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(Signed) R. J. Hursey, Jr.

PERMANENT ADDRESS:

Dept. of Mathematics
East Carolina University
Greenville, N.C., USA
27834

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ON ORDERED GROUPS SATISFYING THE MAXIMAL CONDITION LOCALLY

by

© ROBERT JACKSON HURSEY, JR.

A THESIS

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies for acceptance, a thesis entieled "ON ORDERED GROUPS SATISFYING THE MAXIMAL CONDITION LOCALLY", submitted by Robert Jackson Hursey in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

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Supervisor

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External Examiner

Date Spring, 1971

ABSTRACT

It is the purpose of this manuscript to study the class of orderable groups which satisfy the maximal condition for subgroups locally. As such, it is, therefore, a study of a class of torsion-free, generalized solvable groups on which there is imposed a finiteness condition. The choice of the local satisfaction of the maximal condition as a vehicle of potential interest and utility in the study of 0-groups was suggested by (1) a careful investigation of torsion-free, locally nilpotent groups, and by (2) the realization that the imposition of this finiteness condition upon an 0-group necessitates the normality of each of the group's convex subgroups.

Chapter 0 is expository in nature and serves to introduce the concepts and results employed throughout this manuscript.

Chapter I is devoted to a study of torsion-free, locally nilpotent groups. The main result of this chapter is that the convex families of a torsion-free, locally nilpotent group are central systems of the group. Necessary and sufficient conditions that a family of subgroups of a torsion-free, locally nilpotent group G be the family of all convex subgroups with respect to some order on G are also derived. The chapter is concluded with the proof of a condition which guarantees that certain members of the upper central series of a torsion-free, nilpotent group G be absolutely convex in G.

Polycyclic 0- and polycyclic R-groups are investigated in Chapter II. Herein is stated a necessary and sufficient condition that a polycyclic 0-group be nilpotent. We also find that the derived group of an 0-group which satisfies the maximal condition locally, which

satisfies the maximal condition, or which is polycyclic is a Z-group, is a ZD-group, or is nilpotent, respectively. The fact, proved herein, that a group G is a supersolvable, R-group if and only if G is a finitely generated, torsion-free, nilpotent group yields a number of interesting corollaries. Examples of polycyclic R-groups which are neither nilpotent nor orderable conclude this chapter.

In Chapter III, we give counterexamples of various erroneous assertions appearing in the literature and present substitute theorems for these false claims. We also prove that the group of o-automorphisms of a polycyclic O-group is nilpotent by abelian and polycyclic.

In Chapter IV, the omnipresent condition of the local satisfaction of the maximal condition joins forces with yet another finiteness condition, viz., the condition that the O-groups under consideration admit only finitely many different orders. Such groups are shown to be locally polycyclic; moreover, if such a group is nonabelian, then the Fitting subgroup exists, is absolutely convex, and coincides with the isolator of the derived group of the given group. We end Chapter IV by demonstrating that a torsion-free, nonabelian, locally nilpotent group admits infinitely many different orders.

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TABLE OF CONTENTS

CHAPTER 0

PRELIMINARIES

A group G on which there can be defined a relation $\leq$ which is reflexive, antisymmetric, and transitive and which has the additional property that $a,b,x,y \in G$ and $a \leq b$ imply $xay \leq xby$ is said to be a partially ordered group, $\leq$ is said to be a partial order on G. and G is said to be partially ordered (with respect to $\leq$). Associated with each partial order $\leq$ on a group G is the positive cone P(G) of G, where $P(G) = \{x | x \in G \text{ and } 1 \leq x\}$. It follows easily that the subset P(G) of the partially ordered group G possesses the following properties:

(i) $P(G) \cap P^{-1}(G) = \{1\}$, where $P^{-1}(G) = \{x^{-1} | x \in P(G)\}$;

(ii) $P(G)P(G) \subseteq P(G)$; and

(iii) $x^{-1}P(G)x \subseteq P(G)$ for each $x \in G$.

In other words, P(G) is a normal subsemigroup of G, containing no element together with its inverse other than the identity 1 of G.

Conversely, if G is a group possessing a subset A which satisfies properties (i) - (iii), then G is a partially ordered group with positive cone A with respect to the relation $\leq$ given by

$$a \leq b \text{ if and only if } a^{-1}b \in A.$$

A group G which is partially ordered with respect to $\leq$ is an ordered group if and only if $\leq$ is complete in the sense that $a,b \in G$ imply $a \leq b$ or $b \leq a$. This is easily seen to be equivalent to the assertion that G possesses a subset P(G) with properties (i) - (iii) and, in addition, $P(G) \cup P^{-1}(G) = G$. If G is an ordered group with respect to $\leq$, we say that $\leq$ is an order on G, that G is ordered (with respect to $\leq$), and that G is <u>orderable</u> (with respect to $\leq$). Following Neumann [14], we shall denote the class of orderable groups by "0" and shall say that <u>G is an 0-group</u> if and only if $G \in 0$. It is easily seen that any (partial) order on G is uniquely determined by its positive cone, so that, for the sake of brevity, we shall say "the (partial) order $P(G)$ on G", instead of "the (partial) order on G with positive cone $P(G)$."

A subgroup H of a group G ordered with respect to $P(G)$ is itself an ordered group with respect to the induced order $P(H) = P(G) \cap H$ on H.

A subgroup C of a group G ordered with respect to $\leq$ is <u>convex</u> (with respect to $\leq$) if and only if $x \in G$, $c \in C$, and $1 \leq x \leq c$ imply $x \in C$, whereas a subgroup A of G is <u>absolutely convex</u> if and only if A is convex with respect to each order on G. If $G \in 0$ and $A \subseteq G$, then the <u>convex subgroup of G generated by A</u> is the intersection of all convex subgroups of G containing A. If A is a subgroup of G, then it follows that the convex subgroup of G generated by A is $P(G)A \cap P^{-1}(G)A$.

Suppose now that G is ordered with respect to $\leq$ and that N is a normal convex subgroup of G. The factor group G/N can be ordered by the relation, which we also denote by $\leq$, given by

$$aN \leq bN \text{ if and only if } a \leq bn \text{ for some } n \in N:$$

It is clear that $a \leq a \cdot 1$, whence $aN \leq aN$; if $aN \leq bN$ and $bN \leq aN$, then $a \leq bn_1$ and $b \leq an_2$ for some $n_1, n_2 \in N$. Thus, $(a^{-1}b)^{-1} \leq n_1$ and $a^{-1}b \leq n_2$, while $1 \leq (a^{-1}b)^{-1}$ or $1 \leq a^{-1}b$ as $\leq$ is an order on G. By the convexity of N, $a^{-1}b \in N$, whence $aN = bN$; if $aN \leq bN$ and $bN \leq cN$, then $a \leq bn_1$ and $b \leq cn_2$ for some $n_1, n_2 \in N$, so

$a \leq c(n_2 n_1)$. Thus, $aN \leq cN$; finally, if $a, b \in G$, then $a \leq bn$ for some $n \in N$ or $b < an$ for all $n \in N$. Thus, $aN \leq bN$ or $bN \leq aN$. The order on G/N defined above is called the induced order on G/N by G.

A result of Levi [11] which is relevant to this discussion and which shall later prove useful is

Theorem 0.1: Suppose N is a normal subgroup of a group G and that both N and G/N are orderable. Then there exists an order on G inducing the orders on N and G/N if and only if $P(N)$ is invariant under the inner automorphisms of G.

Proof: If $P(G)$ is an order on G inducing a given order $P(N)$ on N, then $P(G) \cap N = P(N)$, from which the invariance of $P(N)$ follows as a consequence of the invariance of $P(G)$ and N.

On the other hand, if $P(N)$ and $P(G/N)$ denote orders on N and G/N, respectively, and if $P(N)$ is invariant under conjugation by elements of G, then it is easily verified that $P(G) = P(N) \cup \{x | x \in G-N \text{ and } xN \in P(G/N)\}$ defines an order on G inducing the orders $P(N)$ and $P(G/N)$ on N and G/N, respectively.

This line of thought has led us to the salient and useful

Theorem 0.2: If $G \in 0$ and H is a normal subgroup of G, then there exists an order on G with respect to which H is convex if and only if $G/H \in 0$.

Proof: Clearly, if there exists an order on G with respect to which H is convex, then G/H is ordered with respect to the order induced on G/H by G.

Conversely, if $P(G/H)$ denotes an order on G/H, then $P_H(G) =$

$(P(G) \cap H) \cup \{x | x \in G-H$ and $xH \in P(G/H)\}$ is easily shown to be an order on G with respect to which H is convex.

If $\leq$ is an order on a group G, then $\leq$ is an Archimedean order on G and G is an Archimedean ordered group (with respect to $\leq$) if and only if $a,b \in G$, $1 < a$, $1 < b$ imply the existence of a positive integer n such that $b \leq a^n$. An interesting observation to be immediately made is that an Archimedean ordered group is "convex simple" in the sense that such a group contains no proper, nontrivial, convex subgroups.

If G and H are O-groups and f is a mapping of G into H, then f is an o-homomorphism of G into H if and only if f is a group homomorphism of G into H and f is order-preserving in the sense that $a,b \in G$ and $a \leq_1 b$ imply $f(a) \leq_2 f(b)$, where $\leq_1$ and $\leq_2$ denote the orders on G and H, respectively. Furthermore, if f is a one-to-one o-homomorphism of G onto H and if f^{-1} is an o-homomorphism of H onto G, then f is an o-isomorphism of G onto H. Two ordered groups, say G and H, are o-isomorphic if and only if there exists an o-isomorphism of G onto H.

At this time, it is now possible to state a well known, indispensable result of Holder [7], a proof of which can also be found in [2], pp. 45-46:

Theorem 0.3: A group G is an Archimedean ordered group if and only if G is o-isomorphic to a subgroup of the additive group of real numbers with the natural ordering.

As an immediate consequence of Theorem 0.3, we see that any archimedean ordered group is abelian.

Another result of interest and utility is due to Hion, the proof given here being essentially that given in [2], pp. 46-47:

Theorem 0.4: Suppose $A \neq \{0\}$ and B are subgroups of the additive group of real numbers, endowed with the natural ordering, and suppose f is an o-homomorphism of A into B. Then there exists a nonnegative real number r such that $f(a) = ar$ for each $a \in A$.

Proof: Let us first assume there exists $a_o \in A$ such that $0 < a_o$ and such that $f(a_o) = 0$. Let $a \in A$ such that $0 < a$. By the Archimedean property for the reals, there exists a positive integer n such that $0 < a \leq na_o$, whence $0 \leq f(a) \leq nf(a_o) = 0$. Thus, $f(a) = 0$ for all $0 < a \in A$. In this case, therefore, $f(a) = 0$ for all $a \in A$, so $r = 0$.

Let us now assume that $0 < a \in A$ implies $f(a) \neq 0$. Let $0 < a_1$, $0 < a_2$, where $a_1, a_2 \in A$. Suppose, by way of a contradiction, that $f(a_1)/f(a_2) \neq a_1/a_2$. Without loss of generality, assume that $f(a_1)/f(a_2) < a_1/a_2$. Let m/n be a rational number, with $0 < m$ and $0 < n$, chosen so that $f(a_1)/f(a_2) < m/n < a_1/a_2$. Then $ma_2 < na_1$ and $nf(a_1) = f(na_1) < mf(a_2) = f(ma_2)$, which is impossible, since f is order-preserving. Therefore, $f(a_1)/f(a_2) = a_1/a_2$; i.e., $f(a)/a$, $a \in A$, is constant for $0 < a \in A$. Thus, $f(a) = ar$ for some real number r, where $a \in A$. Clearly, r is nonnegative.

When Theorem 0.4 is applied to the group of o-automorphisms of an Archimedean ordered group, it -- together with Theorem 0.3 -- yields the following interesting and useful

Corollary 0.5: The o-automorphisms of an Archimedean ordered group form an abelian group which is isomorphic to a subgroup of the multiplicative group of positive real numbers.

In other words, an o-automorphism of an Archimedean ordered group is essentially nothing more than multiplication by a positive real number.

Needless to say, the class of torsion-free groups is an immensely wide class of groups. One method of studying this monstrosity is, of course, to study certain of its subclasses. In particular, we shall be most interested in that class of torsion-free groups known as the class of orderable groups. Many prominent mathematicians have found the study of 0-groups an interesting study of torsion-free groups; conspicuously among those earliest investigators of 0-groups was B. H. Neumann, for it was Neumann who first proved that the property of being an 0-group is a property of finite character; that is, if we agree to say that a group G <u>possesses a property P locally</u> if and only if each finitely generated subgroup of G possesses property P, then Neumann was first to show that G is an 0-group if and only if G is an 0-group locally (see [15]). An application of Neumann's result to abelian groups renders transparent the fact that a nontrivial, abelian group G is an 0-group if and only if G is torsion-free. As is often the case, Neumann's result can be drawn as an easy corollary to a later theorem of Los [12] and Ohnishi [17]:

<u>Theorem 0.6</u>: A group G is an 0-group if and only if for every finite set $a_1, a_2, \ldots, a_n$, $1 \neq a_i$ for $i = 1,2,\ldots,n$, of elements of G, the signs $\mathcal{E}_i = \pm 1$, $i = 1,2,\ldots,n$, can be chosen so that $1 \notin S(a_1^{\mathcal{E}_1}, a_2^{\mathcal{E}_2}, \ldots, a_n^{\mathcal{E}_n})$, where $S(a_1^{\mathcal{E}_1}, a_2^{\mathcal{E}_2}, \ldots, a_n^{\mathcal{E}_n})$ denotes the normal subsemigroup of G generated by $\{a_1^{\mathcal{E}_1}, a_2^{\mathcal{E}_2}, \ldots, a_n^{\mathcal{E}_n}\}$. Fuchs (see [2], pp. 34-35) has demonstrated the fundamentally important role played by the normal subsemigroups of orderable groups.

