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AMENABILITY AND INVARIANCE PROPERTIES OF GROUP ALGEBRAS OF LOCALLY COMPACT GROUPS

by

Ross Stokke ©

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research for acceptance, a thesis entitled **Amenability and Invariance Properties of Group Algebras of Locally Compact Groups** submitted by **Ross Thomas Stokke** in partial fulfillment of the requirements for the degree of **Doctor of Philosophy** in Mathematics.

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To Anna and Ava

ABSTRACT

A net in the group algebra of a locally compact group which commutes asymptotically with elements from the measure algebra is called quasi-central. In this thesis we provide new characterizations of locally compact groups whose group algebras possess quasi-central bounded approximate units. Reiter-type and structural conditions for such groups are obtained which indicate that these groups behave much like the tractable [SIN]-groups. A general notion of an amenable action on the predual of a von Neumann algebra is developed to prove these theorems. Applications to the Fourier algebra are discussed.

We study the relationship between the classical invariance properties of amenable locally compact groups G and the approximate diagonals possessed by their associated group algebras $L^1(G)$. From the existence of a weak form of approximate diagonal for $L^1(G)$ we provide a direct proof that G is amenable. Conversely, we give a formula for constructing a strong form of approximate diagonal for any amenable locally compact group. In particular we have a new proof of Johnson's Theorem: A locally compact group G is amenable precisely when $L^1(G)$ is an amenable Banach algebra. Several structural Følner-type conditions are derived, each of which is shown to correctly reflect the amenability of $L^1(G)$. We show that a semigroup algebra is 1-amenable precisely when the semigroup is an amenable group. We obtain Følner conditions characterizing semigroups with 1-amenable semigroup algebras.

We consider amenable representations, introduce a notion of complete-amenability for representations, and examine the relationship between the two concepts. Several

C^*-algebraic characterizations of amenable and completely-amenable representations are obtained. We define versions of the Fourier and Fourier-Stieltjes algebras for an arbitrary representation. We show that a representation is amenable whenever an associated Fourier algebra has a bounded approximate unit, and that a representation is amenable precisely when an associated Fourier-Stieltjes algebra has an identity.

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Contents

Chapter 1

Introduction

A locally compact group G is called amenable if the space $L^\infty(G)$, of essentially bounded functions on G, possesses a translation-invariant functional called an invariant mean. The notion of amenability traces its origins back to the study of finitely additive measures which are invariant under groups of isometries, a theory which led to the celebrated Banach-Tarski paradox. Given the history of the subject, the prominence of invariance properties in the theory of amenable groups is of no surprise. Among these properties is the Reiter condition from which the very deepest of the classical invariance properties, the Følner conditions, can be derived. The Følner conditions are, in a sense, especially nice because they provide the only known descriptions of amenability in terms of the internal structure of the group itself, rather than in terms of a related Banach algebra.

A Banach algebra is called amenable if it possesses a particular cohomological property. The theory of amenable Banach algebras was born in 1972 when B.E. Johnson proved his famous theorem which states that a locally compact group G is amenable precisely when its associated group algebra $L^1(G)$ is amenable [24]. Shortly thereafter, Johnson proved his fundamental characterization of amenable Banach algebras in terms of the existence of virtual and approximate diagonals [25]. Virtual diagonals are often said to play the role in the theory of amenable Banach algebras that invariant means play in the theory of amenable groups. Indeed, the existence of approximate diagonals may be interpreted as an invariance property for amenable Banach algebras.

An elegant theory of amenable representations was developed in 1990 by M.E.B. Bekka [3]. In this general context Bekka proved analogues of all of the classical invariance properties including the Følner conditions, and interpreted several amenability theories in terms of amenable representations. In particular he proved that a locally compact group is amenable if and only if all of its representations are amenable.

A bounded net in a Banach algebra $\mathcal{A}$ which behaves asymptotically like an identity for $\mathcal{A}$ is called a bounded approximate unit. A bounded approximate unit for $L^1(G)$ is called quasi-central if it commutes asymptotically with elements from the measure algebra $M(G)$. In [48] A. Sinclair asked when group algebras possess quasi-central bounded approximate units. V. Losert and H. Rindler [36] have shown that group algebras of amenable groups always have quasi-central bounded approximate units and in Chapter 3 we provide new answers to Sinclair's question.

A locally compact group G is called a [SIN]-group (small invariant neighbourhood group) if it has a base for the neighbourhood system at the identity comprised of compact sets which are invariant under inner automorphisms. The [SIN]-groups are necessarily unimodular and are precisely those groups whose group algebras possess central bounded approximate units [37]. In Chapter 3 we call a locally compact group a quasi-[SIN]-group if it possesses a base for the neighbourhood system at the identity which is asymptotically invariant under inner automorphisms. We prove that for a group to be quasi-[SIN], it is both necessary and sufficient that it is unimodular and its group algebra has a quasi-central bounded approximate unit. This structural theorem combines with Losert and Rindler's theorem to say that unimodular amenable locally compact groups behave much like the very tractable class of [SIN]-groups.

In Chapter 4 we study the relationship between the classical invariance properties of amenable locally compact groups and the approximate diagonal invariance property of their associated group algebras. An integral part of this endeavour is our work on quasi-central bounded approximate units from Chapter 3. We give an explicit formula for constructing a compactly-invariant approximate diagonal (a very strong form of approximate diagonal) from Reiter's condition and a nice form of quasi-central bounded approximate unit possessed by amenable groups. Conversely, we show how Reiter's condition and strong forms of quasi-central bounded approximate units can be obtained from the existence of compactly-invariant approximate diagonals. From weaker forms of approximate diagonals we show how to construct nets converging to topological invariance. One corollary of these results is a new proof of Johnson's theorem. The existence of a compactly-invariant approximate diagonal can be interpreted as a Reiter condition for amenable group algebras, and doing this we derive new Følner conditions for amenable locally compact groups which have a very different flavour from their classical counterparts. We then show how related Følner conditions can be obtained from the classical Følner condition and our work from Chapter 3 on quasi-[SIN]-groups. All of our Følner conditions are shown to correctly reflect the amenability of $L^1(G)$ in the sense that they naturally yield compactly-invariant approximate diagonals comprised of normalized characteristic functions.

The definition of amenability also makes sense in the more general context of semigroups, and it is known that amenability of a semigroup algebra implies that the semigroup itself is amenable. However, unlike the case for groups, the converse does not hold. Thus a theme of many papers has been to address the problem of describing those semigroups, in terms of the internal structure of the semigroup itself, which carry amenable semigroup algebras. The problem, now a quarter century in age, has only been completely settled in special cases. The general trend however, is that amenability of the semigroup algebra imposes very strong conditions upon the semigroup. We show that semigroup algebras are 1-amenable, (that is have an approximate diagonal bounded in norm by 1) precisely when the semigroup is an amenable *group*. When searching for internal properties of a semigroup related to amenability it is natural to look for structural Følner-type conditions. We obtain a Følner condition which, with no extraneous conditions imposed upon the semigroup, characterizes those semigroups with 1-amenable semigroup algebras.

In the final chapter we turn our attention to Bekka's amenable representations and introduce a notion of complete-amenability for representations. We provide the relationship between these two concepts of amenability and interpret amenability, inner amenability, and amenable group actions in terms of completely-amenable representations. We describe complete-amenability through a weak-containment property and provide several characterizations of complete-amenability of a representation in terms of the existence of certain states upon C^*-algebras. For each representation π of G we define a Fourier algebra $A(\pi)$ and show that when an associated Fourier algebra $A(\pi \otimes \overline{\pi})$ has a bounded approximate unit, the representation π is necessarily amenable.

Chapter 2

Preliminaries

2.1 Amenable locally compact groups and semi-groups

Throughout this thesis G will denote a locally compact group with identity element e, left Haar measure λ, and modular function Δ. We will refer to a set or function which is Borel measurable simply as measurable. If A is a measurable subset of G, then $|A|$ will denote its Haar measure. The characteristic function of A is 1_A and if $0 < |A| < \infty$, ϕ_A is its normalized characteristic function $\frac{1}{|A|}1_A$. Integration of a Borel measurable function f taken with respect to Haar measure is written

$$\int_G f(x)\, dx.$$

Let $L^p(G)$ be the space of all complex-valued measurable functions f on G such that

$$\int_G |f(x)|^p dx < \infty, \qquad (1 \leq p < \infty).$$

Identifying functions that are equal λ-almost everywhere, $L^p(G)$ is a Banach space with norm

$$\|f\|_p = (\int |f(x)|^p\, dx)^{\frac{1}{p}}, \qquad (f \in L^p(G)).$$

The C^*-algebra of all essentially bounded complex-valued Borel measurable functions on G equipped with the essential supremum norm is denoted by $L^\infty(G)$. The C^*-

subalgebras of continuous bounded functions and the functions vanishing at infinity on G are respectively $CB(G)$ and $C_0(G)$. The space of continuous functions with compact support is denoted by $C_{00}(G)$. For $1 \leq p < \infty$ we let

$$L^p(G)_1^+ = \{f \in L^p(G) : f \geq 0 \text{ and } \|f\|_p = 1\}.$$

When G is discrete we will use the notation $l^p(G)$.

With convolution product

$$f * g(t) = \int_G f(s)g(s^{-1}t)\, ds$$

and involution

$$f^*(t) = \frac{1}{\Delta(t)}\overline{f(t^{-1})}, \qquad (f, g \in L^1(G),\ s \in G),$$

$L^1(G)$ becomes an involutive Banach algebra, called the *group algebra* of G. The Banach space $M(G)$ of all complex regular Borel measures on G may be identified with the dual of $C_0(G)$ through the pairing

$$\langle h, \mu \rangle = \int_G h(s)\, d\mu(s), \qquad (\mu \in M(G),\ h \in C_0(G)).$$

With convolution product defined by

$$\langle h, \mu * \nu \rangle = \int_G h(st)\, d\mu(s)d\nu(t)$$

and involution

$$\langle h, \mu^* \rangle = \overline{\int_G \overline{h(s^{-1})}\, d\mu(s)}, \quad (\mu, \nu \in M(G),\ h \in C_0(G)),$$

$M(G)$ also becomes an involutive Banach algebra called the *measure algebra* of G. The group algebra $L^1(G)$ is identified with a closed ideal in $M(G)$ through

$$f \mapsto \mu_f : L^1(G) \to M(G)$$

where

$$\langle h, \mu_f \rangle = \int h(s)f(s)\, ds, \qquad (f \in L^1(G), h \in C_0(G)).$$

If δ_x denotes the Dirac measure at $x \in G$ and $f \in L^1(G)$, we have

$$\delta_x * f(s) = f(x^{-1}s) \text{ and } f * \delta_x(s) = \frac{1}{\Delta(x)} f(sx^{-1}), \quad (s \in G).$$

Let H be a closed subgroup of G and let $X = G/H$ be the space of left cosets of H in G with the quotient topology induced by the canonical map $G \to X$. Let G act on X through $a \cdot xH = (ax)H$. Two measures are said to be *equivalent* if they share the same collections of null sets. A positive regular Borel measure μ on X is called *quasi-invariant* if the measure $a \cdot \mu$ is equivalent to μ where for $a \in G$ and E a Borel measurable subset of X, $a \cdot \mu(E) := \mu(a \cdot E)$. If μ is quasi-invariant, let $\sigma(a, xH)$, $(a, x \in G)$, denote the Radon-Nikodym derivative of $a \cdot \mu$ taken with respect to μ. The quasi-regular Borel measure μ is called *strongly continuous* if the map

$$(a, xH) \mapsto \sigma(a, xH) : G \times X \to [0, \infty)$$

is jointly continuous. Every coset space admits a strongly continuous quasi-invariant positive regular Borel measure. For $s \in G$ and $f \in L^1(X, \mu)$ we write

$$\delta_a * f(xH) = \sigma(a, xH) f(a^{-1}xH).$$

If ϕ is any complex-valued function on G and $a \in G$, we write

$$(l_a\phi)(s) = \phi(as) \text{ and } (r_a\phi)(s) = \phi(sa), \quad (s \in G).$$

A functional $m \in L^\infty(G)^*$ satisfying $\|m\| = m(1_G) = 1$ is called a *mean.* The locally compact group G is called *amenable* if there exists a mean m on $L^\infty(G)$ such that

$$m(l_a\phi) = m(\phi), \qquad (a \in G, \ \phi \in L^\infty(G)).$$

Such a mean is said to be *left invariant.* We call G *inner-amenable* if $L^\infty(G)$ has an *inner-invariant mean;* that is, if there is a mean m on $L^\infty(G)$ such that

$$m(l_a r_a^{-1}\phi) = m(\phi), \qquad (a \in G, \ \phi \in L^\infty(G)).$$

The components of the following two theorems will be referred to in Chapter 4 as the *classical invariance properties* of amenable locally compact groups. The various parts of these theorems are due to M. Day, E. Følner, I. Namioka, and H. Reiter. Proofs may be found in [18], [39], and [40].

Theorem 2.1.1. *The following are equivalent for a locally compact group G.*

(1) G is amenable.

(2) There exists a net $(f_\alpha) \subset L^1(G)_1^+$ such that $\|\delta_x * f_\alpha - f_\alpha\|_1 \to 0, \quad (x \in G)$.

(3) There exists a net $(f_\alpha) \subset L^1(G)_1^+$ such that $\|g * f_\alpha - f_\alpha\|_1 \to 0, \quad (g \in L^1(G)_1^+)$.

(4) There exists a net $(f_\alpha) \subset L^1(G)_1^+$ such that $\|\delta_x * f_\alpha - f_\alpha\|_1 \to 0$ uniformly in x on compact subsets of G.

Theorem 2.1.2. *Let G be a locally compact group. Then G is amenable if and only if it satisfies the Følner condition*

(FC): For every $\epsilon > 0$ and every compact subset K of G there exists a compact subset A of G with $|A| > 0$ such that

$$|xA \triangle A| < \epsilon|A|, \qquad \text{for every } x \in K.$$

A net as in part (3) of Theorem 2.1.1 is said to *converge to topological invariance* and the existence of a net as in part (4) of Theorem 2.1.1 is called *Reiter's condition.* It is not difficult to see that (4) $\Rightarrow$ (3) $\Rightarrow$ (2) $\Rightarrow$ (1) in Theorem 2.1.1. The condition (FC) says that the net (f_α) in part (4) of Theorem 2.1.1 may be taken to be comprised of normalized characteristic functions $\phi_A = \frac{1}{|A|}1_A$, $A \subset G$.

Let S be a discrete semigroup. With the convolution product

$$(f * g)(t) = \sum_{xy=t} f(x)g(y), \qquad (f, g \in l^1(S),\ t \in S),$$

$l^1(S)$ is a Banach algebra called the *semigroup algebra of S*. As with locally compact groups, we call S *amenable* if there is a left invariant mean on $l^\infty(S)$.

2.2 Amenable Banach algebras

Throughout this section let A be a fixed Banach algebra. A Banach space E is called a left Banach A-module if E is a left A-module, and there exists a number $K > 0$ such that

$$\|a \cdot x\| \leq K\|a\|\,\|x\|, \qquad (a \in A,\ x \in E).$$

Similarly one defines right Banach A-modules, and Banach A-bimodules. It is easy to see that if E is a left Banach A-module, then its continuous dual E^* becomes a right Banach A-module with module multiplication given by

$$(\phi \cdot a)(x) = \phi(a \cdot x), \qquad (\phi \in E^*,\ a \in A,\ x \in E).$$

We will refer to E^* as the *dual Banach A-module of E.*

Let E be a Banach A-bimodule. An E-derivation is a linear mapping $D : A \to E$ such that

$$D(ab) = D(a) \cdot b + a \cdot D(b), \qquad (a, b \in A).$$

For example if $x \in E$, a simple calculation shows that ad_x is a bounded derivation where

$$ad_x(a) := a \cdot x - x \cdot a, \qquad (a \in A).$$

The map ad_x is called an *inner derivation.* The Banach algebra A is called *amenable* if for every Banach A-bimodule E, every bounded derivation $D : A \to E^*$ is inner. The reason for this use of terminology is the following theorem due to Barry Johnson [24].

Theorem 2.2.1. (Johnson's Theorem) *Let G be a locally compact group. Then G is amenable if and only if its group algebra $L^1(G)$ is amenable.*

The projective tensor product $A\hat{\otimes}A$ is a Banach A-bimodule with products determined by $(a \otimes b) \cdot c = a \otimes bc$ and $c \cdot (a \otimes b) = ca \otimes b$. Let π denote the canonical homomorphism determined by $\pi(a \otimes b) = ab$. An *approximate diagonal* for A is a bounded net (m^γ) in $A\hat{\otimes}A$ such that for each $a \in A$,

$$\lim_\gamma (m^\gamma \cdot a - a \cdot m^\gamma) = 0 \ \text{ and } \ \lim_\gamma \pi(m^\gamma)a = a.$$

A *virtual diagonal* for A is an element M of the dual Banach A-bimodule $(A\hat{\otimes}A)^{**}$ such that for each $a \in A$, $M \cdot a = a \cdot M$ and $(\pi^{**}M)a = a$. In [25] Barry Johnson proved that the Banach algebra A is amenable if and only if it possesses an approximate diagonal which is true if and only if it has a virtual diagonal. The Banach algebra

A is called k-amenable, where k is a positive constant, if it has an approximate (equivalently virtual) diagonal with bound k [26]. Clearly A is amenable precisely when it is k-amenable for some $k > 0$. Proof of these statements and much more about amenable Banach algebras may be found for example in [45].

2.3 Representations of locally compact groups

If $\mathcal{H}$ is a Hilbert space, $B(\mathcal{H})$ is the von Neumann algebra of bounded linear operators on $\mathcal{H}$. For $\xi \in \mathcal{H}$ we let ω_ξ be the vector state on $B(\mathcal{H})$ associated to ξ which is defined by $\omega_\xi(T) = \langle T\xi, \xi\rangle$, $(T \in B(\mathcal{H}))$. For $\xi, \eta \in \mathcal{H}$, $\xi \otimes \eta^*$ is the rank-one operator on $\mathcal{H}$ defined by $(\xi \otimes \eta^*)(\zeta) = \langle \zeta, \eta\rangle \xi$, $(\zeta \in \mathcal{H})$. We note that the linear span of a subset F of a linear space E will be denoted by $\langle F\rangle$. General references for the material on operator algebras and their representations needed in this thesis are [8], [35], and [49].

Let G be a locally compact group. A *continuous unitary representation* of G is a pair $\{\pi, \mathcal{H}\}$ where π is a homomorphism of G into the group of unitary operators on the Hilbert space $\mathcal{H}$ which is continuous with respect to the weak operator topology (WOT) on $B(\mathcal{H})$. In this thesis we will refer to continuous unitary representations of G simply as representations of G.

Important examples of representations of G are $\{\lambda_2, L^2(G)\}$ and $\{\rho_2, L^2(G)\}$ respectively defined by

$$\lambda_2(s)\xi(t) = \xi(s^{-1}t) \text{ and } \rho_2(s)\xi(t) = \Delta(s)^{\frac{1}{2}}\xi(ts), \quad (\xi \in L^2(G),\ s, t \in G).$$

The representations $\{\lambda_2, L^2(G)\}$ and $\{\rho_2, L^2(G)\}$ are called the *left and right regular representations of* G. The *conjugation representation* $\{\beta, L^2(G)\}$ may be defined by

$$\beta(s) = \lambda_2(s)\rho_2(s), \qquad (s \in G).$$

If H is a closed subgroup of G and $X = G/H$ has strongly continuous quasi-invariant positive regular Borel measure μ, then the *quasi-regular representation*

$\{Ind_H^G 1_H, L^2(X,\mu)\}$ is defined by

$$Ind_H^G 1_H(s)\xi(xH) = \sigma(a, xH)^{\frac{1}{2}}\xi(s^{-1}xH), \quad (s, x \in G,\ \xi \in L^2(X,\mu).$$

Two representations $\{\pi, \mathcal{H}\}$ and $\{\gamma, \mathcal{K}\}$ of G are said to be *unitarily equivalent* if there exists a unitary operator $U : \mathcal{H} \to \mathcal{K}$ such that for every $s \in G$, $U\pi(s) = \gamma(s)U$. The set of equivalence classes of unitarily equivalent representations of G is denoted by $\Sigma(G)$.

If A is an involutive Banach algebra, a $*$-*representation* of A is a pair $\{\pi, \mathcal{H}\}$ where $\mathcal{H}$ is a Hilbert space and $\pi : A \to B(\mathcal{H})$ is a $*$-homomorphism. A $*$-representation $\{\pi, \mathcal{H}\}$ is called *non-degenerate* if $\langle \pi(a)\xi : a \in A,\ \xi \in \mathcal{H}\rangle$ is dense in $\mathcal{H}$.

There is a one-to-one correspondence between the representations $\{\pi, \mathcal{H}\}$ of G and the $*$-representations of the measure algebra $M(G)$ given by the formula

$$\langle \pi(\mu)\xi, \eta\rangle = \int_G \langle \pi(s)\xi, \eta\rangle \, d\mu(s), \qquad (\mu \in M(G),\ \xi, \eta \in \mathcal{H}). \tag{2.1}$$

Restricting equation (2.1) to $L^1(G)$ yields a one-to-one correspondence between the representations of G and the non-degenerate $*$-representations of $L^1(G)$. The induced representations of $\{\pi, \mathcal{H}\}$ on $L^1(G)$ and $M(G)$ given by equation (2.1) will also be denoted by $\{\pi, \mathcal{H}\}$. Now any representation $\{\pi, \mathcal{H}\}$ of G satisfies

$$\|\pi(f)\| \le \|f\|_1, \qquad (f \in L^1(G)),$$

so we may define a new norm on $L^1(G)$ by

$$\|f\|_{\Sigma(G)} = \sup\{\|\pi(f)\| : \{\pi, \mathcal{H}\} \in \Sigma(G)\}, \qquad (f \in L^1(G)).$$

The completion of $(L^1(G), \|\cdot\|_{\Sigma(G)})$ is a C^*-algebra called the *group C^*-algebra of G* and is denoted by $C^*(G)$. There is a one-to-one correspondence between the non-degenerate $*$-representations of $L^1(G)$ and $C^*(G)$, and therefore a one-to-one correspondence between the representations of G and $C^*(G)$.

For a representation $\{\pi, \mathcal{H}\}$ of G, let $\ker(\pi) = \{x \in C^*(G) : \pi(x) = 0\}$ denote the kernel of π in $C^*(G)$. A representation $\{\pi, \mathcal{H}\}$ is said to be *weakly contained* in another representation $\{\gamma, \mathcal{K}\}$ of G, (and we write $\pi \preceq \gamma$), if $\ker(\pi) \supset \ker(\gamma)$.

References for the material which follows are [13] and [2], respectively due to P. Eymard and his student G. Arsac.

If $\{\pi, \mathcal{H}\} \in \Sigma(G)$ and $\xi, \eta \in \mathcal{H}$, then $\xi *_\pi \eta$ will denote the *coefficient of* π defined by

$$\xi *_\pi \eta(s) = \langle \pi(s)\xi, \eta\rangle, \qquad (s \in G).$$

The set of *continuous positive-definite functions on* G

$$P(G) = \{\xi *_\pi \xi : \{\pi, \mathcal{H}\} \in \Sigma(G),\ \xi \in \mathcal{H}\}$$

corresponds perfectly with the set of positive linear functionals on $C^*(G)$ and it follows that $B(G) = \langle P(G)\rangle$ may be identified with the dual of $C^*(G)$. The pairing of these spaces satisfies

$$\langle f, u\rangle = \int_G f(s)u(s)\, ds, \qquad (f \in L^1(G),\ u \in B(G)).$$

It can be seen that $B(G)$ is precisely the set of all coefficients of representations $\{\pi, \mathcal{H}\} \in \Sigma(G)$. With respect to its dual norm and pointwise defined operations, $B(G)$ is a Banach algebra, called the *Fourier-Stieltjes algebra of* G. If $u \in B(G)$ is positive-definite, then $\|u\|_{B(G)} = \|u\|_\infty = u(e)$. The state space of $C^*(G)$ is thus $P_1(G) = \{u \in P(G) : u(e) = 1\}$.

Let $\{\pi, \mathcal{H}\}$ be a representation of G. Let G_d be the group G endowed with the discrete topology and let π_d be the representation π viewed as a representation of G_d. We will be concerned with the C^*-algebras associated to $\{\pi, \mathcal{H}\}$

$$C^*_\pi = \overline{\pi(L^1(G))}^{\|\cdot\|_{B(\mathcal{H})}} = \pi(C^*(G)) = C^*(G)/\ker(\pi) \ \text{ and } \ C^*_{\delta,\pi} = C^*_{\pi_d}.$$

The von Neumann algebra generated by π is

$$VN_\pi = \overline{\langle \pi(G)\rangle}^{WOT} = \overline{\pi(L^1(G))}^{WOT} = \overline{C^*_\pi}^{WOT}.$$

The commutant of a subset $\mathcal{E}$ of $B(\mathcal{H})$ is the set

$$\mathcal{E}' = \{B \in B(\mathcal{H}) : BE = EB \text{ for every} E \in \mathcal{E}\}.$$

From the von Neumann bicommutant theorem we also have $VN_\pi = \pi(G)''$, the bicommutant of $\pi(G)$ in $B(\mathcal{H})$.

We define

$$A_\pi = \overline{\langle \xi *_\pi \eta : \xi, \eta \in \mathcal{H}\rangle}^{\|\cdot\|_{B(G)}}$$

and let B_π denote the w^*-closure of A_π in $B(G)$. It can be shown that A_π may be identified with the predual $(VN_\pi)_*$ of VN_π and B_π may be identified with the dual of C^*_π. The pairings satisfy

$$\langle u, \pi(f)\rangle = \int_G f(s)u(s)\,ds, \qquad (f \in L^1(G),\ u \in A_\pi) \tag{2.2}$$

and

$$\langle \pi(f), u\rangle = \int_G f(s)u(s)\,ds, \qquad (f \in L^1(G),\ u \in B_\pi). \tag{2.3}$$

The space B_π may also be described in the following two ways. If $u \in P(G)$, let $\{\pi_u, \mathcal{H}_u\} \in \Sigma(G)$ be the cyclic representation of G associated to u (see [8]), and let

$$P_\pi = \{u \in P(G) : \pi_u \preceq \pi\}.$$

Then P_π may be identified with the set of positive linear functionals on C^*_π and $B_\pi = \langle P_\pi\rangle$, the linear span of P_π in $B(G)$. Also

$$\begin{aligned} B_\pi &= (C^*_\pi)^* = (C^*(G)/\ker(\pi))^* = \ker(\pi)^\perp \\ &= \{u \in B(G) : x \in C^*(G) \text{ and } \pi(x) = 0 \Rightarrow \langle x, u\rangle = 0\}. \end{aligned}$$

We will call A_π the *Fourier space* associated to the representation $\{\pi, \mathcal{H}\}$ and we will refer to B_π as the *Fourier-Stieltjes space* associated to $\{\pi, \mathcal{H}\}$. If $\{\lambda_2, L^2(G)\}$ is the left regular representation of G, the space A_{λ_2} is a closed ideal of $B(G)$, called the *Fourier algebra* of G and is usually denoted by $A(G)$. The space B_{λ_2} is also an ideal in $B(G)$, often denoted $B_r(G)$ and called the *reduced Fourier-Stieltjes algebra of* G. The *reduced group C^*-algebra of* G is $C^*_r(G) = C^*_{\lambda_2}$ and the *group von Neumann algebra of* G is $VN(G) = VN_{\lambda_2}$.

