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THE UNIVERSITY OF ALBERTA

On Geometric Properties of Operators

BY
JON DWIGHT VANDERWERFF

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research for acceptance, a thesis entitled **On Geometric Properties of Operators** submitted by **Jon Dwight Vanderwerff** in partial fulfillment of the requirements for the degree of **Master of Science**.

Supervisor

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ABSTRACT

In this thesis, we study operator generalizations of certain geometric properties of Banach spaces. The definitions are extended in such a manner that the identity map on a space possesses the operator generalization of a given property precisely when the space itself has this said property.

The first part of the thesis presents the definitions and basic analogical features of some geometric properties of operators. In particular, we first study strictly, locally uniformly and uniformly convex operators. Subsequently, the notions of Gâteaux differentiability, Fréchet differentiability and uniform smoothness for operators are discussed. Finally, the concepts of extreme, denting, exposed and strongly exposed points are extended.

Much of the latter portion of this thesis is devoted to the study of the extremal structure of operators. Specifically, we show that locally uniformly convex operators have operator weak-star strongly exposed points in weak-star compact sets. It is also shown that an operator whose image of the unit ball has the Asplund property can be nicely characterized by the extremal structure of its dual operator.

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NOTATION

$B(X) = \{x \in X : \|x\| \le 1\}$

$B(X, Y)$ = the set of bounded linear operators from X into Y

$\text{conv}(S)$ = the convex hull of S

$\overline{\text{conv}}(S)$ = the norm closed convex hull of S

$\overline{\text{conv}}^{w^*}(S)$ = the weak*-closed convex hull of S

$e_i = \{\delta_{ij}\}_{j=1}^{\infty}$ where $\delta_{ij} = 1$ if $i = j$ and 0 otherwise

$M(S) = \sup\{\|x\| : x \in S\}$

$M(g, S) = \sup\{g(x) : x \in S\}$

$\mathbb{N} = \{1, 2, 3, \dots\}$

$\mathbb{R}$ = the real numbers

$\overline{S}$ = the norm closure of S

$S(g, \beta, C) = \{x \in C : g(x) \ge M(g, C) - \beta\}$

$S(X) = \{x \in X : \|x\| = 1\}$

$S_1 \sim S_2 = \{x \in S_1 : x \notin S_2\}$

$x_\alpha \xrightarrow{w} x$ means x_α converges to x in the weak topology

$x_\alpha \xrightarrow{w^*} x$ means x_α converges to x in the weak* topology

$x_n \to x$ means x_n converges to x in the norm

$x_n \not\to x$ means x_n does not converge to x

X^* = the set of continuous linear functionals on X endowed with the usual supremum norm

$(X_1 \oplus X_2)_{\ell_p} = \{(x_1, x_2) \in X_1 \oplus X_2 : \|(x_1, x_2)\| = \|\{\|x_1\|_{X_1}, \|x_2\|_{X_2}\}\|_{\ell_p}\}$

CONVENTIONS

In this thesis, any Banach space is a real Banach space. All operators are assumed to be bounded linear operators. The sequence spaces ℓ_p for $1 \leq p \leq \infty$ and c_0 have the natural numbers for their index sets.

INTRODUCTION

Given a property of a Banach space, one may ask if a corresponding generalization of this property can be defined for bounded linear operators between Banach spaces. Such a generalization would at least require the identity operator on X to possess the generalized property as defined for operators precisely when the space X has the given property. A common and important (topological) example of such a situation is that of compact operators which generalize locally compact spaces. One can also extend geometric notions to the setting of operators, in fact, B. Beauzamy introduced uniformly convex and uniformly convexifiable operators in [1]. It is from this work of Beauzamy that the initial motivation for this thesis arose. In addition to investigating operator analogues of convexity, this thesis will also study properties of smoothness and extremal points in the context of operators.

The material presented in Chapters One, Two and Three is quite straightforward, nevertheless, much of this material is referred to in latter chapters. Moreover, when developing a parallel theory there are certain results which should be verified. We hope that the first three chapters have accomplished this with some degree of success. More specifically, in the first chapter, strictly, locally uniformly and uniformly convex operators are discussed. Smoothness is the dual notion of convexity in Banach spaces, this serves as our motivation for the second chapter which gives generalizations of some notions of smoothness and shows them to satisfy certain duality relationships with convex operators. The third chapter introduces various extremal properties of operators and provides some examples of results concerning extremal properties which do not have analogues for operators.

Many of the theorems on extreme points in Banach spaces require completeness, however, the preimage topology generated by a bounded linear operator on a Banach space is, in general, not complete. For instance, we cannot have an operator analogue of Phelps' result [12, Thm. 9] which shows that subset dentability is equivalent to having strongly exposed points in bounded closed convex sets. However, in our main result of Chapter Four, using techniques of J. Lindenstrauss and R.R. Phelps we show that for a locally uniformly convex operator, say T, on a dual space, every w^*-compact and convex set is the w^*-closed convex hull of its w^*-T-strongly exposed points, that is, points which are strongly exposed with respect to T by w^*-continuous functionals.

In the fifth and final chapter, Asplund operators (operators whose images of the unit ball have the Asplund property) are studied in connection with many of the notions introduced in the first three chapters. In particular, an Asplund operator can be characterized by the strongly exposed points of its dual operator. Moreover, for a given separable operator T, it is shown that T being an Asplund operator is equivalent to notions of Fréchet differentiability with respect to T and separability of the range of its dual operator T^*. Although these are analogues of theorems from the theory of Banach spaces, they do provide us with new characterizations of Asplund operators.

Chapter One

CONVEX OPERATORS

In the paper [1], B. Beauzamy introduced uniformly convexifiable and uniformly convex operators. This chapter, in addition to discussing uniformly convex operators as defined by Beauzamy, also introduces locally uniformly convex and strictly convex operators. The definitions are given in Section 1.1. Sequence characterizations of convex operators are given and used in the second section to prove various facts about convex operators. A brief discussion on the possibility of non-trivial convex operators constitutes the third section. In Section 4.1, we discuss some elementary renormings via operators to obtain convex operators.

The list of people who have contributed to the study of convex spaces is large. Unfortunately, we have not gone to the original sources of the theorems for which we present operator analogues. Because of this, the authors of the ideas appearing here have not been credited; this is by no means an attempt by us to receive such credit.

1.1. Basic Notions.

DEFINITION 1.1.1. Let $T \in \mathcal{B}(X,Y)$. We shall say T is *strictly convex* if $\|x\| = \|y\| = \left\|\frac{x+y}{2}\right\| = 1$ implies $Tx = Ty$.

In the case that for any fixed $\varepsilon > 0$ and any fixed $y \in S(X)$ there exists $\delta = \delta(\varepsilon, y) > 0$ so that $\left\|\frac{x+y}{2}\right\| < 1 - \delta$ whenever $x \in S(X)$ and $\|Tx - Ty\| \geq \varepsilon$, then T is said to be *locally uniformly convex.*

The operator T will be called *uniformly convex* if for any $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that $\left\|\frac{x+y}{2}\right\| \leq 1 - \delta$ if $x, y \in S(X)$ and $\|Tx - Ty\| \geq \varepsilon$.

Remark 1.1.2. (a) From the definitions it is clear that T is uniformly convex $\Rightarrow$ T is locally uniformly convex $\Rightarrow$ T is strictly convex.

(b) Observe that the identity operator $I : X \to X$ is strictly convex if and only if X is strictly convex; this observation is also true for local uniform and uniform convexity. For Banach spaces, X is strictly convex $\not\Rightarrow$ X is locally uniformly convex $\not\Rightarrow$ X is uniformly convex. Hence, T is strictly convex $\not\Rightarrow$ T is locally uniformly convex $\not\Rightarrow$ T is uniformly convex.

(c) If X is uniformly convex, it is easy to see that $T : X \to Y$ is uniformly convex. Similarly for local uniform and strict convexity.

(d) If $T : X \to Y$ is uniformly convex, then $T : X \to Z$ is uniformly convex for $Z \supset Y$; the same is true for strictly and locally uniformly convex operators.

<u>Example</u> 1.1.3. Let $B = \{(x, y) \in \mathbb{R} \times \mathbb{R} : |y| \leq 1 + \sqrt{1 - x^2},\ |x| \leq 1\}$. Let $\|\cdot\|$ be the non-strictly convex norm on $\mathbb{R} \times \mathbb{R}$ whose closed unit ball is B. Define $T : (\mathbb{R} \times \mathbb{R}, \|\cdot\|) \to \mathbb{R}$ by $T(x, y) = x$. Then T is strictly convex since

$$\|(x, y) + (v, w)\| = 2 \quad \text{if and only if} \quad x = v,\ |x| = 1 \text{ and } y, w \in [-1, 1],$$

and for such (x, y) and (v, w), $T(x, y) - T(v, w) = x - v = 0$.

1.2. Some Properties of Convex Operators.

We begin this section with some useful sequence characterizations of convex operators.

PROPOSITION 1.2.1. *For $T \in B(X,Y)$, the following are equivalent:*

(a) T is uniformly convex.

(b) $2\|x_n\|^2 + 2\|y_n\|^2 - \|x_n + y_n\|^2 \to 0$ implies $\|Tx_n - Ty_n\| \to 0$ for all bounded sequences $\{x_n\}$ and $\{y_n\}$ in X.

(c) $\|x_n\| \le 1$, $\|y_n\| \le 1$ and $\|x_n + y_n\| \to 2$ implies $\|Tx_n - Ty_n\| \to 0$.

Proof. (a) $\Rightarrow$ (b): Suppose (b) is not true, that is, we can find sequences $\{x_n\}$ and $\{y_n\}$ in X with $\|x_n\|, \|y_n\| \le M$ for all n and some $M > 0$ so that $2\|x_n\|^2 + 2\|y_n\|^2 - \|x_n + y_n\|^2 \to 0$ but $\|Tx_n - Ty_n\| \not\to 0$. By passing to subsequences, we may assume that $\|Tx_n - Ty_n\| \ge \varepsilon$ for some $\varepsilon > 0$ and $\|x_n\| \to r$ for some $r \in \mathbb{R}$. Now, $(\|x_n\| - \|y_n\|) \to 0$ so $\|y_n\| \to r$, which in turn implies $r \ge \frac{\varepsilon}{2\|T\|} > 0$. Hence we may assume $x_n \ne 0$ and $y_n \ne 0$ for all n; since

$$2\left\|\frac{x_n}{r}\right\|^2 + 2\left\|\frac{y_n}{r}\right\|^2 - \left\|\frac{x_n}{r} + \frac{y_n}{r}\right\|^2 \to 0$$

we have,

$$2\left\|\frac{x_n}{\|x_n\|}\right\|^2 + 2\left\|\frac{y_n}{\|y_n\|}\right\|^2 - \left\|\frac{x_n}{\|x_n\|} + \frac{y_n}{\|y_n\|}\right\|^2 \to 0.$$

Hence $\left\|\frac{x_n}{\|x_n\|} + \frac{y_n}{\|y_n\|}\right\|^2 \to 4$. However,

$$\liminf_n \left\|T\left(\frac{x_n}{\|x_n\|} - \frac{y_n}{\|y_n\|}\right)\right\| = \frac{1}{r}\ \liminf_n \|Tx_n - Ty_n\| \ge \frac{\varepsilon}{r} > 0.$$

Therefore T is not uniformly convex.

(b) $\Rightarrow$ (c) If $\|x_n\| \le 1$, $\|y_n\| \le 1$ and $\|x_n + y_n\| \to 2$, then $2\|x_n\|^2 + 2\|y_n\|^2 - \|x_n + y_n\|^2 \to 0$ and so $\|Tx_n - Ty_n\| \to 0$ by (b).

(c) $\Rightarrow$ (a). If T is not uniformly convex, find $\{x_n\}, \{y_n\} \subset S(X)$ so that $\|x_n + y_n\| \to 2$ but $\|Tx_n - Ty_n\| \not\to 0$. Hence (c) fails. ∎

PROPOSITION 1.2.2. *For $T \in B(X, Y)$, the following are equivalent:*

(a) T is locally uniformly convex.

(b) For fixed $y \in X$ and $\{x_n\} \subset X$, $2\|x_n\|^2 + 2\|y_n\|^2 - \|x_n + y_n\|^2 \to 0$ implies $\|Tx_n - Ty_n\| \to 0$.

(c) $\|x_n\| \le \|y\| = 1$ and $\|x_n + y\| \to 2$ implies $\|Tx_n - Ty\| \to 0$.

Proof. Set $y_n = y$ for all n in the proof of 1.2.1. ∎

PROPOSITION 1.2.3. *Let $T \in B(X, Y)$, then the following are equivalent:*

(a) T is strictly convex.

(b) $2\|x\|^2 + 2\|y\|^2 - \|x + y\|^2 = 0$ implies $Tx = Ty$ (for $x, y \in X$).

(c) $2\|x\|^2 + 2\|y\|^2 - \|x + y\|^2 = 0$ implies $Tx = Ty$ (for $x, y \in S(x)$).

Proof. (Outline) Observe that $2\|x\|^2 + 2\|y\|^2 - \|x + y\|^2 = 0$ if and only if $\|x\| = \|y\|$ and $\|x + y\| = \|x\| + \|y\|$. Using the aforementioned fact, linearity and homogeneity it is easy to establish the above equivalencies. ∎

From Propositions 1.2.1(c) and 1.2.2(c) we immediately have the

COROLLARY 1.2.4. *(a) An operator T is uniformly convex if and only if for each $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ so that $\left\|\frac{x+y}{2}\right\| < 1 - \delta$ whenever $x, y \in B(X)$ satisfy $\|Tx - Ty\| \ge \varepsilon$.*

(b) The operator T is locally uniformly convex if and only if for fixed $\epsilon > 0$ and $y \in S(X)$, there exists $\delta = \delta(\epsilon, y) > 0$ so that $\|\frac{x+y}{2}\| < 1 - \delta$ whenever $x \in B(X)$ and $\|Tx - Ty\| > \epsilon$.

Another result which can be quite easily proved using Propositions 1.2.1 and 1.2.3 is:

PROPOSITION 1.2.5. *Let X be finite dimensional. If $T \in B(X, Y)$ is strictly convex, then T is uniformly convex.*

<u>Proof</u>. Suppose T is not uniformly convex. Then we can find bounded sequences $\{x_n\}$ and $\{y_n\}$ in X so that $2\|x_n\|^2 + 2\|y_n\|^2 - \|x_n + y_n\|^2 \to 0$ but $\|Tx_n - Ty_n\| \geq \epsilon$ for all n. Since $B(X)$ is compact, we choose subsequences $\{x_{n_k}\}$ and $\{y_{n_k}\}$ so that $x_{n_k} \to x$ and $y_{n_k} \to y$ for some x and y in X. Now

$$2\|x\|^2 + 2\|y\|^2 - \|x + y\|^2 = \lim_k (2\|x_{n_k}\|^2 + 2\|y_{n_k}\|^2 - \|x_{n_k} + y_{n_k}\|^2) = 0,$$

however, $\|T(x - y)\| = \lim_k \|Tx_{n_k} - Ty_{n_k}\| \geq \epsilon$. Therefore T is not strictly convex, as condition (c) of Proposition 1.2.3 fails. ∎

<u>Example</u> 1.2.6. The operator given in Example 1.1.3 is uniformly convex since it is strictly convex and $\mathbb{R} \times \mathbb{R}$ is finite dimensional.

Locally uniformly convex operators possess some nice properties on dual spaces. Two such properties are demonstrated in the following propositions.

