On sonobuoy placement for submarine tracking

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ABSTRACT

This paper addresses the problem of detecting and tracking an unknown number of submarines in a body of water using a known number of moving sonobuoys. Indeed, we suppose there are N submarines collectively maneuvering as a weakly interacting stochastic dynamical system, where N is a random number, and we need to detect and track these submarines using M moving sonobuoys. These sonobuoys can only detect the superposition of all submarines through corrupted and delayed sonobuoy samples of the noise emitted from the collection of submarines. The signals from the sonobuoys are transmitted to a central base to analyze, where it is required to estimated how many submarines there are as well as their locations, headings, and velocities. The delays induced by the propagation of the submarine noise through the water mean that novel historical filtering methods need to be developed. We summarize these developments within and give initial results on a simplified example.

Keywords: target tracking, sonobuoy, submarine, nonlinear filtering

1. INTRODUCTION

Although passive maneuverable sonobuoys are a very effective countermeasure against today's stealthy submarines, they have critical drawbacks in use. Measurements from the sonobuoys' hydrophone arrays are very sporadic due to sonobuoy deployment patterns and limited battery life. Moreover, the data sampled by the sonobuoys are distorted by the propagation loss and corrupted by ambient noise factors like temperature, current and pressure variations. Finally, both localizing relative direction from the sonobuoys and processing of the three dimensional measurements are inherently complicated. Our solution to resolving these problems is to employ mathematical models and methods. In particular, the position-dependent propagational delay of the sonobuoy sound pressure forces us to develop a novel historical filtering approach.

1.1. Water body

We model the ocean as the negative half space

$$\mathbb{R}^3_- \doteq \{\xi = (x, y, z) \in \mathbb{R}^3 : z < 0\}, \tag{1}$$

so points in the ocean are constrained to have a negative vertical component. Moreover, for exhibition purposes, we assume that the ocean is completely homogeneous and that sound pressure hitting the surface of the ocean will reflect back in a lossless manner.

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1.2. Submarine signal

We consider an interacting multiple target tracking problem of a random number N of submarines, where each submarine is constrained to remain under the ocean surface. For simplicity, we take each submarine to be a sphere with radius $\varepsilon > 0$. This means that the center of mass of the submarine is constrained to be in $\mathbb{R}^3_{-\varepsilon}$, where

$$\mathbb{R}^3_{\pm\varepsilon} \doteq \{\xi = (x, y, z) \in \mathbb{R}^3 : z < \pm\varepsilon\}. \tag{2}$$

We also force the submarines to stay above a deepest value $z_{\min} < -2\varepsilon$ to reflect the reality that submarines can not go arbitrarily deep due to pressure constraints.

The three dimensional position, orientation, and forward speed of the i^{th} submarine is modeled by a diffusion process designed to keep the position of all submarines in $\mathbb{R}^3_{-\varepsilon}$ with their depth above $z_{\min}$, their velocities v within physical constraints $v_{\min} \le v \le v_{\max}$ for some $0 \le v_{\min} \le v_{\max}$, their angles of attack within physical constraints $\phi_{\min} \le \phi \le \phi_{\max}$ for some $-\frac{\pi}{2} \le \phi_{\min} < 0 < \phi_{\max} \le \frac{\pi}{2}$, and their orientations adjusted to avoid collisions. In particular, suppose $X_t^i = \begin{bmatrix} x_t^i \\ y_t^i \\ z_t^i \end{bmatrix}$, v_t^i, θ_t^i, and $\phi_t^i \in [\phi_{\min}, \phi_{\max}]$ represent the three dimensional position, forward speed, horizontal bearing, and angle of attack of the i^{th} submarine. Then, we define the interaction terms for the ith submarine with respect to the other submarines to be

$$\Delta_t^i = \sum_{n=1, n\neq i}^{N} \left[\frac{(y_t^i - y_t^n)\cos(\theta_t^i)}{(x_t^i - x_t^n)^2 + (y_t^i - y_t^n)^2 + (z_t^i - z_t^n)^2} - \frac{(x_t^i - x_t^n)\sin(\theta_t^i)}{(x_t^i - x_t^n)^2 + (y_t^i - y_t^n)^2 + (z_t^i - z_t^n)^2} \right] \tag{3}$$

and

$$\Gamma_t^i = \sum_{n=1 n\neq i}^{N} \left[\frac{(z_t^i - z_t^n)\cos(\phi_t^i)}{(x_t^i - x_t^n)^2 + (y_t^i - y_t^n)^2 + (z_t^i - z_t^n)^2} - \frac{\sqrt{(x_t^i - x_t^n)^2 + (y_t^i - y_t^n)^2}\sin(\phi_t^i)}{(x_t^i - x_t^n)^2 + (y_t^i - y_t^n)^2 + (z_t^i - z_t^n)^2} \right]. \tag{4}$$

