University of Alberta

Numerical Simulation of an Elastic String

by

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A thesis is submitted to the Faculty of Graduate Studies and Research in partial fulfillment of the requirement for the degree of Master of science

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DEDICATION

This manuscript is dedicated to the memory of my friend Wubeshet Bekele

ABSTRACT

A numerical method for solving nonlinear, unsteady partial differential equations is developed to simulate the motion of an elastic string. The equations that govern this motion are reconstructed in conservative form to take advantage of a conservative discretization. This conservative model is required to capture the propagation of the wave generated by the motion of the string.

The partial differential equations that describe the motion of a string are discretized by an implicit finite difference approximation. These implicit equations are time advanced by a fictitious first order partial differential equation, whose steady state solutions is equivalent to the solution of the string motion at the next time step. The fictitious first order partial differential equation is integrated to its steady state by an explicit Runge-Kutta method.The resulting numerical method is extremely robust and will be used in future investigation of fluid structural interaction problems.

This numerical method is evaluated for five test cases. The first test case is a comparison between analytical and numerical solutions. For the rest four test cases the solutions obtained are analyzed for their ability to maintain the conservation of the total energy in the system and in the case of low amplitude waves the maintenance of a constant propagation speed.

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CHAPTER ONE

INTRODUCTION

1-1 Historical Background

The motion of an elastic string has been the subject of interest since the sixteen century. In 1741 Taylor took the first steps towards formulating the equations for a string motion by describing it as the force proportional to a normal acceleration. In 1743 D'Alembert derived an explicit partial differential equation for small motions of a heavy vibrating string and in 1751 Euler formulated the correct equation of motion for large vibrations of a string. A good historical presentation can be found in "The Rational Mechanics of Flexible or Elastic Bodies" by C.Truesdell [1].

The intellectual journey begun in the sixteen century has not finalized investigations of an elastic string. The equation of string motion, which is mainly dependent on the constitutive relations, is still under study for different material properties. For material that are elastic, thermostatic and viscoelastic the constitutive relations depend on stretch, stretch and temperature, and finally stretch or the time derivative of stretch, respectively.

Physical systems are often modeled mathematically by nonlinear equation that are difficult to solve, as a result these equations are often linearized for simplicity. In some cases a linearization causes the mathematical model to lose its intended feature, on the other hand one could simply formulate a linear model. This can be achieved by assuming a purely transverse motion which results in a constitutive relation as discussed by J.B.Keller [2].

Any mathematical model of significant sophistication is sure to result in a series of nonlinear equations.Solving these nonlinear, hyperbolic partial differential equations is difficult by direct methods. It is also difficult to find solutions to these equations, by the traditional method of characteristics. However, through the years there have been many numerical methods suggested to approximate the solutions of these problems. The accuracy of which are not the same. Some of these different methods are presented below to show both current progress and the need for alternative.

1) Von Neumann's Method: A method of solution for wave propagation in a continuous media was presented in 1944 by John Von Neumann [3]. Von Neumann proposed that the treatment of hydrodynamic shock problems corresponds to a return from the continuum theory to a kinetic theory using a very simplified quasi-molecular model. Unfortunately this method doesn't produce an equivalent discrete mathematical model of the continuum media when there is a discontinuity in velocity or strain. Applying this method to the solution of an elastic string problem is discussed by S.S. Antman[4] as follows: "using the principle of classical particle mechanics, the equation of motion of K beads are joined in sequence by massless, non-linearly elastic springs with both ends fixed.

The solutions for the positions, velocity, and strain of this discrete model converges to the string model only in the classical (smooth) solution, not to the one which suffers jump discontinuity in its velocity and strain. The numerical method Von Neumann proposed can be used to find a smooth solution only."

2) Galerkin method: An approximate solution of the equation of motion of an elastic string can be produced from a Faedo-Galerkin method. In this method the presence of a mass matrix (a matrix which is time dependent) makes it difficult to find the solution of the system of ordinary differential equations[5].The analysis of the above method on a quasi-linear model of an elastic string was carried out in 1973 by R.W.Dickey[6]. He proved that the solution of the system of ordinary differential equations converges to the classical (smooth) solutions of the partial differential equations if there is no shock (discontinuity) in the propagation of the wave. This method requires the governing equations to be written in the first order differential equation and imposes the condition that the stretch should be greater or equal to one. Like Von Neumann's method, the Faedo-Galerkin method is also valid only for smooth solutions.

3) Godunov scheme: One of the most current method for solving the equation of an elastic string is a Godunov scheme. For a good work using this method on elastic string motion see J.Wegner, J.B.Haddow and R.J.Tait in 1989[7], also K.Abdella in 1989[8] and J.L.Zhong in 1994 [9]. This method is based on the Riemann problem. A comparison between different Riemann solvers, starting from the exact solver to different ways of approximations was investigated in 1991 by R.J.Tait et.al. [10]. Unfortunately this method requires a significant amount of computing resources and is not easily extended to

multidimension. This method also requires the governing equations to be written in the first order differential equation and imposes the condition of stretch to be greater or equal to one.

While there has been significant development in the modeling of an elastic string, there is still the need for a multidimensional model that dose not impose the condition of stretch to be greater or equal to one.

1-2 Objectives

The objectives of this thesis are to develop a numerical scheme which will provide a solution for the dynamical response of the motion of a string with out imposing the condition that stretch to be greater or equal to one. The objectives can be broken down into three main parts.

First, the numerical method should be stable. It should also be capable of simulating long time wave motion with possible wave reflections from a boundary. These requirements are fulfilled by the following actions: (i) Undertake no linearization of the equation of string motion or it's numerical construction; (ii) Imposing no condition on stretch; (iii) Ensure at least Von Neumann stability [11]; (iv)The method will be tested for number of wave reflections to ensure consistency.

Second, the numerical method should be accurate in space and time as well as economical in computer time and memory requirements.

The discretization is performed in conservative form, which is similar to a finite volume approach where the equation of motion is satisfied in each computational grid cell. To be efficient in computer time and memory, a modified explicit, four stage Runge-Kutta method is used time advance the unsteady model. This approach has the advantage of controlling the stability and accuracy of the scheme, which is achieved by controlling stage coefficients in the Runge-Kutta formulation.

Third, the numerical method should be able to be extended to solve a mathematical model of a fluid structural interaction problem. This will be fulfilled since the numerical method is not constructed to be valid only for an elastic string. The approach can be used to solve nonlinear unsteady partial differential equations. For future analysis of the wave propagation in the face of fluid and elastic structural interactions, an external force is considered which will couple the two media.To test the resulting method, a number of different test cases will be simulated.

1-3 Outline

In Chapter Two the nonlinear equations that describe the unsteady motion of an elastic string are developed. This construction is divided into three parts: (1) a kinematics of deformation, which specifies the length, the shape, the boundary and initial conditions of the string; (2) a dynamical equation of motion, which considers all forces acting on the string and utilizes the relation of unbalanced forces with acceleration; (3) a constitutive relation, which embodies the material property of a perfectly flexible elastic string in the mathematical model.

Formulation of the governing equation in conservative form is necessary to take the advantage of conservative discretization. In Chapter Three the conservative version of the governing equation is constructed from the principle of linear momentum. Hyperbolic, nonlinear equations that have been investigated by researchers for years; some of these works are briefly cited in Chapter Three. The formation of shock and its relationship to the type of constitutive relation used is discussed in this Chapter. From this discussion it will be shown that a linear constitutive relation can not produce shocks.

The velocity of the wave is calculated using a traveling wave model. The total energy of the system is a summation of kinetic energy, stored energy, and potential energy. An expression for total energy is constructed and used to validate our numerical simulation. In most previous investigation a condition for stretch to be greater or equal to one is imposed. However, this condition is only required for stable equilibrium configuration, and need not be imposed for the dynamical problem.

A conservative numerical discretization of the conservative equation of motion is presented in Chapter Four. Construction of a conservative discretization, the effect of nonlinear mathematical model on the discretized form, and the problem in handling the boundary condition is discussed in Chapter Four. Based on a linearized version of the governing equation and using a Von Neumann stability analysis as a guide, a discrete form of the governing equation, which fulfills the Lax equivalence theorem [11], is chosen. The numerical discretization used is an implicit treatment that is second order accurate both spatially and temporally.

One of the objective of this these is to develop a numerical method which can be used in the future fluid structure interaction problem. Thus we require a method that is capable of simulating the long time response of a dynamical motion. Thus to show why a first order numerical method is not suitable for simulating a large time response we will illustrate the performance of several TVD method.

In Chapter Five we will construct a new method for solving the equation. A fictitious first order partial differential equation is defined in such a way that the steady state solutions of this equation is equivalent to one time step advancement of the discretized implicit equation of string motion. To find the steady state solution of this first order partial differential equation, an explicit, four stage Runge-kutta method is chosen. This method is adapted from the work of Jameson [12].

The steady state solution of the defined fictitious first order partial differential equation is taken as one time step advancement of the unsteady equation of string motion. By repeated operation of the above scheme, the solution is obtained for a desired length of time.

In Chapter Six results for several test cases are presented. The test cases include: (1) a Comparison of a numerical and analytical solution. (2) a dynamical response in the absence of external pressure; (3) a dynamical response to impact only. (4) a large amplitude simulation to initiate nonlinear behavior.

For the first case the position of the sting obtained form the numerical as well as the analytical solution are shown on the same graph. For the last four test cases, the position of the string before and after reflections and a comparison of two wave positions at every

25,000 time steps are presented. Characteristics of the residuals of first order partial differential equation, which shows how many time steps are taken in the explicit Runge-Kutta scheme, are plotted. The percentage of normalized total energy is plotted to show the characteristics of the numerical solutions in conservation of the total energy of the system. For large amplitude motion of the string the stretch for different time levels are plotted to show the variation of the streach.The motion of the string for 100,000 time steps, which is equivalent to fifty reflections, gives encouraging results of the accuracy of the system for the tested period.

CHAPTER TWO

MATHEMATICAL MODEL

In this Chapter, an equation of motion for a perfectly flexible string is formulated. Even though the string under consideration is one dimensional, the particles (material points) occupy one Euclidean space and therefore the movements of these particles are observed in the 2D plane. A detailed description of the geometrical configuration is given under the section on the kinematics of deformation.

A motion can be initiated in a system due to unbalanced forces acting on the system.These forces can be constant or variable. The interaction and the relationship of these forces with acceleration is presented in the section on the dynamic equation of motion. and finally the material property of the considered material is modeled by an appropriate constitutive relation. The constitutive relation will determine the type of response the material renders to the subjected forces and geometrical conditions. In the last section of this Chapter, a non-dimensional form of the governing equation is constructed to investigate the characteristics of the string equation.

2 - 1 Kinematics of Deformation

Consider the space in which the string is observed in an Euclidean point spaces, $\mathcal{E}$, and the time it is observed in real time, R. A reference configuration is defined as the configuration in which the string lies in the $\mathbf{i}$- direction along the interval of [0, L]. A string consists of a continuously distributed material. Each material point in the string is distinguished by its coordinate S in the reference (un deformed) configuration.

Fig. 2.1 Reference configuration

After the string undergoes some motion, its particles occupy a different positions in the Euclidean point space. The new shape of the string is known as the deformed configuration.

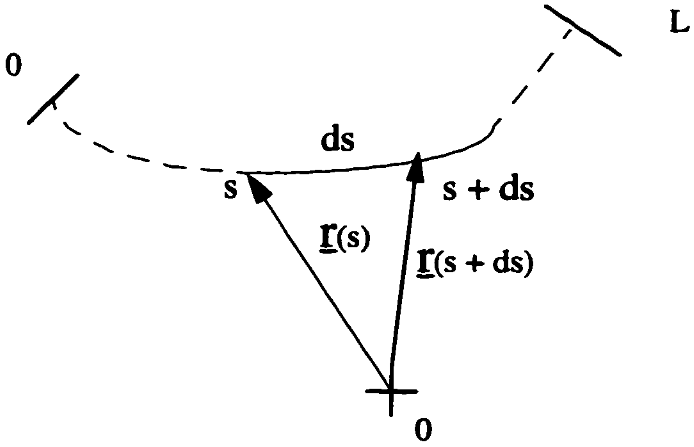


Fig. 2.2 Deformed configuration at fixed time

Let

S = reference arc length

s(S) = arc length along the deformed string

The relation between the reference and deformed configuration is expressed by stretch, which is a deformation gradient.Here

$$\lambda(S) = \frac{ds}{dS}$$ is a local "stretch" of the string at a fixed time.

For the dynamical case,

$$\lambda(S,t) = \frac{\partial s}{\partial S}$$

When $\lambda(S,t) > 1$ the string is elongated, and when $\lambda(S,t) < 1$ the string is compressed.Most investigation, impose the condition $\lambda(S,t) > 1$. But this condition is not imposed in our investigation for the reason presented in Chapter Three.The strain, which is defined as a ratio of a change in deformed length over a reference or original length, is related to stretch as:

$$\delta = \frac{\partial s - \partial S}{\partial S} = \lambda - 1$$

where δ is the strain.

Boundary Conditions

Let the ends of the string be fixed at the points 0 and L, and let $\underline{\mathbf{r}}$ be continuous on [0 1] for all t > 0. Then prescribing $\underline{\mathbf{r}}$ at S = 0 and S = 1 provides the following boundary conditions:

$$\underline{r}_1(0,t) = 0 \qquad \underline{r}_1(1,t) = L$$

$$\underline{r}_2(0,t) = 0 \qquad \underline{r}_2(1,t) = 0$$

Initial Condition

Assume that the string is initially in an unstretched configuration with no initial velocity. This means that the initial conditions are as follows:

$$\underline{r}_1(S,0) = S \qquad \underline{r}_2(S,0) = 0$$

$$\frac{\partial \underline{r}_1(S,0)}{\partial t} = 0 \qquad \frac{\partial \underline{r}_2(S,0)}{\partial t} = 0$$

2 - 2 Dynamic Equation of Motion

Forces acting on a system are external forces and contact forces, which are the internal resistance of the material to deformation. The configuration of the forces acting on the system is shown in Fig.2. 3.

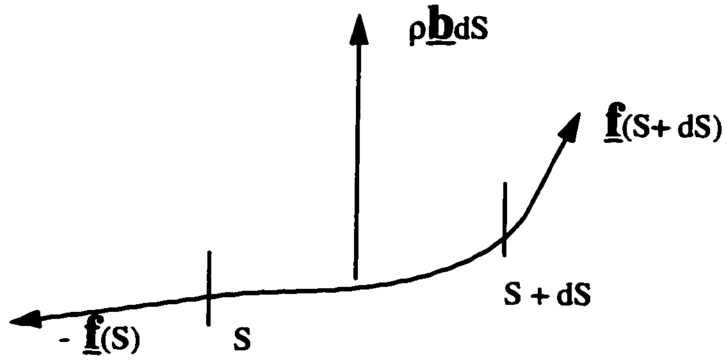


Fig.2. 3 Configuration of forces

Consider all acting forces on an elemental part of a string. Using a relation of unbalanced force and acceleration, the dynamic equation of motion is derived as follows.

Let $\underline{\mathbf{f}}(S,t)$ be the force exerted by the part (S , L] on the remaining part [0 , S) at time t; let $\underline{\mathbf{b}}$ (S,t) be the force applied externally per unit mass; and let ρ(S,t) be the mass per unit initial length. At a fixed time,

$$\underline{f}(S+dS) - \underline{f}(S) + \rho\underline{b}\, dS = \rho \frac{\partial^2 \underline{r}}{\partial t^2}\, dS$$

This is equivalent to

$$\frac{\partial \underline{f}}{\partial S} + \rho\underline{b} = \rho \frac{\partial^2 \underline{r}}{\partial t^2} \qquad (2.1)$$

2 - 3 Constitutive Relation

Those material properties which relate the contact force $\underline{\mathbf{f}}(S,t)$ to the motion of the string are called the constitutive relations. These constitutive relations distinguish the different responses of different materials. Thus a rubber band, a steel wire, a cotton thread, a thermoelastic or viscoelastic material would all require different constitutive relations.

A defining property of a string is its perfect flexibility, which is expressed mathematically by the following requirement: that $\underline{\mathbf{f}}(S,t)$ be tangent to the curve at $\underline{\mathbf{r}}(S,t)$ for each S and t.

Let $\underline{l} = \frac{\partial \underline{r}}{\partial s}$ be a unit tangent to deformed string; then a unit tangent can be related to stretch as follows.

$$\underline{l} = \frac{\partial \underline{r}}{\partial S} \quad \frac{\partial S}{\partial s} = \frac{\partial \underline{r}}{\partial S} \quad \lambda^{-1} \tag{2.2}$$

Assuming that there is no shear force on the string so that $\underline{f}$ (S,t) is parallel to $\underline{l}$, then the contact force can be written as

$$\underline{f}(S,t) = f(S,t)\,\underline{l} \tag{2.3}$$

Here f(S,t) is the intensity of the force. Assuming that the intensity of force is some function of stretch (λ), f(S,t) = f(λ) (this is the elastic case, similar to a rubber band), then

$$f(S,t)\underline{l} = \lambda^{-1} f(\lambda) \frac{\partial \underline{r}}{\partial S} \tag{2.4}$$

if $\quad g(\lambda) = \lambda^{-1} f(\lambda)$. $\quad$ (2.5)

Then Eqn. 2.3 can be written as

$$\underline{f}(S,t) = g(\lambda) \frac{\partial \underline{r}}{\partial S} \tag{2.6}$$

Using the above constitutive relation in Eqn.2.1, the equation of motion will have the following form:

$$\frac{\partial}{\partial S}\left(g(\lambda) \frac{\partial \underline{r}}{\partial S}\right) + \rho \underline{b} = \rho \frac{\partial^2 \underline{r}}{\partial t^2} \tag{2.7}$$

2 - 4 Governing Equation of Motion

Considering an elemental length of a deformed string with the external force applied on it, the configuration is shown in Fig. 2.4 below. The equation of motion, Eqn. 2.7 can be further analyzed as follows.

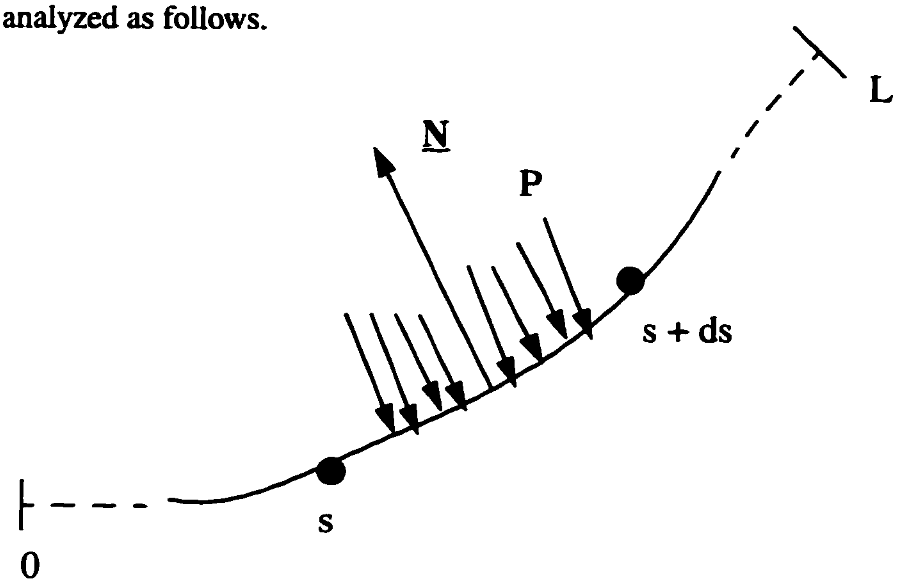


Fig. 2.4 pressure distribution on deformed string

Now $\rho\, \underline{\mathbf{b}}$ is the applied force per unit initial length; to find the force per unit deformed length, multiply $\rho\, \underline{\mathbf{b}}$ by λ.Then

$\rho\underline{\mathbf{b}}\lambda^{-1}$ = force per unit deformed length.

Let P be pressure intensity and $\underline{\mathbf{N}}$ a unit normal to the deformed string. The relationship between pressure and external force can be written as

$$\lambda^{-1}\rho\underline{\mathbf{b}} = -P\underline{\mathbf{N}}$$

then $$\rho\underline{\mathbf{b}} = -P\underline{\mathbf{N}}\lambda \qquad (2.8)$$

If $\underline{\mathbf{k}}$ is a unit vector perpendicular to the page then

$$\underline{\mathbf{N}} = \underline{\mathbf{k}} \; \mathrm{X} \, \underline{\mathbf{l}}$$

$$\lambda \underline{\mathbf{N}} = \underline{\mathbf{k}} \; \mathrm{X} \, (\lambda \underline{\mathbf{l}}) \tag{2.9}$$

where 'X' represents a cross product. Using the relationship of a unit tangent and the concept of stretch of Eqn. 2.2 in Eqn. 2.9

$$\lambda \underline{\mathbf{N}} = \underline{\mathbf{k}} \, \mathrm{X} \frac{\partial \underline{r}}{\partial S} \tag{2.10}$$

Eqn. 2.8 and Eqn. 2.10 can be combined to form the following equation.

$$\rho \underline{\mathbf{b}} = -\mathrm{P}(\underline{\mathbf{k}} \, \mathrm{X} \frac{\partial \underline{r}}{\partial S}) = \mathrm{P}(\frac{\partial \underline{r}}{\partial S} \; \mathrm{X} \, \underline{\mathbf{k}}) \tag{2.11}$$

Eqn. 2.11 can be written in tensor notation as

$$\rho \underline{\mathbf{b}} = \mathrm{P}(\frac{\partial r_i}{\partial S} e_i \;\; \mathrm{X} \, \underline{\mathbf{k}}) \tag{2.12}$$

For the system under consideration, **k** has only one component; thus Eqn. 2.12 can be written as

$$\rho \underline{\mathbf{b}} = \mathrm{P} \, \varepsilon_{i\,3j} \frac{\partial r_i}{\partial S} e_j \tag{2.13}$$

$$= \mathrm{P} \, \varepsilon_{ji\,3} \frac{\partial r_i}{\partial S} e_j \qquad\qquad = \mathrm{P} \, \varepsilon_{ji} \frac{\partial r_i}{\partial S} e_j \tag{2.14}$$

ε_{ij} in the above equation is an Alternating tensor that can have the values

$$\varepsilon_{11} = \varepsilon_{22} = 0 \qquad \varepsilon_{12} = 1 \qquad \varepsilon_{21} = -1$$

Using the relation of Eqn. 2.14 for external force in Eqn. 2.7, the equation of motion in two-dimensional Euclidean space will be:

$$g(\lambda)\frac{\partial^2 r_i}{\partial S^2} + \lambda^{-1}\frac{\partial g}{\partial \lambda}\frac{\partial r_i}{\partial S}\frac{\partial r_j}{\partial S}\frac{\partial^2 r_j}{\partial S^2} + P\,\varepsilon_{ji}\frac{\partial r_j}{\partial S} = \rho\frac{\partial^2 r_i}{\partial t^2} \qquad (2.15)$$

where i = 1, 2 and j = 1, 2.