PROPOSITION 1.2.7. *If $T : X^* \to Y$ is locally uniformly convex and $x_\alpha \xrightarrow{w^*} x$ where $x_\alpha \in B(X^*)$ all α and $x \in S(X^*)$, then $\|Tx_\alpha - Tx\| \to 0$.*

<u>Proof</u>. First, $\frac{x_\alpha + x}{2} \xrightarrow{w^*} x$ since $x_\alpha \xrightarrow{w^*} x$. Thence $\|\frac{x_\alpha + x}{2}\| \to \|x\| = 1$ because dual norms are w^*-lower semicontinuous. Hence we can find α_k so that

$\left\|\frac{x_\alpha + x}{2}\right\| > 1 - \frac{1}{k}$ for all $\alpha > \alpha_k$. Therefore, $\|Tx_\alpha - Tx\| \leq \varepsilon\left(\frac{1}{k}, x\right)$ for all $\alpha \geq \alpha_k$ and $\varepsilon\left(\frac{1}{k}, x\right) \to 0$ since T is locally uniformly convex. ■

PROPOSITION 1.2.8. *If X is separable and $T : X^* \to Y$ is locally uniformly convex, then TX^* is separable.*

Proof. The separability of X implies that X^* is w^*-separable, whence we can choose $D = \{d_k : k = 1, 2, \dots\} \subseteq S(X^*)$ so that D is w^*-dense in $S(X^*)$. It suffices to show that $T(D)$ is dense in $T(S(X^*))$; to this end we let $y = Tx$ for some $x \in S(X^*)$ and $\{x_k\} \subset D$ so that $x_k \xrightarrow{w^*} x$. By Proposition 1.2.7 $\|Tx_k - Tx\| \to 0$, therefore $T(D)$ is dense in $T(S(X^*))$. ■

The final result of this section is a proposition about convex operators on direct sums whose proof is expedited by the sequential characterizations of convex operators given at the beginning of this section.

PROPOSITION 1.2.9. *Let $T_1 \in B(X_1, Y_1)$, $T_2 \in B(X_2, Y_2)$ and define $T : (X_1 \oplus X_2)_{\ell_2} \to (Y_1 \oplus Y_2)_{\ell_2}$ by $T(x, y) = (T_1 x, T_2 y)$.*

(a) If T_1 and T_2 are strictly convex, then T is strictly convex.

(b) If T_1 and T_2 are (locally) uniformly convex, then T is (locally) uniformly convex.

Proof. (b) Suppose T_1 and T_2 are uniformly convex. Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in $(X_1 \oplus X_2)_{\ell_2}$ where $x_n = (x_{n,1}, x_{n,2})$ and $y_n = (y_{n,1}, y_{n,2})$. If

$$2\|x_n\|^2 + 2\|y_n\|^2 - \|x_n + y_n\|^2 \to 0,$$

then

$$2(\|x_{n,1}\|^2 + \|x_{n,2}\|^2) + 2(\|y_{n,1}\|^2 + \|y_{n,2}\|^2) - (\|x_{n,1} + y_{n,1}\|^2 + \|x_{n,2} + y_{n,2}\|^2) \to 0.$$

It follows that

$$2\|x_{n,1}\|^2 + 2\|y_{n,1}\|^2 - \|x_{n,1} + y_{n,1}\|^2 \to 0,$$

and

$$2\|x_{n,2}\|^2 + 2\|y_{n,2}\|^2 - \|x_{n,2} + y_{n,2}\|^2 \to 0.$$

By the uniform convexity of T_1 and T_2, we have that $\|T_1 x_{n,1} - T_1 y_{n,1}\| \to 0$ and $\|T_2 x_{n,2} - T_2 y_{n,2}\| \to 0$. Therefore $\|Tx_n - Ty_n\| \to 0$, which in turn implies that T is uniformly convex.

The proofs for the strict and local uniform convexity cases are similar – thus the reason for their conspicuous absence. ■

1.3. On Non-trivial Convex Operators.

Heretofore we have not demonstrated the existence of a uniformly convex operator such that both X and $X/\ker T$ are not uniformly convex, *a fortiori*, not uniformly convexifiable. However, in this section, a one-to-one uniformly convex operator on a space isomorphic to c_0 will be exhibited.

Before proving a proposition which will help us in constructing uniformly and locally uniformly convex operators, two definitions should be given. Firstly, a space X is said to be ***weakly locally uniformly convex*** if $(x_n - y) \xrightarrow{w} 0$ whenever $\|x_n + y\| \to 2$ for $\{x_n\}$, $y \subset B(X)$. Secondly, if $(x_n - y_n) \xrightarrow{w} 0$ whenever $\|x_n + y_n\| \to 2$ where $\{x_n\}, \{y_n\} \subset B(X)$, then X is said to be ***weakly uniformly convex***.

PROPOSITION 1.3.1. *(a) Let X be weakly locally uniformly convex. If $T : X \to Y$ is a compact operator, then T is locally uniformly convex.*

(b) Let X be weakly uniformly convex. If $T : X \to Y$ is compact, then T is uniformly convex.

Proof. (a) Let $\{x_n\}$, $y \subset B(X)$. If $\|x_n + y\| \to 2$ then $(x_n - y) \xrightarrow{w} 0$ and so $\|Tx_n - Ty\| \to 0$ by the compactness of T. Therefore T is locally uniformly convex.

(b) Suppose $\{x_n\}, \{y_n\} \subset B(X)$ and $\|x_n + y_n\| \to 2$. Then $(x_n - y_n) \xrightarrow{w} 0$ and so $\|Tx_n - Ty_n\| \to 0$ by the compactness of T. Therefore T is uniformly convex. ∎

Remark 1.3.2 (a) Weak local uniform convexity cannot be replaced by strict convexity in Proposition 1.3.1(a).

Consider the strictly convex norm $|\!|\!| \cdot |\!|\!|$ on $C[0,1]$ defined by $|\!|\!| x |\!|\!| = (\|x\|_\infty^2 + \|x\|_2^2)^{1/2}$. Let $T : (C[0,1], |\!|\!| \cdot |\!|\!|) \to \mathbb{R}$ be defined by $Tf = f(0)$. This map is compact since it is a bounded functional, however, T is not locally uniformly convex. Let $f \equiv \frac{1}{2}$ and for $n = 2, 3, 4, \ldots$, define

$$f_n(x) = \begin{cases} \frac{n}{2}x & \text{if } 0 \le x \le \frac{1}{n}, \\ \frac{1}{2} & \text{if } \frac{1}{n} < x \le 1. \end{cases}$$

Observe that $|\!|\!| f_n |\!|\!| \le 1$, $|\!|\!| f |\!|\!| = 1$ and $|\!|\!| f_n + f |\!|\!| \to 2$. On the other hand, $Tf_n = f_n(0) = 0$ for $n = 2, 3, 4, \ldots$ and $Tf = f(0) = \frac{1}{2}$. Therefore $\|Tf_n - Tf\| = \frac{1}{2}$ for all n, so T is not locally uniformly convex.

(b) In Proposition 1.3.1(b) we cannot replace weak uniform convexity with local uniform convexity.

Consider $(c_0, \|\cdot\|_D)$ where $\|\cdot\|_D$ denotes Day's norm defined by

$$\|(a_i)\|_D = \left(\sum\left(\frac{|\pi(a_i)_n|}{2^n}\right)^2\right)^{1/2} \quad \text{where} \quad \pi(a_i)_1 \geq \pi(a_i)_2 \geq \ldots$$

and π is a permutation of $\mathbb{N}$. In [13] it is shown that Day's norm is locally uniformly convex on c_0. Now let

$$a_n = \sum_{i=2}^{n+1} \sqrt{3}\, e_i \qquad \text{and} \qquad b_n = \sum_{i=1}^{n} \sqrt{3}\, e_i$$

and suppose $T : (c_0, \|\cdot\|_D) \to \ell_2$ is given by $T(a_i) = (\frac{1}{2^i} a_i)$. Observe that T is compact, $\|a_n\|_D \leq 1$ and $\|b_n\|_D \leq 1$. Finally, $\|Ta_n - Tb_n\|_2 = \|\frac{\sqrt{3}}{2}\, e_1\|_2 = \frac{\sqrt{3}}{2}$ for all n whereas $\|a_n + b_n\|_D \geq \left(\sum_{i=1}^{n} \frac{12}{4^i}\right)^{1/2} \to 2$ as $n \to \infty$. Therefore T is not uniformly convex.

As promised, we now give an example of a one-to-one uniformly convex operator on a space isomorphic to c_0.

<u>Example</u> 1.3.3. Since $c_0^* = \ell_1$ is separable, there is an equivalent norm, say $\|\cdot\|$ on c_0 such that $(c_0, \|\cdot\|)$ is weakly uniformly convex (see [16, p. 200]). Now, $T : (c_0, \|\cdot\|) \to c_0$ given by $T(a_i) = (\frac{1}{i}\, a_i)$ is compact, so by Proposition 1.3.1(b), T is uniformly convex. We point out that T is one-to-one, so $c_0/\ker T = c_0$. Moreover, c_0 is not reflexive so it has no equivalent uniformly convex norms.

As was just shown, there are one-to-one uniformly convex operators on non-uniformly convex spaces. This is not the case for strictly convex operators.

<u>Fact</u> 1.3.4. A space X is strictly convex if and only if there is a one-to-one strictly convex operator on X.

<u>Proof</u>. $\Rightarrow$: If X is strictly convex, then $I : X \to X$ is a one-to-one strictly convex operator where I is the identity operator.

$\Leftarrow$: Suppose $T : X \to Y$ is one-to-one and strictly convex. If $\|x\| = \|y\| = \|\frac{x+y}{2}\| = 1$ then $Tx = Ty$ since T is strictly convex. Moreover, $x = y$ since T is one to one. ∎

We will not abandon our study of convex operators on account of the restrictions of Fact 1.3.4. We hasten to add that in general, the operators studied will not be one-to-one, moreover, locally uniformly convex operators will receive more attention.

1.4. Renorming and Convex Operators.

In the previous section we considered the convexity of a given operator with respect to a given norm. However, for the operator $T : X \to Y$, one may ask if there are conditions on TX for which X has an equivalent norm $\|\!| \cdot |\!\|$ so that $T : (X, \|\!| \cdot |\!\|) \to Y$ has some prescribed convexity. This is precisely the type of question investigated in the present section.

PROPOSITION 1.4.1. *Suppose $T : X \to (Y, \|\cdot\|)$ and let $(\overline{TX}, \|\cdot\|)$ be the Banach space with norm $\|\cdot\|$ inherited from Y.*

(a) *If $(\overline{TX}, \|\cdot\|)$ has an equivalent norm which is strictly convex, then there is an equivalent norm $\|\!| \cdot |\!\|$ on X so that $T : (X, \|\!| \cdot |\!\|) \to Y$ is strictly convex.*

(b) *If $(\overline{TX}, \|\cdot\|)$ has an equivalent (locally) uniformly convex norm, then X has an equivalent norm $\|\!| \cdot |\!\|$ so that $T : (X, \|\!| \cdot |\!\|) \to Y$ is (locally) uniformly convex.*

Proof. Since all arguments are similar and easy, only the proof for uniform convexity will be given here. Let $\|\cdot\|_1$ be an equivalent norm on $(\overline{TX}, \|\cdot\|_1)$ which is uniformly convex. Now, we define $|\!|\!| \cdot |\!|\!|$ on X by $|\!|\!| x |\!|\!| = (\|x\|^2 + \|Tx\|_1^2)^{1/2}$. If $2|\!|\!| x_n |\!|\!|^2 + 2|\!|\!| y_n |\!|\!|^2 - |\!|\!| x_n + y_n |\!|\!|^2 \to 0$ for the bounded sequences $\{x_n\}, \{y_n\}$ in X, then $2\|Tx_n\|_1^2 + 2\|Ty_n\|_1^2 - \|Tx_n + Ty_n\|_1^2 \to 0$ and thus $\|Tx_n - Ty_n\|_1 \to 0$ by the uniform convexity of $\|\cdot\|_1$. Therefore $\|Tx_n - Ty_n\| \to 0$, so T is uniformly convex.

By a theorem of S.L. Troyanski in [15] and a theorem of P. Enflo in [7] respectively, we obtain

COROLLARY 1.4.2. *Let $T : X \to Y$ be a continuous linear operator.*

(a) *If $\overline{TX}$ is weakly compact generated, then there is an equivalent norm $|\!|\!| \cdot |\!|\!|$ on X such that $T : (X, |\!|\!| \cdot |\!|\!|) \to Y$ is locally uniformly convex.*

(b) *If $\overline{TX}$ is uniformly convexifiable, then $T : (X, |\!|\!| \cdot |\!|\!|) \to Y$ is uniformly convex for some equivalent norm $|\!|\!| \cdot |\!|\!|$ on X.*

It should be noted that, albeit with more effort, B. Beauzamy [1, p. 121] proved a stronger result than Corollary 1.4.2(b), namely

THEOREM 1.4.3. (Beauzamy) *For $T \in \mathcal{B}(X, Y)$ there exists an equivalent norm on which T is uniformly convex if and only if $T(\mathcal{B}(X))$ does not have the finite tree property.*

If a set A fails the finite tree property, then A is relatively weakly compact, so from Beauzamy's work we also have

COROLLARY 1.4.4. *Uniformly convex operators are weakly compact.*

For the purposes of this thesis, Proposition 1.4.1 with its corollaries will be enough. Moreover, the norm given in Proposition 1.4.1 is quite easy to work with.

<u>Example</u> 1.4.5. Let $T : c_0 \to \ell_p$ for $p \in (1,\infty)$ be defined by $T(a_i) = \left(\frac{1}{2^i}\, a_i\right)$ and $|\!|\!| x |\!|\!| = (\|x\|_\infty^2 + \|Tx\|_p^2)^{1/2}$. Since ℓ_p is uniformly convex, from the proof of Proposition 1.4.1 we see that $|\!|\!| \cdot |\!|\!|$ is an equivalent norm on c_0 for which $T : (c_0, |\!|\!| \cdot |\!|\!|) \to \ell_p$ is uniformly convex.

LEMMA 1.4.6. *Let $T : X^* \to Y^*$ be w^*-w^* continuous. Then the norm $|\!|\!| x |\!|\!| = (\|x\|_1^2 + \|Tx\|_2^2)^{1/2}$ is an equivalent dual norm on X^* if the norms $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent dual norms on X^* and Y^*.*

<u>Proof</u>. Suppose $x_\eta \xrightarrow{w^*} x$. Then $Tx_\eta \xrightarrow{w^*} Tx$. Now

$$\|x\|_1 \leq \liminf \|x_\eta\|_1 \quad \text{and} \quad \|Tx\|_2 \leq \liminf \|Tx_\eta\|_2$$

since dual norms are w^*-lower semicontinuous. Thus $|\!|\!| x |\!|\!| \leq \liminf |\!|\!| x_\eta |\!|\!|$, that is $|\!|\!| \cdot |\!|\!| : X \to \mathbb{R}$ is w^*-lower semicontinuous. Therefore, $|\!|\!| \cdot |\!|\!|$ is a dual norm. ∎

<u>Example</u> 1.4.7. Consider $T : \ell_\infty \to \ell_2$ defined by $T(a_i) = \left(\frac{1}{2^i}\, a_i\right)$. Observe that T is w^*-w^* continuous. By Lemma 1.4.6, $|\!|\!| x |\!|\!| = (\|x\|_\infty^2 + \|Tx\|_2^2)^{1/2}$ is an equivalent dual norm on ℓ_∞. By the proof of Proposition 1.4.1(b), $T : (\ell_\infty, |\!|\!| \cdot |\!|\!|) \to \ell_2$ is uniformly convex. We remark that this is a one-to-one uniformly convex operator on a space which does not have any equivalent weakly locally uniformly convex norm, see [5, p. 120].

