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SOME INEQUALITIES IN APPROXIMATION THEORY

BY

WEIYU CHEN

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THE UNDERSIGNED CERTIFY THAT THEY HAVE READ, AND RECOMMEND TO THE FACULTY OF GRADUATE STUDIES AND RESEARCH FOR ACCEPTANCE, A THESIS ENTITLED **SOME INEQUALITIES IN APPROXIMATION THEORY** SUBMITTED BY **WEIYU CHEN** IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE DEGREE OF **DOCTOR OF PHILOSOPHY** IN MATHEMATICS.

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TO MY WIFE AND MY PARENTS.

ABSTRACT

This thesis deals with three kinds of inequalities, the Landau-Kolmogorov-type inequality, inequalities involving mixed and directional derivatives, and direct and converse inequalities of Bernstein-Durrmeyer operators.

The Landau-Kolmogorov-type inequality will be discussed in Chapter 2. We will use Chebyshev-Euler splines to obtain some new results of the Landau-Kolmogorov-type inequality on a finite interval. These results can be applied not only to finite intervals but to R or R^+ as well.

In Chapter 3, estimates of mixed partial derivatives by iterated directional derivatives of the same order in R^d are given. This inequality will be proved for $1 \leq p \leq \infty$ on any open set in R^d with the best constant 1. This result can be extended to other function spaces which satisfy certain conditions.

The last three chapters, that is, Chapter 4, 5 and 6 are devoted to the study of the multivariate Bernstein-Durrmeyer operators. Some basic properties and strong converse inequalities of type C and D of these operators will be given in Chapter 4. The strong converse inequality of type A for $1 \leq p \leq \infty$, $d \leq 3$ and $1 < p < \infty$, and any d will be proved in Chapter 5. For $p = 1$ (or $p = \infty$) and $d > 3$, a somewhat weaker result, namely a strong converse inequality of type B will be obtained. In Chapter 6, we will establish the weak-type direct and converse

inequalities between the approximation of the multivariate Bernstein-Durrmeyer operators and the best algebraic polynomial approximation.

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CHAPTER 1

INTRODUCTION

1.1. Preamble

In this thesis, we consider three kinds of inequalities, the Landau-Kolmogorov-type inequality, inequalities involving mixed and directional derivatives, and direct and converse inequalities of Bernstein-type operators.

It is well known that the Landau-Kolmogorov inequality was initiated by Landau in 1913. It can be written as

$$(1.1.1) \qquad \left\|f^{(\ell)}\right\|_I \le C_{n,\ell} \left\|f\right\|_I^{1-\frac{\ell}{n}} \left\|f^{(n)}\right\|_I^{\frac{\ell}{n}}, \qquad 1 \le \ell \le n-1.$$

Landau [10] proved the above inequality with sharp constants for $n = 2$, $I = R$ or $I = R^+$. In 1939, Kolmogorov [9] proved (1.1.1) on R for all n and determined the best constants $C_{n,\ell}$. There are several alternate proofs of Kolmogorov's theorem available in the literature.

Several papers dealt with (1.1.1) for $I = R^+$, but their constants are not the best possible when $n \ge 4$. In 1970, Schoenberg and Cavaretta [11] gave a procedure to compute the best constants of (1.1.1) for $I = R^+$ for all n. However, the constants are given as limits of some sequences and are not explicit.

When we consider the Landau-Kolmogorov inequality on a finite interval (1.1.1) is not valid. Instead of I on the left hand side of (1.1.1) we need either

I', a subinterval of I, or an additional term on the right hand side. Gorny [8] started the study of the above inequality on a finite interval in 1939. This problem is still being investigated and I hope to make a contribution towards its solution.

One can consider the Landau-Kolmogorov inequality in other metric spaces. In this thesis, we will consider this inequality only in uniform norm on the real line R or part of R. We will use Chebyshev-Euler splines to obtain some new results for the Landau-Kolmogorov inequality on a finite interval. These results can be applied not only to finite intervals, but to R, or R^+, as well.

When we deal with multivariate differentiation in R^d a natural question arises: how can one estimate mixed partial derivatives by iterated directional derivatives of the same order in R^d? Estimates were proved implicitly in the effort to characterize the K-functional of the pair (L_p, W_p^r) [1, Ch. 5]. Though this fact is not stated explicitly, the Kemperman Lemma [1, Lemma 4.11] actually implies

$$\left\| \frac{\partial^k f}{\partial \xi_1 \cdots \partial \xi_k} \right\|_{L_p(D)} \le C(k) \sup_{\xi} \left\| \frac{\partial^k f}{\partial \xi^k} \right\|_{L_p(D)} \tag{1.1.2}$$

(with $C(k)$ that increases geometrically when $k \to \infty$ [1, Ch. 5]) for a domain D which is nice enough and any k directions $\xi_1, \ldots, \xi_k$ in R^d. (This will be shown explicitly in Chapter 3.) This inequality may be also accomplished by resorting to the Laplacian and the Riesz transforms, where p is restricted to

$1 < p < \infty$ and the constant is of the order $(dp/(p-1))^k$ (see [12]). We will prove the above inequality for $1 \le p \le \infty$ with the best constant, that is, 1.

We will then consider the Bernstein-Durrmeyer operators. These operators were introduced by Durrmeyer [7] in 1967 (for the one dimensional case) and studied by Derriennic [3] (1981) who also introduced the multivariate analogue in [4]. Their approximation behaviour was discussed extensively by Derriennic [3], and by Ditzian and Ivanov [5].

Recently, Berens and Xu [2] found a useful relation between the eigenvalues of M_n and M_{n-1}. From that, they obtained a strong converse inequality of type D, in the terminology of [6], that was given later.

The Bernstein-Durrmeyer operators possess some desirable properties, most notably, commutativity (between themselves and with the Voronovskaja-type differential operator), self-adjointness, and simple expansion by Legendre polynomials. These properties of the Bernstein-Durrmeyer operators make them simpler in some respects than Bernstein polynomial operators. Therefore, we may be able to prove some approximation results for them that we are not able to prove for Bernstein polynomial approximation, or its L_p analogue, the Kantorovich operator.

Derriennic [4] (1985) generalized the above operators to the multivariate case. In this thesis, we will study these multivariate Bernstein-Durrmeyer operators, give some of their basic properties and prove the strong converse inequality of type A for $1 \le p \le \infty$, $d \le 3$ or $1 < p < \infty$, any d and of type B for $p = \infty$, 1 and $d > 3$. Finally, we will compare this approximation process with the best algebraic polynomial approximation using the weak-type inequalities in both directions.

1.2. Outline of the thesis

The Landau-Kolmogorov inequality will be discussed in Chapter 2. We will use the Chebyshev-Euler splines to prove that if $\|f\|_I \le \|T_{n,k}\|_I$, $\|f^{(n)}\|_I \le \|T_{n,k}^{(n)}\|_I$ and $n + k + \ell$ is even, then

$$|f^{(\ell)}(0)| \le |T_{n,k}^{(\ell)}(0)|, \qquad 1 \le \ell \le n-1 \tag{1.2.1}$$

where $T_{n,k}(x)$ is the Chebyshev-Euler spline of degree n with k knots and $I = [-1,1]$. Aided by the above inequality we can prove that

$$\|f^{(\ell)}\|_{I'} \le C'_{n,\ell} \|f\|_I^{1-\frac{\ell}{n}} \|f^{(n)}\|_I^{\frac{\ell}{n}} \tag{1.2.2}$$

with the best constant $C'_{n,\ell}$, where I' is a subinterval of I, but we do not know whether or not this is the biggest subinterval with the above property. In the last section of Chapter 2, we will replace I by R, the real line, and obtain a new proof of Kolmogorov's theorem.

In Chapter 3, we will prove that

$$\left\|\frac{\partial}{\partial \xi_1} \cdots \frac{\partial}{\partial \xi_k} f\right\|_L \le \sup_{\xi} \left\|\left(\frac{\partial}{\partial \xi}\right)^k f\right\|_L \tag{1.2.3}$$

where ξ and $\xi_1, \ldots, \xi_k$ are unit vectors in R^d, $L = L_p(D)$, $1 \le p < \infty$, or $L = C(D)$, D being an open set in R^d. From the validity of (1.2.3) for $C(R^d)$, we may deduce the result for other function spaces, namely any Banach space of functions or generalized functions on R^d for which translation is a strong, weak, or weak* continuous isometry.

In the following three chapters, we will study the multivariate Bernstein-Durrmeyer operators. Some basic properties and strong converse inequalities of type C and D (in the terminology of [6]) of these operators will be given in Chapter 4. The strong converse inequality of type A for $1 \le p \le \infty$, $d \le 3$ and $1 < p < \infty$, and any d will be proved in Chapter 5. For $p = 1$ or $p = \infty$, and $d > 3$, we will prove a somewhat weaker result, namely a strong converse inequality of type B in the terminology of [6]. In the last chapter, that is, Chapter 6, we will obtain the weak-type direct and converse inequalities between the approximation of the multivariate Bernstein-Durrmeyer operators and the best algebraic polynomial approximation.

References will be given at the end of each chapter.

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CHAPTER 2

LANDAU-KOLMOGOROV INEQUALITY ON A FINITE INTERVAL

2.1. Introduction

In 1913, Landau [11] proved that

$$\text{(2.1.1)} \qquad \left\|f^{(\ell)}\right\|_I \leq C_{n,\ell} \|f\|_I^{1-\frac{\ell}{n}} \left\|f^{(n)}\right\|_I^{\frac{\ell}{n}}, \qquad 1 \leq \ell \leq n-1$$

for $n = 2$, $I = R$ or $I = R^+$ with the sharp constants $\sqrt{2}$ and 2, respectively. In 1939, Kolmogorov [10] solved (2.1.1) on R for all n and ℓ and determined the best constants. There are several alternate proofs of (2.1.1) for $I = R$ of which we mention those by Bang [1] (1941), Cavaretta [3] (1974), and de Boor and Schoenberg [2] (1976).

Hadamard [7] (1914), Gorny [6] (1939) and Matorin [12] (1958) were concerned with (2.1.1) for $I = R^+$, but their constants were not optimal when $n \geq 4$. In 1970, Schoenberg and Cavaretta [14] gave a procedure to find the best constant for the inequality for $I = R^+$, and all n and ℓ. The constants were given as limits of some sequences and are not explicit.

Several papers have dealt with inequalities similar to (2.1.1) on a finite interval. Of these, we mention Gorny [6] (1939), Kallioniemi [8] (1976), Pinkus [13] (1978) and Fabry [5] (1987). In the present work, Chebyshev-Euler splines are used to prove the inequality generalizing the Landau-Kolmogorov-Gorny inequality with the best constant in some sense. These results are generalizations of works by Fabry [5] and Kallioniemi [8]. We will prove that

$$\|f^{(\ell)}\|_{[-1+\delta,1-\delta]} \leq \frac{|T_{n,k}^{(\ell)}(0)|}{\rho_{n,k}^{1-\frac{\ell}{n}}(2^{n-1}\cdot n!)^{\frac{\ell}{n}}} \|f\|_{[-1,1]}^{1-\frac{\ell}{n}} \|f^{(n)}\|_{[-1,1]}^{\frac{\ell}{n}}, \tag{2.1.2}$$

where $T_{n,k}(x)$ is the Chebyshev-Euler spline of degree n with k knots, $\rho_{n,k} = \|T_{n,k}\|_{[-1,1]}$ and $\delta = \left(\frac{2^{n-1}\cdot n!\,\|f\|_{[-1,1]}}{\rho_{n,k}\|f^{(n)}\|_{[-1,1]}}\right)^{1/n}$. The constant $\frac{|T_{n,k}^{(\ell)}(0)|}{\rho_{n,k}^{1-\frac{\ell}{n}}(2^{n-1}\cdot n!)^{\frac{\ell}{n}}}$ can not be replaced by any smaller one.

If we use a sequence of intervals $[-A_\ell, A_\ell]$ such that $A_\ell \to \infty$, we can derive a new proof of Kolmogorov's theorem for R. Therefore, one obtains a uniform approach to the Landau-Kolmogorov problem by using the Chebyshev-Euler splines (see also Schoenberg and Cavaretta [14] for $I = R^+$).

2.2. Properties of the Chebyshev-Euler splines

In order to solve the Landau problem on a finite interval we consider the following perfect splines defined on the interval $I = [-1, 1]$:

$$T(x) = 2^{n-1}x^n + \sum_{i=1}^{k}(-1)^i 2^n (x - \xi_i)_+^n + \sum_{j=0}^{n-1} a_j x^j \tag{2.2.1}$$

where a_j, $0 \le j < n$ and ξ_i, $1 \le i \le k$ are free parameters, and

$$-1 < \xi_1 < \xi_2 < \cdots < \xi_k < 1. \tag{2.2.2}$$

Let T be the collection of all perfect splines of the form (2.2.1).

DEFINITION 2.2.1: We define the perfect spline $T_{n,k}(x)$ as the function of form (2.2.1) such that

$$\|T_{n,k}\|_I = \inf_{T \in \mathrm{T}} \|T\|_I \, . \tag{2.2.3}$$

We call $T_{n,k}(x)$ the Chebyshev-Euler spline of degree n with k knots (see [4] and [14]).

If for $T(x) \in \mathrm{T}$ there are m points $-1 \le t_1 \le t_2 \le \cdots \le t_m \le 1$ such that

$$T(t_i) = (-1)^{i_0 + i} \|T\|_I \, , \qquad 1 \le i \le m$$

for some fixed i_0 (0 or 1), we say that $T(x)$ has m points of equioscillation.

Now, we cite an important theorem from [4], yielding some basic properties of the Chebyshev-Euler splines. In the next section, we will use those properties to prove our main results. This theorem guarantees the existence and uniqueness of $T_{n,k}(x)$.

THEOREM 2.2.2 (CAVARETTA [4]). *There is a unique perfect spline* $T_{n,k}(x)$ *of degree* n *with* k *simple knots satisfying (2.2.3).* $T_{n,k}(x)$ *has precisely* $n+k+1$ *points of equioscillation, and is in fact the Chebyshev-Euler spline.*

The following proposition was stated in [14] but no proof was given there. For the sake of completeness, we will prove it here.

PROPOSITION 2.2.3. *For* $T_{n,k}(x)$ *given in Definition 2.2.1,*

$$T_{n,k}(-x) = (-1)^{n+k} T_{n,k}(x).$$

PROOF: Suppose $-1 < \xi_1 < \xi_2 < \cdots < \xi_k < 1$ are the k simple knots of $T_{n,k}(x)$, and

$$T_{n,k}(x) = 2^{n-1}x^n + \sum_{i=1}^{k}(-1)^i\, 2^n(x-\xi_i)_+^n + \sum_{\ell=0}^{n-1} a_\ell\, x^\ell.$$

Since

$$(-x-\xi_i)_+^n = (-1)^n(x+\xi_i)^n - (-1)^n(x+\xi_i)_+^n,$$

we have

$$\begin{aligned} T_{n,k}(-x) &= (-1)^n \Bigg[2^{n-1}x^n + \sum_{i=1}^{k} 2^n(-1)^i x^n + \sum_{i=1}^{k}(-1)^i \sum_{\ell=0}^{n-1} \binom{n}{\ell} \xi_i^{n-\ell} x^\ell \\ &\qquad + \sum_{\ell=0}^{n-1}(-1)^{n+\ell} a_\ell \, x^\ell \Bigg] \\ &= (-1)^{n+k} \Bigg[2^{n-1}x^n + \sum_{j=1}^{k}(-1)^j \, 2^n (x-\eta_j)_+^n + P_{n-1}(x) \Bigg] \\ &\equiv (-1)^{n+k} \widehat{T}_{n,k}(x) \end{aligned}$$

where $j = k - i + 1$, $\xi_i = -\eta_{k-i+1} = -\eta_j$, and

$$P_{n-1}(x) = (-1)^k \sum_{\ell=0}^{n-1} \Bigg[(-1)^{n+\ell} a_\ell + 2^n \binom{n}{\ell} \sum_{i=1}^{k} (-1)^i \, \xi_i^{n-\ell} \Bigg] x^\ell$$

is a polynomial of degree $n-1$. Thus $\widehat{T}_{n,k}(x)$ is a perfect spline of the form (2.2.1), and $\|T_{n,k}\|_I = \|\widehat{T}_{n,k}\|_I$. Therefore, by the uniqueness of $T_{n,k}(x)$, we have

$$T_{n,k}(x) = \widehat{T}_{n,k}(x),$$

and

$$\xi_i = -\xi_{k-i+1}, \qquad i = 1, 2, \ldots, k.$$

This completes the proof of Proposition 2.2.3. □

PROPOSITION 2.2.4 (KARLIN [9]). *Suppose* $\rho_{n,k} \equiv \|T_{n,k}\|_I$ *with* $T_{n,k}(x)$ *satisfying (2.2.3). Then* $\rho_{n,k}$ *is strictly decreasing in* k *and*

$$\lim_{k \to +\infty} \rho_{n,k} = 0$$

([9, p. 409, Lemma 5.7]).

2.3. The main results

In this section we discuss the main results of the paper. First we prove (2.1.2) and give another version of the Landau-Kolmogorov inequality on the finite interval. Then we derive a new proof of Kolmogorov's theorem on the real line R.

In order to prove (2.1.2), we need the following key result, which was proved in [8] for $k = 0$. In that case, $T_{n,k}(x)$ is exactly the Chebyshev polynomial of degree n.

THEOREM 2.3.1. *Let $f(x) \in C^{n-1}[-1,1]$, and $f^{(n-1)}(x)$ be absolutely continuous such that*

$$\|f\| \le \rho_{n,k}, \qquad \|f^{(n)}\| \le 2^{n-1} \cdot n!.$$

Then, for even $n+k+\ell$ and $1 \le \ell \le n-1$, we have

$$|f^{(\ell)}(0)| \le |T_{n,k}^{(\ell)}(0)|. \tag{2.3.1}$$

The constant $|T_{n,k}^{(\ell)}(0)|$ on the right hand side of (2.3.1) cannot be replaced by any smaller one.

PROOF: Without loss of generality, we assume that $n+k$ and ℓ are both odd. (The case where both $n+k$ and ℓ are even can be treated in a similar manner.) Set

$$F(x) = (f(x) - f(-x))/2.$$

Then $F(x)$ and $T_{n,k}(x)$ are both odd functions, and

$$F^{(i)}(x) = \big(f^{(i)}(x) - (-1)^i f^{(i)}(-x)\big)/2, \qquad 0 \le i \le n.$$

Hence

$$|F^{(\ell)}(0)| = |f^{(\ell)}(0)|,$$

and

$$\|F\| \le \rho_{n,k}, \qquad \|F^{(n)}\| \le 2^{n-1} \cdot n!.$$

We now have only to show that

$$|F^{(\ell)}(0)| \le |T_{n,k}^{(\ell)}(0)|.$$

Assuming this is not so, there exists a constant α, $\alpha > 1$, or $\alpha < -1$, such that

$$F^{(\ell)}(0) = \alpha T_{n,k}^{(\ell)}(0).$$

We assume $\alpha > 1$ and the case $\alpha < -1$ can be treated in a similar manner. Define $h(x) : [-1,1] \to R$ by

$$h(x) \equiv \alpha T_{n,k}(x) - F(x),$$

then $h(x)$ is an odd function.

Since $\|F\| \le \rho_{n,k}$ and $T_{n,k}(x)$ has $n+k+1$ points of equioscillation by Theorem 2.2.2, $h(x)$ must have at least $n+k$ zeros in $[-1,1]$. By the Rolle theorem, $h^{(\ell-1)}(x)$ must then have at least $n+k+1-\ell$ simple zeros in $(-1,1)$. Observing also that $h^{(\ell-1)}(x)$ is an odd function, $h^{(\ell-1)}(0) = 0$. Thus, by the Rolle theorem again, $h^{(\ell)}(x)$ must have at least $n+k-\ell$ zeros in $(-1,0)\cup(0,1)$. On the other hand, by the definition of $h(x)$, $h^{(\ell)}(0)=0$. Therefore, $h^{(\ell)}(x)$ has at least $n+k-\ell+1$ zeros in $(-1,1)$ and $h^{(n-1)}(x)$ will have at least $k+2$ zeros in $(-1,1)$. This implies that there exists an integer i_0, $1\le i_0\le k-1$, such that $h^{(n-1)}(x)$ has at least two zeros in $[\xi_{i_0},\xi_{i_0+1}]$. We select two of these zeros, say η_1 and η_2, and assume $\eta_1<\eta_2$. Thus,

$$\begin{aligned}0=\left|h^{(n-1)}(\eta_2)\right| &= \left|h^{(n-1)}(\eta_2)-h^{(n-1)}(\eta_1)\right| \\ &= \left|\int_{\eta_1}^{\eta_2}\left(\alpha T_{n,k}^{(n)}(x)-F^{(n)}(x)\right)dx\right| \\ &\ge \alpha(\eta_2-\eta_1)2^{n-1}\cdot n!-(\eta_2-\eta_1)2^{n-1}\cdot n!>0,\end{aligned}$$

which is a contradiction. □

THEOREM 2.3.2. *Let $f(x)\in C^{n-1}[-1,1]$, and $f^{(n-1)}(x)$ be absolutely continuous, then for an even integer $n+k+\ell$,*

$$\left\|f^{(\ell)}\right\|_{[-1+\delta,1-\delta]}\le\frac{|T_{n,k}^{(\ell)}(0)|}{\rho_{n,k}^{1-\frac{\ell}{n}}(2^{n-1}\cdot n!)^{\frac{\ell}{n}}}\,\|f\|^{1-\frac{\ell}{n}}\left\|f^{(n)}\right\|^{\frac{\ell}{n}} \tag{2.3.2}$$

where $\delta=\left(\dfrac{2^{n-1}\cdot n!\,\|f\|}{\rho_{n,k}\left\|f^{(n)}\right\|}\right)^{1/n}$ and $1\le\ell\le n-1$. Furthermore, the constant on the right hand side of (2.3.2) can not be replaced by any smaller one.

