\sigma}_{\rho} L - \sum_{n=0}^{\infty} (n+1) \psi_{A|(\rho\underset{\sim}{\alpha}(n))} L^{A\sigma\underset{\sim}{\alpha}(n)} \quad . \tag{4.4}$$

The condition (2.11) that $L(\Psi_{\underset{\sim}{A}})$ be a scalar density is

$$\frac{\partial L}{\partial \Psi_{A}} (I_{A}{}^{B})_{\rho}{}^{\sigma} \Psi_{B} + \delta_{\rho}^{\sigma} L = 0 \quad . \tag{4.5}$$

With $\Psi_{A} = (\psi_{A|\alpha(n)},\ n = 0, 1, \ldots)$ this is

$$\sum_{n=0}^{\infty} L^{A\alpha(n)} (I_{A\alpha(n)}{}^{B\beta(n)})_{\rho}{}^{\sigma} \psi_{B|\beta(n)} + \delta_{\rho}^{\sigma} L = 0 \quad . \tag{4.6}$$

This generalizes (2.15). The two steps leading to (2.16) must now be generalized. First note that repeated use of (2.5) gives

$$L^{A\alpha(n)} (I_{A\alpha(n)}{}^{B\beta(n)})_{\rho}{}^{\sigma} \psi_{B|\beta(n)} = L^{A\alpha(n)} (I_{A}{}^{B})_{\rho}{}^{\sigma} \psi_{B|\alpha(n)} - n L^{A\sigma\alpha(n-1)} \psi_{A|(\rho\alpha(n-1))} \quad , (n=1,2,\ldots). \tag{4.7}$$

Next, we rewrite the first term on the right-hand side of the above equation as a divergence plus a term not involving derivatives of ψ_{A}. The simplest method of achieving this is to note that (4.2) implies

$$L_{*}^{A\tau\alpha(n)}{}_{|\tau} = L^{A\alpha(n)} - L_{*}^{A\alpha(n)} \quad . \tag{4.8}$$

Hence

$$\sum_{n=0}^{\infty} L^{A\alpha(n)} (I_{A}{}^{B})_{\rho}{}^{\sigma} \psi_{B|\alpha(n)}$$

$$= \sum_{n=0}^{\infty} (L_{*}^{A\tau\alpha(n)}{}_{|\tau} + L_{*}^{A\alpha(n)}) (I_{A}{}^{B})_{\rho}{}^{\sigma} \psi_{B|\alpha(n)}$$

$$= \Big(\sum_{n=0}^{\infty} L_{*}^{A\tau\alpha(n)} (I_{A}{}^{B})_{\rho}{}^{\sigma} \psi_{B|\alpha(n)} \Big)_{|\tau} - \sum_{n=0}^{\infty} L_{*}^{A\tau\alpha(n)} (I_{A}{}^{B})_{\rho}{}^{\sigma} \psi_{B|\alpha(n)\tau} +$$

$$+ \sum_{n=0}^{\infty} L_*^{A\underset{\sim}{\alpha}(n)} (I_A{}^B)_\rho{}^\sigma \psi_{B|\underset{\sim}{\alpha}(n)}$$

$$= \bar{U}^{\tau\sigma}{}_{\rho|\tau} + L_*^A (I_A{}^B)_\rho{}^\sigma \psi_B \quad . \qquad (4.9)$$

(4.6), (4.7), (4.9) together yield

$$\bar{U}^{\tau\sigma}{}_{\rho|\tau} + \bar{t}_\rho{}^\sigma + L_*^A (I_A{}^B)_\rho{}^\sigma \psi_B = 0 \quad . \qquad (4.10)$$

It will become clear in the next section that, for variational purposes, equation (4.10) is more useful when written in terms of $U^{\tau\sigma}{}_\rho$ and $t_\rho{}^\sigma$ defined by

$$U^{\tau\sigma}{}_\rho \equiv \bar{U}^{\tau\sigma}{}_\rho - Y^{\tau\sigma}{}_\rho , \quad Y^{\tau\sigma}{}_\rho \equiv \sum_{n=0}^{\infty} (n+1) \psi_{A|(\rho\underset{\sim}{\alpha}(n))} L_*^{A\sigma\tau\underset{\sim}{\alpha}(n)} , \qquad (4.11)$$

$$t_\rho{}^\sigma \equiv \delta_\rho^\sigma L - \sum_{n=0}^{\infty} \psi_{A|(\rho\underset{\sim}{\alpha}(n))} L_*^{A\sigma\underset{\sim}{\alpha}(n)} \quad . \qquad (4.12)$$

(4.12) is the natural definition of a canonical energy tensor for Lagrangians containing higher derivatives (cf. [29], [15, p.122]).

From (4.8), (4.11) we have $Y^{\tau\sigma}{}_{\rho|\tau} = t_\rho{}^\sigma - \bar{t}_\rho{}^\sigma$ so that

$$\bar{U}^{\tau\sigma}{}_{\rho|\tau} + \bar{t}_\rho{}^\sigma = U^{\tau\sigma}{}_{\rho|\tau} + t_\rho{}^\sigma \quad . \qquad (4.13)$$

Inserting (4.13) into (4.10), we obtain the generalized form of identity (2.16)

$$U^{\tau\sigma}{}_{\rho|\tau} + t_\rho{}^\sigma + L_*^A (I_A{}^B)_\rho{}^\sigma \psi_B = 0 \qquad (4.14)$$

4.2 Generalized Action Integral; Variation of Tetrad

As a generalization of the action integral (3.10) we now consider

$$I = \int L(\psi_{A|\underset{\sim}{\alpha}(n)})\ d^4x + (16\pi)^{-1} \int \sqrt{-g}\ R\ d^4x \ , \qquad (4.15)$$

$$\psi_A = (a^m,\ e_\alpha^{(a)},\ \phi_A,\ R^\alpha{}_{\beta\gamma\delta}) \quad ; \qquad (4.16)$$

in which arbitrary symmetrized derivatives of the field variables ϕ_A, $R^\alpha{}_{\beta\gamma\delta}$ are now permitted to appear, but we still assume that second and higher derivatives of a^m and $e_\alpha^{(a)}$ are absent.

Under a variation of the tetrad field we have (Appendix 4)

$$\delta_{(e)} L = L_*^A \delta_{(e)} \psi_A + U^{\tau\sigma}{}_\rho \delta\Gamma^\rho_{\sigma\tau} + (\text{div}) \qquad (4.17)$$

with L_*^A, $U^{\tau\sigma}{}_\rho$ defined in (4.2), (4.3) and (4.11).

Define the gravitational multipole moments $q_\alpha{}^{\beta\gamma\delta\underset{\sim}{\lambda}(n)}$ and associated quantities $Q^{A\underset{\sim}{\lambda}(n)}$ as special cases of general definitions (4.1), (4.2):

$$q_\alpha{}^{\beta\gamma\delta\underset{\sim}{\lambda}(n)} = \sqrt{-g}\,q^{A\underset{\sim}{\lambda}(n)} \equiv \partial L/\partial R_{A|\underset{\sim}{\lambda}(n)} = \partial L/\partial R^\alpha{}_{\beta\gamma\delta|\underset{\sim}{\lambda}(n)} \qquad (4.18)$$

$$Q_\alpha{}^{\beta\gamma\delta\underset{\sim}{\lambda}(n)} = \sqrt{-g}\,Q^{A\underset{\sim}{\lambda}(n)} = \sum_{m=0}^{\infty} (-1)^m q^{A\underset{\sim}{\lambda}(n)\underset{\sim}{\beta}(m)}{}_{|\beta(m)} \ . \qquad (4.19)$$

From (4.16) we have

$$L_{*}^{A}\delta_{(e)}\psi_A = (\delta L/\delta e_{\sigma}^{(a)})\delta e_{\sigma}^{(a)} + Q_{\alpha}{}^{\beta\gamma\delta}\delta R^{\alpha}{}_{\beta\gamma\delta}$$

giving (4.17) in the form $\delta_{(e)}L =$

$$(\delta L/\delta e_{\sigma}^{(a)})\delta e_{\sigma}^{(a)} + U^{\tau\sigma}{}_{\rho}\delta\Gamma^{\rho}_{\sigma\tau} + Q_{\alpha}{}^{\beta\gamma\delta}\delta R^{\alpha}{}_{\beta\gamma\delta} + (\mathrm{div}) \ . \quad (4.20)$$

This generalizes (3.13), and a calculation patterned after (3.15) to (3.20) gives

$$(8\pi)^{-1}\sqrt{-g}\ G^{\rho\sigma} = (\delta L/\delta e_{\sigma}^{(a)})e^{(a)\rho}$$
$$+ \left(\tfrac{1}{2}(S^{\rho\tau\sigma} + S^{\sigma\tau\rho}) - U^{\tau(\rho\sigma)}\right)_{|\tau} + 4Q^{\lambda(\rho\sigma)\mu}{}_{|\mu\lambda} \ . \quad (4.21)$$

We thus recover the spin equation in the form (3.21).

With $\psi_A = (a^m, e_{\alpha}^{(a)}, \phi_A, R^{\alpha}{}_{\beta\gamma\delta})$ the fundamental identity (4.14) is

$$(\delta L/\delta e_{\sigma}^{(a)})e_{\rho}^{(a)} =$$
$$U^{\tau\sigma}{}_{\rho|\tau} + t_{\rho}{}^{\sigma} + Q^{A}(I_A{}^{B})_{\rho}{}^{\sigma}R_B + L_{*(\phi)}^{A}(I_A{}^{B})_{\rho}{}^{\sigma}\phi_B \quad (4.22)$$

where $$L_{*(\phi)}^{A\underset{\sim}{\alpha}(n)} \equiv \sum_{m=0}^{\infty}(-1)^m(\partial L/\partial\phi_{A|\underset{\sim}{\alpha}(n)\underset{\sim}{\beta}(m)})_{|\underset{\sim}{\beta}(m)} \ . \quad (4.23)$$

Inserting (4.22) into (4.21) gives the *gravitational field equations*

$$(8\pi)^{-1}\sqrt{-g}\ G^{\rho\sigma} = \sqrt{-g}\ T^{\rho\sigma} \equiv \mathbf{t}^{\rho\sigma} + \tfrac{1}{2}(S^{\rho\tau\sigma} + S^{\sigma\tau\rho} + S^{\sigma\rho\tau})_{|\tau}$$

$$+\ 4\mathbf{Q}^{\lambda(\rho\sigma)\mu}{}_{|\mu\lambda} + \mathbf{Q}^{A}(I_A{}^{B})^{\rho\sigma}R_B + \mathbf{L}^{A}_{*(\phi)}(I_A{}^{B})^{\rho\sigma}\phi_B\ . \quad (4.24)$$

The $\phi_{A|\underset{\sim}{\lambda}(n)}$ represent the external fields and their derivatives. Although ϕ_A may be chosen so that their field equations take the form $\mathbf{L}^{A}_{*(\phi)} = 0$ we still include the last term of (4.24) for the following reason: for the electromagnetic field it is more convenient to choose $\phi_A = (A_\alpha, F_{\alpha\beta})$ (with $\phi_{A|\underset{\sim}{\lambda}(n)} = F_{\alpha\beta|\underset{\sim}{\lambda}(n)}$ for $n=1,2..$) rather than $\phi_{A|\underset{\sim}{\lambda}(n)} = A_{\alpha|\underset{\sim}{\lambda}(n)}$ $(n=0,1...)$; even though $L^{A}_{*(\phi)} = 0$ only for the latter choice. When L depends only on first derivatives of A_α, as in chapter 3, both conventions for ϕ_A give the same results with little difference in ease of derivation. Chapter 4 followed the more traditional method of setting $\phi_A = A_\alpha$. The gauge dependent $t_{\rho(\phi)}^{\sigma}$ was then combined with the gauge dependent field spin flux terms in the usual manner to obtain gauge invariance. When L depends on higher derivatives of A_α the first advantage of using $\phi_A = (A_\alpha, F_{\alpha\beta})$ is the gauge invariance from the outset of $t_{\rho(\phi)}^{\sigma}$ and $S^{\rho\sigma\tau}_{(\phi)}$ (only step (3.59) is needed for complete gauge invariance of all parts of $T^{\rho\sigma}$). The second advantage is that the multipole moments $\partial L/\partial\phi_{A|\underset{\sim}{\lambda}(n)} = \partial L/\partial F_{\alpha\beta|\underset{\sim}{\lambda}(n)}$ have the same symmetries as the special relativistic definitions of the next chapter.

