Exotic Fusion Categories and Their Modular Data

by

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Abstract

The majority of known examples of fusion categories come directly from classical structures – vector spaces, groups, representations, and the like. In recent years the technique of constructing fusion categories as endomorphisms on Cuntz algebras was developed and has already lead to completely new examples of fusion categories. Somewhat surprisingly, the fusion categories found seem to belong to infinite families. We push the Cuntz construction further and find more examples within two potentially infinite families, the near group fusion categories and the Haagerup-Izumi fusion categories.

For all of the newly cataloged fusion categories, we also compute their modular data. In the case of the Haagerup-Izumi series, we find that all new examples satisfy the conjecture of [10], which posits an unexpectedly simple form for the modular data in terms of certain bilinear forms. In the case of near group categories associated to an odd ordered abelian group, we find that the modular data of new examples also satisfy a similar conjecture (found in [6]). When the order of the group is even, no such conjecture existed; we provide a new conjecture which predicts the modular data for all current examples.

Dedicated to my wife, whose encouragement and willingness to subject herself to years of incomprehensible white-board sketches enabled this thesis.

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Chapter 1

Introduction

Math and physics have a long history, with advances in one often spurring breakthroughs in the other. Recently, conformal field theories (CFTs) have sparked a great deal of completely new mathematics. CFTs are two-dimensional quantum field theories which exhibit conformal symmetry – they describe the motion of subatomic particles in two dimensions (one spatial, one time), roughly speaking. The best mathematical description of CFTs are vertex operator algebras (VOAs) which axiomatize the key notions of a CFT. They were first introduced by Borcherds in [2] in the context of CFTs and the monstrous moonshine conjectures, however they have found a life of their own in numerous areas. Despite their usefulness, VOAs are still poorly understood in the grand scheme of things, partly because they are so new. However, one can define the representation category of a VOA (similarly to how one defines group representation) which provides a more down-to-Earth approach to their study. The representations of a VOA are relatively well behaved, and when the VOA satisfies some additional conditions (specifically if the VOA is rational and C_2-cofinite), then the associated representation category will be a modular tensor category (MTC). They are special for many reasons – they yield numerical invariants for knots, for one. But the key property of an MTC is a piece of associated combinatorial data, called modular data, which amounts to a representation of SL$(2, \mathbb{Z})$. As far as invariants go, it is a good one (although it is not a complete invariant), so understanding modular data is an important part of understanding a given MTC.

The current problem with MTCs is that the best current examples are also the most mundane, in the sense that they are built using classical techniques.

They come directly from lattices, groups, Lie algebras, etc, which is somewhat surprising given their applicability to mathematics which should, by all rights, be completely new. The solution, of course, is to find exotic examples of MTCs or at least their modular data, which is the goal of this thesis. Our plan is first to construct exotic fusion categories $\mathcal{C}$ by means of the Cuntz algebra construction. Then we make use of the tube algebra associated to each $\mathcal{C}$ to find the modular data of $\mathcal{Z}(\mathcal{C})$, the center of $\mathcal{C}$, which is an MTC. In this way we can avoid dealing explicitly with the details of the MTCs.

Before explaining further details, we would be remiss to ignore the importance of fusion categories in their own right. Historically, fusion categories were (and are) studied for more than their association to physics. They play an important role in the theory of Von Neumann algebras too. In particular, any finite depth, finite index subfactor yields two related fusion categories. We will not make use of this correspondence directly, but it is important to mention as much of the literature uses the language of subfactors rather than fusion categories. See [22] for more details on the correspondence.

The fusion categories we construct will fall into two classes: near-group fusion categories and Haagerup-Izumi fusion categories. Both classes were proposed by Izumi in [16] where he also constructed the first two examples of each. Initially it was expected that both series would terminate quickly, although Evans and Gannon released a series of work ([10], [7], [6]) in which they constructed more examples in each class and now the prevalent notion is that both series are infinite. A portion of our novel results (found in Chapters 3 and 4) find further examples in each class as well, which strongly supports the notion that these are in fact infinite classes of fusion categories.

The method we use to construct the fusion categories is called the Cuntz algebra construction. The idea is to interpret the objects of the fusion category as algebra endomorphisms on a particular infinite dimensional algebra, and use the natural structure imposed by the (potential) fusion category to determine the endomorphisms in terms of complex solutions to a system of polynomial equations. The method was pioneered by Izumi in [15] for specific small index subfactors at which point it was noticed that the construction would generalize to hypothetical higher index subfactors at the cost of more equations in more variables. From a theoretical point of view, therefore, the problem was reduced to only checking for solutions to equations, but the problem became necessarily more complex from a computational standpoint.

With the fusion categories found, the next problem is to describe the associated MTC. This is done through Ocneanu's tube algebra (See [9]). In this thesis, we focus on the most important part of the MTC, the modular data. In the tube algebra setting, this is done by solving another (much simpler) system of polynomial equations.

The modular data of our new fusion categories is our second and more important result. In addition to the numerical data, we also conjecture a simple form for the modular data which is not at all expected from its tube algebra definition.

The plan for the remainder of this thesis is as follows. Chapter 2 covers the essential background – we define the relevant categorical notions of fusion categories and MTCs and outline both the Cuntz algebra construction and the tube algebra method. We end the section by briefly summarizing key results of symmetric forms which play a role in later chapters. Chapter 3 details our results for near group categories, and chapter 4 does the same for the Haagerup-Izumi categories. Finally in chapter 5 we pose a few outstanding questions which we hope to answer in future works.

Throughout this paper, we assume basic knowledge of category theory. (A good introduction for those unfamiliar is [21]). We also assume that the reader is familiar with Galois theory of fields as it plays a small but significant role in the classification of near group categories.

Chapter 2

Background

2.1 Fusion Categories

As one of the primary goals of this thesis is the construction of new fusion categories, it is important to give sufficient background. The formal definition of fusion categories is a long list of conditions but they are all motivated by finite dimensional vector spaces, as we will see. For most purposes it is enough to know that the definition exists and we will occasionally make use of some of the properties, but as with many categorical concepts we try to avoid getting too involved with the details. The purpose of the remainder of the subsection will be to familiarize the reader with some of the relevant concepts from the basic theory of fusion categories, starting from the definition. To make things more concrete we provide plenty of examples in the latter part of the subsection. This section primarily follows [1], particularly chapters 1 and 2.

Definition 2.1.1 (Fusion Category)**.** *Let k be an algebraically closed field with $Char(k) = 0$. A k-linear, semisimple, rigid monoidal category $\mathcal{C}$ is called a fusion category if there are finitely many isomorphism classes of simple objects, the spaces of morphisms are finite dimensional, and $End(1) = k$.*

Throughout the remainder of this thesis, we will specialize to the case $k = \mathbb{C}$. The various conditions required of a fusion category are defined as follows:

Definition 2.1.2 ($k-$linear)**.** *A $k-$linear category is one in which all Hom-sets have the structure of a k vector space.*

We denote the identity morphism of an object X by 1_X.

Definition 2.1.3 (Isomorphism). *A morphism $f : X \to Y$ is an isomorphism if there exists another morphism $g : X \to Y$ such that $fg = 1_X$ and $gf = 1_Y$.*

Definition 2.1.4 (Monomorphism). *A morphism $f : A \to B$ is called a monomorphism if for all morphisms $g, h : C \to B$, $f \circ g = f \circ h$ implies $h = g$.*

A monomorphism is the categorical generalization of an injection and behaves in much the same way.

Definition 2.1.5 (Direct Sum). *Let $\mathcal{C}$ be a k-linear category and $X, Y \in \mathcal{C}$ objects. Then an object $Z \in \mathcal{C}$ is said to be a direct sum of X and Y if there are morphisms*

$$\begin{aligned} u &\in Hom(X, Z), \quad u' \in Hom(Z, X), \\ v &\in Hom(Y, Z), \quad v' \in Hom(Z, Y), \end{aligned}$$

such that

$$\begin{aligned} u \circ u' + v \circ v' &= 1_Z, \\ u' \circ u = 1_X, \quad v' \circ v &= 1_Y. \end{aligned}$$

Write $Z = X \oplus Y$.

It is best to think about u and v as embeddings and u', v' as projections.

Definition 2.1.6 (Semisimple). *A semisimple category is one in which each object is a direct sum of simple objects. An object X is simple if the only monomorphisms into X come from 0 and X itself. That is, if $i : Y \hookrightarrow X$ is a monomorphism, then $Y \cong X$ or $Y \cong 0$.*

Intuitively speaking, a simple object is one with no nontrivial "sub" objects. The lack of a monomorphism is the lack of an embedding of a smaller object into the simple one. For example, $\mathbb{C}$ considered as a vector space over itself has no nontrivial subspaces, and as we will see, it is a simple object in the category of vector spaces over $\mathbb{C}$.

Definition 2.1.7 (Natural Transformation/Natural Isomorphism). *Let $\mathcal{F}, \mathcal{G} : \mathcal{C} \to \mathcal{D}$ be functors. A natural transformation $\nu : \mathcal{F} \to \mathcal{G}$ is a family $\{\nu_X\}$ of morphisms indexed by objects X in $\mathcal{C}$ satisfying two properties. First, ν_X is a morphism $\mathcal{F}(X) \to \mathcal{G}(X)$. Second, for every morphism $f \in Hom_{\mathcal{C}}(X, Y)$, the equality*

$$\nu_Y \circ \mathcal{F}(f) = \mathcal{G}(f) \circ \nu_X$$

holds. If for all $X \in \mathcal{C}$, ν_X is an isomorphism in $\mathcal{D}$, then ν is called a natural isomorphism.

Less formally speaking, a natural transformation is a map of functors which respects their structure. It is no coincidence that this looks similar to the definition of every other structure preserving map (i.e. morphism) and we will make use of this fact in a key example later. It is common to define only the morphisms of a natural transformation, leaving the functors implicit, which we will often do moving forward.

Definition 2.1.8 (Monoidal Category). *A category $\mathcal{C}$ is called monoidal if it contains the following data:*

1. *A functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, called the tensor product.*
2. *For all objects $U, V, W \in \mathcal{C}$, a natural isomorphism*

   $$\alpha_{U,V,W} : (U \otimes V) \otimes W \to U \otimes (V \otimes W)$$

   called the associativity isomorphism.
3. *For each $V \in \mathcal{C}$, natural isomorphisms*

   $$\begin{aligned} \lambda_V :& \mathbf{1} \otimes V \to V; \\ \rho_V :& V \otimes \mathbf{1} \to V \end{aligned}$$

In addition, the following diagrams must commute:

$$\begin{tikzcd}
 & X \otimes Y & \\
(X \otimes 1) \otimes Y \arrow[ur, "\rho_X \otimes 1_Y"] \arrow[rr, "\alpha_{X,1,Y}"] & & X \otimes (1 \otimes Y) \arrow[ul, "1_X \otimes \lambda_Y"']
\end{tikzcd}$$

$$\begin{tikzcd}
 & (W \otimes X) \otimes (Y \otimes Z) \arrow[dl, "\alpha_{W \otimes X,Y,Z}"'] \arrow[dr, "\alpha_{W,X,Y \otimes Z}"] & \\
((W \otimes X) \otimes Y) \otimes Z \arrow[d, "\alpha_{W,X,Y} \otimes 1_Z"] & & W \otimes (X \otimes (Y \otimes Z)) \\
(W \otimes (X \otimes Y)) \otimes Z \arrow[rr, "\alpha_{W,X \otimes Y,Z}"] & & W \otimes ((X \otimes Y) \otimes Z)) \arrow[u, "1_W \otimes \alpha_{X,Y,Z}"']
\end{tikzcd}$$

They are called the triangle and pentagon identities, respectively.

The necessity of the triangle and pentagon identities is not immediately apparent, but they are needed for the following reason. In a more concrete setting, say a ring, the associativity axiom parallels property (2) of a monoidal category. That is, associativity of multiplication tells us that $(ab)c = a(bc)$, for all a, b, c in the ring. It's easy to extend this inductively to any finite product, so that any bracketing is equal. We can therefore drop the bracketing entirely. The issue in a monoidal category is that $\alpha_{U,W,V}$ is not equality, but natural isomorphism. It is still possible to show that any word $X_1 \otimes X_2 \otimes \cdots \otimes X_n$, bracketed any way you like, is naturally isomorphic to any other bracketing of the same string. What the pentagon axiom adds to this is the statement that any such isomorphism (in the case $n = 4$) is identical. That is, there is a unique isomorphism $(W \otimes X) \otimes (Y \otimes Z) \to W \otimes (X \otimes (Y \otimes Z))$.

Similar issues occur since the isomorphisms λ_V and ρ_V are not equalities. The triangle condition ensures that given a word $X \otimes 1 \otimes Y$, we can collapse the unit object either to the right or the left, and that the choice yields the same isomorphism.

The well known Mac Lane coherence theorem [23] essentially states that the triangle and pentagon identities are necessary and sufficient to ensure that we can freely rearrange brackets and collapse unit elements in words of any length. That is, given any choice of valid bracketing, it is possible to rearrange it to some other bracketing in many (possibly) different ways by applying α to different objects in different orders, but the coherence theorem ensures that the way you move from one word to another doesn't matter. The isomorphisms you get will be the same.

Definition 2.1.9 (Dualizable object). *An object X is called (right) dualizable if there exists another object X^* and morphisms*

$$\epsilon_X : X^* \otimes X \to 1, \quad \eta_X : 1 \to X \otimes X^*$$

such that the following compositions are the identity:

$$X \longrightarrow 1 \otimes X \longrightarrow (X \otimes X^*) \otimes X \longrightarrow X \otimes (X^* \otimes X) \longrightarrow X \otimes 1 \longrightarrow X$$
$$X^* \longrightarrow X^* \otimes 1 \longrightarrow X^* \otimes (X \otimes X^*) \longrightarrow (X^* \otimes X) \otimes X^* \longrightarrow 1 \otimes X^* \longrightarrow X^*$$

Left duals are defined analogously and denoted ${}^*X$. The morphisms ϵ_X and η_X are called evaluation and coevaluation respectively. If X and Y are objects with duals, and $f \in Hom(X, Y)$ then there is a morphism $f^* \in Hom(Y^*, X^*)$ defined as the composition

$$Y^* \xrightarrow{1_{Y^*} \otimes \eta_X} Y^* \otimes X \otimes X^* \xrightarrow{1_{Y^*} \otimes f \otimes 1_{X^*}} Y^* \otimes Y \otimes X^* \xrightarrow{\epsilon_Y \otimes 1_{X^*}} X^*.$$

Definition 2.1.10 (Rigid). *A category is called rigid if every object has a left and right dual.*

Rigidity is compatible with the a monoidal category in the following way:

Lemma 2.1.11. *Let $\mathcal{C}$ be a rigid, monoidal category. Then there are unique isomorphisms*

$$1^* \to 1, \quad {}^*1 \to 1$$
$$(X \otimes Y)^* \to Y^* \otimes X^*$$

and

$$\alpha^*_{XYZ} = \alpha_{Z^* Y^* X^*}$$

Proof. The first map is evidently

$$1^* \xrightarrow{\rho^{-1}} 1^* \otimes 1 \xrightarrow{\epsilon_1} 1$$

and the second is the analogous statement for left duals. For the second statement, we see that

$$(Y^* \otimes X^*) \otimes (X \otimes Y) \to (Y^* \otimes (X^* \otimes X)) \otimes Y \to (Y^* \otimes 1) \otimes Y \to 1$$

defines a morphism $\epsilon_{X \otimes Y} : (Y^* \otimes X^*) \otimes (X \otimes Y) \to 1$. A similar composition works to define $\eta_{X \otimes Y}$.

Finally, the following commutative diagram shows the last statement:

$$\begin{tikzcd}
(X \otimes (Y \otimes Z))^* \arrow[r, "\alpha^*_{XYZ}"] \arrow[d] & ((X \otimes Y) \otimes Z)^* \\
(Z^* \otimes Y^*) \otimes X^* \arrow[r, "\alpha_{Z^*Y^*X^*}"] & Z^* \otimes (Y^* \otimes Z^*) \arrow[u]
\end{tikzcd}$$

The vertical arrows are the isomorphisms found earlier. □

There are plenty of examples of fusion categories, most of which are constructed directly from classical structures, such as vector spaces, groups, representations, etc. The most fundamental is $\mathbf{Vect_k}$, as many of its properties serve as the inspiration behind the abstract definitions given above. As a result, the definitions can be understood very concretely as they apply to $\mathbf{Vect_k}$, so they provide useful intuition before seeing more complicated examples.

Example 2.1.1 ($\mathbf{Vect_k}$). $\mathbf{Vect_k}$ is the category of finite dimensional vector spaces over a field k. The morphisms are linear maps. It is well known that for all vector spaces U, V, $\{f : U \to V | f \text{ is linear}\}$ is itself a vector space under pointwise addition and scalar multiplication, so $\mathbf{Vect_k}$ is $k-$linear.

The next condition to check is semisimplicity. If $dim(V) = n$, then $V \cong k^n$, so V is a direct sum of n copies of k. k is simple, because any non-trivial vector space either has dimension 1 and is isomorphic to k, or has dimension > 1, in which case there is no injection into k. This also shows that k is the only simple object, up to isomorphism.

The tensor product is exactly the tensor product on vector spaces and we can use it to build up the associativity isomorphism as follows. Starting with vector spaces X, Y, Z, there is a map $t_x : Y \times Z \to (X \otimes Y) \otimes Z$ defined by $t_x(y, z) = (x \otimes y) \otimes z$. This map is obviously bilinear (and bijective) and so by the universal property of the tensor product, it induces a linear map $T_x : Y \otimes Z \to (X \otimes Y) \otimes Z$. Then define a new map $T : X \times (Y \otimes Z) \to (X \otimes Y) \otimes Z$ by $T(x, \sum y \otimes z) = T_x(\sum y \otimes z)$, which is again bilinear and bijective. Applying the universal property once more, we get a linear bijection of vector spaces $\alpha_{X,Y,Z} : X \otimes (Y \otimes Z) \to (X \otimes Y) \otimes Z$, which is the desired isomorphism.

We took the tensor unit to be the base field k, and we know that as vector spaces $k \otimes V \cong V$ and $V \otimes k \cong V$. Here again we just take λ_V, ρ_V to be the following isomorphisms, defined on basis elements,

$$\lambda_V(1 \otimes v) = cv, \quad \rho_V(v \otimes 1) = cv. \quad (v \in V, c \in k)$$

and extend linearly to arbitrary elements. Verifying naturality and the triangle and pentagon identities are tedious diagram chases. Roughly speaking, naturality holds because the isomorphisms are entirely independent of basis or ground field choice. The triangle identity holds because the tensor product is $k-$bilinear, so $cx \otimes y = x \otimes cy$.

We are left to check rigidity. It is no surprise that the dual of a vector space is also its (left and right) dual in $\mathbf{Vect_k}$ as well. The evaluation map is equally clear, since it is where the name comes from:

$$\epsilon_X(f \otimes x) = f(x), \quad (x \in X, f \in X^*).$$

It extends to arbitrary tensors linearly. Coevaluation is defined by

$$\eta_X(1) = \sum_i x_i \otimes x_i^*,$$

where the x_i form a basis of X and the x_i^* form the corresponding dual basis. The best way see why this is the correct map is to use the fact that $X \otimes X^* \cong Hom(X, X)$ by the isomorphism $x \otimes f \mapsto f(-) \cdot x$. Passing $\sum_i x_i \otimes x_i^*$ through this isomorphism yields the identity transformation, so η_X is defined by sending 1 to 1_X. For this reason, it is also sometimes called the unit, with η_X then being the counit. It is not obvious from the definition that η_X should be independent of choice of basis, but under this perspective it is.