PROOF: For any $x_0\in[-1+\delta,1-\delta]$, define $F(x):[-1,1]\to R$ by

$$F(x)=\rho_{n,k}f(x_0+\delta x)/\,\|f\|\,.$$

Then

$$\|F\| \le \rho_{n,k}, \qquad \left\|F^{(n)}\right\| \le 2^{n-1} \cdot n!,$$

and

$$\left|F^{(\ell)}(x)\right| = \rho_{n,k}\delta^{\ell} f^{(\ell)}(x_0 + \delta x) / \|f\| \, .$$

Applying Theorem 2.3.1, we have

$$\begin{aligned} \left|f^{(\ell)}(x_0)\right| &= \left|F^{(\ell)}(0)\right| \|f\| / (\rho_{n,k} \cdot \delta^{\ell}) \\ &\le \frac{\left|T_{n,k}^{(\ell)}(0)\right|}{\rho_{n,k}^{1-\frac{\ell}{n}} (2^{n-1} \cdot n!)^{\frac{\ell}{n}}} \|f\|^{1-\frac{\ell}{n}} \left\|f^{(n)}\right\|^{\frac{\ell}{n}}. \end{aligned}$$

This completes the proof. □

For the general finite interval $[a, b]$, using linear transformation, we have

COROLLARY 2.3.3. *Let* $f(x) \in C^{(n-1)}[a, b]$, $f^{(n-1)}(x)$ *be absolutely continuous, then for even* $n + k + \ell$,

$$\left\|f^{(\ell)}\right\|_{[a+\delta, b-\delta]} \le \frac{\left|T_{n,k}^{(\ell)}(0)\right|}{\rho_{n,k}^{1-\frac{\ell}{n}} (2^{n-1} \cdot n!)^{\frac{\ell}{n}}} \|f\|_{[a,b]}^{1-\frac{\ell}{n}} \left\|f^{(n)}\right\|_{[a,b]}^{\frac{\ell}{n}} \tag{2.3.3}$$

where $\delta = \left(\dfrac{2^{n-1} \cdot n! \, \|f\|_{[a,b]}}{\rho_{n,k} \left\|f^{(n)}\right\|_{[a,b]}} \right)^{1/n}$ *and* $1 \le \ell \le n - 1$.

In Theorem 2.3.1 we use $|T_{n,k}^{(\ell)}(0)|$ to estimate $|f^{(\ell)}(0)|$. Actually, using the same argument, we can estimate $|f^{(\ell)}(\pm 1)|$ by $|T_{n,k}^{(\ell)}(\pm 1)|$. This is a generalization of Theorem 1 in [5], that theorem was proved only for the Chebyshev polynomials.

THEOREM 2.3.4. *Suppose* $f(x)$ *satisfies the conditions in Theorem 2.3.1. Then, for* $1 \leq \ell \leq n-1$, *we have*

$$|f^{(\ell)}(\pm 1)| \leq |T_{n,k}^{(\ell)}(\pm 1)|. \tag{2.3.4}$$

The constant $|T_{n,k}^{(\ell)}(\pm 1)|$ *can not be replaced by any smaller one.*

REMARK . A stronger result than Theorem 2.3.4 was obtained by Schoenberg and Cavaretta in [14]. In fact, the interval can be a little smaller, but the proof there is quite complicated and only a sketch of the proof is given.

Using Theorem 2.3.4, we can also estimate the two parts of the interval $[-1, 1]$ adjacent to ± 1. Thus, combining with Theorem 2.3.1, we will obtain another version of the Landau-Kolmogorov inequality on the finite interval. This improves the result of Theorem 2 in [5], in particular, for the middle part of the interval.

THEOREM 2.3.5. *Let* $f(x) \in C^{n-1}[-1,1]$, *and* $f^{(n-1)}(x)$ *be absolutely continuous, then for* $n+k+\ell$ *even and* $1 \le \ell \le n-1$,

$$\|f^{(\ell)}\|_{I_i} \le |T^{(\ell)}_{n,k}(i)| \left(\frac{\|f\|}{\rho_{n,k}}\right)^{1-\frac{\ell}{n}} \left[\max\left\{\frac{\|f^{(n)}\|}{2^{n-1}\cdot n!}, \left(\frac{3}{2}\right)^n \frac{\|f\|}{\rho_{n,k}}\right\}\right]^{\frac{\ell}{n}} \tag{2.3.5}$$

where $I_i = [-1+2(i+1)/3, -1+2(i+2)/3]$, $i = -1, 0, 1$.

PROOF: For $i=-1,0,1$, let $x_0 \in I_i$, define $F_i(x): [-1,1] \to R$ by

$$F_i(x) = \rho_{n,k} f(x_0 + (x-i)\mu) / \|f\|$$

where $\mu = \min\{2/3, [2^{n-1}\cdot n! \|f\| / (\rho_{n,k}\|f^{(n)}\|)]^{\frac{1}{n}}\}$. Then, $F_i(x)$ is well defined, and

$$\|F_i\| \le \rho_{n,k}, \qquad \|F_i^{(n)}\| \le 2^{n-1}\cdot n!, \qquad i=-1,0,1.$$

Applying Theorem 2.3.4 or Theorem 2.3.1 and observing that

$$|f^{(\ell)}(x_0)| = \|f\| \, |F^{(\ell)}(i)| / (\rho_{n,k}\mu^\ell), \qquad i=-1,0,1,$$

we have

$$|f^{(\ell)}(x_0)| \le |T^{(\ell)}_{n,k}(i)| \left(\frac{\|f\|}{\rho_{n,k}}\right)^{1-\frac{\ell}{n}} \left[\max\left\{\frac{\|f^{(n)}\|}{2^{n-1}\cdot n!}, \left(\frac{3}{2}\right)^n \frac{\|f\|}{\rho_{n,k}}\right\}\right]^{\frac{\ell}{n}}.$$

This completes the proof of Theorem 2.3.5. □

REMARK . Since $n+k+\ell$ can be any integer (even or odd) in Theorem 2.3.4, $n+k+\ell$ can be odd in the inequality (2.3.5) for $i=\pm 1$. It is also unnecessary to divide $[-1,1]$ into three equal parts, but in this case, the constant $(3/2)^n$ in front of $\|f\|/\rho_{n,k}$ will be replaced by a different constant.

In Corollary 2.3.3, one can obtain the inequality (2.3.5) by linear transformation for a general finite interval $[a,b]$. Now we can derive a new proof of the Landau-Kolmogorov inequality on R.

For convenience, we normalize $T_{n,k}(x)$ first, writing

$$S_{n,k}(x) = \rho_{n,k}^{-1} T_{n,k}(\rho_{n,k}^{\frac{1}{n}} x). \tag{2.3.6}$$

Clearly $S_{n,k}(x)$ is defined on $[-\rho_{n,k}^{-\frac{1}{n}}, \rho_{n,k}^{-\frac{1}{n}}]$, and satisfies

$$\|S_{n,k}\| = 1, \qquad \|S_{n,k}^{(n)}\| = 2^{n-1} \cdot n!.$$

LEMMA 2.3.6. *For $S_{n,k}(x)$ defined in (2.3.6), we have*

$$\left|S_{n,0+i}^{(\ell)}(0)\right| \geq \left|S_{n,2+i}^{(\ell)}(0)\right| \geq \cdots \geq \left|S_{n,2k+i}^{(\ell)}(0)\right| \geq \cdots, \quad i=0 \text{ or } 1 \tag{2.3.7}$$

where $1 \leq \ell \leq n-1$ and $n+\ell+i$ is even.

PROOF: Without loss of generality, assume that $i = 0$ and $n + \ell$ is even. Set

$$F_{n,2k+2}(x) = \frac{\rho_{n,2k}}{\rho_{n,2k+2}} T_{n,2k+2}\left(\left(\frac{\rho_{n,2k+2}}{\rho_{n,2k}}\right)^{1/n} x\right).$$

Since $\rho_{n,2k+2}/\rho_{n,2k} \leq 1$, $F_{n,2k+2}(x)$ is well defined on $[-1,1]$, and

$$\|F_{n,2k+2}\| \leq \rho_{n,2k}, \qquad \|F^{(n)}_{n,2k+2}\| \leq 2^{n-1} \cdot n!.$$

By Theorem 2.3.1,

$$\left|F^{(\ell)}_{n,2k+2}(0)\right| = \frac{\rho_{n,2k}}{\rho_{n,2k+2}} \left(\frac{\rho_{n,2k+2}}{\rho_{n,2k}}\right)^{\frac{\ell}{n}} \left|T^{(\ell)}_{n,2k+2}(0)\right| \leq \left|T^{(\ell)}_{n,2k}(0)\right|,$$

or

$$\frac{\left|T^{(\ell)}_{n,2k}(0)\right|}{\rho_{n,2k}^{1-\frac{\ell}{n}}} \geq \frac{\left|T^{(\ell)}_{n,2k+2}(0)\right|}{\rho_{n,2k+2}^{1-\frac{\ell}{n}}}.$$

Thus,

$$\left|S^{(\ell)}_{n,2k}(0)\right| \geq \left|S^{(\ell)}_{n,2k+2}(0)\right|.$$

□

THEOREM 2.3.7. *Let $f(x) \in C^{n-1}(-\infty,\infty)$, $f^{(n-1)}(x)$ be absolutely continuous, then*

(2.3.8) $$\|f^{(\ell)}\|_{(-\infty,\infty)} \leq C_{n,\ell} \, \|f\|^{1-\frac{\ell}{n}}_{(-\infty,\infty)} \, \|f^{(n)}\|^{\frac{\ell}{n}}_{(-\infty,\infty)}$$

where $C_{n,\ell} = \lim_{k\to\infty} |S^{(\ell)}_{n,2k+i}(0)|/(2^{n-1}\cdot n!)^{\frac{\ell}{n}}$, *and* $i=0$ *or* 1 *such that* $n+\ell+i$ *is even. Moreover,* $C_{n,\ell}$ *is the Kolmogorov's constant for* R.

PROOF: Suppose that $i=0$ and $n+\ell$ is even. Applying Corollary 2.3.3, we have

$$\|f^{(\ell)}\|_{(-\infty,\infty)} \le \frac{|S^{(\ell)}_{n,2k}(0)|}{(2^{n-1}\cdot n!)^{\frac{\ell}{n}}}\,\|f\|^{1-\frac{\ell}{n}}_{(-\infty,\infty)}\,\|f^{(n)}\|^{\frac{\ell}{n}}_{(-\infty,\infty)}.$$

Since k is arbitrary, and by Lemma 2.3.6,

$$\|f^{(\ell)}\|_{(-\infty,\infty)} \le C_{n,\ell}\,\|f\|^{1-\frac{\ell}{n}}_{(-\infty,\infty)}\,\|f^{(n)}\|^{\frac{\ell}{n}}_{(-\infty,\infty)}.$$

Now, consider the function sequence $\{S_{n,2k}(x)\}_{k=0}^{\infty}$. Let N be any integer. By Proposition 2.2.4, there exists an integer K such that

$$\rho^{-\frac{1}{n}}_{n,2k} \ge N+1, \qquad \text{for } k \ge K.$$

Using definition of $S_{n,2k}(x)$ and applying Theorem 2.3.4, we now have

$$\|S^{(\ell)}_{n,2k}\|_{[-N,N]} \le \left|T^{(\ell)}_{n,0}(\pm 1)\right|, \qquad 0\le \ell \le n, \quad k\ge K.$$

Hence, for any $x_1,\, x_2 \in [-N,N]$, we have

$$\left|S^{(\ell)}_{n,2k}(x_1) - S^{(\ell)}_{n,2k}(x_2)\right| \le \left|T^{(\ell+1)}_{n,0}(\pm 1)\right|\,|x_1-x_2|, \qquad 0\le \ell\le n-1,\ k\ge K.$$

Therefore the functions $\{S_{n,2k}^{(\ell)}(x)\}_{k=0}^{\infty}$ $(0 \le \ell \le n-1)$ are uniformly bounded and equicontinuous on $[-N, N]$.

Using the Arzela-Ascoli theorem, we can find a subsequence $\{S_{n,2k_i}(x)\}_{i=1}^{\infty}$ of $\{S_{n,2k}(x)\}_{k=K}^{\infty}$, such that $\{S_{n,2k_i}^{(\ell)}(x)\}_{i=1}^{\infty}$ $(0 \le \ell \le n-1)$ are all uniformly convergent on $[-N, N]$. By the diagonalization process, we pick a subsequence $\{S_{n,2k_j}(x)\}_{j=1}^{\infty}$ of $\{S_{n,2k_i}(x)\}_{i=1}^{\infty}$, such that $\{S_{n,2k_j}^{(\ell)}(x)\}_{j=1}^{\infty}$ $(0 \le \ell \le n-1)$ are all uniformly convergent on any finite interval.

The limit function of the above process, $E_n(x)$, satisfies $E_n(x) \in C^{n-1}(-\infty, \infty)$, $E_n^{(n-1)}(x)$ is absolutely continuous,

$$\|E_n\|_{(-\infty,\infty)} \le 1, \qquad \|E_n^{(n)}\|_{(-\infty,\infty)} \le 2^{n-1} \cdot n!,$$

and

$$\left|E_n^{(\ell)}(0)\right| = \lim_{k\to\infty} \left|S_{n,2k}^{(\ell)}(0)\right|, \qquad 0 \le \ell \le n-1.$$

Therefore, $E_n(x)$ is an extremal function of (2.3.8), and $C_{n,\ell}$ should be the Kolmogorov's constant for R. This completes the proof. □

By Kolmogorov's theorem, we know $C_{n,\ell}$ explicitly, but it is difficult to calculate $S_{n,2k+i}^{(\ell)}(0)$ for large n and k. However, Theorem 2.3.7 established the relation between the Kolmogorov's constant $C_{n,\ell}$ and $\{S_{n,2k+i}^{(\ell)}(0)\}_{k=0}^{\infty}$. For

$n = 2$ or 3, we can calculate $S^{(\ell)}_{n,2k+i}$, which yields exactly the Kolmogorov's constants $C_{n,\ell}$. Actually all terms in (2.3.7) have the same value for $n = 2$ and $n = 3$.

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CHAPTER 3

MIXED AND DIRECTIONAL DERIVATIVES*

3.1. Introduction

Estimates of mixed derivatives by directional derivatives were proved (implicitly) in the effort to characterize the K-functional of the pair (L_p, W_p^r) (see [1, Ch. 5]). Though this fact is not stated explicitly, the Kemperman Lemma (see [1, Lemma 4.11, p. 338]) actually implies for a domain D which is nice enough

$$(3.1.1) \qquad \left\| \frac{\partial^k f}{\partial \xi_1 \ldots \partial \xi_k} \right\|_{L_p(D)} \le C(k) \sup_{\xi} \left\| \frac{\partial^k f}{\partial \xi^k} \right\|_{L_p(D)}$$

with $C(k)$ that increases geometrically with k (see [1, Ch. 5]). In this paper we will show that $C(k)$ can be replaced by 1, which is obviously the best possible constant, as the directions $\xi_1, \ldots \xi_k$ are any k directions, or unit vectors, in R^d. We remark also that here the assumption that D is open in R^d is sufficient. Furthermore, the result is valid for many other spaces, as will be shown in Section 3.5. We hope that this inequality will help in settling other problems, for example the investigation of the still open problem of the best constant for the multivariate Landau-Kolmogorov inequality (see [2]). We further believe that (3.1.1) with the elegant best constant $C(k) = 1$ is desirable by itself.

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3.2. The local result

The local version of the result of this paper is given in the following theorem, which is essentially the basis for all other results of the paper.

THEOREM 3.2.1. *Suppose* $\frac{\partial^k}{\partial \xi^k} f(x)$ *exists and is continuous in a neighbourhood* $U \subset R^d$ *of* x_0 *for any direction* ξ. *Then the mixed derivative* $\frac{\partial^k}{\partial \xi_1 \dots \partial \xi_k} f(x)$ *exists, is continuous in a neighbourhood of* x_0, *and*

$$\left|\frac{\partial}{\partial \xi_1} \cdots \frac{\partial}{\partial \xi_k} f(x_0)\right| \leq \sup_{\xi}\left|\frac{\partial^k}{\partial \xi^k} f(x_0)\right|. \tag{3.2.1}$$

REMARK. We note that ξ_i are any k directions in R^d.

We observe that Theorem 3.2.1 implies for an open domain D and $\frac{\partial^k}{\partial \xi^k} f(x) \in C(D)$ that $\frac{\partial}{\partial \xi_1} \cdots \frac{\partial}{\partial \xi_k} f(x) \in C(D)$ and

$$\left\|\frac{\partial}{\partial \xi_1} \cdots \frac{\partial}{\partial \xi_k} f(x)\right\|_{C(D)} \leq \sup_{\xi}\left\|\left(\frac{\partial}{\partial \xi}\right)^k f(x)\right\|_{C(D)}. \tag{3.2.2}$$

We will actually prove the result for the spaces of functions $L_p(D)$ and $C(D)$ simultaneously.

THEOREM 3.2.2. *Suppose* $f \in B$, $\frac{\partial^k}{\partial \xi^k} f \in B$ *for all* ξ *and* $\left\|\frac{\partial^k f}{\partial \xi^k}\right\|_B \leq M$, *where either* $B = L_p(D)$ $1 \leq p < \infty$ *or* $B = C(D)$, D *being an open set in* R^d. *Then* $\frac{\partial}{\partial \xi_1} \cdots \frac{\partial}{\partial \xi_k} f \in B$ *and*

$$\left\|\frac{\partial}{\partial \xi_1} \cdots \frac{\partial}{\partial \xi_k} f(x)\right\|_B \leq \sup_{\xi}\left\|\left(\frac{\partial}{\partial \xi}\right)^k f(x)\right\|_B. \tag{3.2.3}$$

REMARK 3.2.3. It sufficient to prove that $\frac{\partial}{\partial \xi_1} \cdots \frac{\partial}{\partial \xi_k} f \in B_\varepsilon$ and

$$\left\| \frac{\partial}{\partial \xi_1} \cdots \frac{\partial}{\partial \xi_k} f(x) \right\|_{B_\varepsilon} \leq \sup_{\xi} \left\| \frac{\partial^k}{\partial \xi^k} f(x) \right\|_B \tag{3.2.4}$$

where $B_\varepsilon = L_p(D_\varepsilon)$ or $B_\varepsilon = C(D_\varepsilon)$ and where

$$D_\varepsilon \equiv \{x;\ x + B(\varepsilon) \subset D\}, \quad B(\varepsilon) \equiv \{y; \|y\| \leq \varepsilon\} \tag{3.2.5}$$

and $\|y\|$ is the Euclidean norm in R^d. That the inequality (3.2.4) implies (3.2.3) is clear for $B = C(D)$ and follows from the theorem on monotone convergence for $B = L_p(D)$.

For the proof which will be carried through in Section 3.4 using several lemmas stated and proved in Section 3.3, we need the following concepts. For a function space on domain D, $S(D)$, we define the transformations

$$T(y)f(x) = f(x+y), \quad T(y):\ S(D) \to S(D-y) \tag{3.2.6}$$

and

(3.2.7)

$$\Delta_y f(x) = \big(T(y) - I\big) f(x) = f(x+y) - f(x), \quad \Delta_y:\ S(D) \to S(D \cap (D-y)).$$

In the next section, and when proving Theorem 3.2.2, $S(D)$ will be $L_p(D_*)$ or $C(D_*)$ where D_* would be D_ε and its translates. We give (3.2.6) and (3.2.7) in the present generality to accommodate some remarks in Section 3.5.

3.3. Some crucial lemmas

We first prove the following equivalence result.

LEMMA 3.3.1. *Suppose* $f \in L_p(D)$ *or* $f \in C(D)$. *Then the derivative* $\frac{\partial}{\partial \xi_1} \cdots \frac{\partial}{\partial \xi_k} f(x)$ *exists in* $L_p(D)$ *or* $C(D)$ *as a local strong derivative if and only if*

$$h^{-k} \Delta_{h\xi_1} \dots \Delta_{h\xi_k} f(x) \to g(x) \quad \text{in} \quad L_p(D_\varepsilon) \quad \text{or} \quad C(D_\varepsilon) \tag{3.3.1}$$

for every $\varepsilon > 0$. *In this case*

$$g(x) = \frac{\partial}{\partial \xi_1} \cdots \frac{\partial}{\partial \xi_k} f(x)$$

a.e. *in* D_ε *for* $B = L_p(D)$ *and everywhere in* D_ε *for* $B = C(D)$.

REMARK. For h small enough, which depends on ε, $\Delta_{h\xi_1} \dots \Delta_{h\xi_k}$ is defined on $L_p(D_\varepsilon)$.

