University of Alberta

Discrete point sets with long-range order and associated point processes.

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Abstract

Let Λ be a uniformly discrete point set in $\mathbb{R}^d$ with multiple colours. Then its weighted two-point correlation measure and its diffraction measure are determined by its structure. The inverse problem for Λ is to determine the structure of the point set Λ by using its weighted two-point correlation measure. However, the fact is that knowing only the two-point correlation measure usually is not enough to do this. One principal purpose of this thesis is to understand what underlies this fact.

We will show that if the frequency of every local patterns of $\mathcal{D}_r^{(m)}$ (the space of all r-uniformly discrete m-coloured point sets) exists at Λ, then Λ uniquely determines a stationary probability measure μ on $\mathcal{D}_r^{(m)}$. This measure μ contains the basic information of the structure of Λ.

Moreover, μ uniquely determines a stationary point process $(X, \mathbb{R}^d, \mu)$, where $X := \mathrm{supp}(\mu)$. In the case that μ is ergodic, we will prove that the $n+1$-point correlation measure of the point process is equal to the n-th moment of the Palm measure for $n = 2, 3, \ldots$. This result generalizes Gouéré's argument for the case that $n = 2$. Meanwhile, basing on Steven Dworkin's argument that the diffraction of typical point sets comprising X is related to the dynamical spectrum of X, we will prove that there exists an $\mathbb{R}^d$-equivariant, isometric embedding that takes the L^2-space of $\mathbb{R}^d$ under the diffraction measure into $L^2(X, \mu)$ and the algebra generated in $L^2(X, \mu)$ by the image of this embedding is dense in $L^2(X, \mu)$. It will follow that the full information about μ is available from the weights and the set of all correlations (that is the two-point, three-point,..., correlations). This thesis will end with a discussion about two particular point processes.

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This thesis is typeset using LaTeX.

Contents

Introduction

Discrete closed point sets are natural idealized abstractions of atomic structures of physical materials. We consider the space of discrete closed point sets satisfying the hard core condition that there is a positive lower bound to the separation distance between the individual points. In practice, such a set would be in 2 or 3 dimensional space, but for our purposes we shall simply assume that it lies in some Euclidean space $\mathbb{R}^d$. The main objective of this thesis work is to study the space of r-uniformly discrete point sets with long range order.

The existence of long range order aperiodic structure in the physical world was first established by D. Shechtman and his coworkers in 1982, by presenting a point-like diffraction picture (similar to Figure 1) with an unusual fivefold symmetry (impossible for crystals) produced by samples from an Al-Mn alloy which had been rapidly cooled after melting. Such materials are now called **quasicrystals**.

In mathematics the phenomenon of the long range order was investigated much earlier than the discovery of the quasicrystals. However, these investigations mainly focused on almost periodic functions in harmonic analysis and on aperiodic tilings. The study of the point sets with long range order was inspired by the discovery of quasicrystals and has been supported by harmonic analysis and the theory of aperiodic tilings since its birth.

One ultimate goal of this study is to determine the structure of a point set with long range order based on the information from its diffraction picture. This is the so-called **inverse problem for quasicrystals**. However, this problem is not well formulated since in many cases the information from the diffraction picture of a unknown point set is not enough to determine the structure of the point set. One main task of this thesis is to address this problem by reformulating the inverse problem in terms of stochastic processes and seek an answer for it.

We start this with a mathematical model for diffraction. From the point view of physics, the diffraction of a point set is a result of the interference of waves originating, by scattering, from points in this point set. Formally, we think of each

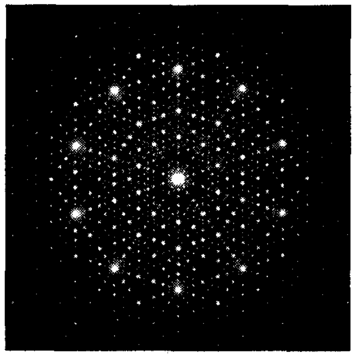

Figure 1: A diffraction picture of Aluminium 70-Cobalt 11-Nickel 19 (Al-70, Co-11 Ni-19), C. Beeli

contributing wave function as the Fourier transform of a Dirac measure supported at some point of the point set. In the case of a lattice, the intensity of the interference of these waves can be modeled as multiplied by the density of the lattice with the sum of all Dirac measures corresponding to the points in the dual lattice. However, many mathematical problems arise when we move from crystals to quasicrystals. Specifically, the sum of the Dirac measures supported on a aperiodic point set might not be Fourier transformable. Even if it is Fourier transformable, it is hard to assign any physical meaning to it.

The current generally accepted mathematical formulation of the diffraction was proposed by A. Hof [23, 24]. In his papers, the notion of an averaged two-point correlation measure of a point set was introduced to avoid these problems. If a two-point correlation exists, then it is a positive and positive definite measure. By Bochner's theorem [46], such a measure is Fourier transformable and the Fourier transformation is a positive measure called the **diffraction measure**. By [6], a diffraction measure is always translation bounded. Moreover, it can be decomposed into three parts with respect to the Lebesgue measure: a pure point measure, an absolutely continuous measure, and a singular continuous measure.

We call a point set admitting a diffraction measure a **diffractive** point set. In particular, if the diffraction measure is discrete (or equivalently pure point[1]), then we say that the point set is **pure point diffractive**.

It is quite general that two distinct point sets may have the same diffraction measure. A known example is the Rudin-Shapiro sequence and a point set from the

[1]It implies that a point-like diffraction picture can be expected to be observed in a lab.

Bernoulli system (with $p = \frac{1}{2}$), see [1]. They are very different[2]. But their diffraction measures are both composed of a discrete part and a fraction of Lebesgue measure and are exactly the same. This happens also even among pure point diffractive point sets, see [3]. In each of these cases, the diffraction measure does not uniquely determine a point set.

Thus, we turn to consider another measure produced by a point set with long range order which mainly characterizes the point set. But before this, let us lay down the general setting for this thesis first. We start this by introducing the notion of r-uniform discreteness. Let $r > 0$ and let C_r denote the cube

$$C_r = \{x = (x_1, \dots, x_d) \in \mathbb{R}^d : |x_i| < \frac{r}{2} \text{ for all } i\}.$$

$\Lambda \subset \mathbb{R}^d$ is r-uniformly discrete if for all $x \in \mathbb{R}^d$, $\mathrm{card}((x + C_r) \cap \Lambda) \leq 1$. Let $\mathcal{D}_r$ denote the set of all r-uniformly discrete subsets of $\mathbb{R}^d$. Often r-uniform discreteness is stated in terms of balls instead of cubes, but this makes no intrinsic change to the concept and cubes are generally more convenient for us in this thesis.

We also wish to consider point sets in which the points can be of various types or colours. We define this precisely later in Section 1.4, but the idea is intuitively obvious. When we speak of r-uniform discreteness for coloured sets, we mean r-uniform discreteness when the colour is ignored.

From now on, we consider those point sets constituted from a finite number of different types, or colours, of points, each of which has its own scattering strength. Such a point set is called an **m-multi-colour set**, or simply, a **m-multiset**. The set of all r-uniformly discrete m-multisets is denoted by $\mathcal{D}_r^{(m)}$. Then the set $\mathcal{D}_r$ can be thought of as the set of all r-uniformly discrete single-colour point sets.

It is known that $\mathcal{D}_r^{(m)}$ is a compact topological space when equipped with the natural topology (local topology), which is used for the construction of dynamical systems in the theory of tilings and Delone point sets. Moreover, the translation action of $\mathbb{R}^d$ is continuous on the space $\mathcal{D}_r^{(m)}$ and $\mathcal{D}_r^{(m)}$ is an invariant of it. Thus, the couple $(\mathcal{D}_r^{(m)}, \mathbb{R}^d)$ forms a topological dynamical system, which serves as a general background for all measure-theoretic dynamical systems of point sets from $\mathcal{D}_r^{(m)}$.

Now we come back to a point set with long range order in $\mathcal{D}_r^{(m)}$. The measure that we need characterizes the structure of the point set by signifying the collection of all local clusters occurring in the point set and their frequencies. It will also appear as an invariant probability measure for a dynamical system of point sets inside $\mathcal{D}_r^{(m)}$. To determine this measure, we consider the dynamical system as a

[2]The Rudin-Shapiro sequence is deterministic and the Bernoulli system is random.

point process, in which we think in terms of a random variable whose outcomes are the various (single colour or m-multicolour) point sets.

It is standard in the theory of point processes to model the point sets involved as point measures $\lambda = \sum_{x\in\Lambda}\delta_x$, so that it is the support of the measures that correspond to the actual point sets. This turns out to be very convenient for several reasons. The most natural topology for measures, the vague topology, exactly matches the local topology (Prop. 2.3.1). Ultimately, to discuss diffraction, one ends up in measures and the vague topology anyway, so having them from the outset is useful. It is easy to build in the notion of colouring and weightings into measures. Most of all, according to the theory of point processes, the probability measure (the **law**) of a point process is determined by a family of finite dimensional distributions. We will use this fact to define an $\mathbb{R}^d$−invariant probability measure directly from a specific point set $\Lambda \in \mathcal{D}_r^{(m)}$. The main procedure and the basic idea are outlined as follows (the details are left to Chapter 2). For simplicity, here we assume Λ is a single colour uniformly discrete point set, i.e., $\Lambda \in \mathcal{D}_r$.

Let $\{\rho(A_1,\dots,A_n;k_1,\dots,k_n)\}$ be a family of values for each finite sequence $(A_1,\dots,A_n)$ of pairwise disjoint n semi-open rectangles in $\mathbb{R}^d$ and each sequence $(k_1,\dots,k_n)$ of n nonnegative integers, $n \in \mathbb{N}$. According to [37], there are six conditions on the family of values $\{\rho(A_1,\dots,A_n;k_1,\dots,k_n)\}$ that are necessary and sufficient for it to uniquely determine a family of finite dimensional distributions (these are given in Proposition 2.4.1). We will prove that for a point set $\Lambda \in \mathcal{D}_r$, the limit values

$$\rho(A_1,\dots,A_n;k_1,\dots,k_n) := \lim_{R\to\infty}\frac{1}{l(C_R)}l\left(\{t \in C_R : \lambda(t+A_j) = k_j, j = 1,\dots,n\}\right), \tag{0.0.1}$$

satisfy these six conditions, if they exist. Thus, these values define a probability measure on $\mathcal{D}_r$, being determined by the point set Λ. We denote this measure μ_Λ. Just as we expected, μ_Λ is an invariant of the translation action group $\mathbb{R}^d$ and hence it determines a measure-theoretic dynamical system on $\mathcal{D}_r$. In particular, when Λ is a uniformly recurrent point set with respect to $(\mathcal{D}_r, \mathbb{R}^d)$, the resulting measure-theoretic dynamical system is minimal.

In the case that Λ is of finite local complexity (FLC) (see Section 1.1.1 for the definition), the knowledge of μ_Λ is effectively the same as the knowledge of the frequencies of the local clusters of Λ. This shows that in this case the frequencies of local clusters do essentially characterize the structure of the point set Λ.

Now let $\hat{\gamma}_\Lambda$ denote the diffraction measure of Λ. The inverse problem for Λ is essentially the question how to determine the measure μ_Λ by knowing $\hat{\gamma}_\Lambda$. Since the diffraction measure $\hat{\gamma}_\Lambda$ is determined by the two-point correlation measure, we equivalently look at the relationship between the two-point correlation measure γ_Λ and the measure μ_Λ.

This relationship is indirectly illustrated by Steven Dworkin's argument, see [16] and corollary 4.1.4. The main idea of this argument is the following. Let $X = \mathrm{supp}(\mu_\Lambda)$. X is a compact subset of $\mathcal{D}_r$. The triple $(X, \mathbb{R}^d, \mu_\Lambda)$ forms a topological and measure theoretical dynamical system. We assume that the dynamical system $(X, \mathbb{R}^d, \mu_\Lambda)$ is ergodic, i.e., it is measure-theoretically irreducible. Steven Dworkin's argument relates the diffraction of the typical point sets comprising X to the dynamical spectrum of X, which is determined by the measure μ and the group action of $\mathbb{R}^d$.

Gouéré [21] proved that for a realization Λ (point set) in an ergodic point process $(X, \mathbb{R}^d, \mu_\Lambda)$, the two-point correlation measure γ_Λ exists μ_Λ-almost surely, and if it exists, then it is equal to the first moment measure of the Palm measure of μ_Λ, which can be regarded as the conditional probability measure of μ_Λ on the subset $\{\Lambda' \in X : 0 \in \Lambda'\}$. We denote by $\dot{\mu}_1$ the first moment measure of the Palm measure of μ_Λ and think of the measure $\dot{\mu}_1$ as the two-point correlation measure of the point process itself instead of individual point sets in the point process. By this, we can avoid to use the word "almost surely" so often. Moreover, this provides a better setting for us to reconsider the question about the relationship of the two measures.

We shall show that what underlies Steven Dworkin's argument is a certain isometric embedding θ of the Hilbert space $L^2(\mathbb{R}^d, \hat{\dot{\mu}}_1)$ into $L^2(X, \mu_\Lambda)$. Both of these Hilbert spaces afford natural representations of $\mathbb{R}^d$, call them U_t and T_t respectively ($t \in \mathbb{R}^d$). Representation T arises from the translation action on $\mathbb{R}^d$ and U is a multiplication action which we define in (4.1.2). The embedding θ intertwines the representations. However, θ is not in general surjective, and in fact it can fail to be surjective quite badly. However, one fundamental result of this thesis is that the algebra generated by the image of $L^2(\mathbb{R}^d, \hat{\dot{\mu}}_1)$ under the embedding mapping is dense in $L^2(X, \mu)$. Applying this theorem, we will easily give another proof to the main theorem of the paper [34] that if $\hat{\dot{\mu}}_1$ is pure point, then the linear span of the eigenfunctions of T is dense in $L^2(X, \mu)$.

Obviously, the measure $\dot{\mu}_1$ is determined by the law μ. But to determine μ, we need other information besides $\dot{\mu}_1$ in most cases. This comes in the form of the so-called higher correlation measures of the point process.

To help the reader get a basic idea about these correlation measures, we introduce a higher correlation measure of a point set first and it will lead to the definition of the correlation measure of the point process, just as the two-point correlation measure does. For simplicity, we consider a point set $\Lambda \in \mathcal{D}_r$ having finite local complexity.

For $n > 1$, if a $(n+1)$**-point correlation measure** of Λ exists, it has the form that

$$\gamma_\Lambda^{n+1} := \sum_{(x_1,\ldots,x_n)\in\Xi^n} \eta(x_1,\ldots,x_n)\delta_{x_1,\ldots,x_n},$$

where $\Xi^n := \{(x_1,\ldots,x_n) : \exists x_0 \in \Lambda, x_i + x_0 \in \Lambda, i = 1,\ldots,n\}$ and $\eta(x_1,\ldots,x_n)$ is equal to the frequency of the occurrence in Λ of local cluster $\{0, x_1,\ldots,x_n\}$ under translation. When $n > 2$, we call such a measure a **higher correlation measure**. Similar to $\gamma_\Lambda = \dot{\mu}_1$, μ-almost surely for $\Lambda \in X$, we have $\gamma_\Lambda^{n+1} = \dot{\mu}_n$, where $\dot{\mu}_n$ is the nth moment measure of the Palm measure. Thus, we call $\dot{\mu}_n$ the $(n+1)$-point correlation measure of the point process. By the Palm measure theory, the $(n+1)$-point correlation measure $\dot{\mu}_n$ is the $(n+1)$th reduced moment measure of the measure μ. The first moment measure of the law is just the intensity of the point process times Lebesgue measure and is derivable from the higher moment measures.

Finally, we will prove that for a known (or given) ergodic point process, its law is uniquely determined by all its correlation measures. This result not only demonstrates the relationship between correlation measures of a point process and the law, but also indicates a direction to the inverse problem for uniformly discrete point sets with long range order; that is, to determine the structure of such a point set, we may need only to consider the higher correlation measures.

Generally, for a collection of correlation measures of some point set, there is an issue about the existence and uniqueness of an ergodic point process admitting the same correlation measures. We call it the **determination problem for the correlations**. In this thesis, we will focus on the uniqueness of an ergodic point process for a collection of correlation measures and call it the **unique determination problem for the correlations**. In particular, we will consider the uniqueness of an ergodic point process for a finite set of correlation measures and call it the **unique finite determination problem for the correlations**.

The best situation is that a point process can be uniquely determined by the two-point correlation measure of the point process (i.e., a two-point correlation measure). However, as we have pointed out, this is not true in general. For pure point diffractive point sets, physicist D. Mermin [38] argued that its structure ought to be determined by its two-point and three-point correlation measures. This statement has been confirmed by the recent work of D. Lenz and R.V. Moody based on the results of this thesis with one more assumption on the diffraction, namely that there is no extinction of the Bragg peaks. In this thesis work, in Section 6, we show that the first $k+1$ correlation measures of a point process uniquely determine an k-step Markov process. We will also prove that a point process generated by a model set is uniquely determined by its 2-point and 3-point correlation measures.

This thesis is organized as follows. In Chapter 1, we will give a basic introduction to diffraction theory by emphasizing several important types of point sets. In Chapter 2, we will build up a dynamical system of point sets after we define an invariant probability measure of the translation group $\mathbb{R}^d$ on $\mathcal{D}_r^{(m)}$ determined by a specific point set. In Chapter 3, we will discuss the relationship between the diffraction measure and the law of the point processes. In Chapter 4, Dworkin's argument, which was briefly stated in Section 2.6, will be revisited. In Chapter 5, the square-mean Bomberi-Taylor conjecture and a strange inequality will be presented. This thesis will finish up with some discussion about the unique finite determination for correlation measures known to be produced by a special point process: a k-step Markov process.

The content of Chapter 3, Chapter 4 and Chapter 5 come from my joint work [14] with my supervisor Dr. Moody. The content of Chapter 6 comes from my paper [13]. These constitute the original research contributions of the thesis.

Chapter 1

Discrete closed point sets and the diffraction theory

Introduction

Start with $\mathbb{R}^d$ endowed with its usual inner (dot) product, and metric given by the Euclidean distance $|x-y|$ between points $x, y \in \mathbb{R}^d$. For $x \in \mathbb{R}^d$, we let $B_R(x), C_R(x)$ denote the open ball of radius R and the open cube of edge length R and centered at x in $\mathbb{R}^d$. For simplicity, we often write B_R, C_R for $B_R(0), C_R(0)$. Lebesgue measure will be indicated by ℓ. Closures of sets in $\mathbb{R}^d$ is denoted by overline symbols. The overline also represents complex conjugation in this thesis, but there is little risk of confusion.

The main purpose of this chapter is to give a basic introduction to the theory of the diffraction on discrete closed point sets. This includes the r-uniformly discrete point sets. In general, a discrete closed point set of long range order is mainly characterized by its density, clusters and frequencies of the occurrence of its clusters, and so on. Local isomorphism is an equivalence relation defined on the space of discrete closed point sets.

Although in many articles, the notions of cluster and pattern are treated as the same, they have distinct meanings in this thesis. A cluster P is a finite set of points $P := \{0, p_1, \dots, p_n\} \subset \mathbb{R}^d$ and a typical pattern is a pair (P, V), where P is a cluster and V is a neighbourhood of the origin. More details on patterns will be discussed in Chapter 2.

Several important types of point sets with long range order including model sets and substitution point sets will be discussed in this chapter to help a reader who is new to this field. We will also discuss the diffraction theory of two random systems:

the Bernoulli system on $\mathbb{Z}$ and a random system on a model set. This will help us to move from a deterministic point setting to a random point setting. The Fibonaaci set will appear in several sections in this chapter and basically it will serve as a typical example both of a model set and a substitution point set.

1.1 Discrete closed point sets

For $s \in \mathbb{R}^d$ and $A, B \subset \mathbb{R}^d$, define $s + A := \{s + x : x \in A\}$ and $A + B := \{z \in \mathbb{R}^d : z = x + y, x \in A, y \in B\}$. Denote by card$(A)$ the cardinality of the set A.

Let Λ be an infinite point set on $\mathbb{R}^d$. If for all $R > 0, a \in \mathbb{R}^d$, $\text{card}((C_R + a) \cap \Lambda) < \infty$, then we call Λ a **discrete closed point set**[1]. We define the **density** of Λ by

$$\text{dens}(\Lambda) := \lim_{R\to\infty} \frac{\text{card}(C_R \cap \Lambda)}{l(C_R)},$$

if the limit exists.

Λ is **uniformly discrete** if and only if there is a $r' > 0$ so that Λ is r'-uniformly discrete, i.e., for all $a \in \mathbb{R}^d$, $\text{card}((C_{r'} + a) \cap \Lambda) \leq 1$. Λ is **relatively dense** if and only if there is a compact set $K \subset \mathbb{R}^d$ so that $K + \Lambda = \mathbb{R}^d$, or equivalently, there is a $R > 0$ such that for $a \in \mathbb{R}^d$, $\text{card}((C_R + a) \cap \Lambda) \geq 1$. If Λ is uniformly discrete and relatively dense, then we call Λ a **Delone set**.

1.1.1 Clusters and their frequencies

Let Λ be a discrete closed point set.

Definition 1.1.1. *For a relatively compact set $K \subset \mathbb{R}^d$ containing the origin and $t \in K \cap \Lambda$, the finite set $P := (-t + \Lambda) \cap K$ is called a* ***cluster*** *of Λ.*

It is clear that P is a finite subset of $(\Lambda - \Lambda)$ and $0 \in P$. For such a cluster P, if there is a point $s \in \Lambda$ such that $(-s + \Lambda) \cap K = P$, then s is called a **translation vector** of P.

If for all $R > 0$, $\text{card}\{P' : P' = (-t + \Lambda) \cap \overline{C_R}, t \in \Lambda\} < \infty$, (there are a finite number of clusters appearing in the cube C_R relative to the positions of Λ), or equivalently, $\text{card}((\Lambda - \Lambda) \cap \overline{C_R}) < \infty$, then we say that Λ has **finite local complexity** (FLC).

If for each cluster P of Λ (or equivalently, for each cluster P of the form $\Lambda \cap C_R, R > 0$), the set of all translation vectors of P is relatively dense, then we say that Λ is **repetitive**.

[1]This inequality is called the local finiteness condition.

Let Λ be a discrete closed point set with FLC and let P be a cluster of Λ.

Definition 1.1.2. *If the limit*

$$\lim_{R\to\infty} \frac{\mathrm{card}\{t \in B_R \cap \Lambda : P \subset (-t+\Lambda)\}}{l(B_R)}$$

exists, then we call it the ***frequency*** *of P and denote it by* $\mathrm{freq}(P, \Lambda)$.

In particular, if the limit

$$\lim_{R\to\infty} \frac{\mathrm{card}\{t \in (a+B_R) \cap \Lambda : P \subset (-t+\Lambda)\}}{l(B_R)}$$

exists *uniformly* for all $a \in \mathbb{R}^d$, then we say that P has a **uniform frequency** in Λ. If all clusters of Λ have uniform cluster frequency, then we say that the point set Λ has **uniform cluster frequencies** (UCF).

1.1.2 Local indistinguishability

Let Λ be a discrete closed point set in $\mathbb{R}^d$. We define a set $Y_\Lambda := \{\Lambda' \subset \mathbb{R}^d :$ for all compact sets $K \subset \mathbb{R}^d$, there exists $t \in \mathbb{R}^d$, such that $\Lambda' \cap K = (-t+\Lambda) \cap K\}$. Two point sets Λ_1, Λ_2 is **locally indistinguishable**[2] **(LI)** if and only if $\Lambda_1 \in Y_{\Lambda_2}$ and $\Lambda_2 \in Y_{\Lambda_1}$. This defines an equivalence relation on all discrete closed point sets. The equivalence class of Λ with respect to the locally indistinguishability is called the LI class of Λ and is denoted by $[\Lambda]$.

For example, we consider the point set $\Lambda := \mathbb{Z}^+ \cup (2\mathbb{Z}^-)$, where $\mathbb{Z}^+$ denotes the set of nonnegative integers and $\mathbb{Z}^-$ denotes the set of negative integers. By observation, $Y_\mathbb{Z} \subset Y_\Lambda$, but $Y_\mathbb{Z} \not\supseteq Y_\Lambda$. So, Λ is not LI to $\mathbb{Z}$.

1.1.3 Example: the Fibonacci chain

The **Fibonacci symbol sequence** $\alpha = \{\alpha_n\}_\mathbb{N}$ can be given as a limit[3] of a sequence of words $\{\beta_n\}_\mathbb{N}$ of two symbols $\{a, b\}$ satisfying the following induction rules:

1. $\beta_1 := \{a\}, \beta_2 := \{ab\}$;
2. $\beta_n := \beta_{n-1} + \beta_{n-2}$, for $n > 2$,

[2] Or locally isomorphic

[3] With respect to the partial order given by that $x < y$ if y contains x as its beginning.

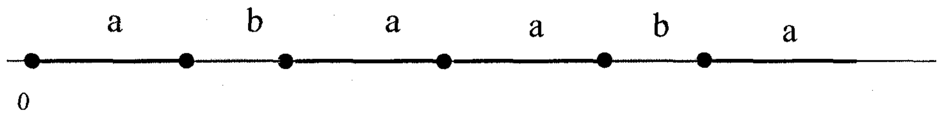

Figure 1.1: The one sided Fibonacci chain

where "+" is the operator catenating two words. The first few terms of the sequence $\{\beta_n\}_{\mathbb{N}}$ are:

$$a, ab, aba, abaab, abaababa, \ldots$$

Note that $\{z_n : z_n = \mathrm{card}(\beta_n)\}$ is the famous Fibonacci number sequence and we have that

$$\lim_{n\to\infty} \frac{z_n}{z_{n-1}} = \tau, \tag{1.1.1}$$

where $\tau = \frac{1+\sqrt{5}}{2}$ and $\frac{1}{\tau}$ is the golden ratio.

The Fibonacci symbol sequence is a recursive sequence. Let ι be a finite subword of it. There exists a positive integer N such that $\iota \subset \beta_N$. If we rewrite the sequence α as follows,

$$\alpha = \beta_{N+1} + \beta_N + \beta_{N+1} + \beta_{N+1} + \beta_N + \ldots,$$

then it is easy to observe that there is at least one β_N in each subword with length z_{N+2}. As a consequence, ι appears in every subword with length z_{N+2}. We have that the Fibonacci symbol sequence is repetitive.

Definition 1.1.3. *The* ***one sided Fibonacci chain*** $\{t_n\}_0^\infty$ *on the real line is defined on* $[0, \infty)$ *by that:* $t_0 = 0$, *and for* $n \in \mathbb{N}$

$$t_n := t_{n-1} + \begin{cases} \tau, & \text{if } \alpha_n = a \\ 1, & \text{if } \alpha_n = b. \end{cases} \tag{1.1.2}$$

It is demonstrated by Figure 1.1, where $a = \tau, b = 1$.

Because τ and 1 are integrally independent, i.e., the equation $x\tau + y = 0$ has only the trivial solution on $\mathbb{Z}^2$, there is a one-to-one correspondence between the clusters of Λ and the finite subsequences of the Fibonacci symbol sequence $\{\alpha_n\}_{\mathbb{N}}$. It is convenient to represent a cluster of the Fibonacci chain by a subword from the symbolic sequence. By observation, these subwords of the Fibonacci symbolic sequence form a tree demonstrated by Figure 1.2.

Definition 1.1.4. *A* ***two sided Fibonacci chain*** *is a bi-infinite sequence from* $\{a, b\}$ *which is LI to the 1-sided Fibonacci chain.*

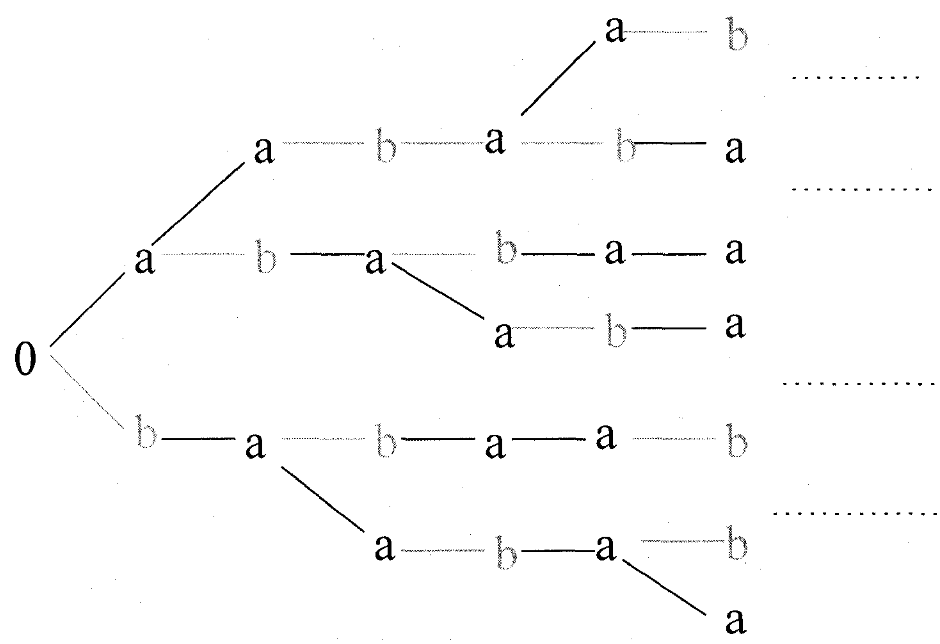

Figure 1.2: The local clusters tree

Lemma 1.1.5. *A two sided Fibonacci chain is a repetitive Delone set with FLC.*

PROOF: Let Λ be a two sided Fibonacci chain. Obviously, Λ is a Delone set. It has FLC since the distance between any two points is of the form $m\tau + n, m, n \in \mathbb{Z}^+$, and hence $\Lambda - \Lambda$ is a uniformly discrete set. Finally, Λ is repetitive because the Fibonacci symbol sequence is repetitive. □

It is known that a two sided Fibonacci chain has UCF. The frequency of the occurrence of each cluster in two sided Fibonacci chain is equal to the frequency of the occurrence of this cluster in the one sided Fibonacci chain. We will see this later from the aspect of model sets.

Let Λ be a two sided Fibonacci chain. Now we compute the density of it and the frequencies of two clusters: $\{0, \tau\}$ and $\{0, 1\}$ represented by a and b respectively. Note that $\text{card}(\beta_n) = z_n$, $\text{card}\{a \in \beta_n\} = z_{n-1}$, $\text{card}\{b \in \beta_n\} = z_{n-2}$ and $z_n = z_{n-1} + z_{n-2}$. By (1.1.1), we have

$$\begin{aligned}
\text{dens}(\Lambda) &= \lim_{n \to \infty} \frac{z_n}{z_{n-2} + z_{n-1} \cdot \tau} \\
&= \frac{\tau}{2\tau - 1}.
\end{aligned}$$

Similarly, we have

$$\begin{aligned} \text{freq}(a) &= \lim_{n\to\infty} \frac{z_{n-1}}{z_{n-2} + z_{n-1} \cdot \tau} \\ &= \frac{\tau}{\tau + 2}, \end{aligned} \tag{1.1.3}$$

and

$$\begin{aligned} \text{freq}(b) &= \lim_{n\to\infty} \frac{z_{n-2}}{z_{n-2} + z_{n-1} \cdot \tau} \\ &= \frac{2}{\tau + 2}. \end{aligned} \tag{1.1.4}$$

Note that we have chosen a special sequence $\{z_n\}$ in above limits. However, these limits also exist with respect to the sequence $\mathbb{Z}$. At the end of this chapter, we will prove that the one sided Fibonacci chain is actually the right side of a model set and we have a much easier way to compute the frequencies of its clusters, which leads to the same result.

1.2 The introduction of diffraction theory

1.2.1 The diffraction of crystals

In general, a lattice in $\mathbb{R}^d$ is a discrete subgroup that spans $\mathbb{R}^d$. Thus it has the form $\{\sum_{i=1}^{d} a_i v_i | a_i \in \mathbb{Z}\}$, where $\{v_1, \cdots, v_d\}$ is a basis of $\mathbb{R}^d$. Let $\mathcal{L}$ be a lattice in $\mathbb{R}^d$. The dual group of $\mathcal{L}$ or the reciprocal lattice of $\mathcal{L}$, denoted by $\mathcal{L}^\circ$, is defined by

$$\mathcal{L}^\circ := \{k \in \mathbb{R}^d | k \cdot s \in \mathbb{Z}, \text{ for all } s \in \mathcal{L}\}. \tag{1.2.1}$$

From the point view of physics, the diffraction phenomenon is a result of interference of waves with a given wavelength along all unobstructed paths. A simple wave function propagating along the direction represented by a unit vector k with the initial phase zero is of the form

$$\varphi(r,t) = A \exp\{i(2\pi k \cdot r - \omega t)\}, r \in \mathbb{R}^d,$$

where A is the amplitude of the wave, ω is frequency of the wave, and t denotes the time. The interference of two waves originating from two lattice points is determined by their phase difference, which is caused by the relative positions of the two points.

As a consequence, the **intensity** of the interference of all waves, i.e., the space average of $|\sum_{r\in\mathcal{L}}\varphi(r,0)|^2$, along the direction $x\in\mathbb{R}^d$ is

$$\begin{aligned} I(x): &= \lim_{R\to\infty}\frac{1}{l(C_R)}A\sum_{r\in(\mathcal{L}-\mathcal{L})}\exp\{2\pi i x\cdot r\} \\ &= A\,\mathrm{dens}(\mathcal{L})\sum_{r\in\mathcal{L}}\exp\{2\pi i x\cdot r\}. \end{aligned} \tag{1.2.2}$$

We point out that the right hand side of equation (1.2.2) is just a formal expression. It should be treated as a measure and the limit is taken in the vague topology. The justification will be given after the Poisson summation formula (1.2.4).

For a general discrete closed point set Λ, we define δ_Λ as the countable sum $\sum_{x\in\Lambda}\delta_x$, where δ_x is the delta measure supported at $\{x\}$. δ_Λ is called the **Dirac comb** of Λ. Similarly, let $\delta_{\mathcal{L}^\circ}:=\sum_{x\in\mathcal{L}^\circ}\delta_x$. By the **Poisson summation formula for lattices** [2],

$$\hat{\delta}_{\mathcal{L}}=\mathrm{dens}(\mathcal{L})\delta_{\mathcal{L}^\circ}, \tag{1.2.3}$$

where $\hat{\delta}_{\mathcal{L}}$ denotes the Fourier transform of the measure $\delta_{\mathcal{L}}$. Meanwhile the Fourier transform term-by-term of $\delta_{\mathcal{L}}$ is equal to

$$\hat{\delta}_{\mathcal{L}}(x)=\sum_{y\in\mathcal{L}}\exp\{2\pi i y\cdot x\}. \tag{1.2.4}$$

To understand the right side of the above equation, we need to treat the measure $\delta_{\mathcal{L}}$ as a tempered distribution, i.e., $\delta_{\mathcal{L}}$ is a linear functional of the Schwartz space $\mathbb{S}(\mathbb{R}^d)$ of rapidly decreasing C^∞ functions[4]. The Fourier transform of a tempered distribution ν is defined by $\hat{\nu}(f):=\nu(\hat{f})$ for all Schwartz functions f. An important property of the Fourier transform on tempered distributions is that if $\{\nu_n\}\to\nu_0$ in vague topology[5], then $\{\widehat{\nu_n}\}\to\widehat{\nu_0}$ in the vague topology. Since for all Schwartz functions f,

$$\begin{aligned} \delta_{x_0}(\hat{f}): &= \hat{f}(x_0)=\int_{\mathbb{R}^d}e^{-2\pi x_0\cdot y}f(y)dy \\ &= e^{-2\pi x_0\cdot(\cdot)}(f). \end{aligned}$$

Hence, $\widehat{\delta_{x_0}}(x)$ is equal to $e^{-2\pi x_0\cdot x}$ as a measure. Now we take $\nu_n:=\sum_{x\in C_{R_n}\cap\mathcal{L}}\delta_x$, where $R_n\to\infty$ and $\nu_0:=\sum_{x\in\mathcal{L}}\delta_x$. Then $\{\nu_n\}\to\nu_0$ vaguely. Then we get the

[4] $\mathbb{S}(\mathbb{R}^d)\cap C_c(\mathbb{R}^d)$ is densely embedded in the space of continuous functions of compact support, $C_c(\mathbb{R}^d)$.

[5] It means that $\{\nu_n\}(f)\to\nu_0(f)$, for all functions $f\in\mathbb{S}(\mathbb{R}^d)$.

formula (1.2.4) and as we pointed out before, the right side of (1.2.4) makes sense as being a limit of measures under the vague topology.

Combining equation (1.2.3) and equation (1.2.4), we obtain that

$$I = A(\mathrm{dens}(\mathcal{L}))^2 \delta_{\mathcal{L}^\circ}. \tag{1.2.5}$$

It says that there are sharp peaks (Bragg peaks) at every point of the dual lattice, $\mathcal{L}^\circ$, and all of them have the same weight $A(\mathrm{dens}(\mathcal{L}))^2$. We define the **diffraction measure** of $\mathcal{L}$ as the measure $(\mathrm{dens}(\mathcal{L}))^2\delta_{\mathcal{L}^\circ}$. The big difficulty for us to extend this definition to an aperiodic point set is that there is no Poisson summation formula for a general aperiodic point set.

1.2.2 The diffraction of a discrete closed point set

Let Λ be an infinite discrete closed point set. We define the operator "$\tilde{\ }$" acting on all functions on $\mathbb{R}^d$ as $\tilde{f}(x) := \overline{f(-x)}$ and acting on all measures on $\mathbb{R}^d$ as $\tilde{\nu}(f) := \overline{\nu(\tilde{f})}$.

Definition 1.2.1. *(See [24].) If the limit*

$$\lim_{R\to\infty} \frac{1}{l(C_R)} \left(\sum_{x,y\in\Lambda\cap C_R} \delta_x * \widetilde{\delta_y} \right) \tag{1.2.6}$$

exists[6] *in vague topology, then we call this limit the **2-point correlation measure** (or the **autocorrelation measure**) of Λ relative to $\{C_R\}$ and denote it by γ_Λ.*

Although we won't need them in this thesis, it is possible to average over more general sequences than cubes. One commonly used class of type of averaging sequences are the van Hove sequences (of which the cubes are an example). Denote A^0 the interior of A and A^c the complement set of A in $\mathbb{R}^d$.

Definition 1.2.2. *If A and K are compact subsets, then the K-**boundary** of A is defined by*

$$\partial^K A := (K + A) \setminus A^0 \cup ((-K + \overline{A^c}) \cap A). \tag{1.2.7}$$

[6]Usually, a two-point correlation measure depends on the averaging sequence it chosen. Two different averaging sequences may yield two different two-point correlation measures. But we will not worry about such an issue in this thesis. Simply we will just use one fixed averaging sequence in this thesis and if the limit exists with respect to that averaging sequence, then we call it the two-point correlation measure of the point set.

Remark 1.2.3. *Both* $(K + A)$ *and* $-K + \overline{A^c}$ *are compact because a compact set "plus" a closed set is a compact set. As a consequence,* $\partial^K A$ *is compact. We point out that it is not always true that the sum of two "closed" sets is a closed set. For example, let* $A := \mathbb{Z}\sqrt{2}$ *and* $K := \mathbb{Z}$*. Both* A *and* K *are closed, but* $A+K = \mathbb{Z}+\mathbb{Z}\sqrt{2}$ *is not closed.*

Definition 1.2.4. *A sequence of compact subsets* $\{V_n \subset \mathbb{R}^d\}$ *is called a van Hove sequence if it satisfies that*

1. $V_n \subset V^0_{n+1}$ *for* $n = 1, 2, \cdots$;
2. $l(V_n) > 0$, $l(V_n) \to \infty$ *as* $n \to \infty$;
3. *For all compact sets* K, $\lim_{n\to\infty} \frac{l(\partial^K(V_n))}{l(V_n)} = 0$.

Now we are ready to define a more general 2-point correlation measure of Λ as follows.

Definition 1.2.5. *Suppose* $\{V_n\}$ *is a van Hove sequence in* $\mathbb{R}^d$*. If the limit*

$$\gamma_{\Lambda} := \lim_{n\to\infty} \frac{1}{l(V_n)} \left(\sum_{x,y\in\Lambda\cap V_n} \delta_x * \widetilde{\delta_y} \right) \tag{1.2.8}$$

exist in vague topology, then we call this limit the ***2-point correlation measure*** *(or the* ***autocorrelation measure****) of* Λ *relative to* $\{V_n\}$*.*

For a fixed Λ, not all van Hove sequences necessarily give a limit and it is possible that we may obtain two distinct limits according to two distinct van Hove sequences. But throughout this thesis work, we always consider the 2-point correlation measure relative to the sequence $\{C_R\}$ and we simply call it the 2-point correlation measure of Λ.

A function f, defined on $\mathbb{R}^d$, is said to be **positive definite** if the inequality

$$\sum_{m,n=1}^{N} c_n \overline{c_m} f(x_n - x_m) \geq 0,$$

holds for every choice of $x_1, \ldots, x_N$ in $\mathbb{R}^d$ and for every choice of complex numbers $c_1, \ldots, c_N$. See [46]. There are two important properties of the positive definite function f:

- for all $t \in \mathbb{R}^d$, $|f(t)| \leq f(0)$,

- $$|f(t+s)-f(t)| \leq 2f(0)\mathrm{Re}[f(0)-f(s)]. \tag{1.2.9}$$

For a regular measure ν on $\mathbb{R}^d$, ν is **positive** if and only if for an arbitrary positive function f, $\nu(f) \geq 0$. On the other hand, ν is a **positive definite measure** if and only if for any continuous function f with compact support, i.e., $f \in C_c(\mathbb{R}^d)$, $\nu * f * \tilde{f}$ is a **positive definitive function**. Obviously, the 2-point correlation measure γ_Λ is positive. Furthermore, we know that the function of the form $g * \tilde{g}, g \in L^1(\mathbb{R}^d)$ is positive definitive. Taking $g_R = [(\sum_{x\in\Lambda\cap C_R} \delta_x) * f]/\sqrt{l(C_R)}$, we have that γ_Λ is positive definitive by the observation,

$$\frac{1}{l(C_R)}\Big(\sum_{x,y\in\Lambda\cap C_R} \delta_x * \widetilde{\delta_y}\Big) * (f * \tilde{f}) = g_R * \widetilde{g_R},$$

and the fact that the limit of a sequence of positive definitive functions is also positive definitive.

By the Bochner theorem (see [46]), γ_Λ is Fourier transformable and its Fourier transform is also a positive and positive definite measure. This new measure is called the **diffraction measure** of the point set and is denoted by $\hat{\gamma}_\Lambda$. Since a diffraction pattern is mostly distinguished by its pure point part, it is interesting to know that under what conditions, $\hat{\gamma}_\Lambda$ is pure point. The following theorem is an important result of the diffraction theory.

Theorem 1.2.6. *If γ_Λ is almost periodic, then $\hat{\gamma}_\Lambda$ is pure point.*

PROOF: We only give a sketch of the proof here. For the details, we refer to [32].

First, for a function g on $\mathbb{R}^d$ and $\epsilon > 0$, we define a set $\mathscr{T}_g^\epsilon$ as

$$\mathscr{T}_g^\epsilon := \{x \in \mathbb{R}^d : \sup_{y\in\mathbb{R}^d} |g(y) - g(x+y)| \leq \epsilon\}.$$

We call $\mathscr{T}_g^\epsilon$ the set of **ϵ-almost periods** of g. A function g is called **ϵ-almost periodic** if $\mathscr{T}_g^\epsilon$ is relatively dense in $\mathbb{R}^d$. If for every choice of $\epsilon > 0$, g is ϵ-almost periodic, then we call g an **almost periodic function**.

Next, for a measure ν, ν is an **almost periodic measure** if and only if for an arbitrary continuous function f with compact support, the convolution $\nu * f$ is an almost periodic function.

The main theorem in the theory of almost-periodic functions states that every almost-periodic function is the uniform limit of trigonometric polynomials. Since the Fourier transform of a trigonometric polynomial is a finite sum of Dirac measures, intuitively the Fourier transform of an almost-periodic function is a finite or a countable sum of Dirac measures.