Assume a linear relationship between contact force and stretch as shown in the following equations:

$$f(\lambda) = E(\lambda - 1)$$

$$g(\lambda) = E\left(1 - \frac{1}{\lambda}\right) \qquad (2.16)$$

where E is an elastic constant.

after substituting the above relation in Eqn. 2.15, the equation of string motion in expanded form will be:

$$E\left(1 - \frac{1}{\lambda}\right)\frac{\partial^2 r_1}{\partial S^2} + \frac{E}{\lambda^3}\left(\frac{\partial r_1}{\partial S}\frac{\partial^2 r_1}{\partial S^2} + \frac{\partial r_2}{\partial S}\frac{\partial^2 r_2}{\partial S^2}\right)\frac{\partial r_1}{\partial S} - P\frac{\partial r_2}{\partial S} = \rho\frac{\partial^2 r_1}{\partial t^2}$$

$$E\left(1 - \frac{1}{\lambda}\right)\frac{\partial^2 r_2}{\partial S^2} + \frac{E}{\lambda^3}\left(\frac{\partial r_1}{\partial S}\frac{\partial^2 r_1}{\partial S^2} + \frac{\partial r_2}{\partial S}\frac{\partial^2 r_2}{\partial S^2}\right)\frac{\partial r_2}{\partial S} + P\frac{\partial r_1}{\partial S} = \rho\frac{\partial^2 r_2}{\partial t^2}$$

(2.17)

Using the matrix representation of the governing equation, it is important to find the equation type and characteristics of the string. In matrix form, the governing equation can be written as:

$$\underline{\underline{\mathbf{A}}}\,\frac{\partial^2 \underline{r}}{\partial S^2} + \underline{\underline{\mathbf{B}}}\,\frac{\partial \underline{r}}{\partial S} = \frac{\partial^2 \underline{r}}{\partial t^2} \tag{2.18}$$

where

$$\underline{\underline{\mathbf{A}}} = \begin{bmatrix} \frac{E}{\rho}\left(1-\frac{1}{\lambda}\right) & 0 \\ 0 & \frac{E}{\rho}\left(1-\frac{1}{\lambda}\right) \end{bmatrix}$$

$$\underline{\underline{\mathbf{B}}} = \begin{bmatrix} \frac{E}{\rho\lambda^3}\left(\frac{\partial r_1}{\partial S}\frac{\partial^2 r_1}{\partial S^2} + \frac{\partial r_2}{\partial S}\frac{\partial^2 r_2}{\partial S^2}\right) & -\frac{P}{\rho} \\ \frac{P}{\rho} & \frac{E}{\rho\lambda^3}\left(\frac{\partial r_1}{\partial S}\frac{\partial^2 r_1}{\partial S^2} + \frac{\partial r_2}{\partial S}\frac{\partial^2 r_2}{\partial S^2}\right) \end{bmatrix}$$

$$\frac{\partial \underline{r}}{\partial S} = \begin{bmatrix} \frac{\partial r_1}{\partial S} \\ \frac{\partial r_2}{\partial S} \end{bmatrix}, \quad \frac{\partial^2 \underline{r}}{\partial S^2} = \begin{bmatrix} \frac{\partial^2 r_1}{\partial S^2} \\ \frac{\partial^2 r_2}{\partial S^2} \end{bmatrix}, \quad \frac{\partial^2 \underline{r}}{\partial t^2} = \begin{bmatrix} \frac{\partial^2 r_1}{\partial t^2} \\ \frac{\partial^2 r_2}{\partial t^2} \end{bmatrix}$$

2-5 Dimensional Analysis

Dimensional analysis is used to determine the dimensionless characteristics of the equation of string motion.The non-dimensional form of the equation is obtained by choosing non-dimensional quantities as follows (the hat representing the non-dimensional quantity).

$$\hat{P} = \frac{PL}{E}$$

$$\hat{S} = \frac{S}{L} \qquad \varepsilon[0 \quad 1]$$

$$\hat{r}_i = \frac{r_i}{L}$$

$$\hat{\lambda} = \lambda$$

$$\hat{t} = t\sqrt{\frac{E}{\rho L^2}}$$

$$\frac{\partial}{\partial S} = \frac{\partial}{\partial \hat{S}} \frac{\partial \hat{S}}{\partial S} = \frac{1}{L} \frac{\partial}{\partial \hat{S}}$$

$$\frac{\partial^2}{\partial S^2} = \frac{1}{L^2} \frac{\partial^2}{\partial \hat{S}^2} \qquad (2.19)$$

$$\frac{\partial^2}{\partial t^2} = \frac{E}{\rho L^2} \frac{\partial^2}{\partial \hat{t}^2}$$

Let $$\hat{C}_0 = \left(\frac{L}{t}\right) \Big/ \left(\sqrt{\frac{E}{\rho}}\right) \qquad (2.20)$$

Where $\hat{C}_o$, the propagation speed, is a dimensionless characteristic of the equation of motion and depends on the property of the material, E and ρ. Using the above relations in eqn. 2.17, the non-dimensional form of the governing equation, after dropping the hat, will be

$$\left(1 - \frac{1}{\lambda}\right)\frac{\partial^2 r_1}{\partial S^2} + \frac{1}{\lambda^3}\left(\frac{\partial r_1}{\partial S}\frac{\partial^2 r_1}{\partial S^2} + \frac{\partial r_2}{\partial S}\;\frac{\partial^2 r_2}{\partial S^2}\right)\frac{\partial r_1}{\partial S} - P\frac{\partial r_2}{\partial S} = \frac{\partial^2 r_1}{\partial t^2}$$

$$\left(1 - \frac{1}{\lambda}\right)\frac{\partial^2 r_2}{\partial S^2} + \frac{1}{\lambda^3}\left(\frac{\partial r_1}{\partial S}\frac{\partial^2 r_1}{\partial S^2} + \frac{\partial r_2}{\partial S}\;\frac{\partial^2 r_2}{\partial S^2}\right)\frac{\partial r_2}{\partial S} + P\frac{\partial r_1}{\partial S} = \frac{\partial^2 r_2}{\partial t^2} \qquad (2.21)$$

CHAPTER THREE

CONSERVATION LAWS & PROPERTIES OF MOTION

There are different types of waves propagated in an elastic media, one of which is shock wave. A shock wave is defined as a surface whose motion is continuous, but whose velocity, deformation gradient, and higher order derivatives suffer finite jump discontinuities [13]. A substantial amount of work has been done in shock wave propagation in gas dynamics.However wave propagation in elastic material varies significantly from wave motion in gases. In elastic media, shock waves and continuous simple waves occur in both expansive and compressive motion as discussed by Courant - Friedrichs [14]:

> "there is always a sonic discontinuity at the head of a rarefaction wave entering a zone in which the material is unstrained. In contrast to a gas, which expands indefinitely under zero pressure, an elastic material assumes a well defined original state when it suffers no stress."

Another type of elastic wave is an acceleration wave. An acceleration wave is defined as a surface whose motion and first order derivatives are continuous, but the acceleration, the second and the higher order derivatives of the motion suffer finite jump discontinuities [15]. In the section of shock and constitutive relation we will discuss the possibility of shock formation for the constitutive relation used in this thesis.

It has been shown by Lax and Wendroff [16] that a discontinuous solution can be computed without special treatment of the discontinuity if both the partial differential equation, and its numerical approximation, are constructed in conservative form.

In the following Chapter a conservative form of the governing equation of string motion, based on the conservation of momentum, is constructed. Other properties of motion which will help in the understanding of wave motion; i.e, wave velocity and equation for the total energy of the system will also be investigated.At last section of Chapter Three we present energy criteria for stability, which will show why we don't require that a condition on stretch be imposed in our investigation.

3 - 1 Conservation Laws

The conservation of momentum for a moving string requires that the total change of force flux across the boundary is equal to the total rate of change of momentum.

For a string which is contained in the closed interval of [0 L], the total amount of momentum is given by

$$\int_0^L \rho \frac{\partial \underline{r}}{\partial t} \, dS$$

Let $\underline{F}$ be the flux across the boundary. From conservation of momentum, the following relation is obtained.

$$\frac{\partial}{\partial t}\int_0^L \rho \frac{\partial \underline{r}}{\partial t}\, dS = \int_0^L \frac{\partial \underline{F}}{\partial S}\, dS \qquad (3.1)$$

The non-dimensional form of Eqn. 3.1 is constructed using the following non-dimensional terms (the hat represents the non-dimensional terms);

$$\hat{\underline{r}} = \frac{\underline{r}}{L}$$

$$\hat{S} = \frac{S}{L}$$

$$\frac{\partial^2}{\partial t^2} = \frac{E}{\rho L^2}\left(\frac{\partial^2}{\partial \hat{t}^2}\right)$$

$$\hat{\underline{F}} = \frac{\underline{F}}{E}$$

$$\int_0^L \rho L \left(\frac{E}{\rho L^2}\right)\frac{\partial^2 \hat{\underline{r}}}{\partial \hat{t}^2}\, d\hat{S} = \int_0^L \frac{E}{L}\,\frac{\partial \hat{\underline{F}}}{\partial \hat{S}}\, d\hat{S} \qquad (3.2)$$

Dropping the hat, the integral form of the equation of motion become:

$$\int_0^L \left(\frac{\partial}{\partial t}\left(\frac{\partial \underline{r}}{\partial t}\right) - \frac{\partial \underline{F}}{\partial S}\right) dS = 0 \qquad (3.3)$$

If every point where all partial derivatives of $\frac{\partial \underline{r}}{\partial t}$ and $\underline{\mathbf{F}}$ exist, the differential equation form of the conservation law will be

$$\frac{\partial \underline{F}}{\partial S} = \frac{\partial}{\partial t}\left(\frac{\partial \underline{r}}{\partial t}\right) \tag{3.4}$$

We now connect this conservation form of the equation with the previously derived governing equation in Chapter Two.

Let $\underline{\mathbf{F}} = (\underline{\mathbf{F}}_1 \ , \ \underline{\mathbf{F}}_2)^T$

$\underline{\mathbf{r}} = (\underline{\mathbf{r}}_1 \ , \ \underline{\mathbf{r}}_2)^T$

where

$$\underline{F}_1 = \frac{\partial \underline{r}_1}{\partial S} - \frac{1}{\lambda}\left(\frac{\partial \underline{r}_1}{\partial S}\right) - P\,\underline{r}_2 \tag{3.5a}$$

$$\underline{F}_2 = \frac{\partial \underline{r}_2}{\partial S} + \frac{1}{\lambda}\left(\frac{\partial \underline{r}_2}{\partial S}\right) + P\,\underline{r}_1 \tag{3.5b}$$

$$\frac{\partial}{\partial t}\left(\frac{\partial \underline{r}}{\partial t}\right) = \left[\frac{\partial^2 r_1}{\partial t^2}, \frac{\partial^2 r_2}{\partial t^2}\right]^T \tag{3.5c}$$

Using the above relation, Eqn. 3.4 can be written in expanded form as

$$\frac{\partial}{\partial S}\left(\frac{\partial \underline{r}_1}{\partial S} - \frac{1}{\lambda}\left(\frac{\partial \underline{r}_1}{\partial S}\right) - P\,\underline{r}_2\right) = \frac{\partial}{\partial t}\left(\frac{\partial \underline{r}_1}{\partial t}\right)$$

$$\frac{\partial}{\partial S}\left(\frac{\partial \underline{r}_2}{\partial S} + \frac{1}{\lambda}\left(\frac{\partial \underline{r}_2}{\partial S}\right) + P\,\underline{r}_1\right) = \frac{\partial}{\partial t}\left(\frac{\partial \underline{r}_2}{\partial t}\right) \tag{3.6}$$

3 - 2 Wave Velocity

The velocity of a propagating wave is dependent of the deformation gradient, elastic modulus and mass of the material [17]. Traveling waves are waves whose solutions have the form of $\{\underline{r} = \underline{r}(S - Ct)\}$, where C is a constant speed of the wave. The transformation of the governing equation of string motion from two independent variables S and t to one independent variable $\{\ \xi = S - Ct\ \}$ is done as follows.

Let

C = constant speed of the wave

at $t = 0$ $\qquad \xi = S_o$

$$\xi = S - Ct$$

$$S = \xi + Ct$$

Transforming each term of the governing equation separately

$$\frac{\partial r_i}{\partial S} = \frac{\partial r_i}{\partial \xi}\frac{\partial \xi}{\partial S} = \frac{\partial r_i}{\partial \xi} = r_i'$$

$$\frac{\partial^2 r_i}{\partial S^2} = \frac{\partial}{\partial S}\left(\frac{\partial r_i}{\partial \xi}\frac{\partial \xi}{\partial S}\right) = \frac{\partial}{\partial S}\left(\frac{\partial r_i}{\partial \xi}\right) = \frac{\partial}{\partial \xi}\left(\frac{\partial r_i}{\partial S}\right) = \frac{\partial^2 r_i}{\partial \xi^2} = r_i''$$

$$\frac{\partial r_i}{\partial t} = \frac{\partial r_i}{\partial \xi}\frac{\partial \xi}{\partial t} = -C\frac{\partial r_i}{\partial \xi} = -Cr_i'$$

$$\frac{\partial^2 r_i}{\partial t^2} = \frac{\partial}{\partial S}\left(\frac{\partial r_i}{\partial \xi}\frac{\partial \xi}{\partial t}\right) = \frac{\partial}{\partial t}\left(-C\frac{\partial r_i}{\partial \xi}\right) = C^2\frac{\partial^2 r_i}{\partial \xi^2} = C^2 r_i'' \tag{3.7}$$

$$\frac{\partial}{\partial S}\left(g(\lambda)\frac{\partial r_i}{\partial S}\right) = g(\lambda)\frac{\partial^2 r_i}{\partial S^2} + \frac{\partial g(\lambda)}{\partial S}\frac{\partial r_i}{\partial S} \tag{3.8}$$

But $$\frac{\partial g(\lambda)}{\partial S} = \frac{\partial g(\lambda)}{\partial \lambda}\frac{\partial \lambda}{\partial S}$$

Form Eqn. 2.2 using the relation of stretch

$$\lambda^2 = \frac{\partial r_j}{\partial S}\frac{\partial r_j}{\partial S}$$

then $$\frac{\partial \lambda}{\partial S} = \frac{1}{\lambda}\frac{\partial r_j}{\partial S}\frac{\partial^2 r_j}{\partial S^2} \tag{3.9}$$

Eqn. 3.9 can be written as an equation dependent on a single variable using the transformation given in eqn. 3.7:

$$\frac{\partial \lambda}{\partial S} = \frac{1}{\lambda} r_j' r_j''$$

and $$\frac{\partial g(\lambda)}{\partial S} = \frac{1}{\lambda}\frac{\partial g(\lambda)}{\partial \lambda} r_j' r_j'' \tag{3.10}$$

Thus after substituting Eqn. 3.10 in Eqn. 3.8 we get:

$$g(\lambda)\frac{\partial^2 r_i}{\partial S^2}+\frac{\partial g(\lambda)}{\partial S}\ \frac{\partial r_i}{\partial S} = g(\lambda)r_i\,'' + \frac{1}{\lambda}\frac{\partial g(\lambda)}{\partial\lambda}\ r_i\,'r_j\,'r_j\,''$$

$$= \left(g(\lambda)\delta_{ij} + \frac{1}{\lambda}\frac{\partial}{\partial\lambda}g(\lambda)r_i\,'r_j\,'\right)r_j\,''$$

$$= A_{ij}\ r_j\,''$$

where $$A_{ij} = g(\lambda)\delta_{ij} + \frac{1}{\lambda}\frac{\partial}{\partial\lambda}g(\lambda)r_i\,'r_j\,' \tag{3.11}$$

and δ_{ij} is Kronecker delta, which can have the values $\delta_{ij} = 1$ if $i = j$

$\delta_{ij} = 0$ if $i \neq j$

The governing equation of string motion written in Eqn. 2.15 can be transformed to a single independent variable ξ using the above equations:

$$\rho C^2 r_i\,'' = A_{ij}\ r_j\,'' + P\,\varepsilon_{ji}\,r_j\,' \tag{3.12}$$

The primes in Eqn. 3.12 are a partial differential with respect to ξ.

Recall the relation of Eqn. 2.5 that

$$g(\lambda) = \frac{f(\lambda)}{\lambda}$$

then $$\frac{\partial g}{\partial\lambda} = \frac{1}{\lambda}\frac{\partial f}{\partial\lambda} - \frac{1}{\lambda^2}f(\lambda) \tag{3.13}$$

A_{ij} of Eqn. 3.11, after substituting Eqn. 3.13, can be written as

$$A_{ij} = \frac{f(\lambda)}{\lambda}\left(\delta_{ij} - \frac{1}{\lambda^2} r_i' r_j'\right) + \frac{1}{\lambda^2}\frac{\partial f}{\partial \lambda} r_i' r_j' \qquad (3.14)$$

But for a unit tangent the relation from Eqn. 2.2 can be written as

$$\frac{\partial r_i}{\partial S} = r_i' = \lambda l_i \qquad (3.15)$$

Substituting Eqn. 3.15 in Eqn. 3.14 the equation of A_{ij} will be

$$A_{ij} = \frac{f(\lambda)}{\lambda}(\delta_{ij} - l_i l_j) + \frac{\partial f}{\partial \lambda} l_i l_j \qquad (3.16)$$

To find the speed of propagation, let the wave be an acceleration wave, where the first order differentials are continuous but the second order and higher differential are discontinuous.

Thus at discontinuity the equation of string motion, Eqn. 3.12, will have the following two forms, one from the left side 'L' and the other from the right side 'R' of the discontinuity as follows:

$$\rho C^2 (r_i'')_L = A_{ij} \; (r_j'')_L + P \, \varepsilon_{ji} \; r_j' \qquad (3.17a)$$

$$\rho C^2 (r_i'')_R = A_{ij} \; (r_j'')_R + P \, \varepsilon_{ji} \; r_j' \qquad (3.17b)$$

$(A_{ij})_L = (A_{ij})_R = A_{ij}$ because A_{ij} depends on r' which is continuous.

Subtracting Eqn. 3.17b from Eqn. 3.17a we get

$$\rho C^2 ((r_i'')_L - (r_i'')_R) = A_{ij} \; ((r_j'')_L - (r_j'')_R) \qquad (3.18)$$

But Eqn. 3.18 can be written as

$$\rho C^2 d_i = A_{ij}\ d_j \tag{3.19}$$

where $d_i = (r_i'')_L - (r_i'')_R$

From Eqn. 3.19 it can be seen that ρC^2 is an eigen value of A_{ij}

To find the eigen value of A_{ij} multiply Eqn. 3.16 by l_j

$$A_{ij}\ l_j = \frac{f(\lambda)}{\lambda}(\ \delta_{ij}\, l_j\ - l_i l_j l_j\ ) + \frac{\partial f}{\partial \lambda} l_i l_j l_j$$

(3.20)

But $\quad l_j l_j = 1 \qquad$ (3.21a)

$$\delta_{ij}\, l_j = l_i \tag{3.21b}$$

Then using Eqn 3.21a and 3.21b, Eqn. 3.20 is reduced to

$$A_{ij}\ l_j = \frac{\partial f}{\partial \lambda} l_i \tag{3.22}$$

Thus $\frac{\partial f}{\partial \lambda}$ is one of the eigenvalues of A_{ij} . Using a linear relationship between stretch and contact force (Eqn. 2.16), this eigenvalue is equal to E (elastic constant).

Since A_{ij} is a symmetric two by two matrix, it has two real eigenvalues.

To find the other eigenvalue of A_{ij} , multiply Eqn. 3.16 by N_j , a unit normal to the deformed string.

$$A_{ij}\ N_j = \frac{f(\lambda)}{\lambda}(\delta_{ij}N_j\ - l_i l_j N_j) + \frac{\partial f}{\partial\lambda} l_i l_j N_j \tag{3.23}$$

since $l_j N_j = 0$ and $\delta_{ij}N_j = N_i$ (3.24)

Eqn. 3.23 after substituting Eqn. 3.24 will be

$$A_{ij}\ N_j = \frac{f(\lambda)}{\lambda}N_i \tag{3.25}$$

thus from Eqn. 3.25, $\frac{f(\lambda)}{\lambda}$ is the other eigen value of A_{ij} . Using the linear relationship of stretch and contact force, this eigen value is equal to $E\left(1-\frac{1}{\lambda}\right)$.

The speed of the wave in the string can be found from Eqn 3.19 and the two eigen values of A_{ij} as follows

$$C = \pm\sqrt{\frac{E}{\rho}} \quad \text{and} \quad C = \pm\sqrt{\frac{E}{\rho}\left(1-\frac{1}{\lambda}\right)} \tag{3.26}$$

The first speed in Eqn. 3.26 is the fast wave, which depends on the material property i.e., the elastic constant and mass per unit length of the material. For the string under consideration $\lambda = 1$ is the reference configuration and for real eigen values $\lambda \geq 1$; thus the second wave speed in Eqn. 3.26 is less than the first wave speed, which is called the slow wave speed. The '+' and '-' sign shows a wave moving to the left or right direction, respectively.

To calculate the time (T) which the wave will take to move from S = 0 to S = 1 based on the fastest wave speed (this wave speed is equal to the longitudinal wave speed calculated by M.F. Beatty & J.B. Haddow [18]).

$$T = \frac{1}{C_f} = L\sqrt{\frac{\rho}{E}} \tag{3.28}$$

where, C_f is the fast wave speed. The non dimensional time it takes for the wave to travel from one boundary point to another is $\hat{T} = 1$ (using Eqn. 2.19 for non dimensioning).

3 - 3 Shock & Constitutive Relations

It is possible to produce large amplitude waves using a linear stress and stretch relationship.But from experimental investigations we know that high stretch values are not possible with a linear constitutive relation. To mimic physically plausibly often the Ogden's or Mooney-Rivlin model for constitutive relation is used. These constitutive relations have a non-linear relation between stress and stretch. However this constitutive relation will allow shocks to form during the propagation of the wave. In this section, the relation between shock formation and constitutive relation will be constructed.

The relations between shock formation and material properties are discussed by J.D.Achenbach [19]. To show these relations of material property and shock formation, a general equation of wave propagation without external force is presented as follows.

$$C(F)\frac{\partial^2 r}{\partial S^2} = \frac{\partial^2 r}{\partial t^2} \quad (3.29)$$

where $F = \lambda l$ is the deformation gradient and $f(F)$ is stress

The general relation between stress and deformation gradient can be represented graphically as

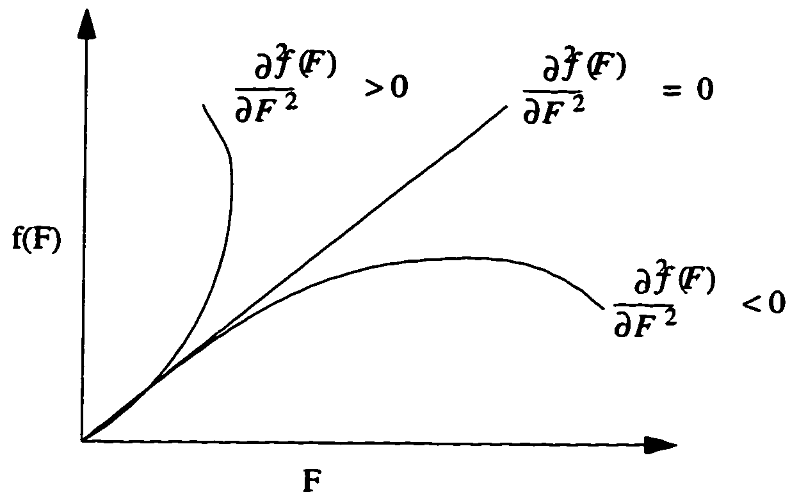


Fig. 3.1 stress and deformation gradient

To analyze Eqn. 3.29 lets consider the following initial and boundary conditions

$$t < 0 \qquad \lambda = \lambda_0 = 0 \qquad F = F_0 \quad (3.30)$$

$$t > 0 \qquad S = 0 \qquad \frac{\partial r}{\partial t} = v\ (0,t) = h\ (t) \quad (3.31)$$

where h(0) = 0

Eqn. 3.29 can be written as

$$\frac{\partial V}{\partial t} - C(F)^2\frac{\partial F}{\partial S} = 0$$

$$\frac{\partial V}{\partial S} - \frac{\partial F}{\partial t} = 0 \tag{3.32}$$

where $V = \frac{\partial r}{\partial t}$

The characteristic curve of Eqn. 3.32 in (S,t) plane is

$$\frac{dS}{dt} = \pm C(F) \tag{3.33}$$

If a new coordinate ξ and η are introduced in such a way that

ξ = constant along $+C\ (F)$ curves

η = constant along $-C\ (F)$ curves and introducing a quantity Q(F) as

$$Q(F) = \int_{F_0}^{F} C(F)dF \tag{3.34}$$

Eqn. 3.32 can be transformed to a new equation along the introduced new coordinates system as

$$\frac{\partial V}{\partial \eta} - \frac{\partial Q}{\partial \eta} = 0 \qquad \text{along} \quad +C\ (F)$$

$$\frac{\partial V}{\partial \xi} + \frac{\partial Q}{\partial \xi} = 0 \qquad \text{along} \quad -C\ (F) \tag{3.35}$$

Eqn. 3.35 can be integrated to yield

$$V\ (\xi,\eta) - Q\ (\xi,\eta) = g_1(\xi) \qquad \text{along} \ +C\ (F)$$

$$V\ (\xi,\eta) + Q\ (\xi,\eta) = g_2(\xi) \qquad \text{along} \ -C\ (F) \tag{3.36}$$

Applying the initial condition on Eqn. 3.36 and at S = 0, $\xi = \eta = t$, we will have the following condition in the (ξ, η) plane.

$$\xi < 0 \qquad V = 0 \qquad Q = 0$$

$$\xi = \eta = t \qquad V = h(t) = h(\xi) \tag{3.37}$$

Thus the solution for $V(\xi, \eta)$ and $Q(\xi, \eta)$ are

$$V = h(\xi)$$

$$Q = -h(\xi) = \int_{F_0}^{F} C(F)dF \tag{3.38}$$

in Eqn.3.38 V and Q are function of ξ only.This implies that $F = F(\xi)$ and $C = C(\xi)$ Thus integration of

$$\frac{dS}{dt} = +C(\xi) \quad \text{subjected to the condition } \xi = t \text{ at } S = 0 \text{ gives}$$

$$S = C(\xi)(t - \xi) \tag{3.39}$$

The solution given by Eqn. 3.36 are simple wave solutions and are valid as long as the $+C(F)$ characteristic do not intersect. If we consider the case that $h(\xi)$ increases monotonically, when C(F) > 0 thus from Eqn. 3.38 F decreases in magnitude. If

$\frac{\partial \mathfrak{f}(F)}{\partial F^2} > 0$, $C(\xi)$ decreases monotonically as F decreases, Thus $+C(F)$ characteristics then never intersect. Thus the simple wave solution presented are valid. If

$\frac{\partial \mathfrak{f}(F)}{\partial F^2} < 0$, $C(\xi)$ increases as F decreases thus an intersection will occur, which is a shock formation. From the above we can see that shock formation is a nonlinear effect in a concave curve of stress and stretch relation. For a linear stress and stretch relation there will not be a shock formation. There for we believe that non linear behavior observed when a high amplitude wave propagated in the string is not due to shock formation, but instead a consequence of stretch being less than one.

