University of Alberta

POLYNOMIAL REPRESENTATIONS OF THE GENERAL LINEAR, SYMPLECTIC, AND QUANTUM GENERAL LINEAR GROUPS

by

Anna Stokke ©

A thesis submitted to the Faculty of Graduate Studies and Research
in partial fulfillment of the requirements for the degree of
Doctor of Philosophy

in

Mathematics

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Anna Stokke

Department of Mathematical and Statistical Sciences
University of Alberta
Edmonton, Alberta
Canada, T6G 2G1

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UNIVERSITY OF ALBERTA

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research for acceptance, a thesis entitled **Polynomial Representations of the General Linear, Symplectic, and Quantum General Linear Groups** submitted by **Anna Stokke** in partial fulfillment of the requirements for the degree of **Doctor of Philosophy** in Mathematics.

Bruce N Allison

Dr. Bruce Allison (Chair)

Gerald Cliff

Dr. Gerald Cliff (Supervisor)

Dr. Terry Gannon

Hans H. Brungs

Dr. Hans Brungs

Joseph Culberson

Dr. Joe Culberson

Bruce N Allison for A. Kleshchev

Dr. Alexander Kleshchev (University of Oregon)

November 26, 2002

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To my precious daughter Ava

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ABSTRACT

The polynomial representations of the general linear, symplectic, and quantum general linear groups are studied. Three methods which yield spanning sets for the irreducible polynomial $GL(n, K)$-module $L(\lambda)$ are discussed. It is shown that the spanning sets given by the first two methods are the same, up to sign, and are related to the third spanning set by the Désarménien matrix.

A symplectic version of the Désarménien matrix is defined and it is shown that this matrix gives a symplectic straightening algorithm. A standard basis is given for the symplectic Weyl module and as a corollary to this basis theorem, a spanning set for the irreducible polynomial $Sp(2m, K)$-module, $\overline{L(\lambda)}$, is obtained. The basis given for the symplectic Weyl module is shown to be related to the basis for the symplectic Schur module by the symplectic Désarménien matrix.

Quantum analogues of these results are also given. In particular, a quantized version of the Désarménien matrix is defined which proves to give a quantum straightening algorithm. A new proof of the standard basis theorem for the q-Weyl module is given and it is shown that the standard basis for the q-Weyl module is related to the basis for the q-Schur module via the quantized Désarménien matrix.

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Chapter 1

Introduction

Polynomial representations of $GL(n, \mathbb{C})$ were first studied by Isaai Schur in his doctoral thesis of 1901, [S]. There he proved that the irreducible polynomial representations of $GL(n, \mathbb{C})$ of homogeneity r are in one-to-one correspondence with the partitions of r into at most n parts. Many others have worked to give characteristic-free versions of his results. One of the first major works in this direction was that of Carter and Lusztig [CL]. There they work over an infinite field K and use the hyperalgebra for $GL(n, K)$ to construct for each partition λ of $r \geq 0$ a $GL(n, K)$-module $\Delta(\lambda)$, called the Weyl module, which is a subspace of tensor space. Each Weyl module $\Delta(\lambda)$ possesses a unique maximal submodule M so the quotient module $L(\lambda) = \Delta(\lambda)/M$ is irreducible. Moreover, the set of $GL(n, K)$-modules $L(\lambda)$, where λ is a partition of r into not more than n parts, forms a complete list of non-isomorphic irreducible polynomial $GL(n, K)$-modules. When K has characteristic zero, $\Delta(\lambda)$ and $L(\lambda)$ are isomorphic and Weyl's dimension formula ([H], §24.3) gives the dimension of $\Delta(\lambda)$. When K has non-zero characteristic the dimensions of the irreducible $GL(n, K)$-modules $L(\lambda)$ are in general not known.

In the celebrated work of Green [G], Green maps out the entire polynomial representation theory of $GL(n, K)$ without the use of the hyperalgebra. There he replaces Carter and Lusztig's hyperalgebra with certain finite dimensional K-algebras called Schur algebras. The Schur algebras may be used to study polynomial representations

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of $GL(n, K)$, where K is any infinite field, from a combinatorial perspective.

Throughout this thesis, we assume that K is an infinite field of arbitrary characteristic. We begin with a preliminary chapter, which follows [G], where we discuss the theory of Young tableaux and polynomial representations of $GL(n, K)$. In Chapter 3 we address the problem of finding the dimensions of the elusive irreducible $GL(n, K)$-modules $L(\lambda)$. We give three different methods which yield spanning sets for $L(\lambda)$. We prove that the first two spanning sets coincide up to sign and show that they are related to the third via an upper triangular, unimodular matrix known as the Désarménien matrix. Various versions of this matrix play a significant role throughout this dissertation.

From there we shift our attention to the symplectic group, $Sp(2m, K)$. Using the hyperalgebra for $GL(n, K)$ and semistandard Young tableaux of shape λ, Carter and Lusztig proved the standard basis theorem for the Weyl module, $\Delta(\lambda)$, in [CL]. There are also Weyl modules for the symplectic group $Sp(2m, K)$, but a symplectic version of the standard basis theorem has not been given. In Chapter 4 we provide a standard basis theorem for the symplectic Weyl module using the symplectic tableaux of R. C. King [K] and operators in the hyperalgebra for $Sp(2m, K)$.

To accomplish our goal, we develop a symplectic version of the Désarménien matrix. The Désarménien matrix is defined in [De] and [DKR] using what they call Capelli operators, which can be viewed as elements of the hyperalgebra for $GL(n, K)$. This matrix is interesting in its own right and may be looked at from three different viewpoints. It is a combinatorial object in that the entries of the matrix may be determined in a purely combinatorial manner, it provides a straightening algorithm for writing a given bideterminant as a linear combination of bideterminants given by semistandard λ-tableaux, and it provides the connection between the Carter-Lusztig basis for the Weyl module, $\Delta(\lambda)$, and the basis of bideterminants for the Schur module, $\nabla(\lambda)$. We prove that our matrix is upper triangular and unimodular, and that it gives a straightening algorithm for bideterminants in the symplectic Schur module. We proceed to employ this matrix to prove our symplectic standard basis

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theorem and our proof shows, as in the original setting, that the basis we give and the basis of semistandard symplectic bideterminants for the symplectic Schur module are connected via our matrix. We close the chapter by giving a spanning set for the irreducible polynomial representation, $\overline{L(\lambda)}$, of $Sp(2m, K)$.

In Chapter 5, we work in the quantized version of the hyperalgebra for $GL(n, K)$. Given an indeterminate q, we have the $\mathbb{C}(q)$-algebra $U_{\mathbb{C}(q)}$, called the quantized universal enveloping algebra. Let $\mathcal{A}$ denote $\mathbb{Z}[q, q^{-1}]$, the ring of Laurent polynomials in q. There is an integral form $U_{\mathcal{A}}$ for $U_{\mathbb{C}(q)}$, where $U_{\mathcal{A}}$ is an $\mathcal{A}$-subalgebra of $U_{\mathbb{C}(q)}$. The quantum hyperalgebra, U_q, is defined by base change; $U_q = U_{\mathcal{A}} \otimes_{\mathcal{A}} K$. This is an interesting object of study and when one sets $q = 1$, the classical theory is recovered. Thus, results about the classical hyperalgebra for $GL(n, K)$ are special cases of results about the quantized hyperalgebra. Even more interesting is the case where q is taken to be a pth root of unity in K, a field of characteristic zero. In this case, the representation theory of U_q bears a striking resemblance to the representation theory of the classical hyperalgebra for $GL(n, F)$ when F has characteristic p. Thus, it is natural to see if our results in the previous chapters quantize.

We begin by defining a quantized version of the Désarménien matrix. Our matrix is an upper triangular, invertible matrix and gives a straightening algorithm for quantum bideterminants in the q-Schur module. We also use this matrix to prove the standard basis theorem for the q-Weyl module. This is not the first proof of this theorem; it was proved previously by Dipper and James [DJ] and by R. Green [Gr]. Our proof is substantially different from the former two and as a consequence of our proof, we see that the basis of semistandard quantum bideterminants for the q-Schur module is related to the standard basis for the q-Weyl module by our quantized Désarménien matrix. We conclude the dissertation with a discussion of the spanning set for the irreducible U_q-module $L_q(\lambda)$ that arises as a consequence of the standard basis theorem.

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Chapter 2

Preliminaries

2.1 Young tableaux

Throughout the thesis we take K to be an infinite field and n and r fixed positive integers. If m is a positive integer, let $\mathbf{m} = \{1, \ldots, m\}$. Most of this chapter follows Green [G].

A *partition* of r is a k-tuple $\lambda = (\lambda_1, \ldots, \lambda_k)$ where $\lambda_i \in \mathbb{N}$, $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k$, and $\sum_{i=1}^{k} \lambda_i = r$. Throughout the chapter we work with a fixed partition λ of r.

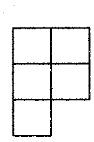
The *Young diagram* of shape λ is the set

$$[\lambda] = \{(i, j) : i \geq 1, 1 \leq j \leq \lambda_i\}.$$

The Young diagram of shape λ is usually depicted in the plane by an arrangement of r boxes in k left-justified rows with the ith row consisting of λ_i boxes.

Example 2.1

If $r = 5$, then $\lambda = (2, 2, 1)$ is a partition of r and the Young diagram of shape λ is


.

The *conjugate* of λ is the s-tuple $\mu = (\mu_1, \mu_2, \ldots, \mu_s)$ where μ_i is the length of the ith column of the Young diagram of shape λ. As can been seen by examining Example 2.1, the conjugate of the partition $\lambda = (2, 2, 1)$ is $\mu = (3, 2)$.

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A λ-*tableau* is a map $T : [\lambda] \to \mathbf{n}$ depicted in the plane by filling the boxes of the Young diagram of shape λ with numbers from the set $\mathbf{n}$. The map T is not necessarily bijective.

Example 2.2

If $\lambda = (2, 2, 1)$, the following are λ-tableaux:

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & 4 \\ \hline 3 \\ \cline{1-1}\end{array}, \quad \begin{array}{|c|c|}\hline 3 & 1 \\ \hline 4 & 3 \\ \hline 6 \\ \cline{1-1}\end{array}.$$

The *content* of a λ-tableau T is the n-tuple $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ where α_i is the number of entries equal to i in T. In Example 2.2, if we let $n = 6$, the first tableau has content $\alpha = (2, 1, 1, 1, 0, 0)$ and the second has content $\alpha = (1, 0, 2, 1, 0, 1)$.

Definition 2.1 *A λ-tableau is called semistandard if the entries in each row increase weakly from left to right and the entries in each column increase strictly from top to bottom.*

In Example 2.2, the first tableau is semistandard while the second is not.

A *basic* λ-*tableau* is a bijection $\widehat{T} : [\lambda] \to \mathbf{r}$. There are many choices for $\widehat{T}$; in this work we shall always take $\widehat{T}$ to be the λ-tableau which is obtained by filling the Young diagram of shape λ with the numbers $1, \ldots, r$ canonically across the rows. For a given partition λ of r, we denote this basic λ-tableau by $\widehat{T}_\lambda$.

Let $I(n, r)$ denote the set of r-tuples $I = (i_1, \ldots, i_r)$ with $i_\rho \in \mathbf{n}$ for $1 \leq \rho \leq r$. Each $I \in I(n, r)$ is in fact a function $I : \mathbf{r} \to \mathbf{n}$. Given a λ-tableau T, we will sometimes associate a unique $I \in I(n, r)$ with T where $T = I \circ \widehat{T}_\lambda$. We denote T by T_I in this case. Thus, T_I is the λ-tableau that arises from filling the Young diagram of shape λ canonically across the rows with the numbers in I.

Example 2.3

If $I = (1, 3, 4, 2, 5)$, then $T_I = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 4 & 2 \\ \hline 5 \\ \cline{1-1}\end{array}$.

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Let S_r denote the symmetric group on r letters. We define a right action of S_r on $I(n,r)$ by

$$I \cdot \sigma = (i_{\sigma 1}, \ldots, i_{\sigma r}). \tag{2.1}$$

Then S_r acts on the set of λ-tableaux with entries from $\mathbf{n}$ by $T_I\sigma = T_{I\cdot\sigma}$. The *column stabilizer* of $\widehat{T}_\lambda$, denoted $C(\widehat{T}_\lambda)$, is the set of $\sigma \in S_r$ which preserve the columns of $\widehat{T}_\lambda$ under this action and the *signed column sum* for $\widehat{T}_\lambda$ is the sum

$$\{C(\widehat{T}_\lambda)\} = \sum_{\sigma \in C(\widehat{T}_\lambda)} sgn(\sigma)\sigma.$$

Example 2.4

Let $\lambda = (2,2)$. Then $\widehat{T}_\lambda = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array}$ and $C(\widehat{T}_\lambda) = \{(1),(13),(24),(13)(24)\}$.

2.2 The Schur algebra and polynomial $GL(n,K)$-modules

Let $1 \leq i,j \leq n$ and define $x_{ij} : GL(n,K) \to K$ to be the function which associates each matrix $g = (g_{ij})_{1\leq i,j\leq n}$ with its ijth entry g_{ij}. The function x_{ij} is called the ijth *coordinate function* of $GL(n,K)$.

Definition 2.2 *Define $A(n)$ to be the K-algebra generated by the n^2 coordinate functions x_{ij}, $1 \leq i,j \leq n$.*

Since K is infinite, we may regard $A(n)$ as the algebra of all polynomials over K in the n^2 indeterminates x_{ij}, $1 \leq i,j \leq n$. Let $A(n,r)$ be the subspace of $A(n)$ consisting of polynomials in the x_{ij} which are homogeneous of degree r. The K-algebra $A(n)$ has grading

$$A(n) = \oplus_{r\geq 0} A(n,r)$$

where $A(n,1) = K1$.

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Given $I = (i_1, \dots, i_r)$ and $J = (j_1, \dots, j_r)$ in $I(n,r)$, we let $x_{I,J}$ denote the element $x_{i_1 j_1} x_{i_2 j_2} \cdots x_{i_r j_r}$ in $A(n,r)$. The symmetric group S_r acts on $I(n,r) \times I(n,r)$ by $(I,J) \cdot \sigma = (I \cdot \sigma, J \cdot \sigma)$ where $I \cdot \sigma$ is the right action defined in (2.1). Given $I, J, I', J' \in I(n,r)$, we write $I \sim J$ if there exists a permutation $\sigma \in S_r$ with $J = I \cdot \sigma$ and $(I,J) \sim (I',J')$ if there exists a permutation $\sigma \in S_r$ with $(I',J') \cdot \sigma = (I,J)$. Let Γ be a set of representatives of the S_r-orbits of $I(n,r) \times I(n,r)$. Clearly $x_{I,J} = x_{I',J'}$ if and only if $(I,J) \sim (I',J')$ and the set

$$\{x_{I,J} : I, J \in \Gamma\}$$

is a K-basis of $A(n,r)$.

The K-algebra $A(n)$ is a coalgebra with comultiplication $\delta : A(n) \to A(n) \otimes_K A(n)$ given by

$$\delta(x_{ij}) = \sum_{\rho=1}^{n} x_{i\rho} \otimes x_{\rho j},$$

and counit $\varepsilon : A(n) \to K$ given by

$$\varepsilon(x_{ij}) = \delta_{ij}$$

and $A(n,r)$ is a sub-coalgebra of $A(n)$. Since the dual of a coalgebra is an associative algebra, the dual of $A(n,r)$ is a finite dimensional associative K-algebra.

Definition 2.3 *The Schur algebra $S(n,r)$ is the dual space of $A(n,r)$;*

$$S(n,r) = (A(n,r))^* = Hom_K(A(n,r), K).$$

Multiplication in $S(n,r)$ is given by

$$\xi \cdot \eta(x_{I,J}) = \sum_{M \in I(n,r)} \xi(x_{I,M}) \eta(x_{M,J}), \ \xi, \eta \in S(n,r), \ I, J \in I(n,r).$$

For $I, J \in I(n,r)$, define

$$\xi_{I,J}(x_{I',J'}) = \begin{cases} 1 & \text{if } (I,J) \sim (I',J') \\ 0 & \text{otherwise} \end{cases}.$$

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Then the set

$$\{\xi_{I,J} : (I,J) \in \Gamma\} \tag{2.2}$$

is the dual basis for $S(n,r)$.

We now discuss some of the important $GL(n,K)$-modules. All of the modules in this thesis are assumed to be finite dimensional.

Definition 2.4 *A $GL(n,K)$-module V with basis $v_1,\ldots,v_n$ is said to afford a polynomial representation of $GL(n,K)$ if for each $g \in GL(n,K)$,*

$$gv_j = \sum_{i=1}^{m} c_{ij}(g)v_i, \text{ where each } c_{ij}(g) \in A(n). \tag{2.3}$$

We say that V is a *polynomial module of degree r* if each $c_{ij}(g)$ in (2.3) is in $A(n,r)$. Let $M(n)$ denote the category of left polynomial $GL(n,K)$-modules and $M(n,r)$ the category of left polynomial $GL(n,K)$-modules of degree r. The following theorem ([G], 2.2c.) shows that it is enough to study polynomial $GL(n,K)$-modules of degree r.

Theorem 2.1 *Let V be a polynomial $GL(n,K)$-module with $V \in M(n)$. Then*

$$V = \oplus_{r\geq 0} V_r$$

where each submodule V_r of V belongs to $M(n,r)$.

Given $A \in GL(n,K)$, define $e_A \in S(n,r)$ by $e_A(P) = P(A)$ where $P \in A(n,r)$. One can extend the map $A \to e_A$ linearly to get a map $e : KGL(n,K) \to S(n,r)$ which is a morphism of K-algebras. Let $mod(S(n,r))$ denote the category of left $S(n,r)$-modules. A module V in either of the categories $M(n,r)$ or $mod(S(n,r))$ can be studied as a module of the other category via the rule

$$\kappa v = e(\kappa)v, \text{ for all } \kappa \in KGL(n,K),\ v \in V. \tag{2.4}$$

We have the following theorem ([G], Proposition 2.4c)).

Theorem 2.2 *The categories $M(n,r)$ and $mod(S(n,r))$ are equivalent.*

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Example 2.5

Let V be an n-dimensional vector space with basis $v_1, \ldots, v_n$. Then $GL(n, K)$ acts on V by

$$gv_j = \sum_{i=1}^{n} g_{ij} v_i = \sum_{i=1}^{n} x_{ij}(g) v_i, \;\; g \in GL(n, K). \tag{2.5}$$

Thus V is a polynomial $GL(n, K)$-module of degree one; that is $V \in M(n, 1)$. The $GL(n, K)$-module V is called the *natural module.* By (2.4), the $S(n, r)$-action on V is given by

$$\xi v_j = \sum_{i=1}^{n} \xi(x_{ij}) v_i, \;\; \xi \in S(n, r), \;\; 1 \leq j \leq n.$$

Let $V^{\otimes r}$ denote the r-fold tensor power $V \otimes V \otimes \cdots \otimes V$ (r times). For $I \in I(n, r)$, we let v_I denote the element $v_{i_1} \otimes \cdots \otimes v_{i_r} \in V^{\otimes r}$. The following set is a K-basis for $V^{\otimes r}$:

$$\{v_I = v_{i_1} \otimes \cdots \otimes v_{i_r} : I = (i_1, \ldots, i_r) \in I(n, r)\}.$$

We define a $GL(n, K)$-action on $V^{\otimes r}$ by extending the action in (2.5):

$$\begin{aligned} gv_J &= (gv_{j_1}) \otimes \cdots \otimes (gv_{j_r}) \\ &= \sum_{I \in I(n,r)} g_{i_1 j_1} \cdots g_{i_r j_r} v_I \\ &= \sum_{I \in I(n,r)} x_{I,J}(g) v_I, \;\; g \in G, \;\; J \in I(n, r). \end{aligned}$$

Thus $V^{\otimes r} \in M(n, r)$. The $S(n, r)$-action on $V^{\otimes r}$ is given by

$$\xi v_J = \sum_{I \in I(n,r)} \xi(x_{I,J}) v_I, \;\; \xi \in S(n, r), \;\; J \in I(n, r).$$

□

2.3 The Schur module

The general linear group $GL(n, K)$ acts on $A(n)$ by

$$gP(X) = P(Xg), \;\; g \in GL(n, K), \;\; P \in A(n), \;\; X = (x_{ij})_{1 \leq i,j \leq n}.$$

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Moreover, $A(n,r)$ belongs to $M(n,r)$.

Given an $n \times n$ matrix $A = (a_{ij})_{1 \leq i,j \leq n}$, and subsequences I, J of $(1,2,\ldots,n)$ let A_J^I denote the determinant of the minor of A whose rows are indexed by I and columns indexed by J. If $I = (i_1, i_2, \ldots, i_k)$, $J = (j_1, j_2, \ldots, j_k)$, we shall also denote A_J^I by

$$A^{i_1,i_2,\ldots,i_k}_{j_1,j_2,\ldots,j_k}.$$

Suppose that $\lambda_1 = s$, so the Young diagram of shape λ has s columns. For a tableau T, let $T(j)$ denote its jth column. Given two λ-tableaux S and T the *bideterminant* $(S:T) \in A(n,r)$ is given by

$$(S:T) = X^{S(1)}_{T(1)} X^{S(2)}_{T(2)} \cdots X^{S(s)}_{T(s)}.$$

Example 2.6

Let $\lambda = (2,1)$. Then

$$\begin{aligned} \left(\begin{array}{|c|c|}\hline 2 & 4 \\ \hline 3 \\ \cline{1-1} \end{array} : \begin{array}{|c|c|}\hline 4 & 1 \\ \hline 2 \\ \cline{1-1} \end{array} \right) &= X^{2,3}_{4,2} X^{4}_{1} \\ &= (x_{24}x_{32} - x_{22}x_{34})x_{41}. \end{aligned}$$

□

Let T_λ denote the λ-tableau whose entries in the ith row are all i's. In this dissertation, we shall mainly be concerned with bideterminants $(T_\lambda : T)$ and we let $[T] = (T_\lambda : T)$. In this notation, $[T_\lambda] = (T_\lambda : T_\lambda)$ is then the product of the determinants of the principal minors of X of sizes $\mu_1, \mu_2, \ldots, \mu_s$ where $\mu = (\mu_1, \mu_2, \ldots, \mu_s)$ is the conjugate of λ.

Example 2.7

Let $\lambda = (3,2)$. Then $T_\lambda = \begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 2 & 2 \\ \cline{1-2} \end{array}$ and $[T_\lambda] = \left(\begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 2 & 2 \\ \cline{1-2} \end{array} : \begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 2 & 2 \\ \cline{1-2} \end{array} \right)$.

If $T = \begin{array}{|c|c|c|}\hline 2 & 3 & 4 \\ \hline 5 & 6 \\ \cline{1-2} \end{array}$, then

$$\begin{aligned} [T] &= (T_\lambda : T) \\ &= X^{1,2}_{2,5} X^{1,2}_{3,6} X^{1}_{4} \\ &= (x_{12}x_{25} - x_{15}x_{22})(x_{13}x_{26} - x_{16}x_{23})x_{14} \end{aligned}$$

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Let λ be a partition of r such that the Young diagram of shape λ has at most n rows. Let $g = (g_{ij})_{1 \leq i,j \leq n} \in GL(n,K)$ and $J = (j_1, j_2, \ldots, j_r) \in I(n,r)$. It can be shown that

$$g \cdot [T_J] = \sum_{I \in I(n,r)} g_{I,J}[T_I] = \sum_{I \in I(n,r)} x_{I,J}(g)[T_I]. \tag{2.6}$$

Thus the K-span of the bideterminants $[T]$ where each T is a λ-tableau is a non-zero $GL(n,K)$-invariant submodule of $A(n,r)$ which belongs to $M(n,r)$.

Definition 2.5 *The Schur module $\nabla(\lambda)$ is the K-span of the set*

$$\{[T] : T \textit{ is a } \lambda\textit{-tableau}\}.$$

There is a very nice basis for $\nabla(\lambda)$ given by the set of bideterminants which come from semistandard λ-tableaux. The proof of this result involves a so-called *straightening algorithm* which allows one to write a bideterminant $[T]$ as a K-linear combination of bideterminants $[S]$ where each S is a semistandard λ-tableau with the same content as T. There are a number of versions but we shall use the method given in [F], §8.1, p. 108 - 110, (see also [To], p. 420) which we now describe.

Theorem 2.3 (Straightening algorithm) *The following relations hold in $\nabla(\lambda)$:*

1. *If T is a λ-tableau with two repeated entries in a column then $[T] = 0$.*

2. *Suppose that $I \in I(n,r)$. Then if $\sigma \in C(\widehat{T}_\lambda)$ we have $[T_{I\sigma}] = sgn(\sigma)[T_I]$.*

3. *Let J be a fixed subsequence of column $j+1$ of a λ-tableau T, and let I be a subsequence of column j of T, having the same size as J; we denote this size by $|I|$. Let $T^*(I,J)$ be the tableau obtained by interchanging the elements in I and J, maintaining the ordering of the elements. Let $T(I,J)$ be the column increasing tableau obtained from $T^*(I,J)$ by applying a suitable column permutation; we will denote this permutation by σ_I, since we keep J fixed and vary I. Then we have*

$$[T] = \sum_{\substack{|I|=|J| \\ I \subseteq T(j)}} [T^*(I,J)] = \sum_{\substack{|I|=|J| \\ I \subseteq T(j)}} sgn(\sigma_I)[T(I,J)], \tag{2.7}$$

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where the sum runs over the subsequences I of the jth column $T(j)$ of T with the same cardinality as J.

Proof. Since $[T]$ is the product of determinants corresponding to each of its columns, it suffices to prove 1. and 2. for a one-column tableau. Both follow immediately from the definition of bideterminant. If $[T]$ has repeated entries in a column then $[T]$ is the determinant of a matrix which has two identical columns so clearly $[T] = 0$. To see 2., suppose that $[T_I]$ is the determinant of the matrix A. Then if σ is a two-cycle with $\sigma \in C(\widehat{T}_\lambda)$, $T_{I\sigma}$ is the same as T_I except that two distinct column entries have been interchanged. Thus $[T_{I\sigma}]$ is the determinant of B where B is the same as A except that two columns have been exchanged. Since $\det A = -\det B$, we have $[T_{I\sigma}] = \mathrm{sgn}(\sigma)[T_I]$ and the result for a general permutation σ now follows.

Part 3. of the theorem follows from a linear algebraic identity that was proved by Sylvester in 1851, see [F], §8.1. Namely for $k \times k$ matrices A and B, we have

$$\det A \cdot \det B = \sum \det A' \cdot \det B'$$

where the sum is over all pairs (A', B') of matrices obtained from A and B by interchanging a fixed set of m rows of A with m rows of B preserving the ordering of the rows. □

To derive a straightening algorithm, order the set of λ-tableaux by $S \succ T$ if, in the right-most column which is different in the two tableaux, the lowest box in which they differ has a larger entry in S. If T is column increasing but not semistandard, suppose that the entry in the kth row of the column j is larger than the entry in the kth row of the column $j+1$. Then if J is taken to be the sequence of entries in column j of T which occur in rows 1 through k, and I is any subsequence of column j having the same size as J, we have

$$T(I, J) \succ T. \tag{2.8}$$

Combined with (2.7), this gives a straightening algorithm, by downward induction on $\succ$.

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Example 2.8

$$
\begin{aligned}
\left[\begin{array}{|c|c|c|}\hline 1&3&2\\\hline 4&5\\\cline{1-2}\end{array}\right] &= \left[\begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4&5\\\cline{1-2}\end{array}\right] + \left[\begin{array}{|c|c|c|}\hline 1&3&5\\\hline 4&2\\\cline{1-2}\end{array}\right] \\
&= \left[\begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4&5\\\cline{1-2}\end{array}\right] - \left[\begin{array}{|c|c|c|}\hline 1&2&5\\\hline 4&3\\\cline{1-2}\end{array}\right] \\
&= \left[\begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4&5\\\cline{1-2}\end{array}\right] - \left[\begin{array}{|c|c|c|}\hline 2&1&5\\\hline 3&4\\\cline{1-2}\end{array}\right] \\
&= \left[\begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4&5\\\cline{1-2}\end{array}\right] - \left[\begin{array}{|c|c|c|}\hline 1&2&5\\\hline 3&4\\\cline{1-2}\end{array}\right] + \left[\begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2&4\\\cline{1-2}\end{array}\right].
\end{aligned}
$$

□

We now state the basis theorem for $\nabla(\lambda)$. There are several proofs, see for instance [G], 4.5a, [F], Theorem 1, p. 110, or [DKR], p. 78.

Theorem 2.4 (Basis theorem for $\nabla(\lambda)$) *The Schur module $\nabla(\lambda)$ has K-basis consisting of bideterminants $[T]$ where each T is a semistandard λ-tableau.*

When the characteristic of K is zero, $\nabla(\lambda)$ is irreducible. When K has non-zero characteristic, this may no longer be the case. The Schur module $\nabla(\lambda)$ does, however, possess a remarkable $GL(n,K)$-submodule.

Definition 2.6 *Let $L(\lambda)$ denote the $GL(n,K)$-submodule of $\nabla(\lambda)$ generated by $[T_\lambda]$.*

We shall study $L(\lambda)$ in more detail in Chapter 3. For a proof of the following, see [G], 5.4c, 3.5a, where $L(\lambda)$ is denoted by $D^{\min}_{\lambda,K}$ or [Ma], Theorem 3.4.1, where $\nabla(\lambda)$ is denoted by $M(\lambda)$.