Suppose that a measure ν is Fourier transformable. Recall that the Fourier transform of the measure ν can be given by

$$\hat{\nu}(f) = \nu(\hat{f}),$$

for $f \in \mathbb{S}(\mathbb{R}^d)$. Moreover, we have that

$$\widehat{\nu * f} = \hat{\nu} \cdot \hat{f}. \tag{1.2.10}$$

By the assumption, γ_Λ is almost periodic, i.e., $\gamma_\Lambda * f$ is an almost periodic function for $f \in \mathbb{S}(\mathbb{R}^d)$. For a compact set $K \subset \mathbb{R}^d$, there is a Fourier transformable function $f \in S(\mathbb{R}^d)$ such that $\hat{f}(x) \neq 0$, for $x \in K$. Thus, the restriction of the measure $\hat{\nu}$ on K is equal to $(\widehat{\gamma_\Lambda * f}|_K)/(\hat{f}|_K)$. Since $\widehat{\gamma_\Lambda * f}|_K$ is pure point, $\hat{\gamma}_\Lambda|_K$ is also pure point for any K. Therefore, $\hat{\gamma}_\Lambda$ is pure point. □

In general, it is not easy to verify a 2-point correlation measure is almost periodic. However, in the case of a FLC point set, there is a condition on the 2-point correlation measure coefficient which is sufficient and necessary for a 2-point correlation measure being almost periodic.

Let Λ be an FLC point set. Since $(\Lambda - \Lambda) \cap C_R$ is a finite set and $\delta_x * \widetilde{\delta_y} = \delta_{x-y}$, the definition of the 2-point correlation of Λ can be rewritten as

$$\gamma_\Lambda := \lim_{R\to\infty} \frac{1}{l(C_R)} \sum_{x,y \in \Lambda \cap C_R} \delta_{x-y}.$$

A consequence of the van Hove property of cubes is

$$\gamma_\Lambda := \sum_{z \in (\Lambda - \Lambda)} \eta_z \delta_z = \sum_{z \in (\mathbb{R}^d)} \eta_z \delta_z, \tag{1.2.11}$$

where the coefficient η_z is the frequency of the occurrence of the cluster $P := \{0, z\}$ in Λ if $z \in (\Lambda - \Lambda)$ and $\eta_z = 0$ otherwise. Then $\gamma_\Lambda = \eta \cdot \sum_{z \in (\Lambda - \Lambda)} \delta_z$. By observation, for all Fourier transformable functions $f \in C_c(\mathbb{R}^d)$, $\gamma_\Lambda * f$ are almost periodic if and only if the function η is almost periodic. Since η is positive definite and hence it satisfies the inequality (1.2.9), we can equivalently define a ϵ-almost period of η by that $\{z : |\eta_0 - \eta_z| \leq \epsilon\}$. As a consequence, η is almost periodic if and only if for all $\epsilon > 0$, the set $\{z : |\eta_0 - \eta_z| \leq \epsilon\}$ is relatively dense.

Putting all this together, we have the following theorem.

Theorem 1.2.7. *[23] Suppose that Λ has FLC and γ_Λ exists. Then γ_Λ is almost periodic if and only if η is almost periodic, i.e., the set $\{z : |\eta_0 - \eta_z| \le \epsilon\}$ is relatively dense for $\epsilon > 0$. In particular, if η is almost periodic, then $\hat{\gamma}_\Lambda$ is a pure point measure.*

1.2.3 The Wiener diagram

It is interesting to see that the new definition of diffraction measures for general discrete closed point sets is consistent with the old definition for crystals. For that, we draw the following diagram

$$\begin{tikzcd} \omega \arrow[r, "*"] \arrow[d, "FT"'] & \gamma_\omega \arrow[d, "FT"] \\ \widehat{\omega} \arrow[r, "|.|^2"] & \widehat{\gamma_\omega} \end{tikzcd}$$

Wiener Diagram

where $\omega := \delta_\Lambda$ and FT is the abbreviation for Fourier transform. This diagram is often called the **Wiener diagram**. One can verify that the two-point correlation of the lattice Λ is equal to $\mathrm{dens}(\Lambda)\delta_\Lambda$. By the Poisson summation formula for lattices, its Fourier transform is equal to $\mathrm{dens}(\Lambda)^2\delta_{\Lambda^\circ}$, just as the old definition. So crystals make the Wiener diagram commute.

1.3 Regular model sets and their diffraction measures

In this subsection we offer a famous construction of pure point diffraction Delone sets by way of an example.

A **locally compact Abelian group** is an abelian group G equipped with a topology such that G is a Hausdorff space, each point has a compact neighbourhood, and the mapping $G \times G \to G$ defined by $(x, y) \mapsto x - y$ is continuous. A lattice $\mathcal{L}$ in G is a discrete subgroup which is cocompact, i.e., $G/\mathcal{L}$ is compact. This definition is equivalent to our previous definition for lattices in $\mathbb{R}^d$.

Definition 1.3.1. *A **cut and project scheme** consists of a direct product $\mathbb{R}^d \times H$ of a real space and a locally compact abelian group H, and a lattice $\mathcal{L}$ in $\mathbb{R}^d \times H$ such that with respect to the natural projections $p_1 : \mathbb{R}^d \times H \to \mathbb{R}^d, p_2 : \mathbb{R}^d \times H \to H$.*

1. p_1 restricted to $\mathcal{L}$ is 1-1.

2. $p_2(\mathcal{L})$ is dense in H.

Let $M := p_1(\mathcal{L})$. Denoting by $(\cdot)^*$ the mapping $p_2 \cdot (p_1|_{\mathcal{L}})^{-1}$, we have $(\cdot)^* : M \longrightarrow H$. The mapping $(\cdot)^*$ is a group homomorphism, but in most situations that we deal with it is not continuous (anywhere). According [39], we have the following definition.

Definition 1.3.2. *Let $(\mathbb{R}^d \times H, \mathcal{L})$ be a cut and project scheme and let Ω be a relatively compact set of H with nonempty interior. Suppose Λ is the point set defined by*

$$\Lambda := \Lambda(\Omega) = \{p_1(z) | z \in \mathcal{L}, p_2(z) \in \Omega\}.$$

Ω is called the window of Λ. We usually write this in the slightly simpler form

$$\Lambda = \{x \,|\, x^* \in \Omega\}.$$

For every choice of $t \in \mathbb{R}^d$, the point set $-t + \Lambda$ is called a ***model set*** *or a* ***cut and project set****.*

In particular, if Λ is a model set defined by a window Ω whose boundary has measure 0 with respect to the Haar measure on H, then we call Λ a **regular model set**.[7]

Lemma 1.3.3. *([41], Lemma 2.5) Let $(\mathbb{R}^d \times H, \mathcal{L})$ be a cut and project scheme. Let $U \subset H$ be a nonempty open set. Then there is a compact set K in $\mathbb{R}^d$ so that*

$$\mathbb{R}^d \times H = \mathcal{L} + (K \times U). \tag{1.3.1}$$

Proposition 1.3.4. *([41], Proposition 2.6) Model sets are Delone sets.*

PROOF: It suffices to prove that for a model set of the form $\Lambda = \Lambda(\Omega)$, where Ω is some relatively compact set with nonempty interior. By the lemma 1.3.3, there is a compact set K in $\mathbb{R}^d$ so that $\mathbb{R}^d \times H = \mathcal{L} + (K \times U)$, where $U := \Omega^0$, the interior set of Ω. For $x \in \mathbb{R}^d$,

$$(x, 0) = (d, d^*) + (k, -w),$$

for some $d \in M, k \in K, w \in \Omega$. Then $d^* = w \in \Omega$ gives $d \in \Lambda$, and $x = d+k \in \Lambda+K$. Thus, $\mathbb{R}^d = \Lambda + K$ and hence Λ is relatively dense.

[7] Although the construction and result below are valid for arbitrary locally compact Abelian groups H, the case $H = \mathbb{R}^n$ for some n provides numerous useful examples. In these cases Haar measure is Lebesgue measure.

Meanwhile, because Ω is a relatively compact set and $\mathcal{L}$ is a lattice (hence is discrete), there is a $s > 0$ such that for all $0 < t < s$, $K_t := \overline{B_t} \times (\overline{\Omega} - \overline{\Omega})$ satisfies $K_t \cap \mathcal{L} = \{0\}$. Thus $(\varLambda - \varLambda) \cap B_s = \{0\}$, i.e., $\varLambda$ is uniformly discrete. □

Let $\varLambda := \varLambda(\Omega)$ be a regular model set. Since the mapping $(\cdot)^*$ is $\mathbb{Z}$-linear, $\varLambda - \varLambda \subset \varLambda(\Omega - \Omega)$. So $\varLambda - \varLambda$ is uniformly discrete and hence $\varLambda$ is a FLC point set. As we pointed out before, this implies that the 2-point correlation measure of Λ, γ_Λ has the form

$$\gamma_\Lambda(x) = \sum_{x \in (\Lambda - \Lambda)} \eta_x \delta_x = \sum_{x \in \mathbb{R}^d} \eta_x \delta_x,$$

where the function η is defined in (1.2.11). Denote the Haar measure of the locally compact abelian group H by θ_H. The 2-point correlation coefficient η_x can be calculated to be

$$\eta_x = \mathrm{freq}(P) = \frac{\theta_H(\Omega \cap (-x^* + \Omega))}{\theta_H(\Omega)}, \tag{1.3.2}$$

for $P = \{0, x\}$, see [40]. Obviously, $\eta_0 = 1$.

Therefore, the 2-point correlation measure $\gamma_\varLambda$ exists as a limit. Moreover, for an arbitrary ball B_ϵ, the model set $\varLambda(B_\epsilon)$ is relatively dense. By equation (1.3.2), for $\epsilon > 0$, the set $z : |\eta_0 - \eta_z| \le \epsilon$ is relatively dense. According to Theorem 1.2.7, we have that the diffraction measure of $\varLambda$ is pure point. Since translation does not affect correlation, all regular model sets are pure point diffractive.

In particular, when $H = \mathbb{R}^k$, k is a positive integer, the diffraction measure is given by

$$\hat{\gamma}_\varLambda = \sum_{k \in \mathcal{L}^\circ} |a(k)|^2 \delta_{p_2(k)}, \tag{1.3.3}$$

where $\mathcal{L}^\circ$ is the dual group of the lattice $\mathcal{L}$ and $a(k)$ is the so-called Fourier-Bohr coefficient (or amplitude)

$$a(k) = \frac{\mathrm{dens}(\varLambda)}{\theta_H(\Omega)} \int_\Omega e^{-2\pi i (p_2(k) \cdot y)} dy. \tag{1.3.4}$$

Note that $a(0) = \mathrm{dens}(\varLambda)$.

It says that the Bragg peaks lie in the projection of the lattice $\mathcal{L}^\circ$ into the physical space. With appropriate definition of dual lattices, this result applies as well for an arbitrary locally compact abelian group. This is due to A. Hof [23], M. Schlottmann [47], and [7] in the present proof.

Theorem 1.3.5. *Let $\varLambda$ be a regular model set defined by the cut and project scheme of Definition 1.3.1 and with window Ω. Then $\varLambda$ is pure point diffractive with the square root of the intensities of the Bragg peaks given by (1.3.4).*

1.4 Multisets and their diffraction

1.4.1 General multisets

In this section we add the notion of "colour" to our point sets. The resulting objects are called multisets[8]. To specify a multiset, we introduce $\boldsymbol{m} := \{1, \ldots, m\}$, $m = 1, 2, 3, \ldots$, which one may intuitively think of as labels for different colours, with the discrete topology and take as our basic space the set $\mathbb{E} := \mathbb{R}^d \times \boldsymbol{m}$ with the product topology. Thus, any point $(x, i) \in \mathbb{E}$ refers to the point x of $\mathbb{R}^d$ with colour i. When $m = 1$ we simply identify $\mathbb{E}$ and $\mathbb{R}^d$. Closures of sets in $\mathbb{E}$ are denoted by overline symbols. Denote the image of a discrete closed point set $\Lambda_i \subset \mathbb{R}^d \times \{i\}$ under the canonical projection from $\mathbb{R}^d \times \{i\} \to \mathbb{R}^d$ by $\Lambda_i^{\downarrow}$, for $i = 1, \ldots, m$. A discrete closed point set Λ in $\mathbb{E}$ is called a **m-multiset** if it is of the form: $\Lambda := \dot{\bigcup}_{i=1}^{m} \Lambda_i$, where $\Lambda_i \subset \mathbb{R}^d \times \{i\}$ is a discrete closed set such that the flattening of Λ defined by $\Lambda^{\downarrow} := \bigcup_{(x,i)\in\Lambda}\{x\} \subset \mathbb{R}^d$ is a disjoint union of $\Lambda_i^{\downarrow}$, i.e., $\Lambda^{\downarrow} := \dot{\bigcup}_{i=1}^{m} \Lambda_i^{\downarrow}$. For convenience, we often write $\Lambda_i^{\downarrow}$ as Λ_i too. But one can tell which one we mean from the context.

There is the natural translation action of $\mathbb{R}^d$ on $\mathbb{E}$ given by

$$T_t : (t, (x, i)) \mapsto t + (x, i) := (t + x, i)\,.$$

Given $\Lambda \subset \mathbb{E}$, and $B \subset \mathbb{R}^d$, we define

$$\begin{aligned} B + \Lambda \quad &:= \textstyle\bigcup_{b\in B} T_b \Lambda &&\subset \mathbb{E} \\ B \cap \Lambda \quad &:= \{(x, i) \in \Lambda \,:\, x \in B, i \in \boldsymbol{m}\} &&\subset \mathbb{E}. \end{aligned} \tag{1.4.1}$$

Let $\mathcal{O} := \{(0, 1), \ldots, (0, m)\} \subset \mathbb{E}$. Then $C_R^{(m)} := C_R + \mathcal{O}$ is a 'rainbow' cube that consists of the union of the cubes (C_R, i), $i = 1, \ldots, m$. Its closure is $\overline{C_R^{(m)}}$.

Let $r > 0$. An m-multiset $\Lambda \subset \mathbb{E}$ is said to be r-**uniformly discrete** if for all $a \in \mathbb{R}^d$,

$$\operatorname{card}((a + C_r) \cap \Lambda) \le 1\,. \tag{1.4.2}$$

In particular this implies that points of distinct colours cannot coincide. The family of all the r-uniformly discrete subsets of $\mathbb{E}$ will be denoted by $\mathcal{D}_r^{(m)}$. In the case that $m = 1$, we simply write $\mathcal{D}_r^{(1)}$ as $\mathcal{D}_r$.

At the beginning of this section, we will give a general formula to compute the 2-point correlation measure of a multiset if the 2-point correlation measures of each component and the so-called 2-point inter-correlation measures of every pair of components are given. Treated as a typical example of multisets, the general

[8]This is not an entirely standard use of the word multiset

substitution Delone sets will be considered. We finish the section by an application of the general formula for the 2-point correlation measure of multisets of two random systems.

1.4.2 Diffraction measure of a general multiset

Let Λ be an m-multiset and $\Lambda = \dot{\cup}_{i=1}^{m} \Lambda_i$, where $\Lambda_i \subset \mathbb{R}^d \times \{i\}$. Let $w := (w_1, \ldots, w_m)$ be the vector of m real numbers denoting the scattering weights of $(\Lambda_1, \ldots, \Lambda_m)$ respectively. Correspondingly, **the weighted Dirac comb** of Λ is given as $\delta_\Lambda^w := \sum_{i=1}^{m} w_i \delta_{\Lambda_i}$.

Analogous to the definition of the 2-point correlation measure of a single colour point set, we define the 2-point weighted correlation measure of two point sets with distinct colours as follows.

Definition 1.4.1. *For given $i, j \in \mathbf{m}$, if the limit*

$$\lim_{R \to \infty} \frac{1}{l(C_R)} (\delta_{\Lambda_i \cap C_R} * \widetilde{\delta_{\Lambda_j \cap C_R}}) \tag{1.4.3}$$

exists in vague topology, then it is called the ***2-point correlation measure of*** *Λ_i* ***and*** *Λ_j. For convenience, we denote it by $\gamma_{\Lambda_i, \Lambda_j}$.*

When $i = j$, this definition turns out to be the **2-point correlation measure of a single colour point set.**

To compute the 2-point weighted correlation measure of the m-multiset Λ, we assume that for each pair of components $(\Lambda_i, \Lambda_j), i, j \in \boldsymbol{m}$, the 2-point correlation measure of Λ_i and Λ_j exists and define the **2-point weighted correlation measure** of Λ as

$$\gamma_\Lambda = \sum_{i,j=1}^{m} w_i w_j \gamma_{\Lambda_i, \Lambda_j}. \tag{1.4.4}$$

1.4.3 Substitution Delone multisets and their diffraction

A natural way in which multisets arise is on substitution point sets. We say that a linear map $Q : \mathbb{R}^d \to \mathbb{R}^d$ is **expansive** if there is a $c > 1$ with

$$|Qx - Qy| \geq c \cdot |x - y|,$$

for all $x, y \in \mathbb{R}^d$, see [31]. This is equivalent to saying that all eigenvalues of Q lie outside of the closed unit disk of $\mathbb{C}$.

Definition 1.4.2. $\Lambda = \dot{\bigcup}_{i=1}^{m} \Lambda_i$ *is called a substitution Delone multiset (SDM) if* Λ *is a Delone multiset and there exist an expansive map* $Q : \mathbb{R}^d \to \mathbb{R}^d$ *and finite sets* D_{ij} *for* $i, j \leq m$ *such that*

$$\Lambda_i^{\downarrow} = \bigcup_{j=1}^{m} (Q\Lambda_j^{\downarrow} + D_{ij}), i = 1, \ldots, m, \tag{1.4.5}$$

The unions on the right-hand side are to be disjoint.

An $m \times m$ **matrix function system** (MFS) on a non-empty set Y in $\mathbb{R}^d$ is an $m \times m$ matrix $\Phi = (\Phi_{ij})$, where each Φ is a finite set (the empty set is allowed) of mappings from Y to Y, see [35]. Take $\Phi_{ij} := \{f : x \mapsto Qx + a, a \in D_{ij}\}$. We define $\Phi(\Lambda) := \dot{\cup}_{i=1}^{m} \Lambda_i'$, where $\Lambda_i' \subset \mathbb{R}^d \times \{i\}$ and $\Lambda_i'^{\downarrow} := \bigcup_{j=1}^{m} \Phi_{ij} \Lambda_j^{\downarrow}$. With this terminology, Λ of Definition 1.4.2 can be defined as a fixed point of the MFS Φ, i.e., $\Lambda = \Phi(\Lambda)$.

Define a square matrix $B := (b_{ij})_{m \times m}$ by $b_{ij} := \text{card}(D_{ij})$. If for each $i = 1, \ldots, m$, $\sum_{j=1}^{m} D_{ij}$ is the same integer, then we say that Λ is a **constant length substitution** point set. B is called **the substitution matrix** of Λ. If there exists a positive integer k such that all entries of the kth power of the matrix B are positive, then we say that B is **primitive**.

We assume that the substitution matrix B is primitive. By the Perron-Frobenius theorem [22], B has the following properties:

1. one of its eigenvalues is positive and greater than (in absolute value) all other eigenvalues;

2. there is a positive left eigenvector and also a positive right eigenvector corresponding to that eigenvalue;

3. that eigenvalue is a simple root of the characteristic equation of B.

Such an eigenvalue is called the **dominant (or Perron-Frobenius) eigenvalue** of the matrix M. The following lemma (Theorem 2.3 of [31]) shows a geometrical meaning of the dominant eigenvalue.

Proposition 1.4.3. *If* Λ *is a primitive SDM with expansive map* Q*, then the dominant eigenvalue of its substitution matrix* B *is equal to* $|det(Q)|$.

Generally, it is a challenge to compute the diffraction measure of a SDM. We do know that for a representable SDM, i.e., one that can be represented by a tiling on $\mathbb{R}^d$, it is pure point diffractive only if the dominant eigenvalue of its substitution matrix is a Pisot number, see [20].

In the remainder of this section, we will introduce substitution tilings and discuss under what conditions, a SDM is representable. Since it is not important for what follows, the reader can safely skip this part and move to the next section directly.

A tiling on $\mathbb{R}^d$ is a set of tiles covering the whole space $\mathbb{R}^d$ but overlapping only on the edges. There are many ways to define a tiling for a given Delone point set. Since we are considering a multiset, we focus on those tilings $\mathbb{R}^d$ consisting of a finite number of types of tiles. Let Γ be such a tiling. A tile τ of Γ is often specified by a pair (O, i) where $O = \mathrm{supp}(\tau)$ is a compact set in $\mathbb{R}^d$ and i denotes the tile type. A finite set P of tiles is called a **patch** if the tiles of P have mutually disjoint interiors.

Analogous to substitution of a Delone multiset, we define a tile-substitution as follows:

Definition 1.4.4. *Let $A = \{\tau_1, \ldots, \tau_m\}$ be a finite set of tiles, $\tau_i = (O_i, i)$. We denote by $\mathcal{P}_A$ the set of all patches composed of translated copies of the τ_i's. We say that $\Phi : A \longrightarrow \mathcal{P}_A$ is a tile-substitution with expansive mapping Q if there exist finite sets $D_{ij} \subset \mathbb{R}^d$ for $1 \leq i, j \leq m$, such that*

$$\Phi(\tau_j) = \{u + \tau_i : u \in D_{ij}, i = 1, \ldots, m\}$$

with

$$Q(O_j) = \bigcup_1^m (D_{ij} + O_i) \text{ for } 1 \leq j \leq m.$$

A substitution tiling on $\mathbb{R}^d$ is an invariant of a tile-substitution.

Definition 1.4.5. *A Delone multiset $\Lambda = \bigcup_1^m \Lambda_i$ is called representable (by tiles) if there exists $\tau_i = (O_i, i), 1 \leq i \leq m$, so that $\{x + \tau_i : x \in \Lambda_i, i = 1, \ldots, m\}$ is a tiling of $\mathbb{R}^d$.*

In general, an SDM is not representable. The following proposition provides a sufficient condition to make Λ representable.

Proposition 1.4.6. *([35], Theorem 3.7) Let Λ be a primitive SDM such that each cluster of it is a translate of a subcluster of $\Phi^k(x_j)$ for some $x_j \in \Lambda_j, j = 1, \ldots, m$, and some positive integer k (such clusters are called **legal**). Then Λ is representable.*

1.4.4 The Bernoulli system

The Bernoulli system is a random system on $\mathbb{Z}$ which can be specified by a family of independently and identically distributed random variables $\{t_i\}_{\mathbb{Z}}$ that can take any of the m distinct real numbers $\{w_1, \ldots, w_m\}$, with attached probabilities $\{p_1, \ldots, p_m\}, p_i > 0$ such that $\sum_1^m p_i = 1$. It can also be regarded as a system of m-multiset with the vector of weights $w = \{w_1, \ldots, w_m\}$. Let Λ be a typical outcome of this random system. Then it has the form that $\Lambda = \dot{\cup}\Lambda_i, \Lambda_i \subset \mathbb{Z} \times \{i\}$ where the flattening of each component $\Lambda_i^{\downarrow}$ is the set of index numbers of those random variables which take the real number w_i, and the flattening of Λ is equal to $\mathbb{Z}$. Moreover, for every $k \in \mathbb{Z}$, we have that

$$\mathbb{P}\{k \in \Lambda_i^{\downarrow}\} = p_i.$$

Obviously, every $\Lambda_i^{\downarrow}$ has FLC. So the 2-point weighted correlation measure of Λ_i is of the form

$$\gamma_{\Lambda_i} := \sum_{n \in \mathbb{Z}} \eta_i(n)\delta_n$$

with 2-point correlation coefficients

$$\begin{aligned}
\eta_i(-n) &= \lim_{k\to\infty} \frac{|w_i|^2}{2k} \sum_{x \in [-k,k]} \mathbf{1}_{\Lambda_i}(x)\mathbf{1}_{\Lambda_i}(x-n) \\
&= \begin{cases} |w_i|^2\mathbb{P}(\{n \in \Lambda_i\}), & \text{if } n = 0, \\ |w_i|^2\mathbb{P}(\{n \in \Lambda_i\})^2, & \text{if } n \neq 0. \end{cases} \\
&= \begin{cases} p_i|w_i|^2, & \text{if } n = 0, \\ p_i^2|w_i|^2, & \text{if } n \neq 0. \end{cases}
\end{aligned}$$

The limit exists because of the law of the large numbers. As a consequence, γ_{Λ_i} exists with probabilistic certainty. Similarly, we can prove that for $i \neq j$, the weighted correlation measure $\gamma_{\Lambda_i,\Lambda_j}$ exists with probabilistic certainty and has the form

$$\gamma_{\Lambda_i,\Lambda_j} := \sum_{n \in \mathbb{Z}} \eta_{i,j}(n)\delta_n,$$

with 2-point correlation coefficients

$$\eta_{i,j}(-n) = \begin{cases} 0, & \text{if } n = 0, \\ p_ip_j|w_i||w_j|, & \text{if } n \neq 0. \end{cases}$$

By (1.4.4),

$$\begin{aligned}
\gamma_\Lambda &= \sum_{i=1}^m p_i^2|w_i|^2\delta_{\mathbb{Z}\setminus\{0\}} + \sum_{i=1}^m p_i|w_i|^2\delta_0 + \sum_{i,j=1,i\neq j}^m p_ip_j|w_i||w_j|\delta_{\mathbb{Z}\setminus\{0\}}. \\
&= \left(\langle|w|\rangle^2 \sum_{n\in\mathbb{Z}} \delta_n\right) + (\langle|w|^2\rangle - \langle|w|\rangle^2)\delta_0.
\end{aligned}$$

Note that $\hat{\delta}_0 = l$ and $\hat{\delta}_{\mathbb{Z}} = \delta_{\mathbb{Z}}$ by the Poisson summation formula for lattices. Putting all this together, we have the following proposition:

Proposition 1.4.7. *([4]) The 2-point correlation γ_Λ exists with probabilistic certainty and has the form $\gamma_\Lambda = \sum_{n\in\mathbb{Z}} \eta(n)\delta_n$ with 2-point correlation coefficients*

$$\eta(n) = \begin{cases} \langle |w|^2 \rangle, & \text{if } n = 0, \\ \langle |w| \rangle^2, & \text{if } n \neq 0, \end{cases} \tag{1.4.6}$$

where $\langle |w|^2 \rangle = \sum_{i=1}^m p_i |w_i|^2$ and $\langle |w| \rangle^2 = (\sum_{i=1}^m p_i |w_i|)^2$. Consequently, the diffraction measure is, with probability one, $\mathbb{Z}$-periodic and given by

$$\hat{\gamma}_\Lambda = \left(\langle |w| \rangle^2 \sum_{n\in\mathbb{Z}} \delta_n \right) + \left(\langle |w|^2 \rangle - \langle |w| \rangle^2 \right) l. \tag{1.4.7}$$

□

Thus the diffraction of the Bernoulli system is a mix of a pure point part with Bragg peaks of intensities $\langle |w| \rangle^2$ at the points of $\mathbb{Z}$ and an absolutely continuous part of weight $(\langle |w|^2 \rangle - \langle |w| \rangle^2)$.

Remark 1.4.8. *We will see in the future that for all the m-multisets which are realizations of the Bernoulli system, the 2-point correlation measures exist with probabilistic certainty. In fact, the Bernoulli system is an ergodic point process and we will prove in Chapter 3 that for point sets which are outcomes of a given ergodic point process with a probability distribution μ, the 2-point correlation measure exists μ-almost surely.*

1.4.5 A random system on a model set

We are going to define a random system on a model multiset analogous to the Bernoulli system. First, let $\Lambda_0 = \dot{\cup}_{i=1}^m \bar{\Lambda}_i$ be an m-multiset such that the flattening of every component $\bar{\Lambda}_i$ is a regular model set resulting from one cut and project scheme $(\mathbb{R}^d \times H, \mathcal{L})$ with a distinct window set[9]. Then let p be a vector of m positive numbers, $p = (p_1, \ldots, p_m)$ such that $0 \leq p_i \leq 1, \sum_{i=1}^m p_i = 1$. Finally, we define a random system of point sets $\{\tilde{\Lambda} = \dot{\cup}_1^m \tilde{\Lambda}_i : \tilde{\Lambda}_i \subset \bar{\Lambda}_i\}$ by insisting that every m-multiset $\tilde{\Lambda}$ is an outcome of a random process for which

$$\mathbb{P}(\{x \in \tilde{\Lambda}_i^{\downarrow}\}) = p_i, \text{ if } \quad x \in \bar{\Lambda}_i, \tag{1.4.8}$$

[9]The overline symbols here are just denotations to distinguish different related sets.

for $i = 1, \ldots, m$. We call this system **a random system on the model set Λ_0**.

Let Λ be a typical point set of the random process on $\bar{\Lambda}$ with the probability distribution vector p and $\Lambda = \dot{\cup}_{i=1}^{m} \Lambda_i$, where $\Lambda_i \subset \bar{\Lambda}_i$. Evidently, $(\Lambda^{\downarrow} - \Lambda^{\downarrow}) \subset (\bar{\Lambda}^{\downarrow} - \bar{\Lambda}^{\downarrow})$, so $\Lambda^{\downarrow}$ and all the flattenings of its components $\Lambda_i^{\downarrow}$ have FLC. Thus, the 2-point correlation measures of Λ and its components exist with probabilistic certainty. For simplicity, here we choose the vector of the weights to be $(1, \ldots, 1)$ (unweighted).

As usual, let γ_{Λ_i} be the 2-point correlation measure of the model set Λ_i and let $\gamma_{\bar{\Lambda}_i}$ be the 2-point correlation measure of the model set $\bar{\Lambda}_i$ for $i = 1, \ldots, m$. The 2-point correlation measures of γ_{Λ_i} can be expressed by the 2-point correlation measure of $\gamma_{\bar{\Lambda}_i}$ as follows.

$$\gamma_{\Lambda_i} = p_i^2 \gamma_{\bar{\Lambda}_i} + (p_i - p_i^2)\delta_0. \tag{1.4.9}$$

Therefore,

$$\hat{\gamma}_{\Lambda_i} = p_i^2 \hat{\gamma}_{\bar{\Lambda}_i} + (p_i - p_i^2) l, \tag{1.4.10}$$

This says that the diffraction measure of each colour component contains a pure point part plus a multiple of Lebesgue measure.

To compute the 2-point correlation of Λ, we need to consider the 2-point correlation measures $\gamma_{\Lambda_i, \Lambda_j}$, for $i, j \in \{1, \ldots, m\}, i \neq j$.

$$\gamma_{\Lambda} = \sum_{i,j=1}^{m} \gamma_{\Lambda_i, \Lambda_j} = \sum_{i=1}^{m} \gamma_{\Lambda_i} + \sum_{1 \leq i < j \leq m} 2\gamma_{\Lambda_i, \Lambda_j}. \tag{1.4.11}$$

By the law of the large numbers, we have that

$$\gamma_{\Lambda_i, \Lambda_j} = p_i p_j \gamma_{\bar{\Lambda}_i, \bar{\Lambda}_j} \quad \text{a.s.} \tag{1.4.12}$$

Since

$$\gamma_{(\Lambda_i \dot{\cup} \Lambda_j)} = \gamma_{\bar{\Lambda}_i} + \gamma_{\bar{\Lambda}_j} + 2\gamma_{\bar{\Lambda}_i, \bar{\Lambda}_j}, \tag{1.4.13}$$

so

$$\gamma_{\Lambda_i, \Lambda_j} = \frac{1}{2}\left(\gamma_{\bar{\Lambda}_i \dot{\cup} \bar{\Lambda}_j} - (\gamma_{\bar{\Lambda}_i} + \gamma_{\bar{\Lambda}_j})\right). \tag{1.4.14}$$

Plugging (1.4.14) into the formula (1.4.11),

$$\begin{aligned} \gamma_{\Lambda} &= \sum_{i=1}^{m} \left(p_i^2 \gamma_{\bar{\Lambda}_i} + (p_i - p_i^2)\delta_0\right) + \sum_{1 \leq i < j \leq m} p_i p_j \left(\gamma_{\bar{\Lambda}_i \dot{\cup} \bar{\Lambda}_j} - \gamma_{\bar{\Lambda}_i} - \gamma_{\bar{\Lambda}_j}\right) & (1.4.15) \\ &= \sum_{i=1}^{m} (p_i - \sum_{j \neq i} p_j) p_i \gamma_{\bar{\Lambda}_i} + \sum_{1 \leq i < j \leq m} p_i p_j \gamma_{\bar{\Lambda}_i \dot{\cup} \bar{\Lambda}_j} + \sum_{i=1}^{m} (p_i - p_i^2)\delta_0. & (1.4.16) \end{aligned}$$

Since the sets $\bar{\Lambda}_i \dot{\cup} \bar{\Lambda}_j, i \neq j$, are also regular model sets, all components in the above equation are positive definite. Therefore, the Fourier transformation γ_Λ exists and it is equal to

$$\hat{\gamma}_\Lambda = \sum_{i=1}^{m}(p_i - \sum_{j \neq i} p_j) p_i \hat{\gamma}_{\bar{\Lambda}_i} + \sum_{1 \leq i < j \leq m} p_i p_j \hat{\gamma}_{\bar{\Lambda}_i \dot{\cup} \Lambda_j} + \sum_{i=1}^{m}(p_i - p_i^2) l. \tag{1.4.17}$$

Note that the first two items on the right side of the above equation are pure point by Theorem 1.3.5. We conclude that the diffraction measure of Λ exists with probabilistic certainty and it consists of a pure point part plus a multiple of Lebesgue measure.

Proposition 1.4.9. *The 2-point correlation γ_Λ exists with probabilistic certainty and has the form (1.4.15). Consequently, the Fourier transformation γ_Λ exists with probabilistic certainty. It consists of a pure point part plus a multiple of Lebesgue measures and it is given by (1.4.17).*

1.4.6 Example: Revisiting the Fibonacci chain

We return to the Fibonacci symbolic sequence. Note that it can also be generated by a substitution ψ on sequences of the two symbols $\{a, b\}$ defined by

$$\psi(a) = ab, \quad \psi(b) = a.$$

In fact, the fixed point $\psi^\infty(a)$ can be approximated by a sequence of words $\{\psi^n(a)\}_1^\infty$ which satisfies an induction rule that for $n \in \mathbb{N}$,

$$\psi^n(a) = \psi^{n-1}(ab) = \psi^{n-1}(a)\psi^{n-1}(b) = \psi^{n-1}(a)\psi^{n-2}(a).$$

This is consistent with the induction rules $\beta_n = \beta_{n-1} + \beta_{n-2}$ introduced in Section 1.1.3.

It is known that ψ is related to a tile (interval)-substitution and correspondingly the symbolic Fibonacci sequence is related to a substitution tiling (of intervals) on the real line by replacing a, b by two intervals with appropriate lengths. Suppose that the two intervals we desire are $[0, x_1), [0, x_2), x_1, x_2 > 0$. We are going to find out the appropriate values for x_1 and x_2.

Let B be the substitution matrix of ψ

$$B = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

B is primitive because

$$B^2 = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix},$$

is a positive matrix. By Perron-Frobenius theorem, B has one dominant eigenvalue. By simple computation, we have that the dominant eigenvalue is τ and the positive left eigenvector corresponding to τ is $(\tau, 1)$.

On the real line, every linear expansive map Q of a tile-substitution is of the form $Qx = c_0 x, c_0 > 1$, where c_0 is called the expansive constant. Suppose the expansive constant of the tile-substitution is c. Then we have

$$c[0, x_1) = [0, x_1) \cup ([0, x_2) + x_1) = [0, x_1 + x_2) \text{ and } c[0, x_2) = [0, x_1).$$

Equivalently,

$$cx_1 = x_1 + x_2, \quad cx_2 = x_1.$$

This can be rewritten as

$$(x_1, x_2) \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = c(x_1, x_2).$$

So c is the dominant eigenvalue of the matrix B and the pair (x_1, x_2) is the left eigenvector of B. From above, we have $c = \tau$ and $(x_1, x_2) = (\tau, 1)$ is the unique solution up to scaling. If we start the tile-substitution with the interval $[0, \tau)$, then the entire set of left end points of the intervals in the resulting tiling on $\mathbb{R}^+$ gives exactly the one sided Fibonacci chain.

For convenience, we denote the one sided Fibonacci chain by Λ_0 here. We set out to show that Λ_0 is the right side of a model set on the real line.

First, we let Γ_1 denote the set of the left end points of the tiles marked by a and let Γ_2 denote the left end points of the tiles marked by b. Evidently, $\Lambda_0 = \Gamma_1 \cup \Gamma_2$.

We define a $\mathbb{Z}$-linear mapping ϕ on the module $\mathbb{Z}[\tau]$ by $\phi(m+n\tau) = m - \frac{n}{\tau}$, where $m, n \in \mathbb{Z}, \mathbb{Z}[\tau] := \mathbb{Z} + \mathbb{Z}\tau$. Since $(m + n\tau, \phi(m + n\tau)) = m(1, \phi(1)) + n(\tau, \phi(\tau))$, the set $\mathcal{L} := \{(x, \phi(x)) : x \in \mathbb{Z}[\tau]\} = \{(m+n\tau, m - \frac{n}{\tau}) : m, n \in \mathbb{Z}\}$ is a lattice in $\mathbb{R}^2$ with a basis $\{(1,1), (\tau, -\frac{1}{\tau})\}$. See the lattice in Figure 1.3. The reason for choosing ϕ as we have done is that $-\frac{1}{\tau}$ is the algebraic conjugate of τ (it also satisfies the equation $x^2 = x + 1$ and so ϕ: $\mathbb{Z}[\tau] \to \mathbb{Z}[\tau]$ is actually a ring homomorphism). Let π_1, π_2 be the two canonical projections of the lattice $\mathcal{L}$ and let the mapping $(\cdot)^*$ be the mapping ϕ. Then we get a cut and project scheme, which is shown in the following diagram.

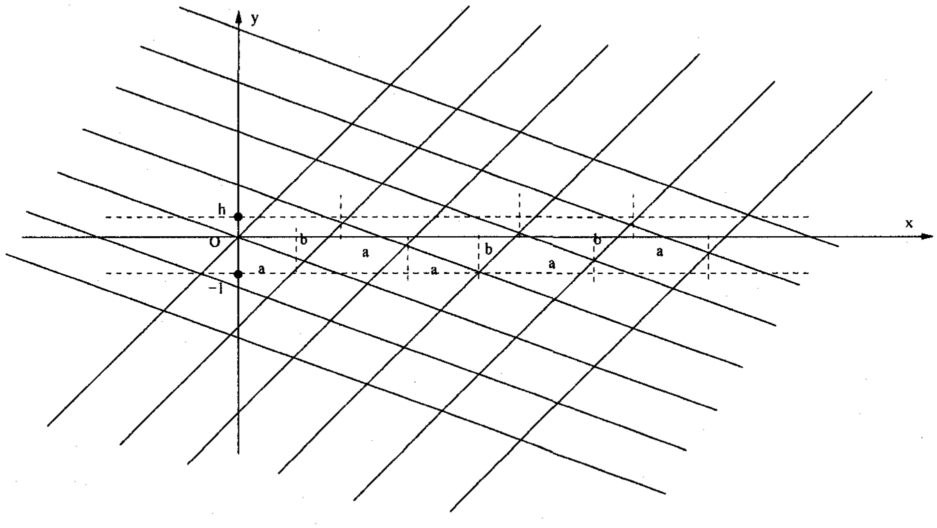

Figure 1.3: The lattice $\mathcal{L}$ spanned by $(1,1)$ and (τ,τ^*)

$$
\begin{array}{ccccc}
\mathbb{R} & \longleftarrow & \mathbb{R}\times\mathbb{R} & \longrightarrow & \mathbb{R} \\
\bigcup & & \bigcup & & \bigcup \\
\mathbb{Z}[\tau] & \xleftarrow{\pi_1} & \mathcal{L} & \xrightarrow{\pi_2} & \mathbb{Z}[\tau] \\
\bigcup & & & & \bigcup \\
\Lambda & & \xrightarrow{*} & & \Omega
\end{array}
\tag{1.4.18}
$$

We are going to show that the image of the one sided Fibonacci chain under the mapping $(\cdot)^*$ is dense in a relatively compact set in the internal space $\mathbb{R}$ and hence it can be embedded into a model set.

Next, observe that (Γ_1, Γ_2) satisfying the following equation.

$$\Gamma_1 = (\tau\cdot\Gamma_1)\cup(\tau\cdot\Gamma_2), \quad \Gamma_2 = \tau\cdot\Gamma_1 + \tau.$$

Since $(\cdot)^*$ is a ring homomorphism, (Γ_1^*, Γ_2^*) (the images of the two sets (Γ_1, Γ_2) under the $(\cdot)^*$ mapping) satisfying

$$\Gamma_1^* = (\frac{-1}{\tau}\cdot\Gamma_1^*)\cup(\frac{-1}{\tau}\cdot\Gamma_2^*), \quad \Gamma_2^* = \frac{-1}{\tau}\cdot\Gamma_1^* + \frac{-1}{\tau}. \tag{1.4.19}$$

Plugging Γ_2^* into the first equality, we obtain

$$\Gamma_1^* = (\frac{-1}{\tau}\cdot\Gamma_1^*)\cup(\frac{1}{\tau^2}\cdot\Gamma_1^* + \frac{1}{\tau^2}). \tag{1.4.20}$$

The right side of the equation defines a contractive mapping of a set. By Hutchinson's contractive mapping theorem [25] there exists a unique compact set, which is fixed under the contractive mapping. One can verify that the set $W_1 = [-\frac{1}{\tau^2}, \frac{1}{\tau}]$ is a solution of the equation (1.4.20). It implies that $\bar{\Gamma}_1 = W_1$. Putting $\bar{\Gamma}_1 = W_1$ back to the second equality in (1.4.19), we get that $\bar{\Gamma}_2 = [-1, -\frac{1}{\tau^2}]$.

Note that $\overline{\Gamma_2} \cup \overline{\Gamma_2} = [-1, \frac{1}{\tau}]$. We point out that the lattice $\mathcal{L}$ intersects with the two lines $y = -1$ and $y = \frac{1}{\tau}$, which means that adding or deleting one of the two boundary points $\{-1, \frac{1}{\tau}\}$ will change the corresponding model set by one point.

Here we consider a model set Λ_1 with a window $[-1, \frac{1}{\tau})$ first. It is a Delone set on $\mathbb{R}$ by theorem 1.3.4 and $0 \in \Lambda_1$. Define $\Lambda_1^+ := \{x \in \Lambda_1 : x \geq 0\}$. The following proposition says that Λ_1^+ is the same as Λ_0.

Proposition 1.4.10. $\Lambda_0 = \Lambda_1^+$

PROOF: $\{0, \tau\}$ generates Λ_0 under the substitution rule. But $0, \tau \in \Lambda_1^+$ and Λ_1^+ is closed under the substitution. Thus, $\Lambda_0 \subset \Lambda_1^+$.

Obviously, the minimum separation distance between points of Λ_0 are 1 respectively. If one inserts any other point into Λ_0, then the minimum separation distance of the resulting point set is less than 1. As a consequence, if we prove that the minimum separation distance among points of Λ_1 is 1, then Λ_1^+ should be a subset of Λ_0 and the proof is completed.

Suppose $p_1 = (x_1 + y_1\tau), p_2 = (x_2 + y_2\tau)$, $x_1, x_2, y_1, y_2 \in \mathbb{Z}$, are two points in Λ_1 such that

$$|p_1 - p_2| < 1. \tag{1.4.21}$$

By the assumption $p_1, p_2 \in \Lambda_1$,

$$-(1 + \frac{1}{\tau}) < (p_1 - p_2)^* < 1 + \frac{1}{\tau}. \tag{1.4.22}$$

Let $m = x_1 - x_2$ and $n = y_1 - y_2$. Without lost of generality, we can assume that $p_1 > p_2$. Then the two inequalities, (1.4.21) and (1.4.22) can be rewritten as

$$\begin{array}{rcl} 0 & < m + n\tau & < 1, \\ -(1 + \frac{1}{\tau}) & < m - \frac{n}{\tau} & < (1 + \frac{1}{\tau}). \end{array}$$

These are equivalent to

$$\begin{array}{rcl} -n\tau & < m & < 1 - n\tau, \\ -(1 + \frac{1}{\tau}) + \frac{n}{\tau} & < m & < \frac{n}{\tau} + (1 + \frac{1}{\tau}). \end{array}$$

We aim to show that the two inequalities have no integer solution, or equivalently, the intersection of two intervals $J_n := (-n\tau, 1 - n\tau) \cap (-(1 + \frac{1}{\tau}) + \frac{n}{\tau}, (1 + \frac{1}{\tau}) + \frac{n}{\tau})$ contains no integers for all $n \in \mathbb{Z}$. Clearly the intersection J_n may be nonempty only in the following two situations.

