ABSTRACT HARMONIC ANALYSIS ON LOCALLY COMPACT RIGHT TOPOLOGICAL GROUPS

by

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Abstract

Abstract harmonic analysis is well established on compact Hausdorff admissible right topological (CHART) groups. Specifically these groups are one-sided analogues of topological groups, where the elements that multiply continuously on the other side are dense in the group. The analytic theory of such groups was facilitated by strong topological results that lead to the existence of a Haar measure. However, not much is known about whether analogues hold in the non-admissible case, and the locally compact setting has been left untouched. In this thesis our goal is to broaden the scope of the current literature by considering these cases.

In particular, we first establish theory on locally compact right topological groups, including a sufficient condition for the existence of a Haar measure. We then consider analogues of various classical function algebras on these groups and discuss their properties. Then we introduce various measure algebra analogues on compact right topological groups and use their properties to characterize the existence of a Haar measure. These are also used to obtain hereditary properties - relating existence of a Haar measure on substructures to that on the group itself. In the process we provide sufficient conditions that do not rely on admissibility. The main challenge in this work lies in the lack of nice algebro-topological properties on right topological groups, which makes classical abstract harmonic analytic techniques unavailable for application. The lack of examples of such groups also impacts empirical evidence to draw inspiration from.

Dedicated to Matt, Buki and my family

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Contents

Chapter 1

Introduction

A right topological group is a group equipped with a topology that makes its right multiplication continuous. This is a natural generalization of the extensively studied classical case of locally compact topological groups. The theory of abstract harmonic analysis on locally compact groups was initially inspired by traditional Fourier analysis which is ubiquitously applied in various areas of science. Such an analysis is possible due to the inherent existence of a Haar measure, a non-zero left-invariant Radon measure, on these groups. Abstract harmonic analysis has since vastly expanded to include the more general structures of semigroups, locally compact quantum groups, hypergroups and semi-hypergroups, and now right topological groups.

Interest in right topological groups however, arose not from harmonic analysis, but from the theory of topological dynamics, which studies dynamical systems or flows. A flow consists of a pair (S, X), where S is a semigroup, X is a compact Hausdorff space and S acts continuously on X. Topological dynamics makes connections between various algebraic and topological properties of the action and S itself, and studies the asymptotic properties of the (S, X). In particular, distal flows form a special class with nice asymptotic properties. For a flow (S, X), the closure of S in X^X forms a semigroup Σ, known as the enveloping semigroup of (S, X). Then, (S, X) is said to be **distal** if the flow (Σ, X) satisfies cancellativity in Σ. Ellis beautifully characterized distal flows by proving that these are exactly the flows whose enveloping semigroups are compact Hausdorff (admissible) right topological groups [7]. An interest in such groups has since followed in an attempt to better understand distal flows. Furstenberg, for example, used Σ to give a fundamental structure theorem on minimal and distal flows [10].

A second reason Ellis inadvertently provides for the study of such groups, is due to his work on joint and separate continuity in[8]. Here he showed that any locally compact Hausdorff semitopological group is topological, i.e. separate continuity on locally compact groups naturally implies joint continuity of multiplication and continuity of the inverse map. From an abstract harmonic perspective, if we consider weakening the continuity requirements on locally compact topological groups and studying this generalization, we naturally skip semitopological groups and jump to right topological groups.

Motivated by Furstenberg's work, Namioka pioneered the study of right topological groups from an analytic perspective in [37]. He introduced the σ-topology for a right topological group, the quotient topology induced by the multiplication map, and showed that the topology characterizes (in some sense) the degree to which the group has left-continuity properties. Special focus has been given in the literature to admissible groups. The topological center of a right topological group G, denoted by $\Lambda(G)$, is defined to be the set of those elements with respect to which left multiplication is continuous. Admissible right topological groups are precisely those that have a dense topological center. Namioka showed that compact Hausdorff admissible right topological (**CHART**) groups satisfy an analogue of Furstenberg's structure theorem. This theorem later turned out to be key in Milnes and Pym's construction of a Haar measure on these groups [31][32], which built the foundation for conducting abstract harmonic analysis on them.

Lau and Loy initiated this abstract harmonic analytic study of compact right topological groups in [20]. They constructed various measure and function algebra analogues on such groups, defined convolution on these and studied their properties, comparing them to the classical topological case. Further, in [21], they studied extensively the Fourier algebra of CHART groups, proving that this is isomorphic to the Fourier algebra of a compact topological group (thereby proving that these algebras are not complete invariants for right topological groups).

Other selected works on compact right topological groups include: work on representation theory of these groups in [28][36], equicontinuous and distal compact right topological groups in [30][29], generalizations of strong normal systems of subgroups in [35] and group extensions in [38].

Our work is divided primarily into two points of focus. Firstly, we are interested in generalizing the existing theory on compact right topological groups to locally compact right topological groups. This includes the works of Namioka [37], of Milnes and Pym [31][32], and of Lau and Loy [20][21]. The difficulty in this generalization lies in the fact that techniques from topological dynamics, including Furstenberg's theorem, have been developed for compact spaces and cannot be used in our work. Moreover, many of the ideas used in these citations that do not rely on flows, strongly rely on compactness anyway. As such, we often work with σ-locally compact right topological groups to provide some extra structure.

For our second point of focus, we attempt to characterize the existence of a Haar measure on compact right topological groups. The current literature only provides admissibility as a sufficient condition. We therefore try to circumvent around this condition. The challenge in dealing with the non-admissible case is the lack of pleasant topological properties - in view of Ellis' theorem, admissibility brings a right topological group close to being topological (in fact, one may note from the literature that tacking on additional nice properties almost always makes the group trivially topological). We introduce new measure algebra analogues and use fixed point theory to link their non triviality to the existence of a Haar measure.

In Chapter 2, we give basic definitions and notations that will be used in this thesis. Further, we introduce background theory on locally compact topological groups and existing results in the literature on compact right topological groups.

Chapter 3 generalizes some of Namioka's work to locally compact right topological groups. We then give a sufficient condition for the existence of a Haar measure on these groups.

In Chapter 4, we discuss classical function algebras generalized to locally compact right topological groups and their properties, including analogues of the Fourier and Fourier-Steiltjes algebra.

We begin Chapter 5 by generalizing existing measure algebra analogues on compact right topological groups to the locally compact case. We then introduce several new measure algebra analogues on compact right topological groups and characterize the existence of a Haar measure using their non-triviality. This characterization provides an alternative sufficient condition to admissibility for the existence of a Haar measure. We also discuss various properties of these measure algebras.

Chapter 6 explores some hereditary properties of right topological groups, linking the existence of a Haar measure on sub-structures to that on the group itself. Here we apply some of the results from Chapter 5.

Lastly, we conclude this work with some open problems in Chapter 7.

Chapter 2

Preliminaries

In this chapter, we will present the standard definitions and notation we will be using in this thesis. In addition, we present the basic results that motivate our work as well as the current research on right topological groups.

Note that in this work, we do not assume that topological spaces are Hausdorff unless specified.

For any $E \subset X$, we will denote by $\overline{E}$, the closure of E in τ. Further, if X is a locally convex space, $\text{conv}(E)$ will denote the convex hull of E.

Throughout, $C(X)$ will indicate the continuous complex-valued functions on X. Subspaces of $C(X)$ we will be interested in include $C_b(X)$, $C_c(X)$ and $C_0(X)$, functions in $C(X)$ that respectively, are bounded, have compact support, and vanish at infinity (given X is locally compact). Here, for any $f \in C(X)$, the support of f is defined to be $\text{supp}(f) = \overline{\{x \in X \mid |f(x)| > 0\}}$, while f is said to vanish at infinity if, for every $\epsilon > 0$, there exists $K \subset X$ compact such that $|f(x)| < \epsilon$ for all $x \in X \setminus K$.

Equipped with $\| \cdot \|_\infty$, the the supremum norm, $C_b(X)$ and $C_0(X)$ form C*-algebras, while $\overline{C_c(X)} = C_0(X)$. Clearly $C_c(X) \subset C_0(X) \subset C_b(X) = C(X)$. When X is compact, this inclusion collapses to $C(X) = C_b(X) = C_0(X) = C_c(X)$. We will also frequently use $M(X)$, the space of all complex Radon measures on X. By the Riesz representation theorem, this can be identified as the dual of $C_c(X)$.

Algebraic structures such as groups, and more generally semigroups may be equipped with topologies that make the multiplication continuous to different degrees. A semigroup S equipped with a topology τ is said to be

- **Right topological** if the right multiplication maps $r_s : S \to S$, $x \mapsto xs$ are continuous for all $s \in S$.
- **Semitopological** if the multiplication map $m : S \times S \to S$, $(s,t) \mapsto st$ is separately continuous
- **Topological** if the multiplication map m is is jointly continuous

The right topological definition has an obvious left analogue. Throughout, for each $s \in S$ we use l_s and r_s to denote the left, respectively right multiplication by s on S, and L_s, R_s to similarly denote the left, right translations of functions on S by s. For a measure $\mu \in M(G)$, we shall use the notation ${}_s\mu$ and μ_s to denote the left and right translations by s, respectively.

The above definitions apply ad verbum to groups, except a **topological group** G is additionally required to have a continuous inverse map, $G \to G$, $g \mapsto g^{-1}$. Due to Ellis' celebrated result however [6], this property is obtained for free for the most interesting cases of groups; in fact, the semitopological and topological definitions coincide:

Theorem 2.1 (Ellis)**.** *Every locally compact Hausdorff semitopological group is topological.*

For a right topological semigroup S, we denote by $\Lambda(S)$ the **topological center** of S i.e.

$$\Lambda(S) = \{s \in S \mid l_s : S \to S \text{ is continuous } \}$$

Here S is said to be **admissible** if $\Lambda(S)$ is dense in it. If S is a group, this essentially makes it almost topological in a weak sense, wherein small assumptions about structures on S force it to be topological (see [37], [20]).

Due to Theorem 2.1, interest in generalizing the harmonic theory of locally compact groups to less topologically restrictive structures naturally becomes focused on right topological groups, the most specific and structured case being that of compact Hausdorff admissible right topological (**CHART**) groups. That being said, the study of such groups was initiated in topological dynamics due to their natural occurrence in the theory of distal flows.

A **flow** is a pair (S, X) consisting of a semigroup S and a non-empty compact Hausdorff space X, along with a map $p : S \times X \to X$, satisfying the following conditions:

- $p(s, \cdot)$ is continuous for all $s \in S$
- $p(st, x) = p(s, p(t, x))$

We simplify this notation by using $p(s, x) = sx$.

For every flow (S, X), S is naturally embedded in X^X as a subsemigroup and thus obtains the topology induced by the product topology on X^X. The closure $\Sigma(S) := \bar{S} \subset X^X$ is then known as the **enveloping semigroup** of (S, X), and is easily checked to be a CHART semigroup with topological center $S \subset X^X$.

A flow (S, X) is said to be **distal** if for every net $\{s_\alpha\}_{\alpha \in A}$, and any $x, y \in X$, $\lim_\alpha s_\alpha x = \lim_\alpha s_\alpha y$ implies $x = y$. In other words the flow $(\Sigma(S), X)$ satisfies cancellativity in $\Sigma(S)$. Ellis proved the following fundamental theorem characterizing these flows [7]:

Theorem 2.2 (Ellis)**.** *A flow is distal if and only if its enveloping semigroup is a group.*

It follows that $\Sigma(S)$ is a CHART group for every distal flow (S, X). This has made CHART groups the focus of the study of right topological groups. In this thesis however, we try to be as general as possible, dealing with locally compact admissible right topological groups, as well as compact right topological groups that are not admissible.

2.1 Locally Compact Groups

Part of the existing literature on right topological groups attempts to generalize the standard results in abstract harmonic analysis of locally compact topological groups. To give a better motivation of these concepts, we briefly introduce abstract harmonic analysis on locally compact groups. Some excellent introductory references on this topic include [9], [25], and [14].

A topological space is said to be **locally compact** if every point has a base of neighbourhoods consisting of compact sets. In case of ambiguity in the non-hausdorff case, this is the definition we will always assume. A **locally compact topological group** then is just a topological group (G, τ), where τ is locally compact.

Central to the building of harmonic analysis on locally compact groups is the notion of a Haar measure. Let (G, τ) denote a locally compact Hausdorff topological group. A measure λ on G is said to be a right **Haar measure** if it is a non-zero Radon measure satisfying right invariance i.e. $\lambda(E) = \lambda(Eg)$ for all $g \in G$, and $E \subset G$ Borel. It is well known that every locally compact topological group has a right (or left) invariant Haar measure that is unique up to scalar multiplication. For a compact group, one usually normalizes the Haar measure so that $\lambda(G) = 1$, and with this convention, the Haar measure is uniquely determined. Additionally, the Haar measure on a compact group is both left and right invariant. Familiar examples of locally compact groups include $\mathbb{R}$ or $\mathbb{C}$ with the usual topology, where the Lebesgue measure forms a Haar measure, and the (compact) circle group $\mathbb{T}$ with its usual topology and measures. A non-abelian example of a (compact) topological group is given by $SO_3(\mathbb{R})$, the special unitary matrices on $\mathbb{R}$.

The existence of a right Haar measure allows us to define convolution of functions. Given f, g, Borel measurable functions on G, the **convolution** of f and g is defined by

$$f * g(x) = \int f(xy^{-1})g(y)d\lambda(y)$$

The map $f * g$ need not be bounded in general, however, the following results are standard [9]:

Proposition 2.3. *Let G be a locally compact topological group and $1 \leq p \leq \infty$. Then, for any $f \in L_p(G)$, and $g \in L_1(G)$, the following hold*

1. *$f * g \in L_p(G)$ and $\|f * g\|_p \leq \|f\|_1 \|g\|_p$*
2. *If $1 < p < \infty$ and $1/p + 1/q = 1$, then for any $h \in L_q(G)$, $f * h \in C_0(G)$. Further, $\|f * g\|_\infty \leq \|f\|_p \|g\|_q$.*

From 1, it immediately follows that $(L_1(G), *)$ is a Banach algebra.

Note that the order of convolution in 1 may be reversed to the same effect if f has compact support or if G satisfies a property called unimodularity, where the left and right Haar measure coincide (see [9]). Unimodularity trivially holds for compact topological groups.

Convolutions generalize nicely to measures in $M(G)$. The convolution of measures $\mu, \nu \in M(G)$ is given by

$$\langle \mu * \nu, f \rangle = \int \int f(xy) d\mu(x) d\nu(y)$$

for each $f \in C_b(G)$. The Haar measure of a locally compact topological group lies in $M(G)$ if and only if G is compact. However, we do have $L_1(G)d\lambda \subset M(G)$.

With the nice properties of convolution, $(M(G), *)$ is a unital Banach algebra with identity δ_e and $(L_1(G), *)$ forms a two-sided ideal in $M(G)$. Usual notions for Banach Algebras may then be considered for $L_1(G)$ and $M(G)$. For example, $L_1(G)$ is unital if and only if G is discrete. However, any base of compact (symmetric) neighbourhoods of $\mathcal{U}$ of e provides a bounded approximate identity for $L_1(G)$ via $\{\chi_U/\lambda(U) \mid U \in \mathcal{U}\}$. The vital property that allows one to prove this is as follows;

Proposition 2.4. *Suppose G is a locally compact topological group. Then,*

1. *If $1 \leq p < \infty$, and $f \in L_p(G)$, then $\|L_y f - f\|_p, \|R_y f - f\|_p \to 0$ as $y \to e$.*

2. *For all $f \in C_0(G)$, the same convergences hold in the uniform norm.*

Unfortunately, Proposition 2.4 does not hold for right topological groups. In fact, an inherent property of topological groups that does not hold for right topological groups is the existence of a symmetric base of neighbourhoods of e (i.e. sets $U \subset G$ satisfying $U^{-1} = U$). If G is topological, a function in $C_b(G)$ is called **left** or **right uniformly continuous** if the respective one-sided convergence in 2 of Proposition 2.4 is satisfied and **uniformly continuous** if both are. By 2, it follows that uniformly continuous functions are topologically abundant, as $C_0(G)$ separates points from closed sets. Part of the challenge in developing the research on right topological is to work around an absence of this property.

2.2 Compact right topological groups

In this section we will introduce the existing results on compact right topological groups. Most of this theory is focused specifically on CHART groups. We attempt to generalize much of this in the upcoming chapters.

Fundamental to studying the analytic theory of right topological groups is Namioka's pioneering work on the σ-topology [37]. Let (G, τ) denote a compact right topological group. The **σ-topology** on (G, τ) is defined to be the quotient topology induced by the multiplication map

$$\begin{aligned} (G, \tau) \times (G, \tau) &\to G \\ (x, y) &\mapsto x^{-1}y \end{aligned}$$

Namioka intended the σ in the nomenclature to indicate "symmetry". The justification for this is given by the following result:

Theorem 2.5 (Namioka). *Let (G, τ) be a right topological group. Then,*

- *$\sigma \subset \tau$ and equality holds if and only if (G, τ) is topological*
- *(G, σ) is a semitopological group with a continuous inverse map*
- *(G, σ) is T_1 if and only if (G, τ) is Hausdorff. Further, (G, σ) is Hausdorff if and only if (G, τ) is topological*

Observe that if G is locally compact, a lack of being Hausdorff is all that holds (G, σ) (and therefore (G, τ)) from being topological. This also produces an interesting class of semitopological groups.

One of the advantages of working with admissible right topological groups is the following theorem [37].

Theorem 2.6 (Namioka). *Let (G, τ) be a admissible right topological group and let $\mathcal{U}$ be a base of open neighbourhoods of e in τ. Then,*

1. *The quotient map $\phi : (G, \tau) \times (G, \tau) \to (G, \sigma)$ is open*
2. *The family $\{U^{-1}U \mid U \in \mathcal{U}\}$ forms a base of σ-open neighbourhoods of e in (G, σ)*

The second part clearly follows from the first. For general right topological groups, it becomes hard to explicitly find the open sets in the σ-topology. For $U, W \in \tau$, $\phi(U \times W) = U^{-1}W$ is open in the σ-topology if and only if $\Delta \cdot (U \times W)$ is open in $(G \times G, \tau \times \tau)$, where $\Delta = \{(g, g) \mid g \in G\}$ denotes the diagonal of G. As we shall see, the known examples of compact right topological groups are either "twisted" products of topological groups or enveloping semigroups of a flow, so that checking whether a set is σ-open in these spaces becomes cumbersome or intractable. The result above comes in handy.