4.3 Matter and Field Decomposition

We continue in similar fashion to chapter 3, decomposing L as

$$L = L_1(\psi_{A|\underset{\sim}{\alpha}(n)}) + L_2(\psi_{A|\underset{\sim}{\alpha}(n)}) \quad . \tag{4.25}$$

Since $\psi_{A|\underset{\sim}{\alpha}(n)}$ contains only first derivatives of a^m and $e_\alpha^{(a)}$, it follows that

$$t_\rho{}^\sigma = t_{\rho(mat)}{}^\sigma + t_{\rho(\phi)}{}^\sigma - \sum_{n=0}^{\infty} R_{A|(\rho\underset{\sim}{\alpha}(n))} Q^{A\sigma\underset{\sim}{\alpha}(n)} \tag{4.26}$$

where

$$t_{\rho(mat)}{}^\sigma = \delta_\rho^\sigma L_1 - (\partial_\rho a^m)\frac{\partial L}{\partial(\partial_\sigma a^m)} - e_{\alpha|\rho}^{(a)}\frac{\partial L}{\partial e_{\alpha|\sigma}^{(a)}} \tag{4.27}$$

and

$$t_{\rho(\phi)}{}^\sigma = \delta_\rho^\sigma L_2 - \sum_{n=0}^{\infty} \phi_{A|(\rho\underset{\sim}{\alpha}(n))} L_{*(\phi)}^{A\sigma\underset{\sim}{\alpha}(n)} \quad . \tag{4.28}$$

The discussion of the structure of $t_{\rho(mat)}{}^\sigma$ in chapter 3, equations (3.29) to (3.38), applies without change: in terms of $L_1 = (-g)^{-\frac{1}{2}} L_1$ and $\sqrt{-g}\, L(u^\alpha, \ldots, \dot{e}_\alpha^{(a)}, \ldots) =$

$$L'(u^\alpha, a^m, \partial_\alpha a^m, e_\alpha^{(a)}, e_{\alpha|\beta}^{(a)}, \ldots) \equiv L(a^m, \Delta_\alpha^\beta \partial_\beta a^m, \ldots) \tag{4.29}$$

we have

$$t_{\rho(mat)}{}^\sigma = (-g)^{-\frac{1}{2}} t_{\rho(mat)}{}^\sigma = P_\rho u^\sigma + p_\rho{}^\sigma \tag{4.30}$$

where

$$P_\rho = \frac{\partial L}{\partial u^\rho} - L_1 u_\rho \,, \quad p_\rho{}^\sigma = L_1 \Delta_\rho^\sigma - (\partial_\rho a^m)\frac{\partial L}{\partial(\partial_\sigma a^m)} \quad . \tag{4.31}$$

In (4.26) $t_\rho{}^\sigma$ has split into matter, field and curvature parts. The quantities $Y^{\tau\sigma}{}_\rho$, $U^{\tau\sigma}{}_\rho$, $S^{\rho\sigma\tau}$ also split: if we define

$$U^{\tau\sigma}_{(\phi)\rho} \equiv (I_A{}^B)_\rho{}^\sigma \sum_{n=0}^{\infty} L^{A\tau\underset{\sim}{\alpha}(n)}_{*(\phi)} \phi_{B|\underset{\sim}{\alpha}(n)} - \sum_{n=0}^{\infty} (n+1)\phi_{A|(\rho\underset{\sim}{\alpha}(n))} L^{A\sigma\tau\underset{\sim}{\alpha}(n)}_{*(\phi)}, \quad (4.32)$$

$$U^{\tau\sigma}_{(R)\rho} \equiv (I_A{}^B)_\rho{}^\sigma \sum_{n=0}^{\infty} Q^{A\tau\underset{\sim}{\alpha}(n)} R_{B|\underset{\sim}{\alpha}(n)} - \sum_{n=0}^{\infty} (n+1)R_{A|(\rho\underset{\sim}{\alpha}(n))} Q^{A\sigma\tau\underset{\sim}{\alpha}(n)} \quad (4.33)$$

$$S^{\rho\sigma\tau}_{(\phi)} \equiv 2U^{\tau[\rho\sigma]}_{(\phi)} \quad S^{\rho\sigma\tau}_{(R)} \equiv 2U^{\tau[\rho\sigma]}_{(R)}, \quad S^{\rho\sigma} \equiv 2e^{(a)[\rho}\partial L/\partial \dot{e}^{(a)}_{\sigma]} \quad (4.34)$$

then (4.3), (4.11) give

$$U^{\tau\sigma}{}_\rho = U^{\tau\sigma}_{(\phi)\rho} + U^{\tau\sigma}_{(R)\rho} - \sqrt{-g}\,(\partial L/\partial \dot{e}^{(a)}_\sigma)e^{(a)}_\rho u^\tau \quad (4.35)$$

and

$$S^{\rho\sigma\tau} = \sqrt{-g}\, S^{\rho\sigma}u^\tau + S^{\rho\sigma\tau}_{(\phi)} + S^{\rho\sigma\tau}_{(R)} \quad . \quad (4.36)$$

As in chapter 3 the decomposition of $S^{\rho\sigma\tau}$ is independent of the split $L = L_1 + L_2$. This split is only needed in order to partition the term $\delta^\sigma_\rho L$ of $t_\rho{}^\sigma$ and thus only affects the diagonal parts of (4.27) and (4.28).

To summarize, in equations (4.24) generalizing (3.24) the total energy tensor $T^{\rho\sigma}$ is found to consist of:

1. A material energy-momentum tensor $t^{\rho\sigma}_{(mat)} = P^\rho u^\sigma + p^{\rho\sigma}$ written as a convective four-momentum flux and a pressure tensor.
2. A canonical energy tensor for the fields ϕ_A, given by (4.28) as a certain combination of field derivatives $\phi_{A|\underset{\sim}{\lambda}(n)}$ and quantities $L^{A\underset{\sim}{\lambda}(n)}_{*(\phi)}$ formed from the multipole moments $\partial L/\partial\phi_{A|\underset{\sim}{\lambda}(n)}$ according to (4.23).
3. A divergence formed from the total spin flux which

consists of matter, field and curvature parts.

4. Contributions from gravitational multipole moments.
5. A term proportional to $L^{A}_{*(\phi)}$ which is usually zero by virtue of the field equations satisfied by ϕ_A and if non-zero may be included in $t_{\rho(\phi)}^{\sigma}$.

4.4 Einstein-Lorentz Theory

Set $\phi_A = (A_\alpha, F_{\alpha\beta})$ in the previous section (see discussion that follows (4.24)).

The electromagnetic field equations are obtained by variation of A_α keeping $g_{\alpha\beta}$ fixed. The variation in L is

$$\delta L = \frac{\partial L}{\partial A_\alpha}\delta A_\alpha + \sum_{n=0}^{\infty}\frac{\partial L}{\partial F_{\alpha\beta|\underset{\sim}{\lambda}(n)}}\delta(F_{\alpha\beta|\underset{\sim}{\lambda}(n)})$$

$$= \frac{\partial L}{\partial A_\alpha}\delta A_\alpha + \sum_{n=0}^{\infty}(-1)^n\left(\frac{\partial L}{\partial F_{\alpha\beta|\underset{\sim}{\lambda}(n)}}\right)_{|\underset{\sim}{\lambda}(n)}\delta F_{\alpha\beta} + (\mathrm{div}) \quad .$$

Noting that $\delta F_{\alpha\beta} = 2\nabla_{[\alpha}\delta A_{\beta]}$ and that $\partial L/\partial F_{\alpha\beta|\underset{\sim}{\lambda}(n)}$ is antisymmetric in α,β, we replace $\delta F_{\alpha\beta}$ with $-2\nabla_\beta\delta A_\alpha$ to obtain

$$\delta L = \frac{\partial L}{\partial A_\alpha}\delta A_\alpha + 2\sum_{n=0}^{\infty}(-1)^n\left(\frac{\partial L}{\partial F_{\alpha\beta|\underset{\sim}{\lambda}(n)}}\right)_{|\underset{\sim}{\lambda}(n)\beta}\delta A_\alpha + (\mathrm{div}) \quad . \tag{4.37}$$

Denoting

$$J^\alpha \equiv \frac{\partial L}{\partial A_\alpha} \;, \quad H^{\alpha\beta} \equiv -8\pi\sum_{n=0}^{\infty}(-1)^n\left(\frac{\partial L}{\partial F_{\alpha\beta|\underset{\sim}{\lambda}(n)}}\right)_{|\underset{\sim}{\lambda}(n)} \;, \tag{4.38}$$

from (4.37), (4.38) the variational principle $\delta I = 0$ gives the *electromagnetic field equations*:

$$H^{\alpha\beta}{}_{|\beta} = 4\pi J^{\alpha} \quad . \tag{4.39}$$

In terms of $L_0 = -(16\pi)^{-1} F_{\alpha\beta}F^{\alpha\beta}$ (cf. (3.54)) and L define *multipole moment* densities

$$m^{\alpha\beta\underset{\sim}{\lambda}(n)}(x) \equiv 2\partial(L - L_0)/\partial F_{\alpha\beta|\underset{\sim}{\lambda}(n)} \quad , \tag{4.40}$$

polarization tensor $M^{\alpha\beta}$

$$M^{\alpha\beta} \equiv \sum_{n=0}^{\infty} (-1)^n m^{\alpha\beta\underset{\sim}{\lambda}(n)}{}_{|\underset{\sim}{\lambda}(n)} \tag{4.41}$$

and associated tensors

$$M^{\alpha\beta\gamma(m)} \equiv \sum_{n=0}^{\infty} (-1)^n m^{\alpha\beta\underset{\sim}{\gamma}(m)\underset{\sim}{\lambda}(n)}{}_{|\underset{\sim}{\lambda}(n)} \tag{4.42}$$

which imply polarization relations

$$H^{\alpha\beta} = F^{\alpha\beta} - 4\pi M^{\alpha\beta} \quad . \tag{4.43}$$

Assume that the material is non-conducting by setting $J^{\alpha} = eN^{\alpha}$ (cf. (3.57)). Again this implies that both $P_{\rho}{}^{\sigma}$ and the "kinetic" momentum p_{ρ}

$$p_{\rho} \equiv P_{\rho} - enA_{\rho} \tag{4.44}$$

are gauge invariant.