In particular, Δ_t^i and Γ_t^i are used to control the bearing and angle of attack, respectively, of the submarines in such a way as to try to avoid collisions with other submarines. We also use control of the angle of attack to ensure that the submarine does not leave the water or go too deep. The total control on the angle of attack is then given by

$$\gamma_t^i = \begin{cases} (\overline{\gamma}_t^i)^{\frac{1}{2}} \left(\frac{\phi_{\max} - \phi_t^i}{\phi_{\max} - \phi_{\min}} \right)^{\frac{3}{4}} & \overline{\gamma}_t^i \ge 0 \\ -(-\overline{\gamma}_t^i)^{\frac{1}{2}} \left(\frac{\phi_t^i - \phi_{\min}}{\phi_{\max} - \phi_{\min}} \right)^{\frac{3}{4}} & \overline{\gamma}_t^i < 0 \end{cases}, \text{ with } \overline{\gamma}_t^i = \frac{c_z}{z_t^i - z_{\min}} + \frac{c_z}{z_t^i + \varepsilon} + c_\phi \Gamma_t^i \tag{5}$$

and the dynamics of the i^{th} submarine are given by

$$\begin{bmatrix} dX_t^i \\ dV_t^i \end{bmatrix} = \begin{bmatrix} dx_t^i = v_t^i \cos(\theta_t^i)\cos(\phi_t^i)dt + \sigma_x dW_t^{x,i} \\ dy_t^i = v_t^i \sin(\theta_t^i)\cos(\phi_t^i)dt + \sigma_y dW_t^{y,i} \\ dz_t^i = v_t^i \sin(\phi_t^i)dt + \sigma_z \sqrt{(-\varepsilon - z_t^i)(z_t^i - z_{\min})}dW_t^{z,i} \\ d\theta_t^i = c_\theta \Delta_t^i dt + \sigma_\theta dW_t^{\theta,i} \\ d\phi_t^i = \gamma_t^i dt + \sigma_\phi \sqrt{(\phi_{\max} - \phi_t^i)(\phi_t^i - \phi_{\min})}dW_t^{\phi,i} \\ dv_t^i = c_v(v_{AVG} - v_t^i)dt + \sigma_v \sqrt{(v_{\max} - v_t^i)(v_t^i - v_{\min})}dW_t^{v,i} \end{bmatrix}, \tag{6}$$

where $\left\{W^{x,i}, W^{y,i}, W^{z,i}, W^{\theta,i}, W^{\phi,i}, W^{v,i}\right\}_{i=1}^{\infty}$ are all independent Brownian motions, the σ's and c's are all positive constants, and $v_{AVG} = \frac{v_{\min} + v_{\max}}{2}$. We are really interested in the counting measure that keeps track of the state of all submarines but does not differentiate which one is which, so we define E as the collection of finite counting measures over $\mathbb{R}^6$ and $S_t \in E$ by

$$S_t = \sum_{j=1}^{N} \delta_{X_t^i, V_t^i}, \tag{7}$$

where δ_x puts a point mass at the location $x \in \mathbb{R}^6$. Now, since the sound pressure in the system depends upon the current and past states of the signal S_t, we define the pathspace version of this process $S_{[0,t]} \in C_E[0,\infty)$ by

$$S_{[0,t]}(\tau) \doteq \begin{cases} S_\tau & \tau < t \\ S_t & \tau \geq t \end{cases} . \tag{8}$$