Theorem 2.5 *The submodule $L(\lambda)$ is the unique irreducible $GL(n,K)$-submodule of $\nabla(\lambda)$. Moreover, every irreducible polynomial representation of $GL(n,K)$ is afforded by $L(\lambda)$ for some partition λ.*

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2.4 Weights and weight spaces

Let $D(n) \subset GL(n, K)$ be the subgroup of diagonal matrices and $B(n) \subset GL(n, K)$ the subgroup of upper triangular matrices. If V is a polynomial representation of $GL(n, K)$, $v \in V$ is called a *weight vector* of *weight* $\chi = (\chi_1, \ldots, \chi_n)$, $\chi_i \in \mathbb{N}_0$, if

$$d \cdot v = d_{11}^{\chi_1} \cdots d_{nn}^{\chi_n} \cdot v$$

for all $d = diag(d_{11}, \cdots, d_{nn}) \in D(n)$.

Example 2.9

A bideterminant $[T]$ in $\nabla(\lambda)$ has weight $\chi = (\chi_1, \ldots, \chi_n)$ where χ is the content of T; that is χ_i is equal to the number of i's in T. This can easily be seen using (2.6), for if $d = diag(d_{11}, \ldots, d_{nn})$, we have

$$d \cdot [T_J] = \sum_{I \in I(n,r)} d_{i_1 j_1} \cdots d_{i_r j_r} [T_I] = d_{j_1 j_1} \cdots d_{j_r j_r} [T_J].$$

For instance, $d \cdot \left[\begin{array}{|c|c|c|}\hline 1 & 2 & 2 \\ \hline 4 \\ \cline{1-1}\end{array}\right] = d_{11} d_{22}^2 d_{44} \left[\begin{array}{|c|c|c|}\hline 1 & 2 & 2 \\ \hline 4 \\ \cline{1-1}\end{array}\right].$ □

Definition 2.7 *Let V be a $GL(n, K)$-module. A vector $v \in V$ is a highest weight vector if $B(n) \cdot v = K^* \cdot v$.*

Example 2.10

Let $\lambda = (2, 1)$. Then $[T_\lambda] = \left[\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array}\right]$ is a highest weight vector since

$$g \cdot [T_\lambda] = \sum_{I \in I(n,r)} g_{i_1 1} g_{i_2 1} g_{i_3 2} [T_I] = g_{11} g_{11} g_{22} [T_\lambda],$$

if $g \in B(n)$. □

In general, equation (2.6) shows that the bideterminant $[T_\lambda] \in \nabla(\lambda)$ has weight λ and is the unique highest weight vector in $\nabla(\lambda)$, up to multiplication by a non-zero scalar.

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Given a polynomial $GL(n, K)$-module V and an n-tuple $\chi = (\chi_1, \ldots, \chi_n)$ of non-negative integers, the *weight space* associated to χ is defined as

$$V^\chi = \{v \in V : d \cdot v = d_{11}^{\chi_1} \cdots d_{nn}^{\chi_n} \cdot v \text{ for all } d \in D(n)\}.$$

We have the following theorem; see [G], Proposition 3.3f.

Theorem 2.6 *A polynomial $GL(n, K)$-module V is the direct sum $\oplus_\chi V^\chi$ of its weight spaces.*

2.5 The Weyl module

In this section we give Green's definition of the Weyl module, see [G], 5.1. The Weyl module may also be defined using the hyperalgebra for $GL(n, K)$. We will discuss this equivalent definition in Chapter 4. We first discuss contravariant duality.

The dual space V^* of a module $V \in M(n, r)$ becomes a left $GL(n, K)$-module in the usual way by defining $(g \cdot f)(v) = f(g^{-1}v)$, for $g \in GL(n, K)$, $f \in V^*$, $v \in V$, but V^* equipped with this action will not necessarily belong to the category $M(n, r)$. Instead, we define

$$(g \cdot f)(v) = f(g^t v). \tag{2.9}$$

where g^t denotes the transpose of the matrix g. The dual V^* equipped with this new action, denoted V°, is called the *contravariant dual* to V. The contravariant dual of a module $V \in M(n, r)$ does belong to $M(n, r)$.

To express the action (2.9) in terms of the action of $S(n, r)$, define an involutory anti-automorphism $J : S(n, r) \to S(n, r)$ by

$$J(\xi)(x_{I,J}) = \xi(x_{J,I}), \; I, J \in I(n, r).$$

Then the contravariant dual V° is the $S(n, r)$-module with action

$$(\xi \cdot f)(v) = f(J(\xi)v), \; \xi \in S(n, r), \; f \in V^\circ, \; v \in V.$$

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Given modules V and W in $M(n,r)$, a K-bilinear form $(,) : V \times W \to K$ is *contravariant* if

$$(\xi v, w) = (v, J(\xi)w), \ \xi \in S(n,r), \ v \in V, \ w \in W.$$

We have the following basic result.

Theorem 2.7 *Let V and W be modules in $M(n,r)$. There is a one-to-one correspondence between contravariant forms $(\ ,\ ) : V \times W \to K$ and morphisms $\psi : V \to W^\circ$ given by*

$$\psi(v)(w) = (v, w), \ v \in V, \ w \in W.$$

The form is non-degenerate if and only if ψ is an isomorphism.

Our aim is to define the contravariant dual of the Schur module, $\nabla(\lambda)$. To do so, we begin with an $S(n,r)$-module epimorphism $\phi : V^{\otimes r} \to \nabla(\lambda)$ defined by

$$\phi(v_I) = [T_I].$$

Let N denote the kernel of ϕ.

Given $I, J \in I(n,r)$ define $\delta_{I,J} = \prod_{k=1}^r \delta_{i_k, j_k}$. Define the canonical form

$$\langle , \rangle : V^{\otimes r} \times V^{\otimes r} \to K$$

by

$$\langle v_I, v_J \rangle = \delta_{I,J}. \tag{2.10}$$

The form $\langle , \rangle$ is contravariant and non-degenerate so the orthogonal complement to N with respect to $\langle , \rangle$ is an $S(n,r)$-invariant submodule of $V^{\otimes r}$.

Definition 2.8 *The Weyl module $\Delta(\lambda)$ is the orthogonal complement to N with respect to $\langle , \rangle$;*

$$\Delta(\lambda) = \{x \in V^{\otimes r} : \langle x, N \rangle = 0\}.$$

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Now define a form $(,) : \Delta(\lambda) \times \nabla(\lambda) \to K$ by

$$(x, \phi(y)) = \langle x, y \rangle, \; x \in \Delta(\lambda), \; y \in V^{\otimes r}.$$

The form $(,)$ is non-degenerate and contravariant, so by Theorem 2.7,

$$\Delta(\lambda) \cong (\nabla(\lambda))^{\circ}.$$

When the characteristic of K is zero, $\Delta(\lambda) \cong \nabla(\lambda)$ and $\Delta(\lambda)$ is irreducible. If the characteristic of K is not zero, this is not necessarily the case. However, the Weyl module $\Delta(\lambda)$ has a unique maximal submodule M and the irreducible $GL(n, K)$-module $L(\lambda)$ of Definition 2.6 is isomorphic to $\Delta(\lambda)/M$.

Define a right action of S_r on $V^{\otimes r}$ by $v_I\sigma = v_{I \cdot \sigma}$ where $I \in I(n,r)$ and $\sigma \in S_r$. Let $I(\lambda) \in I(n,r)$ denote the r-tuple which satisfies $T_{I(\lambda)} = T_\lambda$ and define

$$v_\lambda = v_{I(\lambda)}\{C(\widehat{T}_\lambda)\}. \tag{2.11}$$

For example, if $\lambda = (2,1)$, then $T_\lambda = T_{I(\lambda)}$ where $I(\lambda) = (1,1,2)$, so

$$v_\lambda = v_1 \otimes v_1 \otimes v_2 - v_2 \otimes v_1 \otimes v_1.$$

We now state Green's version of the standard basis theorem for $\Delta(\lambda)$. The theorem was originally proved by Carter and Lusztig in [CL]. In the form given here, it appears in [G], Theorem 5.3b.

Theorem 2.8 *The set*

$$\{\xi_{I,I(\lambda)}v_\lambda : I \in I(n,r), \; T_I \textit{ semistandard}\}$$

is a K-basis for $\Delta(\lambda)$.

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Chapter 3

Determining irreducible $GL(n, K)$-modules

3.1 Introduction

If K has characteristic zero, it is well known that the Schur modules $\nabla(\lambda)$ are irreducible; that is $\nabla(\lambda) = L(\lambda)$. The dimension of $\nabla(\lambda)$ is given by Weyl's dimension formula. When the characteristic of K is $p > 0$, $L(\lambda)$ may or may not be isomorphic to $\nabla(\lambda)$, and in general the dimension of $L(\lambda)$ and the dimensions of its weight spaces are not known.

In this chapter we discuss three methods which yield K-spanning sets for $L(\lambda)$ from which its dimension may be determined. We first give a new method which is based on the Binet-Cauchy formula for expanding the minor of a product of matrices. The second method we discuss is due to Pittaluga and Strickland [PS]. The third spanning set we give, originally due to Clausen [C], consists of the set of sums $\widehat{R}(T)$ where T is a semistandard λ-tableau and $\widehat{R}(T)$ denotes the sum of bideterminants corresponding to tableaux S which are row equivalent to T. The proof we give

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The results in this chapter were obtained in collaboration with Gerald Cliff and a version of this chapter shall appear in [CS].

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that these elements form a spanning set is different from Clausen's and uses the Schur algebra and Green's version of the Carter-Lusztig standard basis theorem for the Weyl module $\Delta(\lambda)$. In Section 3.3 we show that the first two spanning sets are the same up to sign and in Section 3.4 we show that the first two spanning sets are related to the third by a triangular, unimodular matrix called the Désarménien matrix, Ω [De].

3.2 The first spanning set

Fix $\lambda = (\lambda_1, \ldots, \lambda_k)$ a partition of r and suppose that $\mu = (\mu_1, \ldots, \mu_s)$ is the conjugate of λ. In this section we present a new method for obtaining a spanning set for $L(\lambda)$ which is based on linear algebraic techniques.

We know that $L(\lambda)$ is K-spanned by all $A \cdot [T_\lambda]$ where A varies over $GL(n, K)$. In [Mu], §222 a formula is given for evaluating a minor of a product of matrices. In particular, if I and J are two subsequences of $(1, 2, \ldots, n)$ of size m, then

$$(XA)^I_J = \sum_H X^I_H A^H_J$$

where H varies over all subsequences of $(1, 2, \ldots, n)$ of size m. This follows from the Binet-Cauchy formula, [P], 2.3, p. 10 or [Mu], §217. Let $X = (x_{ij})_{1 \le i,j \le n}$. Then

$$\begin{aligned} A \cdot [T_\lambda](X) = [T_\lambda](XA) &= \prod_{k=1}^{s} (XA)^{1,2,\ldots,\mu_k}_{1,2,\ldots,\mu_k} = \prod_{k=1}^{s} \sum_{I_k} X^{1,2,\ldots,\mu_k}_{I_k} A^{I_k}_{1,2,\ldots,\mu_k} \\ &= \sum_{I_1, I_2, \ldots, I_s} \left(\prod_{k=1}^{s} X^{1,2,\ldots,\mu_k}_{I_k} \right) \left(\prod_{k=1}^{s} A^{I_k}_{1,2,\ldots,\mu_k} \right) \end{aligned}$$

where for each k, I_k varies over all subsequences of $(1, 2, \ldots, n)$ of size μ_k. For each s-tuple $(I_1, I_2, \ldots, I_k)$, $\prod_{k=1}^{s} X^{1,2,\ldots,\mu_k}_{I_k}$ is a bideterminant $[T] = (T_\lambda : T)$, and $\prod_{k=1}^{s} A^{I_k}_{1,2,\ldots,\mu_k}$ is a bideterminant $(T : T_\lambda)$ evaluated at the matrix A; we denote the latter object by $T'(A)$. (We have written T' because the rows and columns of the bideterminant T are switched in evaluating $(T : T_\lambda)$ at A.) So

$$A \cdot [T_\lambda] = \sum_T [T] \cdot T'(A) \tag{3.1}$$

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where T varies over the set $\mathcal{C}_\lambda$ of all column increasing λ-tableaux T.

Write the bideterminant $[T]$ as a K-linear combination of bideterminants given by semistandard tableaux;

$$[T] = \sum_{S \in \mathcal{T}_\lambda} \gamma_{TS}[S]$$

where $\mathcal{T}_\lambda$ denotes the set of semistandard λ-tableaux. Apply the K-algebra automorphism on $A(n)$ which takes x_{ij} to x_{ji}. Then we get

$$T'(A) = \sum_{S \in \mathcal{T}_\lambda} \gamma_{TS} S'(A)$$

where $S'(A)$ is the bideterminant $(S : T_\lambda)$ evaluated at A. Then $A \cdot [T_\lambda]$ can be written as

$$A \cdot [T_\lambda] = \sum_{T \in \mathcal{C}_\lambda} \Big(\sum_{S \in \mathcal{T}_\lambda} \gamma_{TS}[S]\Big)\Big(\sum_{U \in \mathcal{T}_\lambda} \gamma_{TU} U'(A)\Big) = \sum_{U \in \mathcal{T}_\lambda} U'(A)\Big(\sum_{T \in \mathcal{C}_\lambda, S \in \mathcal{T}_\lambda} \gamma_{TU}\gamma_{TS}[S]\Big). \tag{3.2}$$

Define

$$\mathcal{B} = \left\{ \sum_{T \in \mathcal{C}_\lambda, S \in \mathcal{T}_\lambda} \gamma_{TU}\gamma_{TS}[S] : U \in \mathcal{T}_\lambda \right\}. \tag{3.3}$$

We have shown that every element of $L(\lambda)$ is a K-linear combination of elements of $\mathcal{B}$. Let $M(\lambda)$ be the K-span of the set $\mathcal{B}$. We have

$$L(\lambda) \subseteq M(\lambda) \subseteq \nabla(\lambda).$$

We want to show that $L(\lambda) = M(\lambda)$. Define

$$P_U = \sum_{T \in \mathcal{C}_\lambda, S \in \mathcal{T}_\lambda} \gamma_{TU}\gamma_{TS}[S], \qquad U \in \mathcal{T}_\lambda.$$

We must show that for each semistandard λ-tableau U_0, P_{U_0} is a linear combination

$$\sum_A c_A A \cdot [T_\lambda] = \sum_{A,U} c_A U'(A) P_U,$$

for some elements A of $GL(n, K)$ and some scalars $c_A \in K$.

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Lemma 3.1 *Suppose that $f_1, f_2, \dots, f_k$ are linearly independent polynomials, over K, in variables $x_1, x_2, \dots, x_m$. Then there exist m-tuples $A_1, A_2, \dots, A_k \in K^m$ such that*

$$\det\big(f_j(A_i)_{1\le i,j\le k}\big) \neq 0.$$

Proof. We use induction on k. Suppose that for every linearly independent set of r polynomials with $1 \le r \le k$ there exist m-tuples $A_1, \dots, A_r$ with

$$\det(f_j(A_i)_{1\le i,j\le r}) \neq 0$$

and assume that

$$\det\big(f_j(A_i)_{1\le i,j\le k}\big) = 0$$

for all m-tuples $A_1, A_2, \dots, A_k \in K^m$. Expand this determinant along the last row. Let G_j be the $(k-1)\times(k-1)$ matrix obtained from $\big(f_j(A_i)\big)$ by deleting the last row and j-th column. Then

$$\sum_j (-1)^{j+k} f_j(A_k) \det G_j$$

is the zero polynomial in A_k. The set $\{f_j : j = 1, \dots k\}$ is linearly independent so each $\det G_j = 0$, for all choices of m-tuples $A_1, A_2, \dots, A_{k-1}$. However, by induction, there exist $A_1, A_2, \dots, A_{k-1}$ such that $\det G_1 \neq 0$. This is a contradiction, and the proof is complete. □

Lemma 3.2 *Suppose that $\{f_1, f_2, \dots, f_k\}$ and $\{p_1, p_2, \dots, p_k\}$ are sets of polynomials in m variables over K, and that $\{f_i\}_{1\le i\le k}$ is linearly independent. Then for each l, there exist m-tuples $A_1, A_2, \dots, A_k \in K^m$ and scalars $c_1, c_2, \dots, c_k \in K$ such that*

$$p_l = \sum_{1\le i,j\le k} c_i f_j(A_i) p_j.$$

Proof. From the previous lemma there exist $A_1, A_2, \dots A_m$ satisfying $\det\big(f_j(A_i)\big) \neq 0$. Consider the system of k equations in the k-unknowns c_i, $1 \le i \le k$:

$$\begin{aligned} \sum_{i=1}^{k} c_i f_j(A_i) &= 0, \qquad j \neq l \\ \sum_{i=1}^{k} c_i f_l(A_i) &= 1. \end{aligned}$$

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Since $\det(f_j(A_i)) \neq 0$, this system has a (unique) solution $c_1, c_2 \ldots c_k \in K$. Multiply the jth equation by p_j for each $1 \leq j \leq k$ and add, giving

$$p_l = \sum_{1 \leq i,j \leq k} c_i f_j(A_i) p_j.$$

$\square$

Theorem 3.3 *The set $\mathcal{B}$ is a spanning set for $L(\lambda)$.*

Proof. We must show that for each $U_0 \in \mathcal{T}_\lambda$, there exist elements $A \in GL(n, K)$ and scalars $c_A \in K$ such that

$$P_{U_0} = \sum_A c_A A \cdot [T_\lambda] = \sum_{A,U} c_A U'(A) P_U,$$

where the sum runs over the $U \in \mathcal{T}_\lambda$, by (3.2) and the definition of P_U. Enumerate the elements of $\mathcal{T}_\lambda$ as $U_1, U_2, \ldots U_k$. Then for integers i, $1 \leq i \leq k$ define

$$p_i = P_{U_i} = \sum_{T \in \mathcal{C}_\lambda, S \in \mathcal{T}_\lambda} \gamma_{TU_i} \gamma_{TS}[S].$$

Let $f_i = (U_i : T_\lambda)$. The set $\{f_i\}_{1 \leq i \leq k}$ is linearly independent, so applying the previous two lemmas, we find A_i and scalars c_i such that for each l,

$$p_l = \sum_{1 \leq i,j \leq k} c_i f_j(A_i) p_j$$

and $\det(f_j(A_i)) \neq 0$. Each f_i and p_i are polynomials in the n^2 variables x_{ij} so each A_i can be regarded as an $n \times n$ matrix A_i. We want the A_i to be in $GL(n, K)$.

First suppose that $\mu_1 = n$ so that λ is a partition of $r \geq n$. Since $\det(f_j(A_i)) \neq 0$, for each i there must exist j such that $f_j(A_i) \neq 0$. By definition

$$f_j = (U_j : T_\lambda) = X^{I_1}_{1,2,\ldots,\mu_1} \cdots X^{I_s}_{1,2,\ldots,\mu_s}$$

where $I_1, \ldots, I_s$ are the columns of U_i. Since $\mu_1 = n$, we have

$$(A_i)^{I_1}_{1,\ldots,\mu_1} = (A_i)^{1,\ldots,n}_{1,\ldots,n} = \det(A_i) \neq 0.$$

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Thus

$$p_l = \sum_{i,j} c_i f_j(A_i) p_j \in L(\lambda)$$

which proves that $L(\lambda) = M(\lambda)$ in this case.

In the general case, let λ' be the partition obtained from λ by placing a column of length n to the left of the Young diagram of λ; thus the conjugate μ' of λ' is $(n, \mu_1, \mu_2, \ldots, \mu_k)$. Consider $L(\lambda') \subseteq M(\lambda') \subseteq \nabla(\lambda')$. Since $\mu'_1 = n$, it follows from the previous paragraph that $L(\lambda') = M(\lambda')$. But all the elements in each of $L(\lambda)$ and $M(\lambda)$ can be obtained from those of $L(\lambda')$ and $M(\lambda')$, respectively, be dividing by $\det(X)$. Hence $L(\lambda) = M(\lambda)$ and the proof is complete. $\square$

The spanning set $\mathcal{B}$ is well adapted to the weight space decomposition of $L(\lambda)$. Each bideterminant $[T]$ has a well defined weight χ, and if T is not semistandard, the straightening procedure gives us $[T]$ as a linear combination of semistandard bideterminants, each of which also has weight χ. Let $\mathcal{T}_\lambda^\chi$ be the set of semistandard λ-tableaux of weight χ, and define

$$\mathcal{B}^\chi = \Big\{ \sum_{T \in \mathcal{C}_\lambda, S \in \mathcal{T}_\lambda} \gamma_{TU}\gamma_{TS}[S] : U \in \mathcal{T}_\lambda^\chi \Big\}.$$

The following result follows by projecting onto weight spaces.

Corollary 3.4 *The weight space $L(\lambda)^\chi$ has spanning set $\mathcal{B}^\chi$.*

Example 3.1

Take $n = 4$, $\lambda = (2, 1)$.

$$\begin{aligned} A \cdot [T_\lambda] &= (XA)^{1,2}_{1,2}(XA)^1_1 \\ &= (X^{1,2}_{1,2}A^{1,2}_{1,2} + X^{1,2}_{1,3}A^{1,3}_{1,2} + X^{1,2}_{1,4}A^{1,4}_{1,2} + X^{1,2}_{2,3}A^{2,3}_{1,2} + X^{1,2}_{2,4}A^{2,4}_{1,2} + X^{1,2}_{3,4}A^{3,4}_{1,2}) \cdot \\ &\qquad (X^1_1A^1_1 + X^1_2A^2_1 + X^1_3A^3_1 + X^1_4A^4_1) \end{aligned}$$

Let $\chi = (1, 1, 0, 1)$ and consider the projection $(A \cdot [T_\lambda])^\chi$ of $A \cdot [T_\lambda]$ onto the χ-weight space of $L(\lambda)$.

$$\begin{aligned} (A \cdot [T_\lambda])^\chi &= \left[\begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 \\ \cline{1-1}\end{array}\right] A^{1,2}_{1,2}A^4_1 + \left[\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 4 \\ \cline{1-1}\end{array}\right] A^{1,4}_{1,2}A^2_1 + \left[\begin{array}{|c|c|}\hline 2 & 1 \\ \hline 4 \\ \cline{1-1}\end{array}\right] A^{2,4}_{1,2}A^1_1 \\ &= \left[\begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 \\ \cline{1-1}\end{array}\right] A^{1,2}_{1,2}A^4_1 + \left[\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 4 \\ \cline{1-1}\end{array}\right] A^{1,4}_{1,2}A^2_1 \end{aligned}$$

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$$+\left(\left[\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 4 \\ \cline{1-1}\end{array}\right]-\left[\begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 \\ \cline{1-1}\end{array}\right]\right)\left(A^{1,4}_{1,2}A^{2}_{1}-A^{1,2}_{1,2}A^{4}_{1}\right)$$

$$=A^{1,2}_{1,2}A^{4}_{1}\left(2\left[\begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 \\ \cline{1-1}\end{array}\right]-\left[\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 4 \\ \cline{1-1}\end{array}\right]\right)+A^{1,4}_{1,2}A^{2}_{1}\left(-\left[\begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 \\ \cline{1-1}\end{array}\right]+2\left[\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 4 \\ \cline{1-1}\end{array}\right]\right)$$

Thus the weight space $L(\lambda)^{\chi}$ is spanned over K by the elements

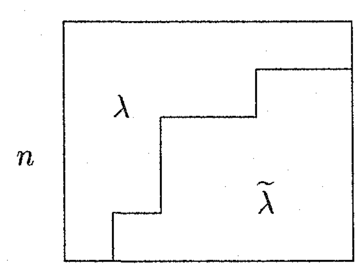
$$2\left[\begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 \\ \cline{1-1}\end{array}\right]-\left[\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 4 \\ \cline{1-1}\end{array}\right],\quad -\left[\begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 \\ \cline{1-1}\end{array}\right]+2\left[\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 4 \\ \cline{1-1}\end{array}\right].$$

Note that these two elements are linearly independent unless the characteristic of K is 3, when they are equal; so $\dim L(\lambda)^{\chi}$ is 1 if the characteristic of K is 3, and is 2 otherwise. □

3.3 The Pittaluga-Strickland method

In this section we present a method due to Pittaluga and Strickland [PS] for finding a spanning set for $L(\lambda)$. Our use of rows and columns of tableaux is reversed from that in [PS].

Let $\lambda = (\lambda_1, \ldots, \lambda_k)$ be a partition of r with conjugate $\mu = (\mu_1, \ldots, \mu_s)$. Define $\widetilde{\mu}$ to be the partition given by $\widetilde{\mu}_1 = n - \mu_s, \ldots, \widetilde{\mu}_s = n - \mu_1$, and let $\widetilde{\lambda}$ be the conjugate of $\widetilde{\mu}$. For example, if $\lambda = (3,2)$ and $n = r = 5$, then $\mu = (2,2,1)$, so $\widetilde{\mu} = (4,3,3)$ and $\widetilde{\lambda} = (3,3,3,1)$. Pictorially, the Young diagrams for λ and $\widetilde{\lambda}$ form an $n \times s$ rectangle when placed side by side with $\widetilde{\lambda}$ rotated by 180°.



We shall define an $SL(n,K)$-equivariant map from the dual $\nabla(\widetilde{\lambda})^*$ to $\nabla(\lambda)$. Since $\mathrm{Hom}_K(\nabla(\widetilde{\lambda})^*, \nabla(\lambda))$ is naturally isomorphic to $\nabla(\lambda) \otimes \nabla(\widetilde{\lambda})$, we first find an $SL(n,K)$-invariant element of $\nabla(\lambda) \otimes \nabla(\widetilde{\lambda})$.

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Consider the rectangular-shaped Young diagram with n rows and s columns; the top part of this is the Young diagram associated to λ and the bottom is associated to $\widetilde{\lambda}$. Fill column k of the λ part of the diagram consecutively with the numbers $1, 2, \ldots, \mu_k$; fill each column of the $\widetilde{\lambda}$ portion consecutively with the numbers $n + 1, n+2, \ldots, 2n - \mu_k$. This gives us a rectangular tableau R. In the following example $n = 4$, λ is the partition $(3, 1)$, and R is

1	1	1
2	5	5
5	6	6
6	7	7

In this section we replace our $n \times n$ matrix X of indeterminates by a $2n \times n$ matrix $X = (x_{ij})_{1 \le i \le 2n, 1 \le j \le n}$. Let $C(n)$ be the polynomials $K[x_{ij} : 1 \le i \le 2n, 1 \le j \le n]$. Then $GL(n, K)$ acts on $C(n)$ by $g \cdot p(X) = p(Xg)$, for $p \in C(n)$, $g \in GL(n, K)$.

Let $R(k)$ denote the determinant of the minor of X whose rows are indexed by column k of R, and whose columns are $1, 2, \ldots, s$. Expand $R(k)$ using Laplace expansion on the first μ_k rows (see [Mu], P. 80 or [P], 2.4.1, P. 11):

$$R(k) = \sum_{I_k} (-1)^{\nu(I_k)} X_{I_k}^{1,2,\ldots,\mu_k} X_{I_k'}^{n+1,n+2,\ldots,2n-\mu_k}$$

where I_k varies over all subsequences of $(1, 2, \ldots, n)$ of size μ_k, I_k' is the complement of I_k in $(1, 2, \ldots, n)$, and

$$\nu(I_k) = \sum_{i \in I_k} i - \frac{\mu_k(\mu_{k+1})}{2}.$$

Now define

$$\alpha = \prod_{k=1}^{s} R(k).$$

Then

$$\alpha = \sum_{I_1, I_2, \ldots I_s} \left(\prod_{k=1}^{s} (-1)^{\nu(I_k)} X_{I_k}^{1,2,\ldots,\mu_k} \right) \left(\prod_{k=1}^{s} X_{I_k'}^{n+1,n+2,\ldots,2n-\mu_k} \right). \tag{3.4}$$

Let $A'(n)$ be the polynomials $K[x_{ij} : n+1 \le i \le 2n, 1 \le j \le n]$ which again is a $GL(n, K)$-module via right translation. There is a $GL(n, K)$-isomorphism σ from $C(n)$ to $A(n) \otimes A'(n)$ given by $\sigma(x_{ij}) = x_{ij} \otimes 1$ if $1 \le i \le n$ and $\sigma(x_{ij}) = 1 \otimes x_{ij}$ if

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$n+1 \leq i \leq 2n$. There is also a $GL(n,K)$-isomorphism $\tau : A'(n) \to A(n)$ given by $\tau(x_{i+n,j}) = x_{ij}$. Applying σ and then $1 \otimes \tau$ to α we get the element

$$\beta = \sum_{I_1, I_2, \ldots I_s} \left(\prod_{k=1}^{s} (-1)^{\nu(I_k)} X_{I_k}^{1,2,\ldots,\mu_k} \right) \otimes \left(\prod_{k=1}^{s} X_{I'_k}^{1,2,\ldots,n-\mu_k} \right)$$

in $A(n) \otimes A(n)$.

For each s-tuple $(I_1, I_2, \ldots I_s)$, $\prod_{k=1}^{s} X_{I_k}^{1,2,\ldots,\mu_k}$ is a bideterminant $[T]$ such that the jth column of the λ-tableau T is $I_j = T(j)$. As well, $\prod_{k=1}^{s} X_{I'_k}^{1,2,\ldots,n-\mu_k}$ is a bideterminant $[\overline{T}]$ where $\overline{T}$ is a $\widetilde{\lambda}$-tableau whose jth column is I'_{s-j}.