From (4.2), (4.23) and (4.43) we obtain

$$(-g)^{-\frac{1}{2}} L_{*(\phi)}^{A} (I_A{}^B)_\rho{}^\sigma \phi_B = (4\pi)^{-1} H^{\alpha\sigma} F_{\alpha\rho} - J^\sigma A_\rho \quad ,$$

$$t_{\rho(\phi)}^{\sigma} = \delta_\rho^\sigma L_2 - \tfrac{1}{2}\sqrt{-g} \sum_{n=0}^{\infty} F_{\alpha\beta|(\rho\underset{\sim}{\alpha}(n))} M^{\alpha\beta\sigma\underset{\sim}{\alpha}(n)} \quad ,$$

so that

$$t_{\rho(\phi)}^{\sigma} + L_{*(\phi)}^{A} (I_A{}^B)_\rho{}^\sigma \phi_B = \sqrt{-g}\; T_{\rho(em)}^{\sigma} - A_\rho J^\sigma \tag{4.45}$$

where we have defined a gauge-invariant electromagnetic energy-momentum tensor by

$$T_{\rho(em)}^{\sigma} \equiv \delta_\rho^\sigma L_2 + \frac{1}{4\pi} F_{\alpha\rho} H^{\alpha\sigma} - \frac{1}{2} \sum_{n=0}^{\infty} F_{\alpha\beta|(\rho\underset{\sim}{\lambda}(n))} M^{\alpha\beta\sigma\underset{\sim}{\lambda}(n)} \quad . \tag{4.46}$$

Inserting (4.26), (4.30), (4.44), (4.45) into (4.24) gives the *gravitational field equations* as

$$(8\pi)^{-1} G^{\rho\sigma} = T^{\rho\sigma} \equiv T^{\rho\sigma}_{(mat)} + T^{\rho\sigma}_{(em)} + \tfrac{1}{2}(S^{\rho\tau\sigma} + S^{\sigma\tau\rho} + S^{\sigma\rho\tau})_{|\tau}$$

$$+ 4Q^{\lambda(\rho\sigma)\mu}{}_{|\mu\lambda} + Q^A (I_A{}^B)^{\rho\sigma} R_B - \sum_{n=0}^{\infty} R_{A|(\cdot\underset{\sim}{\alpha}(n))}^{\;\;\rho} Q^{A\sigma\underset{\sim}{\alpha}(n)} \tag{4.47}$$

where
$$T^{\rho\sigma}_{(mat)} = p^\rho u^\sigma + p^{\rho\sigma} \tag{4.48}$$

represents the flux of material four-momentum, $T^{\rho\sigma}_{(em)}$ is given by (4.46) and the total spin flux $S^{\rho\sigma\tau} = (-g)^{-\frac{1}{2}} \mathfrak{S}^{\rho\sigma\tau}$ is given by (4.36). $T^{\rho\sigma}_{(em)}$, depending on all derivatives of $F_{\alpha\beta}$ and all multipole moments, generalizes the (generalized) Minkowski tensor (3.62).

In terms of the multipole moments (4.40), (4.41), (4.42) the field spin flux is given by

$$S^{\rho\sigma\tau}_{(em)} = (-g)^{-\frac{1}{2}} S^{\rho\sigma\tau}_{(\phi)} =$$

$$2\sum_{n=0}^{\infty} F^{[\rho}{}_{\lambda|\underset{\sim}{\alpha}(n)} M^{\sigma]\lambda\tau\underset{\sim}{\alpha}(n)} + \sum_{n=0}^{\infty} (n+1) F^{[\rho}{}_{\lambda\mu|(\cdot\underset{\sim}{\alpha}(n)\cdot)} M^{\lambda\mu\dot{\sigma}]\tau\underset{\sim}{\alpha}(n)} \quad . \tag{4.49}$$

5. MULTIPOLE EXPANSION OF ELECTRIC CURRENT AND ENERGY-MOMENTUM

5.1 Introduction

We have seen in the earlier chapters how the variational approach may be used to derive the *fully covariant* dynamics of spinning polarized media. By not specifying any particular Lagrangian we have created a framework into which any detailed model must fit. Of course, the price we pay for this is the lack of physical insight to be gained from the formal definitions such as $P_\alpha = \partial L/\partial v^\alpha$ and $M^{\alpha\beta} = 2\partial(L-L_0)/\partial F_{\alpha\beta}$. The objectives of this chapter are to gain a clear understanding of the meaning of the various dynamical quantities such as p^α, $S^{\alpha\beta}$ and $M^{\alpha\beta}$ appearing in earlier chapters and to explore the concept of "gravitational polarization" (polarization of energy-momentum) in detail.

We shall derive the same dynamical laws, albeit in special relativity, by showing first that the structure of charge and current for a composite particle (extended body) may be summarized in certain uniquely defined quantities called multipole moments. These are tensors generalizing the electric and magnetic dipole, quadrupole moments etc. of non-relativistic electrodynamics and they combine in a certain way to form the polarization tensor.

Starting from the microscopic form for the current of a system of point charges, we shall show that a multipole expansion results in the decomposition of current into an overall flow and a polarization part. This is one of the two steps in the construction of the macroscopic Maxwell equations for polarized media from point charge theory. The other step is to suitably define an averaging scheme to obtain smoothly varying quantities. (This thesis explores only the multipole formalism and not the averaging procedures.)

A non-relativistic derivation of the macroscopic Maxwell equations using spatial averaging may be found in [31] (reviewed in Jackson [32], see also references cited there). A detailed analysis of the relativistic formulation is contained in de Groot and Suttorp's book [5] (chapter V). They achieve smooth averaged quantities by assuming the composite particle density is large enough that a smooth distribution function exists. This reference also derives composite particle equations of motion, balance laws and discusses in detail the 3+1 representation of the various dynamical quantities.

We extend the multipole expansion method by showing that it may be applied to fluxes more general than just the electric 4-vector current. Applied specifically to the microscopic energy-momentum tensor (4-momentum flux), the method leads to a gravitational polarization tensor $N^{\alpha\beta\gamma}$ completely analogous to the electromagnetic polarization tensor $M^{\alpha\beta}$. (The relation between $N^{\alpha\beta\gamma}$ and the spin terms of the gravitational field equations will be discussed in section 6.)

In formulating the single particle dynamics, definitions of four-momentum and spin arise that make the equations of motion and balance equations appear in their most simple form (the same equations as those of the Lagrangian formulation). Finally, the expansion of the microscopic energy-momentum is shown to give the same decomposition of total energy-momentum into material, electromagnetic and polarization parts as the Lagrangian result, eq.(4.47).

We will therefore see, for media whose spin is ultimately orbital in nature, that the microscopic (symmetric) total energy-momentum leads to a macroscopic total (symmetric) energy-momentum tensor of which the asymmetric four-momentum flux is only a part. This part describes the overall ("gross") flow of four-momentum just as the phenomenological electric current describes the overall flow of charge, while the total energy-momentum and the total electric current also contain polarization parts.

5.2 Classical Microscopic Model of Matter

The microscopic picture is a cloud of structureless, charged, point particles i, each with rest mass m_i, charge e_i, world-line $z_i^\alpha(s_i)$, normalized four-velocity $u_i^\alpha(s_i) = dz_i^\alpha/ds_i$ and four-momentum $p_i^\alpha = m_i u_i^\alpha$. The (special relativistic) microscopic electromagnetic field equations are

$$\partial_\beta f^{\alpha\beta} = j^\alpha(x) = \sum_i \int e_i u_i^\alpha \, \delta^4(x - z_i(s_i)) \, ds_i \qquad (5.1)$$

where $\delta^4(x)$ denotes the four-dimensional delta function. The symmetric microscopic material energy-momentum tensor (four--momentum flux) is

$$t^{\alpha\beta}_{(mat)}(x) = \sum_i \int m_i u^\alpha_i u^\beta_i \ \delta^4(x-z_i(s_i)) \ ds_i \quad . \tag{5.2}$$

If the point particles are bunched together into distinct stable groups , so that each one is part of a composite particle k (extended body), then our physical picture is a medium consisting of particles k with spin and other structure. Let $Z^\alpha_k(\tau_k)$ denote some choice of central (reference) world-line for particle k, with four-velocity $U^\alpha_k = dZ^\alpha_k/d\tau_k$. If we neglect the charge structure of each particle k by assuming that the total charge $e_k = \sum_{i \text{ of } k} e_i$ lies entirely on Z^α_k, we have the microscopic current $j^\alpha(x)$ approximated by

$$J^\alpha(x) = \sum_k \int e_k U^\alpha_k \ \delta^4(x-Z_k(\tau_k)) \ d\tau_k \quad . \tag{5.3}$$

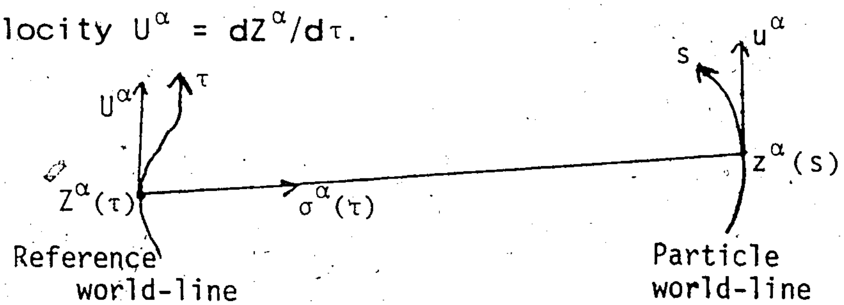
This part of $j^\alpha(x)$ is due to the motion as a whole of the composite particles. $j^\alpha(x)$ consists of $J^\alpha(x)$ and a "polarization" part that describes the contribution from the detailed charge structure of each particle. We may also approximate $t^{\alpha\beta}_{(mat)}(x)$ with

$$T^{\alpha\beta}_{(mat)}(x) = \sum_k \int p^\alpha_k U^\beta_k \ \delta^4(x-Z_k(\tau_k)) \ d\tau_k \tag{5.4}$$

in terms of the total four-momentum $p_k^\alpha = \sum_{i \text{ of } k} m_i u_i^\alpha$ of each k. The meaning of the above expression is clear: it is the flux of four-momentum (four-momentum current) of the composite particles idealized as point particles.[1]

5.3 Notation

From (5.1) to (5.4) it is sufficient to consider the multipole expansion of the current and energy-momentum due to a single particle. Many particle results are then inferred simply by summation, first over i for fixed k to obtain composite particle quantities, then over k. We therefore consider a single point particle with charge e, rest mass m, world-line $z^\alpha(s)$, normalized four-velocity $u^\alpha = dz^\alpha/ds$ and four-momentum mu^α. We investigate the particle's "eccentric dynamics" with respect to some (arbitrarily chosen) time-like reference world-line $Z^\alpha(\tau)$ with normalized four-velocity $U^\alpha = dZ^\alpha/d\tau$.



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[1](5.3) and (5.4) must be averaged to obtain smoothly varying spacetime functions. For example, whenever a smooth distribution function $\mu(x,\Omega)$ depending on phase space variables Ω may be defined, averages of (5.3) and (5.4) are obtained by replacing the summation over k and integration along Z_k^α with integration over phase space:

$$<J^\alpha(x)> = \int \mu(x,\Omega)\, e(\Omega)\, U^\alpha(\Omega)\; d\Omega \quad ,$$

$$<T^{\alpha\beta}_{(mat)}> = \int \mu(x,\Omega)\, p^\alpha(\Omega)\, U^\beta(\Omega)\; d\Omega \quad ,$$

(See [5,6c]).

Any monotonic function $s = s(\tau)$ defines a connecting vector

$$\sigma^\alpha(\tau) = z^\alpha(\tau) - Z^\alpha(\tau) \quad . \tag{5.5}$$

(We do not restrict $s(\tau)$ with conditions such as $\sigma^\alpha U_\alpha = 0$ since the multipole expansion may be carried through for *any* central world-line and *any* world-tube "slicing".