The following lemma records some elementary facts which will be used in Chapter 5. Each part of the lemma is almost certainly well-known.

Lemma 2.3.1. *Let $\{\pi, \mathcal{H}\}$ be a representation of G.*

(1) If $x \in C_\pi^* \subset VN_\pi$ and $u \in A_\pi \subset B_\pi$, then $\langle x, u\rangle = \langle u, x\rangle$ where on the left we view $x \in C_\pi^*$, $u \in B_\pi$ and on the right we view $x \in VN_\pi$, $u \in A_\pi$. That is, the pairing is unambiguous.

(2) If $u = \xi *_\pi \eta$ and $f \in L^1(G)$, then $\langle \pi(f), u\rangle = \langle \pi(f)\xi, \eta\rangle$.

(3) For any $u \in A_\pi$ and $t \in G$, $\langle u, \pi(t)\rangle = u(t)$.

(4) The set of states on C_π^* is $B_\pi \cap P_1(G) = \{u \in P_\pi : u(e) = 1\}$.

(5) The set of normal, (that is w^*-continuous) states on VN_π is $A_\pi \cap P_1(G)$.

Proof. Part (1) follows from the density of $\pi(L^1(G))$ in C_π^* and equations (2.2) and (2.3).

Part (2) follows from equations (2.1) and (2.2).

Part (3) is proved as in the case of the Fourier algebra $A(G)$. We may assume that $u = \xi *_\pi \eta$. Let $t \in G$ and take a net (f_α) in $L^1(G)$ such that $\pi(f_\alpha) \to \pi(t)\ \sigma(VN_\pi, A_\pi)$. Then we also have $\pi(f_\alpha) \to \pi(t)\ WOT$, so from part (2)

$$\begin{aligned} \langle u, \pi(t)\rangle &= \lim\langle u, \pi(f_\alpha)\rangle = \lim\langle \pi(f_\alpha)\xi, \eta\rangle \\ &= \langle \pi(t)\xi, \eta\rangle = u(t). \end{aligned}$$

Parts (4) and (5) are obvious. □

Finally we record the following result due to P. Eymard [14, page 48,49]. Eymard states the theorem only for quasi-regular representations, but it can readily be seen that his proof works in the general case. We let 1_G denote the trivial representation of G which is also the constant function $1_G : G \to \mathbf{C} : s \to 1$. A coefficient of the form $\xi *_\pi \xi$ is called a *positive-definite function associated to π.*

Theorem 2.3.2. *Let $\{\pi, \mathcal{H}\}$ be a representation of G. Then $1_G \preceq \pi$ if and only if there is a net of positive-definite functions in $P_1(G)$ associated to π which converges uniformly to 1_G on compact subsets of G.*

Chapter 3

Quasi-Central Bounded Approximate Units in Group Algebras of Locally Compact Groups

3.1 Introduction

Let G be a locally compact group. A net (u_α) in $L^1(G)$ is called *weakly asymptotically central* if $\delta_x * u_\alpha - u_\alpha * \delta_x \to 0, \quad (x \in G)$ where convergence is with respect to the weak topology in $L^1(G)$. A net (u_α) in $L^1(G)$ is called *quasi-central* if $\|\mu * u_\alpha - u_\alpha * \mu\|_1 \to 0, \quad (\mu \in M(G))$. We will use the notation $\mathcal{K}(e) = \{U :$ U is a compact neighbourhood of $e\}$.

Locally compact groups G whose group algebras $L^1(G)$ possess quasi-central bounded approximate units (bau) have been studied by several authors; see for example [36], [48], [50], [51], [54]. In particular A. Sinclair first asked the question, 'when does $L^1(G)$ have a quasi-central bounded approximate identity?' [48, Problem A3.4]. V. Losert and H. Rindler addressed this problem in [36] and among other things, they showed that the existence of a weak asymptotically central bau in $L^1(G)$ is equivalent to the existence of a quasi-central bau, and that group algebras of amenable groups always possess quasi-central bau, [36, Theorem 3]. We note that important use of [36, Theorem 3] was made in the papers [42], [43] and we shall use it extensively in this thesis. In this chapter we will provide new answers to Sinclair's question.

In section two we develop an amenability theory in the very general context of a group action on the predual of a von Neumann algebra. The machinery developed in section two is used in section three to prove Theorem 3.3.4, which is an analogue of Reiter's condition [18, 3.2.1] for groups whose group algebras possess quasi-central bau. This result includes the converse direction of [36, Theorem 2].

A locally compact group G is called a [SIN]-*group* (*small invariant neighbourhood group*) if there is a base for the neighbourhood system at the identity comprised of compact sets which are invariant under inner automorphisms. A well-known theorem due to R. Mosak [37] states that $G \in$[SIN] if and only if $L^1(G)$ possesses a central bau. Moreover every [SIN]-group is unimodular.

In section four we define quasi-[SIN]-groups to be those locally compact groups for which there is a base for the neighbourhood system at the identity which is asymptotically invariant under inner automorphisms. We prove the main result of this chapter, Theorem 3.4.3, which states that G is a quasi-[SIN]-group if and only if G is unimodular and $L^1(G)$ possesses a quasi-central bounded approximate unit.

Applications of this work are discussed in section five. We begin by characterizing locally compact groups G with group algebras admitting quasi-central bounded approximate units in terms of the Fourier and Fourier-Stieltjes algebras of G. We then discuss applications to the cohomology of the Fourier algebra.

Our main application of the results found in this chapter appear in Chapter 4.

3.2 Amenable action on the predual of a W^*-algebra

In this section we briefly outline a unified approach under which the standard techniques used to develop the basic theory of amenable groups, up to and including Reiter's condition, may be used to develop the theory of several types of amenability.

We omit most proofs as they may all be adapted from their classical counterparts, (see for example [18], [39], [40]). In the special case of amenable representations (Example 3.2.2 , part (3) below) this theory was developed by M. Bekka [3, Sections 2, 3, and 4], and the details found in this paper may be helpful. For sections three, four, and five of this chapter we only need up to Lemma 3.2.9 (3) in the special case of Example 3.2.2 part (2). We have chosen to set our presentation in this more general context because it is no more difficult to do so and because this approach does not seem to exist elsewhere in the literature.

Let $\mathcal{M}$ be a W^*-algebra with predual $\mathcal{M}_*$. Let $S(\mathcal{M})$ denote the state space of $\mathcal{M}$, $(\mathcal{M}_*)_1^+$ the normal states of $\mathcal{M}$. References for Banach G, $L^1(G)$, and $M(G)$-modules are [24, Chapter 2] and [40, Section 11].

Definition 3.2.1. A locally compact group G will be said to have *positive action* on $\mathcal{M}_*$, if $\mathcal{M}_*$ is a left Banach G-module such that
(i) $\|s \cdot \phi\| \leq \|\phi\|$, $\quad (\phi \in \mathcal{M}_*,\ s \in G)$, and
(ii) $s \cdot \phi \in (\mathcal{M}_*)_1^+$ whenever $s \in G$, $\phi \in (\mathcal{M}_*)_1^+$.

Example 3.2.2. (1) Let $\mathcal{M} = L^\infty(G)$, $\quad \mathcal{M}_* = L^1(G)$, with

$$s \cdot f = \delta_s * f, \quad (f \in L^1(G), s \in G).$$

(2) Let $\mathcal{M} = L^\infty(G)$, $\quad \mathcal{M}_* = L^1(G)$, with

$$s \cdot f = \delta_s * f * \delta_{s^{-1}}, \quad (f \in L^1(G), s \in G).$$

(3) Let $\{\pi, \mathcal{H}\}$ be a continuous unitary representation of G, $\mathcal{M} = B(\mathcal{H})$ the bounded linear operators on $\mathcal{H}$, $\mathcal{M}_* = T(\mathcal{H})$ the trace class operators on $\mathcal{H}$, and define

$$s \cdot T = \pi(s) T \pi(s^{-1}), \quad (T \in T(\mathcal{H}), s \in G).$$

(4) Let G be a locally compact group, H a closed subgroup of G, $X = G/H$ the left coset space of G modulo H. Let $\mathcal{M} = L^\infty(X, \nu), \mathcal{M}_* = L^1(X, \nu)$, where ν is a strongly continuous quasi-invariant positive Borel measure on X, (see for example [14]). Define

$$s \cdot f = \delta_s * f, \quad (f \in L^1(X, \nu), \quad s \in G).$$

(5) Let $(\mathcal{M}, G, \alpha)$ be a W^*-dynamical system. That is, $\mathcal{M}$ is a W^*-algebra, G is a locally compact group, and $\alpha : G \to Aut(\mathcal{M})$ is a homomorphism of G into the group of *-automorphisms of $\mathcal{M}$, such that for each $x \in \mathcal{M}$, $s \to \alpha_s(x) : G \to (\mathcal{M}, \sigma(\mathcal{M}, \mathcal{M}_*))$ is continuous. Define

$$s \cdot \phi = (\alpha_{s^{-1}})^*(\phi), \quad (s \in G,\ \phi \in \mathcal{M}_*),$$

where $(\alpha_s)^* : \mathcal{M}^* \to \mathcal{M}^*$ is the adjoint map of $\alpha_s : \mathcal{M} \to \mathcal{M}$. In fact each of our first four examples is a special case of this last example.

For the remainder of this section, G is a locally compact group, and $\mathcal{M}$ is a W^*-algebra such that G has positive action on $\mathcal{M}_*$. Note that $\mathcal{M}_*$ is a left Banach $M(G)$-module (and essential Banach $L^1(G)$-module) through the action defined by the weak integral

$$\mu \cdot \phi = \int_G s \cdot \phi \, d\mu(s), \quad (\phi \in \mathcal{M}_*,\ \mu \in M(G)).$$

Dual module operations on $\mathcal{M}$ and $\mathcal{M}^*$ are defined in canonical fashion. This next lemma is often required in the proofs of the statements which follow.

Lemma 3.2.3. *Let $e_{\mathcal{M}}$ be the identity of $\mathcal{M}$, and let $M(G)_1^+$ denote the set of probability measures in $M(G)$. The following statements hold:*
(1) $(\mathcal{M}_*)_1^+$ is w^*-dense in $S(\mathcal{M})$.
(2) For each $\mu \in M(G)_1^+$, $e_{\mathcal{M}} \cdot \mu = e_{\mathcal{M}}$.
(3) $(\mathcal{M}_*)_1^+ = G \cdot (\mathcal{M}_*)_1^+ = M(G)_1^+ \cdot (\mathcal{M}_*)_1^+$.
(4) $S(\mathcal{M}) = G \cdot S(\mathcal{M}) = M(G)_1^+ \cdot S(\mathcal{M})$.

Proof. (1) This is standard and may be found for example in [49].
(2) Let $\mu \in M(G)_1^+$. Then for any $\phi \in (\mathcal{M}_*)_1^+$,

$$\langle \phi, e_{\mathcal{M}} \cdot \mu \rangle = \langle \mu \cdot \phi, e_{\mathcal{M}} \rangle = \int_G \langle s \cdot \phi, e_{\mathcal{M}} \rangle \, d\mu(s) = 1 = \langle \phi, e_{\mathcal{M}} \rangle$$

because the action of G on $\mathcal{M}_*$ is positive. But $(\mathcal{M}_*)_1^+$ separates points of $\mathcal{M}$, so $e_{\mathcal{M}} \cdot \mu = e_{\mathcal{M}}$.

(3) The first equality is obvious. For the second one, note that if $\mu \in M(G)_1^+$ and $\phi \in (\mathcal{M}_*)_1^+$, then $\|\mu \cdot \phi\| \leq 1$, and from part (2), $\mu \cdot \phi(e_{\mathcal{M}}) = \phi(e_{\mathcal{M}} \cdot \mu) = \phi(e_{\mathcal{M}}) = 1$. Hence $\mu \cdot \phi$ is a normal state on $\mathcal{M}$.

(4) The dual module actions on $\mathcal{M}^*$ are $w^* - w^*$ continuous, so this follows from parts (1) and (3). $\square$.

Definition 3.2.4. We will say that G *acts amenably on* $\mathcal{M}_*$ if there exists a state m on $\mathcal{M}$ such that

$$m(x \cdot s) = m(x), \quad (s \in G,\ x \in \mathcal{M}).$$

The state m will be called a *G-invariant mean* (G-IM) for the action.

The interpretation of this definition in Example 3.2.2 parts (1)-(4) is as follows:

(1) G acts amenably on $\mathcal{M}_*$ $\Leftrightarrow$ G is amenable.

(2) G acts amenably on $\mathcal{M}_*$ $\Leftrightarrow$ G is inner amenable.

(3) G acts amenably on $\mathcal{M}_*$ $\Leftrightarrow$ $\{\pi, \mathcal{H}\}$ is amenable [3].

(4) G acts amenably on $\mathcal{M}_*$ $\Leftrightarrow$ G acts amenably on X [14].

Definition 3.2.5. An element $x \in \mathcal{M}$ will be called *uniformly continuous* if $s \mapsto x \cdot s : G \to (\mathcal{M}, \|\cdot\|)$ is continuous. Let $UC(\mathcal{M}) = \{x \in \mathcal{M} : x \text{ is uniformly continuous}\}$.

Remarks 3.2.6. (1) For Examples 3.2.2 (1), (3), and (4), we respectively have $UC(\mathcal{M}) = C_{ru}(G)$ as defined in [23], $UC(\mathcal{M}) = X(\mathcal{H})$ as defined in [3], and $UC(\mathcal{M}) = UCB(X)$ as defined in [14].

(2) In the case of Example 3.2.2 (2), $UC(\mathcal{M})$ may contain functions which are not continuous on G. For example if there exists $U \in \mathcal{K}(e)$ which is invariant under inner automorphisms (that is if G is an [IN]-group) then it is clear that $1_U \in UC(\mathcal{M})$.

(3) $UC(\mathcal{M})$ is always a ($\|\cdot\|$-closed) right Banach G-submodule of $\mathcal{M}$ containing $e_{\mathcal{M}}$. In the case of Example 3.2.2 (5), (and hence in all of our examples), it is easy to see that $UC(\mathcal{M})$ is a C^*-subalgebra of $\mathcal{M}$, (and $(UC(\mathcal{M}), G, \alpha|_{UC(\mathcal{M})})$ is a 'C^*-system').

Lemma 3.2.7. *We always have* $UC(\mathcal{M}) = \mathcal{M} \cdot L^1(G)$.

Definition 3.2.8. A state m on $\mathcal{M}$ is called a *topological invariant mean* (TIM) if

$$m(x \cdot u) = m(x), \quad (x \in \mathcal{M},\ u \in L^1(G)_1^+).$$

An element $m \in UC(\mathcal{M})^*$ such that $\|m\| = m(e_{\mathcal{M}}) = 1$ will be called a *mean.* A mean m is a TIM on $UC(\mathcal{M})$ if

$$m(x \cdot u) = m(x), \quad (x \in UC(\mathcal{M}),\ u \in L^1(G)_1^+).$$

Lemma 3.2.9. *The following statements hold.*

(1) If m is a TIM on $\mathcal{M}$ (respectively $UC(\mathcal{M})$), then m is a G-IM on $\mathcal{M}$ (respectively $UC(\mathcal{M})$).

(2) If m is a G-IM on $UC(\mathcal{M})$, then m is a TIM on $UC(\mathcal{M})$.

(3) If m is a G-IM on $UC(\mathcal{M})$ and $u \in L^1(G)_1^+$, then m_u is a TIM on $\mathcal{M}$, where

$$m_u(x) := m(x \cdot u), \quad (x \in \mathcal{M}).$$

Proposition 3.2.10. *The following statements are equivalent.*

(1) G acts amenably on $\mathcal{M}_*$.

(2) There is a TIM on $\mathcal{M}$.

(3) There is a G-IM on $UC(\mathcal{M})$.

(4) There is a TIM on $UC(\mathcal{M})$.

Corollary 3.2.11. *The following are equivalent for a locally compact group G.*

(1) G is amenable.

(2) Every positive action of G on the predual of a W^*-algebra $\mathcal{M}$ is amenable.

Proof. (2) $\Rightarrow$ (1) is obvious. For (1) $\Rightarrow$ (2) apply Day's fixed point theorem [18, 3.3.5] to the natural action of G on the set $\mathcal{S}$ of means on $UC(\mathcal{M})$. □

Corollary 3.2.12. *The following statements are equivalent.*

(1) G acts amenably on $\mathcal{M}_*$.

(2) There is a net $(\phi_\alpha) \subset (\mathcal{M}_*)_1^+$ such that $\|s \cdot \phi_\alpha - \phi_\alpha\| \to 0, \quad (s \in G)$.

(3) There is a net $(\phi_\alpha) \subset (\mathcal{M}_*)_1^+$ such that $\|u \cdot \phi_\alpha - \phi_\alpha\| \to 0, \quad (u \in L^1(G)_1^+)$.

Proposition 3.2.13. (Reiter's condition) *The following statements are equivalent.*

(1) G acts amenably on $\mathcal{M}_*$.

(2) For any $\epsilon > 0$ and any compact subset K of G there exists $\phi \in \mathcal{M}_*$ such that

$$\|s \cdot \phi - \phi\| < \epsilon, \quad (s \in K).$$

(3) There is a net $(\phi_\alpha) \subset (\mathcal{M}_*)_1^+$ such that $\|\mu \cdot \phi_\alpha - \phi_\alpha\| \to 0, \quad (\mu \in M(G)_1^+)$.

This is precisely Reiter's condition in each of our Examples 3.2.2 (1)-(4).

3.3 A Reiter Condition

Let G be a locally compact group. Throughout the sequel we will restrict our attention to the positive action

$$x \cdot f := \delta_x * f * \delta_{x^{-1}}, \quad (x \in G,\ f \in L^1(G))$$

of G on $L^1(G)$, (Example 3.2.2 (2)). All references to TIM, G-IM, $UC(L^\infty(G))$, etc are with respect to this action. It is easy to see that

$$\mu \cdot f(y) = \int_G \Delta(x) f(x^{-1}yx)\, d\mu(x) \quad \text{a.e. } y \quad (\mu \in M(G), \quad f \in L^1(G))$$

and

$$\phi \cdot \mu(y) = \int_G \phi(xyx^{-1})\, d\mu(x) \quad \text{locally a.e. } y \quad (\mu \in M(G), \quad \phi \in L^\infty(G))$$

describe the induced $M(G)$-module and dual $M(G)$-module operations on $L^1(G)$ and $L^\infty(G)$ respectively. In particular we have

$$\phi \cdot x(y) = \phi \cdot \delta_x(y) = \phi(xyx^{-1}) \quad (x \in G,\ \phi \in L^\infty(G)).$$

A mean m on $L^\infty(G)$ is called *inner invariant* if

$$m(\phi \cdot x) = m(\phi), \quad (\phi \in L^\infty(G),\ x \in G)$$

and is called *an extension of the Dirac measure δ_e (from $CB(G)$ to $L^\infty(G)$)* if

$$m(\phi) = \phi(e), \qquad (\phi \in CB(G)).$$

In [36, Lemma 3] it is shown that m extends the Dirac measure at e if and only if $m(\phi) = m(\phi 1_V)$ for any $\phi \in L^\infty(G)$, $V \in \mathcal{K}(e)$, which in turn holds if and only if $m(\phi) = 0$ for any $\phi \in L^\infty(G)$ which vanishes locally a.e on a neighbourhood of e. The following is contained in [36, Theorem 5].

Lemma 3.3.1. *For $L^1(G)$ to have a quasi-central bau it is necessary and sufficient that $L^\infty(G)$ has an inner invariant mean which extends the Dirac measure at e.*

Lemma 3.3.2. *If $L^1(G)$ has a quasi-central bau, then there is a TIM on $L^\infty(G)$ which extends the Dirac measure at e.*

Proof. Direct $\mathcal{K}(e)$ by reverse inclusion and consider the bau $\{\phi_U : U \in \mathcal{K}(e)\}$ for $L^1(G)$, where $\phi_U := \frac{1}{|U|}1_U$. Let m be an inner invariant mean for $L^\infty(G)$ extending δ_e. By Lemma 3.2.9 (2) and (3), m_U is a TIM for $L^\infty(G)$ where $m_U(\psi) = m(\psi \cdot \phi_U)$, $(\psi \in L^\infty(G))$. Let m_0 be a w^*-limit point of (m_U) in $L^\infty(G)^*$; without loss of generality assume that $m_U \to m_0$ w^*. Clearly m_0 is a TIM on $L^\infty(G)$. Suppose that $\phi \in L^\infty(G)$ and $\phi(x) = 0$ locally a.e. on a neighbourhood V of e. By [36, Lemma 3] we only need to show that $m_0(\phi) = 0$. To this end take $U_0 \in \mathcal{K}(e)$ which is symmetric and satisfies $U_0^3 \subset V$. Then for any $U \subset U_0$ and almost every $x \in U_0$

$$\phi \cdot \phi_U(x) = \frac{1}{|U|} \int_U \phi(yxy^{-1})\, dy = 0.$$

That is, $(\phi \cdot \phi_U)|_{U_0} = 0$ a.e. for $U \subset U_0$. But m extends δ_e, so

$$m_0(\phi) = \lim_U m_U(\phi) = \lim_U m(\phi \cdot \phi_U) = \lim_{U \subset U_0} m(\phi \cdot \phi_U) = 0.$$

$\square$

Notation For any $U \in \mathcal{K}(e)$ let

$$\Psi(U) := \{v \in L^1(G)_1^+ : \text{support}(v) \subset U\} \cap L^\infty(G).$$

Lemma 3.3.3. *Let m be a mean on $L^\infty(G)$ extending δ_e. Then for any $U \in \mathcal{K}(e)$, $m \in \overline{\Psi(U)}^{w^*}$.*

Proof. If not, then by the Hahn-Banach separation theorem we may find $f \in L^\infty(G)$ and $\epsilon > 0$ such that

$$Re\langle f, m\rangle > \epsilon + Re\langle f, v\rangle, \quad (v \in \Psi(U)).$$

Letting $g = (Ref)1_U$ we have

$$(*) \qquad \langle v, g\rangle + \epsilon < m(g), \quad (v \in \Psi(U)),$$

where we have used [36, Lemma 3]. Let $\alpha = ess\ sup\{g(x) : x \in U\}$ and $A = \{x \in U : g(x) > \alpha - \frac{\epsilon}{2}\}$. Then $|A| > 0$ and $\phi_A \in \Psi(U)$. Observe that if $g' = g + \alpha 1_{G\setminus U}$ then (again by use of [36, Lemma 3])

$$m(g) = m(g1_U) = m(g'1_U) = m(g') \le ess\ sup(g') = \alpha.$$

Hence by $(*)$

$$\alpha - \frac{\epsilon}{2} < \langle \phi_A, g\rangle < m(g) - \epsilon \le \alpha - \epsilon,$$

a contradiction. □

We may now prove the following version of Reiter's condition for groups whose group algebras possess quasi-central bounded approximate units. This may be seen as an improvement on the converse direction of [36, Theorem 2].

Theorem 3.3.4. *Let G be a locally compact group such that $L^1(G)$ has a quasi-central bau. Then for any $\epsilon > 0$, any compact subset K of G, and any compact neighbourhood U of e there is some $u \in \Psi(U)$ such that*

$$\|\delta_x * u * \delta_{x^{-1}} - u\|_1 < \epsilon, \quad (x \in K).$$

In particular, if $L^1(G)$ has a quasi-central bau (u_β), then (u_β) may be chosen so that

$$\|\delta_x * u_\beta - u_\beta * \delta_x\|_1 \to 0$$

uniformly on compact subsets of G, and for any neighbourhood U of e, there exists β_0 such that $u_\beta \in \Psi(U)$ whenever $\beta \succeq \beta_0$.

Proof. Choose a symmetric set $V \in \mathcal{K}(e)$ such that $V^3 \subset U$. Choose $E \in \mathcal{K}(e)$ such that

$$\|\phi_E * \phi_V - \phi_V\|_1 < \epsilon \text{ and } \|\delta_x * \phi_V - \phi_V\|_1 < \epsilon, \quad (x \in E).$$

Take $x_1, ..., x_k \in K$ such that $K \subset \bigcup_{k=1}^n x_k E$. For $k = 1, ..., n$ let $\psi_k = \delta_{x_k} * \phi_E$. Using Lemmas 3.3.2, 3.3.3, and an idea due to Namioka [38, 2.2] one can obtain a net $(\phi_\alpha) \subset \Psi(V)$ such that $\|\phi \cdot \phi_\alpha - \phi_\alpha\|_1 \to 0 \quad (\phi \in L^1(G)_1^+)$. In particular, for some α

$$\|\phi_V \cdot \phi_\alpha - \phi_\alpha\|_1 < \epsilon \text{ and } \|\psi_k \cdot \phi_\alpha - \phi_\alpha\|_1 < \epsilon, \quad (k = 1, .., n).$$

Let $\phi = \phi_V \cdot \phi_\alpha$, that is

$$\phi(y) = \frac{1}{|V|} \int_V \Delta(x) \phi_\alpha(x^{-1}yx)\, dx, \quad \text{a.e. } y.$$

Then by Lemma 3.2.3 (3), $\phi \in L^1(G)_1^+$, support$(\phi) \subset V^3 \subset U$, and it is clear that $\phi \in L^\infty(G)$. Thus $\phi \in \Psi(U)$.

As in the proof of the classical version of Reiter's condition [18, 3.2.1] one can now show that

$$\|\delta_x * \phi * \delta_{x^{-1}} - \phi\|_1 = \|x \cdot \phi - \phi\|_1 < 5\epsilon, \quad (x \in K).$$

□

Remarks 3.3.5. (1) By Lemmas 3.2.9 (1), 3.3.1, and 3.3.2, $L^1(G)$ has a quasi-central bau if and only if there is a TIM on $L^\infty(G)$ extending the Dirac measure at e.
(2) A net (u_α) satisfying the convergence property of Theorem 3.3.4 is necessarily a quasi-central bau. This can be seen by arguing as in [39, 4.3].

3.4 The Main Theorem

We begin with a definition.

Definition 3.4.1. A net (U_α) of measurable subsets of G with $0 < |U_\alpha| < \infty$ will be called *asymptotically invariant* (under inner automorphisms) if

$$\frac{|xU_\alpha \triangle U_\alpha x|}{|U_\alpha|} \to 0, \qquad (x \in G).$$

We will call G a *quasi-*[SIN]*-group* if it possesses an asymptotically invariant net $(U_\alpha) \subset \mathcal{K}(e)$ which comprises a base for the neighbourhood system at e.

We remark that in [1], a [QSIN]-group (standing for quasi-[SIN]-group) is defined to be any locally compact group whose group algebra has a quasi-central bau. Theorem 3.4.3 shows that our definitions do not quite coincide.