Define

$$\nu(T) = \sum_{k=1}^{s} \nu(T(k)).$$

Then

$$\prod_{k=1}^{s} (-1)^{\nu(I_k)} = \prod_{k=1}^{s} (-1)^{\nu(T(k))} = (-1)^{\sum_{k=1}^{s} \nu(T(k))} = (-1)^{\nu(T)} \text{ and}$$

$$\beta = \sum_{T \in \mathcal{C}_\lambda} (-1)^{\nu(T)} [T] \otimes [\overline{T}] \in \nabla(\lambda) \otimes \nabla(\widetilde{\lambda}). \tag{3.5}$$

Suppose that the entries in column k of R are $r_1, r_2, \ldots, r_n$. Then for $g \in GL(n,K)$ we have

$$g \cdot R(k) = (Xg)_{1,2,\ldots,n}^{r_1,r_2,\ldots r_n} = X_{1,2,\ldots,n}^{r_1,r_2,\ldots r_n} g_{1,2,\ldots,n}^{1,2,\ldots n} = (\det g) R(k).$$

Hence $g \cdot \alpha = (\det g)^s \alpha$, and $g \cdot \beta = (\det g)^s \beta$. Now β gives us ϕ in $\mathrm{Hom}_K(\nabla(\widetilde{\lambda})^*, \nabla(\lambda))$ given by

$$\phi(f) = \sum_{T \in \mathcal{C}_\lambda} (-1)^{\nu(T)} f([\overline{T}])[T], \qquad f \in \nabla(\widetilde{\lambda})^*.$$

Since $g\beta = (\det g)^s \beta$, then

$$\phi(gf) = \sum_{T \in \mathcal{C}_\lambda} (-1)^{\nu(T)} f(g^{-1}[\overline{T}])[T] = g \sum_{T \in \mathcal{C}_\lambda} (-1)^{\nu(T)} f(g^{-1}[\overline{T}]) g^{-1}[T] = (\det g)^{-s} g\phi(f).$$

Thus the image of ϕ is a $GL(n,K)$-submodule of $\nabla(\lambda)$. It can be shown, as in [PS], that $\mathrm{im}\,\phi$ is $L(\lambda)$; this also follows from Theorem 3.9 below.

In the sum (3.5) for β, write the tensor factors in $[T] \otimes [\overline{T}]$ as linear combinations of semistandard bideterminants. Let $\mathcal{T}_{\widetilde{\lambda}}$ denote the set of semistandard $\widetilde{\lambda}$-tableaux.

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Then

$$\beta = \sum_{\substack{S \in \mathcal{T}_\lambda \\ U \in \mathcal{T}_{\widetilde{\lambda}}}} a_{SU}[S] \otimes [U] \in \nabla(\lambda) \otimes \nabla(\widetilde{\lambda})$$

for some integers a_{SU} regarded as elements of K. The basis $\{[U] \in \mathcal{T}_{\widetilde{\lambda}}\}$ of $\nabla(\widetilde{\lambda})$ gives rise to the dual basis $\{[U]^* : U \in \mathcal{T}_{\widetilde{\lambda}}\}$ of $\nabla(\widetilde{\lambda})^*$, and

$$\phi([U]^*) = \sum_{S \in \mathcal{T}_\lambda} a_{SU}[S].$$

Define

$$\mathcal{S} = \{\phi([U]^*) : U \in \mathcal{T}_{\widetilde{\lambda}}\},$$

which is a spanning set of $\operatorname{im}\phi$.

Let $\mathcal{C}_\lambda^\chi$ denote the set of all column increasing λ-tableaux of weight χ. For $T \in \mathcal{C}_\lambda^\chi$ the bideterminant $[\widetilde{T}]$ where T is a $\widetilde{\lambda}$-tableau has a certain weight, which we shall call $\overline{\chi}$. Define

$$\beta^\chi = \sum_{T \in \mathcal{C}_\lambda^\chi} (-1)^{\nu(T)}[T] \otimes [\widetilde{T}] \in \nabla(\lambda)^\chi \otimes \nabla(\widetilde{\lambda})^{\overline{\chi}}.$$

Straightening $[T]$ gives us a linear combination of semistandard bideterminants of the same weight χ, hence

$$\beta^\chi = \sum_{\substack{S \in \mathcal{T}_\lambda^\chi \\ U \in \mathcal{T}_{\widetilde{\lambda}}^{\overline{\chi}}}} a_{SU}[S] \otimes [U] \in \nabla(\lambda)^\chi \otimes \nabla(\widetilde{\lambda})^{\overline{\chi}}$$

and if $[U]$ with $U \in \mathcal{T}_{\widetilde{\lambda}}$ has weight $\overline{\chi}$ then

$$\phi([U]^*) = \sum_{S \in \mathcal{T}_\lambda^\chi} a_{SU}[S].$$

Define

$$\mathcal{S}^\chi = \{\phi([U]^*) : U \in \mathcal{T}_{\widetilde{\lambda}}^{\overline{\chi}}\},$$

which is a spanning set for $(\operatorname{im}\phi)^\chi$.

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Example 3.2

Suppose that $n = 4$ and $\lambda = (2,1)$. Then

$$R = \begin{ytableau} 1 & 1 \\ 2 & 5 \\ 5 & 6 \\ 6 & 7 \end{ytableau}$$

$$\begin{aligned} \alpha &= (X^{1,2}_{1,2}X^{5,6}_{3,4} - X^{1,2}_{1,3}X^{5,6}_{2,4} + X^{1,2}_{1,4}X^{5,6}_{2,3} + X^{1,2}_{2,3}X^{5,6}_{1,4} - X^{1,2}_{2,4}X^{5,6}_{1,3} + X^{1,2}_{3,4}X^{5,6}_{1,2}) \cdot \\ &\qquad (X^{1}_{1}X^{5,6,7}_{2,3,4} - X^{1}_{2}X^{5,6,7}_{1,3,4} + X^{1}_{3}X^{5,6,7}_{1,2,4} - X^{1}_{4}X^{5,6,7}_{1,2,3}) \end{aligned}$$

Expand α as a sum of monomials in $X^{1,2}_{i,j}X^{1}_{k}X^{5,6,7}_{a,b,c}X^{5,6}_{d,e}$ where $\{i,j,d,e\} = \{k,a,b,c\} = \{1,2,3,4\}$; consider the sum of the monomials for which $\{i,j,k\} = \{1,2,4\}$ that is, consider the sub-sum α^{χ} where $\chi = (1,1,0,1)$, giving

$$\alpha^{\chi} = X^{1,2}_{1,2}(-X^{1}_{4})X^{5,6,7}_{1,2,3}X^{5,6}_{3,4} + X^{1,2}_{1,4}(-X^{1}_{2})X^{5,6,7}_{1,3,4}X^{5,6}_{2,3} + (-X^{1,2}_{2,4})X^{1}_{1}X^{5,6,7}_{2,3,4}X^{5,6}_{1,3}.$$

Then

$$\begin{aligned} \beta^{\chi} &= -\left[\begin{ytableau} 1 & 4 \\ 2 \end{ytableau}\right] \otimes \left[\begin{ytableau} 1 & 3 \\ 2 & 4 \\ 3 \end{ytableau}\right] - \left[\begin{ytableau} 1 & 2 \\ 4 \end{ytableau}\right] \otimes \left[\begin{ytableau} 1 & 2 \\ 3 & 3 \\ 4 \end{ytableau}\right] - \left[\begin{ytableau} 2 & 1 \\ 4 \end{ytableau}\right] \otimes \left[\begin{ytableau} 2 & 1 \\ 3 & 3 \\ 4 \end{ytableau}\right] \\ &= -\left[\begin{ytableau} 1 & 4 \\ 2 \end{ytableau}\right] \otimes \left[\begin{ytableau} 1 & 3 \\ 2 & 4 \\ 3 \end{ytableau}\right] - \left[\begin{ytableau} 1 & 2 \\ 4 \end{ytableau}\right] \otimes \left[\begin{ytableau} 1 & 2 \\ 3 & 3 \\ 4 \end{ytableau}\right] \\ &\quad - \left(\left[\begin{ytableau} 1 & 2 \\ 4 \end{ytableau}\right] - \left[\begin{ytableau} 1 & 4 \\ 2 \end{ytableau}\right]\right) \otimes \left(\left[\begin{ytableau} 1 & 2 \\ 3 & 3 \\ 4 \end{ytableau}\right] - \left[\begin{ytableau} 1 & 3 \\ 2 & 4 \\ 3 \end{ytableau}\right]\right) \\ &= \left(-2\left[\begin{ytableau} 1 & 4 \\ 2 \end{ytableau}\right] + \left[\begin{ytableau} 1 & 2 \\ 4 \end{ytableau}\right]\right) \otimes \left[\begin{ytableau} 1 & 3 \\ 2 & 4 \\ 3 \end{ytableau}\right] \\ &\quad + \left(\left[\begin{ytableau} 1 & 4 \\ 2 \end{ytableau}\right] - 2\left[\begin{ytableau} 1 & 2 \\ 4 \end{ytableau}\right]\right) \otimes \left[\begin{ytableau} 1 & 2 \\ 3 & 3 \\ 4 \end{ytableau}\right] \end{aligned}$$

Hence

$$\phi\left(\left[\begin{ytableau} 1 & 3 \\ 2 & 4 \\ 3 \end{ytableau}\right]^{*}\right) = -2\left[\begin{ytableau} 1 & 4 \\ 2 \end{ytableau}\right] + \left[\begin{ytableau} 1 & 2 \\ 4 \end{ytableau}\right] \quad \text{and}$$

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$$\phi\left(\left[\begin{array}{cc} 1 & 2 \\ 3 & 3 \\ 4 & \end{array}\right]^*\right) = \left[\begin{array}{cc} 1 & 4 \\ 2 & \end{array}\right] - 2\left[\begin{array}{cc} 1 & 2 \\ 4 & \end{array}\right].$$

So $\mathcal{S}^\chi$ consists of the two elements

$$-2\left[\begin{array}{cc} 1 & 4 \\ 2 & \end{array}\right] + \left[\begin{array}{cc} 1 & 2 \\ 4 & \end{array}\right], \quad \left[\begin{array}{cc} 1 & 4 \\ 2 & \end{array}\right] - 2\left[\begin{array}{cc} 1 & 2 \\ 4 & \end{array}\right].$$

Compare with Example 3.1 □

We want to show that the elements of $\mathcal{S}$ are the same, up to sign, as those in $\mathcal{B}$ of (3.3). In the expression (3.5) for β, we want to examine what happens to $[\overline{T}]$ when we write $[\overline{T}]$ as a linear combination of semistandard bideterminants. Consider the following example.

Example 3.3

Let $n = 4$. If $T = \begin{array}{cc} 1 & 4 \\ 2 & \end{array}$, then $\overline{T} = \begin{array}{cc} 1 & 3 \\ 2 & 4 \\ 3 & \end{array}$. The first column $\overline{T}(1)$ of $\overline{T}$ is the complement of the second column $T(2)$ of T and $\overline{T}(2)$ is the complement of $T(1)$. Both tableaux are semistandard. □

We first show that T is semistandard if and only if $\overline{T}$ is.

Theorem 3.5 *If T is a semistandard λ-tableau, then $\overline{T}$ is a semistandard $\widetilde{\lambda}$-tableau.*

Proof. It is enough to prove the result for a two column tableau. Suppose that the entries in columns one and two of T are $a_1 < a_2 < \ldots < a_m$ and $b_1 < b_2 < \ldots < b_r$ respectively. Since T is semistandard, $a_j \leq b_j$ for $1 \leq j \leq r$. Let $\beta_1 < \beta_2 < \ldots < \beta_{n-r}$ and $\alpha_1 < \alpha_2 < \ldots < \alpha_{n-m}$ be the entries in columns one and two of $\overline{T}$. By definition, $\overline{T}(2)$ is the complement of $T(1)$ and $\overline{T}(1)$ is the complement of $T(2)$.

We shall use induction to show that $\beta_j \leq \alpha_j$ for $1 \leq j \leq n-m$. Suppose that $\beta_1 > \alpha_1$ and let $\alpha_1 = l$. By definition of $\overline{T}$, α_1 is the minimal entry in $\overline{T}(2)$, $a_1 = 1, a_2 = 2, \ldots a_{l-1} = l-1$, and $a_l > l$ (since l does not occur in $T(1)$). The minimal number which does not occur in $T(2)$ is $\beta_1 > \alpha_1 = l$, so $b_1 = 1, b_2 = 2, \ldots, b_{l-1} = l-1$, and $b_l = l > a_l$. Since this contradicts the fact that T is semistandard, $\beta_1 \leq \alpha_1$.

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Now assume that $\beta_{j-1} \leq \alpha_{j-1}$ and suppose that $\beta_j > \alpha_j$. Then, since $\beta_{j-1} \leq \alpha_{j-1} < \alpha_j < \beta_j$, α_j must occur in $T(2)$ for there is no number which is in $\overline{T}(1)$ that falls between β_{j-1} and β_j. Since there are j numbers less than or equal to α_j in $\overline{T}(2)$, there are $s = \alpha_j - j$ numbers less than α_j which are not in $\overline{T}(2)$. These s numbers must occur in the first s rows of $T(1)$. Since α_j is not in $T(1)$, $a_{s+1} > \alpha_j$. We will show that $b_{s+1} = \alpha_j$.

Since $\beta_{j-1} < \alpha_j < \beta_j$, there are $j-1$ numbers less than or equal to $\alpha_j - 1$ which occur in $\overline{T}(1)$, so there are $s = \alpha_j - j$ numbers less than α_j which occur in $T(2)$. Again, they occur in the first s rows of $T(2)$. Since α_j occurs in $T(2)$, so $b_{s+1} = \alpha_j < a_{s+1}$ which contradicts the fact that T is semistandard. Consequently, $\beta_j \leq \alpha_j$. This completes the proof. $\square$

Next, if T is not semistandard, we apply the straightening algorithm described in (2.7) to $[T]$. We want to see what then happens to $[\overline{T}]$ in the sum β when we straighten $[T]$, so we consider tableaux of the form $T(I, J)$ and determine what happens to $\overline{T}$ when I and J are interchanged in T. We first illustrate with an example.

Example 3.4

Let $n = 4$ and let $T = \begin{array}{|c|c|}\hline 2 & 1 \\ \hline 4 & \\ \cline{1-1}\end{array}$. Then $\overline{T} = \begin{array}{|c|c|}\hline 2 & 1 \\ \hline 3 & 3 \\ \hline 4 & \\ \cline{1-1}\end{array}$. Also, $[T] = \left[\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 4 & \\ \cline{1-1}\end{array}\right] - \left[\begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 & \\ \cline{1-1}\end{array}\right]$

and $[\overline{T}] = \left[\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 3 \\ \hline 4 & \\ \cline{1-1}\end{array}\right] - \left[\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline 3 & \\ \cline{1-1}\end{array}\right]$. If we write $[T] = [T_1] - [T_2]$, then $[\overline{T}] = [\overline{T}_1] - [\overline{T}_2]$.

Lemma 3.6 *Suppose that T is a tableau with two columns and that I and J are subsets of the same cardinality of the first and second columns of T respectively.*

1. *If $I \cap J \neq \emptyset$, then $T(I, J) = T(I - I \cap J, J - I \cap J)$.*

2. *If σ and θ are permutations such that $T^*(I, J) = sgn(\sigma)T(I, J)$ and $\overline{T}^*(I, J) = sgn(\theta)\overline{T}(I, J)$ then $sgn(\sigma) = sgn(\theta)$.*

Proof. (i) We will show that if $I \cap J \neq \emptyset$, then $T(I, J) = T(I - I \cap J, J - I \cap J)$. Suppose that $I \cap J = \{x\}$, $I = (i_1, \ldots, i_m, x, \ldots)$, and $J = (j_1, \ldots, j_k, x, \ldots)$. Since

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the entries in T which are not members of I or J are irrelevant to our proof, we consider the following tableaux where $T^* = T^*(I,J)$ and $T^{**} = T^*(I-\{x\}, J-\{x\})$:

$$T = \begin{array}{cc} i_1 & j_1 \\ \vdots & \vdots \\ i_k & j_k \\ i_{k+1} & x \\ \vdots & \vdots \\ i_m & j_m \\ x & j_{m+1} \\ \vdots & \vdots \end{array}, \quad T^* = \begin{array}{cc} j_1 & i_1 \\ \vdots & \vdots \\ j_k & i_k \\ x & i_{k+1} \\ \vdots & \vdots \\ j_m & i_m \\ j_{m+1} & x \\ \vdots & \vdots \end{array}, \quad T^{**} = \begin{array}{cc} j_1 & i_1 \\ \vdots & \vdots \\ j_k & i_k \\ j_{k+2} & x \\ \vdots & \vdots \\ j_{m+1} & i_m \\ x & i_{m+1} \\ \vdots & \vdots \end{array}$$

Now, $T^*(I,J) = sgn(\sigma)T^*(I-\{x\}, J-\{x\})$ where σ is the product of the two permutations which make $T^*(I,J)$ and $T^*(I-\{x\}, J-\{x\})$ identical. It is clear from the above tableaux that these permutations have the same length, so $sgn(\sigma) = 1$, and $T^*(I,J) = T^*(I-\{x\}, J-\{x\})$. By induction on the size of $I \cap J$, $T(I,J) = T(I - I \cap J, J - I \cap J)$ where $I \cap J$ is of any size.

(ii) By part 1 we may assume that $I \cap J = \emptyset$. Then, $I \cap T(2) = \emptyset$ and $J \cap T(1) = \emptyset$, for otherwise $T(I,J) = 0$. It follows that $I \subseteq \overline{T}(1)$ and $J \subseteq \overline{T}(2)$ so $\overline{T}^*(I,J)$ is well-defined. We will show that if $|I| = |J| = 1$, then the permutations under consideration have the same sign. The result then follows for subsets I and J of any size since one may interchange the corresponding elements in I and J one at a time.

Let $I = \{a\}$, $J = \{b\}$, $T^* = T^*(I,J)$ and $\overline{T}^* = \overline{T}^*(I,J)$. Suppose that $T^*(1)$ is not column increasing and suppose that there is an $x \in T^*(1)$ with $x > b$ but $x < a$. (The argument is essentially the same if there is an $x \in T^*(1)$ with $x < b$ but $x > a$).

Let $\mathcal{X}_1 = \{x \in T^*(1) | x > b \text{ but } x < a\}$ and suppose that

$$\alpha_s < \alpha_{s+1} < \ldots < \alpha_t$$

are the elements of $\mathcal{X}_1$. Then $T^*(1)$ becomes column increasing after one applies the cycle σ_1 which places b in the row in which α_s occurs and moves α_i down a row for $s \leq i \leq t$. There is a similar cycle σ_2 which makes $T^*(2)$ column increasing, and

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cycles $\overline{\sigma}_1$ and $\overline{\sigma}_2$ which make $\overline{T}^*(1)$ and $\overline{T}^*(2)$ column increasing.

Let $A = \{x \in \mathcal{X}_1 : x \notin \overline{T}\}$ so that $\mathcal{X}_1 = (\mathcal{X}_1 \cap \overline{T}) \cup A$. Let $\overline{\mathcal{X}}_1 = \{x \in \overline{T}^*(1) : x > b \text{ but } x < a\}$ and $B = \{x \in \overline{\mathcal{X}}_1 : x \notin T\}$. Then, $\overline{\mathcal{X}}_1 = (\mathcal{X}_1 \cap \overline{T}) \cup B$, and if $\sigma_1 \neq \sigma_2$, then $A \neq \emptyset$ or $B \neq \emptyset$, or both.

Suppose that $A \neq \emptyset$ and let $x \in \mathcal{X}_1$, $x \notin \overline{T}$. Then $x \in T^*(2)$, and since $x > b$ but $x < a$, $T^*(2)$ is not column increasing. It follows that $x \in \mathcal{X}_2 = \{x \in T^*(2) : x > b \text{ but } x < a\}$ and $\mathcal{X}_2 = (\mathcal{X}_2 \cap \overline{T}) \cup A$. Similarly, if $B \neq \emptyset$, the set $\overline{\mathcal{X}}_2 = \{x \in \overline{T}^*(2) : x > b \text{ but } x < a\} = (\mathcal{X}_2 \cap \overline{T}) \cup B$.

Let $|\mathcal{X}_1 \cap \overline{T}| = l_1$, and $|\mathcal{X}_2 \cap \overline{T}| = l_2$. Then the length of σ_1 is $l(\sigma_1) = l_1 + |A|$ and $l(\sigma_2) = l_2 + |A|$. Since $l(\overline{\sigma_1}) = l_1 + |B|$ and $l(\overline{\sigma_2}) = l_2 + |B|$, $sgn(\sigma_1\sigma_2) = sgn(\overline{\sigma_1}\,\overline{\sigma_2})$.
□

Lemma 3.7 *If T is a two-column tableau, and J is an ordered subset of the second column of T, then*

$$[\overline{T}] = \sum_{\substack{|I|=|J| \\ I \subseteq T(1)}} sgn(\sigma_I)[\overline{T(I,J)}].$$

Proof. We will prove that

$$\sum_{\substack{|I|=|J| \\ I \subseteq T(1)}} sgn(\sigma_I)[\overline{T(I,J)}] = \sum_{\substack{|I|=|J| \\ I \subseteq \overline{T}(1)}} sgn(\sigma_I)[\overline{T}(I,J)],$$

from which the statement follows, since the right-hand side is certainly equal to $[\overline{T}]$ by (2.7) applied to $\overline{T}$.

Applying Lemma 3.6, part 1 to T, we may assume that $I \cap J = \emptyset$. As noted at the beginning of the proof of Lemma 3.6, part 2., we have $J \subseteq \overline{T}(2)$ if and only if $J \subseteq T(2)$, and $I \subseteq \overline{T}(1)$ if and only if $I \subseteq T(1)$ so $\overline{T}(I,J)$ is well-defined. Furthermore,

$$\{I : |I| = |J|,\ I \subseteq T(1),\ I \cap J = \emptyset\} = \{I : |I| = |J|,\ I \subseteq \overline{T}(1),\ I \cap J = \emptyset\}.$$

Since $I \cap J = \emptyset$, $\overline{T(I,J)} = \overline{T}(I,J)$. Since the permutation which makes $T^*(I,J)$ column increasing has the same sign as the permutation which makes $\overline{T}^*(I,J)$ column increasing, the two sums are identical. □

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Theorem 3.8 *Suppose that* $\{T_i : 1 \le i \le m\}$ *is the set of semistandard* λ*-tableau. If* $[T] = \sum_{i=1}^{m} a_i[T_i]$ *, then* $[\overline{T}] = \sum_{i=1}^{m} a_i[\overline{T_i}]$.

Proof. We apply downward induction on the ordering $\succ$ given in (2.7). If T is semistandard, then so is $\overline{T}$ by Theorem 3.5, so the result holds in this case. In particular it holds for the largest tableau T in the ordering, since if this T were not semistandard one could write $[T]$ as a sum of bideterminants $[T']$ with each $T' \succ T$ by (2.7) and (2.8).

Suppose that the conclusion holds for all $S \succ T$. Suppose that T is not semistandard. Write

$$[T] = \sum_{\substack{|I|=|J| \\ I \subseteq T(k-1)}} sgn(\sigma_I)[T(I,J)] \tag{3.6}$$

where J is a subsequence of $T(k)$, chosen as in (2.7). Then, by Lemma 3.7,

$$[\overline{T}] = \sum_{\substack{|I|=|J| \\ I \subseteq T(k-1)}} sgn(\sigma_I)[\overline{T(I,J)}].$$

Write each $[T(I,J)]$ in the right side of (3.6) as a sum of semistandard bideterminants:

$$[T(I,J)] = \sum_i a_{I,i}[T_i].$$

From (2.8), each $T(I,J)$ in (3.6) satisfies $T(I,J) \succ T$, so by induction, for each $T(I,J)$ on the right of (3.6) we have

$$[\overline{T(I,J)}] = \sum_i a_{I,i}[\overline{T_i}].$$

so that

$$[T] = \sum_{i=1}^{m} \sum_{\substack{|I|=|J| \\ I \subseteq T(k-1)}} sgn(\sigma_I) a_{I,i}[T_i], \qquad [\overline{T}] = \sum_{i=1}^{m} \sum_{\substack{|I|=|J| \\ I \subseteq T(k-1)}} sgn(\sigma_I) a_{I,i}[\overline{T_i}].$$

This completes the proof. $\square$

Due to the above theorem, we may write β as follows:

$$\begin{aligned}\beta = \sum_T (-1)^{\nu(T)}[T] \otimes [\overline{T}] &= \sum_{T \in \mathcal{C}_\lambda} (-1)^{\nu(T)} \left(\sum_{S \in \mathcal{T}_\lambda} \gamma_{TS}[S] \right) \otimes \sum_{U \in \mathcal{T}_\lambda} \gamma_{TU}[\overline{U}] \\ &= \sum_{S,U \in \mathcal{T}_\lambda} \left(\sum_{T \in \mathcal{C}_\lambda} (-1)^{\nu(T)} \gamma_{TS}\gamma_{TU} \right) [S] \otimes [\overline{U}].\end{aligned}$$

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We know that $\{\overline{U} : U \in \mathcal{T}_{\widetilde{\lambda}}\}$ is the set of semistandard $\widetilde{\lambda}$-tableaux. Hence

$$\phi([\overline{U}]^*) = \sum_{S,U \in \mathcal{T}_\lambda} \sum_{T \in \mathcal{C}_\lambda} (-1)^{\nu(T)} \gamma_{TS}\gamma_{TU}[S].$$

We now have that

$$\mathcal{S} = \{ \sum_{T \in \mathcal{C}_\lambda, S \in \mathcal{T}_\lambda} (-1)^{\nu(T)} \gamma_{TS}\gamma_{TU}[S] : U \in \mathcal{T}_\lambda\}.$$

The following theorem explains the relationship between the two spanning sets.

Theorem 3.9 *The elements of the Pittaluga-Strickland spanning set $\mathcal{S}$ are, up to sign, the same as those in the first spanning set $\mathcal{B}$.*

Proof. Note that we need only show that for each $U \in \mathcal{T}_\lambda$, the sign $(-1)^{\nu(T)}$ is the same for each $T \in \mathcal{C}_\lambda$. Then $\sum (-1)^{\nu(T)} \gamma_{TS}\gamma_{TU}[S] = \pm \sum \gamma_{TS}\gamma_{TU}[S]$ which is the same as the element $P_U \in \mathcal{B}$ up to sign. Given $T \in \mathcal{C}_\lambda$, $T = \sum_{S \in \mathcal{T}_\lambda} \gamma_{TS}[S]$, where all bideterminants $[S]$ in the sum have the same weight as $[T]$. So, for each $U \in \mathcal{T}_\lambda$, each $[S]$ in the sum $\sum_{\substack{T \in \mathcal{C}_\lambda \\ S \in \mathcal{T}_\lambda}} \gamma_{TU}\gamma_{TS}[S]$ has the same weight as $[T]$. If suffices to prove, then, that if $[T]$ and $[S]$ are two bideterminants with the same weight, then $(-1)^{\nu(T)} = (-1)^{\nu(S)}$. But,

$$\begin{aligned}
\nu(T) &= \sum_{k=1}^{s} \nu(T(k)) \\
&= \left(\sum_{t \in T(1)} t - \frac{\mu_1(\mu_1+1)}{2} \right) + \ldots + \left(\sum_{t \in T(s)} t - \frac{\mu_s(\mu_s+1)}{2} \right) \\
&= \left(\sum_{t \in T} t \right) - \left(\sum_{i=1}^{k} \frac{\mu_i(\mu_i+1)}{2} \right) \\
&= \left(\sum_{t \in S} t \right) - \left(\sum_{i=1}^{k} \frac{\mu_i(\mu_i+1)}{2} \right), \text{ since [S] and [T] have the same weight} \\
&= \sum_{k=1}^{s} \nu(S(k)) = \nu(S)
\end{aligned}$$

from which the result follows. $\square$

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3.4 A third spanning set and the Désarménien matrix

Suppose that S and T are two λ-tableaux. We say that S and T are row equivalent, denoted $S \sim_r T$ if they are equal up to a permutation of the rows. Define

$$\widehat{R}(T) = \sum_{S \sim_r T} [S].$$

The following theorem is also proved in [C], 6.7(2). The proof we give here uses the Schur algebra and the Carter-Lusztig standard basis theorem, Theorem 2.8, for the Weyl module, $\Delta(\lambda)$.

Theorem 3.10 *The set $\mathcal{A} = \{\widehat{R}(T) : T$ is a semistandard λ-tableau$\}$ is a spanning set for $L(\lambda)$.*

Proof. Since $\{\xi_{I,J} : (I,J) \in \Gamma\}$ forms a basis for $S(n,r)$, $L(\lambda)$ is generated by the set $\{\xi_{I,J} \cdot [T_\lambda] : (I,J) \in \Gamma\}$. But if $I, J \in I(n,r)$, then

$$\xi_{I,J}[T_\lambda] = \sum_{M \in I(n,r)} \xi_{I,J}(x_{M,I(\lambda)})[T_M]$$

and $\xi_{I,J}(x_{M,I(\lambda)}) = 0$ unless $J \sim I(\lambda)$. If $J = I(\lambda)\sigma$ for some $\sigma \in S_r$ then

$$\xi_{I,I(\lambda)\sigma}[T_\lambda] = \xi_{I\sigma,I(\lambda)}[T_\lambda]$$

so $\{\xi_{I,J}[T_\lambda] : I, J \in I(n,r)\}$ is K-spanned by the set $\{\xi_{I,I(\lambda)}[T_\lambda] : I \in I(n,r)\}$. But

$$\xi_{I,I(\lambda)}[T_\lambda] = \sum_M \xi_{I,I(\lambda)}(x_{M,I(\lambda)})[T_M]$$

and $\xi_{I,I(\lambda)}(x_{M,I(\lambda)}) = 0$ unless $I = M\sigma$ where $\sigma \in S_r$ is such that $I(\lambda)\sigma = I(\lambda)$. But this is true if and only if σ permutes the entries in the rows of T_I. Thus,

$$\xi_{I,I(\lambda)}[T_\lambda] = \sum_{T_M \sim_r T_I} (T_{I(\lambda)} : T_M) = \widehat{R}(T_I).$$

This shows that $L(\lambda)$ is K-spanned by the $\widehat{R}(T)$ where T is a λ-tableau.