The final results of this section deal with dual renormings for dual operators with separable range. First, we need a result which is a slight modification of a theorem proved independently by E. Asplund and M. Kadec. We, for the sake of completeness, include the proof which utilizes techniques of K. John and V. Zizler.

LEMMA 1.4.8. *Let X^* be a dual space and Z a separable subspace of X^*. Then there exists an equivalent dual norm $|||\cdot|||$ on X^* so that $|||x_n - z||| \to 0$ whenever $\{x_n\} \subset X$, $z \in Z$ are such that $|||x_n||| \le |||z||| = 1$ but $|||x_n + z||| \to 2$.*

<u>Proof</u>. Define $|||\cdot|||$ on X^* by

$$|||x||| = \left(\|x\|^2 + \sum_{i=1}^{\infty} \frac{1}{2^i}\, \rho^2(x, \ell(y_i))\right)^{1/2}$$

where $\{y_i\}$ is a dense subset of $B(X) \cap Z$, $\ell(y_i) = \{ay_i : a \in \mathbb{R}\}$ and $\rho(x, \ell(y_i)) = \inf\{\|x - y\| : y \in \ell(y_i)\}$. The usual dual norm on X^* is w^*-lower semicontinuous so $x \mapsto \rho(x, \ell(y_i))$ is a w^*-lower semicontinuous mapping. Therefore, $|||\cdot|||$ is w^*-lower semicontinuous and hence an equivalent dual norm on X^*.

Suppose $\{x_n\} \subset X$ satisfies $|||x_n||| \le 1$, $z \in Z$ with $|||z||| = 1$ and $|||x_n + z||| \to 2$. Then $2|||x_n|||^2 + 2|||z|||^2 - |||z_n + z|||^2 \to 0$, and so

(1) $$2\sum_{i=1}^{\infty} \frac{\rho^2}{2^i}(x_n, \ell(y_i)) + 2\sum_{i=1}^{\infty} \frac{\rho^2}{2^i}(z, \ell(y_i)) - \sum_{i=1}^{\infty} \frac{\rho^2}{2^i}(z + x_n, \ell(y_i)) \to 0$$

and

(2) $$2\|z\|^2 + 2\|x_n\|^2 - \|x_n + z\|^2 \to 0 \quad \text{as} \quad n \to \infty.$$

For each fixed i, it follows from (1), that

(3) $$\rho(x_n, \ell(y_i)) \to \rho(z, \ell(y_i)) \quad \text{as} \quad n \to \infty.$$

From (2) it follows that

(4) $$\|x_n\| \to \|z\|.$$

Using (3), (4) and the triangle inequality we show that $\|x_n - z\| \to 0$. Let $\varepsilon > 0$ be given, $K = \min\{1, \inf\{\|x\| : \|x\| = 1\}\}$, and $\delta = \min\left\{\frac{\varepsilon}{8}, \frac{K}{8}\right\}$. By (3), (4) and the fact that $\|x_n + z\| \to 2$, we can choose m so that

$$\rho(x_n, \ell(y_j)) < 2\delta, \quad \big|\|x_n\| - \|z\|\big| < \delta \quad \text{and} \quad \|x_n + z\| \geq 2K - \delta > 8\delta$$

for all $n \geq m$ where y_j is chosen so that $\rho(z, \ell(y_j)) < \delta$. For any fixed $n \geq m$, choose $z_0, z_1 \in \ell(y_i)$ so that

$$\|x_n - z_0\| \leq 2\delta \quad \text{and} \quad \|z - z_1\| \leq \delta.$$

Now, $\big|\|z_0\| - \|z_1\|\big| \leq \big|\|z_0\| - \|x_n\|\big| + \big|\|x_n\| - \|z\|\big| + \big|\|z\| - \|z_1\|\big| \leq 4\delta$. Because $z_0, z_1 \in \ell(y_j)$ the above inequality implies that either $\|z_1 - z_0\| \leq 4\delta$ or $\|z_1 + z_0\| \leq 4\delta$. Finally, observing

$$\|x_n \pm z\| \leq \|x_n - z_0\| + \|z_0 \pm z_1\| + \|z - z_1\|,$$

we see that $\|x_n + z\| \leq 7\delta$ or $\|x_n - z\| \leq 7\delta$. However, $\|x_n + z\| > 8\delta$ since $n \geq m$, therefore $\|x_n - z\| \leq 7\delta < \varepsilon$ for any $n \geq m$. ∎

COROLLARY 1.4.9. *(a) If $T : X^* \to Y^*$ is w^*-w^* continuous and TX^* is separable, then there exists an equivalent dual norm $\|\cdot\|$ on X^* for which $T : (X^*, \|\cdot\|) \to Y^*$ is locally uniformly convex.*

(b) Suppose X is separable and $T : X^* \to Y^*$ is w^*-w^* continuous. Then TX^* is separable if and only if there exists an equivalent dual norm $\|\cdot\|$ on X^* so that $T : (X^*, \|\cdot\|) \to Y^*$ is locally uniformly convex.

Proof. (a) By Lemma 1.4.8 let $\|\cdot\|_1$ be an equivalent dual norm on Y^* which is locally uniformly convex on $\overline{TX^*}$. From Lemma 1.4.6 it follows that

$|\!|\!| x |\!|\!| = (\|x\|^2 + |\!|\!| Tx |\!|\!|_1^2)^{1/2}$ is an equivalent dual norm on X^*. Now $T : (X^*, |\!|\!| \cdot |\!|\!|) \to Y^*$ is locally uniformly convex as was seen in Proposition 1.4.1(b).

(b) This follows from part (a) and Proposition 1.2.8. ■

Chapter Two

SMOOTHNESS THROUGH OPERATORS

In Chapter One we saw that convex operators can be defined quite naturally. In the present chapter we define the notions of Gateaux differentiability, smoothness, Fréchet differentiability and uniform smoothness of a space with respect to the range of an operator. The results presented here are analogues of standard results on smoothness of Banach spaces with particular interest in the theorems pertaining to the duality of smoothness and convexity.

The motivation for the selection of the material presented here is based mostly on Chapter Two of [5]. The techniques used are the same as those which have been used in Banach spaces, many of which were introduced by D.F. Cudia in [3].

2.1. Smoothness and Gâteaux Differentiability.

This section introduces the concept of T-smoothness and shows the partial duality of strictly convex operators and T-smoothness. Finally, T-Gâteaux differentiability is defined and shown to be equivalent to T-smoothness.

DEFINITION 2.1.1. Let $T \in B(X,Y)$ and $y \neq 0$, $y \in Y$. If $f \circ T = g \circ T$ whenever $f, g \in S(Y^*)$ and $f(y) = g(y) = \|y\|$, then y is said to be a *T-smooth point*. We shall say Y is *T-smooth* if each $y \in S(Y)$ is a T-smooth point.

Example 2.1.2. (a) Let $T : c_0 \to c_0$ be defined by $T(a_i) = (a_1, 0, a_2, 0, a_3, 0, \ldots)$. Let $a = (a_i) \in B(c_0)$ be given by $a_2 = a_4 = 1$, $a_i = 0$ for all $i \notin \{2,4\}$.

Taking $f_1 = e_2$ and $f_2 = \frac{1}{2}e_1 + \frac{1}{2}e_2$ we have $f_1(a) = f_2(a) = 1$ so a is not a smooth point. However, a is a T-smooth point: let $f_1, f_2 \in S(\ell_1)$ be such that $f_1(a) = f_2(a) = 1$. Now $f_1 = \sum_{i=1}^{\infty} x_i e_i$ and $f_2 = \sum_{i=1}^{\infty} y_i e_i$ where $\sum |x_i| = \sum |y_i| = 1$. Since $f_1(a) = f_2(a)$, we have $x_2 + x_4 = y_2 + y_4 = 1$ and thus $x_i = y_i = 0$ for all $i \notin \{2,4\}$. Therefore, $f_1(Tx) = f_2(Tx)$ for all $x \in c_0$.

(b) Consider $T : \ell_\infty \to \ell_1$ defined by $T(a_i) = \sum_{i=1}^{\infty} \frac{a_i}{2^i} e_{2i-1}$. Let $p = \sum_{i=1}^{\infty} \frac{1}{2^i} e_{2i-1} = T(1,1,1,\dots)$. Observe that p is not a smooth point of ℓ_1; however, p is T-smooth: let $h = (a_i) \in S(\ell_\infty)$ and suppose $h(p) = 1$, then $a_i = 1$ for all $i = 2n-1$ where $n \in \mathbb{N}$. Therefore $f(p) = g(p) = 1$ implies $f(T(a_i)) = g(T(a_i)) = \sum_{i=1}^{\infty} \frac{a_{2i-1}}{2^i}$ for all $(a_i) \in \ell_\infty$, $f, g \in S(\ell_\infty)$.

(c) Consider $T : \mathbb{R} \to (\mathbb{R} \oplus \mathbb{R})_\infty$ for which $T(x) = (x, 0)$. Observe that $p = (1,1)$ is not a T-smooth point, since $f_1(p) = f_2(p) = 1$ for $f_1 = \left(\frac{1}{2}, \frac{1}{2}\right)$ and $f_2 = (1,0)$ whereas $f_1(T(1)) = \frac{1}{2} \neq 1 = f_2(T(1))$. Therefore $(\mathbb{R} \oplus \mathbb{R})_\infty$ is not T-smooth. However, $\mathbb{R} = T(\mathbb{R})$ is a uniformly smooth space, so it is certainly T-smooth, in fact, $\mathbb{R}$ is T-uniformly smooth (see Definition 2.3.1).

<u>Remark</u> 2.1.3. As Example 2.1.2(c) illustrates, there is a $T \in \mathcal{B}(X, Y)$ such that Y is T-smooth but for some $Z \supset Y$, Z may not be T-smooth. Therefore, T-smoothness is a property of the codomain space with respect to the image of T. This is the reason we have chosen to call Y a T-smooth space rather than calling T a smooth operator. This remark will also apply to T-Fréchet differentiability and T-uniform smoothness.

The following generalizes a well known result of V. Klee to the setting of linear operators.

PROPOSITION 2.1.4. *Let $T \in B(X,Y)$ and T^* denote the dual operator of T.*

(a) If T^* is strictly convex, then Y is T-smooth.

(b) If X^* is T^*-smooth, then T is strictly convex.

Proof. (a) Suppose Y is not T-smooth, that is, there exists $y_0 \in S(Y)$ and $f, g \in S(Y^*)$ so that $f(y_0) = g(y_0) = 1$ but $f(Tx_1) \neq g(Tx_1)$ for some $x_1 \in B(X)$. Now, $1 = \|\frac{f+g}{2}\| \geq (\frac{f+g}{2})(y_0) = 1$, while $\|T^*f - T^*g\| \geq \|T^*(f-g)(x_1)\| > 0$. Therefore, T^* is not strictly convex.

(b) Suppose T is not strictly convex. Hence we can find $x, y \in S(X)$ so that $\|\frac{x+y}{2}\| = 1$ but $T(x-y) \neq 0$. Choose $f \in S(X^*)$ so that $f(\frac{x+y}{2}) = 1$. Observe that $f(x) = f(y) = 1$, that is $x(f) = y(f) = 1$ where we consider $x, y \in X^{**}$. Since $T(x-y) \neq 0$, for some $h \in Y^*$ we have $0 \neq h(T(x-y)) = x \circ T^*(h) - y \circ T^*(h)$. Therefore, f is not a T^*-smooth point. ■

DEFINITION 2.1.5. Let $T \in B(X,Y)$, we will say the norm of Y is T-*Gâteaux differentiable* at $y_0 \in S(Y)$ if

$$\lim_{t \to 0} \frac{\|y_0 + tTx\| + \|y_0 - tTx\| - 2}{t} = 0$$

for any fixed $x \in B(X)$. Moreover, Y is said to be T-*Gâteaux differentiable* if the norm of Y is T-Gâteaux differentiable at each $y \in S(Y)$.

PROPOSITION 2.1.6. *Let $T \in B(X,Y)$ and $y_0 \in S(Y)$. The norm of Y is T-Gâteaux differentiable at y_0 if and only if y_0 is a T-smooth point.*

Proof. $\Rightarrow$: Suppose y_0 is not a T-smooth point. Then there exists $f, g \in S(Y^*)$ and $x \in B(X)$ such that $f(y_0) = g(y_0) = 1$ but $f(Tx) - g(Tx) = \delta > 0$, for some $\delta > 0$. Now, for $t > 0$,

$$\frac{\|y_0 + tTx\| + \|y_0 - tTx\| - 2}{t} \geq \frac{f(y_0 + tTx) + g(y_0 - tTx) - 2}{t}$$
$$\geq \frac{t(f(Tx) - g(Tx))}{t} \geq \frac{\delta t}{t} = \delta > 0.$$

Therefore, the norm of Y is not T-Gâteaux differentiable at y_0.

$\Leftarrow$: Suppose the norm of Y is not T-Gâteaux differentiable at y_0. Hence, we find $\lambda_n \downarrow 0$, $x_0 \in S(X)$ and some $\delta > 0$ so that

$$(1) \qquad \frac{\|y_0 + \lambda_n Tx_0\| + \|y_0 - \lambda_n Tx_0\| - 2}{\lambda_n} \geq \delta > 0.$$

Pick $f_n, g_n \in S(Y^*)$ so that $f_n(y_0 + \lambda_n Tx_0) = \|y_0 + \lambda_n Tx_0\|$ and $g_n(y_0 - \lambda_n Tx_0) = \|y_0 - \lambda_n Tx_0\|$ for all n. Observe that $f_n(y_0) \geq \|y_0\| - \lambda_n\|Tx_0\| - \lambda_n\|Tx_0\|$ and so $f_n(y_0) \to 1$. Similarly, $g_n(y_0) \to 1$. By Alaoglu's theorem, there are $f, g \in B(Y^*)$ so that f is a w^* cluster point of $\{f_n\}$ and g is a w^* cluster point of $\{g_n\}$. Therefore, $f(y_0) = g(y_0) = 1$. Choose subsequences $\{f_{n_k}\}$ and $\{g_{n_k}\}$ so that $f_{n_k}(Tx_0) \to f(Tx_0)$ and $g_{n_k}(Tx_0) \to g(Tx_0)$. Thus, recalling (1),

$$\delta \leq \frac{\|y_0 + \lambda_{n_k} Tx_0\| + \|y_0 - \lambda_{n_k} Tx_0\| - 2}{\lambda_{n_k}}$$
$$= \frac{f_{n_k}(y_0 + \lambda_{n_k} Tx_0) + g_{n_k}(y_0 - \lambda_{n_k} Tx_0) - 2}{\lambda_{n_k}}$$
$$\leq \frac{\lambda_{n_k}(f_{n_k} - g_{n_k})(Tx_0)}{\lambda_{n_k}}.$$

Taking the limit as $k \to \infty$, we have $(f - g)(Tx_0) \geq \delta$. Therefore, y_0 is not a T-smooth point. ∎

2.2. Fréchet Differentiability.

As was the case with Gâteaux differentiability, there is a natural way to define Fréchet differentiability in the setting of operators. A particular interest of this section is the (partial) duality between T-Fréchet differentiability and locally uniformly convex operators.

DEFINITION 2.2.1. Let $T \in B(X,Y)$. We say the norm of Y is *T-Fréchet differentiable at* $y_0 \in S(Y)$ if

$$\lim_{t \to 0} \frac{\|y_0 + tTx\| + \|y_0 - tTx\| - 2}{t} = 0$$

uniformly for $x \in B(X)$. If the norm of Y is T-Fréchet differentiable at every $y \in S(Y)$, then Y is said to be *T-Fréchet differentiable.*

An element $y \in S(Y)$ is said to be a *T-strongly smooth point* if $\|T^* f_n - T^* f\| \to 0$ whenever f, $\{f_n\} \subset S(Y^*)$ satisfy $f_n(y) \to f(y) = 1$.