Lemma 3.4.2. *If G possesses an asymptotically invariant net of subsets, then G is unimodular.*

Proof. For each $x \in G$ and each α,

$$\begin{aligned}\Delta(x) &= \frac{|U_\alpha x|}{|U_\alpha|} = \frac{1}{|U_\alpha|}[|U_\alpha x \backslash xU_\alpha| + |xU_\alpha| - |xU_\alpha \backslash U_\alpha x|] \\ &= 1 + \frac{|U_\alpha x \backslash xU_\alpha|}{|U_\alpha|} - \frac{|xU_\alpha \backslash U_\alpha x|}{|U_\alpha|}\end{aligned}$$

Taking the limit of the final term of the above equation we obtain $\Delta(x) = 1$ for each $x \in G$. □

Theorem 3.4.3. *The following are equivalent for a locally compact group G.*
(1) G is unimodular and $L^1(G)$ has a quasi-central bau.
(2) There exists a net $(U_\alpha) \subset \mathcal{K}(e)$ comprising a base for the neighbourhood system at e such that

$$\frac{|xU_\alpha \bigtriangleup U_\alpha x|}{|U_\alpha|} \to 0$$

uniformly on compact subsets of G. The sets U_α may be chosen to be symmetric.
(3) G is a quasi-[SIN]-group.
(4) $L^1(G)$ has a quasi-central bau comprised of normalized characteristic functions (of compact symmetric neighbourhoods of the identity).

Note that unimodularity does not follow from the existence of a a quasi-central bau alone. Indeed the group algebra of any amenable group has a quasi-central bau [36, Theorem 3].

Proof. (1) $\Rightarrow$ (2) We begin by proving some lemmas, in which we assume that condition (1) is satisfied and $U \in \mathcal{K}(e)$ is *fixed.* If v is a function on G, $\tilde{v}(x) := v(x^{-1})$.

Lemma 3.4.4. *There is a net $(\phi_\alpha) \subset \Psi(U) \cap C_{00}(G)$ such that for each α, $\|\phi_\alpha\|_\infty = \phi_\alpha(e)$, $\widetilde{\phi_\alpha} = \phi_\alpha$, and $\|\delta_x * \phi_\alpha - \phi_\alpha * \delta_x\|_1 \to 0$ uniformly on compact subsets of G.*

Proof. Let $V \in \mathcal{K}(e)$ be symmetric and such that $V^2 \subset U$. Using Theorem 3.3.4 choose a net $(v_\alpha) \subset \Psi(V)$ such that $\|\delta_x * v_\alpha - v_\alpha * \delta_x\|_1 \to 0$ uniformly on compacta. Let $\phi_\alpha = v_\alpha * \widetilde{v_\alpha}$. It is then easy to see that for each α, $\phi_\alpha \in \Psi(U)$, $\|\phi_\alpha\|_\infty = \phi_\alpha(e)$ (for example ϕ_α is positive definite), $\widetilde{\phi_\alpha} = \phi_\alpha$, and because $\Psi(V) \subset L^2(G)$, $\phi_\alpha \in C_{00}(G)$. Finally

$$\begin{aligned}
\|\delta_x * \phi_\alpha - \phi_\alpha * \delta_x\|_1 &\leq \|\delta_x * v_\alpha * \widetilde{v_\alpha} - v_\alpha * \delta_x * \widetilde{v_\alpha}\|_1 + \|v_\alpha * \delta_x * \widetilde{v_\alpha} - v_\alpha * \widetilde{v_\alpha} * \delta_x\|_1 \\
&\leq \|\delta_x * v_\alpha - v_\alpha * \delta_x\|_1 + \|(v_\alpha * \delta_{x^{-1}} - \delta_{x^{-1}} * v_\alpha)^{\sim}\|_1 \\
&= \|\delta_x * v_\alpha - v_\alpha * \delta_x\|_1 + \|v_\alpha * \delta_{x^{-1}} - \delta_{x^{-1}} * v_\alpha\|_1
\end{aligned}$$

from which the uniform convergence on compacta follows. We note that unimodularity was used in this proof. $\square$

Notation We denote the convex hull of a subset S of a linear space by $co(S)$. Let

$$\Phi(U) = co\{\phi_K : K \subset U, K \text{ a compact symmetric neighbourhood of } e\}.$$

Lemma 3.4.5. *There is a net $(\phi_\beta) \subset \Phi(U)$ such that $\|\delta_x * \phi_\beta - \phi_\beta * \delta_x\|_1 \to 0$ uniformly on compact subsets of G.*

Proof. Let (ϕ_α) be a net as in Lemma 3.4.4 and *fix* α. As $\phi_\alpha \in \Psi(U) \cap C_{00}(G)$, $\|\phi_\alpha\|_\infty = \phi_\alpha(e)$, and $\widetilde{\phi_\alpha} = \phi_\alpha$, it follows that for each positive integer n, and each $k = 0, 1, ..., n-1$

$$A_k^\alpha = \{x \in U : \phi_\alpha(x) \geq \frac{k}{n}\phi_\alpha(e)\}$$

is a compact symmetric neighbourhood of e. Note that $A_{n-1}^\alpha \subset ... \subset A_0^\alpha = U$. Let

$$\phi'_{\alpha,n} := \sum_{k=0}^{n-1} \frac{\phi_\alpha(e)}{n} 1_{A_k^\alpha}.$$

Then

$$\phi_{\alpha,n} := \frac{1}{\|\phi'_{\alpha,n}\|_1}\phi'_{\alpha,n} = \sum_{k=0}^{n-1} \lambda_k \phi_{A_k^\alpha}, \quad \text{where } \lambda_k = \frac{\phi_\alpha(e)}{n\|\phi'_{\alpha,n}\|_1}|A_k^\alpha|.$$

Observe that

$$\sum_{k=0}^{n-1} \lambda_k = \sum_{k=0}^{n-1} \lambda_k \int_G \phi_{A_k^\alpha} = \|\phi_{\alpha,n}\|_1 = 1$$

so $\phi_{\alpha,n} \in \Phi(U)$. Now it is not difficult to see that $\|\phi_\alpha - \phi'_{\alpha,n}\|_\infty \leq \frac{\phi_\alpha(e)}{n}$, so

$$\|\phi_\alpha - \phi'_{\alpha,n}\|_1 \leq \frac{\phi_\alpha(e)}{n}|U| \to 0 \text{ as } n \to \infty.$$

Therefore $lim_{n\to\infty}\|\phi'_{\alpha,n}\|_1 = \|\phi_\alpha\|_1 = 1$ and it follows that $lim_{n\to\infty}\|\phi_\alpha - \phi_{\alpha,n}\|_1 = 0$. Let $\mathcal{F} = \{(\epsilon, K) : \epsilon > 0,\ K \subset G \text{ is compact}\}$. For each $\beta = (\epsilon, K) \in \mathcal{F}$ take ϕ_α such that $\|\delta_x * \phi_\alpha - \phi_\alpha * \delta_x\|_1 < \frac{\epsilon}{3}$, and take n such that $\|\phi_\alpha - \phi_{\alpha,n}\|_1 < \frac{\epsilon}{3}$. Then letting $\phi_\beta = \phi_{\alpha,n}$, we have $\|\delta_x * \phi_\beta - \phi_\beta * \delta_x\|_1 < \epsilon, \quad (x \in K)$. Thus $(\phi_\beta)_{\beta\in\mathcal{F}}$ is the net we want. □

Observe that in establishing Lemma 3.4.5, we showed that each ϕ_β may be written in the form

$$(*) \qquad \phi_\beta = \sum_{k=1}^{n} \lambda_k \phi_{A_k},$$

where each $\lambda_k > 0$, $\sum_{k=1}^n \lambda_k = 1$, and $U \supset A_1 \supset A_2 \supset \ldots \supset A_n$, with each set A_k a compact symmetric neighbourhood of e.

Lemma 3.4.6. *Let $\phi \in \Phi(U)$ be written in the form $(*)$. Then*

$$\|\delta_x * \phi - \phi * \delta_x\|_1 = \sum_{k=1}^{n} \lambda_k \frac{|xA_k \triangle A_k x|}{|A_k|}, \quad (x \in G).$$

Proof. This is similar to the proof of [38, 3.3]. For any Borel measurable set A, $x \in G$,

$$(\delta_x * \phi_A - \phi_A * \delta_x)(y) = \begin{cases} \frac{1}{|A|} & \text{if } y \in xA \backslash Ax \\ \frac{-1}{|A|} & \text{if } y \in Ax \backslash xA \\ 0 & \text{otherwise.} \end{cases}$$

Thus, noting that the sets $\bigcup_{k=1}^n xA_k \backslash A_k x$, $\bigcup_{k=1}^n A_k x \backslash xA_k$ are disjoint, it is clear that
$P = \{y : (\delta_x * \phi - \phi * \delta_x)(y) > 0\} = \bigcup_{k=1}^n (xA_k \backslash A_k x)$, and
$N = \{y : (\delta_x * \phi - \phi * \delta_x)(y) < 0\} = \bigcup_{k=1}^n (A_k x \backslash xA_k)$.

Hence

$$\begin{aligned}
\|\delta_x * \phi - \phi * \delta_x\|_1 &= \sum_{k=1}^{n} \lambda_k \Big[\int_P (\delta_x * \phi_{A_k} - \phi_{A_k} * \delta_x)(y)\, dy \\
&\quad - \int_N (\delta_x * \phi_{A_k} - \phi_{A_k} * \delta_x)(y)\, dy\Big] \\
&= \sum_{k=1}^{n} \lambda_k \Big[\int_{xA_k \backslash A_k x} \frac{1}{|A_k|} - \int_{A_k x \backslash x A_k} \frac{-1}{|A_k|}\Big] \\
&= \sum_{k=1}^{n} \lambda_k \frac{|xA_k \triangle A_k x|}{|A_k|}
\end{aligned}$$

We note that unimodularity was used in this proof. □

We can now prove (1) ⇒ (2) of the theorem:

Let $\mathcal{T} = \{(\epsilon, K, U) : \epsilon > 0,\ K \subset G \text{ is compact},\ U \in \mathcal{K}(e)\}$. It suffices to prove that the following statement holds:

(†) For every $(\epsilon, K, U) \in \mathcal{T}$ there is a compact symmetric neighbourhood A of e such that $A \subset U$ and

$$\frac{|xA \triangle Ax|}{|A|} < \epsilon, \quad (x \in K).$$

This will be established from Lemmas 3.4.5 and 3.4.6 by use of an argument similar to the usual proof of the classical Følner condition as found for example in [18, 3.6.2, 3.6.4]. We first show that the statement (†*) holds:

(†*) For every $(\epsilon, K, U) \in \mathcal{T}$, and every $\delta > 0$, there is a compact symmetric neighbourhood A of e with $A \subset U$ and a measurable set $N \subset K$ with $|N| < \delta$ such that

$$\frac{|xA \triangle Ax|}{|A|} < \epsilon, \quad (x \in K \backslash N).$$

To see this let $(\epsilon, K, U) \in \mathcal{T}$, $\delta > 0$ and choose $\phi \in \Phi(U)$ such that for all $x \in K$, $\|\delta_x * \phi - \phi * \delta_x\|_1 < \frac{\epsilon\delta}{|K|}$. If we write ϕ in the form $(*)$, and then integrate the continuous function $x \mapsto \|\delta_x * \phi - \phi * \delta_x\|_1$ over K we obtain

$$\sum_{k=1}^{n} \lambda_k \int_K \frac{|xA_k \triangle A_k x|}{|A_k|}\, dx < \epsilon\delta.$$

As $\sum_{k=1}^{n} \lambda_k = 1$ and each $\lambda_k > 0$ we must have

$$\int_K \frac{|xA \triangle Ax|}{|A|}\, dx < \epsilon\delta$$

for some $A = A_k$. Letting $N = \{x \in K : \frac{|xA \triangle Ax|}{|A|} \geq \epsilon\}$, the sets A and N satisfy ($\dagger^*$) for $(\epsilon, K, U) \in \mathcal{T}$ and δ. We will now show that ($\dagger^*$) $\Rightarrow$ ($\dagger$). Given $(\epsilon, K, U) \in \mathcal{T}$, apply ($\dagger^*$) to the triple $(\frac{\epsilon}{2}, L = K \cup K^2, U) \in \mathcal{T}$ and $\delta = \frac{1}{2}|K|$ to obtain sets A and N. Let $M = L \backslash N$. Observe that for any $k \in K$, $kL \cap L \subset (kM \cap M) \cup (L \backslash M) \cup (kL \backslash kM)$; also $kK \subset kL \cap L$, so $|kL \cap L| \geq |K|$. Therefore

$$2\delta = |K| \leq |kM \cap M| + 2|N| < |kM \cap M| + 2\delta,$$

whence $kM \cap M \neq \emptyset$, $(k \in K)$. Thus $K \subset MM^{-1}$. But for any $x, y \in M = L \backslash N$,

$$\begin{aligned}
\frac{|xy^{-1}A \triangle Axy^{-1}|}{|A|} &= \|\delta_x * \delta_{y^{-1}} * \phi_A - \phi_A * \delta_x * \delta_{y^{-1}}\|_1 \\
&\leq \|\delta_x * (\delta_{y^{-1}} * \phi_A - \phi_A * \delta_{y^{-1}})\|_1 + \|(\delta_x * \phi_A - \phi_A * \delta_x) * \delta_{y^{-1}}\|_1 \\
&= \frac{|Ay \triangle yA|}{|A|} + \frac{|xA \triangle Ax|}{|A|} < \epsilon.
\end{aligned}$$

(2) $\Rightarrow$ (3) is obvious.

(3) $\Rightarrow$ (1) Let (U_α) be an asymptotically invariant base for the neighbourhood system at e, and consider the net of normalized characteristic functions $\phi_\alpha = \phi_{U_\alpha}$. By Lemma 3.4.2, G is unimodular so $\|\delta_x * \phi_\alpha - \phi_\alpha * \delta_x\|_1 = \frac{|xU_\alpha \triangle U_\alpha x|}{|U_\alpha|}$ which converges to zero. Hence (ϕ_α) is an asymptotically central bau and so, by [36, Theorem 2], $L^1(G)$ has a quasi-central bounded approximate unit.

(2) $\Rightarrow$ (4) This follows from Remark 3.3.5(2) and the argument used in (3) $\Rightarrow$ (1)

(4) $\Rightarrow$ (1) Let (ϕ_{U_α}) be such a bau. By Lemma 3.4.2 we only need to show that the net (U_α) is asymptotically invariant. Observe that

$$\begin{aligned}
\|\delta_x * \phi_{U_\alpha} - \phi_{U_\alpha} * \delta_x\|_1 &= \frac{1}{|U_\alpha|}\int |1_{xU_\alpha}(y) - \frac{1}{\Delta(x)} 1_{U_\alpha x}(y)|\, dy \\
&= \frac{|xU_\alpha \backslash U_\alpha x|}{|U_\alpha|} + |1 - \frac{1}{\Delta(x)}| \frac{|xU_\alpha \cap U_\alpha x|}{|U_\alpha|} + \frac{1}{\Delta(x)} \frac{|U_\alpha x \backslash xU_\alpha|}{|U_\alpha|}.
\end{aligned}$$

As $\|\delta_x * \phi_{U_\alpha} - \phi_{U_\alpha} * \delta_x\|_1 \to 0$ and the modular function Δ is always positive (nonzero), it follows that for each $x \in G$

$$\frac{|xU_\alpha \backslash U_\alpha x|}{|U_\alpha|} \to 0 \quad \text{and} \quad \frac{|U_\alpha x \backslash xU_\alpha|}{|U_\alpha|} \to 0.$$

Hence (U_α) is asymptotically invariant. $\square$

Remarks 3.4.7. (1) In the proof of (4) $\Rightarrow$ (1) we only needed $\|\delta_x * \phi_{U_\alpha} - \phi_{U_\alpha} * \delta_x\|_1 \to 0$, $(x \in G)$ and [36, Theorem 2].

(2) By Remark 3.3.5(2) the nets of Lemmas 3.4.4 and 3.4.5 are necessarily quasi-central bau. Hence the existence of such nets in the group algebra of a unimodular group also characterize quasi-[SIN]-groups.

(3) If G is σ-compact and first countable (ie metrizable) then the net in part (2) of Theorem 3.4.3 may be taken to be a sequence.

(4) In [16], the authors define two generalizations of [SIN]-groups, and show that for such groups the so-called inner derivation problem for $L^1(G)$ has a positive solution. It would be interesting to determine the relationship between their $[\mathrm{WSIN}]_1$ and $[\mathrm{WSIN}]_2$-groups and our quasi-[SIN]-groups. If it could be shown that quasi-[SIN]-groups are $[\mathrm{WSIN}]_1$-groups, then combined with [36, Theorem 3] such a result would answer the question [16, 8.2(ii)].

(5) The following statement can be proved by use of Proposition 3.2.13, [38, 3.1], Lemma 3.4.2, and the arguments used in the proofs of Lemma 3.4.6 and Theorem 3.4.3, implications (1) $\Rightarrow$ (2) and (4) $\Rightarrow$ (1):

Proposition 3.4.8. *The following are equivalent for a locally compact group G.*

(1) G is inner amenable (defined in the second section of this chapter) and unimodular.

(2) The following Følner-type condition is satisfied:

For every $\epsilon > 0$ and every compact subset K of G, there is a compact subset A of G such that

$$\frac{|xA \bigtriangleup Ax|}{|A|} < \epsilon, \quad (x \in K).$$

(3) G has an asymptotically-invariant net of subsets.

(4) There is a net of normalized characteristic functions (ϕ_{A_β}) in $L^1(G)$ such that $\|\delta_x * \phi_{A_\beta} - \phi_{A_\beta} * \delta_x\|_1 \to 0$, $(x \in G)$, (or uniformly on compacta of G).

3.5 Applications

In this section we characterize locally compact groups whose group algebras possess quasi-central bounded approximate units in terms of the Fourier and Fourier-Stieltjes algebras. We then present an application of our work to the cohomology of the Fourier algebra. In the next chapter we will exhibit cohomological applications to group algebras.

The Fourier and Fourier-Stieltjes algebras of G are denoted by $A(G)$ and $B(G)$ respectively [13], (also see section three of Chapter 2). Let $\{\lambda_2, L^2(G)\}$ and $\{\rho_2, L^2(G)\}$ respectively denote the left and right regular representations of G. Then the conjugation representation $\{\beta, L^2(G)\}$ of G is defined by $\beta(s) = \lambda_2(s)\rho_2(s), \quad (s \in G)$. For $\xi \in L^2(G)$ we will denote the coefficient $\xi *_\beta \xi$ of ξ with respect to β by e_ξ. That is,

$$e_\xi(s) = \langle \beta(s)\xi, \xi \rangle, \qquad (s \in G).$$

Note that by definition, any $e_\xi \in B(G)$. Let (f_α) be a net of complex-valued functions on G. We will write support$(f_\alpha) \to \{e\}$ if for each neighbourhood U of e, there is some α_0 such that support$(f_\alpha) \subset U$ whenever $\alpha \succeq \alpha_0$. The following result describes when $L^1(G)$ has a quasi-central bau in terms of $A(G)$ and $B(G)$.

Proposition 3.5.1. *The following are equivalent for a locally compact group G.*
(1) $L^1(G)$ has a quasi-central bounded approximate unit.
(2) There exists a net (ξ_α) in $L^2(G)^+_1$ such that support$(\xi_\alpha) \to \{e\}$, and

$$\|ve_{\xi_\alpha} - v\|_{A(G)} \to 0, \qquad (v \in A(G)).$$

Proof. (1) $\Rightarrow$ (2) Let (u_α) be a quasi-central bau for $L^1(G)$ as described in Theorem 3.3.4. Let $\xi_\alpha := u_\alpha^{\frac{1}{2}}$. Then $(\xi_\alpha) \subset L^2(G)^+_1$, support$(\xi_\alpha) \to \{e\}$, and by a standard inequality, (see for example [45, Exercise 4.4.5]), $\|\beta(s)\xi_\alpha - \xi_\alpha\|_2^2 \leq \|\delta_x * u_\alpha - u_\alpha * \delta_x\|_1$ which converges to 0 uniformly on compact subsets of G. It follows that $e_{\xi_\alpha} \to 1$ uniformly on compact subsets of G. The conclusion is now a consequence of [17, Theorem B2].

(2) $\Rightarrow$ (1) Let (ξ_α) be a net as described in statement (2). Let K be any compact subset of G and choose $v \in A(G)$ so that v is identically 1 on K. Then

$$sup\{|e_{\xi_\alpha}(s) - 1| : s \in K\} \leq \|ve_{\xi_\alpha} - v\|_{A(G)} \to 0.$$

Observe that

$$\|\beta(s)\xi_\alpha - \xi_\alpha\|_2^2 = 2|1 - Re\langle\beta(s)\xi_\alpha, \xi_\alpha\rangle| \leq 2|1 - e_{\xi_\alpha}(s)|,$$

so $\|\beta(s)\xi_\alpha - \xi_\alpha\|_2 \to 0$ uniformly on compact subsets of G. Now let $u_\alpha := \xi_\alpha^2$. Then $(u_\alpha) \subset L^1(G)_1^+$ and support$(u_\alpha) \to \{e\}$, so (u_α) is a bounded approximate identity for $L^1(G)$. Moreover by a standard inequality

$$\|\delta_x * u_\alpha - u_\alpha * \delta_x\|_1 = \|(\beta(x)\xi_\alpha)^2 - (\xi_\alpha)^2\|_1 \leq 4\|\beta(x)\xi_\alpha - \xi_\alpha\|_2 \to 0$$

uniformly on compact subsets of G. □

In [42] Z.-J. Ruan proved that a locally compact group G is amenable precisely when its associated Fourier algebra $A(G)$ is operator amenable. We will now indicate how Theorem 3.3.4 allows for a simplification of Ruan's proof. References for the terminology used below are [42] and [45].

The operator projective tensor product $A(G)\hat{\otimes}A(G)$ can be identified with $A(G \times G)$ through the identity

$$(u \otimes v)(s,t) = u(s)v(t), \qquad (u, v \in A(G),\ s,t \in G).$$

Doing this, $A(G \times G)$ has canonical operator $A(G)$-bimodule operations defined by

$$(u \cdot w)(s,t) = u(s)w(s,t) \text{ and } (w \cdot u)(s,t) = w(s,t)u(t),$$

where $w \in A(G \times G)$, $u \in A(G)$, and $s,t \in G$. The multiplication operator

$$\Pi : A(G \times G) \to A(G)$$

is given by restricting functions in $A(G \times G)$ to the diagonal $\{(s,s) : s \in G\}$. In an obvious way one can extend these module operations on $A(G \times G)$ to module operations on $B(G \times G)$, and one can extend Π to a map $\Pi : B(G \times G) \to B(G)$.

It is easy to see that $\{\gamma, L^2(G)\}$ defines a continuous unitary representation of $G \times G$ where $\gamma(s,t) := \lambda_2(s)\rho_2(t)$. For $\xi \in L^2(G)$ we denote the coefficient $\xi *_\gamma \xi$ of ξ with respect to γ by m_ξ. That is,

$$m_\xi(s,t) = \langle \gamma(s,t)\xi, \xi \rangle, \quad (s,t) \in G \times G.$$

Proposition 3.5.2. *The following are equivalent for a locally compact group G.*
(1) $L^1(G)$ has a quasi-central bounded approximate unit.
(2) There is a net (ξ_α) in $L^2(G)_1^+$ with support$(\xi_\alpha) \to \{e\}$ such that

$$(*) \quad \|u \cdot m_{\xi_\alpha} - m_{\xi_\alpha} \cdot u\|_{B(G\times G)} \to 0 \text{ and } \|u\Pi(m_{\xi_\alpha}) - u\|_{B(G)} \to 0, \quad (u \in A(G)).$$

Define $W \in \mathcal{B}(L^2(G \times G))$ by

$$W\xi(s,t) = \xi(s, st), \qquad (\xi \in L^2(G \times G),\ (s,t) \in G \times G).$$

In [42] it is shown that when G is amenable, there is a net $(\xi_\alpha) \subset L^2(G)_1^+$ such that (m_{ξ_α}) satisfies the condition $(*)$. A major part of the proof of this fact is the following nontrivial lemma which is proved for amenable groups in [42]. As stated below the following is [45, Lemma 7.4.2] where V. Runde observed that the amenability condition may be dropped.

Lemma 3.5.3. *Let G be a locally compact group and suppose that there is a net of unit vectors (ξ_α) in $L^2(G)$ such that*

$$\|W(\xi_\alpha \otimes \eta) - (\xi_\alpha \otimes \eta)\|_2 \to 0, \quad (\eta \in L^2(G))$$

and

$$\|\gamma(s,s)\xi_\alpha - \xi_\alpha\|_2 \to 0$$

uniformly on compact subsets of G. Then the net (m_{ξ_α}) in $B(G\times G)$ satisfies condition $(*)$ of Proposition 3.5.2.

We will now show how the existence of a net (ξ_α) as described in Lemma 3.5.3 follows easily from our Theorem 3.3.4.

Proof of Proposition 3.5.2. (1) $\Rightarrow$ (2) Let (u_α) be a net as described in Theorem 3.3.4 and let $\xi_\alpha := u_\alpha^{\frac{1}{2}}$. As shown in the proof of Proposition 3.5.1, $(\xi_\alpha) \subset L^2(G)_1^+$, support$(\xi_\alpha) \to \{e\}$ and $\|\gamma(s,s)\xi_\alpha - \xi_\alpha\|_2 = \|\beta(s)\xi_\alpha - \xi_\alpha\|_2 \to 0$ uniformly on compact subsets of G. Now let $\eta \in L^2(G)$ be arbitrary. Let U be a symmetric neighbourhood of e such that $\|\lambda_2(s)\eta - \eta\|_2 < \epsilon$ whenever $s \in U$, and take α_0 such that support$(\xi_\alpha) \subset U$ whenever $\alpha \succeq \alpha_0$. Then for $\alpha \succeq \alpha_0$

$$\begin{aligned}\|W(\xi_\alpha \otimes \eta) - (\xi_\alpha \otimes \eta)\|_2^2 &= \iint |\xi_\alpha(s)(\eta(st) - \eta(t))|^2 \, dtds \\ &= \int_U \xi_\alpha^2(s)\|\lambda_2(s^{-1})\eta - \eta\|_2^2 \, ds \leq \epsilon^2.\end{aligned}$$

(2) $\Rightarrow$ (1) Observe that $\Pi m_{\xi_\alpha}(s) = e_{\xi_\alpha}(s)$, $(s \in G)$. Now the implication follows from Proposition 3.5.1. $\square$

Remarks 3.5.4. (1) Using [36, Theorem 3] Ruan proved that when G is amenable one can construct a net (ξ_α) as described in Lemma 3.5.3. To accomplish this, Ruan required Losert and Rindler's *explicit construction* of a quasi-central bau for $L^1(G)$ from the Reiter condition characterizing amenable locally compact groups.
(2) By [36, Theorem 3], condition (2) of Proposition 3.5.2 is satisfied when G is amenable. That amenability combined with condition (2) of Proposition 3.5.2 implies that $A(G)$ is operator amenable follows very easily from Leptin's theorem [34]. The details are found in [42, Lemma 3.1].

Chapter 4

F$\emptyset$lner Conditions for Amenable Group and Semigroup Algebras

4.1 Introduction

Amenable Banach algebras were introduced by B.E. Johnson in [24] where he proved that a locally compact group G is amenable if and only if its associated group algebra $L^1(G)$ is amenable, (Johnson's Theorem). Shortly thereafter, Johnson proved his fundamental characterization of amenable Banach algebras in terms of the existence of virtual and approximate diagonals [25]. Virtual diagonals are often said to play the role in the theory of amenable Banach algebras that invariant means play in the theory of amenable groups.