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We know that $\Delta(\lambda)$ has a unique maximal submodule M and $\Delta(\lambda)/M$ is irreducible with $\Delta(\lambda)/M \cong L(\lambda)$. Thus we have an $S(n,r)$-isomorphism $\phi : \Delta(\lambda)/M \to L(\lambda)$. Since ϕ must preserve weight spaces and $[T_\lambda]$ is the unique vector (up to a scalar) in $L(\lambda)$ with weight λ, using the Carter-Lusztig basis for $\Delta(\lambda)$, we have $\phi(\xi_{I,I(\lambda)}v_\lambda + M) = \xi_{I,I(\lambda)}[T_\lambda]$. Since ϕ is surjective and $\{\xi_{I,I(\lambda)}v_\lambda : T_I \text{ semistandard}\}$ is a basis for $\Delta(\lambda)$, $\{\xi_{I,I(\lambda)}[T_\lambda] : T_I \text{ semistandard}\} = \{\widehat{R}(T) : T \text{ semistandard}\}$ generates $L(\lambda)$. □

Example 3.5

Let $n = 3$, $\lambda = (2,1)$ and $\chi = (1,1,1)$. There are two semistandard λ-tableau which give bideterminants of weight χ; $T_1 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$ and $T_2 = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}$. The elements in the spanning set $\mathcal{A}$ are

$$\widehat{R}(T_1) = \left[\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}\right] + \left[\begin{array}{|c|c|}\hline 2 & 1 \\ \hline 3 \\ \cline{1-1}\end{array}\right] = 2\left[\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}\right] + \left[\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}\right] \quad \text{and}$$

$$\widehat{R}(T_2) = \left[\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}\right] + \left[\begin{array}{|c|c|}\hline 3 & 1 \\ \hline 2 \\ \cline{1-1}\end{array}\right] = 2\left[\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}\right] - \left[\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}\right].$$

□

In order to investigate the relationship between $\mathcal{A}$ and $\mathcal{B}$ we introduce a definition and lemma. Given $T \in \mathcal{C}_\lambda$ and $S \in \mathcal{T}_\lambda$, let γ_{TS} denote the *straightening coefficient* of $[S]$ in the straightening decomposition of $[T]$. Define

$$g(S) = \sum_{T \in \mathcal{C}_\lambda} \gamma_{TS}[T].$$

For example, if $\lambda = (2,1)$ and $\chi = (1,1,1)$, there are three column increasing λ-tableaux:

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}, \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}, \quad \begin{array}{|c|c|}\hline 2 & 1 \\ \hline 3 \\ \cline{1-1}\end{array}.$$

Then since

$$\left[\begin{array}{|c|c|}\hline 2 & 1 \\ \hline 3 \\ \cline{1-1}\end{array}\right] = \left[\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}\right] - \left[\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}\right],$$

$$g\left(\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}\right) = \left[\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}\right] - \left[\begin{array}{|c|c|}\hline 2 & 1 \\ \hline 3 \\ \cline{1-1}\end{array}\right].$$

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Lemma 3.11 *Each element in the spanning set $\mathcal{B}$ corresponds to a $g(S)$ where $S \in \mathcal{T}_\lambda$;*

$$\mathcal{B} = \{g(S) : S \in \mathcal{T}_\lambda\}.$$

Proof. We showed in (3.1) that $A \cdot [T_\lambda] = \sum_{T \in \mathcal{C}_\lambda} [T] \cdot T'(A)$. Write $T'(A)$ as a K-linear combination of semistandard bideterminants. This yields

$$\begin{aligned} A \cdot [T_\lambda] &= \sum_{T \in \mathcal{C}_\lambda} [T] \cdot T'(A) \\ &= \sum_{T \in \mathcal{C}_\lambda} [T] \cdot \left(\sum_{S \in \mathcal{T}_\lambda} \gamma_{TS} S'(A) \right) \\ &= \sum_{S \in \mathcal{T}_\lambda} S'(A) \left(\sum_{T \in \mathcal{C}_\lambda} \gamma_{TS} [T] \right) \\ &= \sum_{S \in \mathcal{T}_\lambda} S'(A) g(S) \end{aligned}$$

It follows that $\mathcal{A} = \{g(S) : S \in \mathcal{T}_\lambda\}$. ☐

To prove our next result, we state some results from [De]. Let ${}^R T$ (respectively ${}^C T$) be the tableau obtained by writing T so that its rows (respectively columns) are weakly increasing (respectively increasing). If ${}^C U$ is the image of U under the action of the permutation σ, then let $s(U) = sgn(\sigma)$. Given two column increasing λ-tableau T and T', define

$$\Omega(T, T') = \sum \{s(U) : {}^C U = T, \ {}^R U = T'\}. \tag{3.7}$$

Example 3.6

Consider the tableaux

$$T = \begin{array}{|c|c|c|} \hline 1 & 2 & 2 \\ \hline 3 & 3 & 4 \\ \hline 5 & 5 \\ \cline{1-2} \end{array}, \quad T' = \begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 2 & 3 & 5 \\ \hline 3 & 5 \\ \cline{1-2} \end{array}.$$

There are exactly two tableaux U which satisfy ${}^C U = T$ and ${}^R U = T'$. They are as follows:

$$U = \begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 5 & 3 & 2 \\ \hline 3 & 5 \\ \cline{1-2} \end{array}, \quad \text{and} \quad U' = \begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 3 & 5 & 2 \\ \hline 5 & 3 \\ \cline{1-2} \end{array}.$$

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Since $s(U) = s(U') = 1$, $\Omega(T, T') = 2$.

Write each λ-tableau T as T_I for some $I \in I(n, r)$. We order the set of λ-tableau by declaring

$$T_I < T_J \text{ if } I < J \text{ with respect to the lexicographic order on } I(n, r). \tag{3.8}$$

In other words, $T_I < T_J$ if the first entry in the first row in which they differ is smaller in T_I than in T_J. Let $\mathcal{I}_\lambda = \{I \in I(n,r) : T_I \text{ is a semistandard } \lambda\text{-tableau}\}$. The *Désarménien matrix* is the matrix

$$\Omega = [\Omega(T_I, T_J)]_{I,J \in \mathcal{I}_\lambda}.$$

It is proved in [De] that Ω is a unimodular triangular matrix. Moreover, if S is a column increasing tableau, then Ω bears the following relationship to the straightening coefficients of $[S]$ (recall that γ_{ST_J} is the coefficient of $[T_J]$ in the straightening decomposition of $[S]$):

$$(\gamma_{ST_J})_{J \in \mathcal{I}_\lambda} \cdot \Omega = (\Omega(S, T_I))_{I \in \mathcal{I}_\lambda}. \tag{3.9}$$

Theorem 3.12 *The spanning sets $\mathcal{B}$ and $\mathcal{A}$ are related via the Désarménien matrix. In particular,*

$$\mathcal{B} \cdot \Omega = \mathcal{A}.$$

Proof. Fix $S \in \mathcal{T}_\lambda$. It will be shown that $\sum_{U \in \mathcal{T}_\lambda} g(U)\Omega(U, S) = \widehat{R}(S)$. By definition, $\widehat{R}(S) = \sum_{U' \sim_r S} [U']$, and for each U' in the sum we have $[U'] = sgn(\sigma_{U'})[U]$ where U is a column increasing tableau. So we may write

$$\widehat{R}(S) = \sum_{U' \sim_r S} [U'] = \sum_{\substack{U' \sim_r S \\ U' \sim_c U}} sgn(\sigma_{U'})[U],$$

where all U in the sum are column increasing. Let T be a column increasing tableau, and let a_T be the coefficient of $[T]$ in $\widehat{R}(S)$. Then

$$a_T[T] = \sum_{\substack{T' \sim_r S \\ T' \sim_c T}} sgn(\sigma_{T'})[T] = \Omega(T, S)[T].$$

On the other hand, if b_T is the coefficient of $[T]$ in the sum $\sum_{U \in \mathcal{T}_\lambda} g(U)\Omega(U, S) = \widehat{R}(S)$, then $b_T = \sum_{U \in \mathcal{T}_\lambda} \gamma_{TU}\Omega(U, S) = \Omega(T, S)$, by (3.9). Hence, $a_T = b_T$, and the result now follows. $\square$

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Chapter 4

The symplectic Weyl module

4.1 Introduction

In this chapter, we work with the symplectic analogues of the Schur and Weyl modules. Following [D1], we place a bar over a $GL(n, K)$-module to represent its symplectic analogue. The symplectic Schur module for instance is denoted $\overline{\nabla(\lambda)}$.

In Section 2.5 we discussed Green's version of the Weyl module for $GL(n, K)$. In [G], Green works exclusively with the Schur algebra to give results about polynomial $GL(n, K)$-modules and his version of the standard basis theorem for the Weyl module is given in terms of the Schur algebra. A symplectic version of the Carter-Lusztig standard basis theorem for the symplectic Weyl module has not been given and one of our goals in this chapter is to provide one. There is a Schur algebra for the symplectic group (see [D3] or [Do]) but the theory surrounding it is not fully developed. For instance, there is not a nice combinatorial description of the basis elements as there is in the traditional Schur algebra. Thus we cannot give a symplectic analogue of the standard basis theorem using the symplectic Schur algebra.

There is an alternative approach to using the Schur algebra for the study of the representations of the classical groups. Many authors work with the hyperalgebras of $GL(n, K)$ or $Sp(2m, K)$ to study their representations. There is an equivalence of categories similar to that given in Theorem 2.2. Let V be a module for $GL(n, K)$

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with basis $v_1, \ldots, v_m$. Then V is said to be a *rational* module if for each $g \in GL(n, K)$

$$gv_j = \sum_i c_{ij}(g) v_i$$

where each $c_{ij}(g)$ is a rational function in the n^2 indeterminates x_{ij}, $1 \leq i, j \leq n$. Let U_K denote the hyperalgebra for $GL(n, K)$. The category of finite dimensional U_K-modules is equivalent to the category of finite dimensional rational $GL(n, K)$-modules. Furthermore, every finite dimensional rational $GL(n, K)$-module is isomorphic to a polynomial module tensored with a suitable negative power of the determinant representation. There are symplectic analogues of these results as well. This explains why the hyperalgebras for $GL(n, K)$ and $Sp(2m, K)$ play such a critical role in the theory of their representations.

Carter and Lusztig define the Weyl module in [CL] using the hyperalgebra for $GL(n, K)$ and the standard basis theorem they give involves certain elements from the hyperalgebra of $GL(n, K)$. To give a symplectic version of the standard basis theorem it is natural then to work with the hyperalgebra for the symplectic group.

Although we use the hyperalgebra instead of the Schur algebra, we are still able to use an approach similar to that taken by Green in the case of the general linear group to prove a symplectic standard basis theorem. The main tools in the proof of Green's version of the standard basis theorem are the Carter-Lusztig Lemma, the Désarménien matrix, and the fact that $\nabla(\lambda)$ has a basis consisting of bideterminants given by semistandard λ-tableaux. In [D1], a symplectic version of the Carter-Lusztig lemma is given. It is also shown in that work that the symplectic Schur module, $\overline{\nabla(\lambda)}$, has K-basis consisting of bideterminants given by semistandard symplectic King-tableaux. Thus if one defines a symplectic version of Green's module V_λ, many of the tools needed to prove the symplectic basis theorem are already at one's disposal. The missing piece in the puzzle is the Désarménien matrix so we develop a symplectic version of this matrix along the way.

The Désarménien matrix is defined in [De] and [DKR] using operators which they call Capelli operators. The Capelli operators are actually elements of the hyperalgebra, although the authors do not make use of this fact. We use operators in the

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hyperalgebra for $Sp(2m, K)$ to define a symplectic version of the Désarménien matrix. As is the case with the usual Désarménien matrix, our matrix may be used to give a straightening algorithm for bideterminants in $\overline{\nabla(\lambda)}$. Donkin provides an alternative symplectic straightening algorithm in [D1].

We define an $Sp(2m, K)$-module $\overline{V}_\lambda$ which is the symplectic analogue of Green's V_λ in Section 4.4. Namely, the module we define is the contravariant dual to the symplectic Schur module $\overline{\nabla(\lambda)}$. The symplectic Weyl module $\overline{\Delta(\lambda)}$ is usually defined in terms of the symplectic hyperalgebra. We prove that the module $\overline{V}_\lambda$ we define is isomorphic to $\overline{\Delta(\lambda)}$ in Corollary 4.12. We use the symplectic Désarménien matrix and the symplectic Carter-Lusztig lemma to give a symplectic version of the Carter-Lusztig standard basis theorem in Theorem 4.11. As in the original setting, the symplectic Désarménien matrix is the connection between this basis and the basis of bideterminants given by semistandard symplectic λ-tableaux for $\overline{\nabla(\lambda)}$.

4.2 The hyperalgebras for $GL(n, K)$ and $Sp(2m, K)$

We begin by defining the hyperalgebra for $GL(n, K)$. A nice reference for this section is [JK], §8.2. Let the matrix $X_{ij}, 1 \le i, j \le n$, in the Lie algebra $\mathfrak{gl}(n, \mathbb{C})$ denote the $n \times n$ matrix with a 1 in the $ijth$ position and zeros elsewhere. The *universal enveloping algebra*, $U_{\mathbb{C}}$, of $\mathfrak{gl}(n, \mathbb{C})$ is the associative $\mathbb{C}$-algebra generated by the set $\{X_{ij} : 1 \le i, j \le n\}$ subject to the relations

$$X_{ij}X_{kl} - X_{kl}X_{ij} = \delta_{jk}X_{il} - \delta_{li}X_{kj}, \quad 1 \le i, j, k, l \le n.$$

Define elements e_{ij}, f_{ij}, and h_i, where $1 \le i, j \le n$, in $U_{\mathbb{C}}$ by

$$\begin{aligned} e_{ij} &:= X_{ij} \quad & f_{ij} &:= X_{ji} \quad & h_i &:= X_{ii} \\ e_i &:= e_{i,i+1} \quad & f_i &:= f_{i,i+1} \quad & h_{ij} &:= h_i - h_j. \end{aligned} \tag{4.1}$$

Given $X \in U_{\mathbb{C}}$, and α a positive integer, let

$$X^{(\alpha)} = \frac{X^\alpha}{\alpha!} \text{ and } \binom{X}{\alpha} = \frac{1}{\alpha!}X(X-1)(X-2)\cdots(X-\alpha+1).$$

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Kostant's $\mathbb{Z}$-form for $U_{\mathbb{C}}$, denoted $U_{\mathbb{Z}}$, is the subring of $U_{\mathbb{C}}$ with $\mathbb{Z}$-basis given by

$$\left\{ \prod_{1\leq i<j\leq n} e_{ij}^{(\gamma_{ij})} \prod_{1\leq i\leq n} \binom{h_i}{\beta_i} \prod_{1\leq i<j\leq n} f_{ij}^{(\alpha_{ij})} : \gamma_{ij}, \beta_i, \alpha_{ij} \in \mathbb{Z}_{\geq 0} \right\}. \tag{4.2}$$

The first product is ordered $e_{n-1,n}^{(\gamma_{n-1n})} \cdots e_{2n}^{(\gamma_{2n})} \cdots e_{23}^{(\gamma_{23})} e_{1n}^{(\gamma_{1n})} \cdots e_{13}^{(\gamma_{13})} e_{12}^{(\gamma_{12})}$ and the order in the third is the opposite of this. This basis is known as the *PBW-basis* for $U_{\mathbb{Z}}$.

Definition 4.1 *The hyperalgebra of $GL(n, K)$, denoted U_K, is defined as*

$$U_K = U_{\mathbb{Z}} \otimes_{\mathbb{Z}} K.$$

We write $u \in U_K$ to mean the image in U_K of the element $u \in U_{\mathbb{Z}}$ under the map $\phi : U_{\mathbb{Z}} \to U_K$ defined by $\phi(u) = u \otimes 1$. Note also that if V is $U_{\mathbb{Z}}$-module, then $V \otimes_{\mathbb{Z}} K$ is a U_K-module, and we often drop the tensor when discussing U_K-modules.

The operators in U_K act as derivations on $V^{\otimes r}$, the rth tensor power of V, with e_{ij}, f_{ij}, and h_i acting as follows

$$e_{ij} v_k = \delta_{jk} v_i, \quad f_{ij} v_k = \delta_{ik} v_j, \quad h_i v_k = \delta_{ik} v_k \tag{4.3}$$

and $u(v_{i_1} \otimes \cdots \otimes v_{i_r}) = (uv_{i_1}) \otimes \cdots \otimes v_{i_r} + v_{i_1} \otimes (uv_{i_2}) \cdots \otimes v_{i_r} + \cdots + v_{i_1} \otimes \cdots \otimes (uv_{i_r})$ where $u \in U_K$.

We may define a similar U_K-action on $A(n, r)$;

$$e_{ij} x_{lk} = \delta_{jk} x_{li}, \quad f_{ij} x_{lk} = \delta_{ik} x_{lj}, \quad h_i x_{lk} = \delta_{ik} x_{lk}, \tag{4.4}$$

and $u(x_{l_1 k_1} \cdot \ldots \cdot x_{l_r k_r}) = (u x_{l_1 k_1}) \cdot \ldots \cdot x_{l_r k_r} + \cdots + x_{l_1 k_1} \cdot \ldots \cdot (u x_{l_r k_r})$ where $u \in U_K$. The Schur module $\nabla(\lambda)$ is a U_K-invariant submodule of $A(n, r)$. Indeed, if $[T] \in \nabla(\lambda)$, then $e_{ij}[T] = \sum_k [T_k]$ where the sum runs over the distinct λ-tableaux T_k which are obtained from T by replacing a j in T by an i. The operator f_{ij} acts on a bideterminant $[T]$ in a similar manner and $h_i[T] = \alpha[T]$ where α is the number of i's in T.

Recall from (2.11) that $I(\lambda)$ denotes the subsequence which satisfies $T_{I(\lambda)} = T_\lambda$ and $v_\lambda = v_{I(\lambda)}\{C(\widehat{T}_\lambda)\}$. Let $\Delta(\lambda)_{\mathbb{Z}}$ be the left $U_{\mathbb{Z}}$-module generated by v_λ. We now state the definition of the Weyl module as given by Carter and Lusztig.

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Definition 4.2 *The Weyl module, $\Delta(\lambda)$, is the left U_K-module defined as*

$$\Delta(\lambda) = \Delta(\lambda)_{\mathbb{Z}} \otimes_{\mathbb{Z}} K.$$

Green proves in [G], 5.2b, that the module V_λ of Section 2.5 and $\Delta(\lambda)$ coincide. To give the Carter-Lustzig version of the standard basis theorem we define certain operators in U_K. Let T be a semistandard λ-tableau. Define $F_T \in U_K$ by

$$F_T = \prod_{1 \leq i < j \leq n} f_{ij}^{(\gamma_{ij})} \tag{4.5}$$

where γ_{ij} is the number of entries equal to j in row i of T and product is ordered as in (4.2);

$$F_T = f_{12}^{(\gamma_{12})} f_{13}^{(\gamma_{13})} \cdots f_{1n}^{(\gamma_{1n})} f_{23}^{(\gamma_{23})} \cdots f_{2n}^{(\gamma_{2n})} \cdots f_{n-1,n}^{(\gamma_{n-1,n})}.$$

We now state the Carter-Lusztig standard basis theorem, [CL], P. 118, Theorem.

Theorem 4.1 (Standard basis theorem) *The set*

$$\{F_T v_\lambda : T \textit{ is a semistandard } \lambda\textit{-tableau}\}$$

is a K-basis for $\Delta(\lambda)$.

To define the symplectic group, we let $n = 2m$ where m is a positive integer and introduce symbols $\bar{1}, \bar{2}, \ldots, \bar{m}$. Following [D1], we let $\bar{\mathbf{m}} = \{\bar{1}, \bar{2}, \ldots, \bar{m}\}$. Throughout the remainder of the chapter, we shall identify the set $\mathbf{n} = \{1, \ldots, n\}$ with the set $\mathbf{m} \cup \bar{\mathbf{m}}$ via the map $\phi : \mathbf{n} \to \bar{\mathbf{m}} \cup \mathbf{m}$ defined by $\phi(i) = i$ and $\phi(2m + i - 1) = \bar{i}$, $1 \leq i \leq m$.

Let V be a finite dimensional vector space over K with $\dim V = n$ and let $\{v_1, \ldots v_m, v_{\bar{m}}, \ldots, v_{\bar{1}}\}$ be a basis for V. Define a non-degenerate bilinear form

$$(,) : V \times V \to K$$

by

$$(v_i, v_{\bar{i}}) = 1 = -(v_{\bar{i}}, v_i), \ 1 \leq i \leq m, \text{ and } (v_i, v_j) = 0 \text{ otherwise.}$$

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The *symplectic group*, $Sp(2m, K)$, is as follows:

$$Sp(2m, K) = \{g \in GL(2m, K) : (gx, gy) = (x, y) \text{ for all } x, y \in V\}.$$

Note that there are other forms that can be used to define $Sp(2m, K)$. We are, for instance, using a different form than the one used in [H].

The Lie algebra $\mathfrak{sp}(2m, K)$ is the subalgebra of $\mathfrak{gl}(n, K)$ given by

$$\mathfrak{sp}(2m, K) = \{\phi \in \mathfrak{gl}(n, K) : (\phi(v), w) = -(v, \phi(w))\}.$$

If $i, j \in \mathbf{m}$ with $j \neq \bar{i}$, then X_{ij}, $X_{i\bar{j}}$ and $X_{\bar{j}i}$ do not belong to $\mathfrak{sp}(2m, K)$. For instance,

$$(X_{i\bar{j}} v_{\bar{j}}, v_{\bar{i}}) = (v_i, v_{\bar{i}}) = 1$$

while

$$(v_{\bar{j}}, X_{i\bar{j}} v_{\bar{i}}) = 0.$$

However, $X_{i\bar{j}} + X_{j\bar{i}}$ does belong to $\mathfrak{sp}(2m, K)$. The above problem is remedied since

$$((X_{i\bar{j}} + X_{j\bar{i}}) v_{\bar{j}}, v_{\bar{i}}) = 1$$

and

$$-(v_{\bar{j}}, (X_{i\bar{j}} + X_{j\bar{i}}) v_{\bar{i}}) = -(v_{\bar{j}}, v_j) = 1.$$

It can be shown that $\mathfrak{sp}(2m, K)$ is generated by the matrices $\overline{X}_{ij} \in \mathfrak{gl}(n, K)$, $1 \leq i, j \leq m$, where

$$\begin{array}{ll}
\overline{X}_{i\bar{i}} = X_{i\bar{i}} & \overline{X}_{\bar{i}i} = X_{\bar{i}i} \\
\overline{X}_{ii} = X_{ii} - X_{\bar{i}\bar{i}} & \overline{X}_{i\bar{j}} = X_{i\bar{j}} + X_{j\bar{i}},\ j \neq i \\
\overline{X}_{ij} = X_{ij} - X_{\bar{j}\bar{i}},\ j \neq i & \overline{X}_{\bar{j}i} = X_{\bar{j}i} + X_{\bar{i}j},\ j \neq i.
\end{array}$$

The universal enveloping algebra, $\overline{U}_{\mathbb{C}}$, for $\mathfrak{sp}(2m, K)$ is the subalgebra of $U_{\mathbb{C}}$ generated by the above elements.

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We adopt similar conventions to those in (4.1) for labelling elements of $\overline{U}_{\mathbb{C}}$;

$$\overline{e}_{ij} := \overline{X}_{ij} \quad \overline{f}_{ij} := \overline{X}_{ji} \quad \overline{h}_i := \overline{X}_{ii}$$
$$\overline{e}_i := \overline{e}_{i,i+1} \quad \overline{f}_i := \overline{f}_{i,i+1} \quad \overline{h}_{ij} := \overline{h}_i - \overline{h}_j$$

where $i, j \in \mathbf{m} \cup \overline{\mathbf{m}}$. Kostant's $\mathbb{Z}$-form for $\overline{U}_{\mathbb{C}}$, denoted $\overline{U}_{\mathbb{Z}}$, has PBW-basis as in (4.2), simply bar all operators in the product.

Definition 4.3 *The hyperalgebra of $Sp(2m, K)$ is defined as $\overline{U}_K = \overline{U}_{\mathbb{Z}} \otimes_{\mathbb{Z}} K$.*

The modules $V^{\otimes r}$ and $A(n,r)$ are both $\overline{U}_K$-modules via the actions (4.3) and (4.4).

4.3 The symplectic Désarménien matrix

In order to define the concept of a semistandard symplectic λ-tableau, we reorder the set $\mathbf{m} \cup \overline{\mathbf{m}}$ by

$$\overline{1} < 1 < \overline{2} < 2 < \cdots < \overline{m} < m. \tag{4.6}$$

A version of *symplectic λ-tableaux* were introduced by King, [K]. A symplectic λ-tableau T is obtained by filling the boxes of the Young diagram of shape λ with elements from the set $\mathbf{m} \cup \overline{\mathbf{m}}$. A symplectic tableau T is called *semistandard symplectic* if it is semistandard in the usual sense (with respect to the ordering given in (4.6)) and satisfies the additional property that the elements of row i are all greater than or equal to $\overline{i}$ for each i.

Example 4.1

Let $\lambda = (2,2,1)$, $T = \begin{array}{|c|c|} \hline \overline{1} & \overline{2} \\ \hline 2 & 3 \\ \hline \overline{3} \\ \cline{1-1} \end{array}$, and $S = \begin{array}{|c|c|} \hline \overline{1} & \overline{2} \\ \hline 1 & 2 \\ \hline \overline{3} \\ \cline{1-1} \end{array}$.

Then T is a semistandard symplectic λ-tableau, while S is not. $\square$

The coordinate ring $K[Sp(2m, K)]$ is the restriction to $Sp(2m, K)$ of the algebra of polynomial functions $K[GL(n, K)]$ on $GL(n, K)$. The symplectic Schur module,

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denoted $\overline{\nabla(\lambda)}$, is a subspace of $K[Sp(2m,K)]$ defined as the restriction of $\nabla(\lambda)$ to $Sp(2m,K)$. Given a symplectic tableau T, the bideterminant $\overline{[T]}$ in $\overline{\nabla(\lambda)}$ is the restriction of the bideterminant $[T]$ to $Sp(2m,K)$. We refer to the bideterminant $\overline{[T]}$ where T is a symplectic tableau as a symplectic bideterminant. We will omit the bar on $\overline{[T]}$ and it shall be understood that unless we state otherwise, we are working with symplectic bideterminants in this chapter. As one would hope, we have the following theorem.

Theorem 4.2 *The set*

$$\{[T] : T \textit{ is a semistandard symplectic } \lambda\textit{-tableau}\}$$

forms a K-basis for $\overline{\nabla(\lambda)}$.

A version of Theorem 4.2 was first proved by DeConcini in [Dc] where he used his own version of symplectic tableaux. In [B], the theorem is proved using the King tableaux in the case where K has characteristic zero. The result was later shown to hold true for arbitrary infinite fields in [D1].

Given a semistandard symplectic tableau T with $k \leq m$ rows, we define operators $\overline{E}_T$ and $\overline{F}_T$ in $\overline{U}_K$ as follows:

$$\overline{E}_T = \prod_{\substack{j\neq i \\ 1\leq i\leq k}} \overline{e}_{ij}^{(\gamma_{ij})} \text{ and } \overline{F}_T = \prod_{\substack{j\neq i \\ 1\leq i\leq k}} \overline{f}_{ij}^{(\gamma_{ij})}$$

where γ_{ij} is the number of entries equal to j in row i of T and the product runs over all $j \in \mathbf{m} \cup \overline{\mathbf{m}}$. Note that for a fixed i, we must have $i < j$ or $j = \overline{i}$ for all $j \in \mathbf{m} \cup \overline{\mathbf{m}}$ since T is semistandard symplectic. We order the product $\overline{E}_T$ as

$$\overline{e}_{m\overline{m}}^{(\gamma_{m\overline{m}})} \cdots \overline{e}_{2m}^{(\gamma_{2m})} \cdots \overline{e}_{23}^{(\gamma_{23})} \overline{e}_{2\overline{2}}^{(\gamma_{2\overline{2}})} \overline{e}_{1m}^{(\gamma_{1m})} \cdots \overline{e}_{12}^{(\gamma_{12})} \overline{e}_{1\overline{2}}^{(\gamma_{1\overline{2}})} \overline{e}_{1\overline{1}}^{(\gamma_{1\overline{1}})}$$

and we order $\overline{F}_T$ in the opposite way.

Given a semistandard symplectic λ-tableau T and a column increasing tableau S, we may write $\overline{E}_T[S]$ as a unique linear combination of semistandard bideterminants $[U]$ by Theorem 4.2. We make the following definitions.