Again, it was Neumann [14] who was first to prove the important property that if $G \in 0$, $a,b \in G$, and $a^n b^m = b^m a^n$ for some nonzero integers m and n, then $ab = ba$. However, both results of Neumann listed above and, in general, many of the properties of 0-groups are enjoyed by even a wider class of torsion-free groups known as "R-groups", a class which has received the careful attention of Kontorovic (see [8] and [9]) and Plotkin (see [18] and [19]): A group G is an R-group if and only if $a,b \in G$, n a positive integer and $a^n = b^n$ imply $a = b$. It is clear that an (nontrivial) R-group is torsion-free, since $a \in G$ and $1^n = a^n$ imply $1 = a$. Even of greater significance to us is the fact that each 0-group is an R-group: For, if $a,b \in G$ and $a < b$, then $a^n < b^n$ for all positive integers n. R-groups will prove to be of interest in that which subsequently follows; therefore, we document here some of their most important properties, following a paper of Kontorovic [8].

An important concept in the study of certain classes of torsion-free groups is that of "isolated subgroups:" A subgroup A of a group G is an isolated subgroup of G if and only if $g \in G$, n a positive integer, and $g^n \in A$ imply $g \in A$. It follows readily that the intersection of isolated subgroups is isolated. The isolator of a subset S of a group G is the intersection of all isolated subgroups of G containing S. Isolated subgroups are familiar entities in 0-groups, for if G is ordered with respect to $\leq$, if C is a convex subgroup of G, and if $g \in G$ such that $g^n \in C$ for some positive integer n, then, without loss of generality, we may assume $1 \leq g$, whence $1 \leq g \leq g^n$; therefore, by the convexity of C, $g \in C$. In other words, any convex subgroup of an ordered group is an isolated subgroup of the group. If G is a group and S is a subset of G, let us denote the centralizer of S in G by $C_G(S)$. We can now prove

Theorem 0.7: If G is an R-group and $S \subseteq G$, then $C_G(S)$ is an isolated subgroup of G.

Proof: Let n be a positive integer and let $g \varepsilon G$ such that $g^n \varepsilon C_G(S)$. Then $s^{-1}g^n s = g^n$ for each $s \varepsilon S$. Thus, $(s^{-1}gs)^n = g^n$ for each $s \varepsilon S$, whence, since G is an R-group, $s^{-1}gs = g$ for each $s \varepsilon S$. Therefore, $g \varepsilon C_G(S)$.

Corollary 0.8: The center $Z(G)$ of an R-group is an isolated subgroup of G.

Proof: Take $S = G$ in Theorem 0.7.

Theorem 0.9. If G is an R-group, if $a,b \varepsilon G$, and if $[a^n,b^m] = 1$ for some nonzero integers n and m, then $[a,b] = 1$.

Proof: It suffices to prove the assertion for $m = 1$. Now, $[a^n,b] = 1$ implies $(b^{-1}ab)^n = b^{-1}a^n b = a^n$. Since, by hypothesis, G is an R-group, $b^{-1}ab = a$, whence $[a,b] = 1$.

Theorem 0.10: A torsion-free group G is an R-group if and only if $G/Z(G)$ is an R-group.

Proof: Suppose G is an R-group and let $a,b \varepsilon G$ such that $(aZ(G))^n = (bZ(G))^n$ for some positive integer n. Then $a^n = b^n z$ for some $z \varepsilon Z(G)$, whence $[a^n,b^n] = 1$. Thus, by Theorem 0.9, $ab = ba$. But $a^n = b^n z$, so $(ab^{-1})^n = z \varepsilon Z(G)$. As $Z(G)$ is isolated, $ab^{-1} \varepsilon Z(G)$, so $aZ(G) = bZ(G)$. Therefore, $G/Z(G)$ is an R-group.

Suppose now that $G/Z(G)$ is an R-group and that $a,b \varepsilon G$ such that $a^n = b^n$ for some positive integer n. Then $(aZ(G))^n = (bZ(G))^n$, so that $aZ(G) = bZ(G)$. Thus, $a = bz$ for some $z \varepsilon Z(G)$. But, $a^n = b^n$, so that $1 = z^n$. As G is torsion-free, $z = 1$, whence $a = b$. Therefore, G is an R-group.

The chain $\{1\} = Z_0 \subseteq Z_1 \subseteq \ldots \subseteq Z_i \subseteq \ldots$ of subgroups of a group

G, where $Z_{i+1}/Z_i = Z(G/Z_i)$ for $i = 0, 1, 2, \ldots,$ is called the upper central chain of G. Using Theorem 0.10, an easy inductive argument establishes

Theorem 0.11: If G is an R-group, then each term of the upper central chain, $\{1\} = Z_0 \subseteq Z_1 \subseteq Z_2 \subseteq \cdots \subseteq Z_i \subseteq \ldots,$ of G is isolated in G, each of the factors Z_{i+1}/Z_i of the upper central chain is a torsion-free, abelian group, and each of the factors Z_{i+1}/Z_i is an R-group.

Theorem 0.12: If G is an R-group and A is a maximal abelian subgroup of G, then A is an isolated subgroup of G.

Proof: Suppose $g \varepsilon G$, n is a positive integer, and $g^n \varepsilon A$. Then $[g^n,a] = 1$ for $a \varepsilon A$, whence, by Theorem 0.9, $[g,a] = 1$ for all $a \varepsilon A$. Thus, $< g,A >$ is an abelian subgroup of G and, if $g \notin A$, A would be properly contained in $< g,A >$; however, by the maximality of A, this is not possible. Thus, $g \varepsilon A$.

To conclude our discussion of R-groups, we prove

Theorem 0.13: If G is an R-group and $1 \neq x \varepsilon G$, then the isolator I(x) of x in G is a torsion-free, locally cyclic subgroup of G.

Proof: Since G is torsion-free and $1 \neq x \varepsilon G$, $< x >$ is a torsion-free cyclic -- thus, locally cyclic -- subgroup of G. Also, the union of an ascending chain of torsion-free, locally cyclic groups is a torsion-free, locally cyclic group. Therefore, x is contained in some maximal torsion-free, locally cyclic subgroup A of G. Let $a \varepsilon A$. Then $x,a \varepsilon A$, so $< x,a > = < y >$ for some $y \varepsilon A$. Therefore, $x = y^m$, $a = y^n$, and, hence, $a^m = x^n \varepsilon < x > \subseteq I(x)$. Since I(x) is isolated, $a \varepsilon I(x)$, whence $A \subseteq I(x)$.

By reasoning entirely analogous to that above, x is also contained in some maximal abelian subgroup B of G. By Theorem 0.12, B is isolated in G and $x \in B$, whence $I(x) \subseteq B$. Moreover, since B is isolated in G, the isolator $I_B(x)$ of x in B contains the isolator $I(x)$ of x in G. But B is a torsion-free, abelian group and the isolator of any nonidentity element of such a group is a torsion-free, locally cyclic group (see [10], p. 210). Thus, $I(x) \subseteq I_B(x)$ and $I_B(x)$ is locally cyclic, whence $I(x)$ is a torsion-free, locally cyclic subgroup of G. As $A \subseteq I(x)$ and as A is a maximal torsion-free, locally cyclic group, $A = I(x)$.

An interesting subclass of the class of R-groups is the class of "R*-groups": A group G is an R*-group if and only if $a, b, x_1, x_2, \ldots, x_n \in G$ and $a^{x_1}a^{x_2}\ldots a^{x_n} = b^{x_1}b^{x_2}\ldots b^{x_n}$ imply $a = b$, where by a^x, $x \in G$, we mean $x^{-1}ax$. R*-groups have been studies by Plotkin [18], and Fuchs [3] has used the concept to pose an intriguing, unresolved problem: Is every R*-group an 0-group? It is known that not all R-groups are 0-groups (see, for example, Chapter II). According to Fuchs [3], a group G is <u>generalized torsion-free</u> if and only if $a \in G$ and $1 \in S(a)$ imply $a = 1$. It follows easily (see [3]) that a group is an R*-group if and only if G is generalized torsion-free. It is interesting to observe that any ordered group G is an R*-group: For, if G is ordered with respect to $\leq$ and $a \in G$ such that $1 < a$, then $a^g > 1$ for each $g \in G$. Thus any finite product of conjugates of a is greater than 1.

Prior to undertaking the final topic of this chapter, we digress here to discuss an important subclass of 0-groups known as "0*-groups:" A group G is an <u>0*-group</u> if and only if each partial order on G can be extended to an order on G. An 0*-group is, therefore, characterized by the property that each maximal partial order on G is an order on G.

An 0*-group G is easily seen to be an 0-group, for $\{1\} = P(G)$ is a partial order on G, whence $\{1\}$ can be extended to an order on G. However, only recently has it been shown that not all 0-groups are 0*-groups (see [4]). As in the case for 0-groups, let us note that a group G is an 0*-group if G is an 0*-group locally, a fact manifested by a result of Ohnishi [16]: A group G is an 0*-group if and only if (i) $a,b,c \in G$ and $b,c \in S(a)$ imply $S(b) \cap S(c) \neq \emptyset$ and (ii) G is an R*-group.

Numerous unresolved problems concerning 0*-groups are posed by Fuchs (see [2], pp. 209-213), one of the most interesting being, "Are subgroups of 0*-groups, 0*-groups?" An affirmative answer to this question is given in this manuscript for the case when the 0*-group is locally supersolvable, this result being a consequence of the fact that a torsion-free, locally nilpotent group is an 0*-group (see [3]). We now proceed with a discussion of the final topic of this chapter -- "generalized solvable groups."

Since solvable groups form such a wide generalization of abelian groups, it is not surprising that but a few of the nontrivial properties of abelian groups can be carried over to solvable groups. More interesting in this respect are certain classes of groups intermediate between the classes of abelian and solvable groups -- nilpotent, supersolvable, and polycyclic groups.

By a <u>normal</u> (<u>invariant</u>) <u>series</u> of a group G is meant a finite chain $\{1\} = A_0 \subseteq A_1 \subseteq \cdots \subseteq A_{n-1} \subseteq A_n = G$ of subgroups of G such that A_i is normal in A_{i+1} (G) for $i = 0, 1, \ldots, n-1$. An invariant series $\{1\} = A_0 \subseteq A_1 \subseteq \cdots \subseteq A_n = G$ of a group G is a <u>central series</u> of G if and only if $A_{i+1}/A_i \subseteq Z(G/A_i)$, $i = 0, 1, \ldots, n-1$. A normal (invariant) series $\{1\} = A_0 \subseteq A_1 \subseteq \cdots \subseteq A_n = G$ of a group G is a cyclic normal (invariant) series of G if and only if A_{i+1}/A_i is cyclic for each $i = 0, 1, \ldots, n-1$. A group G is nilpotent if and only if G possesses a cyclic invariant series; finally, G is polycyclic if and only if G possesses a cyclic normal series. By the length of a polycyclic group G with a cyclic normal series $\{1\} = A_0 \subseteq A_1 \subseteq \cdots \subseteq A_n = G$ is meant the number of infinite cyclic factors A_{i+1}/A_i of the given cyclic normal series. It is not difficult to prove that the length of a polycyclic group is an invariant of that group. It is well known that a polycyclic group satisfies the maximal condition for subgroups and that a finitely generated nilpotent group is supersolvable, whence any member of any one of the three classes defined immediately above satisfies the maximal condition for subgroups locally. The importance and relevance of these three classes of groups is manifested by the knowledge that it is precisely the subsequent investigations of those ordered groups which satisfy the maximal condition locally that constitute the bulk of this manuscript.

Just as the desire to generalize the concept of an abelian group ultimately led to the development of new and important classes of solvable groups, so too has the desire to extend the notion of solvable group led to a corresponding development of various classes of generalized solvable groups. The generalizations which have apparently been most fruitful are those which extend the concept of a normal series to that of a "normal system" of a group. In a brillant paper of Cernikov and Kuros [1] -- which we follow here -- the notion of a normal system is deftly employed to generate numerous important classes of generalized solvable groups.