In this chapter we will study the exact relationship between the classical invariance properties possessed by amenable locally compact groups and the virtual/approximate diagonal invariance properties of their amenable group algebras. We will also address the problem of determining which discrete semigroups carry amenable semigroup algebras. The deepest of the invariance properties characterizing amenable locally compact groups are the combinatorial Følner conditions, which have also proven to be interesting and useful in the study of amenable semigroups [18], [38], [39], [40], [53]. We will establish Følner-type conditions characterizing discrete semigroups and locally compact groups whose associated L^1-algebras are 1-amenable. Thus one of our purposes here is to give internal properties of semigroups and locally compact groups which reflect the *Banach algebra* amenability of their associated L^1-algebras.

In section two we investigate the relationship between the invariance properties of amenable locally compact groups and the virtual/approximate diagonal invariance properties of their amenable group algebras. Using Reiter's condition and the existence of a strong form of quasi-central bau for $L^1(G)$ (guaranteed by [36] and Theorem 3.3.4) we are able to explicitly write down a formula which gives a strong form of approximate diagonal for $L^1(G)$, (Theorem 4.2.9). Moreover we directly show that weaker forms of approximate diagonals for $L^1(G)$ are sufficient for G to be amenable, (Theorem 4.2.2, Proposition 4.2.4). From the existence of an approximate diagonal of norm one, we show how to construct a net converging to topological invariance, (Corollary 4.2.3). Corollary 4.2.12 is a formulation of our results in terms of compactly-invariant and measure-invariant approximate diagonals, (see Definition 4.2.6). One immediate consequence of this work is a new proof of Johnson's theorem given entirely in terms of approximate diagonals.

The results given in section two set the groundwork for section three in which we derive our Følner conditions. We begin by interpreting Corollary 4.2.12 as a Reiter condition for amenable group algebras and use it to derive Følner conditions (F) and ($\mathrm{F}_\mathcal{V}$) for unimodular amenable locally compact groups. Theorem 4.3.6 combines Theorem 3.4.3 with (FC) to obtain structural properties ($\mathrm{A}_\mathcal{V}$), ($\mathrm{B}_\mathcal{V}$), and ($\mathrm{C}_\mathcal{V}$) for unimodular amenable groups. Theorem 4.3.7 shows that each of our properties (F), ($\mathrm{F}_\mathcal{V}$), ($\mathrm{A}_\mathcal{V}$), ($\mathrm{B}_\mathcal{V}$), and ($\mathrm{C}_\mathcal{V}$) is a 'correct' Følner condition reflecting the amenability of $L^1(G)$ in the sense that each one yields a (compactly-invariant) approximate diagonal for $L^1(G)$ comprised of normalized characteristic functions of subsets of $G \times G$.

It is well known that if S is a semigroup, then amenability of the semigroup algebra $l^1(S)$ implies the amenability of S. However, unlike the case in which S is a group, the converse does not hold. The articles [9], [10], [12], [19], [20] and [32] have addressed the problem of describing, in terms of the internal structure of the semigroup itself, those semigroups carrying amenable semigroup algebras. These papers show that amenability of $l^1(S)$ imposes very strong conditions on S, especially when additional algebraic conditions are placed upon the semigroup S. Indeed when $l^1(S)$ is amenable, it can be said that S is 'close' to a group. For a survey of what is known to date, see [21]. In section four, we show that for a discrete semigroup S, $l^1(S)$ is 1-amenable if and only if S is an amenable *group*. Moreover we provide a Følner-type condition which, with no extraneous conditions placed upon S, characterizes 1-amenability of $l^1(S)$.

4.2 Virtual and approximate diagonals for group algebras

In this section we will begin our investigation of the relationship between the classical invariance properties of amenable locally compact groups and the approximate/virtual diagonals possessed by their associated amenable group algebras. For certain discrete semigroups this relationship was considered in [9].

Throughout, G will again denote a locally compact group with left Haar measure λ, modular function Δ, and identity e. Recall that $\lambda \times \lambda$ is a Haar measure, and $(x,y) \mapsto \Delta(x)\Delta(y)$, $(x,y) \in G \times G$, is the associated modular function, on $G \times G$. We will denote the Haar measure of a Borel subset A of either G or $G \times G$, by $|A|$. As usual, $L^1(G \times G)$ and $L^1(G)\hat{\otimes}L^1(G)$ are identified through

$$(h \otimes k)(x,y) = h(x)k(y), \quad (h, k \in L^1(G), \quad x, y \in G).$$

References for Banach G, $L^1(G)$, and $M(G)$-modules are Chapter two of [24] and section eleven of [40]. The following dual-module and $M(G)$-module operations are easily verified.

The space $L^1(G \times G)$ becomes an essential Banach $L^1(G)$-module with operations defined through

$$f \cdot (h \otimes k) = (f * h) \otimes k, \quad (h \otimes k) \cdot f = h \otimes (k * f), \quad (h, k, f \in L^1(G)).$$

The corresponding $M(G)$-module operations are given by

$$\mu \cdot (h \otimes k) = (\mu * h) \otimes k, \quad (h \otimes k) \cdot \mu = h \otimes (k * \mu), \quad (h, k \in L^1(G), \mu \in M(G)).$$

Thus $L^1(G \times G)$ has Banach G-module operations

$$a \cdot m(x,y) = m(a^{-1}x, y), \quad m \cdot a(x,y) = \Delta(a^{-1}) m(x, ya^{-1}), \quad (m \in L^1(G \times G), a \in G).$$

Now

$$a \cdot \phi = r_{(e,a)}\phi \quad \text{and} \quad \phi \cdot a = l_{(a,e)}\phi, \qquad (a \in G, \ \phi \in L^\infty(G \times G))$$

define the dual G-module structure of $L^\infty(G \times G)$. As usual the group algebra $L^1(G)$ is often viewed as a Banach G-module through the operations

$$x \cdot f = \delta_x * f \ \text{ and } \ f \cdot x = f * \delta_x, \quad (f \in L^1(G), \ x \in G).$$

Let $\pi : L^1(G \times G) \to L^1(G)$ be the multiplication operator, $\pi^* : L^\infty(G) \to L^\infty(G \times G)$ its adjoint map. Let $p : G \times G \to G : (s,t) \mapsto st$.

Lemma 4.2.1. *(1) For any $\phi \in L^\infty(G)$, $\pi^*(\phi) = \phi \circ p$.*
(2) π maps $L^1(G \times G)_1^+$ into $L^1(G)_1^+$, and π^{**} maps the set of means on $L^\infty(G \times G)$ into the set of means on $L^\infty(G)$.
(3) If $(m^\gamma) \subset L^1(G \times G)$ is an approximate diagonal for $L^1(G)$, then $\lim \langle 1_{G \times G}, m^\gamma \rangle =$ 1.

Proof. (1) Let $h, k \in L^1(G)$, $\phi \in L^\infty(G)$. Then

$$\langle h \otimes k, \pi^*\phi \rangle = \langle h * k, \phi \rangle = \iint h(s)k(t)\phi(st)dtds = \langle h \otimes k, \phi \circ p \rangle.$$

(2) Note that $\|\pi\| = 1$ and from part (1) $\pi^*(1_G) = 1_{G \times G}$. Therefore, if $f \in L^1(G \times G)_1^+$, $\|\pi(f)\| \le 1$ and $\langle 1_G, \pi(f) \rangle = \langle 1_{G \times G}, f \rangle = 1$ whence $\pi(f)$ is a normal state on $L^\infty(G)$; that is $\pi(f) \in L^1(G)_1^+$. The second statement follows from the w^*-continuity of π^{**} and the w^*-density of the normal states within the state space of a von Neumann algebra.

(3) Let (m^γ) be an approximate diagonal for $L^1(G)$. If $f \in L^1(G)_1^+$, observe that $\langle g, f \cdot 1_G \rangle = \langle g * f, 1_G \rangle = \langle g, 1_G \rangle \langle f, 1_G \rangle = \langle g, 1_G \rangle$, $(g \in L^1(G))$, so $f \cdot 1_G = 1_G$. Therefore

$$\begin{aligned}\lim \langle 1_{G\times G}, m^\gamma \rangle &= \lim \langle \pi^*(1_G), m^\gamma \rangle = \lim \langle f \cdot 1_G, \pi(m^\gamma) \rangle \\ &= \lim \langle 1_G, \pi(m^\gamma) * f \rangle = \langle 1_G, f \rangle = 1.\end{aligned}$$

□

We will now show that for G to be amenable, it is sufficient that there exist weaker forms of virtual and approximate diagonals for $L^1(G)$. First we introduce some notation. For $\phi \in L^\infty(G)$, define $\phi^\sharp(s,t) = \phi(s)$, $(s,t) \in G \times G$. Then it is a simple matter to see that $\phi \mapsto \phi^\sharp : L^\infty(G) \to L^\infty(G \times G)$ is linear and isometric. For $m \in L^1(G \times G)$ we define $g_m \in L^\infty(G)^*$ by

$$\langle \phi, g_m \rangle = \langle \phi^\sharp, m \rangle, \qquad (\phi \in L^\infty(G)).$$

Observe that the map $m \mapsto g_m$ is linear and contractive. We note here that the proof of the next theorem will show that g_m belongs to $L^1(G)$.

Theorem 4.2.2. *The following are equivalent for a locally compact group G.*
(1) G is amenable.
(2) There exists $M \in L^\infty(G \times G)^*$ such that $M(1_{G\times G}) = 1$ and $f \cdot M = M \cdot f$, $(f \in L^1(G))$.
(3) There exists a bounded net $(m^\gamma) \subset L^1(G \times G)$ such that $\lim \langle m^\gamma, 1_{G\times G} \rangle = 1$ and $\|f \cdot m^\gamma - m^\gamma \cdot f\|_1 \to 0$, $(f \in L^1(G))$.
The functional M in statement (2) may be chosen to be a mean, and the net (m^γ) in statement (3) may be chosen from $L^1(G \times G)_1^+$.

Proof. (2) ⇔ (3) The argument used to prove [38, Theorem 2.2] yields this equivalence.
(1) ⇒ (3) A stronger result will be proved in Theorem 4.2.9.
(3) ⇒ (1) Let $m \in L^1(G \times G)$ and let $g = g_m$. We *claim* that $g \in L^1(G)$. By the

Krein-Smulyan theorem it suffices to show that g is w^*-continuous on the unit ball of $L^\infty(G)$. So suppose that $\phi_\alpha \to \phi \ \ \sigma(L^\infty(G), L^1(G))$, with (ϕ_α) bounded by 1 in $L^\infty(G)$. If $m = h \otimes k$ with $h, k \in L^1(G)$, then

$$\langle \phi_\alpha - \phi, g\rangle = \langle \phi_\alpha^\sharp - \phi^\sharp, m\rangle = \langle \phi_\alpha - \phi, h\rangle \int k(t)dt \to 0. \qquad (\dagger)$$

By linearity, density, and the assumption that (ϕ_α) is bounded, $(\dagger)$ now holds for general $m \in L^1(G\times G)$. Thus $g \in L^1(G)$ as claimed.

We now show that for any $f \in L^1(G)_1^+$ and $\phi \in L^\infty(G)$,

$$\text{(i) } \langle \phi^\sharp, m\cdot f\rangle = \langle \phi, g\rangle \quad \text{and} \quad \text{(ii) } \langle \phi^\sharp, f\cdot m\rangle = \langle \phi, f * g\rangle.$$

We may suppose that $m = h\otimes k$, $h, k \in L^1(G)$. Equation (i) is then an easy calculation. To establish equation (ii), we first define $\psi \in L^\infty(G)$ by $\psi(t) = \int \phi(st)f(s)ds$, $(t \in G)$. Viewing f, g, and h as elements of $M(G)$ we have

$$\begin{aligned}
\langle \phi, f * g\rangle &= \iint \phi(st)f(s)g(t)dsdt = \langle \psi, g\rangle = \langle \psi^\sharp, m\rangle \\
&= \iint \psi(s)h(s)k(t)dsdt = \iiint \phi(rs)f(r)h(s)k(t)drdsdt \\
&= \int \langle \phi, f*h\rangle k(t)dt = \iint \phi^\sharp(s,t)(f*h\otimes k)(s,t)dsdt \\
&= \langle \phi^\sharp, f\cdot m\rangle.
\end{aligned}$$

Suppose now that (m^γ) is as in statement (3) of the theorem and let $g_\gamma = g_{m^\gamma}$. Then for any $f \in L^1(G)_1^+$,

$$\begin{aligned}
\|f * g_\gamma - g_\gamma\|_1 &= \sup\{|\langle \phi, f*g_\gamma - g_\gamma\rangle| : \phi \in L^\infty(G) \text{ and } \|\phi\|_\infty \le 1\} \\
&= \sup\{|\langle \phi^\sharp, f\cdot m^\gamma - m^\gamma \cdot f\rangle| : \phi \in L^\infty(G) \text{ and } \|\phi\|_\infty \le 1\} \\
&\le \|f\cdot m^\gamma - m^\gamma\cdot f\|_1
\end{aligned}$$

which converges to 0. Let n be a w^*-limit point of (g_γ); assume without loss of generality that $g_\gamma \to n \ \ w^*$. Then $n(1_G) = \lim\langle 1_G, g_\gamma\rangle = \lim\langle 1_{G\times G}, m^\gamma\rangle = 1$, so n is non-zero. Moreover, if we fix $f \in L^1(G)_1^+$, then for any $x \in G$ and any $\phi \in L^\infty(G)$, $n(l_x\phi) = \lim\langle l_x\phi, g_\gamma\rangle = \lim\langle l_x\phi, f*g_\gamma\rangle = \lim\langle \phi, \delta_x * f * g_\gamma\rangle = \lim\langle \phi, g_\gamma\rangle = n(\phi)$. Thus $n \in L^\infty(G)^*$ is non-zero and left-invariant from which it follows that G is amenable. $\square$

Observe that it follows from part (3) of Lemma 4.2.1 that any approximate diagonal for $L^1(G)$ satisfies the properties of the net described in part (3) of Theorem 4.2.2. In the course of the proof of Theorem 4.2.2, the following fact emerged.

Corollary 4.2.3. *Suppose that G is an amenable locally compact group with a net (m^γ) as described in Theorem 4.2.2 part (3). Let $g_\gamma = g_{m^\gamma}$. Then for every $f \in L^1(G)_1^+$, we have $\|f * g_\gamma - g_\gamma\|_1 \to 0$. Thus, if $(m^\gamma) \subset L^1(G \times G)_1^+$, then $(g_\gamma) \subset L^1(G)_1^+$ is a net converging to topological invariance.*

With respect to the G-module action on $L^1(G \times G)$ we have the following proposition.

Proposition 4.2.4. *The following are equivalent for a locally compact group G.*
(1) G is amenable.
(2) There exists a mean $M \in L^\infty(G \times G)^*$ such that $x \cdot M = M \cdot x$, $(x \in G)$.
(3) There exists a non-zero element $M \in L^\infty(G \times G)^*$ such that $x \cdot M = M \cdot x$, $(x \in G)$.
(4) There exists a net $(m^\gamma) \subset L^1(G \times G)_1^+$ such that $\|x \cdot m^\gamma - m^\gamma \cdot x\|_1 \to 0$, $(x \in G)$.

Proof. (1) $\Rightarrow$ (2) If G is amenable, then so is $G \times G$. Any two-sided invariant mean M on $L^\infty(G \times G)$ satisfies $x \cdot M = M \cdot x$, $(x \in G)$.
(2) $\Rightarrow$ (1) Take M as in condition (2). Then $m(\phi) := M(\phi^\sharp)$, $(\phi \in L^\infty(G))$ defines a mean on $L^\infty(G)$. Moreover, for any $\phi \in L^\infty(G)$ and any $a \in G$, $m(l_a\phi) = M((l_a\phi)^\sharp) = M(\phi^\sharp \cdot a) = M(a \cdot \phi^\sharp) = M(\phi^\sharp) = m(\phi)$, so m is a left invariant mean on $L^\infty(G)$.
(2) $\Rightarrow$ (3) is obvious.
(3) $\Rightarrow$ (2) It is easy to see that we may suppose that M is self-adjoint. Let M have Jordan decomposition $M = M^+ - M^-$. Now the G-module operations are isometric and preserve positivity in $L^\infty(G \times G)^*$, so it follows from the uniqueness of the Jordan decompositions of $x \cdot M$ and $M \cdot x$ that $x \cdot M^+ = M^+ \cdot x$ and $x \cdot M^- = M^- \cdot x$. If $M^+ \neq 0$ (say) then $M' = \frac{1}{M^+(1_{G \times G})} M^+$ is a mean on $L^\infty(G \times G)$ such that $x \cdot M' = M' \cdot x$.
(2) $\Leftrightarrow$ (4) The argument from [38, Theorem 2.2] yields the equivalence of (2) and (4). $\square$

Remarks 4.2.5. Proposition 4.2.4 also holds for any discrete semigroup. In this situation, (2) $\Rightarrow$ (1) is [9, Lemma 3], and for inverse semigroups (1) $\Rightarrow$ (2) is [9, Lemma 4]. If S has an identity, our argument above proves (1) $\Rightarrow$ (2), however for an arbitrary semigroup, the implication may still be obtained using an argument similar to that of [23, 17.18 (b)].

Let G be a locally compact group. In terms of the dual module action of G on $L^\infty(G)$ which was introduced at the beginning of this chapter, a mean m on $L^\infty(G)$ is inner invariant if $m(x \cdot \phi \cdot x^{-1}) = m(\phi)$, $(\phi \in L^\infty(G),\ x \in G)$. As well, recall from Chapter 3 that m is called an extension of the Dirac measure δ_e (from $CB(G)$ to $L^\infty(G)$) if $m(\phi) = \phi(e)$, $(\phi \in CB(G))$.

Definition 4.2.6. We will say that an approximate diagonal $(m^\gamma) \subset L^1(G \times G)_1^+$ is *compactly-invariant* [respectively *measure-invariant*] if $\|x \cdot m^\gamma - m^\gamma \cdot x\|_1 \to 0$ uniformly on compact subsets of G, [respectively $\|\mu \cdot m^\gamma - m^\gamma \cdot \mu\|_1 \to 0$, $(\mu \in M(G))$]. A virtual diagonal M for $L^1(G)$ is *measure-invariant* if M is a mean and $\mu \cdot M = M \cdot \mu$, $(\mu \in M(G))$.

The following proposition contains some simple observations.

Proposition 4.2.7. *(1) Let (m^γ) be an approximate diagonal for $L^1(G)$. If (m^γ) is compactly-invariant, then (m^γ) is measure-invariant, and $(\pi(m^\gamma))$ is a quasi-central bau contained in $L^1(G)_1^+$ such that $\|\delta_x * \pi(m^\gamma) - \pi(m^\gamma) * \delta_x\|_1 \to 0$ uniformly on compact subsets of G. If (m^γ) is measure-invariant, then $(\pi(m^\gamma))$ is a quasi-central bau in $L^1(G)_1^+$.*
(2) Let M be a measure-invariant virtual diagonal for $L^1(G)$. Then $\pi^{**}M$ is an inner-invariant mean on $L^\infty(G)$ which extends the Dirac measure at e.

Proof. (1) If $m \in L^1(G \times G)$ and $\mu \in M(G)$, then by [24, 2.1] $\mu \cdot m$ and $m \cdot \mu$ are given by the weak integrals

$$\mu \cdot m = \int t \cdot m \, d\mu(t) \qquad m \cdot \mu = \int m \cdot t \, d\mu(t).$$

We may suppose that μ has compact support C. Then for $\phi \in L^\infty(G \times G)$ and $m \in L^1(G \times G)$,

$$\begin{aligned} |\langle \phi, \mu \cdot m - m \cdot \mu \rangle| &= |\int \langle \phi, t \cdot m - m \cdot t \rangle d\mu(t)| \\ &\leq \int_C \|\phi\|_\infty \|t \cdot m - m \cdot t\|_1 d|\mu|(t) \\ &\leq \|\phi\|_\infty \|\mu\| \sup\{\|t \cdot m - m \cdot t\|_1 : t \in C\}. \end{aligned}$$

Therefore if (m^γ) is compactly-invariant, $\|\mu \cdot m^\gamma - m^\gamma \cdot \mu\|_1 \leq \|\mu\| \sup\{\|t \cdot m^\gamma - m^\gamma \cdot t\|_1 : t \in C\}$, which converges to zero. The balance of (1) now follows from Lemma 4.2.1 and the fact that π is an $M(G)$-module morphism.

(2) The fact that $\pi^{**}(M)$ is a mean is from Lemma 4.2.1. Now π^{**} is an $M(G)$-module morphism so $\pi^{**}(M)$ is inner-invariant; as $\pi^{**}(M)$ is a weak*-limit point of a bounded approximate unit, it extends the Dirac measure at e, [36]. □

We will need some facts which we summarize in the following theorem; part one is [36, Theorem 3], part two is just a restatement of Theorem 3.3.4.

Theorem 4.2.8. *Let G be a locally compact group.*

(1) If G is amenable, then $L^1(G)$ has a quasi-central bounded approximate unit.

(2) Suppose that $L^1(G)$ has a quasi-central bau (e_β). Then (e_β) may be chosen so that $(e_\beta) \subset L^1(G)^+_1$, $\|\delta_x * e_\beta - e_\beta * \delta_x\|_1 \to 0$ uniformly on compact subsets of G, and for any neighbourhood U of e, there exists β_0 such that support$(e_\beta) \subset U$ whenever $\beta \succeq \beta_0$.

We will now show how to construct a compactly-invariant approximate diagonal for $L^1(G)$ from Reiter's condition for amenable groups and a quasi-central bau as described in the above theorem. We remark that on page 319 of [7] it is incorrectly stated that when G is an amenable locally compact group, one can construct a virtual diagonal for $L^1(G)$ as follows: Letting m be an invariant mean on $L^\infty(G)$, define

$$M(h) = \int_G h(t, t^{-1}) dm(t), \qquad (h \in L^\infty(G \times G)),$$

where the formal integral represents the action of m on the function $t \mapsto h(t,t^{-1})$. However unless G is discrete, the set $\{(t,t^{-1}) : t \in G\}$ may have measure zero so the map M is not well-defined.

Theorem 4.2.9. *Let G be an amenable locally compact group. Let $(f_\alpha) \subset L^1(G)_1^+$ be a net such that $\|\delta_x * f_\alpha - f_\alpha\|_1 \to 0$ uniformly on compact subsets of G. Let $(e_\beta) \subset L^1(G)_1^+$ be a quasi-central bau for $L^1(G)$ as in Theorem 4.2.8 (2). For each $\gamma = (\alpha, \beta)$, define*

$$m^\gamma(s,t) = f_\alpha(s)e_\beta(st), \quad (s,t) \in G \times G.$$

Then (m^γ) is a compactly-invariant approximate diagonal for $L^1(G)$ contained in $L^1(G \times G)_1^+$.

Proof. It is easy to see that $(m^\gamma) \subset L^1(G \times G)_1^+$. Observe that for any $x \in G$ and any $(s,t) \in G \times G$,

$$x \cdot m^\gamma(s,t) = m^\gamma(x^{-1}s,t) = f_\alpha(x^{-1}s)e_\beta(x^{-1}st) = \delta_x * f_\alpha(s)\delta_x * e_\beta(st),$$

and

$$m^\gamma \cdot x(s,t) = \Delta(x^{-1})m^\gamma(s,tx^{-1}) = f_\alpha(s)e_\beta * \delta_x(st).$$

Therefore

$$\begin{aligned}
\|x \cdot m^\gamma - m^\gamma \cdot x\|_1 &= \iint |\delta_x * f_\alpha(s)\delta_x * e_\beta(st) - f_\alpha(s)e_\beta * \delta_x(st)|dtds \\
&= \iint |\delta_x * f_\alpha(s)\delta_x * e_\beta(t) - f_\alpha(s)e_\beta * \delta_x(t)|dtds \\
&\le \iint |\delta_x * f_\alpha(s) - f_\alpha(s)|\delta_x * e_\beta(t)dtds \\
&\quad + \iint f_\alpha(s)|\delta_x * e_\beta(t) - e_\beta * \delta_x(t)|dtds \\
&= \|\delta_x * f_\alpha - f_\alpha\|_1 + \|\delta_x * e_\beta - e_\beta * \delta_x\|_1 \to 0
\end{aligned}$$

uniformly on compact subsets of G.

Now by Lemma 4.2.1 (2), $\pi(m^\gamma) \subset L^1(G)_1^+$. Let U be any neighbourhood of e and take β_0 such that support$(e_\beta) \subset U$ whenever $\beta \succeq \beta_0$. Fix α_0, let $\gamma_0 = (\alpha_0, \beta_0)$, and suppose that $\gamma \succeq \gamma_0$. Then support$(m^\gamma) \subset p^{-1}(U)$, so by Lemma 4.2.1 (1)

$$\begin{aligned}\int_{G\setminus U} \pi(m^\gamma)(x)dx &= \langle \pi(m^\gamma), 1_{G\setminus U}\rangle = \langle m^\gamma, 1_{G\setminus U}\circ p\rangle \\ &= \int_{p^{-1}(U)} m^\gamma(s,t) 1_{G\setminus U}\circ p(s,t)d(s,t) = 0.\end{aligned}$$

It follows that $\pi(m^\gamma) = 0$ a.e on $G\setminus U$. Hence $(\pi(m^\gamma))$ is a (quasi-central) bau for $L^1(G)$. □

Remarks 4.2.10. Suppose conversely that (m^γ) is a compactly-invariant approximate diagonal for $L^1(G)$, and let $g_\gamma = g_{m^\gamma}$. Then a simpler version of the argument used to establish (3) ⇒ (1) of Theorem 4.2.2 shows that $(g_\gamma) \subset L^1(G)_1^+$ and $\|\delta_x * g_\gamma - g_\gamma\|_1 \to 0$ uniformly on compact subsets of G. This is Reiter's condition. Moreover, by Proposition 4.2.7, $(\pi(m^\gamma)) \subset L^1(G)_1^+$ is a quasi-central bau for $L^1(G)$ such that $\|\delta_x * \pi(m^\gamma) - \pi(m^\gamma) * \delta_x\|_1 \to 0$ uniformly on compact subsets of G.

One immediate consequence of this work is a new proof of Johnson's Theorem given entirely in terms of approximate diagonals.

Corollary 4.2.11. (Johnson's Theorem) *The following are equivalent for a locally compact group G.*

(1) G is amenable.

(2) $L^1(G)$ is 1-amenable.

(3) $L^1(G)$ is amenable.

Proof. (1) ⇒ (2) is contained in Theorem 4.2.9 and (2) ⇒ (3) is trivial. (3) ⇒ (1) follows from Lemma 4.2.1 (3) and Theorem 4.2.2. □

In the language of Definition 4.2.6 we have proved

Corollary 4.2.12. *The following are equivalent for a locally compact group G.*

(1) G is amenable.

(2) $L^1(G)$ has a compactly-invariant approximate diagonal.

(3) $L^1(G)$ has a measure-invariant approximate diagonal.

(4) $L^1(G)$ has a measure-invariant virtual diagonal.

Remarks 4.2.13. (1) In the next section we will interpret condition (2) of Corollary 4.2.12 as a Reiter condition characterizing amenability of $L^1(G)$ and use it to obtain structural properties for $G \times G$ which reflect the amenability of $L^1(G)$.
(2) In this section we have examined the relationship between nets converging to topological invariance, quasi-central bounded approximate units and approximate diagonals for $L^1(G)$. Both nets converging to invariance and quasi-central bounded approximate units are clearly less complicated than approximate diagonals and have the advantage that they can be studied separately, so it would be interesting to see what can be said along these lines in the context of other amenable Banach algebras.