It is easy to see that T-strongly smooth points are T-smooth points and T-Fréchet differentiability immediately implies T-Gâteaux differentiability. A less obvious but useful relation which is analogous to the situation in Banach spaces is

PROPOSITION 2.2.2. *Let $y \in S(Y)$, then y is a T-strongly smooth point if and only if the norm of Y is T-Fréchet differentiable at y.*

Proof. $\Rightarrow$: If the norm of Y is not T-Fréchet differentiable at y, then there exists $\{t_n\} \subset \mathbb{R}$, $t_n \downarrow 0$, $\{x_n\} \subset B(X)$ and $\varepsilon > 0$ so that

$$\|y + t_n T x_n\| + \|y - t_n T x_n\| \geq \varepsilon t_n + 2.$$

Choose $f_n, g_n \in S(Y^*)$ which for all $n \in \mathbb{N}$ satisfy

$$f_n(y + t_n Tx_n) = \|y + t_n Tx_n\| \quad \text{and} \quad g_n(y + t_n Tx_n) = \|y + t_n Tx_n\|.$$

Observe that $f_n(y) \to 1$ and $g_n(y) \to 1$ since $t_n \downarrow 0$. However,

$$f_n(y + t_n Tx_n) + g_n(y - t_n Tx_n) \geq 2 + \varepsilon t_n,$$

and therefore,

$$t_n \|T^* f_n - T^* g_n\| \geq (f_n - g_n)(t_n Tx_n) \geq \varepsilon t_n.$$

So for any $f \in S(X^*)$, in particular for f such that $f(y) = 1$, we have

$$\|T^* f - T^* f_n\| + \|T^* f - T^* g_n\| \geq \|T^* f_n - T^* g_n\| \geq \varepsilon.$$

Now, either $\|T^* f - T^* f_n\| \not\to 0$ or $\|T^* f - T^* g_n\| \not\to 0$, therefore g is not a T-strongly smooth point.

$\Leftarrow$: Let $\varepsilon > 0$. If the norm of Y is T-Fréchet differentiable at y, then we can find $\delta > 0$ so that

$$\|y + hTx\| + \|y - hTx\| \leq 2 + \frac{\varepsilon}{2}\,\delta$$

for all $x \in \hat{B}(X)$ and all $h \in \mathbb{R}$ satisfying $|h| \leq \delta$. Thus,

$$f_n(y + Tx) + f(y - Tx) \leq 2 + \frac{\varepsilon}{2}\,\delta$$

for all f_n, $f \in S(Y)$ and $|h| \leq \delta$. Suppose that $f_n(y) \to f(y) = 1$. Choosing m so that $n \geq m$ implies $f_n(y) \geq 1 - \frac{\varepsilon}{2}\delta$ for all $n \geq m$, we obtain

$$(f_n - f)(hTx) \leq 2 - f_n(y) - f(y) + \frac{\varepsilon}{2}\,\delta \leq \varepsilon\delta.$$

Taking the supremum over $x \in B(X)$ and $|h| \leq \delta$ yields

$$\|T^* f_n - T^* f\| \leq \varepsilon$$

for all $n \geq m$. Therefore, y is a T-strongly smooth point. ∎

Corollary 2.2.3. *If $T \in B(X,Y)$ and T^* is locally uniformly convex, then Y is T-Fréchet differentiable.*

Proof. Let $y \in S(Y)$ be arbitrary. Choose $f \in S(Y^*)$ so that $f(y) = 1$. If $\{f_n\} \subset S(Y^*)$ and $f_n(y) \to 1$, then $\|f_n + f\| \geq (f_n + f)(y) \to 2$. Hence $\|T^*f - T^*f_n\| \to 0$ by the local uniform convexity of T^*; therefore y is a T-strongly smooth point. ∎

We have now developed enough machinery to easly prove the main and final result of this section.

Theorem 2.2.4. *Let $T \in B(X,Y)$ and suppose TX is separable. Then the following are equivalent:*

(a) T^*Y^* is separable.

(b) Y^* has an equivalent dual norm on which T^* is locally uniformly convex.

(c) There is an equivalent norm on Y which is T-Fréchet differentiable.

(d) There is an equivalent norm on Y so that y is a T-strongly smooth point for every $y \in S(Y)$.

Proof. By the virtues of Corollary 1.4.9, Proposition 2.2.2 and Corollary 2.2.3, even when TX is not separable, we have (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Leftrightarrow$ (d). We now show (d) $\Rightarrow$ (a). Since separability is isomorphism invariant we assume that every $y \in S(Y)$ is a T-strongly smooth point. Observe further that if $X_1 = \overline{TX} \subset Y$ then $T^*X_1^* = T^*Y^*$, thus we may assume that Y is separable. Let $D = \{y_k\}$ be a dense set in $S(Y)$. Choose $f_k \in S(Y^*)$ so that $f_k(y_k) = 1$ for all k. It suffices to show $T^*(D_1)$ is dense in $T^*(S(Y^*))$ where $D_1 = \{f_k : f_k \text{ is chosen as above}\}$.

By the Bishop–Phelps theorem, $D_2 = \{f \in S(Y^*) : f(y) = 1 \text{ for some } y \in S(Y)\}$ is dense in $S(Y^*)$. Let $f_0 \in D_2$, then $f_0(y_0) = 1$ for some $y_0 \in S(Y)$. Choose $\{y_n\} \subset D_1$ so that $y_n \to y_0$. Now $f_n(y_0) \to f_0(y_0)$, hence $\|T^* f_n - T^* f_0\| \to 0$ since y_0 is a T-strongly smooth point. Therefore $T^* D_1$ is dense in $T^*(S(Y^*))$ since it is dense in $T^*(D_2)$. ∎

2.3. Uniform Smoothness.

The notion of T-uniform smoothness is defined and shown to be the dual notion of uniformly convex operators.

DEFINITION 2.3.1. Let $T \in \mathcal{B}(X,Y)$, then Y is said to be T-*uniformly smooth* if given $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ so that

$$\|y + \alpha Tx\| + \|y - \alpha Tx\| < 2 + \varepsilon\alpha$$

for all $0 < \alpha < \delta$, $y \in S(Y)$ and $x \in B(X)$.

LEMMA 2.3.2. *Suppose $T \in \mathcal{B}(X,Y)$ is uniformly convex, then X^* is T^*-uniformly smooth.*

Proof. Suppose X^* is not T^*-uniformly smooth, that is, there exists $\{f_n\} \subset S(X^*)$, $\{g_n\} \subset B(Y^*)$, and $\{t_n\} \subset \mathbb{R}$, $t_n \downarrow 0$ for which

$$\|f_n + t_n T^* g_n\| + \|f_n - t_n T^* g_n\| \geq 2 + \varepsilon t_n.$$

Now choose $x_n, y_n \in B(X)$ so that

$$(f_n + t_n T^* g_n)(x_n) \geq \|f_n + t_n T^* g_n\| - \frac{\varepsilon t_n}{4},$$

and

$$(f_n - t_n T^* g_n)(y_n) > \|f_n - t_n T^* g_n\| - \frac{\varepsilon t_n}{4}$$

are satisfied for all n, in which case we have

$$(1) \qquad (f_n + t_n T^* g_n)(x_n) + (f_n - t_n T^* g_n)(y_n) \geq 2 + \frac{\varepsilon}{2} t_n.$$

By (1), $f_n(x_n + y_n) \to 2$ since $t_n \to 0$, and thus $\|x_n + y_n\| \to 2$. Moreover, it also follows from (1) that $t_n(T^* g_n(x_n - y_n)) \geq t_n \frac{\varepsilon}{2}$. Therefore, T is not uniformly convex since $\|Tx_n - Ty_n\| \geq T^* g_n(x_n - y_n) \geq \frac{\varepsilon}{2}$. ∎

LEMMA 2.3.3. *Let $T \in B(X, Y)$. If Y is T-uniformly smooth, then T^* is uniformly convex.*

<u>Proof</u>. Let $g_1, g_2 \in S(Y^*)$ and $\varepsilon > 0$ be given. Suppose $\|T^* g_1 - T^* g_2\| \geq \varepsilon$. Observe that

$$\begin{aligned}
\|g_1 + g_2\| &= \sup\{[g_1(y) + g_2(y)] : y \in S(Y)\} \\
&= \sup\{[g_1(y + \eta Tx) + g_2(y - \eta Tx) - (g_1 - g_2)(\eta Tx)] : y \in S(Y)\} \\
&\leq \sup\{[\|y + \eta Tx_0\| + \|y - \eta Tx_0\| - \eta \frac{\varepsilon}{2}]\} \\
&\leq 2 + \frac{\varepsilon\eta_0}{4} - \frac{\varepsilon\eta_0}{2} = 2 - \frac{\varepsilon\eta_0}{4}
\end{aligned}$$

where $x_0 \in B(X)$ is chosen so that $(y_1^* - y_2^*)(Tx_0) \geq \frac{\varepsilon}{2}$ and η_0 is chosen so that $\|y + \eta Tx\| + \|y - \eta Tx\| \leq 2 + \frac{\varepsilon}{4}\eta_0$ for all $0 < \eta < \eta_0$, $y \in S(Y)$ and $x \in B(X)$ by the T-uniform smoothness of Y. From the above inequality, as desired, T^* is uniformly convex. ∎

Now it is easy to prove

THEOREM 2.3.4. *(a) For $T \in \mathcal{B}(X,Y)$, the following are equivalent:*

(i) T is uniformly convex,

(ii) X^* is T^*-uniformly smooth,

(iii) T^{**} is uniformly convex.

(b) For $T \in \mathcal{B}(X,Y)$, the following are equivalent:

(i) Y is T-uniformly smooth,

(ii) T^* is uniformly convex,

(iii) Y^{**} is T^{**}-uniformly smooth.

Proof. (a) From Proposition 2.3.2 we have (i) $\Rightarrow$ (ii). Using $T = T^* \in \mathcal{B}(Y^*, X^*)$ in Proposition 2.3.3 it follows that (ii) $\Rightarrow$ (iii). We obtain (iii) $\Rightarrow$ (i) as a consequence of the fact $T^{**}x = Tx$ for $x \in X$ under the natural canonical embedding of X into X^{**}.

The proof of (b) is equally easy. ∎

We should remark that moduli of convexity and smoothness can be defined for operators.

DEFINITION 2.3.5. Let $T \in \mathcal{B}(X,Y)$. The ***modulus of convexity*** of T is defined as

$$\delta_T(\varepsilon) = \inf\{1 - \frac{1}{2}\|x+y\| : \|x\| = \|y\| = 1, \quad \|Tx - Ty\| \geq \varepsilon\}.$$

The ***modulus of smoothness*** for T is defined by

$$\rho_T(\tau) = \sup\left\{\frac{\|y + \tau Tx\| + \|y - \tau Tx\| - 2}{2} : x \in S(X), \quad y \in S(Y)\right\}.$$

Proceeding in the same manner as J. Lindenstrauss did for Banach spaces, we can prove the

THEOREM 2.3.0. *Let $T \in B(X, Y)$, then*

(a) $\rho_{T^*}(\tau) = \sup\left\{\frac{\varepsilon\tau}{2} - \delta_T(\varepsilon) : 0 < \varepsilon \leq 2\|T\|\right\}$

(b) $\rho_T(\tau) = \sup\left\{\frac{\varepsilon\tau}{2} - \delta_{T^*}(\varepsilon) : 0 \leq \varepsilon \leq 2\|T^*\|\right\}$.

From this one can also deduce the duality Theorem 2.3.4. We have not included the details here, because, subsequent to verifying this result, we found it in the Appendix of [8].

Chapter Three

EXTREMAL PROPERTIES OF OPERATORS

This chapter serves as an introduction to our study of extremal properties of operators. Specifically, we will define extreme, denting, exposed and strongly exposed points of operators and discuss some of their elementary properties in the first section. In Section 3.2 we present examples which show that operator analogues of some theorems on extreme points fail. The third section briefly examines the extremal structure of convex operators.

3.1. Definitions and Basic Facts.

We begin this chapter by introducing the notions of extreme, denting, exposed and strongly exposed points for operators.

DEFINITION 3.1.1. For the following, let C be a bounded subset of X and as usual $T \in \mathcal{B}(X,Y)$.

The T-diameter of $C = T\text{-diam}(C) = \sup\{\|Tx - Ty\| : x, y \in C\}$.

An element $x \in C$ will be called a *T-extreme point of C* if $T(y) = T(z)$ whenever $x = \frac{y+z}{2}$ and $y, z \in C$.

We shall say that x_0 is a *T-denting point of C* if given any $\varepsilon > 0$ there is some $g \in X^*$, $\beta > 0$ and a slice $S = S(g, \beta, C) = \{x \in C : g(x) \geq M(g, C) - \beta\}$ where $M(g, C) = \sup\{g(x) : x \in C\}$ so that $x \in S$ and T-diam$(S) < \varepsilon$. The set C is said to be *T-dentable* if it has slices of arbitrarily small T-diameter.

The element $x \in C$ is called a *T-exposed point of C* if for some $f \in X^*$ we have $f(x) > f(y)$ whenever $y \in C$ and $Tx \neq Ty$, in this case, the functional f is said to *T-expose x in C*.

If there exists $f \in X^*$ such that $f(x) \geq f(y)$ for all $y \in C$ and $\|Tx - Tx_n\| \to 0$ whenever $f(x_n) \to f(x)$ for $\{x_n\} \subset C$ then $x \in C$ is a *T-strongly exposed point of C*; such an f is said to *T-strongly expose $x \in C$*.

The terminology *w^*-T-denting point* will be used if the slices are given by w^*-continuous functionals. Similarly for w^*-T-(strongly) exposed points.

Remark 3.1.2. (a) In [2, p. 42,43] there are examples showing x is an extreme point $\not\Rightarrow$ x is an exposed point $\not\Rightarrow$ x is a strongly exposed point. Therefore letting T be the identity operator on X, we have that x is T-extreme $\not\Rightarrow$ x is T-exposed $\not\Rightarrow$ x is T-strongly exposed.

(b) Clearly if x is an extreme point of C, then x is a T-extreme point of C for any T; similarly for denting, exposed and strongly exposed points.

Fact 3.1.3. (a) If T is 1–1, then x is an extreme point of C if and only if x is a T-extreme point of C.

(b) Let $x \in C$. Then x is a T-strongly exposed point of $C \Rightarrow x$ is a T-exposed point of $C \Rightarrow x$ is a T-extreme point of C.

(c) Suppose $\varepsilon_n \downarrow 0$ and the T-diameter of the slices $S_n = S(g, \varepsilon_n, C) \to 0$. If $x_0 \in \bigcap_n S(g, \varepsilon_n, C)$, then x_0 is T-strongly exposed in C by g.

(d) Suppose $T : X \to Y$ and y_0 is an extreme (resp. denting) point of $TC \subset Y$. If $x_0 \in C$ and $Tx_0 = y_0$, then x_0 is a T-extreme (resp. T-denting) point of C.

(e) If $T(B(X))$ is subset dentable, then X is subset T-dentable.

(f) Suppose $T : X \to Y$ and f (strongly) exposes $y_0 = Tx_0 \in TC$ for $C \subset X$. Then x_0 is T-(strongly) exposed in C by $\lambda = f \circ T$.

Proof. (a) This is clear since $Tx = Ty$ if and only if $x = y$.