If G is a group and Σ is a family of subgroups of G which is a chain with respect to set theoretical inclusion, then a <u>jump</u> in Σ, denoted by $A \prec B$, is a pair A,B of elements of Σ such that A is a proper subset of B and no element of Σ lies strictly between A and B. A family Σ of subgroups of a group G which contains $\{1\}$ and G, which is a chain with respect to set inclusion, and which is closed with respect to arbitrary unions and intersections is a <u>normal system</u> for G if and only if for each jump $A \prec B$ in Σ, A is a normal subgroup of B. A normal system Σ is an <u>invariant system</u> for G if and only if $A \in \Sigma$ implies A is a normal subgroup of G. An invariant system Σ of G is a <u>central system</u> for G if and only if $B/A \subseteq Z(G/A)$ for each jump $A \prec B$ in Σ. Naturally, therefore, a group is a <u>generalized solvable</u> group if and only if it contains a normal system Σ for which B/A is abelian for each jump $A \prec B$ in Σ. A group G is a <u>Z-group</u> if and only if G possesses a central system. Z-groups are, therefore, generalized nilpotent groups. Finally, a group G is a <u>ZD-group</u> if and only if G possesses a descending, well-ordered central system. The remarkable, underlying fact which makes all this discussion of generalized solvable groups immediately relevant to 0-groups is

<u>Theorem 0.14</u>: Every 0-group is a generalized solvable group. Moreover, if G is a group ordered with respect to $\leq$ and Σ is the family of subgroups of G which are convex with respect to $\leq$, then

(i) $\{1\}, G \in \Sigma$, Σ is a chain with respect to set inclusion, and Σ is closed with respect to arbitrary unions and intersections;

(ii) $C \in \Sigma$ and $g \in G$ imply $C^g = g^{-1}Cg \in \Sigma$;

(iii) If $D \prec C$ is a jump in Σ, then D is a normal subgroup

C and C/D is isomorphic to a subgroup of the additive group of real numbers;

(iv) If $D \prec C$ is a jump in Σ, then $[N_G(D), N_G(D), C] \subseteq D$, where $N_G(D)$ denotes the normalizer of D in G and $[N_G(D), N_G(D), C] = \langle [x,y,z] \mid x,y \in N_G(D)$ and $z \in C \rangle$;

(v) $C \in \Sigma$, $a \in G$, and $S(a) \cap C \neq \phi$ imply some conjugate of a belongs to C.

Proof: (i) Clearly, $\{1\}, G \in \Sigma$. Now suppose $A,B \in \Sigma$ and $A \not\subseteq B$. Then there exists $a \in A-B$. Without loss of generality, let us assume that $1 < a$. If $b \in B$, then $b < a$; otherwise, $1 < a \leq b$ implies, by the convexity of B, that $a \in B$. Thus, by the convexity of $A, B \subseteq A$. Thus, Σ is a chain with respect to set inclusion. The remaining assertions of (i) follow easily.

(ii) Suppose $c \in C$, $x,g \in G$, and $1 \leq x \leq c^g$. Then $1 \leq gxg^{-1} \leq c$ and, by the convexity of C, $gxg^{-1} \in C$; i.e., $x \in C^g$. Therefore, $C^g \in \Sigma$.

(iii) If $D \prec C$ is a jump in Σ and $g \in G$, then D^g is a proper subgroup of C^g and, if $K \in \Sigma$ such that $D^g \subset K \subset C^g$, then $D \subset gKg^{-1} \subset C$, which is impossible as $gKg^{-1} \in \Sigma$ and as $D \prec C$ is a jump in Σ. Thus, if $D \prec C$ is a jump in Σ and $g \in G$, then $D^g \prec C^g$ is also a jump in Σ. Now suppose $D \prec C$ is a jump in Σ and $x \in N_G(D)$. Then $D = D^x \prec C^x$ is a jump in Σ, and Σ is a chain with respect to set inclusion, whence $C^x \subseteq C$ or $C \subseteq C^x$. If $C^x \subseteq C$, then $D \subset C^x \subseteq C$, so that $C^x = C$. Similarly, if $C \subseteq C^x$, then we again find $C^x = C$. An analogous argument shows $N_G(C) \subseteq N_G(D)$. Therefore, if $D \prec C$ is a jump in Σ, then $N_G(D) = N_G(C)$. We have, therefore, that the normality of either member of a jump in Σ guanantees the normality in G of the other.

The factor group C/D is an ordered group with respect to the order induced on C/D by G. Since $D \prec C$, C/D can contain no nontrivial, proper, convex subgroups. But any ordered group containing no nontrivial, proper, convex subgroups is an Archimedean ordered group (see [2], p. 47). Application of Theorem 0.3 completed the proof of (iii).

(iv) Since $N_G(D) = N_G(C)$, each element $a \in N_G(D)$ induces an o-automorphism $\overline{a}$ on C/D given by

$$(Dc)^{\overline{a}} = Dc^a.$$

Thus, if $a,b \in N_G(D)$, then, by Corollary 0.3, $\overline{a}\,\overline{b} = \overline{b}\,\overline{a}$; i.e., $(Dc)^{\overline{a}\,\overline{b}} = (Dc)^{\overline{b}\,\overline{a}}$, so $Db^{-1}a^{-1}cba$, whence $[a^{-1}, b^{-1}, c^{-1}]$ $= bab^{-1}a^{-1}caba^{-1}b^{-1}c^{-1} \in D$ for $a,b \in N_G(D)$, $c \in C$.

(v) Let $a \in G$, $C \in \Sigma$, and suppose, without loss of generality, that $1 < a$. If no conjugate of a belongs to C, then, by the convexity of C, $c < a^g$ for each $g \in G$ and each $c \in C$. Thus, any finite product of conjugates of a is greater than each element of C, whence $S(a) \cap C = \phi$. To conclude this first chapter, we mention now the important theorem of Podderyugin and Rieger, the statement and proof of which can be found in [2], pp. 51 - 52:

<u>Theorem 0.15</u>: A group G is an 0-group if and only if G possesses a family Σ of subgroups satisfying conditions (i) - (v) of Theorem 0.14. Furthermore, if Σ is a family of subgroups of G satisfying these five conditions, then there can be defined an order on G with respect to which each element of Σ is convex.

CHAPTER I

TORSION-FREE, LOCALLY NILPOTENT GROUPS

The literature devoted to the study of torsion-free, locally nilpotent groups is of gigantic proportions. Even so, these groups continue to command the continuing interests of many noteworthy mathematicians, and the existing knowledge of torsion-free, locally nilpotent groups is apparently far from complete. To those primarily interested in 0-groups, the torsion-free, locally nilpotent groups are of extreme importance, for each such group is an example not only of an 0-group, but even an 0*-group. History readily verifies that the concept of isolated subgroup, which arises naturally in the study of 0-groups, has proved to be an invaluable tool in many investigations of torsion-free, locally nilpotent groups. It would, therefore, seem reasonable to attempt to broaden the existing body of knowledge of torsion-free, locally nilpotent groups by investigating these groups as 0- or 0*-groups. One of the many interesting and unresolved problems posed by Fuchs (see [2], Problem 9 (a), p. 209), which is relevant to this line of thought, is the determination of those subgroups of an 0-group G which appear as convex subgroups with respect to some order on G. In the case when G satisfies the maximal condition for subgroups locally, it is not too difficult to now resolve this problem. We begin by establishing a result which has application to a class of groups even wider than the class of torsion-free, locally nilpotent groups.

<u>Lemma 1.1</u>: If G is an ordered group satisfying the maximal condition for subgroups locally, then each convex subgroup of G is normal

in G.

Proof: Suppose C is a convex subgroup of G, $g \in G$, and $c \in C$. Let $K = \langle g, c \rangle$. Then K is an ordered group satisfying the maximal condition for subgroups, and $K \cap C$ is a convex subgroup of K with respect to the order $P(K) = P(G) \cap K$ induced by G on K. If $K \cap C$ is not normal in K and $k \in K$ such that $k^{-1}(K \cap C)k \neq K \cap C$, then, without loss of generality, we may assume that $K \cap C \subset k^{-1}(K \cap C)k$, since $K \cap C$ and $k^{-1}(K \cap C)k$ are both convex in K and the convex subgroups of K form a chain with respect to set inclusion. Thus, $K \cap C \subset k^{-1}(K \cap C)k \subset k^{-2}(K \cap C)k^2 \subset k^{-3}(K \cap C)k^3 \subset \ldots\ldots$ is an infinite ascending chain of subgroups of K, contrary to the fact that K satisfies the maximal condition for subgroups. Therefore, $K \cap C$ is normal in K, whence, as $g \in K$ and $c \in K \cap C$, $c^g \in K \cap C \subseteq C$. Thus, C is normal in G.

Corollary 1.2: If G is an ordered group satisfying the maximal condition for subgroups locally and if H is a subgroup of G, then there exists an order on G with respect to which H is convex if and only if H is normal in G and $G/H \in 0$.

Proof: The convex subgroups of G are normal by Lemma 1.1. An appeal to Theorem 0.2 completes the proof.

Corollary 1.3: If G is a torsion-free, locally nilpotent group, then each convex subgroup of G is normal in G. Furthermore, if H is a subgroup of G, then necessary and sufficient conditions that G admits an order with respect to which H is convex are that H be normal and isolated in G.

Proof: Since G is torsion-free, locally nilpotent, $G \in 0$; G also satisfies the maximal condition for subgroups locally, whence each convex subgroup of G is normal in G.

If H is a subgroup of G which is convex with respect to some order on G, then we have seen that H is isolated in G.

Now, suppose that H is a normal, isolated subgroup of G. Then G/H is a torsion-free, locally nilpotent group, whence $G/H \in 0$. Theorem 0.2 guarantees that there exists an order on G with respect to which H is convex.

If $G' = [G,G]$ is the derived group of G, then, as an immediate consequence of Lemma 1.1 and condition (iv) of Theorem 0.14, we have

Corollary 1.4: If G is an ordered group satisfying the maximal condition for subgroups locally, then $[G',C] \subseteq D$ for each jump $D \prec C$ in the family of convex subgroups of G with respect to the given order.

Proof: As D,C are normal in G, condition (iv) of Theorem 0.14 is $[G,G,C] \subseteq D$; i.e., $[G',C] \subseteq D$.

As we shall soon see, the conclusion of Corollary 1.4 can be strengthened to read $[G,C] \subseteq D$ by requiring that G be torsion-free, locally nilpotent. To prepare for the demonstration of the truth of this assertion, we prove

Lemma 1.5: Suppose G is an ordered group, $D \prec C$ is a jump in the family of convex subgroups of G, either D or C is normal in G, $D \neq Dc_o \in C/D$, and $Dc_o^g = Dc_o$ for some $g \in G$. Then $Dc^g = Dc$ for all $c \in C$.

Proof: Let $\bar{C} = C/D$. By (iii) of Theorem 0.14, $\bar{C}$ is o-isomorphic to a subgroup of the additive group of real numbers. It follows easily that the mapping $\bar{c} \to (\bar{c})^g$ (i.e., $Dc \to Dc^g$) is an o-isomorphism of $\bar{C}$ onto $\bar{C}$, where we note that, by the normality of either D or C, both D and C are normal in G. By Theorem 0.4, this mapping is, therefore, given by $\bar{c} \to ct$, where t is a positive real constant. Thus, $\bar{c}_o \to c_o t$. But, by hypothesis, $\bar{c}_o = c_o t$, whence $t = 1$. Thus, $Dc^g = Dc$ for all $c \in C$.

We are now ready to prove

<u>Theorem 1.6</u>: If G is a torsion-free, locally nilpotent group and Σ is the family of convex subgroups of G with respect to some order on G, then Σ is a central system of G.

<u>Proof</u>: By Theorem 0.14 and Corollary 1.3, Σ is an invariant system of G. It remains only to prove that $C/D \subseteq Z(G/D)$ for each jump $D \prec C$ in Σ. Let $\bar{G} = G/D$, $\bar{C} = C/D$, $\bar{g} \in \bar{G}$, $\bar{1} \neq \bar{c} \in \bar{C}$, and let $\bar{K} = \langle \bar{g},\bar{c} \rangle \cap \bar{C}$. Then $\{1\} \neq \bar{K}$, $\langle \bar{g},\bar{c} \rangle$ is nilpotent, and $\bar{K}$ is normal in $\langle \bar{g},\bar{c} \rangle$, whence $\bar{K} \cap Z(\langle \bar{g},\bar{c} \rangle) \neq \{1\}$; i.e., there exists $\bar{k} \in \bar{K} \cap Z(\langle \bar{g},\bar{c} \rangle)$ such that $\bar{k} \neq \bar{1}$. Thus, there exists $Dk \in C/D$ such that $D \neq Dk$ and $Dk^g = Dk$. Lemma 1.5 guarantees that $Dc^g = Dc$ for all $c \in C$, whence $[c,g] \in D$ for each $c \in C$ and each $g \in G$. Therefore, $[G,C] \subseteq D$; i.e., $C/D \subseteq Z(G/D)$.