1.3. Noise emission

Each submarine is assumed to fit in a three dimensional ε-ball $B(X_t^i, \varepsilon)$ centered at the location X_t^i of its centre of mass. It then emits noise pressure distributed over $B(X_t^i, \varepsilon)$ according to some infinitely continuously differentiable $\mathbb{R}$-valued function Θ of $\mathbb{R}^3$, i.e. $\Theta \in C^\infty(\mathbb{R}^3)$. We assume that Θ is even with respect to z, i.e. $\Theta(x,y,z) = \Theta(x,y,-z)$, and satisfies

$$\Theta \geq 0,\ \Theta 1_{B(0,\varepsilon)^C} \equiv 0, \text{ and } \int_{B(0,\varepsilon)} \Theta(\Xi) d\Xi = 1, \tag{9}$$

where $B(0,\varepsilon)^C$ is the complement of $B(0,\varepsilon)$ and $\Xi = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$. For example, Θ could be

$$\Theta(\Xi) \doteq \begin{cases} c_\Theta \exp\left\{ -\frac{1}{\varepsilon^2 - |\Xi|^2} \right\} & |\Xi| < \varepsilon \\ 0 & \text{otherwise} \end{cases}, \text{ where } c_\Theta = \frac{1}{\int_{B(0,\varepsilon)} \exp\left\{ -\frac{1}{\varepsilon^2 - |\Xi|^2} \right\} d\Xi} \tag{10}$$

is chosen so that $\int_{\mathbb{R}^3} \Theta(\Xi) d\Xi = \int_{B(0,\varepsilon)} \Theta(\Xi) d\Xi = 1$. Then, the noise emitted by the i^{th} submarine at time t is assumed to be $\Theta(\Xi - X_t^i)$.

To approximate the noise generated by all N submarines, we let $T > 0$ be chosen so that all noise generated prior time $-T$ has left the area of interest by time $t = 0$ and define the noise generation function

$$f(t,\Xi) = \sum_{i=1}^N \Theta(\Xi - X_t^i) = \int_{\mathbb{R}^6} \Theta(\Xi - \pi^X(x)) dS_t(x) \text{ in } (-T,\infty) \times \mathbb{R}^3_-, \tag{11}$$

where π^X denotes the 3 dimensional projection onto the positional component, i.e. $\pi^X(\begin{bmatrix} X \\ V \end{bmatrix}) = X$. This function $f(t,\Xi)$ represents the superposition of noise emitted from all submarines at time t and at the location Ξ.

1.4. Noise propagation

By our assumption that the ocean is homogeneous and with perfectly reflecting surface boundary, the sound pressure $u(t,x)$ from the N submarines follows the wave equation

$$\begin{aligned} \frac{d^2 u(t;x,y,z)}{dt^2} &= \frac{1}{c} \Delta u(t;x,y,z) + f(t;x,y,z) \text{ in } (-T,\infty) \times \mathbb{R}^3_- \\ u(-T;x,y,z) &= 0, \frac{d}{dt} u(-T;x,y,z) = 0 \text{ in } \mathbb{R}^3_- \\ \frac{d}{dz} u(t;x,y,0) &= 0 \text{ in } (-T,\infty) \times \mathbb{R}^2, \end{aligned} \tag{12}$$

where c is a known speed constant. The ocean is considered to be infinitely deep to avoid further reflections, which is a usable approximation in the middle of the oceans. We let the unique solution to this equation be denoted $u(t,\Xi;S_{[0,t]})$ to highlight the dependence of the noise on the historical states of submarines through the forcing term $f(t;x,y,z)$.

1.5. Sonobuoy locations

The sonobuoy positions $\{\eta_t^i\}_{i=1}^M$ are maneuverable within the constraints that $\left|\frac{d\eta_t^i}{dt}\right| \le K_v$ and $\left|\frac{d^2\eta_t^i}{dt^2}\right| \le K_a$. For simplicity in this exhibition, we take the sonobuoy paths to be known, i.e. deterministic. Future work will consider the case where the sonobuoys can be controlled based upon the observations that arrive from previous sonobuoy measurements.