PROOF: By the repeated application of the mean value theorem, we see that the first assumption implies the second for the space of continuous functions. For the space L_p the same argument is used essentially, and we write for $|h| \le \varepsilon/k$, $x \in D_\varepsilon$

$$(3.3.2) \quad h^{-k} \Delta_{h\xi_1} \dots \Delta_{h\xi_k} f(x)$$

$$= h^{-k} \int_0^h T(u_k \xi_k) \int_0^h \cdots \int_0^h T(u_1 \xi_1) \frac{\partial^k}{\partial \xi_1 \dots \partial \xi_k} f(x) du_1 \dots du_k \quad \text{a.e.}$$

Obviously, the operator $O_h : L_p(D) \to L_p(D_\varepsilon)$ given by

$$O_h \equiv h^{-k} \int_0^h T(u_k\xi_k) \int_0^h \cdots \int_0^h T(u_1\xi_1)du_1 \dots du_k$$

is bounded, and

$$\|O_h\varphi - \varphi\|_{L_p(D_\varepsilon)} \to 0 \quad \text{for all} \quad \varphi \in L_p(D).$$

We now show that for $g(x)$ given by (3.3.1) and $|t_i| < \varepsilon/k$

$$\text{(3.3.3)} \quad \frac{1}{t_1 \dots t_k} \Delta_{t_1\xi_1} \dots \Delta_{t_k\xi_k} f(x) = \frac{1}{t_1 \dots t_k} \int_0^{t_k} T(u_k\xi_k) \int_0^{t_{k-1}} \cdots \int_0^{t_1} T(u_1\xi_1) g(x) du_1 \dots du_k$$

a.e. in $D_{2\varepsilon}$ when g in (3.3.1) is in $L_p(D_\varepsilon)$ $1 \le p < \infty$, and everywhere in $D_{2\varepsilon}$ when $g \in C(D_\varepsilon)$. To show (3.3.3) we observe that for $|t_i| < \frac{\varepsilon}{k}$

$$O_{t_1 \dots t_k} \equiv \frac{1}{t_1 \dots t_k} \int_0^{t_k} T(u_k\xi_k) \cdots \int_0^{t_1} T(u_1\xi_1) du_1 \dots du_k$$

is a bounded transformation from $L_p(D_\varepsilon)$ to $L_p(D_{2\varepsilon})$ $(1 \le p < \infty)$ or from $C(D_\varepsilon)$ to $C(D_{2\varepsilon})$. In fact for t_i small enough

$$\text{(3.3.4)} \qquad \|O_{t_1 \dots t_k}\varphi\|_{B_\varepsilon} \le \|\varphi\|_B$$

and

$$\text{(3.3.5)} \qquad \lim_{t_i \to 0} \|O_{t_1 \dots t_k}\varphi(x) - \varphi(x)\|_{B_\varepsilon} = 0,$$

no matter what order of t_i we use in the limit. This implies

$$O_{t_1 \dots t_k} g(x) = \lim_{h \to 0} h^{-k} O_{t_1 \dots t_k} \Delta_{h\xi_1} \dots \Delta_{h\xi_k} f(x)$$
$$= \lim_{h \to 0} O_{h,\dots,h} \{ \frac{1}{t_1 \dots t_k} \Delta_{t_1\xi_1} \dots \Delta_{t_k\xi_k} f(x) \}$$
$$= \frac{1}{t_1 \dots t_k} \Delta_{t_1\xi_1} \dots \Delta_{t_k\xi_k} f(x)$$

for $x \in D_{2\varepsilon}$. We now write

$$g(x) = \lim_{t_1 \to 0} \dots \lim_{t_k \to 0} O_{t_1 \dots t_k} g(x)$$
$$= \lim_{t_1 \to 0} \dots \lim_{t_k \to 0} \frac{1}{t_1 \dots t_k} \Delta_{t_1\xi_1} \dots \Delta_{t_k\xi_k} f(x)$$
$$= \frac{\partial}{\partial \xi_1} \cdots \frac{\partial}{\partial \xi_k} f(x)$$

in $L_p(D_{2\varepsilon})$ or in $C(D_{2\varepsilon})$, which implies our result. □

LEMMA 3.3.2. *Theorem 3.2.2 is valid with a constant* $C(k)$ *instead of* 1, *that is*

(3.3.6) $$\left\| \frac{\partial}{\partial \xi_1} \cdots \frac{\partial}{\partial \xi_k} f(x) \right\|_B \le C(k) \sup_{\xi} \left\| \left(\frac{\partial}{\partial \xi} \right)^k f(x) \right\|_B .$$

PROOF: We have to prove the analogue of (3.2.4) with a constant $C(k)$ that does not depend on ε. This follows the Kemperman lemma ([1, p. 338]) which yields

$$\prod_{i=1}^{k} \Delta_{h\xi_i} = \sum_{S\subset\{1,\ldots,k\}} (-1)^{|S|} T(h_S^*) \Delta_{h_S}^{k} \tag{3.3.7}$$

where the sum is on all the subsets S of $\{1,\ldots,k\}$, $|S|$ is the number of elements in S

$$h_S^* \equiv h \sum_{j\in S} \xi_j \qquad \text{and} \qquad h_S \equiv h \sum_{j\in S} j^{-1}\xi_j.$$

We choose h such that $2kh < \varepsilon$ so that the transformation on the right and on the left of (3.3.5) is from B to B_ε. The constant can now be estimated, but there is no use for it as eventually the constant 1 will be achieved. □

LEMMA 3.3.3. *Under the assumptions of Theorem 3.2.2,* $\left\| \dfrac{\partial^k}{\partial\xi_1 \ldots \partial\xi_k} f \right\|_B$ *is continuous in* ξ_i.

PROOF: For our purposes it is probably sufficient to prove that $\left\| \dfrac{\partial^k}{\partial\xi_1 \ldots \partial\xi_k} f \right\|_{B_\varepsilon}$ is continuous in ξ_i, but the result is valid as stated and may be useful in the future. We can assume the change is in ξ_1. Let $\xi_1^* - \xi_1 = \delta\eta$ where η is a unit vector. By earlier consideration

$$\left| \left\| \frac{\partial}{\partial\xi} \cdots \frac{\partial}{\partial\xi_k} f(x) \right\|_B - \left\| h^{-k} \Delta_{h\xi} \Delta_{h\xi_2} \ldots \Delta_{h\xi_k} f(x) \right\|_{B_\varepsilon} \right| < \frac{\varepsilon_1}{3}$$

for $\xi = \xi_1$ and $\xi = \xi_1^*$ for ε and h small enough. Using (3.3.4), we now write

$$\begin{aligned}\left\|h^{-k}\Delta_{h\xi_1}\ldots\Delta_{h\xi_k}f(x)\right\|_{B_\epsilon} - \left\|h^{-k}\Delta_{h\xi_1^*}\Delta_{h\xi_2}\ldots\Delta_{h\xi_k}f(x)\right\|_{B_\epsilon} \\ \leq \left\|\delta\frac{1}{h\delta}\frac{1}{h^{k-1}}\Delta_{h\delta\eta}\Delta_{h\xi_2}\ldots\Delta_{h\xi_k}f(x)\right\|_{B_\epsilon} \\ \leq \delta\left\|O_{\delta h,h,\ldots,h}\frac{\partial}{\partial\eta}\frac{\partial}{\partial\xi_2}\cdots\frac{\partial}{\partial\xi_k}f\right\|_{B_\epsilon} \\ \leq \delta\left\|\frac{\partial}{\partial\eta}\frac{\partial}{\partial\xi_2}\cdots\frac{\partial}{\partial\xi_k}f\right\|_{B} \\ \leq \delta C(k)\sup\left\|\frac{\partial^k f}{\partial\xi^k}\right\|_{B}.\end{aligned}$$

The last inequality follows from Lemma 3.3.2. The result is easily concluded from the above estimate. □

3.4. Proof of the main result

We are now able to prove the main result, i.e. Theorem 3.2.2.

Proof of Theorem 3.2.2: The proof follows by induction on k. For $k = 2$ we use the identity

$$\Delta_{h\xi}\Delta_{h\eta} = \Delta^2_{h(\xi+\eta)/2} - T(h\eta)\Delta^2_{h(\xi-\eta)/2}, \tag{3.4.1}$$

set (for $\xi \neq \eta$) $\xi_1 = (\xi+\eta)/\|\xi+\eta\| = \frac{\xi+\eta}{2a}$, $\eta_1 = (\xi-\eta)/\|\xi-\eta\| = \frac{\xi-\eta}{2b}$ (where the norms are the Euclidean norm of R^d), and recall

$$(3.4.2) \qquad a^2+b^2 = \left\|\frac{\xi+\eta}{2}\right\|^2 + \left\|\frac{\xi-\eta}{2}\right\|^2 = 2\left(\left\|\frac{\xi}{2}\right\|^2 + \left\|\frac{\eta}{2}\right\|^2\right) = 1.$$

Using Lemma 3.1, (3.4.1), (3.4.2) and $h < \varepsilon/4$, we have

$$\begin{aligned}
\left\|\frac{\partial}{\partial\xi}\frac{\partial}{\partial\eta}f\right\|_{B_\varepsilon} &= \lim_{h\to 0}\|h^{-2}\Delta_{h\xi}\Delta_{h\eta}f\|_{B_\varepsilon} \\
&\le \lim_{h\to 0}\{a^2\|h^{-2}\Delta^2_{h\xi_1}f\|_{B_\varepsilon} + b^2\|h^{-2}\Delta^2_{h\eta_1}f\|_{B_{\varepsilon/2}}\} \\
&\le \left\{a^2\left\|\frac{\partial^2 f}{\partial\xi_1^2}\right\|_B + b^2\left\|\frac{\partial^2 f}{\partial\eta_1^2}\right\|_B\right\} \\
&\le \sup_\xi\left\|\frac{\partial^2 f}{\partial\xi^2}\right\|_B,
\end{aligned}$$

and therefore,

$$(3.4.3) \qquad \left\|\frac{\partial}{\partial\xi}\frac{\partial}{\partial\eta}f\right\|_B \le \left\{a^2\left\|\frac{\partial^2 f}{\partial\xi_1^2}\right\|_B + b^2\left\|\frac{\partial^2 f}{\partial\eta_1^2}\right\|_B\right\} \le \sup_\xi\left\|\frac{\partial^2 f}{\partial\xi^2}\right\|_B.$$

We proceed with the induction hypothesis on $0 < m < k$ to obtain

$$\begin{aligned}
I &\equiv \sup_{\xi_1,\dots,\xi_k}\left\|\frac{\partial}{\partial\xi_1}\cdots\frac{\partial}{\partial\xi_k}f\right\|_B \\
&\le \sup_{\xi,\xi_{m+1},\dots,\xi_k}\left\|\left(\frac{\partial}{\partial\xi}\right)^m\frac{\partial}{\partial\xi_{m+1}}\cdots\frac{\partial}{\partial\xi_k}f\right\|_B \\
&\le \sup_{\xi,\eta}\left\|\left(\frac{\partial}{\partial\xi}\right)^m\left(\frac{\partial}{\partial\eta}\right)^{k-m}f\right\|_B \\
&\equiv I(m) \le I.
\end{aligned}$$

We observe, using Lemma 3.3.3, that maximum can replace supremum in both $I(m)$ and I. We need to show that for some direction ζ

$$I = I(m) = \left\| \frac{\partial^k}{\partial \zeta^k} f \right\|_B .$$

Given $0 < m < k$, we have

$$I = I(m) = \left\| \frac{\partial^m}{\partial \xi^m} \frac{\partial^{k-m}}{\partial \eta^{k-m}} f \right\|_B .$$

If $\xi = \eta$, the theorem is proved, and therefore, we assume $\xi \neq \eta$. If k is even, we choose m so that $2m = k$, and using the first inequality in (3.4.3) repeatedly, we have

$$\begin{aligned} I = I\Big(\frac{k}{2}\Big) &= \left\| \frac{\partial^m}{\partial \xi^m} \frac{\partial^m}{\partial \eta^m} f \right\|_B \\ &\leq \sum_{\ell=0}^{m} \binom{m}{\ell} a^{2\ell} b^{2m-2\ell} \left\| \frac{\partial^{2\ell}}{\partial \xi_1^{2\ell}} \frac{\partial^{2m-2\ell}}{\partial \eta_1^{2m-2\ell}} f \right\|_B \\ &\leq a^{2m} \left\| \frac{\partial^{2m}}{\partial \xi_1^{2m}} f \right\|_B + \Big(\sum_{\ell=0}^{m-1} \binom{m}{\ell} a^{2\ell} b^{2m-2\ell} \Big) I, \end{aligned}$$

and as $a^2 + b^2 = 1$, we have

$$\sum_{\ell=0}^{m-1} \binom{m}{\ell} a^{2\ell} b^{2m-2\ell} = 1 - a^{2m},$$

and therefore,

$$I = \left\| \frac{\partial^{2m}}{\partial \xi_1^{2m}} f \right\|_B .$$

For odd k we construct a sequence $\{m_i, \xi_i, \eta_i\}$ such that $0 < m_i < k$,

$$I = I(m_i) = \left\| \frac{\partial^{m_i}}{\partial \xi_i^{m_i}} \frac{\partial^{k-m_i}}{\partial \eta_i^{k-m_i}} f \right\|_B \tag{3.4.4}$$

and the angle between ξ_{i+1} and η_{i+1} is half the angle between ξ_i and η_i. For $m_i < \frac{k}{2}$ we choose $\xi_{i+1} = (\xi_i + \eta_i)/\|\xi_i + \eta_i\|$, $\eta_{i+1} = \eta_i$ and $m_{i+1} = 2m_i$, while for $m_i > \frac{k}{2}$ we choose $\xi_{i+1} = \xi_i, \eta_{i+1} = (\xi_i + \eta_i)/\|\xi_i + \eta_i\|$ and $k - m_{i+1} = 2(k - m_i)$. To show that (3.4.4) is valid for $i + 1$, we write for $m_i < \frac{k}{2}$

$$\zeta \equiv (\xi_i - \eta_i)/\|\xi_i - \eta_i\| \quad \text{and} \quad a_i \equiv \frac{1}{2}\|\xi_i + \eta_i\|,$$

and hence

$$\begin{aligned} I = I(m_i) &\le \sum_{\ell=0}^{m_i} \binom{m_i}{\ell} a_i^{2\ell}(1 - a_i^2)^{m_i-\ell} \left\| \frac{\partial^{2\ell}}{\partial \xi_{i+1}^{2\ell}} \frac{\partial^{2m_i-2\ell}}{\partial \zeta^{2m_i-2\ell}} \frac{\partial^{k-2m_i}}{\partial \eta_{i+1}^{k-2m_i}} f \right\|_B \\ &\le a_i^{2m_i} \left\| \frac{\partial^{2m_i}}{\partial \xi_{i+1}^{2m_i}} \frac{\partial^{k-2m_i}}{\partial \eta_{i+1}^{k-2m_i}} f \right\|_B + (1 - a_i^{2m_i})I. \end{aligned}$$

The case $m_i > \frac{k}{2}$ is symmetric. As $\|\xi_i - \eta_i\| \to 0$, Lemma 3.3.3 yields our theorem. □

3.5. Extensions to other function spaces and other domains

As Theorem 3.2.2 is valid for $C(D)$ for all open sets D, it is valid for $C(R^d)$. From this we may deduce the result to a Banach space of functions or generalized functions on R^d for which translation is an isometry, that is

$$\|T(y)f(\cdot)\|_B = \|f(\cdot + y)\|_B = \|f(\cdot)\|_B. \tag{3.5.1}$$

Translation is called strongly, weakly or weakly* continuous if $T(y)f - f \to 0$ (as $\|y\| \to 0$) in the strong, weak or weak* topology respectively (the last only when $X^* = B$, for some Banach space X).

THEOREM 3.5.1. *Suppose B is a Banach space of functions or generalized functions on R^d for which translation is a strongly, weakly or weakly* continuous isometry. Then $\frac{\partial^k f}{\partial \xi^k} \in B$ for all ξ implies $\frac{\partial^k f}{\partial \xi_1 \ldots \partial \xi_k} \in B$ where derivatives are taken in the strong, weak or weak* sense respectively, and*

$$\left\| \frac{\partial^k}{\partial \xi_1 \ldots \partial \xi_k} f(\cdot) \right\|_B \leq \sup_\xi \left\| \frac{\partial^k}{\partial \xi^k} f(\cdot) \right\|_B . \tag{3.5.2}$$

PROOF: We define

$$F(x) = \langle f(x + \cdot), g(\cdot) \rangle$$

for $g \in B^*$ in case the strong or weak result is proved, and for $g \in X$, $X^* = B$ in case the weak* result is proved. We set $\|g\|_{B^*} = 1$ or $\|g\|_X = 1$

respectively, and write

$$\left\|\frac{\partial^k F(\cdot)}{\partial\xi_1\ldots\partial\xi_k}\right\|_{C(R^d)} \le \sup_{\xi}\left\|\frac{\partial^k F(\cdot)}{\partial\xi^k}\right\|_{C(R^d)} \le \sup_{\xi}\left\|\frac{\partial^k f(\cdot)}{\partial\xi^k}\right\|_B .$$

As $\frac{\partial^k}{\partial\xi_1\ldots\partial\xi_k}F(0)$ is given for every g, $\frac{\partial^k f}{\partial\xi_1\ldots\partial\xi_k}$ exists and we may choose g_ε such that

$$\begin{aligned}\left\|\frac{\partial^k}{\partial\xi_1\ldots\partial\xi_k}F\right\|_{C(R^d)} &\ge \left|\frac{\partial^k}{\partial\xi_1\ldots\partial\xi_k}F(0)\right| \\ &= \left|\langle\frac{\partial^k}{\partial\xi_1\ldots\partial\xi_k}f(\cdot), g_\varepsilon(\cdot)\rangle\right| \\ &\ge \left\|\frac{\partial^k}{\partial\xi_1\ldots\partial\xi_k}f\right\|_B - \varepsilon\end{aligned}$$

with any $\varepsilon > 0$. □

REMARK 3.5.2. Theorem 3.5.1 applies to $L_\infty(R^d)$ as translation is weakly* continuous isometry in that space.

REMARK 3.5.3. For an open set Ω for which a function $f \in W_p^r(\Omega)$ can be extended to $F \in W_p^r(R^d)$, that is $F(x) = f(x)$ for $x \in \Omega$ and for which $m(\partial\Omega) = 0$, we have

$$\left\|\frac{\partial^k}{\partial\xi_1\ldots\partial\xi_k}F\right\|_{L_p(\bar\Omega)} \le \sup_{\xi}\left\|\frac{\partial^k f}{\partial\xi^k}\right\|_{L_p(\Omega)}, \quad 0 < k \le r. \tag{3.5.3}$$

The same is valid for the norms $C(\bar{\Omega})$ and $C(\Omega)$ if Ω is such that $f \in C^r(\Omega)$ can be extended to $F \in C^r(R^d)$. Such extensions are discussed extensively elsewhere (see [3] and [4]).

REMARK 3.5.4. In Theorem 3.2.2, $L_p(\Omega)$ can be replaced by any Banach space of functions on Ω, $B(\Omega)$ which satisfy:

(1) $$|f(x)| \geq |g(x)| \quad \text{for} \quad x \in \Omega \quad \text{implies} \quad \|f\|_{B(\Omega)} \geq \|g\|_{B(\Omega)},$$

(2) $$\|O_{t_1 \dots t_k} f\|_{B(\Omega_\epsilon)} \leq \|f\|_{B(\Omega)} \quad \text{for} \quad t_i \quad \text{small enough}$$

and

(3) $$\|f\|_{B(\Omega_\epsilon)} \leq M \quad \text{implies} \quad \lim_{\epsilon \to 0+} \|f\|_{B(\Omega_\epsilon)} = \|f\|_{B(\Omega)}.$$

It is easy to see that properties (1), (2) and (3) are satisfied by many spaces and are not particular to L_p.

REMARK 3.5.5. As $(R^d_+)^0 = \Omega$ satisfies the condition in Remark 3.5.3, (3.5.3) is valid for $C(R^d_+)$, and therefore, the method of Theorem 3.5.1 will imply validity for any Banach spaces on R^d_+ for which translation $T(y)$ for $y \in R^d_+$ is a contraction which is strongly, weakly or weakly* continuous.

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CHAPTER 4

MULTIVARIATE DURRMEYER-BERNSTEIN OPERATORS*

4.1. Introduction

The Bernstein polynomial approximation process given by

$$(4.1.1) \qquad B_n(f,x) \equiv \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\Big(\frac{k}{n}\Big) \equiv \sum_{k=0}^{n} P_{n,k}(x) f\Big(\frac{k}{n}\Big)$$

has served as a guide for theorems that can be proved for a whole class of non-convolution linear approximation processes. The Durrmeyer modification of Bernstein approximation [10] given by

$$(4.1.2) \qquad M_n(f,x) \equiv (n+1) \sum_{k=0}^{n} P_{n,k}(x) \int_0^1 P_{n,k}(t) f(t)\,dt,$$

appears more complicated and maybe is more difficult to compute but possesses some desirable properties, most notably, commutativity, self-adjointness and simple expansion by Legendre polynomials. It is the above-mentioned properties of $M_n f$ that makes it simpler than Bernstein polynomial approximation. Therefore, we may be able to prove for (4.1.2) approximation results that we are not able to

*Joint work with Z. Ditzian.

prove (either at this time or ever) for Bernstein polynomial approximation or its L_p analogue, the Kantorovich operator.

The basic properties of the Durrmeyer-Bernstein operator were given by Derriennic [2], [3] and [4]. Being a new operator, there were many articles obtaining for the Durrmeyer-Bernstein polynomials what was already known for other approximation processes with similar techniques used for those processes.

In [6] Ditzian and Ivanov obtained among other theorems a result [6, Th. 7.1] for $M_n f$, the analogue of which has not yet been proved for Bernstein polynomials (and it is not sure that the analogous result is indeed valid for Bernstein polynomials, though we believe it is). A more recent attempt at achieving better results for $M_n f$ than those known for Bernstein polynomials or their L_p version, the Kanotorovich polynomials, was made by Berens and Xu [1, Th. 3] who achieved an equivalence result between the rate of approximation, $\|M_n f - f\|_p$ and the appropriate K-functional. This result has been exceeded now by a better result for Bernstein polynomials by Ditzian and Ivanov [7, section 8].

For the multivariate case an equivalence between the rate of convergence of the Bernstein polynomials and the appropriate K-functional is still some distance away, and techniques in [7] would require proof of some difficult inequalities. Even for the special case of the saturation theorem the equivalence results that are available [9] are less elegant than desired.

We feel, therefore, that it is worthwhile to have the equivalence between the rate of convergence of $M_n f - f$ on a simplex and the appropriate K -functional both as model for further results and as the first multivariate non-convolution theorem of its type.

4.2. Notations and preliminary results

The multivariate Bernstein type, or Bernstein-Durrmeyer-type operator was introduced by Derriennic [3] and is given by

$$M_n(f,u) = \frac{(n+d)!}{n!} \sum_{\frac{\beta}{n} \in T} P_{n,\beta}(u) \int_T P_{n,\beta}(x) f(x)dx \tag{4.2.1}$$

where

$$T = \Big\{v = (v_1, \ldots, v_d) : \ 0 \le v_i, \ \sum_{i=1}^{d} v_i \le 1\Big\},$$

$x = (x_1, \ldots, x_d)$, $u = (u_1, \ldots, u_d)$, $\beta = (k_1, \ldots, k_d)$ with k_i integers,

$$P_{n,\beta}(u) = \frac{n!}{\beta!(n-|\beta|)!} u^{\beta}(1-|u|)^{n-|\beta|} \tag{4.2.2}$$

and

$$\beta! = k_1! \cdots k_d!, \quad |\beta| = \sum_{i=1}^{d} k_i, \quad |u| = \sum_{i=1}^{d} u_i \tag{4.2.3}$$

$$\text{and} \quad u^{\beta} = u_1^{k_1} \cdots u_d^{k_d} \quad \text{where } u_i^{k_i} = 1 \text{ if } k_i = u_i = 0.$$

Derriennic observed and proved the following important properties.