Following Namioka, for subgroups K of G, we denote by G/K the left cosets of K i.e. $\{xK \mid x \in G\}$. Namioka showed the following [37]

Proposition 2.7. *G/K is Hausdorff if and only if K is σ-closed.*

Recall that in the topological case it suffices for K to be closed (see [9]). We will use $(G/K, \tau)$ and $(G/K, \sigma)$ to mean the quotient topology on G/K induced by (G, τ) and (G, σ) respectively.

Let $L \subset G$ be a closed normal subgroup. Let us denote by (L, σ), the relative topology induced by (G, σ). We warn the reader that this does not always coincide with the (finer) σ-topology of (L, τ) (work on this topology may be found in [35]). We define $N(L)$ to be the intersection of all σ-closed σ-neighbourhoods of e in G. The following is a key result of Namioka that highlights the properties of $N(L)$ [37];

Proposition 2.8. *Let G be a compact Hausdorff right topological group. Then,*

1. *$N(L)$ is a σ-closed normal subgroup of L*
2. *$(L/N(L), \tau) = (L/N(L), \sigma)$ and the resulting group is a compact topological group*
3. *The action map*

$$\begin{aligned}(G/N(L), \tau) \times (L/N(L), \tau) &\to (G/N(L), \tau)\\ ([x], [y]) &\mapsto [xy]\end{aligned}$$

is jointly continuous.

Via a quotient, one therefore obtains a topological group $G/N(G)$ from the compact right topological group G. However, when is $L/N(L)$ non-trivial? Namioka showed the following result in the countably admissible case (i.e. when $\Lambda(G)$ has a countable subsemigroup that is dense in G) [37], and the result was later generalized to arbitrary compact admissible groups by Pym and Milnes [32].

Theorem 2.9. *Suppose G is a CHART group and $L \lhd G$ is a closed subgroup satisfies $L \neq \{e\}$. Then, $N(L) \neq L$.*

An alternate proof that does not use flows may be found in [34].

Central to the existence of a Haar measure on compact right topological groups is the idea of a **strong normal system of subgroups**. A right topological group (G, τ) is said to have such a system if there exists a family $\{L_\xi\}_{\xi<\xi_0}$ of σ-closed normal subgroups of G, indexed by some ordinal $\xi_0 > 0$, satisfying the following conditions:

1. $L_{\xi_0} = G$, $L_0 = \{e\}$
2. $L_\xi \supset L_{\xi+1}$ and for a limit ordinal $\xi < \xi_0$, $L_\xi = \cap_{\eta<\xi} L_\eta$;
3. $L_\xi / L_{\xi+1}$ is a compact Hausdorff topological group;
4. the action map

$$\begin{aligned} G/L_{\xi+1} \times L_\xi/L_{\xi+1} &\to G/L_{\xi+1} \\ ([x],[y]) &\mapsto [xy] \end{aligned}$$

   is jointly continuous.

Pym and Milnes exploited Proposition 2.8 and Theorem 2.9 to obtain the following nice theorem [31];

Theorem 2.10 (Milnes, Pym)**.** *Every compact Hausdorff right topological group G with a strong normal system of subgroups has a unique right invariant Haar measure that is left invariant with respect to $\Lambda(G)$.*

They then highlighted the importance of a strong normal system of subgroups is highlighted by providing the following result [31], using Namioka's work in [37]:

Theorem 2.11 (Furstenberg-Namioka-Ellis Structure Theorem)**.** *Every CHART group has a strong normal system of subgroups.*

The flavor of the proof is inspired by Furstenberg and Ellis' work on flows. From this, we have the obvious corollary:

Corollary 2.12. *Every CHART group has a unique right invariant Haar measure that is left invariant under* $\Lambda(G)$.

so that analysis may be considered on CHART groups.

The construction of a strong normal system of subgroups was done by using transfinite induction, taking $G \supset N(G) \supset N(N(G)) \supset \ldots \supset \{e\}$, highlighting the usefulness of Namioka's construction of the subgroup $N(G)$. Note that Theorem 2.10 is the only known sufficient condition for the existence of a Haar measure on compact right topological groups.

Remark 2.13. It is important to note that the existence of a Haar measure on such groups is highly non-trivial. The usual construct for a Haar measure on a compact topological group makes use of continuity properties such as Proposition 2.4 that do not hold for right topological groups. In fact, in [19], under the assumption of the continuum hypothesis, Kunen proves the existence of a non-admissible compact Hausdorff right topological group that cannot hold a right Haar measure. Explicit examples of such groups are hard to obtain.

We end this section with some examples of compact right topological groups referenced from [28], [31], [30].

Example 2.14. Consider $G = \mathbb{T} \times E$, where E is the group of all endomorphisms on T (equipped with pointwise multiplication). We consider $\mathbb{T}$ with its usual topology and E with the topology induced by the product topology on $\mathbb{T}^{\mathbb{T}}$. G is then equipped with the product of these topologies. For $(u, h), (w, g) \in G$, multiplication is defined as follows

$$(u, h)(w, g) = (uwR_u h \circ g(e^2 i), hg)$$

One easily checks that equipped with this, G forms a compact right topological group. Continuity of left multiplication by (u, h) here requires h to be continuous, i.e.

$$\Lambda(G) = \mathbb{T} \times \hat{\mathbb{T}} = \{(u, h) \in G \mid h : t \mapsto t^n \text{ for some } n \in \mathbb{N}\}$$

Note that E may be identified with the Bohr compactification of $\mathbb{Z}$, and the topological center is dense in G, i.e. G is a CHART group. The unique Haar measure on G is easily given by the product of the Haar measures on $\mathbb{T}$ and E respectively. The subgroup $N(G)$ is given by $\mathbb{T} \times \{1\}$. A strong normal system of subgroups on G as specified by Theorem 2.9 is given by $G \supset N(G) \supset \{e\}$, where we observe that $G/N(G) \cong \{1\} \times E$ is a topological group.

The next example provides a compact right topological group that is not admissible.

Example 2.15. Consider $\mathbb{T}$ again and let ϕ be a discontinuous automorphism on it with $\phi \circ \phi = 1$. Milnes [30] constructs an example of such a map by considering $\mathbb{T} \cong \mathrm{Tor}(\mathbb{T}) \oplus c\,\mathbb{Q}$, where $\mathrm{Tor}(\cdot)$ indicates the torsion subgroup, and c is the cardinality of the reals. Then ϕ may be defined to be the map that keeps $\mathrm{Tor}(\mathbb{T})$ fixed but switches two fixed copies of $\mathbb{Q}$ (see also Example 6e of [29]). Here discontinuity occurs because $Tor(\mathbb{T})$ is dense in $\mathbb{T}$. Now we define $G = \mathbb{T} \times \{\phi, 1\}$ with the multiplication $(u, h)(v, g) = (g(u)v, h \circ g)$, and the product topology obtained by equipping $\mathbb{T}$ with its usual topology, and $\{\phi, 1\}$ with the discrete topology. Then, G is a compact Hausdorff right topological group with the topological center $\mathbb{T} \times \{1\}$, clearly not admissible. A Haar measure still exists on G; if λ denotes the Haar measure of $\mathbb{T}$, then, the Haar measure on G is given by $\lambda_\phi/2 + \lambda_1/2$, where λ_ϕ denotes a copy of λ on $\mathbb{T} \times \{\phi\}$ and λ_1 is one on $\mathbb{T} \times \{1\}$. In fact, we have a strong normal system of subgroups given by $G \supset N(G) \supset \{e\}$, where $N(G) = \mathbb{T} \times \{1\}$. We remark that admissibility is not a necessary condition for the existence of a strong normal system or a Haar measure.

Lastly, we give an example where a non-admissible right topological group does not possess a strong normal system of subgroups.

Example 2.16 ([28],[31])**.** Consider the set $G = \mathbb{T} \times \{-1, 1\}$, with the group multiplication $(u, \epsilon)(v, \delta) = (u^\delta v, \epsilon\delta)$ We equip G with the topology which has the closed and open neighbourhood base for each $(e^{ia}, 1)$, $(e^{ib}, -1)$, $a, b \in \mathbb{R}$, $a < b$ of the form

$$U_{a,b} = \{(e^{ia}, 1), (e^{ib}, -1)\} \cup \{(e^{i\theta}, \epsilon) \mid a < \theta < b, \epsilon = \pm 1\}$$

Then, G forms a compact right topological group with the trivial topological center $\{(1, 1)\}$ and thus is not admissible. Further, it was shown in [31] that G cannot possess closed normal subgroups $N \subset G$ satisfying G/N is topological (i.e. satisfying Proposition 2.8). It follows that G has no strong normal system of subgroups. Additionally, as it turns out, $N(G) = G$ in this case so that $G/N(G)$ is trivial. Despite this, G has a unique Haar measure given by $\min(1, (b - a)/2\pi)$ on each neighbourhood $U_{a,b}$.

Taking advantage of Milnes and Pym's construction of the Haar measure, Lau and Loy undertook abstract harmonic analysis on compact right topological groups [20][21]. They defined the analogues of the group algebras and measure algebras, and discussed the function algebras on these groups, including the Fourier algebra. We will discuss and generalize this work in Chapter 5.

Chapter 3

Locally compact right topological groups and the Haar measure

Analysis on compact Hausdorff right topological groups, particularly CHART groups is well established. Motivated by the initial work of Namioka [37], Pym and Milnes proved the existence of a unique Haar measure on these groups [31]. In this chapter we generalize some of this theory to the less topologically restrictive but still well-structured case of σ-locally compact Hausdorff spaces, still working mostly with admissible groups.

We begin by generalizing some of Namioka's fundamental result from [37], followed by a proof of the existence of a Haar measure given the existence of a compact strong normal system of subgroups.

A **σ-locally compact** right topological group is a right topological group (G, τ) with a σ-compact, locally compact topology τ. Here we use the usual definition of σ-compactness i.e. a space which can be written as a countable union of compact subsets.

Remark 3.1. In order to avoid confusion with the σ-topology, we shall always refer to a compact set in this topology by referring to it as being compact in the σ-topology, or (G, σ) (as opposed to σ-compact).

As we shall see, admissibility plays a much more vital role in the locally compact case compared to compact right topological groups.

We begin by establishing a standard result we will require, which will allow us to use the σ-local compactness of the space in a similar manner to compactness in the original work.

Lemma 3.2 (Open mapping theorem for LCH right topological groups)**.** *Suppose* (G, τ) *a* σ*-compact right topological group, and* H *is a Hausdorff right topological group with the Baire property. If* $f : G \to H$ *is a continuous surjective homomorphism, then it is open.*

Proof. As G is σ-compact, $G = \cup_{n\in\mathbb{N}} C_n$ for a sequence $\{C_n\}_{n\in\mathbb{N}}$ of compact subsets of G. As f is surjective, $H = \cup_{n\in\mathbb{N}} f(C_n)$. Without loss of generality, we assume that f is injective (otherwise we consider the quotient homomorphism on $G/\mathrm{Ker}(f)$, where we observe that the quotient map $G \to G/\mathrm{Ker}(f)$ is open). Clearly for every $n \in \mathbb{N}$, $f|_{C_n} : C_n \to f(C_n)$ is a homeomorphism, as a bijective map from a compact space into a Hausdorff space. Since H has the Baire property, it follows that $f(C_n)$ cannot be nowhere dense for all $n \in \mathbb{N}$. Therefore, there exists some $m \in \mathbb{N}$ such that $f(C_m)$ has a non-empty interior; let $f(x)$ be an interior point for $x \in C_m$. It follows that C_m being homeomorphic to $f(C_m)$ has a non-empty interior as well.

We shall show that f^{-1} is continuous. Suppose now that $f(g_\alpha) \to f(g)$ for some net $\{g_\alpha\} \subset G$, where $g \in C_n$ for some $n \in \mathbb{N}$. By continuity of right multiplication,

$$f(g_\alpha)f(g)^{-1}f(x) = f(g_\alpha g^{-1}x) \to f(x)$$

where, since $f(x)$ lies in the interior of $f(C_m)$, it follows that a tail of $\{f(g_\alpha g^{-1}x)\}$ is contained in $f(C_m)$. However, as $f|_{C_m}$ is a homeomorphism, we have, $[f|_{C_m}]^{-1}$ is continuous so that $g_\alpha g^{-1}x \to x$ whence $g_\alpha \to g$ holds by continuity of right multiplication. This concludes the proof. □

Lemma 3.3. *If* (G, τ) *is a locally compact Hausdorff admissible right topological group, then* (G, σ) *is a locally compact semitopological group that has the Baire property.*

Proof. As (G, τ) is admissible, by Theorem 2.6, the continuous quotient map $\phi : (G \times G, \tau \times \tau) \to (G, \sigma)$ is open. Suppose $U \in \sigma$ is an open neighbourhood of e. Then, $\phi^{-1}(U)$ is an open neighborhood of (e, e) in $\tau \times \tau$, so that by local compactness of the topology, there exists a compact neighborhood $K \in \tau \times \tau$ of (e, e) such that $K \subset \phi^{-1}(U)$. it follows that $\phi(K) \subset U$, where by the openness and continuity of ϕ,

$\phi(K)$ is a compact neighbourhood of e in (G, σ). It follows that (G, σ) is locally compact.

Suppose $\{U_n\}_{n\in\mathbb{N}} \subset \sigma$ are σ-open sets dense in the σ topology. We claim that $\cap_{n\in\mathbb{N}} U_n$ is dense in (G, σ). Indeed, $\{\phi^{-1}(U_n)\}_{n\in\mathbb{N}}$ are open and dense in $G \times G$, so that by $G \times G$ being locally compact Hausdorff (whence Baire), $\cap_{n\in\mathbb{N}} \phi^{-1}(U_n)$ is dense in $G \times G$. Applying ϕ, by surjectivity, $\cap_{n\in\mathbb{N}} U_n$ is dense in (G, σ) as well. □

The following proposition is a straightforward generalization of Namioka's result for the compact case [37].

Proposition 3.4. *Suppose (G, τ) is a locally compact admissible Hausdorff metrizable right topological group. Then, G is topological.*

Proof. Suppose $y \in G$. Since G is admissible and metrizable, there exists a sequence $\{y_n\}_{n\in\mathbb{N}} \subset \Lambda(G)$, such that $y_n \to y$. Consider $l_y : G \to G, x \mapsto yx$. For any $x \in G$, by the right continuity of G,

$$l_y(x) = yx = \lim_{n\in\mathbb{N}} y_n x = \lim_{n\in\mathbb{N}} l_{y_n}(x)$$

where $\{l_{y_n}\}_{n\in\mathbb{N}}$ are continuous on G. However, by Osgood's theorem (see 9.5 in [18]), by the metrizability of G, it follows that $\{l_{y_n}\}_{n\in\mathbb{N}}$ is equicontinuous at some residual set $A \subset G$. By virtue of being locally compact Hausdorff, G is Baire, and thus, A has non-empty interior. It follows that the set of points of continuity of l_y is non-empty. However, this then gives a common point of continuity for $r_z \circ l_y$, for all $z \in G$. It thus follows that l_y is continuous on G. Since $y \in G$ was arbitrary, we conclude that G is semitopolological. By Theorem 2.1 then, G is topological. □

This is one of the examples of instances where adding extra structure to an admissible group makes it trivially topological.
Note however, that Ruppert [44] has given examples of metrizable non-admissible right topological groups so that the above result need not hold without admissibility.

Ellis' work on separate and joint continuity in [6], i.e. Theorem 2.1 was beautifully generalized by Namioka to groups actions [38]. Namioka does this is a more general setting than we state, but this version will suffice for our needs.

Theorem 3.5 (Namioka)**.** *Let X be locally compact Hausdorff space and (G,τ) be a locally compact Hausdorff right topological group acting on X. If the map*

$G \times X \to X$, $(g,x) \to gx$ is separately continuous, then it is jointly continuous.

Proposition 3.6. *Let (G,τ) be a σ-locally compact admissible Hausdorff right topological group. If $L \subset G$ is a σ-closed subgroup, then*

1. *$N(L)$ is a normal subgroup that is closed in (L,σ)*
2. *$(L/N(L),\tau) = (L/N(L),\sigma)$ is a σ-locally compact Hausdorff topological group*
3. *$G/N(L) \times L/N(L) \to G/N(L)$, $([x],[y]) \mapsto [xy]$ is jointly continuous.*

Proof. Part 1 follows easily as $N(L)$ is the intersection of σ-closed neighbourhoods, and normality is guaranteed by Corollary 1.1 in [37]. Further, this result also guarantees that $(L/N(L),\sigma)$ is Hausdorff, while local compactness of the space follows from Lemma 3.3 as $(L,\sigma) \subset (G,\sigma)$ is σ-closed while $\pi_{N(L)} : L \mapsto L/N(L)$ is open and continuous.

Since (G,τ) is σ-compact, for any union of compact sets $\cup_{n\in\mathbb{N}} C_n = G$, by continuity and openness of the quotient map $(G,\tau) \to (G/N(L),\tau)$, $\{(L \cap C_n)/N(L)\}$ is a sequence of compact sets making $(L/N(L),\tau)$ σ-locally compact. Now the identity map $(L/N(L),\tau) \to (L/N(L),\sigma)$ is a continuous map from a σ-compact right topological group into a locally compact Hausdorff right topological group, so that by Lemma 3.2, it follows that the map is also open. Thus, $(L/N(L),\tau) = (L/N(L),\sigma)$ as claimed. As $(L/N(L),\sigma)$ is a locally compact Hausdorff semitopological group, by Ellis' theorem, it is a σ-locally compact topological group.

Lastly, let us prove 3. By continuity of right multiplication of $G/N(L)$, it is clear that $G/N(L) \times L/N(L) \to G/N(L)$, $([x],[y]) \mapsto [xy]$ is continuous in the first variable. We will thus show continuity in the second variable. Let us fix $x \in G$. Then, the map $l_x : L/N(L) \to G/N(L)$, $[y] \to [xy]$ is a $\sigma - \sigma$ homeomorphism so that by $(L/N(L),\tau) = (L/N(L),\sigma)$, $(xL/N(L),\tau)$ is a σ-compact Hausdorff space. It follows now that the following composition of open maps is also open:

$$\begin{array}{ccccc}
(L/N(L),\tau) & & & & \\
\| & & & & \\
(L/N(L),\sigma) & \longrightarrow & (xL/N(L),\sigma) & \longrightarrow & (xL/N(L),\tau) \\
[y] & \longmapsto & [xy] & \longmapsto & [xy]
\end{array}$$

We claim that this composition is continuous and prove this analogously to the open mapping theorem. Indeed consider the inverse map $p : (xL/N(L), \tau) \to (L/N(L), \tau)$, $[xy] \mapsto [y]$. The domain and co-domain spaces are both σ-locally compact Hausdorff and thus have the Baire property. It follows (by a similar argument to Lemma 3.2) that there exists a compact set $C \subset xL/N(L)$, such that $p(C)$ has an interior point $p(c)$, with $c \in C$. If $\{g_\alpha\} \subset xL/N(L)$ is a net such that $x^{-1}g_\alpha$ to $x^{-1}g$ for some $g \in L/N(L)$, it follows by continuity of right multiplication of $(G/N(L), \tau)$ that $(x^{-1}g_\alpha)(x^{-1}g)^{-1}p(c) \to p(c)$ and $p(C)$ contains a tail of $\{(x^{-1}g_\alpha)(x^{-1}g)^{-1}p(c)\}$. However, $p|_C : (C, \tau) \to (x^{-1}C, \tau)$ is a homeomorphism so that applying $[p|_C]^{-1}$, $g_\alpha g^{-1}c \to c$. By continuity of right multiplication, clearly $g_\alpha \to g$ and thus, p^{-1} is continuous. This concludes the proof of the claim. We have thus shown that the map in 3. is separately continuous. Joint continuity of the map then follows from Namioka's Theorem 3.5. □

We also obtain the following generalization from [37].