3 - 4 Energy of the System

The total energy of the system is composed of kinetic energy, strain energy (stored energy), and the potential energy due to the external force. The total energy of the system for a conservative system should be constant with respect to time.

The total kinetic energy of the system, represented by K is calculated over the region [0 1] and is given by the relation,

$$K = \int_0^1 \rho \left| \frac{\partial r}{\partial t} \right|^2 dS \tag{3.40}$$

for the same domain the strain energy density is given by

$$\wp = \int_1^{\lambda} f(\lambda) d\lambda \tag{3.41}$$

The total stored energy at a given time t is given by:

$$\Psi = \int_0^1 \wp(\lambda, S) dS \tag{3.42}$$

if the external force has the form

$$\underline{b}\,(S,t) = \hat{g}\,(r\,(S,t\,), S\,) \tag{3.43}$$

Let ω be a function of r and S, then the potential energy density of the external force is represented by

$$\hat{g} = \frac{\partial \omega}{\partial r}$$

if $\hat{g}$ is conservative, then the total potential energy of the system due to external force is represented by the following relation:

$$\Omega = \int_0^1 \omega(r,S)\,dS \tag{3.44}$$

using the initial conditions specified in Chapter Two, the total energy of the system from initial time t = 0 to t = τ is given by

$$E_T = \int_0^1 \left[\Psi + \frac{1}{2} \left| \frac{\partial r}{\partial t} \right|^2 + \Omega \right] dS \tag{3.45}$$

For the case when there is no external force applied, Eqn. 3.45 will be

$$E_T = \int_0^1 \left[\Psi + \frac{1}{2} \left| \frac{\partial r}{\partial t} \right|^2 \right] dS \tag{3.46}$$

but using the linear relationship between contact force and stretch the total stored energy can written as:

$$\wp(\lambda) = \int_1^{\lambda} E(\lambda - 1)\,d\lambda \tag{3.47}$$

after substituting Eqn. 3.47 and applying the dimensional analysis of Chapter Two, the total energy of the system (Eqn. 3.46)in non-dimensional form can be written as:

$$E_T = \int_0^1 \left(\frac{\lambda^2}{2} - \lambda + \frac{1}{2} + \frac{1}{2} \left| \frac{\partial r}{\partial t} \right|^2 \right) dS \tag{3.48}$$

This equation will be used in the numerical approximation section to check the obtained numerical solutions for maintaining the conservation of the total energy.

3 - 5 Energy Criteria

In this section, the energy criterion for the stability of a string in equilibrium is derived from a variational principle. We will show that the stable equilibrium configuration exists when the local stretch is every where greater or equal to one, to minimize the potential energy.

Lets assume that $\mathbf{r}(S)$ be an energy minimizing configuration (original configuration). The original configuration is given a small virtual displacement. The new displaced position is assumed to fulfill the boundary condition of the original configuration.Virtual work has been done by the external force to produce the new displaced configuration. For a stable string configuration, the virtual work done by the external force should not exceed the increase of the stored strain energy. However if the stored strain energy is less than the virtual work done by the external force, the excess energy will appear as kinetic energy and ultimately produce a motion.Thus a stable configuration is defined as an equilibrium configuration with minimum potential energy.

If the admissible configuration (displaced position which fulfill the boundary condition of the original configuration) represented by **r***(S). For equilibrium configuration from the principle of minimum potential energy

$$\Psi(r')_{admissible} \geq \Psi(r^{*\prime})_{orginal} \tag{3.49}$$

Ψ is a potential energy and primes represent derivation with respect to S

The proof for Eqn. 3.49 can be found in K.Washizu [20]. The energy criterion of elastic stability for network has been derived by A.A. Atai and D.J.Steigmann [21]. Here, their derivation is adapted for elastic string problem.

Let $$r^*(S) = r(S) + a\phi(S) \tag{3.50}$$

where a is a fixed vector and S_1, S_2, S_3 are three arbitrary points in the interval [0 1] with $S_1 < S_2 < S_3$ then $\phi(S)$ is defined as

$$\phi(S) = \begin{cases} \phi_0(S) & (S \in [0 \;\; S_1] \cup [S_3 \;\; 1]) \\ \phi_1(S) & (S \in [S_1 \;\; S_2]) \\ \phi_2(S) & (S \in [S_2 \;\; S_3]) \end{cases} \tag{3.51}$$

Let $\theta = (S_2 - S_1)/(S_3 - S_1)$

and

$$\phi_0(S) = 0 \;,\; \phi_1(S) = S - S_1 \,,\; \phi_2(S) = (-(\theta/(1-\theta)))(S - S_3) \tag{3.52}$$

Integrating Eqn. 3.49 in the region $[S_1 \;\; S_3] \subset [0 \;\; 1]$

thus

$$\int_{S_1}^{S_3} (\Psi(r^{*\prime}) - \Psi(r'))dS \geq 0 \tag{3.53}$$

Let $\Delta = S_3 - S_1$, $\Delta_1 = S_2 - S_1$ and $\Delta_2 = S_3 - S_2 = (1-\theta)\Delta$ (3.54)

substituting Eqn. 3.50, 3.51 and 3.54 in Eqn. 3.53 and dividing by Δ

$$\frac{\theta}{\Delta_1}\int_{S_1}^{S_2} \Psi(r' + a)dS \quad + \frac{1-\theta}{\Delta_2}\int_{S_2}^{S_3} \Psi\left(r' - \frac{\theta}{1-\theta}a\right)dS \; -$$

$$\frac{1}{\Delta}\int_{S_1}^{S_3} \Psi(r')dS \geq 0 \tag{3.55}$$

Let $\Delta \to 0$ thus $\Delta_1, \Delta_2 \to 0$ with θ fixed and use of mean value theorem also dividing by $\theta > 0$ gives the algebraic Weierstrass inequality

$$\Psi(r' + a) + \frac{1-\theta}{\theta}\Psi\left(r' - \frac{\theta}{1-\theta}a\right) - \frac{1}{\theta}\Psi(r') \geq 0 \tag{3.56}$$

For small θ, the second term in Eqn. 3.56 is given the leading order by

$$\frac{1-\theta}{\theta}\Psi\left(r' + \frac{\theta}{1-\theta}a\right) = \frac{1}{\theta}\Psi(r') - \Psi(r') - a \bullet \hat{f}(r') + \frac{(1-\theta)}{\theta}o(\theta) \tag{3.57}$$

where $\hat{f}(r') = \frac{\partial \Psi(r')}{\partial r} = f(\lambda)l$ (3.58) substituting Eqn.3.57 in Eqn. 3.56 and when $\theta \rightarrow 0^+$

$$\Psi(r' + a) - \Psi(r') - a \bullet \hat{f}(r') \geq 0 \tag{3.59}$$

If $\lambda = |r'|, \mu = |r' + a|$ and $l = \lambda^{-1} r'$ (3.60)

also from Eqn. 3.42

$$\Psi'(r') = \wp(\lambda) \tag{3.61}$$

substituting Eqn. 3.58,3.61 and 3.59 in Eqn. 3.56

$$\wp(\mu) - \wp(\lambda) \geq f(\lambda) l \bullet a \tag{3.62}$$

If $a = (\mu - \lambda) l$ since a is arbitrary fixed vector, then Eqn.3.62 will be

$$\wp(\mu) - \wp(\lambda) \geq (\mu - \lambda) f(\lambda) \quad , \ \mu > 0 \tag{3.63}$$

Eqn. 3.63 implies that $f'(\lambda) \geq 0$, thus the tangent modulus of the string is non negative at every point in the minimum potential energy configuration.

If $a = a\,m$, where $\mathbf{m}$ is any unit vector perpendicular to $\mathbf{l}$ then

$\mu = |\lambda l + a\,m| = \sqrt{\lambda^2 + a^2}$ which give Eqn.3.62 as

$$\wp(\mu) - \wp(\lambda) \geq 0 \qquad \mu > \lambda \tag{3.64}$$

for small $|a|$ Eqn. 3.63 gives

$$(\mu - \lambda)[f(\lambda) + (\mu - \lambda)^{-1} o\,(\mu - \lambda))] \geq 0 \tag{3.65}$$

dividing Eqn. 3.64 by $(\mu - \lambda) > 0$ and with the limit $(\mu - \lambda) \rightarrow 0^+$ Eqn. 3.65 will became

$$f(\lambda) \geq 0 \tag{3.66}$$

The linear constitutive relation from Eqn. 2.16

$$f(\lambda) = E(\lambda - 1) \tag{3.67}$$

form Eqn. 3.66 and Eqn. 3.67 it can be deduced that

$$\lambda \geq 1 \tag{3.68}$$

Eqn. 3.68, the direct consequence of Eqn. 3.66 is a sufficient condition for stable configuration of a string. This condition is not valid for motion of a string which experience instability. When a high amplitude wave initiated in the string there is a possibility for the string to experience all kind of behavior, stable as well as un stable configuration. Thus it is our conjecture that one doesn't need to impose Eqn. 3.68 in the dynamical investigation of an elastic string.

CHAPTER FOUR

DISCRITIZATION AND

EFFECT OF APPROXIMATION

4 - 1 Conservative form of Discretization

To construct a conservative discretization in general, let **F**(S) be a function defined in the range $B \leq S \leq E$, where the interval [B E] is discretized by considering the set of points as a cell center. these cell center points are S_1, S_2,...S_{i-1}, S_i, S_{i+1}...S_{m-2},S_{m-1} and the boundary represented by S_0 = B and S_m = E. The discrete representation of a function is (**F**(B), **F**(S_1), **F**(S_2),.....**F**(S_{i-1}), **F**(S_i), **F**(S_{i+1})....**F**(S_{m-2}),**F**(S_{m-1}), **F**(E). Half way between the cell center is represented as the cell boundary of the individual cells. These boundaries are $S_{1/2}, S_{3/2}, \ldots\ldots S_{i-1/2}, S_{i+1/2}, \ldots\ldots S_{m-3/2}, S_{m-1/2}$ as shown in the fig. 4.1

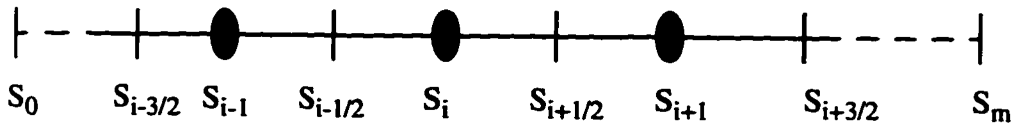


Fig. 4.1 Conservative Discretization

If the quantity $(S_{i+1} - S_i)$ is the mesh size which is taken as constant given by

$\Delta S = (S_m - S_0)/m$ and $S_i = S_0 + i\Delta S$ for $i = 0,1.......m$.

The differential of $\mathbf{F}(S)$ is approximated at cell center by a conservative discretization form as

$$\frac{\partial F}{\partial S}(S_i) = \frac{F_{i+1/2} - F_{i-1/2}}{\Delta S} \tag{4.1}$$

where $F_{i+1/2}$ and $F_{i-1/2}$ are the values of a function F at the cell boundaries. This approximation insures that the flux at the right side of one cell is equal to the left side of the neighboring cell at their common boundary.

The selected form of discretization approximate the flux term at two time levels, at n+1 and n. This implicit form of discretization with implicitness control parameter applied on the conservative form of the partial differential equation of the string motion; Eqn. 3.4; is given in conservative form as follows.

$$\mu\left(\frac{F_{i+1/2}^{n+1} - F_{i-1/2}^{n+1}}{\Delta S}\right) + (1-\mu)\left(\frac{F_{i+1/2}^{n} - F_{i-1/2}^{n}}{\Delta S}\right) = \frac{r_i^{n+1} - 2r_i^{n} + r_i^{n-1}}{\Delta t^2} \tag{4.2}$$

where $F_{i+1/2}^{n} = (F_1^{n} \quad , \quad F_2^{n})_{1+1/2}^{T} \quad , \quad r_i^{n} = (r_1^{n} \quad , \quad r_2^{n})_i^{T}$

n is time level and μ is implicitness control parameter range from 0 to 1. When $\mu = 0$ the scheme is fully explicit and when $\mu = 1$ it became fully implicit, the value in between will makes it partial implicit.

4 - 2 Effect of Non-Conservative Form

One of the advantages of a conservative discretization is that it can be used to calculate the solution of a partial differential equation which has a discontinuity in its first order term. However this advantage as shown by Lax and Wendroff [16], will not be utilized unless both the differential and discretized equations are in conservative form.

To show how various partial differential equation forms will lead to different discretized equation, a conservative and a quasi-linear equation forms are considered. For both equation forms each term is approximated by conservative discretization for a length of three grid cells for illustration purpose.

Lets consider three grid cells A,B,C as shown in Fig. 4.2 below

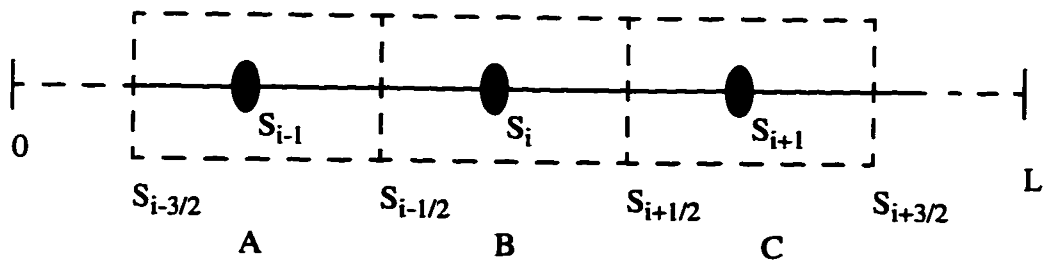


Fig.4.2 Three grid cells.

S_i, S_{i+1} and S_{i-1} are points at the cell center and $S_{i-3/2}$, $S_{i-1/2}$, $S_{i+1/2}$ and $S_{i+3/2}$ are cell boundaries.

The conservative mathematical model which is valid for each cells can be written as

$$\frac{\partial F}{\partial S} - \frac{\partial V}{\partial t} = 0 \qquad (4.3)$$

where $V = \frac{\partial r}{\partial t}$

The conservative discrete form for each cell can be written as:

for cell A

$$\frac{F_{i-1/2} - F_{i-3/2}}{\Delta S} - \frac{\partial V_{i-1}}{\partial t} = 0 \qquad (4.4a)$$

for cell B

$$\frac{F_{i+1/2} - F_{i-1/2}}{\Delta S} - \frac{\partial V_i}{\partial t} = 0 \qquad (4.4b)$$

for Cell C

$$\frac{F_{i+3/2} - F_{i+1/2}}{\Delta S} - \frac{\partial V_{i+1}}{\partial t} = 0 \qquad (4.4c)$$

The sum of the three cells (Eqn. 4.4a, 4.4b, 4.4c) average is equivalent to one cell whose center is at i and it's cell boundary at $S_{i+3/2}$ and $S_{i-3/2}$

$$\frac{F_{i+3/2} - F_{i-3/2}}{3\Delta S} - \frac{\partial}{\partial t}\left(\frac{V_{i+1} + V_i + V_{i-1}}{3}\right) = 0 \qquad (4.5)$$

The above construction shows that the conservative discretization on conservative mathematical form leads to a cancellation of the internal fluxes and the final discretized form (Eqn. 4.5) is in conservative form.

If the mathematical model is written in quasi-linear (non-conservative) form, Eqn. 4.3 will became

$$\Phi(r)\frac{\partial r}{\partial S} - \frac{\partial V}{\partial t} = 0 \tag{4.6a}$$

where $$\Phi(r)\frac{\partial r}{\partial S} = \frac{\partial F}{\partial S} \tag{4.6b}$$

Applying conservative discretization on the three cells using Eqn. 4.6,

for cell A:

$$\Phi(r)_{i-1}\frac{r_{i-1/2} - r_{i-3/2}}{\Delta S} - \frac{\partial V_{i-1}}{\partial t} = 0 \tag{4.7a}$$

but $\Phi(r)_{i-1}$ can be approximated as the average of $\Phi(r)_{i-1/2}$ and $\Phi(r)_{i-3/2}$ then

$$\left(\frac{\Phi(r)_{i-1/2} + \Phi(r)_{i-3/2}}{2}\right)\left(\frac{r_{i-1/2} - r_{i-3/2}}{\Delta S}\right) - \frac{\partial V_{i-1}}{\partial t} = 0 \tag{4.7b}$$

Similarly for cell B and C are

for cell B

$$\left(\frac{\Phi(r)_{i+1/2} + \Phi(r)_{i-1/2}}{2}\right)\left(\frac{r_{i+1/2} - r_{i-1/2}}{\Delta S}\right) - \frac{\partial V_i}{\partial t} = 0 \tag{4.7c}$$

for cell C

$$\left(\frac{\Phi(r)_{i+3/2}+\Phi(r)_{i+1/2}}{2}\right)\left(\frac{r_{i+3/2}-r_{i+1/2}}{\Delta S}\right)-\frac{\partial V_{i+1}}{\partial t} = 0 \qquad (4.7d)$$

The sum of the three cells average (Eqn. 4.7b, 4,7c, 4,7d) is given as

$$\left(\frac{\Phi(r)_{i+3/2}+\Phi(r)_{i-3/2}}{2}\right)\left(\frac{r_{i+3/2}-r_{i-3/2}}{3\Delta S}\right)-\frac{\partial}{\partial t}\left(\frac{V_{i+1}+V_i+V_{i-1}}{3}\right) =$$

$$(\Phi(r)_{i+3/2}-\Phi(r)_{i-1/2})\left(\frac{r_{i+1/2}-r_{i-3/2}}{6\Delta S}\right) -$$

$$(\Phi(r)_{i+1/2}-\Phi(r)_{i-3/2})\left(\frac{r_{i+3/2}-r_{i-1/2}}{6\Delta S}\right) \qquad (4.7e)$$

But discretizing Eqn 4.6b in one cell, i as the cell center and i+3/2, i-3/2 cell boundaries

$$\Phi(r)_i\left(\frac{r_{i+3/2}-r_{i-3/2}}{3\Delta S}\right) = \frac{F_{i+3/2}-F_{i-3/2}}{3\Delta S} \qquad (4.8)$$

but $\Phi(r)_i$ can be approximated as the average of $\Phi(r)_{i+3/2}$ and $\Phi(r)_{i-3/2}$, then Eqn. 4.8 can be written as

$$\left(\frac{\Phi(r)_{i+3/2}+\Phi(r)_{i-3/2}}{2}\right)\left(\frac{r_{i+3/2}-r_{i-3/2}}{3\Delta S}\right) = \frac{F_{i+3/2}-F_{i-3/2}}{3\Delta S} \qquad (4.9)$$

substituting Eqn. 4.9 in Eqn. 4.7e the discretized form of the non conservative equation will became

$$\frac{F_{i+3/2} - F_{i-3/2}}{3\Delta S} - \frac{\partial}{\partial t}\left(\frac{V_{i+1} + V_i + V_{i-1}}{3}\right) =$$

$$(\Phi(r)_{i+3/2} - \Phi(r)_{i-1/2})\left(\frac{r_{i+1/2} - r_{i-3/2}}{6\Delta S}\right) -$$

$$(\Phi(r)_{i+1/2} - \Phi(r)_{i-3/2})\left(\frac{r_{i+3/2} - r_{i-1/2}}{6\Delta S}\right) \qquad (4.10)$$

The final form of the discretized equation of the quasi-linear partial differential equation is not in conservative form. since extra terms are produced on the right hand side of Eqn.4.10, a conservative discretization of a differential equation is equivalent to the integral (weak) form of the conservation law (the proof for this can found in Lax and Wendroff [16] or G.A.Sod[22]). Thus the conservative form is the preferred approach.

4 - 3 Approximation at The Boundary

In general a numerical approximation requires more boundary conditions than its partial differential counter part [22]. The additional boundary conditions, can be another source of instability.

(1) General Finite Deference Discretization

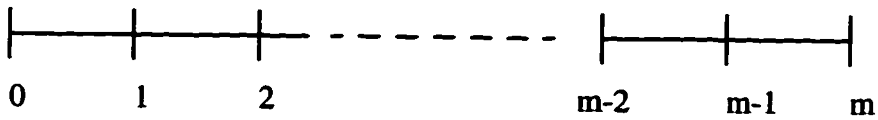


Fig. 4.3 General Finite Difference Boundary Discretization

Applying a central difference discretization at the interior points, each point needs information from both of its neighbors. If point 1 is considered, the discrete form needs information from 0 and 2. However the physical boundary condition gives information at point 0 on the position r_1 and r_2, but not on their spacial derivatives. Stretch at the boundaries are unknown, but are required by the finite deference scheme.

(2) Full Cell Type of Discretization

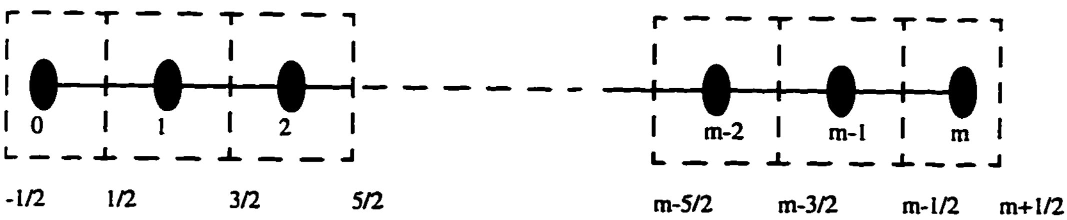


Fig.4.4 Full Cell Type of Discretization

A full cell type of discretization, considering each cell centers as evaluating points, will need values at half cell at both sides of the boundary. To evaluate at the cell center 0 and m an information at boundaries -1/2 and m+1/2 is required in addition to the information in the internal computational domain.

(3) Internal Computational Domain

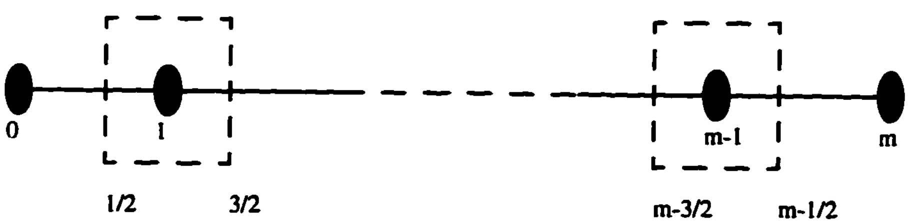


Fig.4.5 Internal Computational Domain

In internal computational domain discretization, evaluation of the differential equation is done in the internal points between 1 and m-1. The boundary conditions of the problem is included in the calculation by averaging at the boundaries 1/2 and m-1/2 of the geometrical quantities at a point 0,1 and m-1, m respectively. This kind evaluation is adapted in this work.

4 - 4 Stability Analysis

When constructing of a finite difference approximation, one must exercise choices of grid size, time step size, and time advancement. These choices shouldn't allow a truncation error to grow indefinitely (amplified) as the solution progress from one time step to another. To make decision on the above choices, a stability analysis is used as a guide.

In this thesis a Von Neumann method of stability analysis is used. Von Neumann method is a linear stability analysis based on a finite Fourier analysis.This method investigates the characteristic of the error as a function of the frequency content of the initial condition and of the solution.