- In the case that
$$-n\tau < -(1 + \frac{1}{\tau}) + \frac{n}{\tau} < 1 - n\tau, \tag{1.4.23}$$
the inequality (1.4.23) gives $1 < n < \tau$. Since $(1, \tau)$ contains no integers, there is no J_n.

- In the case that
$$-(1 + \frac{1}{\tau}) + \frac{n}{\tau} < -n\tau < (1 + \frac{1}{\tau}) + \frac{n}{\tau}. \tag{1.4.24}$$
The inequality (1.4.24) follows that $-\frac{\tau+1}{\tau+2} < n < \frac{\tau+1}{\tau+2}$. Thus, $n = 0$. Plugging $n = 0$ into (1.4.23), we have $J_0 = (0, 1)$, which contains no integers.

Therefore, we conclude that the inequalities (1.4.23) has no integer solution. □

Remark 1.4.11. *One can also prove this in a geometric way. First, if we replace the two symbols $\{a, b\}$ by two vectors $\vec{a} = (\tau, \tau^*)$ and $\vec{b} = (1, 1)$, then we get a substitution $\bar{\psi}$ with $\bar{\psi}(\{\vec{a}\}) = \{\vec{a}\vec{b}\}, \bar{\psi}(\vec{b}) = \vec{a}$. As a consequence, the Fibonacci symbolic sequence corresponds to a fixed point of the new substitution $\bar{\psi}^\infty\{\vec{a}\}$. For convenience, we denote it by $\{\alpha'_n\}_1^\infty$. This sequence has a geometric meaning that the sequence $\{c_i := \sum_1^i \alpha'_j\}_1^\infty$ specifies a path on the lattice $\mathcal{L}$, which starts from the zero and goes rightwards. One can verify that the first coordinates $\{P_1(c_i)\}_1^\infty$ of this sequence are the same as the one sided Fibonacci chain, i.e., $\Lambda_0 = \{P_1(c_i)\}_1^\infty$. Therefore, $\{P_2(c_i)\}_1^\infty = \Lambda_0^* \subset [-1, \frac{1}{\tau})$. This says that this path winds through the strip bounded by $y = -1$ and $y = \frac{1}{\tau}$, see Figure 1.4. Moreover, from the picture, one can see that the sequence $\{c_i\}_1^\infty$ defines a rhomb chain $\{O_i\}_1^\infty$ such that each rhomb O_i is specified by the three vector (c_{i-1}, c_i, c_{i+1}).*

Meanwhile, this rhomb chain can also be generated as follows. We start from the first rhomb O_1. We denote it by O'_1. It can be specified by three vector (c_0, c_1, c_2). There are two rhombs on its right side. However, there is only one rhomb that has its edge entirely in the strip next to the point c_2. We denote this rhomb by O'_2. Repeating this infinitely, we obtain one and only one rhomb chain $\{O'_i\}_1^\infty$. Since we showed that $\{O_i\}_1^\infty$ is such a rhomb chain, so we have that $O_i = O'_i$, for $i = 1, 2, 3, \ldots$. We denote the sequence of vertexes specifying the rhomb chain $\{O'_i\}_1^\infty$ by $\{v_i\}_1^\infty$. Then, $v_i = c_i$, for $i = 1, 2, 3, \ldots$. The set, $\{0\} \cup \{P_1(v_i)\}_1^\infty$ provides all points of Λ_1^+. So $\Lambda_0 = \Lambda_1^+$. This is shown in Figure 1.5.

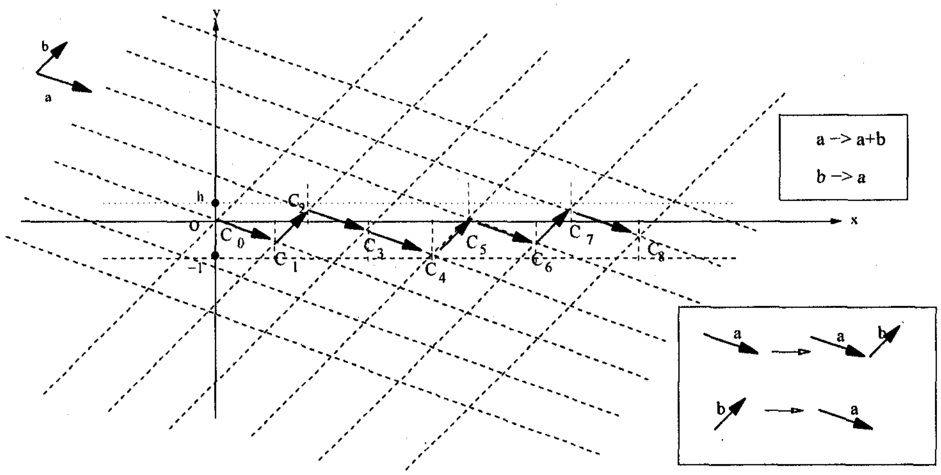

Figure 1.4: The new substitution and the path defined by $\{c_i\}_1^\infty$

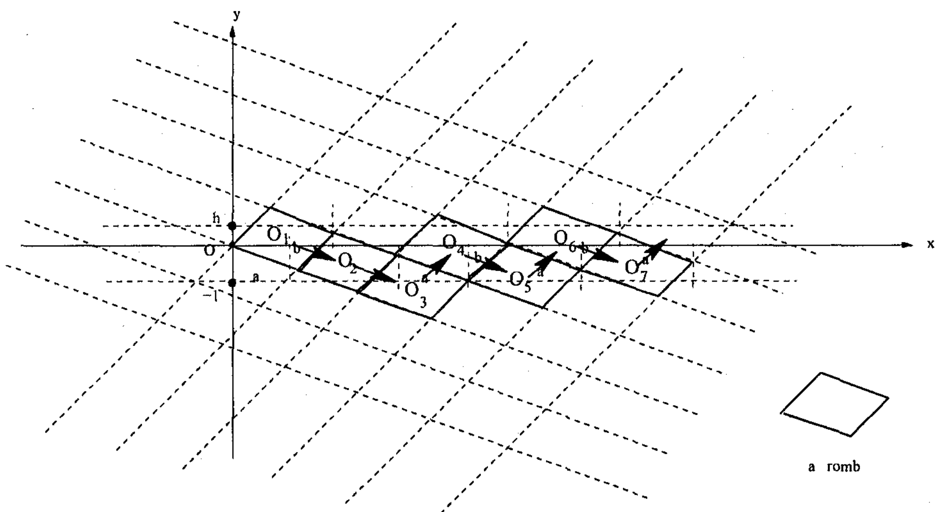

Figure 1.5: The rhomb chain $\{O_i\}_1^\infty$

Now we are going to show that the model set Λ_1, which is 2-sided, is LI to Λ_0. For that, we need the following lemma.

Lemma 1.4.12. *Λ_1 is repetitive.*

PROOF: Let P be an arbitrary cluster of Λ_1 and $P := \{0, p_1, \ldots, p_n\}$. Recall that $W := [-1, \frac{1}{\tau})$. Consider $W_P := W \cap (\cap_1^n(-p_i^* + W))$. Then W_P is also a half open and half closed interval. Thus, if W_P is nonempty, then it has nonempty interior (note that $0 \in W_P$), which implies that the model set Λ_P with the window W_P is a Delone set. Note that every point in Λ_P is a translation vector of the cluster P. Since Λ_P is Delone set, the set of translation vectors of the cluster P is relatively dense. It follows that Λ_1 is repetitive. □

Proposition 1.4.13. $[\Lambda_1] = [\Lambda_0]$.

PROOF: Recall that $\Lambda_0 \subset \Lambda_1$. Hence, $\Lambda_0 \subset [\Lambda_1]$. It suffices to show that $\Lambda_1 \in [\Lambda_0]$.

By the lemma 1.4.12, for each cluster of Λ_1, the translation number of this cluster in Λ_1 is relatively dense. This implies that this cluster appears in Λ_1^+ (equivalently in Λ_0) infinitely times, which implies that $\Lambda_1 \in [\Lambda_0]$. □

Remark 1.4.14. *Similarly we can prove that the model set Λ_2 with the window $(-1, \frac{1}{\tau}]$ which is LI to Λ_0 too. Since the two windows $(-1, \frac{1}{\tau}]$ and $(-1, \frac{1}{\tau}]$ are different only at the boundary points, the two model sets Λ_1 and Λ_2 are the same everywhere except two points $-1, -\tau$. Obviously, $-1 \in \Lambda_1$ but not in Λ_2, and $-\tau \in \Lambda_2$ but not in Λ_1.*

Consider the model set with windows $[-1, \frac{1}{\tau}]$ (respectively $(-1, \frac{1}{\tau})$). *We point out that it is not LI to Λ^+. In fact, it is not repetitive since the interval $[0, \tau - 1)$ ($[0, 1 + \tau)$) appear only once at $x = -\tau$ ($x = -1 - \tau$).*

By a suitable generalization of Theorem 1.3.2, the frequency of the occurrence of a cluster $P := \{0, p_1, \ldots, p_n\}$ in Λ_1 is equal to

$$\mathrm{freq}(P) = \frac{l(W \cap_1^n (p_i^* + W))}{l(W)},$$

where $p_i \in \Lambda_1, i = 1, \ldots, n$.

For instance, let P be the cluster corresponding to the word "bb". Then $P = \{0, 1, 2\}$ and $W \cap (-1 + W) \cap (-2 + W) = \emptyset$ because the total length of the window W is $1 + \frac{1}{\tau} < 2$. Therefore, $\mathrm{freq}(P) = 0$. When this happens, we say that the word "bb" is forbidden in the Fibonacci symbolic sequence. Similarly, one can verify that another word, "aaa", is forbidden also. In fact, there are an infinite number of forbidden words in the Fibonacci symbolic sequence.

Chapter 2

Point set dynamical systems

Introduction

Let $\Lambda \in \mathcal{D}_r^{(m)}$ be a given point set. In Chapter 1, we introduced the 2-point correlation measure and the diffraction measure of a given point set on $\mathbb{R}^d$. The main purpose of this chapter is to define a probability measure on $\mathcal{D}_r^{(m)}$ determined by Λ, which can serve as an invariant probability measure for a dynamical system of point sets in $\mathcal{D}_r^{(m)}$.

For this, we will introduce a topological space $\mathscr{M}_r^{(m)}$ of measures on $\mathbb{R}^d$ and show that it is homeomorphic to $\mathcal{D}_r^{(m)}$. Then we will think of a dynamical system as a point process, which can actually be defined by the one-to-one correspondence between $\mathcal{D}_r^{(m)}$ and $\mathscr{M}_r^{(m)}$. The law of a point process, which corresponds to the dynamical system measure on $\mathcal{D}_r^{(m)}$, has a nice property that it can be determined by a collection of values defined on a semi-ring of the σ-algebra of $\mathscr{M}_r^{(m)}$.

Such a collection of values may be given by the point set Λ. So the point set Λ may determine a probability measure on $\mathcal{D}_r^{(m)}$. Denote by μ_Λ the law determined by Λ and let X be the support of μ_Λ. Then the triple $(X, \mathbb{R}^d, \mu_\Lambda)$ defines a dynamical system on $\mathcal{D}_r^{(m)}$.

In particular, we prove that if the point set Λ is uniformly recurrent in $\mathcal{D}_r^{(m)}$, then the resulting point set dynamical system $(X, \mathbb{R}^d, \mu_\Lambda)$ is minimal.

Finally, we will introduce the Dworkin argument aiming at the relationship between the 2-point correlation measure of Λ and the measure μ_Λ.

2.1 Preliminaries

Let G be an abelian group. A dynamical system (X, G) consists of a set X and a group action of G on X. A subset $B \subset X$ is **invariant** if and only if $B = g^{-1}(B)$ for all $g \in G$.

If X is a compact topological space and G is a topological group acting continuously on X, then (X, G) is called a **topological dynamical system**. On the other hand, if X is a measure space and X is equipped with an G-invariant probability measure μ, i.e., for all measurable sets $B \subset X$ and all $g \in G$, $\mu(B) = \mu(g^{-1}(B))$, then the triple (X, G, μ) is called a **measure theoretical dynamical system.** In this thesis, we usually have both happening at the same time.

A topological dynamical system (X, G) is **minimal** if and only if for all $\xi \in X$, the orbit of ξ under the G-action is dense in X. Equivalently, the only closed invariant subsets of X are $\emptyset$ and X. The measure μ and the measure theoretical dynamical system (X, G, μ) are called **ergodic** if and only if every measurable invariant set $B \subset X$ satisfies $\mu(B) = 0$, or 1. A dynamical system (X, G) may admit many ergodic measures. If it admits only one ergodic measure, then (X, G, μ) is called a **uniquely ergodic dynamical system.**

In particular, when G is an abelian topological group (like $\mathbb{R}^d$), for a point set $x \in X$, the closed set $\overline{Gx}$ is called the orbit closure of x. Evidently, $\overline{Gx}$ is a G-invariant set. Thus, the couple $(\overline{Gx}, G)$ gives a sub-dynamical system of (X, G). We will show that for special kinds of points $x \in X$, these sub-dynamical systems are minimal.

In these definitions, we have used multiplication notation, which is standard for the general theory of dynamical systems. But in this thesis, G is always $(\mathbb{R}^d, +)$, so in the sequel these concepts occur in their additive notation.

Definition 2.1.1. *A subset S of an abelian topological group G is* ***relatively dense*** *if there exists a compact set $K \subset G$ such that $S + K = G$.*

Definition 2.1.2. *A point $x \in X$ is called* ***uniformly recurrent*** *if for any neighbourhood $x \in V$, the set $\{g \in G : g + x \in V\}$ is relatively dense.*

Theorem 2.1.3. *([19], Theorem 1.17) If x is a uniformly recurrent point of a dynamical system (X, G), then the orbit closure $\overline{Gx}$ is a minimal G-invariant closed subset of X and hence $(\overline{Gx}, G)$ is a minimal dynamical system.*

2.2 The topological measure space $\mathscr{M}_r^{(m)}$

Let $\mathbb{E} = \mathbb{R}^d \times \boldsymbol{m}$ be as in Section 1.4.1. A very convenient way to deal with countable point subsets Λ of $\mathbb{E}$ and families of them is to put them into the context of measures by replacing them by pure point measures, where the atoms correspond to the points of the set(s) in question. To this end we introduce the following objects on any locally compact space S :

- $\mathcal{S}$, the set of all Borel subsets of S;
- $\mathcal{B}(S)$, the set of all relatively compact Borel subsets of S;
- $BM_c(S)$, the space of all bounded measurable $\mathbb{C}$-valued functions of compact support on S;
- $C_c(S)$, the continuous $\mathbb{C}$-valued functions with compact support on S. If S is known to be compact, we can write $C(S)$ instead.

Following Karr [26] we let $\mathscr{M}$ denote the set of all **positive Radon measures** on $\mathbb{E}$, that is all positive regular Borel measures λ on $\mathbb{E}$ for which $\lambda(A) < \infty$ for all $A \in \mathcal{B}(\mathbb{E})$. Equivalently, we may view these measures as linear functionals on the space $C_c(\mathbb{E})$. We give $\mathscr{M}$ the vague topology. This is the topology for which a sequence $\{\lambda_n\} \in \mathscr{M}$ converges to $\lambda \in \mathscr{M}$ if and only if $\{\lambda_n(f)\} \to \lambda(f)$ for all $f \in C_c(\mathbb{E})$. This topology has a number of useful characterizations, some of which we give below.

Within $\mathscr{M}$ we have the subset $\mathscr{M}_p$ of **point measures** λ, those for which $\lambda(A) \in \mathbb{N}$ for all $A \in \mathcal{B}$. (Here $\mathbb{N}$ is the set of natural numbers, $\{0, 1, 2, \dots\}$.) These measures are always pure point measures in the sense that they are countable (possibly finite) sums of delta measures:

$$\lambda = \sum a_x \delta_x, \qquad x \in \mathbb{E}, a_x \in \mathbb{N}\,.$$

Within $\mathscr{M}_p$ we also have the set $\mathscr{M}_s$ of **simple point measures** λ, those satisfying $\lambda(\{x\}) \in \{0, 1\}$, which are thus of the form

$$\lambda = \sum_{x \in \Lambda} \delta_x,$$

where the support Λ is a countable subset of $\mathbb{E}$. Evidently for these measures, for $x \in \mathbb{E}$,

$$\lambda(\{x\}) > 0 \Leftrightarrow \lambda(\{x\}) = 1 \Leftrightarrow x \in \Lambda\,.$$

The Radon condition prevents the support of a point measure from having accumulation points in $\mathbb{E}$. Thus, the correspondence $\Lambda \longleftrightarrow \lambda$ provides a bijection between the closed discrete point sets of $\mathbb{E}$, i.e. the discrete point sets with no accumulation points, and $\mathscr{M}_s$. This is the connection between point sets and measures that we wish to use.[1] We note that the translation action of $\mathbb{R}^d$ on $\mathbb{E}$ produces an action of $\mathbb{R}^d$ on functions by $T_t f(x) = f(T_{-t}x)$, and on the spaces $\mathscr{M}$, $\mathscr{M}_p$, $\mathscr{M}_s$ of measures by $(T_t(\lambda))(A) = \lambda(-t + A), (T_t\lambda)(f) = \lambda(T_{-t}(f))$ for all $A \in \mathcal{B}(\mathbb{E})$, and for all measurable functions f on $\mathbb{E}$.

Here are some useful characterizations of the vague topology and some of its properties. These are cited in [26], Appendix A and appear with proofs in [11], Appendix A2.

Proposition 2.2.1. *(The vague topology)*

(i) For $\{\lambda_n\}, \lambda \in \mathscr{M}$ the following are equivalent:

(a) $\{\lambda_n(f)\} \to \lambda(f)$ for all $f \in C_c(\mathbb{E})$ (definition of vague convergence).

(b) $\{\lambda_n(f)\} \to \lambda(f)$ for all $f \in BM_c(\mathbb{E})$ for which the set of points of discontinuity of f has λ-measure 0.

(c) $\{\lambda_n(A)\} \to \lambda(A)$ for all $A \in \mathcal{B}(\mathbb{E})$ for which λ vanishes on the boundary of A, i.e. $\lambda(\partial A) = 0$.

(ii) In the vague topology, $\mathscr{M}$ is a complete separable metric space and $\mathscr{M}_p$ is a closed subspace.

(iii) A subspace L of $\mathscr{M}$ is relatively compact in the vague topology if and only if for all $A \in \mathcal{B}(\mathbb{E})$, $\{\lambda(A) : \lambda \in L\}$ is bounded, which again happens if and only if for all $f \in C_c(\mathbb{E})$, $\{\lambda(f) : \lambda \in L\}$ is bounded.

Note that $\mathscr{M}_s$ is not a closed subspace of $\mathscr{M}_p$: a sequence of measures in $\mathscr{M}_s$ can converge to point measure with multiplicities.

Proposition 2.2.2. *(The Borel sets of $\mathscr{M}$) The following σ-algebras are equal:*

(i) The σ-algebra $\mathscr{B}(\mathscr{M})$ of Borel sets of $\mathscr{M}$ under the vague topology.

[1] Note that the point sets that we are considering here are simple in the sense that the multiplicity of each point in the set is just 1. However, it is not precluded that the same point in $\mathbb{R}^d$ may occur more than once in such a point set, though necessarily it would have to occur with different colours. Very soon, however, we shall also preclude this.

(ii) *The σ-algebra generated by requiring that all the mappings $\lambda \mapsto \lambda(f)$, $f \in C_c(\mathbb{E})$ are measurable.*

(iii) *The σ-algebra generated by requiring that all the mappings $\lambda \mapsto \lambda(A)$, $A \in \mathcal{B}(\mathbb{E})$ are measurable.*

(iv) *The σ-algebra generated by requiring that all the mappings $\lambda \mapsto \lambda(f)$, $f \in BM_c(\mathbb{E})$ are measurable.*

A measure $\lambda \in \mathscr{M}$ is **translation bounded** if for all bounded sets $K \in \mathcal{B}(\mathbb{E})$, $\{\lambda(a+K) : a \in \mathbb{R}^d\}$ is bounded. In fact, a measure is translation bounded if this condition holds for a single set of the form $K = K_0 \times \boldsymbol{m}$ where $K_0 \subset \mathbb{R}^d$ has a non-empty interior. For such a K and for any positive constant n, we define the space $\mathscr{M}_p(K,n)$ of translation bounded measures $\lambda \in \mathscr{M}_p$ for which

$$\lambda(a+K) \le n$$

for all $a \in \mathbb{R}^d$. Evidently $\mathscr{M}_p(K,n)$ is closed if K is open, and by Prop. 2.2.1 it is relatively compact, hence compact. See also [6], where this is proved in a more general setting.

If $r > 0$ then $\mathscr{M}_p(C_r^{(m)},1)$ is the set of point measures λ whose support Λ is 1-uniformly discrete. In particular, the set $\mathscr{M}_s(C_r^{(m)},1) := \mathscr{M}_p(C_r^{(m)},1) \cap \mathscr{M}_s$ is closed in $\mathscr{M}_p(C_r^{(m)},1)$ and hence it is compact. For convenience, we simply rewrite $\mathscr{M}_s(C_r^{(m)},1)$ as $\mathscr{M}_r^{(m)}$ and in particular, we rewrite $\mathscr{M}_s(C_r^{(1)},1)$ as $\mathscr{M}_r$. By the above paragraph, $\mathscr{M}_r^{(m)}$ and $\mathscr{M}_r$ are compact.

Remark 2.2.3. *By Proposition 2.2.2 (iii), the σ-algebra $\mathscr{B}(\mathscr{M}_r)$ of Borel sets of $\mathscr{M}_r$ can also be generated by a family of subsets of the form*

$$\{\lambda \in \mathscr{M}_r : \lambda(A_i) = k_i, i = 1, \dots, n\}, \tag{2.2.1}$$

where A_i is a Borel set in $\mathbb{R}^d$ and k_i is a nonnegative integers, for $n \in \mathbb{N}$. Since the σ-algebra of Borel sets of $\mathbb{R}^d$ can be generated by the semi-open rectangles in $\mathbb{R}^d$, the Borel sets $\{A_i\}_1^n$ in the equation (2.2.1) can be restricted to these semi-open rectangles.

Similarly, the σ-algebra $\mathscr{B}(\mathscr{M}_r^{(m)})$ of Borel sets of $\mathscr{M}_r^{(m)}$ can be generated by a family of subsets of the form

$$\{\lambda \in \mathscr{M}_r^{(m)} : \lambda = \sum_{i=1}^{m} \lambda_i, \lambda_i(A_j^i) = k_j^i, i,j = 1, \dots, n\}, \tag{2.2.2}$$

where λ_i is a uniform discrete measure on the space $\mathbb{R}^d \times \{i\}$, A_j^i is a semi-open rectangle in $\mathbb{R}^d \times \{i\}$ and k_j^i is a nonnegative integer, for $n \in \mathbb{N}$.

If $\lambda \in \mathscr{M}_s$ is a translation bounded measure on $\mathbb{R}^d$ we shall often write expressions like $\sum_{x\in B} \lambda(\{x\})$ where B is some uncountable set (like $\mathbb{R}^d$ itself). Such sums only have a countable number of terms and so sum to a non-negative integer if B is bounded, or possibly to $+\infty$ otherwise.

2.3 The local topology on $\mathcal{D}_r^{(m)}$

Note that there is a one-to-one correspondence $\lambda \leftrightarrow \Lambda$ between measures in $\mathscr{M}_r^{(m)}$ and the point sets in $\mathcal{D}_r^{(m)}$. Thus, the point sets space $\mathcal{D}_r^{(m)}$ can be assigned a topology transferred by the correspondence. However, the most commonly used topology in the study of point set dynamical systems on $\mathcal{D}_r^{(m)}$ is the so-called local topology since it implies a notion of closeness that depends on the local configuration of points (as opposed to other topologies that depend only on the long-range average structure of the point set).

The local topology is most easily described as the uniform topology on $\mathcal{D}_r^{(m)}$ generated by the entourages

$$U(C_R, \epsilon) := \{(\Lambda', \Lambda'') \in \mathcal{D}_r^{(m)} \, : \, C_R \cap \Lambda' \subset C_\epsilon + \Lambda'', \quad C_R \cap \Lambda'' \subset C_\epsilon + \Lambda'\}\,, \tag{2.3.1}$$

where R, ϵ run over the positive real numbers.

Note that in (2.3.1), Λ and Λ' are subsets of $\mathbb{E}$. Intuitively two sets are close if on large cubes their points can be paired, taking colour into account, so that they are all within ϵ-cubes of each other. It is easy to see that $\mathcal{D}_r^{(m)}$ is closed in this topology.

Given any $\Lambda' \in \mathcal{D}_r^{(m)}$ we define the open set

$$U(C_R, \epsilon)[\Lambda'] := \{\Lambda'' \in \mathcal{D}_r^{(m)} \, : \, (\Lambda', \Lambda'') \in U(C_R, \epsilon)\}\,.$$

Proposition 2.3.1. (See also [6]) *Under the correspondence $\lambda \leftrightarrow \Lambda$ between measures in $\mathscr{M}_r^{(m)}$ and the point sets in $\mathcal{D}_r^{(m)}$, the vague and local topologies are the same. As a consequence, $\mathcal{D}_r^{(m)}$ is compact.*

Proof: Let $\{\lambda_n\}$ be a sequence of elements of X for which the corresponding sequence $\{\Lambda_n\} \subset \mathcal{D}_r^{(m)}$ converges in the local topology to some point set $\Lambda \in \mathcal{D}_r^{(m)}$. Choose any positive function $f \in C_c(\mathbb{E})$ and suppose that its support is in C_R, and choose any $\epsilon > 0$. Let $N_0 := 1 + \sup_{n\in\mathbb{N}} \lambda_n(C_R)$ and find $\eta > 0$ so that $\eta < r$ and for all $x, y \in C_R$,

$$|x - y| < \eta \Longrightarrow |f(x) - f(y)| < \epsilon/N_0\,.$$

Let $\{x_1,\dots,x_N\} = C_R \cap \Lambda \subset \mathbb{E}$. Then for all large n, $C_R \cap \Lambda_n \subset \{C_\eta + x_1, \dots C_\eta + x_N\}$ with exactly one point in each of these cubes. Then

$$|\lambda_n(f) - \lambda(f)| = \left| \sum_{y \in C_R \cap \Lambda_n} f(y) - \sum_{x \in C_R \cap \Lambda} f(x) \right| \leq N_0 \epsilon / N_0 = \epsilon \,.$$

Thus $\{\lambda_n(f)\} \to \lambda(f)$, and since $f \in C_c(\mathbb{E})$ was arbitrary, $\{\lambda_n\} \to \lambda$ in X.

Now, going the other way, suppose that $\{\lambda_n\} \to \lambda$ in X. Let $R > 0$ and let $C_R \cap \Lambda = \{x_1, \dots, x_N\}$. Choose any $0 < \epsilon < r$, small enough that for all $i \leq N$, $C_\epsilon + x_i \subset C_R^{(m)}$, and let

$$f_\epsilon := \sum_{i=1}^{N} \mathbf{1}_{C_\epsilon + x_i} \,.$$

Then $\{\lambda_n(f_\epsilon)\} \to \lambda(f_\epsilon) = N = \lambda(C_R) \leftarrow \{\lambda_n(C_R)\}$, so for all $n >> 0$, $\lambda_n(f_\epsilon) = N = \lambda_n(C_R)$ (see Prop. 2.2.1). Since each cube $C_\epsilon + x_i$ can contain at most one point of any element of $\mathcal{D}_r^{(m)}$, then for all $n >> 0$, and for all $i \leq N$, there is a $y_i^{(n)} \in (C_\epsilon + x_i) \cap \Lambda_n$. This accounts for all the points of $C_R \cap \Lambda_n$. Thus $\Lambda_n \in U(C_R, \epsilon)[\Lambda]$. This proves that $\{\Lambda_n\} \to \Lambda$. □

2.4 A probability measure on $\mathcal{D}_r^{(m)}$

For simplicity, we consider the case $m = 1$ first. Let Λ be a given point set in $\mathcal{D}_r$ and λ be the Dirac comb of Λ, i.e., $\lambda = \delta_\Lambda$. Denote by $\mathcal{R}$ the set of all semi-open d-dimensional rectangles, i.e., the sets of the form $I_1 \times \cdots \times I_d \subset \mathbb{R}^d$, where $I_1, \dots, I_d$ are of the form $[a, b)$, or $(a, b] \subset \mathbb{R}$.

By Remark 2.2.3, the σ-algebra $\mathscr{B}(\mathscr{M}_r)$ of Borel sets of $\mathscr{M}_r$ can be generated by a family of subsets of the form

$$\{\Lambda' \in \mathcal{D}_r : \lambda'(A_i) = k_i, i = 1, \dots, n, \text{ where } \lambda' \text{ is the Dirac comb of } \Lambda'\}, \qquad (2.4.1)$$

where $A_i \in \mathcal{R}, k_i \in \mathbb{Z}^+$, for $i = 1, \dots, n$ and $n \in \mathbb{N}$. For convenience, we rewrite the subset given by (2.4.1) as $X_{(A_1,\dots,A_n;k_1,\dots,k_n)}$ and call $(A_1, \dots, A_n; k_1, \dots, k_n)$ a pattern of $\mathcal{D}_r$[2].

Definition 2.4.1. *For a pattern $(A_1, \dots, A_n; k_1, \dots, k_n)$ of $\mathcal{D}_r$, its* ***frequency*** *on the point set Λ is defined by*

$$\rho_\Lambda(A_1, \dots, A_n; k_1, \dots, k_n) := \lim_{R \to \infty} \frac{1}{l(C_R)} l\left(\{t \in C_R : \lambda(t + A_j) = k_j, j = 1, \dots, n\}\right), \qquad (2.4.2)$$

[2]The patterns of Λ defined in Section 3.4 are special types of patterns of $\mathcal{D}_r$

if the limit exists[3].

Evidently, this is equivalent to

$$\rho_\Lambda(A_1,\ldots,A_n;k_1,\ldots,k_n) := \lim_{R\to\infty}\frac{1}{l(C_R)} l\left(\{t\in C_R : t+\Lambda \in X_{(A_1,\ldots,A_n;k_1,\ldots,k_n)}\}\right), \tag{2.4.3}$$

Applying the standard "$\epsilon - R$" argument, we can prove the following proposition.

Proposition 2.4.2. *Let $\Lambda_1,\Lambda_2 \in \mathcal{D}_r$ be two arbitrary point sets such that the set $(\Lambda_1\triangle\Lambda_2)$ has density 0, where $\triangle$ is the symmetric difference. Suppose that for a pattern $(A_1,\ldots,A_n;k_1,\ldots,k_n)$ of $\mathcal{D}_r$, $\rho_{\Lambda_1}(A_1,\ldots,A_n;k_1,\ldots,k_n)$ exists. Then we have that $\rho_{\Lambda_2}(A_1,\ldots,A_n;k_1,\ldots,k_n)$ exists also and it is equal to $\rho_{\Lambda_1}(A_1,\ldots,A_n;k_1,\ldots,k_n)$.*

Example 2.4.3. *Let Λ be the point set $\mathbb{Z}$ and let Λ' be the point set $\Lambda\setminus\{0\}$. By Proposition 2.4.2, for all patterns $(A_1,\ldots,A_n;k_1,\ldots,k_n)$ of $\mathcal{D}_r$,*

$$\rho_\Lambda(A_1,\ldots,A_n;k_1,\ldots,k_n) = \rho_{\Lambda'}(A_1,\ldots,A_n;k_1,\ldots,k_n). \tag{2.4.4}$$

Especially,

$$\rho_{\Lambda'}(-1+(-\frac{1}{2},\frac{1}{2}),(-\frac{1}{2},\frac{1}{2}),1+(-\frac{1}{2},\frac{1}{2});1,0,1) = 0,$$

contrasting to

$$\rho_{\Lambda'}(-1+(-\frac{1}{2},\frac{1}{2}),(-\frac{1}{2},\frac{1}{2}),1+(-\frac{1}{2},\frac{1}{2});1,1,1) = 1.$$

We assume that the point set $\Lambda\in\mathcal{D}_r$ satisfies the following assumption:

FI: The frequency of each pattern $(A_1,\ldots,A_n;k_1,\ldots,k_n)$ exists on Λ.

Let $p(A_1,\ldots,A_n;k_1,\ldots,k_n)$ be a value for a pattern $(A_1,\ldots,A_n;k_1,\ldots,k_n)$ of $\mathcal{D}_r$. There is a fundamental question: are there some conditions on p can guarantee that it uniquely determines a probability measure μ on $(\mathscr{M}_r,\mathscr{B}(\mathscr{M}_r))$ satisfying

$$\mu(X_{A_1,\ldots,A_n;k_1,\ldots,k_n}) = p(A_1,\ldots,A_n;k_1,\ldots,k_n)? \tag{2.4.5}$$

This question is answered by the following proposition.

Proposition 2.4.4. *([37]) p uniquely determines a probability measure μ on $(\mathscr{M}_r,\mathscr{B}(\mathscr{M}_r))$ satisfying (2.4.5) if and only if the following six conditions are fulfilled:*

[3]Note that the special averaging sequence $\{C_R : R>0\}$ is used again and the convergence or the limiting value of the two averages may not be the same if it is replaced by a different averaging sequence.

1. $p(A_1, \ldots, A_n; k_1, \ldots, k_n) \geq 0$;

2. $p(A_1, \ldots, A_n; k_1, \ldots, k_n) = p(A_{i_1}, \ldots, A_{i_n}; k_{i_1}, \ldots, k_{i_n})$, *for every permutation* $(i_1, \ldots, i_n)$ *of numbers* $1, \cdots, n$;

3. $\sum_{k=0}^{\infty} p(A, k) = 1$;

4. $\sum_{k_1=0}^{\infty} p(A_1, \ldots, A_n; k_1, \ldots, k_n) = p(A_2, \ldots, A_n; k_2, \ldots, k_n)$, *for* $n \in \mathbb{N}, n > 1$;

5. $p(A_1, \ldots, A_n, B; k_1, \ldots, k_n, s)$

$$= \sum_{s_1+\cdots+s_j=s} p(A_1, \ldots, A_n, B_1, \ldots, B_j; k_1, \ldots, k_n, s_1, \ldots, s_j), \tag{2.4.6}$$

   where $A_1, \ldots, A_n, B_1, \cdots, B_j \in \mathcal{R}$ *are pairwise disjoint and* $B = \cup_{i=1}^{j} B_i \in \mathcal{R}$, $j, s \in \mathbb{Z}^+$;

6. $p(A_1^{(j)}, \ldots, A_{n(j)}^{(j)}; 0, \ldots, 0) \to 1$, *as* $j \to \infty$ *for every finite sequence* $(A_1^{(j)}, \ldots, A_{n(j)}^{(j)})$, $A_i^{(j)} \in \mathcal{R}$, *with* $A^{(j)} = \bigcup_{i=1}^{n(j)} A_i^{(j)} \searrow \emptyset$.

Theorem 2.4.5. *The function* ρ_Λ *in equation (2.4.2) uniquely determines a probability measure*[4] μ_λ.

PROOF: We show that the value ρ_Λ satisfies the six conditions in the Proposition 2.4.4.

Obviously, the function ρ_Λ satisfies the first two conditions. The third condition is satisfied by ρ_Λ because

$$\begin{aligned}
\sum_{k=0}^{\infty} \rho_\Lambda(A, k) &= \sum_{k=0}^{\infty} \lim_{R\to\infty} \frac{1}{l(C_R)} l\left(\{t \in C_R : \lambda(t + A) = k\}\right) \\
&= \lim_{R\to\infty} \frac{1}{l(C_R)} l\left(\cup_{k=0}^{\infty} \{t \in C_R : \lambda(t + A) = k\}\right) \\
&= \lim_{R\to\infty} \frac{1}{l(C_R)} l\left(\{t \in C_R\}\right) \\
&= 1.
\end{aligned}$$

Note that we switch the order of the limit and the infinite sum in the second equation. This can be justified by the fact that $0 \leq \frac{1}{l(C_R)} l\left(\{t \in C_R : \lambda(t + A) = k\}\right) \leq 1$, $0 \leq \sum_{k=0}^{\infty} \frac{1}{l(C_R)} l\left(\{t \in C_R : \lambda(t + A) = k\}\right) = \frac{1}{l(C_R)} l\left(\cup_{k=0}^{\infty} \{t \in C_R : \lambda(t + A) = k\}\right) \leq 1$, and the monotone convergence theorem.

[4]We use this notation to make it clear that the probability measure depends on λ.

Similarly, we can prove that conditions (4) and (5) are fulfilled by ρ_Λ. Especially, equation (2.4.6) follows from the combinatorial identity

$$\{t \in C_R : \lambda(t+B) = s\} = \bigcup_{s_1+\cdots+s_m=s} \{t \in C_R : \lambda(t+B_j) = s_j, j = 1, \ldots, m\},$$

where the union is disjoint.

Now we prove that ρ_Λ satisfies condition (6). First of all, the assumption $A^{(j)} \searrow \emptyset$ means that: (i) $A^{(j+1)} \subset A^{(j)}, j = 1, 2, \ldots$; (ii) $\cap_{j=1}^{\infty} A^{(j)} = \emptyset$. It is a simple property of measures that $l(A^{(j)}) \to l(\bigcap_{j=1}^{\infty} A^{(j)}) = l(\emptyset) = 0$. Without lost generality, we assume that $A^{(1)}$ is bounded and hence it has a finite partition that $A^{(1)} = \bigcup_{k=1}^{K} C'_k$ C'_k is a translated copy of some semi-open cube C_s, where $s \le r$ (and s depends on k).

Next, we point out that $\bigcap_{i=1}^{n(j)}\{t \in C_R : \lambda(t + A_i^{(j)}) = 0\} = \{t \in C_R : \lambda(t + A^{(j)}) = 0\}$. As a consequence, $\rho_\Lambda(A_1^{(j)}, \ldots, A_{n(j)}^{(j)}; 0, \ldots, 0) = \rho_\Lambda(A^j; 0)$. Hence, it is equivalent to show that $\rho_\Lambda(A^{(j)}; 0) \to 1$.

For that, we consider a refined partition of $A^{(j)}$ such as $A^{(j)} = \bigcup_{i=1}^{n(j)} \bigcup_{k=1}^{K} A_{i_k}^{(j)}$, where $A_{i_k}^{(j)} = A_i^{(j)} \cap C'_k$. Because every $A_{i_k}^{(j)}$ is embedded in a translated copy of C_r, we have

$$C_R \setminus \{t \in C_R : \lambda(t + A_i^{(j)}) = 0\} = \{t \in C_R : \lambda(t + A_i^{(j)}) = 1\}.$$

Because $\Lambda \cap (a + C'_r) \le 1$ for $a \in \mathbb{R}^d$,

$$\begin{aligned} l(\{t \in C_R : \lambda(t + A_i^{(j)}) = 1\}) &\le \operatorname{card}(t \in C_R \cap \Lambda) \cdot l(A_i^{(j)}) \\ &\le c \cdot l(C_R) \cdot l(A_i^{(j)}), \end{aligned} \tag{2.4.7}$$

where $c = (\frac{1}{r} + 1)^d$. As a consequence,

$$\begin{aligned} C_R \setminus \{t \in C_R : \lambda(t + A^{(j)}) = 0\} &= C_R \setminus \bigcap_{i=1}^{n(j)} \{t \in C_R : \lambda(t + A_i^{(j)}) = 0\} \\ &= \bigcup_{i=1}^{n(j)} C_R \setminus \{t \in C_R : \lambda(t + A_i^{(j)}) = 0\} \\ &= \bigcup_{i=1}^{n(j)} \{t \in C_R : \lambda(t + A_i^{(j)}) = 1\}. \end{aligned}$$

Thus,

$$\begin{aligned}
l(C_R \setminus \{t \in C_R : \lambda(t + A^{(j)}) = 0\}) &\leq \sum_{i=1}^{n(j)} l(\{t \in C_R : \lambda(t + A_i^{(j)}) = 1\}) \\
&\leq c \cdot l(C_R) \cdot \sum_{i=1}^{n(j)} l(A_i^{(j)}) \\
&\leq c \cdot l(C_R) l(A^{(j)}).
\end{aligned}$$

Finally, putting all this together, we have

$$\begin{aligned}
\rho_A(A(j); 0) &= \lim_{R\to\infty} \frac{l(\{t \in C_R : \lambda(t + A^{(j)}) = 0\})}{l(C_R)} \\
&\geq \lim_{R\to\infty} \frac{l(C_R) - c \cdot l(C_R) l(A^{(j)})}{l(C_R)} \\
&= 1 - c \cdot l(A^{(j)}).
\end{aligned}$$

So $\rho_A(A^{(j)}; 0) \to 1$ as $l(A^{(j)}) \to 0$. □

Proposition 2.4.6. *μ_λ is invariant with respect to the $\mathbb{R}^d$-action.*

PROOF: For an arbitrary n semi-open rectangles $A_1, \ldots, A_n$, n nonnegative integers $k_1, \ldots, k_n$ and $s \in \mathbb{R}^d$,

$$\begin{aligned}
&\mu_\lambda(s + A_1, \ldots, s + A_n; k_1, \ldots, k_n) \\
&= \lim_{R\to\infty} \frac{1}{l(C_R)} l(\{t \in C_R : \lambda(s + t + A_i) = k_i, i = 1, \ldots, n\}) \\
&= \lim_{R\to\infty} \frac{1}{l(C_R)} l(\{t' \in (s + C_R) : \lambda(t' + A_i) = k_i, i = 1, \ldots, n\}) \\
&(t' = t + s) \\
&= \lim_{R\to\infty} \frac{1}{l(C_R)} l(\{t' \in (C_R) : \lambda(t' + A_i) = k_i, i = 1, \ldots, n\}) \\
&= \mu_\lambda(A_1, \ldots, A_n; k_1, \ldots, k_n).
\end{aligned}$$

We use the fact that $\lim_{R\to\infty} \frac{l((s+C_R)\triangle C_R)}{l(C_R)} = 0$ in the third step, where $\triangle$ is the symmetric difference. So we conclude that ρ_A is an invariant of $\mathbb{R}^d$-action. Since μ_λ is generated by ρ_A from equation (2.4.5), μ_λ is also an invariant of $\mathbb{R}^d$-action. □

Through the one-to-one correspondence $\lambda \leftrightarrow \Lambda$, the measure μ_λ induces a probability measure on $\mathcal{D}_r$. We denote this induced measure by μ_Λ. Evidently, the measure μ_Λ is an invariant of $\mathbb{R}^d$-action

Lemma 2.4.7. *For all $s \in \mathbb{R}^d$, $\mu_{s+\Lambda}$ exists and it is equal to μ_Λ.*

This follows from the invariance of μ_Λ. It says that the probability measure is fixed on the orbit of Λ.

The physical meaning of the probability measure μ_Λ is easy to understand in the case that Λ has FLC. See the following proposition.