1.6. Sonobuoy observations

At each measurement instant $0 < t_1 < t_2 < ...$, there are M measurements, one from each of M sonobuoys. The measurement $Y_k^i(\eta_{t_k}^i)$ received in the command center from the i^{th} sonobuoy at the position $\eta_{t_k}^i$ and at the time t_k is a function of the sound pressure distributed over the sonobuoy and then corrupted by noise:

$$Y_k = \begin{bmatrix} Y_k^1(\eta) \\ Y_k^2(\eta) \\ \vdots \\ Y_k^M(\eta) \end{bmatrix}, \; Y_k^i(\eta) = \int_{\mathbb{R}^3} \phi^i(\Xi - \eta_{t_k}^i) u(t_k, \Xi; S_{[0,t_k]}) d\Xi + V_k^i, \quad 0 \le i \le M, \tag{13}$$

where $\{V_k\}$ is a $\mathbb{R}^M$-valued independent $\Sigma\mathcal{N}(0, I)$ sequence. Here, the $\{\phi^i(\cdot)\}_{i=1}^M$ are some given generalized functions that represent the density of the hydrophone array as well as its sensitivity to sound pressure, and $\{\eta_{t_k}^i\}_{i=1}^M \subset \mathbb{R}^3$ are the sonobuoy locations at time t_k. For simplicity, the ϕ^i can be taken to be

$$\phi^i(\Xi) = \sum_{l=0}^{L} \delta_{(0,0,-\beta l)} \tag{14}$$

for some small $\beta > 0$, in which case

$$Y_k^i(\eta) = \sum_{l=0}^{L} u(t_k, \eta_{t_k}^i - (0, 0, \beta l)^T; S_{[0,t_k]}) + V_k^i, \quad 0 \le i \le M. \tag{15}$$

This would represent the situation where we are sensing the sound pressure at L equally spaced pinpoint locations directly below each sonobuoy.

1.7. Objectives

The objective is to calculate the best mean-square estimate of the submarine paths given the sonobuoy observations

$$E\left[S_{[0,t_k]} \middle| \sigma\{Y_1, ..., Y_k\} \right], \tag{16}$$

efficiently and accurately on a computer.

2. METHOD

2.1. Notation

For any topological space $\mathcal{X}$, we let $B(\mathcal{X})$, $C(\mathcal{X})$, $\overline{C}(\mathcal{X})$ denote respectively the bounded, measurable; continuous; and bounded, continuous functions $f : \mathcal{X} \to \mathbb{R}$. Next, recalling that E is our space of counting measures, we let $C_E[0, \infty)$ denote the E -valued continuous functions of (time) $[0, \infty)$. Moreover, we let π_t be the projection function from $C_E[0, \infty)$ to E at time t, meaning that $\pi_t(S) = S_t$ for $S \in C_E[0, \infty)$. Finally, we let $C_c^k(\mathbb{R}^n)$ be the k times continuously differentiable $\mathbb{R}$-valued functions with compact support on $\mathbb{R}^n$. To keep track of all of the information generated by the submarine signal and sonobuoy observations, we define $\mathcal{F}_t^{SY} \doteq \sigma\{S_s, Y_k : s \le t \text{ and } t_k \le t\}$ and $\mathcal{F}_k^{SY} \doteq \mathcal{F}_{t_k}^{SY}$. Moreover, to keep track of the information given by the observations alone, we define $\mathcal{F}_k^Y \doteq \sigma\{Y_1, ..., Y_k\}$ for $k = 1, 2, ...$

2.2. Mathematical equation for the historical signal

To handle the reflections at the surface, we let $\overline{X}_t^i, \overline{V}_t^i \in \mathbb{R}^3_{-\varepsilon}$ be the mirror image of X_t^i, V_t^i over the plane $z = 0$, set $\widehat{S}_t = \sum_{j=1}^N \delta_{X_t^i, V_t^i} + \sum_{j=1}^N \delta_{\overline{X}_t^i, \overline{V}_t^i}$, and take $(\mathcal{D}(\mathcal{L}), \mathcal{L}) \subset \overline{C}(E) \times \overline{C}(E)$ to be the weak generator for the Markov process $\{S_t, t \geq 0\}$, so that

$$M_t^S(\varphi) \doteq \varphi(S_t) - \varphi(S_0) - \int_0^t \mathcal{L}\varphi(S_s)ds, \tag{17}$$