Let $\dot{\sigma}^\alpha$ denote $d\sigma^\alpha/d\tau$. Let

$$I_n(f(\tau)) \equiv \int f(\tau)(\sigma.\partial)^n \, \delta^4(x-Z(\tau)) \, d\tau \tag{5.6}$$

where $\partial_\alpha g = \partial g/\partial x^\alpha = g_{,\alpha}$ and $(\sigma.\partial)^n = \sigma^{\alpha_1}\ldots\sigma^{\alpha_n}\partial_{\alpha_1}\ldots\partial_{\alpha_n}$.
I has the following three properties:

$$I_n(c_1 f(\tau) + c_2 g(\tau)) = c_1 I_n(f(\tau)) + c_2 I_n(g(\tau)) \quad , \tag{5.7}$$

$$\partial_\alpha I_n(f(\tau)\sigma^\alpha) = I_{n+1}(f(\tau)) \quad , \tag{5.8}$$

$$\partial_\alpha I_n(f(\tau)U^\alpha) = I_n(df(\tau)/d\tau) + n\partial_\alpha I_{n-1}(f(\tau)\dot{\sigma}^\alpha) \quad . \tag{5.9}$$

Proof of (iii):

$$\partial_\alpha I_n(f(\tau)U^\alpha) = \int f(\tau)(\sigma.\partial)^n((-\partial\ /\partial Z^\alpha)\delta^4(x-Z(\tau)))\ U^\alpha \ d\tau$$

$$= \int \frac{d}{d\tau}(f(\tau)\sigma^{\alpha_1}\ldots\sigma^{\alpha_n})\partial_{\alpha_1}\ldots\partial_{\alpha_n}\delta^4(x-Z(\tau)) \ d\tau$$

$$= I_n(df(\tau)/d\tau) + n\partial_\alpha I_{n-1}(f(\tau)\dot{\sigma}^\alpha) \quad .$$

According to (5.9),

$$\partial_\beta I_n(f(\tau)\sigma^\alpha U^\beta) = I_n(\dot{f}\sigma^\alpha) + I_n(f\dot{\sigma}^\alpha) + n\partial_\beta I_{n-1}(f\sigma^\alpha\dot{\sigma}^\beta)$$

where the dot denotes differentiation w.r.to τ.
Using (5.8) and summing gives

$$\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{(n+1)!}\,\partial_\beta I_n(f\sigma^\alpha U^\beta) = \sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{(n+1)!}\,I_n(\dot{f}\sigma^\alpha)$$

$$- I_0(f\dot{\sigma}^\alpha) + \partial_\beta\left[\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(n+1)!}\,I_{n-1}(f(\dot{\sigma}^\alpha\sigma^\beta + n\sigma^\alpha\dot{\sigma}^\beta))\right]$$

or

$$0 = - I_0(f\dot{\sigma}^\alpha) + \sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{(n+1)!}\,I_n(\dot{f}\sigma^\alpha)$$
$$+ \partial_\beta\left[\sum_{n=0}^{\infty}\frac{(-1)^{n}}{(n+1)!}\,I_n(f(\sigma^\alpha U^\beta + \frac{1}{(n+2)}\dot{\sigma}^\alpha\sigma^\beta + \frac{(n+1)}{(n+2)}\sigma^\alpha\dot{\sigma}^\beta))\right] . \tag{5.10}$$

5.4 Multipole Expansion of Current and Energy-Momentum

Having developed some notation, we now proceed to expand the microscopic fluxes in powers of σ^α to produce a splitting into "gross" and polarization parts.

The current and energy-momentum of the particle are given by

$$j^\alpha(x) = \int eu^\alpha(s)\ \delta^4(x-z(s))\ ds \quad , \tag{5.11}$$

$$t^{\alpha\beta}_{(mat)}(x) = \int mu^{\alpha}u^{\beta}\ \delta^4(x-z(s))\ ds \quad . \tag{5.12}$$

These are examples of a general flux

$$f^{\alpha}(x) = \int f(s)u^{\alpha}(s)\ \delta^4(x-z(s))\ ds \quad . \tag{5.13}$$

From $z^{\alpha} = Z^{\alpha}+\sigma^{\alpha}$ we have

$$u^{\alpha}ds = dz^{\alpha} = (dz^{\alpha}/d\tau)d\tau = (U^{\alpha}+\dot{\sigma}^{\alpha})d\tau \tag{5.14}$$

and

$$u^{[\alpha}U^{\beta]} = -u^{[\alpha}\dot{\sigma}^{\beta]} \quad . \tag{5.15}$$

Inserting (5.14) into (5.13) gives

$$f^{\alpha}(x) = \int f(\tau)(U^{\alpha}(\tau)+\dot{\sigma}^{\alpha}(\tau))\ \delta^4(x-z(\tau))\ d\tau \quad .$$

Expanding $\delta^4(x-z) = \delta^4(x-Z-\sigma)$ as

$$\delta^4(x-z(\tau)) = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}(\sigma.\partial)^n\delta^4(x-Z(\tau)) \tag{5.16}$$

gives, in the notation of (5.6)

$$f^{\alpha}(x) = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}I_n(f(\tau)(U^{\alpha}+\dot{\sigma}^{\alpha})) \quad .$$

Let F^{α} denote the lowest order (σ^0) term in the above:

$$F^{\alpha}(x) = I_0(fU^{\alpha}) = \int f(\tau)\ \delta^4(x-Z(\tau))\ d\tau \quad .$$

Then, with (5.8),

$$f^{\alpha}(x) = F^{\alpha}(x) + I_0(f\dot{\sigma}^{\alpha}) + \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(n+1)!} \partial_{\beta} I_n(f(U^{\alpha}+\dot{\sigma}^{\alpha})\sigma^{\beta}) .$$

(5.10) then implies

$$f^{\alpha}(x) = F^{\alpha}(x) + \partial_{\beta} X^{\alpha\beta} + \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(n+1)!} I_n(\dot{f}\sigma^{\alpha})$$

where

$$X^{\alpha\beta} = \sum_{n=0}^{\infty} \frac{(-1)^{n}}{(n+1)!} I_n(2f\sigma^{[\alpha}(U^{\beta]}+\frac{(n+1)}{(n+2)}\dot{\sigma}^{\beta]})) .$$

Setting $f = e$ and $f = mu^{\alpha}$ respectively gives

$$j^{\alpha}(x) = J^{\alpha}(x) + \partial_{\beta} M^{\alpha\beta} , \qquad (5.17)$$

$$t^{\alpha\beta}_{(mat)}(x) = T^{\alpha\beta}(x) + \partial_{\gamma} N^{\alpha\beta\gamma} + A^{\alpha\beta} , \qquad (5.18)$$

with

$$J^{\alpha}(x) = I_0(eU^{\alpha}) = \int eU^{\alpha}\ \delta^4(x-Z(\tau))\ d\tau \qquad (5.19)$$

$$T^{\alpha\beta}(x) = I_0(mu^{\alpha}U^{\beta}) = \int mu^{\alpha}U^{\beta}\ \delta^4(x-Z(\tau))\ d\tau , \qquad (5.20)$$

$$M^{\alpha\beta}(x) = e\sum_{n=0}^{\infty} \frac{(-1)^{n}}{(n+1)!} I_n(2\sigma^{[\alpha}(U^{\beta]}+\frac{(n+1)}{(n+2)}\dot{\sigma}^{\beta]})) , \qquad (5.21)$$

$$N^{\alpha\beta\gamma}(x) = \sum_{n=0}^{\infty} \frac{(-1)^{n}}{(n+1)!} I_n(2mu^{\alpha}\sigma^{[\beta}(U^{\gamma]}+\frac{(n+1)}{(n+2)}\dot{\sigma}^{\gamma]})) \qquad (5.22)$$

and

and
$$A^{\alpha\beta} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(n+1)!} I_n\left(\frac{d(mu^{\alpha})}{d\tau}\sigma^{\beta}\right) \quad . \tag{5.23}$$

$M^{\alpha\beta}$ is the *polarization* tensor. It is constructed from the particle *multipole moments*

$$m^{\alpha\beta\underset{\sim}{\lambda}(n)}(\tau) \equiv \frac{2e}{(n+1)!}\,\sigma^{[\alpha}(U^{\beta]} + \frac{(n+1)}{(n+2)}\dot{\sigma}^{\beta]})\sigma^{\underset{\sim}{\lambda}(n)} \tag{5.24}$$

and associated multipole moment densities

$$m^{\alpha\beta\underset{\sim}{\lambda}(n)}(x) = I_0(m^{\alpha\beta\underset{\sim}{\lambda}(n)}(\tau)) = \int m^{\alpha\beta\underset{\sim}{\lambda}(n)}(\tau)\,\delta^4(x-Z(\tau))\,d\tau \tag{5.25}$$

according to

$$M^{\alpha\beta} = \sum_{n=0}^{\infty}(-1)^n \partial_{\underset{\sim}{\lambda}(n)} m^{\alpha\beta\underset{\sim}{\lambda}(n)}(x) \tag{5.26}$$

which is equivalent to (5.21).

(5.26) is the special relativistic form of (4.41). (5.24) and (5.25) therefore give a clear physical meaning to (4.40) in terms of the detailed particle structure.

$A^{\alpha\beta}$ in (5.18) arises from the integration by parts in identity (5.9). It has no counterpart in (5.17) since $de/d\tau = 0$. If the particles are in free motion so that $A^{\alpha\beta}$ vanishes, then (5.17) and (5.18) are the decomposition of j^{α} and $t^{\alpha\beta}_{(mat)}$ into "gross" and "polarization" currents. In general when matter and field interact we have $d(mu^{\alpha})/d\tau$ and $A^{\alpha\beta}$ non zero. $A^{\alpha\beta}$ is a field-matter interaction term that itself must be split three ways into material, field and polarization parts.

5.5 Spinning Multipole in an External Electromagnetic Field, Balance Laws.

The Lorentz force equation is

$$d(mu^{\alpha})/d\tau = ef^{\alpha}{}_{\beta}(z)dz^{\beta}/d\tau \quad . \tag{5.27}$$

In the expansion (5.17) the multipole moments (5.24) appear as the basic entities summarizing the four current "structure". It is possible to express the above Lorentz force in terms of these and derivatives of $f_{\alpha\beta}$ at Z^{α}, together with a total time derivative along the reference world-line. This can be seen from the following identity (Appendix 5) valid for a function f(z):

$$f(z)dz^{\beta}/d\tau =$$

$$f(Z)U^{\beta} - e^{-1}\sum_{n=0}^{\infty} f(Z)_{,\gamma\underset{\sim}{\lambda}(n)}m^{\beta\gamma\underset{\sim}{\lambda}(n)} + \frac{d}{d\tau}\left(\sum_{n=0}^{\infty}\frac{1}{(n+1)!}\sigma^{\beta}(\sigma.\partial)^{n}f(Z)\right) \tag{5.28}$$

Setting $f(z) = ef^{\alpha}{}_{\beta}(z)$ then gives the multipole expansion of (5.27) as

$$dp^{\alpha}/d\tau = ef^{\alpha}{}_{\beta}(Z)U^{\beta} - \sum_{n=0}^{\infty} f^{\alpha}{}_{\beta}(Z)_{,\gamma\underset{\sim}{\lambda}(n)}m^{\beta\gamma\underset{\sim}{\lambda}(n)} \tag{5.29}$$

where

$$p^{\alpha} \equiv mu^{\alpha} - \sum_{n=0}^{\infty} \frac{e}{(n+1)!} f^{\alpha}{}_{\beta}(Z)_{,\underset{\sim}{\lambda}(n)} \sigma^{\beta} \sigma^{\underset{\sim}{\lambda}(n)} \quad . \tag{5.30}$$

The total time derivative of (5.28) has been combined with mu^{α} in (5.29). This gives the equations in their simplest form, with the force determined from the fundamental quantities U^{α} and $m^{\alpha\beta\underset{\sim}{\lambda}(n)}$ of the previous section. The resulting equation is the special relativistic limit of the Lagrangian equation (2.29).

To obtain spin equations of motion con der

$$\frac{1}{2}\frac{d}{d\tau}(2\sigma^{[\alpha}mu^{\beta]}) = \dot{\sigma}^{[\alpha}mu^{\beta]} + \sigma^{[\alpha}\frac{d}{d\tau}(mu^{\beta]})$$

$$= mu^{[\alpha}U^{\beta]} + e\sigma^{[\alpha}f^{\beta]}{}_{\gamma}(z)(dz^{\gamma}/d\tau) \tag{5.31}$$

from (5.15) and (5.27). The multipole expansion of the second term in the above may be found in Appendix 6. With spin angular momentum defined as

$$S^{\alpha\beta} \equiv 2\sigma^{[\alpha}(mu^{\beta]} - \sum_{n=0}^{\infty} \frac{e(n+1)}{(n+2)!} f^{\beta]}{}_{\gamma}(Z)_{,\underset{\sim}{\lambda}(n)} \sigma^{\gamma} \sigma^{\underset{\sim}{\lambda}(n)}) \tag{5.32}$$

the expansion of (5.31) takes the form

$$\frac{1}{2}dS^{\alpha\beta}/d\tau = p^{[\alpha}U^{\beta]} \tag{5.33}$$

$$- \sum_{n=0}^{\infty} f^{[\alpha}{}_{\gamma}(Z)_{,\underset{\sim}{\lambda}(n)} m^{\beta]\gamma\underset{\sim}{\lambda}(n)} + \sum_{n=0}^{\infty} (n+1) f^{[\dot{\alpha}}{}_{\gamma}(Z)_{,\delta\underset{\sim}{\lambda}(n)} m^{\gamma\delta\dot{\beta}]\underset{\sim}{\lambda}(n)}$$

which is the special relativistic limit of (2.30). In similar fashion to the definition (5.30) of four momentum, some interaction angular momentum that appeared in the spin equations as a total time derivative has been designated as "material". The couple acting is then determined from U^{α} and $m^{\alpha\beta\underset{\sim}{\lambda}(n)}$ for a given electromagnetic field.