Proposition 2.4.8. *Suppose the point set Λ has FLC. When the measure μ_Λ exists, the frequencies of all local patterns of Λ exist and they can be expressed by μ_Λ.*

PROOF: Let $P := \{0, p_1, \ldots, p_n\}$ be a $(n+1)$-point patch of the point set Λ, i.e., $p_j \in \Lambda - \Lambda$ for $i = 1, \ldots, n$. Because Λ has FLC and hence $\Lambda - \Lambda$ is locally finite, there is a minimum separation distance $r' > 0$ among points in $(\Lambda - \Lambda) \cap C_{R'}$, where $R' = \max\{|p_1|, \ldots, |p_n|\}$. Let V be a neighbourhood of the origin, $V \subset C_{r'}$. Define

$$X_{P,V} := \{\Lambda' \in \mathcal{D}_r : P \subset \Lambda' + V\}. \tag{2.4.8}$$

It is quite straightforward to see that

$$\begin{aligned}\mu_\Lambda(X_{P,V}) &= \lim_{R\to\infty} \frac{l\left(\{t \in C_R : P \subset (-t+\Lambda) + V\}\right)}{l(C_R)} \\ &= \lim_{R\to\infty} \frac{\operatorname{card}\left(\{t \in C_R \cap \Lambda : P \subset (-t+\Lambda)\}\right) \cdot l(V)}{l(C_R)}\end{aligned}$$

By Definition 1.1.2, freq(P, V) exists and it is equal to $\frac{\mu_\Lambda(X_{P,V})}{l(V)}$. □

The following proposition is a simple consequence of Proposition 2.4.2.

Proposition 2.4.9. *Let Λ_1, Λ_2 be two point sets in $\mathcal{D}_r$. Suppose dens$(\Lambda_1 \triangle \Lambda_2) = 0$ and the measure μ_{Λ_1} exists. Then μ_{Λ_2} exists also and it is equal to μ_{Λ_1}.*

Now we modify this to treat the multicolour case. Let Λ be a given a point set in $\mathcal{D}_r^{(m)}$. We are going to define a probability measure on $\mathcal{D}_r^{(m)}$ determined by Λ. Recall that Λ has the form $\cup_{i=1}^m \Lambda_i \times \{i\}$. So the Dirac comb of Λ, denoted by λ as usual, is in $\mathscr{M}_r^{(m)}$ and it can be expressed by a n-tuple $(\delta_{\Lambda_1}, \ldots, \delta_{\Lambda_m})$, which satisfies that $\sum_{i=1}^m \delta_{\Lambda_i}(-t + C_r) \leq 1$ for $t \in \mathbb{R}^d$. As before, λ^i denotes the restriction of the Dirac comb λ to the space $\mathbb{R}^d \times \{i\}$, for $i = 1, \ldots, m$. Define $\vec{\lambda} := \{\lambda^1, \ldots, \lambda^m\}$. We call a subset $\mathbf{A} \subset \mathbb{E}$ a semi-open rectangle in $\mathbb{E}$ if it has the form that $\bigcup_{i=1}^m A_i \times \{i\}$, where $A_1, \ldots, A_m$ are semi-open rectangles in $\mathbb{R}^d$. We denote the set of all semi-open rectangles in $\mathbb{E}$ by $\vec{\mathcal{R}}$.

Analogous to the case that $m = 1$, for n arbitrary semi-open rectangles, $\mathbf{A}_1, \ldots, \mathbf{A}_n$, in $\mathbb{E}$ and n arbitrary vectors $\mathbf{K}_1, \ldots, \mathbf{K}_n \in (\mathbb{Z}^+)^m$, we call the sequence

$$(\mathbf{A}_1, \ldots, \mathbf{A}_n; \mathbf{K}_1, \ldots, \mathbf{K}_n)$$

a **pattern of** $\mathcal{D}_r^{(m)}$ and denote by $X_{(\mathbf{A}_1,\dots,\mathbf{A}_n;\mathbf{K}_1,\dots,\mathbf{K}_n)}$ the subset of $\mathscr{M}_r^{(m)}$ defined by $\{\lambda' \in \mathscr{M}_r^{(m)} : \vec{\lambda}'(t+\mathbf{A}_j) = \mathbf{K}_j, j=1,\dots,n\}$. Similarly, the family of subsets of this form generates the σ-algebra of $\mathscr{M}_r^{(m)}$.

Definition 2.4.10. *Let* $(\mathbf{A}_1,\dots,\mathbf{A}_n;\mathbf{K}_1,\dots,\mathbf{K}_n)$ *be a pattern of* $\mathcal{D}_r^{(m)}$. *If the limit*

$$\begin{aligned}&\rho_\Lambda(\mathbf{A}_1,\dots,\mathbf{A}_n;\mathbf{K}_1,\dots,\mathbf{K}_n)\\&:= \lim_{R\to\infty}\frac{1}{l(C_R)}l\left(\{t\in C_R : \vec{\lambda}(t+\mathbf{A}_j)=\mathbf{K}_j, j=1,\dots,n\}\right),\end{aligned} \tag{2.4.9}$$

exists, then we say that the frequency of the pattern $(\mathbf{A}_1,\dots,\mathbf{A}_n;\mathbf{K}_1,\dots,\mathbf{K}_n)$ *on the point set* Λ *exists and it is equal to* $\rho_\Lambda(\mathbf{A}_1,\dots,\mathbf{A}_n;\mathbf{K}_1,\dots,\mathbf{K}_n)$.

Similarly, the limit defined by (2.4.2) is equivalent to

$$\begin{aligned}&\rho_\Lambda((\mathbf{A}_1,\dots,\mathbf{A}_n;\mathbf{K}_1,\dots,\mathbf{K}_n))\\&:= \lim_{R\to\infty}\frac{1}{l(C_R)}l\left(\{t\in C_R : t+\Lambda\in X_{(\mathbf{A}_1,\dots,\mathbf{A}_n;\mathbf{K}_1,\dots,\mathbf{K}_n)}\}\right),\end{aligned} \tag{2.4.10}$$

Now we assume that the point set $\Lambda\in\mathcal{D}_r^{(m)}$ satisfies the following assumption:
FI′: The frequency of each pattern of $\mathcal{D}_r^{(m)}$ exists on Λ.

We wish to show that the values of ρ_Λ uniquely determine a probability measure μ on $(\mathscr{M}_r, \mathscr{B}(\mathscr{M}_r))$ fulfilling the condition

$$\begin{aligned}&\mu(\{\lambda'\in\mathscr{M}_r^{(m)} : \vec{\lambda}'(\mathbf{A}_j)=\mathbf{K}_j, j=1,\dots,n\})\\=\;&\rho_\Lambda(\mathbf{A}_1,\dots,\mathbf{A}_n;\mathbf{K}_1,\dots,\mathbf{K}_n).\end{aligned} \tag{2.4.11}$$

for n arbitrary semi-open rectangles $\mathbf{A}_1,\dots,\mathbf{A}_n$ and n arbitrary vectors $\mathbf{K}_1,\dots,\mathbf{K}_n\in(\mathbb{Z}^+)^m$.

For that, we need a generalization of Proposition 2.4.4 as follows. Using the same technique to prove Proposition 2.4.4 [37], we can also show the following Proposition. When come to condition 6, we need the fact that $\sum_{i=1}^m \delta_{\Lambda_i}(-t+C_r)\le 1$.

Proposition 2.4.11. *Let* $p(\mathbf{A}_1,\dots,\mathbf{A}_n;\mathbf{K}_1,\dots,\mathbf{K}_n)$ *be a value for* n *arbitrary disjoint semi-open rectangles and* n *arbitrary vectors* $\mathbf{K}_1,\dots,\mathbf{K}_n\in(\mathbb{Z}^+)^m$. *Then* p *uniquely determines a probability measure* μ *on* $(\mathscr{M}_r^{(m)},\mathscr{B}(\mathscr{M}_r))$ *satisfying (2.4.11) if and only if the following six conditions are fulfilled:*

1. $p(\mathbf{A}_1,\dots,\mathbf{A}_n;\mathbf{K}_1,\dots,\mathbf{K}_n)\ge 0$;

2. $p(\mathbf{A}_1,\dots,\mathbf{A}_n;\mathbf{K}_1,\dots,\mathbf{K}_n) = p(\mathbf{A}_{i_1},\dots,\mathbf{A}_{i_n};\mathbf{K}_{i_1},\dots,\mathbf{K}_{i_n})$, *for every permutation* $(i_1,\dots,i_n)$ *of numbers* $1,\cdots,n$;

3. $\sum_{k_1,\ldots,k_m=0}^{\infty} p(\mathbf{A},\mathbf{K}) = 1$, *where* $\mathbf{K} = (k_1,\ldots,k_m)$*;*

4. $\sum_{k_1,\ldots,k_m=0}^{\infty} p(\mathbf{A}_1,\ldots,\mathbf{A}_n;\mathbf{K}_1,\ldots,\mathbf{K}_n) = p(\mathbf{A}_2,\ldots,\mathbf{A}_n;\mathbf{K}_2,\ldots,\mathbf{K}_n)$, *where* $\mathbf{K}_1 = (k_1,\ldots,k_m)$, *for* $n \in \mathbb{N}, n > 1$*;*

5. $\rho_\Lambda(\mathbf{A}_1,\ldots,\mathbf{A}_n,\mathbf{B};\mathbf{K}_1,\ldots,\mathbf{K}_n,\mathbf{S})$
$$= \sum_{\mathbf{S}_1+\cdots+\mathbf{S}_j=\mathbf{S}} \rho_\Lambda(\mathbf{A}_1,\ldots,\mathbf{A}_n,\mathbf{B}_1,\ldots,\mathbf{B}_j;\mathbf{K}_1,\ldots,\mathbf{K}_n,\mathbf{S}_1,\ldots,\mathbf{S}_j), \quad (2.4.12)$$
where $\mathbf{A}_1,\ldots,\mathbf{A}_n,\mathbf{B}_1,\cdots,\mathbf{B}_j \in \vec{\mathcal{R}}$ *are pairwise disjoint and* $\mathbf{B} = \cup_{i=1}^{j}\mathbf{B}_i \in \vec{\mathcal{R}}$, $\mathbf{S},\mathbf{S}_j \in (\mathbb{Z}^+)^m$*;*

6. $p(\mathbf{A}_1^{(j)},\ldots,\mathbf{A}_{n(j)}^{(j)};\vec{0},\ldots,\vec{0}) \to 1$, *as* $j \to \infty$, *for every finite sequence* $(\mathbf{A}_1^{(j)},\ldots,\mathbf{A}_{n(j)}^{(j)})$, $\mathbf{A}_i^{(j)} \in \vec{\mathcal{R}}$, *with* $\mathbf{A}^{(j)} = \bigcup_{i=1}^{n(j)} \mathbf{A}_i^{(j)} \searrow \emptyset$.

□

Analogously to Theorem 2.4.5 and Proposition 2.4.6, we have the following result.

Theorem 2.4.12. *The function* ρ_Λ *in equation (2.4.10) uniquely determines a probability measure* μ_Λ*, which is invariant with respect to* $\mathbb{R}^d$*-action.*

Also we have the following proposition as a generalization of Proposition 2.4.13.

Proposition 2.4.13. *Let* Λ_1,Λ_2 *be two point sets in* $\mathcal{D}_r^{(m)}$*. Suppose* $dens(\Lambda_1\triangle\Lambda_2) = 0$ *and the measure* μ_{Λ_1} *exists. Then* μ_{Λ_2} *exists also and it is equal to* μ_{Λ_1}.

2.5 The point set dynamical systems on $\mathcal{D}_r^{(m)}$

Let Λ be a point set in $\mathcal{D}_r^{(m)}$ satisfying the assumption **FI′**. By Theorem 2.4.12, the probability measure μ_Λ exists and it is an invariant of $\mathbb{R}^d$-action. Let $X := \text{supp}(\mu_\Lambda)$. Then X has the following properties:

- X is a compact subspace of $\mathcal{D}_r^{(m)}$;
- X is an invariant set with respect to the $\mathbb{R}^d$-action.

As a consequence, the triple $(X,\mathbb{R}^d,\mu_\Lambda)$ defines a topological dynamical system and a measure theoretic dynamical system at the same time. In particular, $(X,\mathbb{R}^d)$ is a sub-dynamical system of $(\mathcal{D}_r^{(m)},\mathbb{R}^d)$ and in most cases of interest, the topological space X is considerably smaller than $\mathcal{D}_r^{(m)}$.

2.5.1 Minimality

To define a uniformly recurrent point set, we need to consider the topological dynamical system $(\mathcal{D}_r^{(m)}, \mathbb{R}^d)$. According to Definition 2.1.2, a point set $\Lambda' \in \mathcal{D}_r^{(m)}$ is uniformly recurrent if and only if for every neighbourhood Θ of Λ', the set $\{t \in \mathbb{R}^d : t + \Lambda' \in \Theta\}$ is relatively dense.

Proposition 2.5.1. *If Λ' has FLC, then Λ' is uniformly recurrent if and only if Λ' is repetitive.*

PROOF: Without loss of generality, we consider neighbourhoods Θ of Λ' of the form $U(C_R, \epsilon)[\Lambda']$, where $R > 0, \epsilon > 0$ and ϵ is expected to be very small. By the assumption that Λ' has FLC, $\Lambda' - \Lambda'$ is locally finite. Thus, there is a positive number $b > 0$ such that $\{q_i + C_b\}$ are pairwisely disjoint, where $q_i \in (\Lambda' - \Lambda') \cap C_R$. We choose ϵ to satisfy that $\epsilon < \min\{\frac{b}{2}, r\}$ and $((\Lambda' - \Lambda') \cap C_R) \cap (\partial C_R + C_\epsilon) = \emptyset$ (no points in $((\Lambda' - \Lambda') \cap C_R)$ are in the C_ϵ-neighbourhood of the boundary of C_R). We claim that for each $t \in \mathbb{R}^d$, $t + \Lambda' \in U(C_R, \epsilon)[\Lambda']$, there exists a vector $s \in C_\epsilon$ such that $(s + t + \Lambda') \cap C_R = (\Lambda' \cap C_R)$. Recall that $s + t$ is called a translation vector of the cluster $(\Lambda' \cap C_R)$. This follows that the set $\{t \in \mathbb{R}^d : t + \Lambda' \in U(C_R, \epsilon)[\Lambda']\}$ is relatively dense if and only if the translation vectors of the cluster $(\Lambda' \cap C_R)$ are relatively dense, which implies that Λ' is uniformly recurrent if and only if Λ' is repetitive.

So it suffices to show that the claim is true. Recall that $t + \Lambda' \in U(C_R, \epsilon)[\Lambda']$ if and only if $(t + \Lambda') \cap C_R \subset \Lambda' + C_\epsilon$ and $\Lambda' \cap C_R \subset (t + \Lambda') + C_\epsilon$. It means that the set $(t + \Lambda') \cap C_R$ is in the C_ϵ-neighbourhood of the set $\Lambda' \cap C_R$. Suppose the set $\{p'_1, \ldots, p'_{n'}\} = (t + \Lambda') \cap C_R$ and the set $\{p_1, \ldots, p_n\} = \Lambda' \cap C_R$. Evidently, $n = n'$. Without lost of generality, we assume that for each $i \in \{1, \ldots, n\}$, $|p'_i - p_i| < \epsilon$. Then for $i \neq j, i, j \in \{1, \ldots, n\}$, $|(p'_i - p'_j) - (p_i - p_j)| < 2\epsilon \leq b$. Note that $(p'_i - p'_j), (p_i - p_j) \in (\Lambda - \Lambda) \cap C_R$. By the definition of b, this follows that for $i \neq j, i, j \in \{1, \ldots, n\}$, $(p'_i - p'_j) = (p_i - p_j)$. Thus, for $i \in \{1, \ldots, n\}$, $p'_i - p_i$ is a constant. Take $s = p_1 - p'_1$. Then s satisfies that $s \in C_\epsilon$ and $(s + t + \Lambda') \cap C_R = \Lambda' \cap C_R$. □

By Theorem 2.1.3, we know that if Λ is uniformly recurrent, then the orbit closure of Λ denoted by X_Λ is minimal. We wish to use this result to prove that the dynamical system $(X, \mathbb{R}^d, \mu_\Lambda)$ is minimal by showing that $X = X_\Lambda$.

However, it is still unstated that for a point set $\Lambda' \in \mathcal{D}_r^{(m)}$, what is the meaning that $\Lambda' \in X$? Recall that for a regular positive measure ν on some topological space, a point z is in the support of ν if and only if for all open neighbourhoods N_z of z, $\nu(N_z) > 0$. Further, if there exists a neighbourhood base $\{N_z^{(n)}\}_1^\infty$ of the point

z, then it is equivalent to say that $z \in \text{supp}(\nu)$ if and only if for $n = 1, 2, 3, \ldots$, $\nu(N_z^{(n)}) > 0$.

Now we turn back to the point set Λ' and the space $\mathcal{D}_r^{(m)}$. Let $\{(R_n, \epsilon_n)\}_1^\infty$ be a sequence such that $R_n \nearrow \infty, \epsilon_n \searrow 0$ as $n \to \infty$. Then the set $\{U(C_{R_n}, \epsilon_n)[\Lambda']\}$ is a neighbourhood base of the point set Λ'. Therefore, $\Lambda' \in X$ if and only if $\mu_\Lambda(U(C_{R_n}, \epsilon_n)[\Lambda']) > 0$, for $n = 1, 2, 3, \ldots$.

We wish to use the definition of μ_Λ to compute $\mu_\Lambda(U(C_{R_n}, \epsilon_n)[\Lambda'])$ for $n = 1, 2, 3, \ldots$. For this, we need an alternative description of $U[C_{R_n}, \epsilon_n][\Lambda']$ in terms of measures. This is provided by the following lemma.

Lemma 2.5.2. *Suppose R_n, ϵ_n are chosen so that no point of $C_{R_n} \cap \Lambda'$ is in the $\overline{C_{\epsilon_n}}$-boundary of C_{R_n}, i.e., $(C_{R_n} \cap \Lambda') \cap (\partial C_{R_n} + \overline{C_{\epsilon_n}}) = \emptyset$. Then*

$$\mu_\Lambda(U(C_{R_n}, \epsilon_n)[\Lambda']) = \lim_{R \to \infty} \frac{1}{l(C_R)} l(\{t \in C_R : (t + \Lambda) \in U(C_{R_n}, \epsilon_n)[\Lambda']\}). \quad (2.5.1)$$

PROOF: Denote by $\{p_1, \ldots, p_{k(n)}\}$ the finite set $C_{R_n} \cap \Lambda'$. For all point $p \in \Lambda'$, let C_p be the index number of the colour of p, i.e., $C_p = j$ if and only if p is a point in the j-th component of Λ'. Recall that $\Lambda'' \in U(C_{R_n}, \epsilon_n)[\Lambda']$ if and only if $(\Lambda', \Lambda'') \in U(C_{R_n}, \epsilon_n)$, i.e., $\Lambda' \cap C_{R_n} \subset \Lambda'' + C_{\epsilon_n}$ and $\Lambda'' \cap C_{R_n} \subset \Lambda' + C_{\epsilon_n}$. This implies that there exists one point of Λ'' in each small cube $p_i + C_{\epsilon_n}$ for $i = 1, \ldots, k(n)$ and no points from Λ'' appear in the remainder of C_{R_n}, see the Figure 2.1. In terms of vector measures, we have that $\vec{\lambda''}(p_i + C_{\epsilon_n}) = \mathbf{1}_{C_{p_i}}$, for $i = 1, \ldots, k(n)$, and $\vec{\lambda''}(C_{R_n} \setminus \bigcup_{i=1}^{k(n)} (p_i + C_{\epsilon_n})) = \mathbf{0}$, where $\mathbf{1}_{C_{p_i}}$ is the unit vector which has 1 at index C_{p_i}.

By condition (5) in Proposition 2.4.11, equation (2.4.11) holds also for patterns defined by rectangles. Since the set $C_{R_n} \setminus \bigcup_{i=1}^{k(n)} (p_i + C_{\epsilon_n})$ can be decomposed into a disjoint union of rectangles of $\mathbb{R}^d$, we can think of

$$\left(p_1 + C_{\epsilon_n}, \ldots, p_{k(n)} + C_{\epsilon_n}, C_{R_n} \setminus \bigcup_{i=1}^{k(n)} (p_i + C_{\epsilon_n}); \mathbf{1}_{c(p_1)}, \ldots, \mathbf{1}_{c(p_1)}, \mathbf{0} \right)$$

as a finite disjoint union of patterns defined by rectangles and rewrite the set $U[C_{R_n}, \epsilon_n][\Lambda']$ as

$$X_{\left(p_1 + C_{\epsilon_n}, \ldots, p_{k(n)} + C_{\epsilon_n}, C_{R_n} \setminus \bigcup_{i=1}^{k(n)} (p_i + C_{\epsilon_n}); \mathbf{1}_{c(p_1)}, \ldots, \mathbf{1}_{c(p_1)}, \mathbf{0}\right)}.$$

Moreover, by (2.4.11) and (2.4.10), we have equation (2.5.1). $\square$

Now we set out to prove that $X = X_\Lambda$ in the case that Λ is uniformly recurrent. The following lemma is useful to estimate the measure value $\mu_\Lambda(U(C_{R_n}, \epsilon_n)[\Lambda]')$.

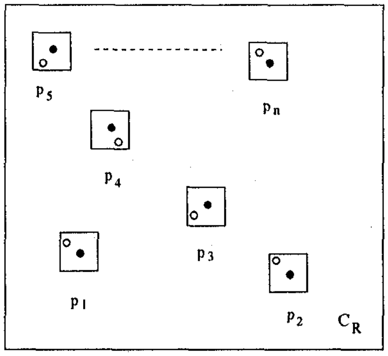

Figure 2.1: The black solid dots represent points from the set $\Lambda' \cap C_R$ whereas the blank dots represent $\Lambda'' \cap C_R$, where Λ'' is an arbitrary point set in $U(C_{R_n}, \epsilon_n)[\Lambda']$. The whole picture shows a pattern defined by $U(C_{R_n}, \epsilon_n)[\Lambda']$.

Lemma 2.5.3. *Suppose for each $R_n > 0$, ϵ_n is chosen to satisfy that there is no point of $C_{R_n} \cap \Lambda$ in the $\overline{C_{\epsilon_n}}$-boundary of C_{R_n}, i.e., $(C_{R_n} \cap \Lambda) \cap (\partial C_{R_n} + \overline{C_{\epsilon_n}}) = \emptyset$. For a point set $\Lambda' \in U(C_{R_n}, \frac{1}{2}\epsilon_n)[\Lambda]$, $U(C_{R_n}, \frac{1}{2}\epsilon_n)[\Lambda'] \subset U(C_{R_n}, \epsilon_n)[\Lambda]$.*

PROOF: By the assumption, $\Lambda' \cap C_{R_n} \subset \Lambda + C_{\frac{1}{2}\epsilon_n}, \Lambda \cap C_{R_n} \subset \Lambda' + C_{\frac{1}{2}\epsilon_n}$, and $\Lambda'' \cap C_{R_n} \subset \Lambda' + C_{\frac{1}{2}\epsilon_n}, \Lambda' \cap C_{R_n} \subset \Lambda'' + C_{\frac{1}{2}\epsilon_n}$. Note that the boundary effect is eliminated by how R_n, ϵ_n is chosen. Therefore, $\Lambda'' \cap C_{R_n} \subset (\Lambda' \cap C_{R_n}) + C_{\frac{1}{2}\epsilon_n} \subset \Lambda + C_{\epsilon_n}, \Lambda \cap C_{R_n} \subset (\Lambda' \cap C_{R_n}) + C_{\frac{1}{2}\epsilon_n} \subset \Lambda'' + C_{\epsilon_n}$, i.e. $U(C_{R_n}, \frac{1}{2}\epsilon_n)[\Lambda'] \subset U(C_{R_n}, \epsilon_n)[\Lambda]$. This is shown in Fig. 2.2. □

Lemma 2.5.4. $X \subset X_\Lambda$

PROOF: It suffices to show that for a point set $\Lambda' \in X$, there is a neighbourhood basis $U(C_{R_n}, \epsilon_n)[\Lambda']$ of Λ' and $t_n \in \mathbb{R}^d$ such that $t_n + \Lambda \in U(C_{R_n}, \epsilon_n)[\Lambda']$. Since $\Lambda' \in X$, $\mu_\Lambda(U(C_{R_n}, \epsilon_n)[\Lambda']) > 0$. Using the limit defined by equation (2.5.1), we have that $l(\{t \in C_{R_n} : t + \Lambda \in U(C_{R_n}, \epsilon_n)[\Lambda']\}) > 0$ and hence the set $\{t \in C_{R_n} : t + \Lambda \in U(C_{R_n}, \epsilon_n)[\Lambda']\}$ is nonempty. Taking t_n as a point of this set, we have that $(t_n + \Lambda) \in U(C_{R_n}, \epsilon_n)[\Lambda']$. Therefore, $\Lambda' \in X_\Lambda$. Since Λ' is an arbitrary point set in X, we have $X \subset X_\Lambda$. □

Finally, we are ready to show the following proposition.

Proposition 2.5.5. *Suppose Λ is uniformly recurrent, then $X = X_\Lambda$.*

PROOF: We have proved that $X \subset X_\Lambda$ in Lemma 2.5.4. So it suffices to show that $X_\Lambda \subset X$.

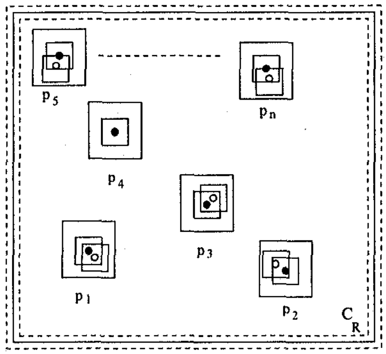

Figure 2.2: The black solid dots represent points from the set $\Lambda \cap C_R$ whereas the blank dots represent $\Lambda' \cap C_R$, where Λ' is an arbitrary point set in $U(C_{R_n}, \frac{1}{2}\epsilon_n)[\Lambda]$. This shows $U(C_{R_n}, \frac{1}{2}\epsilon_n)[\Lambda'] \subset U(C_{R_n}, \epsilon_n)[\Lambda]$.

In this proof, for each n, we choose R_n, ϵ_n in the same way as we did the previous lemma. As usual, let $\{p_1, \ldots, p_{k(n)}\} = \Lambda \cap C_{R_k}$.

First, we prove $\Lambda \in X$. It suffices to show that $\mu_\Lambda(U(C_{R_n}, \epsilon_n)[\Lambda]) > 0$, for $n = 1, 2, 3, \ldots$. By Lemma 2.5.2, it suffices to show that

$$\lim_{R \to \infty} \frac{\{t \in \mathbb{R}^d : t + \Lambda \in U(C_{R_n, \epsilon_n}[\Lambda])\}}{l(C_R)} > 0,$$

for $n = 1, 2, 3, \ldots$.

Since Λ is assumed to be uniformly recurrent, the set $\{t \in \mathbb{R}^d : t + \Lambda \in U(C_{R_n}, \frac{1}{2}\epsilon_n)[\Lambda]\}$ is relatively dense, i.e., there exists a positive number $Y_n > 0$ such that for all $a \in \mathbb{R}^d$, $\mathrm{card}((a + C_{Y_n}) \cap U(C_{R_n}, \frac{1}{2}\epsilon_n)[\Lambda]\} \geq 1$. Suppose $s \in (a + C_{Y_n}) \cap \{t \in \mathbb{R}^d : t + \Lambda \in U(C_{R_n}, \frac{1}{2}\epsilon_n)[\Lambda]\}$ and $s' \in s + C_{\frac{1}{2}\epsilon_n}$. Then $s + \Lambda \in U(C_{R_n}, \frac{1}{2}\epsilon_n)[\Lambda]$ and $s' + \Lambda \in U(C_{R_n}, \frac{1}{2}\epsilon_n)[s + \Lambda]$. By Lemma 2.5.3, we have $s' + \Lambda \in U(C_{R_n}, \epsilon_n)[\Lambda]$ and hence $s + C_{\frac{1}{2}\epsilon_n} \subset ((a + C_{Y_n}) \cap \{t \in \mathbb{R}^d : t + \Lambda \in U(C_{R_n}, \epsilon_n)[\Lambda]\}$.

Thus, ignoring the boundary effect which vanishes as $n \to \infty$, we have $\mu_\Lambda(U(C_{R_n}, \epsilon_n)[\Lambda]) \geq \frac{l(C_{\epsilon_n}) l(C_R)}{2^d l(C_{Y_n})}$. Therefore, when the limit exists, we have that $\mu_\Lambda(U(C_{R_n}, \epsilon_n)[\Lambda]) \geq \frac{l(C_{\epsilon_n})}{2^d l(C_{Y_n})} > 0$. By Lemma 2.5.2, $\Lambda \in X$.

Since X is closed and $\mathbb{R}^d$-invariant, $X_\Lambda \subset X$. $\square$

Combining Theorem 2.1.3 and Proposition 2.5.5, we obtain the following result.

Theorem 2.5.6. *Suppose that a point set $\Lambda \in \mathcal{D}_r^{(m)}$ is uniformly recurrent. If the probability measure μ_Λ exists, then the point set dynamical system $(X, \mathbb{R}^d, \mu_\Lambda)$ is minimal.* $\square$

Corollary 2.5.7. *([47]) Suppose $\Lambda \in \mathcal{D}_r^{(m)}$ has FLC and it is repetitive. If the probability measure μ_Λ exists, then the point set dynamical system $(X, \mathbb{R}^d, \mu_\Lambda)$ is minimal.* $\square$

We point out that in spite of the fact that the measure μ_Λ originates from the point set Λ, Λ may not be in the support of the measure μ_Λ. For instance, we consider the point set $\mathbb{Z} \setminus \{0\}$. By Proposition 2.4.13, $\mu_{\mathbb{Z}\setminus\{0\}}$ exists and it is equal to $\mu_{\mathbb{Z}}$. Note that for $0 < \epsilon < \frac{1}{2}$ and $R = 2, 3, 4, \ldots$, $U(C_R, \epsilon)[\mathbb{Z} \setminus \{0\}] = \bigcup_{i \in (\mathbb{Z}\setminus\{0\}) \cap (1-R, R-1)} (i + (-\epsilon, \epsilon))$ and by the definition of μ_Λ, $\mu_{\mathbb{Z}}(U(C_R, \epsilon)[\mathbb{Z}\setminus\{0\}]) = 0$. Therefore, $(\mathbb{Z} \setminus \{0\}) \notin \operatorname{supp}(\mu_{(\mathbb{Z}\setminus\{0\})})$.

2.5.2 Ergodicity

Theorem 2.5.8. *(Theorem 2.6, [34]) Let $(X, \mathbb{R}^d)$ be the dynamical system in Theorem 2.5.6. It is uniquely ergodic if and only if for all functions $f \in C(X)$,*

$$\frac{1}{l(C_R)} \int_{C_R} f(-t + \xi) dt \to \textit{constant}, \quad \textit{as } R \to \infty, \tag{2.5.2}$$

uniformly in $\xi \in X$, with the constant depending on f. $\square$

The main purpose of this section is to show that under an appropriate condition, the dynamical system $(X, \mathbb{R}^d)$ is uniquely ergodic and hence the invariant measure μ_Λ is uniquely ergodic.

It has been proved in (Theorem 2.7, [34]) that in the case that Λ is a Delone multiset with FLC, the dynamical system $(X_\Lambda, \mathbb{R}^d)$ is uniquely ergodic if and only if Λ has UCF. We wish to get the same result in a more general setting $\mathcal{D}_r^{(m)}$. For this, we need to generalize the notion UCF to a point set $\Lambda \in \mathcal{D}_r^{(m)}$. This is given as follows.

Definition 2.5.9. *For a pattern $F := (\mathbf{A}_1, \ldots, \mathbf{A}_n; \mathbf{k}_1, \ldots, \mathbf{k}_n)$ of $\mathcal{D}_r^{(m)}$, if the limit*

$$I(\mathbf{1}_{X_F}, a + \Lambda) := \lim_{R \to \infty} \frac{1}{l(C_R)} \int_{C_R} \mathbf{1}_{X_F}(-t + a + \Lambda) dt, \tag{2.5.3}$$

exists uniformly for $a \in \mathbb{R}^d$, then we say that this pattern has ***uniform frequency***[5] *at Λ. Further, if all patterns of $\mathcal{D}_r^{(m)}$ have uniform frequencies at Λ, then we say that Λ has* ***uniform pattern frequencies (UPF)***.

[5] The limit $I(\mathbf{1}_{X_F}, \Lambda)$ is equal to the frequency $\rho_\Lambda(F)$

In general, we define that for a function f defined on X and a point set $\xi \in X$,

$$I_R(f,\xi) := \frac{1}{l(C_R)} \int_{C_R} f(-t+\xi)dt. \tag{2.5.4}$$

and

$$I(f,\xi) := \lim_{R\to\infty} I_R(f,\xi), \tag{2.5.5}$$

if the limit exists.

Lemma 2.5.10. *The function $I_R(\mathbf{1}_{X_F}, \Lambda')$ is continuous with respect to Λ'.*

PROOF: We are going to show that for a $\epsilon > 0$, there is exists a $\delta > 0$, such that for all $\Lambda'' \in U(C_R, \delta)[\Lambda']$,

$$|I_R(\mathbf{1}_{X_F}, \Lambda') - I_R(\mathbf{1}_{X_F}, \Lambda'')| < \epsilon. \tag{2.5.6}$$

Suppose for $i = 1, \ldots, n$, $\mathbf{A}_i := \bigcup_1^d A_i^j \times \{j\}$ and $\Lambda' = \bigcup_1^d \Lambda_j' \times \{j\}$. We define $\partial F_\delta = (\partial C_R \cup \bigcup_{i=1}^n \bigcup_{j=1}^d \partial A_i^j) + C_\delta$, $Z(\Lambda') := \{t \in C_R : -t + \Lambda' \in X_F\}$ and $Y^{(\delta)}(\Lambda') := \{t \in C_R : (-t+\Lambda') \cap (\partial F_\delta) = \emptyset\}$. Moreover, we define $Z^{(\delta)}(\Lambda') := Z(\Lambda') \cap Y^{(\delta)}(\Lambda')$.

From the proof of Theorem 2.4.5 (see inequality (2.4.7)),

$$l\{t \in C_R : \mathrm{card}((-t+\Lambda') \cap (\partial F_\delta) > 0) \leq c \cdot l(C_R) l(\partial F_\delta),$$

where $c = (1+\frac{1}{r})^d$. It is clear that $l(\partial F_\delta) \to 0$, as $\delta \to 0$. So there is a $\delta_1 > 0$ such that for $\delta < \delta_1$, $l(C_R \setminus Y^{(\delta)}(\Lambda')) \leq \epsilon \cdot l(C_R)$.

Note that for a $\Lambda'' \in U(C_R, \delta)$, $Z^{(\delta)}(\Lambda') \subset Z(\Lambda'')$. Furthermore,

$$Z^{(\delta)}(\Lambda') = Z(\Lambda') \setminus (Z(\Lambda') \cap (C_R \setminus Y^{(\delta)}(\Lambda'))),$$

it follows that

$$l(Z^{(\delta)}(\Lambda')) = l(Z(\Lambda')) - l((Z(\Lambda') \cap (C_R \setminus Y^{(\delta)}(\Lambda')))) \geq l(Z(\Lambda')) - l((C_R \setminus Y^{(\delta)}(\Lambda'))).$$

Therefore,

$$\begin{aligned} l(Z(\Lambda'')) - l(Z(\Lambda')) &\geq -l(Z^{(\delta)}(\Lambda')) - l(Z(\Lambda')) \\ &\geq -l(C_R \setminus Y^{(\delta)}(\Lambda')) \geq -\epsilon \cdot l(C_R). \end{aligned}$$

Similarly, we can show that there exists $\delta_2 > 0$ such that for $\delta < \delta_2$, and a $\Lambda'' \in U(C_R, \delta)[\Lambda']$,

$$l(Z(\Lambda')) - l(Z(\Lambda'')) \geq -\epsilon \cdot l(C_R).$$

Taking $\delta = \min\{\delta_1, \delta_2\}$, then for a $\Lambda'' \in U(C_R, \delta)$, the inequality (2.5.6) follows from the above two inequalities. $\square$

Lemma 2.5.11. *Let f be a continuous function on X, i.e., $f \in C(X)$. Then f can be uniformly approximated by simple functions of the form*

$$\sum c_{(\mathbf{A}_1,\ldots,\mathbf{A}_n;\mathbf{k}_1,\ldots,\mathbf{k}_n)} \mathbf{1}_{X_{(\mathbf{A}_1,\ldots,\mathbf{A}_n;\mathbf{k}_1,\ldots,\mathbf{k}_n)}}.$$

PROOF: Recall that X is compact. Hence, f is a uniformly continuous function. So for a $\epsilon > 0$, there is an entourage $U(C_{R'}, \delta)$ such that for any two point sets, ξ and η, $(\xi, \eta) \in U(C_{R'}, \delta)$, $|f(\xi) - f(\eta)| < \epsilon$. Now we take a finite set of disjoint semi-open rectangles $\{A_1, \ldots, A_n\}$ such that each of them can be embedded into a small cube $a + C_s$, where $a \in \mathbb{R}^d$, $s < \min\{r, \frac{\delta}{2}\}$ and $C_{R'} \subset \bigcup_{i=1}^n A_i$. For $i = 1, \ldots, n$, we define $\mathbf{A}_i := \cup_{j=1}^m A_i \times \{j\}$. Then for $\{\mathbf{A}_1, \ldots, \mathbf{A}_n\}$, the space X has a partition

$$X = \bigcup_{\mathbf{k}_1,\ldots,\mathbf{k}_n \in \{\mathbf{0}, e_1, \ldots, e_m\}} X_{(\mathbf{A}_1,\ldots,\mathbf{A}_n;\mathbf{k}_1,\ldots,\mathbf{k}_n)}, \tag{2.5.7}$$

where $e_1, \ldots, e_m$ are unit vectors of $\mathbb{R}^m$.

Note that the pattern $(\mathbf{A}_1, \ldots, \mathbf{A}_n; \mathbf{k}_1, \ldots, \mathbf{k}_n)$ can be simply specified by $(\mathbf{k}_1, \ldots, \mathbf{k}_n)$. Evidently, for such a pattern $(\mathbf{k}_1, \ldots, \mathbf{k}_n)$, if $\xi, \eta \in X_{(\mathbf{k}_1,\ldots,\mathbf{k}_n)}$, then $(\xi, \eta) \in U(C_{R'}, \delta)$. So for a $\xi_{(\mathbf{k}_1,\ldots,\mathbf{k}_n)} \in X_{(\mathbf{k}_1,\ldots,\mathbf{k}_n)}$,

$$||f - f(\xi_{(\mathbf{k}_1,\ldots,\mathbf{k}_n)}) \mathbf{1}_{X_{(\mathbf{k}_1,\ldots,\mathbf{k}_n)}}||_\infty < \epsilon,$$

where f is restricted to $X_{(\mathbf{k}_1,\ldots,\mathbf{k}_n)}$.

It is clear that $\operatorname{card}\{(\mathbf{k}_1, \ldots, \mathbf{k}_n) \in \{\mathbf{0}, e_1, \ldots, e_m\}^n\} = (m+1)^n$. Therefore, f can be approximated by a simple function as follows,

$$||f - \sum_{\mathbf{k}_1,\ldots,\mathbf{k}_n \in \{\mathbf{0}, e_1, \ldots, e_m\}} f(\xi_{(\mathbf{k}_1,\ldots,\mathbf{k}_n)}) \mathbf{1}_{X_{(\mathbf{k}_1,\ldots,\mathbf{k}_n)}}||_\infty < \epsilon.$$

□

Theorem 2.5.12. *Let Λ be a point set in $\mathcal{D}_r^{(m)}$ having UPF. Then the measure μ_Λ exists and it is uniquely ergodic.*

PROOF: First of all, μ_Λ exists because Λ has UPF.

Next, by Lemma 2.5.11, it suffices to prove it for the functions f of the form $\mathbf{1}_{X_{(\mathbf{A}_1,\ldots,\mathbf{A}_n;\mathbf{k}_1,\ldots,\mathbf{k}_n)}}$.

By the definition of μ_Λ, for a pattern $F := (\mathbf{A}_1, \ldots, \mathbf{A}_n; \mathbf{k}_1, \ldots, \mathbf{k}_n)$, $I(\mathbf{1}_{X_F}, \Lambda)$ exists and it is equal to $\mu_\Lambda(X_F)$. Since Λ has UPF, the limit $I(\mathbf{1}_{X_F}, a + \Lambda)$ exists uniformly for $a \in \mathbb{R}^d$. Moreover, as a consequence of van Hove sequence property, we have that $I(\mathbf{1}_{X_F}, a + \Lambda) = I(\mathbf{1}_{X_F}, \Lambda)$, for $a \in \mathbb{R}^d$. We aim to show that $I(\mathbf{1}_{X_F}, \Lambda')$

exists uniformly and $I(\mathbf{1}_{X_F}, \Lambda') = \mu_\Lambda(X_F)$, for $\Lambda' \in X$. Recall that $X \subset X_\Lambda$ (see Lemma 2.5.4). So it suffices to show it for $\Lambda' \in X_\Lambda$.

Let Λ' be a limit point set of the orbit of Λ, i.e., there is a sequence $\{s_m : s_m \in \mathbb{R}^d\}_1^\infty$ such that $s_m + \Lambda \to \Lambda'$ in the local topology.

By Lemma 2.5.10, the function $I_R(\mathbf{1}_{X_F}, \Lambda')$ is continuous with respect to Λ'. Thus, $I_R(\mathbf{1}_{X_F}, \Lambda') = \lim_{m\to\infty} I_R(\mathbf{1}_{X_F}, s_m+\Lambda)$. Further, for all s_m, $I_R(\mathbf{1}_{X_F}, s_m+\Lambda) \to I(\mathbf{1}_{X_F}, s_m + \Lambda)$ uniformly as $R \to \infty$ and $\lim_{R\to\infty} I_R(\mathbf{1}_{X_F}, s_m + \Lambda) \equiv I(\mathbf{1}_{X_F}, \Lambda) = \mu_\Lambda(X_F)$. Thus, for a $\epsilon > 0$, there exists $R' > 0$, such that for $R > R'$, and $m \in \mathbb{N}$,

$$|I_R(\mathbf{1}_{X_F}, s_m + \Lambda) - \mu_\Lambda(X_F)| < \frac{\epsilon}{2}.$$

Meanwhile, for the same ϵ, there exists $N \in \mathbb{N}$ such that for $m > N$,

$$|I_R(\mathbf{1}_{X_F}, \Lambda') - I_R(\mathbf{1}_{X_F}, s_m + \Lambda)| < \frac{\epsilon}{2}.$$

As a consequence, for $R > R'$,

$$|I_R(\mathbf{1}_{X_F}, \Lambda') - \mu_\Lambda(X_F)| < \epsilon.$$

Therefore, $I(\mathbf{1}_{X_F}, \Lambda') = \lim_{R\to\infty} I_R(\mathbf{1}_{X_F}, \Lambda') = \mu_\Lambda(X_F)$. □

2.6 Spectral measures and the diffraction measure

Consider the Hilbert space $L^2(X, \mu)$. Along with the action of T on X there is an unitary action (also denoted by T) on the space of square integrable functions on X. For any function $f \in L^2(X, \mu)$, $T_t f(\Lambda) := f(-t + \Lambda)$ for all $t \in \mathbb{R}^d$. It is easy to verify that T is well defined.

2.6.1 Spectral measures

That the spectral theory of such a unitary representation is useful in understanding the geometry of the structures which are encoded in the dynamical system goes back to O. Koopman [29] in the 1930's, and is well-known in mathematical physics. Since usually it is hard to compute the spectrum of T directly, one looks at spectral measures of all the functions in $L^2(X, \mu)$. For any function $f \in L^2(X, \mu)$, $(f, T_t f)$ can be verified to be a positive definite function on $\mathbb{R}^d$. By Bochner's theorem [46], there exists a positive measure μ_f such that

$$(f, T_t f) = \int e^{2\pi i t} d\mu_f.$$

μ_f is called the **spectral measure** of the function f.