is a continuous $\{\mathcal{F}_t^S\}_{t\geq 0}$-martingale for each $\varphi \in \mathcal{D}(\mathcal{L})$. Now, since we must work with the pathspace version of this process, it is convenient to define a pathspace variant of our operator that will be the weak generator for $t \to S_{[0,t]} \in C_E[0,\infty)$. With this in mind, we let $I^m = \{\tau_i^m\}_{i=1}^m$ be such that $I^m \subset I^{m+1}$, $\mathcal{I} \doteq \cup_{m=1}^\infty I^m$ is dense in $[0,\infty)$, and $0 = \tau_0^m \leq \tau_1^m \leq \cdots \leq \tau_{m-1}^m \leq \tau_m^m$. Then, we define $\mathcal{D}(\mathcal{L}_{[0,s]})$ to be the linear span

$$sp\left\{\varphi_1(\pi_{\tau_1^m}) \cdots \varphi_m(\pi_{\tau_m^m}) : \varphi_i \in \mathcal{D}(\mathcal{L}),\, m = 1, 2, ...\right\} \tag{18}$$

for $s \in [0,\infty)$ and take $\mathcal{L}_{[0,s]}$ to be the operators on $B(C_E[0,\infty))$ defined on $\mathcal{D}(\mathcal{L}_{[0,s]})$ by

$$\mathcal{L}_{[0,s]}\varphi_1(\pi_{\tau_1^m}) \cdots \varphi_m(\pi_{\tau_m^m}) = \varphi_1(\pi_{\tau_1^m}) \times \cdots \times \varphi_{j-1}(\pi_{\tau_{j-1}^m}) \times \mathcal{L}(\varphi_{j,m})(\pi_s) \tag{19}$$

for $\tau_{j-1}^m \leq s < \tau_j^m$, where $\varphi_{j,m}(x) = \varphi_j(x) \cdots \varphi_m(x)$. This operator is necessarily time-inhomogeneous even when the original operator is homogeneous. Since these domains are only measurable functions, we can not immediately use the fact that these functions separate points on $D_E[0,\infty)$ to conclude that they separate probability measures. However, we can show that if two probability measures μ^1, μ^2 agree on $\mathcal{D}(\mathcal{L}_{[0,s]})$ they agree on the cylinder sets and hence on $\mathcal{B}(D_E[0,\infty))$. With this *pathspace* operator, we can define the historical martingale problem

$$\mathcal{M}_t^S(\Phi) \doteq \Phi(S_t) - \Phi(S_0) - \int_0^t \mathcal{L}_{[0,s]}\Phi(S_{[0,s]})ds, \tag{20}$$

a continuous $\{\mathcal{F}_t^S\}_{t\geq 0}$-martingale for each $\Phi \in \mathcal{D}(\mathcal{L}_{[0,s]})$. In particular, one can readily verify that

$$\mathcal{M}_t^S(\varphi_1(\pi_{\tau_1^m}) \cdots \varphi_m(\pi_{\tau_m^m})) = \sum_{j=1}^m \int_{\tau_{j-1}^m \wedge t}^{\tau_j^m \wedge t} \varphi_0(S_{\tau_0}) \cdots \varphi_{j-1}(S_{\tau_{j-1}})dM_s^S(\varphi_{j,m}). \tag{21}$$

Equation (20) uniquely characterizes the distribution of the submarine paths and will be useful in characterizing the optimal estimate of the submarine locations given the observations.

2.3. Explicit solution for the wave equation

To process the sonobuoy observations, it is convenient to derive an explicit solution for the sound pressure u in terms of the historical submarine paths. The solution to the wave equation (12) at time t and point (x,y,z) is

$$u(t;x,y,z) = \frac{1}{4\pi}\int_{B(\sqrt{c}(x,y,z);t+T)} \frac{g(t+T-|\sqrt{c}(x,y,z)-(\xi,\theta,\zeta)|;\xi,\theta,\zeta)}{|\sqrt{c}(x,y,z)-(\xi,\theta,\zeta)|}d\xi d\theta d\zeta, \tag{22}$$

where

$$g(t;x,y,z) = \begin{cases} f(t;x,y,z) & z \leq 0 \\ f(t;x,y,-z) & z > 0 \end{cases}. \tag{23}$$