A. $M_n f$ is self-adjoint. That is, for $f,\ g \in L_1(T)$ one has

$$\langle M_n f, g\rangle = \int_T M_n(f,u)g(u)du = \int_T M_n(g,u)f(u)du = \langle f, M_n g\rangle. \tag{4.2.4}$$

B. The space L_m defined by $\Pi_m = L_m \oplus \Pi_{m-1}$ where Π_ℓ is the collection of all polynomials of total degree ℓ , and

$$\langle f, g\rangle = \int_T f(u)g(u)du$$

satisfies

$$M_n q = \lambda_{n,m} q, \quad \lambda_{n,m} = \frac{(n+d)!\, n!}{(n+m+d)!\,(n-m)!}, \quad q \in L_m. \tag{4.2.5}$$

That is, L_m is an eigenspace of M_n with eigenvalue $\lambda_{n,m}$.

C. Denoting the orthogonal projection of f on L_m by $P_m f$ we have

$$M_n f = \sum_{m=0}^{n} \lambda_{n,m} P_m f. \tag{4.2.6}$$

D. The expansion (4.2.6) implies

$$M_n M_\ell f = \sum_{m=0}^{\min(\ell,n)} \lambda_{n,m}\, \lambda_{\ell,m}\, P_m f = M_\ell M_n f. \tag{4.2.7}$$

E. $M_n f$ is a contraction in L_p

$$\|M_n f\|_p \leq \|f\|_p. \tag{4.2.8}$$

While this last assertion was not explicitly stated, it is clear. In fact, for $p = \infty$

$$\|M_n(f,u)\|_\infty \leq \|f\|_\infty \frac{(n+d)!}{n!} \left\| \sum_{\frac{\beta}{n} \in T} P_{n,\beta}(u) \int_T P_{n,\beta}(x)dx \right\|_\infty = \|f\|_\infty,$$

for $p = 1$

$$\|M_n(f,u)\|_1 \leq \frac{(n+d)!}{n!} \sum_{\frac{\beta}{n} \in T} \int_T P_{n,\beta}(u)du \int_T P_{n,\beta}(x)|f(x)|dx$$

$$= \int \sum_{\frac{\beta}{n} \in T} P_{n,\beta}(x)|f(x)|dx = \int |f(x)|dx,$$

and for $1 < p < \infty$ one obtains the estimate E by the Riesz-Thorin interpolation theorem.

We now introduce the self-adjoint partial differential operator

(4.2.9)

$$P(D) = \left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_d}\right) \begin{pmatrix} x_1(1-x_1) & -x_1x_2 & \cdots & -x_1x_d \\ -x_1x_2 & & & \\ \vdots & & \ddots & \\ -x_1x_d & & & x_d(1-x_d) \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \vdots \\ \vdots \\ \frac{\partial}{\partial x_d} \end{pmatrix}.$$

For this differential operator one can show the following Voronovskaja-type result.

LEMMA 4.2.1. *For* $f \in C^2(T)$

$$\lim_{n\to\infty} n\,(M_n f(x) - f(x)) = P(D)f(x), \qquad x \in T. \tag{4.2.10}$$

PROOF: Actually this is just a new form of a result proved by Derriennic [4, Theorem 2] who showed

$$\lim_{n\to\infty} n(M_n f(x) - f(x)) = \sum_{i=1}^{d}(1-(d+1)x_i)\frac{\partial f(x)}{\partial x_i}$$
$$+\sum_{i=1}^{d} x_i(1-x_i)\frac{\partial^2 f(x)}{\partial x_i^2} - \sum_{i=1}^{d}\sum_{\substack{j=1\\ j\neq i}}^{d} x_i x_j\,\frac{\partial^2 f(x)}{\partial x_i \partial x_j},$$

which is (4.2.10) as inspection shows. □

We feel that the form (4.2.9), which is explicitly self-adjoint, is more convenient than the summation. Of course, the self-adjointness itself follows from the self-adjointness of the left hand side of (4.2.10).

We define the K -functional

$$K(f,t)_p = \inf_{g\in C^2(T)} \left(\|f-g\|_p + t\|P(D)g\|_p\right). \tag{4.2.11}$$

We note that if we were less restrictive on g and took L_p in the restriction on g in (4.2.11) into account, we would not get a smaller K -functional, and if we restrict g more (like $g \in C^{17}(T)$), we would not get a bigger K -functional.

We can now deduce immediately the following results:

LEMMA 4.2.2. *For* $q \in L_m$

$$P(D)q = -m(m+d)q. \tag{4.2.12}$$

PROOF: We only have to show

$$\lim_{n\to\infty} n(\lambda_{n,m} - 1) = -m(m+d). \tag{4.2.13}$$

This is evident, as

$$n(\lambda_{n,m} - 1) = n\Big(\frac{n\cdots(n-m+1) - (n+m+d)\cdots(n+1+d)}{(n+m+d)\cdots(n+1+d)}\Big) =$$

$$= \frac{\Big[-\sum_{i=1}^{m-1} i - \sum_{i=1}^{m}(i+d)\Big] + O\Big(\frac{1}{n}\Big)}{1 + O\Big(\frac{1}{n}\Big)} = -m(m+d) + O\Big(\frac{1}{n}\Big).$$

□

LEMMA 4.2.3. *For* $f \in L_1(T)$ *we have*

$$P(D)M_n f = n(n+d)(M_{n-1}f - M_n f). \tag{4.2.14}$$

PROOF: Using straightforward computations, we have

$$\lambda_{n,m} = -\frac{n(n+d)}{m(m+d)}(\lambda_{n-1,m} - \lambda_{n,m}). \tag{4.2.15}$$

We now recall (4.2.6) and (4.2.12) to write

$$P(D)M_n f = P(D)\sum_{m=0}^{n} \lambda_{n,m} P_m f = -\sum_{m=0}^{n} m(m+d)\lambda_{n,m} P_m f,$$

in which we substitute (4.2.15) to obtain (4.2.14). □

From Lemma 4.2.3 we deduce the following useful corollary:

COROLLARY 4.2.4. *For* $f \in L_1(T)$ *we have*

$$M_n f - f = \sum_{k=n+1}^{\infty} \frac{1}{k(k+d)} P(D) M_k f. \tag{4.2.16}$$

PROOF: For $\ell > n$ (4.2.14) implies

$$M_n f - M_\ell f = \sum_{k=n+1}^{\ell} \frac{1}{k(k+d)} P(D) M_k f.$$

As for $f \in L_1(T)$, $\|M_\ell f - f\|_{L_1(T)} \to 0$, we have (4.2.16). □

LEMMA 4.2.5. *For* $f \in C^2(T)$ *we have*

$$P(D)M_n f = M_n P(D) f. \tag{4.2.17}$$

PROOF: We first note that for $f \in \Pi_\ell$

$$f = \sum_{i=0}^{\ell} P_i f \quad \text{and} \quad P_i f \in L_i,$$

and hence (4.2.6) and (4.2.12) imply

$$P(D)M_n f = -\sum_{i=0}^{\min(n,\ell)} i(i+d)\lambda_{n,i}P_i f = M_n P(D) f.$$

We now observe that any $f \in C^2(T)$ and $\varepsilon > 0$ can be approximated by $p \in \Pi_\ell$, $\ell \equiv \ell(f,\varepsilon)$ so that

$$\|f - p\|_{C(T)} + \|P(D)(f-p)\|_{C(T)} < \varepsilon.$$

As p already satisfies (4.2.17), we estimate

$$\|M_n P(D)(f-p)\|_{C(T)} \le \|P(D)(f-p)\|_{C(T)} \le \varepsilon,$$

and using Lemma 4.2.3,

$$\begin{aligned}\|P(D)M_n(f-p)\| &\le n(n+d)\big(\|M_{n-1}(f-p)\| + \|M_n(f-p)\|\big) \le 2(n(n+d))\|f-p\| \\ &\le 2n(n+d)\varepsilon.\end{aligned}$$

Since n is fixed and ε arbitrary, we obtain (4.2.17). □

4.3. The main result

We denote as usual, $A_n \sim B_n$, that is the sequence A_n and B_n are equivalent iff

$$c^{-1}A_n < B_n < cA_n.$$

The main theorem of this paper is the following equivalence between $K\left(f, \frac{1}{n}\right)_p$ and an expression involving the rate of convergence of $M_k f$ to f .

THEOREM 4.3.1. *For* $f \in L_p(T)$, $1 \leq p \leq \infty$

$$K\left(f, \frac{1}{n}\right)_p \sim \sup_{k \geq n} \|M_k f - f\|_p \sim \sup_{n \leq k \leq 2n} \|M_k f - f\|_p \tag{4.3.1}$$

where $K\left(f, \frac{1}{n}\right)_p$ *is given by (4.2.11).*

PROOF: We first prove the direct result

$$\|M_n f - f\|_p \leq 4K\left(f, \frac{1}{n}\right)_p. \tag{4.3.2}$$

We choose $g \in C^2(T)$ such that

$$\|f - g\|_p \leq (1+\eta)K\left(f, \frac{1}{n}\right)_p \quad \text{and} \quad \frac{1}{n}\|P(D)g\|_p \leq (1+\eta)K\left(f, \frac{1}{n}\right)_p,$$

which is possible for any $\eta > 0$ (g depends on f, n, p and η). Using E of section 4.2, we have

$$\|M_n(f-g)-(f-g)\|_p \leq 2\|f-g\|_p \leq 2(1+\eta)K\Big(f,\ \frac{1}{n}\Big)_p.$$

Using Corollary 4.2.4, Lemma 4.2.5 and (4.2.8) we have

$$\begin{aligned}
\|M_n g - g\|_p &\equiv \Big\| \sum_{k=n+1}^{\infty} \frac{1}{k(k+d)} P(D) M_k g \Big\|_p \leq \sum_{k=n+1}^{\infty} \frac{1}{k(k+d)} \|M_k P(D) g\|_p \\
&\leq \sum_{k=n+1}^{\infty} \frac{1}{k(k+d)} \|P(D)g\|_p = \Big[\frac{1}{d} \sum_{k=n+1}^{n+d} \frac{1}{k}\Big] \|P(D)g\|_p \\
&\leq \frac{1}{n}\|P(D)g\|_p \leq (1+\eta)K\Big(f,\ \frac{1}{n}\Big)_p.
\end{aligned}$$

Choosing $\eta = \frac{1}{3}$, we have (4.3.2).

Since $K\big(f,\ \frac{1}{n}\big)_p$ is monotone in n and

$$\sup_{n \leq k \leq 2n} \|M_k f - f\|_p \leq \sup_{k \leq n} \|M_k f - f\|_p,$$

we have to show only

$$K\Big(f,\ \frac{1}{n}\Big)_p \leq C \sup_{n \leq k \leq 2n} \|M_k f - f\|_p. \tag{4.3.3}$$

We now construct

$$g = \Big[\sum_{k=n+1}^{2n} \frac{1}{k(k+d)}\Big]^{-1} \sum_{k=n+1}^{2n} \frac{1}{k(k+d)} M_k f,$$

and observe that $g \in C^2(T)$

$$\|g - f\|_p \le \sup_{n \le k \le 2n} \|M_k f - f\|_p$$

and

$$\begin{aligned}
\frac{1}{n}\|P(D)g\|_p &\le \frac{1}{n}\Big[\sum_{k=n+1}^{2n} \frac{1}{k(k+d)}\Big]^{-1} \Big\|\sum_{k=n+1}^{2n} \frac{1}{k(k+d)} P(D)M_k f\Big\|_p \\
&\le \frac{2(2n+d)}{n}\Big\|\sum_{k=n+1}^{2n} (M_{k-1}f - M_k f)\Big\|_p \\
&\le C\big(\|M_n f - f\|_p + \|M_{2n} f - f\|_p\big) \le C \sup_{n \le k \le 2n} \|M_k f - f\|_p .
\end{aligned}$$

□

REMARK 4.3.2: Examining the direct proof and the choice of g, we observe that (4.3.1) can be replaced by

$$K\Big(f, \frac{1}{n}\Big)_p \sim \sup_{n \le k \le [An]} \|M_k f - f\|_p \tag{4.3.4}$$

for any $A > 1$ and $n \ge n_0(A)$. With a little more work we obtain:

THEOREM 4.3.3. *For* $f \in L_p(T)$

$$K\Big(f, \frac{1}{n}\Big)_p \sim n^{-1} \sum_{k=n}^{[An]} \|M_k f - f\|_p \tag{4.3.5}$$

for any fixed $A > 1$ *and* $n \geq n_0(A)$.

PROOF: The direct result follows (4.3.2). We then use (4.3.4) with B instead of A such that $B^2 \leq A$ and average the estimate that was obtained for $K\left(f, \frac{1}{\ell}\right)_p$ where we take the actual inequality (and not the final result). □

We did not prove Theorem 4.3.3 in full detail as Theorem 4.3.1 provides a nice (and easy to prove) equivalence result, and we believe that in the future

$$\|M_n f - f\|_p \sim K\left(f, \frac{1}{n}\right)_p$$

will be proved and that will supercede the results here. This last result eludes us at present for the multivariate case.

The saturation result can now be observed from the saturation of the K - functional $K\left(f, \frac{1}{n}\right)_p$.

THEOREM 4.3.4. *Suppose* $f \in L_p(T)$ $1 \leq p \leq \infty$. *Then for* $K\left(f, \frac{1}{n}\right)_p = O\left(\frac{1}{n}\right)$ $n \to \infty$ *it is necessary and sufficient that* $P(D)f$ *defined in the distribution (Sobolev) sense is in* $L_p(T)$ *for* $1 < p \leq \infty$ *and in* $M(T)$ *(finite measures on* T *) for* $p = 1$.

PROOF: The necessity is the routine weak * convergence argument. The sufficiency follows the fact that $C^2(T)$ (and even the space of polynomials) is dense in the appropriate space. □

4.4. Iterations

In this section we will demonstrate how to iterate the result of the last section and obtain the following result.

THEOREM 4.4.1. *For* $f \in L_p(T)$

$$K_r\Big(f,\ \frac{1}{n^r}\Big)_p \sim \sup_{n\le k_i\le 2n} \Big\| \Big[\prod_{i=1}^{r}(M_{k_i} - I)\Big] f \Big\|_p \tag{4.4.1}$$

where

$$K_r(f,t)_p = \inf_{C^{2r}(T)} \big(\|f-g\|_p + t\|P(D)^r g\|_p\big) \tag{4.4.2}$$

and $P(D)^r$ *is the* r^{th} *iterate of* $P(D)$ *given in (4.2.9).*

PROOF: For the direct inequality we write $f = f - g + g$ where

$$\|f-g\|_p \le 2K_r\Big(f,\ \frac{1}{n^r}\Big)_p \quad \text{and} \quad n^{-r}\|P(D)^r g\|_p \le 2K_r\Big(f,\ \frac{1}{n^r}\Big)_p.$$

Obviously,

$$\left\| \left[\prod_{i=1}^{r} (M_{k_i} - I) \right] (f - g) \right\|_p \le 2^r \|f - g\|_p \le 2^{r+1} K_r\Big(f, \frac{1}{n^r}\Big)_p,$$

and using the estimate in Theorem 4.3.1,

$$\left\| \left[\prod_{i=1}^{r} (M_{k_i} - I) \right] g \right\|_p \le \prod_{i=1}^{r} k_i^{-1} \|P(D)^r g\|_p \le Cn^{-r} \|P(D)^r g\|_p$$

from which the direct inequality

$$\left\| \left[\prod_{i=1}^{r} (M_{k_i} - I) \right] f \right\|_p \le C_1 \; K_r\Big(f, \frac{1}{n^r}\Big)_p$$

follows. For the other direction we define

$$O_n f = \left[\sum_{k=n+1}^{2n} \frac{1}{k(k+d)} \right]^{-1} \left(\sum_{k=n+1}^{2n} \frac{1}{k(k+d)} M_k f \right)$$

as we have defined g in Theorem 4.3.1. We now define

(4.4.3) $$O_{n,r} = I - (I - O_n^r)^r$$

where O_n^r is the r^{th} iterate of O_n . We now choose

$$g = O_{n,r} f.$$

Using (4.2.8),

$$\|O_n f\|_p \le \|f\|_p,$$

and hence

$$\|f-g\|_p \le \|(I-O_n^r)^r f\|_p \le r^r \|(I-O_n)^r f\|_p \le r^r \sup_{n\le k_i\le 2n} \left\| \left[\prod_{i=1}^{r} (I-M_{k_i}) \right] f \right\|_p .$$

We now write

$$\begin{aligned} n^{-r}\left\|P(D)^r g\right\|_p &\le n^{-r}\sum_{\ell=1}^{r}\binom{r}{\ell}\left\|P(D)^r O_n^{r\ell} f\right\|_p \\ &= n^{-r}\sum_{\ell=1}^{r}\binom{r}{\ell}\left\|O_n^{r(\ell-1)}P(D)^r O_n^r f\right\|_p \\ &\le n^{-r}(2^r-1)\left\|P(D)^r O_n^r f\right\|_p \\ &\le C n^{-r+1}\sup_{n\le k\le 2n}\left\|(M_k-I)P(D)^{r-1}O_n^{r-1} f\right\|_p \\ &\le C(r)\sup_{n\le k_i\le 2n}\left\|\left[\prod_{i=1}^{r}(M_{k_i}-I)\right] f\right\|_p . \end{aligned}$$

□

ADDED IN PROOF: We have just learned that two additional papers on the subject by Berens *et al.* and by Derriennic are in preparation. There should not be much overlap as Berens received this manuscript (in the present form) at the Jakimovski conference in Tel-Aviv, and Derriennic's main interest is in a class of interesting new operators $D_n^{(s)}$ which extend M_n. The inverse results here for M_n are better than Derriennic's but we do not treat $D_n^{(s)}$ at all.

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CHAPTER 5

STRONG CONVERSE INEQUALITY FOR THE BERNSTEIN-DURRMEYER OPERATOR*

5.1. Introduction

The Bernstein-Durrmeyer operator (see [10] and [3]) is given by

$$M_n(f,x) = (n+1)\sum_{k=0}^{n} P_{n,k}(x)\int_0^1 P_{n,k}(y)f(y)\,dy \tag{5.1.1}$$

where

$$P_{n,k}(x) = \binom{n}{k} x^k(1-x)^{n-k}.$$

We will prove a strong converse inequality of type A, in the terminology of [8], that is, we will show

$$\|M_n f - f\|_p \sim \inf\left(\|f-g\|_p + \frac{1}{n}\left\|(\varphi^2 g')'\right\|_p\right) \tag{5.1.2}$$

for $1 \le p \le \infty$ with $\varphi(x)^2 = x(1-x)$. For $1 < p < \infty$, we will prove an analogue of (5.1.2) for the multivariate Bernstein-Durrmeyer operator introduced by Derriennic [4]. In case $p = 1$ or $p = \infty$ and the higher dimensional analogue of (5.1.1), we will prove a somewhat weaker result (that is, a strong converse inequality of type B in the terminology of [8]). Several recent articles [1], [2] and

*Joint work with Z. Ditzian and K. Ivanov.

[6] achieved (among other results) converse inequalities for these operators that are obviously weaker than those in the present paper.

5.2. Notations

The Multivariate Bernstein-Durrmeyer operator was introduced by Derriennic [4] as

$$M_n(f,x) = \frac{(n+d)!}{n!} \sum_{\frac{\beta}{n} \in T} P_{n,\beta}(x) \int_T P_{n,\beta}(u) f(u)\, du \tag{5.2.1}$$

where $x, u \in R^d$ $\left(x = (x_1, \ldots, x_d)\right)$, $\beta = (k_1, \ldots, k_d)$ with k_i integers, and $T = \{u : 0 \le u_i,\ \sum_{i=1}^{d} u_i \le 1\}$. The polynomial $P_{n,\beta}(u)$ is given by

$$P_{n,k_1,\ldots,k_d}(u_1, \ldots, u_d) \equiv P_{n,\beta}(u) = \frac{n!}{\beta!(n-|\beta|)!} u^\beta (1-|u|)^{n-|\beta|} \tag{5.2.2}$$

where $\beta! = k_1! \cdots k_d!$, $u^\beta = u_1^{k_1} \cdots u_d^{k_d}$ $(u_i^{k_i} = 1 \text{ if } k_i = u_i = 0)$, $|u| = \sum_{i=1}^{d} u_i$ and $|\beta| = \sum_{i=1}^{d} k_i$.

Many properties were proven about the operators $M_n f$ which will be quoted as we use them. We define

$$P(D) = \sum_{i=1}^{d} \frac{\partial}{\partial x_i} x_i (1-|x|) \frac{\partial}{\partial x_i} + \sum_{i<j} \Big(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j}\Big) x_i x_j \Big(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j}\Big) \tag{5.2.3}$$

and recall that for $f \in C^2(T)$, it was proved in [5] that

$$n\{M_n(f,x) - f(x)\} \to P(D) f(x). \tag{5.2.4}$$

(The operator $P(D)$ given by (5.2.3) may take other forms as can be seen in [4] and [2].)

5.3. Estimate of $\|P(D)M_n f\|_p$

It follows from Derriennic's research [6], detailed only for $d = 1$ and $d = 2$, that

$$\|P(D)^r M_n f\|_p \le C n^r \|f\|_p . \tag{5.3.1}$$

We will need for $r = 1$ the following better estimate on the constant C.

THEOREM 5.3.1. *For $f \in L_p(T)$, where T is the d-dimensional simplex given in Section 5.2, and for $P(D)$ given by (5.2.3), we have*

$$\|P(D)M_n f\|_p \le 2dn \|f\|_p . \tag{5.3.2}$$

PROOF: First we show that it is sufficient to prove (5.3.2) for $p = \infty$ (or $p = 1$). Assume (5.3.2) for $p = \infty$. We take $g \in C^2(T)$ and $f \in L_1(T)$ and then use [2, Lemma 2.5]

$$P(D)M_n g = M_n P(D) g, \qquad g \in C^2(T). \tag{5.3.3}$$

We recall from [4] the self-adjointness of M_n and $P(D)$ with respect to the scalar product $\langle f, g\rangle = \int_T f(u)g(u)\,du$ to obtain

(5.3.4)
$$|\langle P(D)M_n f, g\rangle| = |\langle f, P(D)M_n g\rangle| \le \|f\|_{L_1(T)} \|P(D)M_n g\|_{L_\infty(T)}$$
$$\le 2dn \|f\|_{L_1(T)} \|g\|_{L_\infty(T)} \,.$$

As (5.3.4) is valid for all $g \in C^2(T)$, we have (5.3.2) for $p = 1$. The inequality (5.3.2) for $p = \infty$ and $p = 1$ implies now (5.3.2) for $1 < p < \infty$ via the Riesz-Thorin interpolation theorem.