Example 2.1.2 (G graded vector spaces)**.** For a fixed group G and field k, consider the category whose objects are G-graded finite dimensional vector spaces $V = \bigoplus_{g\in G} V_g$. We call $g = |V_g|$ the degree of V. The morphisms are given by linear maps which respect the grading. That is, the morphisms are all linear transformations $f : V \to W$ such that $f(V_g) \subset W_g$. We denote it by $\mathbf{Vect^G}$. Define a tensor product on $\mathbf{Vect^G}$ by

$$(V \otimes W)_g = \bigoplus_{h\in G} V_h \otimes V_{h^{-1}g}.$$

The tensor identity here is just k itself, viewed as a vector space. With these choices, we can define λ_V and ρ_V by

$$\lambda_V(a \otimes v) = av = \rho_V(v \otimes a)$$

and the associativity isomorphism is just shifting brackets. Then this defines a monoidal category. The category is clearly $k-$linear. Additionally, we have simple objects given by k_g, $g \in G$. We now have to find duals for every object. We have

$$\begin{aligned}(V^*)_g &= Hom_g(V, k)\\ &\cong Hom(V_{g^{-1}}, k)\\ &= (V_{g^{-1}})^*.\end{aligned}$$

In the above, $Hom_g(V, k)$ denotes the degree g linear transformations, which are linear transformations $f : V \to W$ such that $f(V_h) \subseteq V_g h$. The isomorphism follows because the only degree g maps $V \to k$ are those whose domain is V_{-g}. Since G is finite, there are finitely many simple objects, and clearly all objects are finite direct sums of these. Since the identity object is $k_e = k$, we have $End(1) = End(k) = k$.

Example 2.1.3 ($\mathbf{Rep(G)}$)**.** Let G be a finite group and k an algebraically closed field of characteristic 0. Then the following data defines a fusion category, called $\mathbf{Rep(G)}$. The objects are pairs (V, ρ) of vector spaces V and representations $\rho : G \to GL(V)$. To see what the morphisms should be, consider G as a category with one object, $*$, and morphisms $g : * \to *$ which correspond to group elements and obey the structure of the group, in the sense that composition of morphisms is the group multiplication. Then a representation ρ is just a functor $G \to \mathbf{Vect_k}$, sending $*$ to V and group elements to linear maps in $GL(V)$. In this context, we think of $\mathbf{Rep(G)}$ as a category whose objects are functors, making the morphisms natural transformations. So, to find all the morphisms, we have to describe the natural transformations between two representations. Since G only has one object, any natural transformation η will be completely described by the corresponding morphism η_*, so we are free to identify them. All that is left is to apply the naturality condition. For any morphism g and representations ρ, ρ', the following diagram must commute:

$$\begin{tikzcd}
\rho(*) \arrow[r, "\eta"] \arrow[d, "\rho(g)"] & \rho'(*) \arrow[d, "\rho'(g)"] \\
\rho(*) \arrow[r, "\eta"] & \rho'(*)
\end{tikzcd}$$

In other words, we are looking for all linear transformations η such that $\eta \circ \rho(g) = \rho'(g) \circ \eta$, for all $g \in G$. Such maps are called intertwiners.

Now it remains to see why $\mathbf{Rep(G)}$ is a fusion category. It is k-linear because the set of morphisms have a natural vector space structure. Simple objects correspond to irreducible representations (irreps). By Maschke's theorem, every representation is a direct sum of irreps, so $\mathbf{Rep(G)}$ is semisimple.

The tensor product is the tensor product of representations, and the tensor unit $\mathbf{1} = (k, 1)$ is the base field together with the trivial representation.

Other properties all work out in essentially the same way as in $\mathbf{Vect_k}$.

The last definition of this section plays an important role in simplifying our final results.

Definition 2.1.12 (Equivalence of (fusion) categories)**.** *Let $\mathcal{C}, \mathcal{D}$ be (fusion) categories. Then $\mathcal{C}$ and $\mathcal{D}$ are equivalent as categories if there exist functors $\mathcal{F} : \mathcal{C} \to \mathcal{D}$,*

$\mathcal{G} : \mathcal{D} \to \mathcal{C}$ such that $\mathcal{F} \circ \mathcal{G} \cong 1_{\mathcal{D}}$ and $\mathcal{G} \circ \mathcal{F} \cong 1_{\mathcal{C}}$, where the isomorphisms in both cases are natural.

If in addition $\mathcal{F}$ and $\mathcal{G}$ both preserve duals, direct sums, and tensor products, then $\mathcal{C}$ and $\mathcal{D}$ are equivalent as fusion categories.

2.2 Modular Tensor Categories

This subsection lays the groundwork necessary to define Modular Tensor Categories (MTCs). As motivation, turn back to vector spaces: Two finite dimensional vector spaces V and W always satisfy $V \otimes W \cong W \otimes V$, however as objects in $\mathbf{Vect_k}$, they are distinct. This suggests that we should define a commutativity isomorphism, similar to the associativity isomorphism defined on monoidal categories. It takes some work to ensure that everything still works coherently, so we have to wade through that. One result of that work will be the twist, which is the key feature of a ribbon category. It also allows us to define the categorical notion of trace, which generalizes the trace of a linear map on a vector space. This in turn is key to the long sought definition of an MTC.

Definition 2.2.1 (Braided Monoidal Category)**.** *A braided monoidal category is a monoidal category $\mathcal{C}$ with natural isomorphisms*

$$\sigma_{X,Y} : X \otimes Y \to Y \otimes X, \quad (X, Y \in Ob(\mathcal{C}))$$

called the braiding for which the following diagrams commute:

$$\begin{tikzcd}
(X \otimes Y) \otimes Z \arrow[r, "\alpha_{X,Y,Z}"] \arrow[d, "\sigma_{X,Y} \otimes Id"] & X \otimes (Y \otimes Z) \arrow[r, "\sigma_{X,Y\otimes Z}"] & (Y \otimes Z) \otimes X \arrow[d, "\alpha_{Y,Z,X}"] \\
(Y \otimes X) \otimes Z \arrow[r, "\alpha_{Y,X,Z}"] & Y \otimes (X \otimes Z) \arrow[r, "Id \otimes \sigma_{X,Z}"] & Y \otimes (Z \otimes X)
\end{tikzcd}$$

$$\begin{tikzcd}
X \otimes (Y \otimes Z) \arrow[r, "\alpha^{-1}_{X,Y,Z}"] \arrow[d, "Id \otimes \sigma_{Y,Z}"] & (X \otimes Y) \otimes Z \arrow[r, "\sigma_{X\otimes Y,Z}"] & Z \otimes (X \otimes Y) \arrow[d, "\alpha^{-1}_{Z,X,Y}"] \\
X \otimes (Z \otimes Y) \arrow[r, "\alpha^{-1}_{X,Z,Y}"] & (X \otimes Z) \otimes Y \arrow[r, "\sigma_{X,Z} \otimes Id"] & (Z \otimes X) \otimes Y
\end{tikzcd}$$

These diagrams say that the isomorphisms built up by applying associativity and commutativity in different orders will be equal.

At this point, if we were to continue relying on the example of finite dimensional vector spaces, we would be led astray. In $\mathbf{Vect_k}$, the braiding satisfies $\sigma_{Y,X} \circ \sigma_{X,Y} = Id_{X \otimes Y}$, but this is not true for a general braided monoidal category. The ones for which that equality holds are called symmetric monodial categories.

Example 2.2.1. Take the category of $\mathbb{Z}$-graded vector spaces. We have the usual tensor product of graded vector spaces, but we can define a grading which is not the obvious one. Define

$$\sigma_{X,Y}(X \otimes Y) = \lambda^{|X||Y|}(Y \otimes X),$$

where λ is a non-zero element of the underlying field, and $|X|$, $|Y|$ denote the degrees of X and Y respectively. It is easy to chase this σ through the diagrams in Definition 2.2.1 and see that it satisfies the coherence conditions. Evidently, $\sigma^2 = Id$ if and only if $\lambda^2 = 1$, so this braiding is generally not symmetric.

We move forward by again considering vector spaces. In $\mathbf{Vect_k}$ there is a canonical map $\delta : X \to X^{**}$ given by $x \mapsto ev_x$. This map satisfies the following properties:

$$\delta_{X \otimes Y} = \delta_X \otimes \delta_Y, \quad \delta_1 = 1, \quad \delta_{X^*} = (\delta_X^*)^{-1}. \tag{2.2.1}$$

Continuing with the theme of generalizing key properties on maps of vector spaces, it is natural to ask when braided categories have maps satisfying these properties, as they are not in any way guaranteed. Before answering that question, there is another similar map which is guaranteed, given by

$$\begin{tikzcd} X^{**} \arrow[r, "id \otimes \eta_X"] \arrow[rrr, bend right, "\psi"'] & X^{**} \otimes X \otimes X^* \arrow[r, "id \otimes \sigma_{X,X^*}"] & X^{**} \otimes X^* \otimes X \arrow[r, "\epsilon_{X^{**}} \otimes id"] & X. \end{tikzcd}$$

Since it goes in the reverse direction and is also an isomorphism it makes sense to compose the two and then see how close the resulting map is to the identity. It is not hard to see that this composition is the identity if and only if the category is symmetric, as in the case of vector spaces. This leads to the following important definition.

Definition 2.2.2 (Twist)**.** *Let $\mathcal{C}$ be a braided monoidal category with natural isomorphisms $\{\delta\}$ whose components satisfy (2.2.1) and $\{\phi\}$ whose components are defined as above. A twist is a natural isomorphism $\{\theta_V : V \to V\}$ whose components are $\theta_V = \phi_V \delta_V$*

Lemma 2.2.3. *The twist satisfies the following properties:*

$$\begin{aligned} \theta_{X \otimes Y} &= \sigma_{YX}\sigma_{XY}(\theta_X \otimes \theta_Y), \\ \theta_1 &= id \end{aligned}$$

To prove this now we would require a tedious diagram chase, so instead we refer to Chapter 2 of [1] where somewhat more theory is developed, including a graphical depiction of morphisms in a braided category. In that framework it becomes easy to prove many properties of braided categories, as it translates commutative diagrams into so-called braid diagrams, which are extremely intuitive.

Definition 2.2.4 (Ribbon Category)**.** *A ribbon category is a rigid braided monoidal category with a twist that additionally satisfies*

$$(\theta_V)^* = \theta_{V^*}$$

for all objects V.

The twist in a ribbon category allows us to define the notion of trace, which as we will see is useful in multiple contexts.

Definition 2.2.5 (Trace)**.** *Let $f \in Hom(V, V)$. Then define*

$$tr(f) = \eta_V \sigma_{VV^*} \circ (\theta_V f \otimes 1) \circ \epsilon_V.$$

To understand the categorical trace, it is best to consider its relation to the trace of a matrix.

Example 2.2.2. Let V be a vector space with $dim(V) = n$ and A an $n \times n$ matrix. Then, $tr(A) = \sum_{i=1}^{n} A_{ii}$.

Let $T : V \to V$ be a linear transformation with A as its matrix, after choosing a basis $\mathcal{B} = \{e_i\}_{i=1}^{n}$. We define $tr(T) = tr(A)$ and notice that this definition is independent of basis. But, we can define $tr(T)$ in a more complicated way as well. Let $\{e_i^*\}_{i=1}^{n}$ be the dual basis for $\mathcal{B}$. Then,

$$T(e_i) = \sum_{j=1}^{n} A_{ij} e_j$$

so that $A_{ij} = e_i^*(T(e_j))$. Substituting in to the original formula, we find

$$tr(T) = \sum_{i=1}^{n} e_i^*(T(e_i)).$$

Now recalling how evaluation and coevaluation worked in **Vect**, it is easy to see why the categorical trace has the definition it does.

The presence of the twist in the categorical definition ensures compatibility and the braid ensures that the objects appear in the correct order, but details such as those are less important for us as we will not be dealing explicitly with them.

One important feature of the usual trace of matrices is that $tr(1) = dim(V)$ where 1 is the identity map on V. This motivates the following definition:

Definition 2.2.6 (Quantum/Global Dimension)**.** *Let $\mathcal{C}$ be a ribbon category and $X \in \mathcal{C}$. Let $Irr(\mathcal{C})$ be the set of isomorphism classes of simple objects and suppose that $|Irr(\mathcal{C})| < \infty$. Then the quantum dimension of X is given by $dim(X) = tr(1_X)$. The global dimension of the ribbon category $\mathcal{C}$ is $dim(\mathcal{C}) = \sum_{X \in Irr(\mathcal{C})} dim(X)^2$.*

This is a good notion of dimension, not only because it is the "correct" dimension in $\mathbf{Vect_k}$, but also because it obeys some of the expected properties of a dimension. That is, $dim(X \otimes Y) = dim(Y \otimes X)$ and $dim(X \oplus Y) = dim(X) + dim(Y)$.

Definition 2.2.7 (Modular Tensor Category (MTC))**.** *A category $\mathcal{C}$ is called a modular tensor category if it is a semisimple ribbon category with $Irr(\mathcal{C})$ finite, and if* *the matrix S defined by*

$$S_{ij} = tr(\sigma_{V_iV_j} \circ \sigma_{V_jV_i}), \quad V_i, V_j \in Irr(\mathcal{C}),$$

is invertible.

The motivation behind this definition (in particular requiring non-degeneracy of the S matrix) comes from the physical interpretation of MTCs as representation categories of VOAs. A second combinatorial invariant works with the S matrix to form the so called modular data of the MTC. It is defined by

$$T = (t_{ij}) = \delta_{ij}\theta_i.$$

Finding modular data is one of the primary goals of this thesis, but we will cover exactly what it is in section 2.3.

2.3 Modular Data

From this point on we can safely ignore MTCs and restrict ourselves to their modular data, which is their most important feature. As a simple combinatorial invariant, it captures some of the information about an MTC without digging too deeply into the blood and guts of the category itself. This section follows the exposition of [11].

Definition 2.3.1 (Modular Data)**.** *Modular data consists of a finite set Φ containing an identity element 1, together with complex matrices $(S_{ab})_{a,b\in\Phi}$ and $(T_{ab})_{a,b\in\Phi}$, satisfying the following properties:*

1. *S is unitary and symmetric, and T is diagonal of finite order.*

2. *$S_{1,b} > 0$ for all $b \in \Phi$.*

3. *$S^2 = (ST)^3$.*

4. *The numbers*

$$N_{a,b}^{c} = \sum_{d\in\Phi} \frac{S_{ad}S_{bd}\overline{S_{cd}}}{S_{1d}} \tag{2.3.1}$$

are non-negative integers for all $a, b, c \in \Phi$.

Proposition 2.3.2. *For* S *and* T *defined as in Definition 2.3.1, the maps*

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mapsto S, \quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \mapsto T$$

define a representation of $SL(2,\mathbb{Z})$.

Proposition 2.3.3. *The* S *and* T *matrices of a unitary MTC define modular data.*

We define unitary in section 2.4, and it can be dropped if one is willing to loosen the definition of modular data slightly by removing condition 2. For a proof of this proposition, see chapter 3 of [1]. Further exposition on the relationship between modular data and MTCs can be found both there and in [20]. It is important to summarize the key properties of modular data that we will be making use of, but in order to do that, we need the following standard terminology.

- The numbers $N_{a,b}^{c}$ are called *fusion coefficients*, and (2.3.1) is called *Verlinde's formula*.
- An element of Φ is called a *primary*.

The best way to think about the fusion coefficients is as matrices. That is, for a fixed primary a we can form the matrix N_a defined by $(N_a)_{b,c} = N_{a,b}^{c}$. This interpretation, together with (2.3.1), tells us that the N_a are simultaneously diagonalizable by S and that the columns of S are eigenvectors of the N_a, each with eigenvalue $S_{a,b}/S_{1,b}$.

Definition 2.3.4 (Simple Currents)**.** *A simple current is a primary* $a \in \Phi$ *such that* $S_{a,1} = S_{1,1}$.

To each simple current j, we can associate a phase $\varphi_j : \Phi \to \mathbb{C}^*$ and a permutation J of Φ such that

$$\begin{aligned} J0 &= j \\ S_{Ja,b} &= \varphi_j(b) S_{ab} \\ T_{Ja,Ja}\overline{T}_{a,a} &= \overline{\varphi_j(a)} T_{j,j} \overline{T_{1,1}} \\ (T_{j,j}\overline{T_{1,1}})^2 &= \overline{\varphi_j(j)}. \end{aligned}$$

We also call the permutation J associated to j a simple current. The set of simple currents J forms an abelian group under composition of permutations, called the center of the modular data.

In some sense, simple currents are the most interesting primaries. For a general primary a, $S_{a,1} \geq S_{1,1}$. Since $S_{a,1}/S_{1,1}$ is always an eigenvalue of N_a (the maximal real eigenvalue, called the Perron-Frobenius eigenvalue), we see that if a is a simple current, then the maximal eigenvalue of N_a is just 1.

2.4 Center

Although exotic fusion categories are interesting in their own right, our main use for them is to construct exotic MTCs using the Drinfel'd double construction. Some of the details are technical, but the idea itself is quite simple. We can think of a fusion category as the categorization of a ring, and a braided fusion category (e.g., an MTC) as the categorification of a commutative ring. Every ring has a center which is commutative, so at first glance it seems reasonable to expect an analogous result from fusion categories, and indeed there is. It is also called the center, or the Drinfel'd double.

It is easy to define the center of any ring R as

$$Z(R) = \{r \in R \mid rs = sr \ \forall s \in R\}.$$

The most obvious categorization of this definition is

$$\mathcal{Z}(\mathcal{C}) = \{r \in \mathcal{C} \mid r \otimes s \cong s \otimes r \ \forall s \in \mathcal{C}\},$$

for $\mathcal{C}$ a category, is essentially correct up to some technical details. The biggest issue is the isomorphism $r \otimes s \cong s \otimes r$. In principal there are many isomorphisms which may work, but which would define entirely different categories, so it is important to resolve that ambiguity.