In terms of the submarines, the solution is

$$\begin{aligned} u(t;x,y,z) &= \int_{B(\sqrt{c}(x,y,z);t+T)} \frac{\sum_{i=1}^N \left[\Theta((\xi,\theta,\zeta) - X^i_{t-|\sqrt{c}(x,y,z)-(\xi,\theta,\zeta)|}) + \Theta((\xi,\theta,\zeta) - \overline{X}^i_{t-|\sqrt{c}(x,y,z)-(\xi,\theta,\zeta)|})\right]}{4\pi|\sqrt{c}(x,y,z)-(\xi,\theta,\zeta)|}d\xi d\theta d\zeta \\ &= \int_{B(\sqrt{c}(x,y,z);t+T)} \frac{\sum_{i=1}^N \left[\Theta(\Xi - X^i_{t-|\sqrt{c}(x,y,z)-\Xi|}) + \Theta(\Xi - \overline{X}^i_{t-|\sqrt{c}(x,y,z)-\Xi|})\right]}{4\pi|\sqrt{c}(x,y,z)-\Xi|}d\Xi \\ &\doteq u(t;x,y,z;S_{[0,t]}). \end{aligned} \tag{24}$$

This explicit representation will ease the computer evaluation of the conditional distribution for the historical signal given the observations.

2.4. Computation via Bayes' rule

In order to calculate the distribution of the signal given the observations it is easier to first calculate this distribution with an artificial (reference) probability measure and then to convert back using Bayes rule. With this in mind, we define

$$h_k = \begin{bmatrix} h_k^1 \\ h_k^2 \\ \vdots \\ h_k^M \end{bmatrix}, \text{ where } h_k^i = \int_{\mathbb{R}^3} \phi_k^i(\Xi - \eta_{t_k}^i) u(t_k, \Xi; S_{[0,t_k]}) d\Xi, \tag{25}$$

and

$$A_k = \prod_{j=1}^{k} \zeta_j, \text{ where } \zeta_j \doteq \exp\left[-h_j^T \Sigma^{-2} V_j - \frac{1}{2} h_j^T \Sigma^{-2} h_j\right] = \exp\left[-h_j^T \Sigma^{-2} Y_j + \frac{1}{2} h_j^T \Sigma^{-2} h_j\right]. \tag{26}$$

In the case that $\phi^i(\Xi) = \sum_{l=0}^{L} \delta_{(0,0,-\beta l)}$, we would have that

$$h_k^i = \sum_{l=0}^{L} u(t_k, \eta_{t_k}^i - (0,0,\beta l)^T; S_{[0,t_k]}), \text{ for } i = 1, 2, ..., M. \tag{27}$$

In either the general or specific case, we set

$$Z_k \doteq A_k^{-1} = \prod_{j=1}^{k} \zeta_j^{-1}. \tag{28}$$

Now, $\{A_k\}$ is a $\left\{\mathcal{F}_k^{SY}\right\}_{k\geq 1}$-martingale and we can use Girsanov's theorem (and extension) to define a new probability measure on $\mathcal{F}_\infty^{SY}$

$$\overline{P}(\Gamma) = P(\Gamma A_k) \qquad \forall \Gamma \in \mathcal{F}_k^{SY} \tag{29}$$

for all $t > 0$. Therefore, letting $E, \overline{E}$ denote expectation with respect to $P, \overline{P}$ respectively, using the fact that $S_{[0,t]}$ is $\mathcal{F}_t^{SY}$-measurable and recalling Bayes' rule, we have that

$$E[\varphi(S_{[0,t_k]})|\mathcal{F}_k^Y] = \frac{\bar{E}[\varphi(S_{[0,t_k]}) Z_k | \mathcal{F}_k^Y]}{\bar{E}[Z_k | \mathcal{F}_k^Y]} \tag{30}$$

for $\varphi \in B(D_E[0,\infty))$. Since the denominator and numerator of (30) are both calculated from $\bar{E}[g(S_{[0,t_k]}) Z_k | \mathcal{F}_k^Y]$ with $g = 1$ and $g = \varphi$ respectively, we only need a method of computation for

$$\mu_{[0,t]}\varphi \doteq \bar{E}[\varphi(S_{[0,t]}) Z_k | \mathcal{F}_k^Y] \text{ for } t \in [t_k, t_{k+1}) \tag{31}$$

over a rich enough class of functions $\varphi : C_E[0,\infty) \to \mathbb{R}$, such as $\mathcal{D}(\mathcal{L}_{[0,s]})$.