4.3 Følner Conditions

The Følner condition (FC) was proved for discrete groups by E. Følner in [15]. An elegant proof due to I. Namioka appears in [38]. Employing Namioka's method, (FC) is proved in [18] for amenable locally compact groups by use of [38, 3.1] and Reiter's condition [18, 3.6.2] . We begin this section with Theorem 4.3.3 where we derive our Følner conditions (F) and ($\mathrm{F}_\mathcal{V}$) from condition (2) of Corollary 4.2.12 and Lemma 4.3.1. We then derive from the classical Følner condition (FC) and Theorem 3.4.3 Følner-type conditions ($\mathrm{A}_\mathcal{V}$), ($\mathrm{B}_\mathcal{V}$), and ($\mathrm{C}_\mathcal{V}$). Finally we show that all of our conditions are the correct Følner conditions reflecting the amenability of $L^1(G)$ in the sense that they yield (compactly-invariant) approximate diagonals for $L^1(G)$ comprised of normalized characteristic functions.

Let $\mathcal{B}$ denote the σ-algebra of Borel subsets of G, and let $\mathcal{A}$ be the algebra of subsets of $G \times G$ generated by $\mathcal{B} \times \mathcal{B} = \{\mathrm{B} \times \mathrm{C} : \mathrm{B}, \mathrm{C} \in \mathcal{B}\}$. As $\mathcal{B} \times \mathcal{B}$ is a semi-algebra, [41, page 303], $A \in \mathcal{A}$ if and only if A may be written as a finite disjoint union of sets in $\mathcal{B} \times \mathcal{B}$. If A is any measurable subset of $G \times G$ with $|A| < \infty$, then by regularity of Haar measure and a simple compactness argument, for any $\epsilon > 0$ there is $A' \in \mathcal{A}$ such that $|A \bigtriangleup A'| < \epsilon$. Thus the $\mathcal{A}$-simple functions are dense in $L^1(G \times G)$. In particular part (2) of the following lemma follows from Corollary 4.2.12; part (1) is essentially [38, 3.1], (one just needs to check that the sets $A_i \in \mathcal{A}$).

Lemma 4.3.1. *(1) If $m \in L^1(G \times G)_1^+$ is an $\mathcal{A}$-simple function, then m may be written in the form*

$$m = \sum_{i=1}^{n} \lambda_i \phi_{A_i},$$

where each $\lambda_i > 0$, $\sum_{i=1}^{n} \lambda_i = 1$, $A_1 \supset \ldots \supset A_n$, with each $A_i \in \mathcal{A}$, $0 < |A_i| < \infty$ and $\phi_A := \frac{1}{|A|} 1_A$.

(2) If G is amenable then $L^1(G)$ has a compactly-invariant approximate diagonal $(m^\gamma) \subset L^1(G \times G)_1^+$ comprised of $\mathcal{A}$-simple functions. $\square$

For any $C \subset G$, we will use the notation

$$\nabla(C) = p^{-1}(C) = \{(x, y) : xy \in C\},$$

and for $A \subset G \times G$ we write

$$x \cdot A = \{(xs, t) : (s, t) \in A\} \quad \text{and} \quad A \cdot x = \{(s, tx) : (s, t) \in A\}.$$

Observe that if $A \subset G \times G$, then $x \cdot 1_A = l_{(x^{-1},e)} 1_A = 1_{(x,e)A} = 1_{x \cdot A}$, and $1_A \cdot x = \frac{1}{\Delta(x)} r_{(e,x^{-1})} 1_A = \frac{1}{\Delta(x)} 1_{A \cdot x}$. We will now define our first two structural conditions.

(F) For every $\epsilon > 0$, every $\delta > 0$, and every $K, L \subset G$ with K compact and L measurable and of finite measure, there exist sets $A \in \mathcal{A}$ and $N \subset L$ with $0 < |A| < \infty$, $|N| < \delta$ such that

(i) $|x \cdot A \,\triangle\, A \cdot x| < \epsilon |A|, \quad (x \in K)$, and

(ii) $|A \setminus \nabla(Lx^{-1})| + |A \setminus \nabla(xL^{-1})| < \epsilon |A|, \quad (x \in L \setminus N)$.

For each base $\mathcal{V}$ for the neighbourhood system at e we define a corresponding Følner condition ($\mathrm{F}_\mathcal{V}$) as follows:

($\mathrm{F}_\mathcal{V}$) For every $\epsilon > 0$, every $\delta > 0$, every compact $K \subset G$, and every $V \in \mathcal{V}$, there exist sets $A \in \mathcal{A}$ and $N \subset V$ with $0 < |A| < \infty$, $|N| < \delta$ such that

(i) $|x \cdot A \,\triangle\, A \cdot x| < \epsilon |A|, \quad (x \in K)$, and

(ii) $|A \setminus \nabla(Vx^{-1})| + |A \setminus \nabla(xV^{-1})| < \epsilon |A|, \quad (x \in V \setminus N)$.

Conditions (ii) of (F) and ($\mathrm{F}_{\mathcal{V}}$) say that the set A lies very close to the reverse-diagonal $\nabla(e) = \{(x, x^{-1}) : x \in G\}$. Lemma 4.3.2 will show that the unimodularity condition cannot be omitted from the statements of the theorems in this section.

Lemma 4.3.2. *Suppose that there exists a net (A_α) of measurable subsets of $G \times G$ with $0 < |A_\alpha| < \infty$ such that*

$$\frac{|x \cdot A_\alpha \,\Delta\, A_\alpha \cdot x|}{|A_\alpha|} \to 0, \qquad (x \in G).$$

Then G is unimodular.

Proof. For each $x \in G$ and each α,

$$\Delta(x) = \frac{|A_\alpha(e, x)|}{|A_\alpha|} = \frac{|A_\alpha \cdot x|}{|A_\alpha|}$$

so the argument is the same as that which is given in the proof of Lemma 3.4.2. □

Theorem 4.3.3. *The following are equivalent for a locally compact group G.*

(1) G is amenable and unimodular.

(2) G satisfies the Følner condition (F).

(3) For every base $\mathcal{V}$ for the neighbourhood system at e, G satisfies the Følner condition ($\mathrm{F}_{\mathcal{V}}$).

Proof. In order to prove (1) $\Rightarrow$ (2) we need two lemmas.

Lemma 4.3.4. *Let $m \in L^1(G \times G)_1^+$ be an $\mathcal{A}$-simple function, written as in Lemma 4.3.1 (1). Then for any $x \in G$,*

$$\|x \cdot m - m \cdot x\|_1 = \sum_{i=1}^{n} \lambda_i \frac{|x \cdot A_i \,\Delta\, A_i \cdot x|}{|A_i|}.$$

Proof. For $A \in \mathcal{A}$ and $x \in G$ we have

$$(x \cdot \phi_A - \phi_A \cdot x)(s, t) = \begin{cases} \frac{1}{|A|} & \text{if } (s,t) \in x \cdot A \setminus A \cdot x \\ \frac{-1}{|A|} & \text{if } (s,t) \in A \cdot x \setminus x \cdot A \\ 0 & \text{otherwise.} \end{cases}$$

The lemma is now established by arguing as in the proof of Lemma 3.4.6. □

Though we are dealing only with unimodular groups in the theorem, the following lemma works equally well (and is no more difficult to prove) for any locally compact group, so we prove it accordingly.

Lemma 4.3.5. *Let G be a locally compact group, (not necessarily unimodular) and let $m \in L^1(G \times G)_1^+$ be an $\mathcal{A}$-simple function, written as in Lemma 4.3.1 (1). Let $L \subset G$ be a Borel set with $|L| < \infty$. Then*

$$\|\pi(m) * 1_L - 1_L\|_1 = \sum_{i=1}^{n} \frac{\lambda_i}{|A_i|} \int_L (|A_i \setminus \bigtriangledown (Lx^{-1})| + |A_i \setminus \bigtriangledown (xL^{-1})|)dx.$$

Proof. We'll first show that for any $A \in \mathcal{A}$,

$$(i)\ \pi(1_A) * 1_L(x) = |A \cap \bigtriangledown(xL^{-1})| \qquad (x \in G), \quad \text{and}$$

$$(ii) \int_{G \setminus L} \pi(1_A) * 1_L(x)dx = \int_L |A \setminus \bigtriangledown (Lx^{-1})|dx.$$

If $A \in \mathcal{A}$ is arbitrary, then A is a disjoint union $\bigcup_{i=1}^n A_i$, where each $A_i = B_i \times C_i \in \mathcal{B} \times \mathcal{B}$. Thus $1_A = \sum_{i=1}^n 1_{A_i}$ and it suffices to demonstrate (i) and (ii) for $A = B \times C$, where B, C are Borel subsets of G of finite measure. In this case

$$\begin{aligned} \pi(1_A) * 1_L(x) &= \pi(1_B \otimes 1_C) * 1_L(x) = 1_B * 1_C * 1_L(x) \\ &= \iint 1_B(s) 1_C(t) 1_L(t^{-1}s^{-1}x)dtds \\ &= \iint 1_{(B \times C) \cap \bigtriangledown(xL^{-1})}(s,t)dtds = |A \cap \bigtriangledown(xL^{-1})| \end{aligned}$$

which is (i).

Now

$$\begin{aligned} \int_{G \setminus L} \pi(1_A) * 1_L(x)dx &= \int_{G \setminus L} 1_B * 1_C * 1_L(x)dx \\ &= \int_{G \setminus L} \int_G \frac{1}{\Delta(y)} (1_B * 1_C)(xy^{-1}) 1_L(y)dydx \\ &= \int_L \int_{G \setminus L} \frac{1}{\Delta(y)} (1_B * 1_C)(xy^{-1})dxdy \end{aligned}$$

by use of Fubini's theorem. But

$$
\begin{aligned}
\int_{G\setminus L} \frac{1}{\Delta(y)}(1_B * 1_C)(xy^{-1})dx &= \int_G \frac{1}{\Delta(y)}(1_B * 1_C)(xy^{-1})dx \\
&\quad - \int_L \frac{1}{\Delta(y)}(1_B * 1_C)(xy^{-1})dx \\
&= |B||C| - \int_G \frac{1}{\Delta(y)}(1_B * 1_C)(xy^{-1})1_L(x)dx \\
&= |A| - \iint 1_B(z)\frac{1}{\Delta(y)}1_C(z^{-1}xy^{-1})1_L(x)dxdz \\
&= |A| - \iint 1_B(z)1_C(x)1_L(zxy)dxdz \\
&= |A| - \iint 1_{(B\times C)\cap\bigtriangledown(Ly^{-1})}(z,x)dxdz \\
&= |A| - |A \cap \bigtriangledown(Ly^{-1})| = |A\setminus \bigtriangledown(Ly^{-1})|
\end{aligned}
$$

which gives equation (ii).

Finally, if we express $m = \sum_{i=1}^{n} \lambda_i \phi_{A_i}$ as in Lemma 4.3.1 (1), we obtain

$$
\begin{aligned}
\|\pi(m) * 1_L - 1_L\|_1 &= \int_L |\pi(m) * 1_L(x) - 1|dx + \int_{G\setminus L} \pi(m) * 1_L(x)dx \\
&= \int_L |\sum_{i=1}^{n} \frac{\lambda_i}{|A_i|}(\pi(1_{A_i}) * 1_L(x) - |A_i|)| \\
&\quad + \sum_{i=1}^{n} \frac{\lambda_i}{|A_i|}\int_L |A_i\setminus \bigtriangledown(Lx^{-1})|dx \\
&= \sum_{i=1}^{n} \frac{\lambda_i}{|A_i|}\int_L (|A_i\setminus \bigtriangledown(xL^{-1})| + |A_i\setminus \bigtriangledown(Lx^{-1})|)dx.
\end{aligned}
$$

□

We can now prove (1) ⇒ (2) of the theorem, by use of an argument similar to the proof of the classical Følner condition (FC) as found for example in [18, 3.6.2, 3.6.4]. We first show that condition (F^*) holds

(F^*) : For every $\epsilon > 0, \delta > 0$, $K, L \subset G$ with K compact and L measurable and of finite measure, there exist sets $A \in \mathcal{A}$ and $N_1 \subset K$, $N_2 \subset L$ with $0 < |A| < \infty$, $|N_i| < \delta$, $i = 1, 2$ such that

(i) $|x \cdot A \bigtriangleup A \cdot x| < \epsilon|A|, \quad (x \in K\setminus N_1)$, and

(ii) $|A\setminus \bigtriangledown(Lx^{-1})| + |A\setminus \bigtriangledown(xL^{-1})| < \epsilon|A|, \quad (x \in L\setminus N_2)$.

To see this let $\epsilon, \delta > 0$, $K, L \subset G$ be as in (F^*). By Lemma 4.3.1 (2) we can find an $\mathcal{A}$-simple function $m \in L^1(G \times G)_1^+$ such that for each $x \in K$

$$\|x \cdot m - m \cdot x\|_1 < \frac{\epsilon\delta}{2|K|} \quad \text{and} \quad \|\pi(m) * 1_L - 1_L\|_1 < \frac{\epsilon\delta}{2}.$$

Writing m as in Lemma 4.3.1 (1), and integrating the continuous function $x \mapsto \|x \cdot m - m \cdot x\|_1$ over K we obtain

$$\sum_{i=1}^{n} \lambda_i \int_K \frac{|x \cdot A_i \triangle A_i \cdot x|}{|A_i|} dx < \frac{\epsilon\delta}{2} \qquad (\dagger)$$

from Lemma 4.3.4. By Lemma 4.3.5 we have

$$\sum_{i=1}^{n} \lambda_i \int_L \frac{|A_i \backslash \nabla (Lx^{-1})| + |A_i \backslash \nabla (xL^{-1})|}{|A_i|} dx < \frac{\epsilon\delta}{2}. \qquad (\dagger\dagger)$$

Adding $(\dagger)$ and $(\dagger\dagger)$ we obtain

$$\sum_{i=1}^{n} \lambda_i [\int_K \frac{|x \cdot A_i \triangle A_i \cdot x|}{|A_i|} dx + \int_L \frac{|A_i \backslash \nabla (Lx^{-1})| + |A_i \backslash \nabla (xL^{-1})|}{|A_i|} dx] < \epsilon\delta.$$

As $\sum_{i=1}^{n} \lambda_i = 1$ and each $\lambda_i > 0$, we must have

$$\int_K \frac{|x \cdot A \triangle A \cdot x|}{|A|} dx + \int_L \frac{|A \backslash \nabla (Lx^{-1})| + |A \backslash \nabla (xL^{-1})|}{|A|} dx < \epsilon\delta$$

for some $A = A_i$. Letting $N_1 = \{x \in K : |x \cdot A \triangle A \cdot x| \geq \epsilon|A|\}$ and $N_2 = \{x \in L : |A \backslash \nabla (Lx^{-1})| + |A \backslash \nabla (xL^{-1})| \geq \epsilon|A|\}$, the sets A, N_1, and N_2 satisfy (F^*).

We now establish $(F^*) \Rightarrow (F)$. Let $\epsilon, \delta > 0$, $K, L \subset G$ be as in (F). Apply (F^*) to $H = K \cup K^2$, L, $\frac{\epsilon}{2}$ and $\delta' = \min\{\frac{1}{2}|K|, \delta\}$ to obtain sets A, N_1 and N_2. Let $M = H \backslash N_1$. Observe that for any $k \in K$, $kH \cap H \subset (kM \cap M) \cup (H \backslash M) \cup (kH \backslash kM)$; also $kK \subset kH \cap H$, so $|kH \cap H| \geq |K|$. Therefore

$$2\delta' \leq |K| \leq |kM \cap M| + 2|N_1| < |kM \cap M| + 2\delta',$$

whence $kM \cap M \neq \emptyset$, $(k \in K)$. Thus $K \subset MM^{-1}$. But for any $x, y \in M = H \backslash N_1$,

$$\begin{aligned}
\frac{|(xy^{-1}) \cdot A \triangle A \cdot (xy^{-1})|}{|A|} &= \|(xy^{-1}) \cdot \phi_A - \phi_A \cdot (xy^{-1})\|_1 \\
&\leq \|x \cdot (y^{-1} \cdot \phi_A - \phi_A \cdot y^{-1})\|_1 + \|(x \cdot \phi_A - \phi_A \cdot x) \cdot y^{-1}\|_1 \\
&= \|\phi_A \cdot y - y \cdot \phi_A\|_1 + \|x \cdot \phi_A - \phi_A \cdot x\|_1 \\
&= \frac{|A \cdot y \triangle y \cdot A|}{|A|} + \frac{|x \cdot A \triangle A \cdot x|}{|A|} < \epsilon.
\end{aligned}$$

(2) $\Rightarrow$ (3) is obvious.

(3) $\Rightarrow$ (1) For each pair $\alpha = (\epsilon, K)$ choose $A_\alpha \in \mathcal{A}$ to satisfy condition (i) of ($\mathrm{F}_\mathcal{V}$); let $m^\alpha = \phi_{A_\alpha}$. From Lemma 4.3.2 we know that G is unimodular, so that by Lemma 4.3.4 $\|x \cdot m^\alpha - m^\alpha \cdot x\|_1 \to 0$ uniformly on compact subsets of G. That G is amenable now follows from Proposition 4.2.4. $\square$

Let $\mathcal{V}$ be any base for the neighbourhood system at e. In the following definitions of conditions ($\mathrm{A}_\mathcal{V}$), ($\mathrm{B}_\mathcal{V}$), and ($\mathrm{C}_\mathcal{V}$), the subsets A of $G \times G$ and K of G are to be assumed *compact* and of *positive measure.*

($\mathrm{A}_\mathcal{V}$) : For every $\epsilon > 0$, $K \subset G$, and $V \in \mathcal{V}$, there exists $A \subset G \times G$, such that

$$\text{(i) } A \subset \nabla(V) \quad \text{and} \quad \text{(ii) } |x \cdot A \,\triangle\, A \cdot x| < \epsilon|A|, \quad (x \in K).$$

($\mathrm{B}_\mathcal{V}$) : For every $\epsilon > 0$, $K \subset G$, and $V \in \mathcal{V}$, there exists $A \subset G \times G$, such that

$$\text{(i) } |A \setminus \nabla(V)| < \epsilon|A| \quad \text{and} \quad \text{(ii) } |x \cdot A \,\triangle\, A \cdot x| < \epsilon|A|, \quad (x \in K).$$

($\mathrm{C}_\mathcal{V}$) : For every $\epsilon > 0$, $K \subset G$, and $V \in \mathcal{V}$, there exists $A \subset G \times G$, such that

$$\text{(i) } |A \cap \nabla(V)| > (1-\epsilon)|A| \quad \text{and} \quad \text{(ii) } |x \cdot A \cap A \cdot x| > (1-\epsilon)|A|, \quad (x \in K).$$

The next result is proved by use of the Følner condition (FC) and Theorem 3.4.3.

Theorem 4.3.6. *Let G be a locally compact group, and let $\mathcal{V}$ be any base for the neighbourhood system at e. The following are equivalent.*

(1) G is amenable and unimodular.

(2) G satisfies the condition ($\mathrm{A}_\mathcal{V}$).

(3) G satisfies the condition ($\mathrm{B}_\mathcal{V}$).

(4) G satisfies the condition ($\mathrm{C}_\mathcal{V}$).

Proof. By Lemma 4.3.2 each of the conditions ($\mathrm{A}_\mathcal{V}$), ($\mathrm{B}_\mathcal{V}$), and ($\mathrm{C}_\mathcal{V}$) imply unimodularity. It is clear that (2) $\Rightarrow$ (3), and (3) $\Leftrightarrow$ (4) follows from the identity $|x \cdot A \,\triangle\, A \cdot x| = 2(|A| - |x \cdot A \cap A \cdot x|)$. We have (4) $\Rightarrow$ (1) by the argument given to prove (3) $\Rightarrow$ (1) of Theorem 4.3.3. Only (1) $\Rightarrow$ (2) remains.

Let $0 < k < 1$, $K \subset G$ compact, and $V \in \mathcal{V}$. It is sufficient to find a compact set A such that $A \subset \triangledown(V)$ and $|x \cdot A \cap A \cdot x| > k|A|, \quad (x \in K)$. By use of the condition (FC), Theorem 4.2.8 (1), and Theorem 3.4.3 we can find compact subsets B and U of G with U a neighbourhood of e contained in V such that

$$|xB \cap B| > \sqrt{k}|B| \text{ and } |xU \cap Ux| > \sqrt{k}|U|, \quad (x \in K).$$

Let $A = \{(s,t) : s \in B \text{ and } st \in U\}$. Obviously $A \subset \triangledown(V)$. Now it is easy to see that for any $x \in G$, $x \cdot A = \{(s,t) : s \in xB \text{ and } st \in xU\}$, and $A \cdot x = \{(s,t) : s \in B \text{ and } st \in Ux\}$. It follows that $x \cdot A \cap A \cdot x = \{(s,t) : s \in xB \cap B \text{ and } st \in xU \cap Ux\}$. But if C and W are any two measurable subsets of G with finite measure, and $E = \{(s,t) : s \in C \text{ and } st \in W\}$, then

$$|E| = \iint 1_E(x,y)dydx = \iint 1_E(x, x^{-1}y)dydx = \iint 1_{C\times W}(x,y)dydx = |C||W|.$$

Thus for any $x \in K$,

$$|x \cdot A \cap A \cdot x| = |xB \cap B||xU \cap Ux| > \sqrt{k}|B|\sqrt{k}|U| = k|A|.$$

$\square$

The final result of this section shows that all of our Følner conditions correctly reflect the amenability of $L^1(G)$ in the sense that each yields a compactly-invariant approximate diagonal comprised of normalized characteristic functions. Before stating the result we fix some notation. Suppose that G is an amenable and unimodular locally compact group and let $\mathcal{V}$ be any base for the neighbourhood system at e. Let

$$\mathcal{T} = \{\alpha = (\epsilon, K, L) : \epsilon > 0, \ K, L \subset G \text{ compact}, \ |K|, |L| > 0\},$$

$$\mathcal{T}_{\mathcal{V}} = \{\beta = (\epsilon, K, V) : \epsilon > 0, \ K \subset G \text{ compact with } |K| > 0, \ V \in \mathcal{V}\}.$$

Direct $\mathcal{T}$ by putting $\alpha_0 = (\epsilon_0, K_0, L_0) \preceq \alpha_1 = (\epsilon_1, K_1, L_1)$ if and only if $\epsilon_1 \le \epsilon_0$, $K_1 \supset K_0$, $L_1 \subset L_0$; direct $\mathcal{T}_{\mathcal{V}}$ analogously. For each $\alpha = (\epsilon, K, L) \in \mathcal{T}$, take A_α a Borel measurable subset of $G \times G$ (and $N_\alpha \subset L$) to correspond to (ϵ, K, L) and $\delta = \frac{1}{2}|L|$ as in condition (F). For each $\beta = (\epsilon, K, V) \in \mathcal{T}_{\mathcal{V}}$ take a subset A_β of $G \times G$ to correspond to (ϵ, K, V) as in either condition ($\mathrm{F}_{\mathcal{V}}$), ($\mathrm{A}_{\mathcal{V}}$), ($\mathrm{B}_{\mathcal{V}}$) or ($\mathrm{C}_{\mathcal{V}}$), (if A_β is chosen using ($\mathrm{F}_{\mathcal{V}}$), assume that $N_\beta \subset V$ is chosen with respect to $\delta = \frac{1}{2}|V|$).

Theorem 4.3.7. *Let G be a unimodular amenable locally compact group. The nets (ϕ_{A_α}) and (ϕ_{A_β}) of normalized characteristic functions of subsets A_α, A_β of $G \times G$ are compactly-invariant approximate diagonals for $L^1(G)$.*

Proof. For notational convenience we write $\phi_\alpha = \phi_{A_\alpha}$, $\phi_\beta = \phi_{A_\beta}$. Observe that by Lemma 4.3.4, (where the assumption that the functions were $\mathcal{A}$-simple was unnecessary), in all cases we have $\|x \cdot \phi_\alpha - \phi_\alpha \cdot x\| \to 0$ and $\|x \cdot \phi_\beta - \phi_\beta \cdot x\| \to 0$ uniformly on compacta of G.

Claim 1: Assuming that the sets $A_\alpha, A_\beta \in \mathcal{A}$, the nets $(\pi(\phi_\alpha))$ and $(\pi(\phi_\beta))$ are bounded approximate units for $L^1(G)$.

This will follow if we can show that for any compact subset L_0 of G with $|L_0| > 0$, $\|\pi(\phi_\alpha) * 1_{L_0} - 1_{L_0}\|_1 \to 0$ and $\|\pi(\phi_\beta) * 1_{L_0} - 1_{L_0}\|_1 \to 0$. To this end, let $\epsilon_0 > 0$, and take U an open neighbourhood of L_0 such that $|U \backslash L_0| < \frac{\epsilon_0}{6}$. By [23, 4.10] we can find a compact neighbourhood W_0 of e such that $W_0 W_0^{-1} L_0 \subset U$. Choosing K_0 compact with $|K_0| > 0$ arbitrarily, let $\alpha_0 = (\frac{\epsilon_0}{3|L_0|}, K_0, W_0)$, and suppose that $\alpha = (\epsilon, K, W) \succeq \alpha_0$. Then for any $x \in L_0$, $WW^{-1}x \subset U$, so $WW^{-1} \subset Ux^{-1}$ and $WW^{-1} \subset xU^{-1}$. In particular, for any $x \in L_0$, taking any $y_x \in W \backslash N_\alpha$ one has $Ux^{-1} \supset Wy_x^{-1}$ and $xU^{-1} \supset y_x W^{-1}$. Thus

$$\begin{aligned} |A_\alpha \backslash \bigtriangledown (Ux^{-1})| + |A_\alpha \backslash \bigtriangledown (xU^{-1})| &\leq |A_\alpha \backslash \bigtriangledown (Wy_x^{-1})| + |A_\alpha \backslash \bigtriangledown (y_x W^{-1})| \\ &< \epsilon |A_\alpha|, \qquad (x \in L_0). \end{aligned}$$

Using this and Lemma 4.3.5 we obtain

$$\begin{aligned} \|\pi(\phi_\alpha) * 1_{L_0} - 1_{L_0}\|_1 &= \|\pi(\phi_\alpha) * (1_U - 1_{U\backslash L_0}) - (1_U - 1_{U\backslash L_0})\|_1 \\ &\leq \|\pi(\phi_\alpha) * 1_U - 1_U\|_1 + \|\pi(\phi_\alpha)\|_1 \|1_{U\backslash L_0}\|_1 + \|1_{U\backslash L_0}\|_1 \\ &\leq \int_U \frac{|A_\alpha \backslash \bigtriangledown (Ux^{-1})| + |A_\alpha \backslash \bigtriangledown (xU^{-1})|}{|A_\alpha|}\, dx + \frac{2\epsilon_0}{6} \\ &\leq \int_{U\backslash L_0} 2\, dx + \int_{L_0} \epsilon\, dx + \frac{\epsilon_0}{3} \\ &\leq \frac{2\epsilon_0}{3} + \epsilon |L_0| \leq \epsilon_0. \end{aligned}$$

If the sets A_β correspond to $(\mathrm{F}_\mathcal{V})$ the argument given above works provided that W_0 is chosen from $\mathcal{V}$. Suppose now that the sets A_β are chosen to correspond to $(\mathrm{B}_\mathcal{V})$.