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Definition 4.4 *Let S and T be λ-tableaux and suppose that T is semistandard symplectic. Define $\overline{\Omega}(S,T) = c$ where c is the coefficient of $[T_\lambda]$ in the sum $\overline{E}_T[S]$.*

Example 4.2

Let $T = \begin{array}{|c|c|c|}\hline 1 & \bar{2} & 2 \\ \hline \bar{2} & 2 \\ \cline{1-2}\end{array}$ and let $S = \begin{array}{|c|c|c|}\hline \bar{1} & 1 & 2 \\ \hline 1 & \bar{2} \\ \cline{1-2}\end{array}$. Then

$$\begin{aligned}
\overline{E}_T[S] &= \bar{e}_{2\bar{2}}\bar{e}_{12}\bar{e}_{1\bar{2}}[S] \\
&= e_{2\bar{2}}(e_{12} - e_{\bar{2}\bar{1}})(e_{1\bar{2}} + e_{2\bar{1}}) \left[\begin{array}{|c|c|c|}\hline \bar{1} & 1 & 2 \\ \hline 1 & \bar{2} \\ \cline{1-2}\end{array}\right] \\
&= e_{2\bar{2}}(e_{12} - e_{\bar{2}\bar{1}}) \left[\begin{array}{|c|c|c|}\hline 2 & 1 & 2 \\ \hline 1 & \bar{2} \\ \cline{1-2}\end{array}\right] \\
&= e_{2\bar{2}} \left[\begin{array}{|c|c|c|}\hline 2 & 1 & 1 \\ \hline 1 & \bar{2} \\ \cline{1-2}\end{array}\right] \\
&= \left[\begin{array}{|c|c|c|}\hline 2 & 1 & 1 \\ \hline 1 & 2 \\ \cline{1-2}\end{array}\right] = -[T_\lambda],
\end{aligned}$$

so $\overline{\Omega}(S,T) = -1$. $\square$

We order the symplectic λ-tableaux as in (3.8); that is $T_I < T_J$ if $I < J$ with respect to the lexicographic order on $I(n,r)$. Let

$$\overline{\mathcal{I}}_\lambda = \{I \in I(n,r) : T_I \text{ is a semistandard symplectic } \lambda\text{-tableau}\}.$$

Definition 4.5 *The symplectic Désarménien matrix is the matrix*

$$\overline{\Omega} = [\overline{\Omega}(T_I, T_J)]_{I,J \in \overline{\mathcal{I}}_\lambda}.$$

We want to prove that the matrix $\overline{\Omega}$ is a unimodular upper triangular matrix. The following example should serve to motivate Theorem 4.4.

Example 4.3

Let $T = \begin{array}{|c|c|c|}\hline 1 & \bar{2} & 2 \\ \hline \bar{2} & 2 \\ \cline{1-2}\end{array}$. Then $\overline{E}_T = \bar{e}_{2\bar{2}}\bar{e}_{12}\bar{e}_{1\bar{2}} = e_{2\bar{2}}(e_{12} - e_{\bar{2}\bar{1}})(e_{1\bar{2}} + e_{2\bar{1}})$ and

$\overline{E}_T[T] = e_{2\bar{2}}(e_{12} - e_{\bar{2}\bar{1}}) \left[\begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline \bar{2} & 2 \\ \cline{1-2}\end{array}\right] = e_{2\bar{2}} \left[\begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline \bar{2} & 2 \\ \cline{1-2}\end{array}\right] = \left[\begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 2 & 2 \\ \cline{1-2}\end{array}\right] = [T_\lambda]$

so $\overline{\Omega}(T,T) = 1$.

$\square$

In the proofs of the theorems that follow, we will often make use of the following lemma which we state without proof. Its validity may easily be shown using a simple inductive argument.

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Lemma 4.3 *Suppose that $\lambda = (r)$ and suppose that the one-row λ-tableau T contains only j's. Then $e_{ij}^{(r)}[T] = [S]$ where S is a one-row λ-tableau which contains only i's.*

Theorem 4.4 *If T is a semistandard symplectic λ-tableau, then $\overline{\Omega}(T,T) = 1$.*

Proof. We need to prove that $\overline{E}_T[T] = [T_\lambda]$. Since T is semistandard symplectic, there are exactly $\gamma_{1\bar{1}}$ $\bar{1}$'s in T and they all occur in the first row. By Lemma 4.3, $\bar{e}_{1\bar{1}}^{(\gamma_{1\bar{1}})}[T] = [U]$ where U is identical to T except that the entries in the first $\gamma_{1\bar{1}}$ columns of the first row of U are equal to one. Let $\kappa \in \mathbf{m} \cup \overline{\mathbf{m}}$ with $\kappa < m$ and suppose that $\bar{e}_{1\kappa}^{(\gamma_{1\kappa})} \cdots \bar{e}_{1\bar{1}}^{(\gamma_{1\bar{1}})}[T] = [U]$ where the first $\gamma_{1\bar{1}} + \gamma_{11} + \cdots + \gamma_{1\kappa}$ columns of the first row of U contain ones while the remainder of the tableau is identical to T. Note that there are no $\bar{1}$'s in U as they were all changed to ones by the operator $\bar{e}_{1\bar{1}}^{(\gamma_{1\bar{1}})}$. Let η be minimal in $\mathbf{m} \cup \overline{\mathbf{m}}$ with $\eta > \kappa$ so that $\bar{e}_{1\eta}^{(\gamma_{1\eta})}$ occurs in the product $\overline{E}_T$ directly after $\bar{e}_{1\kappa}^{(\gamma_{1\kappa})}$. If $\eta = \bar{j} \in \overline{\mathbf{m}}$, then $\bar{e}_{1\eta} = e_{1\bar{j}} + e_{j\bar{1}}$ and since there are no $\bar{1}$'s in U, any product of operators which contains $e_{j\bar{1}}$ takes $[U]$ to zero. Thus $(e_{1\bar{j}} + e_{j\bar{1}})^{(\gamma_{1\bar{j}})}[U] = e_{1\bar{j}}^{(\gamma_{1\bar{j}})}[U]$. Since U is semistandard symplectic, any $\bar{j}$'s which appear in U below the first row must occur in the first $\gamma_{1\bar{1}} + \cdots + \gamma_{1\kappa}$ columns. Since a one occurs in the first row of U above each of these $\bar{j}$'s, changing any of them to a one results in a zero bideterminant. So by Lemma 4.3 $e_{1\bar{j}}^{(\gamma_{1\bar{j}})}[U] = [U']$ where U' is identical to T except that the entries in the first $\gamma_{1\bar{1}} + \gamma_{11} + \cdots + \gamma_{1\kappa} + \gamma_{1\eta}$ columns of the first row of U' are equal to one. If $\eta = j \in \mathbf{m}$, then $\bar{e}_{1\eta} = e_{1j} - e_{\bar{j}\bar{1}}$ and we form the same conclusion. By induction we have shown that $\bar{e}_{1m}^{(\gamma_{1m})}\bar{e}_{1\bar{m}}^{(\gamma_{1\bar{m}})} \cdots \bar{e}_{1\bar{1}}^{(\gamma_{1\bar{1}})}[T] = [U]$ where U is identical to T except that the first row of U consists entirely of ones. A similar argument shows that $\bar{e}_{2m}^{(\gamma_{2m})}\bar{e}_{2\bar{m}}^{(\gamma_{2\bar{m}})} \cdots \bar{e}_{2\bar{2}}^{(\gamma_{2\bar{2}})} \cdots \bar{e}_{1\bar{1}}^{(\gamma_{1\bar{1}})}[T] = [U]$ where the first two rows of U coincide with T_λ and the remainder of the tableau coincides with T. Inductively, we get the general result.

$\square$

Let V be a $\overline{U}_K$-module and let $\chi = (\chi_1, \ldots, \chi_n)$ be an n-tuple of non-negative integers. The *weight space* V^χ is defined

$$V^\chi = \{v \in V : h_i v = \chi_i v, \ i = 1, \ldots, n\}.$$

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The vectors in V^χ are called *weight vectors* of weight χ. If $[T] \in \overline{\nabla(\lambda)}$ and $\chi_{T,i}$ denotes the number of i's in T, the weight of $[T]$ is $\chi = (\chi_{T,1} - \chi_{T,\bar{1}}, \ldots, \chi_{T,m} - \chi_{T,\bar{m}})$. For instance, if $m = 2$, the bideterminant $[T]$ where T is as in Example 4.3, has weight $\chi = (1,0)$. Up to multiplication by a non-zero scalar, the bideterminant $[T_\lambda]$ is the unique vector of weight λ in $\overline{\nabla(\lambda)}$.

Given a symplectic λ-tableau T with k rows, the row sequence associated to T is the subsequence I which satisfies $T = T_I$. The tableaux T and S in Example 4.1 have row sequences $I_1 = (\bar{1}, \bar{2}, 2, 3, \bar{3})$ and $I_2 = (\bar{1}, \bar{2}, 1, 2, \bar{3})$ respectively so we see that $S < T$.

Theorem 4.5 *Suppose that S and T are symplectic λ-tableaux and that T is semistandard.*

1. *If $\overline{\Omega}(S,T) \neq 0$ then $[S]$ and $[T]$ have the same weight.*

2. *If S is semistandard and $\overline{\Omega}(S,T) \neq 0$ then $S \leq T$.*

Proof. If $[U]$ is a bideterminant with weight $\chi = (\chi_1, \ldots, \chi_m)$ then it is easy to see that $\overline{e}_{i\bar{j}} = e_{i\bar{j}} + e_{j\bar{i}}$ takes $[U]$ to the weight space $\overline{\nabla(\lambda)}^\alpha$ where

$$\alpha = (\chi_1, \ldots, \chi_i + 1, \ldots, \chi_j + 1, \ldots, \chi_m);$$

that is every bideterminant in the sum $\overline{e}_{i\bar{j}}[U]$ has weight α. Similarly, $\overline{e}_{ij} = e_{ij} - e_{\bar{j}\bar{i}}$ takes $[U]$ to $\overline{\nabla(\lambda)}^\alpha$ where $\alpha = (\chi_1, \ldots, \chi_i + 1, \ldots, \chi_j - 1, \ldots, \chi_m)$ and clearly $\overline{e}_{i\bar{i}}$ takes $[U]$ to $\overline{\nabla(\lambda)}^\alpha$ where $\alpha = (\chi_1, \ldots, \chi_i + 2, \ldots, \chi_m)$. So if $[T_1]$ and $[T_2]$ are bideterminants that do not have the same weight, and $1 \leq i \leq k$, $\kappa \in \mathbf{m} \cup \overline{\mathbf{m}}$, then $\overline{e}_{i\kappa}[T_1]$ and $\overline{e}_{i\kappa}[T_2]$ lie in different weight spaces. To prove 1., suppose that $[S]$ and $[T]$ do not have the same weight. Inductively we see that $\overline{E}_T$ takes $[S]$ and $[T]$ to different weight spaces. But $\overline{E}_T[T] = [T_\lambda]$ by Theorem 4.4, so $[T_\lambda]$ does not appear in the sum $\overline{E}_T[S]$. Thus $\overline{\Omega}(S,T) = 0$.

To prove 2., we may assume by 1. that $[S]$ and $[T]$ have the same weight. Suppose that $S > T$ and let the first place in the row sequences where the two tableaux differ be at t in T and s in S where t is in the ith row of T. We shall consider two cases. Suppose

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first that $t = \bar{\imath}$. Then since S and T are identical in the first $i-1$ rows, an argument similar to that used to prove Theorem 4.4 shows that $\bar{e}_{i-1,m}^{(\gamma_{i-1,m})} \cdots \bar{e}_{1\bar{1}}^{(\gamma_{1\bar{1}})}[S] = [U]$ where the first $i-1$ rows of U are identical to T_λ and the remainder of U coincides with S. Since $s > \bar{\imath}$, S contains fewer $\bar{\imath}$'s than T in the ith row; that is S has fewer than $\gamma_{i\bar{\imath}}$'s in the ith row. But S is semistandard symplectic so contains no $\bar{\imath}$'s below the ith row. Thus the only $\bar{\imath}$'s in U are those that occur in the ith row so U contains fewer than $\gamma_{i\bar{\imath}}$ $\bar{\imath}$'s. Thus $e_{i\bar{\imath}}^{(\gamma_{i\bar{\imath}})}[U] = 0$ and it follows that $E_T[S] = 0$.

Now suppose that $t > \bar{\imath}$. We first argue that $t \neq i$. Suppose that $t = i$ so that the two row sequences first differ at i in the ith row of T and at $s > i$ in the ith row of S. Then T contains more i's than S and since $[S]$ and $[T]$ have the same weight, T also contains more $\bar{\imath}$'s than S. Since both are semistandard symplectic, this contradicts the fact that S and T are identical in their row sequences prior to $i > \bar{\imath}$ in T. So we may assume that $t > i$. Let $\kappa \in \mathbf{m} \cup \overline{\mathbf{m}}$ be maximal with $\kappa < t$ so that $\overline{E}_T = \cdots \bar{e}_{it}^{(\gamma_{it})} \bar{e}_{i\kappa}^{(\gamma_{i\kappa})} \cdots \bar{e}_{1\bar{1}}^{(\gamma_{1\bar{1}})}$. Since S and T are identical in their row sequences prior to t in T we have that $\bar{e}_{i\kappa}^{(\gamma_{i\kappa})} \cdots \bar{e}_{1\bar{1}}^{(\gamma_{1\bar{1}})}[S] = [U]$ where the first $i-1$ rows of U coincide with T_λ, the first $\gamma_{i\bar{\imath}} + \cdots + \gamma_{i\kappa}$ columns of the ith row of U contain i's and the remainder of U is identical to S. The next operator to appear in the product $\overline{E}_T$ is $\bar{e}_{it}$. Since S is semistandard symplectic and $t > \bar{\imath}$, there are no $\bar{\imath}$'s in U. Thus we may assume that $\bar{e}_{it}^{(\gamma_{it})} = e_{it}^{(\gamma_{it})}$. Indeed, if $t = \bar{\jmath} \in \overline{\mathbf{m}}$, then $\bar{e}_{it} = e_{i\bar{\jmath}} + e_{j\bar{\imath}}$ and the latter operator plays no role since U contains no $\bar{\imath}$'s. We form the same conclusion if $t = j \in \mathbf{m}$. Again, we see that there are less than γ_{it} t's in the ith row of U and since S is semistandard symplectic, any t's which appear below the ith row of U must appear in the first $\gamma_{i\bar{\imath}} + \gamma_{ii} + \cdots + \gamma_{i\kappa}$ columns. Since each of these columns contains an i, changing such a t to an i results in a zero bideterminant. Thus, $e_{it}^{(\gamma_{it})}[U] = 0$ from which it follows that $\overline{E}_T[S] = 0$. □

We have the following immediate corollary to Theorems 4.4 and 4.5.

Corollary 4.6 *The matrix $\overline{\Omega}$ is an upper triangular unimodular matrix.*

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Proof. By Theorem 4.4, $\overline{\Omega}$ has ones on the diagonal and by Theorem 4.5, the entries below the diagonal are zero. □

As in the case of the usual Désarménien matrix, this matrix does provide an algorithm for writing a symplectic bideterminant as a linear combination of bideterminants given by semistandard symplectic tableaux. Before deriving the algorithm, we prove the following lemma.

Lemma 4.7 *Suppose that S is a column increasing λ-tableau and T a semistandard λ-tableau and suppose that $[S]$ and $[T]$ have the same weight. Then*

$$\overline{E}_T[S] = \overline{\Omega}(S,T)[T_\lambda].$$

Proof. Since $[S]$ and $[T]$ have the same weight, $\overline{E}_T$ takes $[S]$ and $[T]$ to the same weight space. By Theorem 4.4, $\overline{E}_T[T] = [T_\lambda]$, so every bideterminant in the sum $\overline{E}_T[S]$ has weight λ. Since $[T_\lambda]$ is the unique vector of weight λ in $\overline{\nabla(\lambda)}$ up to multiplication by a scalar, we have $\overline{E}_T[S] = c[T_\lambda] = \overline{\Omega}(S,T)[T_\lambda]$. □

We now discuss a method for "straightening" a symplectic bideterminant. If S is a symplectic λ-tableau then by Theorem 4.2 we can write $[S]$ as a sum of semistandard bideterminants, and it can easily be shown that these bideterminants must have the same weight as $[S]$. Let χ be the weight of $[S]$ and let

$$\overline{\mathcal{I}}_\lambda^\chi = \{I \in \overline{\mathcal{I}}_\lambda : [T_I] \text{ has weight } \chi\}.$$

Let $\overline{\Omega}_\chi = [\overline{\Omega}(T_I,T_J)]_{I,J\in\overline{\mathcal{I}}_\lambda^\chi}$. Then $[S] = \sum_{I\in\overline{\mathcal{I}}_\lambda^\chi} a_I[T_I]$ and given $J \in \overline{\mathcal{I}}_\lambda^\chi$ we have

$$\overline{E}_{T_J}[S] = \sum_{I\in\overline{\mathcal{I}}_\lambda^\chi} a_I \overline{E}_{T_J}[T_I] = \sum_{I\in\overline{\mathcal{I}}_\lambda^\chi} a_I \overline{\Omega}(T_I,T_J)[T_\lambda] \text{ by Lemma 4.7.}$$

But

$$\overline{E}_{T_J}[S] = \overline{\Omega}(S,T_J)[T_\lambda]$$

so we have

$$\overline{\Omega}(S,T_J)[T_\lambda] = \sum_{I\in\overline{\mathcal{I}}_\lambda^\chi} a_I \overline{\Omega}(T_I,T_J)[T_\lambda].$$

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Thus $\overline{\Omega}(S,T_J) = \sum_{I\in\overline{\mathcal{I}}^\chi_\lambda} a_I\overline{\Omega}(T_I,T_J)$ for all $J \in \overline{\mathcal{I}}^\chi_\lambda$. It follows that

$$[\overline{\Omega}(S,T_I)]_{I\in\overline{\mathcal{I}}^\chi_\lambda} = [a_I]_{I\in\overline{\mathcal{I}}^\chi_\lambda}\overline{\Omega}_\chi$$

and since $\overline{\Omega}_\chi$ is invertible by Corollary 4.6, we have

$$[a_I]_{I\in\overline{\mathcal{I}}^\chi_\lambda} = [\overline{\Omega}(S,T_I)]_{I\in\overline{\mathcal{I}}^\chi_\lambda}(\overline{\Omega}_\chi)^{-1}.$$

Example 4.4

Let $m = 2$, $\lambda = (2,1)$. We would like to apply the straightening algorithm to $[T] = \left[\begin{array}{|c|c|}\hline 1 & \overline{1} \\ \hline \overline{2} \\ \cline{1-1}\end{array}\right]$. There are two semistandard symplectic λ-tableaux which yield bideterminants of weight $\chi = (0,-1)$; $T_1 = \begin{array}{|c|c|}\hline \overline{1} & 1 \\ \hline \overline{2} \\ \cline{1-1}\end{array}$, $T_2 = \begin{array}{|c|c|}\hline \overline{2} & \overline{2} \\ \hline 2 \\ \cline{1-1}\end{array}$.

$\overline{E}_{T_2} = \frac{1}{2}(e_{1\overline{2}}+e_{2\overline{1}})^2$ and $\overline{E}_{T_2}[T_1] = -[T_\lambda]$, so $\overline{\Omega}(T_1,T_2) = -1$. Thus, $\overline{\Omega}_\chi = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$. Also $\overline{E}_{T_1}[T] = [T_\lambda]$ and $\overline{E}_{T_2}[T] = 0$ so $\overline{\Omega}(T,T_1) = 1$ and $\overline{\Omega}(T,T_2) = 0$. If a_1 is the coefficient of $[T_1]$ in the straightening decomposition of $[T]$ and a_2 that of $[T_2]$ then

$$(a_1,a_2) = (1,0)\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = (1,1).$$

□

4.4 A basis for the symplectic Weyl module

Define $\overline{\Delta(\lambda)}_\mathbb{Z}$ to be the left $\overline{U}_\mathbb{Z}$-module generated by v_λ, where v_λ is as defined in (2.11).

Definition 4.6 *The symplectic Weyl module is the left $\overline{U}_K$-module defined as*

$$\overline{\Delta(\lambda)} = \overline{\Delta(\lambda)}_\mathbb{Z} \otimes_\mathbb{Z} K.$$

We now construct the contravariant dual of the Schur module $\overline{\nabla(\lambda)}$ which we will call $\overline{V_\lambda}$ since it is the symplectic analogue of Green's $GL(n,K)$-module V_λ, defined

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in Section 2.5. The contravariant dual of $\overline{\nabla(\lambda)}$ was also defined in [W], where it is denoted W_λ, using a somewhat different approach. It can be shown using the theory of algebraic groups that $\overline{\nabla(\lambda)}^\circ \cong \overline{\Delta(\lambda)}$. We will show this directly in Corollary 4.12 so we refer to $\overline{V_\lambda}$ as the symplectic Weyl module throughout the remainder of the chapter.

Define an anti-automorphism $J : \overline{U}_K \to \overline{U}_K$ by $J(\overline{e}_{ij}) = \overline{f}_{ij}$, $J(\overline{f}_{ij}) = \overline{e}_{ij}$, and $J(\overline{h}_i) = \overline{h}_i$. The dual module V^* becomes a left $\overline{U}_K$-module via the action

$$(gf)(v) = f(J(g)v), \ f \in V^*, \ g \in \overline{U}_K, \ v \in V.$$

The left $\overline{U}_K$-module V^* with the above action is the contravariant dual to V, denoted V°.

Given two $\overline{U}_K$-modules V and W, a K-bilinear form $(,) : V \times W \to K$ is called *$Sp(2m, K)$-contravariant* if

$$(uv, w) = (v, J(u)w), \text{ for all } u \in \overline{U}_K, \ v \in V, \ w \in W.$$

As in Section 2.5 define the map $\phi : V^{\otimes r} \to \nabla(\lambda)$ by $\phi(v_I) = [T_I]$. Let

$$\psi : V^{\otimes r} \to \overline{\nabla(\lambda)}$$

denote the composition of ϕ with restriction to $Sp(2m, K)$. Then ψ is a well-defined $\overline{U}_K$-epimorphism so $N = \ker\psi$ is a $\overline{U}_K$-module.

Let $\langle , \rangle : V^{\otimes r} \times V^{\otimes r} \to K$ be the canonical form on $V^{\otimes r}$ defined in (2.10);

$$\langle v_I, v_J \rangle = \delta_{IJ}$$

where $I = (i_1, \ldots, i_r)$, $J = (j_1, \ldots, j_r) \in I(n, r)$ and $\delta_{IJ} = \prod_{\rho=1}^r \delta_{i_\rho, j_\rho}$.

Define $\overline{V_\lambda}$ to be the orthogonal complement to $N = \ker\psi$ in $V^{\otimes r}$ with respect to the above form. In other words,

$$\overline{V_\lambda} = \{x \in V^{\otimes r} : \langle x, N \rangle = 0\}.$$

It is easy to see that $\langle \overline{e}_{ij} v, w \rangle = \langle v, \overline{f}_{ij} w \rangle$ and $\langle \overline{h}_i v, w \rangle = \langle v, \overline{h}_i w \rangle$ for all $v, w \in V^{\otimes r}$, so the form $\langle , \rangle$ is $Sp(2m, K)$-contravariant. Furthermore, if $u \in \overline{U}_K, v \in \overline{V_\lambda}$, and

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$x \in N$, we have

$$\langle uv, x\rangle = \langle v, J(u)x\rangle = 0$$

since $J(u)x \in N$. Thus $\overline{V_\lambda}$ is a $\overline{U}_K$-submodule of $V^{\otimes r}$. The form

$$(,) : \overline{V_\lambda} \times \overline{\nabla(\lambda)} \to K$$

defined by

$$(x, \psi(y)) = \langle x, y\rangle, \ x \in \overline{V_\lambda}, \ y \in V^{\otimes r} \tag{4.7}$$

is a non-degenerate, $\overline{U}_K$-contravariant form since $\langle,\rangle$ is non-degenerate and $\overline{U}_K$-contravariant. To see that the form is $\overline{U}_K$-contravariant, let $u \in \overline{U}_K$, $x \in \overline{V_\lambda}$, and $y \in V^{\otimes r}$. Then $(ux, \psi(y)) = \langle ux, y\rangle = \langle x, uy\rangle = (x, \psi(uy)) = (x, u\psi(y))$ since ψ is a $\overline{U}_K$-epimorphism. It follows from a symplectic analogue of Theorem 2.7 that $\overline{V_\lambda} \cong (\overline{\nabla(\lambda)})^\circ$.

In [D1], Donkin proves a symplectic version of the Carter-Lusztig lemma ([CL], Lemma 3.3 or [G], 4.6a.). The Carter-Lusztig lemma is usually stated in terms of a so-called Garnir identity as in [G], 4.6a. In the following version of the lemma, we replace this condition with one which comes from Sylvester's identity. Recall from (2.7) that if J is a fixed subsequence of column $j+1$ of a λ-tableau T, and I a subsequence of column j of T, having the same size as J we let $T^*(I, J)$ be the tableau obtained by interchanging the elements in I and J, maintaining the ordering of the elements. We let $\mathcal{T}(\lambda)$ denote the set of λ-tableau.

Lemma 4.8 (Carter-Lusztig Lemma) *Let $f : \mathcal{T}(\lambda) \to F$ be a map into an abelian group F which satisfies the following conditions:*

1. *$f(T) = 0$ if T has equal entries at two distinct places in the same column.*

2. *$f(S) = -f(T)$ if S is obtained from T by interchanging two distinct entries in the same column.*

3. *For any subsequence J of column $j+1$ of T, we have*

$$f(T) = \sum_I f(T^*(I, J))$$

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where the sum is over the subsequences I of the jth column of T that have the same cardinality as J.

Then the image of $\mathcal{T}(\lambda)$ under f is contained in the set

$$\{f(T) : T \textit{ is semistandard }\}.$$

Proof. The lemma follows immediately from the argument in (2.8). □

The symplectic version of the Carter-Lusztig lemma that is proved in [D1] does involve the traditional Garnir identity, but replacing this identity with the one we give above does not require alteration of the proof given there. Before stating the lemma, we define the necessary notation. In order to avoid confusion with the notation used in (2.7), we use different notation from that used in [D1]. In particular, if $Q = (q_1, q_2, \ldots, q_k)$ is a fixed subsequence of a fixed column of T and $I = (i_1, i_2, \ldots, i_k)$ is a subsequence with entries from $\mathbf{m} \cup \overline{\mathbf{m}}$ with the same cardinality as Q, we let $T\langle Q, I\rangle$ denote the tableau which arises by replacing the elements in Q with those in I maintaining the ordering of the elements.

Example 4.5

Let $T = \begin{array}{|c|c|}\hline \bar{2} & 2 \\ \hline 2 & 3 \\ \hline \bar{3} \\ \cline{1-1} 3 \\ \cline{1-1}\end{array}$ and let $Q = (\bar{2}, 2, \bar{3}, 3)$ be the subsequence of the first column of T.

If $I = (\bar{1}, 1, \bar{4}, 4)$, then $T\langle Q, I\rangle = \begin{array}{|c|c|}\hline \bar{1} & 2 \\ \hline 1 & 3 \\ \hline \bar{4} \\ \cline{1-1} 4 \\ \cline{1-1}\end{array}$.

Let $\overline{\mathcal{T}(\lambda)}$ denote the set of symplectic λ-tableau.

Lemma 4.9 (Symplectic Carter-Lusztig Lemma) *Let F be an abelian group and let $f : \overline{\mathcal{T}(\lambda)} \to F$ be a map satisfying 1 - 3 of Lemma 4.8 and the following additional condition:*

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4. For any subsequence $Q = (\overline{q_1}, q_1, \ldots, \overline{q_a}, q_a)$ of even cardinality of the jth column of T, with $q_k \in \mathbf{m}$ for $1 \leq k \leq a$, and $q_1 < q_2 < \cdots < q_a$ we have

$$f(T) = (-1)^a \sum_I T\langle Q, I\rangle$$

where the sum runs over all sequences $I = (\overline{i_1}, i_1, \ldots, \overline{i_a}, i_a)$ with $i_k \in \mathbf{m}$ and $i_1 < i_2 < \cdots < i_a$ which are disjoint from Q.

Then the image of $\overline{\mathcal{T}(\lambda)}$ under f lies in the subgroup of F generated by

$$\{f(T) : T \text{ is semistandard symplectic}\}.$$

We give an example to illustrate property 4.

Example 4.6

Suppose that $m = 4$ and $\lambda = (1,1,1,1)$. For a map $f : \overline{\mathcal{T}(\lambda)} \to F$ to satisfy property 4. of the lemma, we need for instance

$$f\left(\begin{array}{|c|}\hline \bar{1}\\\hline 1\\\hline \bar{2}\\\hline 2\\\hline\end{array}\right) = f\left(\begin{array}{|c|}\hline \bar{3}\\\hline 3\\\hline \bar{4}\\\hline 4\\\hline\end{array}\right), \quad f\left(\begin{array}{|c|}\hline \bar{1}\\\hline 1\\\hline \bar{2}\\\hline 2\\\hline\end{array}\right) = -f\left(\begin{array}{|c|}\hline \bar{2}\\\hline 2\\\hline \bar{2}\\\hline 2\\\hline\end{array}\right) - f\left(\begin{array}{|c|}\hline \bar{3}\\\hline 3\\\hline \bar{2}\\\hline 2\\\hline\end{array}\right) - f\left(\begin{array}{|c|}\hline \bar{4}\\\hline 4\\\hline \bar{2}\\\hline 2\\\hline\end{array}\right),$$

and

$$f\left(\begin{array}{|c|}\hline \bar{1}\\\hline 1\\\hline \bar{2}\\\hline 2\\\hline\end{array}\right) = -f\left(\begin{array}{|c|}\hline \bar{1}\\\hline 1\\\hline \bar{3}\\\hline 3\\\hline\end{array}\right) - f\left(\begin{array}{|c|}\hline \bar{1}\\\hline 1\\\hline \bar{4}\\\hline 4\\\hline\end{array}\right) - f\left(\begin{array}{|c|}\hline \bar{1}\\\hline 1\\\hline \bar{1}\\\hline 1\\\hline\end{array}\right).$$

□

The symplectic Carter-Lusztig lemma allows us to describe $N = \ker\psi$ more precisely. We give a brief proof of the next result which follows the proof of [G], 5.2a.