Equations (5.29) and (5.33) may be cast in the form of balance laws. Define a material energy-momentum tensor $T^{\alpha\beta}_{(mat)}$:

$$T^{\alpha\beta}_{(mat)}(x) = I_0(p^{\alpha}U^{\beta}) = \int p^{\alpha}U^{\beta}\ \delta^4(x-Z(\tau))\ d\tau \quad . \qquad (5.34)$$

(5.9) and (5.29) immediately give

$$\partial_{\beta}T^{\alpha\beta}_{(mat)} = f^{\alpha}{}_{\beta}(x)J^{\beta}(x) - \sum_{n=0}^{\infty} f^{\alpha}{}_{\beta}(x)_{,\gamma\underset{\sim}{\lambda}(n)}m^{\beta\gamma\underset{\sim}{\lambda}(n)}(x). \qquad (5.35)$$

The right hand side may be written as a divergence (Appendix 7), giving

$$\partial_{\beta}(T^{\alpha\beta}_{(mat)} + T^{\alpha\beta}_{(em)}) = 0 \qquad (5.36)$$

where

$$T^{\alpha\beta}_{(em)} \equiv (4\pi)^{-1}(f^{\alpha}{}_{\gamma}f^{\beta\gamma} - \tfrac{1}{4}g^{\alpha\beta}f^{\gamma\delta}f_{\gamma\delta}) - f^{\alpha}{}_{\gamma}M^{\beta\gamma}$$

$$- \frac{1}{2}\sum_{n=0}^{\infty} f_{\gamma\delta}(x)_{,\underset{\sim}{\lambda}(n)}{}^{\alpha}M^{\gamma\delta\underset{\sim}{\lambda}(n)\beta} \quad . \qquad (5.37)$$

(5.37) is the special relativistic limit of (4.46) where $L_2 = L_0$.

The angular momentum balance law is obtained in the same way. Let

$$S^{\alpha\beta\gamma}_{(mat)} \equiv I_0(S^{\alpha\beta}U^{\gamma}) = \int S^{\alpha\beta}U^{\gamma}\ \delta^4(x-Z(\tau))\ d\tau \quad . \qquad (5.38)$$

Then from (5.9) and (5.33)

$$\frac{1}{2}\partial_{\gamma}S^{\alpha\beta\gamma}_{(mat)} = \frac{1}{2}I_0(dS^{\alpha\beta}/d\tau) = T^{[\alpha\beta]}_{(mat)}$$

$$-\sum_{n=0}^{\infty} f^{[\alpha}{}_{\gamma}(x)_{,\underset{\sim}{\lambda}(n)}m^{\beta]\gamma\underset{\sim}{\lambda}(n)}(x) + \sum_{n=0}^{\infty}(n+1)f^{[\dot{\alpha}}{}_{\gamma}(x)_{,\delta\underset{\sim}{\lambda}(n)}m^{\gamma\delta\underset{\sim}{\lambda}(n)\dot{\beta}]}(x)$$

The last two terms are expressible as the sum of $T^{[\alpha\beta]}_{(em)}$ and a divergence (Appendix 8). In terms of

$$S^{\alpha\beta\gamma}_{(em)} \equiv \qquad (5.39)$$

$$2\sum_{n=0}^{\infty}\left\{f^{[\alpha}{}_{\kappa,\underset{\sim}{\lambda}(n)}M^{\beta]\kappa\underset{\sim}{\lambda}(n)\gamma} - (n+1)f^{[\dot{\alpha}}{}_{\kappa,\mu\underset{\sim}{\lambda}(n)}M^{\kappa\mu\underset{\sim}{\lambda}(n)\gamma\dot{\beta}]}\right\}$$

we have

$$\frac{1}{2}\partial_{\gamma}(S^{\alpha\beta\gamma}_{(mat)} + S^{\alpha\beta\gamma}_{(em)}) = (T_{(mat)} + T_{(em)})^{[\alpha\beta]} \quad . \qquad (5.40)$$

(5.39) is the special relativistic form of (4.49).

5.6 Localization of Energy Momentum.

Having obtained the same equations of motion and balance laws as in the Lagrangian theory, we now consider the derivation of energy-momentum localization as embodied in the gravitational field equations (4.47). The microscopic total energy tensor

$$t^{\alpha\beta} = t^{\alpha\beta}_{(mat)} + t^{\alpha\beta}_{(em)} \qquad (5.41)$$

is given by summing (5.12) and the microscopic electromagnetic energy-momentum tensor

$$t^{\alpha\beta}_{(em)}(x) = (4\pi)^{-1}(f^{\alpha\gamma}f^{\beta}{}_{\gamma} - \tfrac{1}{4}g^{\alpha\beta}f^{\gamma\delta}f_{\gamma\delta}) \qquad (5.42)$$

All three tensors in (5.41) are symmetric. We wish to show that the multipole expansion of (5.41) is the special relativistic form of (4.47).

The total energy-momentum tensor is often defined *from* the balance equations. Any conserved tensor, for example $T^{\alpha\beta}_{(mat)}+T^{\alpha\beta}_{(em)}$, is then a candidate. The balance laws merely state however that certain tensors have vanishing divergence. To these an arbitrary curl may be added without affecting the balance law, so the balance laws do not provide a unique prescription for localization.

At the microscopic level equation (5.41) uniquely specifies the total energy momentum distribution. Only the (averaged) multipole expansion of $t^{\alpha\beta}$ may be legitimately referred to as the true distribution of total energy-momentum, which will be symmetric and conserved li[illegible] $t^{\alpha\beta}$. From (5.36) and (5.40) we have that $T^{\alpha\beta}_{(mat)} + T^{\alpha\beta}_{(em)}$ [illegible]s conserved but asymmetric, while one may easily verify [illegible] the tensor $J^{\alpha\beta}$ defined by

$$J^{\alpha\beta} \equiv (T_{(mat)} + T_{(em)})^{\alpha\beta} + \frac{1}{2}\partial_\gamma(S^{\alpha\gamma\beta} + S^{\beta\gamma\alpha} + S^{\beta\alpha\gamma}) \;, \quad (5.43)$$

$$S^{\alpha\beta\gamma} \equiv S^{\alpha\beta\gamma}_{(mat)} + S^{\alpha\beta\gamma}_{(em)} \;, \quad (5.44)$$

is both conserved and symmetric. This is the simplest[1] symmetric conserved tensor one may construct from the balance laws. Since $T^{\alpha\beta}_{(mat)}+T^{\alpha\beta}_{(em)}$, $J^{\alpha\beta}$ and $t^{\alpha\beta}$ are all conserved they must differ from each other by a curl. We have, for some $A^{\alpha(\beta\gamma)} = 0$,

$$t^{\alpha\beta} = (T_{(mat)} + T_{(em)})^{\alpha\beta} + \partial_\gamma A^{\alpha\beta\gamma} \quad (5.45)$$

(5.45), (5.44) and (5.40) imply $\partial_\gamma A^{[\alpha\beta]\gamma} = -\frac{1}{2}\partial_\gamma S^{\alpha\beta\gamma}$, giving

$$A^{[\alpha\beta]\gamma} = -\frac{1}{2}S^{\alpha\beta\gamma} + \frac{1}{2}\partial_\delta Q^{\alpha\beta\gamma\delta} \quad (5.46)$$

for some $Q^{\alpha\beta\gamma\delta}$ satisfying $Q^{(\alpha\beta)\gamma\delta} = 0 = Q^{\alpha\beta(\gamma\delta)}$.

Any third rank tensor that is antisymmetric in its last two indices, such as $A^{\alpha\beta\gamma}$, may be rewritten as

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[1] The definition of $J^{\alpha\beta}$ contains only those tensors that appear in (5.36) and (5.40).

$$A^{\alpha\beta\gamma} = A^{[\alpha\beta]\gamma} + A^{[\gamma\alpha]\beta} + A^{[\gamma\beta]\alpha} \quad . \tag{5.47}$$

(5.45), (5.47), (5.46) and (5.43) imply that the required expansion of $t^{\alpha\beta}$ takes the form

$$t^{\alpha\beta} = J^{\alpha\beta} + Q^{\gamma(\alpha\beta)\delta}{}_{,\gamma\delta} \tag{5.48}$$

(5.48) is the special relativistic limit of (4.47) and completes the comparison with Chapter 4. This chapter has provided physical interpretation of the formal definitions of the Lagrangian theory, albeit in special relativity. The fluxes of four-momentum and spin acquire a direct operational significance in terms of the distribution of spinning particles. Localization (4.47), formally defined by variation of the metric, has been shown to be the (averaged) multipole expansion of the microscopic $t^{\alpha\beta}$. The present derivation of (5.48) is tailored for those not working in general relativity, for whom the definition of energy-momentum as the variational derivative of L with respect to the metric may have little physical appeal. The concept of "gravitational polarization" has been investigated by comparison with the traditional account of electromagnetic polarization. The Belinfante-Rosenfeld spin terms, together with the gravitational multipole contributions, have been shown to be the counterpart of $M^{\alpha\beta}$.

6. MULTIPOLE ANALYSIS IN CURVED SPACETIME

6.1 Introduction

This chapter will generalize the multipole analysis of the previous chapter to curved spacetime. Trying to generalize, step by step, each equation of the previous chapter proves to be an extremely arduous task because the chapter makes great use of the simplicities afforded by special relativity (commuting of partial derivatives, vector nature of (Minkowski) co-ordinate differences, "non-local" character of vectors). The following questions must be answered before one can even consider a Taylor expansion and subsequent manipulation of derivatives:

1. how is the "connecting" vector σ^{α} to be defined?
2. how is the particle four-momentum (a vector field along the *central* world-line Z^{α}) to be defined in terms of the energy-momentum distribution (p^{μ} along z^{μ}) and the applied fields?

In special relativity the connecting vector may be identified with the geodesic (straight line) path joining the two points. To generalize this unambiguously to curved spacetime one must consider only points $z^{\mu}(t)$ in a normal neighbourhood of $Z^{\alpha}(t)$ (non-focusing of geodesics) and then set σ^{α} equal to σl^{α} where l^{α} is the unit tangent vector at Z^{α} to the geodesic Zz (of length σ).

In the definition of four-momentum (and spin) one needs a criterion for selecting the type of transport of p^{μ} from

z^{μ} to Z^{α}, from all possible choices that reduce to parallel propagation for vanishing curvature [12]. As Dixon has shown[1], parallel propagation is not the ideal choice.

The approach of Chapter four does not generalize easily to curved spacetime because their is no simple fully covariant way of expanding (and then manipulating) *tensors*. A suitable method may be inferred by noting that scalars are the only tensorial objects one may expand simply in a fully covariant way, and the relevant scalar from which the dynamics may be derived is of course the Lagrangian. The first effect of considering curved spacetime is therefore the identification of "eccentric Lagrangian dynamics" (expansion of L) as the only conceptually simple approach. The second effect is that $q^{\alpha\beta\gamma\delta\underset{\sim}{\varepsilon}(n)}$ (and p^{α}, $S^{\alpha\beta}$, $m^{\alpha\beta\underset{\sim}{\gamma}(n)}$) no longer needs to be "synthesized" as in the previous chapter, it is given by differentiation of the expanded form of L. One merely turns the Lagrangian "crank".