Proposition 3.7. *If G is a σ-locally compact admissible right topological group and $f : G \to H$ is a continuous homomorphism into a Hausdorff topological group H, then f factors through $G/N(G)$.*

Proof. Observe that $f \circ \phi : (G \times G, \tau \times \tau) \to H, (x, y) \mapsto f(x^{-1}y) = f(x)^{-1}f(y)$ is continuous as f is continuous and H is topological. By definition this implies σ-continuity of f. By Namioka's Corollary 1.1 in [37], the factorization follows. □

Note here that homomorphisms into topological groups can thus never separate points in G.

3.1 Existence of Haar measure

In this section, we will present the proof of the existence of a Haar measure provided a compact strong normal system of subgroups exists.

A locally compact right topological group (G, τ) is said to have a **compact strong normal system of subgroups** if there exists a family $\{L_\xi\}_{\xi<\xi_0}$ of σ-closed normal subgroups of G, indexed by some ordinal $\xi_0 > 0$, satisfying the following conditions:

1. $L_{\xi_0} = G$, $L_0 = \{e\}$
2. L_ξ are compact for all $\xi \neq 0$
3. $L_\xi \supset L_{\xi+1}$ and for a limit ordinal $\xi < \xi_0$, $L_\xi = \cap_{\eta<\xi} L_\eta$;
4. $L_\xi / L_{\xi+1}$ is a compact Hausdorff topological group;
5. the action map

$$\begin{aligned} G/L_{\xi+1} \times L_\xi/L_{\xi+1} &\to G/L_{\xi+1} \\ ([x],[y]) &\mapsto [xy] \end{aligned}$$

   is jointly continuous.

We begin by presenting a lemma.

Lemma 3.8. *Let G be a locally compact Hausdorff right topological group. Suppose L, M are normal subgroups of G (not equal to G) satisfying the following conditions*

- *L, M are compact in (G, σ)*
- *$M \subset L$ and L/M is a non-trivial topological group*
- *$G/M \times L/M \to G/M$, $([x],[y]) \to [xy]$ is continuous.*

Let ν be the unique Haar measure on L/M. Then, the map $\phi : C_c(G/M) \to C_c(G/L)$ given by $\phi(f) = \int_{L/M} f(\cdot t)\, d\lambda(t)$, is a positive retraction such that if $f \in C_c(G/M)$ and $\operatorname{supp}(f) = K/M$, *then* $\operatorname{supp}(\phi(f)) \subset K/L$. *Moreover, for each $g \in G$, $R_g \circ \phi = \phi \circ R_g$.*

Proof. By the continuity of the map $G/M \times L/M \to G/M$, $([x],[y]) \to [xy]$, it is clear that the map $\int_{L/M} f(\cdot t)\, d\lambda(t)$ is well-defined. Moreover, if $x \in L$, by the left-invariance of ν,

$$s \mapsto \phi(f)(sx) = \int_{L/M} f(sxt) d\lambda(t) = \int_{L/M} f(st) d\lambda(t) = \phi(f)(s)$$

so that $\phi(f)$ is constant on cosets of L implying that it is in $C(G/L)$. Moreover, if $f \in C_c(G/L) \subset C_c(G/M)$,

$$s \mapsto \phi(f)(s) = \int_{L/M} f(st) d\lambda(t) = \int_{L/M} f(s) d\lambda(t) = f(s)$$

so that the map is a retraction.

If $\text{supp}(f) = K/M$, for $s \in G$,

$$|\phi(f)(s)| = \left| \int_{L/M} f(st)\ d\lambda(t) \right| \leq \|f\|_\infty \int_{L/M} \mathbf{1}_{K/M}(st)\ d\lambda(t) = \lambda((s^{-1}K \cap L)/M)$$

Here $s^{-1}K \cap L$ is non-empty implies that $s \in KL$. In other words, the above quantity is non-zero only when $s \in KL$, so that if we consider $\phi(f) \in C(G/L)$, it follows that $[s] \in K/L$. This proves that $supp(\phi(f)) \subset K/L$, and that $f \in C_c(G/L)$.

Now, note that $L/M \to G/M \to L/M$, $[t] \mapsto [gt] \mapsto [gtg^{-1}]$ is continuous, the first map being continuous by assumption, and the second being continuous due to G/L being a right topological group. It is easy to observe that the map $C(L/M) \to \mathbb{C}$, $f \mapsto \int f(gtg^{-1}) d\mu(t)$ is right-translation invariant. However, since the Haar measure on L/M is unique, it follows that $\int f(gtg^{-1}) d\mu(t) = \int f(t) d\mu(t)$. Using this fact, if $g, s \in G$,

$$\begin{aligned} R_g[\phi(f)](s) = \phi(f)(sg) = \int_{L/M} f(sgt) d\mu(t) &= \int_{L/M} f(sgtg^{-1}g) d\mu(t) \\ &= \int_{L/M} f(stg) d\mu(t) \\ &= \int_{L/M} R_g f(st) d\mu(t) \\ &= \phi(R_g f)(s) \end{aligned}$$

thereby proving the last claim.

□

We now present the main result of this chapter, a generalization of the theorem in [31].

Theorem 3.9. *Suppose G is a locally compact Hausdorff right topological group that has a compact strong normal system of subgroups. Then, G has a right invariant Haar measure.*

Proof. Let $\{L_\xi\}_{\xi \le \xi_0}$ be the given strong normal system of subgroups. For each $\xi > 0$, we denote by $\phi_\xi : C_c(G/L_{\xi+1}) \to C_c(G/L_\xi)$, the map from Lemma 3.8, and by ν_ξ the Haar measure on $L_\xi/L_{\xi+1}$. Using transfinite induction, we construct for each $\xi > 0$, a linear functional $\psi_\xi : C_c(G/L_\xi) \to \mathbb{C}$, satisfying the following conditions:

1. ψ_ξ is positive
2. ψ_ξ is right-invariant
3. $\psi_\xi(f) = \psi_\eta(f)$, for all $f \in C_c(G/L_\eta)$, for all $0 < \eta \le \xi$.
4. For each K compact, $\psi_\xi(K/L_\xi) \le \psi_1(K/L_1) < \infty$

By the Riesz representation theorem, each ψ_ξ on $C_c(G/L_\xi)$, corresponds to a unique regular Borel measure on G/L_ξ, and 4 implies the existence of a common upper bound for these applied to every fixed compact set of G. We shall show that ψ_1 is non-zero, so that by 3, it follows that for some $f \in C_c(G/L_1)$, $0 < \psi_\eta(f) = \psi_\xi(f)$, i.e. ψ_ξ is non-zero.

For the base case, we observe that $G/L_1 = L_0/L_1$ by assumption is a locally compact Hausdorff topological group. Thus, we may fix a Haar measure ψ_1 on G/L_1, so that the map $\psi_1 : C_c(G/L_1) \to \mathbb{C}$ is the desired linear functional satisfying 1-4.

Suppose for $\xi < \xi_0$, there exists a functional $\psi_\xi : C_c(G/L_\xi) \to \mathbb{C}$ of the desired form. Then, we define $\psi_{\xi+1} : C_c(G/L_{\xi+1}) \to \mathbb{C}$ to be given by $\psi_{\xi+1} = \psi_\xi \circ \phi_\xi$. Positivity and right invariance are clear from Lemma 3.8 and the right invariance of ψ_ξ. For any $0 < \eta \le \xi + 1$, $f \in C_c(G/L_\eta)$,

$$\psi_{\xi+1}(f) = \psi_\xi \circ \phi_\xi(f) = \psi_\xi(f) = \psi_\eta(f)$$

Here, since $\xi+1$ is the smallest ordinal following ξ, any ordinal $\eta < \xi+1$, satisfies $\eta \le \xi$, so that the second equality follows from the retraction property of ϕ_ξ (Lemma 3.8), and the third equality follows from the induction assumption. Lastly, for any $f \in C_c(G/L_{\xi+1})$ with support K in G and $\|f\|_\infty \le 1$, by Lemma 3.8, $\phi_\xi(f)$ has support K/L_ξ, so that

$$\psi_{\xi+1}(f) = \psi_\xi \circ \phi_\xi(f) \le \|f\|_\infty \psi_\xi(K/L_\xi) < \psi_1(K)$$

and 4 holds for the successor case.

Suppose $\xi \le \xi_0$ is a limit ordinal so that $L_\xi = \cap_{\eta<\xi} L_\eta$. Then, consider the subalgebra $D = \cup_{\eta<\xi} C_c(G/L_\eta) \subset C_c(G/L_\xi) \subset C_0(G/L_\xi)$. If $[x] \neq [y]$, for $[x],[y] \in G/L_\xi$, then $x^{-1}y \notin L_\xi = \cap_{\eta<\xi} L_\eta$, so that for some $\eta < \xi$, $[x] \neq [y]$ in G/L_η. By local compactness then, there exists some $f \in C_c(G/L_\eta)$, such that $f([x]) \neq f([y])$. It follows that D separates points in G/L_ξ. Moreover, it is clear that D is vanishing nowhere. By the Stone-Weierstrass theorem, D is dense in $C_0(G/L_\xi)$. Now for each compact set K, we fix an open neighbourhood U_K of K such that $\overline{U_K}$ is compact. Then, by Urysohn's lemma, for each $\xi \le \xi_0$, there exists a function $p_\xi^K : G/L_{\xi_n} \to [0,1]$, such that $\mathbf{1}_{K/L_{\xi_n}} \le p_\xi^K \le \mathbf{1}_{U_K/L_{\xi_n}}$. Consider any $f \in C_c(G/L_\xi)$ and let $\{f_n\}_{n\in\mathbb{N}} \subset D$ be such that $f_n \to f$. Without loss of generality, we assume $\|f\|_\infty \le 1$ so that $\{f_n\}_{n\in\mathbb{N}}$ may be chosen to satisfy $\|f_n\|_\infty \le 1$. Let us denote by ξ_n, the ordinal corresponding to each $f_n \in C_c(G/L_{\xi_n})$. Suppose $C = \mathrm{supp}(f)$ in G. Then, $\{f_n p_{\xi_n}^C\}_{n\in\mathbb{N}} \subset D$, which we will write as $\{f_n p_n^C\}_{n\in\mathbb{N}}$ have supports contained in U_C/L_{ξ_n}, for each $n \in \mathbb{N}$, and clearly, converge to f. Since $\xi = \sup_{\eta<\xi} \eta$, and $C_c(G/L_{\eta_1}) \subset C_c(G/L_{\eta_2})$, for $\eta_1 \ge \eta_2$, we may assume that $\{\xi_n\}$ is monotone increasing, so that for $m \ge n$, by the induction assumption,

$$\begin{aligned}|\psi_{\xi_n}(p_n f_n) - \psi_{\xi_m}(p_m f_m)| = |\psi_{\xi_m}(p_n f_n - p_m f_m)| &\le \psi_{\xi_m}(U_C/L_{\xi_m})\|f_n - f_m\|_\infty \\ &\le \psi_{\xi_1}(U_C/L_1)\|f_n - f_m\|_\infty \to 0\end{aligned}$$

as $m, n \to \infty$. Thus, the sequence $\{\psi_{\xi_n}(p_n f_n)\}_{n\in\mathbb{N}}$ is Cauchy, so that we define $\psi_\xi(f) = \lim_{n\in\mathbb{N}} \psi_{\xi_n}(p_n f_n)$. It is easily checked that ψ is well-defined and does not depend on the choice of specific choice of U_K or the corresponding maps p_ξ^K.

That ψ_ξ is positive and linear follows from its definition and the induction assumption. Right-invariance of ψ_ξ also follows as $\{f_n\}_{n\in\mathbb{N}}$ from above satisfies, $\{R_g[f_n p_n]\}_{n\in\mathbb{N}} \subset D$, $R_g[f_n p_n] \to R_g f$, so that

$$\psi_\xi(R_g f) = \lim_{n\in\mathbb{N}} \psi_{L_{\xi_n}}(R_g[f_n p_n]) = \lim_{n\in\mathbb{N}} \psi_{L_{\xi_n}}(f_n p_n) = \psi_\xi(f)$$

Regarding 3, one notes that if $0 < \eta \le \xi$, then, for $f \in C_c(G/L_\eta)$, one may consider an arbitrary sequence $\{\xi_n\} \ge \eta$, and take $f_n = f$ trivially, so that by the induction assumption,

$$\psi_\xi(f) = \lim_{n\in\mathbb{N}} \psi_{\xi_n}(f_n) = \lim_{n\in\mathbb{N}} \psi_\eta(f_n) = \psi_\eta(f)$$

Lastly, $\psi_\xi(K) \leq \psi_1(K) < \infty$, follows similarly from the induction assumption.

Since $L_{\xi_0} = \{e\}$, there exists a linear functional ψ_{ξ_0} on $C_c(G)$ satisfying the criteria 1-4. This provides the desired Haar measure. $\square$

Uniqueness of the theorem follows similarly.

Theorem 3.10. *The Haar measure in theorem Theorem 3.9 is unique up to scalar multiplication.*

Proof. Let $\{\psi_\xi\}_{\xi \leq \xi_0}$ be the Haar measures on G/L_ξ, $0 < \xi \leq \xi_0$ and suppose μ is a Haar measure on G. As before we denote by μ_ξ the unique Haar measure on $L_\xi/L_{\xi+1}$. We shall show that there exists $c > 0$ such that $c\mu(f) = \psi_\xi(f)$, for all $f \in C_c(G/L_\xi)$, for all $\xi \leq \xi_0$ using transfinite induction.

Since G/L_1 is a locally compact topological group, and μ forms a Haar measure on it, it follows that there exists some $c > 0$, such that $c\mu(f) = \psi_1(f)$.

Assume the same induction hypothesis holds for $\xi < \xi_0$, with the same constant c. Then, for any $f \in C_c(G/L_{\xi+1})$,

$$\begin{aligned}
\psi_{\xi+1}(f) = \psi_\xi \circ \phi_\xi(f) &= \int \phi_\xi(s) d\psi_\xi(s) \\
&= c \int \phi_\xi(s) d\mu(s) \\
&= c \int \int_{L_\xi/L_{\xi+1}} f(st) d\mu_\xi(t) d\mu(s) \\
&= c \int \int_{L_\xi/L_{\xi+1}} f(s) d\mu_\xi(t) d\mu(s) \\
&= c \int f(s) d\mu(s) \\
&= c\mu(f)
\end{aligned}$$

where we made use of the right-translation invariance of μ. The successor case is hence justified.

Now suppose ξ is a limit ordinal and $f \in C_c(G)$. Recall from the proof of theorem, that $\psi_\xi(f) = \lim_{n\in\mathbb{N}} \psi_{\xi_n}(f_n)$, for $f_n \in C_c(G/L_{\xi_n})$, where $\cup_{n\in\mathbb{N}}\text{supp}(f_n) \subset U$ for some compact set $U \subset G$ and $\|f_n\| \leq \|f\|_\infty$ for all $n \in \mathbb{N}$. By the induction hypothesis, therefore, $\psi_\xi(f) = c\lim_{n\in\mathbb{N}} \mu(f_n) = c\mu(f)$, by the dominated convergence theorem. This concludes the proof. $\square$

We shall conclude this chapter with some examples of σ-locally compact right topological groups. However, we shall first discuss a fundamental idea which has not been touched upon in the current literature. Given a right topological group, one might question when its quotient groups are topological (and thus automatically have a Haar measure). In particular, when do results like Proposition 3.6 hold for general subgroups? To answer this question, we need the $\sigma\sigma$-topology of (G, σ).

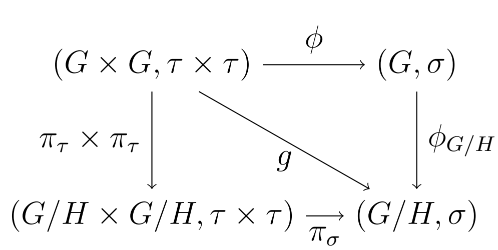
We define the $\sigma\sigma$ **topology** on G, to be the σ topology of the compact semi-topological group (G, σ) i.e. induced by the quotient map $(G, \sigma) \times (G, \sigma) \to G$, $(g, h) \mapsto g^{-1}h$.

By Theorem 2.5, $\sigma\sigma \subsetneq \sigma$ and $(G, \sigma\sigma)$ is a compact semitopological group with continuous inverse. Furthermore, by [37], Corollary 1.1, $N(G)$ is precisely $\overline{\{e\}}^{\sigma\sigma}$.

Let $H \subset G$ be a closed normal subgroup. Recall that $(G/H, \tau)$, $(G/H, \sigma)$ denotes G/H with the quotient topologies induced by (G, τ) and (G, σ) respectively, where we denote the respective quotient maps by π_τ, π_σ. Let us further define $(G/H, \sigma_{G/H})$ to be the σ topology induced by $(G/H, \tau)$ i.e. by the quotient map $\phi_{G/H} : (G/H, \tau) \times (G/H, \tau) \to G/H$, $([x], [y]) \mapsto [x^{-1}y]$.