By Theorems 0.14 and 1.6, necessary conditions that a family Σ of subgroups of a torsion-free, locally nilpotent group be the family of convex subgroups of G with respect to some order on G are that Σ be a central system of G and that C/D be isomorphic to a subgroup of the additive group of real numbers for each jump $D \prec C$ in Σ. While it is true that the five conditions of Theorem 0.14 are sufficient conditions on a family Σ of subgroups of any group G to guarantee that G can be ordered and that each member of Σ be convex with respect to this forementioned order, these five conditions--as was suggested by Fuchs (see [2], p. 52)--are not sufficient to ensure that Σ coincides with the family of <u>all</u> subgroups of G which are convex with respect to this order; i.e., it is possible that certain subgroups of G, not in Σ, will appear as convex subgroups with respect to the order on G which is constructed by the utilization of Σ (see the proof of Theorem 11 in [2], pp. 51 - 52). That this situation can actually arise has been demonstrated by Graham [5], who has an example to show that the five conditions of Theorem 0.14 do not characterize the families of convex subgroups of 0-groups. We mention here that Graham [5] has also obtained independent proofs of part of Corollary 1.3, and Theorems 1.6 and 1.7.

If however, we reimpose the condition that G be torsion-free, locally nilpotent, then the two necessary conditions stated above are sufficient to guarantee that an order can be defined on G with respect to which only those elements of Σ will be convex. We state this result as

Theorem 1.7: If G is torsion-free, locally nilpotent and Σ is a family of subgroups of G such that

(i) Σ is a central system of G, and

(ii) C/D is isomorphic to a subgroup of the additive group of real numbers for each jump $D \prec C$ in Σ,

then Σ is the family of all subgroups of G which are convex with respect to some order on G.

Proof: The proof will parallel that of Theorem 0.15 found in [2], pp. 51 - 52 with only slight modifications (which are made possible by the normality of the elements of Σ and by the fact that $[G,C] \subseteq D$ for each jump $D \prec C$ in Σ).

If $D \prec C$ is a jump in Σ, then, by hypothesis, C/D is a subgroup of the additive group of real numbers and, thus, there is defined on C/D a natural Archimedean order which we shall denote by $P(C,D)$. We are now ready to define an order on G: For $1 \neq x \in G$, let $D \prec C$ be the jump in Σ determined by x (i.e., D is the union of all elements of Σ not containing x, while C is the intersection of all elements of Σ containing x. It follows readily that $D, C \in \Sigma$, that $D \prec C$ is a jump in Σ, and that $x \in C-D$). Then $x \in P(G)$ if and only if $xD \in P(C,D)$ in C/D. We now verify that $P(G)$ is an order on G: (i) If $1 \neq g \in P(G) \cap P^{-1}(G)$, then both gD and $g^{-1}D = (gD)^{-1}$ belong to $P(C,D)$, where $D \prec C$ is the jump determined by g. As $P(C,D)$ is an order on C/D, $gD = D$, whence $g \in D$. But this is impossible as $g \in C-D$. Thus, $P(G) \cap P^{-1}(G) = \{1\}$; (ii) now suppose $a,b \in P(G)$ and $D_1 \prec C_1$, $D_2 \prec C_2$ are the jumps in Σ determined by a and b, respectively. Since Σ is a chain with respect to set inclusion, let us assume $C_1 \subseteq C_2$. If $C_1 = C_2$, then $D_1 = D_2$ and, as $P(C_2,D_2) = P(C_1,D_1)$ is an order on $C_2/D_2 = C_1/D_1$, $abD_2 = aD_2bD_2 \in P(C_2,D_2)$, whence $ab \in P(G)$. Moreover, in the case $C_1 = C_2$, $D_1 \prec C_1$ (or, $D_2 \prec C_2$) is the jump determined by ab, for, if $ab \in C_1$ and $ab \in D_1$, then $aD_1 = b^{-1}D_1$, contradicting the fact that both a and b are positive. If C_1 is properly contained in C_2, then $D_1 \subset C_1 \subseteq D_2 \subset C_2$. Thus, $abD_2 = bD_2 \in P(C_2,D_2)$, whence $ab \in P(G)$; the same proof shows that $ba \in P(G)$; (iii) if $x \in P(G)$, with $D \prec C$ the jump in Σ determined by x, and if $g \in G$, then $x^g = x[x,g]$, where $[x,g] \in D$ since $[G,D] \subseteq D$, and, thus, $x^gD = x[x,g]D = xD \in P(C,D)$. Therefore, $x^g \in P(G)$ if $x \in P(G)$; (iv) if $1 \neq x \in G$ and $D \prec C$ is the jump in Σ determined by x, then $xD \in P(C,D)$ or $x^{-1}D \in P(C,D)$, whence $x \in P(G)$ or $x^{-1} \in P(G)$. Therefore, $P(G)$ is an order on G, and each element of Σ is convex with respect to $P(G)$.

Now let K be a subgroup of G which is convex with respect to $P(G)$. Let B be the union of all elements of Σ contained in K, and let A be the intersection of all elements of Σ containing K. Suppose that $K \notin \Sigma$; i.e., suppose that $B \neq A$. We assert that $B \prec A$ is a jump in Σ: For, B is properly contained in A and, if $D \in \Sigma$ and $B \subset D \subseteq A$, then $D \subseteq K$ or $K \subseteq D$, since both D and K are convex and the family of convex subgroups is a chain with respect to set inclusion. If $D \subseteq K$, then $D \subseteq B$, and this is not possible as $B \subset D$. If $K \subseteq D$, then $A \subseteq D$, whence $A = D$. Therefore, if $K \notin \Sigma$, then $B \prec A$ is a jump in Σ. Now each of B, K, and A is convex with respect to $P(G)$ and $B \subset K \subset A$, so that K/B is a proper, nontrivial, convex subgroup of A/B with respect to the order on A/B induced by G, which is precisely $P(A,B)$. But, this is not possible as A/B is Archimedean ordered by $P(A,B)$ and, therefore, contains no proper, nontrivial, convex subgroups. Thus, $A = B = K$ and, hence, $K \in \Sigma$.

The results of Theorem 1.7 make it clear that if G is a torsion-free, locally nilpotent group and Σ is a central system of G such that C/D is isomorphic to a subgroup of the additive group of reals for each jump $D \prec C$ in Σ, then each element of Σ is isolated in G; however, this same conclusion can be drawn from much weaker hypotheses, namely: If G is a group and Σ is a normal system of G such that C/D is torsion-free for each jump $D \prec C$ in Σ, then each element of Σ is isolated in G. To prove this assertion, suppose $K \in \Sigma$, $g \in G$, $g^n \in K$ for some positive integer n, but $g \notin K$. Let $D \prec C$ be the jump in Σ determined by g. Then $g^n \in K \subseteq D$, so $D = g^nD = (gD)^n$. Thus, $gD \neq D$ and gD is of finite order in C/D, a contradiction.

To conclude this first chapter, we turn briefly to a consideration of the existence of absolutely convex subgroups. Little is apparently known about such subgroups, a conclusion manifested by the generality of a suggested problem of Fuchs (see [2], Problem 11, p. 210). As we shall later see, the problem of the existence of absolutely convex subgroups in 0-groups arises in the consideration of those 0-groups satisfying the maximal condition for subgroups locally and which admit only finitely many different orders. For the present, however, we shall be content to establish a sufficient condition that a torsion-free nilpotent group admits certain members of its upper central series as absolutely convex subgroups.

Theorem 1.8: If G is a torsion-free, nilpotent group with upper central series $\{1\} = Z_0 \subset Z_1 \subset \ldots \subset Z_n = G$ and Z_{i+1}/Z_i is locally cyclic for $i = 0,1,\ldots,s$, then $\{1\}, Z_1, Z_2,\ldots,Z_{s+1}$ are absolutely convex subgroups of G and $\{1\} \prec Z_1 \prec Z_2 \prec \ldots \prec Z_s \prec Z_{s+1}$ are jumps in the family of convex subgroups of any order on G.

Proof: We induct on n, the nilpotency class of G. If G is abelian, we assert that $\{\{1\},G\}$ is the convex family of G with respect to any order on G; i.e., that $\{1\} \prec G$; otherwise, there would exist a convex subgroup C such that $\{1\} \neq C \subset G$. Suppose $1 \neq c \in C$ and $g \in G-C$. Then $< g,c > = < g_0 >$ for some $g_0 \in G$ as G is locally cyclic. Thus, there exists an integer k such that $g_0^k = c \in C$, whence, as C is isolated in G, $g_0 \in C$. Thus, as $g = g_0^t$ for some integer t, $g \in C$, a contradiction. Suppose, therefore, that G is nonabelian and let there be given an order $P(G)$ on G. Suppose $Z_1 = Z(G)$ is locally cyclic and let $\{1\} \neq C$ be a convex subgroup of G (if no convex subgroup C of G exists such that $\{1\} \subset C \subset G$, then G is o-isomorphic to a subgroup of the additive group of reals, hence abelian). Then C is normal in G and, since G is nilpotent, $\{1\} \neq Z_1 \cap C$. Suppose $Z_1 \not\subseteq C$, so there exists $z \in Z_1 - C$. Let $1 \neq x \in Z_1 \cap C$ and consider $< z,x >$. Now, $< z,x > \subseteq Z_1$ and Z_1 is locally cyclic, whence for some $a \in Z_1$ and some integers m and n, $z = a^m$ and $x = a^n$. Thus, $z^n = a^{mn}$ and $x^m = a^{mn}$, so $z^n = x^m \in C$. But, C is isolated in G, whence $z \in C$, a contradiction. Therefore, $Z_1 \subseteq C$ for any nontrivial convex subgroup C of G. Now let A be the intersection of all the nontrivial convex subgroups of G (with respect to $P(G)$). Then $Z_1 \subseteq A$ and A is convex. Moreover, A is a minimal convex subgroup of G; i.e., $\{1\} \prec A$ is a jump. By Theorem 1.6, $A \subseteq Z(G) = Z_1$. Thus, $A = Z_1$, and, therefore, Z_1 is convex with respect to $P(G)$, while $\{1\} \prec Z_1$ is a jump in the family of subgroups of G which are convex with respect to $P(G)$.

If $G = Z_2$, we are finished. If not, we consider the torsion-free, nilpotent group G/Z_1. Since Z_1 is convex, the given order $P(G)$ induces an order on G/Z_1. Now $Z(G/Z_1) = Z_2/Z_1$ and Z_2/Z_1 is locally cyclic, so by applying the above argument for G and Z_1 to G/Z_1 and Z_2/Z_1, we have that Z_2/Z_1 is a convex subgroup of G/Z_1 and that $\{1\} \prec Z_2/Z_1$ is a jump in the family of convex subgroups of G/Z_1. But there is a one-to-one correspondence between the convex subgroups of G/Z_1 and the convex subgroups of G containing Z_1. Thus, both Z_1 and Z_2 are convex subgroups of G with respect to $P(G)$ and $\{1\} \prec Z_1$, $Z_1 \prec Z_2$ are jumps. Repeated application of this argument to $G/Z_2, \ldots, G/Z_s$ completes the proof.

CHAPTER II

ORDERED, POLYCYCLIC GROUPS

This chapter is devoted primarily to a study of those 0-groups which are solvable and which satisfy the maximal condition for subgroups--ordered, polycyclic groups. At times, however, the condition that the groups in question be orderable will be supplanted by the weaker condition that the groups be R-groups.

It is well known that the union of a chain of normal, nilpotent subgroups of a group G is a normal, but not necessarily nilpotent, subgroup of G. If, however, G satisfies the maximal condition for normal subgroups, which it clearly does if it satisfies the maximal condition for subgroups, then infinite ascending chains of normal (nilpotent) subgroups cannot occur and, therefore, by Fitting's Theorem, G contains a maximum normal, nilpotent subgroup called the Fitting subgroup of G. In particular, the Fitting subgroup of a polycyclic group G always exists and is a characteristic subgroup of G. Pertinent to this line of thought is

Theorem 2.1: If G is a polycyclic, R-group and if F denotes the Fitting subgroup of G, then F is isolated in G.

Proof: Plotkin [18] has proved that if G is an R-group and H is a locally nilpotent subgroup of G, then the isolator I(H) of H in G is also locally nilpotent. Now I(F) is the intersection of all isolated subgroups of G containing F. But, if S is isolated in G and $F \subseteq S$, then S^g is also isolated in G and contains F for each $g \in G$, whence I(F) is normal in G. Furthermore, as G is polycyclic and I(F) is a subgroup of G, I(F) is finitely generated, whence I(F) is nilpotent. But, F is the maximum normal nilpotent subgroup of G; hence, $I(F) \subseteq F$. Clearly, $F \subseteq I(F)$. Thus, $F = I(F)$, and F is isolated in G.

The result just established strongly suggests that the Fitting subgroup of an ordered polycyclic group can be made convex with respect to some order on G. All that needs be done to establish this conjecture is to prove $G/F \in 0$. With this in mind, we continue with

Theorem 2.2: If G is an ordered, polycyclic group of length $L(G) = r$, then the number of subgroups of G which are convex with respect to any (fixed) order on G is at most $r + 1$.