Take a neighbourhood $W_0 \in \mathcal{V}$ such that $W_0 L_0 \subset U$ and choose $V_0 \in \mathcal{V}$ so that $V_0 \subset W_0 \cap W_0^{-1}$. Suppose that $\beta = (\epsilon, K, V) \succeq \beta_0 = (\frac{\epsilon_0}{6|L_0|}, K_0, V_0)$. Then for each $x \in L_0$, $V \subset Ux^{-1}$ and $V \subset xU^{-1}$, so again we have

$$|A_\beta \backslash \nabla (Ux^{-1})| + |A_\beta \backslash \nabla (xU^{-1})| \leq 2|A_\beta \backslash \nabla (V)| < 2\epsilon |A_\beta|.$$

That $\|\pi(\phi_\beta) * 1_{L_0} - 1_{L_0}\|_1 < \epsilon_0$ now follows as before. This argument also works when the sets A_β are chosen to correspond to $(\mathrm{A}_\mathcal{V})$ or $(\mathrm{C}_\mathcal{V})$.

Claim 2: Without the assumption that the sets $A_\alpha, A_\beta \in \mathcal{A}$, the nets $(\pi(\phi_\alpha))$ and $(\pi(\phi_\beta))$ are bounded approximate units for $L^1(G)$.

We prove this in the case in which the sets A_β are chosen to correspond to the condition $(\mathrm{F}_\mathcal{V})$. The other cases all follow from a similar argument. For each $\beta = (\epsilon, K, V) \in \mathcal{T}_\mathcal{V}$, take $A'_\beta \in \mathcal{A}$ such that $|A'_\beta \triangle A_\beta| < \min\{\epsilon |A_\beta|, \frac{1}{2}|A_\beta|\}$. Let $\phi'_\beta = \phi_{A'_\beta}$. Then

$$\begin{aligned}
\|\phi'_\beta - \phi_\beta\|_1 &\leq \|\phi_{A'_\beta} - \frac{1}{|A_\beta|} 1_{A'_\beta}\|_1 + \|\frac{1}{|A_\beta|} 1_{A'_\beta} - \phi_{A_\beta}\|_1 \\
&= \frac{||A_\beta| - |A'_\beta||}{|A_\beta|} + \frac{|A_\beta \triangle A'_\beta|}{|A_\beta|} \\
&\leq 2\frac{|A_\beta \triangle A'_\beta|}{|A_\beta|} < 2\epsilon. \qquad (4.3.7.1)
\end{aligned}$$

Now $|A_\beta| = |A_\beta \cap A'_\beta| + |A_\beta \backslash A'_\beta| \leq |A'_\beta| + \frac{1}{2}|A_\beta|$, so $|A_\beta| \leq 2|A'_\beta|$. It follows that for any $x \in V \backslash N_\beta$,

$$\begin{aligned}
|A'_\beta \backslash \nabla (Vx^{-1})| + |A'_\beta \backslash \nabla (xV^{-1})| &= |(A'_\beta \cap A_\beta) \backslash \nabla (Vx^{-1})| \\
&\quad + |(A'_\beta \backslash A_\beta) \backslash \nabla (Vx^{-1})| \\
&\quad + |(A'_\beta \cap A_\beta) \backslash \nabla (xV^{-1})| \\
&\quad + |(A'_\beta \backslash A_\beta) \backslash \nabla (xV^{-1})| \\
&\leq |A_\beta \backslash \nabla (Vx^{-1})| + |A_\beta \backslash \nabla (xV^{-1})| \\
&\quad + 2|A'_\beta \triangle A_\beta| \\
&< \epsilon |A_\beta| + 2\epsilon |A_\beta| \leq 6\epsilon |A'_\beta|. \qquad (4.3.7.2)
\end{aligned}$$

It follows from (4.3.7.2) and Claim 1 that $(\pi(\phi'_\beta))$ is a bounded approximate unit for

$L^1(G)$. But for any $f \in L^1(G)$,

$$\|\pi(\phi_\beta) * f - f\|_1 \leq \|\pi(\phi_\beta - \phi'_\beta) * f\|_1 + \|\pi(\phi'_\beta) * f - f\|_1$$

which by use of inequality (4.3.7.1) converges to zero. Thus Claim 2 holds. □

Remarks 4.3.8. (1) It follows from the proof of Theorem 4.3.7 that we do not need to assume that the sets A_α and A_β belong to $\mathcal{A}$, (as in the official statements of (F) and ($\mathrm{F}_\mathcal{V}$)), in order to obtain compactly-invariant approximate diagonals comprised of normalized characteristic functions. Moreover, for any unimodular amenable group, the conditions ($\mathrm{A}_\mathcal{V}$), ($\mathrm{B}_\mathcal{V}$), and ($\mathrm{C}_\mathcal{V}$) hold even if we demand that the sets $A_\beta \in \mathcal{A}$.
(2) When the sets A_β correspond to condition ($\mathrm{A}_\mathcal{V}$), one can prove that the net $(\pi(\phi_{A_\beta}))$ is a bounded approximate diagonal for $L^1(G)$ by means of a simpler argument which is similar to the one found in the last paragraph of the proof of Theorem 4.2.9.
(3) Let G be any unimodular locally compact group which is amenable. Let $\mathcal{U} = (U_\delta)$ be a base for the neighbourhood system at e as in Theorem 3.4.3, and let (K_γ) be a Følner net for G in the sense that it satisfies condition (3) of [39, Definition 4.15]. Let

$$A_{\gamma,\delta} = \{(s,t) : s \in K_\gamma \text{ and } st \in U_\delta\}.$$

Then the argument used in the proof of Theorem 4.3.6 together with Theorem 4.3.7 show that the net $(\phi_{A_{\gamma,\delta}})$ is a compactly-invariant approximate diagonal for $L^1(G)$. If G is σ-compact and first countable, (that is metrizable), then this approximate diagonal may be chosen to be a sequence. We remark that a great deal of research has been done regarding the explicit construction of Følner nets for certain classes of locally compact groups, [39, Chapter 6]. It would be interesting if for such groups one could similarly construct asymptotically invariant nets as described in Theorem 3.4.3. If so, we would have a method for constructing compactly-invariant approximate diagonals, comprised of normalized characteristic functions, for group algebras of such groups. When G is abelian, any base $\mathcal{U}$ for the neighbourhood system at e satisfies condition (2) of Theorem 3.4.3 so it is particularly easy to construct approximate diagonals for $L^1(G)$. For example, if G is the additive group of real numbers, letting $A_n = \{(s,t) : s \in [-n,n] \text{ and } s+t \in [\frac{-1}{n}, \frac{1}{n}]\}$, the sequence (ϕ_{A_n}) is a compactly-invariant approximate diagonal for $L^1(\mathbf{R})$.

(4) In light of Proposition 4.2.4, it is clear that the condition $(F^\sharp)$ also characterizes unimodular amenable locally compact groups, where

$(F^\sharp)$: For every $\epsilon > 0$ and every compact subset K of G, there exists a set $A \in \mathcal{A}$ such that $|x \cdot A \triangle A \cdot x| < \epsilon |A|, \quad (x \in K)$.

(5) Theorems 2.1.1 and 2.1.2 provide a hierarchy of nets converging to invariance, each one characterizing amenable locally compact groups. We have now provided a corresponding hierarchy of approximate diagonals for amenable group algebras.

(6) We leave open the question of whether the subsets N of L and V can be omitted from the conditions (F) and ($\mathrm{F}_\mathcal{V}$).

4.4 1-Amenability of semigroup algebras

In this final section of Chapter 4 we turn our attention to the problem of determining which semigroups carry amenable semigroup algebras. Throughout, S will denote a (discrete) semigroup. The most complete solutions to this problem have been obtained by placing additional algebraic conditions on the semigroup [9], [10], [19], [20], and [32]. We will see that with no extraneous conditions placed upon our semigroup S, if its semigroup algebra $l^1(S)$ is 1-amenable, then S is necessarily an amenable group. We will also provide Følner conditions (A), (B), and (C) on S which correspond exactly to 1-amenability of $l^1(S)$.

If E is a subset of S, $|E|$ is its cardinality. We write

$$[s^{-1}t] = \{x \in S : sx = t\} \quad \text{and} \quad [st^{-1}] = \{x \in S : s = xt\}, \quad (s,t \in S).$$

As in the case for groups, we may identify $l^1(S)\hat{\otimes} l^1(S)$ with $l^1(S \times S)$ and we shall identify the Dirac function at an element s of S with s itself. Doing this, a function

$m^\gamma \in l^1(S \times S)$ may be written in the form

$$m^\gamma = \sum_{S\times S} \beta^\gamma_{s,t}(s,t)$$

and one sees that the net (m^γ) is an approximate diagonal for $l^1(S)$ if and only if it is bounded, and for each $v \in S$

$$\lim_\gamma \sum_{S\times S} \beta^\gamma_{s,t}[(s,tv)-(vs,t)] = 0, \quad \lim_\gamma \sum_{S\times S} \beta^\gamma_{s,t} stv = v.$$

Further details regarding these identifications may be found in [9].

Definition 4.4.1. The semigroup S is *left cancellative* if for any $s,t \in S$, $|[s^{-1}t]| \leq 1$. If for any $s \in S$, $sup\{|[s^{-1}t]| : t \in S\} < \infty$, then S will be called *left subcancellative.*

Recall from [32], that S is *left weakly cancellative,* if for any $s,t \in S$, $|[s^{-1}t]| < \infty$. It is clear that left cancellativity implies left subcancellativity, which in turn implies left weak cancellativity. That these implications cannot be reversed is fairly easy to show.

For $A \subset S \times S$ and $v \in S$, $v \cdot A$, $A \cdot v$ are as defined in the previous section, and we write

$$\nabla(v) = \{(s,t) \in S \times S : stv = v\}.$$

In the following definitions of Følner-type conditions (A), (B), and (C), the subsets A of $S \times S$, and F of S are to be assumed *finite and non-empty:*

(A) : For every $\epsilon > 0$, $F \subset S$, there exists $A \subset S \times S$, such that

$$A \subset \nabla(v), \quad |v \cdot A \triangle A \cdot v| < \epsilon|A|, \quad (v \in F).$$

(B) : For every $\epsilon > 0$, $F \subset S$, there exists $A \subset S \times S$, such that

$$|A \setminus \nabla(v)| \quad + \quad |v \cdot A \triangle A \cdot v| < \epsilon|A|, \quad (v \in F).$$

(C) : For every $\epsilon > 0$, $F \subset S$, there exists $A \subset S \times S$, such that

$$|A \cap \nabla(v)| \quad + \quad |v \cdot A \cap A \cdot v| > (2-\epsilon)|A|, \quad (v \in F).$$

We note that when S is a group, for any $v \in S$ we have $\triangledown(v) = \{(s, s^{-1}) : s \in S\}$, so the conditions ($\mathrm{A}_\mathcal{V}$), ($\mathrm{B}_\mathcal{V}$), and($\mathrm{C}_\mathcal{V}$) of Section 4.3 are non-discrete analogues of these new conditions.

Theorem 4.4.2. *The following are equivalent for a semigroup S:*

(1) $l^1(S)$ is 1-amenable.

(2) S is an amenable group.

(3) S is one-sided subcancellative, and satisfies condition (A).

(4) S is one-sided subcancellative, and satisfies condition (B).

(5) S satisfies condition (C).

Example 4.4.3. Let S be the semigroup of positive integers with multiplication given by $n \cdot m = \min\{n, m\}$. Then S satisfies condition (A), (and therefore condition (B)), yet S is not a group. To see this let $\epsilon > 0$, and let F be a finite subset of S. Let $m = \max(F)$, and put $B = \{m+1, ..., m+n\}$, where $2/n < \epsilon$. Then for any $v \in F$, $A = B \times B \subset \triangledown(v)$, and

$$|v \cdot A \triangle A \cdot v| \;/\; |A| \;= (|v \cdot A| + |A \cdot v|) \;/\; |A| \;=\; 2n/n^2 \;< \epsilon.$$

This example shows that at least some form of cancellativity is needed in conditions (3) and (4) of the theorem. However note that our example is not even weakly cancellative, (for any $n \in S$, $[n^{-1}n]$ is infinite). It would be interesting to find an example of a weakly cancellative semigroup which is not a group, yet satisfies (A).

To prove the theorem we need some preliminary results.

Lemma 4.4.4. *If $v \in S$, and sup$\{|[v^{-1}t]| : t \in S\} \leq n$, then for any finite $A \subset S \times S$, $|v \cdot A| \geq \frac{1}{n}|A|$.*

Proof. Write A as $A = \bigcup_{i=1}^m \{x_i\} \times C_i$, where $x_i \neq x_j$ whenever $i \neq j$. Let $B = \{x_1, ..., x_m\}$, $vB = \{y_1, ..., y_k\}$, and for $1 \leq j \leq k$, let $M_j = \{i : vx_i = y_j\}$. Then $|M_j| \leq n$, and

$$v \cdot A = \bigcup_{i=1}^{m} \{vx_i\} \times C_i = \bigcup_{j=1}^{k} (\{y_j\} \times \bigcup_{i \in M_j} C_i).$$

Thus

$$|v \cdot A| = \sum_{j=1}^{k} |\bigcup_{i \in M_j} C_i| \geq \sum_{j=1}^{k} \frac{1}{|M_j|} \sum_{i \in M_j} |C_i| \geq \frac{1}{n} \sum_{j=1}^{k} \sum_{i \in M_j} |C_i| = \frac{1}{n}|A|.$$

□

A right simple semigroup which contains an idempotent is called a *right group*. A discussion of right groups can be found in section 1.11 of [6].

Lemma 4.4.5. *A semigroup S is a right group if and only if*

(†) for any $v_1, v_2 \in S$, (not necessarily distinct), there exists $(s,t) \in \nabla(v_1)$ such that $s \in v_2 S$.

Proof. Assume that S satisfies (†) and suppose that I is a (non-empty) proper right ideal in S. Take $v_1 \in S \backslash I$, $v_2 \in I$ and choose $(s,t) \in \nabla(v_1)$ such that $s \in v_2 S$. Then $s \in I$, so $v_1 = stv_1 \in I$, a contradiction. Thus S is right simple. It now suffices to exhibit an idempotent in S. To this end, let $v \in S$ and take $(s,t) \in \nabla(v)$ with $s = vx$ for some $x \in S$. Then $(xtv)^2 = (xt)(vx)tv = (xt)(stv) = xtv$. The converse, (which we don't need), follows easily from the fact that right groups are regular and right simple. □

Lemma 4.4.6. *If S satisfies condition (C), OR if S is left subcancellative and satisfies condition (B), then S is a right group.*

Proof. Suppose first that S is left subcancellative and satisfies condition (B). We will show that S satisfies condition (†) of Lemma 4.4.5. Let $v_1, v_2 \in S$ and suppose that $sup\{|[v_2^{-1}t]| : t \in S\} \leq n$. Take A to be a finite non-empty subset of $S \times S$ such that

$$|A \backslash \nabla(v_i)| + |v_i \cdot A \bigtriangleup A \cdot v_i| < \frac{1}{2n}|A|, \quad (i = 1, 2).$$

Then using Lemma 4.4.4 we have

$$\frac{1}{2n}|A| > |v_2 \cdot A \backslash A \cdot v_2| = |v_2 \cdot A| - |v_2 \cdot A \cap A \cdot v_2| \geq \frac{1}{n}|A| - |v_2 \cdot A \cap A \cdot v_2|,$$

and so $|v_2 \cdot A \cap A \cdot v_2| > \frac{1}{2n}|A|$. Suppose that for every $(s,t) \in A \cap \nabla(v_1)$, $(s, tv_2) \notin v_2 \cdot A$. Then $v_2 \cdot A \cap A \cdot v_2 \subset \{(s, tv_2) : (s,t) \in A \setminus \nabla(v_1)\}$, so

$$\frac{1}{2n}|A| < |v_2 \cdot A \cap A \cdot v_2| \leq |A \setminus \nabla(v_1)| < \frac{1}{2n}|A|,$$

a contradiction. Thus for some $(s,t) \in A \cap \nabla(v_1)$, $(s, tv_2) \in v_2 \cdot A$; in particular $(s,t) \in \nabla(v_1)$, and $s \in v_2 S$. To see that condition (C) implies that property (†) holds is similar, but easier: arguing as above, let $\epsilon = 1$ in condition (C) applied to $v_1, v_2 \in S$. □

If A is a finite subset of $S \times S$, let ϕ_A denote the normalized characteristic function $\frac{1}{|A|}1_A$ of A.

Lemma 4.4.7. *Let S be a left cancellative semigroup which satisfies the following condition in which the sets F and A are to be assumed finite and non-empty:*
(*) For every $\epsilon > 0$, $F \subset S$, there exists $A \subset S \times S$, such that

$$|A \setminus \nabla(v)| \quad + \quad |v \cdot A \setminus A \cdot v| < \epsilon |A|, \quad (v \in F).$$

Then $l^1(S)$ has an approximate diagonal comprised of normalized characteristic functions of finite non-empty subsets of $S \times S$.

Proof. Fix $A \subset S \times S$, $v \in S$. If for $(x,y) \in S \times S$ we write $[(x,y) \cdot v^{-1}] = \{(s,t) : (s,tv) = (x,y)\}$, then it is easy to see that A is the disjoint union

$$A = \bigcup_{(x,y) \in A \cdot v} (A \cap [(x,y) \cdot v^{-1}]) \qquad (4.4.7.1).$$

Now

$$\phi_A \cdot v = \frac{1}{|A|} \sum_{(s,t) \in A} (s, tv), \quad v \cdot \phi_A = \frac{1}{|A|} \sum_{(s,t) \in A} (vs, t),$$

so

$$(\phi_A \cdot v - v \cdot \phi_A)(x,y) = \begin{cases} \frac{1}{|A|}|A \cap [(x,y) \cdot v^{-1}]| & \text{if } (x,y) \in A \cdot v \setminus v \cdot A \\ \frac{-1}{|A|} & \text{if } (x,y) \in v \cdot A \setminus A \cdot v \\ \frac{1}{|A|}[|A \cap [(x,y) \cdot v^{-1}]| - 1] & \text{if } (x,y) \in A \cdot v \cap v \cdot A \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, using (4.4.7.1) and left cancellativity we obtain

$$\begin{aligned}
\|\phi_A \cdot v - v \cdot \phi_A\|_1 &= \sum_{(x,y)} |(\phi_A \cdot v - v \cdot \phi_A)(x,y)| \\
&= \frac{1}{|A|}[\sum_{(x,y)\in A\cdot v\backslash v\cdot A} |A \cap [(x,y)\cdot v^{-1}]| + \sum_{(x,y)\in v\cdot A\backslash A\cdot v} 1 \\
&\quad + \sum_{(x,y)\in A\cdot v\cap v\cdot A} (|A \cap [(x,y)\cdot v^{-1}]| - 1)] \\
&= \frac{1}{|A|}[\sum_{(x,y)\in A\cdot v} |A \cap [(x,y)\cdot v^{-1}]| + |v\cdot A\backslash A\cdot v| - |A\cdot v \cap v \cdot A|] \\
&= \frac{1}{|A|}[|A| - |A\cdot v\cap v\cdot A| + |v\cdot A\backslash A\cdot v|] \\
&= 2\frac{|v\cdot A\backslash A\cdot v|}{|A|}. \qquad (4.4.7.2)
\end{aligned}$$

Also $\pi(\phi_A)v = \frac{1}{|A|}\sum_{(s,t)\in A} stv$, so

$$\begin{aligned}
\|\pi(\phi_A)v - v\|_1 &= |(\pi(\phi_A)v - v)(v)| + \sum_{x\neq v} \pi(\phi_A)v(x) \\
&= |\frac{|A\cap\nabla(v)|}{|A|} - 1| + \frac{|A\backslash\nabla(v)|}{|A|} = 2\frac{|A\backslash\nabla(v)|}{|A|}. \qquad (4.4.7.3)
\end{aligned}$$

(Observe that left cancellativity was not used in the calculation of (4.4.7.3)). Let $\mathcal{F} = \{(F,\epsilon) : F \subset S \text{ finite}, \epsilon > 0\}$ and for each $\gamma = (F,\epsilon) \in \mathcal{F}$, choose $A_\gamma \subset S \times S$ to correspond to γ as in $(*)$; let $m^\gamma = \phi_{A_\gamma}$. Then

$$\|v\cdot m^\gamma - m^\gamma \cdot v\|_1 \to 0 \quad \text{and} \quad \|\pi(m^\gamma)v - v\|_1 \to 0$$

follow from the calculations (4.4.7.2) and (4.4.7.3). □

Put $\Gamma(v) = (vS \times Sv) \cap \{(s,t) \in S \times S : stv = vst = v\}, \quad (v \in S)$.

Lemma 4.4.8. *If $l^1(S)$ is 1-amenable, then for any $v_1, v_2 \in S$, $\Gamma(v_1) \cap \Gamma(v_2) \neq \emptyset$.*

Proof. It will in fact be shown that $\Gamma(v_1)\cap\Gamma(v_2)$ supports an approximate diagonal for $l^1(S)$. Let $(m^\gamma) \subset l^1(S\times S)$ be an approximate diagonal for $l^1(S)$ with $\|m^\gamma\|_1 \leq 1$, and let $M \in l^\infty(S\times S)^*$ be a weak*-limit point of (m^γ). Then $\|M\| \leq 1$ and as M

is a virtual diagonal for $l^1(S)$, $M(1_{S\times S}) = 1$. Thus M is a mean on $l^\infty(S \times S)$. By the weak*-density of $\mathcal{F} = \{f \in l^1(S \times S) : f \geq 0, \|f\|_1 = 1, \text{support}(f) \text{ finite}\}$ in the set of means on $l^\infty(S \times S)$, there exists a net $(f^\delta) \subset \mathcal{F}$ such that (f^δ) converges to M in the weak*-topology of $l^\infty(S \times S)$; this net is necessarily a "weak" approximate diagonal for $l^1(S)$. Standard methods (due to Namioka, [38, Theorem 2.2]), yield an approximate diagonal (p^α) in $\mathcal{F}$ for $l^1(S)$. We write $p^\alpha = \sum \beta^\alpha_{s,t}(s,t)$. For $v \in S$, let

$$Z(v) = (vS \times S) \cap \{(s,t) : stv = v\} \quad \text{and} \quad W(v) = (S \times Sv) \cap \{(s,t) : vst = v\}.$$

Then

$$\lim_\alpha \sum_{(s,t)\in Z(v)} \beta^\alpha_{s,t} = 1 \quad \text{and} \quad \lim_\alpha \sum_{(s,t)\in W(v)} \beta^\alpha_{s,t} = 1,$$

where the first limit is calculated in the proof of [10, Theorem 1] and the second limit follows from a symmetric argument. Fix $v_1, v_2 \in S$. We *claim* that $p^\alpha_0 := \sum\{\beta^\alpha_{s,t}(s,t) : (s,t) \in Z(v_1)\}$ is an approximate diagonal for $l^1(S)$. For given $v \in S$, $p^\alpha_0 \cdot v - v \cdot p^\alpha_0 = p^\alpha \cdot v - v \cdot p^\alpha - n^\alpha$, where

$$n^\alpha = \sum_{(s,t)\notin Z(v_1)} \beta^\alpha_{s,t}[(s,tv) - (vs,t)].$$

Now $\|n^\alpha\|_1 \leq 2\sum\{\beta^\alpha_{s,t} : (s,t) \notin Z(v_1)\} = 2(\|p^\alpha\|_1 - \sum\{\beta^\alpha_{s,t} : (s,t) \in Z(v_1)\}) \to 2(1-1) = 0$, so $\|p^\alpha_0 \cdot v - v \cdot p^\alpha_0\| \to 0$. Similarly $\|\pi(p^\alpha_0)v - v\|_1 \to 0$, which proves the claim.

Now $\|p^\alpha_0\|_1 \to 1$, so $q^\alpha := \frac{1}{\|p^\alpha_0\|_1} p^\alpha_0$ is also an approximate diagonal for $l^1(S)$, and (q^α) is in $\mathcal{F}$ with support$(q^\alpha) \subset$ support$(p^\alpha) \cap Z(v_1)$. Working now with (q^α) and $W(v_1)$ we obtain an approximate diagonal $(r^\alpha) \subset \mathcal{F}$ with support$(r^\alpha) \subset$ support$(q^\alpha) \cap W(v_1) \subset$ support$(p^\alpha) \cap \Gamma(v_1)$. Finally working with (r^α) and $\Gamma(v_2)$ the above argument yields an approximate diagonal $(s^\alpha) \subset \mathcal{F}$ with support$(s^\alpha) \subset$ support$(r^\alpha) \cap \Gamma(v_2) \subset \Gamma(v_1) \cap \Gamma(v_2)$. $\square$

We may now prove Theorem 4.4.2:

(1) $\Rightarrow$ (2) From [10], S is a regular amenable semigroup so it suffices to prove that S contains a unique idempotent, (by regularity it has at least one). So suppose $e, f \in S$

are idempotents. From Lemma 4.4.8 we may take $(s,t) \in \Gamma(e) \cap \Gamma(f)$. Then $es = s$ and $tf = t$, so $e = est = st = stf = f$.

(2) $\Rightarrow$ (3) Let $\epsilon > 0$, and let F be a finite non-empty subset of S. From the classical Følner condition (FC) which characterizes amenable groups, there is a finite subset B of S such that

$$|vB \cap B| > (1 - \epsilon/2)|B|, \qquad (v \in F).$$

Let $A = \{(s, s^{-1}) : s \in B\}$. Then $A \subset \nabla(v) = \{(x, x^{-1}) : x \in S\}$, $(v \in S)$, and it is readily verified that $x \mapsto (x, x^{-1}v)$ defines a bijection from $vB \cap B$ onto $v \cdot A \cap A \cdot v$. Therefore $|v \cdot A \cap A \cdot v| = |vB \cap B| > (1-\epsilon/2)|B| = (1-\epsilon/2)|A|$, whence $|v \cdot A \bigtriangleup A \cdot v| < \epsilon|A|$, $(v \in F)$.

(3) $\Rightarrow$ (4) is obvious.

(4) $\Rightarrow$ (1) We assume that S is left subcancellative, the other case following by a symmetric argument. By Lemma 4.4.6, S is left cancellative and S satisfies condition $(*)$ of Lemma 4.4.7; therefore S is 1-amenable.

(2) $\Rightarrow$ (5) is proved by the argument given in (2) $\Rightarrow$ (3).

(5) $\Rightarrow$ (1) By Lemma 4.4.6, S is left cancellative and condition (C) implies condition $(*)$ of Lemma 4.4.7; therefore S is 1-amenable. $\square$

Remarks 4.4.9. (1) Let S be a discrete group. In Lemma 4.4.7 we saw that condition (B) naturally yields an approximate diagonal comprised of normalized characteristic functions of finite subsets A of $S \times S$. Similarly, the Følner condition (FC) gives rise to a net comprised of normalized characteristic functions of finite subsets A of S which converges to left (or right) invariance in $l^1(S)$. This suggests that our conditions (A), (B), and (C) are the "correct" Følner conditions corresponding to (1-)amenability of $l^1(S)$.