Lemma 3.11. *Let G be a locally compact right topological group and let H be a σ-closed normal subgroup of G. Then, $(G/H, \sigma_{G/H}) = (G/H, \sigma)$, and $(G/H, \tau) = (G/H, \sigma)$ holds if and only if $(G/H, \tau)$ is topological.*

Proof. To prove the first claim, we consider the following commutative diagram



One easily checks that the diagram commutes and a function $f : (G/H, \tau) \to X$, where X is any topological space, is continuous on either of $(G/H, \sigma_{G/H})$, $(G/H, \sigma)$ if and only if $f \circ g$ is continuous. It follows that the two topologies coincide. Now by Theorem 2.5, $(G/H, \tau)$ is topological if and only if $(G/H, \tau) = (G/H, \sigma_\tau)$, where the latter space coincides with $(G/H, \sigma)$. The conclusion follows. □

Lemma 3.12. *Let G be a σ-locally compact Hausdorff right topological group and $H \subset G$ be a closed normal subgroup. If G is either compact or admissible, then, $(G/H, \tau)$ is a Hausdorff topological group if and only if H is $\sigma\sigma$-closed.*

Proof. By Proposition 2.7 $(G/H, \tau)$ is Hausdorff if and only if H is σ-closed, so that we may restrict the proof to this case. Suppose H is also $\sigma\sigma$-closed. Then, by Lemma 3.3, $(G/H, \sigma)$ is locally compact Hausdorff. However, by Lemma 3.11, $(G/H, \sigma) = (G/H, \sigma_\tau)$, so that $(G/H, \tau) \to (G/H, \sigma)$, $g \mapsto g$, is a continuous homomorphism from a σ-compact group into a Hausdorff Baire group, thus a homeomorphism by the open mapping theorem. It follows by Lemma 3.11 that $(G/H, \tau)$ is topological. Conversely, if $(G/H, \tau)$ is a Hausdorff topological group, we have $(G/H, \tau) = (G/H, \sigma) = (G/H, \sigma_{G/H})$ is Hausdorff so that by Proposition 2.7 H must be $\sigma\sigma$-closed. This concludes the proof. □

An excellent source of examples in the compact case is [28], and our own examples have been inspired by this paper. Additionally, [29] provides a general framework for constructing such examples via the concept of Schreier products of groups.

Example 3.13. Consider the group $\mathbb{C}_* \times \mathbb{C}^{AP} = \mathbb{C}_* \times \hat{\mathbb{C}}_d$. We equip this group with the following multiplication:

$$(w, h)(v, g) = (wv, R_v hg)$$

If G is further equipped with the product topology, one obtains a σ-locally compact right topological group. Since $\mathbb{C} \hookrightarrow \hat{\mathbb{C}}$ via $z \mapsto [t \mapsto e^{2\pi izt}]$ sits inside $\lambda(C^{AP})$, i.e is the set of continuous characters on $\mathbb{C}$, $\Lambda(G) \cong \mathbb{C} \times \mathbb{C}$. Further one notes that G is admissible.
Consider the normal subgroup $H = \{1\} \times \mathbb{C}^{AP}$. Taking the quotient of G with respect to this subgroup, one obtains an algebraic isomorphism onto $\mathbb{C} \times 1$, in fact, we get the following composition

$$\mathbb{C}^* \times \mathbb{C}^{AP} \to \mathbb{C}^* \times \mathbb{C}^{AP}/H \to \mathbb{C}$$

$$(w,h) \mapsto [w,1] \to w$$

Since the above diagram commutes, it is clear that $G/H \to \mathbb{C}$ is a continuous bijective homomorphism. By Lemma 3.2, $G/H \cong \mathbb{C}$, whence by Lemma 3.12, H is $\sigma\sigma$-closed and thus contains $N(G)$. One notes however, that no proper subgroup $N \subset H$ provides a topological quotient G/N so that $H = N(G)$ as $N(G) = \overline{\{e\}}^{\sigma\sigma}$. The hypothesis of Theorem 3.9 is satisfied for G. Indeed, one gets a compact strong normal system of subgroups given by $G \supset N(G) \supset \{e\}$ as $G \times N(G) \to G$, $(w,h)(1,g) = (w,hg)$ is clearly separately continuous, whence jointly continuous by Theorem 3.5. A Haar measure on G is simply given by the product of Haar measures on $\mathbb{C}$ and $\mathbb{C}^{AP}$ respectively.

Here we observe that $G' = \mathbb{T} \times \mathbb{C}^{AP}$ forms a subgroup of G that is also an admissible right topological group (discussed in [28]) and further, $N(G') = N(G)$. An open question is determining when subgroups of a σ-locally compact right topological group satisfy the latter property.

Example 3.14. Let G be the group $\mathbb{T} \times \mathbb{C} \times \mathbb{C}^{AP} = \mathbb{T} \times \mathbb{C} \times \hat{\mathbb{C}}_d$ with multiplication given by:

$$(u,x,h)(v,y,g) = (uvh(y), x+y, hg)$$

Along with the product topology and the specified multiplication, G becomes a σ-locally compact right topological group. The topological center can be verified to consist of the continuous elements of $\hat{\mathbb{C}}_d$ i.e. $\Lambda(G) = \mathbb{T} \times \mathbb{C} \times \mathbb{C}$, where, as before $\mathbb{C} \subset \hat{\mathbb{C}}_d$ via $z \mapsto [t \mapsto e^{2\pi i z t}]$. G is hence admissible.

In this case, we may note that for $H = \mathbb{T} \times \{0\} \times \{1\}$, $G/H \cong \mathbb{C} \times \mathbb{C}^{AP}$ is a locally compact topological group. Further, no proper subgroup of H gives a topological quotient with respect to G, so that by Lemma 3.12, $H = N(G)$. Again, we have $N(G)$ is compact. A compact strong normal system is given by $G \supset N(G) \supset \{e\}$; the multiplication map $[\mathbb{T} \times \mathbb{C} \times \hat{\mathbb{C}}_d] \times [\mathbb{T} \times \{0\} \times \{1\}] \to \mathbb{T} \times \mathbb{C} \times \hat{\mathbb{C}}_d$ is clearly separately continuous as $N(G) \subset \Lambda(G)$, whence jointly continuous by Theorem 3.5.

Unlike the last example, $N(G) \subset \Lambda(G)$ in this case. Observe that in both examples, $N(G)$ is a compact topological group.

General examples of locally compact admissible topological groups are plentiful via taking products of admissible compact right topological groups and locally compact topological groups. In particular, for a locally compact topological group G, $G \times G^{D(G)}$ is usually non-trivially right topological, locally compact and admissible. Here by $G^{D(G)}$, we refer to the distal compactification of G and the reader is referred to [3] for more details.

If G is σ-compact, $G \times G^{D(G)}$ is also σ-locally compact. In this case, note that a strong normal system of subgroups and thus the existence of a Haar measure is guaranteed as $G^{D(G)}$ is a CHART group (Theorem 3.9). However, it still remains an open question as to whether every admissible σ-locally compact group possesses a Haar measure or a strong normal system of subgroups. Further, is $N(G)$ always compact for such a group?

Chapter 4

Function algebras

In this chapter, we will discuss familiar function algebras from harmonic analysis, including Fourier algebras, in the setting of locally compact right topological setting. These algebras form an important part of the analytic theory over such groups and their structure is fundamentally linked to the topological structure of the group itself (see [14],[17]). As we shall see, unfortunately in the right topological setting, the theory of these algebras is not as rich and these algebras may be degenerate in some sense (not separating points for example).

Let G be a locally compact Hausdorff right topological group. Further, let $A \subset C_0(G)$ be a non-trivial translation-invariant C*-algebra of $C_0(G)$. We then define

$$\text{Fix}(A) = \{g \in G \mid L_g f = f, \text{ for all } f \in A\}$$

For every subgroup L of G, we shall denote by $\tilde{\pi}_L : C_0(G/L) \to C_0(G)$ the map $f \mapsto f \circ \pi_L$, where π_L is the quotient map from G into G/L.

Here we generalize a result in [20]. The proof follows with mild modifications and is presented below.

Lemma 4.1. *Let G be a locally compact Hausdorff right topological group. If $A \subset C_0(G)$ is a non-trivial translation invariant C*-algebra, then $F = Fix(A)$ satisfies the following;*

1. *F is a closed normal subgroup of G*

2. $\tilde{\pi}_F : C_0(G/F) \to A$ *is an isometric isomorphism, i.e.*

$$A = \{f \in C_0(G) \mid L_y f = f \text{ for all } y \in F\}$$

3. G/F *is a locally compact Hausdorff topological group, so that* F *is* σ*-closed.*

Proof. It is straightforward to observe that F is a closed subgroup. Normality also follows, as for any $x, y \in G$, $g \in F$,

$$L_{xgx^{-1}} f(y) = f(xgx^{-1}y) = L_x f(gx^{-1}y) = L_g[L_x f](x^{-1}y) = L_x f(x^{-1}y) = f(y)$$

where the second last inequality follows because $L_x f \in A$ by the translation invariance of A, and $g \in F$.

To prove 2, we consider the map $\psi : A \to C_0(G/F)$, $f \mapsto \tilde{f}$, where $\tilde{f} \in C_0(G/F)$ is the unique function satisfying $\tilde{f} \circ \pi_F = f$. Here, $\psi(A) \subset C_0(G/F)$ is clearly a C^*-subalgebra. Moreover, $[x] \neq [y]$ in G/F implies that $x^{-1}y \notin F$, so that by the definition of F, there exist $f \in A$, $g \in G$ such that $L_{x^{-1}y} f(g) \neq f(g)$ and thus, $\psi[R_g f](x^{-1}y) \neq \psi[R_g f](e)$, where $R_g f \in A$ due to the right translation invariance of A. Hence, $\psi(A)$ separates the points of $C_0(G/F)$. It is also clear that $\psi(A)$ is non-vanishing everywhere by translations, as A contains a non-trivial function. By Stone-Weierstrauss theorem, $\psi(A) = C_0(G/F)$, and the map $\tilde{\pi}_F$, the inverse of ψ, is an isometric isomorphism.

To show 3, first observe that for any $\tilde{f} \in C_0(G/F)$, $x, y \in G$,

$$L_x f(y) = f(xy) = \tilde{f}([xy]) = L_{[x]} \tilde{f}([y])$$

where $L_x f \in A$ by assumption, and so, $L_{[x]} \tilde{f} = \psi[L_x f] \in C_0(G/F)$. If $[x] \in G/F$ and $\{[g_a l]\} \to [g]$ in G/F, it then follows for every $f \in C_0(G)$,

$$\tilde{f}([xg_\alpha]) = L_{[x]} \tilde{f}([g_\alpha]) \to L_{[x]} \tilde{f}([g]) = \tilde{f}([xg])$$

Since G/F is locally compact, whence completely regular, $l_{[x]} : G/F \to G/F$ is continuous. By Theorem 2.1, it follows that G/F is topological. Moreover, by Proposition 2.7 F is σ-closed. $\square$

We note here that if we assume that G is σ-compact admissible, by Lemma 3.12, F is additionally $\sigma\sigma$ closed. As a result of this, we obtain the following;

Corollary 4.2. *Given a σ-locally compact Hausdorff right topological group, if G is either compact or admissible,*

$$\mathrm{Fix}(C_0(G,\sigma)) = N(G)$$

Proof. By Lemma 4.1, for $F = \mathrm{Fix}(C_0(G,\sigma))$, $(G/F,\tau)$ is a Hausdorff topological group, so that by Lemma 3.12, F is $\sigma\sigma$-closed. However, $N(G) = \overline{\{e\}}^{\sigma\sigma}$ implies $N(G) \subset F$.

Suppose that G is compact or admissible. If $N(G) = G$, $F \subset N(G)$ is trivial. If $N(G) \neq G$ on the other hand, for any $g \notin N(G)$, there exists some $\tilde{f} \in C_0(G/N(G))$ such that $\tilde{f}([g]) \neq \tilde{f}([e])$. By Proposition 3.6, $f = \tilde{f} \circ \pi_{N(G)} \in C(G,\sigma)$ and separates e and g, which implies $g \notin F$ and thus $N(G) = F$.

□

Let G be a right topological semigroup.

We define the **left continuous functions on** G to be given by

$$LC(G) = \{f \in C_b(G) \mid L_g f \in C_b(G), \text{ for all } g \in G\}$$

and the **left continuous functions vanishing at infinity** to be

$$LC_0(G) = \{f \in C_0(G) \mid L_g f \in C_0(G), \text{ for all } g \in G\}$$

Note that the latter definition is somewhat artificial: $LC_0(G) \subset LC(G) \cap C_0(G)$, however, equality is not known, since our left multiplication is not continuous (and thus compact sets need not be preserved by it).

We also define the standard spaces

$$AP(G) = \{f \in C_b(G) \mid RO(f) \text{ is relatively compact in } C_b(G)\},$$

the **almost periodic functions** on G, and

$$WAP(G) = \{f \in C_b(G) \mid RO(f) \text{ is relatively weakly compact in } C_b(G)\},$$

the **weakly almost periodic functions** on G.

The following proposition is a standard result in harmonic analysis (see Theorem 1.8 of [3]).

Proposition 4.3. *Given a non-empty set S, and a conjugate closed subspace E of $l_\infty(S)$, E^* is the weak*-closed linear span of $\epsilon(S)$, where $\epsilon : S \to E^*$ is the function sending $s \in S$ to the evaluation functional $\epsilon(s) : f \mapsto f(s)$.*

Lemma 4.4. *If G is a right topological semigroup, the following equalities hold*

1. $AP(G) = AP(G_d) \cap C_b(G)$
2. $WAP(G) = WAP(G_d) \cap C_b(G)$

Proof. The first result is obvious. Let us prove 2. Suppose $f \in l_\infty(G)$. Then, By Grothendieck's double limit theorem (see Theorem A.5 in [3]) and Proposition 4.3, for any sequences $\{g_n\}$,$\{h_n\} \subset G$, the following middle limits are equal when they exist

$$\lim_m \lim_n \epsilon(g_n) R_{h_m} f = \lim_m \lim_n f(g_n h_m) = \lim_n \lim_m f(g_n h_m) = \lim_n \lim_m \epsilon(g_n) R_{h_m} f$$

if and only if $RO(f)$ is relatively weakly compact in $l_\infty(G)$, i.e. $f \in WAP(G_d)$. However, if $f \in C_b(G)$, this is clearly equivalent to $f \in WAP(G)$ by another application of Proposition 4.3 and Grothendieck's theorem. □

Theorem 4.5. *Let G be a locally compact Hausdorff right topological group. The following hold:*

1. *$C_b(G, \sigma) \subset LC(G)$ and $C_0(G, \sigma) \subset LC_0(G)$.*
2. *$LC(G)$ separates points from closed sets in G if and only if G is topological.*
3. *If G is admissible, $WAP(G) \subset LC(G)$ and $WAP(G) \cap C_0(G) \subset LC_0(G)$*

If further G is σ-compact admissible or compact, then

4. $LC_0(G) = C_0(G, \sigma)$

5. *If* G *is non-compact,* $AP(G) \cap C_0(G) = \{0\}$ *so that* $AP(G) \oplus C_0(G, \sigma) \subset WAPG(G)$

Proof. We first prove 1 and 4. Since (G, σ) is a semitopological group, it is clear that $C_b(G, \sigma) \subset LC(G)$ and that $C_0(G, \sigma) \subset LC_0(G)$. Suppose in addition that G is σ-compact admissible or compact. Since $LC_0(G)$ is a non-trivial closed, translation-invariant subalgebra of G, by Lemma 4.1 and Lemma 3.12, $\text{Fix}(LC_0(G))$ contains $N(G) = \overline{\{e\}}^{\sigma\sigma}$. On the other hand, $\text{Fix}(LC_0(G)) \subset \text{Fix}(C_0(G, \sigma)) = N(G)$. It follows that $\text{Fix}(LC_0(G)) = N(G)$, so that by Corollary 4.2 and 2 of Lemma 4.1, $C_0(G, \sigma) = LC_0(G)$.

If $LC(G)$ separates points from closed sets, the initial topology of $LC(G)$ coincides with the original topology on G (see 8.15 in [51]). However, for all $f \in LC(G)$, $y_\alpha \to y$ in G implies $f(xy_\alpha) \to f(xy)$, since $L_x f \in C(G)$ for all $x \in G$. Thus, G is a locally compact semitopological group, and by [7], a topological group. The converse is clear as $LC(G) = C_b(G)$ in the topological case.

To prove 3, consider $f \in WAP(G)$; by assumption, $RO(f)$ is relatively weakly compact in $C_b(G)$. By Grothendieck's double limit theorem (see Theorem A.5 in [3]) and Proposition 4.3, for any sequences $\{g_n\}$,$\{h_n\} \subset G$, the following middle limits are equal when they exist

$$\lim_m \lim_n \epsilon(h_m) L_{g_n} f = \lim_m \lim_n \epsilon(g_n) R_{h_m} f = \lim_n \lim_m \epsilon(g_n) R_{h_m} f = \lim_n \lim_m \epsilon(h_m) L_{g_n} f$$

Consider the set $\{L_g f \mid g \in \Lambda(G)\} \subset C_b(G)$. Then, the above equality certainly holds for $\{g_n\} \subset \Lambda(G)$, $\{h_n\} \subset G$ when the limits exist. Therefore, by Grothendieck's theorem again, $\{L_g f \mid g \in \Lambda(G)\}$ is relatively weakly compact in $C_b(G)$. For any $g \in G$ now, by admissibility there exists some net $\{g_\alpha\} \subset \Lambda(G)$ such that $g_\alpha \to g$, and thus $L_{g_\alpha} f \to L_g f$ pointwise. However, by weak compactness, $\{L_{g_\alpha} f\}$ has a weak cluster point in $C_b(G)$, which must coincide with $L_g f$. Thus, $L_g f \in C_b(G)$ for all $g \in G$. If in addition, $f \in C_0(G)$, then, $\{L_{g_\alpha} f\} \subset C_0(G)$ so that its weak limit $L_g f \in C_0(G)$, whence $f \in LC_0(G)$.

Lastly, if G is non-compact, σ-compact admissible, if $f \in AP(G) \cap C_0(G)$, then by 3,4, and Corollary 4.2, $f \in C_0(G, \sigma) = C_0(G/N(G))$. If $N(G) = G$, the claim is now obvious; otherwise by Proposition 3.6, $G/N(G)$ is a non-trivial locally compact topological group and the 5 follows from the classical result (see 4.2.2a of [3]). $\square$

In the case of a CHART group, it was shown in [20], and follows from Theorem 4.5 1 and 3,, that $LC(G) = WAP(G) = AP(G) = C(G, \sigma)$. In fact, even if G is not admissible, $LC(G) = C(G, \sigma)$. We wonder if it is possible for $LC(G)$ to separate points in the locally compact case.

4.1 Fourier Algebras on G

The results presented on function algebras may be used to obtain some interesting properties on the Fourier algebra analogues.

Suppose G is a locally compact Hausdorff right topological group. Let us denote G with the discrete topology by G_d. The unitary representations on G_d correspond one-to-one with the non-degenerate representations on $l_1(G)$, which we denote by Σ. Σ then induces a norm on $l_1(G)$, namely $\|f\|_* = \sup_{\pi \in \Sigma} \|\pi(f)\|$ for $f \in l_1(G)$. The completion of $l_1(G)$ under this norm is then a C*-algebra, denoted by $C^*(G_d)$ and known as the **group C*-algebra** of G_d.