2.5. Weighted particle method

By the method of the previous subsection, we only need to find an approximation for

$$\mu_{[0,t]}\varphi \doteq \bar{E}[\varphi(S_{[0,t]}) Z_k | \mathcal{F}_k^Y] \text{ for } t \in [t_k, t_{k+1}). \tag{32}$$

However, $S_{[0,t]}$ is independent of $\mathcal{F}_k^Y$ with respect to $\overline{P}$. Therefore, we can construct independent particles $t \to S_{[0,t]}^m$ with the same law as $t \to S_{[0,t]}$ and define the weight of the m^{th} particle by

$$Z_k^m \doteq \prod_{j=1}^{k} \exp\left[h_{j,m}^T \Sigma^{-2} Y_j - \frac{1}{2} h_{j,m}^T \Sigma^{-2} h_{j,m}\right], \tag{33}$$

where

$$h^i_{k,m} = \int_{\mathbb{R}^3} \phi^i_k(\Xi - \eta^i_{t_k}) u(t_k, \Xi; S^m_{[0,t_k]}) d\Xi, \text{ for } i = 1, 2, ..., M,\ m = 1, ..., n \tag{34}$$

in the general case or

$$h^i_{k,m} = \sum_{l=0}^{L} u(t_k, \eta^i_{t_k} - (0,0,\beta l)^T; S^m_{[0,t_k]}), \text{ for } i = 1, 2, ..., M,\ m = 1, ..., n \tag{35}$$

in the case that $\phi^i(\Xi) = \sum_{l=0}^{L} \delta_{(0,0,-\beta l)}$. Then, using the law of large numbers, one has that

$$\frac{1}{n} \sum_{m=1}^{n} \varphi(S^m_{[0,t_k]}) Z^m_k \overset{n\to\infty}{\to} \bar{E}[\varphi(S_{[0,t_k]}) Z_k | \mathcal{F}^Y_k] \text{ for } \varphi \in \mathcal{D}(\mathcal{L}_{[0,s]}) \tag{36}$$

and

$$\frac{1}{n} \sum_{m=1}^{n} \delta_{S^m_{[0,t_k]}}(\cdot) Z^m_k \overset{n\to\infty}{\to} \mu_{[0,t_k]}(\cdot). \tag{37}$$

2.6. Resampled particle method

The method of the previous subsection works in theory but suffers in practice due to the fact that the particle locations $\left\{S^m_{[0,t_k]}\right\}_{m=1}^{n}$ are not affected by the observations, but rather only by the particle weights Z^m_k. As a result, the particles do not represent the submarine signal well and the approximation tends to break down in practice. To counteract this problem, our group has previously developed the Selectively Resampling Particle filter. This method does pairwise resampling of the particles, starting with the two with the highest and lowest weights. It replaces these two particles with two particles both having the path of the highest weighted particle with high probability or the path of the lowest weighted particle with small probability, and both with weights equal to the average of the two previous weights. This is done in a manner so as not to introduce bias into the system. The process is repeated until the weights of all particles are within a given bound.

3. RESULTS

Software implementing the above solution has been constructed and shows promise in initial trials with a single submarine. However, it is not yet completed to work with multiple submarines.

3.1. Example problem

The problem has been implemented with concrete values for the many parameters involved. The radius of the submarines ε is taken to be 0.01, and the area of interest is the ocean in $[0,1] \times [0,1] \times [z_{\min} = -1, -\varepsilon]$. Submarines that leave this area of interest are considered, as a simplification, to have absolutely no interaction with the remaining simulation. The submarine constraints $v_{\min}$, $v_{\max}$, $\phi_{\min}$, and $\phi_{\max}$ take the values 0.001, 0.005, $-\frac{\pi}{4}$, and $\frac{\pi}{4}$. Constants of the submarine dynamics c_z and c_v take values 0.000001 and 0.01, while c_θ and c_ϕ are unnecessary with only one submarine. The constants for the noise terms in the submarine dynamics are $\sigma_x = \sigma_y = \sigma_z = 0.001$, $\sigma_\theta = 0.08$, $\sigma_\phi = 0.05$, and $\sigma_v = 0.2$.

For calculation of the generated noise, a value of $\varepsilon = 0.01$ implies that the value for c_Θ must equal approximately 4.75958×10^5. The speed constant c of transmission through the water is taken to be 1, and this means that a value of $T = 4$ is enough to ensure that both the direct and reflected sounds made in the area of interest at time $-T$ will have all left by time 0.