Proof: Let $D \prec C$ be a jump in the family of convex subgroups of G with respect to some (fixed) order on G. Then C/D is a normal, finitely generated subgroup of the ordered, polycyclic group G/D; moreover, C/D is isomorphic to a subgroup of the additive group of reals. Thus, C/D is a torsion-free, finitely generated, abelian group and can, therefore, be decomposed into a direct product of infinite cyclic factors, the number of infinite cyclic factors in the decomposition being at least one. As G satisfies the maximal condition for subgroups, the chain of convex subgroups of G is a descending system of G, $G = C_0 \succ C_1 \succ C_2 \succ \dots \succ C_\alpha \succ C_{\alpha+1} \succ \dots$, where $C_{\alpha+1}$ is the maximal convex subgroup of G contained in C_α. Now, if K is a normal subgroup of G, then, since G is polycyclic, $L(G) = L(K) + L(G/K)$. Each convex subgroup of G is normal in G, whence $r = L(G) = L(C_r) + L(C_{r-1}/C_{r-2}) + L(C_{r-2}/C_{r-3}) + \dots + L(C_1/C_2) + L(G/C_1) \geq L(C_r) + r$. Thus, $L(C_r) = 0$ and, hence, $C_r = \{1\}$. In other words, the maximum possible number of jumps in the family of convex subgroups of G is r, whence the maximum possible number of convex subgroups of G is $r + 1$.

The chain of convex subgroups of an ordered, polycyclic group is, therefore, an invariant series of the group. It is interesting to speculate about the nature of a polycyclic group G of length r whose convex family (with respect to some order) has precisely $r+1$ members. This speculation leads to

Theorem 2.3: An ordered, polycyclic group G is nilpotent if and only if there exists an order on G with respect to which the number of convex subgroups is precisely $L(G) + 1$, where $L(G)$ denotes the length of G

Proof: First, let us suppose that for some order on the polycyclic group G the number of convex subgroup is $r+1$, where $L(G) = r$. Then, for each jump $D \prec C$ in the chain of convex subgroups of G, C/D is an infinite cyclic group. Thus, $Aut(C/D)$ is isomorphic to the cyclic group of order two. Note now that C/D is a normal subgroup of the ordered group G/D and that conjugation of C/D by an element of G/D is an o-automorphism of C/D. Since there are only two automorphisms of C/D and only one--the identity--is order-preserving, we have $Dc^g = Dc$ for each $c \in C$ and each $g \in G$. Thus, $C/D \subseteq Z(G/D)$ for each jump $D \prec C$, whence, as the chain of convex subgroups is finite, G is nilpotent.

Next, let G be nilpotent, so that G is a torsion-free, finitely generated, nilpotent group. We shall induct on the length of G. Let $\{1\} = Z_0 \subseteq Z_1 \cdots \subseteq Z_n = G$ be the upper central series of G. If $L(G) = 1$, then $G = Z_1$ is an infinite cyclic group, and $\{\{1\}, G\}$ is the family of convex subgroups of G with respect to the order on G obtained by requiring $1 \leq g$, where $G = \langle g \rangle$. Let us now assume the theorem true for all torsion-free, finitely generated, nilpotent groups G such that $L(G) < k$; let G be such a group and suppose $L(G) = k$. Let P_1 denote some arbitrary, but fixed, order on G. Choose

$1 \neq z \in Z(G)$, and let $C_1 = I(z)$ be the isolator of z in G. By Theorem 0.13, C_1 is a torsion-free, locally cyclic group. As G is polycyclic, C_1 is finitely generated, whence C_1 is an infinite cyclic, normal, isolated subgroup of G. Thus, $L(G/C_1) = k-1$, whence, by inductive assumption, there exists an order P_2 on G/C_1 such that the number of convex subgroups is k. Therefore, $P(G) = (P_1 \cap C_1) \cup \{x | x \in G-C \text{ and } xC_1 \in P_2\}$ is an order on G with respect to which C_1 is convex and with respect to which the number of convex subgroups is $k+1$. This completes the proof.

It is easily seen that if G is an ordered group, if H is a subgroup of G, and if C is a convex subgroup of G, the $H \cap C$ is a convex subgroup of H with respect to the order on H induced by G. The next result reveals the useful and intuitive fact that any subgroup of H which is convex with respect to the order on H induced by G arises in this manner.

Lemma 2.4: Let $P(G)$ denote an order on a group G. Let Σ denote the corresponding family of convex subgroups of G; suppose H is a subgroup of G. Then the family Σ^* of subgroups of H which are convex with respect to the induced order $P(H) = P(G) \cap H$ on H is the family $\{C \cap H | C \in \Sigma\}$. Moreover, if $D^* \prec C^*$ is a jump in Σ^*, then there exists a jump $D \prec C$ in Σ such that $D^* = H \cap D$ and $C^* = H \cap C$.

Proof: If $C \in \Sigma$, if $a \in C \cap H$, and $h \in H$, and if $1 \leq h \leq a$, then, as C is convex, $h \in C$, whence $h \in H \cap C$ and, thus, $H \cap C \in \Sigma^*$. Let us suppose $K \in \Sigma^*$. Let K_G be the convex subgroup of G generated by K; i.e., K_G is the intersection of all convex subgroups of G containing K. We assert that $H \cap K_G = K$. Note that $H \cap K_G = H \cap (\bigcap_{\substack{C \in \Sigma \\ K \leq C}} C) \supseteq H \cap K = K$. Suppose, by way of a contradiction, that $K \subset H \cap K_G$ and let $a \in (H \cap K_G) - K$. Without loss of generality, we may assume that $1 \leq a$. Now, $K_G = KP(G) \cap KP^{-1}(G)$ (see [2], pp. 18-19) and $a \in K_G$, whence $a = kg$, where $k \in K$ and $g \in P^{-1}(G)$. Thus, $1 \leq a = kg \leq k$ and $a \in H$. But, K is convex in H, so $a \in K$, a contradiction. Thus, $K_G \in \Sigma$ and $H \cap K_G = K$.

Now let $D^* \prec C^*$ be a jump in Σ^*. Let A be the intersection of all convex subgroups C of G such that $C \cap H = C^*$; let B be the union of all convex subgroups D of G such that $D \cap H = D^*$. We assert that $B \prec A$ is a jump in Σ, that $B \cap H = D^*$, and $A \cap H = C^*$: First, $A \cap H = \Big(\bigcap_{\substack{C \in \Sigma \\ C \cap H = C^*}} C\Big) \cap H = \bigcap_C (C \cap H) = \bigcap_C C^* = C^*$.

Next, $B \cap H = \Big(\bigcup_{\substack{D \in \Sigma \\ D \cap H = D^*}} D\Big) \cap H = \bigcup_D (D \cap H) = \bigcup_D D^* = D^*$.

Also, as $D^* \subset C^*$, $B \subset A$. Finally, let $E \in \Sigma$ such that $B \subseteq E \subseteq A$. Then $B \cap H \subseteq E \cap H \subseteq A \cap H$; i.e., $D^* \subseteq E \cap H \subseteq C^*$, and $E \cap H \in \Sigma^*$ as $E \in \Sigma$. But $D^* \prec C^*$ is a jump in Σ^*, whence $D^* = E \cap H$ or $C^* = E \cap H$. Thus, $E \subseteq B$ or $A \subseteq E$ and, hence, $B = E$ or $A = E$. Therefore, $A \prec B$ is a jump in Σ.

Lemma 2.4 finds immediate application in the demonstration of the truth of

Theorem 2.5: If G is an ordered group satisfying the maximal condition for subgroups locally, then the derived group G' of G is a Z-group.

Proof: By Lemma 2.4, each jump $D^* \prec C^*$ in the family Σ^* of convex subgroups of G' with respect to the order on G' induced by G is given by $D \cap G' \prec C \cap G'$, where $D \prec C$ is a jump in the family of convex subgroups of G. By Corollary 1.4, $[G',C] \subseteq D$, whence

$[G',C^*] \subseteq D^*$ for each jump $D^* \prec C^*$ in Σ^*. Therefore, Σ^* is a central system of G'; hence, G' is a Z-group.

If the word "locally" is deleted from the hypotheses of Theorem 2.5, a stronger assertion regarding the nature of G' as a generalized nilpotent group can be made:

Theorem 2.6: If G is an ordered group satisfying the maximal condition for subgroups, then G' is a ZD-group.

Proof: As G satisfies the maximal condition, the chain of convex subgroups of G is a descending system of G, $G = C_0 \succ C_1 \succ C_2 \succ \cdots \succ C_\alpha \succ C_{\alpha+1} \succ \cdots$, where $C_{\alpha+1}$ is the maximal convex subgroup of G contained in C_α. Thus, the family Σ^* of convex subgroups of G' with respect to the order on G' induced by G is a descending system of G'. By Theorem 2.5, Σ^* is a central system of G' and, hence, G' is a ZD-group.

Let us bear in mind that our main concern in this chapter is with polycyclic groups. This being the case, we return to ordered groups with this structure by merely augmenting the hypotheses of Theorem 2.6 with the condition that G be solvable. In so doing, we shall obtain an interesting extension of the well known result that the derived group of a supersolvable group is nilpotent; however, as shall soon be seen, the conditions that G be polycyclic and ordered are not sufficient to guarantee that G be supersolvable.

Theorem 2.7: If G is an ordered, polycyclic group, then G' is nilpotent.

Proof: By Theorem 2.6, G' is a ZD-group with its convex family Σ^* forming a descending central system of G'. But G' is a subgroup of the polycyclic group G and is, therefore, polycyclic. By Theorem 2.2, there are only finitely many convex subgroups of G' with respect to any order on G'. Thus Σ^* is a central series of G', and, thus, G' is nilpotent.

Theorem 2.7 reveals an interesting and useful structure property of ordered, polycyclic groups, namely

Corollary 2.8: If G is an ordered, polycyclic group, then G is nilpotent by abelian.

As a further application of Theorem 2.7, it is now possible to establish the truth of the conjecture suggested by Theorem 2.1:

Corollary 2.9: If G is an ordered, polycyclic group, then there exists an order on G with respect to which the Fitting subgroup F of G is convex.

Proof: G' is nilpotent, so $G' \subseteq F$. Thus, G/F is abelian. By Theorem 2.1, G/F is torsion-free. Therefore, G/F is not only an O-group, but even an O*-group. An appeal to Theorem 0.2 completes the proof.

For the following sequence of results, we shall concentrate our attention upon an important subclass of the polycyclic groups--supersolvable groups. So powerful is this refinement from polycyclic to supersolvable that it allows the replacement of the customary condition of orderability by the R-group condition.

Theorem 2.10: A (nontrivial) group G is a supersolvable R-group if and only if G is a torsion-free, finitely generated, nilpotent group.

Proof: If G is a finitely generated, torsion-free, nilpotent group, then G is supersolvable and orderable, whence G is a supersolvable, R-group.

Let us now suppose G is a supersolvable, R-group. We first prove $Z(G) \neq \{1\}$: Let $\{1\} = G_0 \subset G_1 \subset \dots \subset G_n = G$ be a cyclic invariant series of G. Then G_1 is an infinite cyclic group, and, hence, $Aut(G_1)$ is isomorphic to the cyclic group of order two. Also, G_1 is normal in G, so $N_G(G_1) = G$ and, thus, $G/C_G(G_1) = N_G(G_1)/C_G(G_1)$ is isomorphic to a subgroup of $Aut(G_1)$. Therefore, $g \varepsilon G$ implies $g^2 \varepsilon C_G(G_1)$; i.e., $[g^2,x] = 1$ for each $g \varepsilon G$ and each $x \varepsilon G_1$. But G is an R-group, whence, by Theorem 0.9, $[g,x] = 1$. Thus, $\{1\} \neq G_1 \subseteq Z(G)$.

Next, by Theorem 0.10, $G/Z(G)$ is a supersolvable R-group. As G_1 is an infinite cyclic subgroup of $Z(G)$, the length of $G/Z(G)$ is less than the length of G. By induction on the length of G, it follows that $G/Z(G)$ is nilpotent, whence G is nilpotent. Bearing in mind that a torsion-free, locally nilpotent group is an 0*-group, the following corollaries are immediate consequences of Theorem 2.10:

<u>Corollary 2.11</u>: A nontrivial group G is a locally supersolvable, R-group if and only if G is a torsion-free, locally nilpotent group.

<u>Corollary 2.12</u>: A locally supersolvable group G is an 0*-group if and only if G is an R-group.

<u>Corollary 2.13</u>: If G is a locally supersolvable, R-group, then G and each subgroup of G is an 0*-group.

For the case, therefore, of locally supersolvable groups, Corollary 2.13 answers a question posed by Fuchs (see [2], Problem 20, p. 211): "which subgroups of 0*-groups are again 0*-groups?"

A question arising naturally from Theorem 2.10 is whether the hypothesis that G be supersolvable can be weakened, without affecting the conclusion, to the condition that G be polycyclic. The following example illustrates the necessity of the stronger condition, even when the condition that G be an R-group is strengthened to the condition that G be orderable.