Consider the **positive-definite functions** on G, i.e. functions $f : G \to \mathbb{C}$ satisfying $\sum_{i,j=1}^{n} c_i \bar{c_j} f(x_i x_j^{-1}) \geq 0$ for all $\{c_i\}_{i=1}^{n} \subset \mathbb{C}$ and $\{x_i\}_{i=1}^{n} \subset G$. The span of $P(G)$ can be identified as the dual of $C^*(G_d)$, and under this dual norm, forms a Banach algebra, known as the **Fourier-Stieltjes algebra**, $B(G_d)$ on the discrete group G_d. In the right topological group setting the topological analogue has to be defined more carefully than the classical case, due to a lack of continuous representations. For theory on these algebras in the classical case, see [17].

Following [20], we define the **Fourier-Stieltjes algebra** of (G, τ), $B(G)$ to given by $B(G_d) \cap C_b(G)$. We further define the **Fourier algebra** on G, $A(G)$, to be the closure of $B(G_d) \cap C_c(G)$ in $B(G)$. The σ**- Fourier-Steiltjes and -Fourier algebras** on G, will then respectively $B(G,\sigma) = B(G) \cap C(G,\sigma)$ and $A(G,\sigma) = A(G) \cap C(G,\sigma)$.

Lau and Loy have done some extensive work over the Fourier-Steiltjes algebra in the compact setting in [21]. In particular, they showed the following characterization;

Theorem 4.6 (Lau,Loy)**.** *If G be is an admissible compact Hausdorff right topological group, then*

$$B(G) = B(G,\sigma) \cong B(G/N(G))$$

Unlike the locally compact topological group setting, where $B(G)$ completely identifies G up to topological isomorphism, the above result indicates that this does not hold for right topological groups. It however does follow that the quotient group $G/N(G)$ is uniquely identified by $B(G)$ up to topological isomorphism. In the locally compact setting we are unable to produce an analogue of this strong result and instead have the following;

Corollary 4.7. *Let G be an admissible σ-locally compact right topological group.Then, $B(G)$, $A(G)$ are commutative Banach algebras satisfying the following:*

1. *$B(G)$ is closed under translations on G*
2. *$B(G,\sigma) = B(G/N(G))$ and $A(G) = A(G,\sigma)$*
3. *$B(G)$ separates points from closed sets implies that G is topological*
4. *$A(G)$ separates points in G if and only if G is topological*

Proof. That $B(G)$ is closed in norm in $B(G_d)$ follows from the fact that the norm on latter bounds the uniform norm. Observe that $B(G) = B(G_d) \cap C_b(G) \subset WAP(G_d) \cap C_b(G) = WAP(G)$, where the first containment is a classical result (see [4]), and the equality follows from Lemma 4.4. By 3 of Theorem 4.5, $B(G) \subset LC(G)$ so that invariance under translations follows by 4.2.3 in [3]. Now using Corollary 4.2, $B(G,\sigma) = B(G_d) \cap C(G,\sigma) \subset B(G_d) \cap C(G/N(G)) = B(G/N(G))$. The other inclusion follows by Corollary 4.2 or Proposition 3.6.

For 4, we again obtain the containments $B(G_d) \cap C_c(G) \subset WAP(G) \cap C_c(G) \subset LC_0(G) = C(G,\sigma)$, where we used 3, 4 of Theorem 4.5 and Corollary 4.2. It follows that the closure of $B(G_d) \cap C_c(G)$ in $B(G)$, namely $A(G)$, is contained in $B(G,\sigma)$, whence $A(G) = A(G,\sigma)$. The last two claims follows from 3 of Theorem 4.5. □

Unlike Theorem 4.6, where $B(G) \cong B(H)$ for a topological group H, it is possible that there exist non-compact locally compact admissible, or compact non-admissible right topological groups for which this property never holds. We wonder if such groups possess any special properties.

We conclude with a result that shows the failure of a fundamental property of $A(G)$ occuring in the topological case.

Corollary 4.8. *Let G be a σ-locally compact admissible right topological group. Then $A(G)^*$ is a Von Neumann algebra corresponding to a faithful representation π of G, with $\langle \pi(x), f\rangle = f(x)$ for all $x \in G$, $f \in A(G)$ if and only if G is topological.*

Proof. Suppose $A(G)^*$ is a Von Neumann algebra as described. By Corollary 4.7, $A(G) = A(G, \sigma)$, so that $x \mapsto \langle \pi(x), f\rangle = f(x)$ is σ-continuous and factors through $N(G)$ by Theorem 4.5. Since VN_π is determined by its evaluation on $A(G)$, it follows that π is $N(G)$-invariant. This contradicts faithfulness unless G is topological. The converse is a standard result (see [17]). □

Chapter 5

Measure Algebras

In this chapter, we primarily discuss various measure algebra analogues on right topological groups, and their properties. As discussed in Chapter 1, on locally compact topological groups, the space of complex Radon measures, $M(G)$, naturally forms an algebra under convolution. In the right topological case, due to the lack of continuity properties, and more generally, measurability properties of left translations, a workaround is required for the formulation of a canonical measure algebra. In fact, as we shall see, one obtains many measure algebra analogues which simply collapse to the usual measure algebra for topological groups. Many of these are highly non-equivalent in the (strict) right topological case. A lot of our work here is inspired by Lau and Loy's work on measure algebras in the compact setting in [20]. Much of this generalizes well to the locally compact case and we take advantage of this.

An important goal in introducing the measure algebras that we do, is characterizing the existence of a Haar measure on compact right topological groups, presented in Section 5.1. Currently, admissibility and the existence of a strong normal system of subgroups (Proposition 2.8,Theorem 2.9) are the only known sufficient conditions for this. In view of there existing a compact right topological group with no Haar measure [19], we hope our work will shed some light on the matter.

Suppose $\mu, \nu \in M(G)$; note that for $f \in C_c(G)$, the usual convolution of measures, $\int\int f(xy)d\mu(x)d\nu(y)$, may not be defined, since the function $f \cdot \mu : G \mapsto \mathbb{R}$, $y \mapsto \int f(xy)d\mu(x)$ is not necessarily Borel (let alone continuous like in the topological case).

As such, let us define

$$\mathcal{M}_{Cb}(G) = \{\mu \in M(G) \mid f \cdot \mu \in C_b(G), \text{ for all } f \in C_b(G)\}$$

In the compact case, this reduces to

$$\mathcal{M}(G) = \{\mu \in M(G) \mid f \cdot \mu \in C(G), \text{ for all } f \in C(G)\}$$

as introduced by Lau and Loy in [20].

We may now define the convolution of measures as usual,

$$\langle \mu \Box \nu, f \rangle = \langle \nu, \mu \cdot f \rangle = \int \int f(xy) d\mu(x) d\nu(y)$$

Further, we define $\mathcal{L}(G) = \mathcal{M}_{Cb}(G) \cap L_1(G)$, whenever G has a Haar measure.

Lau and Loy's result on $\mathcal{M}(G)$ for a compact right topological group G, works for $\mathcal{M}_{Cb}(G)$ as well, presented as follows.

Proposition 5.1. *Let G be a locally compact Hausdorff right topological group. Then, $(\mathcal{M}_{Cb}(G), \Box)$ is a Banach algebra that is closed in $M(G)$. Furthermore, $l_1(\Lambda(G)) \subset \mathcal{M}_{Cb}(G)$, so that the space is non-trivial, and these are the only point mass measures in $\mathcal{M}_{Cb}(G)$.*

Proof. It is easily seen that $\mathcal{M}_{Cb}(G)$ is a subspace of $M(G)$. If now $\mu_n \to \mu$ for $\{\mu_n\} \in \mathcal{M}_{Cb}(G)$, $\mu \in M(G)$, for any $f \in C_b(G)$, we have,

$$\begin{aligned} |f \cdot \mu(x) - f \cdot \mu_n(x)| &= \left| \int R_x f(y) d\mu(y) - \int R_x f(y) d\mu(y) \right| \\ &\leq \|R_x f\|_\infty \|\mu_n - \mu\| \\ &= \|f\|_\infty \|\mu_n - \mu\| \longrightarrow 0 \end{aligned}$$

so that $f \cdot \mu_n \to f \cdot \mu$ uniformly. Since $\{f \cdot \mu_n\} \subset C_b(G)$, it is clear that $f \cdot \mu \in C_b(G)$, so that $\mu \in \mathcal{M}_{Cb}(G)$ and $\mathcal{M}_{Cb}(G)$ is closed.

Let us show that $\mathcal{M}_{Cb}(G)$ is closed under $\Box$. Suppose $\mu, \nu \in \mathcal{M}_{Cb}(G)$ and $f \in C_b(G)$.

Then, we have,

$$\begin{aligned} f \cdot [\mu \Box \nu](x) = \langle \mu \Box \nu, R_x f \rangle &= \int \int R_x f(yz) d\mu(y) d\nu(z) \\ &= \int \int R_z x f(y) d\mu(y) d\nu(z) \\ &= \int f \cdot \mu(zx) d\nu(z) \\ &= \int R_x [f \cdot \mu](z) d\nu(z) \\ &= [f \cdot \mu] \cdot \nu(x) \end{aligned}$$

where, since $f \cdot \mu \in C_b(G)$ by assumption, it follows that $[f \cdot \mu] \cdot \nu \in C_b(G)$. This proves the claim.

To show associativity, consider again, $\mu, \nu, \gamma \in \mathcal{M}_{Cb}(G)$ and $f \in C_b(G)$, then,

$$\begin{aligned} \langle [\mu \Box \nu] \Box \gamma, f \rangle = \int \int f(xy) d[\mu \Box \nu](x) d\gamma(y) &= \int \langle \mu \Box \nu, R_y f \rangle \, d\gamma(y) \\ &= \int \int \int R_y f(xz) d\mu(x) d\nu(z) d\gamma(y) \\ &= \int \int \int f(xzy) d\mu(x) d\nu(z) d\gamma(y) \\ &= \int \int [f \cdot \mu](zy) d\nu(z) d\gamma(y) \\ &= \langle \nu \Box \gamma, f \cdot \mu \rangle \\ &= \langle \mu \Box [\nu \Box \gamma], f \rangle \end{aligned}$$

which proves the claim.

Further, also note that

$$|\langle \mu \Box \nu, f \rangle| = \left| \int \int f(xy) d\mu(x) d\nu(y) \right| \leq \|f\|_\infty \|\mu\| \|\nu\|$$

showing that $\|\mu \Box \nu\| \leq \|\mu\| \|\nu\|$.

Now suppose $x \in G$, then

$$g \mapsto f \cdot \delta_x(g) = \langle R_g f, \delta_x \rangle = R_g f(x) = L_x f(g)$$

is continuous for all $f \in C_b(G)$ if and only if $x \in \Lambda(G)$, so that $\Lambda(G) \subset \mathcal{M}_{Cb}(G)$. If $\sum_{n=1}^{\infty} a_n \delta_{x_n} \subset l_1(\Lambda(G))$, then, for each $f \in C_b(G)$, $f \cdot [\sum_{n=1}^{\infty} a_n \delta_{x_n}] = \sum_{n=1}^{\infty} a_n L_{x_n} f$ is clearly contained in $C_b(G)$, so that $l_1(\lambda(G)) \subset \mathcal{M}_{Cb}(G)$ also holds. Since $e \in \Lambda(G)$, the space is always non-trivial. □

Proposition 5.2. *Let G be a locally compact Hausdorff right topological group. Then, $\mathcal{L}(G)$ is a closed right ideal in $\mathcal{M}_{Cb}(G)$ containing all $\mu \in L_1(G)$ such that $x \mapsto |\mu|(Kx^{-1})$ is continuous for all $K \subset G$ compact.*

Proof. The first part of the proof follows similarly to Lau and Loy. It is easy to check that $L_1(G)$ is closed in $M(G)$. If $f \in \mathcal{L}(G)$ and $\mu \in \mathcal{M}_{Cb}(G)$, then, given any $K \subset G$ compact Borel with $\lambda(K) = 0$, there exists an increasing sequence $\{g_n\} \subset C_c(G)$ such that $g_n \to \mathbf{1}_K$ in L_1 norm of $f \Box \mu$, so that

$$\langle \mathbf{1}_K, f \Box \mu \rangle = \lim_{n \to \infty} \int \int g_n(xy) f(x) d\lambda(x) d\mu(y) \leq \int \int \mathbf{1}_E(x) f(xy^{-1}) d\lambda(x) d\mu(y) = 0$$

It follows that $f \Box \mu \in \mathcal{M}_{Cb}(G)$, i.e. $\mathcal{L}(G)$ is a right ideal of $\mathcal{M}_{Cb}(G)$.

Consider $\mu \in L_1(G)$ such that $x \mapsto |\mu|(Kx^{-1})$ is continuous for all $K \subset G$ compact. By the absolute continuity of $|\mu|$, it is of the form $f\lambda$ for some $f \geq 0$ in $L_1(G)$. For any Borel set $E \subset G$, by regularity of the measure, there exists a sequence of compact sets $\{K_n\}$ contained in E such that $\lambda(E \backslash K_n) \to 0$. Then,

$$\begin{aligned} |\mu(Ey) - \mu(K_n y)| &\leq |\int 1_E(xy^{-1}) - 1_{K_n}(xy^{-1}) f(x) d\lambda(x)| \\ &\leq \int |1_E(xy^{-1}) - 1_K(xy^{-1})| f(x) d\lambda(x) \\ &\leq \int |1_{(E \backslash K)y^{-1}}(x)| f(x) d\lambda(x) \longrightarrow 0 \end{aligned}$$

uniformly for in $y \in G$, and thus, the continuous functions $y \mapsto \mu(K_n y)$ converge uniformly to $y \mapsto \mu(Ey)$. It follows that $y \mapsto \langle \mu, R_y f \rangle$ is continuous for all simple functions f on G, whence for all $f \in L_\infty(G)$. This proves the second claim. □

Suppose G is a locally compact Hausdorff right topological group with a Haar measure λ. Let us consider the **right regular representation of** G, i.e. the representation $G \to B(L_2(G))$, $g \mapsto R_g$. In the topological case, this representation encodes important information on the structure of the group. It is continuous in the weak-operator topology, and its coefficient functions can be used to obtain the Fourier algebra of the group, which uniquely identifies the group up to topological isomorphism (see [9]). The following result shows an important divergence from this classical case. We shall use "WOT" and "SOT" to indicate the weak- and strong- operator topologies respectively, on $B(L_2(G))$.

Theorem 5.3. *If G is a locally compact Hausdorff right topological group with a Haar measure, then the following are equivalent:*

1. *The right regular representation of G is continuous*
2. $C_c(G) \subset \mathcal{M}_{Cb}(G)$
3. *G is topological.*

Proof. Suppose 1 holds and the right regular representation is continuous. Since the inverse map on the unitary operators on a Hilbert space is WOT-WOT continuous, it follows that $G \to B(L_2(G))$, $x \mapsto R_{x^{-1}}$ is WOT continuous, whence SOT continuous by equivalence of the two topologies on the unitaries. Suppose $f \in C_c(G)$ with support $K \subset G$, and $h \in l_\infty(G)$; then, if $y_\alpha \to y$ in G,

$$\begin{aligned}
\left| \int h(x)\ (f(xy_\alpha^{-1}) - f(xy^{-1}))\ d\lambda(x) \right| &\leq \int \|h\|_\infty\ |f(xy_\alpha^{-1}) - f(xy^{-1})|\ d\lambda(x) \\
&= \|h\|_\infty \int \mathbf{1}_{Ky_\alpha \cup Ky}\ |f(xy_\alpha^{-1}) - f(xy^{-1})|\ d\lambda(x) \\
&\leq \|h\|_\infty\ \|\mathbf{1}_{Ky_\alpha \cup Ky}\|_2\ \left\|R_{y_\alpha^{-1}}f - R_{y^{-1}}f\right\|_2 \\
&\leq 2\|h\|_\infty\ \lambda(K)\ \left\|R_{y_\alpha^{-1}}f - R_{y^{-1}}f\right\|_2 \longrightarrow 0
\end{aligned}$$

where the convergence follows from the SOT-continuity of the right regular representation of G. It thus follows that $h \cdot f d\lambda \in C_b(G)$, i.e. $C_c(G) \subset \mathcal{M}_{Cb}(G)$.

Suppose now that 2 holds. We claim that $C_c(G) \subset \mathcal{M}_{Cb}(G)$ makes $LC(G)$ into a set that can separate points from closed sets. Consider any closed set M such that $e \notin M$.

Since $G\backslash M$ is open, there exist strict inclusions

$$e \in K \subset V \subset \overline{V} \subset U \subset \overline{U} \subset G\backslash M$$

where V, U are open neighbourhoods of e, and K, $\overline{V}$, $\overline{U}$ are compact neighbourhoods of e. Since G is locally compact Hausdorff, by Urysohn's lemma, there exist continuous functions $f, p : G \to [0,1]$ such that $\mathbf{1}_K \leq f \leq \mathbf{1}_V$, and $\mathbf{1}_{\overline{V}} \leq p \leq \mathbf{1}_U$ so that $h = 1 - p$ satisfies $\mathbf{1}_{U^c} \leq h \leq \mathbf{1}_{\overline{V}^c}$. Note here that $\overline{V}^c \supset U^c \supset \overline{U}^c \supset M$, so that $\overline{V}^c$ is a neighbourhood of M.