To apply the Von Neumann method on our discretization, lets consider the conservative discretized equation 4.2, for clarity rewriting the equation.

$$\mu\left(\frac{F_{i+1/2}^{n+1} - F_{i-1/2}^{n+1}}{\Delta S}\right) + (1-\mu)\left(\frac{F_{i+1/2}^{n} - F_{i-1/2}^{n}}{\Delta S}\right) = \frac{r_i^{n+1} - 2r_i^{n} + r_i^{n-1}}{\Delta t^2} \quad (4.8)$$

In the above equation i is a spacial grid point and n is time level. For linear stability analysis, lets consider the coefficients of $\frac{\partial r_i}{\partial S}$ and r_j in Eqn. 3.5 as constants C_1 and C_2, where $C_1 = 1 - \frac{1}{\lambda}$ and $C_2 = P\varepsilon_{ji}$.Further assuming **F** as a single equation, Eqn. 3.5 will become:

$$F = C_1 \frac{\partial r}{\partial S} - C_2 r \tag{4.9}$$

The discretized form of **F** is given as

$$F_{i+1/2}^{n+1} = C_1\left(\frac{r_{i+1} - r_i}{\Delta S}\right)^{n+1} - C_2 r_{i+1/2}^{n+1} \tag{4.10}$$

applying the same discretization on **F** in Eqn. 4.8, at different grid points the discretized equation in expanded form will become.

$$\mu\left[C_1\left(\frac{r_{i+1} - 2r_i + r_{i-1}}{\Delta S^2}\right) - C_2\left(\frac{r_{i+1/2} - r_{i-1/2}}{\Delta S}\right)\right]^{n+1} +$$

$$(1-\mu)\left[C_1\left(\frac{r_{i+1} - 2r_i + r_{i-1}}{\Delta S^2}\right) - C_2\left(\frac{r_{i+1/2} - r_{i-1/2}}{\Delta S}\right)\right]^{n} =$$

$$\frac{r_i^{n+1} - 2r_i^{n} + r_i^{n-1}}{\Delta t^2} \tag{4.11}$$

The values of **r** at the cell boundary are approximated as the average of the values at the cell center on the left and the right of the considered cell boundary as:

$$r_{i+1/2} = \frac{r_{i+1} + r_i}{2} \text{ and } r_{i-1/2} = \frac{r_i + r_{i-1}}{2} \qquad (4.12)$$

Substituting Eqn. 4.12 in Eqn. 4.11 and rearranging the discretized equation will be.

$$\mu\left(C_1 \frac{\Delta t^2}{\Delta S^2}(r_{i+1} - 2r_i + r_{i-1}) - C_2 \frac{\Delta t^2}{2\Delta S}(r_{i+1} - r_{i-1})\right)^{n+1} +$$

$$(1-\mu)\left(C_1 \frac{\Delta t^2}{\Delta S^2}(r_{i+1} - 2r_i + r_{i-1}) - C_2 \frac{\Delta t^2}{2\Delta S}(r_{i+1} - r_{i-1})\right)^{n} =$$

$$r_i^{n+1} - 2r_i^{n} + r_i^{n-1}$$

(4.13)

If the exact solution of the partial differential equation (2.17) is assumed as $\bar{r}_i^{\,n}$ and the numerical solution by $r_i^{\,n}$ then the difference between these values is the error, represented by ε_i^n . Mathematical relationship between the error and numerical solution is given as

$$r_i^{\,n} = \bar{r}_i^{\,n} + \varepsilon_i^n \qquad (4.14)$$

Substituting Eqn. 4.14 in Eqn. 4.13 the equation for the error is found, which is the same form as the discretized equation

$$\mu\left(C_1 \frac{\Delta t^2}{\Delta S^2}(\varepsilon_{i+1} - 2\varepsilon_i + \varepsilon_{i-1}) - C_2 \frac{\Delta t^2}{2\Delta S}(\varepsilon_{i+1} - \varepsilon_{i-1})\right)^{n+1} +$$

$$(1-\mu)\left(C_1 \frac{\Delta t^2}{\Delta S^2}(\varepsilon_{i+1} - 2\varepsilon_i + \varepsilon_{i-1}) - C_2 \frac{\Delta t^2}{2\Delta S}(\varepsilon_{i+1} - \varepsilon_{i-1})\right)^{n} =$$

$$\varepsilon_i^{n+1} - 2\varepsilon_i^n + \varepsilon_i^{n-1} \tag{4.15}$$

The error term can be decomposed using a Fourier series as

$$\varepsilon_i^n = a^n e^{Iki\,\Delta S} \qquad \varepsilon_{i+1}^n = a^n e^{Ik(i+1)\,\Delta S} \tag{4.16}$$

In Eqn. 4.16 I is a complex number equals to $\sqrt{-1}$, a^n is amplitude, k is a wave number, i is spacial grid point and ΔS is the grid spacing. Substituting Eqn. 4.16 in Eqn. 4.15 and after rearranging, the equation for the error will became

$$\mu a^{n+1}\left(C_1 \frac{\Delta t^2}{\Delta S^2}(e^{Ik\,\Delta S} - 2 + e^{-Ik\,\Delta S}) - C_2 \frac{\Delta t^2}{2\Delta S}(e^{Ik\,\Delta S} - e^{-Ik\,\Delta S})\right) +$$

$$(1-\mu)a^n\left(C_1 \frac{\Delta t^2}{\Delta S^2}(e^{Ik\,\Delta S} - 2 + e^{-Ik\,\Delta S}) - C_2 \frac{\Delta t^2}{2\Delta S}(e^{Ik\,\Delta S} - e^{-Ik\,\Delta S})\right) =$$

$$a^{n+1} - 2a^n + a^{n-1}$$

(4.17)

Amplification factor G is defined as the ratio of an error amplitudes of two consecutive time levels, mathematically written as

$$G = \frac{a^n}{a^{n-1}}, \quad G^2 = \frac{a^{n+1}}{a^{n-1}} \tag{4.18}$$

The error equation in terms of amplification factor can be found after substituting Eqn. 4.18 in Eqn. 4.17 using Euler formula, the equation for amplification factor will be:

$$G^{2}\mu\left\{C_1 \frac{\Delta t^2}{\Delta S^2}(2\cos k\Delta S - 2) - C_2 \frac{\Delta t^2}{\Delta S}(I \sin k\Delta S)\right\} +$$

$$G(1-\mu)\left\{C_1 \frac{\Delta t^2}{\Delta S^2}(2\cos k\Delta S - 2) - C_2 \frac{\Delta t^2}{\Delta S}(I \sin k\Delta S)\right\} =$$

$$G^2 - 2G + 1 \tag{4.19}$$

If Cn is assumed as Courant number represent as

$$Cn = \frac{\sqrt{C_1}\Delta t}{\Delta S} \tag{4.20}$$

and Cc is a constant coefficient represented by

$$Cc = \frac{C_2 \Delta t^{2}}{\Delta S} \tag{4.21}$$

let θ be a phase angle represented by

$$\theta = k\Delta S \tag{4.22}$$

then substituting Eqn. 4.20, 4.21 and 4.22 in Eqn. 4.19 and rearranging, the equation of the amplification factor will be

$$G^2[\ \mu\{Cn^2(2\cos\theta - 2) - Cc(I \sin\theta)\} - 1] +$$

$$G\,[(1-\mu)\{Cn^2(2\cos\theta - 2) - Cc(I \sin\theta)\} + 2] - 1 = 0 \tag{4.23}$$

Von Neumann stability criteria: A finite difference scheme (with constant coefficients) is stable if and only if the modules of the amplification factor is less than one.[11]

$$|G| \leq 1 \tag{4.24}$$

The modules of the amplification factor of Eqn. 4.23 is determined graphically by fixing the value of μ and Cc for different values of Cn. Fixing these two values is chosen since Cc is a constant which is related to external force and varies as a square of the time step, but Cn which depends on the speed of the wave varies in the order of the time step. The variation of Cc is very small compared to Cn. Thus Cc is fixed at 0.1, the implicitness control parameter $\mu = 0.89$, and Cn is varied from 0.5 to 1.5 for the phase angle 0 to pi. For an explicit discretized form of the equation of string motion, the modules of the amplification factor vs. phase angle is plotted in Fig. 4.6. The amplification factor for explicit method is calculated by using $\mu = 0$ in Eqn. 4.23. Thus form Fig. 4.6 we can see that an explicit discretization is unstable under all condtions. For an implicit discretization using the above specified values the plot for the modules of the amplification factor vs. phase angle is shown in Fig. 4.7.Then form fig. 4.7, the modulus of the amplification factor which fulfill the Von Neumann stability criteria and also which give a minimum area under the modulus of the amplification factor vs. phase angle curve is selected. Thus a time step $\Delta t = 1 \times 10^{-4}$ and a grid spacing of $\Delta S = 5 \times 10^{-3}$ for a courant number 0.5 are obtained for the selected modulus of amplification factor curve. These values are used in the numerical approximation of the string motion.

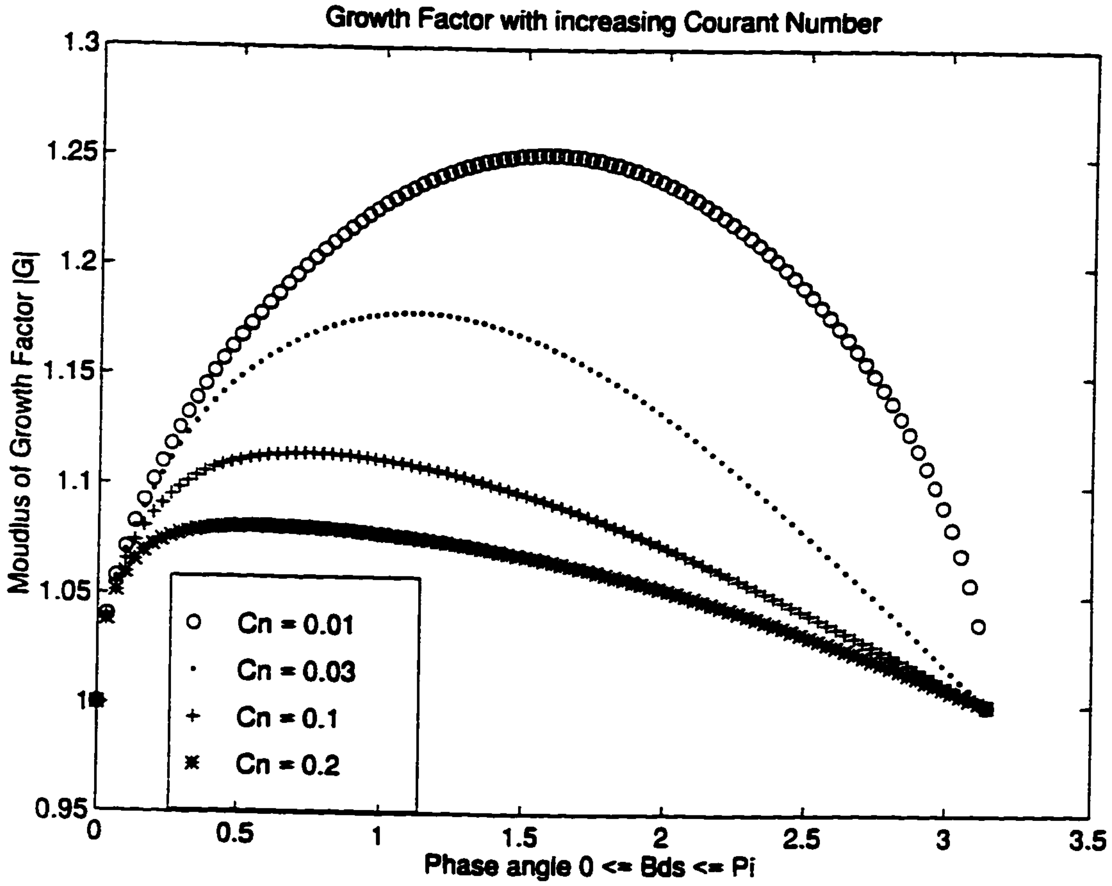


Fig. 4.6 Growth Factor for the explicit Discretization

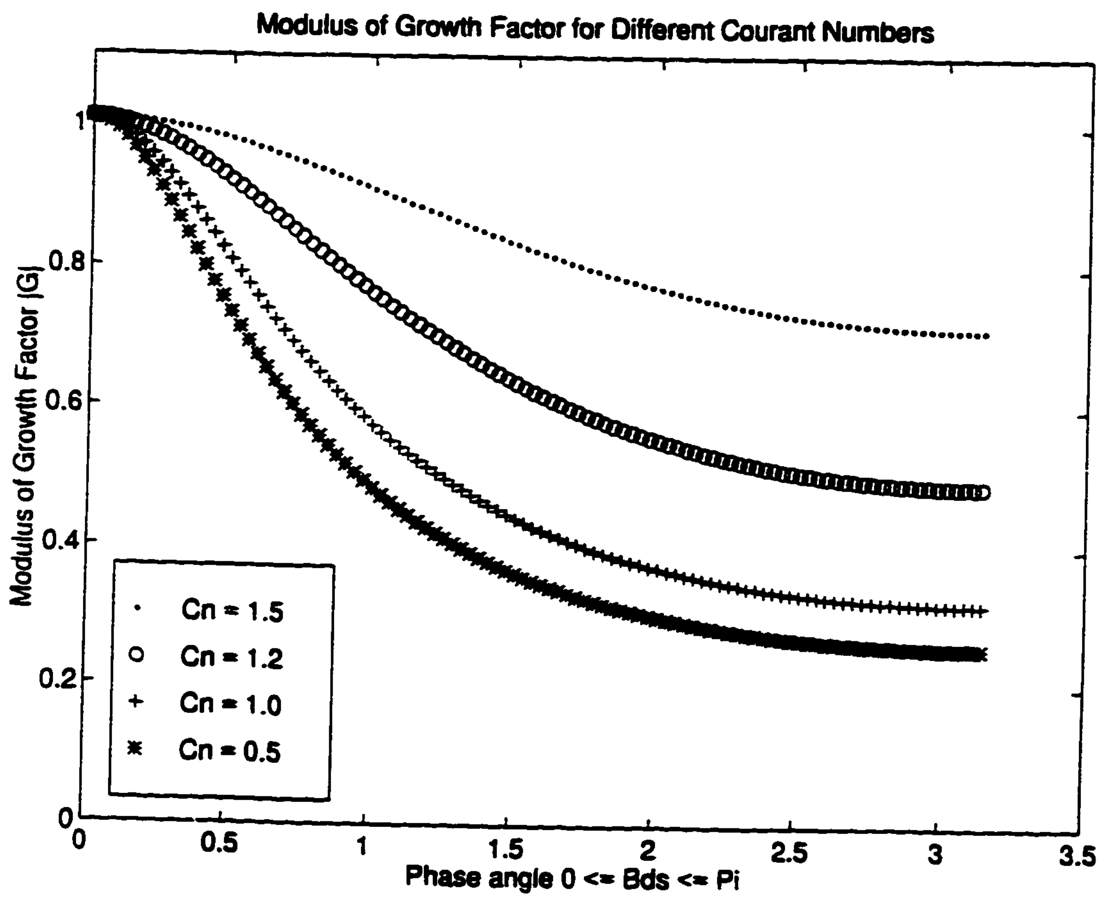


Fig. 4.7 Growth Factor for the implicit Discretization

CHAPTER FIVE

METHOD OF SOLUTION

There are varies types of numerical methods for solving a nonlinear, discretized, partial differential equations. To cite some of these methods (1) Newton's method: For solving systems of n by n nonlinear equations requires a good initial approximation, evaluating n^2 partial differentials and solving n equations at each step of an iteration. Evaluation of a Jacobian is not economical in either computational time or storage. Because of these disadvantages a Newton's method is not preferred. (2) Broyden's method: This method replaces the Jacobian matrix of Newton's method with an approximation matrix which is updated on each iteration.However Broyden's method like Newton's method requires a good initial approximation, and therefore difficulty of converge when there is no close initial approximation.

In this thesis a different numerical method is developed to solve the nonlinear, unsteady, discretized, equation of the string. A pseudo unsteady, first order partial differential equation is defined to integrate the nonlinear discretized equation of the string motion. Steady state solution of this first order partial differential equation is equivalent to the time advancement of the string equation.

To find the steady state solutions of the defined fictitious in time of first order partial differential equation, a four stage Runge-Kutta method is used. The detail implementation of this method is presented in the following sections.

5 - 1 Selected Numerical Methods

Long Time Performance

Many numerical methods preform well for short period investigation of the wave motion, but their dissipative behavior makes them unfit for long time simulation. In this section we will investigate the behavior of a number of total variation diminishing (TVD) numerical methods. Since we are interested in a numerical method which has a good long time response, we will show why these methods were not chosen for our investigation.

First order one dimensional wave equation wave propagation is given by

$$\frac{\partial r}{\partial t} + C\frac{\partial r}{\partial S} = 0 \tag{5.1}$$

where C is a wave speed

with initial and boundary condition as follows

boundary condition, the string is fixed at both ends

$$r(0,t) = 0 \qquad r(1,t) = 2000$$

the initial condition contain three wave shapes, rectangular, triangular and parabolic represented by the following equations

$$r(S,0) = \begin{cases} 0 & (S \leq 20) \\ \dfrac{\sqrt{1-(S-30DS)^2}}{(10DS)^2} & (20 < S \leq 40) \\ 0 & (40 < S \leq 60) \\ \dfrac{S-60DS}{10DS} & (60 < S \leq 70) \\ \left(1-\dfrac{S-70DS}{10DS}\right) & (70 < S \leq 80) \\ 0 & (80 < S \leq 100) \\ 1 & (100 < S \leq 120) \\ 0 & (S > 120) \end{cases} \tag{5.2}$$

The discretized form of Eqn. 5.1 for each method is given as follows.

Let $Cn = \dfrac{CDt}{DS}$ Courant Number

1) **Upwind method**

$$r_i^{n+1} = r_i^n - Cn(r_i^n - r_{i-1}^n) \tag{5.3}$$

2) **Lax Wendroff method**

$$r_i^{n+1} = r_i^n - (Cn(r_i^n - r_{i-1}^n) - ((Cn)/2)(1 - Cn)(r_{i+1}^n - 2r_i^n + r_{i-1}^n))$$

(5.4)

3) **Min Mod method**

$$r_i^{n+1} = r_i^n - (Cn(r_i^n - r_{i-1}^n) - ((Cn)/2)(1 - Cn)\phi(r_{i+1}^n - 2r_i^n + r_{i-1}^n))$$

where

$$\Lambda = \frac{r_{i-1}^n - r_{i-2}^n}{r_i^n - r_{i-1}^n} \qquad \phi = max\,(0, min\,(1, \Lambda)) \qquad (5.5)$$

5) **MUSCL**

$$r_i^{n+1} = r_i^n + (-Cn(r_i^n - r_{i-1}^n)) -$$

$$((Cn)/2)(1 - Cn)(\phi_1(r_{i+1}^n - r_i^n) - \phi_2(r_i^n - r_{i-1}^n))$$

where $\Phi_a = \left|\frac{r_{i-1}^n - r_{i-2}^n}{r_i^n - r_{i-1}^n}\right|$, $\Phi_b = \left|\frac{r_i^n - r_{i-1}^n}{r_{i+1}^n - r_i^n}\right|$

$$\Phi_c = \frac{r_{i-1}^n - r_{i-2}^n}{r_i^n - r_{i-1}^n} \quad , \quad \Phi_d = \frac{r_i^n - r_{i-1}^n}{r_{i+1}^n - r_i^n}$$

$$\phi_1 = \frac{\Phi_d + \Phi_b}{1 + \Phi_b} \quad , \quad \phi_2 = \frac{\Phi_c + \Phi_a}{1 + \Phi_a} \qquad (5.7)$$

The solution obtained using all the above method for the position of the wave at 500 and 2000 time steps are shown in Fig. 5.1 and 5.2. From the result we can observe that all the methods except Lax Wendroff shows a significant amount of dissipation. The initial wave shape is lost due to numerical dissipation. Note that the calculation is done with out reflection. If reflection is included in the calculation all of the wave may have dissipated out. In Lax Wendroff method even though it is not dissipative, the undershoot and overshoot and finally it loses of the initial wave shape is observed. Since for the tested period the performance of these methods are not satisfactory for long period investigation of wave motion. Thus we chose not to purse any of these methods for our investigation.

5 - 2 Defining a Residual

A conservative construction of the governing equation of the string requires that the change in the flux at the boundary to be equal to the rate of change of momentum. A residual is defined as the change between the flux at the boundary of a mesh cell and the rate of change of momentum with in this cell. In a conservative system this residual is zero. The primary objective of this method of solution is to reduce the residual to zero, by doing so the conservation principle will be insured as well as the solving method will deliver the required solutions.

Mathematically the residual is defined by rewriting the discretized Eqn. 4.2 as follows

$$R^*(r_i^{n+1}, r_i^{n}, r_i^{n-1}) \;=0= \frac{r_i^{n+1}}{\Delta t^2} - \mu\left(\frac{F_{i+1/2}^{n+1} - F_{i-1/2}^{n+1}}{\Delta S}\right) -$$

$$\frac{2\, r_i^{n}}{\Delta t^2} - (1-\mu)\left(\frac{F_{i+1/2}^{n} - F_{i-1/2}^{n}}{\Delta S}\right) + \frac{r_i^{n-1}}{\Delta t^2} \tag{5.8}$$

In eqn 5.8 $F_{i+1/2}$ and $F_{i-1/2}$ can be written in expanded form using Eqn.3.5c as follows:

$$F_{i+1/2} = (F_1|_{i+1/2} \;,\; F_2|_{i+1/2})^T \tag{5.9}$$

$$F_{i-1/2} = (F_1|_{i-1/2} \;,\; F_2|_{i-1/2})^T \tag{5.10}$$

$$F_1|_{i+1/2} = \left(1 - \frac{2}{\lambda_{i+1} + \lambda_i}\right)\left(\frac{r_1|_{i+1} - r_1|_i}{\Delta S}\right) - P\left(\frac{r_2|_{i+1} + r_2|_i}{2}\right) \tag{5.11}$$

$$F_1|_{i-1/2} = \left(1 - \frac{2}{\lambda_{i-1} + \lambda_i}\right)\left(\frac{r_1|_i - r_1|_{i-1}}{\Delta S}\right) - P\left(\frac{r_2|_i + r_2|_{i-1}}{2}\right) \tag{5.12}$$

similarly the other components of **F** i.e, $F_2|_{i+1/2}$ and $F_2|_{i-1/2}$ are constructed.

Solution of the nonlinear equation (5.8), when it is done by the traditional method is given in a general form as [23]

$$r_i^{n+1}|_{new} = r_i^{n+1}|_{old} - H_i\, R^*\left(r_i^{n+1}|_{old}, r_i^{n}, r_i^{n-1}\right)\beta_i \tag{5.13}$$

The solution of nonlinear equations using this method is achieved by iterating Eqn. 5.13 until it meet a pre stated tolerance limit. If we denote the Jacobian of **R*** as $A(r_i^{n+1})$, for Newton's method $H_i = A_i^{-1}$ and $\beta_i = 1.0$, where evaluation of A_i inverse is required for each iteration.Thus this method is not used since it requires a good initial approximation, large storage and significant computational time. A method which is developed in this thesis treats eqn. 5.8 as a modified steady state problem in a fictitious time t* and solved using a multi stage Runge-Kutta method.