We observe that

$$(5.3.5) \qquad x_i(1-|x|)\frac{\partial}{\partial x_i}P_{n,\beta}(x) = \big(k_i(1-|x|) - (n-|\beta|)x_i\big)P_{n,\beta}(x)$$

and hence

(5.3.6)
$$\frac{\partial}{\partial x_i}x_i(1-|x|)\frac{\partial}{\partial x_i}P_{n,\beta}(x)$$
$$= \frac{\big(k_i(1-|x|) - (n-|\beta|)x_i\big)^2}{x_i(1-|x|)}P_{n,\beta}(x) - (n - |\beta| + k_i)P_{n,\beta}(x).$$

Similarly

(5.3.7)
$$\Big(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j}\Big)x_i x_j\Big(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j}\Big)P_{n,\beta}(x)$$
$$= \frac{(k_i x_j - k_j x_i)^2}{x_i x_j}P_{n,\beta}(x) - (k_i + k_j)P_{n,\beta}(x).$$

Recalling $M_n(1,x)=1$, we have

$$
\begin{aligned}
0 &= P(D)M_n(1,x) \\
&= \sum_{\frac{\beta}{n}\in T}\left(\left\{\sum_{i=1}^{d}\frac{\left(k_i(1-|x|)-(n-|\beta|)x_i\right)^2}{x_i(1-|x|)}\right.\right. \\
&\qquad\qquad \left.\left.+\sum_{i<j}\frac{(k_ix_j-k_jx_i)^2}{x_ix_j}\right\}-nd\right)P_{n,\beta}(x) \\
&= \sum_{\frac{\beta}{n}\in T}(I_{n,\beta}(x)-nd)P_{n,\beta}(x),
\end{aligned}
\tag{5.3.8}
$$

which implies

$$\sum_{\frac{\beta}{n}\in T} I_{n,\beta}(x)P_{n,\beta}(x) = nd\sum_{\frac{\beta}{n}\in T} P_{n,\beta}(x) = nd.$$

We now estimate

$$b_{n,\beta}\equiv\left|\frac{(n+d)!}{n!}\int_T f(x)P_{n,\beta}(x)\,dx\right|\le\|f\|_{L_\infty(T)} \tag{5.3.9}$$

and use $I_{n,\beta}(x)\ge 0$ to obtain

$$|P(D)M_n(f,x)|\le\sum_{\frac{\beta}{n}\in T}(I_{n,\beta}(x)+nd)P_{n,\beta}(x)\|f\|_{L_\infty(T)}\le 2nd\,\|f\|_{L_\infty(T)}\,.$$

□

We are also able to prove the following useful estimate.

THEOREM 5.3.2. *Under the assumptions of Theorem 5.3.1, we have*

$$\left\|P(D)M_n^2 f\right\|_p \le dn\,\|f\|_p . \tag{5.3.10}$$

PROOF: Following the proof of Theorem 5.3.1, we only have to consider $p = \infty$. We can write

$$\begin{aligned}
|P(D)M_n^2(f,x)| &= |M_n P(D) M_n(f,x)| \\
&\le \left(\frac{(n+d)!}{n!}\right)^2 \sum_{\frac{\gamma}{n}\in T} P_{n,\gamma}(x) \sum_{\frac{\beta}{n}\in T} \left|\int_T P_{n,\gamma}(u) P(D) P_{n,\beta}(u)\,du\right| \\
&\qquad \times \int_T P_{n,\beta}(v)|f(v)|\,dv \\
&\le \frac{(n+d)!}{n!} \|f\|_{L_\infty(T)} \sum_{\frac{\gamma}{n}\in T} P_{n,\gamma}(x) \sum_{\frac{\beta}{n}\in T} \left|\int_T P_{n,\gamma}(u) P(D) P_{n,\beta}(u)\,du\right| .
\end{aligned}$$

We will show

$$\frac{(n+d)!}{n!} \sum_{\frac{\beta}{n}\in T} \left|\int_T P_{n,\gamma}(u) P(D) P_{n,\beta}(u)\,du\right| \le nd \tag{5.3.11}$$

which implies (5.3.10) for $p = \infty$ and hence for $1 \le p \le \infty$. To prove (5.3.11), we write

$$\begin{aligned}
J_{n,\gamma} &= \frac{(n+d)!}{n!} \sum_{\frac{\beta}{n}\in T} \left|\int_T P_{n,\gamma}(u) P(D) P_{n,\beta}(u)\,du\right| \\
&= \frac{(n+d)!}{n!} \sum_{\frac{\beta}{n}\in T} \left|\sum_{i<j} \int_T (L_{i,j}(D) P_{n,\gamma}(u))(L_{i,j}(D) P_{n,\beta}(u))\,du\right|
\end{aligned}$$

where

(5.3.12)

$$L_{i,i}(D)=\sqrt{u_i(1-|u|)}\frac{\partial}{\partial u_i} \qquad \text{and} \qquad L_{i,j}(D)=\sqrt{u_iu_j}\left(\frac{\partial}{\partial u_i}-\frac{\partial}{\partial u_j}\right) \qquad \text{for } i\neq j.$$

Straightforward computation of $L_{i,j}(D)P_{n,\eta}(u)$ (where $\eta=\beta$ or $\eta=\gamma$) leads now to

$$J_{n,\gamma}\le\frac{(n+d)!}{n!}\sum_{\frac{\ell}{n}\in T}\int_T\Bigg\{\sum_{i=1}^{d}\frac{|k_i(1-|u|)-(n-|\beta|)u_i|\,|\ell_i(1-|u|)-(n-|\gamma|)u_i)|}{u_i(1-|u|)}$$
$$+\sum_{i<j}\frac{|k_iu_j-k_ju_i|\,|\ell_iu_j-\ell_ju_i|}{u_iu_j}\Bigg\}P_{n,\gamma}(u)P_{n,\beta}(u)\,du.$$

Recalling $I_{n,\eta}(u)$ (with $\eta=\beta$ and $\eta=\gamma$) given in (5.3.8), we use the Cauchy-Schwartz inequality to obtain

$$\begin{aligned}J_{n,\gamma}&\le\frac{(n+d)!}{n!}\sum_{\frac{\ell}{n}\in T}\int_T I_{n,\beta}(u)^{1/2}I_{n,\gamma}(u)^{1/2}P_{n,\gamma}(u)P_{n,\beta}(u)\,du\\&\le\left\{\frac{(n+d)!}{n!}\sum_{\frac{\ell}{n}\in T}\int_T I_{n,\beta}(u)P_{n,\gamma}(u)P_{n,\beta}(u)\,du\right\}^{1/2}\\&\qquad\times\left\{\frac{(n+d)!}{n!}\sum_{\frac{\ell}{n}\in T}\int_T I_{n,\gamma}(u)P_{n,\gamma}(u)P_{n,\beta}(u)\,du\right\}^{1/2}\\&=J^*_{n,\gamma}\times J^{**}_{n,\gamma}.\end{aligned}$$

The estimate $J^*_{n,\gamma}\le(nd)^{1/2}$ follows from

$$\sum_{\frac{\ell}{n}\in T}I_{n,\beta}(u)P_{n,\beta}(u)=nd$$

which follows from (5.3.8). To estimate $J^{**}_{n,\gamma}$, we write, using (5.3.5),

$$\int_T \frac{(\ell_i(1-|u|)-(n-|\gamma|)u_i)^2}{u_i(1-|u|)} P_{n,\gamma}(u)\,du = \int_T (\ell_i(1-|u|)-(n-|\gamma|)u_i)\frac{\partial}{\partial u_i} P_{n,\gamma}(u)\,du$$
$$= \frac{n!}{(n+d)!}(n-|\gamma|+\ell_i)$$

and

$$\int_T \frac{(\ell_i u_j - \ell_j u_i)^2}{u_i u_j} P_{n,\gamma}(u)\,du = \frac{n!}{(n+d)!}(\ell_i+\ell_j)$$

which imply $J^{**}_{n,\gamma} \le (nd)^{1/2}$. □

5.4. Voronovskaja-type estimates

M. Derriennic [5] proved the Voronovskaja-type estimate (5.2.4). For the converse inequality of the present paper, we need the following stronger result.

THEOREM 5.4.1. *Suppose $f \in C^4(T)$, $M_n f$ is given by (5.2.1) and $P(D)$ is given by (5.2.3). Then we have for $n > 1$*

$$\text{(5.4.1)} \quad \left\| M_n f - f - \frac{\alpha_n(d)}{2} P(D)[M_n f + f] \right\|_p \le \left(\frac{1}{4}\alpha_n(d)^2 + \frac{1}{2}\frac{1}{(n+1)^3}\right) \|P(D)^2 f\|_p$$

where $\alpha_n(d) \equiv \sum_{k=n+1}^{\infty} \frac{1}{k(k+d)} = \frac{1}{d}\left[\frac{1}{n+1} + \cdots + \frac{1}{n+d}\right]$.

PROOF: Using Corollary 2.4 of [2],

$$M_nf - f = \sum_{k=n+1}^{\infty} \frac{1}{k(k+d)} P(D)M_kf,$$

we write

$$\begin{aligned}
I(n) &= \left\| M_nf - f - \frac{\alpha_n(d)}{2} P(D)(f + M_nf) \right\| \\
&= \frac{1}{2} \left\| \sum_{k=n+1}^{\infty} \frac{1}{k(k+d)} P(D)(M_kf - f) + \sum_{k=n+1}^{\infty} \frac{1}{k(k+d)} P(D)(M_kf - M_nf) \right\| \\
&= \frac{1}{2} \left\| \sum_{k=n+1}^{\infty} \frac{1}{k(k+d)} \sum_{j=k+1}^{\infty} \frac{1}{j(j+d)} P(D)^2 M_jf \right. \\
&\qquad \left. - \sum_{k=n+1}^{\infty} \frac{1}{k(k+d)} \sum_{j=n+1}^{k} \frac{1}{j(j+d)} P(D)^2 M_jf \right\| \\
&= \frac{1}{2} \left\| \sum_{j=n+2}^{\infty} \frac{P(D)^2 M_jf}{j(j+d)} \sum_{k=n+1}^{j-1} \frac{1}{k(k+d)} - \sum_{j=n+1}^{\infty} \frac{P(D)^2 M_jf}{j(j+d)} \sum_{k=j}^{\infty} \frac{1}{k(k+d)} \right\| \\
&\le \frac{1}{2} \sup_j \left\| P(D)^2 M_jf \right\| \sum_{j=n+1}^{\infty} \frac{1}{j(j+d)} \left| \sum_{k=n+1}^{j-1} \frac{1}{k(k+d)} - \sum_{k=j}^{\infty} \frac{1}{k(k+d)} \right|
\end{aligned}$$

(with the understanding $\sum_{k=n+1}^{n} \cdots = 0$). Using Lemma 2.5 of [2], we have for $f \in C^4(T)$

$$P(D)^2 M_jf = M_jP(D)^2 f$$

and hence

$$\left\| P(D)^2 M_jf \right\| \le \left\| P(D)^2 f \right\|.$$

We now have

$$I(n) \leq \frac{1}{2} \left\|P(D)^2 f\right\| \sum_{j=n+1}^{\infty} \frac{1}{j(j+d)} |\alpha_n(d) - 2\alpha_{j-1}(d)|$$

$$\equiv \frac{1}{2} \left\|P(D)^2 f\right\| J(n).$$

To estimate $J(n)$, we define j_0 by

$$j_0 = \max\{j : 2\alpha_{j-1}(d) - \alpha_n(d) > 0\}$$

and as $\alpha_j(d)$ is a decreasing sequence in j, we have

$$J(n) = \sum_{j=n+1}^{j_0} \frac{1}{j(j+d)} \big(2\alpha_{j-1}(d) - \alpha_n(d)\big) + \sum_{j=j_0+1}^{\infty} \frac{1}{j(j+d)} \big(\alpha_n(d) - 2\alpha_{j-1}(d)\big)$$

$$\equiv J_1(n) + J_2(n).$$

To estimate $J_1(n)$, we write

$$J_1(n) = \sum_{j=n+1}^{j_0} \big(\alpha_{j-1}(d) - \alpha_j(d)\big)\big(\alpha_{j-1}(d) + \alpha_j(d)\big)$$

$$+ \sum_{j=n+1}^{j_0} \frac{1}{j^2(j+d)^2} - \alpha_n(d)\big(\alpha_n(d) - \alpha_{j_0}(d)\big)$$

$$\leq \alpha_n(d)^2 - \alpha_{j_0}(d)^2 - \frac{1}{2}\alpha_n(d)^2 + \frac{2/3}{(n+1)^3}$$

as the definition of j_0 implies $\alpha_n(d) - \alpha_{j_0}(d) \geq \frac{1}{2}\alpha_n(d)$ and

$$\sum_{j=n+1}^{j_0} \frac{1}{j^2(j+d)^2} \leq \sum_{j=n+1}^{\infty} \frac{1}{j^2(j+d)^2} \leq \frac{2/3}{(n+1)^3} \qquad \text{for } n \geq 1.$$

To estimate $J_2(n)$, we write

$$J_2(n) \le \alpha_n(d)\alpha_{j_0}(d) - \sum_{j=j_0+1}^{\infty} \big(\alpha_{j-1}(d) - \alpha_j(d)\big)\big(\alpha_{j-1}(d) + \alpha_j(d)\big)$$
$$= \alpha_n(d)\,\alpha_{j_0}(d) - \alpha_{j_0}(d)^2.$$

Combining the estimates for $J_1(n)$ and $J_2(n)$ and as $j_0 \ge 2n+1$, we have

$$\begin{aligned} J(n) &\le \frac{1}{2}\alpha_n(d)^2 + \frac{2/3}{(n+1)^3} + \alpha_{j_0}(d)\big(\alpha_n(d) - 2\alpha_{j_0}(d)\big) \\ &\le \frac{1}{2}\alpha_n(d)^2 + \frac{2/3}{(n+1)^3} + 2\alpha_{j_0}(d)\big(\alpha_{j_0-1}(d) - \alpha_{j_0}(d)\big) \\ &\le \frac{1}{2}\alpha_n(d)^2 + \frac{2/3}{(n+1)^3} + 2\frac{1}{(2n+2)^2}\frac{1}{2n+1} \le \frac{1}{2}\alpha_n(d)^2 + \frac{1}{(n+1)^3}, \end{aligned}$$

which combined with the estimate of $I(n)$ concludes the proof. □

REMARK 5.4.2: For most purposes, the slightly easier to prove estimate

$$\|M_n f - f - \alpha_n(d)P(D)f\|_p \le \frac{1}{2n^2}\left\|P(D)^2 f\right\|_p \tag{5.4.2}$$

will be sufficient. In some cases, however, (5.4.1) will yield results which are qualitatively better.

In one result (Theorem 5.7.2), we need the following extension of (5.4.2).

THEOREM 5.4.3. *Suppose* $f \in C^{2r+2}(T)$ *and* M_n, $P(D)$, T *and* d *are as given in section 5.2. Then*

$$\|(M_n - I)^r f - \alpha_n(d)^r P(D)^r f\|_p \le \frac{r/2}{n^{r+1}}\left\|P(D)^{r+1} f\right\|_p . \tag{5.4.3}$$

PROOF: We first observe

$$\begin{aligned}\|M_n f - f - \alpha_n(d)P(D)f\|_p &= \left\| \sum_{k=n+1}^{\infty} \frac{1}{k(k+d)} P(D)(M_k f - f) \right\|_p \\ &\leq \left\| \sum_{k=n+1}^{\infty} \frac{1}{k(k+d)} \sum_{j=k+1}^{\infty} \frac{1}{j(j+d)} P(D)^2 M_j f \right\|_p \\ &\leq \sum_{k=n+1}^{\infty} \frac{1}{k(k+1)} \sum_{j=k+1}^{\infty} \frac{1}{j(j+1)} \left\|P(D)^2 f\right\|_p \\ &\leq \sum_{k=n+1}^{\infty} \frac{1}{k(k+1)^2} \left\|P(D)^2 f\right\|_p \\ &\leq \frac{1}{2}\frac{1}{n^2} \left\|P(D)^2 f\right\|_p .\end{aligned}$$

We prove (5.4.3) by induction. We assume (5.4.3) for $r = \ell$ and write

$$\left\|(M_n - I)^{\ell+1} f - \alpha_n(d)^\ell P(D)^\ell (M_n - I) f\right\|_p \leq \frac{\ell/2}{n^{\ell+1}} \left\|P(D)^{\ell+1}(M_n - I)f\right\|_p .$$

Since we have

$$\left\|P(D)^{\ell+1}(M_n - I)f\right\|_p = \left\|(M_n - I)P(D)^{\ell+1} f\right\|_p \leq \frac{1}{n} \left\|P(D)^{\ell+2} f\right\|_p$$

and since the induction hypothesis for $\ell = 1$ implies

$$\left\|\alpha_n(d)^\ell (M_n - I)P(D)^\ell f - \alpha_n(d)^{\ell+1} P(D)^{\ell+1} f\right\|_p \leq \frac{\alpha_n(d)^\ell}{2n^2} \left\|P(D)^{\ell+2} f\right\|_p ,$$

the result follows. □

5.5. Estimates of $\|P(D)M_n^r f\|_2$ and its consequences

In this section, we will give an estimate of $\|P(D)M_n f\|_{L_2(T)}$ and of $\|P(D)M_n^r f\|_{L_2(T)}$ which will prove useful also for other $L_p(T)$.

THEOREM 5.5.1. *Suppose* $f \in L_2(T)$, $M_n f$, T *and* $P(D)$ *are as defined in section 5.2. Then we have for* $r = 1, 2, \ldots$

$$\|P(D)M_n^r f\|_{L_2(T)} \le \frac{n}{\sqrt{r}} \|f\|_{L_2(T)} . \tag{5.5.1}$$

For the proof we need the following computational lemma.

LEMMA 5.5.2. *For* $\lambda_{n,k}$ *given by*

$$\lambda_{n,k} = \frac{(n+d)!n!}{(n+d+k)!(n-k)!}, \qquad 0 \le k \le n \tag{5.5.2}$$

we have

$$k(k+d)\lambda_{n,k}^r \le n/\sqrt{r}, \qquad 0 \le k \le n. \tag{5.5.3}$$

PROOF: Since $0 \le \lambda_{n,k} \le 1$, (5.5.3) follows immediately when $k(k+d) \le n/\sqrt{r}$. To prove (5.5.3) for k satisfying $k(k+d) > n/\sqrt{r}$, we estimate $\lambda_{n,k}^j$ using

$$\lambda_{n,k}^j = \left(\frac{(n+d)!n!}{(n+d+k)!(n-k)!}\right)^j = \left(\prod_{i=1}^{k} \frac{n-k+i}{n+d+i}\right)^j = \prod_{i=1}^{k}\left(1 - \frac{d+k}{n+d+i}\right)^j$$

$$\le \left(1 - \frac{d+k}{n+d+k}\right)^{kj} = \frac{1}{(1+\frac{d+k}{n})^{kj}} \le \frac{1}{1+k(k+d)jn^{-1}}.$$

For $j = 1$, we have

$$\lambda_{n,k} \le \frac{n}{n + k(k+d)} \le \frac{n}{k(k+d)}.$$

For $k(k+d) \ge n/\sqrt{r}$ and $j = r-1$, we have

$$\lambda_{n,k}^{r-1} \le \frac{1}{1 + k(k+d)(r-1)n^{-1}} \le \frac{1}{1 + (r-1)/\sqrt{r}} \le \frac{1}{\sqrt{r}}$$

and hence

$$k(k+d)\lambda_{n,k}^{r} \le \frac{k(k+d)}{\sqrt{r}} \frac{n}{k(k+d)} \le \frac{n}{\sqrt{r}}.$$

□

Proof of Theorem 5.5.1: The eigenspaces of the self adjoint operators $P(D)f$ and $M_n f$ are the same and f can be expanded by

$$f = \sum_{k=0}^{\infty} P_k f$$

where

$$(5.5.4) \qquad M_n P_k f = \lambda_{n,k} P_k f \qquad \text{and} \qquad P(D) P_k f = -k(k+d) P_k f$$

with $\lambda_{n,k}$ given by (5.5.2) for $k \le n$ and $\lambda_{n,k} = 0$, $k > n$ (see [4]). We now have, using Bessel equality and Parseval inequality,

$$\|P(D) M_n^r f\|_{L_2(T)} = \left\| \sum_{k=1}^{n} k(k+d)\lambda_{n,k}^{r} P_k f \right\|_{L_2(T)}$$

$$= \Big(\sum_{k=1}^{n}\big(k(k+d)\lambda_{n,k}^{r}\big)^2 \|P_k f\|_{L_2(T)}^2\Big)^{1/2}$$

$$\leq \max_k\big(k(k+d)\lambda_{n,k}^{r}\big)\Big(\sum_{k=1}^{n}\|P_k f\|_{L_2(T)}^2\Big)^{1/2}$$

$$\leq \frac{n}{\sqrt{r}}\|f\|_{L_2(T)}.$$

□

The following estimate for $\|P(D)M_n f\|_p$ can now be derived.

COROLLARY 5.5.4. *For* $1 < p < \infty$ *and* $f \in L_p(T)$ *and any* $A > 0$, *there exists* r, $r = r(A,p,d)$, *such that*

$$\|P(D)M_n^r f\|_{L_p(T)} \leq An\,\|f\|_{L_p(T)}. \tag{5.5.5}$$

PROOF: We recall that Theorem 5.3.1 implies

$$\|P(D)M_n^r f\|_{L_p(T)} \leq 2dn\,\|f\|_{L_p(T)}. \tag{5.5.6}$$

We now use the Riesz-Thorin interpolation theorem with (5.5.6) for $p = \infty$ (or $p = 1$) and (5.5.1) to obtain (5.5.5) for $2 \leq p < \infty$ (or $1 < p \leq 2$).