Then what is the relationship between these spectral measures and the spectrum of the operator? The measure μ_f may be pure point, singular continuous, or absolutely continuous, or any mixture of these. The basic spectral decomposition of $L^2(X,\mu)$ is created by splitting it according to those functions f for which μ_f is of these various types:

$$L^2(X,\mu) = L^2(X,\mu)_{pp} \bigoplus L^2(X,\mu)_{ac} \bigoplus L^2(X,\mu)_{sc}.$$

In hand with this we consider the Hilbert space $L^2(\mathbb{R}^d, \hat{\gamma})$ defined by the diffraction of $\Lambda \in X$ ($\gamma = \gamma_\Lambda$ is almost everywhere the same for all $\Lambda \in X$). We also have a unitary representation U of $\mathbb{R}^d$ on $L^2(\mathbb{R}^d, \hat{\gamma})$ defined by

$$U_t f(x) = e^{-2\pi i t.x} f(x) = \chi_{-t}(x) f(x)\,,$$

where the characters $\chi_k : \mathbb{R}^d \to \mathbb{C}$ are defined by $\chi_k(x) = e^{2\pi i k.x}$.

Compared with the spectrum of T in $L^2(X,\mu)$, the basic structure of the spectrum of U on $L^2(\mathbb{R}^d, \hat{\gamma})$ is much simpler. It can be read directly off that of the measure $\hat{\gamma}$. Specifically, let $\hat{\gamma} = (\hat{\gamma})_{pp} + (\hat{\gamma})_{sc} + (\hat{\gamma})_{ac}$ be the decomposition of $\hat{\gamma}$ into its pure point, singular continuous, and absolutely continuous parts. Then

$$L^2(\mathbb{R}^d, \hat{\gamma}) = L^2(\mathbb{R}^d, \hat{\gamma}_{pp}) \bigoplus L^2(\mathbb{R}^d, \hat{\gamma}_{ac}) \bigoplus L^2(\mathbb{R}^d, \hat{\gamma}_{sc}).$$

With $\square$ standing for pp, sc, or ac, the associated spectral measure of $f \in L^2(\mathbb{R}^d, \hat{\gamma}_\square)$ is $|f|^2(\hat{\gamma})_\square$.

2.6.2 Dworkin's argument

We now come to one of the main questions of our work: is there a connection between the diffraction measure and the dynamic system measure determined by the same point set Λ. An affirmative answer was given to this in a short, but often cited paper of S. Dworkin [16]:

Theorem 2.6.1. *Let $f \in C_c(\mathbb{R}^d)$. Then for μ-almost all $\Lambda \in X$, $\widehat{\gamma_{f*\Lambda}}$ is the spectral measure μ_{N_f} on $L^2(X,\mu)$.*

Here $\widehat{\gamma_{f*\Lambda}}$ is the diffraction measure of the point set Λ that has been modified so that the profile of f is attached to each $x \in \Lambda$, and N simply maps each continuous function f with compact support to the function N_f on X such that $N_f(\Lambda) = \sum_{x\in\Lambda} f(x)$.

This theorem has proven to be very useful inferring information about the diffraction from information coming from the dynamics. Arguments using the theorem are usually said to follow by Dworkin's argument, hence the name. However, as it stands, Theorem 2.6.1 leaves the actual connection between the diffraction and the dynamics very unclear. The rest of this thesis is devoted to uncovering what is really happening.

Chapter 3

The relationship between two measures

Introduction

By definition, a **point process** on $\mathbb{E}$ is a measurable mapping

$$\xi : (\Omega, \mathcal{A}, P) \longrightarrow (\mathscr{M}_p, \mathscr{B}(\mathscr{M}_p))$$

from some probability space into $\mathscr{M}_p$ with its σ-algebra of Borel sets $\mathscr{B}(\mathscr{M}_p)$. That is, it is a random point measure. Sometimes, when $m > 1$, it is called a **multivariate** point process. The **law** of the point process is the probability measure which is the image $\mu := \xi(P)$ of P. The point process is **stationary** if μ is invariant under the translation action of $\mathbb{R}^d$ on $\mathscr{M}_p$.

Thus from the stationary point process ξ we arrive at a measure-theoretical dynamical system $(\mathscr{M}_p, \mathbb{R}^d, \mu)$. Conversely, any such system may be interpreted as a stationary point process (by choosing $(\Omega, \mathcal{A}, P)$ to be $(\mathscr{M}_p, \mathbb{R}^d, \mu)$).

In most cases of interest, the support of the law μ of the process is considerably smaller than all of $\mathscr{M}_p$. In the sequel we shall assume that we have a point process $\xi : (\Omega, \mathcal{A}, P) \longrightarrow (\mathscr{M}_p, \mathscr{B}(\mathscr{M}_p))$ that satisfies the following conditions:

(PPI) the support of the measure $\mu = \xi(P)$ is a closed subset X of $\mathscr{M}_p(C_r^{(m)}, 1)$ for some $r > 0$.

(PPII) μ is stationary and has positive intensity (see below for definition).

(PPIII) μ is ergodic.

These are examples of what are called translation bounded measure dynamical systems in [6], although it should be noted that there the space of measures is not restricted to point measures, or even positive measures.

Obviously under (PPI) and (PPII), X is compact, and $(X, \mathbb{R}^d, \mu)$ is both a measure-theoretic and a topological dynamical system.

Condition (PPI) implies that the point process is simple and so we may identify the measures of the point process as the actual (uniformly discrete) point sets in $\mathbb{E}$ that are their supports. Write $\ddot{X}$ for the subset of $\mathcal{D}_r^{(m)}$ given by the supports of the measures of X. We call a point process satisfying (PPI) and (PPII) a **uniformly discrete stationary point process** and call a point process satisfying (PPI), (PPII) and (PPIII) a **uniformly discrete ergodic point process**. We will make considerable use of these two ways of looking at a point process – either as being formed of point measures or of uniformly discrete point sets.

The ergodic hypothesis eventually becomes indispensable, but for our initial results it is not required. Usually we simplify the terminology and speak of a point process ξ and assume implicitly the accompanying notation $(X, \mathbb{R}^d, \mu)$ and so on. We denote the family of all Borel subsets of X by $\mathcal{X}$.

Recall that for a uniformly recurrent point set $\Lambda \in \mathcal{D}_r^{(m)}$, if the frequency of each pattern exist on Λ, then Λ uniquely determines a probability measure μ on $\mathcal{D}_r^{(m)}$, which satisfies (PPI), (PPII) and (PPIII). Hence, we obtain a uniformly discrete ergodic point process by assigning μ to be the law of it. This shows a way to construct an example of a uniformly discrete ergodic point process. For instance, a regular model set on $\mathbb{R}^d$, which is uniformly recurrent because it is repetitive and has FLC, uniquely determines a uniformly discrete ergodic point process according to Corollary 2.5.6.

It is known that the measure μ admits a bounded conditional measure on the space that $\{\Lambda' \in X : 0 \in \Lambda'\}$, which is called the Palm measure of μ. Gouéré [21] first proved that for all point sets in $(X, \mathbb{R}^d, \mu)$, the two-point autocorrelation measure exists μ-almost surely and is equal to the first moment of the Palm measure μ-almost surely. We will give a new proof to this theorem by introducing a so-called average linear functional on X. As we mentioned in the general introduction, we will treat the first moment of the Palm measure as the two-point correlation measure of the point process and shift our attention from the question about the relationship of the two-point correlation measure and the invariant probability measure on $\mathcal{D}_r^{(m)}$ determined by an individual point set to the question about the relationship the two-point correlation measure and the law, belonging to the entire ergodic point process.

3.1 The moments and counting functions

In this section we work in the one colour case $m = 1$. Thus $\mathbb{E} = \mathbb{R}^d$. We let $\xi : (\Omega, \mathcal{A}, P) \longrightarrow (X, \mathcal{X})$ be a uniformly discrete stationary point process on $\mathbb{E}$ with law μ. We assume that $X \subset \mathscr{M}_p(C_r, 1) \subset \mathscr{M}_s$.

According to Prop. 2.2.2, for each $A \in \mathcal{B}(\mathbb{R}^d)$ and for each $f \in BM_c(\mathbb{R}^d)$, the mappings

$$N_A : \mathscr{M}_p(C_r, 1) \longrightarrow \mathbb{Z}, \qquad N_A(\lambda) = \lambda(A) \tag{3.1.1}$$
$$N_f : \mathscr{M}_p(C_r, 1) \longrightarrow \mathbb{C}, \qquad N_f(\lambda) = \lambda(f) \tag{3.1.2}$$

are measurable functions on $\mathscr{M}_p(C_r, 1)$, and by restriction, measurable functions on X. The first of these simply counts the number of points of the support of λ that lie in the set A, and N_f is its natural extension from sets to functions. Whence the name **counting functions**. They may also be considered as functions on $\mathscr{M}_p$. They may also be viewed as functions on the space X viewed as the space of corresponding point sets.

Thus, for example, in this notation we have for all $f \in BM_c(\mathbb{R}^d)$,

$$\begin{aligned} \int_X \lambda(f) d\mu(\lambda) &= \int_X N_f(\lambda) d\mu(\lambda) = \int_X \sum_{x \in \mathbb{R}^d} \lambda(\{x\}) f(x) d\mu(\lambda) \\ &= \int_X N_f(\Lambda) d\mu(\Lambda) = \int_X \sum_{x \in \Lambda} f(x) d\mu(\Lambda) . \end{aligned} \tag{3.1.3}$$

This is the first moment of the measure μ, henceforth denoted μ_1. More generally, the nth **moments**, $n = 1, 2, \ldots$ of a finite positive measure ω on X are the unique measures on $(\mathbb{R}^d)^n$ defined by

$$\begin{aligned} \omega_n(A_1 \times \cdots \times A_n) &= \int_X \lambda(A_1) \ldots \lambda(A_n) d\omega(\lambda) \\ &= \int_X N_{A_1} \ldots N_{A_n} d\omega , \end{aligned}$$

where $A_1, \ldots A_n$ run through all $\mathcal{B}(\mathbb{R}^d)$. Alternatively, for all $f_1, \ldots, f_n \in BM_c(\mathbb{R}^d)$,

$$\omega_n((f_1, \ldots, f_n)) = \int_X N_{f_1} \ldots N_{f_n} d\omega .$$

Since ω is a finite measure and the values of $\lambda(f) = N_f(\lambda)$ are uniformly bounded for any $f \in BM_c(\mathbb{R}^d)$ as λ runs over X, these expressions define translation bounded measures on $(\mathbb{R}^d)^n$.

If the measure ω is stationary (invariant under the translation action of $\mathbb{R}^d$) then the nth moment of ω is invariant under the action of simultaneous translation of all n variables. Thus, if the point process ξ is also stationary then the first moment of the law of ξ is invariant, hence a multiple of Lebesgue measure:

$$\mu_1(A) = \int_X \lambda(A) \mathrm{d}\mu(\lambda) = I\, \ell(A) . \tag{3.1.4}$$

This non-negative constant I, which is finite because of our assumption of uniform discreteness, is the expectation for the number of points per unit volume of λ in A and is called the **intensity** of the point process. We shall always assume (see PPII) that the intensity is positive, i.e. not zero.

The meaning of N_f can be extended well beyond $BM_c(\mathbb{R}^d)$. To make this extension we introduce the usual L^p-spaces $L^p(\mathbb{R}^d, \ell)$, $L^p(X, \mu)$ together with their norms which we shall indicate by $|| \cdot ||_p$ in either case. In fact, we need these only for $p = 1, 2$. We shall also make use of the sup-norms $|| \cdot ||_\infty$.

Proposition 3.1.1. *The mapping* (3.1.1) *uniquely defines a continuous mapping (also called N)*

$$\begin{array}{rcl} N : L^1(\mathbb{R}^d, \ell) & \longrightarrow & L^1(X, \mu) \\ f & \mapsto & N_f \end{array}$$

satisfying $\|N_f\|_1 \leq \sqrt{2}I\, \|f\|_1$. *Moreover, for all* $f \in L^1(\mathbb{R}^d, \ell)$,

$$N_f(\lambda) = \lambda(f) \quad \textit{for } \mu \textit{ almost surely all } \lambda \in X .$$

PROOF: Let $A \subset \mathbb{R}^d$ be a bounded and measurable set, let $\mathbf{1}_A$ be the characteristic function of A on $\mathbb{R}^d$, and define $N_{\mathbf{1}_A}$ on X by $N_{\mathbf{1}_A}(\lambda) = \lambda(\mathbf{1}_A) = \lambda(A) = N_A(\lambda)$, see (3.1.1). From (3.1.4), $||N_{\mathbf{1}_A}||_1 = \int_X N_A(\lambda) d\mu(\lambda) = I\, \ell(A) = I||\mathbf{1}_A||_1$. This shows that the result holds for N defined on these basic functions.

For simple functions of the form $f = \sum_{k=1}^n c_k \mathbf{1}_{A_k}$, where the sets $A_k \subset \mathcal{B}(\mathbb{R}^d)$ are mutually disjoint and the $c_k \geq 0$, define

$$N_f = \sum_{k=1}^n c_k N_{\mathbf{1}_{A_k}} = \sum_{k=1}^n c_k N_{A_k} .$$

Then

$$\|N_f\|_1 = \sum_{k=1}^n c_k \left\|N_{\mathbf{1}_{A_k}}\right\|_1 = \sum_{k=1}^n c_k\, I\ell(A_k) = I\, \|f\|_1 ,$$

and $N_f(\lambda) = \lambda(f)$ for all $\lambda \in X$.

The extension, first to arbitrary positive measurable functions and then to arbitrary real valued functions f goes in the usual measure theoretical way, and need not be reproduced here.

Finally we use linearity to go to complex-valued integrable f. If $f = f_r + \sqrt{-1} f_i$ is the splitting of f into real and imaginary parts, then $N_f = N_{f_r} + \sqrt{-1} N_{f_i}$, so

$$\|N_f\|_1 \le I(\|f_r\|_1 + \|f_i\|_1) = I \int_{\mathbb{R}^d} (|f_r| + |f_i|) d\ell.$$

Using the inequality $(|f_r| + |f_i|)^2 \le 2(|f_r|^2 + |f_i|^2)$, we have

$$\|N_f\|_1 \le \sqrt{2} I \int_{\mathbb{R}^d} \sqrt{|f_r|^2 + |f_i|^2} d\ell = \sqrt{2} I \, \|f\|_1 .$$

It is clear that if f and g differ on sets of measure 0 then likewise so do N_f and N_g, so this establishes the existence of the mapping. □

Proposition 3.1.2. *Let f_n, $n = 1, 2, 3, \ldots$ and f be measurable $\mathbb{C}$-valued functions on $\mathbb{R}^d$ with supports all contained within a fixed compact set K. Suppose that $||f_n||_\infty, ||f||_\infty < M$ for some $M > 0$ and $\{f_n\} \to f$ in the L^1-norm on $\mathbb{R}^d$. Then $\{N_{f_n}\} \to N_f$ in the L^2-norm on X.*

PROOF: Because of the uniform discreteness, $\lambda(K)$ is uniformly bounded on X by a constant $C(K) > 0$. Then for $g = f$ or $g = f_n$ for some n, $|N_g(\lambda)| < MC(K)$.

$$\begin{aligned} ||N_f - N_{f_n}||_2^2 &= \int_X |N_f(\lambda) - N_{f_n}(\lambda)|^2 \, d\mu(\lambda) \\ &\le \int_X (|N_f(\lambda)| + |N_{f_n}(\lambda)|) \, |N_f(\lambda) - N_{f_n}(\lambda)| \, d\mu(\lambda) \\ &\le 2MC(K) \int_X |N_f(\lambda) - N_{f_n}(\lambda)| \, d\mu(\lambda) , \end{aligned}$$

which, by Prop. 3.1.1, tends to 0 as $n \to \infty$. □

3.2 Averages, the Palm measure, and autocorrelation: 1-colour case

In this section we work in the one colour case $m = 1$. Thus $\mathbb{E} = \mathbb{R}^d$. We let $\xi : (\Omega, \mathcal{A}, P) \longrightarrow (X, \mathcal{X})$ be a uniformly discrete stationary point process on $\mathbb{E}$ with law μ.

3.2.1 The Palm measure

The **Campbell measure** is the measure c' on $\mathbb{R}^d \times X$, defined by

$$c'(B \times D) = \int_D \lambda(B)\,\mathrm{d}\mu(\lambda) = \int_X \sum_{x \in B} \lambda(\{x\}) \mathbf{1}_D(\lambda)\,\mathrm{d}\mu(\lambda) \tag{3.2.1}$$

for all $B \times D \in \mathcal{E} \times \mathcal{X}$.

We note that c' is invariant with respect to simultaneous translation of its two variables. By introducing the measurable mapping

$$\phi : \mathbb{R}^d \times X \longrightarrow \mathbb{R}^d \times X : \quad (x, \lambda) \mapsto (x, T_{-x}\lambda)$$

we obtain a twisted version c of c', also defined on $\mathbb{R}^d \times X$:

$$\begin{aligned} c(B \times D) &= \int_X \sum_{x \in B} \lambda(\{x\}) \mathbf{1}_D(T_{-x}\lambda)\,\mathrm{d}\mu(\lambda) \\ &= \int_X \sum_{x \in B \cap \Lambda} (\mathbf{1}_D)(-x + \Lambda)\,\mathrm{d}\mu(\Lambda)\,, \end{aligned}$$

and this is invariant under translation of the first variable:

$$\begin{aligned} c((t + B) \times D) &= \int_X \sum_{x \in (t+B)} \lambda(\{x\}) \mathbf{1}_D(T_{-x}\lambda)\,\mathrm{d}\mu(\lambda) \\ &= \int_X \sum_{y \in B} T_{-t}\lambda(\{y\}) \mathbf{1}_D((T_{-y}T_{-t}\lambda)\,\mathrm{d}\mu(T_{-t}\lambda) \\ &= c(B \times D)\,, \end{aligned}$$

using the translation invariance of μ.

Hence for a fixed $D \subset X$, c is a multiple $\dot{\mu}(D)\ell(B)$ of Lebesgue measure and $D \mapsto \dot{\mu}(D) := c(B \times D)/\ell(B)$ is a non-negative measure on X that is independent of the choice of $B \in \mathcal{B}(\mathbb{R}^d)$ (assuming that B has positive measure). This measure is called the **Palm measure** of the point process. See [11] for more details.

$$\begin{aligned} \dot{\mu}(D) &= \tfrac{1}{\ell(B)} \int_X \sum_{x \in B} \lambda(\{x\}) \mathbf{1}_D(T_{-x}\lambda)\,\mathrm{d}\mu(\lambda) \\ &= \tfrac{1}{\ell(B)} \int_X \sum_{x \in B \cap \Lambda} \mathbf{1}_D(-x + \Lambda)\mathrm{d}\mu(\Lambda)\,. \end{aligned} \tag{3.2.2}$$

We note that $\dot{\mu}(X) = \int_X \lambda(B) d\mu(\lambda)/\ell(B) = I$, which is the intensity of the point process. Some authors normalize the Palm measure by the intensity in order to render it a probability measure, and then call this probability measure the Palm measure. We shall not do this. However, we note that the normalized Palm measure is often viewed as being the conditional probability

$$\frac{1}{I}\dot{\mu}(D) = \mu(\{\lambda \in D \,|\, \lambda(\{0\}) = 1\}),$$

that is, the probability conditioned by the assumption that 0 is in the support of the point measures that we are considering. In fact the conditional probability defined in this way is meaningless in general since the probability that $\lambda(\{0\}) \neq 0$ is usually 0. But the intuition of what is desired is contained in the definition. Taking B as an arbitrarily small neighbourhood of 0 in (3.2.2), we see that in effect we are only looking at points of λ very close to 0 and then translating λ so that 0 is in the support. The result is averaged over the volume of B.

If the point process falls into the subspace X of $\mathcal{M}$ then the support of the Palm measure is also in X. However, the Palm measure is not stationary in general, since the translation invariance of μ has, in effect, been taken out.

The first moment of the Palm measure, sometimes called the intensity of the Palm measure, is

$$\begin{aligned} \dot{\mu}_1 : \dot{\mu}_1(A) &= \int_X \lambda(A) d\dot{\mu}(\lambda) \quad \text{or equivalently} \\ \dot{\mu}_1(f) &= \int_X \lambda(f) d\dot{\mu}(\lambda) = \int_X N_f(\lambda) d\dot{\mu}(\lambda) \,. \end{aligned} \tag{3.2.3}$$

The first moment of the Palm measure, and also the higher moments to be defined later, play a crucial role in the development of the paper, since they are, in an almost sure sense, the 2-point and higher point correlations of the elements of X.

As with μ, we will, consider the Palm measure interchangeably as a measure on X (as we have already done implicitly in Eq. (3.2.2)).

The importance of the Palm measure is its relation to the average value of a function over a typical point set $\Lambda \in X$, and from there to pattern frequencies in Λ and its direct involvement in the autocorrelation of Λ. To explain this we need to develop the Palm theory a little further.

Lemma 3.2.1. *(Campbell formula) For any measurable function $F : \mathbb{R}^d \times X \longrightarrow \mathbb{R}$,*

$$\int_{\mathbb{R}^d} \int_X F(x, \lambda)\, d\dot{\mu}(\lambda)\, \mathrm{d}x = \int_X \sum_{x \in \mathbb{R}^d} \lambda(\{x\}) F(x, T_{-x}\lambda)\, \mathrm{d}\mu(\lambda)\,.$$

PROOF: This can be proven easily by checking it on simple functions. Let $F = \mathbf{1}_B \times \mathbf{1}_D$. Then

$$\begin{aligned}\int_{\mathbb{R}^d}\int_X \mathbf{1}_B(x)\times\mathbf{1}_D(\lambda)\mathrm{d}\dot{\mu}(\lambda)\mathrm{d}x &= \ell(B)\dot{\mu}(D) = c(B\times D)\\ &= \int_X \sum_{x\in\mathbb{R}^d}\lambda(\{x\})\mathbf{1}_B(x)\mathbf{1}_D(T_{-x}\lambda)d\mu(\lambda)\\ &= \int_X \sum_{x\in\mathbb{R}^d}\lambda(\{x\})F(x,T_{-x}\lambda)\,\mathrm{d}\mu(\lambda)\,.\end{aligned}$$

□

Let ν_R be the function on X defined by

$$\nu_R(\lambda) = \frac{1}{\ell(C_R)}N_{C_R}(\lambda)\,,$$

for all $R > 0$. We treat ν_R as the Radon-Nikodym density of an absolutely continuous measure on X (with respect to μ).

Proposition 3.2.2. *In vague convergence,*

$$\{\nu_R\} \to \dot{\mu} \quad as \quad R \to 0\,.$$

PROOF: Use the definition of the Palm measure in (3.2.2) with B replaced by C_R. Then for any continuous function G on X,

$$\dot{\mu}(G) = \frac{1}{\ell(C_R)}\int_X \sum_{y\in C_R}\lambda(\{y\})G(T_{-y}\lambda)d\mu(\lambda).$$

If we require that $R < r$ then

$$\sum_{y\in C_R}\lambda(\{y\})G(T_{-y}\lambda) = N_{C_R}(\lambda)G(T_{-x}\lambda)\,,$$

where x is the unique point in $\Lambda\cap C_R$ when it is not empty, and then

$$\dot{\mu}(G) = \frac{1}{\ell(C_R)}\int_X N_{C_R}(\lambda)G(T_{-x}\lambda)d\mu(\lambda).$$

On the other hand

$$\nu_R(G) = \frac{1}{\ell(C_R)}\int_X N_{C_R}(\lambda)G(\lambda)d\mu(\lambda).$$

Thus

$$
\begin{aligned}
&|\dot{\mu}(G) - \nu_R(G)| \\
= &\left|\frac{1}{\ell(C_R)} \int_X N_{C_R}(\lambda)\{G(T{-}x + \lambda) - G(\lambda)\} d\mu(\lambda)\right| \qquad (3.2.4)\\
\leq &\frac{1}{\ell(C_R)} \int_X N_{C_R}(\lambda)\, |\{G(T_{-x}\lambda) - G(\lambda)\}|\, d\mu(\lambda)\,.
\end{aligned}
$$

The rest follows from the uniform continuity of G (X is compact). From the inequality (3.2.4),

$$
|\dot{\mu}(G) - \nu_R(G)| \leq \frac{\epsilon_R}{\ell(C_R)} \int_X N_{C_R}(\Lambda) d\mu(\Lambda) = \epsilon_R I\,,
$$

for some $\epsilon_R \to 0$ as $R \to 0$, where I is the intensity of the point process.

Therefore, we have that $\nu_R \to \dot{\mu}$ vaguely. □

3.2.2 Averages

Let ξ be a uniformly discrete ergodic stationary point process, with corresponding dynamical system $(X, \mathbb{R}^d, \mu)$. Let $F \in C(X)$. The **average** of F at $\lambda \in X$ is

$$
\mathrm{Av}(F)(\lambda) = \lim_{R\to\infty} \frac{1}{\ell(C_R)} \sum_{x \in C_R} \lambda(\{x\}) F(T_{-x}\lambda)\,,
$$

if it exists. Thus $\mathrm{Av}(F)$ is a function defined at certain points of X. Alternatively, we may think of F as a function on point sets and write this as

$$
\mathrm{Av}(F)(\Lambda) = \lim_{R\to\infty} \frac{1}{\ell(C_R)} \sum_{x \in \Lambda \cap C_R} F(-x + \Lambda)\,.
$$

We will prove the almost-sure existence of averages.

Proposition 3.2.3. *Let $F \in C(X)$. The average value of $\mathrm{Av}(F)(\lambda)$ of F exists μ-almost surely for $\lambda \in X$ and it is almost surely equal to $\dot{\mu}(F)$. In particular $\mathrm{Av}(F)$ exists as a measurable function on X. If μ is uniquely ergodic then the average value always exists everywhere and is equal to $\dot{\mu}(F)$.*

Proof: It is clear that the average value is constant along the orbit of any point λ for which it exists.

Let $\epsilon > 0$. Since F is uniformly continuous, there is a compact set K and an $s > 0$ so that $|F(\lambda') - F(\lambda'')| < \epsilon$ whenever $(\lambda', \lambda'') \in U(K, s)$. In particular $|F(-x + \lambda) - F(-u + \lambda)| < \epsilon$ whenever $|x - u| < s$. We can assume that $s < r$.

Let $\nu_s := \frac{1}{\ell(C_s)} N_{C_s} : X \longrightarrow \mathbb{C}$, as above. For $x \in \mathbb{R}^d$ and $\lambda \in X$, $N_{C_s}(T_{-x}\lambda) = 1$ if and only if $x \in u + C_s$ for some $u \in \Lambda$. Thus

$$\frac{1}{\ell(C_R)} \int_{C_R} F(T_{-x}\lambda)\nu_s(T_{-x}\lambda)dx \tag{3.2.5}$$

$$\sim \frac{1}{\ell(C_R)} \sum_{u \in C_R} \lambda(\{u\}) \frac{1}{\ell(C_s)} \int_{u+C_s} F(T_{-x}\lambda)dx\,, \tag{3.2.6}$$

where the $\sim$ comes from boundary effects only and becomes equality in the limit.

There is a constant $a > 0$ so that $\operatorname{card}(\lambda(C_R))/\ell(C_R) < a$, independent of R or which $\lambda \in X$ is taken. Using this and our choice of s, we obtain

$$\begin{aligned} | \lim_{R\to\infty} \frac{1}{\ell(C_R)} \int_{C_R} F(T_{-x}\lambda)\nu_s(T_{-x}\lambda)dx & \\ - \lim_{R\to\infty} \frac{1}{\ell(C_R)} \sum_{u\in C_R} \lambda(\{u\})F(T_{-u}\lambda)| < a\epsilon\,. \end{aligned} \tag{3.2.7}$$

The right hand term is just the average value of F at λ, if the limit exists. However, by the Birkhoff ergodic theorem the left integral exists almost surely and is equal to $\int_X F\nu_s \mathrm{d}\mu = \nu_s(F)$, ν_s being treated as a measure.

Now making $\epsilon \longrightarrow 0$, so $s \longrightarrow 0$ also, and using Prop. 3.2.2 we have

$$\mu(F) = \lim_{s\to 0} \nu_s(F) = \lim_{R\to\infty} \frac{1}{\ell(C_R)} \sum_{u\in C_R} \lambda(\{u\})F(T_{-u}\lambda) = \mathrm{Av}(F)(\lambda)\,.$$

Thus the average value of F on λ exists almost surely.

In the uniquely ergodic case, the conclusion of Birkhoff's theorem is true everywhere in X. □

3.2.3 The autocorrelation and the Palm measure

Again, let $\xi : (\Omega, \mathcal{A}, P) \longrightarrow (X, \mathcal{X})$ be a uniformly discrete stationary ergodic point process on $\mathbb{R}^d$ with law μ. For each $\lambda \in X$ we define $\tilde{\lambda}$ to be the point measure on $\mathbb{R}^d$ defined by $\tilde{\lambda}(\{x\}) = \overline{\lambda(\{-x\})}$ (though at this point we are only dealing with real measures). Recall that the autocorrelation of λ is defined as

$$\gamma_\lambda := \lim_{R\to\infty} \frac{1}{\ell(C_R)} \left(\lambda|_{C_R} * \tilde{\lambda}|_{C_R}\right) = \lim_{R\to\infty} \frac{1}{\ell(C_R)} \sum_{x,y\in\Lambda\cap C_R} \delta_{y-x}\,,$$

where the limit, which may or may not exist, is taken in the vague topology.

A simple consequence of the van Hove property of cubes is that:

$$\gamma_\lambda = \lim_{R\to\infty} \frac{1}{\ell(C_R)} \sum_{x\in\Lambda\cap C_R, y\in\Lambda} \delta_{y-x}\,. \tag{3.2.8}$$

Namely, for any $f \in C_c(\mathbb{R}^d)$, say with support K, and any $x \in C_R$, $f(y-x) = 0$ unless $y \in C_R + K$, and thus for large R the only relevant y which are not in C_R are in the K-boundary of C_R, which is vanishingly small in relative volume as $R \to \infty$.

Theorem 3.2.4. *The first moment $\dot{\mu}_1$ of the Palm measure is a positive, positive definite, translation bounded measure. Furthermore, μ-almost surely, $\lambda \in X$ admits an autocorrelation γ_λ and it is equal to $\dot{\mu}_1$. If X is uniquely ergodic then $\dot{\mu}_1 = \gamma_\lambda$ for all $\lambda \in X$.*

Proof: We begin with the statement about the autocorrelation measures γ_λ. Let $f \in C_c(\mathbb{R}^d)$. The autocorrelation of λ at f, if it exists, is

$$\begin{aligned}\gamma_\lambda(f) &= \lim_{R\to\infty} \frac{1}{\ell(C_R)} \sum_{x\in C_R, y\in\Lambda} \lambda(\{x\}) f(y-x) \\ &= \lim_{R\to\infty} \frac{1}{\ell(C_R)} \sum_{x\in C_R} \lambda(\{x\}) N_f(T_{-x}\lambda) \\ &= \dot{\mu}(N_f) = \dot{\mu}_1(f)\end{aligned} \tag{3.2.9}$$

for $\lambda \in X$, μ-almost surely, where we have used Prop. 3.2.3 and (3.2.3).

This is basically what we want, but we must show that it holds for *all* $f \in C_c(\mathbb{R}^d, \mathbb{R})$ for almost all $\lambda \in X$. This is accomplished by using a countable dense (in the sup norm) set of elements of $C_c(\mathbb{R}^d, \mathbb{R})$. We can get (3.2.9) simultaneously for this countable set, and this is enough to get it for all $f \in C_c(\mathbb{R}^d, \mathbb{R})$. Then γ_λ exists and is equal to $\dot{\mu}_1$ for almost all $\lambda \in X$. For more details see [21].

Finally, it is clear that γ_λ is a positive and positive definite measure whenever it exists, and hence also $\dot{\mu}_1$ is positive and positive definite. All positive and positive definite measures are translation bounded, [8] Prop. 4.4., or [23]. □

3.3 Adding Colour

We now look at the changes required to Section 3.2 in order to include colour, i.e. to have $m > 1$. The colour enters in two ways. First of all, the dynamics, that is to say the dynamical hull X and the measure μ, depend on colour since closeness in the local topology depends on simultaneous closeness of points of like-colours. Secondly the autocorrelation, and then the diffraction, depends on colour.

Diffraction depends on how scattering waves from different points (atoms) superimpose upon each other. However, physically, different types of atoms will have different scattering strengths, and so we wish to incorporate this into the formalism. This is accomplished by specifying a vector w of weights to be associated with the different colours and introducing for each point measure λ of our hull X a weighted version of it, λ^w. This will be a measure on $\mathbb{R}^d$. It will be important that the weighting is kept totally separate from the topology and geometry of X. The geometry of the configuration and the weighting of points, which enters into the diffraction, are different things. The measures describing our point sets are measures on $\mathbb{E}$, but the diffraction always takes place on the flattened point sets.

On the geometrical side we have treated the full colour situation from the start. In this section we introduce it into the autocorrelation/diffraction side. This affects almost every result in Section 3.2. However, we shall see that every proof then generalizes quite easily, and we simply outline the new situation and the generalized results, leaving the reader to do the easy modifications to the proofs.

3.3.1 Weighting systems

Let $\xi : (\Omega, \mathcal{A}, P) \longrightarrow (X, \mathcal{X})$ be a uniformly discrete stationary multi-variate point process, where $X \subset \mathscr{M}_p$, $\mathcal{X} = \mathcal{M} \cap X$, and $(X, \mathbb{R}^d, \mu)$ is the resulting dynamical system. As in Section 1.4.1, we let $\mathbb{E} = \mathbb{R}^d \times \boldsymbol{m}$, with $\mathbb{E}^i = \mathbb{R}^d \times \{i\}$ and $\mathbb{E} = \bigcup_{i \le m} \mathbb{E}^i$. For each i we have the restriction

$$\mathrm{res}^i : \lambda \mapsto \lambda^i$$

of measures on $\mathbb{E}$ to measures on $\mathbb{E}^i$. We will simply treat these restricted measures as being measures on $\mathbb{R}^d$. If $\lambda \leftrightarrow \Lambda$ then we also think of res^i as the mapping $\Lambda \mapsto \Lambda^i := \{x \in \mathbb{R}^d : (x,i) \in \Lambda\}$. [1]

The same argument that led to (3.1.4) gives

$$\int_X \lambda^i(A) d\mu(\lambda) = I^{(i)} \lambda(A)$$

for some $I^{(i)} \ge 0$, for each i. We shall always assume:

[1] It is also possible to define associated dynamical systems X^i and with them Palm measures. However, it is important here that everything will always refer back to the full colour situation encoded in the geometry of X.

(PPIIw) $\quad I^{(i)} > 0$ for all $i \leq m$.

A **system of weights** is a vector $w = (w_1, \dots w_m)$ of real numbers.[2] We define a mapping

$$X \longrightarrow M_s(\mathbb{R}^d) \quad \lambda \mapsto \lambda^w := \sum_{i \leq m} w_i \lambda^i .$$

We also have the flattening map:

$$X \longrightarrow M_s(\mathbb{R}^d) \quad \lambda \mapsto \lambda^{\downarrow} := \sum_{i \leq m} \lambda^i .$$

First introduce the measure c^w on $\mathbb{R}^d \times X$:

$$c^w(B \times D) = \int_X \sum_{x \in B} \lambda^w(\{x\}) T_x \mathbf{1}_D(\lambda) \mathrm{d}\mu(\lambda) .$$

Since $(T_x\lambda)^w = T_x(\lambda^w)$ this measure is invariant under translation of the first variable and we have

$$c^w(B \times D) = \ell(B) \dot{\mu}^w(D) .$$

This determines the w-**weighted Palm measure** $\dot{\mu}^w$ on X. This is not a Palm measure in the normal sense of the word. However, it plays the same role as the Palm measure in much of what follows. For example, there is a corresponding Campbell formula:

$$\int_{\mathbb{R}^d} \int_X F(x,\lambda) \mathrm{d}\dot{\mu}^w(\lambda) \mathrm{d}x = \int_X \sum_{x \in \mathbb{R}^d} \lambda^w(\{x\}) F(x, T_{-x}\lambda) \mathrm{d}\mu(\lambda)$$

for all measurable $F : \mathbb{R}^d \times X \longrightarrow \mathbb{C}$.

We note the formula for the weighted intensity:

$$\begin{aligned} I^w l(A) &= \int_X \lambda^w d\mu(\lambda) = \int_X \sum_{x \in A} \lambda^w(\{x\}) d\mu(\lambda) \\ &= c^w(A \times X) = l(A) \dot{\mu}^w(X), \end{aligned}$$

whence

$$I^w = \dot{\mu}^w(X). \tag{3.3.1}$$

For all $i \leq m$, for all $A \in \mathcal{B}(\mathbb{R}^d)$, and for all $f \in BM_c(\mathbb{R}^d)$ define

$$\begin{aligned} &N_A^w : X \longrightarrow \mathbb{N} \qquad N_A^w(\lambda) = \lambda^w(A) = \textstyle\sum_{x \in A} \lambda^w(\{x\}) \\ &N_f^w : X \longrightarrow \mathbb{N} \qquad N_f^w(\lambda) = \lambda^w(f) = \textstyle\sum_{x \in \mathbb{R}^d} \lambda^w(\{x\}) f(x) . \end{aligned} \tag{3.3.2}$$

[2]One could have complex numbers here, but it makes things easier, and more natural for higher correlations, if the weights are real.

Thus, for example,

$$N_A^w(\lambda) = \sum w^i \lambda^i(A) = \sum w^i N_A(\mathrm{res}^i(\lambda)) = \sum N_A \circ \mathrm{res}^i(\lambda)\,. \tag{3.3.3}$$

Define

$$\nu_R^w : X \longrightarrow \mathbb{R}, \quad \nu_R^w(\lambda) = \frac{1}{\ell(C_R)} N_{C_R}^w(\lambda)$$

or equivalently, $\nu_R^w(\Lambda) = \frac{1}{\ell(C_R)} N_{C_R}^w(\Lambda)$. In vague convergence,

$$\{\nu_R^w\} \to \dot{\mu}^w \quad \text{as} \quad R \to 0\,.$$

These auxiliary measures are used, as before, to prove the existence of averages. Let $F \in C(X)$. The w-**average value** of F **on** X is

$$\mathrm{Av}^w(F)(\lambda) = \lim_{R\to\infty} \frac{1}{\ell(C_R)} \sum_{x\in C_R} \lambda^w(\{x\}) T_x F(\lambda)$$

$$\mathrm{Av}^w(F)(\Lambda) = \lim_{R\to\infty} \frac{1}{\ell(C_R)} \sum_{x\in C_R} \lambda^w(\{x\}) F(-x+\Lambda)\,,$$

if it exists.

Prop. 3.2.3 becomes:

Proposition 3.3.1. *The w-average value of $F \in C(X)$ is defined at $\Lambda \in X$, μ-almost surely and is almost surely equal to $\dot{\mu}^w(F)$. If μ is uniquely ergodic then the average value always exists and is equal to $\dot{\mu}^w(F)$.*

We now come to the w-weighted autocorrelation. This is the measure on $\mathbb{R}^d$ defined by

$$\begin{aligned}
\gamma_\lambda^w(f) &= \lim_{R\to\infty} \frac{1}{\ell(C_R)} \lambda^w|_{C_R} * \widetilde{\lambda^w}|_{C_R}(f) \\
&= \lim_{R\to\infty} \frac{1}{\ell(C_R)} \sum_{x\in C_R, y\in\mathbb{R}^d} \lambda^w(\{x\})\overline{\lambda^w(\{y\})} f(y-x) \qquad (3.3.4)\\
&= \lim_{R\to\infty} \frac{1}{\ell(C_R)} \sum_{x\in C_R} \lambda^w(\{x\}) N_f^w(T_{-x}\lambda) \\
&= \mathrm{Av}^w(N_f^w) = \dot{\mu}^w(N_f^w) =: \dot{\mu}_1^w(f)
\end{aligned}$$

for all $f \in C_c(\mathbb{R}^d)$ and for μ-almost all $\lambda \in X$.

We call $\dot{\mu}_1^w$ the **weighted first moment of the weighted Palm measure**.

Theorem 3.3.2. *The weighted first moment $\mathring{\mu}_1^w$ of the weighted Palm measure is a positive definite measure. It is Fourier transformable and its Fourier transform $\widehat{\mathring{\mu}_1^w}$ is a positive translation bounded measure on $\mathbb{R}^d$. Furthermore, μ-almost surely, $\lambda \in X$ admits a w-weighted autocorrelation γ_λ^w and it is equal to $\mathring{\mu}_1^w$. If X is uniquely ergodic then $\mathring{\mu}_1^w = \gamma_\lambda^w$ for all $\lambda \in X$.* □

Remark 3.3.3. *Regarding the statements about the transformability and translation boundedness of the Fourier transform, this is a consequence of the positive definiteness of the Palm measure, see [8] Thm. 4.7, Prop. 4.9.*

3.4 Patterns and pattern frequencies

This section is not used in the sequel. However, it serves to illustrate the importance of the Palm measure.

Let $(X, \mathbb{R}^d, \mu)$ be a multi-colour uniformly discrete stationary ergodic point process. It is of interest to define the frequency of finite colour patterns in X. This is made difficult because from the built-in vagueness of the topology of X we know that we should not be looking for exact matches of some given colour pattern F of $\mathcal{D}_r^{(m)}$, but rather close approximations to it. In addition there is the problem of how to anchor F, in order to specify it exactly as we move it around. This leads us to always assume that F contains 0, and then to define a **pattern** in X as a pair (F, V) where $F = \cup_{i=1}^m (F_i, i)$ is a finite subset of $\mathcal{D}_r^{(m)}$ with $0 \in F^\downarrow := \cup F_i$ and V is a bounded measurable neighbourhood of 0 in $\mathbb{R}^d$. For a pattern (F, V) we then define the collection of elements of X that contain it as

$$X_{F,V} = X_{(F,V)} := \{\Lambda \in X \, : \, F \subset V + \Lambda\} ,$$

and write $\mathbf{1}_{F,V}$ for $\mathbf{1}_{X_{F,V}}$. See (2.4.7) in Proposition 2.4.9 where we were assuming FLC.

Throughout one should keep in mind that F and Λ are multi-colour sets, our conventions are that translations are by elements of $\mathbb{R}^d$ and are always on the left, and the inclusions take colour into account.

For any bounded region B define

$$L_{F,V}(\Lambda, B) := \mathrm{card}\{x \in \Lambda^\downarrow \, : \, F \subset -x + V + \Lambda \, , \, x - V + F \subset B\} .$$

An initial idea for the frequency of the pattern (F, V) in a set $\Lambda \in X$ might be:

$$\mathrm{freq}(\Lambda, F, V) = \lim_{R \to \infty} \frac{1}{\ell(C_R)} L_{F,V}(\Lambda, C_R) . \tag{3.4.1}$$

This definition is very sensitive to the boundary of V, as one can see from the simple example below. In general we do not know how to prove that this limit exists, even almost everywhere in X. However, we can prove that for V open or V closed, if the limit does exist then it is, almost surely, given by the Palm measure of $X_{F,V}$, and this we do know exists almost surely. Thus we are led to define:

The **frequency** of the pattern (F, V) in X is $\dot{\mu}(X_{F,V})$.

The connection with Palm measures comes because (as is easy to see from the van Hove property of expanding cubes)

$$\lim_{R\to\infty} \frac{1}{\ell(C_R)} L_{F,V}(\Lambda, C_R) = \lim_{R\to\infty} \frac{1}{\ell(C_R)} \sum_{x\in\Lambda^{\downarrow}\cap C_R} \mathbf{1}_{F,V}(-x+\Lambda) .$$

The latter is the average over Λ (where the weighting system is all 1s) of $\mathbf{1}_{F,V}$, if it exists.

Proposition 3.4.1. *Let (F, V) be a pattern with V an open set. Then*

$$\underline{\mathrm{freq}}(\Lambda, F, V) = \dot{\mu}(X_{F,V})$$

μ-almost surely for $\Lambda \in X$, where $\underline{\mathrm{freq}}$ means that the lim inf *is taken in* (3.4.1). *Similarly, if (F, V) is a pattern with V a closed set, then*

$$\overline{\mathrm{freq}}(\Lambda, F, V) = \dot{\mu}(X_{F,V})$$

μ-almost surely for $\Lambda \in X$, where $\overline{\mathrm{freq}}$ means that the lim sup *is taken in* (3.4.1).