Since $f \in C_c(G)$, the map $\phi : y \mapsto \int h(xy)f(x)d\lambda(x)$ is continuous. Furthermore,

$$\int h(xy)f(x)d\lambda(x) = \int_{V\cap\overline{V}^c y^{-1}} h(xy)f(x)d\lambda(x) \leq \|f\|_\infty \|h\|_\infty \lambda(V \cap \overline{V}^c y^{-1})$$

so that ϕ is non-zero at y implies that $V \cap \overline{V}^c y^{-1} \neq \varnothing$. However, $x \in V \cap \overline{V}^c y^{-1} \implies xy = vy \in \overline{V}^c$ for some $v \in V$, which implies that $y \in V^{-1}\overline{V}^c$, i.e. $\text{supp}(\phi) \subset V \cap \overline{V}^c y^{-1}$. Since $\overline{V}^c \cap V \subset V^c \cap V = \varnothing$, it follows that $e \notin V^{-1}\overline{V}^c$, i.e. $\phi(e) = 0$. On the other hand, if $y \in \overline{U}^c$, $V \cap \overline{V}^c y^{-1} \supset K \cap U^c y^{-1}$ is a neighbourhood of e, and thus, $\lambda(V \cap \overline{V}^c y^{-1}) > 0$, so that $\phi(e) > 0$. In fact, for $y \in \overline{U}^c$,

$$\begin{aligned}\int h(xy)f(x)d\lambda(x) = \int_{V\cap\overline{V}^c y^{-1}} h(xy)f(x)d\lambda(x) &\geq \int_{K\cap\overline{U}^c y^{-1}} h(xy)f(x)d\lambda(x) \\ &= \lambda(K \cap \overline{U}^c y^{-1}) > 0\end{aligned}$$

Suppose, for some $\varepsilon > 0$, $\int h(xy)f(x)d\lambda(x) > \varepsilon$ for all $y \in \overline{U}^c$, whence for all $y \in M$. Then, it is clear that $\phi|_M > \varepsilon$, $\phi(e) = 0$. Thus, ϕ separates e from M. In other words, the set $D = \{h \cdot f d\lambda \mid h \in C_b(G), f \in C_c(G)\} \subset C_b(G)$ separates points from closed sets in G. Note however, that

$$\begin{aligned}L_x[h \cdot f d\lambda](z) = [h \cdot f d\lambda](xz) = \int h(yxz)f(y)\lambda(y) &= \int h(y)f(yz^{-1}x^{-1})\lambda(y) \\ &= \int h(y)R_{x^{-1}}f(yz^{-1})d\lambda(y)\end{aligned}$$

is continuous in z since $R_{x^{-1}}f \in C_c(G)$ still holds. In other words $D \subset LC(G)$, is a set that separates points from closed sets on G. By Theorem 4.5, G is a topological group.

Lastly 3 implies 1 follows from Proposition 2.4. □

In the compact admissible case, it was shown in [28] that no continuous representation of G may be faithful unless G is topological. This was used to prove Theorem 5.3 in [20] for compact groups. We cannot rely on this in our more general setting, and our proof is constructive. Whether faithful representations exist for general locally compact admissible right topological groups remains an open question.

Following Lau and Loy, let us introduce the space

$$\mathcal{L}_c(G) = \{\mu \in M(G) \mid x \mapsto \mu_{x^{-1}} \text{ is norm continuous}\}$$

Then, we obtain the following result.

Proposition 5.4. *Let G be a locally compact Hausdorff right topological group. Then, $\mathcal{L}_c(G)$ is a closed left ideal in $\mathcal{M}_{Cb}(G)$ that is an L-space. Moreover, for all all $\mu \in \mathcal{L}_c(G)$, $y \mapsto \mu(Ey)$ is continuous.*

Proof. It is straightforward to check that $\mathcal{L}_c(G)$ is closed. Suppose $\mu \in \mathcal{L}_c(G)$ and $\nu \in \mathcal{M}_{Cb}(G)$; for any $f \in C_b(G)$, one then has,

$$y \mapsto \langle R_y f, \nu \square \mu \rangle = \int \int f(xzy) d\nu(y) d\mu(z) = \int \int f(xz) d\nu(y) d\mu_{y^{-1}}(x) = \langle R_y f, \nu \square \mu_{y^{-1}} \rangle$$

Uniform continuity of the map then follows by $\|\nu \square \mu_{y^{-1}}\| \leq \|\nu\| \|\mu_{y^{-1}}\|$. Thus, $\mathcal{L}_c(G)$ is a closed left ideal.

Clearly $\mathcal{L}_c(G)$ is a partially ordered vector space under the usual order of $M(G)$. Furthermore, it is a lattice, as $\mu \in \mathcal{L}_c(G)$ gives

$$\| |\mu| - |\mu_{x^{-1}}| \| = ||\mu| - |\mu_{x^{-1}}||(G) = \leq |\mu - \mu_{x^{-1}}|(G) = |\mu - \mu_{x^{-1}}| \to 0$$

as $x \to e$. It follows that $\mathcal{L}_c(G)$ is an L-space (as $M(G)$ is one).

If $\mu \in \mathcal{L}_c(G)$ and $E \subset G$ is Borel, there exists a sequence $\{f_n\} \subset C_c(G)$ such that $f_n \to f$ in μ. We then have,

$$\mu(Ey) = \int 1_{Ey}(x) d\mu(x) = \int 1_E(x) d\mu_{y^{-1}}(x) \leq \|\mu_{y^{-1}}\|$$

so that continuity clearly follows. $\square$

In the compact case, the presented algebras form analogues of the classical algebras on a locally compact topological group, namely the **group algebra** $L_1(G)$, and the **measure algebra** $M(G)$, respectively. In fact on compact topological groups, $\mathcal{L}_c(G) = \mathcal{L}(G) = L_1(G)$ is a two-sided ideal, while $\mathcal{M}(G) = M(G)$. We therefore have the following generalization of Lau and Loy [20].

Corollary 5.5. *Let G be a locally compact Hausdorff right topological group with a Haar measure. If either one of $\mathcal{L}(G) = L_1(G)$, $\mathcal{L}_c(G) = L_1(G)$ or $\mathcal{M}_{Cb}(G) = M(G)$ hold, then G is topological. If G is compact, the converse holds as well.*

Proof. Since $\mathcal{L}(G), \mathcal{L}_c(G) \subset \mathcal{M}_{Cb}(G)$, and $C_c(G) \subset L_1(G) \subset M(G)$, by Theorem 5.3 the forward implication holds. The converse follows in the compact case from $C(G)$ being contained in the uniformly continuous functions on G when it is topological [9]. $\square$

Remark 5.6. We note here that $\mathcal{M}_{Cb}(G)$ is not necessarily equivalent to $M(G)$ in the locally compact topological case. An appropriate analogue of the Measure algebra in this case is yet to be found. Similarly, while $\mathcal{L}_c(G) = \mathcal{L}(G) = L_1(G)$ in the compact topological case, these do not form analogues of the group algebra in the locally compact case.

Since $\mathcal{L}_c(G) = \mathcal{L}(G)$ in the compact topological case, we may ask what elements lie in their intersection in general. Following Lau and Loy [20], let us denote by

$$D(G) = \{f \in C_b(G) \mid g \mapsto L_y f(g^{-1}) \text{ is continuous for all } y \in G\},$$

. Then we get the following result.

Proposition 5.7. *If G is locally compact then, $L_1(G) \cap WAP(G) \cap D(G) \subset \mathcal{L}(G)$; in particular, $L_1(G) \cap AP(G) \cap D(G) \subset \mathcal{L}(G) \cap \mathcal{L}_c(G)$.*

Proof. Suppose $f \in L_1(G) \cap WAP(G) \cap D(G) \subset \mathcal{L}(G) \cap \mathcal{L}_c(G)$. For each $x \in G$, $y \mapsto f(xy^{-1}) = R_{y^{-1}} f(x)$ is continuous. If $y_\alpha \to y$ in G, then, $f \in WAP(G)$ implies that every subnet of $\{R_{y_\alpha^{-1}}f\}$ has a weak cluster point in $C_b(G)$ which, by pointwise continuity coincides with $R_{y^{-1}}f$. It follows that every subnet of $\{R_{y_\alpha^{-1}}f\}$ has a subnet that converges to $R_{y^{-1}}f$. Thus, $y \mapsto R_{y^{-1}}f$ is weakly continuous. A similar proof shows that for $f \in L_1(G) \cap AP(G) \cap D(G)$, $y \mapsto R_{y^{-1}}f$ is uniformly continuous and the latter claim follows. □

5.1 Characterizing the existence of a Haar measure in the compact case

In this section we shall introduce several subspaces of $M(G)$ that form analogues of the measure algebra in the compact topological case. There are clearly connections between these algebras and the topological structure of the group itself. For the topological case, all these algebras coincide. Further, as we shall see, some of these analogues may be used to characterize the existence of a Haar measure.

In this section we assume that G is a compact Hausdorff right topological group unless stated otherwise.

Let us denote by $B(G)$, the set of bounded **Borel functions** on G with the uniform norm. Further, $Ba_\xi(G)$, the bounded **Baire functions of order** ξ, for an ordinal number ξ, are recursively defined to be functions that are pointwise limits of sequences in the preceding classes, where we assume $B(G) = Ba_0(G)$. The last set in this recursion will be denoted by $Ba(G)$, and is referred to as the bounded **Baire functions**. It is important to note that these spaces form Banach algebras and all coincide when the underlying space is second countable. We refer the reader to [5] for further information on these spaces.

Now let us define

- $\mathcal{M}_b(G) = \{\mu \in M(G) \mid f \cdot \mu \in B(G), \text{ for all } f \in C(G)\}$
- $\mathcal{M}_{Ba}(G) = \{\mu \in M(G) \mid f \cdot \mu \in Ba(G), \text{ for all } f \in C(G)\}$
- $\mathcal{M}_B(G) = \{\mu \in M(G) \mid f \cdot \mu \in B(G), \text{ for all } f \in B(G)\}$

Lemma 5.8. *Suppose $f \in Ba(G)$, and $\mu \in \mathcal{M}_b(G)$ then, $f \cdot \nu \in B(G)$. Further, if $\mu \in \mathcal{M}_{Ba}(G)$, then $f \cdot \nu \in Ba(G)$.*

Proof. We shall show the statement for positive functions in $Ba(G)$ as the complex combinations of these generate $Ba(G)$. We shall prove this by transfinite induction on Baire class functions of ordinal $\xi \leq \omega_0$, denoted by $Ba_\xi(G)$. The base case is clear for both cases.

Suppose now that the statment holds for baire classe $Ba_\xi(G)$. Suppose $f \in Ba_{\xi+1}(G)$. By definition, there exists a sequence $\{f_n\} \subset Ba_\xi(G)$ such that $f_n \to f$ pointwise. Without loss of generality, we may assume that f_n are real and bounded, as $Re(f) \to f$ is clear, and further, as the real part of $B(G)$ is a Banach lattice, $Re(f_n) \vee f_n \to Re(f) \vee f = f$. Applying the dominated convergence theorem to $\{R_y f_n\}$ then, one has, $f_n \cdot \mu(y) = \int f_n(xy) d\mu(x) \to \int f(xy) d\mu(x) = f \cdot \mu(y)$. As the sequence $\{f_n\}$ is in $Ba_{xi}(G)$, by our induction assumption, $\{f_n \cdot \mu\} \subset B(G)$ and as $B(G)$ is closed pointwise, it follows that $f \cdot \mu \in B(G)$.

The limit case is similar. Suppose $\xi_0 \leq \omega_0$ is a limit ordinal. Assume that the statement holds for all $\xi < \xi_0$. Then, for each $f \in Ba_\xi(G)$, f is a pointwise limit of some sequence $\{f_n\} \subset \cup_{\xi<\xi_0} Ba_\xi(G)$. By a similar argument to the previous one, we may assume, $\{f_n\}$ is bounded and real. Then, by dominated convergence theorem, $f_n \cdot \mu \to f \cdot \mu$, where the sequence being in $B(G)$ by assumption, converges in $B(G)$.This concludes the proof of the first statement. The second statement follows similarly from the $Ba(G)$ being pointwise closed. □

Proposition 5.9. *Let G be a compact Hausdorff right topological group. Then, under the usual convolution "□",*

- *$\mathcal{M}_B(G)$ is a closed Banach algebra in $M(G)$*
- *$\mathcal{M}_{Ba}(G)$ is a closed Banach algebra in $M(G)$ containing $\mathcal{M}_\sigma(G)$.*
- *$\mathcal{M}_b(G)$ is a left Banach module of $\mathcal{M}_{Ba}(G)$, containing $\mathcal{M}_{Ba}(G)$ and $\mathcal{M}_B(G)$. $M(G)$ is a left Banach module of $\mathcal{M}_b(G)$. If $Ba(G) = B(G)$, then $\mathcal{M}_b(G)$ is a closed subalgebra of $M(G)$. In particular, if G is second countable, this is the case.*

Proof. Note that if $\mu_n \to \mu$ in $M(G)$, then, for any $f \in B(G)$, $f \cdot \mu_n \to f \cdot \mu$ in the sup norm. Since $Ba(G)$, $B(G)$ and $C(G)$, are norm closed, the norm closure of the three spaces is clear.

Now suppose $\mu, \nu \in M(G)$. Then, (not necessarily defined)

$$f \cdot [\mu \Box \nu](x) = \int \int R_x f(yz) d\mu(y) d\nu(z) = \int f \cdot \mu(zx) d\nu(z) = [f \cdot \mu] \cdot \nu(x)$$

If $\mu, \nu \in \mathcal{M}_B(G)$ and $f \in B(G)$, then, $f \cdot \mu \in B(G)$ so that it is clear from the above that $\mu \Box \nu \in \mathcal{M}_B(G)$ and that $\mathcal{M}_B(G)$ is an algebra. If $\mu, \nu \in \mathcal{M}_b(G)$ and $f \in C(G)$, the $f \cdot \mu \in Ba(G)$, so that by the lemma, $[f \cdot \mu \cdot]\nu \in Ba(G)$, proving that $\mathcal{M}_{Ba}(G)$ is an algebra. Similarly, lemma also shows that $\mathcal{M}_b(G)$ is a left module of $\mathcal{M}_{Ba}(G)$. If $\mu \in \mathcal{M}_b(G)(G)$, $\nu \in M(G)$, $\mu \Box \nu(f) = \int f \cdot \mu(x) d\nu(x)$ is clearly well defined for $f \in C(G)$, so that $M(G)$ is a left-module of $\mathcal{M}_b(G)(G)$. Continuity of convolution is easy to check so that the algebras are Banach algebras and the modules are Banach modules. The other claims are easy to see. □

Corollary 5.10. *If G is Borel, then $M(G) = \mathcal{M}_b(G)$ is a Banach algebra.*

Proof. Let Inv denote the inverse map on G. Suppose G is Borel. Then, $(x, y) \mapsto x^{-1}y$ is a jointly Borel map on $G \times G$. Now this gives us Borel measurability of $x \mapsto x^{-1}$ (set $y = e$). Thus, for any $U \times V \subset G \times G$ open, one has, $[Inv \times Id_G]^{-1}(U \times V) = Inv(U) \times V$ is Borel so that, $Inv \times Id_G$ is a Borel map on $G \times G$. Thus we have that the map $(x, y) \mapsto (x^{-1}, y) \mapsto xy$ is Borel as a composition of Borel maps.

For $f \in C(G)$ and $\mu \in M(G)$, $f \cdot \mu(x) = \int \int f(xy) d\mu(y)$. As the map $x \mapsto f(xy)$ is Borel, by Fubini's theorem, one obtains that the $f \cdot \mu$ is Borel, thus, $M(G) = \mathcal{M}_b(G)$. The proof is then concluded by noting that $M(G)$ is a left module of itself by Proposition 5.9. □

We see that Borel properties are non-trivial and interesting in the right topological case. Further exploration is necessary to see how these characterize the topological properties of the group.

Let us move on to introducing another measure algebra. Inspired by Lau and Loy's work, we define,

$$\mathcal{M}_w(G) = \{\mu \in \mathcal{M}(G) \mid \mu_G \text{ is relatively weakly compact}\}$$

Note that $\mathcal{L}_c(G) \subset \mathcal{M}_w(G) \subset \mathcal{M}(G)$.

Proposition 5.11. *$\mathcal{M}_w(G)$ is a closed, left ideal of $\mathcal{M}(G)$ containing $\mathcal{L}_c(G)$ and left and right invariant under $\Lambda(G)$.*

Proof. It is obvious that $\mathcal{M}_w(G)$ is a subspace of $\mathcal{M}(G)$. Suppose $\{\mu_n\} \subset \mathcal{M}_w(G)$, and $\mu_n \to \mu$. We shall show that μ_G is relatively weakly compact. We shall use Grothendieck's double limit theorem (see Theorem A.5 in [3]) - suppose for a bounded sequence $\{T_n\} \subset M(G)^*$, $\lim_n \lim_m T_n(\mu_(g_m))$ and $\lim_n \lim_m T_m(\mu_(g_n))$ exist.

Note that, for each $T \in M(G)^*$ and $g \in G$, $T \circ R_g^* \in M(G)^*$ and

$$T(\mu_(g)) - T((\mu_k)_g) = TR_g^*(\mu - \mu_k) \leq \|T\|\|\mu - \mu_k\| \to 0$$

uniformly over $g \in G$, and uniformly over T for T contained in some normed bounded subset of $M(G)^*$.

Using this, we have,

$$\begin{aligned} \lim_n \lim_m T_n R_{g_m}^*(\mu) &= \lim_n \lim_m [T_n R_{g_m}^*](\lim_k \mu_k) \\ &= \lim_k \lim_n \lim_m [T_n R_{g_m}^*](\mu_k) \\ &=^{(*)} \lim_k \lim_m \lim_n [T_n R_{g_m}^*](\mu_k) \\ &= \lim_m \lim_n \lim_k [T_n R_{g_m}^*](\mu_k) \\ &= \lim_m \lim_n [T_n R_{g_m}^*](\mu) \end{aligned}$$

where we used at $(*)$ the fact that for each k, $[\mu_k]_G$ is weakly compact, and that $\lim_n \lim_m [T_n R_{g_m}^*](\mu_k)$ and $\lim_m \lim_n [T_n R_{g_m}^*](\mu_k)$ exist (since the limit of each iterated limit exists as $k \to \infty$). The claim thus follows.

Suppose now that $\mu \in \mathcal{M}_w(G)$, $\nu \in \mathcal{M}(G)$, then, and $K \subset G$ is an arbitrary non-empty set so that for each $x \in K$, $T \in M(G)^*$,

$$\langle [\mu \Box \nu]_{x^{-1}}, T \rangle = \langle \mu \Box \nu_{x^{-1}}, T \rangle = \langle \nu_{x^{-1}}, T \Box \mu \rangle$$

Since ν_G is relatively weakly compact, $\{\nu_{x^{-1}} \mid x \in K\}$ has a weak cluster point, say $\gamma \in M(G)$. Then, $\mu \Box \gamma$ is clearly a weak cluster point of $\{[\mu \Box \nu]_{x^{-1}} \mid x \in K\}$. It follows that $[\mu \Box \nu]_G$ is relatively weakly compact and $\mu \Box \nu \in \mathcal{M}_w(G)$.

That $\mathcal{L}_c(G)$ is contained in $\mathcal{M}_w(G)$ is easy to see - for each $\mu \in \mathcal{L}_c(G)$, $x \mapsto \mu_{x^{-1}}$ is norm continuous, thus weakly continuous, so that the image of the map, μ_G is weakly compact.

As $\mathcal{M}(G)$ is closed under right translation by $\Lambda(G)$, and this does not change the right orbit of an element of $\mathcal{M}(G)$, $\mathcal{M}_w(G)$ is closed under right translation. On the other hand, if $\mu \in \mathcal{M}_w(G)$, and $T \in M(G)^*$, for each $g \in G$, $T(g^{-1}\mu) = T \circ L_g^* \in M(G)^*$, one sees that weak convergence of a net in μ_G implies weak convergence of a net in $[{}_x^{-1}\mu]_G$. Thus, ${}_{g^{-1}}\mu \in G$. ∎

We now obtain an important characterization of the existence of a Haar measure.