□

5.6. Strong converse inequalities

In this section, we will prove converse inequalities for the Bernstein-Durrmeyer operator. We will duplicate some arguments from [8] for the sake of completeness. We define the K-functional

$$K_r(f,t^r)_p = \inf_{g\in C^{2r}(T)}\left(\|f-g\|_p + t^r\,\|P(D)^r g\|_p\right). \tag{5.6.1}$$

We note that in this section we are dealing with $r = 1$. Furthermore, we would not get a different K-functional if we relaxed the restriction on the class of functions to which g belongs. We recall that

$$A_n \sim B_n \qquad \text{iff} \qquad C^{-1}A_n \le B_n \le CA_n. \tag{5.6.2}$$

The converse result is given in the following theorem.

THEOREM 5.6.1. *Suppose $P(D)$, $M_n f$ and T are those given in section 5.2 and $K_1(f,t)_p \equiv K(f,t)_p$ is given by (5.6.1). Then we have*

$$\|M_n f - f\|_p + \|M_{dn} f - f\|_p \sim K(f,1/n)_p, \qquad 1 \le p \le \infty \tag{5.6.3}$$

and

$$\|M_n f - f\|_p \sim K(f,1/n)_p, \qquad 1 < p < \infty. \tag{5.6.4}$$

REMARK 5.6.2: In the terminology of [8] the results (5.6.3) and (5.6.4) are strong converse inequalities of type B and type A, respectively. Actually, for $d = 1$, (5.6.3) yields

$$\|M_n f - f\|_p \sim K(f, 1/n)_p \qquad \text{for} \qquad 1 \le p \le \infty,$$

and this type of equivalence will be shown for $d = 2$ and $d = 3$ as well (see Theorem 5.6.3). For $d > 1$ (5.6.3) has an advantage over (5.6.4) only for $p = 1$ and $p = \infty$.

PROOF: It was shown in (3.2) of [2] that

$$\|f - M_n f\|_p \le 2K(f, n^{-1})_p$$

and hence we need only estimate $K(f, \frac{1}{n})_p$ by $\|M_n f - f\|_p + \|M_{nd} f - f\|_p$ or by $\|M_n f - f\|$ to prove (5.6.3) and (5.6.4), respectively. (Of course the conditions are not the same.) We do so by constructing $g \in C^2(T)$ such that both $\|f - g\|$ and $\frac{1}{n}\|P(D)g\|$ will satisfy the appropriate estimate. As the K-functional is given as infimum on all $g \in C^2(T)$, we will have our result. To prove (5.6.3), we choose

$$g = \frac{1}{2}(M_{nd} M_n^2 f + M_n^2 f).$$

Using the commutativity relation $M_n M_m = M_m M_n$, we have

$$\left\| f - \frac{1}{2} M_{nd} M_n^2 f - \frac{1}{2} M_n^2 f \right\|_p \le \frac{1}{2} \left\| M_{nd} M_n^2 f - f \right\|_p + \frac{1}{2} \left\| M_n^2 f - f \right\|_p$$

$$\le \frac{1}{2} \left\| M_{nd} f - f \right\|_p + 2 \left\| M_n f - f \right\|_p .$$

To estimate $P(D)g$ we use (5.4.1) but with nd rather than n, that is, we write

$$\left\| M_{nd}\psi - \psi - \frac{\alpha_{dn}(d)}{2} P(D)(M_{nd}\psi + \psi) \right\|_p \tag{5.6.5}$$

$$\le \left(\frac{1}{4} \alpha_{dn}(d)^2 + \frac{1}{2(dn+1)^3} \right) \left\| P(D)^2 \psi \right\|_p$$

with $\psi = M_n^2 f$. We can write using Theorem 5.3.1

$$\left\| P(D)^2 M_n^2 f \right\|_p \le 2nd \left\| P(D) M_n f \right\|_p$$

$$\le 2nd \left\| P(D) \left(\frac{1}{2} (M_{nd} M_n^2 f + M_n^2 f) \right) \right\|_p$$

$$+ nd \left[\left\| P(D)(M_{dn} M_n^2 f - M_n f) \right\|_p \left\| P(D)(M_n - I) M_n f \right\|_p \right]$$

$$\le 2nd \left\| P(D) g \right\|_p + (2nd)^2 \left\| M_n f - f \right\|_p + 2n^2 d^2 \left\| M_{dn} f - f \right\|_p .$$

(Recall $P(D)(M_{dn} M_n^2 f - M_n f) = P(D) M_n (M_n f - f) + P(D) M_n^2 (M_{dn} f - f)$.)

We now complete the proof using (5.6.5) with $\psi = M_n^2 f$ and the above to write

$$\alpha_{dn}(d)\left\|P(D)g\right\|_p \le \left\|M_{dn}M_n^2 f - M_n^2 f\right\|_p + \left(\frac{1}{4}\alpha_{nd}(d)^2 + \frac{1}{2(dn+1)^3}\right)\left\|P(D)^2 M_n^2 f\right\|_p$$

$$\le 2\left\|M_{dn}f - f\right\|_p + 2\left\|M_n f - f\right\|_p$$

$$+ \left(\frac{1}{2}\alpha_{dn}(d) + \frac{1}{(dn+1)^2}\right)\left\|P(D)g\right\|_p .$$

Since $\dfrac{}{d(n+1)} \le \alpha_{dn}(d) \le \dfrac{1}{dn+1}$, we have

$$\frac{1}{n}\left\|P(D)g\right\|_p \le 8d\left(2\left\|M_{dn}f - f\right\|_p + 2\left\|M_n f - f\right\|_p\right), \qquad \text{for } n \ge 3.$$

To prove (5.6.4) we choose $g = \frac{1}{2}(M_n^{r+2}f + M_n^{r+1}f)$ with $r = r(p,d)$ such that (5.5.5) is satisfied with $A = 2$ (which is possible for $1 < p < \infty$ and any d by Corollary 5.5.4). Obviously

$$\left\|f - g\right\|_p \le \frac{1}{2}\left(\left\|M_n^{r+2}f - f\right\|_p + \left\|M_n^{r+1}f - f\right\|_p\right) \le \frac{1}{2}(2r+3)\left\|M_n f - f\right\|_p .$$

To estimate $\frac{1}{n}\left\|P(D)g\right\|$ we use Theorem 5.4.1 and write

$$\left\|M_n(M_n^{r+1}f) - M_n^{r+1}f - \frac{\alpha_n(d)}{2}P(D)(M_n^{r+2}f + M_n^{r+1}f)\right\|_p$$

$$\le \left(\frac{1}{4}\alpha_n(d)^2 + \frac{1}{2(n+1)^3}\right)\left\|P(D)^2 M_n^{r+1}f\right\|_p$$

and

$$\begin{aligned}\|P(D)^2 M_n^{r+1} f\|_p &\le 2n \|P(D) M_n f\|_p \\ &\le 2n \left\| P(D)\Big(\frac{1}{2} M_n^{r+2} f + \frac{1}{2} M_n^{r+1} f\Big)\right\|_p \\ &\qquad + n \cdot 2dn\big(\|M_n^{r+1} f - f\|_p + \|M_n^r f - f\|_p\big) \\ &\le 2n \|P(D) g\|_p + n^2 2d(2r+1) \|M_n f - f\|_p\end{aligned}$$

and proceed as before to complete th· ; roof. □

THEOREM 5.6.3. *Under the assumptions of Theorem 5.3.1, we have*

$$\|M_n f - f\|_p \sim K(f, 1/n)_p$$

for $1 \le p \le \infty$ *and* $d = 1, 2, 3$.

PROOF: Actually, we only have to prove the equivalence for $p = 1$ and ∞ in case $d = 2$ and $d = 3$. We choose $g = \frac{1}{2}(M_n^4 f + M_n^3 f)$ and use (5.4.1) to write

$$\left\| M_n\psi - \psi - \frac{\alpha_n(d)}{2} P(D)(M_n\psi + \psi)\right\|_p \le \left(\frac{1}{4}\alpha_n(d)^2 + \frac{1}{2(n+1)^3}\right) \|P(D)^2 \psi\|_p$$

with $\psi = M_n^3 f$. The proof now follows the same lines (see also [7]) using the fact that $n\alpha_n(d)$ is close to one for $n \ge n_0$ and using Theorem 5.3.2 instead of Theorem 5.3.1. □

REMARK 5.6.4: It would be desirable to prove Theorem 5.6.3 for all d and we believe that this result is valid. This would follow from the estimate

$$\|P(D)M_n^r f\|_p \le \varepsilon(r)\,\|f\|_p$$

with $\varepsilon(r) = o(1)$, $r \to \infty$. While we believe this last estimate to be true, we are not able to prove it at present.

5.7. Iterations

In this section we will use the results of the last section to obtain Theorems about equivalence to $K_r(f, t^r)$.

THEOREM 5.7.1. *For* $f \in L_p(T)$, $1 < p < \infty$, *or* $f \in L_p(T)$, $\dim T \le 3$ *and* $1 \le p \le \infty$,

$$K_r(f, n^{-r})_p \sim \|(M_n - I)^r f\|_p \tag{5.7.1}$$

where $K_r(f, t^r)_p$ *is given by (5.6.1) and* M_n *by (5.2.1).*

PROOF: The estimate

$$K_r(f, n^{-r})_p \le C(r)\,\|(M_n - I)^r f\|_p \tag{5.7.2}$$

follows from the estimate achieved in Theorems 6.1 and 6.3 and Theorem 10.4 of [8] which use the estimate

$$(5.7.3) \qquad \frac{1}{n}\left\|P(D)(M_n^\ell f)\right\|_p \le B\left\|f - M_n f\right\|_p$$

for some ℓ. We proved for some r

$$\frac{1}{2n}\left\|P(D)(M_n^{r+1}f + M_n^r f)\right\|_p \lesssim B_1\left\|f - M_n f\right\|_p$$

which implies (5.7.3). ((5.7.3) could have been proved directly.) The estimate

$$(5.7.4) \qquad \left\|(M_n - I)^r f\right\|_p \le B(r)K_r(f, n^{-r})_p$$

was shown when proving Theorem 4.1 of [2] and is the easier direction in any case. □

We can also prove the following result which is of interest only for $p = 1$ and $p = \infty$ when $d > 3$, as otherwise it is just a special case of Theorem 5.7.1.

THEOREM 5.7.2. *For* $f \in L_p(T)$, $1 \le p \le \infty$, *we have*

$$(5.7.5) \qquad K_r(f, n^{-r})_p \sim \max_{0\le i\le r}\left\|(M_n - I)^{r-i}(M_{nd} - I)^i f\right\|_p, \qquad n \ge n_0$$

and

$$(5.7.6) \qquad K_r(f, n^{-r})_p \sim \left\|(M_n - I)^r f\right\|_p + \left\|(M_{nm} - I)^r f\right\|_p, \qquad n \ge n_0$$

for some $m = m(r)$.

REMARK 5.7.3: The advantage of (5.7.5) is that it is easier to prove (and d may be smaller than m). The advantage of (5.7.6) is that it yields two terms and hence the iteration is still a strong converse inequality of type B in the terminology of [8]. Moreover, M_{nd} and M_{nm} in (5.7.5) and (5.7.6) can be replaced by M_ℓ, with $nd \le \ell \le nA$ and $nm \le \ell \le nA$, respectively.

PROOF OF THEOREM 5.7.2: The direct inequalities in (5.7.5) and (5.7.6), that is,

$$\left\|(M_n - I)^{r-i}(M_{nd} - I)^i f\right\|_p \le CK_r(f, n^{-r})_p, \qquad 0 \le i \le r$$

and

$$\|(M_{sn} - I)^r f\|_p \le CK_r(f, n^{-r})_p, \qquad s = 1, m$$

follows from earlier results (see for instance the proof of Theorem 4.1 in [2]). For the proof of (5.7.5) we have to show

$$K_r(f, n^{-r})_p \le B \max_{0 \le i \le r} \left\|(M_n - I)^{r-i}(M_{nd} - I)^i f\right\|_p . \tag{5.7.7}$$

To obtain (5.7.7) we choose g as

$$g \equiv O_{n,r} f = \sum_{s=1}^{r} (-1)^{s-1} \binom{r}{s} O_n^{rs} f, \qquad O_n f \equiv \frac{1}{2}(M_{nd} M_n^2 + M_n^2) f. \tag{5.7.8}$$

We estimate $\|f-g\|_p$ by

$$\|f-g\|_p = \|f - O_{n,r}f\|_p = \|(O_n^r - I)^r f\|_p \le r^r \|(O_n - I)^r f\|_p$$

$$\le Ar^r \max_{0\le i\le r} \left\|(M_n - I)^{r-i}(M_{nd} - I)^i\right\|_p .$$

To complete the proof of (5.7.7) we estimate $n^{-r}\|P(D)^r g\|_p$ by

$$n^{-r}\|P(D)^r g\|_p = n^{-r}\|P(D)^r O_{n,r} f\|_p \le 2^r n^{-r} \max_{1\le s\le r} \|P(D)^r O_n^{rs} f\|_p$$

$$\le 2^r n^{-r}\|P(D)^r O_n^r f\|_p$$

$$\le An^{-r+1}\left(\left\|P(D)^{r-1}O_n^{r-1}(M_n - I)f\right\|_p + \left\|P(D)^{r-1}O_n^{r-1}(M_{nd} - I)f\right\|_p\right)$$

$$\le \cdots \le B \max_{0\le i\le r} \left\|(M_n - I)^{r-i}(M_{nd} - I)^i f\right\|_p .$$

To prove (5.7.6) it remains to show that for some integer m we have

$$K_r(f, n^{-r})_p \le B(\|(M_n - I)^r f\|_p + \|(M_{nm} - I)^r f\|_p). \tag{5.7.9}$$

We postpone the choice of m and choose g

$$g = \sum_{s=1}^{r} (-1)^{s+1}\binom{r}{s} M_n^{(r+1)s} f. \tag{5.7.10}$$

The estimate of $\|f-g\|_p$ is given by

$$\|f-g\|_p = \left\|(M_n^{r+1} - I)^r f\right\|_p \le (r+1)^r \|(M_n - I)^r f\|_p .$$

To estimate $n^{-r}\left\|P(D)^r g\right\|_p$ we write

$$n^{-r}\left\|P(D)^r g\right\|_p \leq n^{-r}2^r \sup_{1\leq s\leq r}\left\|P(D)^r M_n^{(r+1)s} f\right\|_p \leq n^{-r}2^r \left\|P(D)^r M_n^{r+1} f\right\|_p$$

and hence it is sufficient to estimate $n^{-r}\left\|P(D)^r M_n^{r+1} f\right\|_p$. Using Theorem 5.4.3 with mn replacing n and m chosen so that $2rd2^r \leq m$, we have

$$\left\|(M_{nm}-I)^r M_n^{r+1} f - \alpha_{nm}(d)^r P(D)^r M_n^{r+1} f\right\|_p$$

$$\leq \frac{r}{2(nm)^{r+1}}\left\|P(D)^{r+1} M_n^{r+1} f\right\|_p$$

$$\leq \frac{rd}{m}\frac{1}{(mn)^r}\left\|P(D)^r M_n^r f\right\|_p$$

$$\leq \frac{rd}{m}\frac{1}{(mn)^r}\left\|P(D)^r M_n^r (M_n - I)^r f\right\|_p + \frac{rd}{m}\frac{1}{(mn)^r}\sum_{s=1}^{r}\binom{r}{s}\left\|P(D)^r M_n^{r+s} f\right\|_p$$

$$\leq \frac{rd}{m}\frac{2^r}{(mn)^r}\left\|P(D)^r M_n^{r+1} f\right\|_p + \frac{r}{2}\left(\frac{2d}{m}\right)^{r+1}\left\|(M_n - I)^r f\right\|_p .$$

Since $\dfrac{rd2^r}{m} \leq \dfrac{1}{2}$ we complete the proof writing

$$\alpha_{nm}(d)^r\left\|P(D)^r M_n^{r+1} f\right\|_p \leq \frac{1}{2}\frac{1}{(mn)^r}\left\|P(D)^r M_n^{r+1} f\right\|_p$$

$$+\left\|(M_{nm} - I)^r f\right\|_p + \frac{r}{2}\left(\frac{2d}{m}\right)^{r+1}\left\|(M_n - I)^r f\right\|_p$$

and recalling $\alpha_{nm}(d)^r = \left(\dfrac{1}{nm}\right)^r + O(n^{-r-1})$. □

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CHAPTER 6

BEST POLYNOMIAL AND DURRMEYER APPROXIMATION IN $L_p(S)$ *

6.1. Introduction

In an earlier article [7], the multivariate Bernstein polynomial approximation was compared to Best polynomial approximation in the uniform norm on the simplex. As the Bernstein polynomial approximation is not a bounded operator on $L_p(S)$, we cannot compare it with the best polynomial approximation to a general function in $L_p(S)$. In the univariate case, it was the Kantorovich-Bernstein operator that took the place of Bernstein operators in the comparison. We find the Durrmeyer-Bernstein polynomial approximation to be the preferred alternative since it is not only bounded on $L_p(S)$, but commutes with other operators of the same family and with the appropriate differential operator as well. Furthermore, the Durrmeyer-Bernstein operators are self-adjoint and can be given by an expansion of orthogonal polynomials. The multivariate Durrmeyer-Bernstein operators were actually introduced by M. Derriennic [4] who gave them as an extension of the one-dimensional case given by Durrmeyer [12].

For the simplex $S = \{u = (u_1, \ldots, u_d) : u_i \geq 0, \ \sum_{i=1}^{d} u_i \leq 1\}$, the

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Durrmeyer-Bernstein operator is given by

$$(6.1.1)\qquad M_n(f,u)=\frac{(n+d)!}{n!}\sum_{\frac{\beta}{n}\in S}P_{n,\beta}(u)\int_S P_{n,\beta}(x)\,f(x)\,dx$$

with $x,\,u,\,\beta\in R^d$ $(\beta=(k_1,\ldots,k_d)$ with integer entries $k_i)$ and

$$(6.1.2)\qquad P_{n,k_1,\ldots,k_d}(u_1,\ldots,u_d)\equiv P_{n,\beta}(u)=\frac{n!}{\beta!\,(n-|\beta|)!}u^\beta(1-|u|)^{n-|\beta|}$$

where $\beta!=k_1!\cdots k_d!$, $u^\beta=u_1^{k_1}\cdots u_d^{k_d}$ $(u_i^{k_i}=1$ if $k_i=u_i=0)$, $|u|=\sum_{i=1}^d u_i$ and $|\beta|=\sum_{i=1}^d k_i$. A key to theorems on these operators is the self-adjoint partial differential operator associated with them and given by

(6.1.3)

$$\begin{aligned}P(D)&\equiv\sum_{i=1}^d\frac{\partial}{\partial x_i}x_i(1-|x|)\frac{\partial}{\partial x_i}+\sum_{i<j}\left(\frac{\partial}{\partial x_i}-\frac{\partial}{\partial x_j}\right)x_ix_j\left(\frac{\partial}{\partial x_i}-\frac{\partial}{\partial x_j}\right)\\&\equiv\sum_{i\le j}P_{i,j}(D).\end{aligned}$$

In this paper, we will show that the behaviour of $\|M_nf-f\|_{L_p(S)}$ is closely related to $E_{[\sqrt{n}]}(f)_{L_p(S)}$ where

(6.1.4)

$$E_k(f)_{L_p(S)}=\inf\{\|f-P\|_{L_p(S)}:P\text{ polynomial of total degree smaller than }k\}.$$

As a result of our investigation, we will have for $0<\alpha<r$,

$$(6.1.5)\qquad \|(M_n-I)^rf\|_{L_p(S)}=O(n^{-\alpha})\iff E_n(f)_{L_p(S)}=O(n^{-2\alpha}).$$

The first result of this type was proved by K. Ivanov [13] who showed for $f \in C[0,1]$ and $0 < \alpha < 1$,

$$\|B_n f - f\|_{C[0,1]} = O(n^{-\alpha}) \iff E_n(f)_{C[0,1]} = O(n^{-2\alpha}) \tag{6.1.6}$$

where B_n is the Bernstein polynomial operator. For $C(S)$, this type of result was proved in [7]. That is, for $f \in C(S)$ and $0 < \alpha < 1$, it was shown in [7] that

$$\|B_n f - f\|_{C(S)} = O(n^{-\alpha}) \iff E_n(f)_{C(S)} = O(n^{-2\alpha}). \tag{6.1.7}$$

Similar to the situation in [7], we will prove a more delicate relation between $E_n(f)$ and the rate of approximation. Here the results are on $L_p(S)$, $1 \le p \le \infty$ or $C(S)$ and not merely on $C(S)$ and are much better even for $C(S)$. Moreover, because of the properties of M_n, we will obtain relations between $P(D)f$ and directional smoothness and apriori estimates. In Section 6.2, we obtain a bound of $\|(M_n - I)^r f\|_{L_p(S)}$ by $E_k(f)_{L_p(S)}$ with $k^2 \le n$. In Section 6.3, we describe the relation between $E_k(f)_{L_p(S)}$ and the related K-functional. In Section 6.5, we estimate $E_n(f)_{L_p(S)}$ with the aid of the sequence $\|(M_k - I)^r f\|_{L_p(S)}$ for $k \le n^2$. In Section 6.6, we derive conclusions from the above results pertaining to directional smoothness, Besov spaces and apriori estimates.

6.2. Estimates of $\|(M_n - I)^r f\|_p$

We recall the identity (see [2] or [3])

$$M_n f - f = \sum_{k=n+1}^{\infty} \frac{1}{k(k+d)} P(D) M_k f. \tag{6.2.1}$$

The identity (6.2.1) combined with $P(D)M_k g = M_k P(D) g$ for $g \in C^{2r}(S)$ [2, Lemma 2.5], $\|M_k f\|_p \le \|f\|_p$ and $\sum_{k=n+1}^{\infty} \frac{1}{k(k+d)} \le \frac{1}{n}$ yields for $g \in C^{2r}(S)$

$$\|(M_n - I)^r g\|_p \le n^{-r} \|P(D)^r g\|_p . \tag{6.2.2}$$

We can now state and prove our estimate.