<u>Example 2.14</u>: Let H be the naturally ordered subgroup of the additive group of reals which is generated by $a = 1$ and $b = 1/2\ (1 + \sqrt{5}\,)$; i.e., $H = \langle a \rangle + \langle b \rangle$. Let θ be the mapping of H into H given by $x^{\theta} = xb$ for each $x \in H$. It is clear that θ is an o-automorphism of H onto H. Let G be the semi-direct product of H by $\langle \theta \rangle$. We observe that H is normal in G, H is polycyclic, and G/H is polycyclic, whence G is polycyclic.

The order $P(H)$ on H is, of course, the set of all positive real numbers in H. Note that G/H is an infinite cyclic group, so $G/H \in \mathcal{O}$. Let $h \in P(H)$. Then $\theta^{-1}h\theta = h^{\theta} = hb = h\cdot 1/2\ (1 + \sqrt{5}\,) > 0$, so that $P(H)$ is invariant under the inner automorphisms of G. Thus, by Theorem 0.1, there exists an order on G, namely

$$P(G) = P(H) \cup \{x \mid x \in G\text{-}H \text{ and } xH = \theta^{k}H \text{ for some integer } k \geq 1\},$$

with respect to which G is an ordered, polycyclic group.

Now, $a^{\theta} = b$ and $b^{\theta} = a + b$. Also, $[a,\theta] = -a + b$, while $[a,\theta^2] = b$. Thus, $[a,\theta]^{-1}\,[a,\theta^2] = a$ and, thus, $a \in [a,G]$, whence G is not even a ZD-group. Therefore, G is not nilpotent. Furthermore, the center $Z(G)$ of G is trivial: For let $z \in Z(G)$. Then $Z = h\theta^{r}$, where $h \in H$ and r is an integer. Since $z \in Z(G)$, $1 = a = z^{-1}\,az = a^{\theta^{r}} = (1/2\ (1 + \sqrt{5}\,))^{r}$, whence $r = 0$. Thus, $z = h \in H$ and, hence, $z = m + (n/2)(1 + \sqrt{5}\,)$ for some integers m and n. Now $0 = [z,\theta] = z^{-1}z^{\theta} = -(n + (m/2)(1 + \sqrt{5}\,)) + (n + (m/2)(1 + \sqrt{5}\,))((1/2)(1 + \sqrt{5}\,))$, from which it follows that $n = m = 0$, whence $z = 0$. Thus, $Z(G) = \{0\}$.

Let us now consider yet another aspect of this example: Let $\leq$ denote an arbitrary order on G. As $\theta \in G$, either θ is positive or θ is negative. Let us assume that $1 < \theta$. Note also that b is a conjugate of a (i.e., $\theta^{-1}\,a\theta = a^{\theta} = b$) and, thus, a is positive if and only if b is positive; let us also assume $1 < a$.

Let $x \in H$. Then $x^\theta = xb$, so $\theta^{-1}x\theta = xb$, whence $x\theta = \theta xb$. As $1 < \theta$, $xb < x\theta$ and, thus, $b < \theta$. Furthermore, $1 < a$ implies $b < ab$, while $a^\theta = b$ and $b^\theta = ab$, whence $a^\theta < b^\theta$ and, thus, $a < b$.

If $a^2 \leq b$, then $(a^2)^\theta \leq b^\theta$ so that $(a^\theta)^2 \leq ab$ and, thus, $b^2 \leq ab$, whence $b \leq a$, a contradiction. Thus, $b < a^2$.

We know that $b < \theta$. Assuming that $b^k < \theta$ for the positive integer k implies $(b^k)^\theta < \theta^\theta = \theta$, so $(b^\theta)^k < \theta$, whence $(ab)^k < \theta$; i.e., $a^k b^k < \theta$ and, as $2 \leq k$, $b < a^k$, whence $b^{k+1} < a^k b^k < \theta$. Therefore, $b^n < \theta$ for all integers n.

Therefore, if $h \in H$, then $h = a^m b^n < b^m b^n = b^{m+n} < \theta$; hence, if $x \in H$ and $\theta^k y \in G\text{-}H$, where k is a positive integer and $y \in H$, then $x < \theta^k y$ since $xy^{-1} \in H$ and $xy^{-1} < \theta < \theta^k$. It is not, therefore, possible that $1 \leq \theta^k y \leq x$, where k is a positive integer and $x,y \in H$, whence H is convex with respect to $\leq$.

It follows, therefore, that H is an absolutely convex subgroup of G. Let us also note that with respect to the arbitrary order $\leq$ the family of convex subgroups of G is (always) $\{\{1\}, H, G\}$. Finally, only four different orders can be defined on G, namely those orders obtained by interchanging either or both of the sets of positive and negative elements of H and G/H.

For the case of a supersolvable group G, a very simple necessary and sufficient condition that $G \in 0$ has been found. One may suspect that, as the R-group condition on a supersolvable group G is sufficent to guarantee that $G \in 0^*$, the R-group condition on a polycyclic group G is sufficient to guarantee that $G \in 0$. Unfortunately, this is not the case as is illustrated by

Example 2.15: Let $H = < a_1 > + < a_2 >$, where $< a_1 >$ and $< a_2 >$ are infinite cyclic groups. Let v be the automorphism of H given by $a_1^v = a_2$ and $a_2^v = -a_1 - 3a_2$. Let $x = \lambda a_1 + \mu a_2 \varepsilon H$. Then $3x^v = -3\mu a_1 + 3\lambda a_2 - 9\mu a_2$ and $x^{v^2} = 8\mu a_2 - 3\lambda a_2 + 3\mu a_1 - \lambda a_1$. Thus, $x + 3x^v + x^{v^2} = 0$ for all $x \varepsilon H$; i.e., $(1 + 3v + v^2)(x) = 0$ for all $x \varepsilon H$.

Let G be the semi-direct product of H by $< v >$, where we note here that v is of infinite order in Aut(H) (the matrix of the transformation v makes this apparent). As noted previously, $x(x^v)^3 x^{v^2} = 1$ for $x \varepsilon H$, whence G is not an R*-group and, hence, $G \notin 0$. We shall now prove that G is an R-group: Let $g = v^n h \varepsilon G$. Then $(v^n h)^t = v^{tn}(h^{v^{(t-1)n}} h^{v^{(t-2)n}} \ldots h^{v^n} h)$ for each positive integer t. Thus, if $(v^n h)^t = (v^m k)^t$, where $h,k \varepsilon H$, then $v^{tn}(h^{v^{(t-1)n}} h^{v^{(t-2)n}} \ldots h^{v^n} h) = v^{tm}(k^{v^{(t-1)m}} k^{v^{(t-2)m}} \ldots k^{v^m} k)$, whence $n = m$. Thus,

$$h^{v^{(t-1)n}} h^{v^{(t-2)n}} \ldots h^{v^n} h = k^{v^{(t-1)n}} k^{v^{(t-2)n}} \ldots k^{v^n} k, \text{ whence}$$

$(hk^{-1})^{v^{(t-1)n}} (hk^{-1})^{v^{(t-2)n}} \ldots (hk^{-1})^{v^n} (hk^{-1}) = 1$. Now let $u \varepsilon H$. We now assert that

$$u u^{v^n} u^{v^{2n}} \ldots u^{v^{(t-1)n}} = 1$$

implies $u = 1$; i.e., in additive notation, that $0 = u + u^{v^n} + u^{v^{2n}} + \ldots + u^{v^{(t-1)n}} = (1 + v^n + v^{2n} + \ldots + v^{(t-1)n})(u)$ implies $u = 0$. Note that $1 + v^n + v^{2n} + \ldots + v^{(t-1)n} = 0$ implies $0 = (1+v^{tn})/(1-v^n)$, so that all the roots of $1 + v^n + v^{2n} + \ldots + v^{(t-1)n} = 0$ are of modulus 1. Since the roots of $1 + 3v + v^2 = 0$ are not of modulus 1, $1 + 3v + v^2$ does not divide $1 + v^n + v^{2n} + \ldots + v^{(t-1)n}$. Let $1 + v^n + v^{2n} + \ldots + v^{(t-1)n} = P(v)(1 + 3v + v^2) + (r + sv)$, where $P(v)$ is a polynomial in v with integer coefficients and where r and s

are integers, not both 0. Now, $0 = (1 + v^n + v^{2n} + \dots + v^{(t-1)n})(u) = (P(v)(1 + 3v + v^2) + (r + sv))(u) = P(v)(1 + 3v + v^2)(u) + (r + sv)(u) = ru + sv(u)$. Let $u = \lambda a_1 + \mu a_2$. Then $0 = ru + sv(u)$ implies $(r\lambda - s\mu)a_1 + (r\mu + s\lambda - 3s\mu)a_2 = 0$, whence

$$\begin{cases} r\lambda - s\mu = 0 \\ s\lambda + (r - 3s)\mu = 0. \end{cases}$$

Thus, $(r^2 - 3rs + s^2)\mu = 0$. Suppose now that $\mu \neq 0$. Then $r^2 - 3rs + s^2 = 0$. Recall that $r \neq 0$ or $s \neq 0$.

Case 1: Suppose $r \neq 0$. Then $s = (3r \pm |r| \sqrt{5})/2$, which is impossible as $(3r \pm |r| \sqrt{5})/2$ is irrational and s in an integer. Thus, $\mu = 0$, whence $r\lambda = 0$. As $r \neq 0$, $\lambda = 0$ and, hence, $u = 0$.

Case 2: Suppose $s \neq 0$. Then, as in Case 1, $r = (3s \pm |s| \sqrt{5})/2$, which is again impossible. Thus, $\mu = 0$, whence $s\lambda = 0$. As $s \neq 0$. $\lambda = 0$ and, thus, $u = 0$.

Therefore, if $x,y \in G$ and $x^t = y^t$ for some positive integer t, then $x = y$, whence G is an R-group. Clearly, G is polycyclic, but $G \notin 0$.

With this example, the formal expositions of this chapter are concluded. However, there remain many fascinating problems concerning orderable, polycyclic groups which have been considered by this author and which have escaped resolution. One such problem evolves from a more general query of Fuchs: Is a polycyclic, R*-group necessarily an 0-group? Another question of considerable interest is: Are polycyclic 0-groups necessarily 0*-groups?

CHAPTER III

SOME REMARKS ON TWO PAPERS OF REE

The purpose of this chapter is to study two of the earliest papers of R. Ree (see [20] and [21]) and to demonstrate that a number of incorrect assertions appear therein. It is interesting and only fair to note that the errors in these papers are a direct consequence of but one erroneous, casually made, and seemingly innocuous statement found in the proof of Theorem 1 of [20], for it is here that the claim is made that any finitely generated subgroup of the additive group of reals is isomorphic to the additive group of integers; i.e., that the additive group of reals is a locally cyclic group. The measure of the magnitude of the erroneousness of this assertion is made manifest upon the observation that this statement is equivalent to the claim that all real numbers are rational. It is, therefore, not surprising that the application of such a statement has begotten the sequence of incorrect conclusions which appear in the works under consideration.

Let us now begin our investigation of the results of the earlier paper [20]: Theorem 1 asserts that if an ordered group G satisfies the maximal condition for subgroups, then G is a ZD-group; Theorem 2 continues with the assertion that an ordered, finitely generated, solvable group is nilpotent if and only if G satisfies the maximal condition for subgroups.

Both of these assertions are false, as is demonstrated by Example 2.14, which exhibits an ordered polycyclic group which is not even a ZD-group. The correct statement of Theorem 1 of [20] is obtained, by Theorem 2.6, by replacing G by G' in the conclusion. Similarly, by Theorem 2.7, the derived group G' of an ordered, polycyclic group

G is necessarily nilpotent, but G need not be nilpotent. Theorem 2.3 provides a correct necessary and sufficient condition that an ordered, polycyclic group be nilpotent.

Now, let G be a torsion-free, finitely generated, nilpotent group with upper central series $\{1\} = Z_o \subseteq Z_1 \subseteq \cdots \subseteq Z_n = G$. Then Z_{i+1}/Z_i is a torsion-free, finitely generated, abelian group, for $i = 1,2,\ldots,n-1$. As Z_{i+1}/Z_i is supersolvable, there exists a cyclic invariant series, $Z_i = A_1/Z_i \subseteq A_2/Z_i \subseteq \cdots \subseteq A_t/Z_i = Z_{i+1}/Z_i$, of Z_{i+1}/Z_i, where each A_j/Z_i is an infinite cyclic group and where the A_j, $j = 1,2,\ldots,t$, are subgroups of Z_{i+1}. Note that, as $[G,Z_{j+1}] \subseteq Z_j$, $[G,A_{j+1}] \subseteq Z_i \subseteq A_j$. Therefore, there exists a central series $\{1\} = F_o \subseteq F_1 \subseteq \cdots \subseteq F_m = G$ of G such that F_{i+1}/F_i is an infinite cyclic group, $i = 0,1,\ldots,m-1$. Ree calls such a series an "F-series of G." Let $F_{i+1}/F_i = < x_{i+1}F_i >$, where $x_{i+1} \in F_{i+1}$, $i = 0,1,\ldots,m-1$. The elements $x_1, x_2, \ldots, x_m$ are called an "F-basis of G." Ree correctly observes that each element $g \in G$ can be written uniquely in the form $g = x_1^{e_1} x_2^{e_2} \ldots x_m^{e_m}$, where $e_1, e_2, \ldots, e_m$ are integers, and that G can be ordered lexicographically with respect to $e_1, e_2, \ldots, e_m$ as follows:

For $g_1 = x_1^{e_1} x_2^{e_2} \ldots x_m^{e_m}$ and $g_2 = x_1^{f_1} x_2^{f_2} \ldots x_m^{f_m}$, we make $g_1 \leq g_2$ if and only if $e_i = f_i$ for $i = 1,2,\ldots,m$, or $e_t < f_t$ for some t such that $1 \leq t \leq m$ and $e_i = f_i$ for $i = 1,2,\ldots,t-1$. This lexicographic order on G is said, by Ree, to be "defined" by the F-basis, $x_1, x_2, \ldots, x_m$.