5 - 3 Time Marching

A fictitious first order partial differential equation is defined as dependent of a fictitious time t* (a time different from the one used in the string equation of motion) and contains the nonlinear equation 5.8 as follows.

$$\frac{\partial \tilde{r}}{\partial t^*} + R^*(\tilde{r_i}, r_i^n, r_i^{n-1}) = 0 \qquad (5.14)$$

where

$$R^*(\tilde{r_i}, r_i^n, r_i^{n-1}) = \frac{\tilde{r_i}}{\Delta t^2} - \mu\left(\frac{\tilde{F}_{i+1/2} - \tilde{F}_{i-1/2}}{\Delta S}\right) -$$

$$\frac{2r_i^n}{\Delta t^2} - (1-\mu)\left(\frac{F_{i+1/2}^n - F_{i-1/2}^n}{\Delta S}\right) + \frac{r_i^{n-1}}{\Delta t^2} \qquad (5.15)$$

In Eqn. 5.15 $\tilde{F}_{i+1/2}$ and $\tilde{F}_{i-1/2}$ are given as

$$\tilde{F}_{i+1/2} = (\tilde{F}_1 \big|_{i+1/2} \;, \; \tilde{F}_2 \big|_{i+1/2} \qquad (5.16)$$

$$\tilde{F}_{i-1/2} = (\tilde{F}_1 \big|_{i-1/2} \; \tilde{F}_2 \big|_{i-1/2})^T \qquad (5.17)$$

where

$$\tilde{F}_1 \big|_{i+1/2} = \left(1 - \frac{2}{\tilde{\lambda}_{i+1} + \tilde{\lambda}_i}\right)\left(\frac{\tilde{r}_1\big|_{i+1} - \tilde{r}_1\big|_i}{\Delta S}\right) - P\left(\frac{\tilde{r}_2\big|_{i+1} + \tilde{r}_2\big|_i}{2}\right)$$

$$\tilde{F}_1 \big|_{i-1/2} = \left(1 - \frac{2}{\tilde{\lambda}_{i-1} + \tilde{\lambda}_i}\right)\left(\frac{\tilde{r}_1\big|_i - \tilde{r}_1\big|_{i-1}}{\Delta S}\right) - P\left(\frac{\tilde{r}_2\big|_i + \tilde{r}_2\big|_{i-1}}{2}\right)$$

Eqn. 5.15 can be written in a compact form as

$$R^*(\tilde{r}_i \;, r_i^{\,n}, r_i^{\,n-1}) = \frac{\tilde{r}}{\Delta t^2} - \mu R(\tilde{r}_i) - S(r_i^{\,n}, r_i^{\,n-1}) \qquad (5.18)$$

where

$$R(\tilde{r}_i) = \mu\left(\frac{\tilde{F}_{i+1/2} - \tilde{F}_{i-1/2}}{\Delta S}\right)$$

$$S(r_i^{\,n}, r_i^{\,n-1}) = \frac{2\, r_i^{\,n}}{\Delta t^2} + (1-\mu)\left(\frac{F_{i+1/2}^{\,n} - F_{i-1/2}^{\,n}}{\Delta S}\right) - \frac{r_i^{\,n-1}}{\Delta t^2} \qquad (5.19)$$

The residual defined in Eqn.5.18 can be considered as a summation of two parts. The first part which is the active part contain terms at n+1 time level. The second part, which is called the source term contain terms at n and n-1 time levels. Thus in marching to steady state, only the variable $\tilde{r_i}$, where $r_i^{\,n}$ and $r_i^{\,n-1}$ are held fixed over the pseudo time integration.

At steady state Eqn. 5.14 will become

$$R^*(\tilde{r_i}\ , r_i^{\,n}, r_i^{\,n-1}) = 0 \qquad (5.20)$$

Comparing Eqn. 5.20 and Eqn.5.8, the two equation are equal when $\tilde{r}_i = r^{n+1}$.

Thus by time marching of Eqn. 5.14 to steady state, the solutions for each time steeps of the unsteady nonlinear equation of the string motion are obtained. This time marching of Eqn. 5.14 to steady state is done using a modified explicit, four stage Runge-Kutta method.

5 - 4 Modified Multi-Stage Runge-Kutta Method

To find steady state solutions of the first order partial differential equation (5.14), a modified four stage Runge-Kutta method is used. In a classical Runge-Kutta method the stage coefficients are evaluated for accuracy by comparing Runge-Kutta expression with a Taylor series. In a modified multi stage Runge Kutta method some of the stage coefficients are used to get the required order of accuracy by comparing with Taylor series, while the rest of the coefficients are determined from a stability analysis.

The general construction of m stages of Runge-Kutta method can been done as follows:

$$\tilde{r} = r^n$$

$$\tilde{r}^{(1)} = r^n - \alpha_1 \Delta t^* R^*(\tilde{r})$$

$$\tilde{r}^{(2)} = r^n - \alpha_2 \Delta t^* R^*(\tilde{r}^{(1)})$$

$$\tilde{r}^{(3)} = r^n - \alpha_3 \Delta t^* R^*(\tilde{r}^{(2)})$$

$$\vdots$$

$$\vdots$$

$$\tilde{r}^{(m)} = r^n - \alpha_m \Delta t^* R^*(\tilde{r}^{(m)}) \quad (5.21)$$

Where $\tilde{r}^{(m)}$ is the value of $\tilde{r}$ at m stage, $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_{m-1}$ are stage coefficients, their values are determined from the point of stability and accuracy, $\alpha_m = 1$ for consistency.

More stages implies more control over stability and accuracy as well as more computational time. Thus for some applications the gain produced by increasing the number of stages may not offset the additional computational time required. In this work a four stage method is used.To perform the linear stability analysis, the active term, which is given in section 5.3 is used after a constant coefficient applied for the equation of **F** from section 4.4. The source term is set equal to zero since only the active term is changing in the fictitious time until it reached steady state. The amplification factor of any stage is a function of amplification factor of the preceding stage [24]. The equation of the amplification factor for each stage is constructed as follows.

The equation for the first stage is written as

$$\tilde{r}^{(1)} = r^{n} - \alpha_1 \Delta t^* R^*(\tilde{r}) \qquad (5.22)$$

equation 5.22 after substituting the linear relation of section 4.4 will be.

$$\tilde{r}_i^{(1)} = r_i^{\,n} - \alpha_1 \Delta t^* \left(\tilde{r}_i - C_1 \frac{\mu \Delta t^2}{\Delta S^2} (\tilde{r}_{i+1} - 2\tilde{r}_i + \tilde{r}_{i-1}) + C_2 \frac{\mu \Delta t^2}{2 \Delta S} (\tilde{r}_{i+1} - \tilde{r}_{i-1}) \right)$$

(5.23)

Let Cr be the Courant number in the Runge-Kutta method given by

$$Cr = \frac{\Delta t^* C_1}{\Delta S} \qquad (5.24)$$

let Ck be a constant which relate the fictitious time step and the external force given by

$$Ck = \frac{\Delta t^* C_2}{\Delta S} \qquad (5.25)$$

Substituting Eqn. 4.14, 5.24 and 5.25 in Eqn. 5.23 and using the relation $\tilde{r} = r^{n}$ the error equation for the first stage will be

$$\tilde{\varepsilon}_i^{(1)} = \tilde{\varepsilon}_i - \alpha_1 \left(\tilde{\varepsilon}_i - Cr \frac{\mu \Delta t^2}{\Delta S} (\tilde{\varepsilon}_{i+1} - 2\tilde{\varepsilon}_i + \tilde{\varepsilon}_{i-1}) + Ck \frac{\mu \Delta t^2}{2} (\tilde{\varepsilon}_{i+1} - \tilde{\varepsilon}_{i-1}) \right)$$

(5.26)

Using the Fourier series decomposition give by Eqn. 4.16, the error Eqn. 5.26 will became

$$\tilde{a}^{(1)} = \tilde{a} - \alpha_1 \tilde{a}\left(1 - Cr\ \frac{\mu \Delta t^2}{\Delta S}(2\cos\theta - 2) + Ck\mu\Delta t^2 (I \sin\theta)\right)$$

(5.27)

where $\tilde{a}^{(1)}$ and $\tilde{a}$ are the amplitudes of first stage and the initial respectively. I is a complex number equals to $\sqrt{-1}$, θ is a phase angle given by $\theta = k\ \Delta S$

The amplification factor for each stage is defined by the ratio of the amplitudes of two consecutive stages as follows

$$G_m = \frac{\tilde{a}^{(m)}}{\tilde{a}^{(m-1)}} \tag{5.28}$$

Substituting Eqn. 5.28 in Eqn. 5.27 the equation for amplification factor for the first stage will be.

$$G_1 = 1 - \alpha_1\left(1 - Cr\ \frac{\mu \Delta t^2}{\Delta S}(2\cos\theta - 2) + Ck\mu\Delta t^2 (I \sin\theta)\right) \tag{5.29}$$

The equation for the second stage is written as

$$\tilde{r}^{(2)} = r^{n} - \alpha_2 \Delta t^* R^*(\tilde{r}^{(1)}) \tag{5.30}$$

Eqn. 5.30 in expanded form will be

$$\tilde{r}_i^{(2)} = r_i^{\ n} - \alpha_2 \Delta t^* \left(\tilde{r}_i^{(1)} - C_1 \frac{\mu \Delta t^2}{\Delta S^2}(\tilde{r}_{i+1} - 2\tilde{r}_i + \tilde{r}_{i-1})\right)$$

$$- \alpha_2 \Delta t^* C_2 \frac{\mu \Delta t^2}{2\Delta S}(\tilde{r}_{i+1} - \tilde{r}_{i-1}) \tag{5.31}$$

Constructing an error equation for the second stage and applying the Fourier series decomposition on Eqn. 5.31 it is found:

$$\tilde{a}^{(2)} = \tilde{a} - \alpha_2 \tilde{a}^{(1)} \left(1 - Cr \frac{\mu \Delta t^2}{\Delta S}(2\cos\theta - 2) + Ck\mu\Delta t^2 (I \sin\theta)\right) \tag{5.32}$$

Using the relation of Eqn. 5.28, the equation for amplification factor of the second stage is written as:

$$G_2 = 1 - \alpha_2 G_1 \left(1 - Cr \frac{\mu \Delta t^2}{\Delta S}(2\cos\theta - 2) + Ck\mu\Delta t^2 (I \sin\theta)\right) \tag{5.33}$$

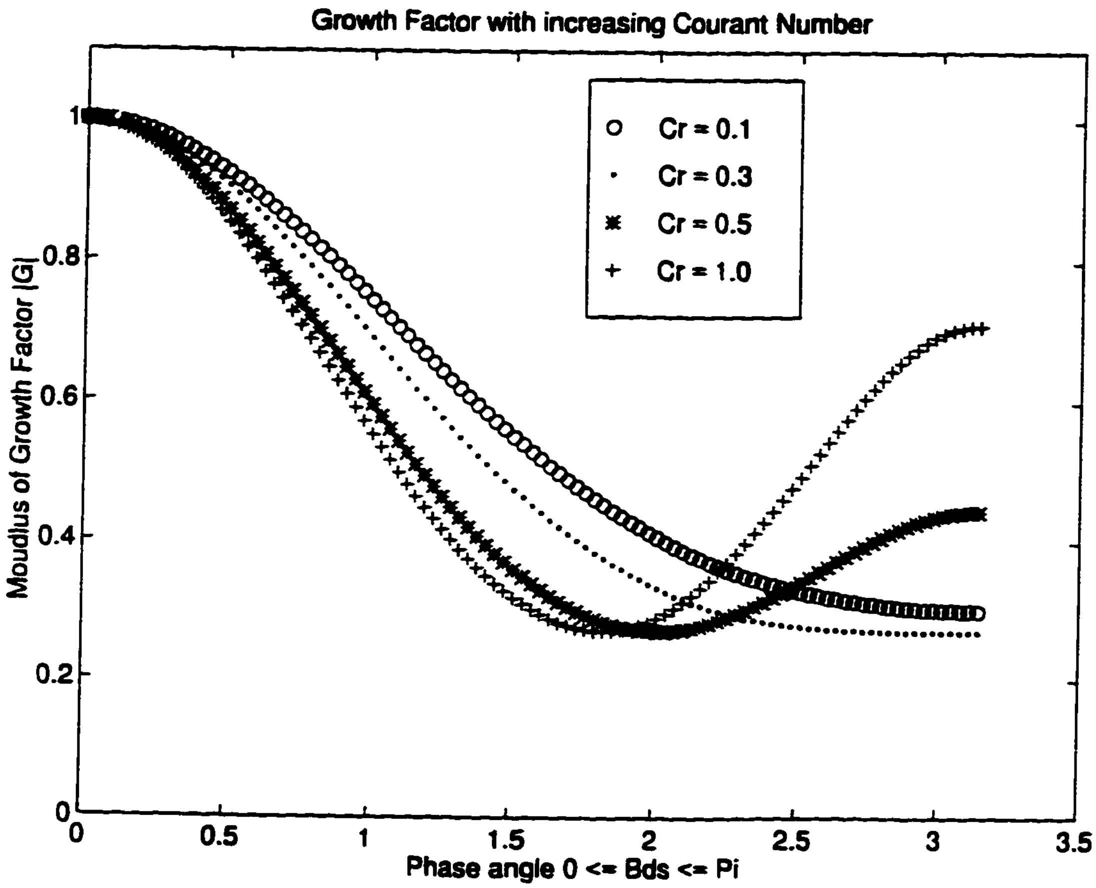
Extending Eqn. 5.33 for m stage the amplification factor at m stage is given by the equation.

$$G_m = 1 - \alpha_m G_{m-1} \left(1 - Cr \frac{\mu \Delta t^2}{\Delta S}(2\cos\theta - 2) + Ck\mu\Delta t^2 (I \sin\theta)\right) \tag{5.34}$$

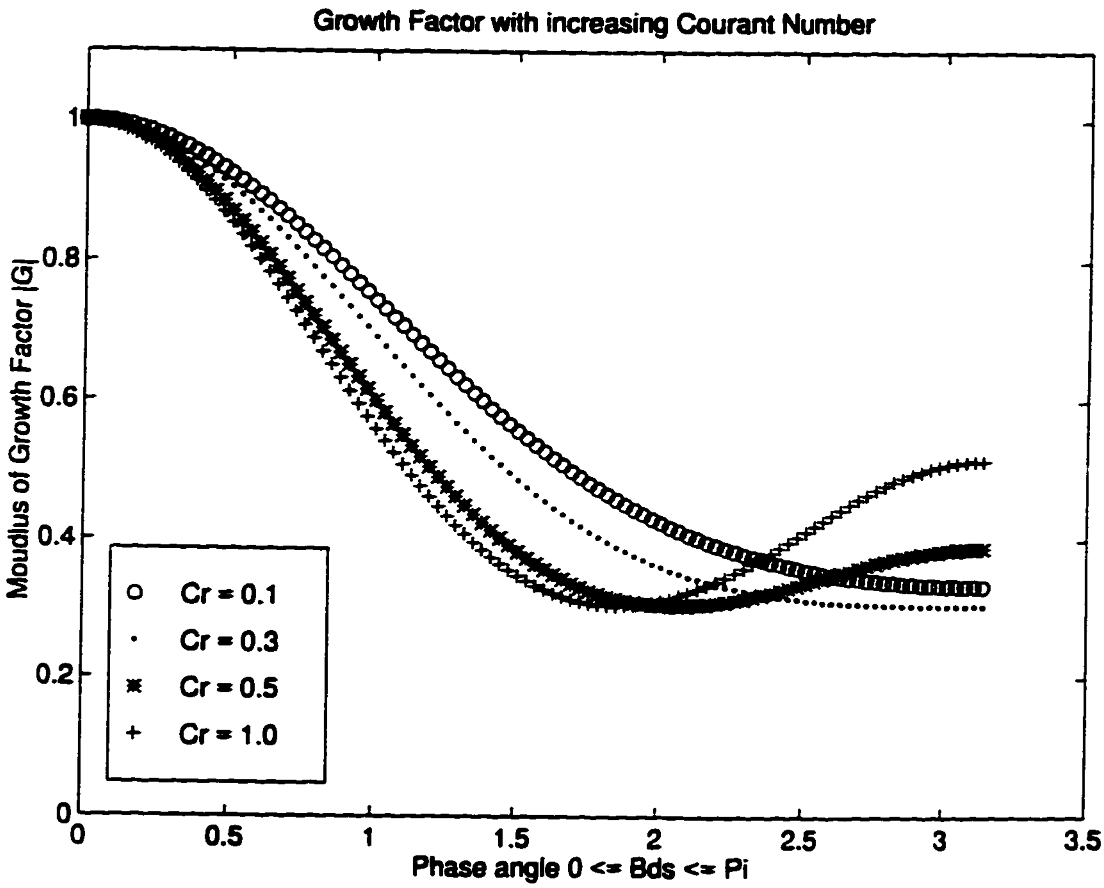
The modulus of amplification factor for four stage is determined graphically for different combinations of stage coefficients as shown from fig. 5.1 to 5.4. Using a Taylor series, two of the stage coefficients are determined for second order accuracy as $\alpha_4 = 1$ and $\alpha_3 = \frac{1}{2}$. For the stability analysis of the Runge-Kutta method the values of Ck is fixed to 0.01.The Courant number is varied from 0.1 to 1 for different combination of the two stage coefficients. From a number of amplification factor curves, the one which fulfills the Von Neumann stability criteria and a minimum area under the curve of the amplification factor vs. phase angle is chosen. Thus from the above analysis $\alpha_2 = \frac{1}{3}$,

$\alpha_1 = \frac{1}{6}$ and the fictitious time step $\Delta t^* = 1 \times 10^{-5}$ of Cr = 1.0 is obtained.

By repeated operation of the modified explicit Runge-Kutta method the solution of the unsteady string equation is obtained for the desired length of time.



5.1 Selected Numerical Method solutions at 500 time steps



5.2 Selected Numerical Method solutions at 2000 time steps

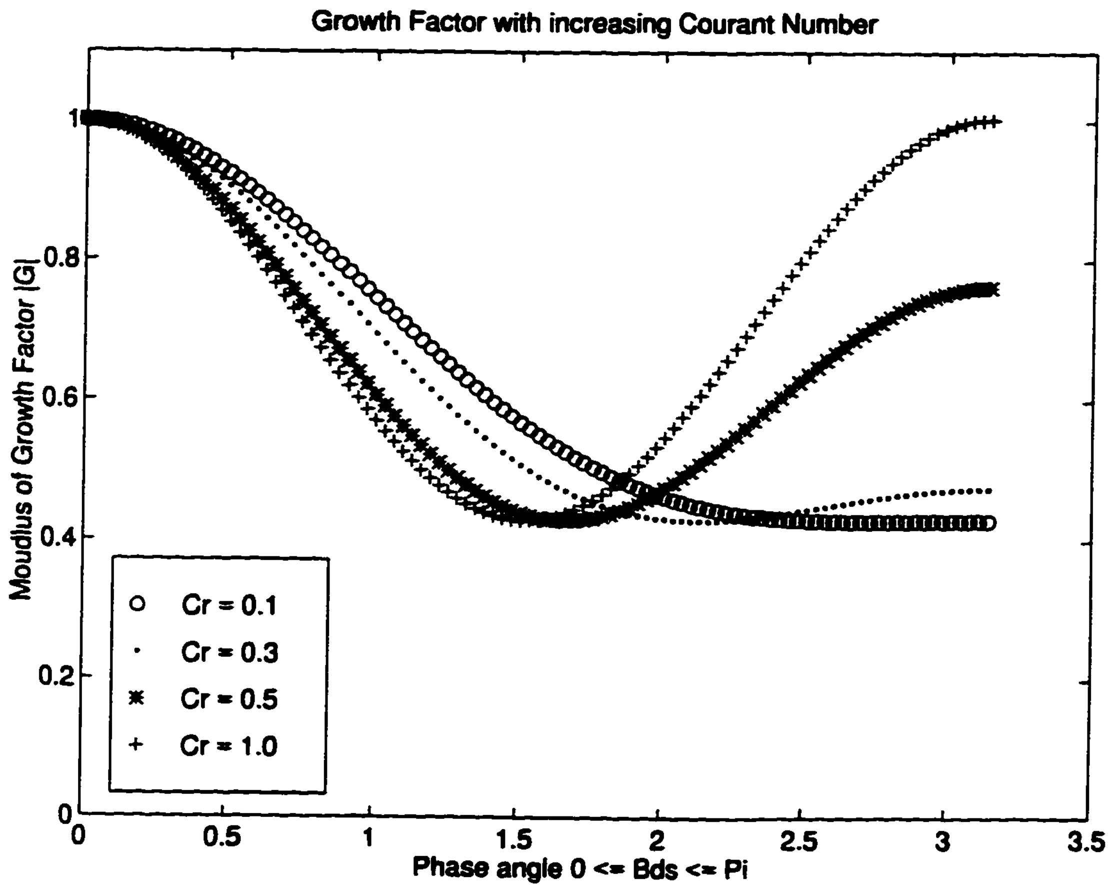


Fig. 5.3 Amplification Factor, $\alpha_1 = \frac{1}{4}$,$\alpha_2 = \frac{1}{3}$, $\alpha_3 = \frac{1}{2}$

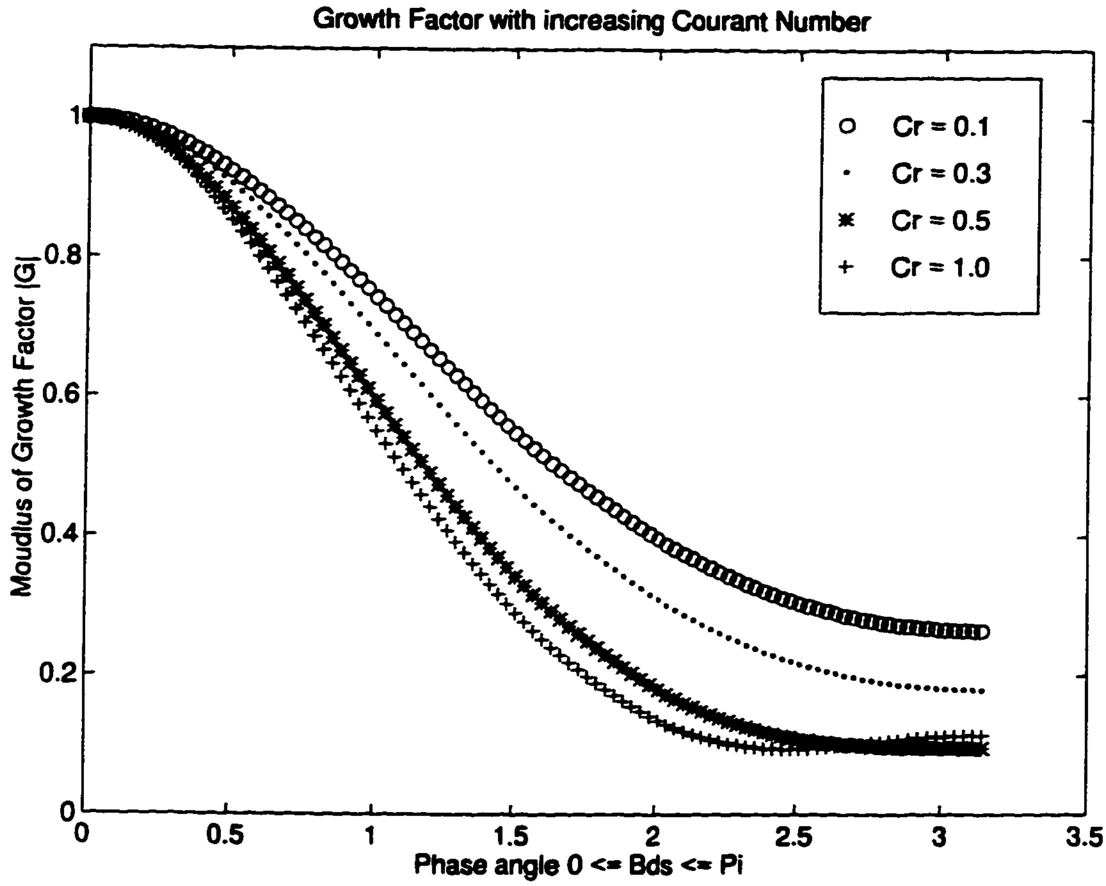


Fig. 5.4 Amplification Factor, $\alpha_1 = \frac{1}{8}$, $\alpha_2 = \frac{1}{8}$, $\alpha_3 = \frac{1}{2}$