Lemma 3.4.2. *$X_{F,V}$ is open if V is bounded and open and closed if V is bounded and closed.*

Proof: Let V be open and let $\Lambda \in X_{F,V}$. Then $F \subset V + \Lambda$. Since V is open and F is finite, there is an $\epsilon > 0$ so that for each $f \in F$, with $f = v + x$, where $v \in V$, $x \in \Lambda$ (there may be choices, but fix one choice x for each f), $f + C_\epsilon \subset V + x$. Choose $R > 0$ so that $-V + F \subset C_R$. Let $\Lambda' \in U(\overline{C_R}, C_\epsilon)[\Lambda]$ and let $f = v + x \in F$, as above. Since $\overline{C_R} \cap \Lambda \subset C_\epsilon + \Lambda'$ and $x \in \overline{C_R} \cap \Lambda$, $x = c + x'$ where $x' \in \Lambda'$, $c \in C_\epsilon$. Then $c + f \in V + x$, so $f \in V + x'$.

Since $f \in F$ was arbitrary, $F \subset V + \Lambda'$ and $\Lambda' \in X_{F,V}$. Thus the open neighbourhood $U(\overline{C_R}, C_\epsilon)[\Lambda]$ of Λ lies in $X_{F,V}$.

The argument for V closed is similar. □

Proof of Prop. 3.4.1 (sketch): Consider the case when V is open. Then $X_{F,V}$ is open and the value of any regular measure at $X_{F,V}$ can be approximated as closely as desired by a compact set $K \subset X_{F,V}$. For any such K we can find a continuous function f with $\mathbf{1}_K \le f \le \mathbf{1}_{X_{F,V}}$. Using Prop. 3.3.1, where all weights are assumed equal to 1, we obtain that $\dot{\mu}(f)$ is almost surely the average of f on Λ and, from the definition of f, that for any $\epsilon > 0$ and for large enough R,

$$\dot{\mu}(K) \le \dot{\mu}(f) \le \underline{\mathrm{freq}}(\Lambda, F, V) \le \frac{1}{\ell(C_R)} \sum_{x \in \Lambda \cap C_R} \mathbf{1}_{F,V}(-x + \Lambda) + \epsilon .$$

Integrating over X and using the Campbell formula we have, independent of R,

$$\dot{\mu}(K) \le \int_X \underline{\mathrm{freq}}(\Lambda, F, V) \mathrm{d}\mu \le \dot{\mu}(X_{F,V}) .$$

Now since we can make $\dot{\mu}(K)$ as close as we wish to $\dot{\mu}(X_{F,V})$, we obtain both

$$\dot{\mu}(X_{F,V}) \le \underline{\mathrm{freq}}(\Lambda, F, V) \quad \text{and} \quad \int_X \underline{\mathrm{freq}}(\Lambda, F, V) \mathrm{d}\mu = \dot{\mu}(X_{F,V})$$

From this $\dot{\mu}(X_{F,V}) = \underline{\mathrm{freq}}(\Lambda, F, V)$, μ-almost everywhere.

The result for V closed is similar, this time approximating by open sets from above. □

Example: Consider the usual dynamical system based on $\mathbb{Z}$: $X(\mathbb{Z}) \simeq \mathbb{R}/\mathbb{Z}$. Let $F := \{0, 1/4\}$ and $V := (-1/4, 1/4)$. For any $\Lambda = t + \mathbb{Z}$, we have

$$\frac{1}{2n} \sum_{u \in (t+\mathbb{Z}) \cap [-n,n]} \mathbf{1}_{F,V}(-u + t + \mathbb{Z}) = 0$$

while

$$\frac{1}{2n} \sum_{u \in (t+\mathbb{Z}) \cap [-n,n]} \mathbf{1}_{F,\overline{V}}(-u + t + \mathbb{Z}) \approx 1 .$$

In this case we have $X_{F,V} = \emptyset$, $X_{F,\overline{V}} = X$ and

$$0 = \mathrm{freq}(\Lambda, F, V) = \dot{\mu}(X_{F,V}) < \dot{\mu}(X_{F,\overline{V}}) = \mathrm{freq}(\Lambda, F, \overline{V}) = 1 .$$

Chapter 4

Dworkin's argument revisited

Introduction

Dworkin's argument (Section 2.4) shows an important connection between the spectrum and the diffraction of the dynamical system $(X, \mathbb{R}^d, \mu)$. In this chapter, we will see that what underlies Dworkin's argument is a certain isometric embedding θ of the Hilbert space $L^2(\mathbb{R}^d, \widehat{\mu_1})$ into $L^2(X, \mu)$. Both these Hilbert spaces afford natural unitary representations of $\mathbb{R}^d$, call them U_t and T_t respectively ($t \in \mathbb{R}^d$). Representation T arises from the translation action of $\mathbb{R}^d$ on X and U is a multiplication action which we define in (4.1.2). The embedding θ intertwines the representations. However, θ is not in general surjective, and in fact it can fail to be surjective quite badly.

The fact is that the diffraction, or equivalently the autocorrelation measure of a typical point set $\Lambda \in X$, does not usually contain enough information to determine the measure μ, even qualitatively; see for example an explicit discussion of this in [49]. We will give a number of other examples which show that outside the situation of pure point diffraction, one must assume that this is the normal state of affairs. In fact, even in the pure point case, θ can fail to be surjective. However, we shall show in Theorem 4.4.3 that, pure point or not, the knowledge of *all* the correlations of Λ (2-point, 3-point, etc.) is enough to determine μ.

Section 4.3 provides a number of examples that fit into the setting discussed here and that illustrate a variety of things that can happen. The reader may find it useful to consult this section in advance, as the chapter proceeds.

4.1 Diffraction and the embedding theorem

4.1.1 The embedding theorem in the unweighted case

We recall that the Fourier transform of such a measure ω on $\mathbb{R}^d$ can be defined by the formula:

$$\widehat{\omega}(f) = \omega(\widehat{f}) \tag{4.1.1}$$

for all f in the space $\mathbb{S}$ of rapidly decreasing functions of $\mathbb{R}^d$. In fact, it will suffice to have this formula on the space $\mathbb{S}_c$ of compactly supported functions in $\mathbb{S}$, since they are dense in $\mathbb{S}$ in the standard topology on $\mathbb{S}$ ([48]). The key point is that if $\{f_n\} \in \mathbb{S}_c$ converges to $f \in \mathbb{S}$, then $\{\widehat{f_n}\}$ converges to $\widehat{f}$ and one can use the translation boundedness of ω to see then that $\{\omega(\widehat{f_n})\}$ converges to $\{\omega(\widehat{f})\}$, i.e. $\widehat{\omega}(f)$ is known from the values of $\{\widehat{\omega}(f_n)\}$.

The measure $\widehat{\gamma_\lambda}$ is the **diffraction** of λ, when it exists. Our results show that the first moment of the Palm measure, $\mathring{\mu}_1$ must also be a positive, positive definite transformable translation bounded measure and that almost surely $\widehat{\mathring{\mu}_1}$ is the diffraction of $\lambda \in X$.

The next result appears, in a slightly different form in [21]. For complex-valued functions h on $\mathbb{E}$ define $\tilde{h}$ by $\tilde{h}(x) = \overline{h(-x)}$. We denote the standard inner product defined by $||\cdot||_2$ on $L^2(X, \mu)$ by $(\cdot,\cdot)$.

Proposition 4.1.1. *Let $g, h \in BM_c(\mathbb{R}^d)$ and suppose that $g * \tilde{h} * \mathring{\mu}_1$ is a continuous function on $\mathbb{R}^d$. Then for all $t \in \mathbb{R}^d$,*

$$g * \tilde{h} * \mathring{\mu}_1(-t) = (T_t N_g, N_h)\,.$$

PROOF: It suffices to prove the result when g, h are real-valued functions. By Prop.2.2.2, N_g, N_h are measurable functions on X, and they are clearly L^1-functions (Prop. 3.1.1).

$$\begin{aligned}
g * \tilde{h} * \mathring{\mu}_1(-t) &= \int_{\mathbb{R}^d} (g * \tilde{h})(-t-u)\mathrm{d}\mathring{\mu}_1(u) = \int_{\mathbb{R}^d} (\tilde{g} * h)(t+u)\mathrm{d}\mathring{\mu}_1(u) \\
&= \int_{\mathbb{R}^d} \widetilde{T_t g} * h(u)\mathrm{d}\mathring{\mu}_1(u) = \int_X \left(\sum_{x \in \mathbb{R}^d} \lambda(\{x\})(\widetilde{T_t g} * h)(x) \right) \mathrm{d}\mathring{\mu}(\lambda) \\
&= \int_{\mathbb{R}^d} \int_X \sum_{x \in \mathbb{R}^d} \lambda(\{x\})(T_t g)(u) h(x+u)\, \mathrm{d}\mathring{\mu}(\lambda)\mathrm{d}u \\
&= \int_{\mathbb{R}^d} \int_X (T_t g)(u) T_{-u} N_h(\lambda)\mathrm{d}\mathring{\mu}(\lambda)\, \mathrm{d}u
\end{aligned}$$

where we have used (3.1.3) and the dominated convergence theorem to rearrange the sum and the integral. Now using the Campbell formula we may continue:

$$\begin{aligned}\dot{\mu}_1 * g * \tilde{h}(-t) &= \int_X \sum_{u\in\mathbb{R}^d} \lambda(\{u\})(T_t g)(u) N_h(\lambda)\mathrm{d}\mu(\lambda) \\ &= \int_X N_{T_t g}(\lambda) N_h(\lambda)\mathrm{d}\mu = (T_t N_g, N_h)\,.\end{aligned}$$

□

We are now at the point where we can prove the embedding theorem. This involves the two Hilbert spaces $L^2(\mathbb{R}^d, \widehat{\dot{\mu}_1})$ and $L^2(X, \mu)$. Since the translation action of $\mathbb{R}^d$ on X is measure preserving, it gives rise to a unitary representation T of $\mathbb{R}^d$ on $L^2(X,\mu)$ by the usual translation action of $\mathbb{R}^d$ on measures.

We also have a unitary representation U of $\mathbb{R}^d$ on $L^2(\mathbb{R}^d, \widehat{\dot{\mu}_1})$ defined by

$$U_t f(x) = e^{-2\pi i t.x} f(x) = \chi_{-t}(x) f(x)\,, \tag{4.1.2}$$

where the characters χ_k are defined by $\chi_k(x) = e^{2\pi i k.x}$. We denote the inner product of $L^2(\mathbb{R}^d, \widehat{\dot{\mu}_1})$ by $\langle\cdot,\cdot\rangle$ and note that with respect to it U is a unitary representation of $\mathbb{R}^d$.

Proposition 4.1.2. *If $g, h \in \mathbb{S}$ are rapidly decreasing functions then*

$$g * \tilde{h} * \dot{\mu}_1(-t) = \langle U_t(\hat{g}); \hat{h}\rangle\,.$$

In particular,

$$\langle U_t(\hat{g}), \hat{h}\rangle = (T_t N_g, N_h)\,.$$

Thus there is an isometric embedding intertwining U and T,

$$\theta : L^2(\mathbb{R}^d, \widehat{\dot{\mu}_1}) \longrightarrow L^2(X,\mu)\,,$$

under which

$$\hat{f} \mapsto N_f$$

for all $f \in \mathbb{S}$.

PROOF: As we have pointed out, it will suffice to show the first result for $g, h \in \mathbb{S}_c$ since it is dense in $\mathbb{S}$ under the standard topology of $\mathbb{S}$. We note that the hypotheses of Prop. 4.1.1 are satisfied, so, starting as in its proof and denoting the inverse Fourier transform by $f \mapsto \check{f}$, we have

$$g * \tilde{h} * \dot{\mu}_1(-t) = \int_{\mathbb{E}} \widetilde{T_t g} * h(u)\mathrm{d}\dot{\mu}_1(u) = \int_{\mathbb{E}} (\widetilde{T_t g})^{\vee} h^{\vee} \mathrm{d}\widehat{\dot{\mu}_1}\,.$$

The first result follows from $\check{h} = \check{\bar{\tilde{h}}} = \bar{\tilde{h}}$ and $(\widetilde{T_t g})^\vee = \widehat{T_t g} = \chi_{-t}\hat{g}$.

The second part of the proposition follows from Prop. 4.1.1 and the observation that $\mathbb{S}_c$ is dense in $C_c(\mathbb{R}^d)$ in the sup norm ([48], Thm. 1), hence certainly in the $||\cdot||_2$-norm, and $C_c(\mathbb{R}^d)$ is dense in $L^2(\mathbb{R}^d, \widehat{\mu_1})$ in the $||\cdot||_2$-norm (see [46], Appendix E).

Thus we have the existence of the embedding on a dense subset of $L^2(\mathbb{R}^d, \widehat{\mu_1})$ and it extends uniquely to the closure. □

4.1.2 The embedding theorem in the weighted case

Prop. 4.1.1 has the weighted form: Let $g, h \in BM_c(\mathbb{R}^d)$ and suppose that $g * \tilde{h} * \dot{\mu}_1^w$ is a continuous function on $\mathbb{R}^d$. Then for all $t \in \mathbb{R}^d$,

$$g * \tilde{h} * \dot{\mu}_1^w(-t) = (T_t N_g^w, N_h^w)\,. \tag{4.1.3}$$

By Theorem 3.3.2, $\dot{\mu}_1^w$ is Fourier transformable and its Fourier transform is a positive measure, which is denoted by $\widehat{\dot{\mu}_1^w}$.

Our interest now shifts to $L^2(\mathbb{R}^d, \widehat{\dot{\mu}_1^w})$, its inner product $\langle\cdot,\cdot\rangle^w$, and the unitary representation U^w of $\mathbb{R}^d$ on it which is given by the same formula as (4.1.2).

From equation (4.1.3) we obtain our embedding theorem, which is the full colour version of Prop. 4.1.2.

Theorem 4.1.3. *For each system of weights $w = (w_1, \dots, w_m)$, the mapping*

$$\hat{f} \mapsto N_f^w\,, \tag{4.1.4}$$

defined for all $f \in \mathbb{S}$, extends uniquely to an isometric embedding

$$\theta^w : L^2(\mathbb{R}^d, \widehat{\dot{\mu}_1^w}) \longrightarrow L^2(X, \mu)$$

which intertwines the representations U and T.

We note here that the space on the left-hand side depends on w while the space on the right-hand side does not. The question of the image of θ^w is then an interesting one. We come to this later.

We also note that the formula for $\theta^w(f)$ in (4.1.4), though true for $f \in \mathbb{S}$, and no doubt many other functions too, is not true in general, and in particularly not true for some functions that we will need to consider in the discussion of spectral properties, e.g. see Cor. 4.1.6.

Theorem 4.1.3 gives an isometric embedding of $L^2(\mathbb{R}^d, \widehat{\dot{\mu}_1^w})$ into $L^2(X, \mu)$ and along with it a correspondence of the spectral components of $L^2(\mathbb{R}^d, \widehat{\dot{\mu}_1^w})$ and its image in $L^2(X,\mu)$. Now the point is that the spectral information of $L^2(\mathbb{R}^d, \widehat{\mu_1^w})$ can be read directly off that of the measure $\widehat{\mu_1^w}$. Specifically, let $\widehat{\mu_1^w} = (\widehat{\mu_1^w})_{pp} + (\widehat{\mu_1^w})_{sc} + (\widehat{\mu_1^w})_{ac}$ be the decomposition of $\widehat{\mu_1^w}$ into its pure point, singular continuous, and absolutely continuous parts. For $f \in L^2(\mathbb{R}^d, \widehat{\mu_1^w})$, the associated spectral measure σ_f^w on $\mathbb{R}^d$ is given by

$$\langle f, U_t f\rangle^w = \int e^{2\pi i x.t} d\sigma_f^w(x) .$$

However,

$$\langle f, U_t f\rangle^w = \int e^{2\pi i x.t} f(x)\overline{f(x)} d\widehat{\mu_1^w}(x) ,$$

so we have

$$\sigma_f^w = |f|^2 \widehat{\mu_1^w} = |f|^2 (\widehat{\mu_1^w})_{pp} + |f|^2 (\widehat{\mu_1^w})_{sc} + |f|^2 (\widehat{\mu_1^w})_{ac} , \tag{4.1.5}$$

which is the spectral decomposition of the measure σ_f. With $\square$ standing for pp, sc, or ac, we have

$$\begin{aligned} L^2(\mathbb{R}^d, \widehat{\mu_1^w})_\square &:= \{f \in L^2(\mathbb{R}^d, \widehat{\mu_1^w}) \;:\; \sigma_f^w \text{ is of type } \square\} \\ &= \{f \in L^2(\mathbb{R}^d, \widehat{\mu_1^w}) \;:\; \operatorname{supp}(f) \subset \operatorname{supp}((\widehat{\mu_1^w})_\square)\} . \end{aligned}$$

This explains how information about the spectrum of the diffraction can be inferred from the nature of the dynamical spectrum and vice-versa. Since the mapping θ depends on w and is not always surjective, the correspondence between the two has to be treated with care. Some examples of what can happen are given in § 4.3.

Combining (4.1.5) with Theorem 4.1.3, we have S. Dworkin's theorem:

Corollary 4.1.4. *Let $f \in C_c(\mathbb{R}^d)$. Then for μ- almost all $\lambda \in X$, $\widehat{\gamma_{f*\lambda}^w}$ is the spectral measure $\sigma_{N_f^w}$ on $L^2(X,\mu)$.*

Proof: $\gamma_{f*\lambda}^w = f * \tilde{f} * \gamma_\lambda^w$, so $\widehat{\gamma_{f*\lambda}^w} = |\widehat{f}|^2 \widehat{\gamma_\lambda^w} = |\widehat{f}|^2 \widehat{\mu_1^w} = \sigma_{\widehat{f}}^w$ almost surely. Now,

$$\langle \widehat{f}, U_t \widehat{f}\rangle^w = (N_f^w, T_t N_f^w)_{L^2(X,\mu)}$$

so the spectral measure $\sigma_{\widehat{f}}^w$ computed for $L^2(\mathbb{R}^d, \widehat{\mu_1^w})$ is the same as the spectral measure $\sigma_{N_f^w}$ computed for $L^2(X,\mu)$. $\square$

Corollary 4.1.5. *For all $f, g \in L^2(\mathbb{R}^d, \widehat{\mu_1^w})$, the spectral measures $(\langle U_t f, g\rangle^w)^\vee$ and $(T_t\theta^w(f), \theta^w(g))^\vee$ on $\mathbb{R}^d$ are equal, and in particular of the same spectral type: absolutely continuous, singular continuous, pure point, or mix of these.*

Corollary 4.1.6. *For $k \in \mathbb{R}^d$, χ_k is in the point spectrum*[1] *of U_t if and only if $\widehat{\mu_1^w}(k) \neq 0$. The corresponding eigenfunction is $\mathbf{1}_{\{-k\}}$. When this holds, χ_k is in the point spectrum of T_t, the eigenfunction corresponding to it is $\theta^w(\mathbf{1}_{\{-k\}})$, and $||\theta^w(\mathbf{1}_{\{-k\}})|| = \dot{\mu}^w(k)^{1/2}$.*

PROOF: The first statement is clear from (4.1.2) and our remarks above. For the second, suppose that $k \in \mathbb{R}^d$ and $\widehat{\mu_1^w}(k) \neq 0$. Let $f \in L^1(\mathbb{R}^d, \widehat{\mu_1^w})$ be an eigenfunction for k. Then

$$\exp(2\pi ik.t)f(x) = U_t f(x) = \exp(-2\pi it.x)f(x)$$

for all $x \in \mathbb{R}^d$. For x with $f(x) \neq 0$, $\exp(2\pi ik.t) = \exp(-2\pi ix.t)$ for all $t \in \mathbb{R}^d$, so $x = -k$. Thus $f = f(-k)\mathbf{1}_{\{-k\}}$. By Thm. 4.1.3, $\theta^w(f) \in L^2(X,\mu)$ with $T_t(\theta^w(f)) = \chi_k(t)\theta^w(f)$ for all $t \in \mathbb{R}^d$. □

Remark 4.1.7. : *One should note that the eigenvalues always occur in pairs $\pm k$ since $\dot{\mu}_1$ is positive-definite and $\widehat{\mu_1^w}(-k) = \widehat{\mu_1^w}(k)$. How does one work out $\theta^w(\mathbf{1}_{\{-k\}})$? This is the content of the L^2-mean form Bombieri-Taylor conjecture that we shall establish in Sec. 5.1.*

4.2 The algebra generated by the image of θ

4.2.1 The density of $\Theta^w(\mathbb{S})$

Theorem 4.2.1. *Let (X,μ) be an m-coloured stationary uniformly discrete ergodic point process and w a system of weights. Suppose that the weights w_i, $i = 1,\ldots,m$ are all different from one another and also none of them is equal to 0. Then the algebra Θ^w generated by $\theta^w(\mathbb{S})$ and the identity function 1_X is dense in $L^2(X,\mu)$.*

Remark 4.2.2. *If $\widehat{\mu_1^w}(0) \neq 0$ then $\theta^w(\mathbb{S})$ already contains 1_X by Cor. 4.1.6.*

The remainder of this subsection is devoted to the proof of this theorem.

We begin with the construction of certain basic types of finite partitions of X. Here we will find it easier to deal with coloured point sets than with their corresponding measures.

Let $r > 0$ be fixed so that $X \subset \mathscr{M}_p(C_r^{(m)}, 1)$. For each pair of measurable sets $K, V \subset \mathbb{R}^d$, with K bounded and V a neighbourhood of 0, we define

[1] One often simply says that k is in the point spectrum, with the understanding that it means χ_k.

$$U(K,V) := \{(\Lambda,\Lambda') \in \mathcal{D}_r : K \cap \Lambda \subset V + \Lambda' \quad \text{and} \quad K \cap \Lambda' \subset V + \Lambda\}, \tag{4.2.1}$$

which is just a variation on (2.3.1), and serves to define another fundamental system of entourages for the same uniformity, and then the same topology, on X as we have been using all along. For any $\Phi \in \mathcal{D}_r^{(m)}$ we define

$$U(K,V)[\Phi] := \{\Lambda \in X : (\Lambda,\Phi) \in U(K,V)\}.$$

We begin by choosing a finite grid in $\mathbb{R}^d$ and partitioning X according to the colour patterns it makes in this grid. Here are the details. Let $K \subset \mathbb{R}^d$ be a half open cube of the form $[a_1, a_1 + R) \times \cdots \times [a_d, a_d + R)$, $R > 0$, and V be an half-open cube of diameter less than r, centred on 0, which is so sized that its translates can tile K without overlaps. The set of translation vectors used to make up this tiling is denoted by Ψ, so in fact this set is the set of centres of the tiles of the tiling. Each centre locates a tile and in each of these tiles we can have at most one coloured point of Λ, that is, at most one pair (x,i) with $x \in \mathbb{R}^d$ and $i \leq m$. Let

$$\begin{aligned}\mathfrak{P} := \{\Phi = (\Phi_1,\ldots,\Phi_m) \; : (\Phi_0,\Phi_1,\ldots,\Phi_m) \\ \text{is an ordered partition of } \Psi\};\end{aligned} \tag{4.2.2}$$

that is, we take all possible ordered partitions of Ψ into $m+1$ pieces, which we interpret as all the various coloured patterns of cells of our tiling. Φ_i designates the cells containing the points of colour i (second component i), $i = 1,\ldots m$, and Φ_0 designates all the cells which contain no points of the pattern.

The inclusion relation $\subset$ on $\mathfrak{P}$ by $\Phi = (\Phi_1,\ldots,\Phi_m) \subset \Phi' = (\Phi'_1,\ldots,\Phi'_m)$ if and only if $\Phi_i \subset \Phi'_i$, for all $1 \leq i \leq m$, provides a natural partial ordering on $\mathfrak{P}$. Using the notation established in (1.4.1), for each $\Phi \in \mathfrak{P}$ define

$$P[\Phi] := \{\Lambda \in X : K \cap \Lambda \subset V + \Phi, K \cap \Lambda \not\subset V + \Phi' \text{ for any } \Phi' \subsetneqq \Phi\}.$$

Because of the choice of V, an element of X can have at most one point in any one of the cubes making up the tiling of K. Each $P[\Phi]$ is the set of elements of X which make the coloured pattern Φ inside the cube K.

Lemma 4.2.3.

$$X = \bigcup_{\Phi \in \mathfrak{P}} P[\Phi]$$

is a partition of X. Furthermore, for all $\Phi \in \mathfrak{P}$,

$$U(K,V^\circ)[\Phi] \cap X \subset P[\Phi] \subset U(K,\overline{V})[\Phi] \cap X.$$

PROOF: By construction the $P[\Phi]$ form a partition of X. Let $\Phi \in \mathfrak{P}$ and let $\Lambda \in P[\Phi]$. Then $K \cap \Lambda \subset V + \Phi$. Also, for each s lying in some component Φ_i of Φ there is $x \in K \cap \Lambda_i$ with $x \in V + s$, whence $s \in -V + x \subset \overline{V} + x$. This shows $K \cap \Phi \subset \overline{V} + \Lambda$, so $\Lambda \in U(K, \overline{V})[\Phi]$.

On the other hand, if $\Lambda \in U(K, V^\circ)[\Phi]$ then $K \cap \Lambda \subset V^\circ + \Phi \subset V + \Phi$, which is the first condition for $\Lambda \subset P[\Phi]$. Since also $\Phi = K \cap \Phi \subset V^\circ + \Lambda$, for each s in some component Φ_i there is $x \in \Lambda_i$ with $x = -v + s \in V^\circ + s \subset V + s$. By the construction of the tiling of K, no other set $V^\circ + t$, $t \in \Psi$, can contain x. Thus Λ meets every tile centred on a point of Φ and $\Lambda \in P[\Phi]$. □

We know that $\theta^w(\mathbb{S})$ contains all the functions N_f^w, $f \in \mathbb{S}$, in particular all the N_f^w, $f \in \mathbb{S}_c$, and so its L^2-closure contains N_f^w, $f \in C_c(\mathbb{R}^d)$ (use Prop. 3.1.2). Again, using Prop. 3.1.2 we can conclude that $\overline{\theta^w(\mathbb{S})}$ contains all the functions N_A^w, where A is an bounded open or closed subset of $\mathbb{R}^d$. We start with these functions and work to produce more complicated ones.

Lemma 4.2.4. *Let $s \in \Psi$ and let $i \leq m$. Then the functions $N_{V^\circ+s} \circ \mathrm{res}^i(\Lambda)$ and $N_{\overline{V}+s} \circ \mathrm{res}^i(\Lambda)$ are in $\overline{\Theta^w}$.*

PROOF: $N_{V^\circ+s}^w \in \overline{\Theta^w}$. From (3.3.3) and $\mathrm{diam}(V) < r$,

$$N_{V^\circ+s}^w(\Lambda) = \sum_{i=1}^m w_i N_{V^\circ+s} \circ \mathrm{res}^i(\Lambda) \tag{4.2.3}$$

$$= \sum_{i=1}^m w_i N_{V^\circ+s}(\Lambda^i) = 0 \text{ or } w_j \tag{4.2.4}$$

according as $(V^\circ + s) \cap \Lambda$ is empty or contains a (necessarily unique) point x of some colour j. Write $\boldsymbol{F}$ for $N_{V^\circ+s}^w$ and F for $N_{V^\circ+s}$. The first is a function on X, the second a function on r-uniformly discrete subsets of $\mathbb{R}^d$ (see (3.1.1)). Then[2] $\boldsymbol{F}^j(\Lambda) = \sum_{i=1}^m w_i^j F(\Lambda_i)$ since always $F^j(\Lambda_i) = F(\Lambda_i)$ and $F(\Lambda_i)F(\Lambda_k) = 0$ whenever $i \neq k$.

Let W be the $m \times m$ matrix defined by $W_{jk} = w_k^j$. By the hypotheses on the weights it has an inverse Y. Then

$$\sum_{j=1}^m Y_{ij} \boldsymbol{F}^j(\Lambda) = \sum_{j=1}^m Y_{ij} \sum_{k=1}^m w_k^j F(\Lambda_k) = F(\Lambda_i).$$

This proves that the functions $\Lambda \mapsto N_{V^\circ+s}(\Lambda_i) = N_{V^\circ+s} \circ \mathrm{res}_i(\Lambda)$ are all in Θ^w. The same argument applies in the case of $\overline{V}$. □

[2] Here the superscripts really mean powers!

Lemma 4.2.5. *For all $\Phi \in \mathfrak{P}$, $\mathbf{1}_{P[\Phi]} \in \overline{\Theta^w}$.*

PROOF: Let $\Phi \in \mathfrak{P}$ and assume $\Phi \neq \emptyset$. Let

$$\begin{aligned} f_1 : &= \Pi_{i=1}^m \Pi_{s\in\Phi_i} N_{V^\circ+s} \circ \mathrm{res}_i \\ f_2 : &= \Pi_{i=1}^m \Pi_{s\in\Phi_i} N_{\overline{V}+s} \circ \mathrm{res}_i \,, \end{aligned}$$

are all in Θ^w. These functions take the value 1 only on sets Λ which hit all the cells $V^\circ + s$ (respectively $\overline{V} + s$) centred on the points and with the colours specified by Φ. However, such Λ may hit other cells also, hence

$$f_1 \leq \sum_{\Phi\subset\Phi'\in\mathfrak{P}} \mathbf{1}_{P[\Phi']} \leq f_2 \,.$$

However, for any fixed i,

$$\begin{aligned} \int |N_{\overline{V}+s}(\Lambda_i) &- N_{V^\circ+s}(\Lambda_i)|^2 \mathrm{d}\mu(\Lambda) = \int |N_{\overline{V}+s}(\Lambda_i) - N_{V^\circ+s}(\Lambda_i)| \mathrm{d}\mu(\Lambda) \\ &= \int |N_{(\overline{V}\backslash V^\circ)+s}(\Lambda_i)| \mathrm{d}\mu(\Lambda) = I^i\ell((\overline{V}\backslash V^\circ + s)) = 0 \end{aligned}$$

showing that $N_{\overline{V}+s}$ and $N_{V^\circ+s}$ are equal as L^2 functions, whence also f_1 and f_2 are equal. This shows that

$$\sum_{\Phi'\supset\Phi} \mathbf{1}_{P[\Phi']} \in \overline{\Theta^w} \,.$$

In the case that Φ is empty,

$$\sum_{\Phi'\supset\Phi} \mathbf{1}_{P[\Phi']} = \sum_{\Phi'\subset\Psi} \mathbf{1}_{P[\Phi']} = \mathbf{1}_X \in \overline{\Theta^w} \,.$$

Now by Möbius inversion on the partially ordered on the subsets of $\mathfrak{P}$, $\mathbf{1}_{P[\Phi]} \in \overline{\Theta^w}$ for all $\Phi \in \mathfrak{P}$. $\square$

Lemma 4.2.6. *Let $F : X \longrightarrow \mathbb{R}$ be a continuous function and let $\epsilon > 0$. Then there exist half-open cubes K, V as above so that for the corresponding partition of X,*

$$||F - \sum_{\Phi\in\mathfrak{P}} m_\Phi \mathbf{1}_{P[\Phi]}||_\infty \leq \epsilon \,,$$

where $m_\Phi := \inf\{F(\Lambda) : \Lambda \subset P[\Phi]\}$.

PROOF: Since X is compact, F is uniformly continuous. Then given $\epsilon > 0$ there exist a compact set $K \subset \mathbb{R}^d$ and a neighbourhood V' of $0 \in \mathbb{R}^d$ so that $|F(\Lambda) - F(\Lambda')| < \epsilon$ for all $(\Lambda, \Lambda') \in U(K, V') \cap (X \times X)$. We can increase K to some half-open cube of the type above without spoiling this and then choose some half-open cube V, centred on 0 and of diameter less than r, which tiles K and also satisfies $2\overline{V} \subset V'$. We let $\mathfrak{P}$ be the corresponding set of partitions.

Let $\Phi \in \mathfrak{P}$. If $\Lambda, \Lambda' \in U(K, \overline{V})[\Phi] \cap X$ then $K \cap \Lambda \subset \overline{V} + \Phi$ and $\Phi \subset \overline{V} + \Lambda'$. Thus for any $x \in K \cap \Lambda$, $x = v + s = v' + v + x'$ where $s \in \Phi$, $x' \in \Lambda'$ (both with the same colour as x), and $v, v' \in \overline{V}$, from which we conclude $K \cap \Lambda \subset 2\overline{V} + \Lambda'$. In the same way $K \cap \Lambda' \subset 2\overline{V} + \Lambda$, so $(\Lambda, \Lambda') \in U(K, V')$ and $|F(\Lambda) - F(\Lambda')| < \epsilon$. In particular this holds for all $\Lambda, \Lambda' \in P[\Phi]$, since it is contained in $U(K, \overline{V})[\Phi]$, and so F varies by less than ϵ on $P[\Phi]$. The result follows at once from this. □

The proof of Thm. 4.2.1 is an immediate consequence of this. $\overline{\Theta^w(\mathbb{S})}$ contains the functions$\mathbf{1}_{P[\Phi]}$ and so also all their limit points, and hence all continuous functions on X. Finally the continuous functions are dense in $L^2(X, \mu)$. □

For the case $m = 1$, recall that the nth moment of μ is the measure μ_n on $(\mathbb{R}^d)^n$ is defined by $\mu_n(f_1, \ldots, f_n) = \mu(N_{f_1} \ldots N_{f_n})$. Since Thm. 4.2.1 says that the linear span of all the product functions $N_{f_1} \ldots N_{f_n}$ is dense in $L^2(X, \mu)$, we see that μ is entirely determined by its moment measures.

In the general case we may define the nth weighted moments by:

$$\mu_n^w(f_1, \ldots, f_n) = \mu(N_{f_1}^w \ldots N_{f_n}^w)\,. \tag{4.2.5}$$

Then the same argument leads to:

Proposition 4.2.7. *Let (X, μ) be an m-coloured stationary uniformly discrete ergodic point process and w a system of weights in which w_i, $i = 1, \ldots, m$ are all different from one another and also none of them is equal to 0. Then the measure μ is determined entirely by its set of nth weighted moments, $n = 1, 2, \ldots$.*

□

We will relate this to higher correlations in the next section.

Corollary 4.2.8. *Let (X, μ) and w be as in Prop. 4.2.7. Then the measure $\widehat{\mu_1^w}$ (which is the almost sure diffraction for the members of X when the weighting is w) is pure point if and only if the dynamical system (X, μ) is pure point, i.e., the linear span of the eigenfunctions is dense in $L^2(X, \mu)$.*

Remark 4.2.9. *This is the principal result of [34]. See also [21].*

PROOF: The 'if' direction is a consequence of Corollary 4.1.5 of Theorem 4.1.3.

The idea behind the 'only if' direction is simple enough. The assumption is that the linear space of the eigenfuctions $L^2(\mathbb{R}^d, \widehat{\mu_1^w})$ is dense in $L^2(\mathbb{R}^d, \widehat{\mu_1^w})$, and eigenfuctions of $L^2(\mathbb{R}^d, \widehat{\mu}_1)$ map to eigenfuctions of $L^2(X, \mu)$ under θ^w. However, products of eigenfunctions of (X, μ) are again eigenfunctions. We know that the algebra generated by the image of $\mathbb{S}(\mathbb{R}^d)$ in $L^2(X, \mu)$ is dense. So the linear space that we get by taking the algebra generated by the eigenfuctions ought also to be both dense and linearly generated by eigenfunctions. The trouble is that the eigenfuctions of $L^2(\mathbb{R}^d, \widehat{\mu_1^w})$ are not in $\mathbb{S}(\mathbb{R}^d)$ and the space $L^2(X, \mu)$ is not closed under multiplication, so we need to be careful.

The set $BL^2(X, \mu)$ of measurable square integrable functions on X that are bounded on a subset of full measure form an algebra (i.e. the product of such functions are also bounded), and $\theta^w(\mathbb{S})$ is contained in it. In fact any bounded function of $L^2(\mathbb{R}^d, \widehat{\mu_1^w})$ is mapped by θ^w into $BL^2(X, \mu)$, as we can see from Theorem 4.1.3 and equation (3.3.2) and taking approximations by elements of $\mathbb{S}$. For $F \subset BL^2(X, \mu)$, let $L(F)$ denote its linear span and $\langle F \rangle_{\text{alg}}$ the subalgebra of $BL^2(X, \mu)$ generated by F.

By Corollary 3, χ_k is in the point spectrum of U_t if and only if $\widehat{\mu_1^w}(k) \neq 0$, and the eigenfunction corresponding to χ_k is $\theta^w(\mathbf{1}_{\{-k\}})$. Denote by E the set of $\{\mathbf{1}_{\{-k\}} : \widehat{\mu_1^w}(k) \neq 0\}$ and by $L(E)$ its linear span. By Theorem 4.1.3, $\theta^w(E)$ is a set of eigenfunctions of T_t, and by what we just saw $\theta^w(L(E)) \subset BL^2(X, \mu)$. By assumption, $L(E)$ is dense in $L^2(\mathbb{R}^d, \widehat{\mu_1^w})$.

Then

$$\overline{L(\theta^w(E))} \supset \theta^w(\overline{L(E)}) \supset \theta^w(\mathbb{S})$$

and

$$BL^2(X, \mu) \supset \langle \theta^w(L(E)) \rangle_{\text{alg}} .$$

Thus,

$$\overline{\langle \theta^w(E) \rangle_{\text{alg}}} = \overline{\langle \theta^w(\overline{L(E)}) \rangle_{\text{alg}}} \supset \overline{\langle \theta^w(\mathbb{S}) \rangle_{\text{alg}}} = L^2(X, \mu) ,$$

which shows that the denseness of the linear span of the eigenfunctions of $L^2(X, \mu)$.

□

4.3 The continuous dynamics of sequences on the real line

4.3.1 Symbolic shifts and sequences on the real line

A good source for examples is to start with symbolic shifts. We start with a finite alphabet $\mathbf{m} = \{1, \dots m\}$ and then define $\mathbf{m}^{\mathbb{Z}}$ to be the set of all bi-infinite sequences $\zeta = \{z_i\}_{-\infty}^{\infty}$, which we supply with the product topology. Along with the usual shift action $(T(\zeta))_i = \zeta_{i+1}$ for all i, $\mathbf{m}^{\mathbb{Z}}$ becomes a dynamical system over the group $\mathbb{Z}$. We are interested in compact $\mathbb{Z}$-invariant subspaces $X_{\mathbb{Z}}$ of $(\mathbf{m}^{\mathbb{Z}}, \mathbb{Z})$. We will assume that $(X_{\mathbb{Z}}, \mathbb{Z})$ is equipped with an invariant and ergodic probability measure $\mu_{\mathbb{Z}}$. Such measures always exist. We call such a system an **ergodic symbolic dynamical system**. We define for all $\underline{i} = (i_0, \dots, i_k) \subset \mathbf{m}^{k+1}$, $k = 0, 1 \dots$, and $p \in \mathbb{Z}$,

$$X_{\mathbb{Z}}[\underline{i}; p] = \{\zeta \in X_{\mathbb{Z}} \,:\, z_{j+p} = i_j, j = 0, \dots k\} .$$

These cylinder sets form a set of entourages for the standard uniform topology on $X_{\mathbb{Z}}$ which defines the product topology. When $p = 0$, we usually leave it out and also leave off the parentheses; so, for example, $X_{\mathbb{Z}}[ij]$ means $X_{\mathbb{Z}}[(ij); 0]$.

We need to move from the discrete dynamics (action by $\mathbb{Z}$) of $(X_{\mathbb{Z}}, \mathbb{Z})$ to continuous dynamics with an $\mathbb{R}$-action. There is a standard way of doing this by creating the suspension flow of $(X_{\mathbb{Z}}, \mathbb{Z})$, and this new dynamical system has a natural invariant and ergodic measure and so satisfies our conditions PPI, PPII, PPIII. Basically each bi-infinite sequence ζ of $(X_{\mathbb{Z}}, \mathbb{Z})$ is converted into a bi-infinite sequence of coloured points on the real line with z_0 being located at 0. The most obvious thing is to space out the other points of the sequence on the integers, so that z_n ends up at position n. The result can be viewed as a tiling of the line with coloured tiles of length 1, the colour of a tile being the colour of the left end point that defines it. However, there are good reasons to allow different colours to have different tile lengths. [3]

For this purpose we take any set $\mathcal{L} = \{L_1, \dots, L_m\}$ of positive numbers as the tile lengths, with an overall scaling so that

$$\sum_{j=1}^{m} L_j \mu_{\mathbb{Z}}(X_{\mathbb{Z}}[j]) = 1 .$$

Let $r = \min\{L_1, \dots, L_m\}/2$.

[3]Readers interested only in the examples germane to this paper may ignore the introduction of different tile lengths that we introduce here.

Given $\zeta = \{z_n\}_{-\infty}^{\infty} \in X_{\mathbb{Z}}$, define the sequence $S = S(\zeta) = \{S_n\}_{-\infty}^{\infty}$ by $S_0 = 0$, $S_n = \sum_{j=0}^{n-1} L_{z_j}$, if $n > 0$, $S_n = -\sum_{j=n}^{-1} L_{z_j}$ if $n < 0$.

Define

$$\begin{aligned} \pi^{\mathcal{L}} : \mathbb{R} \times X_{\mathbb{Z}} &\longrightarrow \mathcal{D}_r^{(m)}(\mathbb{R}) \\ (t, \zeta) &\mapsto \{(t + S_n, z_n)\}_{-\infty}^{\infty}, \end{aligned} \tag{4.3.1}$$

which "locates" the symbols of ζ along the line (including colour information) so that the nth symbol occurs at $t + S_n$. This simultaneously provides us with a tiling of the line by line segments of lengths $\{L_{z_n}\}$. We let $X_{\mathbb{R}}^{\mathcal{L}} := \pi^{\mathcal{L}}(X_{\mathbb{Z}}) \subset \mathcal{D}_r^{(m)}(\mathbb{R})$. Both $\mathbb{R} \times X_{\mathbb{Z}}$ and $\mathcal{D}_r^{(m)}(\mathbb{R})$ have natural $\mathbb{R}$-actions on them, and the mapping $\pi^{\mathcal{L}}$ is $\mathbb{R}$-invariant. It is easy to see that $\pi^{\mathcal{L}}$ is continuous.

Let R be the equivalence relation on $\mathbb{R} \times X_{\mathbb{Z}}$ defined by transitive, symmetric, and reflexive extension of $(t, \zeta) \equiv_R (t + L_{z_0}, T\zeta)$. Evidently pairs are R-equivalent if and only if they have the same image under $\pi^{\mathcal{L}}$. In fact, (t, ζ) is R-equivalent to a unique element of

$$F^{\mathcal{L}} := \bigcup_{i=1}^{m} (-L_i, 0] \times X_{\mathbb{Z}}[i]$$

and the mapping $\pi^{\mathcal{L}}$ is injective on this set. Since $\overline{F^{\mathcal{L}}} = \bigcup [-L_i, 0] \times X_{\mathbb{Z}}[i]$ is compact and $\pi^{\mathcal{L}}$ maps this set onto $X_{\mathbb{R}}^{\mathcal{L}}$, we see that $X_{\mathbb{R}}^{\mathcal{L}}$ is compact and hence $(X_{\mathbb{R}}^{\mathcal{L}}, \mathbb{R})$ is a topological dynamical system.

4.3.2 Measures on the suspension

We define a positive measure $\mu^{\mathcal{L}}$ on $X_{\mathbb{R}}^{\mathcal{L}}$ by

$$\mu^{\mathcal{L}}(B) := (\ell \otimes \mu_{\mathbb{Z}})((\pi^{\mathcal{L}})^{-1}(B) \cap F^{\mathcal{L}}), \tag{4.3.2}$$

for all Borel subsets B of $X_{\mathbb{R}}^{\mathcal{L}}$. We observe that $\mu^{\mathcal{L}}$ is a probability measure since $\mu^{\mathcal{L}}(X_{\mathbb{R}}^{\mathcal{L}}) = (\ell \otimes \mu_{\mathbb{Z}})(F^{\mathcal{L}}) = \sum L_i \mu_{\mathbb{Z}}(X[i]) = 1$.