We therefore consider the eccentric, Lagrangian dynamics of a particle. From a Lagrangian standpoint the expression $p_{\alpha} = \partial L/\partial v^{\alpha}$ enters automatically as the natural definition of four-momentum. This avoids comparing the relative merits of various proposals for p_{α} and $S^{\alpha\beta}$ (and their justifications in special situations), standard Lagrangian formalism by itself *provides* the answer to question 2. A natural definition for $S^{\alpha\beta}$ also appears in the

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[1]Dixon defines four-momentum and spin so that, for each symmetry of spacetime that also preserves the the electromagnetic field, a corresponding component of four-momentum or spin is a constant of the motion.

equations to be derived and is adopted. These definitions are compared with Dixon's in Section 6 and are found to be the same. This establishes the definitions on a completely firm foundation from two viewpoints.

6.2 Expansion of Lagrangian

As long as one is dealing with scalars, values at different spacetime points may be compared: hypersurface integrals are covariant and Taylor expansions may be written covariantly[1]. The scalar of fundamental importance is the Lagrangian. For a point particle in an electromagnetic field the action integral is

$$I = -m\int ds + e\int A_\lambda dz^\lambda = \int L_0 \, dt \qquad (6.1)$$

where

$$L_0 = -m\left(-g_{\lambda\mu}\frac{dz^\lambda}{dt}\frac{dz^\mu}{dt}\right)^{\frac{1}{2}} + eA_\lambda(z)\frac{dz^\lambda}{dt} \qquad (6.2)$$

in terms of a scalar parameter t along world-line z^λ.

L_0 is to be written in terms of a reference world-line Z^α. Consider a two-space $z^\mu(t,\sigma)$ with

$$v^\mu = \partial z^\mu(t,\sigma)/\partial t \quad , \qquad l^\mu = \partial z^\mu(t,\sigma)/\partial\sigma \quad ,$$

---

[1]For points z, Z, in a normal neighbourhood of eachother, one may write the expansion of a scalar field $\phi(x)$ as $\phi(z) = \sum(1/n!)\sigma^{\underline{\alpha}(n)}\phi(Z)|_{\underline{\alpha}(n)}$ where σ^α is the "geodesic" vector "joining" Z to z: $\sigma^\alpha = \sigma l^\alpha$.

$$z^{\alpha}(t, 0) = Z^{\alpha}(t) \quad , \qquad V^{\alpha} = dZ^{\alpha}/dt \quad , \tag{6.3}$$

$$u^{\mu} = v^{\mu}dt/ds \quad , \qquad U^{\alpha} = V^{\alpha}dt/d\tau \quad ,$$

(where τ and s are the proper-times along Z^{α} and $z^{\mu}(t, \sigma = \text{const})$).

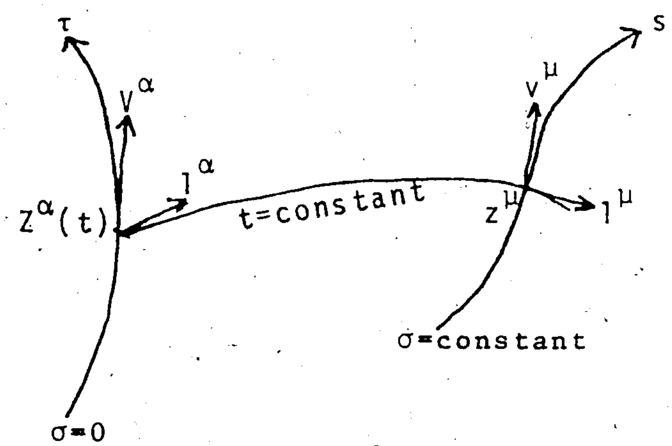

Figure 2

Initially $z^{\mu}(t, \sigma)$ with t constant are an arbitrary set of spacelike curves joining $Z^{\alpha}(t)$ and $z^{\mu}(t, \sigma)$ with σ any parameter along them. Indices α, β, γ,... will be used for tensors at $Z^{\alpha}(t)$ to distinguish them from tensors at z^{λ} which will have indices λ, μ, ν,... . $Z^{\alpha}(t)$ represents the reference world-line and $z^{\lambda}(t, \sigma)$ with σ constant the actual particle world-line. From (6.3) we have

$$\delta v^{\lambda}/\delta\sigma = \delta l^{\lambda}/\delta t \quad . \tag{6.4}$$

To expand (6.1) we need to write the scalars

$$\Psi(\sigma) = \int (-g_{\mu\nu}v^{\mu}v^{\nu})^{\frac{1}{2}}\, dt \quad , \tag{6.5}$$

$$\Phi(\sigma) = \int A_{\mu}(z(t,\sigma))v^{\mu}\, dt \quad , \tag{6.6}$$

in terms of tensors at Z^{α}.

Consider first the expansion of the scalar $\psi(t, \sigma)$ where

$$\psi(t,\sigma) = -(ds/dt)^2 = v_{\mu}v^{\mu}(t,\sigma) \quad , \quad \Psi(\sigma) = \int (-\psi)^{\frac{1}{2}}\, dt \quad . \tag{6.7}$$

Making use of (6.4),

$$\partial\psi/\partial\sigma = 2v_{\mu}\delta v^{\mu}/\delta\sigma = 2v_{\mu}\delta 1^{\mu}/\delta t \quad , \tag{6.8}$$

$$\frac{1}{2}\frac{\partial^2\psi}{\partial\sigma^2} = \frac{\delta v_{\mu}}{\delta\sigma}\frac{\delta 1^{\mu}}{\delta t} + \frac{v_{\mu}\delta^2 1^{\mu}}{\delta\sigma\delta t}$$

$$= \frac{\delta 1_{\mu}}{\delta t}\frac{\delta 1^{\mu}}{\delta t} + v_{\mu}1^{\nu}R_{\nu}{}^{\mu}{}_{\lambda\xi}v^{\lambda}1^{\xi} + v_{\mu}\frac{\delta^2 1^{\mu}}{\delta t\delta\sigma} \quad . \tag{6.9}$$

Now choose the curves $z^{\alpha}(t, \sigma)$, t constant, to be *geodesics* with σ an affine parameter, so that

$$\delta 1^{\mu}(t,\sigma)/\delta\sigma = 0 \quad , \tag{6:10}$$

giving

$$\frac{1}{2}\frac{\partial^2\psi}{\partial\sigma^2} = \frac{\delta l_\lambda}{\delta t}\frac{\delta l^\lambda}{\delta t} + R_{\nu\mu\lambda\xi}v^\mu v^\lambda l^\nu l^\xi \quad . \tag{6.11}$$

Using (6.4) and (6.10) gives

$$\frac{1}{2}\frac{\partial^3\psi}{\partial\sigma^3} = 2\frac{\delta l_\mu}{\delta t}\frac{\delta^2 l^\mu}{\delta\sigma\delta t} + R_{\nu\mu\lambda\xi|\kappa}v^\mu v^\lambda l^\nu l^\xi l^\kappa + R_{\nu\mu\lambda\xi}l^\nu l^\xi \frac{\delta(v^\mu v^\lambda)}{\delta\sigma}$$

$$= 2\frac{\delta l^\mu}{\delta t} l^\nu R_{\nu\mu\lambda\xi} v^\lambda l^\xi + R_{\nu\mu\lambda\xi|\kappa}v^\mu v^\lambda l^\nu l^\xi l^\kappa + 2R_{\nu\mu\lambda\xi} l^\nu l^\xi \frac{\delta l^{(\mu}}{\delta t} v^{\lambda)}$$

Since $R_{\nu\mu\lambda\xi}l^\nu l^\xi$ is symmetric, this is

$$\frac{1}{2}\frac{\partial^3\psi}{\partial\sigma^3} = 4R_{\nu\mu\lambda\xi}l^\nu l^\xi \frac{\delta l^\mu}{\delta t} v^\lambda + R_{\nu\mu\rho\sigma|\lambda}v^\mu v^\rho l^\nu l^\sigma l^\lambda \quad . \tag{6.12}$$

For simplicity, neglect squares and derivatives of the curvature. Then

$$\frac{1}{2}\frac{\partial^4\psi}{\partial\sigma^4} = 4R_{\nu\mu\rho\sigma}l^\nu l^\sigma \frac{\delta l^\mu}{\delta t}\frac{\delta l^\rho}{\delta t} \tag{6.13}$$

and higher derivatives of ψ are all $O(R^2, \nabla R)$. Let $l^\alpha(t)$ denote $l^\mu(t, \sigma=0)$ and define the *"geodesic connecting vector"* (from Z to z) to be $\sigma^\alpha = \sigma l^\alpha$. σ^α is independent of choice of affine parameter σ. In particular, if σ were chosen to be the geodesic distance along Zz then l^α would be the unit tangent vector at Z to Zz and $\sigma = (\sigma_\alpha\sigma^\alpha)^{\frac{1}{2}}$. Collecting together (6.7), (6.8), (6.11), (6.12) and (6.13) gives

$$\psi(t,\sigma) = \sum_{n=0}^{\infty}\frac{\sigma^n}{n!}\left(\frac{\partial^n\psi(t,\sigma)}{\partial\sigma^n}\right)_{|\sigma=0} = V_\alpha V^\alpha + 2V_\alpha\frac{\delta\sigma^\alpha}{\delta t} + \frac{\delta\sigma_\alpha}{\delta t}\frac{\delta\sigma^\alpha}{\delta t} +$$

$$+ R_{\alpha\beta\gamma\delta}\sigma^{\alpha}\sigma^{\delta}\left(V^{\beta}V^{\gamma} + \frac{4}{3}\frac{\delta\sigma^{\beta}}{\delta t}V^{\gamma} + \frac{1}{3}\frac{\delta\sigma^{\beta}}{\delta t}\frac{\delta\sigma^{\gamma}}{\delta t}\right) + O(R^2, \nabla R) \quad . \tag{6.14}$$

Now consider the expansion of

$$\Phi(\sigma) = \int A_{\lambda}(z(t,\sigma))v^{\lambda}(t,\sigma)\, dt \quad . \tag{6.15}$$

Differentiating gives $d\Phi/d\sigma$ =

$$\int (A_{\mu|\nu}l^{\nu}v^{\mu} + A_{\mu}\delta l^{\mu}/\delta t)dt = \int \left(\frac{d(A_{\nu}l^{\nu})}{dt} + (A_{\mu|\nu} - A_{\nu|\mu})l^{\nu}v^{\mu}\right)dt$$

i.e.[1] $\quad - d\Phi/d\sigma = \int F_{\mu\nu}v^{\mu}l^{\nu}\, dt \quad .$

One may show (Appendix 9) that continued differentiation gives, for n = 1, 2,...

$$- d^{n}\Phi/d\sigma^{n} =$$

$$\int \Big[F_{\mu\nu|\underset{\sim}{\lambda}(n-1)}v^{\mu}l^{\nu}l^{\underset{\sim}{\lambda}(n-1)} + (n-1)F_{\mu\nu|\underset{\sim}{\lambda}(n-2)}\frac{\delta l^{\mu}}{\delta t}l^{\nu}l^{\underset{\sim}{\lambda}(n-2)}$$

$$+ \frac{1}{2}(n-1)(n-2)F_{\mu\nu|\underset{\sim}{\lambda}(n-3)}R_{\lambda}{}^{\mu}{}_{\rho\sigma}v^{\rho}l^{\sigma}l^{\nu}l^{\lambda}l^{\underset{\sim}{\lambda}(n-3)}$$

$$+ \frac{1}{6}(n-1)(n-2)(n-3)F_{\mu\nu|\underset{\sim}{\lambda}(n-4)}R_{\lambda}{}^{\mu}{}_{\rho\sigma}\frac{\delta l^{\rho}}{\delta t}l^{\sigma}l^{\nu}l^{\lambda}l^{\underset{\sim}{\lambda}(n-4)}\Big]\, dt$$

$$+ O(R^2, \nabla R) \quad . \tag{6.16}$$

In terms of $\sigma^{\alpha} = \sigma l^{\alpha}$, (6.16) gives

$$\Phi(\sigma) = \sum_{n=0}^{\infty}\frac{\sigma^{n}}{n!}\frac{d^{n}\Phi}{d\sigma^{n}}\Big|_{\sigma=0} = \int \phi(t,\sigma)\, dt$$