Definition 2.4.1 (Half Braidings)**.** *Let $\mathcal{C}$ be a fusion category and $X \in \mathcal{C}$ an object. A half braiding for X is a natural isomorphism $\{\mathcal{E}_X\}$ with $\mathcal{E}_X(-) : X \otimes (-) \to (-) \otimes X$. The naturality assumption is the statement that this diagram commutes, for all $f \in Hom(X, Z)$:*

$$\begin{tikzcd} X \otimes Y \arrow[r, "\mathcal{E}_X(Y)"] \arrow[d, "id_X \otimes f"] & Y \otimes X \arrow[d, "f \otimes id_X"] \\ X \otimes Z \arrow[r, "\mathcal{E}_X(Z)"] & Z \otimes X. \end{tikzcd}$$

The maps must also satisfy the braid condition

$$\begin{tikzcd} X \otimes Y \otimes Z \arrow[r, "\mathcal{E}_X(Y \otimes Z)"] \arrow[d, "\mathcal{E}_X(Y)"'] & Y \otimes Z \otimes X \\ Y \otimes X \otimes Z. \arrow[ur, "\mathcal{E}_X(Z)"'] & \end{tikzcd}$$

Definition 2.4.2 (Center)**.** *Let $\mathcal{C}$ be a fusion category. The center of $\mathcal{C}$, denoted $\mathcal{Z}(\mathcal{C})$ is a category whose objects are pairs $(X, \mathcal{E}_X)$, for $X \in \mathcal{C}$ and $\mathcal{E}_X$ a half braiding. The sets of morphisms are*

$$Hom((X, \mathcal{E}_X), (Y, \mathcal{E}_Y)) = \{f \in Hom(X, Y) | \mathcal{E}_Y(Z) \circ (f \otimes id_X) = id_X \otimes f \circ \mathcal{E}_X(Z))\}.$$

There is a tensor unit given by $(1, \mathcal{E}_1)$ where 1 is the tensor unit in $\mathcal{C}$ and $\mathcal{E}_1(X) = id_X$ for all X. The tensor product is given by

$$(X, \mathcal{E}_X) \otimes (Y, \mathcal{E}_Y) = (X \otimes Y, \mathcal{E}_{X \otimes Y}),$$
$$\mathcal{E}_{X \otimes Y} = (\mathcal{E}_X \otimes Id_Y) \circ (id_X \otimes \mathcal{E}_Y)$$

Finally there is a braiding defined by $\sigma_{(X,\mathcal{E}_X),(Y,\mathcal{E}_Y)} = \mathcal{E}_X(Y)$

With the composition and tensor product of morphisms taken to be the same as in $\mathcal{C}$, $\mathcal{Z}(\mathcal{C})$ inherits the structure of a braided tensor category. With significantly more work, $\mathcal{Z}(\mathcal{C})$ can be shown to be modular as well. For a detailed exposition, see [24].

The half braidings determine a braiding on $\mathcal{Z}(\mathcal{C})$ and as a result we can compute the S matrix entries explicitly in terms of half braidings. The T matrix is found similarly. A convenient way to do that is to use the tube algebra, described in [24]. This realization allows us to find the S and T matrices by solving a set of polynomial equations of degree 1 and 2. The tube algebras for the classes of fusion categories we find in this thesis are worked out in detail in sections 6 and 8 of [16]. In chapters 3 and 4 we give the relevant polynomial equations.

2.5 Cuntz Algebra Construction

As we saw in the previous section, given a fusion category, we can associate an MTC. While this is much simpler than constructing MTCs directly, it comes with the problem that the most well understood examples of fusion categories are the simplest, most classical ones. As a result, the MTCs we can construct using straightforward methods are also only the most obvious examples. One would hope that there are plenty of interesting, truly strange examples of both fusion categories and MTCs.

Perhaps the best approach to find them (that we know) is with the so-called Cuntz algebra construction. It allows us to take advantage of the fact that any fusion category can be interpreted as the category of endomorphisms of some algebra $\mathcal{A}$. (Strictly speaking this is slightly inaccurate, but we will provide a more accurate picture later in this section). The properties of the Cuntz algebra (defined below) allow us to completely determine certain systems of endomorphisms in terms of a relatively small number of complex parameters, and hence completely understand them as a fusion category.

Definition 2.5.1 (Cuntz Algebra)**.** *Let $\mathcal{H}$ be a separable Hilbert space. Let $S_1, S_2, ..., S_n$ be isometries on $\mathcal{H}$; ie,*

$$S_i^\dagger S_j = \delta_{ij}, \qquad \sum_{i=1}^{n} S_i S_i^\dagger = 1. \tag{2.5.1}$$

Then the algebra generated by the S_i is called a Cuntz algebra and is denoted $\mathcal{O}_n$.

Equations (2.5.1) are collectively called the Cuntz relations.

The above definition may at first seem strange, as intuition from finite dimensions implies that, if $S_i^\dagger S_i = 1$, then also $S_i S_i^\dagger = 1$. This fails for infinite dimensions though, as the following example demonstrates.

Example 2.5.1. Let $\mathcal{H} = \ell^2$. Define S_1 and S_2 by

$$S_1((a_1, a_2, ...)) = (a_1, 0, a_3, 0, ...),$$
$$S_2((a_1, a_2, ...)) = (0, a_2, 0, a_4, ...).$$

These obviously satisfy the conditions of the Cuntz algebra.

As the example makes clear, the key difference when moving from a finite dimensional setting to an infinite dimensional one is that we can project onto a proper subspace which is isomorphic to the original space. In the example, S_1 and S_2 do this in two different ways, such that they "add past" each other when being recombined. Intuitively the same thing happens for any isometries satisfying the Cuntz relations. Recalling the definition of direct sum in a $k-$linear category, there is a clear analogy present. This is key in understanding the relevance of the Cuntz algebra $\mathcal{O}_n$ to the construction of fusion categories, as we will see.

The notation $\mathcal{O}_n$ makes no reference to the many possibly different isometries which could generate the Cuntz algebra. Indeed, our use of *the* as opposed to *a* is similarly unjustified. The following theorem ensures this terminology is appropriate.

Theorem 2.5.2. *Let $\mathcal{H}$ be a separable Hilbert space, and let $\{S_i\}_{i=1}^n$ and $\{\hat{S}_i\}_{i=1}^n$ be two sets of isometries, both satisfying (2.5.1). Then the C^*-algebras generated by $\{S_i\}_{i=1}^n$ and $\{\hat{S}_i\}_{i=1}^n$ are naturally isomorphic.*

The proof of this theorem can be found in [3] (Statement 1.9), where further details on Cuntz algebras in general are also given.

The best way to understand the construction is by example. Let G be a finite abelian group of order n. Denote by $[x]$ the isomorphism class of any simple object $x \in \mathcal{C}$, for $\mathcal{C}$ a fusion category. Chapter 3 shows that any (hypothetical) near group category of type $G + n'$ would have to satisfy the fusion rules

$$[g][h] = [gh], \quad [g][\rho] = [\rho][g] = [\rho], \quad [\rho]^2 = n'[\rho] + \sum_{g\in G}[g],$$

for all $g, h \in G$. We can ask which (if any) endomorphisms on some Cuntz algebra satisfy analogous rules. Additionally, we want the endomorphisms to respect the $*$ structure on the Cuntz algebra, and so they should be $*$-endomorphisms. That is, we are looking for algebra endomorphisms f which additionally satisfy $f(x^\dagger) = f(x)^\dagger$ for all $x \in \mathcal{A}$ and a Hermitian conjugate $\dagger$. Although this assumption isn't strictly necessary, it does simplify the working out considerably. Any resultant fusion category will have additional structure as well.

Definition 2.5.3 (Unitary category)**.** *A $\mathbb{C}$- linear category $\mathcal{C}$ is called unitary (or C^*) if there is a conjugate-linear operation $\dagger$: $Hom(X,Y) \to Hom(Y,X)$ such that*

$$\begin{aligned}
(f^\dagger)^\dagger &= f \\
(fg)^\dagger &= g^\dagger f^\dagger \\
(f \otimes g)^\dagger &= f^\dagger \otimes g^\dagger \\
k^\dagger &= \overline{k}, \quad (k \in \mathbb{C}),
\end{aligned}$$

whenever the above expressions make sense. Additionally, we must have $f^\dagger f = 0$ if and only if $f = 0$.

The $\dagger$ operation is often denoted by $*$, although we have reserved that notation for dual objects/morphisms. In our context, every fusion category and indeed MTC will be unitary as well, as a result of the assumption on the endomorphisms which we use to construct them. The benefit of this assumption is ease of calculation and slightly simpler results – for example, dimension is especially well behaved in unitary categories as it is always positive, which may not be the case in nonunitary ones. The drawback is that not every fusion category satisfying a given set of fusion rules will be unitary, so we are necessarily missing some. The generalization to nonunitary categories is a good deal of work (see [7]), but as we will see later, it seems to already be possible to recover the nonunitary numerical data from the unitary data, albeit not in the most useful form.

The most important distinction for the time being between unitary and nonunitary categories is the ease with which the Cuntz algebra construction can be accomplished. It is a theorem of [14] that any unitary fusion category can be realized as the category of endomorphisms on a hyperfinite von Neumann algebra (which the Cuntz algebras are). With that theorem, most of the legwork is done and we can freely ignore any theoretical concerns. Additionally, the assumption that the endomorphisms are $*$-endomorphisms allow for a greater degree of computational simplicity, as we will see in the following example: Zoom in to the simplest possible case, where G is the trivial group:

Example 2.5.2. Suppose we have a unitary fusion category whose simple objects are 1 and ρ, with fusion rules

$$[\rho][\rho] = [1] + [\rho].$$

The goal of the Cuntz algebra construction is to reinterpret 1 and ρ as endomorphisms on an algebra $\mathcal{A}$. The hom-spaces are spaces of intertwiners. That is, for $a \in \mathcal{A}$, $a \in Hom(X, Y)$ if $X(b)a = aY(b)$ for all $b \in \mathcal{A}$. The advantage of this interpretation is that we handle the tensor product easily – it becomes composition of morphisms. However this interpretation makes sums more difficult. The most obvious translation would be to require

$$\rho(\rho(x)) = x + \rho(x), \quad \forall x \in A,$$

but this falls apart. If this were the equation, then applying it to 1, we get

$$\rho(\rho(1)) = 1 = 1 + \rho(1) = 2,$$

which is obviously nonsense. To fix this, we have to consider a slightly more complicated expression which is exactly what we need to define addition and be compatible on the level of the category. Take $s, s^\dagger, t, t^\dagger \in \mathcal{A}$ satisfying

$$ss^\dagger + tt^\dagger = 1, \quad s^\dagger s = t^\dagger t = 1 \quad s^\dagger t = t^\dagger s = 0. \tag{2.5.2}$$

Then suppose ρ and 1 satisfy

$$\rho(\rho(x)) = sxs^\dagger + t\rho(x)t^\dagger. \tag{2.5.3}$$

This solution evidently solves the unit problem caused by the obvious equation $\rho^2(x) = x + \rho(x)$.

The similarity here to the direct sum of objects in a category (Def 2.1.5) is not a coincidence. Equation (2.5.3) is the culmination of the previous discussion about projections in infinite dimensional spaces. Essentially, $\mathcal{A}$ is replaced by two orthogonal copies of itself; one is the image of s and the other is the image of t. Then 1 acts on the first copy and ρ on the second. In this way, on the level of the category, we really are taking direct sums of objects.

Now, it's obvious that $s, s^\dagger, t, t^\dagger$ generate some subalgebra of $\mathcal{A}$, call it $\mathcal{A}_{s,t}$, but it isn't obvious that ρ restricts to an endomorphism of that subalgebra. If it did, then ρ would be determined by how it acts on $s, s^\dagger, t, t^\dagger$ and we could in principle solve for it exactly. The first step is therefore to check that ρ restricts to an endomorphism on $\mathcal{A}_{s,t}$.

It suffices to show that ρ sends $s, s^\dagger, t, t^\dagger$ to elements of $\mathcal{A}_{s,t}$. As ρ is a $*-$endomorphism, it's enough to consider s and t. There are a few intermediate steps to achieve that. First, we'll show that $s, s^\dagger, t, t^\dagger$ are all intertwiners for appropriately chosen endomorphisms. Starting with (2.5.3) and multiplying on the right by s, we get

$$\rho(\rho(x))s = sxs^\dagger s + t\rho(x)t^\dagger s = sx.$$

Similarly, we can show that

$$\rho(\rho(x))t = t\rho(x), \tag{2.5.4}$$

$$s^\dagger\rho(\rho(x)) = xs^\dagger, \tag{2.5.5}$$

$$t^\dagger\rho(\rho(x)) = \rho(x)t^\dagger. \tag{2.5.6}$$

Therefore, on the level of the category, we must have

$$\begin{aligned} s &\in Hom(\rho^2, 1), \\ s^\dagger &\in Hom(1, \rho^2), \\ t &\in Hom(\rho^2, \rho), \\ t^\dagger &\in Hom(\rho, \rho^2). \end{aligned}$$

Next we must show that each of these spaces is in fact generated by $s, s^\dagger, t$ or $t^\dagger$, respectively. The computation is essentially the same in each case, so doing it explicitly for s will suffice.

Suppose that $r \in Hom(\rho^2, 1)$. Then

$$xs^\dagger r = s^\dagger\rho(\rho(x))r = s^\dagger rx,$$

so $s^\dagger r \in Hom(1, 1) = \mathbb{C}$, since 1 is a simple object. The same trick with $t^\dagger$ yields $t^\dagger r \in Hom(\rho, 1) = 0$. Then

$$r = (ss^\dagger + tt^\dagger)r = ss^\dagger r + tt^\dagger r = ss^\dagger r \in \mathbb{C}s,$$

so r is a multiple of $\mathbb{C}$. Since r was arbitrary, it follows that $Hom(\rho^2, 1) = \mathbb{C}s$. Analogous computation shows that

$$\begin{aligned} Hom(1, \rho^2) &= \mathbb{C}s^\dagger, \\ Hom(\rho^2, \rho) &= \mathbb{C}t, \\ Hom(\rho, \rho^2) &= \mathbb{C}t^\dagger. \end{aligned}$$

We now know what all of the intertwiners look like, and there is enough to show that ρ restricts to an endomorphism of $\mathcal{A}_{s,t}$. Consider the following two equations:

$$\begin{aligned} s^\dagger\rho(s)\rho(x) = s^\dagger\rho(sx) = s^\dagger\rho(\rho^2(x)s) &= s^\dagger\rho^2(\rho(x))\rho(s) \\ &= \rho(x)s^\dagger\rho(s), \\ t^\dagger\rho(s)\rho(x) = t^\dagger\rho(sx) &= t^\dagger\rho^2(\rho(x))\rho(s) \\ &= \rho^2(x)t^\dagger\rho(s). \end{aligned}$$

Thus $s^\dagger\rho(s) \in Hom(\rho,\rho) = \mathbb{C}$ and $t^\dagger\rho(s) \in Hom(\rho^2,\rho) = \mathbb{C}t$. So there exist $a, b \in \mathbb{C}$ such that $s^\dagger\rho(s) = a$ and $t^\dagger\rho(s) = bt$. Putting it all together,

$$\rho(s) = (ss^\dagger + tt^\dagger)\rho(s) = as + bt^2,$$

so $\rho(s) \in \mathcal{A}_{s,t}$. Due to unitarity, we know that $\rho(s^\dagger) = \rho(s)^\dagger$, so

$$\rho(s^\dagger) = \rho(s)^\dagger = (as + bt^2)^\dagger = \overline{a}s^\dagger + \overline{b}(t^\dagger)^2.$$

We can compute $\rho(t)$ similarly:

$$\rho(x)s^\dagger\rho(t) = s^\dagger\rho^2(\rho(x))\rho(t) = s^\dagger\rho(\rho^2(x)t) = s^\dagger\rho(t\rho(x)) = s^\dagger\rho(t)\rho^2(x).$$

So, $s^\dagger\rho(t) \in Hom(\rho,\rho^2) = \mathbb{C}t^\dagger$. Hence $s^\dagger\rho(t) = ct^\dagger$ for some $c \in \mathbb{C}$.

As before, we can show that $Hom(\rho^2,\rho^2) = \mathbb{C}ss^\dagger \oplus \mathbb{C}tt^\dagger$. Then,

$$t^\dagger\rho(t)\rho^2(x) = t^\dagger\rho(t\rho(x)) = t^\dagger\rho(\rho^2(x))\rho(t) = t^\dagger\rho^2(\rho(x))\rho(t) = \rho^2(x)t^\dagger\rho(t),$$

so $t^\dagger\rho(t) = dss^\dagger + ett^\dagger$. Putting it all together as before yields $\rho(t) = cst^\dagger + dtss^\dagger + ett^\dagger$, and then $\rho(t^\dagger) = \rho(t)^\dagger$ gives the last result.

The upshot of this working out is that $\rho(\mathcal{A}_{s,t}) \subseteq \mathcal{A}_{s,t}$, and as a result restricts to an endomorphism of $\mathcal{A}_{s,t}$. Moreover, we learn ρ exactly up to 5 complex parameters which are relatively straightforward to solve for. To do so, we use the fact that ρ is a unital algebra endomorphism and so must respect addition and multiplication.

The obvious place to start is by applying ρ to (2.5.2):

$$\rho(s)\rho(s)^\dagger + \rho(t)\rho(t)^\dagger = 1, \quad \rho(s)^\dagger\rho(s) = \rho(t)^\dagger\rho(t) = 1, \quad \rho(s)^\dagger\rho(t) = \rho(t)^\dagger\rho(s) = 0.$$

After substituting in the expressions we found previously, we get numerous identities further restricting the 5 parameters. Additionally using (2.5.3) applied to the generators forces $a = (-1 \pm \sqrt{5})/2$. From there one can show that $b = c = \sqrt{a}$, $e = -a$, and $d = 1$ are the only possible solutions.

The above example generalizes very smoothly to general near group categories, but the detailed proof is more technical and no more enlightening than the example was. It, and other Cuntz algebra constructions for hypothetical fusion categories, can be found in [16], [6], and [7].

2.6 Symmetric Forms, Quadratic forms, and Gauss Sums

Quadratic forms play an important role in Chapter 3, so a slight detour now will be useful. In this section, G is a finite abelian group of order n, $\xi_n = e^{2\pi i/n}$, and $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$.

Definition 2.6.1. *A pairing on G is a map $\langle \cdot, \cdot \rangle : G \times G \to \mathbb{T}$ such that $\langle g, \cdot \rangle$ and $\langle \cdot, g \rangle$ are both 1-dimensional representations of G, for all fixed $g \in G$. That pairing is called symmetric if $\langle g, h \rangle = \langle h, g \rangle$ for all $g, h \in G$. It is called nondegenerate if $\langle g, \cdot \rangle \neq \langle h, \cdot \rangle$, whenever $g \neq h$.*

Pairings are maps to roots of unity. That is, when $G = \mathbb{Z}_n$, any symmetric pairing is $\langle g, h \rangle = \xi_n^{mgh}$ for some $m \in \mathbb{Z}$. Nondegeneracy is the condition that $gcd(m, n) = 1$. When G is noncyclic but still finite and abelian, we can split it into a direct product of cyclic groups, and the pairing will likewise split.

Related to any symmetric form is one or more quadratic forms.

Definition 2.6.2. *A quadratic form on G is a map $q : G \to \mathbb{T}$ such that $q(g) = q(-g)$ and*

$$q(g)q(h)\overline{q(g+h)} = \langle g, h\rangle, \quad (g, h \in G),$$

for some nondegenerate symmetric pairing $\langle , \rangle$.

The question of whether or not quadratic forms and nondegenerate symmetric pairings are in one to one correspondence is natural to ask. Evidently there are at least as many quadratic forms as there are symmetric pairings, so it suffices to see if more than one quadratic form can yield the same pairing. The answer to that is yes, in general there are more quadratic forms than symmetric pairings, but the number of them is under control and there is a complete classification of quadratic forms. We provide only the classification; see [26] for more details.

Fact 2.6.3. *Any quadratic form can be factored into a product of the following four types of indecomposable quadratic form.*

Type p^k: $q(g) = \xi_{p^k}^{mg^2}$*, where $g \in \mathbb{Z}_{p^k}$, p is an odd prime, and $k \in \mathbb{N}$.*

Type 2^k: $q(g) = \xi_{2^{k+1}}^{mg^2}$*, for $g \in \mathbb{Z}_{2^k}$, $k \in \mathbb{N}$, and $m = \pm 1, \pm 3 \mod 8$.*

Type $2^k 2^k_1$: $q(g, h) = \xi_{2^k}^{gh}$*, for $(g, h) \in \mathbb{Z}_{2^k} \times \mathbb{Z}_{2^k}$, $k \in \mathbb{N}$.*

Type $2^k 2^k_2$: $q(g, h) = \xi_{2^k}^{g^2+2gh+h^2}$*, for $(g, h) \in \mathbb{Z}_{2^k} \times \mathbb{Z}_{2^k}$, $k \in \mathbb{N}$.*

The way to build arbitrary quadratic forms on an abelian group G is to first factor G into a direct product of cyclic groups of prime power order. Then any quadratic form on G will factor as a product of indecomposable quadratic forms defined on the factors of G.