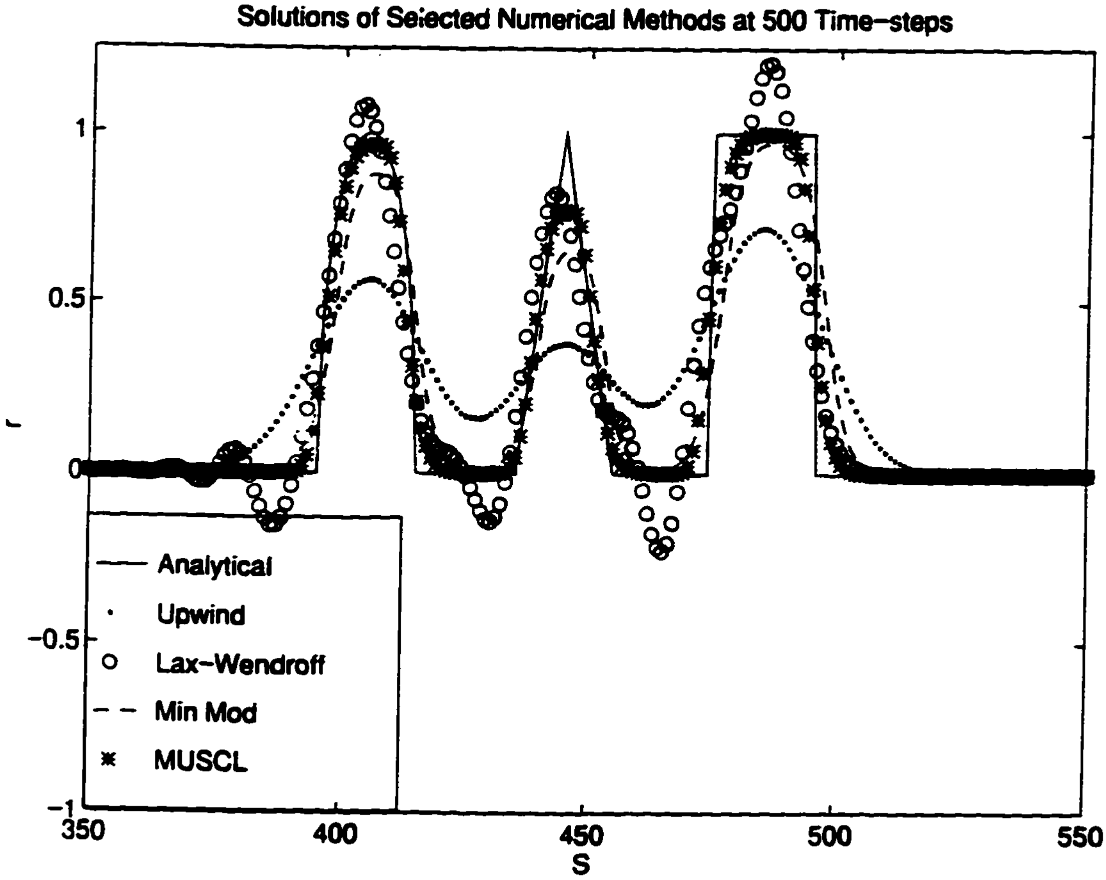


Fig. 5.5 Amplification Factor, $\alpha_1 = \frac{1}{6}$, $\alpha_2 = \frac{1}{3}$, $\alpha_3 = \frac{1}{2}$

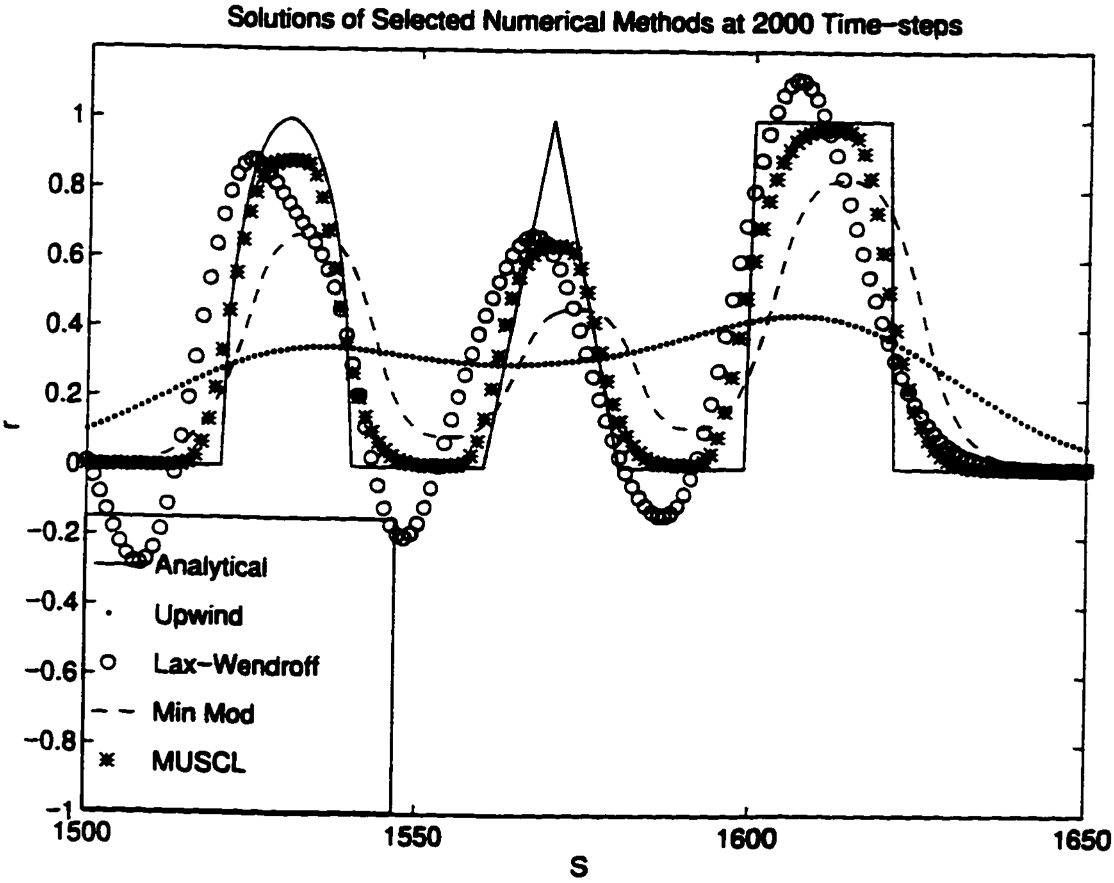


Fig. 5.6 Amplification Factor, $\alpha_1 = \frac{1}{6}$, $\alpha_2 = \frac{1}{4}$, $\alpha_3 = \frac{1}{2}$

CHAPTER SIX

RESULTS

Five test cases are conducted on the developed numerical method of solution. The first test case, a transversely impacted string is investigated using a Mooney-Rivlin constitutive relation. and the solution is compared with analytical results. Then two test cases are conducted by changing the boundary condition at one side to initiate a low and a high amplitude wave motion. The last two test cases are done by applying pressure in five grid points for one time step. As the previous two cases in this case also a low and a high amplitude wave initiated by changing the intensity of the pressure for the two cases. For low amplitude cases the shape of the string at different time levels, the characteristic of the total energy and the history of the residual in marching the fictitious first order partial differential equation to steady state are plotted. For high amplitude waves the shape of the string at different time levels, the characteristic of the total energy, the residual in marching the fictitious first order partial differential equation to steady state and the plot of stretch vs. grid position for different time levels are plotted. All tests are done on 500 grid points with a grid spacing of 5×10^{-3}, a time step of 1×10^{-4} in the unsteady equation of the string, a fictitious time step of 1×10^{-5} in the Runge-Kutta method and implicitness control parameter $\mu = 0.89$ for 100,000 time steps which are equivalent to fifty reflections.

6 - 1 Comparison of Numerical & Analytical Solutions

A comparison of numerical and analytical solutions is performed to show the validity and accuracy of the numerical solutions. Analytical solutions using the method of similarity for the dynamics of transversely impacted string have been investigated by J.B.Haddow [18] and J.L.Wegner[25]. The constitutive relation used in Wegener analytical solution is Mooney-Rivlin constitutive relation. The Mooney-Rivlin constitutive relation is a nonlinear relation between stress and stretch which give a higher stretch value for certain stress relative to the linear constitutive relation.

In the wave motion of an elastic string which has Mooney-Rivlin constitutive relation, the wave motion will experience a shock after the first reflection of the wave at the boundary. During shock formation two characteristic line of the similarity solution will meet. Thus we will have a multiple solution at a point of intersection of the two characteristic lines.Due to multiple solution in the analytical solution will appear after the first reflection, Wegner analytical solution is valid for the wave motion in the string until the first reflection occurs. The calculation for the analytical and numerical solution is preformed as follows.

$$f(\lambda) = E\left(\alpha - \frac{(1-\alpha)}{\lambda}\left(\lambda - \frac{1}{\lambda^2}\right)\right) \qquad 6.1$$

where α = a positive constant in the range of $0 \leq \alpha \leq 1$

The longitudinal and transverse wave speeds are given by

$$C_L = \left\{\frac{1}{\rho}\frac{\partial f(\lambda)}{\partial\lambda}\right\}^{1/2} = \left\{\frac{E}{\rho}\left[\alpha\left(1-\frac{2}{\lambda^3}\right)+\frac{3(1-\alpha)}{\lambda^4}\right]\right\}^{1/2} \qquad 6.2$$

$$C_T = \left\{\frac{1}{\rho}\frac{f(\lambda)}{\lambda}\right\}^{1/2} = \left\{\frac{E}{\rho}\left[\alpha+\frac{(1-\alpha)}{\lambda}\right]\left(1-\frac{1}{\lambda^3}\right)\right\}^{1/2} \qquad 6.3$$

where C_L = Longitudinal wave speed

C_T = Transversal wave speed

For the transversely impacted string the following boundary and initial condition are considered.

Boundary condition

$$r_1(0,t) = 0 \qquad r_1(1,t) = 1$$

$$r_2(0,t) = -Qt \qquad r_2(1,t) = 0 \qquad 6.4$$

where Q is a constant velocity

Initial condition

$$r_1(S,0) = S \qquad r_2(S,0) = 0$$

$$\frac{\partial r_1(S,0)}{\partial t} = 0 \qquad \frac{\partial r_2(S,0)}{\partial t} = Q \qquad 6.5$$

Let λ_f = The maximum stretch after impact

λ_c = The critical stretch in isothermal stress-stretch relation for Mooney-Rivlin material, which separate the region of $C_L > C_T$ form $C_L < C_T$

λ_0 = The initial stretch value

θ = The angle which the string makes with the horizontal

θ_f = The angle which the deformed string makes with the horizontal after impact.

Analytical solutions using the similarity method for the case of $C_L > C_T$ and $\lambda_f < \lambda_c$ with the boundary and initial condition given by Eqn 6.4,6.5 were performed by Wegner [25]. The solution contain four regions in the (S , t) plane as follows.

Region 1

$$\frac{S}{t} \geq C_L(\lambda_0)$$

$$\lambda = \lambda_0 \quad , \qquad \theta = u = v = 0$$

Region 2

$$C_L(\lambda_0) \geq \frac{S}{t} \geq C_L(\lambda_f)$$

$$\theta = v = 0 \quad , \qquad u = -I(\lambda)$$

Region 3

$$C_L(\lambda_f) \geq \frac{S}{t} \geq C_T(\lambda_f)$$

$$\lambda = \lambda_f \quad , \ \theta = v = 0 \ , \ u = -I(\lambda_f)$$

Region 4

$$C_T(\lambda_f) \geq \frac{S}{t} \geq 0$$

$$\lambda = \lambda_f \ , \ \theta = \theta_f \ , \ u = 0 \ , \ v = -Q$$

where u = tangential particle velocity v = vertical particle velocity

$$I(\lambda) = \int_{\lambda_0}^{\lambda} C_L(\eta)d\eta$$

The deformed shape of the string at an arbitrary time t will have the following shape

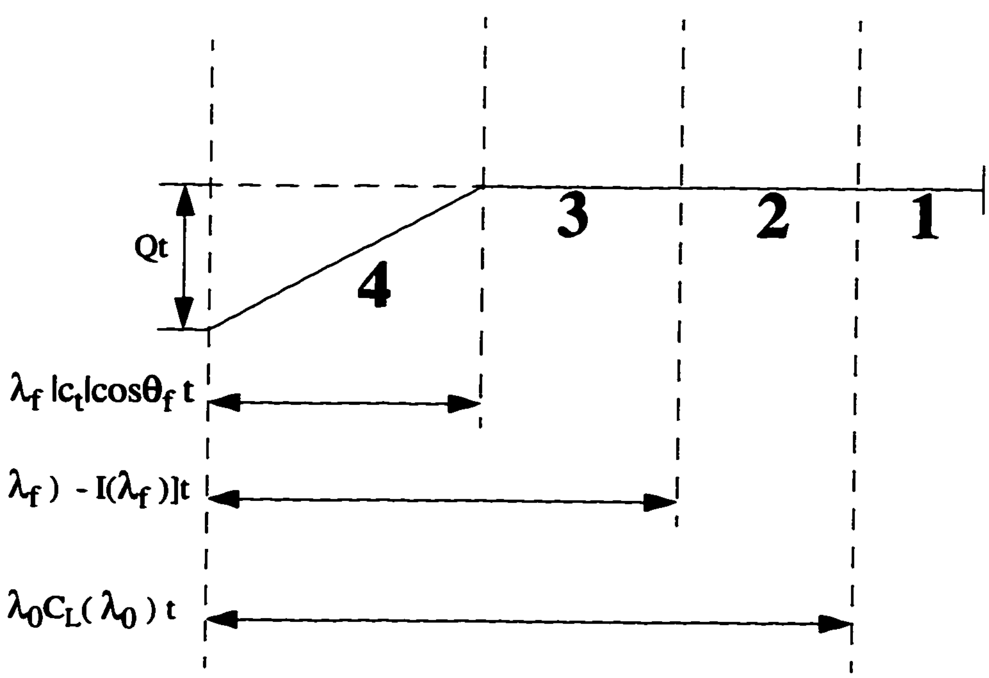


Fig. 6.1 Transversely Impacted String

One need to specify α of Mooney-Rivlin constitutive relation to obtain the numerical solution of the transversely impacted string. The value of $\alpha = 0.6$ gives a close fit with simple tension experimental data for stretch up to 3.5. Thus $\alpha = 0.6$ is used in the numerical as well as the analytical method.For the specified value of α the critical stretch value $\lambda_c = 2.4733$. This critical stretch value determines the region of stretch which makes the longitudinal wave speed grater than the transverse wave speed or vice versa. At critical stretch the two wave speeds are equal.Thus the value of critical stretch is important to determine the appropriate similarity solution to compare with the numerical solution. The other constant need to be specified is Q. The value of Q the constant speed which is applied initially. Thus a non-dimensional constant speed of Q = 1 is used in the numerical analyses. The boundary and the initial condition used for both, numerical and analytical methods are the same.

The analytical and numerical solution calculated are plotted in Fig. 5.2. From the Comparison of the two solutions the following observation are made a) The angle after impact,θ_f for numerical solution is calculated an average error of 1.65% at the pointe where their is a discontinuity of θ. b) From the visual observation of the plots of the two methods, there is no appreciable difference in the position of the string. This observation shows that the numerical method capture the speed of propagation.

From the comparison of the two solutions, the position of the string calculated from the numerical solution is valid and accurate. With this confidence we will now apply our numerical method to a number of other test cases.

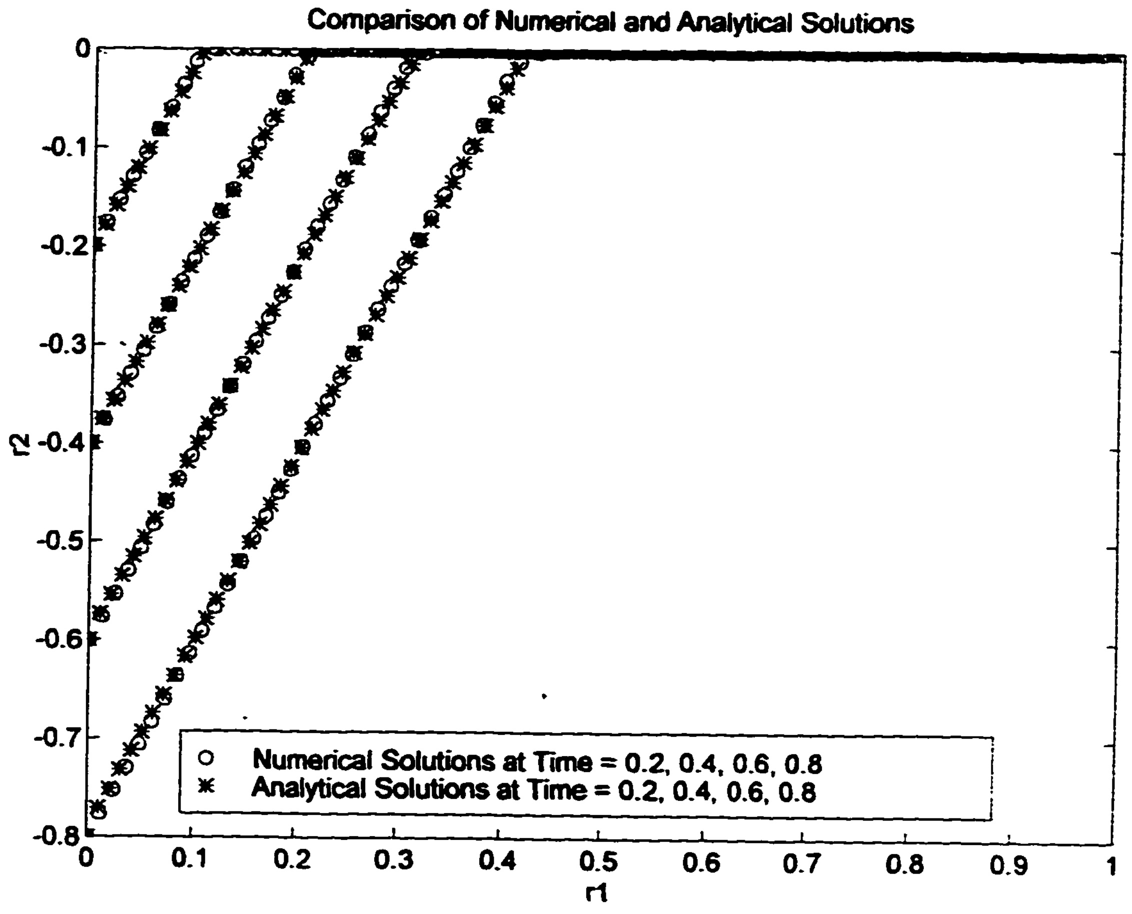


Fig. 6.2 Comparison of Numerical & Analytical Solutions.

6 - 2 Small Amplitude Disturbance by Changing the Boundary Condition at One Side

In this test case the boundary at one side is displaced in the vertical direction by 0.01 non dimensional units ($r_2(S,t) = 0.01$, S = 1) for the entire tested period. The positions of the string at different time levels are shown from Fig.6.1 to Fig. 6.7. Since the amplitude is very small, the wave propagate as a linear wave. Thus the speed of propagation captured by numerical method is constant as expected for the linear range.This property is preserved for the tested length of time, for fifty reflections.Smoothing property observed at the corners of the wave front and 3% wave shift at the center of the string.

The number of time steps needed to march the fictitious first order partial differential equation to study state is the price paid to advance the solution of the unsteady string equation by one time step. The plot of the residual of this fictitious first order partial differential equation vs. time step in Runge-Kutta method at two different time levels of the unsteady equation is shown in Fig.6.8 and Fig.6.9, for the residual in r_1 and r_2. The percentage of the normalized energy vs. time step is plotted from Fig. 6.10 to Fig. 6.12 to show the stability of the scheme and the numerical does not violet the conservation of the total energy.For clarity, the plots for positions of the string are done at an interval of three data points.