These concepts are applied in [20] in the statement of Theorem 3, where it is asserted that any order on a torsion-free, finitely generated, nilpotent group is defined by an F-basis of G. This assertion--even if G is abelian--is false, as is demonstrated by

<u>Example 3.1</u>: Let G be the subgroup of the additive group of real numbers which is generated by $\{1, \sqrt{2}\}$; i.e., $G = < 1 > + < \sqrt{2} >$. Then G is a finitely generated, abelian subgroup of the additive group of reals, and G can, therefore, be ordered by the restriction of the natural ordering on the reals to G. With respect to this induced order, G is, of course, an Archimedean ordered group, and hence G possesses no proper, nontrivial, convex subgroups. Let $\{1\} = F_o \subseteq F_1 \subseteq \cdots \subseteq F_m = G$ be an F-series of G with corresponding F-basis, $f_1, f_2, \ldots, f_m$. Then, as G is polycyclic of length two, $m = 2$, so that $\{1\} = F_o \subset F_1 \subset F_2 = G$, $F_1 = < f_1 >$ and $F_2/F_1 = < f_2F_1 >$. Thus, each element $g \in G$ can be written uniquely in the form $e_1f_1 + e_2f_2$, for some integers e_1, e_2. Let $\leq_1$ denote the order on G defined by the F-basis, f_1, f_2. Now suppose that $g = e_1f_1 + e_2f_2$ and $0 \leq_1 g = e_1f_1 + e_2f_2 \leq_1 e_2'f_2$, where e_2' is an integer. Then $0 \leq_1 - e_1f_1 + (e_2' - e_2)f_2$. Thus, as $0 \leq_1 e_1$ and $0 \leq_1 -e_1$, we have that $0 = e_1$ and, hence, $g = e_2f_2 \in < f_2 >$. Therefore, $< f_2 >$ is a proper, nontrivial, convex subgroup of G with respect to $\leq_1$, whence the F-basis f_1, f_2 cannot define the given Archimedean order on G.

This concludes our study of [20], and we now move to the examination of the later paper [21], where Ree again uses the concepts of F-series and F-bases and where certain results from [20] are cited in order to accomplish the proofs of various assertions. It is, therefore, not surprising that the application of an incorrect statement from [20] leads to an erroneous assertion in [21], and indeed the following is a counterexample to Theorem 2 of [21], which asserts that the group of o-automorphisms (with respect to some order) of a torsion-free, finitely generated, nilpotent group is itself a torsion-free, finitely generated, nilpotent group:

Example 3.2: Let $G = < a_1 > + < a_2 > + < a_3 >$, where $< a_1 >$ is an infinite cyclic group and where $< a_2 > + < a_3 >$ is isomorphic to the subgroup $H = < 1 > + < 1/2(1 + \sqrt{5})>$ of the additive group of real numbers. Let $P(G) = \{na_1 \mid n$ is a nonnegative integer$\} \cup$ $\{x | x \in G- < a_1 >, x = ra_1 + sa_2 + ta_3$, and $s + t/2(1 + \sqrt{5}) > 0\}$. It readily follows that $P(G) \cap P^{-1}(G) = \{0\}$, $P(G) + P(G) \subseteq P(G)$, $-x + P(G) + x \subseteq P(G)$, and that $P(G) \cup P^{-1}(G) = G$, whence $P(G)$ is an order on G. For $x,y \in G$, we define $x \leq y$ if and only if $0 \leq -x + y$. It is easy to see that $0 < a_1$, $0 < a_2$, and $0 < a_3$; also, it follows that the order $P(G)$ is not Archimedean as $a_1 \ll a_2$ and $a_1 \ll a_3$ (i.e., $na_1 < a_2$ and $na_1 < a_3$ for all integers n).

We now define two o-automorphisms of G:

(i) $d: a_1 \to a_1, a_2 \to a_3, a_3 \to a_2 + a_3$:

(ii) $v: a_1 \to a_1, a_2 \to a_2 + ma_1, a_3 \to a_3 + na_1$,

where m,n are arbitrary nonzero integers. Then $d^{-1}: a_1 \to a_1, a_2 \to -a_2 + a_3, a_3 \to a_2$, while $v^{-1} : a_1 \to a_1$, $a_2 \to -ma_1 + a_2, a_3 \to -na_1 + a_3$. It is easily seen that d,v are automorphisms of G. Let us show that d is order-preserving: Let $x = ra_1 + sa_2 + ta_3$ and suppose $0 < x$. Then, if $s = t = 0$, it follows that $x = ra_1$ and $x^d = ra_1 > 0$ since d fixes a_1. If $s \neq 0$ or $t \neq 0$, then the condition that $0 < x$ is independent of a_1 and so, without loss of generality, we may assume that $r = 0$ and, hence, that $x = sa_2 + ta_3$. Then $0 < x$ is equivalent to $0 < s + t/2(1 + \sqrt{5})$, which is equivalent to $-2s/(1 + \sqrt{5}) < t$. On the other hand, $0 < (sa_2 + ta_3)^d$ $= sa_3 + t(a_2 + a_3) = ta_2 + (s + t)a_3$ is equivalent to $t + (s+t)/2(1+\sqrt{5})$

> 0, which is equivalent to $3t + t\sqrt{5} > -s(1+\sqrt{5})$, which is true if and only if $t > -s(1+\sqrt{5})/(3+\sqrt{5}) = -2s/(1+\sqrt{5})$.

Therefore, d is order-preserving. Analogous arguments establish that v, d^{-1}, and v^{-1} are also order-preserving, whence d and v are o-automorphisms of G.

Let $\Delta = \langle d, v \rangle$, so that Δ is a subgroup of the group of o-automorphisms of G. It will be convenient now to represent the elements $ra_1 + sa_2 + ta_3$ as vectors $\begin{pmatrix} r \\ s \\ t \end{pmatrix}$ and the mappings d, v, d^{-1}, v^{-1} as matrices:

$$d = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \quad d^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad v = \begin{pmatrix} 1 & m & n \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{and} \quad v^{-1} = \begin{pmatrix} 1 & -m & -n \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

In general, if a,b are real numbers let $(a,b) = \begin{pmatrix} 1 & a & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. It is easy to check that $(a_1,b_1)(a_2,b_2) = (a_1 + a_2, b_1 + b_2) = (a_2,b_2)(a_1,b_1)$ and that $(a,b)^d = (b, a+b)$. For the elements d and v, we have: $v = (m,n)$, $v^d = (n, m+n)$, and $[v,d]^d = (v^{-1}v^d)^d = ((-m,-n)(n,n+m))^d = (n-m,m)^d = (m,n) = v$. Thus, v is a commutator of every length, whence $v \in \langle [v,\underbrace{d,\ldots,d}_{n}]^{\Delta} \rangle$ for all positive integers n; therefore,

$1 \neq v \in [\underbrace{\Delta,\Delta,\ldots,\Delta}_{n+1}]$ for all nonnegative integers n. Thus, Δ is not nilpotent, and, as Δ is a subgroup of the group of o-automorphisms of G, the group of o-automorphisms of G is not, therefore, nilpotent.

While there is no hope of establishing a result as strong as the one suggested by Ree in Theorem 2, it is possible to establish an analogous result for ordered, polycyclic groups, namely

Theorem 3.3: If G is an ordered, polycyclic group and Δ denotes the group of o-automorphisms of G, then Δ is nilpotent by abelian, and, moreover, Δ is polycyclic.

Proof: Let $\{1\} \prec C_1 \prec C_2 \prec C_3 \cdots \prec C_n = G$ be the chain of convex subgroups of G with respect to the given order on G. Let $\theta \in \Delta$. Then θ induces an o-automorphism θ^* on C_i/C_{i-1} for $i = 1,2,\ldots,n$, given by $(cC_{i-1})^{\theta^*} = c^\theta C_{i-1}$; for, first, we observe that $C_i^\theta = C_i$, $i = 0,1,\ldots,n$, as C_i^θ is convex and as $C_i \subset C_j$ implies $C_i^\theta \subset C_j^\theta$. Next, $c_1^\theta C_{i-1} = c_2^\theta C_{i-1} \Leftrightarrow c_1^\theta = c_2^\theta\, x$, where $x \in C_{i-1} \Leftrightarrow c_1^\theta = c_2^\theta\, x_1^\theta$, where $x_1 \in C_{i-1}$ and $x_1^\theta = x \Leftrightarrow c_1^\theta = (c_2x_1)^\theta$, as θ is an automorphism of $G \Leftrightarrow c_1 = c_2x_1$, where $x_1 \in C_{i-1} \Leftrightarrow c_1C_{i-1} = c_2C_{i-1}$, whence θ^* is one-to-one. Clearly, θ^* is onto C_i/C_{i-1}, and θ^* is a homomorphism of C_i/C_{i-1}, for $i = 1,2,\ldots,n$. Finally, $c_1C_{i-1} \le c_2C_{i-1} \Leftrightarrow$ there exists $a \in C_{i-1}$ such that $c_1 \le c_2a \Leftrightarrow c_1^\theta \le c_2^\theta\, a^\theta$, where $a^\theta \in C_{i-1} \Leftrightarrow (c_1C_{i-1})^{\theta^*} \le (c_2C_{i-1})^{\theta^*}$, whence θ^* is order-preserving. A similar argument shows that $(\theta^*)^{-1}$ is also order-preserving.

For each i such that $1 \le i \le n$, let Δ_i denote the group of all o-automorphisms $\theta \in \Delta$ such that θ^* centralizes C_i/C_{i-1}; i.e., such that $(cC_{i-1})^{\theta^*} = c^\theta C_{i-1} = cC_{i-1}$. Then Δ_i is a normal subgroup of Δ for each $i = 1,2,\ldots,n$, and Δ/Δ_i is isomorphic to a subgroup of the group of o-automorphisms of C_i/C_{i-1}. But, each C_i/C_{i-1} is an Archimedean ordered group, whence the group of o-automorphisms of C_i/C_{i-1} is, by Corollary 0.5, isomorphic to a subgroup of the multiplicative group of positive real numbers. Thus, Δ/Δ_i is abelian for each $i = 1,2,\ldots,n$. Thus, Δ/Δ_0 is abelian, where $\Delta_0 = \bigcap_{i=1}^{n} \Delta_i$. Note that Δ_0 centralizes C_i/C_{i-1} for each $i = 1,2,\ldots,n$, so that $[C_i,\Delta_0] \subseteq C_{i-1}$, $i = 1,2,\ldots,n$, where by $[x,\theta]$ and $[\theta,x]$ we mean $x^{-1}x^\theta$ and $(x^{-1})^\theta x$, respectively. Therefore, $[C_n,\underbrace{\Delta_0,\Delta_0,\ldots,\Delta_0}_{n}] = 1$.

By a result of P. Hall [6, p. 10], which states that if H is a subgroup of a group G, if K is a subgroup of the group of automorphisms of H, and if $[H,\underbrace{K,K,\ldots,K}_{n}] = 1$, then $[\underbrace{K,K,\ldots,K}_{1+\binom{n}{2}},H] = 1$, we have that $[\underbrace{\Delta_0,\Delta_0,\ldots,\Delta_0}_{1+\binom{n}{2}},C_n] = 1$, whence $[\underbrace{\Delta_0,\Delta_0,\ldots,\Delta_0}_{1+\binom{n}{2}}]$ centralizes $C_n = G$, where $[\underbrace{\Delta_0,\Delta_0,\ldots,\Delta_0}_{1+\binom{n}{2}}] \subseteq \Delta$. Thus, $[\underbrace{\Delta_0,\Delta_0,\ldots,\Delta_0}_{1+\binom{n}{2}}]$ is the identity automorphism, and, thus, Δ_0 is nilpotent. Therefore, Δ is nilpotent by abelian.

Smirnov [22] has proved that, for a polycyclic group H, every abelian subgroup of Aut(H) is finitely generated, whereas Mal'cev [13] has proved that any solvable group, all of whose abelian subgroups are finitely generated, is polycyclic. From these results, it readily follows that Δ is polycyclic.