Related to quadratic forms is the notion of a Gauss sum. They will play an important role in the modular data of the near group categories.

Definition 2.6.4. *Given a group G and a quadratic form q on G, define the Gauss sum of q on G as*

$$a_q = \frac{1}{\sqrt{|G|}} \sum_{g \in G} q(g).$$

If a quadratic form q splits as the product of indecomposables, its Gauss sum does as well. That is, $a_q = a_{q_1} a_{q_2} \cdots a_{q_n}$, where $q = q_1 q_2 \cdots q_n$. As a result, we only need the Gauss sums for each type of indecomposible quadratic form to get the full picture.

Fact 2.6.5. *The Gauss sums for each of the indecomposable quadratic forms are*

Type p^k*:*

$$a_q = \begin{cases} \left(\frac{2m}{p}\right) & \text{if } p^k = 1 \mod 4 \\ \left(\frac{2m}{p}\right) i & \text{if } p^k = 3 \mod 4 \end{cases},$$

where $\left(\frac{2m}{p}\right)$ *is the Legendre symbol of* $2m$ *and* p*.*

Type 2^k*:*

$$a_q = \begin{cases} -\xi_8^{-m} & \text{if } m = \pm 3 \text{ and } k \text{ odd} \\ \xi_8^{-m} & \text{else} \end{cases}.$$

Type $2^k 2^k_1$*:* $a_q = 1$.

Type $2^k 2^k_2$*:* $a_q = (-1)^k$.

The proof of this fact is purely number theory, so we omit it here. It can be found in [26].

Specializing to Type p^k for $k = m = 1$ and rearranging yields the classic Gauss sum

$$\sqrt{p} a_q = g_1 = \begin{cases} \sqrt{p} & \text{if } p = 1 \mod 4 \\ i\sqrt{p} & \text{if } p = 3 \mod 4 \end{cases},$$

which we will make some use of as well.

Chapter 3

Near Group Categories

The quintessential example of fusion categories is the representation category of a finite group G, so it is natural to look for fusion categories which are perturbations of these basic ones. One way to achieve this is to add one simple object. So, where the group category has simple objects $[g]$ corresponding to elements $g \in G$, the new category would have those simple objects, together with an additional one called $[\rho]$. The first step to a fusion category is finding a sensible tensor product. Obviously, the objects corresponding to the group should have a tensor product such that $[g][h] = [gh]$. So it remains to find $[\rho][g], [g][\rho]$, and $[\rho]^2$. It turns out that given our choice that the only simple objects are ρ and the g's, there is very little freedom here. First, consider $[\rho][g]$. Since fusion categories are semisimple, $[\rho][g]$ has to be of the form

$$[\rho][g] = n_\rho[\rho] + \sum_{h\in G} n_g[h]$$

If $[\rho][g]$ were to contain any terms with any h, then $[\rho][g] = [h] + ...$, so $[\rho][g][g^{-1}] = [hg^{-1}] + ... = [\rho]$. Since $[\rho]$ is simple, we therefore must have $[\rho] = [hg^{-1}]$. But this breaks the group law, so it is impossible. Therefore, $[\rho][g] = n_\rho[\rho]$. Recall that n_ρ is a non-negative integer though, and by multiplying on the right by g^{-1}, we get $[\rho] = n_\rho[\rho][g^{-1}]$, so $[\rho][g^{-1}] = n_\rho^{-1}$. The only non-negative integer with an inverse is 1, so $[\rho][g] = [\rho]$ for all $g \in G$. Similarly, $[g][\rho] = [\rho]$.

We also see that $[\rho]^* = [\rho]$ since the dual of any simple object is simple. Obviously the only candidate is $[\rho]$ itself.

Finally, we compute $[\rho][\rho]$. By semi-simplicity,

$$[\rho][\rho]^* = [\rho]^2 = n_\rho[\rho] + \sum_{g\in G\setminus\{1\}} n_g[g] + [1].$$

Then we can multiply by any $h \in G$ on both sides, so that

$$[\rho]^2 = n_\rho[\rho] + [h]\sum_g n_g[g] + [h]$$

By the above, some term in the sum has to be exactly 1, and notice that there is only one copy of h. Since h was arbitrary, this has to hold for all $h \in G$, hence $[\rho]^2 = n_\rho[\rho] + \sum_{g\in G}[g]$. The convention is to call $n' := n_\rho$, and a near group category is labeled as being of type $G + n'$, where G is the group and n' is the coefficient of ρ in the expansion of $[\rho][\rho]$. Some effort has already been put into classifying such categories. In particular, Theorem 2 in [6] states that $n' = 0, n-1$, or $n' \in n\mathbb{Z}$ are the only possible values of n' which can yield valid fusion categories. When $n' > n$, there are only a few known examples of near group fusion categories, and the prevailing assumption is that there are only finitely many such cases. The fusion categories of type $G + 0$ are called Tambara Yamagami categories and section 2 of [29] contains a complete classification. Fusion categories of type $G + n - 1$ are classified in [6] (Proposition 5).

Thus the most plentiful remaining source of near group fusion categories are those of type $G + n$, which are the focus of this chapter. We continue the work done by [6], which classified all such categories for $|G| \leq 13$. We also construct the near group categories using the Cuntz algebra construction, however due to both better computing power and different computational approaches, we were able to construct examples for all cyclic G with $|G| < 31$, the data of which are in section 3.3.

3.1 Numerical Data

Throughout this subsection, G is a finite abelian group of order n, and $\widehat{G}$ are the irreps of G. We will also switch to additive notation. Since G is finite and abelian, these are exactly the one dimensional representations of G. In fact, $G \cong \widehat{G}$. The isomorphism $G \cong \widehat{G}$ is then given by $g \mapsto \langle g, \cdot \rangle$, for any nondegenerate pairing in the sense of Definition 2.6.1. When $G = \mathbb{Z}_n$, any $\psi \in \widehat{G}$ is uniquely defined by $\psi(1)$, which will always be equal to ξ_n^m for some $m \in \mathbb{Z}$. As a result, the nondegenerate symmetric pairings are given by $\langle g, h \rangle = \exp[2\pi i m g h / n]$ where $gcd(m, n) = 1$.

The first theorem of this section is that any near group fusion category will yield certain numerical constants, among them a nondegenerate symmetric pairing.

Theorem 3.1.1. *Let G be a finite abelian group with $|G| = n$. Let $\delta = \frac{n+\sqrt{n^2+4n}}{2}$. Let $c \in \mathbb{C}$ and $a : G \to \mathbb{C}, b : G \to \mathbb{C}$ be functions satisfying*

$$a(0) = 1, \tag{3.1.1}$$

$$a(x) = a(-x), \tag{3.1.2}$$

$$a(xy)\langle x, y \rangle = a(x)a(y), \tag{3.1.3}$$

$$\sum_{x \in G} a(x) = \sqrt{n} c^{-3}, \tag{3.1.4}$$

$$b(0) = \frac{-1}{\delta}, \tag{3.1.5}$$

$$\sum_{y \in G} \overline{\langle x, y \rangle} b(y) = \sqrt{n} c \overline{b(x)}, \tag{3.1.6}$$

$$a(x)b(-x) = \overline{b(x)}, \tag{3.1.7}$$

$$\sum_{x \in G} b(x+y)\overline{b(x)} = \delta_{y,0} - \frac{1}{\delta}, \tag{3.1.8}$$

$$\sum_{x \in G} b(x+y)b(x+z)\overline{b(x)} = \overline{\langle y, z \rangle} b(y)b(z) - \frac{c}{\delta\sqrt{n}}. \tag{3.1.9}$$

Then c, a, b determine a near group fusion category of type $G+n$. Two such categories $\mathcal{C}_1$ and $\mathcal{C}_2$ determined by c_1, a_1, b_1 and c_2, a_2, b_2 respectively are equivalent (as fusion categories) if there is $\phi \in Aut(G)$ such that $\langle x, y \rangle_1 = \langle \phi x, \phi y \rangle_2$, $a_1(x) = a_2(\phi x)$, $b_1(x) = b_2(\phi x)$ and $c_1 = c_2$.

This is Corollary 5 of [6], and a proof can be found there.

From this point on, we restrict to $G = \mathbb{Z}_n$. A close look at these equations yields a few simplifying ideas. Consider (3.1.3) and take $y = -x$. Then $\langle x, -x\rangle = a(x)^2$, so $|a(x)| = 1$ and

$$a(x)^2 = e^{-2\pi x^2 im/n}.$$

Solving for $a(x)$ depends on whether n is even or odd. If n is odd, then we must have

$$a(x) = e^{-\pi i x^2 m/n} = \left(e^{2\pi i x^2 m/n}\right)^{\frac{n-1}{2}}.$$

If instead n is even, then we introduce a possible sign and have

$$a(x) = s^x e^{-\pi i x^2 m/n}, \quad s = \pm 1.$$

However we can absorb any potential factor of -1 by taking $m \in \{1, 2, ..., 2n\}$ and coprime to n in this case. In all cases, $a(x)^{-1} = \overline{a(x)}$.

Shifting attention to (3.1.4), we see that the left side will always be a sum of some roots of unity so we can easily find a list of possibilities for c using Gauss sums. We split into cases again. Now it depends on the value of $n \mod 4$. If $n = 1 \mod 4$, then

$$c^{-3}\sqrt{n} = \sum_{x \in G} a(x) = \sum_{x \in G} \left(e^{2\pi i x^2 m/n}\right)^{\frac{n-1}{2}} = \left(\frac{m^{\frac{n-1}{2}}}{n}\right)\sqrt{n} = \pm\sqrt{n}.$$

Rearranging, we see that $c^3 = \pm 1$, so c will be a 6th root of unity. If $n = 3 \mod 4$, similar logic shows that $c^3 = \pm i$, so c is going to be a 12th root of unity. Finally in the even case, c will always be a 24th root of unity.

Now rearranging equation (3.1.7) lets us eliminate half the $b(x)$ values, since we get $b(-x) = \overline{a(x)b(x)}$. And we can do better by eliminating $b(0) = 1/\delta$ since it is always the same. As a consequence of equation (3.1.8), we can write all the data in exponential form so that $b(x) = \frac{1}{\sqrt{n}} \exp(ij(x))$, where $-\pi < j(x) \leq \pi$. This allows an additional simplification when n is even. By equation (3.1.7), when $x = n/2$, we have $x = -x$. Multiplying on both sides by $b(n/2)$ and rearranging yields $b(n/2) = \pm\sqrt{a(n/2)/n}$.

The final simplification comes from (3.1.6) and (3.1.7). Let $\mathbf{b} = (b(0), b(1), ..., b(n-1))^T$. Then, after substituting into (3.1.6) with (3.1.7), we see that $\mathbf{b}$ is an eigenvector for $M_{x,y} = \overline{a(y)}\langle x, y\rangle/\sqrt{n}$ with eigenvalue c. This result is most useful when checking completeness, as we will see.

The best way to solve for the numerical data from this point seems to be to first guess at a possible c, of which there are only ever a few choices. Then iteratively solve the other equations, working from the simplest linear ones to the more complicated nonlinear ones. If ever a choice of c fails, scrap that solution and work on the next one. Of course, there may be more than one c which works, and as mentioned each c will yield a different fusion category, so it is always important to check all possibilities.

Additionally, all values are algebraic, and hence they have an associated minimal polynomial. Because we are only solving the equations numerically, finding the associated minimal polynomial is not always immediate. However, given numerical solutions of a few digits of accuracy, it's easy to use numerical methods to increase the accuracy arbitrarily. Languages such as Mathematica and Maple have in-built functions (FindIntegerNullVector in Mathematica) which can find algebraic approximations of inexact numbers such as the ones appearing as solutions to the defining equations of the Near Group categories. From those we can get minimal polynomials. We will see this in more detail in the next section.

3.2 Near Group Categories and Galois

Fusion categories are algebraic in the sense that their constraints arise as polynomial matrix equations over $\mathcal{C}$. This opens the door for automorphisms σ of $\mathbb{C}$ to act on a fusion category in the following way. Fix $\sigma \in Aut(\mathbb{C})$, and a basis for all Hom spaces in the fusion category. With a choice of basis, the morphisms of the fusion category can be expressed as matrices over $\mathbb{C}$; letting σ act on these matrices entrywise will yield another (probably inequivalent) fusion category which obeys the same fusion rules as the initial category. As a caveat, unitarity is not an algebraic condition in this sense, and in general automorphisms will not preserve unitarity.

To be more explicit, we look at near group categories specifically. If $\sigma \in Aut(\mathbb{C})$ were to send a solution (c, a, b) of the equations in Theorem 3.1.1 to another solution, it must satisfy

$$\sigma(\delta) = \delta \tag{3.2.1}$$

$$\sigma(\overline{b(g)}) = \overline{\sigma(b(g))}. \tag{3.2.2}$$

In this case, we define $b^\sigma(g) = \sigma(b(g))$, and similarly for a^σ. c^σ depends on $\sigma(\sqrt{n})$, since $\sigma(\sqrt{n}) = \pm\sqrt{n}$. c^σ needs to absorb the negative sign if σ acts as multiplication by -1.These satisfy Theorem 3.1.1. It should also be noted that (3.2.2) has been redundant in cases we have seen for cyclic G – whenever $\sigma(\delta) = \delta$, (3.2.2) has always held.

Finding σ for which (3.2.1), (3.2.2) hold therefore allows us to classify fusion categories based on their Galois associates. To achieve such a classification, we first have to find the Galois group associated to a given near group category, and see how its elements σ permute the numerics which define the fusion category. There is a chain of field extensions

$$k \supseteq \mathbb{Q}[\xi_{lcm(24,2n,n+4)}] \supseteq \mathbb{Q}[\sqrt{n^2+4n}] \supseteq \mathbb{Q}$$

whose components work like this: $\mathbb{Q}[\sqrt{n^2+4n}]$ contains δ, and our eventual goal will be for this to be the base field in some Galois extension, guaranteeing (3.2.1) is satisfied by any Galois automorphism of the largeest field extension k. The next extension, $\mathbb{Q}[\xi_{lcm(24,n,n+4)}]$, contains all values $\langle g, h\rangle$, $a(g)$ and c, for all $g, h \in G$. We can see that it is an extension by considering quadratic Gauss sums. Note that this is exactly the reason we've picked this extension – it's in a sense the minimal (ie, roots of unity with the smallest denominators) extension that always contains all of these values. Lastly we define k to be the large extension containing all $b(g)$'s and their Galois associates, in addition to the other ones. In general we don't know what k will look like, only that it is an algebraic extension.

Now, let $\sigma \in Gal(\mathbb{Q}[\xi_{lcm(24,2n,n+4)}] : \mathbb{Q}[\sqrt{n^2+4n}])$. Then we can parameterize

σ by 3 values, $\ell_{24}, \ell_{2n}, \ell_{n+4}$, where $\ell_j \in \mathbb{Z}_j^\times$ for $j = 24, 2n, n+4$. We can see what these numbers mean by looking at how σ acts on certain data. First, consider $\sigma(\xi_{24})$. It has to take ξ_{24} to another 24th root of unity, say $\xi_{24}^{\ell_{24}}$. Similarly, we need $\sigma(\xi_{n+4}) = \xi_{n+4}^{\ell_{n+4}}$. Lastly $\sigma(\xi_{2n}) = \xi_{2n}^{\ell_{2n}}$. If n is odd, we can simplify slightly by considering ξ_n. The difference allows us to deal with a with no ambiguity, so is preferential for that reason. This parameterization tells us exactly what σ does to a general element of $\mathbb{Q}[\xi_{lcm(24,n,n+4)}]$, so importantly it tells us how σ will act on $c, a(g)$, and $\langle g, h\rangle$. That is, c is a 24th root of unity, $\langle g, h\rangle$ are $2n$th roots of unity, and a is determined by the other data. The presence of $n+4$ ensures that this is actually an extension of $\mathbb{Q}[\sqrt{n^2+4n}]$. Put another way, there is a group embedding $Gal(\mathbb{Q}[\xi_{lcm(24,2n,n+4)}] : \mathbb{Q}[\sqrt{n^2+4n}]) \to \mathbb{Z}_{24}^\times \times \mathbb{Z}_{2n}^\times \times \mathbb{Z}_{n+4}^\times$.

There are also constraints on the possible values of ℓ_{24}, ℓ_{2n} and ℓ_{n+4}. In particular, we need $\ell_i = \ell_j \mod \gcd(i,j)$. Another restriction comes from the fact that σ has to fix $\sqrt{n^2+4n}$ (and therefore δ) since it is in the base field. This restriction plays out in the following way:

There are 3 cases, and in all of them we assume n is square-free. If $n = 1 \mod 4$ then we have by Gauss sums,

$$\sqrt{n} = \sum_{j=0}^{n-1} \xi_n^{j^2}.$$

Then

$$\sigma(\sqrt{n}) = \sum_{j=0}^{n-1} \xi_n^{\ell_n j^2} = \left(\frac{\ell_n}{n}\right) \sum_{j=0}^{n-1} \xi_n^{j^2} = \left(\frac{\ell_n}{n}\right)\sqrt{n}.$$

Thus, $\sigma(\sqrt{n})$ depends only on the quadratic residue of $\ell_n \mod n$.

In the next case we have $n = 3 \mod 4$. Then

$$\sigma(\sqrt{n}) = \sigma(-i)\sum_{j=0}^{n-1} \xi_n^{\ell_n j^2} = \left(\frac{\ell_n}{n}\right)(-i)^{\ell_{24}} \sum_{j=0}^{n-1} \xi_n^{j^2} = \left(\frac{\ell_n}{n}\right)(-i)^{\ell_{24}}\sqrt{n}.$$

Here, $\sigma(\sqrt{n})$ depends on the quadratic residue of $\ell_n \mod n$ and on $\ell_{24} \mod 4$.

The final case is if $n = 2$. Then

$$\sqrt{2} = \xi_8 - \xi_8^3,$$

both of which are 24th roots of unity. So σ acts on them by raising them to ℓ^{24}. So the result is only dependent on $\ell_{24} \mod 8$.

So overall, the fact that σ has to fix $\sqrt{n^2+4n}$, together with the above restrictions, gives a small number of options for $(\ell_{2n}, \ell_{n+4}, \ell_{24})$. Note that fixing $\sqrt{n^2+4n}$ is a somewhat artificial constraint arising because we are looking only at unitary categories. If that requirement were dropped, then σ wouldn't have to fix $\sqrt{n^2+4n}$. We don't have to restrict σ so that $\sigma(\sqrt{n}) = \sqrt{n}$ (instead we could have $\sigma(\sqrt{n}) = \pm\sqrt{n}$) because anytime there's a $-\sqrt{n}$ that appears, it's always next to c, so we can absorb the extra -1 by the c.

Now we have $\sigma \in Gal(\mathbb{Q}[\xi_{lcm(n,n+4,24)}] : \mathbb{Q}[\sqrt{n^2+4n}])$, but these σ lift to at least one $\sigma \in Gal(k : \mathbb{Q}[\sqrt{n^2+4n}])$. In practice, one of these obeys $b^\sigma(g) = b(\ell_{2n}g)$. As a result, we have automorphisms which take all the data of one fusion category satisfying some fusion rules to data of another which satisfy the same fusions. That is, applying σ to the fusion equations of one category gives valid fusions which define some other category (in the same Galois orbit).