---

[1] Here we ignore $\int d(A_{\nu}l^{\nu})$ since it does not contribute anything to δI when the world-line is varied.

where

$$\phi(t,\sigma) = A_\alpha V^\alpha - \sum_{n=0}^{\infty} F_{\alpha\beta|\underset{\sim}{\gamma}(n)}\sigma^{\underset{\sim}{\gamma}(n)}\Big(\frac{1}{(n+1)!}V^\alpha\sigma^\beta +$$

$$+ \frac{(n+1)}{(n+2)!}\frac{\delta\sigma^\alpha}{\delta t}\sigma^\beta + \frac{1}{2n!}R_\varepsilon{}^\alpha{}_{\gamma\delta}\sigma^\beta\sigma^\varepsilon\sigma^\delta\Big(\frac{1}{n+3}V^\gamma + \frac{1}{3(n+4)}\frac{\delta\sigma^\gamma}{\delta t}\Big)\Big)$$

$$+ O(R^2, \nabla R) \qquad (6.17)$$

This completes the expansion of (6.1). The action integral has been expressed in terms of *tensors at* Z^α:

$$I = \int L(V^\alpha, g_{\alpha\beta}, \sigma^\alpha, \frac{\delta\sigma^\alpha}{\delta t}, F_{\alpha\beta|\underset{\sim}{\lambda}(n)}, R^\alpha{}_{\beta\gamma\delta})\, dt \qquad (6.18)$$

where

$$L(t) = -m(-\psi(t,\sigma))^{\frac{1}{2}} + e\phi(t,\sigma) \qquad (6.19)$$

and the expanded forms of ψ and ϕ are given in (6.14) and (6.17).

6.3 Equations of Motion in Given External Fields

The previous section concerned itself with the expansion of the specific action (6.1) in powers of σ^α. The expansion may be carried out quite generally for any action integral representing a particle with world-line $z^\mu(t)$ in given external fields ϕ_A.

An action integral

$$I = \int L_0(v^\mu, g_{\mu\nu}, \phi_A(z))\, dt \qquad (6.20)$$

may be written as

$$I = \int L(V^\alpha, g_{\alpha\beta}, \sigma^\alpha, \dot{\sigma}^\alpha, \Psi_{\underset{\sim}{A}}(Z))\, dt + O(R^2, \nabla R) \qquad (6.21)$$

where $\Psi_{\underset{\sim}{A}} = (\phi_{A|\underset{\sim}{\alpha}(n)}, R^\alpha{}_{\beta\gamma\delta})$ and $\dot{\sigma}^\alpha = \delta\sigma^\alpha/\delta t$.

The equations of motion are obtained from variation of $z^\mu(t)$ with fixed endpoints. Also, since $Z^\alpha(t)$ is *any* chosen reference world-line, I must be invariant under arbitrary variation of $Z^\alpha(t)$. This implies that by simultaneous variation in both $z^\mu(t)$ and $Z^\alpha(t)$ it is possible to derive the equations of motion in various equivalent forms.

Consider a one parameter family of infinitesimal displacements in both $z^\mu(t)$ and $Z^\alpha(t)$, $z^\mu(t,\varepsilon) = z^\mu(t)+\varepsilon\eta^\mu(t)$, $Z^\alpha(t,\varepsilon) = Z^\alpha(t)+\varepsilon\xi^\alpha(t)$, with fixed endpoints. We note that since σ^α and $\delta\sigma^\alpha/\delta t$ are both two-point vector fields[1], differentiation must be used with caution, $\delta\sigma^\alpha/\delta t$ and $\sigma^\alpha{}_{|\beta}V^\beta$ are not the same: we have

$$\dot{\sigma}^\alpha = \delta\sigma^\alpha/\delta t = \delta\sigma^\alpha(Z(t),z(t))/\delta t = \sigma^\alpha{}_{|\beta}V^\beta + \sigma^\alpha{}_{|\lambda}v^\lambda . \qquad (6.22)$$

The accompanying absolute variations are

$$\frac{\delta V^\alpha}{\delta\varepsilon} = \frac{\delta}{\delta t}\left(\frac{\partial Z^\alpha}{\partial\varepsilon}\right) , \quad \frac{\delta\Psi_{\underset{\sim}{A}}}{\delta\varepsilon} = \Psi_{\underset{\sim}{A}|\alpha}\frac{\partial Z^\alpha}{\partial\varepsilon} ,$$

---

[1] σ^α **transforms as a vector at Z and as a scalar at z. For a discussion of two-point tensor fields (also called bitensor fields) see [30].**

$$\frac{\delta\sigma^\alpha}{\delta\varepsilon} = \sigma^\alpha{}_{|\beta}\frac{\partial Z^\beta}{\partial\varepsilon} + \sigma^\alpha{}_{|\lambda}\frac{\partial z^\lambda}{\partial\varepsilon} \quad , \quad \frac{\delta}{\delta\varepsilon}\left(\frac{\delta\sigma^\alpha}{\delta t}\right) = \frac{\delta}{\delta t}\left(\frac{\delta\sigma^\alpha}{\delta\varepsilon}\right) + \sigma^\beta R_\beta{}^\alpha{}_{\gamma\delta}V^\gamma\frac{\partial Z^\delta}{\partial\varepsilon} \; . \tag{6.23}$$

Extremizing $I(\varepsilon) = \int_{t_1}^{t_2} L dt$ subject to fixed $z^\mu(t_i, \varepsilon) = z^\mu(t_i, 0)$, $Z^\alpha(t_i, \varepsilon) = Z^\alpha(t_i, 0)$, $(i = 1, 2)$, yields for arbitrary variations $\partial z^\mu/\partial\varepsilon$, $\partial Z^\alpha/\partial\varepsilon$,

$$\frac{dI}{d\varepsilon} = \int_{t_1}^{t_2}\left(\frac{\partial L}{\partial V^\alpha}\frac{\delta}{\delta t}\left(\frac{\partial Z^\alpha}{\partial\varepsilon}\right) + \frac{\partial L}{\partial\sigma^\alpha}\frac{\delta\sigma^\alpha}{\delta\varepsilon} + \frac{\partial L}{\partial\dot\sigma^\alpha}\frac{\delta}{\delta t}\left(\frac{\delta\sigma^\alpha}{\delta\varepsilon}\right)\right.$$
$$\left. + \frac{\partial L}{\partial\dot\sigma^\alpha}\sigma^\beta R_\beta{}^\alpha{}_{\gamma\delta}V^\gamma\frac{\partial Z^\delta}{\partial\varepsilon} + \frac{\partial L}{\partial\Psi_{\underset{\sim}{A}}}\Psi_{\underset{\sim}{A}|\alpha}\frac{\partial Z^\alpha}{\partial\varepsilon}\right) dt = 0 \; .$$

By integration by parts, the action principle then gives

$$0 = \frac{dI}{d\varepsilon} = \tag{6.24}$$

$$\int_{t_1}^{t_2}\left(\frac{\delta L}{\delta\sigma^\alpha}\frac{\delta\sigma^\alpha}{\delta\varepsilon} + \left[-\frac{\delta}{\delta t}\left(\frac{\partial L}{\partial V^\alpha}\right) + \frac{\partial L}{\partial\dot\sigma^\gamma}\sigma^\beta R_\beta{}^\gamma{}_{\delta\alpha}V^\delta + \frac{\partial L}{\partial\Psi_{\underset{\sim}{A}}}\Psi_{\underset{\sim}{A}|\alpha}\right]\frac{\partial Z^\alpha}{\partial\varepsilon}\right) dt$$

for arbitrary $\partial Z^\alpha/\partial\varepsilon$, $\partial z^\mu/\partial\varepsilon$, where

$$\frac{\delta L}{\delta\sigma^\alpha} \equiv \frac{\partial L}{\partial\sigma^\alpha} - \frac{\delta}{\delta t}\left(\frac{\partial L}{\partial\dot\sigma^\alpha}\right) \tag{6.25}$$

and $\delta\sigma^\alpha/\delta\varepsilon$ is given in (6.23).

Choosing $\partial Z^\alpha/\partial\varepsilon = 0$ (reference world-line $Z^\alpha(t)$ held fixed, actual world-line $z^\mu(t)$ varied) therefore gives, from (6.24), the equations of motion in the form

$$\frac{\delta L}{\delta\sigma^\alpha} = 0 \; . \tag{6.26}$$

The equations of motion may also be obtained in an alternate form by simultaneous variation in both $z^\mu(t)$ and $Z^\alpha(t)$, subject to $\delta\sigma^\alpha/\delta\varepsilon = 0$. From (6.24) this immediately gives the *translational equations of motion*

$$\delta P_\alpha/\delta t = \tfrac{1}{2}R_{\beta\gamma\delta\alpha}S^{\beta\gamma}V^\delta + M^{\underset{\sim}{A}}\Psi_{\underset{\sim}{A}|\alpha} \tag{6.27}$$

where

$$P_\alpha \equiv \frac{\partial L}{\partial V^\alpha} \quad , \qquad S^\alpha{}_\beta \equiv 2\sigma^{[\alpha}\frac{\partial L}{\partial\dot{\sigma}^{\beta]}} \quad , \quad M^{\underset{\sim}{A}} \equiv \frac{\partial L}{\partial\Psi_{\underset{\sim}{A}}} \quad . \tag{6.28}$$

Also from (6.24), we see that choosing $\partial z^\mu/\partial\varepsilon = 0$ ($z^\mu(t)$ held fixed, $Z^\alpha(t)$ varied) implies that L satisfies the identity

$$(\delta L/\delta\sigma^\beta)\sigma^\beta{}_{|\alpha} - \delta P_\alpha/\delta t + \tfrac{1}{2}R_{\beta\gamma\delta\alpha}S^{\beta\gamma}V^\delta + M^{\underset{\sim}{A}}\Psi_{\underset{\sim}{A}|\alpha} = 0$$

which confirms the equivalence of (6.26) and (6.27).

To obtain the spin equations of motion, note from (2.11) the following identity satisfied by L:

$$\frac{\partial L}{\partial V^\alpha}V^\beta - 2\frac{\partial L}{\partial g_{\beta\gamma}}g_{\alpha\gamma} + \frac{\partial L}{\partial\sigma^\alpha}\sigma^\beta + \frac{\partial L}{\partial\dot{\sigma}^\alpha}\dot{\sigma}^\beta + \frac{\partial L}{\partial\Psi_{\underset{\sim}{A}}}(I_{\underset{\sim}{A}}{}^{\underset{\sim}{B}})_\alpha{}^\beta\Psi_{\underset{\sim}{B}} = 0 \quad , \tag{6.29}$$

which implies

$$\frac{\delta}{\delta t}\left(\sigma^\beta\frac{\partial L}{\partial\dot{\sigma}^\alpha}\right) = -\frac{\partial L}{\partial V^\alpha}V^\beta + 2\frac{\partial L}{\partial g_{\beta\gamma}}g_{\alpha\gamma} - \frac{\delta L}{\delta\sigma^\alpha}\sigma^\beta - \frac{\partial L}{\partial\Psi_{\underset{\sim}{A}}}(I_{\underset{\sim}{A}}{}^{\underset{\sim}{B}})_\alpha{}^\beta\Psi_{\underset{\sim}{B}}.$$

Antisymmetrization and (6.28), (6.26) then give the *spin equations of motion*

$$\frac{1}{2}\delta S^{\alpha\beta}/\delta t = P^{[\alpha}V^{\beta]} + M^{\underset{\sim}{A}}(I_{\underset{\sim}{A}}{}^{\underset{\sim}{B}})^{[\alpha\beta]}\Psi_{\underset{\sim}{B}} \quad . \qquad (6.30)$$

The derivation of (6.27) and (6.30) closely parallels that of (2.23) and (2.22). However, no introduction of spin coordinates $e_{\alpha}^{(a)}(t)$ and their subsequent variation was needed in this section; the spin equations (6.30) are essentially a consequence of the translational equations in the form (6.26). Appendix 10 formulates the present section in terms of a tetrad field and shows that (2.22) and (2.23) follow from the more general considerations of this section.