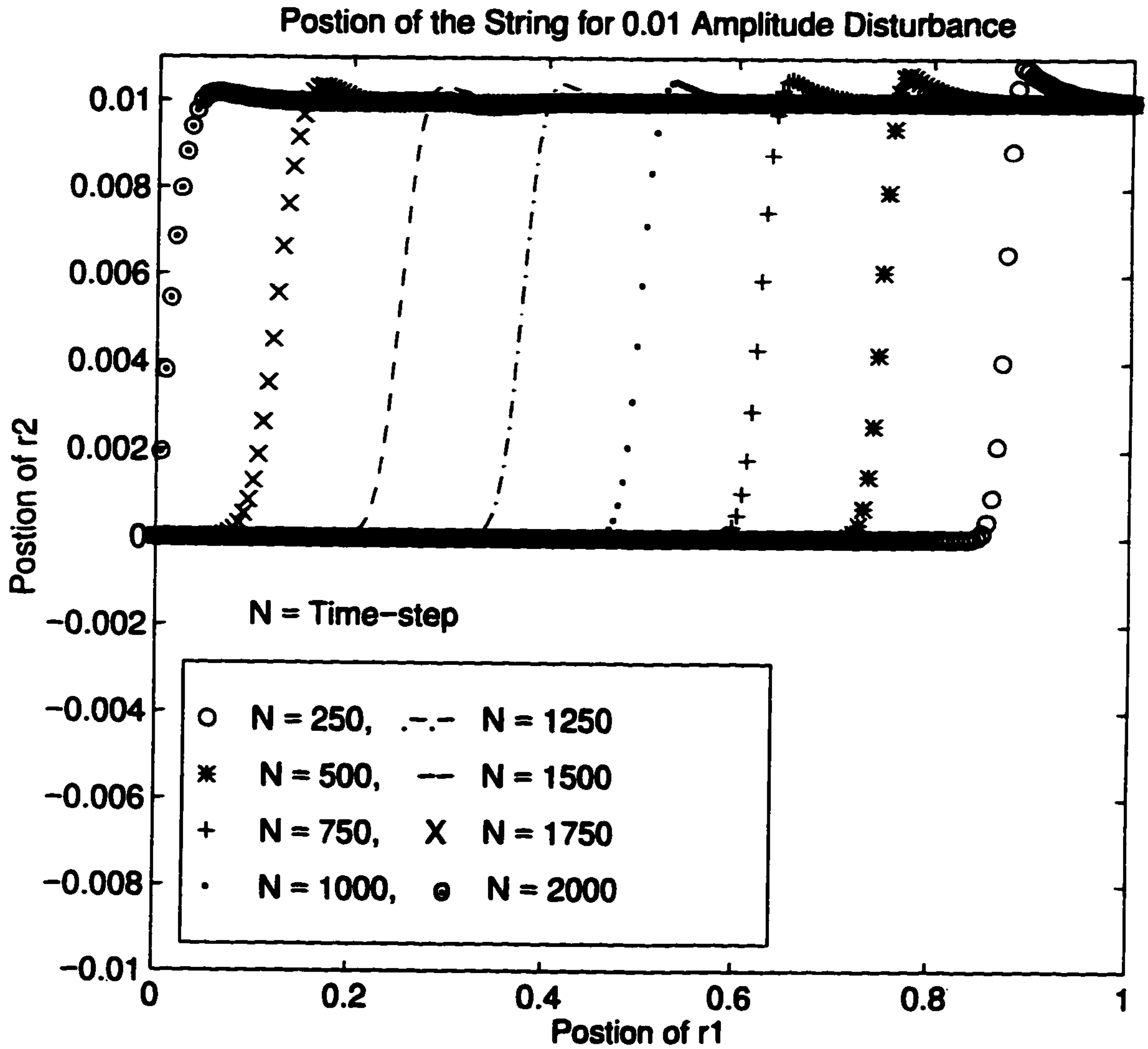


Fig. 6.3 Position of the Sting for the First 2000 Time-steps for Small Amplitude

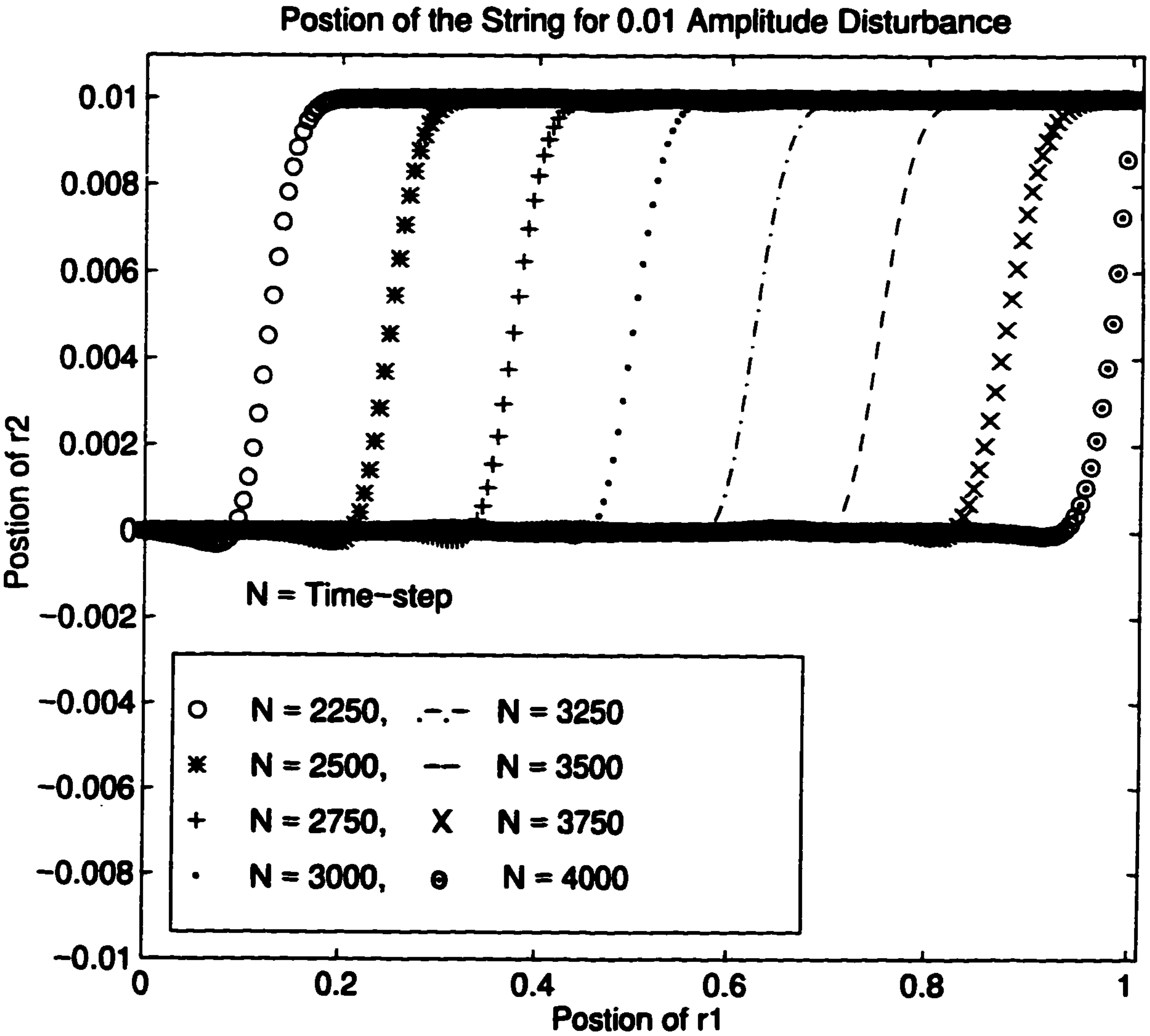


Fig. 6.4 Position of the String from 2000 to 4000 Time-steps for Small Amplitude

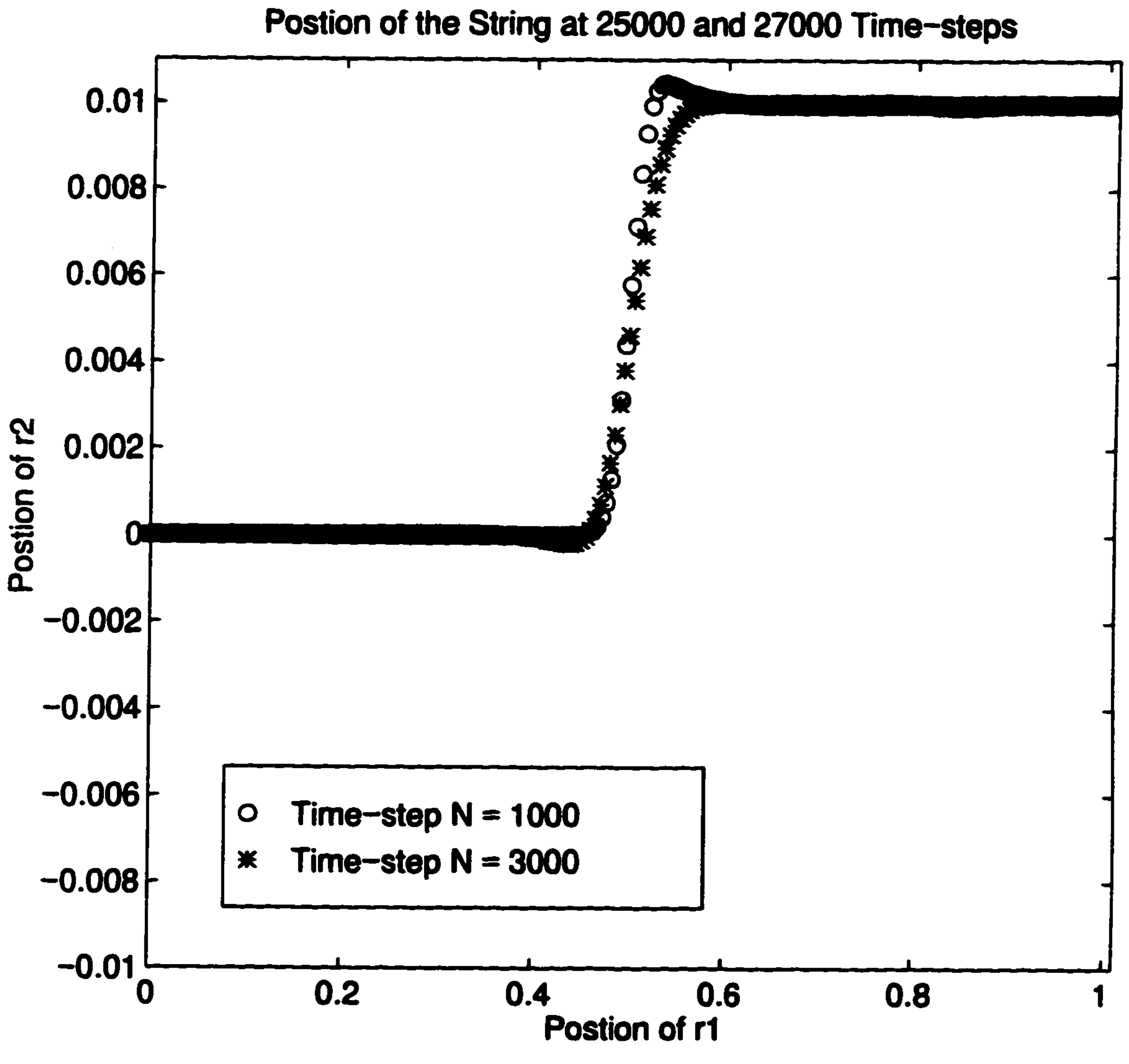


Fig. 6.5 First Comparison of Two Wave Positions for Small Amplitude

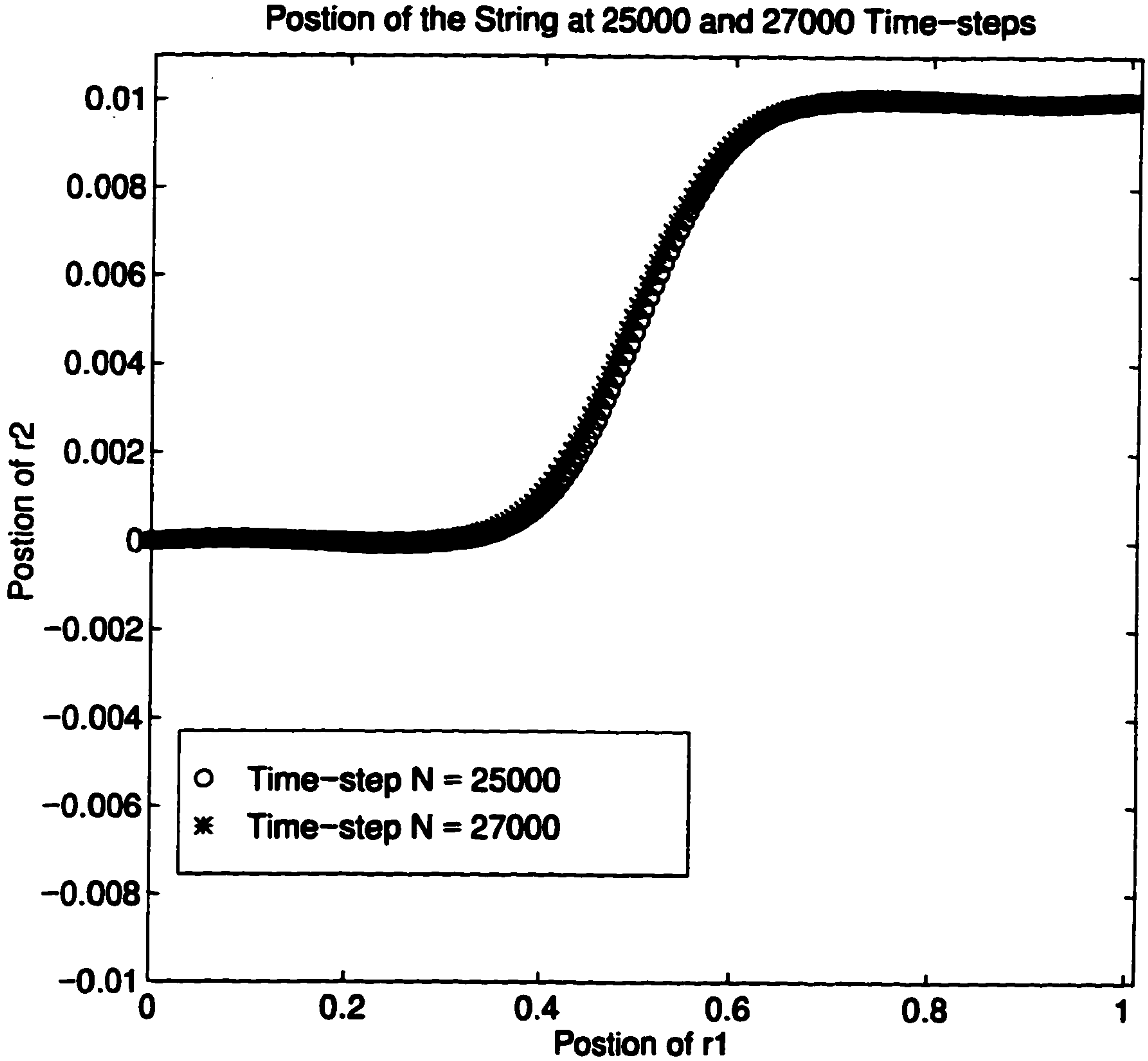


Fig. 6.6 Second Comparison of Two Wave Positions for Small Amplitude

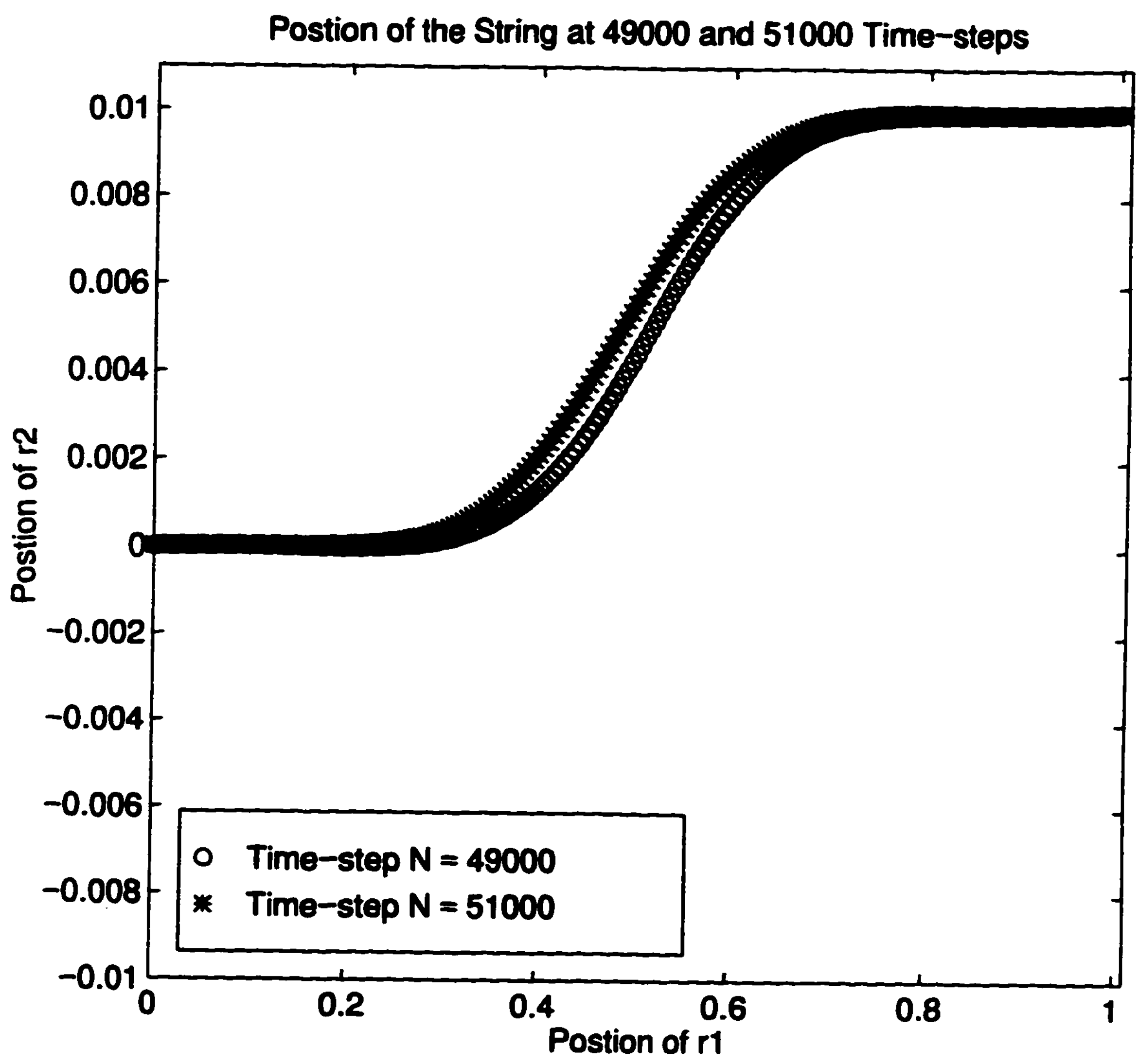


Fig. 6.7 Third Comparison of Two Wave Positions for Small Amplitude

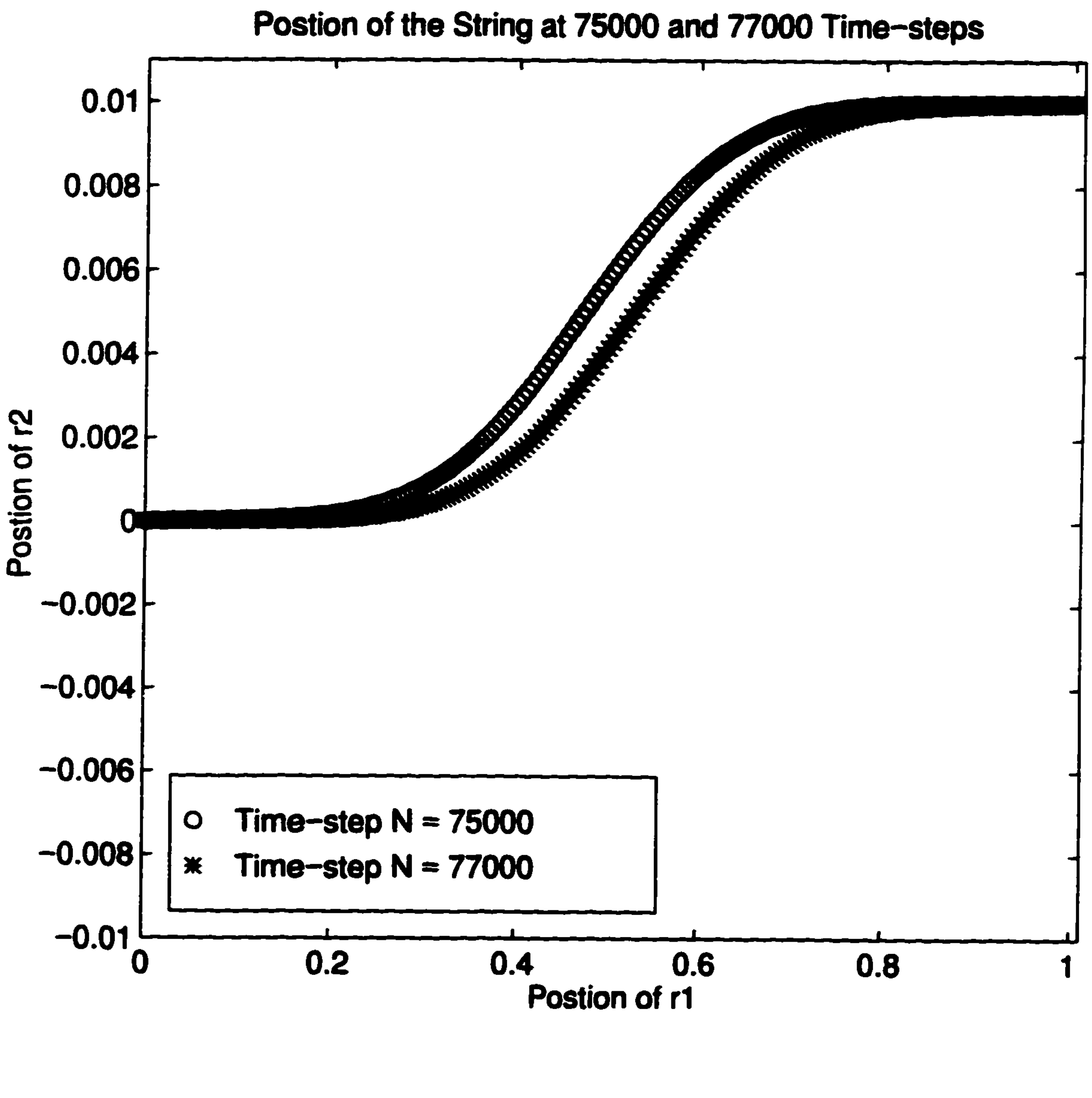


Fig. 6.8 Fourth Comparison of Two Wave Positions for Small Amplitude

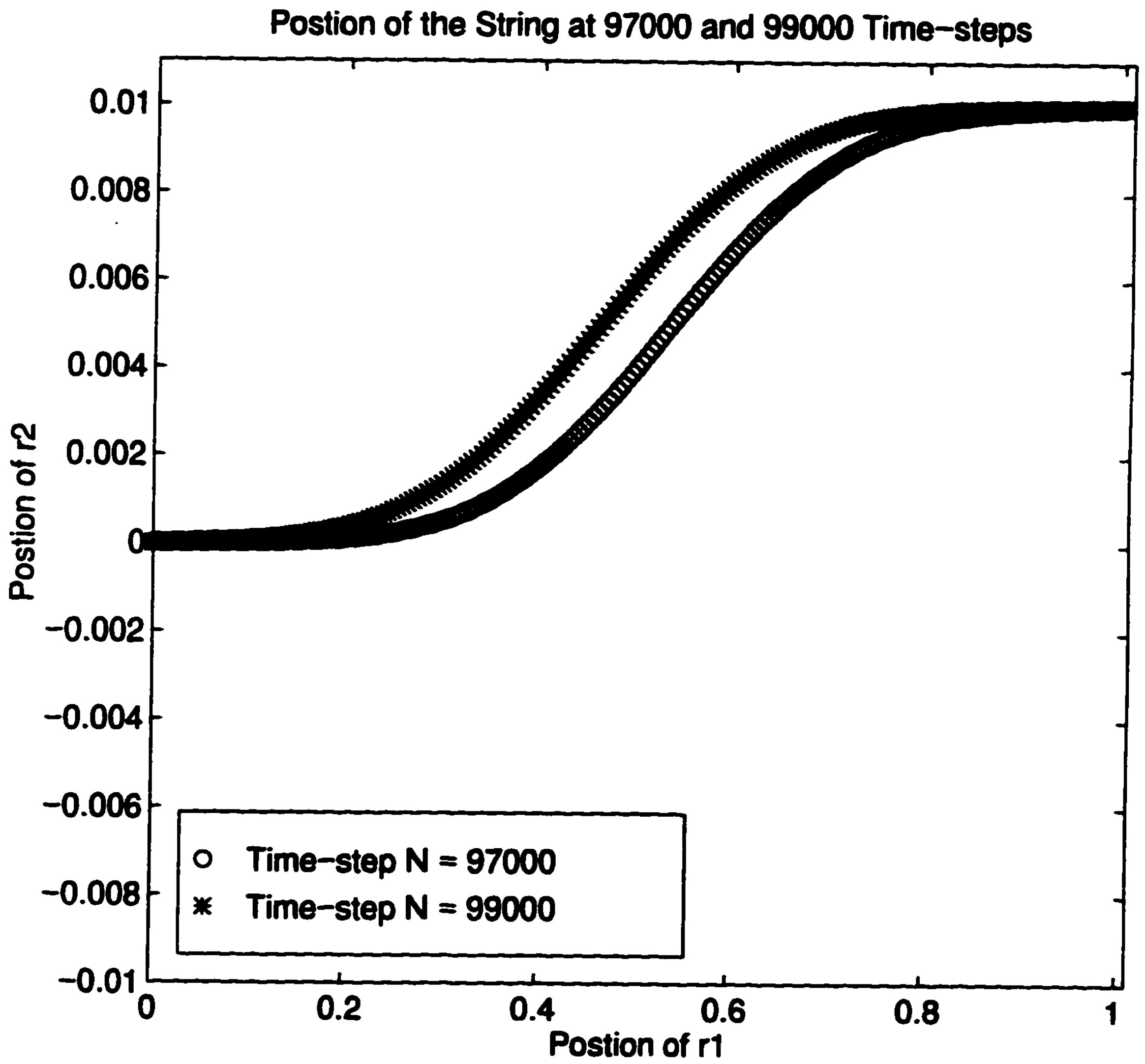


Fig. 6.9 Fifth Comparison of Two Wave Positions for Small Amplitude

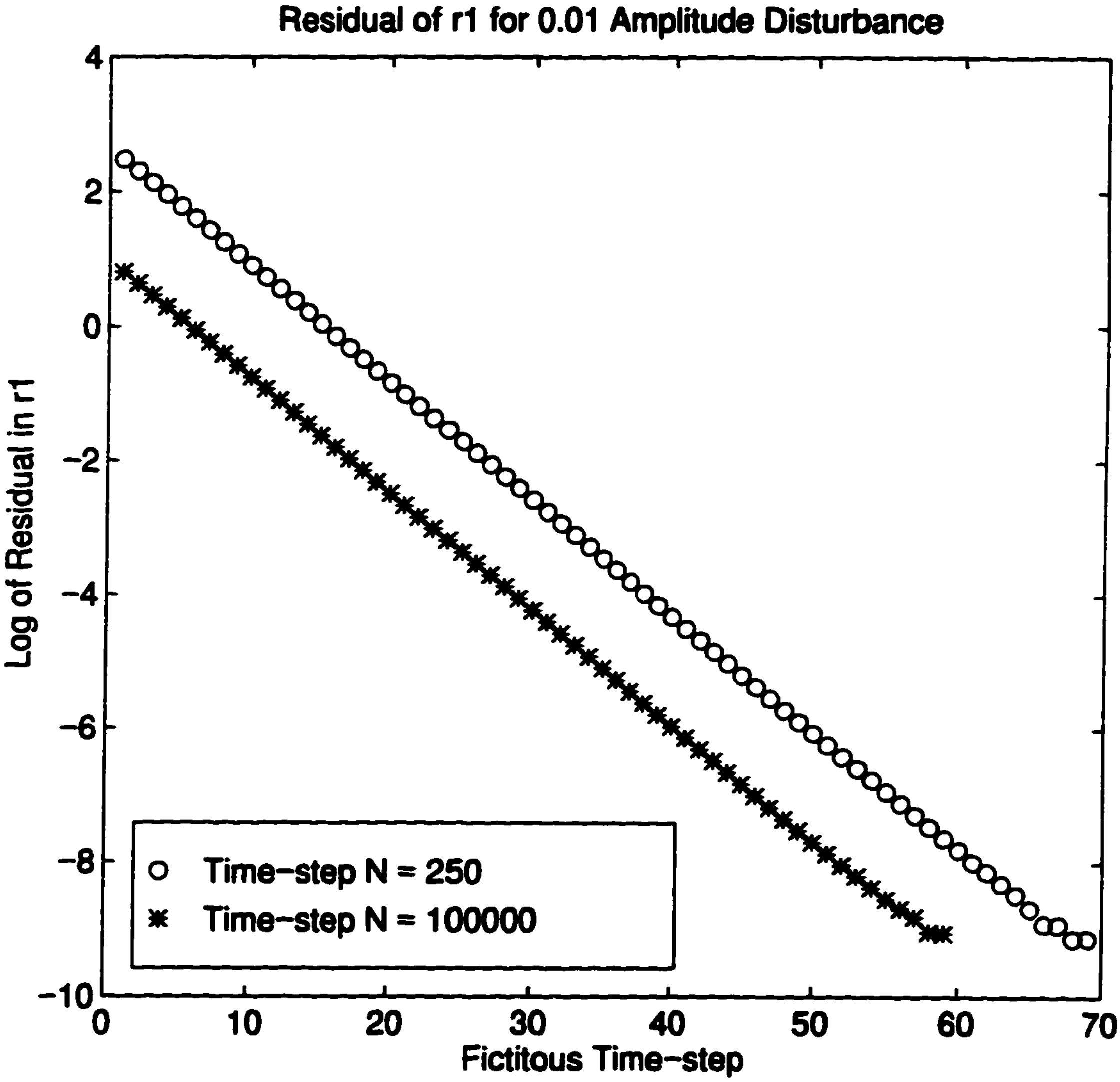


Fig. 6.10 Residual of $\mathbf{r}_1$ for Small Amplitude Case

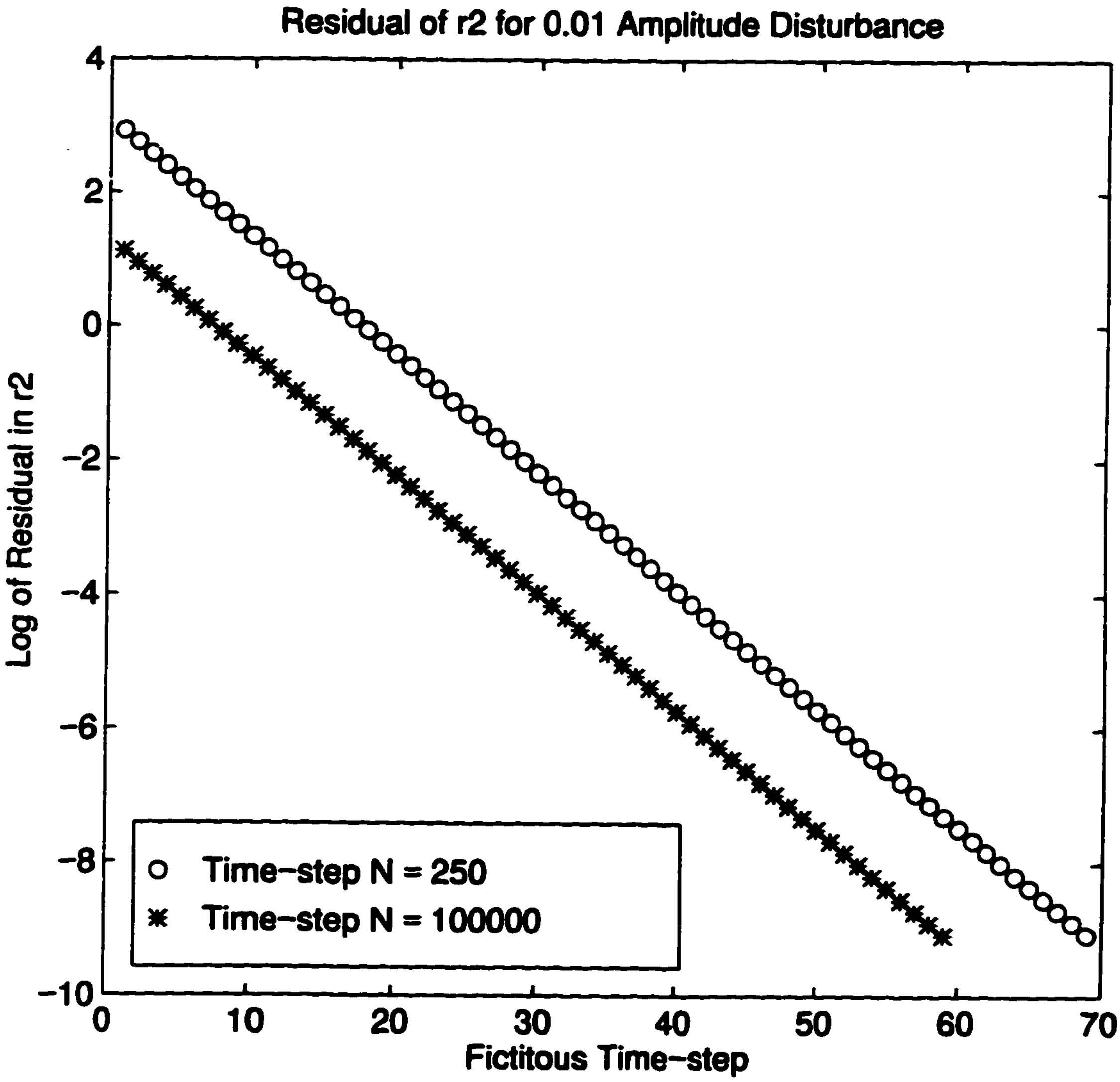


Fig. 6.11 Residual of $\mathbf{r}_2$ for Small Amplitude Case

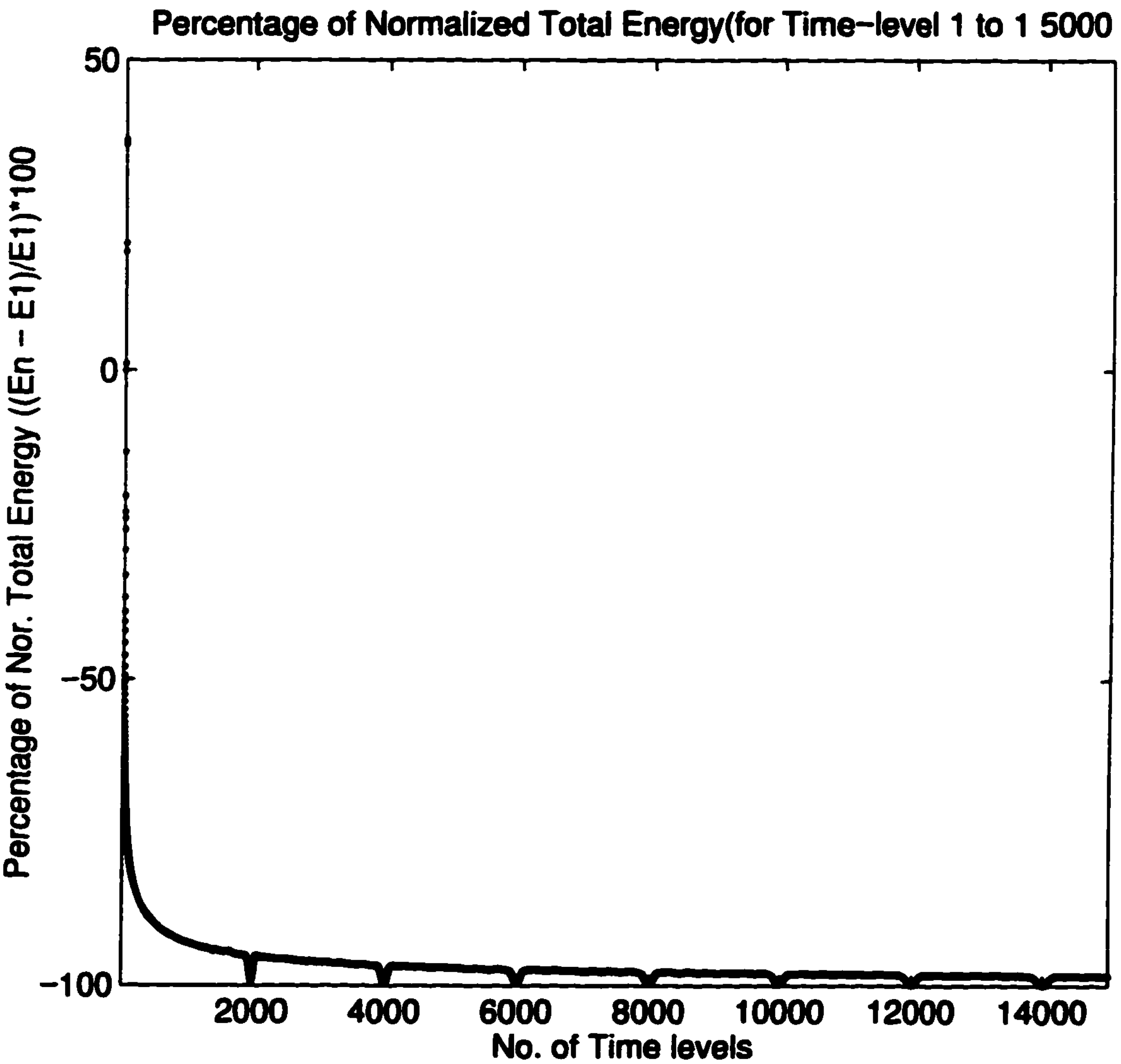


Fig. 6.12 Percentage of Normalized Total Energy in the First 15000 Time-steps for Small Amplitude case

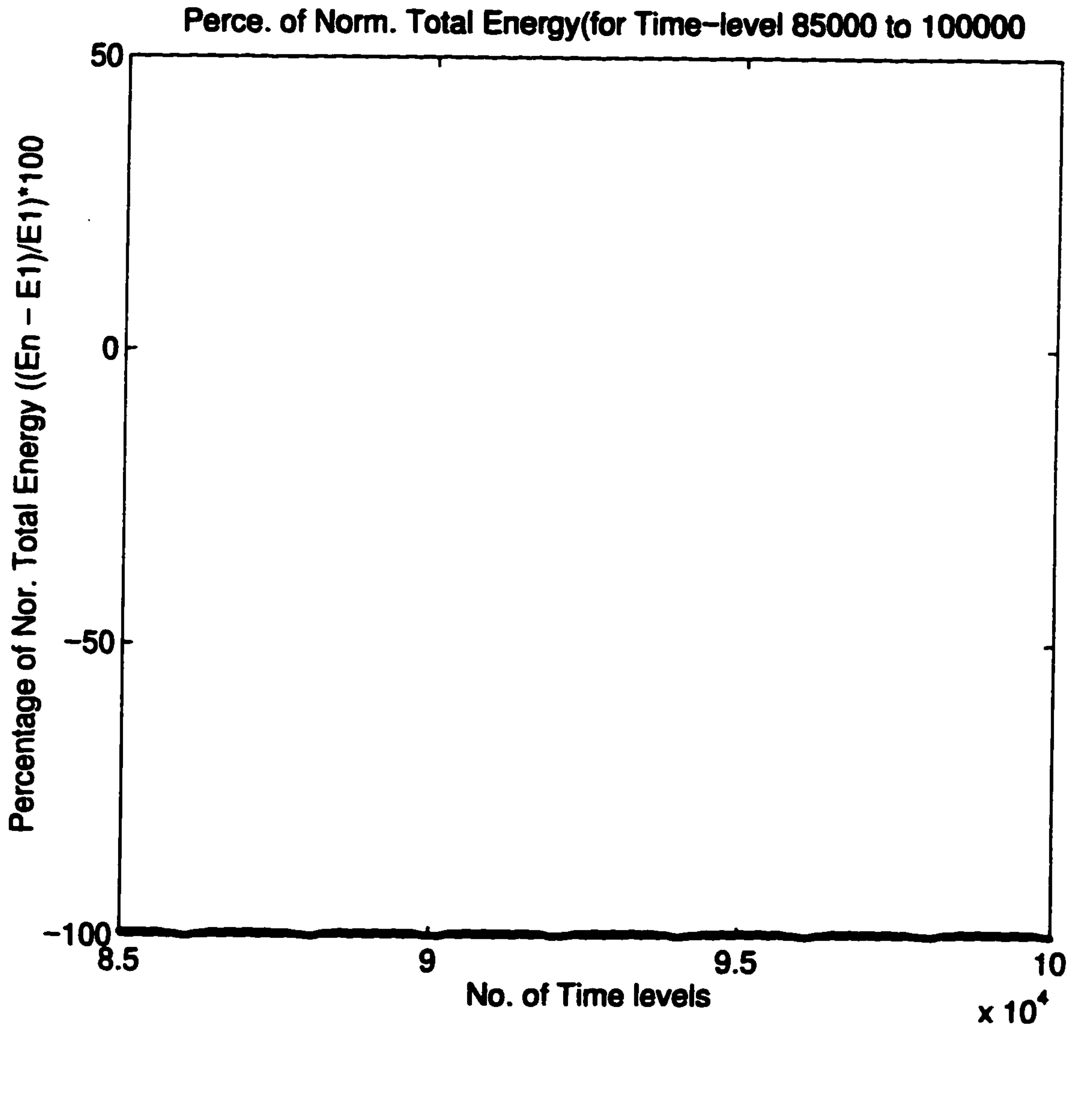


Fig. 6.13 Percentage of Normalized Total Energy from 85000 to 100000 Time-steps for Small Amplitude case

6 - 3 Large Amplitude Disturbance by Changing the Boundary Condition at One Side