The most important and explicit example of $\sigma \in Aut(\mathcal{C})$ which satisfies (3.2.1) and (3.2.2) is complex conjugation. For $n = 1$, this fixes the data, and for $n = 5$, the two complex conjugate fusion categories are equivalent. However, for all other cases we have found, complex conjugation sends a near group category to an inequivalent one.

Example 3.2.1 ($G = \mathbb{Z}_6$)**.** The first step is solving the equations directly in Mathematica. The method is covered in general elsewhere. Recall that we first find all values of m which yield possible non-degenerate bilinear forms $\langle g, h\rangle$ and $a(g)$'s, then we find the possible values of c (which are 24th roots of unity). Then for each pairing of m and c, we try to solve the remaining equations for $b(g)$. Successful solutions are then kept. In the case of $\mathbb{Z}_6$ this method returns 8 solutions over 4 different values of m. The values of m are $\pm1, \pm5$ and are taken $\mod 2n$, which avoids the need to deal explicitly with $a(g)$. We get one possible c value for each m, given by $\xi_{24}^{\pm 1}$, $\xi_{24}^{\pm 5}$ respectively. For each (m, c) pair, there are 2 possible solutions for the $b(g)$, of which we only need to list the $b(1)$, $b(2), b(3)$, since the others are determined from those alone. Note that this is specific to $G = \mathbb{Z}/6\mathbb{Z}$ and in general there won't be 2 solutions for a given (m, c) pair. All the solutions are summarized below, where :

m	c	$\{b(1), b(2), b(3)\}$
1	1	$\{2.95526, 0.0553542, -0.785398\}$
1	1	$\{-2.43166, 2.03904, -0.785398\}$
5	5	$\{2.91503, -1.59091, 2.35619\}$
5	5	$\{-0.29704, -0.503485, 2.35619\}$
-5	-5	$\{-2.91503, 1.59091, -2.35619\}$
-5	-5	$\{0.29704, 0.503485, -2.35619\}$
-1	-1	$\{-2.95526, -0.0553542, 0.785398\}$
-1	-1	$\{2.43166, -2.03904, 0.785398\}$

Since c is always some 24th root of unity, it suffices to record the power mod 24. So, the first entry in the c column is shorthand for ξ_{24}. In the b column, it is sufficient to give the argument of $b(g)$ since the norm is always known, which again helps with economy of notation. We can eliminate half of these by noting that they are equivalent in the usual sense (ie, there is an element of $\mathbb{Z}_6^\times \cong \mathbb{Z}_2$ that permutes $b(g)$'s and corresponds to complex conjugation. So there are 4 inequivalent categories, so we are down to the following collection (which has been reordered conveniently):

m	c	$\{b(1), b(2), b(3)\}$
1	1	$\{2.95526, 0.0553542, -0.785398\}$
-1	-1	$\{-2.95526, -0.0553542, 0.785398\}$
5	5	$\{2.91503, -1.59091, 2.35619\}$
-5	-5	$\{-2.91503, 1.59091, -2.35619\}$

What remains is to find which of the above categories are Galois associates of one another. In order to do so, we find the valid parameterizations of σ. First

ℓ_{24}. Since ℓ_{24} controls c, any σ that sends the first category to the second would have to have $\ell_{24} = -1$, and similar for the other 2 categories (with 5 and -5 respectively). Next, ℓ_6 controls how σ acts on m. That is, $\sigma(m) = \ell_n m$, so again if we want to transform the first category into the second, we need $\ell_6 = -1$, and similar to get to the other 2 categories. Now that ℓ_6 and ℓ_{24} are determined, we compute $\sigma(\sqrt{6})$. By applying the general formulas written above, it isn't hard to see that $\sigma(\sqrt{6}) = \sqrt{6}$. Therefore, whatever value for ℓ_{10} is chosen, it has to result in $\sigma(\sqrt{10}) = \sqrt{10}$.

There are 2 restrictions for ℓ_{10}. We need $\left(\frac{\ell_{10}}{5}\right) = 1$, $\ell_{10} = -1 \mod 2$. Obviously -1 does the job, so we pick that. One does similar computations for each of the other categories in the Galois orbit and we find that each category is actually in the same Galois orbit.

The main purpose of these Galois-theoretic considerations is classification. For example, when G is cyclic, all quadratic forms are Galois associates so we may expect that every near group fusion category for a given group G belongs in the same Galois orbit. However, our requirement that any Galois automorphism fixes δ can occasionally interfere. For example, when $G = \mathbb{Z}_{14}$, quadratic forms $m = 1$ and 5 are associated by a σ with $\ell_{2n} = 5$, but no value of ℓ_{18} will allow σ to fix δ. Therefore these Galois considerations can differentiate, at times, between near group categories for the same group.

A secondary benefit of this classification is that it is in principle always possible to find exact values for the $b(g)$. The $b(g)$ from the same Galois orbit have to be roots of a polynomial with coefficients in $\mathbb{Q}[\delta]$, or equivalently $\mathbb{Q}[1/\delta]$. So we can use the approximate solutions together with a function like FindIntegerNullVector in Mathematica to find the exact polynomial which has the all $b(g)$, for g in a single $Aut(G)$ orbit as solutions. We can make the polynomials smaller by restricting to the ones whose roots are the $b(g)$ which will be permuted to each other by the Galois action. When $G = \mathbb{Z}_n$, the Galois action can relate $b(g)$ to $b(h)$ if $gcd(g,n) = gcd(h,n)$. The coefficients of each such polynomial lie in $\mathbb{Q}[\delta]$. We have explicitly found these polynomials for all Galois orbits for $n \leq 19$, but for reasons of space did not include them here. The fact that the coefficients do lie in $\mathbb{Q}[\delta]$ is a highly nontrivial consistency check.

3.3 Results

3.3.1 New Solutions

The following table contains the data found by solving equations (3.1.1)-(3.1.9). For convenience we have included data for all $n \leq 30$, however the data for $n \leq 13$ was given previously in [6]. All other data is new. The rightmost vector contains the arguments of the values $b(1), ..., b(\lfloor n/2 \rfloor)$ which are sufficient, together with the m and c values, to reconstruct the entire category. We have proven that this is the complete list of near group fusion categories for $n \leq 20$ by using the technique of section 3.3.2 We have not yet applied this technique to the data for higher values of n, and are in fact convinced that our list incomplete for higher n as we have found no fusion categories associated to $G = \mathbb{Z}_{29}$, which was unexpected. The most likely explanation was that the level of precision used at the higher orders was insufficient for Mathematica to find numerical solutions. This could likely be rectified by increasing the precision at the cost of a great deal more time.

The table contains only one representative from each equivalence class of solutions. In addition, it is easy to see that for any set of solutions $\{m, c, b(1), ..., b(\lfloor n/2 \rfloor)\}$, the set $\{-m, \overline{c}, \overline{b(1)}, ..., \overline{b(\lfloor n/2 \rfloor)}\}$ of complex conjugates is also a solution. A pair of complex conjugate solutions will not in general yield equivalent fusion categories, however for economy of notation we do not include the complex conjugate solutions.

The second column of the table labels each category.

Group	**id**	**m**	**c**	j-**values**
$\mathbb{Z}_1$	J_1^1	1	1	$\emptyset$
$\mathbb{Z}_2$	J_2^1	1	$e^{\frac{\pi i 5}{6}}$	(0.78539816)
$\mathbb{Z}_3$	J_3^1	1	$e^{-\frac{\pi i}{6}}$	(-2.8484536)
$\mathbb{Z}_4$	J_4^1	1	$e^{\frac{3\pi i}{4}}$	$(-0.60623837, -1.5707963)$
$\mathbb{Z}_5$	J_5^1	1	-1	$(-1.256637, 1.256637)$
	J_5^2	2	$e^{\frac{2\pi i}{3}}$	$(-1.0071249, 0.3425266)$

$\mathbb{Z}_6$	J_6^1	1	$e^{\frac{\pi i}{12}}$	$(2.9552611, 0.055354168, -0.78539816)$
	J_6^2	5	$e^{\frac{2\pi i}{3}}$	$(2.915033694, -1.5909100, 2.3561944)$
$\mathbb{Z}_7$	J_7^1	1	$-i$	$(1.0516925, -1.793625, 0.31433)$
$\mathbb{Z}_8$	J_8^1	1	$e^{\frac{\pi i}{12}}$	$(-0.87227636, 2.7042615, -2.9767963, 3.1415926)$
	J_8^2	3	$e^{\frac{7\pi i}{12}}$	$(2.46404903, -3.0755747, -0.4918869, 0)$
$\mathbb{Z}_9$	J_9^1	1	$e^{-\frac{2\pi i}{3}}$	$(-2.695680, 1.3670127, 1.418824, -2.383744)$
$\mathbb{Z}_{10}$	J_{10}^1	1	$e^{\frac{3\pi i}{4}}$	$(1.3447773, 2.868685, 1.7756309,$ $-0.64512913, -2.3561944)$
	J_{10}^2	7	$e^{\frac{\pi i}{4}}$	$(-3.077894, 2.519776, -1.424024,$ $3.089463, -0.78539816)$
$\mathbb{Z}_{11}$	J_{11}^1	1	$e^{-\frac{5\pi i}{6}}$	$(1.9464713, 2.0140743, -1.7487929,$ $0.3352432, -0.1427077)$
	J_{11}^2	1	$e^{-\frac{\pi i}{6}}$	$(0.53877136, -2.8317431, 0.2827610,$ $0.46457259, 2.5063157)$
$\mathbb{Z}_{12}$	J_{12}^1	5	$e^{-\frac{7\pi i}{12}}$	$(-3.0822445, 0.34946402, 3.0450322,$ $0.7241984, 0.3823471, -1.570796)$
	J_{12}^2	7	$e^{-\frac{5\pi i}{12}}$	$(2.457353, -2.19152, -0.01465,$ $-2.566258, 0.834821, 1.57079)$
$\mathbb{Z}_{13}$	J_{13}^1	1	-1	$(-2.4521656, 1.9847836, 0.42579608,$ $1.4322079, -1.4550587, 1.1404478)$
	J_{13}^2	1	-1	$(1.4550587, 1.3924399, -1.9847836,$ $-1.2761619, 0.44776608, -1.4322079)$
$\mathbb{Z}_{14}$	J_{14}^1	1	$e^{\frac{3i\pi}{4}}$	$(1.87376, 0.0185449, -3.10904, -1.89339,$ $1.16017, 2.11217, 2.35619)$
	J_{14}^2	1	$e^{\frac{-7i\pi}{12}}$	$(-2.41283, 3.02475, 0.68568 - 2.28102,$ $-0.637960, 2.372581, 2.356194)$
	J_{14}^3	1	$e^{\frac{i\pi}{12}}$	$(-0.558150, 2.97988, -2.424590, 0.936766,$ $0.527876, 1.161870, 2.35619)$
$\mathbb{Z}_{15}$	J_{15}^1	1	$e^{\frac{5i\pi}{6}}$	$(-1.01545, 2.20285, -2.83256, -2.18868,$ $0.316256, 0.200722, -1.23945)$

	J_{15}^2	2	$e^{\frac{i\pi}{6}}$	$(0.401455, -0.88942, -0.444928, -2.93185, 2.25265, -1.13932, -0.192544)$
$\mathbb{Z}_{16}$	J_{16}^1	3	$e^{\frac{i\pi}{4}}$	$(-1.22125, 3.09812, 2.33936, 0.158883, -1.22173, 0.463885, -1.94633, 0)$
	J_{16}^2	7	$e^{-\frac{3i\pi}{4}}$	$(-0.529358, -2.25148, -1.16791, -1.968, 0.778733, -2.75884, 0.604348, 0)$
$\mathbb{Z}_{17}$	J_{17}^1	1	$e^{\frac{2i\pi}{3}}$	$(-1.0887, 2.5331, 2.52182, -3.02391, -0.54935, 0.751107, -1.19756, -1.95719)$
	J_{17}^2	3	$e^{-\frac{i\pi}{3}}$	$(-2.66814, -1.49455, 0.05387, -0.68249, -2.78657, 1.35454, -2.42922, 3.04137)$
$\mathbb{Z}_{18}$	J_{18}^1	1	$e^{\frac{i\pi}{12}}$	$(1.99030, 0.09075, -2.18014, -0.46248, 1.76048, 2.53944, -0.33425, -0.69650, 0.78539)$
	J_{18}^2	1	$e^{\frac{i\pi}{12}}$	$(-2.98189, 2.05308, -0.97584, -0.50514, -3.11965, 0.77820, 2.83684, -1.90854, 0.78539)$
	J_{18}^3	5	$e^{-\frac{7i\pi}{12}}$	$(1.51593, 0.77871, 1.66021, -2.42632, -2.81552, 2.07003, 1.23103, -1.22981, 0.78539)$
	J_{18}^4	5	$e^{-\frac{7i\pi}{12}}$	$(0.12038, -0.22167, -2.47985, -1.26222, 1.69815, -1.19521, -1.69711, 2.45944, 0.78539)$
$\mathbb{Z}_{20}$	J_{19}^1	3	$e^{\frac{7i\pi}{12}}$	$(-1.5127, -0.890643, 1.30931, -1.13839, 0.20962, 0.273548, -2.18102, 2.42318, -0.526725, -1.5708)$
	J_{20}^2	11	$e^{\frac{11i\pi}{12}}$	$(2.72115, 0.992739, -1.17994, -2.84756, -1.0912, 0.858831, 1.16108, 2.03227, -1.83605, 1.5708)$
	J_{20}^3	-11	$e^{\frac{7i\pi}{12}}$	$(-0.494129, -1.55212, -2.32162, 0.84703, -0.501398, -0.192497, 3.10287, -0.98733, 0.99974, 1.5708)$
	J_{20}^4	-13	$e^{-\frac{11i\pi}{12}}$	$(-2.84215, 0.344274, 1.80207, -1.46345, -1.11517, 3.09009, -1.3632, 0.516509, 0.0505006, 1.5708)$
	J_{20}^5	-17	$e^{-\frac{7i\pi}{12}}$	$(-1.98394, -2.7756, -2.93184, -2.39503, 0.995018, 2.1585, -0.138982, -2.60336, -0.997964, 1.5708)$
$\mathbb{Z}_{21}$	J_{21}^1	2	$e^{\frac{2i\pi}{3}}$	$(2.3933, -1.47535, 2.23778, 0.539435, 2.51576, -2.84378, 0.0418671, -1.58297, 0.263239, -0.806736)$

	J_{21}^2	2	$e^{\frac{2i\pi}{3}}$	$(-1.2963, 0.0268746, -1.93267, -1.20054, 1.60254, 2.69878, 2.97437, 2.76233, 0.235483, -3.00924)$
	J_{21}^3	10	$e^{-\frac{2i\pi}{3}}$	$(1.20054, -2.76233, 1.13308, 0.405745, -1.5955, 1.93267, -2.97437, -1.51324, -1.78921, -1.16992)$
$\mathbb{Z}_{22}$	J_{22}^1	7	$e^{-\frac{3i\pi}{4}}$	$(0.755936, -1.14865, -2.06555, 1.44476, 0.551473, 0.700735, -2.83509, -1.27481, -1.92804, 2.87422, 0.785398)$
	J_{22}^2	13	$e^{-\frac{i\pi}{4}}$	$(-1.26051, -2.80831, -1.69908, 0.720007, 1.20588, 0.433743, 2.80737, -2.20926, -0.458365, 2.64826, 2.35619)$
$\mathbb{Z}_{23}$	J_{23}^1	5	i	$(-1.76047, -0.375616, 2.40657, 2.05459, -1.31021, 0.483545, 0.528361, 1.18299, -0.618369, 2.20518, -0.0253942)$
	J_{23}^2	5	$e^{-\frac{5i\pi}{6}}$	$(1.86586, -2.15342, 2.88027, -2.60841, 1.95124, 1.47718, -1.07496, 1.56657, -0.0352662, -1.99727, -2.196)$
	J_{23}^3	7	$e^{-\frac{5i\pi}{6}}$	$(2.10274, -1.5813, 1.64514, 0.617117, -1.65773, -0.00698851, 1.30304, -1.64838, -2.46684, -0.0482917, -1.22686)$
$\mathbb{Z}_{24}$	J_{24}^1	5	$e^{\frac{5i\pi}{12}}$	$(2.63607, 2.98277, -0.0872789, -2.07559, 0.964404, -0.674997, 1.35112, 2.50614, -3.00963, -1.94611, 1.9158, -3.14159)$
	J_{24}^2	11	$e^{-\frac{i\pi}{12}}$	$(2.8668, 2.43273, -2.76482, -0.982608, -2.32205, 1.33428, -0.133956, -2.42274, 0.0471326, 1.07842, 2.59359, 0)$
	J_{24}^3	17	$e^{\frac{5i\pi}{12}}$	$(-0.513063, -2.74273, -1.36543, 1.58574, 1.97488, -0.441133, -1.61338, 1.49992, -2.26249, 2.01451, -0.633811, 0)$
$\mathbb{Z}_{25}$	J_{25}^1	2	1	$(-0.811805, 2.78414, -1.77457, -2.79598, 2.80126, -3.11719, -2.12186, 0.744012, 0.00166425, 0.545179, -0.858711, 1.96473)$

$\mathbb{Z}_{26}$	J^1_{26}	5	$e^{\frac{5i\pi}{12}}$	$(2.2905, 2.85751, 2.46178, -2.68877, -2.33487, 1.48568, -0.440556, -0.870636, 2.50794, -2.96212, -0.236236, -0.355171, 0.785398)$
	J^2_{26}	-5	$e^{-\frac{11i\pi}{12}}$	$(-1.30992, -0.996996, 0.820776, 1.2368, 1.48965, -0.883537, 1.2369, -0.954417, 0.773047, -0.749557, -2.14603, 2.24306, -0.785398)$
$\mathbb{Z}_{28}$	J^1_{28}	5	$e^{\frac{3i\pi}{4}}$	$(0.0150253, -1.48202, -0.535171, 0.374327, 2.17644, 2.68597, 0.523946, 0.378741, -1.07881, -1.87416, 1.90304, -2.23839, -1.49048, -1.5708)$
	J^2_{28}	11	$e^{-\frac{3i\pi}{4}}$	$(2.17012, 1.75765, -1.43258, 1.21786, 1.40787, 1.76022, 0.0256398, 0.867112, -2.34929, 0.4615, -1.59022, -2.21831, 1.57147, 1.5708)$
$\mathbb{Z}_{30}$	J^1_{30}	7	$e^{\frac{7i\pi}{12}}$	$(-0.288937, -1.12986, -1.47505, 1.84459, -1.19226, -1.56207, 1.77504, 0.0821613, 0.30487, -0.858915, 1.62322, 2.74184, -2.38889, -0.663313, -2.35619)$
	J^2_{30}	11	$e^{-\frac{i\pi}{12}}$	$(-1.41372, 1.89281, 0.731926, 1.83499, 0.525567, -3.11733, -0.803273, -2.97259, -0.141535, -0.071548, -2.78467, -2.9665, -1.00904, 0.698041, -2.35619)$
	J^3_{30}	-7	$e^{-\frac{11i\pi}{12}}$	$(0.633287, 1.54544, -0.810544, 2.5788, 1.49368, -1.87101, 2.57613, -1.98519, -1.67296, -1.44005, 1.39576, 1.04269, 0.662849, -0.913612, -0.785398)$
	J^4_{30}	11	$e^{\frac{5i\pi}{12}}$	$(2.58849, 3.11313, -0.0158919, 2.83496, 2.43301, -0.188801, -2.33047, -1.2354, -0.331939, 0.469661, -1.23594, 2.24009, -0.734497, -2.41457, -0.785398)$

TABLE 3.1: Near Group Fusion Categories

It should be noted that even in hindsight, there is no other known way to find these fusion categories, making them truly exotic. The fact that we have found many examples, and that there seem to more examples as n grows, is compelling evidence that there are infinitely many near group categories.