Proposition 5.12. *The following are equivalent:*

1. *G has a Haar measure*
2. *$\mathcal{L}_c(G)$ contains a non-zero measure*
3. *$\mathcal{M}_w(G)$ contains a non-zero measure.*

Proof. That $1 \implies 2$ and $2 \implies 3$ is trivial. We prove $3 \implies 1$.

Suppose $\mu \in \mathcal{M}_w(G)$ is non-zero. Then, by scaling, we may assume $\mu(G) = 1$. By assumption the weak closure of μ_G, is compact, thus, by Krein Smulian, it follows that it's closed convex hull, K, is also weakly compact. Note that for each $g \in G$, $\mu_g(G) = 1$, and thus, for all elements in $\nu \in K$, $\nu(G) = 1$.

We claim that K is closed under translations. Indeed, suppose $\nu \in K$, $h \in G$ and let $\varepsilon > 0$. For each $T \in M(G)^*$ exists some convex combination $\sum_{i=1}^n \alpha_i \mu_{g_i}$ such that $|\langle TR_h^*, \nu - \sum_{i=1}^n \alpha_i \mu_{g_i}, |\rangle < \varepsilon$. Then, for any $h \in G$, $|\langle T, \nu_h - \sum_{i=1}^n \alpha_i (\mu_{g_i})_{h^{-1}} \rangle| = |\langle T, \nu_h - \sum_{i=1}^n \alpha_i \mu_{h^{-1}g_i} \rangle| < \varepsilon$. As ε was arbitrary, it follows $\nu_h \in K$.

Now G acts on $M(G)$ isometrically i.e. for each $g \in G$, $\mu \in M(G)$, $\mu \mapsto \mu_{g^{-1}}$ is a continuous isometry. Thus G is a group of isometries on K. By Ryll-Nardzewski fixed point theorem, K has a fixed point of G. This is clearly a Haar measure on G. The converse is trivial. □

The main takeaway of this result is that the existence of a Haar measure seems equivalent to the existence of a measure with pleasant topological properties.

One of the problems with the previous proposition is that, we are unsure apriori of what elements are contained in $\mathcal{M}_w(G)$. We do however have $C(G, \sigma) d\lambda \subset \mathcal{L}_c(G)$ when G has a Haar measure λ.

We now define,

$$\mathcal{M}_\sigma(G) = \{\mu \in M(G) \mid f \cdot \mu \in C(G, \sigma) \text{ for each } f \in C(G)\}$$

Clearly $\mathcal{M}_\sigma(G) \subset \mathcal{M}(G)$. If G has a Haar measure and thus, $L_1(G)$ is defined, let us set $\mathcal{L}_\sigma(G) = \mathcal{M}_\sigma(G) \cap L_1(G)$.

Proposition 5.13. *$\mathcal{M}_\sigma(G)$ is a closed left-ideal of $\mathcal{M}(G)$ containing the Haar measure on G (if it exists). Furthermore, $\mathcal{M}_\sigma(G)$ and $\mathcal{L}_\sigma(G)$ are closed under right translation.*

Proof. Let us first show that $\mathcal{M}_\sigma(G)$ is closed. Indeed, suppose $\{\mu_\alpha\}_{\alpha \in A} \subset \mathcal{M}_\sigma(G)$ such that $\mu_\alpha \to \mu$, $\mu \in \mathcal{M}(G)$. Then, for any $f \in C(G)$, $\{x_\beta\}_{\beta \in B} \subset G$ such that the convergence $x_\beta \to^\sigma x \in G$ holds in the σ topology,

$$\begin{aligned}
&|f \cdot \mu(x_\beta) - f \cdot \mu(x)| \\
&= \left| \int f(yx_\beta) - f(yx) d\mu(y) \right| \\
&\leq \left| \int f(yx_\beta) d(\mu - \mu_\alpha)(y) \right| + \left| \int f(yx_\beta) - f(yx) d\mu_\alpha(y) \right| + \left| \int f(yx) d(\mu - \mu_\alpha)(y) \right|
\end{aligned}$$

$$\leq 2\|f\|_\infty \|\mu - \mu_\alpha\| + |f \cdot \mu_\alpha(x) - f \cdot \mu_\alpha(x_\beta)|$$

Fix $\varepsilon > 0$. Then, choosing an appropriate $\alpha \in A$ so that the former term is less than $\varepsilon/2$ and an appropriate β for α so that the latter term is less than $\varepsilon/2$ (as $f \cdot \mu_\alpha$ is in $C(G, \sigma)$), we have a bound of ε above which gives us our claim.

To see that $\mathcal{M}_\sigma(G)$ is a left ideal, suppose that $\mu \in \mathcal{M}(G)$ and $\nu \in \mathcal{M}_\sigma(G)$. For any $f \in C(G)$, we have,

$$[\mu \Box \nu] \cdot f(x) = \langle \mu \Box \nu, R_x f \rangle = \langle \nu, R_x f \cdot \mu \rangle = \langle \nu, R_x(f \cdot \mu) \rangle = [f \cdot \mu] \cdot \nu(x)$$

Since $f \cdot \mu \in C(G)$, it follows that $[\mu \Box \nu] \cdot f \in C(G, \sigma)$. This proves our second claim. Note that if λ is a Haar measure on G, then for any $f \in C(G)$, $f \cdot \lambda$ is always a constant and thus, trivially in $C(G, \sigma)$.

We will show the last claim for $\mathcal{M}_\sigma(G)$ and the claim for $\mathcal{L}_\sigma(G)$ will simply follow by absolute continuity of translations of its elements. Suppose $\mu \in \mathcal{M}_\sigma(G)$, $x, g \in G$, and $f \in C(G)$,

$$f \cdot \mu_g(x) = \int f(yx) d\mu_g(y) = \int f(yg^{-1}x) d\mu(y) = \int R_{g^{-1}x} f(y) d\mu(y) = f \cdot \mu(g^{-1}x)$$

Thus, $f \cdot \mu_g = R_g^{-1}[f \cdot \mu]$. Now as $f \cdot \mu \in C(G, \sigma)$ and (G, σ) is a semitopological group, it follows that $f \cdot \mu_g \in C(G, \sigma)$ which concludes our claim. □

For any right topological group G, we denote by G_d, the group with the discrete topology. The **almost periodic (weakly almost periodic)** functions on G_d, $AP(G_d)$ ($WAP(G_d)$), are defined to be those bounded functions whose orbit $\{R_g f \mid g \in G\}$ is relatively compact (relatively weakly compact). Then, the almost periodic (weakly almost periodic functions) on G are simply defined by $AP(G) = C(G) \cap AP(G_d)$ ($WAP(G) = C(G) \cap AP(G_d)$).

In [20], it was shown that, for admissible groups, $WAP(G) = AP(G) = C(G, \sigma)$. Using this, we have the following result.

Proposition 5.14. *If G is admissible, then $\mathcal{M}_w(G) \subset \mathcal{M}_\sigma(G)$.*

Proof. By Lau and Loy, if G is admissible, then $WAP(G) = AP(G) = C(G, \sigma)$. Suppose $\mu \in \mathcal{M}_w(G)$, then, μ_G is relatively weakly compact. Thus, for each $f \in C(G)$,

$$\langle \mu_{g^{-1}}, \nu \cdot f \rangle = \int \int f(xgy) d\mu(x) d\nu(y) = \int L_g[f \cdot \mu](y) d\nu(y) = \langle L_g[f \cdot \mu], \nu \rangle$$

As μ_G is relatively weakly compact, for each subset K of G, $\{\mu_{g^{-1}} \mid g \in K\}$ has a weak limit point, thus $\{\langle \mu_{g^{-1}}, \nu \cdot f \rangle \mid g \in K\}$ has a cluster point over $\nu \in M(G)$, and in particular over $\{\delta_g \mid g \in G\}$. It follows that $\{L_g[f \cdot \mu] \mid g \in G\} \subset l_\infty(G_g)$ is relatively weakly compact in $C(G_d)$, where G_d is G with the discrete topology. In other words, $f \cdot \mu \in WAP(G_d)$ (the left and right weakly almost periodic functions coincide on a topological group, see [3]). As $WAP(G) = WAP(G_d) \cap C(G)$, it follows that $f \cdot \mu \in C(G, \sigma)$, i.e. $\mu \in \mathcal{M}_\sigma(G)$. □

Consider the flow (ρ_G, G), where ρ_G denotes the right translation action of G on itself. An equicontinuous right topological group is defined to be a right topological group G, satisfying the property that it's right flow is equicontinuous i.e. all the maps $\rho_G(g)$, $g \in G$ are equicontinuous. Milnes showed that G is equicontinuous if and only if $AP(G) = WAP(G) = C(G)$ (see [30]). As such, for such groups the previous result looks very different - and we note that our proof of Proposition 5.11 is not redundant to the following result.

Theorem 5.15. *G has a right invariant Haar measure if and only if $\mathcal{M}_\sigma(G)$ contains a non-zero measure.*

Proof. Suppose $\mu \in \mathcal{M}_\sigma(G)$ is non-zero. Then, by scaling it, we may assume that $\mu(G) = 1$. Consider $K = \{\mu_g \mid g \in G\}$. Since $g \mapsto \mu_g$ is w*-continuous, it follows that K is compact. It is clear that K is invariant under right translations of G and that $K \subset \mathcal{M}_\sigma(G)$.

Consider $C = \overline{\text{conv}} K$. Note that since the set is bounded by $\|\mu\|$ and is w*-closed, by the Banach-Alaogu theorem, it is compact. Furthermore, $K \subset C$. Note that R_g still maps C into C for each $g \in G$, and furthermore, that this map is w*-w* -continuous.

It is clear that right translates by G are affine maps on C. We will now proceed to show that right translates by G are also distal on K. Indeed, suppose $\lim_{\alpha\in A}\mu_{g_\alpha} = \lim_{\alpha\in A}\nu_{g_\alpha}$ for some $\mu, \nu \in K$ and some $\{g_\alpha\}_{\alpha\in A} \subset G$.

Since G is compact, $\{g_\alpha^{-1}\}_{\alpha\in A}$ has some subnet $\{g_\beta^{-1}\}_{\beta\in A}$ that converges to some $g^{-1} \in G$. Then, we have,

$$\lim_{\beta\in B}\langle\mu_{g_\beta}, f\rangle = \lim_{\beta\in B}\int f(xg_\beta{}^{-1})d\mu(x) = f\cdot\mu(g_\beta^{-1}) \to f\cdot\mu(g^{-1}) = \langle\mu_g, f\rangle$$

It follows that $\mu_{g_\beta} \to \mu_g$ in the w*-topology. Similarly, $\nu_{g_\beta} \to \nu_g$ in the w*-topology. By $M(G)$ being Hausdorff, we have, in the weak* topology,

$$\lim_{\alpha\in A}\mu_{g_\alpha} = \lim_{\alpha\in A}\nu_{g_\alpha} = \lim_{\beta\in B}\mu_{g_\beta} = \lim_{\beta\in B}\nu_{g_\beta} = \mu_g = \nu_g$$

and thus, $\mu = \nu$.

By Namioka's fixed point theorem then, G has a fixed point in the w*-closed convex hull of K. Since $\mu(G) = 1$, all the elements ν of C satisfy $\nu(1) = 1$. It therefore follows that G has a right invariant Haar measure. □

Proposition 5.16. *Let G be a compact hausdorff right topological group. For $\mu \in \mathcal{M}(G)$, if $R_G\mu \in \mathcal{M}(G)$, then $\mu \in \mathcal{M}_\sigma(G)$.*

Proof. Suppose $\mu \in \mathcal{M}_\sigma(G)$ is such a measure. Then, one has, $f\cdot\mu_x$ is continuous for all $x \in G$. However, $f\cdot\mu_x = L_{x^{-1}}f\cdot\mu$. It thus follows that $f\cdot\mu \in C_c(G) = C(G/N(G))$, so that μ_x is $N(G)$ invariant and in $\mathcal{M}_\sigma(G)$. □

We conclude by summarizing the main theme of this chapter: if a compact right topological group has a non-trivial measure with nice enough topological/continuity properties, then it also possesses a Haar measure. This explains why the non-existence of a Haar measure has been so difficult to prove for right topological groups. In particular the various examples that have been constructed, are some type of twisted (Schreier- see [29]) product of topological groups, and these underlying groups possess Haar measures.

Chapter 6

Hereditary properties of G

Every locally compact topological group possesses a Haar measure. As such, hereditary properties are not sought after in the classical case. However, if one looks instead at the the existence of invariant means on G, i.e. its amenability, hereditary properties do come into play, as the existence of such means is non-trivial for substructures. Every compact topological group in particular, is amenable, as its Haar measure is an invariant mean. It might thus make sense to view compact (and in general locally compact) right topological groups possessing a Haar measure to be satisfying a property analogous to amenability. Amenable locally compact groups have pleasant hereditary properties with amenability passing down to quotients, closed subgroups and directed unions.

The following are some of the easy analogues of this kind that follow; we refer the reader to [40] for the proofs in the topological case;

Proposition 6.1. *Let G be a locally compact Hausdorff right topological group with a right invariant Haar measure. Then,*

1. *Every quotient group of G has a right invariant Haar measure*
2. *If a dense subgroup of G has a Haar measure in $\mathcal{M}(G)$, then so does G.*
3. *If G is compact and a directed union of right topological groups with Haar measures, then G has a Haar measure.*
4. *If G is the image of a right topological group with a Haar measure, then G has a Haar measure.*

5. *If G is the Schreier product of two compact topological groups, then G has a Haar measure.*

Proof. 1 follows trivially by considering the canonical map $C_0(G) \mapsto C_0(G/K)$, for a normal subgroup K.

For the proof of 2, let $H \subset G$ be a dense subgroup with a Haar measure $\lambda_H \in \mathcal{M}(G)$. Then, for any $x \in G$, there exists some net $\{x_\alpha\} \subset H$ converging to x, so that for $f \in C_b(G)$.

$$\int_H R_x f(g) d\lambda_H(g) = \lim_\alpha \int_H R_{x_\alpha} f(g) d\lambda_H(g) = \lim_\alpha \int_H f(g) d\lambda_H(g)$$

so that λ_H is a Haar measure on G.

Statement 3 follows from a standard proof (see Theorem 2.4 [41]) and 4 is similarly easy to check.

Lastly, 5 follows by taking the product of the Haar measures on the underlying groups. □

Here "Schreier product" refers to a generalization of a semidirect product that ensures that the group is right topological. We refer the reader to [29] for details.

We now consider the existence of a Haar measure generalizing from a subgroup to the whole group.

Theorem 6.2. *Suppose G is a σ-locally compact admissible or compact Hausdorff right topological group and $H \subset \Lambda(G)$ is a normal compact topological subgroup. Then, $\lambda_H \in \mathcal{M}(G)$. In the compact case, if H is $\sigma\sigma$-closed, then $\lambda_H \in \mathcal{M}_\sigma(G)$ and G has a Haar measure.*

Proof. As multiplication is separately continuous on H, by Ellis' theorem, H is a compact topological group. Therefore, H has a unique invariant Haar measure and we may define λ_H as above.

We will show that $\lambda_H \in \mathcal{M}(G)$. Firstly, note that $H \times G \to G$, $(h,g) \mapsto hg$ is separately continuous. By Namioka's theorem, it is jointly continuous so that for any $f \in C(G)$, $H \times G \to \mathbb{C}$, $(h,g) \mapsto f(hg)$ is jointly continuous. We claim that $\{L_h f \mid h \in H\} \subset C(G)$ are equicontinuous on G.

As $(h,g) \mapsto f(hg)$ is continuous at every $(m, g_0) \in H \times G$, there exist neighbourhoods $U_m \times V_m \subset H \times G$ of (m, g_0), such that for all $(h,g) \in U_m \times V_m$, $|f(hg) - f(mg_0)| < \varepsilon$. As $\{U_m\}_{m \in H}$ is an open cover for H, there exists a finite subcover $\{U_{m_i}\}_{i=1}^k$. Consider $V = \cap_{i=1}^n V_{m_i}$, also a neighbourhood of g_0. Then, for any $m \in H$, $m \in U_{m_i}$ for some $1 \leq i \leq n$. Thus, for every $g \in V$, $|f(mg) - f(mg_0)| < \varepsilon$, and as this holds for every $m \in H$, $\sup_{m \in H} |f(mg) - f(mg_0)| < \varepsilon$. It follows that, $\sup_{m \in H} |f(mg) - f(mg_0)| \to 0$ as $g \to g_0$. This proves the claim.

Now as $\{L_n f\}_{n \in H}$ are equicontinuous, note that,

$$f \cdot \lambda_H(y) = \int_H R_y f|_H(x) d\lambda(x) = \int_H f|_{x \in H}(xy) d\lambda(x) = \int_H L_x f(y) d\lambda(x)$$

and $y_\alpha \to y$ implies $L_x f(y_\alpha) \to L_x f(y)$ for all $x \in H$, thus, $|L_x f(y_\alpha) - L_x f(y)| \to 0$ for all $x \in H$, and

$$|f \cdot \lambda_H(y) - f \cdot \lambda_H(y_\alpha)| = \left| \int_H L_x f(y) d\lambda(x) - \int_H L_x f(y_\alpha) d\lambda(x) \right| \to 0$$

Thus, $f \cdot \lambda \in C(G)$ and $\lambda \in \mathcal{M}(G)$.

Now note that for $n \in H$ and $f \in C(G)$,

$$\begin{aligned} f \cdot \lambda(ny) = \int_H R_{ny} f|_H(x) d\lambda_H(x) = \int_{Hn} R_y f|_{Hn}(x) d\lambda_H(xn^{-1}) &= \int_{Hn} R_y f|_{Hn}(x) d\lambda_H(xn^{-1}) \\ &= \int_H R_y f|_H(x) d\lambda(x)_H(x) \\ &= f \cdot \lambda(y) \end{aligned}$$

Thus, $f \cdot \lambda(y) \in C(G/H)$.