Theorem 6.2.1. *For $f \in L_p(S)$ and $2^{2(k-1)} < n \le 2^{2k}$ we have*

$$\|(M_n - I)^r f\|_p \le C \sum_{\ell=0}^{k} 2^{2r(\ell-k)} E_{2^\ell}(f)_p. \tag{6.2.3}$$

Using the monotonicity of $E_n(f)_p$ and of $2^{r\ell}$, we can write Theorem 6.2.1 in the following way that some (not us) may find more attractive.

Corollary 6.2.2. *For $f \in L_p(S)$, we have*

$$\|(M_n - I)^r f\|_p \le C n^{-r} \sum_{1 \le k \le \sqrt{n}} k^{2r-1} E_k(f)_p. \tag{6.2.4}$$

PROOF OF THEOREM 6.2.1: We denote by P_n the polynomial of best approximation of total degree smaller than n in $L_p(S)$ and write

(6.2.5)
$$\begin{aligned}\|(M_n - I)^r f\|_p &\leq \|(M_n - I)^r(f - P_{2^k})\|_p + \|(M_n - I)^r P_{2^k}\|_p \\ &\leq 2^r E_{2^k}(f)_p + \|(M_n - I)^r P_{2^k}\|_p .\end{aligned}$$

We now write

$$P_{2^k} = \sum_{\ell=0}^{k}(P_{2^\ell} - P_{2^{\ell-1}}) + P_1.$$

Using (6.2.2) for $g = P_{2^\ell} - P_{2^{\ell-1}}$ and the above expansion, we have

(6.2.6)
$$\|(M_n - I)^r P_{2^k}\|_p \leq n^{-r} \sum_{\ell=1}^{k} \|P(D)^r(P_{2^\ell} - P_{2^{\ell-1}})\|_p .$$

We will show for P_m, a polynomial of total degree m,

(6.2.7)
$$\|P(D)P_m\|_p \leq C_1 m^2 \|P_m\|_p$$

with C_1 that is independent of m and P_m. As $P(D)P_m$ is again a polynomial of total degree smaller than or equal to m, we can iterate the above to obtain

(6.2.8)
$$\|P(D)^r P_m\|_p \leq C_1^r m^{2r} \|P_m\|_p$$

Using (6.2.8) and

$$\|P_{2^\ell} - P_{2^{\ell-1}}\|_p \leq 2E_{2^{\ell-1}}(f)_p$$

in (6.2.6) and recalling (6.2.5), we complete the proof of (6.2.3) pending the proof of (6.2.7). To prove (6.2.7) we recall that $P(D)P_m$ is bounded by a combination of finite number of terms (see (6.1.3)) where each one is exactly like $\left\|\frac{\partial}{\partial x_1}x_1(1-|x|)\frac{\partial}{\partial x_1}P_m(x)\right\|_{L_p(S)} \equiv \|P_{1,1}(D)P_m(x)\|_p$. In fact, one obtains other terms by affine transformations. For example, $\|P_{1,2}(D)f(x)\|_{L_p(S)}$ is $\|P_{1,1}(D)f(u)\|_{L_p(S)}$ with $u_1 = x_1$, $u_2 = 1-x_1\cdots-x_d$, $u_3 = x_3,\cdots,u_d = x_d$. To estimate the above expression, we have

$$\left\|\frac{\partial}{\partial x_1}x_1(1-|x|)\frac{\partial}{\partial x_1}P_m(x)\right\|_{L_p(S)} \leq \left\|x_1(1-|x|)\left(\frac{\partial}{\partial x_1}\right)^2 P_m(x)\right\|_{L_p(S)} + \left\|\frac{\partial}{\partial x_1}P_m(x)\right\|_{L_p(S)}$$

as $|1-|x|-x_1| \leq 1$. The Bernstein inequality

$$\left\|x_1(1-|x|)\left(\frac{\partial}{\partial x_1}\right)^2 P_m(x)\right\|_{L_p(S)} \leq C_2 m^2 \|P_m\|_{L_p(S)} \tag{6.2.9}$$

is a special case of Theorem 2.1 of [8] where we set $r = 2$ and the simplex S for the bounded convex set. The Markov inequality

$$\left\|\frac{\partial}{\partial x_1}P_m(x)\right\|_{L_p(S)} \leq C_3 m^2 \|P_m\|_{L_p(S)} \tag{6.2.10}$$

is a special case of Theorem 4.1 of [8] with the simplex S standing for the bounded convex set. Combining the estimates (6.2.9) and (6.2.10), we derive the desired estimate. □

6.3. Estimate of $E_n(f)_{L_p(S)}$ by an appropriate K-functional

We recall from the monograph [10] by the second author and V. Totik that

$$(6.3.1) \qquad E_n(f)_{L_p(S)} \le M[\omega_S^m(f, 1/n)_{L_p(S)} + n^{-m} \|f\|_{L_p(S)}]$$

which is valid for any simple polytope and hence for the simplex S. The modulus of smoothness $\omega_S^m(f,t)_{L_p(S)}$ was defined by

$$(6.3.2) \qquad \omega_S^m(f,t)_p = \sup_{e \in V_S} \sup_{0<h\le t} \left\| \Delta^m_{h\tilde{d}_S(e,x)^{1/2}e} f(x) \right\|_{L_p(S)}$$

where V_S is the set of unit vectors in the directions of the edges of S

$$(6.3.3) \qquad \tilde{d}_S(e,x) = \left(\min_{x+\lambda e \notin S} d(x, x+\lambda e) \right) \left(\max_{\substack{x+\lambda_i e \in S \\ i=1,2}} d(x+\lambda_1 e, x+\lambda_2 e) \right)$$

and $d(a,b)$ is the Euclidean distance between a and b. We note that the expression in (6.3.2) may look somewhat simpler than $\omega_S^m(f,t)_p$ defined in [10, Chapter 12] but observing (12.2.1) and (12.2.2) of [10], it is clear that these are the same notions with the convention that $\Delta^m_{he} f(x) = 0$ if either of $x \pm \frac{m}{2} he$ does not belong to S. We further note that if we replace $\tilde{d}_S(e,x)$ by $d_S^*(e,x)$, satisfying

$$(6.3.4) \qquad C^{-1} d_S^*(e,x) \le \tilde{d}_S(e,x) \le C d_S^*(e,x) \qquad \text{for } e \in V_S$$

in (6.3.2), we obtain a concept that is equivalent to $\omega_S^m(f,t)_p$. It can be easily seen that for $d_S^*(e_i,x)$ defined by

$$d_S^*(e_i,x)=x_i(1-|x|) \qquad \text{and} \qquad d_S^*((e_j-e_i)/\sqrt{2},x)=x_ix_j \tag{6.3.5}$$

we have

$$d_S^*(e,x)\le \widetilde{d}_S(e,x) \qquad \text{and} \qquad \widetilde{d}_S(e,x)\le 4d_S^*(e,x) \quad \text{for } e\in V_S$$

and hence (6.3.4) is satisfied. Therefore, using $d_S^*(e,x)$ instead of $\widetilde{d}_S(e,x)$ leads to an equivalent expression.

We now define the K-functional

(6.3.6)

$$K_{m,S}(f,t^m)_p=\inf_{g\in C^m(S)}\left(\|f-g\|_{L_p(S)}+t^m\sup_{\xi\in V_S}\left\|\varphi_\xi(\cdot)^m\left(\frac{\partial}{\partial\xi}\right)^m g(\cdot)\right\|_{L_p(S)}\right)$$

where V_S is the set of edges of S and $\varphi_\xi(x)^2=d_S^*(\xi,x)$ for $\xi\in V_S$.

It is implicitly clear from [10, Chapter 12] that we have the following result.

THEOREM 6.3.1. *For* $f\in L_p(S)$

$$\omega_S^m(f,t)_p\le CK_{m,S}(f,t^m)_p. \tag{6.3.7}$$

PROOF: For the sake of completeness, we will give the proof of (6.3.7). For this it would be sufficient to show for any direction $\xi\in V_S$ and function $f\in L_p(S)$

$$\left\|\Delta^m_{h\varphi_\xi(x)\xi}f(x)\right\|_{L_p(S)}\le C\,\|f\|_{L_p(S)} \tag{6.3.8}$$

and for $f \in C^m$

$$\left\|\Delta^m_{h\varphi_\xi(x)\xi} f(x)\right\|_{L_p(S)} \le Ch^m \left\|\varphi_\xi(\cdot)^m \left(\frac{\partial}{\partial \xi}\right)^m f(\cdot)\right\|_{L_p(S)}. \tag{6.3.9}$$

In fact, because of symmetry and linear transformations, it is sufficient to prove (6.3.8) and (6.3.9) for $\xi = e_1$ and $\varphi_\xi(x)^2 = \varphi_{e_1}(x)^2 = x_1(1-|x|)$. We define $t = \frac{x_1}{1-|\tilde{x}|}$ where $\tilde{x} = (x_2, \dots, x_d)$ and write

$$f(x) = f(x_1, x_2, \dots, x_d) = f(t(1-|\tilde{x}|), x_2, \dots, x_d) = f(t(1-|\tilde{x}|), \tilde{x}) \equiv F_{\tilde{x}}(t)$$

$$\Delta^m_{h\varphi_{e_1}(x)e_1} f(x) = \Delta^m_{h\varphi(t)} F_{\tilde{x}}(t) \qquad \text{with} \quad \varphi(t)^2 = t(1-t).$$

We denote $\tilde{S} = \{\tilde{x} = (x_2, \dots, x_d) : x_i \ge 0,\ 1 - x_2 - \cdots - x_d \ge 0\}$ and using the one-dimensional result, that is, (2.4.3) of [10], we write

$$\begin{aligned}
\left\|\Delta^m_{h\varphi_{e_1}(x)e_1} f(x)\right\|^p_{L_p(S)} &= \int_{\tilde{S}} \int_0^{1-|\tilde{x}|} \left|\Delta^m_{h\varphi_{e_1}(x)e_1} f(x)\right|^p dx_1\, d\tilde{x} \\
&= \int_{\tilde{S}} \int_0^1 \left|\Delta^m_{h\varphi(t)} F_{\tilde{x}}(t)\right|^p (1-|\tilde{x}|)\, dt\, d\tilde{x} \\
&\le \int_{\tilde{S}} C^p \int_0^1 |F_{\tilde{x}}(t)|^p (1-|\tilde{x}|)\, dt\, d\tilde{x} \\
&= C^p \int_S |f(x)|^p\, dx = C^p \|f\|^p_{L_p(S)}.
\end{aligned}$$

To prove (6.3.9), we observe that

$$\varphi_{e_1}(x)^m \left(\frac{\partial}{\partial x_1}\right)^m f(x) = \varphi(t)^m \left(\frac{d}{dt}\right)^m F_{\tilde{x}}(t)$$

and using the univariate analogue of (6.3.9) which is (2.4.4) of [10], we have

$$\begin{aligned}\left\|\Delta^m_{h\varphi_{e_1}(x)e_1} f(x)\right\|^p_{L_p(S)} &= \int_{\widetilde{S}} (1-|\widetilde{x}|) \int_0^1 |\Delta^m_{h\varphi(t)} F_{\widetilde{x}}(t)|^p \, dt \, d\widetilde{x} \\ &\le \int_{\widetilde{S}} (1-|\widetilde{x}|) C^p h^{mp} \int_0^1 \left|\varphi(t)^m \left(\frac{d}{dt}\right)^m F_{\widetilde{x}}(t)\right|^p dt\, d\widetilde{x} \\ &\le C^p h^{mp} \int_{\widetilde{S}} \int_0^{1-|\widetilde{x}|} \left|\varphi_{e_1}(x)^m \left(\frac{\partial}{\partial x_1}\right)^m f(x)\right|^p dx\, d\widetilde{x} \\ &\le C^p h^{mp} \left\|\varphi_{e_1}(\cdot)^m \left(\frac{\partial}{\partial x_1}\right)^m f(\cdot)\right\|^p_{L_p(S)}.\end{aligned}$$

For $p=\infty$, the obvious easy modifications are used to prove (6.3.8) and (6.3.9) and this will complete the proof of our theorem. □

6.4. Relation between $\|(M_n-I)^m f\|_p$ and $K_{2r,S}(f,t^r)_p$

The estimate of $\|(M_n-I)^r f\|_p$ by the K-functional given by

$$K_r(f,t^r)_p = \inf_{g\in C^{2r}(S)} \{\|f-g\|_p + t^r \|P(D)^r g\|_p\}, \tag{6.4.1}$$

that is,

$$\|(M_n-I)^r f\|_p \le C K_r(f,n^{-r})_p \tag{6.4.2}$$

was given in [2]. The estimate (6.4.2) has a strong converse, i.e. the terms of (6.4.2) are equivalent, and therefore it is somewhat surprising that we need here a relation with a different K-functional that is not (and can not be) of a strong converse type variety. This estimate is helpful in the next section and will be useful for the main result and comparisons of different K-functionals.

THEOREM 6.4.1. *For* $f \in L_p(S)$, $M_n f$ *defined by (6.1.1) and* $K_{2r,S}(f,t)_p$ *defined by (6.3.6), we have*

$$K_{2r,S}(f,t^r)_p \le \|(M_k - I)^m f\|_p + Ct^r k^r K_{2r,S}(f,k^{-r})_p \tag{6.4.3}$$

for any pair of integers r *and* m.

PROOF: The function f can be written as

$$f = (I - M_k)^m f - \sum_{\ell=1}^{m} \binom{m}{\ell} (-1)^\ell M_k^\ell f$$

and hence, it is sufficient to show that

$$\left\| \varphi_\xi^{2r} \left(\frac{\partial}{\partial \xi} \right)^{2r} M_k f \right\|_{L_p(S)} \le C_1 k^r \|f\|_{L_p(S)} \tag{6.4.4}$$

and

$$\left\| \varphi_\xi^{2r} \left(\frac{\partial}{\partial \xi} \right)^{2r} M_k f \right\|_{L_p(S)} \le C \left\| \varphi_\xi^{2r} \left(\frac{\partial}{\partial \xi} \right)^{2r} f \right\|_{L_p(S)} \tag{6.4.5}$$

where ξ is an edge of S, $\varphi_{e_i}(x)^2 = x_i(1-|x|)$ and $\varphi_{(e_i-e_j)/\sqrt{2}}(x)^2 = x_i x_j$. In fact, symmetry and affine transformations imply that it is sufficient to prove (6.4.5) and (6.4.6) for $\xi = x_1$. These results will be stated in the next two lemmas which, when proved, will imply (6.4.3). □

LEMMA 6.4.2. *For* $f \in L_p(S)$ *and* $f \in C^{2r}(S)$,

$$\left\| (x_1(1-|x|))^r \left(\frac{\partial}{\partial x_1}\right)^{2r} M_k(f,x) \right\|_{L_p(S)} \tag{6.4.6}$$

$$\leq C \left\| (x_1(1-|x|))^r \left(\frac{\partial}{\partial x_1}\right)^{2r} f(x) \right\|_{L_p(S)}.$$

LEMMA 6.4.3. *For* $f \in L_p(S)$,

$$\left\| (x_1(1-|x|))^r \left(\frac{\partial}{\partial x_1}\right)^{2r} M_k(f,x) \right\|_{L_p(S)} \leq Ck^r \|f\|_{L_p(S)}. \tag{6.4.7}$$

REMARK 6.4.4. While Theorem 6.4.1 was proven for any pair of integers r and m, the more interesting case is when $m \geq r$.

In [5], a theorem similar to Lemma 6.4.3 is given but as it is not exactly Lemma 6.4.3 and as the method here can be shown for all d at the same time, we chose to give the proof directly.

PROOF OF LEMMA 6.4.2: The idea of the proof follows the method that was used to prove the univariate analogue (see [9, I, p. 75]). With the conventions $P_{k,\beta}(x) = 0$ whenever $\frac{\beta}{k} \notin S$ and that $\beta + \gamma$ is vector addition in R^d (which we use throughout this paper), we have

$$\frac{\partial}{\partial x_1} P_{k,\beta}(x) = k(P_{k-1,\beta-e_1}(x) - P_{k-1,\beta}(x))$$

and hence

$$(6.4.8)\qquad \left(\frac{\partial}{\partial x_1}\right)^{2r} P_{k,\beta}(x) = \frac{k!}{(k-2r)!}\sum_{i=0}^{2r}(-1)^i\binom{2r}{i}P_{k-2r,\beta-(2r-i)e_1}(x).$$

Using integration by parts, we obtain

$$\begin{aligned}
\left(\frac{\partial}{\partial x_1}\right)^{2r} M_k(f,x) &= \frac{(k+d)!}{k!}\sum_{\frac{\beta}{k}\in S}\left(\frac{\partial}{\partial x_1}\right)^{2r} P_{k,\beta}(x)\int_S P_{k,\beta}(u)f(u)\,du \\
&= \frac{(k+d)!}{(k-2r)!}\sum_{\frac{\beta}{k}\in S}\sum_{i=0}^{2r}(-1)^i\binom{2r}{i}P_{k-2r,\beta-(2r-i)e_1}(x)\int_S P_{k,\beta}(u)f(u)\,du \\
&= \frac{(k+d)!}{(k-2r)!}\sum_{\frac{\beta}{k-2r}\in S} P_{k-2r,\beta}(x)\int_S\sum_{i=0}^{2r}(-1)^i\binom{2r}{i}P_{k,\beta+(2r-i)e_1}(u)f(u)\,du \\
&= \frac{(k+d)!\,k!}{(k-2r)!\,(k+2r)!}\sum_{\frac{\beta}{k-2r}\in S} P_{k-2r,\beta}(x)\int_S f(u)\left(\frac{\partial}{\partial u_1}\right)^{2r} P_{k+2r,\beta+2re_1}(u)\,du \\
&= \frac{(k+d)!\,k!}{(k-2r)!\,(k+2r)!}\sum_{\frac{\beta}{k-2r}\in S} P_{k-2r,\beta}(x)\int_S P_{k+2r,\beta+2re_1}(u)\left(\frac{\partial}{\partial u_1}\right)^{2r} f(u)\,du.
\end{aligned}$$

We can now write

(6.4.9)

$$\begin{aligned}
&[x_1(1-|x|)]^r\left(\frac{\partial}{\partial x_1}\right)^{2r} M_k(f,x) \\
&\qquad = \frac{(k+d)!}{k!}\sum_{\frac{\beta}{k-2r}\in S}\alpha(k,\beta,r)P_{k,\beta+re_1}(x) \\
&\qquad\qquad \times\int_S P_{k,\beta+re_1}(u)[u_1(1-|u|)]^r\left(\frac{\partial}{\partial u_1}\right)^{2r} f(u)\,du
\end{aligned}$$

where

$$\alpha(k,\beta,r) = \frac{k!}{(k-2r)!}\,\frac{k!}{(k+2r)!}\binom{k-2r}{\beta}\binom{k+2r}{\beta+2re_1}\binom{k}{\beta+re_1}^{-2} < 1.$$

Recalling that $\frac{\beta}{k-2r} \in S$ implies $\frac{\beta+re_1}{k} \in S$, we have

$$\begin{aligned}
&\left\|(x_1(1-|x|))^r\left(\frac{\partial}{\partial x_1}\right)^{2r} M_k(f,x)\right\|_{L_p(S)} \\
&\quad\le \left\|\frac{(k+d)!}{k!}\sum_{\frac{\beta}{k-2r}\in S} P_{k,\beta+re_1}(x)\int_S P_{k,\beta+re_1}(u)[u_1(1-|u|)]^r\left|\left(\frac{\partial}{\partial u_1}\right)^{2r} f(u)\right| du\right\|_{L_p(S)} \\
&\quad\le \left\|M_k\left([u_1(1-|u|)]^r\left|\left(\frac{\partial}{\partial u_1}\right)^{2r} f(u)\right|\right)\right\|_{L_p(S)} \\
&\quad\le \left\|[x_1(1-|x|)]^r\left(\frac{\partial}{\partial x_1}\right)^{2r} f(x)\right\|_{L_p(S)},
\end{aligned}$$

which completes the proof of the Lemma. □

The proof of Lemma 6.4.3 requires the following univariate Lemma.

LEMMA 6.4.5. *For* $P_{k,j}(t) = \binom{k}{j}t^j(1-t)^{k-j}$, $\varphi(t)^2 = t(1-t)$ *and integer* r, *we have*

$$\left\|\sum_{j=0}^{k}\varphi(t)^{2r}\left|\left(\frac{d}{dt}\right)^{2r} P_{k,j}(t)\right|\right\|_{L_\infty[0,1]} \le Ck^r \tag{6.4.10}$$

and

$$\left\|\varphi(t)^{2r}\left(\frac{d}{dt}\right)^{2r} P_{k,j}(t)\right\|_{L_1[0,1]} \le Ck^{r-1}. \tag{6.4.11}$$

PROOF: Following the almost standard procedure (see [6], [9] and [10]), we divide $[0,1]$ into $E_k = [\frac{1}{k}, 1-\frac{1}{k}]$ and $E_k^c = [0,1]\backslash E_k$ and give the estimate on $L_p(E_k)$ and $L_p(E_k^c)$ separately. Using (6.4.8), we have

$$\left\|\sum_{j=0}^{k}\varphi(t)^{2r}\left|\left(\frac{d}{dt}\right)^{2r} P_{k,j}(t)\right|\right\|_{L_\infty(E_k^c)} \leq k^{-r}2^{2r}\frac{k!}{(k-2r)!}\sup_{0\leq i\leq 2r}\left\|\sum_{j=0}^{k} P_{k-2r,j-i}(t)\right\|_{L_\infty[0,1]}$$

$$\leq Ck^r,$$

and

$$\left\|\varphi(t)^{2r}\left(\frac{d}{dt}\right)^{2r} P_{k,j}(t)\right\|_{L_1(E_k^c)} \leq \frac{k!}{(k-2r)!}k^{-r}2^{2r}\sup_{0\leq i\leq 2r}\|P_{k-2r,j-i}(t)\|_{L_1[0,1]}$$

$$\leq Ck^{r-1}.$$

To obtain the estimates on $L_\infty(E_k)$ and $L_1(E_k)$, we recall from [6, p. 283] that $P_{k,j}^{(2r)}(t)$ is a sum of terms of the type

$$q_{\ell,m}(t)\frac{(j-kt)^{2r-2\ell-m}}{(t(1-t))^{2r-\ell}}k^\ell P_{k,j}(t)$$

where $\ell \geq 0$, $m \geq 0$, $2r-2\ell-m \geq 0$ and $q_{\ell,m}(t)$ is a polynomial in t which does not depend on k and j (see also [10, Chapter 9]). We now use

$$\sum_{j=0}^{k}|j-kt|^s P_{k,j}(t) \leq Kk^{s/2}\varphi(t)^s \qquad \text{for } t \in E_k,$$

which follows from (3.6) of [6] for even s and the Cauchy-Schwartz inequality for odd s. As $|q_{\ell,m}(t)| \leq C(\ell,m)$, we complete the proof of (6.4.10), writing

$$\left\| q_{\ell,m}(t)(t(1-t))^{\ell-r}k^{\ell}\sum_{j=0}^{k}|j-kt|^{2r-2\ell-m}P_{k,j}(t)\right\|_{L_\infty(E_k)}$$
$$\leq C(\ell,m)Kk^r k^{-m/2}\left\|\varphi(t)^{-m}\right\|_{L_\infty(E_k)} \leq C_1 k^r.$$

Using (4.8) of [9] or (9.4.15) of [10], that is,

$$\int_{E_k}\varphi(t)^{2\ell-2r}k^{\ell}|j-kt|^{2r-2\ell-m}P_{k,j}(t) \leq Ck^{r-1}$$

together with the above description of $P_{k,j}^{(2r)}(t)$, we have (6.4.11). □

We are now ready for the proof of Lemma 6.4.3.