3.3.2 Consistency and Completeness

Before proceeding to find the modular data, there are two questions which need to be addressed. The first was hinted at earlier, being: How do we know whether or not this list is complete? The second is due to the fact that we provide floating point solutions (and indeed solved the equations in floating point). It is not apparent that these floating point solutions correspond to exact solutions. That is, how do we know that the relatively few digits included in the the table are sufficient?

We will answer the latter question first. The process is best explained by example, so we will consider $\mathbb{Z}_{14}$. There are two distinct Galois orbits. in the chart above, the first solution is in a distinct Galois orbit from the second two, which share one. Focusing on the first orbit, we can construct minimal polynomials which have the $b(g)$ as roots. The best way we have found to accomplish this is by first renormalizing the $b(g)$ by considering $\sqrt{n}b(g)$. This causes the $b(g)$ to become algebraic integers, so the resulting minimal polynomials will be monic and generally have smaller coefficients. We then group the $b(g)$ according to which will be permuted to each other by the Galois action. As we discussed above, for any $\ell \in \mathbb{Z}_n^\times$, $b^{(\ell)}(g) := b(\ell g)$ satisfies (3.1.5)-(3.1.9), provided we also replace $a(g)$ by $a^{(\ell)}(g) := a(\ell g)$. In all cases we have seen, there is a lift $\sigma \in Gal(k : \mathbb{Q})$ of that ℓ in $Gal(\mathbb{Q}[\xi_n] : \mathbb{Q})$ such that σ^2 fixes δ and c (this happens automatically), commutes with complex conjugation, and sends $b(g)$ to $b^{(\ell)}(g)$. In the cyclic case, we expect that there will be a Galois automorphism sending $b(g)$ to $b(h)$ exactly when $gcd(g, n) = gcd(h, n)$. This happens in all examples we have seen, but if not it simply means that the polynomial would factor, i.e, would not be minimal. In our example of $\mathbb{Z}_{14}$, the numbers k with $gcd(k, 14) = 1$ are $1, 3, 5, 7, 9, 11, 13$.

Let $p(x)$ be the unique monic polynomial satisfying $p(x) = 0$ if and only if $x = \sqrt{14}b(g)$ for some $g \in \mathbb{Z}_{14}$ with $gcd(g, 14) = 1$. Then $p(x)$ can be written

$$p(x) = 1 + \sqrt{14}a_1 X + a_2 X^2 + \cdots + \sqrt{14}a_{13} X^{13} + X^{14}.$$

We multiply the odd coefficients by $\sqrt{14}$, because this ensures all coefficients

$a_i \in \mathbb{Q} + 1/\delta\mathbb{Q}$. In this particular example, the denominators are 3. There is also a symmetry present whereby $a_i = a_{14-i}$, or more generally $a_i = a_{n-i}$. This is a result of the fact that x is a root if an only if $1/x$ is. For this particular polynomial, the coefficients are

$$(a_1, ..., a_7) = \frac{1}{3}(1 + 5/\delta, 17 - 56/d, 2 + 49/\delta, 46 - 280/\delta, 1 + 128/\delta, 67 - 490/\delta).$$

Note that there will be an integer $N > 0$ (n^2 times the lowest common multiple of the denominators of the a_i will always work) such that $Nb(g)$ is an algebraic integer. Why this is important will be clear shortly.

Next, we redefine our approximate solutions to be the corresponding exact solutions of this polynomial, divided by $\sqrt{n}$. We need to show that (3.1.6) - (3.1.9) are exactly satisfied. Equation (3.1.7) is automatically satisfied by construction. To show (3.1.6), (3.1.8), and (3.1.9) are also satisfied, we multiply them by N, N^2, and N^3 respectively. Then the differences between their left and right sides will be algebraic integers. An algebraic integer is 0 if and only if all of its Galois associates have modulus < 1, and we have complete control over these associates thanks to the polynomial $p(x)$. We also know how the Galois automorphism acts on $a(g)$ thanks to (3.1.7), so it suffices to show that the difference between the left side and right side of (3.1.6), (3.1.8), and (3.1.9) is small for all associates, which is easily done by computer. That is how we show that our solutions are indeed exact.

To ensure that we've found a complete set of solutions for each n, we essentially use Bezout's theorem together with a methodical approach to narrow the number of solutions we are looking for in a given step. To demonstrate, we will take the example of $\mathbb{Z}_{14}$.

Example 3.3.1. There are two potential quadratic forms (up to equivalence) for $\mathbb{Z}_{14}$, determined by $m = 1, 5$. We'll start by looking at $m = 1$. We know that $b(7)$ is fixed up to a sign, and we can use the eigenvalue equation to find eigenvectors and eigenvalues. We find three eigenvalues, the first of which is

$c_1 \approx 0.707 + 0.707i$. For this eigenvalue, we find four eigenvectors, given by

$$E_1 = \begin{gathered}(0, 0.829301, 0.457973 + 0.160252i, 0.365222 + 0.457973i,\\ -0.030148 + 0.267573i, 0.23856 - 0.114884i, 0.1904 + 0.1904i, 0,\\ -0.1904 - 0.1904i, -0.23856 + 0.114884i, 0.030148 - 0.267573i,\\ -0.365222 - 0.457973i, -0.457973 - 0.160252i, -0.829301)\end{gathered}$$

$$E_2 = \begin{gathered}(0, -0.617942 + 0.070785i, -0.368164 - 0.492158i, -0.339626 - 0.26149i,\\ -0.081295 - 0.180726i, 0.039317 + 0.300332i, -0.262135 + 0.109378i, 0,\\ 0.262135 - 0.109378i, -0.039317 - 0.300332i, 0.081295 + 0.180726i,\\ 0.339626 + 0.26149i, 0.368164 + 0.492158i, 0.617942 - 0.070785i)\end{gathered}$$

$$E_3 = \begin{gathered}(-0.39439 + 0.044437i, 0.290314, -0.083359 - 0.029169i,\\ 0.029169 + 0.036576i, 0.014289 - 0.126817i, 0.302773 - 0.145808i,\\ -0.090241 - 0.090241i, 0.39439 - 0.49455i, -0.090241 - 0.090241i,\\ 0.302773 - 0.145808i, 0.014289 - 0.126817i, 0.029169 + 0.036576i,\\ -0.083359 - 0.029169i, 0.290314)\end{gathered}$$

$$E_4 = \begin{gathered}(-0.320498 + 0.004805i, 0.32199 + 0.070785i, -0.240122 - 0.223003i,\\ 0.048552 + 0.188424i, -0.098185 + 0.070764i, 0.292109 - 0.091685i,\\ -0.098522 - 0.190875i, 0.704098 - 0.514291i, -0.098522 - 0.190875i,\\ 0.292109 - 0.091685i, -0.098185 + 0.070764i, 0.048552 + 0.188424i,\\ -0.240122 - 0.223003i, 0.32199 + 0.070785i)\end{gathered}$$

As $\mathbf{b}$ is an eigenvector, we are looking to express it as a linear combination of the E_i. That is, we are trying to find complex numbers e_1, e_2, e_3, e_4 such that

$$e_1E_1 + e_2E_2 + e_3E_3 + e_4E_4 = \mathbf{b}.$$

Since $b(7)$ is known up to a sign, we can fix it by first assuming it is positive. We also know $b(0)$. Thus we can immediately eliminate 2 of the 4 parameters. Writing the remaining parameters in terms of their real and imaginary parts, we are left with four unknown real parameters to solve for, and by using the linear equations we can reduce it to two real parameters. The final step is to use the norm condition on two of the $b(g)$, say $b(1)$ and $b(2)$. Doing so results in two quadratic equations in two variables, so Bezout's theorem tells us there will be exactly four solutions. A solver finds these easily. However, plugging the newly found solutions in for the norm of $b(3)$ results in nonsense, so there are in fact no valid solutions for this choice of sign of $b(7)$, this value of c, and this value of m.

The next step is to back up, and choose $b(7)$ to be negative. Doing the same process, we find valid solutions here. Repeat this process with each possible eigenvalue; for $\mathbb{Z}_{14}$, we found that only the demonstrated one worked.

As the above example shows, the key to check for completeness is to reduce the number of parameters as much as possible and apply Bezout's theorem. After each possibility has been exhausted, we know that we have checked every potential solution and hence our list must be complete. Despite the advancements of computing, modern solvers still miss some solutions due to floating point issues or other problems, so these checks are important.

3.3.3 Modular Data

The following formulae for the S and T matrix were derived in [16] in detail using the tube algebra construction. We only reproduce the relevant results for convenience here. To construct the S and T matrices which define the modular data for the near group fusion categories, we need to find all functions $\xi : G \to \mathbb{T}$ and values $\tau \in G$, $\omega \in \mathbb{T}$ which satisfy

$$\sum_{g\in G} \xi(g) = \sqrt{n}\omega^2 a(\tau)c^3 - n\delta^{-1}, \tag{3.3.1}$$

$$\overline{c}\sum_{g\in G} b(k+g)\xi(g) = \omega^2 c^3 a(\tau)\overline{\xi(k+\tau)} - \sqrt{n}\delta^{-1}, \tag{3.3.2}$$

$$\xi(\tau - g) = \omega c^4 a(g)a(\tau - g)\overline{\xi(g)}, \tag{3.3.3}$$

$$\sum_{g\in G} \xi(g)b(g-k)b(g-h) = c^{-2}b(k-h-\tau)\xi(g)\xi(h)\overline{a(k-h)} - c^2\delta^{-1}. \tag{3.3.4}$$

The S and T matrices for a near group fusion category of type $G+n$ are parameterized by a set of primaries Φ which splits into 4 distinct subsets of primaries given by

- $\mathfrak{a}_g$, $g \in G$,
- $\mathfrak{b}_h$, $h \in G$,
- $\mathfrak{c}_{l,k} = \mathfrak{c}_{k,l}$, $l, k \in G, l \neq k$,
- $\mathfrak{d}_j$, j corresponding to a triple $(\omega_j, \xi_j, \tau_j)$.

We will soon find a more natural parameterization for the $\mathfrak{d}_j$. There are $n(n+3)$ total primaries, with n each of type $\mathfrak{a}$ and $\mathfrak{b}$, $n(n-1)/2$ of type $\mathfrak{c}$, and $n(n+3)/2$ of type $\mathfrak{d}$. In block form the T and S matrices are given by

$$T = \mathrm{diag}(\langle g,g\rangle, \langle h,h\rangle, \langle k,l\rangle, \omega_j) \tag{3.3.5}$$

$$S = \frac{1}{\lambda}\begin{pmatrix} \langle g,g'\rangle^{-2} & (\delta+1)\langle g,h'\rangle^{-2} & (\delta+2)\overline{\langle g,k'+l'\rangle} & \delta\langle g,\tau_{j'}\rangle \\ (\delta+1)\langle h,g'\rangle^{-2} & \langle h,h'\rangle^{-2} & (\delta+2)\overline{\langle h,k'+l'\rangle} & -\delta\langle h,\tau_{j'}\rangle \\ (\delta+2)\overline{\langle k+l,g'\rangle} & (\delta+2)\overline{\langle k+l,h'\rangle} & S_{(k,l),(k',l')} & 0 \\ \delta\langle\tau_j,g'\rangle & -\delta\langle\tau_j,h'\rangle & 0 & S_{j,j'} \end{pmatrix}. \tag{3.3.6}$$

$S_{j,j'}$ is given by

$$\begin{aligned} S_{j,j'} &= \omega_j\omega_{j'}\sum_{g\in G}\langle\tau_j+\tau_{j'}+g,g\rangle \\ &+\delta\omega_j\omega_{j'}a(\tau_j)a(\tau_{j'})n^{-1}\sum_{g,h\in G}\overline{\xi_j(g)\xi_{j'}(h)\langle\tau_j-\tau_{j'}+h-g,h-g\rangle}, \end{aligned} \tag{3.3.7}$$

and

$$S_{(k,l),(k',j')} = (\delta+2)(\overline{\langle k,k'\rangle\langle l,l'\rangle} + \overline{\langle k,l'\rangle\langle l,k'\rangle}).$$

The majority of the T and S matrix entries are easy, however the lower right corner of each is comparatively very complicated. It is for this block that we need to solve (3.3.1)-(3.3.4). This is no great challenge for a computer to solve numerically to a high degree of precision, given high precision values for the data of the fusion category itself. It is possible, in principle, to solve all of the equations exactly, given exact data for the fusion category, however the time cost of solving even small cases exactly was prohibitive. Despite this, it was possible to find some exact values without much effort. As T has finite order, all of its entries must be roots of unity, so ω_j must have an argument which is a rational multiple of π. Given the high precision we can work with, it's easy to ask Mathematica (or another language) to find close rational approximations for the exponent of the ω_j.

Finding modular data numerically using the above formulas is not difficult, but it is also not enlightening. It would be much more preferable to find another expression for the modular data, specifically the lower right corners of the S and T matrices, which do not rely at all on (3.3.1)-(3.3.4), and in turn on the fusion category.

Evans and Gannon provide a conjecture ([6], Conjecture 2) in the cases where n is odd which simplifies these corners considerably, doing away with the need to ever find the fusion categories or solve numerical equations in the first place. We present one form of it below.

Conjecture 3.3.1. *Let G be a finite abelian group of odd order n and let $G+n$ be an associated near group fusion category. Then its modular data is determined by some choice of data $(H, \langle , \rangle, q, \langle , \rangle', q')$, where*

- *H is an abelian group of order $n+4$*
- *$\langle , \rangle$ (resp. $\langle , \rangle'$) is a nondegenerate symmetric form on G (resp. H).*
- *q (resp. q') is an associated quadratic form for $\langle , \rangle$ (resp. $\langle , \rangle'$).*

The S and T matrices are identical to 3.3.5 and 3.3.6, respectively, except for the lower right corners, which are

$$t_{(\tau,\gamma),(\tau,\gamma)} = \langle \tau, \tau \rangle \langle \gamma, \gamma \rangle', \quad (\tau \in G, \gamma \in H \setminus \{0\})$$
$$s_{(\tau,\gamma),(\tau',\gamma')} = -\delta \overline{\langle \tau, \tau' \rangle} \left(\langle \gamma, \gamma' \rangle' + \overline{\langle \gamma, \gamma' \rangle'} \right), \quad (\tau, \tau' \in G, \gamma, \gamma' \in H \setminus \{0\}).$$

This conjecture also offers a new parameterization of the $\mathfrak{d}_j$ primaries. Namely, we now parameterize them as $\mathfrak{d}_{\tau,\gamma} = \mathfrak{d}_{\tau,-\gamma}, \tau \in G$, $\gamma \in H \neq 0$.

Because G has odd order, each quadratic form is compatible with exactly one nondegenerate symmetric form, so there is actually very limited freedom. The quadratic forms are particularly simple. In the case of a cyclic group (in this thesis, all G and nearly all H are cyclic), the associated quadratic form is $q(g) = \exp(2\pi i g^2 m/n)$ for some m coprime with the order of the group.

We have verified that the conjecture holds for $n \leq 29$. It is still an open question whether any choice of (H, q, q') yields the modular data of some category.

Moving on, the question of simplification in the even case is more difficult. This is to be expected for a few reasons. One, there are more quadratic forms on even G. The second comes down to the fact that even G have fixed points under the action of the simple currents whereas odd G do not. We will see both of these factors coming into play later. In the following, we restrict to cyclic groups G of even order. Eventually we would like to generalize our results to noncyclic abelian groups as well.

To state the conjecture we need to first develop some additional notation. As in the odd case, the modular data here is dependent on a small number of parameters which there is some apparent freedom to chose. It will be useful to factor groups into their odd and even part, so in the following, for any group G, denote by G_e the set of elements in G of order a power of 2, and by G_o those elements with odd order, so that $G = G_o \times G_e$.

Let G be a cyclic group of even order and write it as $\mathbb{Z}_{2^s} \times G_o$. Choose a nondegenerate symmetric pairing $\langle , \rangle$ on G and a compatible quadratic form q. Let H be an abelian group of odd order $(n+4)/2^t$, where 2^t is the highest power of 2 dividing $n+4$. Let $\Gamma = \mathbb{Z}_{2^{t+s-1}} \times \mathbb{Z}_2 \times G_o \times H$. Finally let $\alpha : G \to \Gamma$ be an embedding given by $\alpha(1) = (2^{t-1}, 1, 1, 0)$.

Definition 3.3.2. *A triple (H, q', ι) is called compatible with (G, q) if the following properties hold:*

1. *q' is a nondegenerate quadratic form on Γ such that $q'(\alpha(g)) = \langle g, g\rangle$.*

2. *The Gauss sum $a_{q'}$ satisfies $a_{q'} = -1$.*

3. *ι is an involution on Γ satisfying $q'(\iota\gamma) = q'(\gamma)$, for all $\gamma \in \Gamma$, and $\iota(\gamma) = \gamma$ if and only if $\gamma \in \alpha(G)$.*

q' is the product of four (possibly three) quadratic forms, one for each element in the direct product factorization of Γ (or if $s = t = 1$, one for the $\mathbb{Z}_2 \times \mathbb{Z}_2$ component). We will see that the bracketed case never occurs. We fix the notation q'_H for the factor of q' which sees H, and likewise for the other factors.

We can classify Γ by s and t into three distinct cases. If $s = 1$ then $n = 2k$ for some k odd. Hence $n + 4 = 2(k + 2)$, so $t = 1$ as well. If $s = 2$, then $n = 4k$, for k odd, so $n + 4 = 4(k + 1)$. Thus $t \geq 3$. Finally, if $s \geq 3$, then $n = 2^s k$, so $n + 4 = 4(2^{s-2}k + 1)$ and we have $t = 2$. Each of the three cases have slightly different analysis, but in each case it is possible to heavily restrict the possible compatible triples.

Proposition 3.3.3. *Let $G, q,$ and H be as above. Then compatible q', ι are determined entirely by q'_H and possibly a value $a \in \mathbb{Z}_{2^{t+s-1}}$ subject to the following conditions:*

If $s = t = 1$,

$$\begin{aligned} q'_{\mathbb{Z}_2\times\mathbb{Z}_2}(x, y) &= i^{bx^2+by^2}, \\ \iota(x, y, g, h) &= (y, x, g, -h), \end{aligned}$$

for $b \in \{\pm 1\}$ determined by the Gauss sum condition.

If $s = 2$,

$$\begin{aligned} q'_{\mathbb{Z}_{2^{t+1}}}(x) = \xi_{2^{t+2}}^{ax^2}, &\quad q'_{\mathbb{Z}_2}(y) = i^{by^2} \\ \iota(x, y, g, h) = ((4N-1)x + 4y, &\, x + y, g, -h). \end{aligned}$$

Here $a \in \mathbb{Z}_{2^{t+2}}$ is odd and determined up to equivalence by Gauss sums up to a possible shift by 2^{t+1}, b is such that $i^b = \langle 1, 1\rangle_{G_e} i^{2^{t-2}}$, and N is determined by requiring $a(2N - N^2) = b2^{t-3} \mod 2^{t-1}$.