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Theorem 4.10 *The $\overline{U}_K$-module N is the K-span of the subset $R = R_1 \cup R_2 \cup R_3 \cup R_4$ of $V^{\otimes r}$ where*

1. *R_1 consists of all v_I such that T_I has equal entries in a column.*

2. *R_2 consists of all $v_I - sgn(\sigma)v_{I\sigma}$ where $\sigma \in C(\widehat{T}_\lambda)$.*

3. *R_3 consists of all elements $v_K - \sum_I v_{L(I)}$ where if J is a fixed subset of the $(j+1)$th column of T_K, the sum runs over all $L(I) \in I(n,r)$ for which $T_{L(I)} = T_K^*(I,J)$ for some subset I of the jth column of T_K with the same cardinality as J.*

4. *R_4 consists of all elements $v_K - (-1)^a \sum_I v_{L(I)}$ where if $Q = (\overline{q_1}, q_1, \ldots, \overline{q_a}, q_a)$ is a fixed subsequence of a fixed column of T_K with $q_k \in \mathbf{m}$ for $1 \leq k \leq a$ and $q_1 < q_2 < \cdots < q_a$ then the sum runs over all $L(I) \in I(n,r)$ with $T_{L(I)} = T\langle Q, I\rangle$ for some sequence $I = (\overline{i_1}, i_1, \ldots, \overline{i_a}, i_a)$ which is disjoint from Q and satisfies $i_k \in \mathbf{m}$ and $i_1 < i_2 < \cdots < i_a$.*

Proof. It follows from Theorem 2.3 that $R_1 \cup R_2 \cup R_3$ is contained in $\ker\phi$. Since ψ is the composition of ϕ with restriction to $\overline{\nabla(\lambda)}$, the K-span of $R_1 \cup R_2 \cup R_3$ is contained in $\ker\psi = N$. Donkin shows in [D1], Theorem 2.3b, that $[T_K] - (-1)^a \sum_I [T\langle Q, I\rangle] = 0$ where Q and I satisfy the hypotheses of part 4. of Lemma 4.9, so R_4 is contained in $\ker\psi$ as well. Thus if A denotes the K-span of $R_1 \cup R_2 \cup R_3 \cup R_4$, we have $A \subseteq N = \ker\psi$.

Define an epimorphism $g : \overline{T(\lambda)} \to V^{\otimes r}/A$ by $g(T_I) = v_I + A$. Then g satisfies $1-4$ of the symplectic Carter-Lusztig Lemma, so the image of g is contained in the K-span of $\{v_I + A : T_I \text{ semistandard symplectic}\}$. Since the bideterminants given by semistandard symplectic tableaux form a basis for $\overline{\nabla(\lambda)}$ and the map $\rho : V^{\otimes r}/A \to \overline{\nabla(\lambda)}$ defined by $\rho(v_I + A) = \psi(v_I)$ maps $\{v_I + A : T_I \text{ semistandard symplectic}\}$ onto $\{[T_I] : T_I \text{ semistandard symplectic}\}$, ρ is injective. Thus $N \subseteq A$. $\square$

Note that the argument above can be modified to show that $\ker\phi$ is equal to the K-span of $R_1 \cup R_2 \cup R_3$. Simply let A be the K-span of $R_1 \cup R_2 \cup R_3$ in the above

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proof, use Lemma 4.8 and the fact that the bideterminants given by semistandard λ-tableaux form a basis for $\nabla(\lambda)$. We will use this fact in our proof of the symplectic standard basis theorem below.

Theorem 4.11 *The set* $\{\overline{F}_T v_\lambda : T$ *is a semistandard symplectic* λ*-tableau*$\}$ *forms a* K*-basis for* $\overline{V_\lambda}$.

Proof. We first show that $v_\lambda \in \overline{V_\lambda}$. In [G], 5.3a, it is shown that $\langle v_\lambda, x\rangle = 0$ if $x \in \ker\phi$. Since $\ker\phi$ is the K-span of $R_1 \cup R_2 \cup R_3$, we have $\langle v_\lambda, x\rangle = 0$ if $x \in R_1 \cup R_2 \cup R_3$. If $x \in R_4$, then $x = v_K - (-1)^a \sum_I v_{L(I)}$ where by definition, each K and each $L(I)$ contain at least one $\bar{i}$ for some $i \in \mathbf{m}$. Thus $\langle v_\lambda, x\rangle = 0$ which shows that $v_\lambda \in \overline{V_\lambda}$.

Since $\overline{V_\lambda}$ is the contravariant dual to $\overline{\nabla(\lambda)}$, $\overline{V_\lambda}$ and $\overline{\nabla(\lambda)}$ have the same dimension (see [G], Proposition 3.3e). Since $\{[T] : T$ is a semistandard symplectic λ-tableau$\}$ is a basis for $\overline{\nabla(\lambda)}$, we need only show that the set under consideration is linearly independent. Consider the non-degenerate $\overline{U}_K$-contravariant form

$$(\, , \,) : \overline{V_\lambda} \times \overline{\nabla(\lambda)} \to K$$

defined in (4.7). Let $\overline{\mathcal{I}}_\lambda = \{I \in I(n,r) : T_I \text{ semistandard symplectic}\}$. Given $I, J \in \overline{\mathcal{I}}_\lambda$, we have

$$\begin{aligned}
(\overline{F}_{T_I} v_\lambda, \psi(v_J)) &= (v_\lambda, \overline{E}_{T_I}\psi(v_J)) \\
&= (v_\lambda, \overline{E}_{T_I}[T_J]) \\
&= (v_\lambda, \overline{\Omega}(T_J, T_I)[T_\lambda] + \sum_M a_M [T_M])
\end{aligned}$$

where for each M, $a_M \in K$ and $[T_M]$ has weight different from λ. But

$$\begin{aligned}
(v_\lambda, \overline{\Omega}(T_J, T_I)[T_\lambda] + \sum_M a_M [T_M]) &= \overline{\Omega}(T_J, T_I)\langle v_\lambda, v_{I(\lambda)}\rangle + \sum_M a_M \langle v_\lambda, [T_M]\rangle \\
&= \overline{\Omega}(T_J, T_I).
\end{aligned}$$

By Corollary 4.6, $[\overline{\Omega}(T_J, T_I)]_{I,J\in\overline{\mathcal{I}}_\lambda}$ is a triangular, unimodular matrix so the matrix $[(\overline{F}_{T_I} v_\lambda, [T_J])]_{I,J\in\overline{\mathcal{I}}_\lambda} = [(\overline{F}_{T_I} v_\lambda, \psi(v_J))]_{I,J\in\overline{\mathcal{I}}_\lambda}$ is as well. We now have that $\{\overline{F}_{T_I} v_\lambda : I \in \overline{\mathcal{I}}_\lambda\}$ forms a basis for $\overline{V_\lambda}$. □

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Corollary 4.12 *The $\overline{U}_K$-module $\overline{V_\lambda}$ is the symplectic Weyl module; $\overline{V_\lambda} \cong \overline{\Delta(\lambda)}$.*

Proof. Due to the above theorem, $\overline{V_\lambda}$ is a submodule of $\overline{\Delta(\lambda)}$. In [D1], it is shown that the dimension of $\overline{\nabla(\lambda)}$ is given by Weyl's dimension formula. But $\overline{V_\lambda}$ and $\overline{\nabla(\lambda)}$ have the same dimension, and it is well known that the dimension of $\overline{\Delta(\lambda)}$ is also given by Weyl's dimension formula. This completes the proof. □

4.5 A spanning set for the irreducible $\overline{U}_K$-module $\overline{L(\lambda)}$

The Weyl module $\overline{\Delta(\lambda)}$ has a unique maximal $\overline{U}_K$-submodule M (symplectic version of [G], 5.4b) so the quotient module $\overline{\Delta(\lambda)}/M$ is irreducible. It can also be shown that $\overline{\nabla(\lambda)}$ has a unique minimal submodule $\overline{L(\lambda)}$ which is generated as a $\overline{U}_K$-module by $[T_\lambda]$ and $\overline{\Delta(\lambda)}/M \cong \overline{L(\lambda)}$ (see [G], 5.4c, 5.4d). Furthermore, the set

$$\{\overline{L(\lambda)} : \lambda \text{ is a partition of } r \text{ into not more than } m \text{ parts}\}$$

forms a complete list of irreducible non-isomorphic $\overline{U}_K$-modules. We can use the argument we used in the proof of Theorem 3.10 to get a spanning set for the irreducible $\overline{U}_K$-module $\overline{L(\lambda)}$ as a corollary to our standard basis theorem. In that case, we simplified the elements of the spanning set using the basis for the Schur algebra. There is a symplectic version of the Schur algebra (see [D3]), but a combinatorial basis for this algebra like the one in (2.2) has not been given so were are unable to apply the same argument here. We can, however, get a spanning set by applying the elements $\overline{F}_T$ to $[T_\lambda]$.

Corollary 4.13 *The set $\{\overline{F}_T[T_\lambda] : T$ is a semistandard symplectic λ-tableau$\}$ is a K-spanning set for $\overline{L(\lambda)}$.*

Proof. The map

$$\phi : \overline{\Delta(\lambda)}/M \to \overline{L(\lambda)}$$

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defined by $\phi(\overline{F}_T v_\lambda + M) = \overline{F}_T[T_\lambda]$ where T is semistandard symplectic is a surjective $\overline{U}_K$-homomorphism. Since the set $\{\overline{F}_T v_\lambda : T \text{ is a semistandard symplectic } \lambda\text{-tableau}\}$ is a basis for $\overline{\Delta(\lambda)}$, the set $\{\overline{F}_T[T_\lambda] : T \text{ is a semistandard symplectic } \lambda\text{-tableau}\}$ is a spanning set for $\overline{L(\lambda)}$. $\square$

Example 4.7

Let $\lambda = (2,1)$, $m = 2$, $\chi = (0,1)$. There are two semistandard symplectic tableaux which give bideterminants of weight χ;

$$T_1 = \begin{array}{|c|c|}\hline \bar{1} & 1 \\ \hline 2 \\ \cline{1-1}\end{array} \text{ and } T_2 = \begin{array}{|c|c|}\hline \bar{2} & 2 \\ \hline 2 \\ \cline{1-1}\end{array}.$$

Now,

$$\begin{aligned}
\overline{F}_{T_1}[T_\lambda] &= \overline{f}_{1\bar{1}} \left[\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array}\right] \\
&= \left[\begin{array}{|c|c|}\hline \bar{1} & 1 \\ \hline 2 \\ \cline{1-1}\end{array}\right] + \left[\begin{array}{|c|c|}\hline 1 & \bar{1} \\ \hline 2 \\ \cline{1-1}\end{array}\right] \\
&= \left[\begin{array}{|c|c|}\hline \bar{1} & 1 \\ \hline 2 \\ \cline{1-1}\end{array}\right] + \left[\begin{array}{|c|c|}\hline \bar{1} & 1 \\ \hline 2 \\ \cline{1-1}\end{array}\right] + \left[\begin{array}{|c|c|}\hline \bar{2} & 2 \\ \hline 2 \\ \cline{1-1}\end{array}\right], \text{ by the straightening algorithm} \\
&= 2\left[\begin{array}{|c|c|}\hline \bar{1} & 1 \\ \hline 2 \\ \cline{1-1}\end{array}\right] + \left[\begin{array}{|c|c|}\hline \bar{2} & 2 \\ \hline 2 \\ \cline{1-1}\end{array}\right]
\end{aligned}$$

and

$$\begin{aligned}
\overline{F}_{T_2}[T_\lambda] &= (f_{1\bar{2}} + f_{2\bar{1}})(f_{12} - f_{\bar{2}\bar{1}}) \left[\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array}\right] \\
&= (f_{1\bar{2}} + f_{2\bar{1}}) \left[\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array}\right] \\
&= \left[\begin{array}{|c|c|}\hline \bar{2} & 2 \\ \hline 2 \\ \cline{1-1}\end{array}\right] + \left[\begin{array}{|c|c|}\hline 1 & 2 \\ \hline \bar{1} \\ \cline{1-1}\end{array}\right] + \left[\begin{array}{|c|c|}\hline 1 & \bar{1} \\ \hline 2 \\ \cline{1-1}\end{array}\right] \\
&= 3\left[\begin{array}{|c|c|}\hline \bar{2} & 2 \\ \hline 2 \\ \cline{1-1}\end{array}\right] + \left[\begin{array}{|c|c|}\hline \bar{1} & 1 \\ \hline 2 \\ \cline{1-1}\end{array}\right], \text{ by the straightening algorithm}
\end{aligned}$$

and the above two elements give us a spanning set for $\overline{L(\lambda)}^\chi$.

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Chapter 5

A quantum version of the Désarménien matrix

5.1 Introduction

In this chapter, we work in the quantized version of the hyperalgebra for $GL(n, K)$. Given an indeterminate q, we define a $\mathbb{C}(q)$-algebra $U_{\mathbb{C}(q)}$ called the quantized universal enveloping algebra. Let $\mathcal{A} = \mathbb{Z}[q, q^{-1}]$. There is an integral form $U_{\mathcal{A}}$ of $U_{\mathbb{C}(q)}$ due to Lusztig [L] which is an $\mathcal{A}$-subalgebra of $U_{\mathbb{C}(q)}$. As in the classical case, the quantum hyperalgebra U_q is defined by base change; $U_q = U_{\mathcal{A}} \otimes_{\mathcal{A}} K$. When we let $q = 1$, we recover the classical theory so results about U_q give results about the classical case. When q is taken to be a pth root of unity in K, where K is a field of characteristic zero, the representation theory of U_q is similar to that of U_L when L has characteristic p. There are many conjectures regarding this similarity (see [CP] Conj. 11.2.13).

Our main aim is to define a quantized version of the Désarménien matrix using elements in U_q. There are quantum analogues of the familiar Schur and Weyl modules called q-Schur modules and q-Weyl modules and we discuss these modules in some detail. Our quantized Désarménien matrix gives an algorithm for writing a quantum bideterminant in the q-Schur module as a K-linear combination of quantum bideterminants given by semistandard λ-tableaux. We give a quantum version of Green's

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Weyl module in Section 5.4. This requires the introduction of a U_q-contravariant form since the canonical form used in the previous chapter proves not to be U_q-contravariant. We use our matrix and the techniques from Chapter 4 to prove the standard basis theorem for the q-Weyl module. The standard basis theorem was proved previously by Dipper and James [DJ] and then by R. Green [Gr] using a different approach, but the proof we give is substantially different from the former two. A consequence of our proof is that the bases for the q-Schur and q-Weyl modules are connected by the quantized Désarménien matrix. We conclude the chapter by discussing the spanning set for the irreducible U_q-module $L_q(\lambda)$ that arises as a corollary to the standard basis theorem.

5.2 The quantized hyperalgebra

In this section, we construct the quantum analogue $U_{q,K}$ of the hyperalgebra for $GL(n,K)$. We then discuss the quantum analogues of the Schur and Weyl modules. Further details may be found in [J], [CP], or [Ta].

Let q be an indeterminate and let $\mathbb{C}(q)$ be the field of rational functions of q with coefficients in $\mathbb{C}$. The *quantized enveloping algebra*, denoted $U_{\mathbb{C}(q)}$, is the associative $\mathbb{C}(q)$-algebra with generators $E_i,\ F_i,\ K_j,\ K_j^{-1}$ with $1 \le i < n,\ 1 \le j \le n$, subject to the relations that follow. We let $K_{i,i+1}$ denote $K_iK_{i+1}^{-1}$.

$$K_iK_i^{-1} = K_i^{-1}K_i = 1 \qquad K_iK_j = K_jK_i$$
$$K_iE_j = q^{\delta_{i,j}-\delta_{i,j+1}}E_jK_i \qquad K_iF_j = q^{\delta_{i,j+1}-\delta_{ij}}F_jK_i$$
$$E_iE_j = E_jE_i \text{ if } |i-j| > 1 \qquad F_iF_j = F_jF_i \text{ if } |i-j| > 1$$
$$E_iF_j - F_jE_i = \delta_{ij}\frac{K_{i,i+1}-K_{i,i+1}^{-1}}{q-q^{-1}}$$
$$E_i^2E_j - (q+q^{-1})E_iE_jE_i + E_jE_i^2 = 0 \text{ if } |i-j| = 1$$
$$F_i^2F_j - (q+q^{-1})F_iF_jF_i + F_jF_i^2 = 0 \text{ if } |i-j| = 1.$$

The algebra $U_{\mathbb{C}(q)}$ is a Hopf algebra over $\mathbb{C}(q)$ with comultiplication

$$\Delta : U_{\mathbb{C}(q)} \to U_{\mathbb{C}(q)} \otimes U_{\mathbb{C}(q)}$$

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defined by

$$\Delta(E_i) = 1 \otimes E_i + E_i \otimes K_{i,i+1}, \quad \Delta(F_i) = K_{i,i+1}^{-1} \otimes F_i + F_i \otimes 1, \quad \Delta(K_i) = K_i \otimes K_i. \quad (5.1)$$

Note that $U_{\mathbb{C}(q)}$ possesses other comultiplications which are preferred by some authors.

Given $a, c \in \mathbb{N}$, we define

$$[a] = \frac{q^a - q^{-a}}{q - q^{-1}}.$$

The *quantum factorial* is defined as

$$[a]! = [a][a-1] \cdots [1].$$

Given $X \in U_{\mathbb{C}(q)}$ and $\alpha \in \mathbb{N}$, let $X^{(\alpha)}$ denote the divided power $\frac{X^\alpha}{[\alpha]!}$ and define

$$\binom{K_j}{\alpha} = \prod_{s=1}^{\alpha} \frac{q^{-s+1}K_j - q^{s-1}K_j^{-1}}{q^s - q^{-s}}.$$

Let $\mathcal{A} = \mathbb{Z}[q, q^{-1}]$ be the ring of Laurent polynomials in q. The *integral form* $U_{\mathcal{A}}$ of $U_{\mathbb{C}(q)}$ is the Hopf $\mathcal{A}$-subalgebra of $U_{\mathbb{C}(q)}$ generated by the set

$$\left\{ E_i^{(\alpha)}, F_i^{(\alpha)}, K_j, K_j^{-1}, \binom{K_j}{\alpha} : \alpha \in \mathbb{N},\ 1 \le i < n,\ 1 \le j \le n \right\}.$$

If we consider our arbitrary field K with q a unit in K, we may regard K as an $\mathcal{A}$-algebra with $q \in \mathcal{A}$ acting as multiplication by q in K. Thus $U_{\mathcal{A}} \otimes_{\mathcal{A}} K$ is an $\mathcal{A}$-algebra.

Definition 5.1 *The quantum hyperalgebra is defined as* $U_{K,q} = U_{\mathcal{A}} \otimes_{\mathcal{A}} K$.

We shall drop the K and write U_q for $U_{K,q}$. We write $u \in U_q$ to mean the image of u in U_q via the map $\phi : U_{\mathcal{A}} \to U_q$ defined by $\phi(u) = u \otimes 1$.

Let $V_{\mathcal{A}}$ be the $\mathcal{A}$-module generated by $v_1, \cdots, v_n$. Then $V_{\mathcal{A}}$ is a $U_{\mathcal{A}}$-module, called the *natural module*, via

$$E_i v_j = \delta_{i+1,j} v_i, \quad F_i v_j = \delta_{ij} v_{i+1}, \quad K_i v_j = q^{\delta_{ij}} v_j.$$

The coassociative comultiplication Δ defined in (5.1) defines a $U_{\mathcal{A}}$-module action on $V_{\mathcal{A}}^{\otimes r}$. Thus $V_{\mathcal{A}}^{\otimes r} \otimes_{\mathcal{A}} K$ is a U_q-module. We drop the $\mathcal{A}$ and write simply V and $V^{\otimes r}$.

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Example 5.1

Suppose that V has dimension 3 and let $v_3 \otimes v_2 \in V \otimes V$. Then

$$E_2(v_3 \otimes v_2) = v_3 \otimes E_2 v_2 + E_2 v_3 \otimes K_{2,3} v_2 = q v_2 \otimes v_2.$$

□

Define $z_\lambda \in V^{\otimes r}$ by

$$z_\lambda = \sum_{\sigma \in C(\widehat{T}_\lambda)} (-q)^{-l(\sigma)} v_{I(\lambda)\cdot\sigma}$$

where $l(\sigma)$ denotes the length of the permutation $\sigma \in S_r$ and, as always, $I(\lambda) \in I(n,r)$ is the subsequence that satisfies $T_\lambda = T_{I(\lambda)}$. Define $\Delta(\lambda)_{\mathcal{A}}$ to be the left $U_{\mathcal{A}}$-submodule of $V^{\otimes r}$ generated by z_λ.

Definition 5.2 *The q-Weyl module is the left U_q-module*

$$\Delta_q(\lambda) = \Delta(\lambda)_{\mathcal{A}} \otimes_{\mathcal{A}} K.$$

There is also a quantum analogue of $A(n)$. An $n \times n$ matrix A with entries in an $\mathcal{A}$-algebra is a *q-matrix* if its entries satisfy the following relations:

$$\begin{array}{ll} a_{ik}a_{il} = q^{-1}a_{il}a_{ik} & 1 \le i < j \le n \\ a_{ik}a_{jk} = q^{-1}a_{jk}a_{ik} & 1 \le k < l \le n \\ a_{il}a_{jk} = a_{jk}a_{il} & 1 \le i < j \le n, 1 \le k < l \le n \\ a_{ik}a_{jl} - a_{jl}a_{ik} = (q^{-1} - q)a_{il}a_{jk} & 1 \le i < j \le n,\ 1 \le k < l \le n. \end{array} \tag{5.2}$$

Define $A_q(n)$ be the $\mathcal{A}$-algebra with generators x_{ij} subject to the relations which require $X = (x_{ij})_{1 \le i,j \le n}$ to be a q-matrix.

Example 5.2

We have the following relations in $A_q(2)$:

$$\begin{array}{ll} x_{11}x_{12} = q^{-1}x_{12}x_{11} & x_{21}x_{22} = q^{-1}x_{22}x_{21} \\ x_{11}x_{21} = q^{-1}x_{21}x_{11} & x_{12}x_{22} = q^{-1}x_{22}x_{12} \\ x_{12}x_{21} = x_{21}x_{12} & x_{11}x_{22} - x_{22}x_{11} = (q^{-1} - q)x_{12}x_{21}. \end{array}$$

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It can be checked (see [Ta]) that $A_q(n)$ is a $U_{\mathcal{A}}$-module with action

$$E_i x_{kl} = \delta_{i+1,l} x_{ki}, \quad F_i x_{kl} = \delta_{ik} x_{k,i+1}, \quad K_i x_{kl} = q^{\delta_{il}} x_{kl}$$

and $E_i(PQ) = P(E_iQ) + (E_iP)(K_{i,i+1}Q)$, $F_i(PQ) = (K_{i,i+1}^{-1}P)(F_iQ) + (F_iP)Q$, $K_i(PQ) = (K_iP)(K_iQ)$ where $P, Q \in A_q(n)$. These *Leibniz formulas* come from (5.1). An inductive argument shows that

$$\begin{aligned} E_i(P_1P_2\cdots P_s) &= P_1P_2\cdots(E_iP_s) + P_1\cdots(E_iP_{s-1})(K_{i,i+1}P_s) \\ &+ \cdots + (E_iP_1)(K_{i,i+1}P_2)\cdots(K_{i,i+1}P_s) \quad \text{and} \end{aligned} \tag{5.3}$$

$$\begin{aligned} F_i(P_1P_2\cdots P_s) &= (K_{i,i+1}^{-1}P_1)\cdots(K_{i,i+1}^{-1}P_{s-1})(F_iP_s) \\ &+ \cdots + (K_{i,i+1}^{-1}P_1)(F_iP_2)\cdots P_s + (F_iP_1)P_2\cdots P_s, \end{aligned}$$

where $P_1, \ldots, P_s \in A_q(n)$ and $1 \leq i \leq n$. The $U_{\mathcal{A}}$-module $A_q(n)$ becomes a U_q-module via base change.

Example 5.3

Take $x_{12}^2 \in A_q(2)$ and $E_1^2 \in U_{\mathcal{A}}$. Then

$$\begin{aligned} E_1^2 x_{12}x_{12} &= E_1(x_{12}x_{11} + q^{-1}x_{11}x_{12}) \\ &= qx_{11}x_{11} + q^{-1}x_{11}x_{11} \\ &= (q+q^{-1})x_{11}^2 \end{aligned}$$

□

The *quantum determinant* of the matrix $X = (x_{ij})_{1\leq i,j\leq n}$ lies in $A_q(n)$ and is defined by

$$det_qX = \sum_{\sigma\in S_n} (-q)^{-l(\sigma)} x_{1\sigma(1)} \cdots x_{n\sigma(n)}.$$

If $I, J \in I(n,r)$, the *quantum minor* $det_qX_J^I$ is the quantum determinant of the submatrix of X with rows indexed by $I = (i_1, \ldots, i_r)$ and columns by $J = (j_1, \ldots, j_r)$;

$$det_qX_J^I = \sum_{\sigma\in S_r} (-q)^{-l(\sigma)} x_{i_1 j_{\sigma 1}} \ldots x_{i_r j_{\sigma r}}.$$

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Example 5.4

Let $X = (x_{ij})_{1 \le i,j \le 3}$. Then $det_q X^{1,2}_{1,3} = x_{11}x_{23} - q^{-1}x_{13}x_{21}$. $\square$

Given λ-tableaux S and T, the *quantum bideterminant* $(S : T) \in A_q(n)$ is defined as

$$(S : T)_q = det_q X^{S(1)}_{T(1)} det_q X^{S(2)}_{T(2)} \cdots det_q X^{S(s)}_{T(s)}$$

where s is the number of columns in the Young diagram of shape λ and $T(i)$ denotes the subsequence corresponding to the entries in the ith column of T. As always, we are only interested in bideterminants of the form $(T_\lambda : T)_q$ and take $[T]$ to represent such a bideterminant. All bidetermiants $[T]$ in this chapter are taken to represent quantum bideterminants. Using (5.3), we may determine the action of E_i, F_i or K_i on a bideterminant $[T]$. Suppose that T is a λ-tableau with s columns. Let $[T_i]$ be the bideterminant which is given by the one-column tableau that corresponds to the ith column of T so that $[T] = [T_1][T_2] \cdots [T_s]$, where s is the number of columns in T. Then

$$\begin{aligned} E_i[T] &= [T_1] \cdots (E_i[T_s]) + [T_1] \cdots (E_i[T_{s-1}])(K_{i,i+1}[T_s]) + \\ &\quad \cdots + (E_i[T_1])(K_{i,i+1}[T_2]) \cdots (K_{i,i+1}[T_s]), \end{aligned} \tag{5.4}$$

$$\begin{aligned} F_i[T] &= (K^{-1}_{i,i+1}[T_1]) \cdots (K^{-1}_{i,i+1}[T_{s-1}])(F_i[T_s]) + \cdots \\ &\quad + (K^{-1}_{i,i+1}[T_1])(F_i[T_2]) \cdots [T_s] + (F_i[T_1])([T_2]) \cdots [T_s]. \end{aligned}$$

A proof of the following theorem can be found in [NYM]. It is also a consequence of (5.4) combined with Lemma 5.5 which we prove in the following section.

Theorem 5.1 *The K-span of the quantum bideterminants $[T]$, where T is a λ-tableau, is a U_q-invariant submodule of $A_q(n)$.*

Definition 5.3 *The q-Schur module, denoted $\nabla_q(\lambda)$, is the K-span of the quantum bideterminants $[T]$ determined by λ-tableaux T.*

In the following section, we will construct the contravariant dual to $\nabla_q(\lambda)$ as we did in the symplectic case and, as stated in the following theorem, the module we construct will be isomorphic to $\Delta_q(\lambda)$. For a proof see [D2], Proposition 4.1.6.

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Theorem 5.2 *The q-Weyl module is the contravariant dual to $\nabla_q(\lambda)$; that is*

$$(\nabla_q(\lambda))^\circ \cong \Delta_q(\lambda).$$

We shall also use the following quantum analogue of the basis theorem for $\nabla_q(\lambda)$ in Section 2. There are several different proofs of this result; see for instance [Hu] or [HZ].

Theorem 5.3 *The set $\{[T] : T$ is a semistandard λ-tableau$\}$ forms a K-basis for $\nabla_q(\lambda)$.*

5.3 A quantum version of the Désarménien matrix

In this section we use operators in U_q to define a quantum version of the Désarménien matrix. We introduce a number of technical lemmas which will ease our task. Our first lemma may be proved using the relations (5.2).