6.4 Gravitational Field Equations

The curved-spacetime generalization of (5.48) may be obtained by adding free-field Lagrangians for the electromagnetic and gravitational fields to the expanded form (6.19) of the particle Lagrangian and then calculating the gravitational field equations from variation of the metric. This results in the one-particle limit[1] of the equations of Chapter 4 which described a continuous matter distribution.

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[1]We may write I of (6.21) as $I = \int \mathbf{L}_1(x)d^4x$ where $\mathbf{L}_1(x) = \int L(\tau)\delta^4(x-Z(\tau))d\tau$ (τ is proper time along Z^α). If $U^\alpha(x)$, $\sigma^\alpha(x)$ are any smooth functions of x reducing to $U^\alpha(\tau)$, $\sigma^\alpha(\tau)$ on Z^α, and if $L_D(x)$ denotes $L(U^\alpha(x), g_{\alpha\beta}(x), \ldots)$, then $\mathbf{L}_1(x) = \sqrt{-g}n(x)L_D(x)$ where number density $\sqrt{-g}n(x) = \int \delta^4(x-Z(\tau))d\tau$. This is the one particle limit of "dustlike" matter. (We consider $\delta^4(x-Z)$ to be a scalar density at x and a scalar at Z.)

The gravitational field equations are (4.24) with $t_\rho^{\ \sigma}$, $S^{\rho\tau\sigma}$ given by (4.26), (4.36) where $t^{\alpha\beta}_{(mat)}$ and $S^{\alpha\beta\gamma}_{(mat)}$ have the form

$$t^{\alpha\beta}_{(mat)} = \int p^\alpha(\tau) U^\beta(\tau)\ \delta^4(x - Z(\tau))\ d\tau \ , \qquad (6.31)$$

$$S^{\alpha\beta\gamma}_{(mat)} = \int S^{\alpha\beta}(\tau) U^\beta(\tau)\ \delta^4(x - Z(\tau))\ d\tau \ .$$

6.5 Einstein-Lorentz Theory

For a multipole particle in an Einstein-Maxwell ield, equations (2.25) to (2.28) inserted into (6.27) and (6.30) (i.e. (2.23) and (2.22)) give the equations of motion (2.29), (2.30). Specific expressions for p^α, $S^{\alpha\beta}$, $m^{\alpha\beta\underset{\sim}{\gamma}(n)}$ and $q_\alpha^{\ \beta\gamma\delta}$, that generalize (5.24), (5.30) and (5.32) are given simply by differentiation of the Lagrangian (6.19). Noting that $(-\psi)^{\frac{1}{2}} = ds/dt$, equations (6.14), (6.17), (6.19) give the following:

$$p^\alpha = \partial L/\partial V_\alpha - eA^\alpha =$$

$$m\frac{dt}{ds}\left(V^\alpha + \dot{\sigma}^\alpha + R_{\gamma\beta}{}^\alpha{}_\delta \sigma^\gamma \sigma^\delta (V^\beta + \tfrac{2}{3}\dot{\sigma}^\beta)\right) - \sum_{n=0}^{\infty} \frac{e}{(n+1)!} F^\alpha{}_{\beta|\underset{\sim}{\gamma}(n)} \sigma^\beta \sigma^{\underset{\sim}{\gamma}(n)}$$

$$- \sum_{n=0}^{\infty} \frac{e}{2(n+3)n!} F_{\gamma\beta|\underset{\sim}{\gamma}(n)} R_\varepsilon{}^{\gamma\alpha}{}_\delta \sigma^\beta \sigma^\varepsilon \sigma^\delta \sigma^{\underset{\sim}{\gamma}(n)} \ , \qquad (6.32)$$

$$S^{\alpha\beta} = 2\sigma^{[\alpha}\partial L/\partial\dot{\sigma}_{\beta]} =$$

$$2m\frac{dt}{ds}\sigma^{[\alpha}(V^{\beta]} + \dot{\sigma}^{\beta]} + \frac{1}{3}R_{\epsilon}{}^{\beta]}{}_{\gamma\delta}\sigma^{\epsilon}\sigma^{\delta}(2V^{\gamma}+\dot{\sigma}^{\gamma}))$$

$$- \sum_{n=0}^{\infty} \frac{2e(n+1)}{(n+2)!}\,\sigma^{[\alpha}F^{\beta]}{}_{\delta|\underset{\sim}{\gamma}(n)}\sigma^{\delta}\sigma^{\underset{\sim}{\gamma}(n)}$$

$$- \sum_{n=0}^{\infty} \frac{e}{3(n+4)n!}\,\sigma^{[\alpha}F^{\zeta}{}_{\eta|\underset{\sim}{\gamma}(n)}R_{\epsilon\zeta}{}^{\beta]}{}_{\delta}\sigma^{\eta}\sigma^{\epsilon}\sigma^{\delta}\sigma^{\underset{\sim}{\gamma}(n)} \quad , (6.33)$$

$$m^{\alpha\beta\underset{\sim}{\gamma}(n)} = 2\partial L/\partial F_{\alpha\beta|\underset{\sim}{\gamma}(n)} =$$

$$\frac{2e}{(n+1)!}\,\sigma^{[\alpha}(V^{\beta]} + \frac{(n+1)}{(n+2)}\dot{\sigma}^{\beta]})\sigma^{\underset{\sim}{\gamma}(n)} \qquad (6.34)$$

$$- \frac{e}{n!(n+3)}\,R^{[\alpha}_{\epsilon\ \gamma\delta}\sigma^{\beta]}\sigma^{\epsilon}\sigma^{\delta}(V^{\gamma} + \frac{(n+3)}{3(n+4)}\dot{\sigma}^{\gamma})\sigma^{\underset{\sim}{\gamma}(n)} ,$$

and

$$q^{\alpha\beta\gamma\delta} = \partial L/\partial R_{\alpha\beta\gamma\delta} = \qquad (6.35)$$

$$\frac{1}{2}m\frac{dt}{ds}\sigma^{[\alpha}\left(V^{\beta]}V^{[\gamma} + \frac{2}{3}\dot{\sigma}^{\beta]}V^{[\gamma} + \frac{2}{3}V^{\beta]}\dot{\sigma}^{[\gamma} + \frac{1}{3}\dot{\sigma}^{\beta]}\dot{\sigma}^{[\gamma}\right)\sigma^{\delta]}$$

$$- \frac{e}{2}\sum_{n=0}^{\infty}\frac{1}{n!}\,\sigma^{[\dot{\alpha}}F^{(\dot{\beta}]}{}_{\epsilon|\underset{\sim}{\gamma}(n)}\sigma^{\epsilon}\sigma^{\underset{\sim}{\gamma}(n)}\left(\frac{1}{(n+3)}V^{\dot{\gamma})} + \frac{1}{3(n+4)}\dot{\sigma}^{\dot{\gamma})}\right)\sigma^{\dot{\delta}]} .$$

(p^{α} and $S^{\alpha\beta}$ are parameter-independent, $m^{\alpha\beta\underset{\sim}{\gamma}(n)}$ and $q^{\alpha\beta\gamma\delta}$ are not, set $t = \tau$ to fix them uniquely.)

(6.14) and (6.17) are expanded only to "first" order in $R^{\alpha}{}_{\beta\gamma\delta}$, i.e. neglecting squares and derivatives of curvature. It follows that (6.32), (6.33) and (6.34) are to first order in $R^{\alpha}{}_{\beta\gamma\delta}$ while (6.35) is to zero order. An expansion of I to all orders of R would enable $q^{\alpha\beta\gamma\delta\underset{\sim}{\varepsilon}(n)}$ to be calculated for any n.

It was noted in the previous section how the addition of free-field Lagrangians for ϕ_A and $g_{\alpha\beta}$ to a Lagrangian of the form (6.21) gives, on variation of $g_{\alpha\beta}$, the gravitational field equations of Chapter 4 with the material tensors having the one particle forms (6.31). Setting $\phi_A = (A_\alpha, F_{\alpha\beta})$, the whole of Chapter 4, Section 4 applies, with

$$\sqrt{-g}\, J^{\alpha}(x) = \int eU^{\alpha}\, \delta^4(x-Z(\tau))\, d\tau = I_0(eU^{\alpha}) \quad ,$$

$$\sqrt{-g}\, T^{\alpha\beta}_{(mat)}(x) = I_0(p^{\alpha}U^{\beta}),\ \sqrt{-g}\, S^{\alpha\beta\gamma}_{(mat)}(x) = I_0(S^{\alpha\beta}U^{\gamma}) \quad , \tag{6.36}$$

$$\sqrt{-g}\, m^{\alpha\beta\underset{\sim}{\gamma}(n)}(x) \quad I_0(m^{\alpha\beta\underset{\sim}{\gamma}(n)}(\tau))\ ,\ \sqrt{-g}\, q^{\alpha\beta\gamma\delta\underset{\sim}{\varepsilon}(n)}(x) = I_0(q^{\alpha\beta\gamma\delta\underset{\sim}{\varepsilon}(n)}(\tau)) \quad ,$$

where p^{α}, $S^{\alpha\beta}$, etc. are the single particle four-momentum, spin etc. Specific expressions (6.32) to (6.35) may be substituted into (4.47) to give the curved-space generalization of localization (5.48).

For a collection of particles, each moving as a test body in the overall electromagnetic and gravitational fields without direct particle interaction, a sum of actions of the form (6.21) will give the gravitational field equations (4.47) with J^α, $T^{\alpha\beta}_{(mat)}$ etc. given by summation of the right hand sides of (6.36). (4.47) therefore apply to a gaseous assembly of collisionless, charged, spinning particles. In terms of an invariant distribution function $\mu(x,\Omega)$ equations (6.36) are replaced by phase space integrals, for example $J^\alpha(x) = \int \mu(x,\Omega) e(\Omega) U^\alpha(\Omega) d\Omega$[1]. In (3.38) u^α is a timelike left eigenvector of $t^{\alpha\beta}_{(mat)}$ and the material spin flux $S^{\alpha\beta\gamma}_{(mat)} = S^{\alpha\beta}u^\gamma$ is convective, "carried along" by u^α. For a gas the spin flux $S^{\alpha\beta\gamma}_{(mat)} = \int \mu S^{\alpha\beta} U^\gamma d\Omega$ will not, in general, be convective.

6.6 Propagators for Momentum and Spin

To compare with the definitions for momentum and spin given by Dixon [11], in particular (5.1) and (5.2) of [11c], we now show that p_α and $S^{\alpha\beta}$ may be related to $p^\lambda = mu^\lambda$ via two propagators (two-point tensor fields $K^\lambda{}_\alpha$, $H^\lambda{}_\alpha$).

Let $\sigma^{-1\lambda}{}_\alpha(Z,z)$ denote the inverse of $\sigma^\alpha{}_{|\lambda}$:

$$\sigma^{-1\lambda}{}_\alpha \sigma^\alpha{}_{|\mu} = \delta^\lambda_\mu \; , \quad \sigma^\beta{}_{|\lambda} \sigma^{-1\lambda}{}_\alpha = \delta^\beta_\alpha \; . \tag{6.37}$$

Then

$$\dot{\sigma}^\alpha = \sigma^\alpha{}_{|\beta} V^\beta + \sigma^\alpha{}_{|\mu} v^\mu \tag{6.38}$$

gives

$$v^\lambda = K^\lambda{}_\alpha V^\alpha + H^\lambda{}_\alpha \dot{\sigma}^\alpha \tag{6.39}$$

where

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[1]Cf. [6c] which postulates, on the basis of the balance laws, the field equations for a *dipolar* gas.