As a simplification, the sonobuoys are assumed to move along a known path, taken to be a circle of radius 0.3 centered within the area of interest. Four sonobuoys initially take positions at the east, north, west, and south points of this circle and each move counterclockwise about the circle with a speed of one half degree of rotation per unit time. No sonobuoys are added or removed in this stage of the research. Each sonobuoy has $L = 5$ hydrophones which each descend a distance $\beta = 0.01$ below the previous, with the first at the surface. In the noise function V_k for the sonobuoy observations, the noise for each of the sonobuoys is taken to be independent and identically distributed, so that the matrix Σ_k is a diagonal matrix with entries $\sigma_\eta = 0.15$.

While filtering, samples of the signal need only contain a short history of sample submarine states since any noise generated from earlier submarine positions will have left the area of interest. A Selectively Resampling Particle filter, previously called a hybrid weighted interacting particle filter, with $n = 3000$ particles is sufficient to provide excellent tracking results in an efficient computation.

An example simulation is shown in figure 1 and figure 2. In these figures, the ocean truth is at the left, the observations at the sonobuoys are displayed in the middle, and the state of the filter is displayed at the right. The submarine location is shown as a circle trailing the past path of the submarine. Each sonobuoy is shown as a square. Note that the sonobuoys have moved counterclockwise from time 15 to time 100. The four lines in the observation display indicate the sound pressure $Y_k^i,\ i = 1, 2, 3, 4$ experienced at each of the sonobuoys. Positions of particles in the (x, y) plane are shown as white dots in the filter display, and the true location of the submarine is shown as a red circle in order to facilitate visual inspection of filter success. It can be seen that by time 100, the filter has substantially localized the (x, y) position of the submarine.

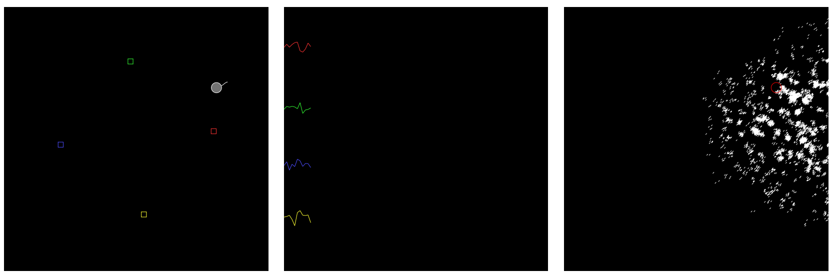

Figure 1. Example simulation at time 15.

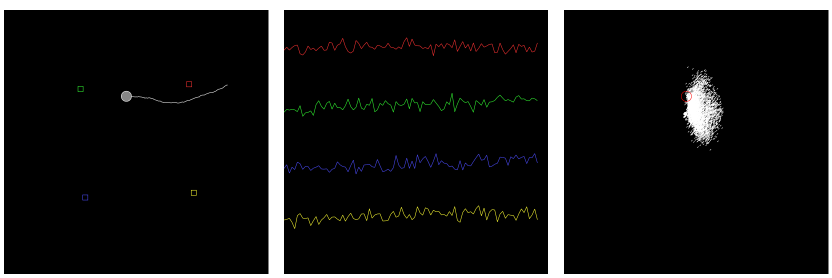

Figure 2. Example simulation at time 100.

3.2. Performance

While the filter does not always hold a close localization of the target submarine, it does track near to the target after an initial detection phase. Also, the target is almost always detected successfully, meaning that eventual localization does occur. Holding a close localization at all times is not to be expected, since the observations are composed of noise pressures in the water that are detected from past positions of the submarine. If the submarine takes an unlikely maneuver, the filter will provide a greater likelihood to states to which the submarine would have been more likely to move until such time as the observations provide correcting information.

Time constraints did not allow the production of graphical analyses of the filter performance before publication.

4. CONCLUSIONS

A promising method has been developed to track submarines using the data from maneuvering sonobuoys. Initial tests with this filter using the parameters given above demonstrate a consistent ability to localize and track one submarine with random initial state. Further work is ongoing to measure the results, to provide tracking of multiple submarines, and to determine optimal control of the sonobuoys.

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