(b) Suppose f T-strongly exposes $x \in C$ and $f(y) = f(x)$ for some $y \in C$. Setting $y_n = y$ for all n yields $f(y_n) \to f(x)$, so $\|Ty_n - Tx\| \to 0$, that is $\|Ty - Tx\| = 0$.

Suppose $x \in C$ is T-exposed by f. If $\frac{y+z}{2} = x$ for some $y, z \in C$, then $f(\frac{y+z}{2}) = f(x)$, so $f(x) = f(y) = f(z)$ and hence $Tx = Ty$.

(c) Notice that $g(x_0) \geq M(g,C) - \varepsilon_n$ for all n. Since $\varepsilon_n \to 0$ we have $g(x_0) = M(g,C)$. Now suppose $g(x_n) \to g(x_0)$ and let $\varepsilon > 0$. Find N_1 so that $n \geq N_1$ implies T-diam$(S_n) < \varepsilon$. Since $g(x_n) \to M(g,C)$, we choose N_2 so that $x_n \in S_{N_1} = S(g, \varepsilon_{N_1}, C)$ whenever $n \geq N_2$. Thus $\|Tx_n - Tx_0\| \leq T$-diam$(S_{N_1}) < \varepsilon$ for $n \geq N_2$. Therefore, $\|Tx_n - Tx_0\| \to 0$.

The proofs of (d) through (f) are all similar and easy so we prove only (f) for $y_0 \in TC$ which is strongly exposed by f. Set $\lambda = f \circ T$ and let $Tx_0 = y_0$. Clearly, $\lambda x \leq \lambda x_0$ for $x \in C$, moreover, if $\lambda x_n \to \lambda x_0$ then $\|f(Tx_n) - f(Tx_0)\| \to 0$. Therefore, $\|Tx_n - Tx_0\| \to 0$ since f strongly exposes Tx_0. ■

Example 3.1.4. (a) There exists $T : X \to Y$ and $x \in C \subset X$ which is a strongly exposed point of C but Tx is not an extreme point of TC. Consider $T : (\mathbb{R} \oplus \mathbb{R})_{\ell_2} \to \mathbb{R}$ given by $T(x,y) = x$. Then $(0,1)$ is a strongly exposed point of the unit ball of $(\mathbb{R} \oplus \mathbb{R})_{\ell_2}$, however, $T(0,1) = 0$ is not a strongly exposed point of $[-1,1] \subset \mathbb{R}$. So the converses of 3.1.3(d) and (f) fail.

(b) Consider $P_n : c_0 \to c_0$ given by $P_n(a_i) = \sum_{i=1}^{n} a_i e_i$. Let $a = \sum_{i=1}^{n} \theta_i e_i + \sum_{i=n+1}^{\infty} a_i e_i$ where $\{\theta_i\}_{i=1}^{n} \in \{-1, 1\}^n$, $|a_i| < 1$ for $i \geq n+1$ and $a_i \to 0$. Now, a is a P_n extreme point of $B(c_0)$ because $x = (x_i)$, $y = (y_i)$, $x, y \in B(c_0)$ and $\frac{x+y}{2} = a$ implies $x_i = y_i = \theta_i$ for $i = 1, 2, \ldots, n$. Therefore $P_n(x - y) = 0$. In fact it is easy to check that $a \in B(c_0)$ is P_n strongly exposed by $f = \sum_{i=1}^{n} \frac{\theta_i}{n} e_i \in B(\ell_1)$. Notice, however, that a is not an extreme point of $B(c_0)$ since $B(c_0)$ has no extreme points.

3.2. Some Remarks on Operator Extreme Points.

Until this juncture, things in the generalized setting of operators have behaved essentially as they do in spaces. However, the first example of this section shows that the obvious operator analogue of the Krein Milman theorem fails.

<u>Example</u> 3.2.1. (a) Let $T : c_0 \to \ell_2$ be the one to one compact operator defined by $T(a_i) = (\frac{1}{2^i} a_i)$. By Fact 3.1.3(a), $a \in B(c_0)$ is a T extreme point of $B(c_0)$ if and only if a is an extreme point of $B(c_0)$. Therefore $B(c_0)$ has no T extreme points.

(b) There are even linear functionals which have no extreme points on $B(c_0)$. Define $T : c_0 \to \mathbb{R}$ by $T(a_i) = \sum_{i=1}^{\infty} \frac{a_i}{2^i}$. Let $a = (a_i) \in B(c_0)$, hence $|a_n| \leq \frac{1}{2}$ for some $n \in \mathbb{N}$. Choose $b = (b_i)$ and $c = (c_i)$ so that $b_i = c_i = a_i$ for $i \neq n$, $b_n = a_n + \frac{1}{4}$ and $c_n = a_n - \frac{1}{4}$. Then, $b, c \in B(c_0)$ and $\frac{b+c}{2} = a$, but

$$Tb - Tc = \frac{1}{2^{n+1}} > 0.$$

The following proposition is an analogue of a result of J. Lindenstrauss [10, Lemma 1].

Proposition 3.2.2. *Let $T \in B(X,Y)$ and suppose K is a closed, bounded and convex set of X. If every closed convex subset of K has a T-extreme point, then K is the closed convex hull of its T-extreme points.*

Proof. Let $T\text{-ext}(K)$ denote the set of extreme points of K. Suppose $D = \overline{\text{conv}}(T\text{-ext}(K))$ and $D \neq K$. By the Hahn-Banach separation and Bishop-Phelps theorems, we can choose $f \in X^*$ so that $f(y) = \sup\{f(x) : x \in K\} > \sup\{f(x) : x \in D\}$ for some $y \in K$. Let $F = \{x \in K : f(x) = f(y)\}$; observe that F is a closed, convex and non-empty subset of K. By the hypothesis, F has a T-extreme point, say x_0. Suppose $x_0 = \frac{v+w}{2}$ for some $v, w \in K$. Then $f(x_0) = f(v) = f(w)$ so $v, w \in F$ and hence $Tv = Tw$. Therefore we have a T-extreme point of K which is not in D, a contradiction. ∎

Corollary 3.2.3. *Suppose $T : X \to Y$ is a weakly compact operator, TC is closed in Y whenever C is closed in X. If F is a closed, bounded and convex subset of X, then $F = \overline{\text{conv}}(T\text{-ext}(F))$.*

Proof. Let F be a closed, bounded and convex subset of X, D a closed convex subset of F. By the Krein-Milman theorem, TD has an extreme point, say $y_0 = Tx_0$. By Fact 3.1.3(d), x_0 is a T-extreme point of D. Finally, from Proposition 3.2.2 it follows that $F = \overline{\text{conv}}(T\text{-ext}(F))$. ∎

Because of the excessively demanding hypotheses of Corollary 3.2.3 there is no claim it has any applicable value. However, contrasting Corollary 3.2.3 with Example 3.2.1 seems to indicate that because the images of complete sets may not be complete for general operators, we cannot expect analogues of theorems which rely on the completeness of a space to hold for operators. In fact, one can check, that if the intersection of nested slices S_n was non empty when T-diam$(S_n) \to 0$, then every closed, bounded and convex subset of X would have a T-extreme point if X was subset T-dentable. We do not include the details here, but provide an example which implicitly shows that if $\cap S_n$ can be empty for the nested slices S_n, then we do not obtain the above analogue of the theorem of J. Lindenstrauss which states that the Radon-Nikodym property implies the Krein-Milman property.

Example 3.2.4. Let $T : c_0 \to \ell_2$ be defined by $T(a_i) = (\frac{1}{2^i}\, a_i)$. Then every bounded subset of c_0 is T-dentable by Fact 3.1.3(e). However, $B(c_0)$ has no T-extreme points as seen in Example 3.2.1.

3.3. Extremal Properties of Convex Operators.

As a prelude to the next chapter, this section presents a few elementary facts concerning convex operators and the extremal properties of operators introduced in Section 3.1.

Fact 3.3.1. (a) Let $T \in B(X, Y)$, then the following are equivalent:

(i) T is strictly convex.

(ii) If $x \in S(X)$, then x is a T-extreme point of $B(X)$.

(iii) If $x \in S(X)$, then x is a T-exposed point of $B(X)$.

(b) If $T \in B(X, Y)$ is a locally uniformly convex operator, then every $x \in S(X)$ is a T-strongly exposed point of $B(X)$.

Proof. (a) (i) $\Rightarrow$ (iii): Let $x \in S(X)$, choose $f \in S(X^*)$ so that $f(x) = 1$. Suppose that $f(y) = 1$ for some $y \in B(X)$. Then $\left\|\frac{x+y}{2}\right\| \geq f\left(\frac{x+y}{2}\right) = 1$. Thus $Tx = Ty$ by the strict convexity of T.

(iii) $\Rightarrow$ (ii): This is in Fact 3.1.3(b).

(ii) $\Rightarrow$ (i): Let $\left\|\frac{z+y}{2}\right\| = 1$ where $y, z \in B(X)$. Now $x = \frac{z+y}{2}$ is a T extreme point, so $Tx = Ty$.

(b) Let $x \in S(X)$, choose $f \in S(X^*)$ so that $f(x) = 1$. If $f(x_n) \to 1$, then $\left\|\frac{x+x_n}{2}\right\| \geq f\left(\frac{x+x_n}{2}\right) \to 1$. Therefore, $\|Tx_n - Tx\| \to 0$ by the local uniform convexity of T. ∎

Before proving the next proposition, we remind the reader that an element $x_0 \in C \subset X$ is a *farthest point* of C if there is a $y \in X$ such that $\|x_0 - y\| \geq \|c - y\|$ for all $c \in C$.

Proposition 3.3.2. *Let $C \subset X$, $T \in B(X, Y)$ and x_0 be a farthest point of C.*

(a) If T is strictly convex, then x_0 is a T-exposed point of C.

(b) If T is locally uniformly convex, then x_0 is T-strongly exposed in C.

Proof. Because the proofs of (a) and (b) are almost identical, we prove only (b). Choose $y \in X$ so that $r = \|y - x_0\| \geq \|y - c\|$ for all $c \in C$. Let $f \in S(X^*)$ be chosen so that $f(x_0 - y) = r$. Now $f(c - y) \leq \|c - y\| \leq r = f(x_0 - y)$ for all $c \in C$, thus $f(c) \leq f(x_0)$. Suppose that $\{c_n\} \subset C$ and $f(c_n) \to f(x_0)$. Note that $\|c_n - y\| \leq r$, $\|x_0 - y\| \leq r$ and

$$\|(c_n - y) + (x_0 - y)\| \geq f(c_n - y) + f(x_0 - y) \to 2(f(x_0 - y)) = 2r.$$

Thus, by the local uniform convexity of T, we have

$$\|T(c_n - x_0)\| = \|T(c_n - y) - T(x_0 - y)\| \to 0.$$

Therefore x_0 is a T-strongly exposed point of C. ∎

For Banach spaces it is known that if X is uniformly convex, then every closed, bounded and convex set in X is the closed convex hull of its strongly exposed points. However, this is not the case for T-strongly exposed points of bounded, closed and convex sets in a space X on which T is uniformly convex as is made manifest by the

Example 3.3.3. In Example 1.4.5 it was shown that $T : (c_0, \|\cdot\|) \to \ell_2$ given by $T(a_i) = \left(\frac{1}{2^i} a_i\right)$ is uniformly convex where $\|(a_i)\| = \left(\left\|(a_i)\right\|_\infty^2 + \left\|\frac{1}{2^i}(a_i)\right\|_2^2\right)^{1/2}$. This is an equivalent norm on c_0, so $B(c_0)$ is a closed, bounded and convex set in $(c_0, \|\cdot\|)$. However, as was shown in Example 3.2.1(a), $B(c_0)$ has no T-extreme points, *a fortiori* no T-strongly exposed points.

Indeed, Example 3.3.3 shows that the convex operators do not possess the extremal structure of convex Banach spaces, however from Proposition 3.3.2 there is hope that if a Banach space has enough extremal structure, then a convex operator on that space may have some extremal properties. In the next chapter we investigate the extremal properties of convex operators on dual spaces.

Chapter Four

EXPOSED POINTS OF CONVEX OPERATORS ON w^*-COMPACT SETS

There are basically two results in this chapter. The first is contained in Section 4.1 where we prove, using farthest point techniques, that for strictly convex operators T on duals to RNP spaces the convex w^*-compact sets are the w^*-closed convex hulls of their T-exposed points. In the second section of this chapter, we utilize techniques of J. Lindenstrauss and R.R. Phelps to show that convex w^*-compact sets are the w^*-closed convex hulls of their w^*-T-strongly exposed points whenever T is locally uniformly convex on the dual space.

4.1. *T*-exposed Points in Duals of RNP Spaces.

Before ambling into the theorem of this section, some notation should be established. For S a subset of some Banach space and T an operator, T-exp(S) denotes the set of all T-exposed points of S, and T-str exp(S) denotes the set of T-strongly exposed points of S. The notation $M(f,S) = \sup\{f(x) : x \in S\}$ and $M(S) = \sup\{\|x\| : x \in S\}$ will also be employed.

The following result of R. Deville and V. Zizler [4, Proposition 3] was proven recently and thus may not be widely known, so for convenience of reference, it is recorded here without proof as

PROPOSITION 4.1.1. *Let X be a Banach space with the RNP (Radon–Nikodym property), X^* be its dual space in its usual dual norm and C be a w^*-compact subset of X^*. Then the set D of all points in X^* which have farthest points in C contains a subset dense and G_δ in X^*.*

Having Proposition 4.1.1 at our disposal, we are now able to prove

THEOREM 4.1.2. *Let X have the RNP and let C be a convex and w^*-compact subset of X^*.*

(a) *If $T \in \mathcal{B}(X^*, Y)$ is strictly convex, then $C = \overline{\text{conv}}^{w^*}(T\text{-exp}(C))$.*

(b) *If $T \in \mathcal{B}(X^*, Y)$ is locally uniformly convex, then $C = \overline{\text{conv}}^{w^*}(T\text{-str exp}(C))$.*

Proof. (a) Let $D = \overline{\text{conv}}^{w^*}(T\text{-exp}(C))$. If $D \neq C$, by the Hahn–Banach separation theorem there exists a w^*-continuous functional f such that $M(f, C) \geq M(f, D) + \delta$ for some $\delta > 0$. Following the technique of J. Lindenstrauss used in Theorem 2 of [9], we define $V : X^* \to (X^* \oplus \mathbb{R})_{\ell_2}$ by $Vx = (x, f(x))$. Now V is w^*–w^* continuous since the identity $I : X^* \to X^*$ is w^*–w^* continuous and f is a w^*-continuous functional. Thus $V(C)$ is a w^*-compact and convex set in $(X^* \oplus \mathbb{R})_{\ell_2}$. Observe further that $(X^* \oplus \mathbb{R})_{\ell_2}$ is a dual to a space with the RNP since $(X^* \oplus \mathbb{R})_{\ell_2} = (X \oplus \mathbb{R})^*_{\ell_2}$ and $X \oplus \mathbb{R}$ has the RNP since X and $\mathbb{R}$ do. Now let $k = \delta^{-1}(M(D) + 1)^2$ where $\delta > 0$ is as above. By Proposition 4.1.1 we can find $(y_0, r) \in (X^* \oplus \mathbb{R})_{\ell_2}$ so that (y_0, r) has a farthest point in $V(C)$ for some y_0 such that $\|y_0\| \leq 1$ and $r < M(f, D)$ which satisfies $M(f, D) - r \geq k$. Let

$k_1 = M(f,D) - r$; for $x \in D$ we have:

$$\begin{aligned}\|(y_0,r) - Vx\| &= \|(y_0,r) - (x, f(x))\| \\ &= \left(\|y_0 - x\|^2 + (r - f(x))^2\right)^{1/2} \\ &\le \left((\|y_0\| + \|x\|)^2 + k_1^2\right)^{1/2} \\ &\le \left((1 + M(D))^2 + k_1^2\right)^{1/2}.\end{aligned}$$