PROOF OF LEMMA 6.4.3: Using the Riesz-Thorin interpolation theorem, it is sufficient to prove our result for $p=1$ and $p=\infty$. Using

$$\left\|(x_1(1-|x|))^r\left(\frac{\partial}{\partial x_1}\right)^{2r}M_k(f,x)\right\|_{L_p(S)}$$
$$\leq\left\|\frac{(k+d)!}{k!}\sum_{\frac{\beta}{k}\in S}(x_1(1-|x|))^r\left|\left(\frac{\partial}{\partial x_1}\right)^{2r}P_{k,\beta}(x)\right|\int_S P_{k,\beta}(u)|f(u)|\,du\right\|_{L_p(S)},$$

it is clear that we need only show

$$\left\|\sum_{\frac{\beta}{k}\in S}(x_1(1-|x|))^r\left|\left(\frac{\partial}{\partial x_1}\right)^{2r}P_{k,\beta}(x)\right|\right\|_{L_\infty(S)}\leq Ck^r \tag{6.4.13}$$

and

$$(6.4.14) \qquad \frac{(k+d)!}{k!}\left\|(x_1(1-|x|))^r\left(\frac{\partial}{\partial x_1}\right)^{2r}P_{k,\beta}(x)\right\|_{L_1(S)} \le Ck^r.$$

To prove (6.4.13) and (6.4.14), we observe that for $\beta = (\ell_1, \ldots, \ell_d)$, one can write

$$P_{k,\beta}(x) = P_{k,(\ell_2,\ldots,\ell_d)}(x_2,\ldots,x_d)P_{k-|\beta|+\ell_1,\ell_1}\left(x_1/(1-\textstyle\sum_{i=2}^d x_i)\right),$$

and

$$(x_1(1-|x|)) = \left(\frac{x_1}{1-\sum_{i=2}^d x_i}\right)\left(1-\frac{x_1}{1-\sum_{i=2}^d x_i}\right)\left(1-\textstyle\sum_{i=2}^d x_i\right)^2.$$

Therefore, we have

$$(x_1(1-|x|))^r\left(\frac{\partial}{\partial x_1}\right)^{2r}P_{k,\beta}(x) = \left(\frac{x_1}{1-\sum_{i=2}^d x_i}\right)^r\left(1-\frac{x_1}{\sum_{i=2}^d x_i}\right)^r$$
$$\times P_{k,\ell_2,\ldots,\ell_d}(x_2,\ldots,x_d)P^{(2r)}_{k-|\beta|+\ell_1,\ell_1}\left(\frac{x_1}{1-\sum_{i=2}^d x_i}\right).$$

We now set $\widetilde{\beta} = (\ell_2,\ldots,\ell_d)$, $\widetilde{x} = (x_2,\ldots,x_d)$, $\widetilde{S} = \{(x_2,\ldots,x_d) : x_i \ge 0, \sum_{i=2}^d x_i \le 1\}$ and $y = \dfrac{x_1}{1-\sum_{i=2}^d x_i} \in [0,1]$. To prove (6.4.13), we write

$$\left\|\sum_{\frac{\beta}{k}\in S}(x_1(1-|x|))^r\left|\left(\frac{\partial}{\partial x_1}\right)^{2r}P_{k,\beta}(x)\right|\right\|_{L_\infty(S)}$$

$$\le \left\|\sum_{\frac{\tilde\beta}{k}\in\widetilde{S}}P_{k,\tilde\beta}(\widetilde{x})\right\|_{L_\infty(\widetilde{S})}\sup_{|\widetilde{\beta}|\le k}\left\|(y(1-y))^r\sum_{\ell=0}^{k-|\widetilde{\beta}|}|P^{(2r)}_{k-|\widetilde{\beta}|,\ell}(y)|\right\|_{L_\infty[0,1]} \le Ck^r.$$

To prove (6.4.14), we write

$$\left\| \frac{(k+d)!}{k!}(x_1(1-|x|))^r \left| \left(\frac{\partial}{\partial x_1}\right)^{2r} P_{k,\beta}(x) \right| \right\|_{L_1(S)}$$

$$\leq \frac{(k+d)!}{k!} \int_{\widetilde{S}} P_{k,\tilde{\beta}}(\widetilde{x}) \left[\int_0^{1-|\tilde{x}|} \left(\frac{x_1}{1-|\widetilde{x}|}\right)^r \left(1-\frac{x_1}{1-|\widetilde{x}|}\right)^r \right.$$

$$\left. \times \left| P^{(2r)}_{k-|\widetilde{\beta}|,\ell_1}\left(\frac{x_1}{1-|\widetilde{x}|}\right) \right| dx_1 \right] d\widetilde{x}$$

$$\leq \frac{(k+d)!}{k!} \int_{\widetilde{S}} P_{k,\widetilde{\beta}}(\widetilde{x})(1-|\widetilde{x}|) \int_0^1 y^r(1-y)^r \left| P^{(2r)}_{k-|\widetilde{\beta}|,\ell_1}(y) \right| dy\, d\widetilde{x}$$

$$\leq \frac{(k+d)!}{k!} \int_{\widetilde{S}} P_{k,\widetilde{\beta}}(\widetilde{x})(1-|\widetilde{x}|) C(k-|\widetilde{\beta}|)^{r-1}\, d\widetilde{x}$$

$$\leq C_1 k^r.$$

□

6.5. The basic estimate of $E_n(f)_p$ by $\|(M_n - I)^r f\|_p$

In this section, we will utilize the results of sections 6.3 and 6.4, in combination with techniques from [10, Section 9.3] and from [14] to obtain the weak-type estimate of $E_n(f)_p$ by $\|(M_n - I)^m f\|_p$.

THEOREM 6.5.1. *For $f \in L_p(S)$, an integer m and $\rho > 0$, we have*

$$(6.5.1)\quad E_n(f)_{L_p(S)} \leq C\left(n^{-\rho} \sum_{0 \leq i, 2^i \leq n} 2^{i\rho} \|(M_{2^{2i}} - I)^m f\|_{L_p(S)} + n^{-\rho} \|f\|_{L_p(S)}\right)$$

and

(6.5.2)

$$E_n(f)_{L_p(S)} \le C\Big(n^{-\rho} \sum_{1\le k\le n^2} k^{(\rho/2)-1} \|(M_k - I)^m f\|_{L_p(S)} + n^{-\rho} \|f\|_{L_p(S)}\Big).$$

REMARK 6.5.2. Theorem 6.5.1 is valid for all integers m and any $\rho > 0$ but seems to be more interesting for $2m \ge \rho$. We note also that the present Theorem is valid even if ρ is an integer.

PROOF OF THEOREM 6.5.1: Using (6.3.1) and Theorem 6.3.1, we have

$$E_n(f)_{L_p(S)} \le C\left(\omega_S^{2r}(f, 1/n)_p + n^{-2r}\|f\|_{L_p(S)}\right) \tag{6.5.3}$$

$$\le C_1\left(K_{2r,S}(f, n^{-2r})_p + n^{-2r}\|f\|_{L_p(S)}\right).$$

We now use Theorem 6.4.1 to obtain

$$K_{2r,S}(f, t^{2r})_p \le \|(M_{k^2} - I)^m f\|_p + Ct^{2r}k^{2r}K_{2r,S}(f, k^{-2r})_p \tag{6.5.4}$$

where C is independent of t, k or f. We choose for a given ρ, $0 < \rho < 2r$, an integer $j > 1$ such that $Cj^{\rho-2r} \le 1$. (For a given ρ, we choose r such that $2r > \rho$ and for this r, (6.5.4) yields a constant C; the integer j depends now on ρ, r and C.) We recall that $K_{2r,S}(f,t)_p \le \|f\|_p$ for any t and this, combined with the above, implies

$$K_{2r,S}(f, n^{-2r})_p \le C_1 n^{-\rho}\Big(\sum_{0\le\ell\le\ell_n} j^{\ell\rho}\|(M_{j^{2\ell}} - I)^m f\|_{L_p(S)} + \|f\|_{L_p(S)}\Big) \tag{6.5.5}$$

where

$$\ell_0 = \max\{\ell : j^\ell \le n\}.$$

At this point, we can combine (6.5.3) and (6.5.5) to derive for some $\rho > 0$ and j (that depends on ρ),

$$(6.5.6) \qquad E_n(f)_{L_p(S)} \le C_2 n^{-\rho}\Big(\sum_{0\le\ell\le\ell_0} j^{\ell\rho} \left\|(M_{j^{2\ell}} - I)^m f\right\|_{L_p(S)} + \|f\|_{L_p(S)}\Big).$$

We note that ρ is any positive number and that for some of the following, (6.5.6) is sufficient. The additional work below is needed for the somewhat simpler form (6.5.1) (where no statement about the existence of j is required) or the form (6.5.2) that is similar to the Stechkin-type sums and so desired by many. The proof for both (6.5.1) and (6.5.2) follows the technique of V. Totik [14].

For the proof of (6.5.1) (and of (6.5.2)), we still use j defined above which depends on C of (6.4.3) and on ρ. We choose i_ℓ to be the biggest integer i for which $\|(M_{2^{2i}} - I)^m f\|_{L_p(S)}$ achieves the minimum in the range $j^{\ell-1} < 2^i \le j^\ell$ for the given f and p. We now use (6.5.4) with $t = n^{-2}$ and $k = 2^{i_{\ell_0}}$, and then with $t = 2^{-i_\ell}$ and $k = 2^{i_{\ell-2}}$ (for ℓ odd or even when ℓ_0 is odd or even, respectively). This implies

$$K_{2r,S}(f, n^{-2r})_p \le \left\|(M_{2^{2i_{\ell_0}}} - I)^m f\right\|_p$$

$$+ C_1 n^{-\rho} \sum_{2<\nu\le\ell_0} 2^{\rho i_\nu} \left\|(M_{2^{2i_{\nu-2}}} - I)^m f\right\|_p + C_1 n^{-\rho}\|f\|_p$$

$$\leq C_2 n^{-\rho}\Big(\sum_{\substack{2^\nu \leq n\\ 0\leq \nu}} 2^{\rho\nu}\,\|(M_{2^{2\nu}}-I)^m f\|_p + \|f\|_p\Big)$$

which in turn implies

$$E_n(f)_p \leq C_2 n^{-\rho}\Big(\sum_{\substack{2^\nu \leq n\\ 0\leq \nu}} 2^{\rho\nu}\,\|(M_{2^{2\nu}}-I)^m f\|_p + \|f\|_p\Big) + C_1 n^{-2r}\,\|f\|_p$$

and since $\rho < 2r$, we have (6.5.1). We note that while the choice of i_ℓ was dependent on f, the final result is not.

To obtain (6.5.2) we use (6.4.3) directly rather than (6.5.4). We define a sequence ν_ℓ depending on f and j such that ν_ℓ is the largest integer satisfying $j^{2\ell-2} < \nu_\ell \leq j^{2\ell}$ such that $\|(M_\nu - I)^m f\|_p$ achieves the minimum in that range. We now apply (6.4.3) first with $t = n^{-2}$ and $k = \nu_{\ell_0}$ and then with $t = \nu_\ell^{-1}$ and $k = \nu_{\ell-2}$. (Of course ℓ is even or odd when ℓ_0 is even or odd.) This consideration implies

$$K_{2r,S}(f, n^{-2r})_p \leq \big\|(M_{\nu_{\ell_0}} - I)^m f\big\|_p + C_1 n^{-\rho}\Big(\sum_{2<\ell\leq \ell_0} \nu_\ell^{\rho/2}\,\big\|(M_{\nu_{\ell-2}} - I)^m f\big\|_p + \|f\|_p\Big)$$

$$\leq C_2 n^{-\rho}\Big(\sum_{1\leq \nu\leq n^2} \nu^{(\rho/2)-1}\,\|(M_\nu - I)^m f\|_p + \|f\|_p\Big).$$

Since the last inequality is independent of the choice of ν_ℓ, it is independent of f. Combining this with (6.5.3), we obtain (6.5.2). □

6.6. Applications and corollaries

Theorems 6.2.1 and 6.5.1 imply immediately

THEOREM 6.6.1. *For* $0 < \alpha < r$, $1 \le p \le \infty$,

(6.6.1)
$$\|(M_n - I)^r f\|_{L_p(S)} = O(n^{-\alpha}) \iff E_n(f)_{L_p(S)} = O(n^{-2\alpha})$$

$$\iff \omega_S^{2r}(f,t)_p = O(t^{2\alpha}).$$

PROOF: $E_n(f)_p = O(n^{-2\alpha})$ implies $E_{2^\ell}(f)_p = O(2^{-2\ell\alpha})$ which implies via (6.2.3) $\|(M_n - I)^r f\| = O(n^{-\alpha})$. The other direction follows from (6.5.6) (or (6.5.1) or (6.5.2)). The second equivalence was proved in [10, Chapter 12].

□

One can also define the appropriate Besov spaces for the sequences $\|(M_n - I)^r f\|_{L_p(S)}$ and $E_n(f)_{L_p(S)}$.

We define for the sequence $\{a_n\}$,

$$\|\{a_n\}\|_{\ell_s} = \begin{cases} \left(\sum_{n=0}^{\infty} |a_n|^s (n+1)^{-1}\right)^{1/s}, & 1 \le s < \infty \\ \sup_n |a_n|, & s = \infty \end{cases}$$

and obtain the following result with the understanding that $E_0(f)_p = \|f\|_p$.

THEOREM 6.6.2. *For* $0 < \alpha < r$, $1 \le p \le \infty$, $1 \le s \le \infty$ *and*

$$b_n(f)_p \equiv \|(M_n - I)^r f\|_{L_p(S)}$$

the norms

$$\left\| \{(n+1)^{2\alpha} E_n(f)_{L_p(S)}\} \right\|_{\ell_s} \qquad \text{and} \qquad \| \{(n+1)^{\alpha} b_n(f)_p\} \|_{\ell_s} + \|f\|_p$$

are equivalent.

Theorem 6.6.2 includes Theorem 6.6.1 when we set $s = \infty$.

PROOF: The proof follows the proof in [11] for a similar situation. Here, however, we have to take care of the fact that $b_n(f)_p$ is not necessarily a monotonic decreasing sequence. This can be done directly by using the Hardy inequality rather than the geometric progression used in [11]. It can also be done by a choice similar to that given in the proof of Theorem 6.5.1. And in fact, a geometric sequence is comparable as it follows from [3] that

$$\|(M_k - I)^r f\|_p \sim \|(M_n - I)^r f\|_p \qquad \text{if } k \sim n$$

at least when $d \leq 3$ or when $1 < p < \infty$. □

We cannot hope to obtain equivalence between the expressions $E_n(f)_p$ and $\|(M_n - I)^m f\|_p$ as the first is an unsaturated approximation process and the second is a saturated approximation process. It is interesting to note the implication of the above on the saturation class.

THEOREM 6.6.3. *If* $\|(M_n - I)^m f\|_p = O(1/n^m)$, *then* $\omega_S^{2m+1}(f,t) = O(t^{2m})$ *and hence* $\left\|\Delta^{2m+1}_{h\varphi_{ij}(x)e_{ij}} f\right\|_{L_p(S)} \leq M h^{2m}$.

PROOF: Theorem 6.5.1 implies $E_n(f)_p = O(1/n^{2m})$ and hence Theorem 12.2.3 of [10] (see (12.2.4) with $r = 2m+1$) implies

$$\omega_S^{2m+1}(f,t)_p = O(t^{2m}).$$

□

In [2] it was shown that $\|(M_n - I)f\|_p = O(n^{-1})$ if and only if $P(D)f$ exists in the weak sense and belongs to $L_p(S)$ for $1 < p \leq \infty$ and to $\mathcal{M}(S)$ (measures on S) for $p = 1$. In fact, we have the following somewhat more general saturation theorem.

THEOREM 6.6.4. *For an integer* m, $\|(M_n - I)^m f\|_p = O(n^{-m})$ *if and only if* $P(D)^m f$ *exists in the weak sense and belongs to* $L_p(S)$ *for* $1 < p \leq \infty$ *and to* $\mathcal{M}(S)$ *(measures on* S*) when* $p = 1$.

PROOF: Suppose $P(D)^m f$ exists in the weak sense and belongs to $L_p(S)$ for $1 < p \leq \infty$ and to $\mathcal{M}(S)$ for $p = 1$. Hence for $g \in \mathcal{D}$ (where $\mathcal{D}$ is the space of L. Schwartz test functions), we have

$$n^{-m}\langle (M_n - I)^m f, g\rangle = \left\langle n^{-m} \sum_{k_1=n+1}^{\infty} \cdots \sum_{k_m=n+1}^{\infty} \frac{P(D)^m M_{k_1} \cdots M_{k_m} f}{k_1(k_1+d)\cdots k_m(k_m+d)}, g\right\rangle$$

$$= n^{-m} \sum_{k_1=n+1}^{\infty} \cdots \sum_{k_m=n+1}^{\infty} \frac{1}{k_1(k_1+d)} \cdots \frac{1}{k_m(k_m+d)}$$
$$\times \langle f, P(D)^m M_{k_1} \cdots M_{k_m} g \rangle .$$

We write

$$\langle f, P(D)^m M_{k_1} \cdots M_{k_m} g \rangle = \langle P(D)^m f, M_{k_1} \cdots M_{k_m} g \rangle$$

and hence

$$|\langle P(D)^m f, M_{k_1} \cdots M_{k_m} g \rangle| \le \|P(D)^m f\| \, \|g\|_q$$

where $\| \ \|$ means $\| \ \|_p$ for $1 < p \le \infty$ and $\| \ \|_{\mathcal{M}(S)}$ for $p = 1$ and $q^{-1} + p^{-1} = 1$. This implies

$$|n^{-m} \langle (M_n - I)^m f, g \rangle| \le \|P(D)^m f\| \, \|g\|_q$$

for all $g \in \mathcal{D}$ and as such g are dense,

$$n^{-m} \|(M_n - I)^m f\|_p \le \|P(D)^m f\|_p .$$

The other direction is even more standard and follows the weak* compactness of the unit ball of a Banach space. The sequence $n^{-m}(M_n - I)^m f$ in L_p, $1 < p \le \infty$ or L_1 has φ as a weak* accumulation point in $L_p(S)$ for $1 < p \le \infty$ or $\mathcal{M}(S)$ respectively. For $g \in \mathcal{D}$, we have

$$n^{-m} \langle (M_n - I)^m f, g \rangle = \langle f, n^{-m} (M_n - I)^m g \rangle \longrightarrow \langle f, P(D)^m g \rangle$$

but

$$n^{-m}\langle (M_n - I)^m f, g\rangle \longrightarrow \langle \varphi, g\rangle$$

which identifies φ as $P(D)^m f$ in the weak sense. □

REMARK 6.6.5. We note that here the direct result causes none of the problems usually encountered in similar cases and this is due to the extremely nice properties of Durrmeyer operators.

We can now deduce from Theorem 6.6.4 and 6.6.3 the following interesting Corollary.

COROLLARY 6.6.6. *If* $P(D)^m f$ *exists in the weak (distributional) sense and belongs to* $L_p(S)$ *for* $1 < p \le \infty$ *or to* $\mathcal{M}(S)$ *for* $p = 1$, *we have*

$$\omega_S^{2m+1}(f,t) \le Ct^{2m}.$$

The following apriori estimate is also a result of the theorems in the present article.

THEOREM 6.6.7. *If* $\omega_S^2(P(D)^m f, t)_p \le Lt^\alpha$ *for* $0 < \alpha < 2$, *then* $\omega_S^{2m+2}(f,t)_p \le L_1 t^{2m+\alpha}$.

PROOF: Using the direct estimate of $(M_n - I)f$ and earlier results, we have

$$\left\|(M_n - I)^{m+1} f\right\|_p \le Cn^{-m} \left\|(M_n - I)P(D)^m f\right\|_p .$$

We further recall that $\omega_S^2(g,t)_p \le Ct^\alpha$ implies $E_n(g)_p \le C_1 n^{-\alpha}$ and hence $\|(M_n - I)g\|_p \le C_2 n^{-\alpha/2}$. With $g = P(D)^m f$, we now have

$$\left\|(M_n - I)^{m+1} f\right\|_p \le C_3 n^{-m-(\alpha/2)}$$

and use (5.1) with $2m + \alpha < \rho$ to obtain

$$E_n(f)_{L_p(S)} \le C_4 n^{-2m-\alpha}.$$

We now use Theorem 12.2.3 of [10], that is, (12.2.4) with $r = 2m + 2$ to obtain our result. □

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