CHAPTER IV

ON O-GROUPS WHICH ADMIT ONLY FINITELY MANY DIFFERENT ORDERS

This chapter is devoted to a study of O-groups which satisfy the maximal condition for subgroups locally and which admit only finitely many different orders. Little is known of O-groups which admit only finitely many different orders, a fact manifested by a question of some apparent difficulty which has been posed by B. H. Neumann (see [2], Problem 18 (c)) and which asks, "What are the O-groups with only a finite number of (different) orders?" Teh [23] has verified that any torsion-free, locally cyclic group admits only finitely many different orders, proving that such a group admits precisely two different orders, both of which are Archimedean. However, Teh has also shown that the torsion-free, locally cyclic groups are singularily unique in the sense that they are the only really "nice" (i.e., abelian) groups possessing this finiteness property, for any torsion-free, abelian group of rank exceeding one is shown by Teh to possess an uncountable number of different orders. We shall extend Teh's result in this chapter by proving that any nonabelian, torsion-free, locally nilpotent group admits infinitely many different orders. Corollary 2.11 indicates, therefore, that examples of O-groups which admit only finitely many different orders are to be found at the earliest among the ordered polycyclic groups. Example 2.14 provides an example of just such a group. It would seem, therefore, that we are confronted with the rather exasperating situation where the absence of a "nice" structure serves to enhance the possibility that an O-group admits only finitely many different orders.

As we shall see, polycyclic O-groups and O-groups which satisfy the maximal condition for subgroups locally and which admit only finitely many different orders share many interesting properties. In support of this assertion, we begin with

Theorem 4.1: Suppose G is an 0-group which satisfies the maximal condition for subgroups locally and which admits only finitely many different orders. Then the number of subgroups of G which are convex with respect to an arbitrary, but fixed, order on G is finite, and G' is nilpotent.

Proof: Let $P(G)$ denote an order on G and let Σ denote the corresponding family of convex subgroups of G with respect to $P(G)$. For each jump $D \prec C$ in Σ, we know, by Lemma 1.1, that D and C are normal in G. We now define a different order $P_C(G)$ on G:

$$P_C(G) = (P(G) \cap D) \cup \{x \mid x \in C-D \text{ and } x < d \text{ for } d \in D\}$$
$$\cup \{x \mid x \in G-C \text{ and } c < x \text{ for } c \in C\}.$$

It is clear that $P_C(G) \neq P(G)$ and that if $D_1 \prec C_1$ and $D_2 \prec C_2$ are jumps in Σ such that $C_1 \neq C_2$, then $P_{C_1}(G) \neq P_{C_2}(G)$. We now verify that $P_C(G)$ is an order on G: Let $1 \neq y \in G$. Then precisely one of the following is true: (I) $c < y$ for all $c \in C$; (II) $y \in C$ and $y < d$ for all $d \in D$; (III) $y \in D$ and $1 < y$; (IV) $y < c$ for all $c \in C$; (V) $y \in C$ and $d < y$ for all $d \in D$; or (VI) $y \in D$ and $y < 1$. Note that $y \in P_C(G)$ if case (I), (II), or (III) holds, while $y \in P_C^{-1}(G)$ if case (IV), (V), or (VI) holds. It is clear that (I) - (VI) are mutually exclusive cases which exhaust G, whence $P_C(G) \cap P_C^{-1}(G) = \{1\}$ and $P_C(G) \cup P_C^{-1}(G) = G$. If $g \in G$, then y^g satisfies (I), (II), or (III) if and only if y does, since D, C are normal subgroups of G. This proves that $g^{-1}P_C(G)g \subseteq P_C(G)$ for each $g \in G$. Finally, let $y,z \in P_C(G)$ and suppose $z \leq y$. If both y and z satisfy (I), then $yz \geq z > c$ for all $c \in C$, whence yz satisfies (I) and, thus, $yz \in P_C(G)$. If y satisfies (I) and z satisfies (II), then yz satisfies (I); otherwise, $yz \leq c$ for some $c \in C$, whence $y \leq cz^{-1} \in C$, a contradiction. Similarily, if y satisfies (I) and z satisfies (III), then yz again satisfies (I). An inspection of the remaining possible, consistent cases shown in each instance that $yz \in P_C(G)$. An analogous argument holds when $y \leq z$. Thus, $P_C(G)$ is an order on G.

Now, if the family Σ were not finite, then infinitely many different orders could be defined on G, one for each jump in Σ. Thus, Σ is finite.

Finally, G' is an 0-group with respect to the induced order $P(G) \cap G'$ on G' with convex family $\Sigma^* = \{C \cap G' | C \in \Sigma\}$. By the proof of Theorem 2.5, Σ^* is a central system of G', and, as Σ is finite, Σ^* is a central series of G', whence G' is nilpotent. Immediate consequences of Theorem 4.1 are

<u>Corollary 4.2</u>: If G is an 0-group which admits only finitely many different orders and which satisfies the maximal condition for subgroups locally, then G is nilpotent by abelian.

Therefore, any group satisfying the hypotheses of Corollary 4.2 is not only a generalized solvable, but even a solvable, group, whence

<u>Corollary 4.3</u>: If G is an 0-group which satisfies the maximal condition for subgroups locally and which admits only finitely many different orders, then G is locally polycyclic.

To state an even stronger conclusion, it is now clear that any group which satisfies the maximal condition for subgroups and which admits only finitely many different orders is necessarily polycyclic. We continue with

Lemma 4.4: Suppose G is a group, H is a normal subgroup of G, and G/H is abelian. Then the isolator $I(H)$ of H in G equals $< g \mid g \in G$ and $g^n \in H$ for some positive integer $n >$.

Proof: Let $A = < g \mid g \in G$ and $g^n \in H$ for some positive integer $n >$. Clearly, $H \subseteq A$. We now assert that A is isolated in G: For, if $g \in G$ and $g^k \in A$ for some positive integer k, then $g^k = u_1 u_2 \ldots u_t$, where $u_1, u_2, \ldots, u_t \in G$ such that $u_i^{a_i} \in H$ for appropriate integers a_i, $i = 1, 2, \ldots, t$. Let $a = a_1 a_2 \ldots a_t$. Then $(g^k H)^a = (u_1 u_2 \ldots u_t H)^a = u_1^a H\, u_2^a H \ldots u_t^a H = H$, as each $u_i^a \in H$. Thus, $g^{ka} \in H$, whence $g \in A$. Thus, A is isolated in G.

Now, if $g \in A$ is a generator of A, then $g^n \in H$ for some positive integer n. If $g \notin I(H)$, then $I(H)$ would not be isolated in G, as $g^n \in H \subseteq I(H)$ would not imply $g \in I(H)$, a contradiction. Thus, $g \in A$ implies $g \in I(H)$. Therefore, $A = I(H)$.

We are now ready to prove

Theorem 4.5: If G is a nonabelian, ordered group which satisfies the maximal condition for subgroups locally and which admits only finitely many different orders, then the Fitting subgroup F of G exists and coincides with the isolator $I(G')$ of G' in G; G/F is nontrivial and locally cyclic; and F is an absolutely convex subgroup of G.

Proof: Let $\leq$ denote an arbitrary, but fixed, order on G and let $\{1\} = C_0 \prec C_1 \prec C_2 \prec \ldots \prec C_{n-1} \prec C_n = G$ denote the corresponding, necessarily finite, chain of convex subgroups of G. By Theorem 0.14, G/C_{n-1} is abelian, whence $G' \subseteq C_{n-1}$, and, thus, as C_{n-1} is isolated, $I(G') \subseteq C_{n-1}$. Since $I(G')$ is isolated, $G/I(G')$ is a torsion-free, abelian group, whence $G/I(G') \in 0$. By Theorem 0.2, there exists an order on G with respect to which $I(G')$ is convex. If the rank of $G/I(G')$ exceeded one, then by Teh's Theorem [23], there would exist infinitely many different orders on $G/I(G')$, and, thus, on G. Therefore, $G/I(G')$ is locally cyclic.

Let us now suppose that $a \in C_{n-1} - I(G')$. We know that there exists an element $b \in G - C_{n-1}$. As $G/I(G')$ is locally cyclic, there exists $g \in G$ such that $< aI(G'), bI(G') > = < gI(G') >$. Thus, $aI(G') = g^n I(G')$ and $bI(G') = g^m I(G')$ for appropriate integers n and m, whence $g^n \in C_{n-1}$; however, this implies that $b = g^m \in C_{n-1}$ as C_{n-1} is isolated. Therefore, $C_{n-1} = I(G')$, and, thus, $I(G')$ is an absolutely convex subgroup of G.

Now let F denote the locally nilpotent radical of G (i.e., F is the largest normal, locally nilpotent subgroup of G) and let $I(F)$ denote the isolator of F in G. As G' is, by Theorem 4.1, nilpotent, $G' \subseteq F$ and $C_{n-1} = I(G') \subseteq I(F)$; also, $[I(F),G] \subseteq C_{n-1}$. We now assert that $[I(F), C_i] \subseteq C_{i-1}$ for $i = 1,2,\ldots,n$: Suppose $s < n$ is the smallest positive integer for which $[I(F), C_s] \not\subseteq C_{s-1}$. Then

$$[C_{s-1}, \underbrace{I(F), I(F),\ldots,I(F)}_{s-1}] = \{1\}.$$

Also, $[C_s,C_s] \subseteq C_{s-1}$ and $C_s \subseteq I(F)$, so

$$[\underbrace{C_s,C_s,\ldots,C_s}_{s+1}] = \{1\},$$

whence C_s is nilpotent. Thus, $C_s \subseteq F$. Restricting the order on G to F makes F an ordered group with convex subgroups $F \cap C_i$, $i = 0,1,\ldots,n$. Since $C_{s-1} \subseteq C_s \subseteq F$, $C_{s-1} \prec C_s$ is a jump in the family of convex subgroups of F. As F is locally nilpotent, $[F,C_s] \subseteq C_{s-1}$ by Theorem 1.6. Since $[I(F),C_s] \not\subseteq C_{s-1}$, there exist elements $x \in I(F)$ and $c \in C_s$ such that $[x,c] \notin C_{s-1}$. But, by Lemma 4.4, $I(F)/F$ is periodic, so there exists a positive integer r such that $[x^r, c] \in C_{s-1}$. However, $G/C_{s-1} \in 0$, so G/C_{s-1} is clearly a torsion-free, R-group, whence, by Theorem 0.9, $[x^r, c] \in C_{s-1}$ implies $[x,c] \in C_{s-1}$, a contradiction. Therefore, $C_i \subseteq I(F)$ and $[I(F), C_{i+1}] \subseteq C_i$ for $i = 0,1,\ldots n-1$, whence $I(F)$ is nilpotent. Therefore, $I(F) \subseteq F$ and, hence, $F = I(F)$ is the Fitting subgroup of G.

Let us now assume that $I(G') \nsubseteq F$. We assert that, in this case, $F = G$: For, if not, there exist elements a and b such that $a \in F - I(G')$ and $b \in G - F$. As G/C_{n-1} is locally cyclic, $< aC_{n-1}, bC_{n-1} > = < gC_{n-1} >$ for some $g \in G$. Thus, $aC_{n-1} = g^k C_{n-1}$ and $bC_{n-1} = g^m C_{n-1}$ for appropriate integers k and m. Thus, $g^k \in F$, so, as F is isolated in G, $g \in F$. However, this implies $b \in F$, a contradiction. Therefore, if $I(G') \nsubseteq F$, then $F = G$, and G is nilpotent. Therefore, $\{1\} = C_0 \prec C_1 \prec C_2 \prec \ldots \prec C_{n-1} \prec G = F$ is, by Theorem 1.6, a central series of G, whence $C_i/C_{i-1} \subseteq Z(G/C_{i-1})$ for $i = 1,2,\ldots,n$. Also, $(G/C_{n-2})/(C_{n-1}/C_{n-2}) \cong G/C_{n-1}$ and G/C_{n-1} is locally cyclic. Thus, G/C_{n-2} is abelian. If the rank of G/C_{n-2} exceeded one, then there could be defined infinitely many different orders on G/C_{n-2} and, thus, on G. Therefore, G/C_{n-2} is locally cyclic. Again, $(G/C_{n-3})/(C_{n-2}/C_{n-3}) \cong G/C_{n-2}$ and G/C_{n-2} is locally cyclic, so G/C_{n-3} is abelian. Repeated application of this argument ultimately yields the conclusion that G/C_1 is locally cyclic, where $C_1 \subseteq Z(G)$, whence G is abelian, a contradiction. Therefore, $C_{n-1} = I(G') = F$.

<u>Corollary 4.6</u>: If G is a nonabelian, torsion-free, locally nilpotent group, then G admits infinitely many different orders.

<u>Proof</u>: The proof of Theorem 4.5 shows that any nonabelian 0-group which satisfies the maximal condition for subgroups locally and which

admits only finitely many different orders has the properties that the locally nilpotent radical F of G coincides with I(G') and that G/I(G') is nontrivial; however, under the hypotheses of this corollary, G is locally nilpotent, so that F = G. Hence, G cannot admit only finitely many different orders.

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