Whence,

$$\sup\{\|(y_0,r) - Vx\| : x \in D\} \le \left((1 + M(D))^2 + k_1^2\right)^{1/2}. \tag{1}$$

On the other hand,

$$\begin{aligned}\sup\{\|(y_0,r) - Vx\| : x \in C\} &\ge \left(0^2 + (r - M(f,C))^2\right)^{1/2} \\ &\ge \left((M(f,D) + \delta - r)^2\right)^{1/2} \\ &= \left((k_1 + \delta)^2\right)^{1/2} = \left(k_1^2 + 2\delta k_1 + \delta^2\right)^{1/2} \\ &\ge \left(k_1^2 + 2(M(D)+1)^2 + \delta^2\right)^{1/2} \qquad \left[\text{since } k_1 \ge \left(\frac{M(D)+1}{\delta}\right)^2\right] \\ &> \left(k_1^2 + (M(D)+1)^2\right)^{1/2} \\ &\ge \sup\{\|(y_0,r) - Vx\| : x \in D\} \qquad [\text{by (1)}].\end{aligned}$$

Thus if $x \in V(C)$ is a farthest point to (y_0,r) we must have $x \in V(C) \sim V(D)$. Because (y_0,r) has a farthest point in $V(C)$, for some $p_0 \in C \sim D$ we have

$$\rho := \|(y_0,r) - Vp_0\| = \sup\{\|(y_0,r) - Vc\| : c \in C\}.$$

We are now in a position to show that p_0 is a T-exposed point of C which will contradict the assumption that $D = \overline{\text{conv}}^{w^*}(T\exp(C))$. Find $g \in (X^* \oplus \mathbb{R})^*_{\ell_2}$ so that $\|g\| = 1$ and $g(Vp_0 - (p_0, r)) = \rho$. Define $\lambda \in (X^*)^*$ by $\lambda = g \circ V$. For $c \in C$, we have

$$g(Vp_0) - g(y_0, r) = \rho \geq \|Vc - (y_0, r)\| \geq g(Vc) - g(y_0, r),$$

hence

$$\lambda(p_0) = g(Vp_0) \geq g(Vc) = \lambda(c) \qquad \text{for all} \quad c \in C.$$

Suppose $\lambda(c) = \lambda(p_0)$ for some $c \in C$. Then $\lambda(c) - g(y_0, r) = \lambda(p_0) - g(y_0, r)$ and so $g(Vc) - g(y_0, r) = g(Vp_0) - g(y_0, r) = \rho$. Whence,

$$\|Vc - (y_0, r) + Vp_0 - (y_0, r)\| = 2\rho.$$

From Proposition 1.2.9 it follows that $T_1 : (X^* \oplus \mathbb{R})_{\ell_2} \to Y$ given by $T_1(x, r) = Tx$ is strictly convex since T is. Therefore,

$$\begin{aligned} 0 &= \|T_1(Vc - (y_0, r)) - T_1(Vp_0 - (y_0, r))\| \\ &= \|T_1(Vc) - T_1(Vp_0)\| \\ &= \|Tc - Tp_0\|. \end{aligned}$$

So indeed, p_0 is T-exposed. ■

The proof of (b) has been omitted, because a stronger result will be presented in the next section; moreover, the reader may have noticed that (b) can be proved with only minor modifications of the proof of (a). It should be emphasized that Y in general need not be a dual space, however, the fact that T is strictly convex on a dual norm was utilized (in the proof of Proposition 4.1.1).

4.2. w^*-T-Strongly Exposed Points in Dual Spaces.

Before attempting to prove that convex w^*-compact sets are the w^*-closed convex hulls of their w^*-T-strongly exposed points for locally uniformly convex operators on the dual norm, some preliminaries are in order.

Terminology. Consider the following condition $(*)$:

$(*)$ For every $\varepsilon > 0$ and $g \in X = (X^*, w^*)^*$ with $\|g\| = 1$ and $D = C \cap g^{-1}(0)$ with $C \sim D \neq \phi$, there exists a w^*-slice S given by $f \in S(X)$ of C with T-diam$(S) < \varepsilon$ and $S \cap D = \phi$ for a given operator T.

We shall say that a Banach space X^* with a $T \in B(X^*, Y)$ has *property* $(*)$ if $(*)$ is satisfied for all convex w^*-compact sets in X^*. In this case we simply write (X^*, T) has property $(*)$.

It should be noted that our motivation to consider the property $(*)$ comes from [12, Lemma 4].

Notation. Similar to the notation of Section 4.1, we denote the set of w^*-T-strongly exposed points of a set S by w^*-T-str exp(S).

The following elementary facts will be used in Lemma 4.2.2.

Fact 4.2.1. (a) If $g \in (X^*, w^*)^*$, A, B are bounded subsets of X^*, and $C = \overline{\text{conv}}^{w^*}(A \cup B)$ then $M(g, C) = \max\{M(g, A), M(g, B)\}$.

(b) Let C be a bounded convex set, g be a functional so that $M(g, C) > 0$, and $C_1 = \{x \in C : g(x) \geq 0\}$. If $S = S(f, \varepsilon, C_1)$ is a slice so that $S \cap g^{-1}(0) = \phi$ and $M(f, C_1) = M(f, C)$, then $S(f, \varepsilon, C_1) = S(f, \varepsilon, C)$.

(c) Suppose $\rho > 0$, $\|f\| = \|g\| = 1$ and $|g(x)| \leq \frac{\rho}{2}$ whenever $x \in B(X) \cap f^{-1}(0)$, then either $\|f - g\| \leq \rho$ or $\|f + g\| \leq \rho$.

The proof of Fact 4.2.1 has been omitted since (a), (b) are easy to observe and well known, while (c) is the clever lemma used in proving the Bishop Phelps theorem and [5, p. 2] is one of many places in which a proof may be found. The next two lemmas are operator analogues of special cases of the beautiful geometric lemmas of R.R. Phelps and E. Bishop respectively, see [12, Lemmas 6 and 7]; the proofs require only cosmetic modifications of those appearing in [12] to accomodate for the added generality of operators. All functionals are assumed to be of norm one in the following lemmas.

LEMMA 4.2.2. *Let X^* be a dual space, $T \in B(X^*, Y)$ and (X^*, T) have property $(*)$. Suppose further that $S(f, \alpha, C)$ is a w^*-slice of the w^*-compact convex set C and $0 < \varepsilon < 1$. Then there exists a slice $S(g, \beta, C)$ of T-diam$(S(g, \beta, C)) < \varepsilon$ such that $g \in S(X)$, $\|f - g\| < \varepsilon$ and $S(g, \beta, C) \subset S(f, \alpha, C)$.*

Proof. By translation we may assume that the origin is in the hyperplane

$$H = \{x \in X^* : f(x) = M(f, C) - \alpha\}.$$

That is, $H = f^{-1}(0)$ and $M(f, C) = \alpha > 0$. Let $M = M(C)$ and let $C_1 = \overline{\text{conv}}^{w^*}[S(f, \alpha, C) \cup (\lambda B(X^*) \cap H)]$ where $\lambda = 2M\varepsilon^{-1} + 1$. Since (X^*, T) has property $(*)$ there is a w^*-slice $S(g, \beta, C_1)$ of T-diameter less than ε which misses $C_1 \cap f^{-1}(0) = \lambda B(X^*) \cap H$. By Fact 4.2.1(a),

$$M(g, C_1) = \max\{M(g, S(f, \alpha, C)),\ M(g, \lambda B(X^*) \cap H)\}.$$

However, $g(x) \leq M(g, C_1) - \beta$ for all $x \in \lambda B(X^*) \cap H$, so

$$M(g, C_1) = M(g, S(f, \alpha, C)).$$

Now, $S(g,\beta,S(f,\alpha,C))$ is a slice of $S(f,\alpha,C)$. Since $S(g,\beta,S(f,\alpha,C))$ misses $f^{-1}(0)\cap C$, Fact 4.2.1(b) implies that

$$S(g,\beta,C) = S(g,\beta,S(f,\alpha,C)).$$

Therefore, $S(g,\beta,C) \subset S(f,\alpha,C)$ and T-diam$(S(g,\beta,C)) \leq T$-diam$(S(g,\beta,C_1)) < \varepsilon$.

To show $\|f-g\| < \varepsilon$, we proceed exactly as in [12] and include the proof here for the sake of completeness. We may choose $z \in S(g,\beta,C)$ so that

$$g(z) > M(g,C_1) - \beta \geq M(g,\lambda B(X^*)\cap H) \geq 0.$$

The symmetry of $\lambda B(X^*)\cap H$ also implies

$$g(\lambda B(X^*)\cap H) \subset [-g(z),g(z)].$$

Equivalently, by linearity, $g[B(X^*)\cap f^{-1}(0)] \subset [-\lambda^{-1}g(z),\lambda^{-1}g(z)]$. Using $\rho = 2\lambda^{-1}g(z)$ in Fact 4.2.1(c), we have either

$$\text{(i)}\quad \|f-g\| \leq 2\lambda^{-1}g(z) \qquad \text{or} \qquad \text{(ii)}\quad \|f+g\| \leq 2\lambda^{-1}g(z).$$

Now, $\|z\| \leq M$ since $z \in C$ and $f(z) = a \geq 0$ since $z \in S(g,\beta,C) \subset S(f,\alpha,C)$. If (ii) occurs, then

$$2\lambda^{-1}g(z) \geq \|f+g\| \geq (f+g)\Big(\frac{z}{\|z\|}\Big) \geq \frac{g(z)}{\|z\|} + \frac{a}{\|z\|} \geq \frac{g(z)}{\|z\|} \geq \frac{g(z)}{M}.$$

Thus we would obtain $2M \geq \lambda$ which contradicts $\lambda = 2M\varepsilon^{-1}+1 > 2M$. Therefore, (i) must occur, that is,

$$\|f-g\| \leq 2\lambda^{-1}g(z) < \frac{2g(z)}{2M}\,\varepsilon \leq \varepsilon.$$

■

LEMMA 4.2.3. *Let X be a Banach space, $T \in \mathcal{B}(X^*, Y)$, and C be a convex w^*-compact subset of X^*. Suppose that for each slice $S(f, \alpha, C)$ where $f \in S(X)$ and for each $\delta > 0$ there exists a slice $S = S(g, \beta, C)$ with $g \in S(X)$ so that T-diam$(S) < \varepsilon$ and moreover*

$$S(g, \beta, C) \subset S(f, \alpha, C) \qquad \text{and} \qquad \|f - g\| < \varepsilon.$$

Then for every $f \in S(X)$, with $\alpha > 0$ and $\beta > 0$ given, there exists $g \in S(X)$ so that $\|f - g\| \leq \beta$ and $x_0 \in S(f, \alpha, C)$ which is T-strongly exposed by g in C.

Proof. We assume without loss of generality that $M(C) = 1$. Let $\alpha, \beta > 0$ be given and consider the slice $S(f, \alpha, C)$ where $f \in S(X)$. Let $g_0 = f$ and $\beta_0 = \min\left\{\frac{\beta}{2}, \alpha\right\}$. Now construct, inductively, sequences $\{g_k\} \subset S(X)$ and $\{\beta_k\} \subset (0, \beta_0)$ which satisfy

$$\|g_{k+1} - g_k\| < 2^{-k}\beta_k \qquad \text{and} \qquad \beta_{k+1} < 2^{-1}\beta_k,$$

by using the hypothesis, g_k, β_k can be chosen so that

$$T\text{-diam}(S(g_k, \beta_k, C)) < 2^{-1}\beta_k \qquad \text{and} \qquad S(g_{k+1}, \beta_{k+1}, C) \subset S(g_k, \beta_k, C).$$

Observe that for all k, j we have

$$\begin{aligned} \|g_{k+j} - g_k\| &= \|g_{k+j} - g_{k+j-1} + g_{k+j-1} - g_{k+j-2} + \cdots + g_{k+1} - g_k\| \\ &\leq \|g_{k+j} - g_{k+j-1}\| + \|g_{k+j-1} - g_{k+j-2}\| + \cdots + \|g_{k+1} - g_k\| \\ &\leq 2^{-(k+j-1)}\beta_{k+j-1} + 2^{-(k+j-2)}\beta_{k+j-2} + \cdots + 2^{-k}\beta_k \\ (1) \qquad &\leq 2^{-k+1}\beta_k. \end{aligned}$$

Therefore, $\{g_k\}$ is norm Cauchy in $S(X)$, and so $g_k \to g$ for some $g \in S(X)$. Taking the limit $j \to \infty$ in (1), we obtain

(2) $$\|g - g_k\| \le 2^{-k+1}\beta_k \qquad \text{for all } k.$$

In particular,

(2′) $$\|g - f\| = \|g - g_0\| \le 2\beta_0 \le \beta.$$

Since we assumed $M(C) = 1$, by (2) we obtain

(3) $$|g(x) - g_k(x)| \le 2^{-k+1}\beta_k \qquad \text{for any} \quad x \in C, \qquad k = 0, 1, 2, \ldots$$

Furthermore, for $x \in C$, $M(g, C) \ge g(x) \ge g_k(x) - 2^{-k+1}\beta_k$, so

(4) $$M(g, C) \ge M(g_k, C) - 2^{-2}\beta_k \quad \text{for} \quad k \ge 3.$$

Whence if $x \in S\big(g, \frac{\beta_k}{4}, C\big)$ and $k \ge 3$, we have

$$\begin{aligned} g_k(x) &\ge g(x) - 2^{-2}\beta_k & \text{[by (3)]} \\ &\ge \left[M(g, C) - \frac{\beta_k}{4}\right] - 2^{-2}\beta_k \\ &\ge \left[\left(M(g_k, C) - \frac{\beta_k}{4}\right) - \frac{\beta_k}{4}\right] - \frac{\beta_k}{4} & \text{[by (4)]} \\ &> M(g_k, C) - \beta_k. \end{aligned}$$

Thus, $S\big(g, \frac{\beta_k}{4}, C\big) \subset S(g_k, \beta_k, C)$ for $k \ge 3$, and therefore T-diam$\big(S(g, \frac{\beta_k}{4}, C)\big) < \beta_k$, that is T-diam$\big(S(g, \frac{\beta_k}{4}, C)\big) \to 0$. Now $S_\delta = \bigcap_{k \ge 3} S\big(g_k, \frac{\beta_k}{4}, C\big)$ is non-empty since it is a nested intersection of w^*-compact sets. Choosing some $x_0 \in S_\delta$, Fact 3.1.3(c) guarantees that x_0 is T-strongly exposed in C by g. Finally, $S_\delta \subset S(g, \beta, C) \subset S(f, \alpha, C)$ and $\|f - g\| \le \beta$ by (2′). ∎

By dint of Lemmas 4.2.2 and 4.2.3 we obtain the following operator analogue of [12, Theorem 10].

PROPOSITION 4.2.4. *Let $T \in B(X^*, Y)$ and suppose (X^*, T) has property $(*)$. Then for any given w^*-compact and convex set K in X^*, the w^*-continuous functionals which T-strongly expose K are dense in X. In particular,*

$$K = \overline{\text{conv}}^{w^*}(w^*\text{-}T\text{-str}\exp(K)).$$

Proof. Since (X^*, T) has property $(*)$, the conclusion of Lemma 4.2.2 applies to K, and hence the hypotheses of Lemma 4.2.3 are satisfied. Notice that Lemma 4.2.3 implies that the T-strongly exposing functionals of K are dense in X; thus, in particular, by the Hahn Banach separation theorem, $K = \overline{\text{conv}}^{w^*}(w^*\text{-}T\text{-str}\exp(K))$. ∎

We should remark that the only reason we have restricted our attention to w^*-slices on w^*-compact sets is because, in general, the intersection of nested slices in Lemma 4.2.3 need not be non-empty. Other than that we have endeavored to avoid excessive tampering with the elegant and lucid proofs in [12].