This is an $\mathbb{R}$-invariant measure on $X_{\mathbb{R}}^{\mathcal{L}}$. It suffices to show the shift invariance for sets of the form $J \times C$ where J is an interval in $(-L_i, 0]$ and C is a measurable subset of some $X_{\mathbb{Z}}[i]$, since these sets generate the σ-algebra of all Borel subsets of $F^{\mathcal{L}}$. We show that shifting of J by $s < 0$ leaves the measure invariant. It is sufficient to do this for $|s| < \min\{L_1, \ldots, L_m\}$, since we can repeat the process if necessary to account for larger s. If $s + J \subset (-L_i, 0]$, then the invariance of ℓ gives what we need immediately. If $s + J \not\subseteq (-L_i, 0]$ then we may break J into two parts; the part which remains in the interval and the part which moves out of it to the left. We can restrict our attention to the part that moves out and then assume that $(s+J)\cap(-L_i,0]=\emptyset$. Then we bring $(s+J)\times C$ back into $F^{\mathcal{L}}$ by writing $C=\bigcup_{j=1}^m C\cap X_{\mathbb{Z}}[ij]$ so that

$$(s+J)\times C\equiv_R\bigcup_{j=1}^m (L_i+s+J)\times T(C\cap X_{\mathbb{Z}}[ij])\,.$$

The measure of this is $\sum_{j=1}^m \ell(J)\mu_{\mathbb{Z}}(T(C\cap X_{\mathbb{Z}}[ij]))=\ell(J)\mu_{\mathbb{Z}}(C)=(\ell\otimes\mu_{\mathbb{Z}})(J\times C)$, which is what we wished to show.

If the original measure $\mu_{\mathbb{Z}}$ on $X_{\mathbb{Z}}$ is ergodic, then so is the measure $\mu^{\mathcal{L}}$. One way to see this is to start with the case when $\mathcal{L}=\{1,\dots,1\}$. In this case we shall denote the objects that we have constructed above with a superscript $\mathbf{1}$ rather than $\mathcal{L}$. It is easy to see that $\mu^{\mathbf{1}}$ is an ergodic measure on $X_{\mathbb{R}}^{\mathbf{1}}$ since the latter can be thought of as $X_{\mathbb{Z}}\times U(1)$, where $U(1)$ is the unit circle in $\mathbb{C}$, with the action of $\mathbb{R}$ being such that going clockwise around the circle once returns one to the same sequence in $X_{\mathbb{Z}}$ except shifted once.

We can define a flow equivalence $\phi: X_{\mathbb{R}}^{\mathbf{1}}\longrightarrow X_{\mathbb{R}}^{\mathcal{L}}$ in the following way. For each $\zeta\in X_{\mathbb{Z}}$ define $f_\zeta:\mathbb{R}\longrightarrow\mathbb{R}$ by

$$f_\zeta(t)=\begin{cases}|S_{-(k-1)}|+L_{z_{-k}}(t-|S_{-(k-1)}|) & \text{if } t\geq 0, k-1\leq t<k\\ -S_{k-1}+L_{z_k}(t-S_{k-1}) & \text{if } t\leq 0, k-1\leq |t|<k\,.\end{cases}$$

This is a strictly monotonic piece-wise linear continuous function which fixes 0. Its intent is clear: if (t,ζ) is understood to represent the sequence ζ placed down in equal step lengths of one unit starting with z_0 at t, then $(f_\zeta(t),\zeta)$ represents the same sequence, now scaled to the new colour lengths L_{z_j} where 0 is the fixed point.

Thus define a mapping $\mathbb{R}\times X_{\mathbb{Z}}\longrightarrow\mathbb{R}\times X_{\mathbb{Z}}$ by $(t,\zeta)\mapsto(f_\zeta(t),\zeta)$. This mapping factors through the equivalence relations that define $X_{\mathbb{R}}^{\mathbf{1}}$ and $X_{\mathbb{R}}^{\mathcal{L}}$ to give the mapping ϕ which is the flow equivalence that we have in mind. For $I\times X_{\mathbb{Z}}[\underline{u}]$, where $I\subset(-L_{u_0},0]$,

$$\phi^{-1}((I\times X_{\mathbb{Z}}[\underline{u}])^{\sim})=\left(\frac{I}{L_{u_0}}\times X_{\mathbb{Z}}[\underline{u}]\right)^{\sim},$$

where the equivalence relations are taken for $\mathcal{L}$ and for $\mathbf{1}$ respectively. Furthermore, $\mu^{\mathcal{L}}((I\times X_{\mathbb{Z}}[\underline{u}])^{\sim})=\ell(I)\mu_{\mathbb{Z}}(X_{\mathbb{Z}}[\underline{u}])$ and $\mu^{\mathbf{1}}((I/L_{u_0}\times X_{\mathbb{Z}}[\underline{u}])^{\sim})=\ell(I/L_{u_0})\mu_{\mathbb{Z}}(X_{\mathbb{Z}}[\underline{u}])$.

Now, if B is an $\mathbb{R}$-invariant subset of $X_{\mathbb{R}}^{\mathcal{L}}$ then $\phi^{-1}(B)$ is an $\mathbb{R}$-invariant subset of $X_{\mathbb{R}}^{\mathbf{1}}$, and so, assuming that $\mu_{\mathbb{Z}}$ is ergodic, $\phi^{-1}(B)$ has measure 1 or 0. If the former, then for all $i\leq m$, $\phi^{-1}(B)\cap((-1,0]\times X_{\mathbb{Z}}[i])$ has $\mu^{\mathbf{1}}$-measure $\mu_{\mathbb{Z}}(X_{\mathbb{Z}}[i])$ from which $B\cap((-L_i,0]\times X_{\mathbb{Z}}[i])$ has measure $L_i\mu_{\mathbb{Z}}(X_{\mathbb{Z}}[i])$, which shows that B is of full measure in $X_{\mathbb{R}}^{\mathcal{L}}$. A similar argument works for the measure 0 case. This shows that $\mu^{\mathcal{L}}$ is ergodic.

4.3.3 Spectral features of the suspension

We assume that $(X_{\mathbb{R}}^{\mathcal{L}}, \mathbb{R}, \mu^{\mathcal{L}})$ is a continuous ergodic dynamical system. Henceforth we shall assume that the set of lengths $\mathcal{L} = \{L_1, \ldots, L_m\}$ is fixed, and drop it from the notation. We may weight the system by choosing any real vector $w = (w_1, \ldots w_m)$ of weights and assigning weight w_i to the colour a_i. According to Prop. 3.3.2 , the weighted first moment $\dot{\mu}_1^w$ of the weighted Palm measure is almost everywhere the weighted autocorrelation of the point sets of $X_{\mathbb{R}}$, and this is everywhere true if the system is uniquely ergodic. We will use the symbol w to also denote the mapping $\mathbf{m} \longrightarrow \{w_1, \ldots w_m\}$, $w(i) = w_i$.

We now come to the autocorrelation. For the purposes of the examples, it is convenient to have all tile lengths equal to 1: $L_j = 1$ for all j, and we shall assume this for the remainder of this section.

Now let $\zeta = \{z_i\}_{-\infty}^{\infty} \in X_{\mathbb{Z}}$. Its autocorrelation, assuming that it exists, is

$$\gamma_\zeta^{w,\mathbb{Z}} = \sum_{k \in \mathbb{Z}} \eta^w(k) \delta_k^{\mathbb{Z}} \,,$$

defined on $\mathbb{Z}$, where

$$\eta^w(k) := \lim_{N \to \infty} \frac{1}{2N+1} \sum_{i=-N}^{N} w(z_i) w(z_{i+k}) \,.$$

Its autocorrelation $\gamma_\zeta^{w,\mathbb{R}}$ when thought of as an element of $X_{\mathbb{R}}$ is defined on $\mathbb{R}$ and is given by

$$\gamma_\zeta^{w,\mathbb{R}} = \sum_{k \in \mathbb{Z}} \eta^w(k) \delta_k^{\mathbb{R}} \,,$$

with the same $\eta^w(k)$.

The difference is in the delta measures, which are defined on $\mathbb{Z}$ and $\mathbb{R}$ respectively. Thus $\widehat{\gamma_\zeta^{w,\mathbb{Z}}}$ is a measure on $\mathbb{T} := \mathbb{R}/\mathbb{Z}$ while $\widehat{\gamma_\zeta^{w,\mathbb{R}}}$ is a measure on $\mathbb{R}$. The relationship between these two measures is simple: for $x \in \mathbb{R}$ and $\dot{x} := x \mod \mathbb{Z}$,

$$\widehat{\delta_k^{\mathbb{Z}}}(\dot{x}) = e^{-2\pi i k.x}, \quad \widehat{\delta_k^{\mathbb{R}}}(x) = e^{-2\pi i k.x} \,.$$

Thus, for all $k \in \mathbb{Z}$, $\widehat{\delta_k^{w,\mathbb{R}}}$ is just the natural periodic extension of $\widehat{\delta_k^{w,\mathbb{Z}}}$ and $\widehat{\gamma_\zeta^{w,\mathbb{R}}}$ is the periodization of $\widehat{\gamma_\zeta^{w,\mathbb{Z}}}$:

$$\widehat{\gamma_\zeta^{w,\mathbb{R}}}(x) = \widehat{\gamma_\zeta^{w,\mathbb{Z}}}(\dot{x}) \,.$$

The latter, hence also the former, exists almost surely.

The pure point, singular continuous, and absolutely continuous parts are also periodized in this process and retain the same types. Thus if the pure point part of $\widehat{\gamma_\zeta^{w,\mathbb{Z}}}$ is $\sum_{\dot{k}\in S} a_{\dot{k}}\delta^{\mathbb{Z}}_{\dot{k}}$ then the pure point part of $\widehat{\gamma_\zeta^{w,\mathbb{R}}}$ is $\sum_{k\in\mathbb{R},\, \dot{k}\in S} a_{\dot{k}}\delta^{\mathbb{R}}_k$, where $\dot{k} = k$ mod $\mathbb{Z}$.

When it comes to $L^2(X_\mathbb{Z}, \mu_\mathbb{Z})$ and $L^2(X_\mathbb{R}, \mu)$ we make the following observation. If $f_{\dot{k}}$ is an eigenfunction for the action of T on $L^2(X_\mathbb{Z}, \mu_\mathbb{Z})$ corresponding to the eigenvalue $\dot{k}$ – that is, $T^n f_{\dot{k}} = \exp(2\pi i k.n) f_{\dot{k}}$ for some (or any) $k \in \mathbb{Z}$ with $\dot{k} = k$ mod $\mathbb{Z}$, we can define a function f_k on $X_\mathbb{R}$ by

$$f_k(t+\zeta) = \exp(-2\pi i k.t) f_{\dot{k}}(\zeta)\,.$$

It is easy to see that this is well-defined and is an eigenfunction for the $\mathbb{R}$-action on $X_\mathbb{R}$ with eigenvalue $-k$ on $\mathbb{R}$. (The change in sign results from the fact that T means shift left by 1, whereas T_t means shift right by t.) This way we see that we have eigenfunctions for $X_\mathbb{R}$ which are all the possible continuous lifts of the eigenfunctions on $\mathbb{R}/\mathbb{Z}$ to eigenfunctions on $\mathbb{R}$.

Unfortunately there does not seem to be any simple connection between the other spectral components of $(X_\mathbb{Z}, \mu_\mathbb{Z})$ and $(X_\mathbb{R}, \mu)$. Thus, for these components, we will be reduced to the consequences that come by the embedding of the diffraction into the dynamics.

4.3.4 The hull of a sequence

We start with an infinite sequence $\xi = (x_1, x_2, \dots)$ of elements of our finite alphabet $\mathbf{m}$ and define $X_\mathbb{Z}(\xi)$ to be the set of all bi-infinite sequences $\zeta = \{z_i\}_{-\infty}^{\infty} \in \mathbf{m}^\mathbb{Z}$ with the property that every finite subsequence $\{z_n, z_{n+1}, \dots, z_{n+k}\}$ (*word*) of ζ is also a word $\{x_p, x_{p+1}, \dots, x_{p+k}\}$ of ξ. Then set $X_\mathbb{Z}(\xi)$ is a closed, hence compact subset of $\mathbf{m}^\mathbb{Z}$, and $(X_\mathbb{Z}(\xi), \mathbb{Z})$ is a dynamical system, called the **dynamical hull** of ξ. $(X_\mathbb{Z}(\xi), \mathbb{Z})$ is mininal (every orbit is dense) if and only if ξ is repetitive (every word reoccurs with bounded gaps).

Given a word $s = \{x_p, x_{p+1}, \dots, x_{p+k}\}$ of ζ, we can ask about the frequency of its appearance (up to translation) in ζ. Let $L(s, [M, N])$ be the number of occurrences of s in the interval $[M, N]$. The frequency of s (relative to $t \in \mathbb{Z}$) is $\lim_{N\to\infty} L(s, t + [-N, N])/2N$, if it exists. It is known that the system $X_\mathbb{Z}(\xi)$ is both minimal and uniquely ergodic (that is, **strictly ergodic**) if and only if for for every $\zeta \in X_\mathbb{Z}(\xi)$ and every word s of ζ the frequency of s exists, the limit is approached uniformly for all in $t \in \mathbb{Z}$, and the frequency is positive. All of this is standard from the theory of sequences and symbolic dynamics [44], Cor. IV.12 .

We can transform $X_{\mathbb{Z}}(\xi)$ into a flow over $\mathbb{R}$ by the technique discussed in the previous subsection and thus obtain $X_{\mathbb{R}}(\xi)$, which will be minimal (respectively ergodic, uniquely ergodic) according as $X_{\mathbb{Z}}(\xi)$ is.

In the next two subsections we consider situations which are derived from two famous sequences, the Thue-Morse and Rudin-Shapiro sequences.

4.3.5 The Thue-Morse sequence

The Thue-Morse sequence can be defined by iteration of the two letter substitution (we use $\{a, b\}$ instead of $\{1, 2\}$)

$$a \longrightarrow ab; \quad b \longrightarrow ba: \quad \xi = abbabaabbaababba\ldots.$$

based on the alphabet $A = \{a, b\}$ (we use $\{a, b\}$ instead of $\{1, 2\}$).

Since the substitution is primitive, it is known that the corresponding dynamical system $X_{\mathbb{Z}} = X_{\mathbb{Z}}(\xi)$, and hence also $X_{\mathbb{R}} = X_{\mathbb{R}}(\xi)$, is minimal and uniquely ergodic.

For an arbitrary weighting system $w = (w_a, w_b)$ we have the diffraction $w_a^2\gamma_{aa} + w_a w_b \gamma_{ab} + w_b w_a \gamma_{ba} + w_b^2 \gamma_{bb}$ where γ_{ij} is the correlation between points of types $i, j \in A$. The natural symmetry $a \leftrightarrow b$ of $X_{\mathbb{Z}}$ gives $\gamma_{aa} = \gamma_{bb}, \gamma_{ab} = \gamma_{ba}$.

Kakutani [27, 28] has determined the diffraction for the weighting system $w = (1, 0)$ and it is

$$\frac{1}{4}\delta_0 + \mathrm{sc}\,,$$

where sc is a non-trivial singular continuous measure on $\mathbb{Z}$. On the other hand, with the weighting $w = (1, 1)$ the elements of $X_{\mathbb{Z}}$ are all just the sequence $\mathbb{Z}$ as far as the autocorrelation is concerned, and the diffraction is $\delta_{\mathbb{Z}}$. From these it follows that the diffraction for a general weighting system is

$$(\frac{w_a + w_b}{2})^2 \delta_0 + (\frac{w_a - w_b}{2})^2 \mathrm{sc}\,.$$

In view of our remarks in §6.1, the diffraction for $X_{\mathbb{R}}$ is

$$(\frac{w_a + w_b}{2})^2 \delta_{\mathbb{Z}} + (\frac{w_a - w_b}{2})^2 \mathrm{scp}$$

where scp is the periodization of $\mathbb{R}$ of the measure sc on $\mathbb{T}$.

The dynamical system is also mixed, pure point plus singular continuous [30]. There is an obvious continuous involution $\sim$ on $X_{\mathbb{Z}}$ that interchanges the a and b symbols. $L^2(X_{\mathbb{Z}}, \mu_{\mathbb{Z}})$ splits into the ±1-eigenspaces for $\sim$: $L^2(X_{\mathbb{Z}}, \mu_{\mathbb{Z}}) = L^2_+(X_{\mathbb{Z}}) \bigoplus L^2_-(X_{\mathbb{Z}})$. $L^2_+(X_{\mathbb{Z}})$ is the pure point part of $L^2(X_{\mathbb{Z}}, \mu)$ and its eigenvalues are all the numbers of the form $k/2^n$, $n = 0, 1, \ldots; 0 \leq k < 2^n$ (literally $\exp(2\pi i k/2^n)$). On the other hand $L^2_-(X_{\mathbb{Z}})$ is singular continuous.

When we move to the suspension of $X_{\mathbb{Z}}$ we obtain $L^2(X_{\mathbb{R}}, \mu)$ which we know certainly retains the eigenvalues of $L^2(X_{\mathbb{Z}}, \mu_{\mathbb{Z}})$ and, due to the embedding of $L^2(\mathbb{R}, \widehat{\dot{\mu}_1^{(1,0)}})$, also retains a singular continuous component.

The dynamical spectrum is, of course, independent of any particular assignments of weights to a and b. We can draw the following conclusions from this:

(i) $w_a = 1, w_b = 0$.

$$\widehat{\dot{\mu}_1^{(1,0)}} = \frac{1}{4}\delta_{\mathbb{Z}} + \mathrm{scp}\,.$$

The eigenfunctions of $L^2(\mathbb{R}, \widehat{\dot{\mu}_1^{(1,0)}})$ are $\mathbf{1}_{\{k\}}, k \in \mathbb{Z}$. It follows that $\theta^w(\mathbf{1}_{\{-k\}})$ is an eigenfunction for eigenvalue k (Thm. 4.1.3, Cor. 4.1.5). Thus θ^w covers only the eigenvalues $k \in \mathbb{Z}$ of $L^2(X_{\mathbb{R}})$ and none of the fractional ones $k/2^n, n > 0$. This shows that θ^w is not surjective. Also θ^w embeds the singular continuous part of $L^2(\mathbb{R}, \widehat{\dot{\mu}_1^{(1,0)}})$ into $L^2_-(\mathbb{R})$, although we do not know the image.

(ii) $w_a = w_b = 1$. In this case the diffraction is $\delta_{\mathbb{Z}}$ (the Thue-Morse sequence with equal weights looks like $\mathbb{Z}$). Although $L^2(\mathbb{R}, \widehat{\dot{\mu}_1^{(1,1)}})$ is pure point, its image does not cover the pure point part of $L^2(X_{\mathbb{R}}, \mu)$, nor does it even generate it as an algebra. This shows that the requirement of unequal weights in Theorem 4.2.1 is necessary.

(iii) $w_a = 1, w_b = -1$. This time the diffraction is singular continuous and θ^w does not even cover anything of the the pure point part of $L^2(X_{\mathbb{R}}, \mu)$.

Cases (ii) and (iii) show that the non-existence of a particular component in the diffraction spectrum implies nothing about its existence or non-existence in the dynamical spectrum.

4.3.6 The Rudin-Shapiro sequence

We define the Rudin-Shapiro sequence using the notation of [43]. Consider the substitution rule s defined on the alphabet $A' := \{1, \bar{1}, 2, \bar{2}\}$ as follows: $s(1) = 1\bar{2}, s(2) = \bar{1}\bar{2}, s(\bar{1}) = \bar{1}2, s(\bar{2}) = 12$. Let ξ be the s-invariant sequence that starts with the symbol 1. We can reduce this to a 2-symbol sequence ξ' with alphabet $\{a, b\}$ by replacing the symbols with no over-bar by the letter a and the others by the letter b. This 2-symbol sequence is usually called the Rudin-Shapiro sequence [44], though Priebe-Frank uses this appellation for the original 4-symbol sequence.

Let us start with the 2-symbol sequence, which results in the 2-coloured minimal and ergodic dynamical hull $(X_{\mathbb{Z}}(\xi'), \mathbb{Z})$, as developed above. There is a natural involution on the dynamical system that interchanges a and b. Once again we introduce a system of weights $w = (w_a, w_b)$.

Under the system of weights $(1, -1)$ it is well known that the diffraction measure of the elements of $X_{\mathbb{Z}}(\xi')$ is the normalized Haar measure on $\mathbb{R}/\mathbb{Z}$ [44], Cor. VIII.5. Thus $L^2(\mathbb{R}, \widehat{\mu_1^{(1,-1)}}) = L^2(\mathbb{R}, \ell)$, where ℓ is Lebesgue measure on $\mathbb{R}$.

On the other hand, the weighting system $(1, 1)$ reduces the elements of $X_{\mathbb{Z}}(\xi')$ to copies of the sequence $\mathbb{Z}$. So, just as in the case of the Thue-Morse sequence, we can deduce the general formula for the diffraction:

$$(\frac{w_a + w_b}{2})^2 \delta_{\mathbb{Z}} + (\frac{w_a - w_b}{2})^2 \ell.$$

The spectral decomposition of $L^2(X_{\mathbb{Z}}(\xi'))$ is of the form

$$L^2(X_{\mathbb{Z}}(\xi')) \simeq H \oplus Z(f)$$

where H is the pure point part with one simple eigenvalue $\exp(2\pi i q)$ for each dyadic rational number $q = a/2^n$, where $a \in \mathbb{Z}$, $n = 0, 1, 2, \ldots$ [12, 36]; and $Z(f)$ is a cyclic subspace which is equivalent to $L^2(\mathbb{R}, \ell)$. In other words, the dynamical spectrum is mixed with a pure point and an absolutely continuous part [4]. In any case, we see that $L^2(X_{\mathbb{R}}(\xi'), \mu)$ contains a pure point part whose eigenvalues include all the dyadic rationals, and also an absolutely continuous part into which the absolutely continuous part of $L^2(\mathbb{R}, \widehat{\mu_1^w})$ must map by θ^w.

The analysis now proceeds exactly as in the case of the Thue-Morse sequence, with the same three types of possibilities except now the singular continuous parts are replaced by absolutely continuous parts.

4.3.7 Regular model sets

In this example we see that even when everything is pure point and there is only one colour, still θ need not be surjective.

Let $(\mathbb{R}^d, \mathbb{R}^d, L)$ be a cut and project scheme with projection mappings π_i, $i = 1, 2$. Thus L is a lattice in $\mathbb{R}^d \times \mathbb{R}^d$, the projection π_1 to the first factor is one-one on L, and the projection $\pi_2(L)$ of L has dense image in the second factor. Let W be a non-empty compact subset which is the closure of its own interior and a subset

[4] Explicitly f is the function on $X_{\mathbb{Z}}(\xi')$ which is defined by $f(\zeta) = 1$ or -1 according as $\zeta(0)$ is a or b. This can be deduced from the main theorem of [43], where the equivalent result for dynamical system arising from the 4 symbol sequence gives two copies of $L^2(\mathbb{R}, \ell)$), and then by dropping to the factor.

of the second factor. We assume that the boundary of W has Lebesgue measure 0. The corresponding model set is

$$\Lambda(W) = \{\pi_1(t) \,:\, t \in L, \pi_2(t) \in W\}.$$

It is a subset of $\mathcal{D}_r$ for some $r > 0$ and it is pure point diffractive [23, 47, 7]. The orbit closure $X = \overline{\mathbb{R}^d + \Lambda(W)}$ is uniquely ergodic. Its autocorrelation γ, and hence its diffraction $\widehat{\gamma}$, is the same for all $\Gamma \in X$. Furthermore, the diffraction is explicitly known:

$$\widehat{\mu_1} = \widehat{\gamma} = \sum_{k \in L^0} a_k \delta_{\pi_1(k)}$$

where L^0 is the $\mathbb{Z}$-dual lattice of L with respect to the standard inner product on $\mathbb{R}^d \times \mathbb{R}^d \simeq \mathbb{R}^{2d}$ and

$$a_k = \left|\widehat{\mathbf{1}_W}(-\pi_2(k))\right|^2.$$

For more on this see [23]. The main point is that $\widehat{\mu_1}(\pi_1(k)) = 0$ if and only if $a_k = 0$.

Likewise $L^2(X, \mu)$ is known and it is isometric in a totally natural way by an $\mathbb{R}^d$-map to $L^2(\mathbb{R}^{2d}/\mathbb{Z}^{2d}, \nu)$, where ν is Haar measure on the torus. Thus the spectrum of X is pure point and the eigenvalues are precisely all the points of L^0. Thus the mapping θ embedding the diffraction into the dynamics will be surjective if and only if for all $k \in L^0$, $a_k \neq 0$.

Now it is easy to see that we can find model sets for our given cut and project scheme for which fail to be surjective at any $k \in L^0$ that we wish, as long as $k \neq 0$. To do this take $W = [-1, 1]$ and for each scaling factor $s > 0$ let $\Lambda^{(s)} := \Lambda(sW)$. The intensities of the Bragg peaks become

$$a_k^{(s)} := s^2 \left|\widehat{\mathbf{1}_W}(-s\pi_2(k))\right|^2.$$

Since $\widehat{\mathbf{1}_W}$ is continuous and takes positive and negative values on every ray through 0 in $\mathbb{R}^d$, but altogether takes the value 0 only on a meagre set, we see that by choosing s suitably we can arrange either that $a_k^{(s)}$ vanishes at any preassigned non-zero $k \in L^0$ (and θ is not surjective) or that alternatively $a_k^{(s)}$ vanishes nowhere on L^0 (and θ is a bijection).

4.3.8 The necessity of non-zero weights in Thm. 4.2.1.

Let $\Lambda = (\Lambda_a, \Lambda_b)$ where

$$\Lambda_a = \{z \in \mathbb{Z} \,:\, z \equiv 0 \text{ or } 2 \mod 4\}, \quad \Lambda_b = \{z \in \mathbb{Z} \,:\, z \equiv 3 \mod 4\}.$$

Then Λ is periodic with period 4 and its hull – that is, the closure of its $\mathbb{R}$ translation orbit – is $X \simeq \mathbb{R}/4\mathbb{Z}$ (a conjugacy of dynamical systems with the standard action of $\mathbb{R}$ on $\mathbb{R}/4\mathbb{Z}$). Thus $L^2(X,\mu)$, where μ is Haar measure on $\mathbb{R}/4\mathbb{Z}$, has pure point spectrum with eigenvalues $\frac{1}{4}\mathbb{Z}$.

Let (w_a, w_b) be a weighting system for Λ. The autocorrelation is everywhere the same and is easily seen to be

$$\mu_1 = \frac{1}{2}w_a^2\delta_{2\mathbb{Z}} + \frac{1}{4}w_a w_b \delta_{1+4\mathbb{Z}} + \frac{1}{4}w_a w_b \delta_{-1+4\mathbb{Z}} + \frac{1}{4}w_b^2\delta_{4\mathbb{Z}}\,.$$

The Fourier transform, that is the diffraction, is then given by

$$\begin{aligned}
\widehat{\mu_1} &= \frac{1}{4}w_a^2\delta_{\frac{1}{2}\mathbb{Z}} + \frac{1}{16}w_a w_b \exp(-2\pi i(\cdot))\delta_{\frac{1}{4}\mathbb{Z}} \\
&+ \frac{1}{16}w_a w_b \exp(2\pi i(\cdot))\delta_{\frac{1}{4}\mathbb{Z}} + \frac{1}{16}w_b^2\delta_{\frac{1}{4}\mathbb{Z}} \\
&= \frac{1}{4}\{(w_a^2 + \frac{1}{2}w_a w_b + \frac{1}{4}w_b^2)\delta_{\mathbb{Z}} + (w_a^2 - \frac{1}{2}w_a w_b + \frac{1}{4}w_b^2)\delta_{\frac{1}{2}+\mathbb{Z}} \\
&+ \frac{1}{4}w_b^2\delta_{\frac{1}{4}+\mathbb{Z}} + \frac{1}{4}w_b^2\delta_{-\frac{1}{4}+\mathbb{Z}}\}\,.
\end{aligned}$$

Now it is clear that the image of θ can only generate eigenfunctions for the eigenvalues $\pm\frac{1}{4} + \mathbb{Z}$ if $w_b \neq 0$ (and then in fact it does so, independent of the value of w_a).

4.4 Higher correlations and higher moments

Let $\xi : (\Omega, \mathcal{A}, P) \longrightarrow (X, \mathcal{X})$ be a uniformly discrete stationary multi-variate point process with accompanying dynamical system $(X, \mathbb{R}^d, \mu)$.

The $n+1$-point **correlation** ($n = 1, 2, \dots$) of $\lambda \in X$ is the measure on $(\mathbb{R}^d)^n$ defined by

$$\begin{aligned}
\gamma_\lambda^{(n+1)}(f) &= \lim_{R\to\infty} \frac{1}{\ell(C_R)} \sum_{y_1,\dots,y_n,x\in C_R} \lambda^w(\{x\})\Pi_{i=1}^n\lambda^w(\{y_i\})T_x f(y_1,\dots,y_n) \\
&= \lim_{R\to\infty} \frac{1}{\ell(C_R)} \sum_{\substack{x\in C_R \\ y_1,\dots y_n\in\mathbb{R}^d}} \lambda^w(\{x\})\Pi_{i=1}^n\lambda^w(\{y_i\})T_x f(y_1,\dots,y_n)\,,
\end{aligned}$$

for all $f \in C_c((\mathbb{R}^d)^n)$. In particular for $f = (f_1, \dots, f_n) \in (C_c(\mathbb{R}^d))^n$, where each

$f_i \in C_c(\mathbb{R}^d)$,

$$\begin{aligned}
\gamma_\lambda^{(n+1)}(f) &= \lim_{R\to\infty} \frac{1}{\ell(C_R)} \sum_{\substack{x\in C_R \\ y_1,\dots y_n \in \mathbb{R}^d}} \lambda^w(\{x\}) \Pi_{i=1}^n \lambda^w(\{y_i\}) T_x f_1(y_1) \dots T_x f_n(y_n) \\
&= \lim_{R\to\infty} \frac{1}{\ell(C_R)} \sum_{x\in C_R} \lambda^w(\{x\}) N_{f_1}^w(T_{-x}\lambda) \dots N_{f_n}^w(T_{-x}\lambda) \\
&= \mathrm{Av}^w(N_{f_1}^w \dots N_{f_n}^w)(\lambda)\,.
\end{aligned}$$

We know that μ-almost surely this exists, it is independent of λ, and

$$\mathrm{Av}^w(N_{f_1}^w \dots N_{f_n}^w)(\lambda) = \dot{\mu}^w(N_{f_1}^w \dots N_{f_n}^w) =: \dot{\mu}_n^w((f_1, \dots, f_n))\,.$$

The measure defined on the right-hand side of this equation is the nth **weighted moment** of the weighted Palm measure at $(f_1, \dots, f_n)$, so we arrive at the useful fact which generalizes what we already know for the 2-point correlation:

Proposition 4.4.1. *The $n+1$-point correlation measure exists almost everywhere on X and is given by $\dot{\mu}_n^w$.*

Of course, in the one colour case where there are no weights (or if the weighting is trivial: $w = (1, \dots, 1)$), then these are ordinary moments.

Lemma 4.4.2. *Assume that the weights are all different and none of them is zero. Then the weighted intensity (and hence the first moment of μ) is determined by the $\dot{\mu}_n^w, n = 1, 2, \dots$.*

PROOF: By (3.3.1), we need to know $\dot{\mu}^w(X)$. Now $\dot{\mu}$ is supported on the X_0 of elements $\lambda \in X$ which have an atom at $\{0\}$ and we have

$$\begin{aligned}
\dot{\mu}^w(X) &= \int_X N_{\mathbf{1}_{\{0\}}}(\lambda^\downarrow) d\dot{\mu}^w(\lambda) \\
&= \int_X \sum_{i=1}^m N_{\mathbf{1}_{\{0\}}}(\lambda^i) d\dot{\mu}^w(\lambda) \\
&= \sum_{i=1}^m \dot{\mu}^w(N_{\mathbf{1}_{\{0\}}} \cdot \mathrm{res}^i).
\end{aligned}$$

From $\dot{\mu}_n^w$ we have

$$\dot{\mu}_1^w(0) = \int_{X_0} N_{\mathbf{1}_{\{0\}}}^w(\lambda) d\dot{\mu}^w(\lambda) = \sum_{i=1}^m w_i \dot{\mu}^w(N_{\mathbf{1}_{\{0\}}} \circ \mathrm{res}^i). \tag{4.4.1}$$

Similarly,

$$\dot{\mu}_2^w(\{(0,0)\}) = \int_{X_0} (N^w_{\mathbf{1}_{\{0\}}})^2(\lambda)d\dot{\mu}^w(\lambda) = \sum_{i=1}^m w_i^2 \dot{\mu}^w(N_{\mathbf{1}_{\{0\}}} \circ \mathrm{res}_i). \tag{4.4.2}$$

Continue this until we get to

$$\dot{\mu}_m^w(\{(0,\ldots,0)\}) = \int_X (N^w_{\mathbf{1}_{\{0\}}})^m(\lambda)d\dot{\mu}^w(\lambda) = \sum_{i=1}^m w_i^m \dot{\mu}^w(N_{\mathbf{1}_{\{0\}}} \circ \mathrm{res}^i). \tag{4.4.3}$$

Using the same argument in Lemma 3, we can solve this system of equations for $\dot{\mu}^w(N_{\mathbf{1}_{\{0\}}} \circ \mathrm{res}^i)$ for $j = 1,\ldots,m$ and hence determine the intensity $\dot{\mu}^w(X)$.

Theorem 4.4.3. *Let (X,μ) be an m-coloured stationary uniformly discrete ergodic point process and w a system of weights in which w_i, $i = 1,\ldots,m$, are all different from one another and also none of them is equal to 0. Then the measure μ is μ-almost surely determined entirely by the weighted $n+1$-point correlations of $\lambda \in X$, $n = 1,2,\ldots$.*

The key to this is the known fact (in the non-weighted case) that the nth moment of the Palm measure, $n = 1,2,\ldots$, is the same as the reduced $(n+1)$st moment of the measure μ itself. Thus knowledge of the correlations gives us the moments $\dot{\mu}_n$ of the Palm measure, which in turn is the same as knowledge of the reduced moments of μ. These in turn determine the moments μ_n, $n = 2,3,\ldots$ of μ. As for μ_1, we already know that it is just the intensity of the point process times Lebesgue measure, and from Lemma 4.4.2, this is derivable from the moments.

First of all we give a short derivation of these facts in the unweighted $m = 1$ case, and then show how to augment these to the weighted case.

Let $g, h_1, \ldots h_n \in C_c(\mathbb{R}^d)$ be chosen freely. Let $G : \mathbb{R}^d \times X \longrightarrow \mathbb{C}$ be defined by

$$G(x,\lambda) = g(x)N_{T_x h_1}(\lambda)\ldots N_{T_x h_n}(\lambda).$$

We use the Campbell formula

$$\int_X \sum_{x\in\mathbb{R}^d} \lambda(\{x\})G(x,\lambda)d\mu(\lambda) = \int_{\mathbb{R}^d}\int_X G(x,T_x\lambda)d\dot{\mu}(\lambda).$$

The left-hand side reads[5]

$$\int_X \sum_{x\in\mathbb{R}^d} \lambda(\{x\}) g(x) N_{T_x h_1}(\lambda)\ldots N_{T_x h_n}(\lambda) d\mu(\lambda) \tag{4.4.4}$$
$$= \int_X \lambda(g)\lambda(T_x h_1)\ldots\lambda(T_x h_n) d\mu(\lambda)$$
$$= \mu_{n+1}(g(T_x h_1)\ldots(T_x h_n)) = \int_{\mathbb{R}^d} g(x)dx\, \mu_{n+1}^{\mathrm{red}}(h_1\ldots h_n)\,,$$

while the right-hand side reads

$$\int_{\mathbb{R}^d}\int_X g(x) N_{T_x h_1}(T_x\lambda)\ldots N_{T_x h_n}(T_x\lambda) d\dot{\mu}(\lambda)dx = \int_{\mathbb{R}^d} g(x)dx\, \dot{\mu}_n(h_1\ldots h_n)\,,$$

since $N_{T_x h}(T_x\lambda) = N_h(\lambda)$. For the reduced moments see [11], Sec. 10.4, especially Lemma 10.4.III and Prop. 10.4.V. The point is that μ_{n+1} is invariant under simultaneous translation of its $n+1$ variables. This invariance can be factored out leading to the rewriting of μ_{n+1} as a product of Lebesgue measure and another measure, which is, *by definition*, the reduced measure. This rewriting is exactly the last part of equation (4.4.4). Thus, $\mu_{n+1}^{\mathrm{red}} = \dot{\mu}_n$ and, using Prop. 4.2.7 and Prop. 4.4.1, Theorem 4.4.3 is proved in the 1-coloured case.

To obtain the weighted version, we use now the functions

$$G^w(x,\lambda) = g(x) N^w_{T_x h_1}(\lambda)\ldots N^w_{T_x h_n}(\lambda)$$

and the weighted form of the Campbell formula. Then the same argument leads to $(\mu^w_{n+1})^{\mathrm{red}} = \dot{\mu}^w_n$ and the proof of Theorem 4.4.3 follows as before. □

Corollary 4.4.4. *For a $\lambda \in X$, the probability measure μ_λ determined by λ (see Chapter 2) exists μ-almost surely and it is equal to μ μ-almost surely.*

PROOF: We define a sequence of properties for point sets in X that for $n \in \mathbb{Z}^+$,

P(n): the $n+1$-point correlation measure exists μ-almost surely and it is equal to $\dot{\mu}^w_n$ μ-almost surely.

For $n \in \mathbb{Z}^+$, define

$$A_n := \{\lambda \in X : \lambda \text{ fulfills P(n)}\},$$

and $A := \bigcap_{n=1}^\infty A_n$. Then by Proposition 4.4.1, $\mu(A_n) = 1$, for $n \in \mathbb{Z}^+$. Thus, $\mu(A) = 1$. Further, for a $\lambda \in X$, by Theorem 4.4.3, the probability measure μ_λ determined by λ exists and it is equal to μ. □

[5] $g(T_x h_1)\ldots(T_x h_n)$ stands for the function whose value on $(x,(y_1,\ldots y_n)) \in \mathbb{R}^d \times (\mathbb{R}^d)^n$ is $g(x)(T_x h_1)(y_1)\ldots(T_x h_n)(y_n)$.

Chapter 5

Other results

5.1 The square-mean Bombieri-Taylor Conjecture

Theorem 5.1.1. *(The square mean Bombieri-Taylor conjecture [23]) Let $(X, \mathbb{R}^d, \mu)$ be a uniformly discrete, multi-coloured stationary ergodic point process, and assume that w is a system of weights. Then the following are equivalent* [1] *:*

(i)

$$\frac{1}{\ell(C_R)} \sum_{x \in C_R} \lambda^w(\{x\}) e^{2\pi i k . x} \not\to 0 \quad \text{as } R \to \infty \, ;$$

(ii) $\widehat{\mu_1^w}(\{k\}) \neq 0$*;*

(iii) k *is an eigenvalue of* U*.*

In the case that k *is an eigenvalue, then*

$$\frac{1}{\ell(C_R)} \sum_{x \in C_R} \lambda^w(\{x\}) e^{2\pi i k . x} \to \theta^w(\mathbf{1}_k) \, .$$

For notational simplicity we shall prove the two technical lemmas that precede the main proof in the 1-dimensional case. However, it is easy to generalize the proof to any dimension d. Throughout, R is assumed to be a positive integer variable.

Lemma 5.1.2. *For all* $\epsilon > 0$*,*

$$\lim_{\epsilon \to 0} \widehat{\mu_1^w}(\mathbf{1}_{[-\epsilon,\epsilon]}) = \widehat{\mu_1^w}(\{0\}) \, ,$$

i.e. $\{\mathbf{1}_{[-\epsilon,\epsilon]}\}_{\epsilon \searrow 0} \longrightarrow \mathbf{1}_{\{0\}}$ *in* $L^2(\mathbb{R}, \widehat{\mu_1^w})$.

[1] Limits here are taken in the L^2-norm on $(X, \mathbb{R}^d, \mu)$. Recently D. Lenz [33] has given a pointwise version of the result.

PROOF: Assume $\epsilon \to 0^+$. Let $F_\epsilon := \mathbf{1}_{[-\epsilon,\epsilon]} - \mathbf{1}_{\{0\}}$. Then for all $x \in \mathbb{R}$, $0 \le F_\epsilon(x) \le 1$ and $F_\epsilon(x) \searrow 0$ pointwise. Since $\widehat{\mu_1^w}$ is a translation bounded positive measure, $\widehat{\mu_1^w}(F_\epsilon) \searrow 0$. Now,

$$\int |\mathbf{1}_{[-\epsilon,\epsilon]} - \mathbf{1}_{\{0\}}|^2 \mathrm{d}\widehat{\mu_1^w} = \int F_\epsilon^2 \mathrm{d}\widehat{\mu_1^w} \le \int F_\epsilon \mathrm{d}\widehat{\mu_1^w} \longrightarrow 0\,.$$

□

Lemma 5.1.3. *As functions of $y \in \mathbb{R}^d$,*

$$\frac{1}{2R}\int_{-R}^{R} e^{2\pi i y.x}\mathrm{d}x \longrightarrow \mathbf{1}_{\{0\}}(y)$$

in $L^2(\mathbb{R}, \widehat{\mu_1^w})$ as $R \to \infty$.

PROOF: Let

$$g_R(y) := \frac{1}{2R}\int_{-R}^{R} e^{2\pi i y.x}\mathrm{d}x = \frac{\sin(2\pi y R)}{2\pi y R}\,.$$

We need to show that $\int_{-\infty}^{\infty} |g_R(y) - \mathbf{1}_{\{0\}}(y)|^2 \mathrm{d}\widehat{\mu_1^w}(y) \longrightarrow 0$. Since $|g_R(y) - \mathbf{1}_{\{0\}}(y)| \le F_\epsilon(y)$ for $-\epsilon \le y \le \epsilon$, we have $\int_{-\epsilon}^{\epsilon} |g_R(y) - \mathbf{1}_{\{0\}}(y)|^2 \mathrm{d}\widehat{\mu_1^w}(y) \longrightarrow 0$ as $\epsilon \to 0$, and the convergence is uniform without reference to R.

For the remaining parts of the integral, we have (the part from $-\infty$ to $-\epsilon$ is the same)

$$\begin{aligned}
\int_{\epsilon}^{\infty} |g_R(y) &- \mathbf{1}_{\{0\}}(y)|^2 \mathrm{d}\widehat{\mu_1^w}(y) = \int_{\epsilon}^{\infty} \frac{\sin^2(2\pi y R)}{(2\pi y R)^2}\mathrm{d}\widehat{\mu_1^w}(y) \\
&\le \frac{1}{(2\pi R\epsilon)^2}\int_{\epsilon}^{\infty} \frac{\epsilon^2}{y^2}\mathrm{d}\widehat{\mu_1^w}(y) \\
&\le \frac{1}{(2\pi R\epsilon)^2}\left\{\int_{\epsilon}^{\epsilon+1} \mathrm{d}\widehat{\mu_1^w}(y) + \sum_{m=1}^{\infty}\frac{\epsilon^2}{m^2}\int_{\epsilon+m}^{\epsilon+m+1}\mathrm{d}\widehat{\mu_1^w}(y)\right\}.
\end{aligned}$$

Since $\int_a^{a+1} \mathrm{d}\widehat{\mu_1^w}(y)$ is uniformly bounded by some constant $C(1)$ (due to the translation boundedness of $\widehat{\mu_1^w}$) we see that $\int_\epsilon^\infty |g_R(y) - \mathbf{1}_{\{0\}}(y)|^2 \mathrm{d}\widehat{\mu_1^w}(y) \longrightarrow 0$ as long as $R\epsilon \to \infty$ as $R \to \infty$. Putting $\epsilon = R^{-1/2}$ gives the necessary convergence of both parts. □

PROOF THEOREM 5.1.1: (ii) $\Leftrightarrow$ (iii): k is an eigenvalue if and only if $-k$ is an eigenvalue, $\widehat{\mu_1^w}(\{k\}) = \widehat{\mu_1^w}(\{-k\})$ for all k, and k is an eigenvalue if and only if $\widehat{\mu_1^w}(\{k\}) \neq 0$.