If H is $\sigma\sigma$-closed, then, by Lemma 3.12, $H \supset N(G)$ - thus for each $f \in C(G)$, $f \cdot \lambda_H \in C(G/N(G))$. As $C(G, \sigma) = C(G/N(G))$ by Lau and Loy, $\lambda_H \in \mathcal{M}_\sigma(G)$. A Haar measure then exists by Theorem 5.15. □

Corollary 6.3. *If G is σ-locally compact admissible or compact, and has a Haar measure, for every compact subgroup $H \subset \Lambda(G)$, it has a left-H invariant right Haar measure. In particular, if $\Lambda(G)$ is closed, then, G has a two-sided invariant Haar measure.*

Proof. By Ellis' theorem (see [7]), being semitopological, H is a compact topological group and thus, has a Haar measure. By Theorem 6.2 then, $\lambda_H \in \mathcal{M}(G)$. Given a Haar measure λ on G, $\lambda_H \Box \lambda$ then gives the desired measure. □

Example 6.4. Consider the group $G = \mathbb{T} \times \{1, \phi\}$ with the multiplication $(u, \varepsilon)(v, \delta) = (u\varepsilon(v), \varepsilon\delta)$ from Example 3.13. This satisfies $\Lambda(G) = N(G) = \mathbb{T} \times \{1\}$, so that $\lambda_{N(G)} \subset \mathcal{M}(G)$ by Theorem 6.2.

Proposition 6.5. *Let G be a σ-locally compact admissible or compact Hausdorff right topological group. If G has a $\sigma\sigma$-closed, compact normal subgroup $H \subset \Lambda_b(G)$ such that H is metrizable and has a right invariant Haar measure, then G has a Haar measure.*

Proof. Suppose $H \subset \Lambda_b(G)$ is as given. By Lemma 3.12, $(G/H, \tau)$ is a locally compact topological group and thus has a Haar measure, say, $\lambda_{G/H}$. Let λ_H be a Haar measure on H.

Fix $f \in C(G)$. Note that the map $H \times G \to \mathbb{C}$, $(x, y) \mapsto f(xy)$ is continuous in the first variable, and Borel in the second variable (since $y \mapsto f(xy)$ is the composition of Borel map $y \mapsto xy$ and continuous map f). Furthermore,H being compact Hausdorff metrizable is separable. Thus, by a standard result in measure theory (see 4.51 of [1]), the map is jointly measurable. Applying Fubni's theorem to the clearly bounded function, $y \mapsto F(y) = \int f|_H(xy) d\lambda_H(x)$ is bounded Borel measurable. It is also clear that this functions is H-invariant. We may thus integrate it with respect to $\lambda_{G/H}$, and define $\lambda \in M(G)$ in the standard way: $\int f(x) d\lambda(x) = \int \int f|_H(xy) d\lambda_H(x) \lambda_{G/H}(x)$. As $\lambda_{G/N(G)}$ is a G-invariant Haar measure, one easily notes that λ is a Haar measure on G. □

Remark 6.6. Note that the above proof still works if one assumes that G/H has a Haar measure instead of H being $\sigma\sigma$ closed. Morever, if H is a compact metrizable topological group, the hypothesis is met. While the hypothesis on H is strong, we note that since admissibility is not assumed, Theorem 3.5 does not come into play - therefore, H may be non-trivially right topological.

Chapter 7

Open Problems and Future Work

Due to one-sided continuity of multiplication, working with right topological groups presents new challenges when generalizing ideas that work well for locally compact topological groups. Moreover, a small list of examples that are challenging to work with makes it difficult to empirically observe the properties of these groups for inspiration. In this chapter we present some open problems in the literature, and some that we have encountered ourselves. Some of these are fundamental in the sense that the analogues in the locally compact case hold quite trivially. We present these in a chapter-wise format, summarizing the contents of the chapter along the way.

7.1 Chapter 2: Locally compact right topological groups and the Haar measure

Compact topological groups and more generally CHART groups always have a unique Haar measure. However, due to the lack of counterexamples, the following problem is still open.

Problem 7.1. If a (locally) compact right topological group G possesses a right Haar measure, is it always unique (up to scalar multiplication)?

We suspect that in light of Kunen's work [19], it may be possible to construct a non-CHART group where the above fails. Further,

Problem 7.2. If a (locally) compact right topological group G possesses a right Haar measure, when is it left $\Lambda(G)$-invariant?

We were able to show in Corollary 6.3 that if G is compact, $\Lambda(G)$ being closed guarantees the existence of such a Haar measure.
In [31], the existence of a strong normal system was proven for CHART groups. Here, we were unable to generalize their result. We thus pose;

Problem 7.3. Does a σ-locally compact admissible right topological group possess a strong normal system of subgroups? Is this always compact (what are sufficient conditions for this)?

Since the construction of a strong normal system of subgroups in [31] is done by considering $N(G)$ repeatedly via transfinite induction, we may generalize the second question as follows;

Problem 7.4. If G is locally compact admissible right topological, is $N(G)$ compact?

In the compact case, since (G, σ) has a weaker topology and is thus compact, every closed set in this space is compact. However, in the locally compact case, we cannot deduce this. Further, while $N(G)$ is the intersection of closed sets in σ, since (G, σ) is non-Hausdorff, compact sets need not be closed and we obtain no information on the compactness of $N(G)$. The examples we have encountered (3.13, 3.14) do have $N(G)$ compact. A counterexample to this problem is thus currently unknown.

7.2 Chapter 3: Function Algebras

The first problem we pose is as follows

Proposition 7.5. *Can Lemma 4.1 be generalized to invariant subalgebras of $C_b(G)$?*

The two cases naturally coincide in the compact case. However, the result relies on the Stone-Weierstrauss theorem which is only applicable to $C_0(G)$ in the non-compact case (a generalization of this is available to $C_b(G)$ but did not work out for us). If a generalization exists, we may give results for $A(G) \subset C_b(G)$, since Lemma 4.1 is not applicable due to 5 of Theorem 4.5.

Again, in light of theorem 4.5, we pose the following problems

Problem 7.6. Can $LC(G)$, $AP(G)$ separate points of G if G is locally compact non-compact and admissible?

In the compact case, the answer to this problem is in the negative as shown in [20].

In the section concerning Fourier Algebras, we were not able to derive much information about $B(G)$. In particular, we do not have a strong result such as theorem 4.6. Therefore,

Problem 7.7. If G is non-compact, locally compact admissible, can $B(G)$ separate points of G? Are there subalgebras of $B(G)$ that correspond to faithful representations of G?

Note that $A(G)$ may never separate points of G so that it does not fulfill the above result (Corollary 4.7).

7.3 Chapter 4: Measure Algebras

Many open problems posed on $\mathcal{M}(G)$ in [20] still remain unanswered. A fundamental one is as follows;

Problem 7.8. Suppose G is a compact right topological group. Is $\mathcal{M}(G)$ a dual Banach space? More generally, are any of our measure algebra analogues dual spaces?

In the topological case, $\mathcal{M}(G) = M(G) = C(G)^*$ (in fact all our measure algebra analogues coincide with $M(G)$). In the admissible case, one easily notes that $\mathcal{M}(G)$ is not weak* closed. Indeed, $\{\delta_g \mid g \in \Lambda(G)\} \subset \mathcal{M}(G)$, and weak*-closedness would imply that $\{\delta_g \mid g \in G\} \subset \mathcal{M}(G)$, which is false unless G is topological.

Another problem we may raise is regarding Theorem 5.3. In the compact case, it was shown in [28] that a right topological group has a continuous faithful representation if and only if it is topological.

Problem 7.9. Can a faithful continuous representation exist on a locally compact right topological group?

In general, we may raise the following problems regarding the algebras $\mathcal{L}(G)$, $\mathcal{L}(G)$;

Problem 7.10. Let G be a locally compact right topological group.

1. When is $\mathcal{L}_c(G) = \mathcal{L}(G)$?

2. When do $\mathcal{L}_c(G)$, $\mathcal{L}(G)$ possess abounded approximate identity?

3. When are $\mathcal{L}(G)$, $\mathcal{L}_c(G)$ amenable?

4. When is $\mathcal{L}(G)$ an L-space?

When G is compact topological, it is well-known that $\mathcal{L}_c(G) = \mathcal{L}(G) = L_1(G)$ (see [42]).

Regarding 2, when G is topological, $L_1(G)$ naturally possesses a bounded approximate identity (see Proposition 2.4 and the discussion prior). However, it was shown in [20] that this construction does not work on right topological groups.

Although amenability is an important topic, we have not delved into this in this thesis. This is partly due to the lack of results in the right topological case. A Banach algebra A is said to be amenable if every derivation $\Delta : A \to B^*$, for a Banach A-bimodule B, Δ is inner, i.e. of the form $a \mapsto ax - xa$, for some $x \in B^*$. Johnson famously showed that a locally compact group G is amenable (i.e. has a left-invariant mean on $L_\infty(G)$, if and only if $L_1(G)$ is amenable [16]. Every compact topological group is amenable (as its Haar measure is the specified mean) and thus has an amenable group algebra. On the other hand, every amenable Banach algebra must possess a bounded approximate identity (see Proposition 2.2.1 of [43]), so that 2 and 3 are related open problems.

Lastly, although $\mathcal{L}_c(G)$ is an L-space, we do not know if the same holds for $\mathcal{L}(G)$.

Problem 7.11. What can we say about the structure of the Banach algebra $\mathcal{M}_b(G) = M(G)$ when G is Borel (Corollary 5.10)?

Since the algebras $\mathcal{M}_\sigma(G)$, $\mathcal{M}_w(G)$ characterize the existence of a Haar measure, it is fundamental to ask:

Problem 7.12. When are $\mathcal{M}_\sigma(G)$, $\mathcal{M}_w(G)$ non-trivial (apriori to the existence of a Haar measure)? What sufficient conditions ensure this?

7.4 Chapter 5: Hereditary Properties

Concerning hereditary properties, the most natural question we may raise is as follows

Problem 7.13. If a right topological group G has a Haar measure, do its closed normal subgroups also possess one? What about open subgroups?

Since it is hard to produce right topological groups that do not possess a Haar measure, this problem naturally remains open. In the case of amenability, even in the non-locally compact topological case, the property passes down to at least open subgroups (see [41]). Unfortunately this proof does not generalize well to our setting.

While we gave a sufficient condition Theorem 6.2, in general;

Problem 7.14. For a closed subgroup $H \subset G$, when is $\lambda_H \in \mathcal{M}(G)$?

Also, since $G/N(G)$ is always topological, it is natural to ask when we may leverage this;

Problem 7.15. If G/H is non-trivial and has a Haar measure for a σ-closed normal $H \subset G$, does G possess a Haar measure?

Problem 7.16. If H, G/H are topological, what conditions ensure that G is topological?

This problem is listed in [20].

For more open problems, the reader is referred to [20], [21].

We hope to answer some of these open questions in our future work.

Bibliography

[1] C. D. Aliprantis and K. C. Border. *Infinite Dimensional Analysis: A Hitchhiker's Guide*. Springer Science & Business Media, 2013.

[2] J. Baker, N. Hindman, and J. Pym. κ-topologies for right topological semigroups. *Proc. Amer. Math. Soc.*, 115(1):251–256, 1992.

[3] J. F. Berglund, H. D. Junghenn, and P. Milnes. *Analysis on semigroups*. Canadian Mathematical Society Series of Monographs and Advanced Texts. John Wiley & Sons, Inc., New York, 1989. Function spaces, compactifications, representations, A Wiley-Interscience Publication.

[4] R. B. Burckel. *Weakly almost periodic functions on semigroups*. Gordon and Breach Science Publishers, New York-London-Paris, 1970.

[5] H. G. Dales, A. T.-M. Lau, and D. Strauss. Second duals of measure algebras. *Dissertationes Math.*, 481:1–121, 2012.

[6] R. Ellis. Locally compact transformation groups. *Duke Mathematical Journal*, 24(2):119–125, 1957.

[7] R. Ellis. Distal transformation groups. *Pacific Journal of Mathematics*, 8(3):401–405, 1958.

[8] R. Ellis. The Furstenberg structure theorem. *Pacific Journal of Mathematics*, 76(2):345–349, 1978.

[9] G. B. Folland. *A Course in Abstract Harmonic Analysis : A Course in Abstract Harmonic Analysis*. Chapman and Hall/CRC, Feb. 2016.

[10] H. Furstenberg. The structure of distal flows. *American Journal of Mathematics*, 85(3):477–515, July 1963.

[11] F. P. Greenleaf. *Invariant means on topological groups and their applications*. Van Nostrand Reinhold Co., 1969.

[12] R. Grigorchuk and P. de la Harpe. Amenability and ergodic properties of topological groups: from Bogolyubov onwards. In *Groups, graphs and random walks*, volume 436 of *London Math. Soc. Lecture Note Ser.*, pages 215–249. Cambridge Univ. Press, Cambridge, 2017.

[13] F. Hahn. A fixed-point theorem. *Mathematical Systems Theory. An International Journal on Mathematical Computing Theory*, 1:55–57, 1967.

[14] E. Hewitt and K. A. Ross. *Abstract harmonic analysis. Vol. I*, volume 115 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin-New York, second edition, 1979. Structure of topological groups, integration theory, group representations.

[15] A. Jabbari and I. Namioka. Ellis group and the topological center of the flow generated by the map $n \mapsto \lambda^{n^k}$. *Milan J. Math.*, 78(2):503–522, 2010.

[16] B. E. Johnson. *Cohomology in Banach algebras*. American Mathematical Society, Providence, R.I., 1972. Memoirs of the American Mathematical Society, No. 127.

[17] E. Kaniuth and A. T.-M. Lau. *Fourier and Fourier-Stieltjes algebras on locally compact groups*, volume 231 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2018.

[18] N.-I. Kelley, J.L. *Linear topological spaces*. Graduate texts in mathematics. Springer-Verlag, 1976.

[19] K. Kunen. Compact \$L\$-spaces and right topological groups. In *Topology Proceedings*, volume 24, pages 295–327 (2001), 1999.

[20] A.-M. Lau and R. Loy. Banach algebras on compact right topological groups. *Journal of Functional Analysis*, 225(2):263–300, Aug. 2005.

[21] A. T.-M. Lau and R. J. Loy. Fourier–Stieltjes algebras and measure algebras on compact right topological groups. *Journal of Functional Analysis*, 266(8):4870–4889, Apr. 2014.

[22] A. T.-M. Lau and J. Ludwig. Unitary closure and Fourier algebra of a topological group. *Studia Math.*, 231(1):1–28, 2015.

[23] A. T.-M. Lau and H. L. Pham. Characterisations of Fourier and Fourier-Stieltjes algebras on locally compact groups. *Adv. Math.*, 289:844–887, 2016.

[24] A. T.-M. Lau and H. L. Pham. On a class of Banach algebras associated to harmonic analysis on locally compact groups and semigroups. *Adv. Oper. Theory*, 3(1):231–246, 2018.

[25] L. H. Loomis. *An introduction to abstract harmonic analysis*. D. Van Nostrand Company, Inc., Toronto-New York-London, 1953.

[26] M. Megrelishvili. Reflexively representable but not Hilbert representable compact flows and semitopological semigroups. *Colloq. Math.*, 110(2):383–407, 2008.

[27] P. Milnes. Continuity properties of compact right topological groups. *Mathematical Proceedings of the Cambridge Philosophical Society*, 86:427–435, 1979.

[28] P. Milnes. Representations of compact right topological groups. *Canad. Math. Bull*, 36(3):314–323, 1993.

[29] P. Milnes. Distal compact right topological groups. *Acta Math. Hungar.*, 65(2):149–164, 1994.

[30] P. Milnes. On distal and equicontinuous compact right topological groups. *International Journal of Mathematics and Mathematical Sciences*, 17(2):379–388, 1994.

[31] P. Milnes and J. Pym. Haar measure for compact right topological groups. *Proceedings of the American Mathematical Society*, 114(2), February 1992.

[32] P. Milnes and J. Pym. Homomorphisms of minimal and distal flows. *Journal of the Nigerian Mathematical Society*, 11:63–80, 1992.

[33] W. B. Moors. Fragmentable mappings and CHART groups. *Fundamenta Mathematicae*, 234:191–200, 2016.

[34] W. B. Moors and I. Namioka. Furstenberg's structure theorem via CHART groups. *Ergodic Theory and Dynamical Systems*, 33(03):954–968, June 2013.

[35] A. Moran. Minimal normal systems of compact right topological groups. *Mathematical Proceedings of the Cambridge Philosophical Society*, 123(2):243–258, Mar. 1998.

[36] A. Moran. The right regular representation of a compact right topological group. *Canadian Mathematical Bulletin*, 41:463–472, 1998.

[37] I. Namioka. Right topological groups, distal flows and a fixed-point theorem. *Mathematical Systems Theory*, 6(1-2):193–209, 1972.

[38] I. Namioka. Ellis groups and compact right topological groups. In *Conference in modern analysis and probability*, volume 26 of *Contemporary Mathematics*, pages 295–300, Providence, RI, 1984. American Mathematical Society.

[39] I. Namioka. Kakutani-type fixed point theorems: a survey. *J. Fixed Point Theory Appl.*, 9(1):1–23, 2011.

[40] A. L. T. Paterson. *Amenability*, volume 29 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1988.

[41] N. W. Rickert. Amenable groups and groups with the fixed point property. *Trans. Amer. Math. Soc.*, 127:221–232, 1967.

[42] W. Rudin. Measure algebras on abelian groups. *Bull. Amer. Math. Soc.*, 65:227–247, 1959.

[43] V. Runde. *Lectures on amenability*, volume 1774 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2002.

[44] W. Ruppert. Rechtstopologische Halbgruppen. *Journal für die Reine und Angewandte Mathematik*, 261:123–133, 1973.

[45] W. Ruppert. Über kompakte rechtstopologische Gruppen mit gleichgradig stetigen Linkstranslationen. *Österreich. Akad. Wiss. Math.-Naturwiss. Kl. S.-B. II*, 184(1-4):159–169, 1975.

[46] W. Ruppert. *Compact semitopological semigroups: an intrinsic theory*, volume 1079 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1984.

[47] W. A. F. Ruppert. Compact semitopological semigroups. In *The analytical and topological theory of semigroups*, volume 1 of *De Gruyter Exp. Math.*, pages 133–170. de Gruyter, Berlin, 1990.

[48] C. Ryll-Nardzewski. Generalized random ergodic theorems and weakly almost periodic functions. *Bulletin de l'Académie Polonaise des Sciences. Série des Sciences Mathématiques, Astronomiques et Physiques*, 10:271–275, 1962.

[49] C. Ryll-Nardzewski. On fixed points of semigroups of endomorphisms of linear spaces. In *Proc. Fifth Berkeley Sympos. Math. Statist. and Probability (Berkeley, Calif., 1965/66), Vol. II: Contributions to Probability Theory, Part 1*, pages 55–61. Univ. California Press, Berkeley, Calif., 1967.

[50] A. I. Shtern. Compact semitopological semigroups and reflexive representability of topological groups. *Russian J. Math. Phys.*, 2(1):131–132, 1994.

[51] S. Willard. *General topology.* Dover Publications, Inc., Mineola, NY, 2004.