If $s \geq 3$,

$$q'_{\mathbb{Z}_{2^{s+1}}}(x) = \xi_{2^{s+2}}^{ax^2}, \quad q'_{\mathbb{Z}_2}(y) = i^{by^2}$$
$$\iota(x, y, g, h) = ((N2^s - 2^{s-1} + 1)x + 2^s y, x + y, g, -h).$$

Here $a \in \mathbb{Z}_{2^{s+2}}$ is odd and determined up to equivalence by Gauss sums up to a possible shift by 2^{s+1}, b is determined by $\xi_{2^s}^{a+b2^s} = \langle 1, 1, \rangle_{G_e}$, and $N = (ab-1)/2+2^{s-3} \mod 2$.

Proof. The proof is split into sections dependent on the value of s, but for each s the mechanics rely on Gauss sums together with property 2. First there are some commonalities which can be dealt with immediately.

By property 1 of compatible triples, q' is fixed on G_o by $\langle , \rangle$. Similarly ι fixes G_o and acts on H as multiplication by -1. The reason for this is straightforward. H is a cyclic group of odd order, and it suffices to consider the case where $|H| = p^k$, for some prime p and $k \in \mathbb{N}$. If $|H|$ contains more than one distinct prime factor, it can be further split into a direct product and the individual factors dealt with as we do in this case. Note that we only need to know $\iota_H(1)$ as ι is a homomorphism and H is generated by 1. Suppose $\iota_H(1) = c$ for some $c \in \mathbb{Z}_{p^k}$. Since ι is an involution, we must have $\iota_H^2(1) = 1 = c^2 \mod p^k$, and the only two solutions are $c = \pm 1$. Since ι fixes an element g if and only if $g \in \alpha(G)$, it follows that $c = -1$, otherwise there would be extraneous elements fixed by ι (for example, $(0, 0, 0, 1)$).

If $s = t = 1$, G_o has order $n/2$ and H has order $(n+4)/2$, both odd. Exactly one of these will be equal to $1 \mod 4$ and the other will be $3 \mod 4$. Thus together their quadratic forms will contribute $\pm i$ to the overall Gauss sum. Therefore the only possibility to define $q'_{\mathbb{Z}_2 \times \mathbb{Z}_2}$ is $q'_{\mathbb{Z}_2 \times \mathbb{Z}_2}(g, g') = i^{bg+bg'}$ with $b = \pm 1$ to ensure the correct overall sum of -1. The definition of α forces $\iota_{\mathbb{Z}_2 \times \mathbb{Z}_2}(x, y) = (y, x)$, which completes the first case.

The final two cases are done similarly, so we focus only on the case $s = 2$. The form of q' is forced in both cases since those are the only quadratic forms for cyclic groups which are a power of 2. That alone limits to $a \in \mathbb{Z}_{2^{s+t}}$ odd. The Gauss sum for $q'_{\mathbb{Z}_{2^{s+t}}}$ depends on $a \mod 4$ which allows us to determine whether

$a = 1$ or $-1 \mod 4$. Any choice of $a \in \mathbb{Z}_{2^{t+1}}$ that is consistent with the correct value of $a \mod 4$ will yield an equivalent quadratic form, however there will still be ambiguity as $q'_{Z_{2^{t+1}}}(x)$ sees $a \mod 2^{t+2}$. As a result, after the choice of $a \mod 2^{t+1}$ is made, it may be necessary to shift it by 2^{t+1}.

The only uncertainty in b is the sign, but property 1 implies $i^b = \langle 1, 1\rangle_{G_e} i^{2^{t-2}}$ and hence fixes it for any choice of symmetric pairing.

Finally it remains to see how ι is fixed, and the only mysterious part is how it acts on Γ_e. In general an automorphism will act on elements$(x, y) \in \Gamma_e$ by $(x, y) \mapsto (c_x x + c_y y, d_x x + d_y y)$ for $c_x, c_y, d_x, d_y \in \mathbb{Z}$. So, to determine ι, it is enough to find those coefficients. For a general automorphism this would be hopeless, but because ι additionally satisfies 3, we can find it exactly. First, looking at d_x and d_y, we see that it only matters whether they are odd or even so we can assume $d_x, d_y \in \{0, 1\}$. Consider $\iota(2^{t-1}, 1) = (2^{t-1}c_x + c_y, d_y)$. This is a fixed point of ι, so $d_y = 1$ follows by comparing the second coordinates. We must also have $c_y = 2^t c'_y$ for $c'_y = 0, 1$. If c'_y is 0, then $\iota(0, 1) = (0, 1)$, which is a contradiction as $(0, 1)$ should not be a fixed point of ι. Thus $c_y = 2^t$. Substituting this back in to $\iota(2^{t-1}, 1)$ we can see that $c_x = 3 \mod 4$, so $c = 4N - 1$ for some N.

To find d_x and to restrict N, we need to look at a norm condition. For d_x, consider

$$\langle (1, 0), (0, 1)\rangle = \langle \iota(1, 0), \iota(0, 1)\rangle.$$

Working this out yields $c_x = 1$. The condition on N comes from the equality $q(1, 0) = q(\iota(1, 0))$, which can easily be worked out as well.

The proof when $s \geq 3$ is essentially the same. □

A different parameterization of the primaries $\mathfrak{d}_j$ is now more natural. It is given by $\mathfrak{d}_\gamma = \mathfrak{d}_{-\gamma}$ for $\gamma \in \Gamma$, $\gamma \neq \alpha(g)$ for any $g \in G$. In the previous notation, we used $\tau \in G$; the new parameterization replaces τ_j by $\alpha^{-1}(-\gamma - \iota\gamma)$.

Conjecture 3.3.4. *Let G be a finite abelian group of even order n and let $G + n$ be an associated near group fusion category. Then its modular data is determined by some choice of compatible triple (H, q', ι).*

The S and T matrices are identical to (3.3.5) *and* (3.3.6)*, respectively, except for the lower right corners, which are*

$$T_{\gamma,\gamma} = q'(\gamma), \quad (\gamma \in \Gamma \setminus \alpha(G))$$
$$S_{\gamma,\gamma'} = \frac{-1}{\sqrt{m}} \left(\langle \gamma, \gamma' \rangle' + \langle \gamma, \iota\gamma' \rangle' \right), \quad (\gamma, \gamma' \in \Gamma \setminus \alpha(G)),$$

where $\langle , \rangle'$ is the symmetric pairing defined by

$$\langle \gamma, \gamma' \rangle' = q'(\gamma) q'(\gamma') \overline{q'(\gamma + \gamma')}.$$

In all cases, the modular data found has agreed with the even or odd conjecture respectively. The following table contains the group H and the quadratic form q'_H which works for each near group category. When a is necessary we also include it. We have not yet found the modular data for even n when $n \geq 18$, but it will be forthcoming.

Group	**id**	H	q'_H	a
$\mathbb{Z}_1$	J_1^1	$\mathbb{Z}_5$	$\xi_5^{2h^2}$	
$\mathbb{Z}_2$	J_2^1	$\mathbb{Z}_3$	$\xi_3^{h^2}$	
$\mathbb{Z}_3$	J_3^1	$\mathbb{Z}_7$	$\xi_7^{h^2}$	
$\mathbb{Z}_4$	J_4^1	$\{0\}$	1	-3
$\mathbb{Z}_5$	J_5^1	$\mathbb{Z}_3^2$	$\xi_3^{h^2+h'^2}$	
	J_5^2	$\mathbb{Z}_9$	$\xi_9^{2h^2}$	
$\mathbb{Z}_6$	J_6^1	$\mathbb{Z}_5$	$\xi_5^{2h^2}$	
	J_6^2	$\mathbb{Z}_5$	$\xi_5^{-h^2}$	
$\mathbb{Z}_7$	J_7^1	$\mathbb{Z}_{11}$	$\xi_{11}^{2h^2}$	
$\mathbb{Z}_8$	J_8^1	$\mathbb{Z}_3$	$\xi_3^{h^2}$	11
	J_8^2	$\mathbb{Z}_3$	$\xi_3^{h^2}$	1
$\mathbb{Z}_9$	J_9^1	$\mathbb{Z}_{13}$	$\xi_{13}^{2h^2}$	
$\mathbb{Z}_{10}$	J_{10}^1	$\mathbb{Z}_7$	$\xi_7^{-k^2}$	
	J_{10}^2	$\mathbb{Z}_7$	$\xi_7^{3k^2}$	

$\mathbb{Z}_{11}$	J_{11}^1	$\mathbb{Z}_{15}$	$\xi_{15}^{2h^2}$	
	J_{11}^2	$\mathbb{Z}_{15}$	$\xi_{15}^{h^2}$	
$\mathbb{Z}_{12}$	J_{12}^1	$\{0\}$	1	3
	J_{12}^2	$\{0\}$	1	-25
$\mathbb{Z}_{13}$	J_{13}^1	$\mathbb{Z}_{17}$	$\xi_{17}^{3h^2}$	
	J_{13}^2	$\mathbb{Z}_{17}$	$\xi_{17}^{3h^2}$	
$\mathbb{Z}_{14}$	J_{14}^1	$\mathbb{Z}_3^2$	$\xi_3^{h^2-h'^2}$	
	J_{14}^2	$\mathbb{Z}_9$	$\xi_9^{-2h^2}$	
	J_{14}^3	$\mathbb{Z}_9$	$\xi_9^{-h^2}$	
$\mathbb{Z}_{15}$	J_{15}^1	Z_{19}	$\xi_{19}^{6h^2}$	
	J_{15}^2	Z_{19}	$\xi_{19}^{6h^2}$	
$\mathbb{Z}_{16}$	J_{16}^1	$\mathbb{Z}_5$	$\xi_5^{2h^2}$	-11
	J_{16}^2	$\mathbb{Z}_5$	$\xi_5^{4h^2}$	-33
$\mathbb{Z}_{17}$	J_{17}^1	Z_{21}	$\xi_{21}^{2h^2}$	
	$J_{17}r$	Z_{21}	$\xi_{21}^{-4h^2}$	
$\mathbb{Z}_{21}$	J_{21}^1	$\mathbb{Z}_{25}$	$\xi_{25}^{h^2}$	
	J_{21}^2	$\mathbb{Z}_{25}$	$\xi_{25}^{2h^2}$	
	J_{21}^3	$\mathbb{Z}_{25}$	$\xi_{25}^{2h^2}$	
$\mathbb{Z}_{23}$	J_{23}^1	$\mathbb{Z}_3 \times \mathbb{Z}_9$	$\xi_9^{2h^2}\xi_3^{2h'^2}$	
	J_{23}^2	$\mathbb{Z}_{27}$	$\xi_{27}^{2h^2}$	
	J_{23}^3	$\mathbb{Z}_{27}$	$\xi_{27}^{h^2}$	
$\mathbb{Z}_{25}$	J_{25}^1	$\mathbb{Z}_{29}$	$\xi_{29}^{2h^2}$	

TABLE 3.2: Near Group Modular Data

All that remains is to show that the conjecture yields valid modular data, which is the content of the next proposition.

Proposition 3.3.5. *The S and T matrices defined by Conjecture 3.3.4 define modular data.*

Proof. The majority of the proof is very straightforward, albeit computationally heavy. We have to show that the four properties which define modular data are satisfied. T is clearly diagonal and of finite order, and S is symmetric. We can check unitarity in blocks, and because the only change is the bottom right corner, there are actually only 4 calculations that need to be done, corresponding to the rightmost column (or bottom row) of S. There is nothing for it but to go through them one by one. Starting with the top right block, it corresponds to the block product

$$S_{11}\overline{S_{14}} + S_{12}\overline{S_{24}} + 0 + S_{14}\overline{S_{44}},$$

where the indices denote which blocks we multiply, according to the 4$\times$ 4 pattern given in (3.3.6). The third summand contains multiplication with a 0 block which explains the presence of 0 in the above. For the remaining three terms, the majority of entries in the resulting matrices are 0. In each of the terms, the nonzero entries are identical, and can be computed easily by simplifying with the fact that $\sum_{j=0}^{n-1} \xi_n^j = 0$. Across the terms the non-zero entries occur at the same place. They work out to

$$-\frac{1}{(n+4)\delta} + \frac{\delta}{(n+4)(n)} - \frac{1}{n+4} = \frac{d^2 - nd - n}{n^2 d + 4nd^2} = 0,$$

as δ is a root of $x^2 - nx - n$. The next block down, $(S\overline{S})_{24}$ is identical, except the signs are switched. The $(S\overline{S})_{34}$ block is slightly different. Only the first two terms, $S_{31}\overline{S_{14}}$, $S_{32}\overline{S_{24}}$ have any non-zero entries. In the former case, they are all $-1/\sqrt{m}$, and the corresponding entries of the second term are $1/\sqrt{m}$. Finally is the bottom right corner. Three of the four terms have non-zero entries, as in the first case. The first two summands have identical entries and all non-zero entries are $1/(n+4)$. In the final term, not all entries are identical (as is expected). The entries along the main diagonal are $\frac{n+2}{n+4}$, as expected, and nonzero entries off the main diagonal correspond to those from the other two terms. They all work out to be $-2/(n+4)$, as needed.

That the topmost row of the S matrix consists of positive real numbers is obvious.

The next step is to check that $S^2 = (ST)^3$, or equivalently $\overline{T}S\overline{T} = STS$, however the computation is nearly identical to the one above, so we omit the details.

The final step is to check that Verlinde's formula holds. Since the only change to the matrix is in the lower right hand corner, we only need to check N_{ab}^c where Verlinde's formula includes entries from that lower right corner. This will happen whenever a, b or c is one of the $\mathfrak{d}_\gamma$ primaries. Since the fusion coefficients satisfy $N_{ab}^c = N_{ba}^c$, this cuts down on the number of cases to consider. There is nothing for it but to start computing. We will start with the case $c = \mathfrak{a}_g$, $g \in G$ and $b = \mathfrak{d}_\gamma$, $\gamma \in \Gamma \setminus \{\alpha(G)\}$.

We have

$$N_{\mathfrak{d}_\gamma \mathfrak{a}_h}^{\mathfrak{a}_g} = \sum_{d \in \Phi} \frac{S_{\mathfrak{d}_\gamma d} S_{\mathfrak{a}_h d} \overline{S_{\mathfrak{a}_g d}}}{S_{1d}} =$$

$$\sum_{\mathfrak{a}_l \in G} \frac{S_{\mathfrak{d}_\gamma \mathfrak{a}_l} S_{\mathfrak{a}_h \mathfrak{a}_l} \overline{S_{\mathfrak{a}_g \mathfrak{a}_l}}}{S_{1\mathfrak{a}_l}} + \sum_{\mathfrak{b}_l \in \Phi} \frac{S_{\mathfrak{d}_\gamma \mathfrak{b}_l} S_{\mathfrak{a}_h \mathfrak{b}_l} \overline{S_{\mathfrak{a}_g \mathfrak{b}_l}}}{S_{1\mathfrak{b}_l}}$$

$$+ \sum_{\mathfrak{c}_{l,k} \in \Phi} \frac{S_{\mathfrak{d}_\gamma \mathfrak{c}_{l,k}} S_{\mathfrak{a}_h \mathfrak{c}_{l,k}} \overline{S_{\mathfrak{a}_g \mathfrak{c}_{l,k}}}}{S_{1\mathfrak{c}_{l,k}}} + \sum_{\mathfrak{d}_\eta \in \Phi} \frac{S_{\mathfrak{d}_\gamma \mathfrak{d}_\eta} S_{\mathfrak{a}_h \mathfrak{d}_\eta} \overline{S_{\mathfrak{a}_g \mathfrak{d}_\eta}}}{S_{1\mathfrak{d}_\eta}}.$$

All of the $S_{\mathfrak{d}_\gamma \mathfrak{c}_{l,k}}$ entries are 0, so the third sum disappears. In what remains, there is a good deal of cancellation. Looking at the first two sums, we notice that $S_{\mathfrak{d}_\gamma} S_{\mathfrak{a}_l} = -S_{\mathfrak{d}_\gamma} S_{\mathfrak{b}_l}$ and $S_{\mathfrak{a}_h \mathfrak{a}_l} = S_{\mathfrak{a}_h \mathfrak{b}_l}/(\delta + 1)$. In all cases, the denominator is the coefficient of the S matrix entry, because the bilinear form will always by 1. We can therefore re-write the sum as

$$N_{\mathfrak{d}_\gamma \mathfrak{a}_h}^{\mathfrak{a}_g} = -\delta \sum_{\mathfrak{a}_l \in \Phi} S_{\mathfrak{d}_\gamma \mathfrak{a}_l} S_{\mathfrak{a}_h \mathfrak{a}_l} \overline{S_{\mathfrak{a}_g \mathfrak{a}_l}} + \frac{1}{\delta} \sum_{\mathfrak{d}_\eta \in \Phi} S_{\mathfrak{d}_\gamma \mathfrak{d}_\eta} S_{\mathfrak{a}_h \mathfrak{d}_\eta} \overline{S_{\mathfrak{a}_g \mathfrak{d}_\eta}}.$$

Every entry of the S matrix is a bilinear form, so we can split each sum into products of sums of roots of unity. By looking at the coefficients, it becomes obvious that the sums cancel, so any potential square roots are eliminated. The other computations are extremely similar. □

One potential value of these conjectures is that they suggest a possible direct construction of MTC's by using well understood ones as building blocks, somewhat akin to the semidirect product of groups.

Chapter 4

Haagerup-Izumi Fusion Categories

As alluded to earlier, there is a relationship between fusion categories and subfactors. In particular, any subfactor satisfying a certain finiteness condition yields two (possibly equivalent) fusion categories, which are generated by a simple object whose dimension δ is the square root of the index of the subfactor. Historically, subfactors were the primary subject of study, and so to motivate the existence of Haagerup-Izumi fusion categories, which were the first examples of exotic fusion categories, we need to digress slightly into the world of subfactors. We endeavor to keep this brief exposition non-technical.

The discovery of the Haagerup Izumi fusion categories were one result of a huge effort by many people to classify all subfactors of small index. The first work towards this goal was done by Jones, specifically the Jones index theorem, which states that the only possible indices < 4 are given by $4\cos^2(\pi/n)$ for $n \in \mathbb{N}$, and that there are subfactors with each of these dimensions. (See [18] for Jones' original statement and proof). Although there are infinitely many subfactors of index D with $4 < D < 5$, one can completely classify the ones satisfying a particular finiteness condition, and as it turns out there are only 10. For a review, see [17].

The construction of these new subfactors (and hence fusion categories) was arduous and an effort to simplify and possibly generalize it, led to the use of the Cuntz algebra construction by Izumi in [16]. In particular, Izumi found a new construction of the Haagerup subfactor (the smallest one of index > 4). This construction fit the Haagerup into a potentially infinite family of subfactors, although it was widely expected to terminate. A similar potential family, proposed by Haagerup, was shown to terminate, and the prevailing expectation was that exotic subfactors, like sporadic groups, were few and far between. However, Izumi showed that the next hypothetical subfactor did exist, and then Evans-Gannon found 7 more in [10]. That work provided evidence that Izumi's family of subfactors might indeed be infinite. In this chaper, we continue this investigation and find the next 11 inequivalent fusion categories, providing yet more evidence supporting the infinitude of the Haagerup-Izumi subfactors.