Lemma 5.4 *Let T a one-column λ-tableau.*

1. *If T contains two entries which are equal, then $[T] = 0$.*
2. *Let S be a λ-tableau which is the same as T except that two of the entries have been interchanged. Then $[T] + (-q)^{-1}[S] = 0$.*

Proof. See [TT], Proposition 2.1. □

Lemma 5.5 *Let T be a one-column tableau for which $[T] \neq 0$.*

1. *If T contains an $i+1$, we have $E_i[T] = [S]$ where S is the λ-tableau that is identical to T except that the $i+1$ has been replaced by an i. If T does not contain an $i+1$, then $E_i[T] = 0$.*
2. *If T contains an i, then $K_i[T] = q[T]$ and $K_i[T] = [T]$ otherwise.*

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Proof. Let $T = T_J$ where $J = (j_1, \ldots, j_r) \in I(n,r)$ and $j_1 < j_2 < \cdots < j_r$. By definition,

$$[T_J] = \sum_{\sigma \in S_r} (-q)^{-l(\sigma)} x_{1j_{\sigma 1}} \cdots x_{rj_{\sigma r}}.$$

If T does not contain an $i+1$, then none of the j_k is equal to $i+1$ so $E_i[T_J] = 0$ since $E_i(x_{1j_{\sigma 1}} \cdots x_{rj_{\sigma r}}) = 0$ for all $\sigma \in S_r$. So suppose that T does contain an $i+1$ and let $j_l = i+1$ where $1 \leq l \leq r$. Since $[T]$ is non-zero, this is the only element in the subsequence which is equal to $i+1$. If none of the j_k is equal to i for $1 \leq k \leq r$, $k \neq l$, then $K_{i,i+1} j_k = j_k$ for $k \neq l$, so $E_i[T_j] = [T_M]$, where M is identical to J except that $j_l = i$, since E_i simply changes an $i+1$ to an i in each of the summands of T_J.

Suppose that one of the $j_k = i$ and $j_l = i+1$ for $1 \leq k, l \leq r$. We will show that in this case $E_i[T_J] = 0$. To accomplish this, we prove that for each $\sigma \in S_r$ there is a unique $\theta \in S_r$ for which

$$E_i(-q)^{-l(\sigma)} x_{1j_{\sigma 1}} \cdots x_{rj_{\sigma r}} + E_i(-q)^{-l(\theta)} x_{1j_{\theta 1}} \cdots x_{j_r \theta r} = 0.$$

From this it certainly follows that $E_i \sum_{\sigma \in S_r} (-q)^{-l(\sigma)} x_{1j_{\sigma 1}} \cdots x_{rj_{\sigma r}} = 0$.

Fix $\sigma \in S_r$ and suppose that $j_{\sigma s} = i$ and $j_{\sigma t} = i+1$. Assume $s < t$ for the alternative case is proved similarly. Let θ be the unique permutation in S_r with $j_{\theta s} = i+1$, $j_{\theta t} = i$, and $j_{\theta m} = j_{\sigma m}$ for $m \neq s,t$. Then

$$\begin{aligned}
E_i(-q)^{-l(\theta)} x_{1j_{\theta 1}} \cdots x_{s,i+1} \cdots x_{ti} \cdots x_{j_{\theta r}} &= (-q)^{-l(\theta)} x_{1j_{\sigma 1}} \cdots (E_i x_{s,i+1}) \cdots (K_{i,i+1} x_{ti}) \cdots \\
&= (-q)^{-(l(\sigma)+1)} q x_{1j_{\sigma 1}} \cdots x_{si} \cdots x_{ti} \cdots x_{j_{\sigma r}} \\
&= -(-q)^{l(\sigma)} x_{1j_{\sigma 1}} \cdots x_{si} \cdots x_{ti} \cdots x_{j_{\sigma r}} \\
&= -E_i(-q)^{-l(\sigma)} x_{1j_{\sigma 1}} \cdots x_{si} \cdots x_{t,i+1} \cdots x_{j_{\sigma r}},
\end{aligned}$$

so the sum of the two is zero. It follows that $E_i[T_J] = 0$ and the proof of 1. is complete.

To prove 2., note that since $[T]$ is non-zero, there can be at most one i in J. The result then follows easily from the definition of $[T_J]$ and the fact that $K_i x_{kl} = q^{\delta_{il}} x_{kl}$. $\square$

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Remark. By altering the proof of 1., we can prove that if T is a one-column tableau that contains an i, then $F_i[T] = [S]$ where S comes from T by changing the i in T to an $i+1$ and $F_i[T] = 0$ otherwise.

Using Lemma 5.5 and (5.4), we can compute the action of an element of U_q on a bideterminant $[T]$.

Example 5.5

$$\begin{aligned}
E_{12}\left[\begin{array}{|c|c|c|}\hline 2&2&1\\\hline 3&3\\\cline{1-2}\end{array}\right] &= \left[\begin{array}{|c|c|}\hline 2&2\\\hline 3&3\\\hline\end{array}\right] E_{12}\left[\begin{array}{|c|}\hline 1\\\hline\end{array}\right] + \left[\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}\right] E_{12}\left[\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}\right] K_{1,2}\left[\begin{array}{|c|}\hline 1\\\hline\end{array}\right] \\
&+ E_{12}\left[\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}\right] K_{1,2}\left[\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}\right] K_{1,2}\left[\begin{array}{|c|}\hline 1\\\hline\end{array}\right] \\
&= q\left[\begin{array}{|c|c|c|}\hline 2&1&1\\\hline 3&3\\\cline{1-2}\end{array}\right] + q^{-1}q\left[\begin{array}{|c|c|c|}\hline 1&2&1\\\hline 3&3\\\cline{1-2}\end{array}\right]
\end{aligned}$$

□

We are interested in working with operators in U_q which serve as the quantum analogues of the operators e_{ij} and f_{ij} in U_K. Let $E_i = E_{i,i+1}$ and $F_i = F_{i,i+1}$ and for $1 \leq i < j \leq n$ with $|i-j| > 1$, define operators E_{ij} and F_{ij} inductively as follows:

$$E_{ij} = q^{-1}E_iE_{i+1,j} - E_{i+1,j}E_i \text{ and } F_{ij} = q^{-1}F_{i+1,j}F_i - F_iF_{i+1,j}.$$

Example 5.6

$$\begin{aligned}
E_{14} &= q^{-1}E_1E_{24} - E_{24}E_1 \\
&= q^{-1}E_1(q^{-1}E_2E_{34} - E_{34}E_2) - (q^{-1}E_2E_{34} - E_{34}E_2)E_1
\end{aligned}$$

□

Due to the recursive nature of the operators E_{ij} where $|i-j| > 1$, it can be quite difficult to compute $E_{ij}[T]$ for an arbitrary λ-tableau T. Fortunately we will frequently be encountering tableaux for which the task becomes simplified. We illustrate the type of tableau in which we are interested in the next example and prove a general result about such tableaux in Lemma 5.6.

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Example 5.7

Let $T = \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 2 & 2 \\ \cline{1-4} 3 & 3 & 6 & 8 \\ \cline{1-4} 4 & 5 \\ \cline{1-2} \end{array}$. Then

$$\begin{aligned} E_{36}[T] &= (q^{-1}E_3E_{46} - E_{46}E_3)[T] \\ &= q^{-1}E_3E_{46}[T], \text{ since } E_3[T] = 0 \\ &= q^{-1}E_3(q^{-1}E_4E_5 - E_5E_4)[T] \\ &= q^{-2}E_3E_4E_5[T] \text{ since } E_3E_5E_4[T] = 0. \end{aligned}$$

Lemma 5.6 *Suppose that T is a column increasing λ-tableau that coincides with T_λ in the first $i-1$ rows. Suppose also that if m is an integer with $i < m < j$, then m does not appear in the ith row of T. Then*

$$E_{ij}[T] = q^{i-j+1}E_iE_{i+1}\cdots E_{j-1}[T].$$

Proof. We fix j and induct on i. For the case where $i = j-1$, the result is trivial. Now assume that the conclusion holds for all k with $i < k \le j-1$; that is if U is a tableau that coincides with T_λ in the first $i-1$ rows and the ith row of U contains no entries between k and j, then $E_{kj}[U] = q^{k-j+1}E_kE_{k+1}\cdots E_{j-1}[U]$. Since the ith row of T contains no $i+1$'s and the first i rows of T are identical to T_λ, any $i+1$'s which occur in T must occur below the ith row. Since T is column increasing and each of $1,\ldots,i-1$ appear in the first $i-1$ rows above such an i, there must be an i above each so we have $E_i[T] = 0$. Thus,

$$\begin{aligned} E_{ij}[T] &= q^{-1}E_iE_{i+1,j}[T] - E_{i+1,j}E_i[T] \\ &= q^{-1}E_i(q^{i+1-j+1}E_{i+1}\cdots E_{j-1})[T] \\ &= q^{i-j+1}E_i\cdots E_{j-1}[T]. \end{aligned}$$

□

The following example should help to motivate Theorem 5.7. Note that a special case of the type of tableau that satisfies the hypotheses of Theorem 5.7 is a one-row tableau with all entries equal to j.

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Example 5.8

Let $T = \boxed{5\,|\,5}$. Then

$$\begin{aligned}
E_{25}^2[T] &= E_{25}q^{-2}E_2E_3E_4[T] \text{ by Lemma 5.6} \\
&= q^{-2}E_{25}E_2E_3\left(\boxed{5\,|\,4} + q^{-1}\boxed{4\,|\,5}\right) \\
&= q^{-2}E_{25}\left(\boxed{5\,|\,2} + q^{-1}\boxed{2\,|\,5}\right) \\
&= (q^{-2})^2E_2E_3E_4\left(\boxed{5\,|\,2} + q^{-1}\boxed{2\,|\,5}\right) \\
&= (q^{-2})^2(q+q^{-1})\boxed{2\,|\,2} \\
&= (q^{-2})^2[2]!\boxed{2\,|\,2}.
\end{aligned}$$

□

For the proof of our next theorem, we introduce some notation. Given $I \in I(n,r)$, define $d(I)$ to be the number of pairs (a,b) which satisfy $a < b$ and $i_a < i_b$. For instance, if $I = (1,2,1,4,3)$, then $d(I) = 7$. In our proof, all r-tuples I contain only i's and j's where $i < j$. For example, if $I = (i,j,i,i,i,i,j,i)$, then $d(I) = 6$.

Theorem 5.7 *Let T be a semistandard λ-tableau that coincides with T_λ in the first $i-1$ rows. Suppose that T contains r j's in the ith row and if m is an integer with $i < m < j$ then m does not occur in the ith row of T. Then*

$$E_{ij}^r[T] = (q^{i-j+1})^r[r]![S]$$

where S is identical to T except that the r j's in the ith row of T have been replaced by i's.

Proof. We want to consider $E_{ij}^s[T]$ for $1 \le s \le r$ and we first argue that in doing so, we may assume that T is a one-row tableau which contains r j's. By Lemma 5.6, $E_{ij}[T] = q^{i-j+1}E_iE_{i+1}\cdots E_{j-1}[T]$. Suppose that the j's in the ith row of T appear in columns ρ through $\rho' = \rho + r$. Only the integers $1,\ldots,i-1$ appear in the first $i-1$ rows and T is semistandard, so if $m > \rho'$ there are no entries between $i-1$ and $j+1$ in the mth column $T(m)$ of T. So if $m > \rho'$ and $i \le l \le j-1$, we have

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$K_{l,l+1}^{-1}[T(m)] = [T(m)]$. Thus the operators $E_i, \cdots, E_{j-1}$ have no effect on any of the columns $T(m)$ for $m > \rho'$.

We now claim that the only non-zero bideterminants in the sum $E_iE_{i+1}\cdots E_{j-1}[T]$ are those that come from applying the operators to the j's in the ith row of T. If m is an integer with $i < m < j$ and if m appears in T then by hypothesis, m must appear below the ith row. Furthermore, since T is column increasing and the integers $1, \ldots, i-1$ appear in the first $i-1$ rows above m and no integer between i and j appears in the ith row of T, there must be an i in the ith row of T above m. Now if $[S]$ is a bideterminant in the sum $E_{j-1}[T]$ that comes from changing a j below the ith row of T to a $j-1$ then we claim that $E_iE_{i+1}\cdots E_{j-2}[S] = 0$. Since there are no $j-1$'s in the ith row of S, any bideterminant in the sum $E_{j-2}[S]$ arises from changing a $j-1$ below the ith row of S to a $j-2$, either the $j-1$ that came from changing the j below the ith row to a $j-1$ or some other $j-1$ that appears below the ith row. Either way, there is an i in the ith row of S above this $j-1$. Continuing, we see that any bideterminant in the sum $E_iE_{i+1}\cdots E_{j-1}[S]$ comes from changing an $i+1$ below the ith row to an i in a bideterminant $[S']$ in the sum $E_{i+1}\cdots E_{j-1}[S]$ which appears below the ith row. Since there is an i in the ith row above such an i, we have $E_i[S'] = 0$ so $E_iE_{i+1}\cdots E_{j-2}[S] = 0$. Thus, we may assume that all bideterminants in the sum $E_{j-1}[T]$ come from changing a j in the ith row of T to a $j-1$.

We then apply the same argument to see that any non-zero bideterminant $[S]$ in the sum $E_{j-2}E_{j-1}[T]$ that comes from a bideterminant $[T']$ in the sum $E_{j-1}[T]$ by replacing a $j-1$ below the ith row of T' with a $j-2$ satisfies $E_iE_{i+1}\cdots E_{j-3}[S] = 0$. Inductively, we see that every bideterminant in the sum $E_iE_{i+1}\cdots E_{j-1}[T]$ comes from changing a j in the ith row to an i and so the operator E_{ij} effects only the portion of T in columns ρ through ρ' of T. We apply the same argument to $E_{ij}^2[T]$; that is if $[U]$ is a bideterminant in the sum $E_{ij}[T]$, then it is enough to consider what E_{ij} does to the entries in columns ρ through ρ' of the ith row of U and inductively we see that in considering $E_{ij}^s[T]$, it is enough to see what happens when T is a one-row tableau which contains r j's; that is $[T] = x_{1j}^r$.

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We will show that $E_{ij}^s[T] = (q^{i-j+1})^s[s]!\sum_I q^{-d(I)}[T_I]$ for $1 \le s \le r$ where the sum runs over all r-tuples I which contain s i's and $r-s$ j's. This will prove the statement, for when we let $s = r$, the only r-tuple I that satisfies the hypotheses of our claim is that which contains r i's and no j's and for this I we have $d(I) = 0$. Then we will have $E_{ij}^r[T] = (q^{i-j+1})^r[r]![S]$.

We first let $s = 1$. By Lemma 5.6, $E_{ij}[T] = q^{i-j+1}E_iE_{i+1}\cdots E_{j-1}[T]$ and

$$\begin{aligned} E_{j-1}[T] &= E_{j-1}x_{1j}^r = x_{1j}\cdots x_{1,j-1} + q^{-1}x_{1j}\cdots x_{1,j-1}x_{1j} + \cdots + q^{-(r-1)}x_{1,j-1}\cdots x_{1j} \\ &= \sum_M q^{-d(M)}[T_M], \end{aligned}$$

where the sum runs over all r-tuples M which contain one $j-1$ and $r-1$ j's. Thus

$$\begin{aligned} E_{ij}[T] &= q^{i-j+1}E_iE_{i+1}\cdots E_{j-1}[T] \\ &= q^{i-j+1}E_iE_{i+1}\cdots E_{j-2}\sum_M q^{-d(M)}[T_M]. \end{aligned}$$

Since $K_{l,l+1}x_{1j} = x_{1j}$ for $i \le l \le j-1$, we have

$$q^{i-j+1}E_iE_{i+1}\cdots E_{j-2}\sum_M q^{-d(M)}[T_M] = q^{i-j+1}\sum_J q^{-d(J)}[T_J],$$

where the sum runs over the r-tuples J which contain one i and $r-1$ j's.

Now suppose that for all $1 \le m \le s$ we have $E_{ij}^m[T] = (q^{i-j+1})^m[m]!\sum_I q^{-d(I)}[T_I]$ where the sum runs over all r-tuples I which contain m i's and $r-m$ j's. Then

$$E_{ij}^s[T] = E_{ij}E_{ij}^{s-1}[T] = (q^{i-j+1})^{s-1}[s-1]!\sum_I q^{-d(I)}E_{ij}[T_I] \tag{5.5}$$

where the sum is over the r-tuples I which contain $s-1$ i's and $r-(s-1)$ j's. Since for each I, every tableau in the sum $E_{ij}[T_I] = q^{i-j+1}E_iE_{i+1}\cdots E_{j-1}[T_I]$ results from changing a j to an i in T_I, we have

$$\sum_I E_{ij}[T_I] = q^{i-j+1}\sum_J \gamma_{IJ}[T_J] \tag{5.6}$$

where $\gamma_{IJ} \in K$ and the sum runs over all J which contain s i's and $r-s$ j's. Furthermore, $\gamma_{IJ} = 0$ in case J cannot be formed by changing a j in I to an i. We will examine the coefficients γ_{IJ} a little later in the proof.

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Substituting (5.6) into (5.5) we have

$$\begin{aligned} E_{ij}^s[T] &= (q^{i-j+1})^{s-1}[s-1]! \sum_I q^{-d(I)} E_{ij}[T_I] \\ &= (q^{i-j+1})^s[s-1]! \sum_I q^{-d(I)} \sum_J \gamma_{IJ}[T_J] \\ &= (q^{i-j+1})^s[s-1]! \sum_J \sum_I q^{-d(I)} \gamma_{IJ}[T_J], \end{aligned}$$

where the sum runs over all J which contain s i's and $r-s$ j's. To complete the induction, then, we need to prove that for each J with s i's and $r-s$ j's, we have

$$\sum_I q^{-d(I)} \gamma_{IJ} = [s] q^{-d(J)},$$

where the sum is over all I which yield J when a j in I is changed to an i. For a fixed J with s i's and $r-s$ j's, , let I_1 in the sum $\sum_I q^{-d(I)}\gamma_{IJ}$ be the r-tuple that comes from changing the first i from the right in J to a j, I_2 the r-tuple that comes from changing the second i from the right in J to a j, etc.

Then

$$\sum_I q^{-d(I)} \gamma_{IJ} = \sum_{k=1}^s q^{-d(I_k)} \gamma_{I_k J}.$$

Let α_1 be the number of j's to the right of the right-most i in J, α_2 the number of j's to the right of the second i from the right in J, etc., so that $d(J) = \sum_{k=1}^s \alpha_k$. Then since the place in which the kth i from the right in J occurs is a j in I_k, we have

$$\begin{aligned} d(I_k) &= \alpha_1 + \cdots + \alpha_{k-1} + (\alpha_{k+1}+1) + \cdots + (\alpha_s + 1) \\ &= \sum_{i=1}^s \alpha_i - \alpha_k + (s-k) \\ &= d(J) - \alpha_k + (s-k). \end{aligned}$$

Now, to examine $\gamma_{I_k J} \in K$ we apply $E_i E_{i+1} \cdots E_{j-1}$ to $[T_{I_k}]$ and look at the coefficient of the portion of this sum that gives J. By definition of I_k, this is the portion of the sum which changes the j in I_k which is in the same position as the kth i from the right in J to an i. We write $\pi_k(E_l)[T]$ to mean the portion of the sum

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$E_l[T]$ that effects the entry in this position of T_{I_k}. Then since there are α_k j's to the right of this j,

$$\pi_k(E_i)\pi_k(E_{i+1})\cdots\pi_k(E_{j-1})[T_{I_k}] = \pi_k(E_i)\pi_k(E_{i+1})\cdots\pi_k(E_{j-2})q^{-\alpha_k}[T_{I'}]$$

where I' is the r-tuple obtained by changing the j in question to a $j-1$. Now,

$$\pi_k(E_i)\pi_k(E_{i+1})\cdots\pi_k(E_{j-2})q^{-\alpha_k}T_{I'} = q^{-\alpha_k}\pi_k(E_i)T_{I''}$$

where I'' is the r-tuple that corresponds to changing the j to an $i+1$ and

$$q^{-\alpha_k}\pi_k(E_i)T_{I''} = q^{-\alpha_k}q^{k-1}T_J$$

since there are $k-1$ i's to the right of the j by definition.

Thus we have

$$\gamma_{I_kJ} = q^{-\alpha_k}q^{k-1}$$

for each k. Then

$$q^{-d(I_k)}\gamma_{I_kJ} = q^{-d(I_k)}q^{-\alpha_k}q^{k-1} = q^{-d(J)+\alpha_k-(s-k)}q^{-\alpha_k}q^{k-1} = q^{-d(J)}q^{-s+2k-1}.$$

Recall that

$$[s] = \frac{q^s - q^{-s}}{q-q^{-1}} = \sum_{k=1}^{s} q^{-s+2k-1}.$$

It follows that

$$\sum_{k=1}^{s} q^{-d(I_k)}\gamma_{I_kJ} = \sum_{k=1}^{s} q^{-d(J)}q^{-s+2k-1} = \sum_{k=1}^{s} q^{-d(J)}[s]$$

and this completes the proof. □

We now define operators $E_{T,q}$ and $F_{T,q}$ in U_q which are the quantum analogues of the operators E_T and F_T that were defined in Chapter 4. The operators $F_{T,q}$ also appear in [Gr] and we note that our definition is slightly different than R. Green's since our definition of F_{ij} differs from the one used there.

Given a semistandard λ-tableau T with $k \leq n$ rows, define

$$E_{T,q} = \prod_{\substack{1\leq i\leq k \\ i<j\leq n}} E_{ij}^{(\gamma_{ij})} \text{ and } F_{T,q} = \prod_{\substack{1\leq i\leq k \\ i<j\leq n}} F_{ij}^{(\gamma_{ij})}$$

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where γ_{ij} is the number of entries equal to j in row i of T. The product $E_{T,q}$ is ordered

$$E_{T,q} = E_{k-1,k}^{(\gamma_{k-1,k})} E_{k-2,k}^{(\gamma_{k-2,k})} \cdots E_{2k}^{(\gamma_{2k})} \cdots E_{23}^{(\gamma_{23})} E_{1k}^{(\gamma_{1k})} \cdots E_{13}^{(\gamma_{13})} E_{12}^{(\gamma_{12})},$$

and we order $F_{T,q}$ in the opposite way.

We are now ready to give quantum analogues of the definitions and theorems we gave in Chapter 4.

Definition 5.4 *Given a semistandard λ-tableau T and a column increasing λ-tableau S, define $\Omega_q(S,T) = c$ where c is the coefficient of $[T_\lambda]$ in the sum $\overline{E}_{T,q}[S]$.*

Definition 5.5 *The quantized Désarménien matrix is the matrix*

$$\Omega_q = [\Omega_q(T_I, T_J)]_{I,J \in \mathcal{I}_\lambda}$$

where $\mathcal{I}_\lambda = \{I \in I(n,r) : T_I$ is a semistandard λ-tableau$\}$.

We begin with an example to motivate the next theorem.

Example 5.9

Let $T = \begin{array}{|c|c|c|}\hline 1 & 1 & 6 \\ \hline 5 & 5 \\ \cline{1-2}\end{array}$. Then $E_{T,q} = E_{25}^{(2)} E_{16}$ and if $i = 1$ and $j = 6$, then T satisfies the hypotheses of Theorem 5.7. Thus,

$$\begin{aligned}
E_{25}^{(2)} E_{16}[T] &= q^{-4} E_{25}^{(2)} \left[\begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 5 & 5 \\ \cline{1-2}\end{array} \right] \\
&= q^{-4} \frac{1}{[2]!} E_{25}^2 \left[\begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 5 & 5 \\ \cline{1-2}\end{array} \right] \\
&= q^{-4} \frac{1}{[2]!} (q^{-2})^2 [2]! \left[\begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 2 & 2 \\ \cline{1-2}\end{array} \right] \quad \text{by Theorem 5.7} \\
&= q^{-4} (q^{-2})^2 \left[\begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 2 & 2 \\ \cline{1-2}\end{array} \right].
\end{aligned}$$

Thus $\Omega_q(T,T) = q^{-4}(q^{-2})^2$. □

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Theorem 5.8 *Suppose that the Young diagram of shape λ has s rows and let T be a semistandard λ-tableau. Then*

$$\Omega_q(T,T) = \prod_{\substack{1\le i\le s\\ i<j\le n}} (q^{i-j+1})^{\gamma_{ij}}$$

where γ_{ij} is the number of entries equal to j in row i of T.

Proof. By Theorem 5.7, we have $E_{12}^{\gamma_{12}}[T] = [\gamma_{12}]![U]$ where U is the single tableau that results from changing the γ_{12} 2's in the first row of T to ones. Thus, $E_{12}^{(\gamma_{12})}[T] = [U]$. Suppose that $E_{1,j-1}^{(\gamma_{1,j-1})}\cdots E_{12}^{(\gamma_{12})}[T] = \prod_{m=1}^{j-1}(q^{1-m+1})^{\gamma_{1m}}[U]$ where the first $\gamma_{11}+\gamma_{12}+\cdots+\gamma_{1,j-1}$ columns of the first row of U contain ones while the remainder of U is identical to T. By Theorem 5.7, we have

$$\begin{aligned}
E_{1j}^{(\gamma_{1j})}\prod_{m=1}^{j-1}(q^{1-m+1})^{\gamma_{1m}}[U] &= \frac{1}{[\gamma_{1j}]!}\prod_{m=1}^{j-1}(q^{1-m+1})^{\gamma_{1m}}E_{1j}^{\gamma_{1j}}[U]\\
&= \frac{1}{[\gamma_{1j}]!}\prod_{m=1}^{j-1}(q^{1-m+1})^{\gamma_{1m}}(q^{1-j+1})^{\gamma_{1j}}[\gamma_{1j}]![U']\\
&= \prod_{m=1}^{j}(q^{1-m+1})^{\gamma_{1m}}[U']
\end{aligned}$$

where the entries in the first $\gamma_{11}+\gamma_{12}+\cdots+\gamma_{1j}$ columns of the first row of U' are ones and the remainder of U coincides with T. This proves that $E_{1n}^{(\gamma_{1n})}\cdots E_{12}^{(\gamma_{12})}[T] = [U]$ where the first row of U consists entirely of ones and the remainder of U' is identical to T. The general result follows by repeating the argument for each of the rows in T. □

If V is a U_q-module and $\chi = (\chi_1,\ldots,\chi_n)$ an n-tuple of non-negative integers, the *weight space* associated to χ is the subspace

$$V^\chi = \{v \in V : K_i v = q^{\chi_i}v,\ 1\le i\le n\}.$$

A vector $v \in V^\chi$ is a *weight vector* of weight χ. It is easy to see, using Lemma 5.5, that the weight of a bideterminant $[T]$ in $\nabla_q(\lambda)$ is $\chi = (\chi_1,\ldots,\chi_n)$ where χ_i is equal to the number of i's that occur in the tableau T. (In other words, χ is the content of T.) In particular, $[T_\lambda]$ is the unique bideterminant in $\nabla_q(\lambda)$ with weight λ, up to multiplication by a scalar.

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Lemma 5.9 *Let T be a λ-tableau. Then $E_{ij}[T] = \sum_k a_k[U_k]$ where each λ-tableau U_k in the sum has one less j and one more i than T and $a_k \in K$.*

Proof. Fix j. The result is trivial for the case $j = i-1$. Suppose that $E_{mj}[T]$ satisfies the hypothesis if $i < m \leq j-1$. Then

$$\begin{aligned} E_{ij}[T] &= q^{-1}E_iE_{i+1,j}[T] - E_{i+1,j}E_i[T] \\ &= q^{-1}E_i\sum a_k[U_k] + E_{i+1,j}\sum b_l[S_l] \end{aligned}$$

where each U_k has one less j and one more $i+1$ than T by the induction hypothesis, each S_l has one more i and one less $i+1$ than T, and $a_k,\ b_l \in K$. But then for each k, $E_i[U_k] = \sum_r \alpha_r[U'_{kr}]$ where each $[U'_{kr}]$ has one more i and one less $i+1$ than U_k, so one less j and one more i than T, and $\alpha_r \in K$. By the induction hypothesis, $E_{i+1,j}[S_l] = \sum \beta_s[S'_{sl}]$ where each $[S'_{sl}]$ in the sum has one more $i+1$ and one less j than S_l. The result now follows. $\square$

Theorem 5.10 *Suppose that S and T are λ-tableaux and suppose that T is semistandard.*

1. *If $\Omega_q(S,T) \neq 0$ then $[S]$ and $[T]$ have the same weight.*

2. *If S is semistandard and $\Omega_q(S,T) \neq 0$ then $S \leq T$.*

Proof. By Lemma 5.9, E_{ij} takes a bideterminant of weight $\chi = (\chi_1, \ldots, \chi_n)$ to the weight space V^α where $\alpha = (\chi_1, \ldots, \chi_i + 1, \ldots, \chi_j - 1, \ldots, \chi_n)$. So if $[S]$ and $[T]$ do not have the same weight, $E_{T,q}$ takes $[S]$ and $[T]$ to different weight spaces, by induction. But $E_{T,q}[T]$ lies in the weight space $\nabla_q(\lambda)^\lambda$ by Theorem 5.8 so there is no bideterminant in the sum $E_{T,q}[S]$ with weight λ and $\Omega_q(S,T) = 0$. This proves 1.