Having proved Proposition 4.2.4, it is of interest to find a sufficiently large class of operators so that (X^*, T) has property $(*)$ for a given T in such a class. Before we show that the locally uniformly convex operators form such a class, we should indicate a result without proof which follows from obvious modifications of [17, Proposition 4].

PROPOSITION 4.2.5. *Let $T : X^* \to Y^*$ be a w^*-w^* continuous operator and K be a w^* compact subset of X^*. For any $\epsilon > 0$, there is a w^*-w^* continuous operator $S : X^* \to Y^*$ such that $\|S - T\| < \epsilon$ and $\|Sx_0\| = \sup\{\|Sx\| : x \in K\}$ for some $x_0 \in K$.*

PROPOSITION 4.2.6. *Let T be a locally uniformly convex operator on X^*, then (X^*, T) has property $(*)$.*

Proof. Let C be a w^* compact and convex subset of X^*. Suppose $g \in S(X)$, $D = C \cap g^{-1}(0)$ with $C \sim D \neq \phi$, and $\epsilon > 0$. We wish to find a w^* slice S of C with T-diam$(S) \leq \epsilon$ such that $S \cap D = \phi$. We may suppose without loss of generality that $T(C) \neq 0$ and $M(g, C) > 0$. Let

$$\text{(i)} \qquad M = \frac{M(C) + 1}{M(g, C)}$$

and using the ingenious method of J. Lindenstrauss [9, Theorem 2] we define $V : X^* \to (X^* \oplus \mathbb{R})_{\ell_2}$ by $Vx = (x, Mg(x))$. Observe that V is w^*-w^* continuous since g and the identity are. Therefore, by Proposition 4.2.5 there is a w^*-w^* continuous operator $\widehat{V}$ so that

$$\text{(ii)} \qquad \|\widehat{V}x_0\| = \sup\{\|\widehat{V}x\| : x \in C\} \qquad \text{and} \qquad \|\widehat{V} - V\| \leq \rho$$

where $\rho = \min\left\{\frac{\epsilon}{4\|T\|M(C)}, \frac{1}{4M(C)}\right\}$. Define $T_1 : (X^* \oplus \mathbb{R})_{\ell_2} \to Y$ by $T_1(x, r) = T(x)$. From Proposition 1.2.9(b) it follows that T_1 is locally uniformly convex. Therefore we can choose δ so that $0 < \delta < \frac{3}{8}$ and $\|T_1\widehat{V}x_0 - T_1 y\| \leq \frac{\epsilon}{4}$ whenever $y \in (X^* \oplus \mathbb{R})_{\ell_2}$, $\|y\| \leq \|\widehat{V}x_0\|$ and $\|\widehat{V}x_0 + y\| \geq 2\|\widehat{V}x_0\| - \delta$. Now take $f \in$

$S((X \oplus \mathbb{R})_{\ell_2})$ so that $f(\widehat{V}x_0) > \|\widehat{V}x_0\| - \frac{\delta}{3}$. Let $\lambda \in X^*$ be defined by $\lambda = f \circ \widehat{V}$ and $S = S(\lambda, C, \frac{\delta}{3})$. Notice that

$$\sup\{\lambda(x) : x \in C\} \geq \lambda(x_0) > \|\widehat{V}x_0\| - \frac{\delta}{3}.$$

Thus, for $y \in S$, we have

$$\lambda(y) > \lambda(x_0) - \frac{\delta}{3} > \|\widehat{V}x_0\| - \frac{2\delta}{3}. \tag{1}$$

Hence,

$$\|\widehat{V}y + \widehat{V}x_0\| \geq f(\widehat{V}y + \widehat{V}x_0) = \lambda(y) + \lambda(x_0) > 2\|\widehat{V}x_0\| - \delta.$$

By our choice of δ, $\|T_1\widehat{V}y - T_1\widehat{V}x_0\| \leq \frac{\varepsilon}{4}$. Therefore,

$$\|T_1\widehat{V}x - T_1\widehat{V}y\| \leq \|T_1\widehat{V}x - T_1\widehat{V}x_0\| + \|T_1\widehat{V}y - T_1\widehat{V}x_0\| \leq \frac{\varepsilon}{2} \tag{2}$$

for all $x, y \in S$. Finally, given any $x, y \in S$

$$\begin{aligned}
\|Tx - Ty\| &= \|T_1Vx - T_1Vy\| \\
&= \|T_1V(x-y) - T_1\widehat{V}(x-y) + T_1\widehat{V}(x-y)\| \\
&\leq \|(T_1V - T_1\widehat{V})(x-y)\| + \|T_1\widehat{V}(x-y)\| \\
&\leq \|T_1\|\,\|V - \widehat{V}\|\,\|x-y\| + \frac{\varepsilon}{2} \qquad \text{[by (2)]} \\
&\leq \frac{\varepsilon}{4\|T\|M(C)}\|T_1\|2M(C) + \frac{\varepsilon}{2} \leq \varepsilon \quad \text{[using (ii) and noting } \|T_1\| \leq \|T\|\text{]}.
\end{aligned}$$

Therefore, T-diam$(S) < \epsilon$. It remains to be shown that $S \cap D = \phi$. Let $x \in D$, then

$$\|\widehat{V}x\| \leq \|(\widehat{V} - V)x\| + \|Vx\|$$

$$\leq \frac{1}{4M(C)}M(C) + M(C) \qquad \text{[by (i) and noting } \|Vx\| = \|x\| \text{ for } x \in D]$$

$$(3) \qquad = M(C) + \frac{1}{4}.$$

However,

$$\|\widehat{V}x_0\| = \sup\{\|\widehat{V}x\| : x \in C\} = \sup\{\|Vx - (V - \widehat{V})x\| : x \in C\}$$

$$\geq \sup\{\|\widehat{V}x\| - \|(V - \widehat{V})x\| : x \in C\}$$

$$\geq \sup\{\|\widehat{V}x\| : x \in C\} - \sup\{\|V - \widehat{V}\| \, \|x\| : x \in C\}$$

$$\geq M \cdot M(g, C) - \frac{1}{4M(C)}M(C) = M \cdot M(g, C) - \frac{1}{4}$$

$$(4) \qquad \geq (M(C) + 1) - \frac{1}{4} = M(C) + \frac{3}{4}.$$

Moreover, if $x \in S$, then

$$\|\widehat{V}x\| \geq f(\widehat{V}x) \geq \|\widehat{V}x_0\| - \frac{2}{3}\delta \qquad \text{[by (1)]}$$

$$(5) \qquad \geq \|\widehat{V}x_0\| - \frac{2}{3}\cdot\frac{3}{8} = \|\widehat{V}x_0\| - \frac{1}{4} \geq M(C) + \frac{1}{2} \qquad \text{[by (4)].}$$

From (3) and (5) we have $S \cap D = \phi$. Now by homogeneity, $S = S(\lambda, C, \frac{\delta}{3}) = S(\frac{\lambda}{\|\lambda\|}, C, \frac{\delta}{3\|\lambda\|})$, therefore (X^*, T) has property $(*)$. ∎

It is clear that Propositions 4.2.4 and 4.2.6 immediately bestow upon us

THEOREM 4.2.7. *Let $T \in B(X^*, Y)$ be a locally uniformly convex operator with respect to a dual norm of X^*. Then for any w^*-compact and convex subset K of X^*, the w^*-continuous functionals which T-strongly expose K are dense in X. In particular, $K = \overline{\text{conv}}^{w^*}(w^*\text{-}T\text{-str exp}(K))$.*

Remark 4.2.8. (a) Theorem 4.2.7 may fail for strictly convex operators $T \in B(X^*, Y)$. Consider $X^* = (\ell_\infty, \|\cdot\|)$ where $\|(x_i)\| = \left(\|(x_i)\|_\infty^2 + \|(\frac{1}{2^i}x_i)\|_2^2\right)^{1/2}$. It follows from Fact 1.3.4 and Example 1.4.7 that $\|\cdot\|$ is a strictly convex dual norm on ℓ_∞. Therefore $I : X^* \to X^*$, where I is the identity, is a strictly convex operator. Since X^* does not have the RNP ($B(c_0) \subset X^*$ has no extreme points), it has a w^*-compact convex set K such that $K \neq \overline{\text{conv}}^{w^*}(w^*\text{-str exp}(K)) = \overline{\text{conv}}^{w^*}(w^*\text{-}I\text{-str exp}(K))$ by the well known results of [12].

(b) It was already noted in Example 3.3.3 that if $T \in B(X, Y)$ is uniformly convex, then there may be a closed convex subset of X which has no T-extreme points.

(c) Let $X^* = (\ell_\infty, \|\cdot\|)$ as in (a). By Example 1.4.7 $T : X^* \to \ell_2$ given by $T(x_i) = (\frac{1}{2^i}x_i)$ is a uniformly convex dual operator. However, $B(c_0)$ has no extreme points. Therefore, even for uniformly convex dual operators there is no guarantee that closed convex sets have T-extreme points.

(d) We cannot replace $K = \overline{\text{conv}}^{w^*}(w^*\text{-}T\text{-str exp}(K))$ in the conclusion of Theorem 4.2.7 by $K = \overline{\text{conv}}(w^*\text{-}T\text{-str exp}(K))$. To see this, recall that H.P. Rosenthal [14, p. 374] has shown that ℓ_∞ has a w^*-compact and convex set K so that

$K \neq \overline{\text{conv}}(\text{ext}(K))$. Hence for T as in (c), $K \neq \overline{\text{conv}}(T\text{-ext}(K))$ since T is one-to-one, *a fortiori*, $K \neq \overline{\text{conv}}(w^*\text{-}T\text{-str exp}(K))$.

(e) We have already seen [Example 3.2.4] that because the preimage topology of T is not complete, we cannot have a precise analogue of [12, Theorem 9], that is, X is subset T-dentable does not imply every closed, bounded and convex subset of X is the closed convex hull of its T-strongly exposed points. We however, have not determined the truth of the statement:

Let $T \in B(X^*, Y)$. If every w^*-compact convex subset of X^* is w^*-T-dentable, then every w^*-compact convex set is the w^*-closed convex hull of its w^*-T-strongly exposed points.

In the next chapter it will be shown that the above statement is true for dual operators. However, for the general case, we only mention that it does not seem like the methods of [12, Lemma 4] can be easily adapted to show that if every w^*-compact convex set is T-dentable, then (X^*, T) has property $(*)$. For instance, consider $C = \{(x, y) \in (\mathbb{R} \oplus \mathbb{R})_{\ell_2} : -1 \leq y \leq 1,\ x = 0\}$ and $T : (\mathbb{R} \oplus \mathbb{R})_{\ell_2} \to \mathbb{R}$ by $T(x, y) = x$. Now $S = \left\{(x, y) \in C : y \geq \frac{1}{2}\right\}$ is a slice of C, and $T\text{-diam}(S) \leq T\text{-diam}(C) = 0$. Consider the isometry $V = (\mathbb{R} \oplus \mathbb{R})_{\ell_2} \to (\mathbb{R} \oplus \mathbb{R})_{\ell_2}$ given by $V(x, y) = (y, x)$. Then $T\text{-diam}(V(S)) = \frac{1}{2}$. So $T\text{-diam}(V(S)) > K \cdot (T\text{-diam}(S))$ for any K in $\mathbb{R}$. This shows that the operator analogue of [12, Lemma 3] fails; note that [12, Lemma 3] is used in proving [12, Lemma 4].

(f) Finally, we mention that if T is uniformly convex, then Proposition 4.2.6 can be proved without using Proposition 4.2.5.

We close this chapter with an application of Theorem 4.2.7 to strongly smooth points. As is our wont, we begin with a

LEMMA 4.2.9. *Let $T \in B(X,Y)$ and $y \in S(Y)$. If y is a w^*-T^*-strongly exposing functional of $B(Y^*)$, then y is a T-strongly smooth point of $B(Y)$.*

Proof. Let g be T^*-strongly exposed in $B(Y^*)$ by $y \in B(Y)$. Then $g(y) = y(g) \geq \sup\{y(f) : f \in B(Y^*)\} = 1$. Now suppose $f_n(y) \to g(y)$ where $\{f_n\} \subset B(Y^*)$, that is $y(f_n) \to y(g)$. Since y T^*-strongly exposes g, we have $\|T^*f_n - T^*g\| \to 0$. Therefore, y is a T-strongly smooth point. ∎

PROPOSITION 4.2.10. *Let $T \in B(X,Y)$ and suppose T^* is locally uniformly convex. If $Y_1 = (Y, \|\cdot\|_1)$ where $\|\cdot\|_1$ is an equivalent norm on Y, then the T-strongly smooth points are dense in $S(Y_1)$.*

Proof. Let $K = B(Y_1^*)$, then K is a convex w^*-compact subset of Y^*. By Theorem 4.2.7 the functionals which T^*-strongly expose K are dense in $S(Y_1)$. Therefore the T-strongly smooth points are dense in $S(Y_1)$ by Lemma 4.2.8. ∎

As an immediate consequence of Proposition 4.2.9 and Theorem 2.3.4(b) we have

COROLLARY 4.2.11. *If Y is T-uniformly smooth and $Y_1 = (Y, \|\cdot\|_1)$ where $\|\cdot\|_1$ is an equivalent norm on Y, then the T-strongly smooth points are dense in $S(Y_1)$.*

Chapter Five

ON CHARACTERIZATIONS OF ASPLUND OPERATORS

This chapter presents some characterizations of Asplund operators, that is, operators $T \in B(X,Y)$ for which $T(B(X))$ has the Asplund property. A well known characterization is: *$T(B(X))$ has the Asplund property if and only if $T^*(B(X^*))$ has the RNP.* Using this, we are able to characterize Asplund operators in terms of extremal properties of T^* on w^*-compact sets (Theorem 5.2.2). In the case that T is an Asplund operator with separable range, using the methods of D. v. Dulst and I. Namioka [6], known facts about w^*-compact RNP sets and results on local uniformly convex operators and T-Fréchet differentiability already presented in this thesis, in Theorem 5.2.4 we also characterize T by the T-Fréchet differentiability of Y and local uniform convexity of T^* under certain equivalent renormings.

5.1. Preliminaries.

The theorems we will prove in the next section utilize many properties of Asplund and RNP sets which are the result of the work of many mathematicians. This section has been set aside to list such properties in an orderly fashion. Our source of enlightenment is Chapters Four and Five of [2].

DEFINITION 5.1.1. Let $D \subset X$ be a bounded set and $F : X \to \mathbb{R}$ any function. We say that F is *D-differentiable at $x \in X$* if there is an $f \in X^*$ such that

$$\lim_{t\to 0^+} \sup_{d\in D} \left| \frac{F(x+td)-F(x)}{t} - f(d) \right| = 0.$$

A bounded $D \subset X$ is said to have the *Asplund property* if each convex continuous function $F : X \to \mathbb{R}$ is D-differentiable on a residual subset of X, that is, a subset that contains a dense G_δ subset of X.

Operators $T \in B(X, Y)$ for which $T(B(X))$ has the Asplund property may be referred to as *Asplund operators*. Notice that if $T(B(X))$ has the Asplund property in $\overline{TX}$ then $T(B(X))$ has the Asplund property in $Y \supset \overline{TX}$ by [2, Theorem 5.2.5].