(2) In [10], J. Duncan and A.L.T Paterson prove that if $l^1(S)$ is amenable then S must be a regular semigroup with finitely many idempotents. We have shown that when S is 1-amenable S has exactly one idempotent. It would be interesting to see what relationship exists between k-amenability for a positive integer k and the number of idempotents of S.

Chapter 5

Amenable and completely-amenable representations

5.1 Introduction

In this chapter we shift our focus to a notion of amenability for unitary representations of locally compact groups. The theory of amenable representations was developed in 1990 by M.E.B. Bekka [3]. Bekka proved analogues of all of the classical invariance properties including the Følner conditions, and interpreted several amenability theories in terms of amenable representations. In particular he proved that a locally compact group is amenable if and only if all of its representations are amenable. Since their introduction amenable representations have attracted much research attention, see for example [5], [27], and [52].

We begin with the introduction of a new notion of amenability for representations, called complete-amenability. We provide the exact relationship between these two concepts of amenability and interpret amenability, inner amenability, and amenable group actions in terms of completely-amenable representations. We describe complete-amenability through a weak-containment property and provide several characterizations of completely-amenable representations in terms of the existence of certain states upon C^*-algebras. For each representation π of G we define a Fourier algebra $A(\pi)$. We prove that when the Fourier algebra $A(\pi \otimes \overline{\pi})$ has a bounded approximate unit, the representation π is necessarily amenable. We conclude the chapter with a discussion of a problem posed in [3] by M.E.B. Bekka. We will make much use here of the material presented in section three of Chapter 2.

5.2 Amenability and complete-amenability for representations

Throughout this chapter G will again denote a locally compact group. A representation of G will always mean a continuous unitary representation of G as defined in Chapter 2. The following definition of an amenable representation was given by M.E.B. Bekka in [3].

Definition 5.2.1. A representation $\{\gamma, \mathcal{K}\}$ of G is called *amenable* if there exists a state ω on $B(\mathcal{K})$ such that

$$\omega(\gamma(s)x\gamma(s^{-1})) = \omega(x), \qquad (x \in B(\mathcal{K}),\ s \in G).$$

The state ω is called a *G-invariant mean* for γ.

Let $TC(\mathcal{K})$ and $HS(\mathcal{K})$ respectively denote the trace-class operators and Hilbert-Schmidt operators on the Hilbert space $\mathcal{K}$. Reiter conditions $(P_1)_\gamma$ and $(P_2)_\gamma$ are defined for a representation $\{\gamma, \mathcal{K}\}$ as follows.

$(P_1)_\gamma$: For every $\epsilon > 0$ and every compact subset K of G there exists a trace-class operator $T \in TC(\mathcal{K})_1^+ = \{R \in TC(\mathcal{K}) : R \geq 0 \text{ and } \|R\|_1 = 1\}$ such that

$$\|\gamma(s)T\gamma(s^{-1}) - T\|_1 < \epsilon \text{ for all } x \in K.$$

$(P_2)_\gamma$: For every $\epsilon > 0$ and every compact subset K of G there exists a Hilbert-Schmidt operator $S \in HS(\mathcal{K})_1^+ = \{R \in HS(\mathcal{K}) : R \geq 0 \text{ and } \|R\|_2 = 1\}$ such that

$$\|\gamma(s)S\gamma(s^{-1}) - S\|_2 < \epsilon \;\text{ for all } x \in K.$$

Recall that the left regular representation $\{\lambda_2, L^2(G)\}$ and the conjugation representation $\{\beta, L^2(G)\}$ of G were defined in section 3 of Chapter 2. The following theorem summarizes the results from [3] that we will need, (also see section 1 of Chapter 3). Respectively the parts of the theorem coincide with Theorems 2.2, 2.3(i), 2.4(i) and 4.3 of [3].

Theorem 5.2.2. *Let G be a locally compact group, let H be a closed subgroup of G and let $\{\gamma, \mathcal{K}\}$ be a representation of G.*
(1) The group G is amenable if and only if the left regular representation $\{\lambda_2, L^2(G)\}$ is amenable which is true if and only if every representation of G is amenable.
(2) The group G acts amenably on the left coset space G/H if and only if the quasi-regular representation $Ind_H^G 1_H$ is amenable.
(3) The group G is inner amenable if and only if the conjugation representation $\{\beta, L^2(G)\}$ is amenable.
(4) The representation $\{\gamma, \mathcal{K}\}$ is amenable if and only if $\{\gamma, \mathcal{K}\}$ satisfies the Reiter condition $(P_1)_\gamma$ which is true if and only if $\{\gamma, \mathcal{K}\}$ satisfies the Reiter condition $(P_2)_\gamma$.

We now introduce a new concept of amenability for representations.

Definition 5.2.3. A representation $\{\pi, \mathcal{H}\}$ will be called *completely amenable* [respectively *completely*-amenable*] if there is a net of unit vectors (ξ_α) in $\mathcal{H}$ such that $\|\pi(s)\xi_\alpha - \xi_\alpha\| \to 0$ uniformly on compact subsets of G [respectively $\|\pi(s)\xi_\alpha - \xi_\alpha\| \to 0, \;(s \in G)$].

Observe that π is completely*-amenable precisely when it is completely-amenable when viewed as a representation of G_d, the group G with the discrete topology. Thus all statements phrased in terms of complete-amenability may be interpreted in terms complete*-amenability. The following remarks record some elementary properties of completely-amenable representations.

Remarks 5.2.4. (1) If $\{\pi, \mathcal{H}\}$ is completely*-amenable then it is amenable. To see this, let (ξ_α) be a net of unit vectors in $\mathcal{H}$ such that $\|\pi(s)\xi_\alpha - \xi_\alpha\| \to 0,\ (s \in G)$. Taking ω to be any w^*-limit point in $B(\mathcal{H})^*$ of the net vector states (ω_{ξ_α}) it is easily seen that ω is a G-invariant mean for π.

(2) In section two of Chapter 3 we interpreted amenability and inner amenability in terms of certain positive actions on $L^1(G)$. Using Reiter's condition, Theorem 3.2.13 applied to these actions together with [45, Exercise 4.4.5] one sees that

(i) G is amenable $\Leftrightarrow$ the left regular representation $\{\lambda_2, L^2(G)\}$ is completely-amenable and

(ii) G is inner amenable $\Leftrightarrow$ the conjugation representation $\{\beta, L^2(G)\}$ is completely-amenable.

When H is a closed subgroup of G we similarly obtain that

(iii) G acts amenably on $G/H \Leftrightarrow$ the quasi-regular representation $Ind_H^G 1_H$ is completely-amenable.

(3) The statements (i), (ii) and (iii) of [3, Remark 1.2] are valid for completely-amenable representations. Indeed

(i) Any representation which is unitarily equivalent to a completely-amenable representation is also completely-amenable.

(ii) If H is a closed subgroup of G and $\{\pi, \mathcal{H}\}$ is a completely-amenable representation of G, then the restriction $\pi|H$ is a completely-amenable representation of H. Moreover, if H is normal and $H \subset \ker(\pi)$, then π is completely-amenable if and only if it is completely-amenable when viewed as a representation of G/H.

(iii) If $\{\pi, \mathcal{H}\}$ is completely-amenable, then so is its conjugate representation $\{\overline{\pi}, \overline{\mathcal{H}}\}$. Moreover the following is clear.

(iv) If $\{\pi, \mathcal{H}\}$ contains a completely-amenable subrepresentation, then $\{\pi, \mathcal{H}\}$ is also completely-amenable.

Example 5.2.5. The following is a simple example of an amenable representation which is not completely-amenable. Let $\mathbf{T} = \{\alpha \in \mathbf{C} : |\alpha| = 1\}$ be the circle group and let $\mathcal{H}$ be any Hilbert space. Consider the representation defined by

$$\gamma : \mathbf{T} \to B(\mathcal{H}) : \alpha \mapsto \alpha \operatorname{id}_{\mathcal{H}}.$$

Observe that any state on $B(\mathcal{H})$ is a G-invariant mean for γ so $\{\gamma, \mathcal{K}\}$ is amenable. However if ξ is any unit vector in $\mathcal{H}$, then $\|\gamma(-1)\xi - \xi\| = 2$ so γ is not completely-amenable.

Thus complete-amenability is strictly a stronger property than amenability for representations. Moreover the above example shows that in contrast to the theory of amenable representations (see [3, Theorems 1.3 and 2.2]), even one-dimensional representations can fail to be completely-amenable, and amenable groups can have representations which are not completely-amenable.

The next result describes the relationship between amenable and completely-amenable representations. It shows that any characterization of completely-amenable representations yields a characterization of amenable representations and hence by [3, Theorem 2.2] of amenable locally compact groups.

Theorem 5.2.6. *The following are equivalent for a representation $\{\gamma, \mathcal{K}\}$ of G.*

(1) $\{\gamma, \mathcal{K}\}$ is amenable.

(2) $\{\gamma \otimes \overline{\gamma}, \mathcal{K} \otimes \overline{\mathcal{K}}\}$ is completely-amenable.

(3) $\{\gamma \otimes \overline{\gamma}, \mathcal{K} \otimes \overline{\mathcal{K}}\}$ is completely-amenable.*

(4) $\{\gamma \otimes \overline{\gamma}, \mathcal{K} \otimes \overline{\mathcal{K}}\}$ is amenable.

Proof. (1) $\Rightarrow$ (2) Define a representation $\{\rho_\gamma, HS(\mathcal{K})\}$ of G by $\rho_\gamma(s)S = \gamma(s)S\gamma(s^{-1})$, $(S \in HS(\mathcal{K}),\ s \in G)$. As γ is amenable, it satisfies the Reiter property $(P_2)_\gamma$ so there is a net of unit vectors (S_α) in $HS(\mathcal{K})$ such that $\|\rho_\gamma(s)S_\alpha - S_\alpha\|_2 \to 0$ uniformly on compact subsets of G. Thus ρ_γ is completely-amenable. As the the map determined by

$$\mathcal{K} \otimes \overline{\mathcal{K}} \to HS(\mathcal{K}) : \xi \otimes \eta \mapsto \xi \otimes \eta^*$$

yields a unitary equivalence of $\{\rho_\gamma, HS(\mathcal{K})\}$ and $\{\gamma \otimes \overline{\gamma}, \mathcal{K} \otimes \overline{\mathcal{K}}\}$, we are done.
(2) $\Rightarrow$ (3) is trivial and (3) $\Rightarrow$ (4) follows from part (1) of Remark 5.2.4.

(4) $\Rightarrow$ (1) Let Ω be a state on $B(\mathcal{K}\otimes\overline{\mathcal{K}})$ such that

$$\Omega((\gamma\otimes\overline{\gamma})(s)x(\gamma\otimes\overline{\gamma})(s^{-1}))=\Omega(x), \qquad (x\in B(\mathcal{K}\otimes\overline{\mathcal{K}}),\ s\in G).$$

Define ω on $B(\mathcal{K})$ by

$$\omega(y)=\Omega(y\otimes \mathrm{id}_{\overline{\mathcal{K}}}), \qquad (y\in B(\mathcal{K})).$$

Then ω is clearly a state on $B(\mathcal{K})$ and for any $y\in B(\mathcal{K}),\ s\in G$ we have

$$\begin{aligned}\omega(\gamma(s)y\gamma(s^{-1})) &= \Omega(\gamma(s)y\gamma(s^{-1})\otimes \mathrm{id}_{\overline{\mathcal{K}}})\\ &= \Omega((\gamma\otimes\overline{\gamma})(s)(y\otimes \mathrm{id}_{\overline{\mathcal{K}}})(\gamma\otimes\overline{\gamma})(s^{-1}))\\ &= \Omega(y\otimes \mathrm{id}_{\overline{\mathcal{K}}})=\omega(y).\end{aligned}$$

□

5.3 Complete-amenability and weak containment

In this section we will characterize complete-amenability of a representation $\{\pi,\mathcal{H}\}$ in terms of both weak containment and the Fourier-Stieltjes spaces B_π.

Theorem 5.3.1. *The following are equivalent for a representation $\{\pi,\mathcal{H}\}$ of G.*

(1) $\{\pi,\mathcal{H}\}$ is completely-amenable.

(2) $1_G\preceq\pi$.

(3) $1_G\in B_\pi$.

Proof. (1) $\Rightarrow$ (2) Let (ξ_α) be a net of unit vectors in $\mathcal{H}$ such that $\|\pi(s)\xi_\alpha-\xi_\alpha\|\to 0$ uniformly on compact subsets of G. Let $u_\alpha=\xi_\alpha *_\pi \xi_\alpha$. Then

$$|u_\alpha(s)-1|=|\langle\pi(s)\xi_\alpha-\xi_\alpha,\xi_\alpha\rangle|\le\|\pi(s)\xi_\alpha-\xi_\alpha\|$$

whence $u_\alpha(s)\to 1$ uniformly on compact subsets of G. It follows from Theorem 2.3.2 that $1_G\preceq\pi$.

(2) $\Rightarrow$ (3) As $B_\pi=\langle P_\pi\rangle$ where $P_\pi=\{u\in P(G):\pi_u\preceq\pi\}$ this is trivial.

(3) $\Rightarrow$ (2) Part (4) of Lemma 2.3.1 says that $B_\pi\cap P_1(G)=\{u\in P_\pi : u(e)=1\}$ so

this is also trivial.

(2) $\Rightarrow$ (1) By Theorem 2.3.2 there is a net (ξ_α) in $\mathcal{H}$ such that $u_\alpha = \xi_\alpha *_\pi \xi_\alpha \in P_1(G)$ and (u_α) converges to 1_G uniformly on compact subsets of G. Observe that $\|\xi_\alpha\|^2 = u_\alpha(e) = 1$. Now

$$\|\pi(s)\xi_\alpha - \xi_\alpha\|^2 = 2|1 - \text{Re } u_\alpha(s)| \leq 2|1 - u_\alpha(s)|$$

from which it follows that $\{\pi, \mathcal{H}\}$ is completely-amenable. □

Corollary 5.3.2. *The following are equivalent for a representation $\{\pi, \mathcal{H}\}$ of G.*

(1) $\{\pi, \mathcal{H}\}$ is completely-amenable.*

(2) $1_{G_d} \preceq \pi_d$.

(3) $1_G \in B_{\pi_d}$.

Leptin's theorem says that a locally compact group G is amenable if and only if its Fourier algebra $A(G)$ has a bounded approximate unit. The following looks a little like a Leptin theorem for completely-amenable representations. In the sixth section of this chapter we will conjecture a more satisfying version of Leptin's theorem for (completely-)amenable representations, (we can only prove one direction).

Corollary 5.3.3. *A representation $\{\pi, \mathcal{H}\}$ is completely-amenable if and only if there is a net $(u_\alpha) \subset A_\pi \cap P_1(G)$ such that for every $v \in A(G)$, $\|u_\alpha v - v\| \to 0$.*

Proof. This follows immediately from Theorem 5.3.1, Theorem 2.3.2, and [17, Theorem B$_2$]. □

As with amenable representations [3, Corollary 5.3] we now have the following result.

Corollary 5.3.4. *Let $\{\pi, \mathcal{H}\}$, $\{\gamma, \mathcal{K}\}$ be representations of G. If $\gamma \preceq \pi$ and γ is completely-amenable, then so is π. In particular, the complete-amenability of π depends only upon its weak equivalence class.*

Let $\{\pi, \mathcal{H}\}$ be a representation of G. In [2] Arsac defines the representation $\omega_\pi = \Sigma \oplus \{\pi_u : u \in P_\pi\}$ and proves that $B_\pi = B_{\omega_\pi} = A_{\omega_\pi}$ [2, Proposition 2.24]. It seems interesting to note that by Proposition 5.3.1 the representation π is completely-amenable precisely when the associated representation ω_π is completely-amenable.

5.4 C^*-algebraic characterizations

We will characterize complete and complete*-amenability of a representation $\{\pi, \mathcal{H}\}$ in terms of the existence of certain states on the C^*-algebras $C^*_{\delta,\pi}$, C^*_π and $B(\mathcal{H})$. The state space of a C^*-algebra $\mathcal{A}$ will be denoted by $S(\mathcal{A})$.

Lemma 5.4.1. *Let $\mathcal{U}$ be a group of unitary operators on a Hilbert space $\mathcal{H}$ and let $\mathcal{A}$ be the C^*-subalgebra of $B(\mathcal{H})$ generated by $\mathcal{U}$.*
(1) If $\phi \in S(\mathcal{A})$, then ϕ is multiplicative on $\mathcal{A}$ if and only if $|\phi(u)| = 1$, $(u \in \mathcal{U})$.
(2) If $\phi \in S(B(\mathcal{H}))$, then $|\phi(u)| = 1$, $(u \in \mathcal{U})$ if and only if

$$\phi(xy) = \phi(yx) = \phi(x)\phi(y), \qquad (x \in B(\mathcal{H}),\ y \in \mathcal{A}). \qquad (*)$$

Proof. We will prove (2), the proof of (1) being similar. If the condition $(*)$ holds, then for any $u \in \mathcal{U}$, $1 = \phi(\mathrm{id}_\mathcal{H}) = \phi(u^*u) = \overline{\phi(u)}\phi(u) = |\phi(u)|^2$. Suppose now that the converse statement holds. Then given $x \in B(\mathcal{H})$ and $u \in \mathcal{U}$, using the Cauchy-Schwartz inequality for states we have

$$\begin{aligned} |\phi(xu) - \phi(x)\phi(u)|^2 &= |\phi(x(u - \phi(u)\mathrm{id}_\mathcal{H}))|^2 \\ &\leq \phi(xx^*)\phi((u - \phi(u)\mathrm{id}_\mathcal{H})^*(u - \phi(u)\mathrm{id}_\mathcal{H})) \\ &= \phi(xx^*)[\phi(\mathrm{id}_\mathcal{H}) - \phi(u)\phi(u^*) - \overline{\phi(u)}\phi(u) + |\phi(u)|^2\phi(\mathrm{id}_\mathcal{H})] \\ &= 0. \end{aligned}$$

Now $\mathcal{A} = \overline{\langle\mathcal{U}\rangle}^{\|\cdot\|_{B(\mathcal{H})}}$ so condition $(*)$ holds by linearity and continuity of ϕ. □

Theorem 5.4.2. *The following are equivalent for a representation $\{\pi, \mathcal{H}\}$ of G.*
(1) $\{\pi, \mathcal{H}\}$ is completely-amenable.*
(2) There is a state ϕ on $C^*_{\delta,\pi}$ such that $\phi(\pi(s)) = 1$, $(s \in G)$. [The state ϕ is necessarily multiplicative].

(3) There is a state ω on $B(\mathcal{H})$ such that $\omega(\pi(s)) = 1$, $(s \in G)$.

(4) There is a state ω on $B(\mathcal{H})$ such that

$$\omega(\pi(s)x) = \omega(x\pi(s)) = \omega(x), \quad (s \in G, \ x \in B(\mathcal{H})).$$

Proof. It follows from equation (2.3) of Chapter 2 that the duality of B_{π_d} with $C^*_{\pi_d} = C^*_{\delta,\pi}$ satisfies

$$\psi(s) = \langle \pi(s), \psi \rangle, \qquad (s \in G, \psi \in B_{\pi_d}).$$

Hence statement (2) is equivalent to saying that $1_G \in B_{\pi_d}$ so (1) $\Leftrightarrow$ (2) is a consequence of Corollary 5.3.2. The equivalence (2) $\Leftrightarrow$ (3) is clear and (3) $\Leftrightarrow$ (4) follows from Lemma 5.4.1. $\square$

Remarks 5.4.3. The invariant mean characterization of completely*-amenable representations given in part (4) of Theorem 5.4.2 suggests that it should be possible to study complete and complete*-amenability in same manner used by Bekka in [3] and which we used in section two of Chapter 3. However unlike the action

$$s \cdot T = \pi(s)T\pi(s^{-1}), \qquad (s \in G, \ T \in TC(\mathcal{H})),$$

the action defined by

$$s \cdot T = \pi(s)T, \qquad (s \in G, \ T \in TC(\mathcal{H})),$$

does not preserve the normal states $TC(\mathcal{H})^+_1$ on $B(\mathcal{H})$. That is, the latter action is not a positive action on $TC(\mathcal{H})$. It is because of this that the familiar techniques breakdown. In particular we have been unable to prove a Reiter-type theorem in this context, namely that complete*-amenability implies complete-amenability. We suspect that the key to such a proof may be the Raikov Theorem which states that on $P_1(G)$ the weak*-topology agrees with the topology of uniform convergence on compact subsets of G.

Let

$$J = \{f \in L^1(G) : \int_G f(s)\, ds = 0\}.$$

Much of the following theorem was proved by C.K. Yuan [54] for the conjugation representation $\{\beta, L^2(G)\}$. Our proof of the equivalence (4) $\Leftrightarrow$ (5) is quite similar to his proof.

Theorem 5.4.4. *The following are equivalent for a representation $\{\pi, \mathcal{H}\}$ of G.*

(1) $\{\pi, \mathcal{H}\}$ is completely-amenable.

(2) There exists a state ω on $B(\mathcal{H})$ such that $\omega(\pi(\mu)) = \mu(G), (\mu \in M(G))$.

(3) There exists a state ω on $B(\mathcal{H})$ such that $\omega(\pi(f)) = \int_G f(s)\ ds, (f \in L^1(G))$.

(4) There exists a state ϕ on C^*_π such that $\phi(\pi(f)) = \int_G f(s)\ ds, (f \in L^1(G))$.

(5) There exists a state ϕ on C^*_π such that $\ker(\phi) = \overline{\pi(J)}^{\|\cdot\|_{B(\mathcal{H})}}$ and $\phi(\pi(f_0)) = 1$ for some $f_0 \in L^1(G)$ with $\int f_0 = 1$.

Proof. (1) $\Rightarrow$ (2) Let (ξ_α) be a net of unit vectors in $\mathcal{H}$ such that $\|\pi(s)\xi_\alpha - \xi_\alpha\| \to 0$ uniformly on compact subsets of G. Let ω be any w^*-limit point in $B(\mathcal{H})^*$ of the net of vector states (ω_{ξ_α}); we may assume that $\omega_{\xi_\alpha} \to \omega\ w^*$ in $B(\mathcal{H})^*$. Let $\mu \in M(G)$ and let $\epsilon > 0$. We can suppose that μ has compact support K. Take α' such that

$$|\omega(\pi(\mu)) - \omega_{\xi_{\alpha'}}(\pi(\mu))| < \frac{\epsilon}{2} \text{ and } \|\pi(s)\xi_{\alpha'} - \xi_{\alpha'}\| < \frac{\epsilon}{2\|\mu\|}, \quad (s \in K).$$

Then

$$\begin{aligned} |\omega(\pi(\mu)) - \mu(G)| &< \frac{\epsilon}{2} + |\omega_{\xi_{\alpha'}}(\pi(\mu)) - \mu(G)| \\ &= \frac{\epsilon}{2} + |\int_G \langle \pi(s)\xi_{\alpha'}, \xi_{\alpha'}\rangle\ d\mu(s) - \int_G d\mu(s)| \\ &\leq \frac{\epsilon}{2} + \int_G |\langle \pi(s)\xi_{\alpha'} - \xi_{\alpha'}, \xi_{\alpha'}\rangle|\ d|\mu|(s) \\ &\leq \frac{\epsilon}{2} + \int_G \frac{\epsilon}{2\|\mu\|}\ d|\mu|(s) = \epsilon. \end{aligned}$$

Hence $\omega(\pi(\mu)) = \mu(G)$.

(2) $\Rightarrow$ (3) is trivial.

(3) $\Rightarrow$ (4) Let ϕ be the restriction of ω to C^*_π. Then $\phi \in (C^*_\pi)^* = B_\pi$ and from equation (2.3) of Chapter 2

$$\int_G f(s)\phi(s)\ ds = \langle \pi(f), \phi\rangle = \int_G f(s)\ ds, \quad (f \in L^1(G)).$$

It follows that $\phi = 1_G \in B_\pi$ which is a state on C^*_π.

(4) $\Rightarrow$ (1) The argument used to establish (3) $\Rightarrow$ (4) shows that $\phi = 1_G \in B_\pi$. By Theorem 5.3.1 $\{\pi, \mathcal{H}\}$ is completely-amenable.

(4) $\Rightarrow$ (5) Let ϕ be a state on C^*_π as in (4) and fix f_0 in $L^1(G)$ with $\int f_0 = 1$. Then $\phi(\pi(f_0)) = 1$ and $f \in J$ if and only if $\phi(\pi(f)) = 0$. Hence $\overline{\pi(J)}^{\|\cdot\|_{B(\mathcal{H})}} \subset \ker(\phi)$. Now suppose that $x \in \ker(\phi)$. Let (f_n) be a sequence in $L^1(G)$ such that $\|\pi(f_n) - x\| \to 0$. As J is an ideal of codimension one in $L^1(G)$ and $f_0 \notin J$, for each n there is some $\alpha_n \in \mathbf{C}$ and $g_n \in J$ such that $f_n = \alpha_n f_0 + g_n$. Now $0 = \lim |\phi(\pi(f_n)) - \phi(x)| = \lim |\alpha_n|$, so

$$\begin{aligned} \|\pi(g_n) - x\| &= \|\pi(f_n) - \alpha_n \pi(f_0) - x\| \\ &\leq \|\pi(f_n) - x\| + |\alpha_n| \|f_0\|_1 \to 0. \end{aligned}$$

Therefore $x \in \overline{\pi(J)}^{\|\cdot\|_{B(\mathcal{H})}}$.

(5) $\Rightarrow$ (4) Let ϕ and f_0 be as statement (5). Then for any $f \in L^1(G)$,

$$g = f - \left(\int_G f(s)\, ds\right) f_0 \in J \quad \text{and} \quad f = g + \left(\int_G f(s)\, ds\right) f_0.$$

Consequently $\phi(\pi(f)) = \int_G f(s)\, ds$. $\square$

If $\{\gamma, \mathcal{K}\}$ is an amenable representation, then by Theorem 5.2.6 we know that all of the conditions from Theorems 5.4.2 and 5.4.4 are equivalent for the associated representation $\{\pi, \mathcal{H}\} = \{\gamma \otimes \overline{\gamma}, \mathcal{K} \otimes \overline{\mathcal{K}}\}$. Beyond this we can extend a result due to E. Bédos which he proved in [4] for the left regular representation $\{\lambda_2, L^2(G)\}$.

Theorem 5.4.5. *The following are equivalent for a representation $\{\gamma, \mathcal{K}\}$ of G.*

(1) $\{\gamma, \mathcal{K}\}$ is amenable.

(2) There is a non-zero multiplicative linear functional ϕ on $C^*_{\delta, \gamma \otimes \overline{\gamma}}$.

(3) There is a state ω on $B(\mathcal{K} \otimes \overline{\mathcal{K}})$ such that $|\omega((\gamma \otimes \overline{\gamma})(s))| = 1, \ (s \in G)$.

Proof. (1) $\Rightarrow$ (2) follows from Theorems 5.2.6 and 5.4.2.

(2) $\Rightarrow$ (3) From Lemma 5.4.1 we have $|\phi((\gamma \otimes \overline{\gamma})(s))| = 1, \ (s \in G)$. Now let ω be any state on $B(\mathcal{K} \otimes \overline{\mathcal{K}})$ which extends ϕ.