Now to prove 2. we may assume that $[S]$ and $[T]$ have the same weight. Suppose that $S > T$ and let t be the first entry in the row sequence of T which differs from the corresponding entry in the row sequence of S. Suppose that the corresponding entry in S is s so that $s > t$. Let i be the row of the Young diagram in which t occurs in T. Suppose that the number of t's in the ith row of S is equal to k. Since

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$S > T$, we have $\gamma_{it} > k$. Thus $E_{T,q}[S] = \cdots E_{it}^{(\gamma_{it}-k)} E_{it}^{(k)} \cdots E_{12}^{(\gamma_{12})}[S]$ where $\gamma_{it} - k > 0$. Repeatedly applying Theorem 5.7 we have $E_{it}^{k} \cdots E_{13}^{\gamma_{13}} E_{12}^{\gamma_{12}}[S] = [k]! \cdots [\gamma_{13}]![\gamma_{12}]![U]$ where the first $i-1$ rows of U and the first $\gamma_{ii} + \gamma_{i,i+1} + \cdots + k$ columns of the ith row of U coincide with T_λ and the remainder of U coincides with S. Now, since S is semistandard, there are no t's in the ith row of U and any which appear below the ith are in the first $\gamma_{ii} + \gamma_{i,i+1} + \cdots + k$ columns. Since each of these columns contains an i in a row above such a t, we have $E_{it}^{\gamma_{it}-k}[U] = 0$ and it follows that $E_{t,q}[S] = 0$. □

Corollary 5.11 *The quantized Désarménien matrix is an invertible upper triangular matrix.*

Proof. Theorem 5.8 shows that the entries on the diagonal of Ω_q are powers of q, so are non-zero. The entries below the diagonal are zero by Theorem 5.10. □

The following lemma allows us to deduce a straightening algorithm for quantum bideterminants. For an alternative straightening algorithm, see [LT].

Lemma 5.12 *Suppose that S and T are λ-tableaux and that T is semistandard. Suppose also that $[S]$ and $[T]$ have the same weight. Then*

$$E_{T,q}[S] = \Omega_q(S,T)[T_\lambda].$$

Proof. Since $[S]$ and $[T]$ have the same weight, $E_{T,q}[S]$ and $E_{T,q}[T]$ lie in the same weight space. By Theorem 5.8, they both lie in the weight space $\nabla_q(\lambda)^\lambda$. Since any bideterminant with weight λ is a scalar multiple of $[T_\lambda]$, we have

$$E_{T,q}[S] = c[T_\lambda] = \Omega_q(S,T)[T_\lambda].$$

□

The method that the quantized Désarménien matrix gives for writing a quantum bideterminant as a linear combination of semistandard bideterminants works in essentially the same way as the method we discussed in Chapter 4. Let $\Omega_{q,\chi}$ be the submatrix of Ω_q that runs over the $\Omega_q(S,T)$ where $[S]$ and $[T]$ have weight χ. Let

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$\mathcal{I}_\lambda^\chi = \{I \in \mathcal{I}_\lambda : T_I \text{ has weight } \chi\}$. If U is a λ-tableau such that $[U]$ has weight χ, then $[U] = \sum_{I\in\mathcal{I}_\lambda^\chi} a_I[T_I]$. If $J \in \mathcal{I}_\lambda^\chi$ then

$$\begin{aligned}\Omega_q(U,T_J)[T_\lambda] &= E_{T_J,q}[U] \quad \text{by Lemma 5.12}\\ &= \sum_{I\in\mathcal{I}_\lambda^\chi} a_I E_{T_J,q}[T_I]\\ &= \sum_{I\in\mathcal{I}_\lambda^\chi} a_I \Omega_q(T_I,T_J)[T_\lambda].\end{aligned}$$

Thus $\Omega_q(U,T_J) = \sum_{I\in\mathcal{I}_\lambda^\chi} a_I\Omega_q(T_I,T_J)$ for all $J \in \mathcal{I}_\lambda^\chi$. In other words,

$$[\Omega_q(U,T_I)]_{I\in\mathcal{I}_\lambda^\chi} = [a_I]_{I\in\mathcal{I}_\lambda^\chi}\Omega_{q,\chi}$$

and since $\Omega_{q,\chi}$ is invertible, we have

$$[a_I]_{I\in\mathcal{I}_\lambda^\chi} = [\Omega_q(U,T_I)]_{I\in\mathcal{I}_\lambda^\chi}(\Omega_{q,\chi})^{-1}.$$

Example 5.10

Let $\lambda = (2,1)$, $n = 3$, and $\chi = (1,1,1)$. There are two semistandard λ-tableaux which give bideterminants of weight χ;

$$T_1 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \text{ and } T_2 = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}.$$

By Theorem 5.8, we have $\Omega_q(T_1,T_1) = 1$ and $\Omega_q(T_2,T_2) = q^{-1}$. To find $\Omega_q(T_1,T_2)$, we compute $E_{T_2,q}[T_1] = E_{13}[T_1] = (q^{-1}E_{12}E_{23} - E_{23}E_{12})[T_1] = 0$. Thus,

$$\Omega_q^\chi = \begin{pmatrix} 1 & 0 \\ 0 & q^{-1}\end{pmatrix}.$$

Let $T = \begin{array}{|c|c|}\hline 2 & 1 \\ \hline 3 \\ \cline{1-1}\end{array}$. Then $E_{T_1,q}[T] = E_{23}E_{12}[T] = qT_\lambda$, so $\Omega_q(T,T_1) = q$. Furthermore, $E_{T_2,q}[T] = (q^{-1}E_{12}E_{23} - E_{23}E_{12})[T] = -q[T_\lambda]$, so $\Omega_q(T,T_2) = -q$. Let a_{T_1} and a_{T_2} be the coefficients of $[T_1]$ and $[T_2]$ respectively in the straightening decomposition of $[T]$. Then

$$(a_{T_1}, a_{T_2}) = (q,-q)\begin{pmatrix} 1 & 0 \\ 0 & q\end{pmatrix} = (q,-q^2), \text{ and}$$

$$[T] = q[T_1] - q^2[T_2].$$

□

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5.4 The standard basis theorem for $\Delta_q(\lambda)$

A quantized version of the Carter-Lusztig basis theorem has been proved by R. Green, [Gr]. We shall prove the standard basis theorem using the quantized Désarménien matrix in Theorem 5.15. Our proof also shows that the standard basis for $\Delta_q(\lambda)$ and the basis of bideterminants for $\nabla_q(\lambda)$ are connected by the quantized Désarménien matrix.

We first construct the contravariant dual to $\nabla_q(\lambda)$. As stated in Theorem 5.2, $\Delta_q(\lambda) \cong (\nabla_q(\lambda))^\circ$. We begin with the map $\phi : V^{\otimes r} \to \nabla_q(\lambda)$ as defined in the classical case.

Proposition 5.13 *The K-linear map $\phi : V^{\otimes r} \to \nabla_q(\lambda)$ given by $\phi(v_I) = [T_I]$ is a U_q-epimorphism.*

Proof. It is clear that ϕ is an epimorphism, so we need only prove that $\phi(uv_J) = u[T_J]$ for each r-tuple J and each $u \in U_q$. Suppose first that $\lambda = 1^r$ so that $\phi(v_J) = [T_J]$ where T_J is a one-column tableau for each J. We will prove that $\phi(K_i v_J) = K_i\phi(v_J)$ and $\phi(E_i v_J) = E_i\phi(v_J)$ for each J and $1 \le i \le n$. The proof that $\phi(F_i v_J) = F_i\phi(v_J)$ is similar to the proof of the latter. If J contains more than one i then by Lemma 5.5 $[T_J] = 0$ since $[T_J]$ has one-column, so if a denotes the number of i's in J ($a \ge 2$) then $\phi(K_i v_J) = q^a[T_J] = 0 = K_i\phi(v_J)$. If J contains less than two i's then $\phi(K_i v_J) = q^a[T_J]$ ($a = 0$ or $a = 1$), but $K_i\phi(v_J) = K_i[T_J] = q^a[T_J]$ as well, so indeed $\phi(K_i v_J) = K_i\phi(v_J)$.

We now prove that $\phi(E_i v_J) = E_i\phi(v_J)$. If J does not contain an $i+1$, then $E_i v_J = 0 = E_i[T_J]$, so we suppose that J contains at least one $i+1$. If J also contains an i, then $E_i v_J = \sum \alpha_M v_M$ where $\alpha_M \in K$ and each r-tuple M contains at least two i's. But then $E_i\phi(v_M) = 0$ for each M and $E_i[T_J] = 0$ as well, since $[T_J]$ is a one-column tableau that contains two $i+1$'s. Further, if J contains more than two $i+1$'s, we also have $\phi(E_i v_J) = 0 = E_i[T_J]$ since $E_i v_J = \sum \alpha_M v_M$ where each v_M in the sum contains at least two $i+1$'s. So the only remaining case is when J contains no i's and one or two $i+1$'s. If J contains exactly one $i+1$, then $E_i v_J = v_M$

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where M is identical to J except that the $i+1$ has been replaced with an i. Thus, $\phi(E_i v_J) = [T_M] = E_i[T_J]$. Now suppose that J contains exactly two $i+1$'s and no i's. Since $E_i\phi(v_J) = E_i[T_J] = 0$, we need to prove that $\phi(E_i v_J) = 0$ as well. Suppose that $J = (j_1, \ldots, i+1, \ldots, i+1, \ldots, j_r)$. Then

$$E_i v_J = v_{j_1} \otimes \cdots \otimes v_{i+1} \otimes \cdots \otimes v_i \otimes \cdots \otimes v_{j_r} + q^{-1} v_{j_1} \otimes \cdots \otimes v_i \otimes \cdots \otimes v_{i+1} \otimes \cdots \otimes v_{j_r}$$

so $\phi(E_i v_J) = [T_M] + q^{-1}[T_{M'}]$ where the tableaux T_M and $T_{M'}$ are the same except that an i and an $i+1$ have been interchanged. But $[T_M] + q^{-1}[T_{M'}] = 0$ by Lemma 5.4. Thus, $\phi(E_i v_J) = 0$.

For the general case, let $\mu = (\mu_1, \ldots, \mu_s)$ be the conjugate of the partition $\lambda = (\lambda_1, \ldots, \lambda_k)$. Given $I \in I(n,r)$, let $I = (i_{11}, i_{12}, \cdots, i_{1\mu_1}, \cdots, i_{s1}, \cdots, i_{s\mu_s})$ and let $I_1 = (i_{11}, i_{12}, \cdots, i_{1\mu_1})$, $I_2 = (i_{21}, i_{22}, \cdots, i_{2\mu_2})$, $\ldots$, $I_s = (s_{11}, s_{12}, \cdots, s_{1\mu_1})$ so that $v_I = v_{I_1} \otimes v_{I_2} \otimes \cdots \otimes v_{I_s}$. It is now easy to check that the map

$$\theta : V^{\otimes r} \to \nabla_q(\mu_1) \otimes \nabla_q(\mu_2) \otimes \cdots \otimes \nabla_q(\mu_s)$$

defined by $\theta(v_{I_1} \otimes v_{I_2} \otimes \cdots \otimes v_{I_s}) = \phi(v_{I_1}) \otimes \phi(v_{I_2}) \otimes \cdots \otimes \phi(v_{I_s})$ is a U_q-homomorphism. As well, the map $\psi : \nabla_q(\mu_1) \otimes \nabla_q(\mu_2) \otimes \cdots \otimes \nabla_q(\mu_s) \to \nabla_q(\lambda)$ given by

$$\psi([T_{I_1}] \otimes [T_{I_2}] \otimes \cdots \otimes [T_{I_s}]) = [T_{I_1}][T_{I_2}] \cdots [T_{I_s}]$$

is a U_q-homomorphism. Thus, $\phi = \psi \circ \theta$ is a U_q-homomorphism as well.

$\square$

We have an anti-automorphism $J : U_q \to U_q$ defined by

$$J(E_i) = F_i, \quad J(F_i) = E_i, \quad J(K_i) = K_i$$

which is a K-algebra homomorphism. If V and W are two U_q-modules, a bilinear form $\langle , \rangle : V \times W \to K$ is said to be U_q-contravariant if for all $u \in U_q$, $v \in V$, and $w \in W$, we have

$$\langle uv, w \rangle = \langle v, J(u)w \rangle.$$

The canonical form $\langle , \rangle$ on $V^{\otimes r}$ defined in (2.10) is not a U_q-contravariant form. For instance,

$$\langle F_1(v_1 \otimes v_2), v_2 \otimes v_2 \rangle = \langle v_2 \otimes v_2, v_2 \otimes v_2 \rangle = 1,$$

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while

$$\langle v_1 \otimes v_2, E_1(v_2 \otimes v_2)\rangle = \langle v_1 \otimes v_2, v_2 \otimes v_1 + q^{-1} v_1 \otimes v_2\rangle = q^{-1}.$$

Thus, if we are to define the quantum analogue of Green's Weyl module, we must define a new form on $V^{\otimes r}$ which is U_q-contravariant. To do so, we introduce the following notation. Given $I \in I(n,r)$, define $\beta(I)$ to be the number of pairs (a,b) for which $a < b$ and $i_a \neq i_b$. For example, if $I = (1,3,2,1)$ then $\beta(I) = 5$. Define a bilinear form $\langle,\rangle_q : V^{\otimes r} \times V^{\otimes r} \to K$ by

$$\langle v_I, v_J\rangle_q = q^{\beta(I)}\delta_{IJ}.$$

For example, $\langle v_1 \otimes v_3 \otimes v_2 \otimes v_1, v_1 \otimes v_3 \otimes v_2 \otimes v_1\rangle = q^5$.

Theorem 5.14 *The form $\langle v_I, v_J\rangle_q : V^{\otimes r} \times V^{\otimes r} \to K$ is a U_q-contravariant form.*

Proof. It suffices to prove that $\langle K_i v_I, v_J\rangle = \langle v_I, K_i v_J\rangle$ and $\langle F_i v_I, v_J\rangle = \langle v_I, E_i v_J\rangle$ for single tensors v_I and v_J where $I, J \in I(n,r)$.

Let $I, J \in I(n,r)$ and let a be the number of i's in I and b the number of i's in J. Using (5.1) we have $K_i v_I = q^a v_I$ and $K_i v_J = q^b v_J$. Thus $\langle K_i v_I, v_J\rangle = q^a\langle v_I, v_J\rangle$ and $\langle v_I, K_i v_J\rangle = q^b\langle v_I, v_J\rangle$. Both are equal to zero if $I \neq J$ and if $I = J$ then $a = b$ so $\langle K_i v_I, v_J\rangle = \langle v_I, K_i v_J\rangle$.

Applying (5.1), we have

$$\begin{aligned} F_i(v_{i_1} \otimes v_{i_2} \cdots \otimes v_{i_r}) &= (K_{i,i+1}^{-1} v_{i_1}) \otimes \cdots \otimes (K_{i,i+1}^{-1} v_{i_{r-1}}) \otimes (F_i v_{i_r}) \\ &+ (K_{i,i+1}^{-1} v_{i_1}) \otimes \cdots \otimes (K_{i,i+1}^{-1} v_{i_{r-2}}) \otimes (F_i v_{i_{r-1}}) \otimes v_{i_r} \\ &+ \cdots + (F_i v_{i_1}) \otimes v_{i_2} \otimes \cdots \otimes v_{i_r} \text{ and} \end{aligned}$$

$$\begin{aligned} E_i(v_{j_1} \otimes v_{j_2} \otimes \cdots \otimes v_{j_r}) &= v_{j_1} \otimes v_{j_2} \otimes \cdots \otimes (E_i v_{j_r}) \\ &+ v_{j_1} \otimes \cdots \otimes v_{j_{r-2}} \otimes (E_i v_{j_{r-1}}) \otimes (K_{i,i+1} v_{j_r}) \\ &+ \cdots + (E_i v_{j_1}) \otimes (K_{i,i+1} v_{j_2}) \otimes \cdots \otimes (K_{i,i+1} v_{j_r}), \end{aligned}$$

For a fixed k with $1 \leq k \leq r$ we will prove that

$\langle K_{i,i+1}^{-1} v_{i_1} \otimes \cdots \otimes K_{i,i+1}^{-1} v_{i_{k-1}} \otimes F_i v_{i_k} \otimes \cdots \otimes v_{i_r}, v_{j_1} \otimes \cdots \otimes v_{j_k} \otimes \cdots \otimes v_{j_r}\rangle_q$ is equal to

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$$\langle v_{i_1} \otimes \cdots \otimes v_{i_k} \otimes \cdots \otimes v_{i_r}, v_{j_1} \otimes \cdots \otimes E_i v_{j_k} \otimes K_{i,i+1} v_{j_{k+1}} \otimes \cdots \otimes K_{i,i+1} v_{j_r} \rangle_q.$$

Since $\langle , \rangle_q$ is bilinear, this will complete the proof. We may assume that the r-tuples I and J coincide before and after the kth place for otherwise both forms above are equal to zero so we replace all such i_l's in the first form by j_l's. Furthermore, we may assume that $v_{i_k} = v_i$ and $v_{j_k} = v_{i+1}$ for again both forms are zero if this is not the case. Suppose that a_1 i's occur in the r-tuple J prior to the kth place and suppose that there are a_2 $i+1$'s in J prior to the kth place. Let b_1 denote the number of i's which appear in J after the kth place and b_2 the number of $i+1$'s that appear after the kth place. Then

$$\langle K_{i,i+1}^{-1} v_{j_1} \otimes \cdots \otimes K_{i,i+1}^{-1} v_{j_{k-1}} \otimes F_i v_i \otimes \cdots \otimes v_{j_r}, v_{j_1} \otimes \cdots \otimes v_{i+1} \otimes \cdots \otimes v_{j_r} \rangle_q = q^{-a_1} q^{a_2} q^{\beta(J')},$$

where $J' = (j_1, \ldots, j_{k-1}, i+1, \ldots, j_r)$ and

$$\langle v_{j_1} \otimes \cdots \otimes v_{j_k} \otimes \cdots \otimes v_{j_r}, v_{j_1} \otimes \cdots \otimes E_i v_{i+1} \otimes K_{i,i+1} v_{j_{k+1}} \otimes \cdots \otimes K_{i,i+1} v_{j_r} \rangle_q = q^{b_1} q^{-b_2} q^{\beta(J)}$$

where $J = (j_1, \ldots, j_{k-1}, i, \ldots, j_r)$. But $\beta(J') = \beta(J) + a_1 + b_1 - a_2 - b_2$ so

$$q^{-a_1} q^{a_2} q^{\beta(J')} = q^{b_1} q^{-b_2} q^{\beta(J)},$$

as desired. $\square$

We now take $V_{\lambda,q}$ to be the orthogonal complement to $N = \ker\phi$ with respect to the form $\langle , \rangle_q$. It is clear that $V_{\lambda,q}$ is a U_q-submodule of $V^{\otimes r}$ and the form

$$(\, , \,)_q : V_{\lambda,q} \times \nabla_q(\lambda) \to K$$

defined by

$$(x, \phi(y))_q = \langle x, y \rangle_q, \;\; x \in V_{\lambda,q}, \;\; y \in V^{\otimes r} \tag{5.7}$$

is a non-degenerate contravariant form since $\langle , \rangle_q$ is non-degenerate and contravariant. Thus $V_{\lambda,q} \cong (\nabla_q(\lambda))^\circ$ via the map $\psi : V_{\lambda,q} \to (\nabla_q(\lambda))^\circ$ defined by

$$\psi(x)(y) = (x, y)_q, \;\; x \in V_{\lambda,q}, \;\; y \in \nabla_q(\lambda).$$

By Theorem 5.2 $V_{\lambda,q} \cong \Delta_q(\lambda)$.

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Theorem 5.15 *The set $\{F_{T,q}z_\lambda : T \text{ is a semistandard } \lambda\text{-tableau}\}$ is a K-basis for $\Delta_q(\lambda)$.*

Proof. By definition $z_\lambda \in \Delta_q(\lambda)$. Let T_I and T_J be semistandard λ-tableaux and let $(\ ,\ )_q$ be the non-degenerate contravariant form on $\Delta_q(\lambda) \times \nabla_q(\lambda)$ defined in (5.7). Then

$$\begin{aligned}(F_{T_I,q}z_\lambda, [T_J])_q &= (z_\lambda, E_{T_I,q}[T_J])_q \\ &= (z_\lambda, \Omega_q(T_J,T_I)[T_\lambda] + \sum_M a_M[T_M])_q\end{aligned}$$

where each $a_M \in K$ and each $[T_M]$ has weight different from λ. But

$$(z_\lambda, [T_M])_q = \langle z_\lambda, v_M\rangle_q = 0$$

for such M, so

$$\begin{aligned}(z_\lambda, \Omega_q(T_J,T_I)[T_\lambda] + \sum_M a_M[T_M])_q &= \Omega_q(T_J,T_I)\langle z_\lambda, v_{I(\lambda)}\rangle_q \\ &= q^{\beta(I(\lambda))}\Omega_q(T_J,T_I)\end{aligned}$$

where by definition $\beta(J)$ is a non-negative integer. Thus for each $J \in \mathcal{I}_\lambda$,

$$(F_{T_I,q}z_\lambda, [T_J])_q = q^{\beta(I(\lambda))}\Omega_q(T_J,T_I).$$

Since $\Omega_q = [\Omega_q(T_J,T_I)]_{I\in\mathcal{I}_\lambda}$ is an upper triangular matrix and the set of semistandard λ-tableaux forms a K-basis for $\nabla_q(\lambda)$, the set

$$\{F_{T,q}z_\lambda : T \text{ is a semistandard } \lambda\text{-tableau}\}$$

is a K-basis for $\Delta_q(\lambda)$. □

As in the classical case, the q-Weyl module has a unique maximal submodule M so the quotient module $\Delta_q(\lambda)/M$ is irreducible. As well, $\Delta_q(\lambda)/M$ is isomorphic to the irreducible submodule $L_q(\lambda)$ of $\nabla_q(\lambda)$ where $L_q(\lambda)$ is generated by $[T_\lambda]$. All irreducible U_q-modules are obtained (up to isomorphism) in this way. The proof of the following corollary is similar to that of Theorem 3.10.

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Corollary 5.16 *The set $\{F_{T,q}[T_\lambda] : T$ is a semistandard λ-tableau$\}$ is a K-spanning set for $L_q(\lambda)$.*

We close with an illustration of this corollary.

Example 5.11

Let $\lambda = (2,1)$, $n = 3$, and $\chi = (1,1,1)$. There are two semistandard tableaux which give bideterminants with weight χ. They are $T_1 = \ytableaushort{12,3}$ and $T_2 = \ytableaushort{13,2}$. The following two elements form a spanning set for $L_q(\lambda)^\chi$:

$$\begin{aligned}
F_{T_1}\left[\ytableaushort{11,2}\right] &= F_{12}F_{23}\left[\ytableaushort{11,2}\right] \\
&= q^{-1}\left[\ytableaushort{12,3}\right] + \left[\ytableaushort{21,3}\right] \\
&= q^{-1}\left[\ytableaushort{12,3}\right] + q\left[\ytableaushort{12,3}\right] - q^2\left[\ytableaushort{13,2}\right] \quad \text{by Example 5.10} \\
&= (q^{-1}+q)\left[\ytableaushort{12,3}\right] - q^2\left[\ytableaushort{13,2}\right] \quad \text{and}
\end{aligned}$$

$$\begin{aligned}
F_{T_2}\left[\ytableaushort{11,2}\right] &= F_{13}\left[\ytableaushort{11,2}\right] \\
&= (q^{-1}F_{23}F_{12} - F_{12}F_{23})\left[\ytableaushort{11,2}\right] \\
&= q^{-1}\left(q^{-1}\left[\ytableaushort{13,2}\right] + \left[\ytableaushort{12,3}\right]\right) - (q^{-1}+q)\left[\ytableaushort{12,3}\right] - q^2\left[\ytableaushort{13,2}\right] \\
&= -q\left[\ytableaushort{12,3}\right] + (q^{-2}+q^2)\left[\ytableaushort{13,2}\right].
\end{aligned}$$

Note that if $q = 1$, we get back the spanning set for $L(\lambda)^\chi$ that we calculated in Example 3.5. The more interesting case is when we let q be a cube root of unity. Then we have

$$\begin{aligned}
q^2F_{T_2}\left[\ytableaushort{11,2}\right] &= -q^3\left[\ytableaushort{12,3}\right] + (1+q^4)\left[\ytableaushort{13,2}\right] \\
&= (q^{-1}+q)\left[\ytableaushort{12,3}\right] - q^2\left[\ytableaushort{13,2}\right] = F_{T_1}\left[\ytableaushort{11,2}\right].
\end{aligned}$$

Thus the two vectors are linearly dependent as in the case where we consider the irreducible U_K-module $L(\lambda)$ over a field K of characteristic 3. It is common for this

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sort of phenomena to occur and it is conjectured (see [CP], Conjecture 11.2.13) that the representation theory of U_q when q is a pth root of unity (and K has characteristic zero) behaves like the representation theory of U_L over a field L of characteristic p under certain conditions.

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Bibliography

[B] A. Berele, *Construction of Sp-modules by tableaux*, Lin. and Multilin. Alg. **19** (1986), 299–307.

[C] M. Clausen, *Letter place algebras and a characteristic-free approach to the representation theory of the general linear and symmetric groups I*, Adv. in Math. **33** (1979), 161–191.

[CS] G. Cliff and A. Stokke, *Determining irreducible $GL(n, K)$-modules*, Math. Proc. Camb. Phil. Soc., to appear.

[CL] R. W. Carter and G. W. Lusztig, *On the modular representations of the general linear and symmetric groups*, Math. Z. **136** (1974), 193–242.

[CP] V. Chari and A. Pressley, *A guide to quantum groups*, Cambridge University Press, Cambridge, 1994.

[Dc] C. De Concini, *Symplectic standard tableaux*, Adv. in Math. **34** (1979), 1–27.

[De] J. Désarménien, *An algorithm for the Rota straightening formula*, Discrete Math. **30** (1980), 51–68.

[DKR] J. Désarménien, J.P.S. Kung, G.-C. Rota, *Invariant theory, young bitableaux, and combinatorics*, Adv. in Math **27** (1978), 63–92.

[DJ] R. Dipper and G. James, *q-tensor space and q-Weyl modules*, Trans. Am. Math. Soc. (1991), 327:251–282.

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[D1] S. Donkin, *Representations of symplectic groups and the symplectic tableaux of R. C. King*, Linear and Multilinear Algebra **29** (1991), 113–124.

[D2] S. Donkin, *The q-Schur algebra*, London Mathematical Society Lecture Note Series **253**, Cambridge University Press, Cambridge, 1998.

[D3] S. Donkin, *On Schur algebras and related algebras I*, J. Algebra **104** (1986), 310–328.

[Do] S. Doty, *Polynomial representations, algebraic monoids, and Schur algebras of classical type*, J. Pure Appl. Algebra **123** (1998), 165–199.

[F] W. Fulton, *Young tableaux*, London Math. Society Student Texts **35**, Cambridge University Press, New York, 1997.

[G] J. A. Green, *Polynomial represenations of GL_n*, Lecture Notes in Mathematics **830**, Springer 1980, Berlin/Heidelberg/New York.

[Gr] R. M. Green, *q-Schur algebras and quuantized enveloping algebras*, Ph.D. thesis, Warwick University, 1995.

[Hu] J. Hu, *A combinatorial approach to representations of quantum linear groups*, Comm. Algebra **26(8)** (1998), 2591-2621.

[H] J. E. Humphreys, *Introduction to Lie algebras and representation theory*, Springer-Verlag, New York, 1972.

[HZ] R.Q. Huang, J.J. Zhang, *Standard basis theorem for quantum linear groups*, Adv. Math. **102** (1993), 202–229.

[JK] G. James, A. Kerber, *The representation theory of the symmetric group*, Enclyclopedia of Mathematics and its Applications, Addison-Wesley, 1981.

[J] J.C. Jantzen, *Lectures on quantum groups*, volume 6 of Graduate studies in math, Am. Math. Soc., 1996.

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[K] R. King, *Weight multiplicities for classical groups,* Group Theoretical Methods in Physics (Fourth International Colloquium Nijmegen 1975), Lecture Notes in Physics **50**, Springer 1975, Berlin/Heidelberg/New York.

[LT] B. Leclerc and J. Thibon, *The Robinson-Schensted correspondence, crystal bases, and the quantum straightening at* $q = 0$, Electron. J. Combin. 3(2):R11 (1996).

[L] G. Lusztig, *Finite dimensional Hopf algebras arising from quantized universal enveloping algebras*, J. Am. Math. Soc. (1990) 3:257–297.

[Ma] S. Martin, *Schur algebras and representation theory*, Cambridge University Press, Cambridge, 1993.

[Mu] T. Muir, *A treatise on the theory of determinants*, Macmillan, London, 1882.

[NYM] M. Noumi, H. Yamada, K. Mimachi, *Finite dimensional representations of the quantum group* $GL_q(n, \mathbb{C})$ *and the zonal spherical functions on* $U_q(n-1)/U_q(n)$, *Japan. J. Math.* **19** (1993), 31–80.

[PS] M. Pittaluga and E. Strickland, *A computer-oriented algorithm for the determination of the dimension and character of a modular irreducible* $SL(n, K)$*-module*, J. Symbolic Comput. **7** (1989), 155–161.

[P] V. V. Prasolov, *Problems and theorems in linear algebra,* Translations of Mathematical Monographs. **134**, American Mathematical Society, Providence, RI, 1994.

[S] I. Schur, *Über eine Klasse von Matrizen die sich einer gegeben Matrix zuordnen lassen*, Inaugural-Dissertation, Berlin, 1901.

[TT] E. Taft and J. Towber, *Quantum deformation of flag schemes and Grassman schemes I-A q-deformation of the shape algebra for* $GL(n)$, *J. Algebra,* **142** (1991), 1–36.

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[Ta] M. Takeuchi, *Some topics on $GL_q(n)$*, J. Algebra **147** (1992), 379–410.

[To] J. Towber, *Young symmetry, the flag manifold, and representations of $GL(n)$*, J. Algebra **61** (1979), 414–462.

[W] B. W. Wetherilt, *A note on a module in tensor space for the symplectic group*, J. Algebra **103** (1986), 294–301.

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