Remark 5.1.2. (a) It is easy to see that a convex function $F : X \to \mathbb{R}$ is $\{d, -d\}$ differentiable if and only if

$$\lim_{t \to 0} \frac{F(x + td) + F(x - td) - 2F(x)}{t} = 0.$$

(b) Observe that $\|\cdot\|_1 : (X, \|\cdot\|_2) \to \mathbb{R}$ where $\|\cdot\|_1$, $\|\cdot\|_2$ are equivalent norms on X is a continuous convex real-valued function. Since $T(B(X))$ is symmetric it follows from (a) and the definitions given in this thesis that Y is T-Fréchet differentiable on a residual set if $T \in B(X, Y)$ is an Asplund operator.

The following result is proved in [2, Section 5.2].

THEOREM 5.1.3. *Let $T \in B(X, Y)$, then $T(B(X))$ has the Asplund property if and only if $T^*(B(Y^*))$ has the RNP.*

One of the reasons for our interest in Asplund operators stems from the following result which is Theorem 5.3.5 in [2].

THEOREM 5.1.4. *Let $T \in B(X, Y)$, if $T(B(X))$ has the Asplund property, then T factors through an Asplund space, that is a space Z for which $B(Z)$ has the Asplund property.*

We will also use some characterizations of w^*-compact convex RNP sets. First, recall that a *tree* in a Banach space E is a sequence $\{x_n : n = 1, 2, \dots\} \subset E$ such that $x_n = \frac{1}{2}(x_{2n} + x_{2n+1})$ for each n; a δ-*tree* is a tree for which

$$\|x_{2n} - x_n\| = \|x_{2n+1} - x_n\| \geq \delta \quad \text{for each } n.$$

The following result is contained in [2, Theorem 4.2.3].

THEOREM 5.1.5. *Let $C \subset X^*$ be a w^*-compact convex set. Then the following are equivalent:*

(a) *C has the RNP.*

(b) *For each w^*-compact convex subset K of C, K is the w^*-closed convex hull of its w^*-strongly exposed points.*

(c) *Each w^*-compact convex subset D of C has w^*-slices of arbitrarily small diameter.*

(d) *C contains no δ-tree for any $\delta > 0$.*

5.2. Characterizations of Asplund Operators.

The interplay between many of the concepts introduced in this thesis will be seen in the two main results of this section which provide characterizations of Asplund and separable Asplund operators respectively. Without further ado, we state without proof a nice lemma which was used in the proof of [11, Thm. 12].

LEMMA 5.2.1. *Let $T : X^* \to Y^*$ be a w^*–w^* continuous operator and C be a w^*-compact and convex subset of $T(B(X^*))$. Then there exists a convex w^*-compact subset K of $B(X^*)$ so that $T(K) = C$ but $T(K_1) \neq C$ for any proper convex w^*-compact subset K_1 of K.*

We remark that Lemma 5.2.1 can be proved using Zorn's lemma and a compactness argument; however, our present interest lies in proving

THEOREM 5.2.2. *Let $T \in \mathcal{B}(X,Y)$, then the following are equivalent:*

(a) $T(B(X))$ has the Asplund property.

(b) $T^*(B(Y^*))$ has the RNP.

(c) $T^*(B(Y^*))$ has no δ-tree for any $\delta > 0$.

(d) $K = \overline{\text{conv}}^{w^*}(w^*\text{-}T^*\text{-str}\exp(K))$ for all w^*-compact convex $K \subset Y^*$.

(e) Every w^*-compact convex set $K \subset Y^*$ has w^*-slices of arbitrarily small T^*-diameter.

Proof. By Theorems 5.1.3 and 5.1.5 we have (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c).

(b) $\Rightarrow$ (d): Let $D = \overline{\text{conv}}^{w^*}(w^*\text{-}T^*\text{-str}\exp(K))$ and suppose $D \neq K$. Once again, the methods of J. Lindenstrauss [9] will be employed. By the Hahn–Banach separation theorem there exists a w^*-continuous functional f such that

$$\sup\{f(x) : x \in K\} \geq \sup\{f(x) : x \in D\} + \delta \qquad \text{for some} \quad \delta > 0.$$

Let $V : Y^* \to (X \oplus \mathbb{R})^*_{\ell_\infty} = (X^* \oplus \mathbb{R})_{\ell_1}$ be defined by $Vx = (T^*x, f(x))$. Of course, V is w^*–w^* continuous since f and T^* are. Moreover, $T(K)$ has the RNP since $T^*(B(Y^*))$ has the RNP and $f(K)$ has the RNP since it is a compact subset of $\mathbb{R}$. Therefore, $V(K) = (T^*(K), f(K))$ has the RNP in $(X^* \oplus \mathbb{R})_{\ell_1}$. Now, $V(D)$ is convex and w^*-compact since V is w^*–w^* continuous. Moreover, $V(D) \neq V(K)$ since $f(D) \neq f(K)$. By Theorem 5.1.4, $V(K)$ is the w^*-closed convex hull of its w^*-strongly exposed points, so there exists $y_0 \in V(K) \sim V(D)$ such that y_0 is strongly exposed by some $g \in (Y \oplus \mathbb{R})_{\ell_\infty}$. But now, $y_0 = Vx_0$ for some $x_0 \in K \sim D$. By Fact 3.1.3(f), x_0 is V-strongly exposed in K by $g \circ V$. Observe that $\|Vx_n - Vx_0\| = \|T^*x_n - T^*x_0\| + |f(x_n) - f(x_0)|$, so $\|Vx_n - Vx_0\| \to 0$ implies $\|T^*x_n - T^*x_0\| \to 0$. Therefore, $x_0 \in K \sim D$ is T^*-strongly exposed by the w^*-continuous functional $g \circ V$, a contradiction which shows that $D = K$.

(d) $\Rightarrow$ (e): Each w^*-T^*-strongly exposed point is a w^*-T^*-denting point.

(e) $\Rightarrow$ (b): Let $\varepsilon > 0$ and C be a convex w^*-compact subset of $T^*(B(Y^*))$. Lemma 5.2.1 asserts that there is a minimal w^*-compact and convex subset K of $B(Y^*)$ so that $T^*(K) = C$. By hypothesis, K is w^*-T^*-dentable, so choose $S = S(f, \alpha, K)$ a w^*-slice of K with T^*-diam$(S) < \varepsilon$. Let $F = \{x \in K : f(x) \leq M(f, K) - \alpha\}$. Then F is a proper subset of K which is w^*-compact and convex, hence $T^*(F) \neq T^*(K) = C$. Now, $T^*(F)$ is a w^*-compact and convex proper subset of C, so by the Hahn–Banach separation theorem there exists a w^*-slice $S(g, \beta, C)$ of C so that $S(g, \beta, C) \cap T(F) = \phi$. Thus diam$(S(g, \beta, C)) \leq$ T^*-diam$(S) < \varepsilon$, which implies C is w^*-dentable. Therefore $T^*(B(Y^*))$ has the RNP by the provisions of Theorem 5.1.4. ∎

COROLLARY 5.2.3. *(a) Let $T \in \mathcal{B}(X,Y)$, if Y^* has an equivalent dual norm so that T^* is locally uniformly convex, then each of conditions (a) to (e) is satisfied in Theorem 5.2.2.*

(b) Let $T \in \mathcal{B}(X,Y)$, if T^* is locally uniformly convex with respect to some equivalent dual norm, then T factors through an Asplund space.

Proof. (a) Observe that if $T^* : Y^* \to X^*$ is locally uniformly convex on a dual norm, hence Proposition 4.2.6 implies that Y^* has w^*-slices of arbitrarily small T^*-diameter. So condition (e) and thus conditions (a) through (e) are satisfied in Theorem 5.2.2.

(b) This follows from part (a) and Theorem 5.1.4. ■

The final objective of this section is to prove the following equivalencies for an Asplund operator with separable range.

THEOREM 5.2.4. *Let $T \in \mathcal{B}(X,Y)$ and suppose that TX is separable, then the following are equivalent:*

(a) $T(B(X))$ has the Asplund property.

(b) $T^*(B(Y^*))$ has the RNP.

(c) $T^*(B(Y^*))$ does not contain a δ-tree for any $\delta > 0$.

(d) $K = \overline{\text{conv}}^{w^*}(w^*\text{-}T^*\text{-str}\exp(K))$ for all w^*-compact convex $K \subset Y^*$.

(e) Each w^*-compact convex $K \subset Y^*$ is w^*-T^*-dentable.

(f) T^*Y^* is separable.

(g) Y^* has an equivalent dual norm on which T^* is locally uniformly convex.

(h) There is an equivalent norm on Y which is T-Fréchet differentiable.

(i) There is an equivalent norm on Y, so that every $y \in S(Y)$ is a T-strongly smooth point.

Proof. From Theorem 5.2.2 it is immediate that (a) ⇔ (b) ⇔ (c) ⇔ (d) ⇔ (e). Also recall that (f) ⇔ (g) ⇔ (h) ⇔(i) is exactly what was proved in Theorem 2.2.4. Moreover, from Corollary 5.2.3 it follows that (g) ⇒ (e). In order to complete the proof it suffices to show that (c) ⇒ (f). We will proceed by producing an operator version of the very elegant construction of D. v. Dulst and I. Namioka in [6]. The construction will be broken into three steps with only technical modifications of the arguments presented in [6].

A. LEMMA. [v. Dulst, Namioka] *Let $\{K_n : n = 1, 2, 3, \ldots\}$ be a family of non-void compact convex subsets of a linear topological space E such that $K_{2n} \cup K_{2n+1} \subset K_n$ for each n. Then there is an infinite tree $\{x_n\}$ in E such that $x_n \in K_n$ for each n.*

Proof. See [6, Lemma 1].

The following is an operator analogue of [6, Proposition 2].

B. PROPOSITION. *Let $T \in \mathcal{B}(X, Y)$. If there exists a bounded set B in Y^* and an $\varepsilon > 0$ such that $\operatorname{diam}(T^*(U)) > \varepsilon$ whenever U is a non-empty relatively w^*-open subset of B, then $T^*(\overline{\operatorname{conv}}^{w^*}(B))$ contains an infinite $\frac{\varepsilon}{2}$-tree.*

Proof. We begin by constructing a sequence U_n of non-empty relatively w^*-open subsets of B and a sequence $\{x_n\} \subset X$ such that

(a) $\|x_n\| = 1 \qquad (n = 1, 2, \ldots)$,

(b) $U_{2n} \cup U_{2n+1} \subset U_n \qquad (n = 1, 2, \ldots)$, and

(c) for each n, if $f \in U_{2n}$ and $g \in U_{2n+1}$, then $T^*(f - g)(x_n) > \epsilon$.

Let $U_1 = B$. Suppose for some positive integer m that U_k is defined for $1 \le k < 2^m$ and x_n for $1 \le n < 2^{m-1}$ so that (a), (b) and (c) are valid for all $1 \le n < 2^{m-1}$. Let k be such that $2^{m-1} \le k < 2^m$; now $\operatorname{diam}(T^*(U_k)) > \epsilon$ by the hypothesis, so we can find h_0 and h_1 in U_k so that $\|x_k\| = 1$ and $T^*(h_0 - h_1)(x_k) = \epsilon + \delta$ for some $\delta > 0$. Let

$$U_{2k} = \left\{f \in U_k : T^*f(x_k) > T^*h_0(x_k) \ldots \right\} = \left\{f \in U_k : Tx_k(f) > Tx_k(h_0) - \frac{\delta}{2}\right\}$$

and

$$U_{2k+1} = \left\{g \in U_k : T^*f(x_k) < T^*h_1(x_k) + \frac{\delta}{2}\right\} = \left\{g \in U_k : Tx_k(g) < Tx_k(h_1) + \frac{\delta}{2}\right\}.$$

These sets are relatively w^* open and non empty subsets of B. Clearly (a) and (b) are satisfied for $n = k$, moreover if $f \in U_{2k}$ and $g \in U_{2k+1}$, then

$$(T^*f - T^*g)(x_k) > \left(T^*h_0(y_k) - \frac{\delta}{2}\right) - \left(T^*h_1(y_k) + \frac{\delta}{2}\right) = \epsilon + \delta - \delta = \epsilon,$$

which shows that (c) is satisfied for $n = k$. As k runs through $\{k : 2^{m-1} \le k < 2^m\}$, $n = 2k$ and $n = 2k+1$ exhausts $\{n : 2^m \le n < 2^{m+1}\}$ so the construction is complete.

For each n, let $K_n = \operatorname{conv}^{w^*}(U_n)$; now K_n is non-empty, w^*-compact and convex. Moreover, (b) implies $K_{2n} \cup K_{2n+1} \subset K_n$. By Lemma A, there is a tree $\{f_n\} \subset X^*$ such that $f_n \in K_n$ for all n. Clearly $f_{2n} - f_{2n+1}$ is the w^*-limit of $\{y_\alpha\}$ for some net $\{g_\alpha\} \subset \operatorname{conv}(U_{2n} - U_{2n+1})$. Therefore,

$$T^*(f_{2n} - f_{2n+1})(x_n) = (f_{2n} - f_{2n+1})(Tx_n)$$

(1) $$= \lim_\alpha g_\alpha(Tx_n) = \lim_\alpha T^*g_\alpha(x_n) \ge \epsilon$$

with the last inequality being true because of (c). Finally, $T^* f_n \in T^*(B)$ and

$$(2) \qquad T^*(f_{2n} - f_n) = \frac{1}{2} T^*(f_{2n} - f_{2n+1}) = T^*(f_{2n+1} - f_{2n}).$$

Thus $\|T^* f_{2n} - T^* f_n\| = \|T^* f_{2n+1} - T^* f_n\| > \frac{\epsilon}{2}$ by (1) and (2). Therefore, $\{T^* f_n\}$ is an infinite $\frac{\epsilon}{2}$-tree in $T^*(\text{conv}^{w^*}(B))$, which completes the proof of Proposition B.

The following operator analogue of [6, Corollary 3] will for all intents and purposes finish the proof of (c) $\Rightarrow$ (f).

C. COROLLARY. *Let $T \in B(X, Y)$ and suppose TX is separable. If $T^* Y^*$ is not separable, then for each $\epsilon \in (0, 1)$, $T^*(B(Y^*))$ has an $\frac{\epsilon}{2}$-tree.*

Proof. Fix ϵ so that $0 < \epsilon < 1$. If $T^*(B(Y^*))$ is not separable, there exists an uncountable set $A \subset B(Y^*)$ such that $\|T^* f - T^* g\| \geq \epsilon$ for any two distinct f and g in A. We may assume Y is separable since if $Y_1 = \overline{TX}$ and $Y_1 \subset Y$, then $T^*(B(Y_1^*)) = T^*(B(Y^*))$. Because Y is separable, $B(Y^*)$ is w^*-separable and w^*-metrizable so all but countably many points of A are w^*-condensation points of A. Let $A_1 = \{x \in A : x \text{ is a } w^*\text{-condensation point of } A\}$, notice that A_1 is uncountable because A is. Moreover, if U is w^*-open and $U_1 = U \cap A_1 \neq \phi$ then there are at least two distinct elements in U_1, this guarantees that $\text{diam}(T^*(U_1)) \geq \epsilon$. Finally, $\overline{\text{conv}}^{w^*}(A_1) \subset B(Y^*)$, so by Proposition B, $T^*(B(Y^*))$ contains on $\frac{\epsilon}{2}$-tree, thus Corollary C is proved.

From Corollary C it is clear that (c) $\Rightarrow$ (f): since TX is separable and $T^*(B(Y^*))$ does not contain any δ-trees for $\delta > 0$, $T^*(B(Y^*))$ must be separable. This completes the proof of Theorem 5.2.4. ∎

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