(iii) $\Leftrightarrow$ (i): Let $f_R := \frac{1}{2R}\chi_k \mathbf{1}_{[-R,R]}$. Then

$$\begin{aligned}\widehat{f_R}(y) &= \frac{1}{2R}\int_{\mathbb{R}} e^{-2\pi i y.x}\chi_k(x)\mathbf{1}_{[-R,R]}(x)\mathrm{d}x \\ &= \frac{1}{2R}\int_{-R}^{R} e^{2\pi i(k-y).x}\mathrm{d}x \\ &\longrightarrow \mathbf{1}_{\{0\}}(k-y) = \mathbf{1}_{\{k\}}(y)\,,\end{aligned}$$

the convergence being as functions of y in $L^2(\mathbb{R}, \widehat{\mu_1^w})$ as $R \to \infty$.

Let $\phi_{-k} = \theta^w(\mathbf{1}_{\{k\}})$. Thus $\widehat{f_R} \to \mathbf{1}_{\{k\}}$ implies that $\theta^w(\widehat{f_R}) \to \phi_{-k}$ in $L^2(X,\mu)$, so

$$\int_X |N^w_{f_R}(\Lambda) - \phi_{-k}(\Lambda)|^2 \mathrm{d}\mu(\Lambda) \to 0\,,$$

which from (3.1.1) gives

$$\int_X \left| \frac{1}{2R} \sum_{x[-R,R]} \lambda^w(\{x\})e^{2\pi i k.x} - \phi_{-k}(\Lambda)\right|^2 \mathrm{d}\mu(\Lambda) \to 0\,.$$

Thus $\frac{1}{2R}\sum_{x\in\Lambda\cap[-R,R]} e^{2\pi i k.x}$ converges in square mean to ϕ_{-k}. Furthermore by Cor. 4.1.6, ϕ_{-k} is a χ_{-k}-eigenfunction for T_t if $\widehat{\mu_1^w}(k) \neq 0$ and is 0 otherwise.

If $\phi_{-k} = 0$ then $\frac{1}{2R}\sum_{x\in\Lambda\cap[-R,R]} e^{2\pi i k.x} = 0$ μ-a.e. If $\phi_{-k} \neq 0$ then $\{\Lambda : \phi_{-k}(\Lambda) = 0\}$ is a measurable T-invariant subset of μ, since ϕ_{-k} is an eigenfuction, so by the ergodicity it is of measure 0 or 1. It must be the former. Now using the Fischer-Riesz theorem [15], there is a subsequence of $\{\frac{1}{2R}\sum_{x\in\cap[-R,R]} \lambda^w(\{x\})e^{2\pi i k.x}\}_R$ which converges pointwise μ-a.e. to ϕ_{-k}. Since ϕ_{-k} is almost everywhere not zero,

$$\frac{1}{2R}\sum_{x\in[-R,R]} \lambda^w(\{x\})e^{2\pi i k.x} \nrightarrow 0.$$

□

5.2 A strange inequality

Let $(X, \mathbb{R}^d, \mu)$ be a uniformly discrete stationary ergodic point process (no colour). Assume that the point sets of X have finite local complexity, μ- a.s. This implies that the autocorrelation measure $\dot\mu_1$ is supported on a closed discrete subset of $\Lambda - \Lambda$ for any Λ whose autocorrelation is $\dot\mu_1$. Thus for $\Lambda \in X$ we have, μ-almost surely,

$$\dot\mu_1(t) = \lim_{R\to\infty} \frac{1}{\ell(C_R)}\mathrm{card}((-t+\Lambda)\cap\Lambda\cap C_R)\,.$$

Proposition 5.2.1. *For all $k, t \in \mathbb{R}^d$,*

$$\left|e^{2\pi i k\cdot t} - 1\right| \widehat{\mu_1}^{1/2}(k) \leq 2(\dot{\mu}_1(0) - \dot{\mu}_1(t))\,.$$

Proof: Let $k \in \mathbb{R}^d$. Then

$$\frac{1}{\ell(C_R)} \sum_{x\in \Lambda\cap C_R} e^{-2\pi i k\cdot x} \longrightarrow g_k$$

in the norm of $L^2(X,\mu)$, where g_k is an eigenfunction of T for the eigenvalue k if $\widehat{\mu_1}(k) \neq 0$ and 0 otherwise (Thm. 5.1.1).

Suppose $\widehat{\mu_1}(k) \neq 0$. Let $t \in \mathbb{R}^d$. Since

$$(T_t g_k)(\Lambda) = g_k(-t+\Lambda) = \lim_{R\to\infty} \frac{1}{\ell(C_R)} \sum_{x\in(-t+\Lambda)\cap C_R} e^{-2\pi i k\cdot x}$$

for almost all $\Lambda \in X$,

$$\begin{aligned}((T_t-1)g_k)(\Lambda) &= \lim_{R\to\infty} \frac{1}{\ell(C_R)} \left\{ \sum_{x\in(-t+\Lambda)\cap C_R} e^{-2\pi i k\cdot x} - \sum_{x\in\Lambda\cap C_R} e^{-2\pi i k\cdot x} \right\} \\ &= \lim_{R\to\infty} h_R(\Lambda)\,,\end{aligned}$$

where

$$h_R(\Lambda) := \frac{1}{\ell(C_R)} \left\{ \sum_{x\in(-t+\Lambda)\backslash\Lambda\cap C_R} e^{-2\pi i k\cdot x} - \sum_{x\in\Lambda\backslash(-t+\Lambda)\cap C_R} e^{-2\pi i k\cdot x} \right\}.$$

Thus $h_R \to (e^{2\pi i k\cdot t} - 1)g_k$ in the L^2-norm on X.

Furthermore,

$$\begin{aligned}|h_R(\Lambda)| &\leq \frac{1}{\ell(C_R)} \left(\sum_{x\in(-t+\Lambda)\backslash\Lambda\cap C_R} \left|e^{-2\pi i k\cdot x}\right| + \sum_{x\in\Lambda\backslash(-t+\Lambda)\cap C_R} \left|e^{-2\pi i k\cdot x}\right| \right) \\ &\leq \frac{1}{\ell(C_R)} \left(\sum_{x\in(-t+\Lambda)\backslash\Lambda\cap C_R} 1 + \sum_{x\in\Lambda\backslash(-t+\Lambda)\cap C_R} 1 \right) \\ &= \frac{1}{\ell(C_R)} \sum_{x\in(-t+\Lambda)\triangle\Lambda\cap C_R} 1 = 2(\dot{\mu}_1(0) - \dot{\mu}_1(t)).\end{aligned}$$

Note that $\dot{\mu}_1(0) \geq \dot{\mu}_1(t)$.

With these preliminaries out of the way, the rest of the proof is straightforward. Since μ is a finite measure, $h_R \to (e^{2\pi i k\cdot t} - 1)g_k$ in the L^1 norm also. Then there is a subsequence $\{h_{R_i}\}$ of $\{h_R\}$ which converges to $(e^{2\pi i k\cdot t} - 1)g_k$ point-wise almost everywhere ([10], Sec. 3.1).

Using the dominated convergence theorem ($|h_R(\Lambda)| \le 2\mathring{\mu}_1(0)$), we have

$$\begin{aligned}
&\int_X \left|e^{2\pi i k\cdot t} - 1\right|^2 |g_k(\Lambda)|^2 \, d\mu(\Lambda) \\
&= \lim_{R_i\to\infty} \int_X |h_{R_i}(\Lambda)|^2 \, d\mu(\Lambda) \le \int_X |2(\mathring{\mu}_1(0) - \mathring{\mu}_1(t))|^2 \, d\mu(\Lambda) \\
&= 4\,|(\mathring{\mu}_1(0) - \mathring{\mu}_1(t))|^2 .
\end{aligned}$$

Meanwhile, from Thm. 4.1.3

$$\begin{aligned}
\int_X \left|e^{2\pi i k\cdot t} - 1\right|^2 |g_k(\Lambda)|^2 \, d\mu(\Lambda) &= \left|e^{2\pi i k\cdot t} - 1\right|^2 \int_{\mathbb{R}^d} \mathbf{1}_k^2 d\widehat{\mathring{\mu}_1} \\
&= \left|e^{2\pi i k\cdot t} - 1\right|^2 \widehat{\mathring{\mu}_1}(k).
\end{aligned}$$

So

$$\left|e^{2\pi i k\cdot t} - 1\right| \widehat{\mathring{\mu}_1}^{\frac{1}{2}}(k) \le 2\,|(\mathring{\mu}_1(0) - \mathring{\mu}_1(t))| = 2(\mathring{\mu}_1(0) - \mathring{\mu}_1(t)) .$$

□

Remark 5.2.2. *This result numerically links three interesting quantities. If $\Lambda \in X$ has autocorrelation $\mathring{\mu}_1$ then for the set $P(\epsilon)$ of ϵ-statistical almost periods of Λ, i.e. t for which $\mathring{\mu}_1(0) - \mathring{\mu}_1(t) < \epsilon$, the Bragg peaks $I(a)$ of intensity greater than $a > 0$, i.e. k for which $\widehat{\mathring{\mu}_1}(k) > a$, can occur only at points k which are $2\epsilon/\sqrt{a}$-dual to $P(\epsilon)$, i.e. k for which $\left|e^{2\pi i k\cdot t} - 1\right| < 2\epsilon/\sqrt{a}$ for all $t \in P(\epsilon)$. If this latter quantity is less than or equal to $1/2$ and either of $P(\epsilon)$ or $I(a)$ is relatively dense, then the other one is a Meyer set [39] Thm. 9.1. Furthermore, Bragg peaks can occur only on the $\mathbb{Z}$-dual of the statistical periods (t for which $\mathring{\mu}_1(t) = \mathring{\mu}_1(0)$), a fact that is of course very familiar in the case of crystals.[2] We note that the inequality seems to be optimal. The maximum values of $\left|e^{2\pi i k\cdot t} - 1\right|$ and $\widehat{\mathring{\mu}_1}^{\frac{1}{2}}(k)$ are 2 and $\mathring{\mu}_1(0)$ respectively, whereas the minimum value of $\mathring{\mu}_1(t)$ is 0.*

[2] We are grateful to Nicolae Strungaru for this last observation.

Chapter 6

How finite correlation measures determine the point process

Introduction

We have proved that for an m-coloured stationary uniformly discrete ergodic point process, the law is uniquely determined by the set of all weighted correlation measures of the point process. In this chapter, we will prove that for a k-step uniformly discrete ergodic simple Markov process, the law μ can be uniquely determined by the first k correlation measures of the point process. We will also prove that a point process generated by a model set is uniquely determined by its 2-point and 3-point correlation measures.

6.1 The correlation measures of a continuous dynamics of intervals

In this section, we only consider uniformly discrete point processes on the real line. From Section 2.4, we can regard a uniformly discrete point process on the real line as a continuous dynamics of intervals and vice versa. We start with an ergodic symbolic dynamical system $(X_0, \mathbb{Z}, \mu_{\mathbb{Z}})$ on the space $A^{\mathbb{Z}}$ equipped with product topology, where $A := \{a_1, \dots, a_m\}$ is an alphabet set. This concept was introduced in Section 4.3.1. Here A replaces $\boldsymbol{m}$.

Let $\mathcal{L}$ be a set of m nonnegative numbers, $\mathcal{L} = \{L_i\}_1^m$. Now we think of a_i as representing a line segment with length L_i, for $i = 1, \dots, m$. $\mathcal{L}$ is said to be **integrally independent** if and only if the equation $\sum_{i=1}^m x_i L_i = 0$, has only the trivial solution (that is $x_i = 0, i = 1, \cdots, m$ on $\mathbb{Z}$). We assume that:

(AI): $\mathcal{L}$ is integrally independent.

Evidently, all elements in $\mathcal{L}$ are distinct. Thus, there is an one-to-one correspondence between "words" in A and "words" in $\mathcal{L}$ and hence we can represents A by $\mathcal{L}$.

By Section 4.3.1 and 4.3.2, to obtain a a uniformly discrete point processes on the real line $(X_{\mathbb{R}}^{\mathcal{L}}, \mu^{\mathcal{L}})$ from the symbolic dynamics $(X_0, \mathbb{Z}, \mu_{\mathbb{Z}})$, we assume that:

$$\textbf{(AII)}: \sum_{j=1}^{m} L_j \mu_{\mathbb{Z}}(X_{\mathbb{Z}}[L_j]) = 1, \tag{6.1.1}$$

where $X_{\mathbb{Z}}[L_j]$ is a cylinder set defined by $X_{\mathbb{Z}}[a_j] := \{\{z_i\} \in X_0 : z_0 = L_j\}$, $j = 1, \ldots, m$. Moreover, since $\mu_{\mathbb{Z}}$ is ergodic, the law $\mu^{\mathcal{L}}$ is also ergodic. We will show that the correlation measures of μ are determined by the probability distribution $\mu_{\mathbb{Z}}$ on these cylinder sets.

Define $\Xi^+ := \{\sum_{j=1}^{m} c_j L_j : c_j \in \mathbb{Z}^+, j = 1, \ldots, m\} \setminus \{0\}$. Note that a typical cluster of a point set in $X_{\mathbb{R}}^{\mathcal{L}}$ is of the form $P := \{t_0 = 0, t_1, \ldots, t_n\}$, $t_0 < t_1 < \cdots < t_n$ and $t_j - t_{j-1} \in \Xi^+$, where $n \in \mathbb{Z}^+$. We can also represent the cluster P by a finite sequence of intervals $[D_1, \ldots, D_n]$, where $D_j = t_j - t_{j-1}$ for $j = 1, \ldots, n$. We define $X_P^0 := \{\Lambda \in X : P \subset X\}$. The characteristic function of X_P^0 can be expressed as a product of finite counting functions (3.1.1) as follows.

$$\mathbf{1}_{X_P^0} = N_{\{0\}} \times N_{\{t_1\}} \times \cdots \times N_{\{t_n\}} \tag{6.1.2}$$

Since $N_{\{0\}}|_{X^0} \equiv 1$, we have

$$\dot{\mu}^{\mathcal{L}}(X_P^0) = \int_{X^0} N_{\{t_1\}} \cdots N_{\{t_n\}} d\dot{\mu} = \dot{\mu}_n^{\mathcal{L}}(\mathbf{1}_{\{t_1\}} \times \cdots \times \mathbf{1}_{\{t_n\}}), \tag{6.1.3}$$

by the definition of $\dot{\mu}_n$ the n-th moment measure of $\dot{\mu}^{\mathcal{L}}$. Recall that the $n+1$-point correlation measure exists and is equal to $\dot{\mu}^{\mathcal{L}n}$ μ-almost surely. Concerning $\mathbf{1}_P = \mathbf{1}_{\{t_1\}} \times \cdots \times \mathbf{1}_{\{t_n\}}$, we can denote $\dot{\mu}_n^{\mathcal{L}}(\mathbf{1}_{\{t_1\}} \times \cdots \times \mathbf{1}_{\{t_n\}})$ equivalently by $\dot{\mu}_n^{\mathcal{L}}([D_1, \cdots, D_n])$.

Continuing with the pattern $P = [D_1, \cdots, D_n]$, we consider the decomposition of D_i into the sum of $L_1, \cdots, L_m$. Suppose that there is some D_i such that $D_i = L_{i_1} + \cdots + L_{i_s}$ (uniquely) where $i_1, \cdots, i_s \in \{1, 2, \cdots, m\}$ and $s \geq 2$. If $\xi \in X_P^0$, then the points $t_{i-1} = D_1 + \cdots + D_{i-1}, t_i = D_1 + \cdots + D_i$ are in ξ and the points of ξ between t_{i-1} and t_i, including t_i, must be $D_1 + \cdots + D_{i-1} + L_{j_1} + \cdots + L_{j_k}, k = 1, 2, \cdots, s$ for some arrangement of $L_{i_1}, \cdots, L_{i_s}$. By observation, X_P^0 is a disjoint union of $X_{P'}^0$ where P' runs over the distinct expansions of P due to the different arrangements of $L_{i_1}, \cdots, L_{i_s}$ that arise in this way. Furthermore, looking at the definition of frequency, we see that $\mathrm{freq}(P, \xi) = \sum \mathrm{freq}(P', \zeta)$. This gives us

$$\dot{\mu}_n^{\mathcal{L}}[D_1, \cdots, D_n] = \sum_{i=1}^{n} \sum_{\sum_1^s L_{i_j} = D_i} \dot{\mu}_{n+s}^{\mathcal{L}}[D_1, \cdots, D_{i-1}, L_{i_1}, \cdots, L_{i_s}, D_{i+1}, \cdots, D_n].$$

From this, we get

Proposition 6.1.1. *For any pattern $P = \{t_0, t_1, \cdots, t_n\} = [D_1, \cdots, D_n]$, inductively, $\dot{\mu}_n^{\mathcal{L}}[D_1, \cdots, D_n]$ can be rewritten as a sum of terms $\dot{\mu}_{n+p}^{\mathcal{L}}[I_1, \cdots, I_p]$ where $I_1, \cdots, I_p \in \mathcal{L}$.* □

When for $j = 1, \ldots, n$, $D_j \in \mathcal{L}$, the sequence $[D_1, \ldots, D_n]$ actually defines a cylinder set $X_{\mathbb{Z}}[D_1, \ldots, D_n]$. In this case, we have

Proposition 6.1.2. $\dot{\mu}_n^{\mathcal{L}}[D_1, \cdots, D_n] = \mu_{\mathbb{Z}}(X_{\mathbb{Z}}[D_1, \ldots, D_n])$.

Proof: We already know that $\dot{\mu}^{\mathcal{L}}(X_P^0) = \dot{\mu}_n^{\mathcal{L}}[D_1, \cdots, D_n]$. By definition of the Palm measure,

$$\dot{\mu}^{\mathcal{L}}(X_P^0) = \frac{1}{l(V)} \int_{X_{\mathbb{R}}^{\mathcal{L}}} \sum_{x \in V \cap \Lambda'} \mathbf{1}_{X_P^0}(-x + \Lambda') d\mu^{\mathcal{L}}(\Lambda'),$$

where V is an arbitrary Borel set of $\mathbb{R}$ of finite positive measure. Here we choose that V is a neighbourhood of the origin such that $V \subset (-r, r)$ and $r := \min\{L_1, \ldots, L_m\}$. Then the sum $\sum_{x \in V \cap \Lambda'} \mathbf{1}_{X_P^0}(-x + \Lambda')$ can only contain one term. We show that it is equal to the characteristic function $\mathbf{1}_{X_{P,V}}$. First, we recall that $X_{P,V}$ is a pattern defined by $X_{P,V} := \{\Lambda' \in X : \exists x \in V \cap \Lambda', P \subset V + \Lambda'\}$. Next, since Ξ^+ is uniformly discrete, every point set in $X_{\mathbb{R}}{}^{\mathcal{L}}$ has FLC, $\Lambda' \in X_{P,V}$ if and only if there is a $x \in V \cap \Lambda'$ such that $P \subset -x + \Lambda'$. It follows that $\mathbf{1}_{X_{P,V}} = \sum_{x \in V \cap \Lambda'} \mathbf{1}_{X_P^0}(-x + \Lambda')$. Therefore, using equation (4.3.2) from Section 4.3.1, we have

$$\begin{aligned}
\dot{\mu}^{\mathcal{L}}(X_P^0) &= \frac{1}{l(V)} \int_{X_{\mathbb{R}}^{\mathcal{L}}} \mathbf{1}_{X_{P,V}}(\Lambda') d\mu^{\mathcal{L}}(\Lambda') = \frac{1}{l(V)} \mu^{\mathcal{L}}(X_{P,V}) \\
&= \frac{1}{l(V)} (l \otimes \mu_{\mathbb{Z}})((\pi^{\mathcal{L}})^{-1}(X_{P,V}) \cap F) \\
&= \frac{1}{l(V)} (l \otimes \mu_{\mathbb{Z}})(V \times X_{\mathbb{Z}}[D_1, \ldots, D_n]) \\
&= \mu_{\mathbb{Z}}(X_{\mathbb{Z}}[D_1, \ldots, D_n]).
\end{aligned}$$

□

Putting all together, then we see that every correlation measure can be expressed as a finite sum of the probability distribution $\mu_{\mathbb{Z}}$ on those cylinder sets.

6.2 Markov processes

In this section, we are interested in a typical kind of uniformly discrete ergodic point process on the real line. To introduce this kind of point process, we start with a stationary symbolic dynamical system $(\mathcal{L}^{\mathbb{Z}}, \mathbb{Z}, \mu_{\mathbb{Z}})$, where $\mathcal{L}$ is a set of m nonnegative numbers as before. Note that $(\mathcal{L}^{\mathbb{Z}}, \mathbb{Z}, \mu_{\mathbb{Z}})$ can also be thought of as a sequence of random variables $\{z_n\}_{\mathbb{Z}}$ such that these random variables take value in $\mathcal{L}$ and the combined probability distribution $\mathbb{P}$ of these random variables is defined by

$$\mathbb{P}(\{\{z_n\}_{\mathbb{Z}} : z_{s+1} = D_1, \cdots, z_{s+n} = D_n\}) = \mu_{\mathbb{Z}}([D_1, \cdots, D_n]),$$

for $s \in \mathbb{Z}$. Thus, the conditional distribution $\mathbb{P}_C$ of $\mathbb{P}$ is given as

$$\mathbb{P}_C(\{z_{n+s} = D_n | z_{s+1} = D_1, \cdots, z_{s+n-1} = D_{n-1}\}) := \frac{\mu_{\mathbb{Z}}([D_1, \cdots, D_n])}{\mu_{\mathbb{Z}}([D_1, \cdots, D_{n-1}])}, \quad (6.2.1)$$

for $s \in \mathbb{Z}$ and $D_j \in \mathcal{L}, j = 1, \cdots, n$, when $n = k+1, k+2, \cdots$.

Definition 6.2.1. *A **k-step time homogeneous Markov chain** (simply a **Markov chain**) is a stationary symbolic dynamical system with an alphabet $\mathcal{L}$ such that the conditional distribution $\mathbb{P}_C$ defined by (6.2.1) satisfies*

$$\begin{aligned}&\mathbb{P}_C(\{z_n = D_n | z_1 = D_1, \cdots, z_{n-1} = D_{n-1}\} \\ &= \mathbb{P}_C(\{z_n = D_n | z_{n-k} = D_{n-k}, \cdots, z_{n-1} = D_{n-1}\},\end{aligned}$$

for $s \in \mathbb{Z}$ and $D_j \in \mathcal{L}, j = 1, \cdots, n$, when $n = k+1, k+2, \cdots$. In particular, if $\mathcal{L}$ satisfies condition (AI) and condition (AII), then we say that the Markov chain is simple.

For a Markov chain, we define a $k \times k$ matrix Q by $Q := (a_{i_1,\ldots,i_k;j_1,\ldots,j_k})$ such that

$$a_{i_1,\ldots,i_k;j_1,\ldots,j_k} = \begin{cases} \mathbb{P}_C(\{z_{k+1} = L_{j_k} \quad |z_1 = L_{i_1}, \cdots, z_k = L_{i_k}\}) \\ \qquad\qquad\qquad\qquad \text{if } j_1 = i_2, \ldots, j_{k-1} = i_k; \\ 0, \qquad\qquad\qquad\quad \text{otherwise.} \end{cases} \quad (6.2.2)$$

We call the matrix Q the **transition matrix** of the Markov chain. Q is **primitive** if and only if there is a positive integer n such that every entries of the power matrix $\mathcal{L}^n$ is positive. It is known that if Q is primitive, then $(\mathcal{L}^{\mathbb{Z}}, \mathbb{Z}, \mu_{\mathbb{Z}})$ is an ergodic symbolic dynamical system.

Definition 6.2.2. *The uniformly discrete simple point process $(X_{\mathbb{R}}^{\mathcal{L}}, \mathbb{R}^d, \mu_{\mathbb{R}}^{\mathcal{L}})$ produced by a Markov chain $(\mathcal{L}^{\mathbb{Z}}, \mathbb{Z}, \mu_{\mathbb{Z}})$ is called an **k-step Markov process**.*

Now we get to the main theorem of this chapter.

Theorem 6.2.3. *Consider the k-step Markov process produced by a simple ergodic Markov chain $(\mathcal{L}^{\mathbb{Z}}, \mathbb{Z}, \mu_{\mathbb{Z}})$. Suppose that $\dot{\mu}_i$ is the (i+1)-point correlation measure of the k-step Markov process for $i = 1, \ldots, k+1$. Then the law μ is uniquely determined by $\{\dot{\mu}_i\}_1^{k+1}$.*

PROOF: For $n \in \mathbb{N}, n > (k+1)$, we take an arbitrary configuration of $[D_1, \ldots, D_n]$, $D_i \in \mathcal{L}, i = 1, \ldots, n$.

$$\begin{aligned}
\dot{\mu}_n[D_1, \cdots, D_n] &= \mu_{\mathbb{Z}}(X_{\mathbb{Z}}[D_1, \ldots, D_n]) \\
&= \mu_{\mathbb{Z}}(X_{\mathbb{Z}}[D_1, \ldots, D_{n_1}])\mathbb{P}_C(D_n|D_1, \cdots, D_{n-1}\} \\
&= \dot{\mu}_{n-1}[D_1, \cdots, D_{n-1}]\mathbb{P}(D_n|D_1, \cdots, D_{n-1}) \\
&= \dot{\mu}_{n-1}[D_1, \cdots, D_{n-1}]\mathbb{P}(D_n|D_{n-k}, \cdots, D_{n-1})
\end{aligned}$$

Therefore,

$$\dot{\mu}_n[D_1, \cdots, D_n] = \frac{\dot{\mu}_{n-1}[D_1, \cdots, D_{n-1}]\dot{\mu}_{k+1}[D_{n-k}, \cdots, D_n]}{\dot{\mu}_k[D_{n-k}, \cdots, D_{n-1}]},$$

from the definition of a k-step Markov process.

Continuing by induction, we prove that $\dot{\mu}_n[D_1, \cdots, D_n]$ is expressible in terms of $\dot{\mu}_1, \cdots, \dot{\mu}_{k+1}$ evaluated on sequences of $\mathcal{L}$.

Next, by Proposition 6.1.1, we know that this claim is also true for an arbitrary configuration of $[D_1, \ldots, D_n]$, $D_i \in \Xi^+$, for $i = 1, \ldots, n$. So, we conclude that $\dot{\mu}_1, \ldots, \dot{\mu}_{k+1}$ determine all other correlation measures. Finally, by Theorem 4.4.3, μ is uniquely determined by $\dot{\mu}_1, \ldots, \dot{\mu}_{k+1}$. □

Remark 6.2.4. *The Theorem 6.2.3 remains true even if $\mathcal{L}$ is not integrally independent. To see this, we just need to reconsider the Proposition 6.1.1. It is still true that every pattern is a disjoint union of some simple patterns due to the different arrangement of those interval lengths $\{L_1, \cdots, L_m\}$ no matter that the decomposition is more complicated. In this case, for a given line segment $D_i \in \Xi^+$, there may exist many distinct combinations of those interval lengths $\{L_1, \cdots, L_m\}$ such that their sum is equal to D_i. Therefore,*

$$C_0[D_1, \cdots, D_n] = \bigcup_{s=1}^{\infty} \bigcup_{\sum_1^s L_{i_j} = D_i} C_0[D_1, \cdots, D_{i-1}, L_{i_1}, \cdots, L_{i_s}, D_{i+1}, \cdots, D_n].$$

6.3 Point processes of Model sets

In this section, we consider an ergodic point process generated by a model set (see Corollary 2.5.6), whose window set is restricted to be a relatively compact subset of $\mathbb{R}^m$. The main purpose of this section is to attack the problem of the uniqueness of such an ergodic point process for given a finite set of correlation measures (2-point and 3-point correlation measures here).

Let Λ be a regular model set such that $\Lambda^* \subset \mathbb{R}^m$. Denote by Ω the closure of the window set Λ^*, i.e., $\Omega = \overline{\Lambda^*}$. Evidently, Ω has nonempty interior and hence $l(\Omega) > 0$.

In general, for $i \in \mathbb{N}$, the $(i+1)$-point correlation measure $\gamma_\Lambda^{(i+1)}$ exists and it has the form

$$\gamma_\Lambda^{(i+1)} = \sum_{x_1,\dots,x_i \in (\mathbb{R}^d)} \eta^{(i)}_{(x_1,\dots,x_i)} \delta_{(x_1,\dots,x_i)}, \tag{6.3.1}$$

where

$$\eta^{(i)}_{(x_1,\dots,x_i)} = \begin{cases} \text{freq}(\{0, x_1, \dots, x_i\}), & \text{if } x_1, \dots, x_i \in (\Lambda - \Lambda), \\ 0, & \text{otherwise.} \end{cases} \tag{6.3.2}$$

It has been proved in [40] that

$$\text{freq}(\{0, x_1, \dots, x_i\}) = \frac{l(\Omega \cap \bigcap_{j=1}^i (-x_j^* + \Omega))}{l(\Omega)}. \tag{6.3.3}$$

It is clear that $\eta_{(0,\dots,0)} = 1$.

Let $f := \mathbf{1}_\Omega$, $E := \{k \in \mathbb{R}^d : \hat{f}(k) = 0\}$, and $D := \mathbb{R}^d \setminus E$. Since f is compactly supported and measurable, its Fourier transform exists and the Fourier transform $\hat{f}$ is continuous. Thus, the set E is closed. It follows that D is a open set. Moreover, since $\hat{f}(0) = l(\Omega) > 0$, we have $0 \in D$. We will show that the set E has no interior.

Note that

$$\hat{f}(k) = \int_\Omega e^{-2\pi i k \cdot x} dx = \int_\Omega \cos(2\pi k \cdot x) dx - i \sin(2\pi k \cdot x) dx.$$

Let g be the real part of the function $\hat{f}$, i.e., $g(k) := \int_\Omega \cos(2\pi k \cdot x) dx$, $k \in \mathbb{R}^d$.

Lemma 6.3.1. *The function g is a real analytic function.*

PROOF: Note that the function $e^{-2\pi i k \cdot x}$ (with the variable x) has a Taylor expansion at the origin as follows,

$$\begin{aligned} e^{-2\pi i k \cdot x} &= e^{-2\pi i \sum_1^d k_j x_j} = a_0 + \\ &\sum_{m=1}^\infty \left[\sum_{i_1,\dots,i_d, i_1+\dots+i_d=m} a_{i_1,\dots,i_d;m} k_1^{i_1} \dots k_d^{i_d} \right], \end{aligned}$$

where

$$\begin{aligned} a_0 &= 1 \\ a_{i_1,\ldots,i_d;m} &= \frac{(-2\pi i)^m x_1^{i_1}\ldots x_d^{i_d}}{i_1!\ldots i_d!}. \end{aligned}$$

Since $e^{-2\pi i k\cdot x} = \cos(2\pi k\cdot x) - i\sin(2\pi k\cdot x)$, $cos(2\pi k\cdot x)$ has a Taylor expansion at the origin as follows,

$$\begin{aligned} \cos(2\pi k\cdot x) &= a_0 + \qquad (6.3.4) \\ & \sum_{m=1}^{\infty}\left[\sum_{i_1,\ldots,i_d,i_1+\cdots+i_d=2m} a_{i_1,\ldots,i_d;2m}k_0^{i_1}\ldots k_d^{i_d}\right], \qquad (6.3.5) \end{aligned}$$

Since Ω is compact, there is a positive number $C > 0$ such that for $i = 1,\ldots,d$ and all $x = (x_1,\ldots,x_d) \in \Omega$, $|x_i| < C$. Hence, $|a_{i_1,\ldots,i_d;2m}| < \frac{(2\pi)^{2m}C^{2m}}{i_1!\ldots i_d!}$. Let $b_0 := l(\Omega)$ and for $m \in \mathbb{N}$,

$$b_{i_1,\ldots,i_d;2m} := \frac{\int_\Omega(-1)^m(2\pi)^{2m}x_1^{i_1}\ldots x_d^{i_d}dx}{i_1!\ldots i_d!} \qquad (6.3.6)$$

It is clear that

$$|b_{i_1,\ldots,i_d;m}| < |a_{i_1,\ldots,i_d;m}|l(\Omega) = \frac{(2\pi)^{2m}C^{2m}}{i_1!\ldots i_d!}l(\Omega).$$

Thus, the function series

$$b_0 + \sum_{m=1}^{\infty}\left[\sum_{i_1,\ldots,i_d,i_1+\cdots+i_d=2m} b_{i_1,\ldots,i_d;2m}k_1^{i_1}\ldots k_d^{i_d}\right], \qquad (6.3.7)$$

converges everywhere.

Furthermore, by the Lebesgue dominated convergence theorem, we have

$$\begin{aligned} g(k) = \int_{x\in\Omega}\cos(2\pi k\cdot x)dx &= b_0 + \qquad (6.3.8) \\ & \sum_{m=1}^{\infty}\left[\sum_{i_1,\ldots,i_d,i_1+\cdots+i_d=2m} b_{i_1,\ldots,i_d;2m}k_1^{i_1}\ldots k_d^{i_d}\right], \end{aligned}$$

Therefore, the function g has a Taylor expansion on $\mathbb{R}^d$. This follows that the function g is an analytic real function. □

It is known that the zero set of a nonzero real analytic function on $\mathbb{R}^d$ has no interior. Since $\hat{f}(k) = 0$ only if $g(k) = 0$, we conclude that the set E has no interior.

Since D is open and $0 \in D$, there is a $r_0 > 0$ such that $B_{r_0} \subset D_0$.

We define a sequence of functions $\{\mathcal{I}^n\}_1^\infty$ by

$$\mathcal{I}^{(n)}(x_1, \dots, x_n) := l(\Omega \cap \bigcap_{j=1}^{n}(x_j + \Omega)),$$

or equivalently,

$$\mathcal{I}^{(n)}(x_1, \dots, x_n) = \int_{\mathbb{R}^d} \prod_{j=1}^{n} \tilde{f}(x_j - t) f(t) dt, \tag{6.3.9}$$

where $f = \mathbf{1}_\Omega$, $\tilde{f}(x) = f(-x)$.

Lemma 6.3.2. *For $n \in \mathbb{N}$, $\mathcal{I}^{(n)}$ is uniquely determined by $\gamma_\Lambda^{(n+1)}$.*

PROOF: It is clear that $\mathcal{I}^{(n)}$ is a continuous function supported on $(\Omega - \Omega)^j$. Recall that Λ is dense in Ω. It follows that $(\Lambda - \Lambda)^*$ is dense in $(\Omega - \Omega)$. Moreover, for $x_1, \dots, x_n \in (\Lambda - \Lambda)$,

$$\mathcal{I}^{(n)}(x_1, \dots, x_n) = l(\Omega) \gamma_\Lambda^{(n+1)}(-x_1^*, \dots, -x_n^*).$$

It implies that $\mathcal{I}^{(n)}$ is uniquely determined by $\gamma_\Lambda^{(n+1)}$, for $n \in \mathbb{N}$. □

Now we are going to show that the function $\hat{f}$ is uniquely determined by $\gamma_\Lambda^{(2)}$ and $\gamma_\Lambda^{(3)}$ on D_0 by using this lemma.

At first, we define ϕ_0 on D by $\phi_0(k) := \frac{\hat{f}(k)}{|\hat{f}(k)|}$. Then $\phi_0(k)$ is a continuous function on D and $|\phi_0(k)| \equiv 1$. Since $\hat{f}(0) = l(\Omega) > 0$, $\phi_0(0) = 1$.

In general, by simple computation, we have

$$\widehat{\mathcal{I}^{(n)}}(k_1, \dots, k_n) = \prod_{j=1}^{n} \bar{\hat{f}}(k_j) \hat{f}(\sum_{j=1}^{n} k_j), \quad n \in \mathbb{N}. \tag{6.3.10}$$

When $n = 1$

$$\mathcal{I}^{(1)}(k) := \int_{\mathbb{R}^d} \tilde{f}(k - t) f(t) dt. \tag{6.3.11}$$

This is the convolution product of the function f and $\tilde{f}$. Thus,

$$\widehat{\mathcal{I}^{(1)}}(k) = \hat{f}(k) \bar{\hat{f}}(k) = |\hat{f}(k)|^2. \tag{6.3.12}$$

When $n = 2$,

$$\widehat{\mathcal{I}^{(2)}}(k_1, k_2) = \overline{\hat{f}}(k_1) \overline{\hat{f}}(k_2) \hat{f}(k_1 + k_2). \tag{6.3.13}$$

Denote by $D^{(2)}$ the set $\{(k_1, k_2) : k_1, k_2, k_1 + k_2 \in D\}$. Note that $B_{\frac{r_0}{2}} + B_{\frac{r_0}{2}} \subset B_{r_0} \subset D_0$. Thus, $B_{\frac{r_0}{2}} \times B_{\frac{r_0}{2}} \subset D^{(2)}$. Then on $D^{(2)}$, we define

$$\psi^{(2)}(k_1, k_2) := \frac{\widehat{\mathcal{I}^{(2)}}(k_1, k_2)}{|\hat{f}(k_1)||\hat{f}(k_2)||\hat{f}(k_1 + k_2)|}. \tag{6.3.14}$$

By (6.3.13), for $(k_1, k_2) \in D^{(2)}$,

$$\phi_0(k_1 + k_2) = \phi_0(k_1)\phi_0(k_2)\psi^{(2)}(k_1, k_2). \tag{6.3.15}$$

This implies that the function ϕ_0 is a particular solution of the following equation:

$$\phi(k_1 + k_2) = \phi(k_1)\phi(k_2)\psi^{(2)}(k_1, k_2), \tag{6.3.16}$$

where ϕ is defined on D and $(k_1, k_2) \in D^{(2)}$. We point out here that this equation is entirely determined by the function $\mathcal{I}^{(1)}, \mathcal{I}^{(2)}$ since the function $\psi^{(2)}$ is given by them.

Equation (6.3.16) is related to the following homogenous equation.

$$\varphi(k_1 + k_2) = \varphi(k_1)\varphi(k_2), \varphi(0) = 1, \tag{6.3.17}$$

where φ is defined on D and $(k_1, k_2) \in D^{(2)}$.

Let ϕ be an arbitrary solution of equation (6.3.16). Then $\frac{\phi}{\phi_0}$ is a solution of equation (6.3.17). We are going to show that every solution of equation (6.3.17) is a character function χ_a on $\mathbb{R}^m$ restricted to D. Hence ϕ is of the form $\phi = \phi_0\chi_a$.

Let H be a locally compact Abelian group. Suppose that Z is a closed subset of H with no interior, and $0 \notin Z$. Let $S := H \backslash Z$ and $S^{(2)} := \{(k_1, k_2) : k_1, k_2, k_1 + k_2 \in S\}$.

Lemma 6.3.3. *$S^{(2)}$ is dense in $H \times H$.*

PROOF: Suppose $S^{(2)}$ is not dense in $H \times H$, i.e., there is a open set $U \times V \subset H \times H \setminus S^{(2)}$, where $U, V \subset H$ are open. Since Z is a closed subset of H with no interior, $S \cap U, S \cap V$ are nonempty open sets. For all $u \in S \cap U, v \in S \cap V$, we have $(u, v) \in H \times H \setminus S^{(2)}$, i.e., $u + v \in Z$. Thus, $S \cap U \subset -v + Z$. This is impossible since on the one hand $S \cap U$ is an open set, on the another hand Z has no interior. □

Proposition 6.3.4. *Let*

$$\varphi : S \longrightarrow U(1)$$

be a continuous mapping satisfying

$$\varphi(s + t) = \varphi(s)\varphi(t)$$

whenever $s, t, s + t \in S$. Then there is a unique character $\chi \in \widehat{H}$ with $\chi | S = \varphi$.

Proof: Let $\mathcal{U}$ be the uniformity on H defined by its structure as a topological group: the basic entourages are the sets

$$U(V) := \{(x, y) : H \times H, x - y \in V\}$$

where V runs through open neighbourhoods of $0 \in H$. Since S is open and $0 \in S$, we can restrict these entourages to those in which $V \subset S$. This uniformity also induces a uniformity on S (which is that of the induced topology on S).

We claim that $\varphi : S \longrightarrow U(1)$ is uniformly continuous. We show that given any $\epsilon > 0$ there is an entourage $U(V(\epsilon)) \cap (S \times S)$ for which $(s, t) \in U(V(\epsilon)) \cap (S \times S)$ implies that $|\varphi(s) - \varphi(t)| < \epsilon$.

In fact $V(\epsilon) := \{s \in S : |\varphi(s) - 1| < \epsilon\}$ works. This is an open subset of S containing 0 and furthermore, $(s, t) \in U(V(\epsilon)) \cap (S \times S)$ implies $s - t \in V(\epsilon)$ and then $s - t \in S$ and $|\varphi(s - t) - 1| < \epsilon$. Using the basic relation satisfied by φ, $|\varphi(s) - \varphi(t)| = |\varphi(s - t)\varphi(t) - \varphi(t)| = |\varphi(s - t) - 1| < \epsilon$, which what we wished to show.

Since H is locally compact, it is complete (see Corollary 1 in Chapter 3.3, [9]). Since Z has no interior, H is the closure of S. Since φ is uniformly continuous on S it extends to a uniformly continuous function $\chi : H \longrightarrow U(1)$. Then the mapping $H \times H \longrightarrow U(1)$ defined by $(x, y) \mapsto \chi(x + y)\chi(x)^{-1}\chi(y)^{-1}$ is continuous and is equal to 1 on all of the set $S^{(2)}$. By Lemma 6.3.3, $S^{(2)}$ is dense in $H \times H$ and so by the continuity, it is identically equal to 1. Thus χ is a character. □

In our particular case, $H = \mathbb{R}^m$, $Z = E$ and $S = D$. The character function χ is of the form $\chi_a := e^{-2\pi i a \cdot (\cdot)}$, for some $a \in \mathbb{R}^d$. Moreover, each solution of (6.3.16) has the form

$$\phi(k) := \phi_0(k)\chi_a(k), \tag{6.3.18}$$

Finally, we get the main result of this section.

Theorem 6.3.5. *Let $\Lambda = \Lambda(\Omega)$ be a regular model set. Then the point process generated by Λ is uniquely determined by the 2-point and 3-point correlation measure of Λ.*

PROOF: It is clear that all correlation measures of the point process are uniquely determined by the set Ω through the formula (6.3.1) and (6.3.3). This implies that the point process itself is uniquely determined by the set Ω by Theorem 4.3.3. We are going to show that any window set generating the same 2-point and 3-point correlation measure is actually a translated copy of the window set Ω.

Recall that equation (6.3.16) is determined by $\widehat{\mathcal{I}^{(1)}}$ and $\widehat{\mathcal{I}^{(2)}}$ and a solution of this equation is of the form

$$\phi(k) := \phi_0(k)\chi_a(k), \tag{6.3.19}$$

where $a \in \mathbb{R}^d$.

Let f' be the characteristic function of another window set generating the same 2-point and 3-point correlation measure. Hence, $|\widehat{f'}| = (\widehat{\mathcal{I}^{(1)}})^{\frac{1}{2}} = |\hat{f}|$. Moreover, the function $\frac{\widehat{f'}}{|\widehat{f'}|}$ is a solution of equation (6.3.16) and hence it is equal to $\phi_0(k)\chi_a(k)$. Putting all this together, we have

$$\widehat{f'}(k) := \hat{f}(k)\chi_a(k). \tag{6.3.20}$$

Taking the inverse Fourier transform on both sides of equation (6.3.21), we have

$$f' := \mathbf{1}_{-a+\Omega} \tag{6.3.21}$$

Therefore, $\widehat{\mathcal{I}^{(1)}}$ and $\widehat{\mathcal{I}^{(2)}}$, or equivalently, $\gamma_\Lambda^{(2)}$ and $\gamma_\Lambda^{(3)}$, uniquely determines the window set Ω up to translation.

A fixed window set uniquely determines a point process through giving all its correlation measures by equation (6.3.3). Furthermore, this equation implies that the correlation measures of this point process and hence the point process will not change if we replace the window set by a translated copy of it. Therefore, $\gamma_\Lambda^{(2)}$ and $\gamma_\Lambda^{(3)}$ uniquely determine the point process generated by the model set Λ. $\square$

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