(3) $\Rightarrow$ (1) For any $x \in B(\mathcal{K} \otimes \overline{\mathcal{K}})$ and $s \in G$ we have

$$\begin{aligned}
\omega((\gamma \otimes \overline{\gamma})(s)x(\gamma \otimes \overline{\gamma})(s^{-1})) &= \omega((\gamma \otimes \overline{\gamma})(s))\omega(x)\omega((\gamma \otimes \overline{\gamma})(s^{-1})) \\
&= \omega((\gamma \otimes \overline{\gamma})(s))\omega(x)\overline{\omega((\gamma \otimes \overline{\gamma})(s))} \\
&= \omega(x)
\end{aligned}$$

where we have used Lemma 5.4.1. Thus ω is a G-invariant mean for $\gamma \otimes \overline{\gamma}$ and it follows from Theorem 5.2.6 that $\{\gamma, \mathcal{K}\}$ is amenable. $\square$

5.5 The Fourier algebra for an arbitrary representation

In Section three of Chapter 2 we defined the Fourier spaces A_π. When our representation is the left regular representation $\{\lambda_2, L^2(G)\}$, A_{λ_2} is the Fourier algebra $A(G)$. It is however quite rare for a Fourier space A_π to be an algebra [2, Proposition 3.26]. In this section we will define and study an analogue of the Fourier algebra for an arbitrary representation.

Let $A(\pi)$ denote the closed subalgebra of $B(G)$ generated by the coefficients $\xi *_\pi \eta$ of π. We will refer to $A(\pi)$ as the *Fourier algebra associated to* π. It is not difficult to see that $A(\pi)$ is closed under left and right translations, so by [2, Theorem 3.17] $A(\pi) = A_{\tau_\pi}$ for some representation τ_π of G. The next proposition gives an explicit description of this representation which we will find useful. Before stating the result we prove a simple lemma which we expect is well-known.

Lemma 5.5.1. *Let $\{\gamma, \mathcal{K}\}$ be a representation of G and suppose that $\mathcal{F} \subset \mathcal{K}$ has dense linear span in $\mathcal{K}$. Then $\langle \xi *_\gamma \eta : \xi, \eta \in \mathcal{F} \rangle$ is dense in A_γ.*

Proof. Let $\xi, \eta \in \mathcal{K}$ and choose sequences (ξ_n), $(\eta_n) \subset \langle \mathcal{F} \rangle$ such that $\|\xi_n - \xi\| \to 0$

and $\|\eta_n - \eta\| \to 0$. Let $u = \xi *_\gamma \eta$ and $u_n = \xi_n *_\gamma \eta_n$. As $u,\ u_n \in B_\gamma = (C^*_\gamma)^*$,

$$\begin{aligned}
\|u - u_n\| &= sup\{|\int_G f(s)(u_n - u)(s)\, ds| : f \in L^1(G),\ \|\gamma(f)\| \le 1\} \\
&= sup\{|\langle \gamma(f)\xi_n, \eta_n\rangle - \langle \gamma(f)\xi, \eta\rangle| : f \in L^1(G),\ \|\gamma(f)\| \le 1\} \\
&= sup\{|\langle \gamma(f)(\xi_n - \xi), \eta_n\rangle + \langle \gamma(f)\xi, \eta_n - \eta\rangle| : f \in L^1(G),\ \|\gamma(f)\| \le 1\} \\
&\le \|\xi_n - \xi\|\|\eta_n\| + \|\xi\|\|\eta_n - \eta\| \to 0.
\end{aligned}$$

Now $(\sum_i \xi_i) *_\gamma (\sum_j \eta_j) = \sum_{i,j}(\xi_i *_\gamma \eta_j)$ so $\langle \xi *_\gamma \eta : \xi, \eta \in \langle \mathcal{F}\rangle\rangle = \langle \xi *_\gamma \eta : \xi, \eta \in \mathcal{F}\rangle$ and we are done. □

Let $\{\pi, \mathcal{H}\}$ be a representation of G. For a positive integer n we employ the notation

$$\pi^{\otimes n} = \otimes_{i=1}^n \pi \quad \text{and} \quad \mathcal{H}^{\otimes n} = \otimes_{i=1}^n \mathcal{H}.$$

Proposition 5.5.2. *Let $\{\pi, \mathcal{H}\}$ be a representation of G and consider the associated representation*

$$\{\tau_\pi, \mathcal{H}_{\tau_\pi}\} = \{\sum_{n=1}^\infty \oplus(\pi^{\otimes n}), \sum_{n=1}^\infty \oplus(\mathcal{H}^{\otimes n})\}.$$

Then $A(\pi) = A_{\tau_\pi} = A(\tau_\pi)$.

Proof. For $\xi_1, \ldots, \xi_k \in \mathcal{H}$, let $\otimes_1^k \xi_i$ denote the associated elementary tensor in $\mathcal{H}^{\otimes k}$ which we view as a subspace of $\mathcal{H}_{\tau_\pi}$. It is clear that $\langle \mathcal{F}\rangle$ is dense in $\mathcal{H}_{\tau_\pi}$ where

$$\mathcal{F} = \{\otimes_1^k \xi_i : \xi_1, \ldots, \xi_k \in \mathcal{H}, k \in \mathrm{N}\}.$$

It follows from Lemma 5.5.1 that to show A_{τ_π} is an algebra, we only need to demonstrate that $\langle \bar{\xi} *_{\tau_\pi} \bar{\eta} : \bar{\xi}, \bar{\eta} \in \mathcal{F}\rangle$ is closed under multiplication. To this end, let $\bar{\xi}, \bar{\eta}, \bar{v}, \bar{\zeta} \in \mathcal{F}$ and let $u = \bar{\xi} *_{\tau_\pi} \bar{\eta}$, $v = \bar{v} *_{\tau_\pi} \bar{\zeta}$. Observe that if $\bar{\xi} = \otimes_1^{k_1} \xi_i$ and $\bar{\eta} = \otimes_1^{k_2} \eta_j$ with $k_1 \neq k_2$, then

$$\bar{\xi} *_{\tau_\pi} \bar{\eta}(s) = (\sum_{n \neq k_1, k_2} 0) + \langle \otimes_1^{k_1} \pi(s)\xi_i, 0\rangle + \langle 0, \otimes_1^{k_2} \eta_j\rangle = 0.$$

Hence we may assume that $\overline{\xi} = \otimes_1^k \xi_i$, $\overline{\eta} = \otimes_1^k \eta_i$, $\overline{v} = \otimes_1^l v_j$, and $\overline{\zeta} = \otimes_1^l \zeta_j$. Then

$$\begin{aligned}
(uv)(s) &= \langle \otimes_1^k \pi(s)\xi_i, \otimes_1^k \eta_i \rangle \langle \otimes_1^l \pi(s) v_j, \otimes_1^l \zeta_i \rangle \\
&= \prod_{i=1}^{k} \langle \pi(s)\xi_i, \eta_i \rangle \prod_{j=1}^{l} \langle \pi(s) v_j, \zeta_j \rangle \\
&= \langle (\otimes_1^{k+l}\pi)(s)(\overline{\xi} \otimes \overline{v}), \overline{\eta} \otimes \overline{\zeta} \rangle = (\overline{\xi} \otimes \overline{v}) *_{\tau_\pi} (\overline{\eta} \otimes \overline{\zeta})(s),
\end{aligned}$$

and $\overline{\xi} \otimes \overline{v}, \overline{\eta} \otimes \overline{\zeta} \in \mathcal{F}$. Thus A_{τ_π} is an algebra. Now it is obvious that $A_\pi \subset A_{\tau_\pi}$ so we have $A(\pi) \subset A_{\tau_\pi}$. Finally if $\overline{\xi} = \otimes_1^k \xi_i$ and $\overline{\eta} = \otimes_1^k \eta_i \in \mathcal{F}$, then

$$\overline{\xi} *_{\tau_\pi} \overline{\eta} = \prod_{i=1}^{k} (\xi_i *_\pi \eta_i) \in A(\pi).$$

Using Lemma 5.5.1 once again we have $A_{\tau_\pi} \subset A(\pi)$. □

In [30] A. T.-M. Lau calls a Banach algebra A an *F-algebra* if A is the predual of some von Neumann algebra $\mathcal{M}$ such that the identity element of $\mathcal{M}$ is a multiplicative linear functional on A. We remark that F-algebras are now commonly referred to as *Lau algebras* and we shall refer to them as such. Observe that if $\{\tau, \mathcal{H}_\tau\}$ is a representation of G for which A_τ is an algebra, then $\tau(e)$ is the identity in $VN_\tau = A_\tau^*$ and $\langle u, \tau(e) \rangle = u(e)$, $(u \in A_\tau)$. It follows that A_τ is a Lau algebra. Thus we can make the following statement.

Proposition 5.5.3. *For any representation $\{\pi, \mathcal{H}\}$ of G, $A(\pi)$ is a Lau algebra.*

We now examine the relationship between the amenability properties of π and τ_π.

Theorem 5.5.4. *Let $\{\pi, \mathcal{H}\}$ be a representation of G. Then $\{\pi, \mathcal{H}\}$ is amenable if and only if $\{\tau_\pi, \mathcal{H}_{\tau_\pi}\}$ is amenable.*

Proof. Suppose first that π is amenable. Then $\tau_\pi = \sum_{n=1}^{\infty} \oplus (\pi^{\otimes n})$ has an amenable subrepresentation and it follows from [3, Theorem 1.3(ii)] that τ_π is amenable. For the converse, suppose that ω is a state on $B(\mathcal{H}_{\tau_\pi})$ such that

$$\omega(\tau_\pi(s) B \tau_\pi(s^{-1})) = \omega(B), \qquad (s \in G,\ B \in B(\mathcal{H}_{\tau_\pi})).$$

Define

$$\Lambda : B(\mathcal{H}) \to B(\mathcal{H}_{\tau_\pi}) : B \mapsto \sum_{n=1}^{\infty} \oplus (B \otimes (\otimes_{k=2}^{n} \mathrm{id}_{\mathcal{H}})).$$

It is easy to see that Λ is a linear isometry such that $\Lambda(\mathrm{id}_{\mathcal{H}}) = \mathrm{id}_{\mathcal{H}_{\tau_\pi}}$. Moreover for any $B \in B(\mathcal{H})$ and $s \in G$,

$$\begin{aligned}
\tau_\pi(s)\Lambda(B)\tau_\pi(s^{-1}) &= \sum_{n=1}^{\infty} \oplus [\pi^{\otimes n}(s)(B \otimes (\otimes_{k=2}^{n} \mathrm{id}_{\mathcal{H}}))\pi^{\otimes n}(s^{-1})] \\
&= \sum_{n=1}^{\infty} \oplus [\pi(s)B\pi(s^{-1}) \otimes (\otimes_{k=2}^{n} \mathrm{id}_{\mathcal{H}})] \\
&= \Lambda(\pi(s)B\pi(s^{-1})).
\end{aligned}$$

Therefore if we define

$$m(B) = \omega(\Lambda(B)), \qquad (B \in B(\mathcal{H})),$$

then it is clear that m is a G-invariant mean on $B(\mathcal{H})$ for $\{\pi, \mathcal{H}\}$. □

Remarks 5.5.5. As in the above proof, τ_π is completely-amenable whenever π is completely-amenable. We have been unable to obtain any form of converse to this statement.

We conclude this section with a definition of the Fourier-Stieltjes algebra associated to an arbitrary representation.

Suppose first that $\{\pi, \mathcal{H}\}$ is a representation of G such that A_π is a subalgebra of $B(G)$; that is, suppose that $A_\pi = A(\pi)$. By [2, Proposition 2.20] an element $u \in B(G)$ belongs to B_π precisely when there is a bounded net in A_π which converges to u uniformly on compact subsets of G. From this it follows easily that B_π is also a Banach subalgebra of $B(G)$.

Now suppose that $\{\pi, \mathcal{H}\}$ is any representation of G and let $A(\pi) = A_{\tau_\pi} = A(\tau_\pi)$ be the Fourier algebra associated to π. From the preceding paragraph $B(\pi) = B_{\tau_\pi}$ is a Banach subalgebra of $B(G)$, which we call the *Fourier-Stieltjes algebra associated to* π. When our representation is the left regular representation $\{\lambda_2, L^2(G)\}$, we have $B(\lambda_2) = B_{\lambda_2} = B_r(G)$, the reduced Fourier-Stieltjes algebra of G.

Proposition 2.20 and Theorem 3.17 of [2] respectively say that B_π is the w^*-closure of A_π in $B(G)$ and that the Fourier spaces A_π are precisely the translation-invariant closed linear subspaces of $B(G)$. It is readily verified that translation is w^*-continuous on $B(G)$, so the spaces B_π are precisely the translation-invariant w^*-closed linear subspaces of $B(G)$. Thus we can make the following statement.

Proposition 5.5.6. *Every translation-invariant closed subalgebra of $B(G)$ is a Fourier algebra $A(\pi)$ associated to a representation $\{\pi, \mathcal{H}\}$ of G. Every translation-invariant w^*-closed subalgebra of $B(G)$ is a Fourier-Stieltjes algebra $B(\pi)$ associated to a representation $\{\pi, \mathcal{H}\}$ of G.*

5.6 Towards Leptin's theorem for amenable representations

Leptin's theorem [34] says that a locally compact group G is amenable if and only if the Fourier algebra $A(G)$ has a bounded approximate unit. In terms of representations this says that (complete-)amenability of the left regular representation $\{\lambda_2, L^2(G)\}$ is characterized by the existence of a bounded approximate unit in $A(\lambda_2)$. We suspect that the corresponding statement holds for any representation but have at present only been able to prove one direction. We begin by proving a simple lemma.

If $\{\pi, \mathcal{H}\}$ is a representation such that $A_\pi = A(\pi)$, let $VN_\pi = A_\pi^*$ have its canonical dual A_π-module structure. We note that because A_π is commutative, the side that we choose to write our module operations does not matter.

Lemma 5.6.1. *Let $\{\pi, \mathcal{H}\}$ be a representation of G such that $A_\pi = A(\pi)$.*
(1) The module action of A_π on VN_π restricted to $\pi(L^1(G))$ is given by pointwise multiplication

$$u \cdot \pi(f) = \pi(uf), \qquad (u \in A_\pi,\ f \in L^1(G)).$$

Consequently C_π^* is a closed A_π-submodule of VN_π.
(2) The dual module action of A_π on $(C_\pi^*)^* = B_\pi$ is given by pointwise multiplication.

That is

$$u \cdot \phi = u\phi, \qquad (u \in A_\pi,\ \phi \in B_\pi).$$

Proof. (1) For $u, v \in A_\pi$ and $f \in L^1(G)$ we have

$$\langle v, u \cdot \pi(f) \rangle = \langle vu, \pi(f) \rangle = \int_G v(s)u(s)f(s)\, ds = \langle v, \pi(uf) \rangle.$$

As $C_\pi^* = \overline{\pi(L^1(G))}^{\|\cdot\|_{B(\mathcal{H})}}$ it follows that C_π^* is an A_π-submodule of VN_π.

(2) Let $u \in A_\pi$, $\phi \in (C_\pi^*)^* = B_\pi$. Then $u\phi \in B_\pi$ and for any $f \in L^1(G)$,

$$\begin{aligned} \langle \pi(f), u \cdot \phi \rangle &= \langle \pi(f) \cdot u, \phi \rangle = \langle \pi(fu), \phi \rangle \\ &= \int_G f(s)u(s)\phi(s)\, ds = \langle \pi(f), u\phi \rangle, \end{aligned}$$

where we have used part (1). □

Theorem 5.6.2. *Let $\{\pi, \mathcal{H}\}$ be a representation of G such that $A_\pi = A(\pi)$ and consider the following statements.*

(1) $\{\pi, \mathcal{H}\}$ is completely-amenable.

(2) $A(\pi)$ has a bounded approximate unit.

(3) The closed ideal $I_\pi = \{u \in A(\pi) : u(e) = 0\}$ of $A(\pi)$ has a bounded approximate unit.

Then (1) $\Leftarrow$ (2) $\Leftrightarrow$ (3)

Proof. By Proposition 5.5.3 $A(\pi)$ is a commutative Lau algebra, so the equivalence of statements (2) and (3) is a direct consequence of [30, Example(1) Page 168] and [30, Theorem 4.10]. Now suppose that $A(\pi) = A_\pi$ has a bau (e_α) and let ϕ be a w^*-limit point of (e_α) in VN_π^*; assume without loss of generality that $e_\alpha \to \phi\ w^*$. Then for any $u \in A_\pi$ and $x \in VN_\pi$ we have

$$\begin{aligned} \langle x, u \cdot \phi \rangle &= \langle x \cdot u, \phi \rangle = \lim \langle x \cdot u, e_\alpha \rangle \\ &= \lim \langle x, ue_\alpha \rangle = \langle x, u \rangle. \end{aligned}$$

Thus $u \cdot \phi = u$, $(u \in A_\pi)$. Letting $\phi_1 \in (C_\pi^*)^* = B_\pi$ denote the restriction of ϕ to C_π^*, Lemma 5.6.1(2) gives

$$u = u \cdot \phi_1 = u\phi_1, \qquad (u \in A_\pi).$$

Suppose that ξ is any unit vector in $\mathcal{H}$. Then given any $s \in G$, letting $\eta = \pi(s)\xi$ and $u = \xi *_\pi \eta$, we have

$$1 = u(s) = u(s)\phi_1(s) = \phi_1(s).$$

Thus $\phi_1 = 1_G \in B_\pi$ and by Theorem 5.3.1 $\{\pi, \mathcal{H}\}$ is completely-amenable. $\square$

If $\{\pi, \mathcal{H}\}$ is any representation of G, then by Proposition 5.5.2 $A_{\tau_\pi} = A(\tau_\pi) = A(\pi)$. Hence the existence of a bounded approximate unit in $A(\pi)$ is sufficient for complete-amenability of $\{\tau_\pi, \mathcal{H}_{\tau_\pi}\}$. For amenable representations we have the following corollary.

Corollary 5.6.3. *Let $\{\gamma, \mathcal{K}\}$ be a representation of G and consider the following conditions.*

(1) $\{\gamma, \mathcal{K}\}$ is amenable.

(2) $A(\gamma \otimes \overline{\gamma})$ has a bounded approximate unit.

(3) The closed ideal $I_{\gamma\otimes\overline{\gamma}} = \{u \in A(\gamma \otimes \overline{\gamma}) : u(e) = 0\}$ of $A(\gamma \otimes \overline{\gamma})$ has a bounded approximate unit.

Then (1) $\Leftarrow$ (2) $\Leftrightarrow$ (3)

Proof. The equivalence of statements (2) and (3) is an immediate consequence of Theorem 5.6.2. For (2) $\Rightarrow$ (1) note that if $A(\gamma\otimes\overline{\gamma})$ has a bounded approximate unit, then by Theorem 5.6.2, $\tau_{\gamma\otimes\overline{\gamma}}$ is amenable and so by Theorem 5.5.4, $\gamma\otimes\overline{\gamma}$ is amenable. Now $\{\gamma, \mathcal{K}\}$ is amenable by Theorem 5.2.6. $\square$

We have unfortunately been unable to prove the implication (1) $\Rightarrow$ (2) of either Theorem 5.6.2 or Corollary 5.6.3. In terms of Fourier-Stieltjes algebras we can make the following statement.

Theorem 5.6.4. *The following are equivalent for a representation $\{\gamma, \mathcal{K}\}$ of G.*

(1) $\{\gamma, \mathcal{K}\}$ is amenable.

(2) The Fourier-Stieltjes algebra $B(\gamma \otimes \overline{\gamma})$ has an identity.

Proof. (1) $\Rightarrow$ (2) If $\{\gamma, \mathcal{K}\}$ is amenable, then by Theorems 5.2.6 and 5.3.1, $1_G \in B_{\gamma\otimes\overline{\gamma}}$. As $B_{\gamma\otimes\overline{\gamma}} \subset B(\gamma \otimes \overline{\gamma})$, (2) follows.

(2) $\Rightarrow$ (1) If $B(\gamma \otimes \overline{\gamma})$ has an identity ϕ, then the argument found in the last few lines of the proof of Theorem 5.6.2 shows that $\phi = 1_G \in B(\gamma \otimes \overline{\gamma})$. It now follows from Theorems 5.2.6, 5.3.1 and 5.5.4 that $\{\gamma, \mathcal{K}\}$ is amenable. $\square$

5.7 Some comments regarding a question posed by M.E.B. Bekka

Let G be a locally compact group. In the previous chapter we discussed Johnson's Theorem which states that G is amenable precisely when its group algebra $L^1(G)$ is amenable. Amenability of G has also been characterized in terms of amenability of the reduced group C^*-algebra $C_r^*(G)$ and Connes-amenability of the group von Neumann algebra $VN(G)$.

Before stating these characterizations in Theorem 5.7.1 we remark that for C^*-algebras, amenability is equivalent to the important C^*-algebraic property of nuclearity. For von Neumann algebras, amenability as defined in Chapter 2 turns out to be too strong of a condition to yield an interesting subclass of von Neumann algebras. The most suitable notion of amenability for von Neumann algebras, called Connes-amenability, takes account of the fact that every von Neumann algebra is the dual space of some Banach space. Connes-amenability of von Neumann algebras is known to be equivalent to each of the conditions of injectivity, semidiscreteness, and Schwartz's property P. The facts stated in this paragraph are of significant depth and are primarily due to M.D Choi, A. Connes, E.G. Effros, E.C. Lance, S. Wassermann and others. A self-contained account may be found in [45]. The following is due to E.G. Effros and E.C Lance [11], A. Guichardet [22], and E.C. Lance [29] in the discrete case; the general case was established by A. T.-M. Lau and A.L.T. Paterson [31], (also see [39] and [44]).

Theorem 5.7.1. *The following are equivalent for a locally compact group G.*

(1) G is amenable.

(2) $C_r^*(G)$ is nuclear and G is inner amenable.

(3) $VN(G)$ is Connes-amenable and G is inner amenable.

We may phrase this in terms of the left regular representation $\{\lambda_2, L^2(G)\}$ and the conjugation representation $\{\beta, L^2(G)\}$ as follows: *λ_2 is amenable precisely when $C^*_{\lambda_2}$ is nuclear and β is amenable which is true precisely when VN_{λ_2} is Connes-amenable and β is amenable.*

In 1990 M. Bekka [3, page 400] asked if it is possible to characterize amenable representations $\{\gamma, \mathcal{K}\}$ in terms of amenability of some Banach algebra associated to γ. Presumably motivated by the above theorem, Bekka suggested $C^*_{\delta,\gamma}$ and VN_γ as two natural candidates.

In [28] E. Kaniuth and A. Markfort showed that if $\{\beta, L^2(G)\}$ is the conjugation representation of G, then G_d is amenable if and only if $C^*_{\delta,\beta}$ is nuclear(=amenable). Thus if we take G to be any non-amenable discrete group, (for example let G be the free group on two generators), then G is trivially inner-amenable, so $\{\beta, L^2(G)\}$ is amenable, but $C^*_{\delta,\beta} = C^*_\beta$ is not nuclear. This suggests that both $C^*_{\delta,\beta}$ and C^*_β are not good candidates when searching for a solution to Bekka's problem. We are unsure as to whether this also implies that VN_β is not Connes-amenable.

According to [3, Corollary 5.5] G is amenable precisely when all of its irreducible representations are amenable. Thus if we take any non-amenable group G, we can find an irreducible representation $\{\gamma, \mathcal{K}\}$ of G which is non-amenable. However by Schur's lemma for irreducible representations

$$VN_\gamma = \gamma(G)'' = \{\alpha \, \mathrm{id}_{\mathcal{K}} : \alpha \in \mathbf{C}\}' = B(\mathcal{K})$$

which is Connes-amenable. Still, for irreducible representations γ we have no related concept of inner amenability so we cannot rule VN_γ out as a contender for the Banach algebra which will solve Bekka's problem.

Recall that

(i) G is amenable if and only if the left regular representation $\{\lambda_2, L^2(G)\}$ is amenable;

(ii) the right regular representation $\{\rho_2, L^2(G)\}$ is unitarily equivalent to $\{\lambda_2, L^2(G)\}$ and $\rho_2(G) \subset \lambda_2(G)'$; and

(iii) the conjugation representation $\{\beta, L^2(G)\}$ is defined by $\beta(s) = \lambda_2(s)\rho_2(s)$, and G is inner amenable if and only if β is amenable.

Based upon an argument that V. Runde has given to prove the equivalence of (1) and (3) in Theorem 5.7.1, (see [44, Theorem 5.3] where Runde actually proves more than this), we can prove the following proposition.

Proposition 5.7.2. *Let $\{\gamma, \mathcal{K}\}$ be a representation of G. Suppose that there is a representation $\{\rho, \mathcal{K}\}$ of G such that $\rho(G) \subset \gamma(G)'$ and let $\{\beta_{\gamma,\rho}, \mathcal{K}\}$ be the (continuous unitary) representation defined by*

$$\beta_{\gamma,\rho}(s) = \gamma(s)\rho(s), \quad (s \in G).$$

If VN_γ is Connes-amenable and $\beta_{\gamma,\rho}$ is amenable, then γ is amenable.

We will omit the proof. Beyond the remarks that we have already made in this section we have the following few remaining comments. First, in light of Theorems 5.2.6 and 5.4.5, we might suggest that the C^*-algebras $C^*_{\delta,\gamma\otimes\overline{\gamma}}$, $C^*_{\gamma\otimes\overline{\gamma}}$ and the von Neumann algebra $VN_{\gamma\otimes\overline{\gamma}}$ are worthy of consideration with regards to Bekka's problem.

Two other possibilities are the Banach algebras $A(\gamma)$ and $B(\gamma)$, (or $A(\gamma\otimes\overline{\gamma})$ and $B(\gamma\otimes\overline{\gamma})$). As mentioned in section 5 of Chapter 3, Ruan's theorem says that G is amenable if and only if the Fourier algebra $A(G) = A(\lambda_2)$ is operator amenable. Operator amenability also makes sense for the Lau algebras $A(\gamma)$ and one might certainly wonder how it relates to the amenability of γ. In [47] V. Runde and N. Spronk introduced a notion of amenability for dual operator Banach algebras called *operator Connes-amenability.* There they proved that G is amenable if and only if the reduced Fourier-Stieltjes algebra $B_r(G) = B(\lambda_2)$ is operator Connes-amenable. Every Fourier-Stieltjes algebra $B(\gamma)$ is a dual operator Banach algebra so it is natural to ask how operator Connes-amenability of $B(\gamma)$ relates to the amenability of γ. In fact, because every operator amenable operator Banach algebra has a bounded approximate unit [42, Proposition 2.3] and every operator Connes-amenable Banach algebra has an identity [47, proof of Theorem 4.4], the following is an immediate consequence of Corollary 5.6.3 and Theorem 5.6.4.

Proposition 5.7.3. *Let $\{\gamma, \mathcal{K}\}$ be a representation of G.*
(1) If $A(\gamma \otimes \overline{\gamma})$ is operator amenable, then $\{\gamma, \mathcal{K}\}$ is amenable.
(2) If $B(\gamma \otimes \overline{\gamma})$ is operator Connes-amenable, then $\{\gamma, \mathcal{K}\}$ is amenable.

As a final remark we note that there is no hope of characterizing amenability of γ in terms of amenability of the Fourier algebras $A(\gamma)$. Indeed, for the Fourier algebra $A(G)$ to be amenable it is both necessary and sufficient that G contain an abelian subgroup of finite index, (see [33] for sufficiency and [46] for necessity).

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