Returning to a categorical perspective, the family consists of fusion categories whose isomorphism classes of simple objects are $[g], g \in G$, where G is a finite abelian group of odd order ν, and $[g\rho], g \in G$. The fusion rules these objects satisfy are

$$\begin{aligned}
[g][h] &= [g+h], \\
[g][h\rho] &= [(g+h)\rho] = [h\rho][-g], \\
[g\rho][h\rho] &= [g-h]\sum_{l\in G}[l\rho]
\end{aligned}$$

for all $g, h \in G$.

It is worth noting that in the simplest case, where G is the trivial group, these fusion rules are the same as the near group rules, and that when $G = \mathbb{Z}_3$, these rules define the Haagerup fusion category.

By using the Cuntz algebra construction in an analogous way to the near group case, one can interpret these hypothetical fusion categories as endomorphisms which are defined by functions $A : G \times G \to \mathbb{C}$ satisfying certain conditions, given in the next section. In the same was as before, any solution to those equations then yields a fusion category whose simple objects obey the above fusion rules.

The previous work by Izumi and Evans-Gannon also included using the Tube algebra to find modular data for each of their new fusion categories in this family. Despite it's complicated appearance, a conjecture made by Evans-Gannon in [10] suggests that it has a much simpler form, which is still mysterious.

Our purpose in this section is to continue this classification effort and lend more evidence to the conjectured form of the modular data. We accomplish this by using slightly faster techniques to push the analysis further. The same technique to prove completeness we outlined in the previous section also works for the new categories, although it has not been done here yet.

For the remainder of this chapter, let G be a group of odd order $\nu = 2n+1$, $n \in \mathbb{N}$, let $\delta = \frac{n+\sqrt{n^2+4n}}{2}$, $\mu = \nu^2 + 4$, $m = (\nu^2+3)/2$, and $\lambda = 2\nu + \nu^2\delta$

4.1 Numerical Data

The equations to be solved are given by the following theorem, the proof of which is in [16].

Theorem 4.1.1. *Let G be a finite abelian group with $|G| = \nu = 2n+1$ for some $n \in \mathbb{N}$. Then for every function $A : G \times G \to \mathbb{C}$ satisfying*

$$A(x,y) = \overline{A(y,x)}, \tag{4.1.1}$$

$$A(x,0) = \delta_{x,0} - \frac{1}{\delta - 1}, \tag{4.1.2}$$

$$A(x,y) = A(-y,x-y) = A(y-x,-x), \tag{4.1.3}$$

$$\sum_{v \in G} A(x+v,y)\overline{A(v,y)} = \delta_{x,0} - \frac{\delta_{y,0}}{\delta}, \tag{4.1.4}$$

$$\sum_{v \in G} A(v,x+y)\overline{A(v+z,x)A(v+w,y)} = A(x+w,z)A(y+z,w) - \frac{\delta_{x,0}\delta_{y,0}}{\delta}, \tag{4.1.5}$$

for all $w,x,y,z \in G$, there is an associated fusion category.

An equivalent interpretation of the above theorem is as a matrix A with entries $A_{x,y} = A(x,y)$.

Evidently, solving these equations quickly becomes impossible by hand. Even asking a computer to solve them for exact values, or without some simplification is unfeasible after a small number of cases. Obviously, (4.1.1) implies that only half the matrix entries need to be solved for. The first row/column as well as the diagonal are also easily handled by (4.1.2) and (4.1.3) when $x = y$. When $x \neq y$, applying (4.1.3) further reduces the number of independent variables, but the number still grows rapidly as G grows in size.

Izumi shows that (4.1.1)-(4.1.4) imply two more key equations. The equations are

$$A(x+y,z)A(z,y) = A(-y,x)A(x,z-y), \tag{4.1.6}$$
$$|A(x,y)| = \sqrt{\delta}, \tag{4.1.7}$$

which hold whenever $x \neq 0, y \neq 0, x \neq y$. By applying (4.1.6) iteratively, we can reduce the number of independent variables to only $(\nu - 1)/2$. At this point, there is little to do but solve the remaining equations (4.1.4)-(4.1.5) by computer. As long as ν is relatively small, doing so numerically with Grobner basis techniques is fast enough, but even that rapidly becomes too slow.

To remedy that, we exploited the fact that Mathematica can quickly find a single solution to an equation if given a near enough guess. Since every solution occurs on the same circle in $\mathbb{C}$, we found that it was very effective to choose $(\nu - 1)/2$ points on the circle $|z| = \sqrt{\delta}$ randomly as a starting guess, and then find a solution to a random subset of equations (4.1.4)-(4.1.5). It is simple to check if a given solution holds for all the equations – if it does, then it must define a valid fusion category, and if not, another guess was made.

The downside to this method is that there is no guarantee that we have found every fusion category for a given G, as it only finds one solution per iteration. Using techniques similar to those discussed in Chapter 3 does allow proof of completeness, although for these categories we have not checked.

In general there are quite a few possible Haagerup-Izumi categories for a given group G, however many of them will be equivalent. Roughly speaking, equivalent categories are ones whose A matrices are permutations of each other. Explicitly,two categories with matrices A, A' are equivalent as fusion categories if there is $\alpha \in Aut(G)$ such that $A'(g,h) = A(\alpha g, \alpha h)$ for all $g, h \in G$. Given any category in one of these equivalence classes, it is easy to recover all others. Therefore we only include the data for one representative in each equivalence class.

There is a convenient way to encode the data which we will make use of. Since all the norms are known, it is convenient to write the solutions in exponential form. Equation (4.1.6) implies that there are $j_2, ..., j_{n+1} \in \mathbb{R}$ so that whenever $0 < g < h < \nu$, we have

$$\begin{aligned} A(g,h) &= \frac{\sqrt{\delta}}{\delta - 1}\exp(i(j_h - j_g - j_{h-g}) \\ j_1 &= 0 \\ j_{n+1+i} &= j_{n+1} + j_n + j_{n-i}, \quad (i = 1, ..., n) \end{aligned}$$

So it is sufficient to record only the real values $j_2, ..., j_{n+1}$. Let J_n be the vector of these values for $\mathbb{Z}_n$.

For groups of order up to and including 19, the data is already known (See [16], [10]), so here we only include the new data. For $n = 21, 23, ..., 29$, the inequivalent Haagerup Izumi fusion categories are defined by the following data:

J_3	(1.292076)
J_5	$(0.1846862, 1.5984702)$
J_7	$(2.471228, 0.51685555, 0.2137724)$
J_9	$(2.396976693, 2.079251103, -0.2079168419, -2.508673987)$
J_9'	$(-2.364737070, 1.031057162, 1.569692175, 0.3383837765)$
J_{11}	$(0.9996507, 2.7258434, -0.5714203, -1.7797340, 1.2675985)$
J_{11}'	$(-2.6444397, -1.7629598, -2.6444440, 2.7572657, 0.1128260)$
J_{13}	$(-3.1050384, 0.5993399, -0.111708,$ $-0.969766, 1.336848, 1.00483129)$
J_{15}	$(-1.0777623, -.7748018, -2.171863, -1.6068402,$ $-.257508, 2.092502, .72289565)$
J_{17}	$(-1.466074, .291489, 3.130735, -2.693185,$ $1.398153, -.611938, -1.667078, -1.754821)$
J_{19}	$(-2.677465, 1.088972, -.899442, .015448, -1.240928,$ $-.493394, 1.839879, -1.525884, -2.084374)$
J_{19}'	$(.896858, -.882585, -2.369855, -1.873294, -1.711620,$ $-.119360, 2.972018, -2.460652, .041334)$

J_{21}	$(0.25589, 2.91947, 2.95031, 1.34158, -0.00657636,$ $1.09014, 0.995409, -1.43622, -3.87381, -6.21603)$
J'_{21}	$(-1.14961, -0.178414, -0.154739, -0.514173, -3.05836,$ $-2.62586, -4.39012, -5.88175, -3.47616, -0.929352)$
J_{23}	$(0.563764, 0.594463, -1.12818, -0.851944, 1.58917, 1.71948,$ $2.46679, 1.88085, 4.76295, 7.7649, 6.02719)$
J'_{23}	$(-2.23747, -3.37377, -5.69085, -6.36094, -4.47943, -3.86521,$ $-3.64149, -6.19999, -4.63387, -5.21694, -6.54144)$
J_{25}	$(-0.512715, -0.641389, -0.589682, 0.488945, -2.54182, -5.31712,$ $-4.83123, -7.63854, -5.66053, -6.35959, -4.72535, -3.70291)$
J'_{25}	$(1.38074, 4.37021, 2.42757, 4.15235, 2.43798, 5.34476,$ $2.52854, 3.17897, 4.43158, 6.18022, 5.84135, 4.62718)$
J_{27}	$(-2.40925, -0.856315, -3.25019, -3.146, -0.992676, -2.73419, -5.34819,$ $-3.31672, -1.22844, -2.07258, -2.88104, -3.97986, -5.94286)$
J'_{27}	$(-3.04159, -1.73766, -3.8587, -6.47742, -3.55205, -2.79902, -2.76745,$ $-3.5036, -1.35632, -2.32267, -0.236228, 1.8383, 2.92363)$
J''_{27}	$(1.42374, 2.46901, 5.48056, 7.80536, 10.8124, 8.50623, 8.33284,$ $7.82228, 10.9555, 12.2511, 11.6376, 13.2468, 11.9087)$
J_{29}	$(-1.18059, -2.08283, -4.94069, -4.67359, -5.71711, -8.06591, -6.40435,$ $-4.1734, -3.39334, -2.56513, -2.82481, -4.5671, -3.34917, -4.91401)$
J_{29}	$(-1.48096, -1.40231, 1.01155, -2.61335, 0.81622, 2.00311, 1.38423,$ $2.69419, -1.27127, -0.461116, 1.49623, 0.390362, 0.604392, -2.95981)$

TABLE 4.1: Haagerup-Izumi Fusion Categories

4.2 Modular Data

For relevant details including the explicit tube algebra construction, see [16]. The process to find the modular data for the Haagerup-Izumi categories is similar to the one to find the Near Group categories. Here we are solving for functions

$C : G \times G \to \mathbb{C}$ and roots of unity ω which satisfy the following equations:

$$\sum_{k \in G} C(0,k) = \omega - \overline{\omega}\delta^{-1}, \tag{4.2.1}$$

$$\omega C(g,h) - \sum_{k \in G} A(g+k, 2h)C(h,k) = \delta_{h,0}\overline{\omega}\delta^{-1}, \tag{4.2.2}$$

$$\begin{aligned} \omega C(p,s)C(h,r)\delta = {} & \delta_{s,h}\delta r, p + \overline{\omega}A(p+h,2s)\delta_{r,s} \\ & + \delta \sum_{k,l \in G} C(k,l)A(h+l-s, r+k-s)A(r-k-s, l-k-s+p) \\ & \qquad A(h+p-k, r+s-k), \end{aligned} \tag{4.2.3}$$

$$\begin{aligned} 0 = {} & 1 + \overline{\phi(g)}\omega \\ & + \delta\omega \sum_{t,k,l \in G} \phi(l)C(k,t)A(t+l+g, k+l)A(t-l-g, k-l). \end{aligned} \tag{4.2.4}$$

$$\frac{\lambda}{\nu} = 1 + \overline{\omega_j}\omega_{j'} + \delta\omega_j \sum_{k,q \in G} C^{j'}(k,l)C^j(k,l) \tag{4.2.5}$$

These equations must hold for all $g, h, p, s, r \in G$ and for all $\phi \in \hat{G}$. For each category we will find exactly m different values for ω and each one will have a corresponding function C. This is guaranteed by the double construction. We denote these by ω_j and C^j respectively, and (4.2.5) is the only equation which involved more than one value of ω. Where only one value is needed, we don't include the additional notation for readability.

Equations (4.2.1) and (4.2.2) can be solved easily by computer as they are linear, and they will often determine ω_j and C^j completely. The remaining equations will be needed if there are still infinitely many solutions after the linear ones are solved. In a typical example, there will be between 0 and 2 free parameters after solving (4.2.1) and (4.2.2), so it is generally sufficient to choose only a few of the supplementary, more complicated equations to check. Of course it is trivial to ensure that the numerical solutions satisfy all necessary equations once they are found. For the proof that these equations will always completely determine the modular data, see [7], Proposition 2.

As in the near group case, we can use these values to construct the T and S matrices. Both matrices are split into 16 blocks according to the combinations of the 4 different types of primaries. Sticking with the notation of [7], the primaries are

- $\mathfrak{0}$ and $\mathfrak{b}$
- $\mathfrak{a}_\psi = \mathfrak{a}_{\overline{\psi}}$, for $\psi \in \hat{G}, \psi \neq 1$
- $\mathfrak{c}_{h,\phi} = \mathfrak{c}_{-h,\phi}$, for $h \in G, h \neq 0, \phi \in \hat{G}$.
- $\mathfrak{d}_l$, for $l = 1, ..., m$.

There are respectively $2, n, n\nu, m$ of each type of primary. Then the matrices are given in block form by

$$T = \text{diag}(1_2; 1_n; \phi(h); \omega_j) \tag{4.2.6}$$

$$S = \frac{1}{\nu}\begin{pmatrix} B & 1_{2\times n} & 1_{2\times n\nu} & C \\ 1_{n\times 2} & 2_{n\times n} & D & 0_{n\times m} \\ 1_{n\nu\times 2} & D^t & E & 0_{n\nu\times m} \\ C^t & 0_{m\times n} & 0_{m\times n\nu} & F \end{pmatrix}. \tag{4.2.7}$$

Whenever a constant is subscripted, that is a block of the size prescribed by the subscript consisting only of that constant. The remaining blocks are as follows:

$$B = \frac{1}{2}\begin{pmatrix} 1 - \frac{\nu}{\sqrt{\mu}} & 1 + \frac{\nu}{\sqrt{\mu}} \\ 1 + \frac{\nu}{\sqrt{\mu}} & 1 - \frac{\nu}{\sqrt{\mu}} \end{pmatrix}, \tag{4.2.8}$$

$$C = \frac{\nu}{\sqrt{\mu}}\begin{pmatrix} 1_{1\times m} \\ -1_{1\times m} \end{pmatrix}, \tag{4.2.9}$$

$$D_{\psi,(h,\phi)} = \psi(h) + \overline{\psi(h)}, \tag{4.2.10}$$

$$E_{(h,\phi),(h',\phi')} = \phi'(h)\phi(h') + \overline{\phi'(h)\phi(h')}, \tag{4.2.11}$$

$$F_{j,j'} = \frac{\nu}{\lambda}\left(\omega_j\omega_{j'} + \delta\sum_{g,p\in G}\overline{C^j(-g,p)C^{j'}g, p+g}\right). \tag{4.2.12}$$

Again the bizarre entries are those in the lower right corner.

As in the near group case, there is numerical evidence suggesting a simple formula for the ω_j and the lower right corner of the S matrix.

Conjecture 4.2.1. *Let G be a group of odd order ν and $\mathcal{C}$ any associated fusion category as defined above. Then there is another abelian group H of order $\nu^2 + 4$ together with a nondegenerate symmetric pairing $\langle \cdot, \cdot \rangle : H \times H \to \mathbb{T}$ such that*

- *The lower corners of S and T are parameterized by $m = (\nu^2 + 4)/2$ elements $h \in H, h \neq 0, h = -h$,*
- *$\omega_h = \exp(2\pi i m \langle h, h \rangle)$,*
- *$S_{h,h'} = -\frac{2}{\sqrt{\lambda}} \cos(2\pi \langle h, h' \rangle)$.*

This conjecture was first stated in [7] and we have found no improvements or refinements. However all of our numerical data does support it. Specifically, the following symmetric pairings generate modular data which is numerically close to the data found directly from the complicated equation (4.2.12).

J_7	$\langle l, l' \rangle = ll'/53$
J_9	$\langle l, l' \rangle = ll'/85$
J_9'	$\langle l, l' \rangle = 12ll'/85$
J_{11}	$\langle l, l' \rangle = ll'/125$
J_{11}'	$\langle (l_1, l_2), (l_1', l_2') \rangle = 2l_1 l_1'/25 + 2l_2 l_2'/5$
J_{13}	$\langle l, l' \rangle = ll'/173$
J_{15}	$\langle l, l' \rangle = ll'/229$
J_{17}	$\langle l, l' \rangle = ll'/293$
J_{19}	$\langle l, l' \rangle = ll'/365$
J_{19}'	$\langle l, l' \rangle = 22ll'/365$
J_{21}	$\langle l, l' \rangle = 3ll'/445$
J_{21}'	$\langle l, l' \rangle = ll'/445$
J_{23}	$\langle l, l' \rangle = 6ll'/533$
J_{23}'	$\langle l, l' \rangle = ll'/533$
J_{25}	$\langle l, l' \rangle = 5ll'/629$
J_{25}'	$\langle l, l' \rangle = ll'/629$
J_{27}	$\langle l, l' \rangle = 2ll'/733$
J_{27}'	$\langle l, l' \rangle = 2ll'/733$

J''_{27}	$\langle l, l' \rangle = 2ll'/733$

TABLE 4.2: Haagerup-Izumi Modular Data

The only cases where there was ambiguity about the group H was when $n = 11, 29$. T he $n = 11$ case was done in [7], as were all cases $n < 21$. When $n = 29$, preliminary checks revealed that in both cases, $H \neq \mathbb{Z}_{845}$. We have not yet checked the other possibility for H, $H = \mathbb{Z}_5 \times \mathbb{Z}_{13} \times \mathbb{Z}_{13}$ but will do so soon.

The most interesting observations regarding the modular data and the fusion categories themselves are the following: First, as was predicted in [10], the modular data is simpler when μ is prime. When it is, the linear equations are enough to determine it without ambiguity, but when μ is composite the situation is more complicated and requires the additional equations. This is, in part, because when μ is composite, there can be more than one root of unity ω which satisfies the same linear equation and this introduces the ambiguity.

The second interesting result is related to the $G = \mathbb{Z}_{27}$ fusion categories and modular data. In particular, it is noteworthy since we have found 3 inequivalent fusion categories, which is more than any other case. However, their modular data is identical. Numerically, the C^j solutions are different, however for all three equivalence classes of fusion category, they generate the same S matrices, and the T matrix is identical in all cases. This was unexpected, as although modular data is not a complete invariant, finding inequivalent fusion categories which share the same modular data is not in general easy. Having found three inequivalent fusion categories is therefore remarkable. We currently have no compelling reason explaining this result, however it is an interesting question to address in the future.

Chapter 5

Future Work

Our key results were the discovery of new fusion categories in two potentially infinite families and the apparently simple forms their modular data takes. The most natural question, therefore, is whether or not these classes of fusion categories are truly infinite.Our data suggests that the existence of these fusion categories is not just an accident of small numbers, but rather that they are part of a larger family, but this is of course not a proof.

A second natural question concerns the modular data. In particular, it is not at all clear why certain symmetric pairings generate modular data corresponding to valid fusion categories and others do not. It seems possible that there is a related class of fusion category which would account for this apparent surplus of modular data. One potential source would be the non-unitary analogues of the fusion categories found here. Some exploration into those has been done in [7] but more is needed. It would also be desirable to predict which symmetric pairing belongs to which fusion category without having to first compute the modular data directly from the tube algebra construction. At present, there is no apparent pattern, so the use of our conjectures is limited by the fact that the longer, more computationally heavy approach is still needed to find the modular data numerically, at which point it is possible to compare all possible conjectured forms to find a match.

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