In this test case the boundary at one side is displaced in the vertical direction by 0.5 non dimensional units ($r_2(S,t) = 0.5$, S = 1) for the entire tested period. The positions of the string at different time levels are shown from Fig.6.1 to Fig. 6.7. The stretch vs. grid poison are plotted next to each plot of the string postion.From the plots of the position of the string it can be seen that the string motion is nonlinear. When the stretch value goes well bellow one the shape of the string corresponds to this low stretch value is different for other regions.But the value of the stretch below one is observed on a short length of the string for short period of time.

The number of time steps needed to march the fictitious first order partial differential equation to study state is the price paid to advance the solution of the unsteady string equation by one time step. The plot of the residual of this fictitious first order partial differential equation vs. time step in Runge-Kutta method at two different time levels of the unsteady equation is shown in Fig.6.8 and Fig.6.9, for the residual in r_1 and r_2.The percentage of the normalized energy vs. time step is plotted from Fig. 6.10 to Fig. 6.12 to show the stability of the scheme and the numerical does not violet the conservation of the total energy For clarity, the plots for positions of the string are done at an interval of three data points.

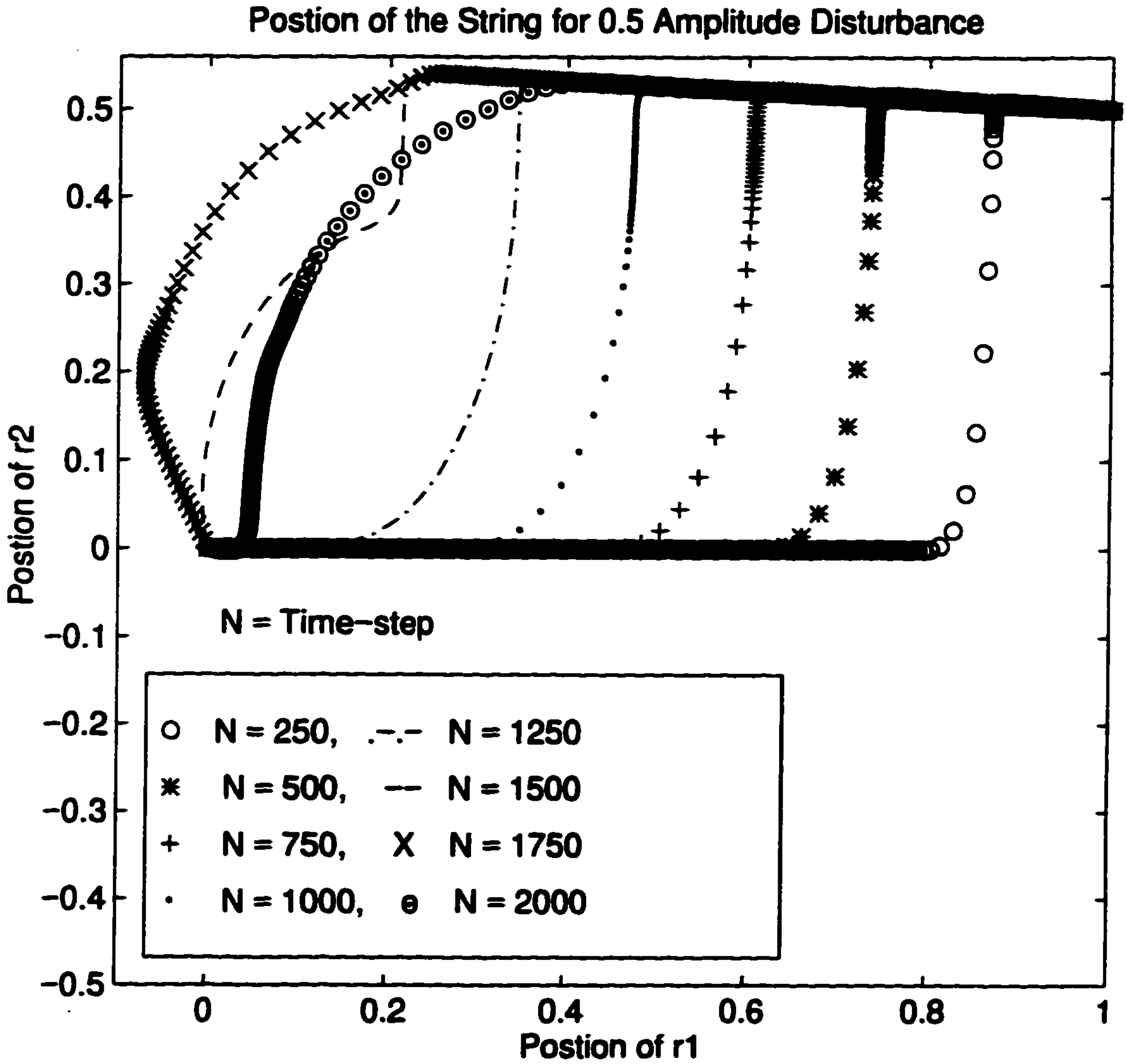


Fig. 6.14 Position of the Sting for the First 2000 Time-steps for Large Amplitude

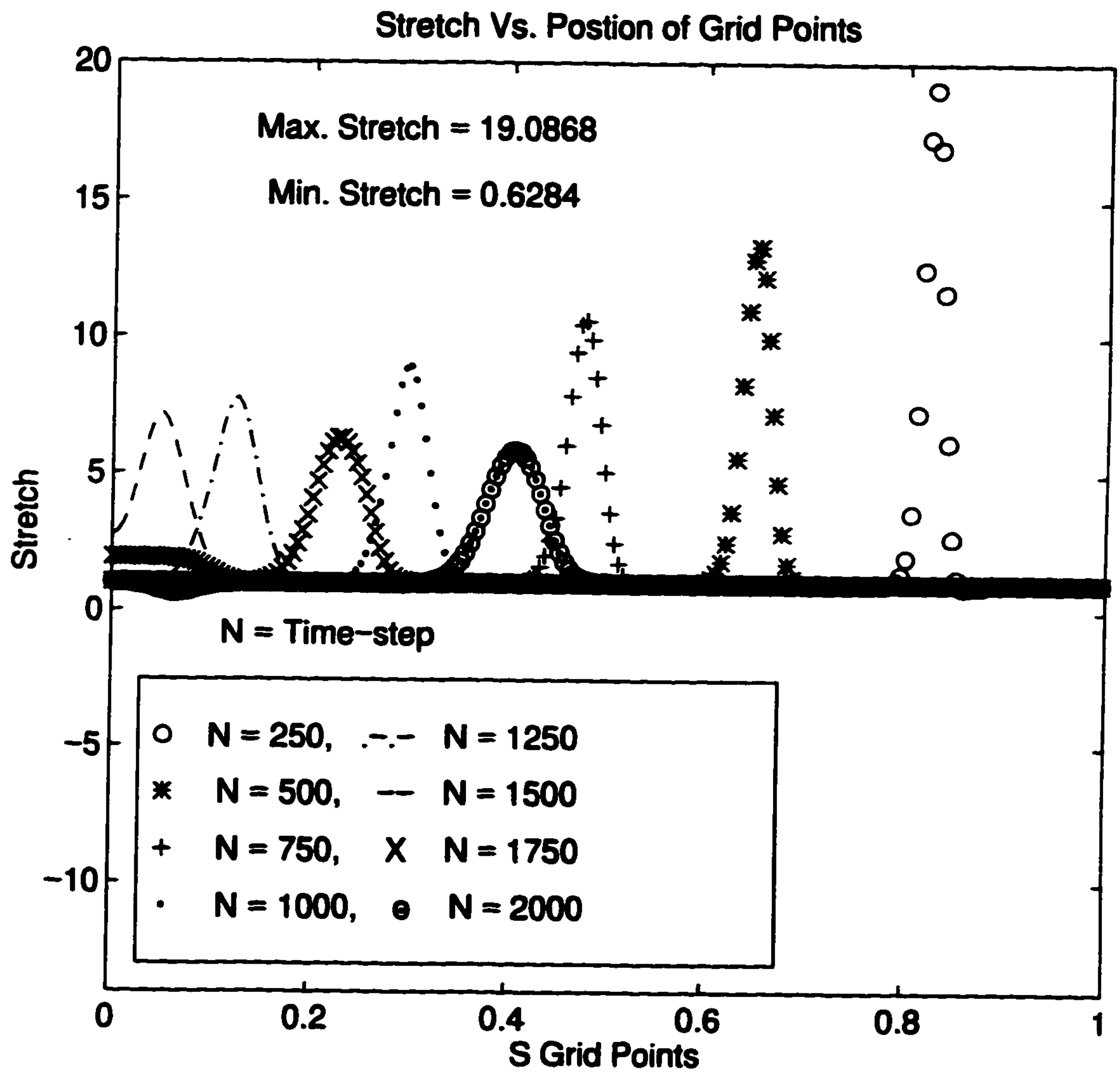


Fig. 6.15 Stretch Value for the First 2000 Time-steps for Large Amplitude

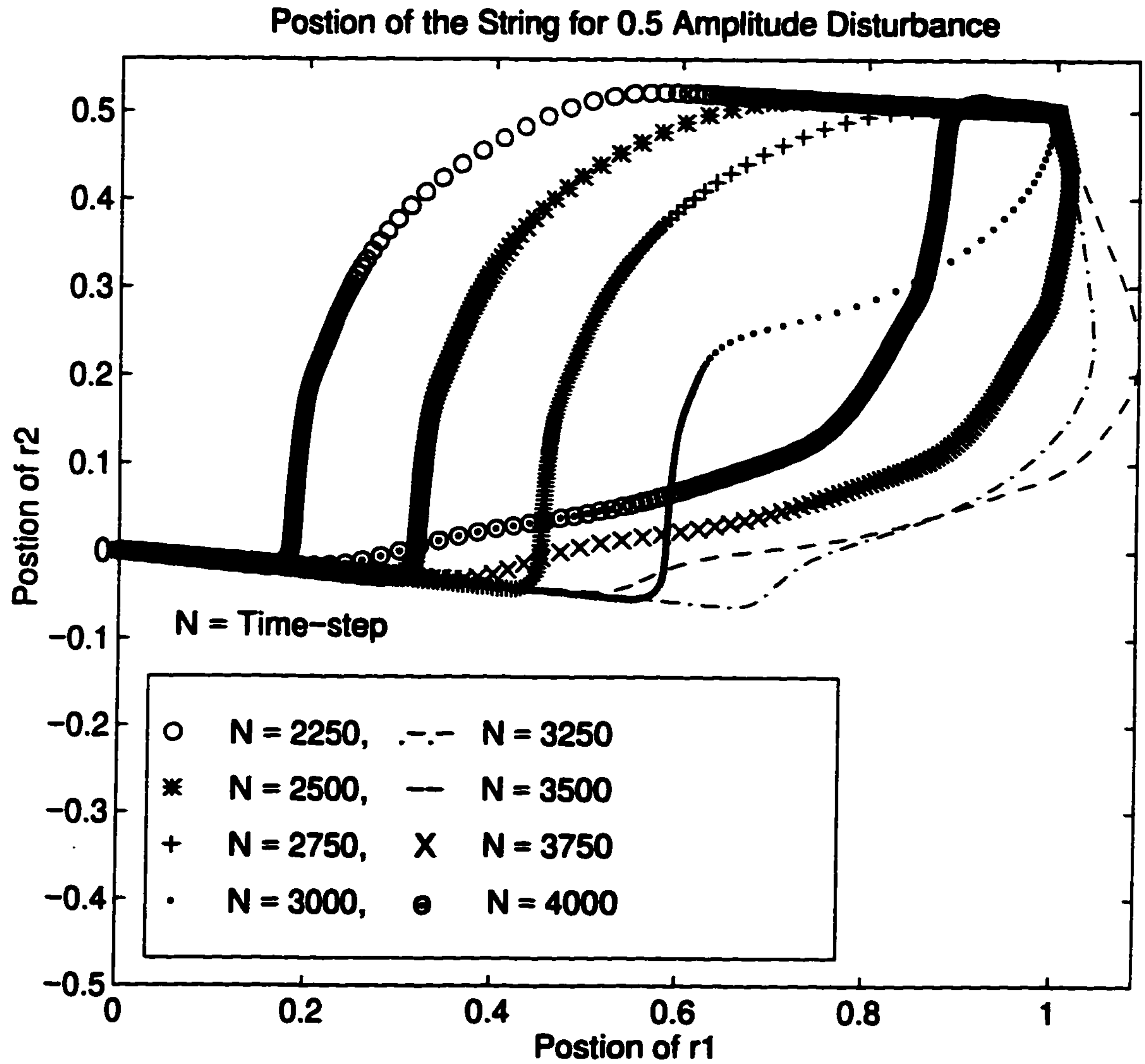


Fig. 6.16 Position of the String from 2000 to 4000 Time-steps for Large Amplitude

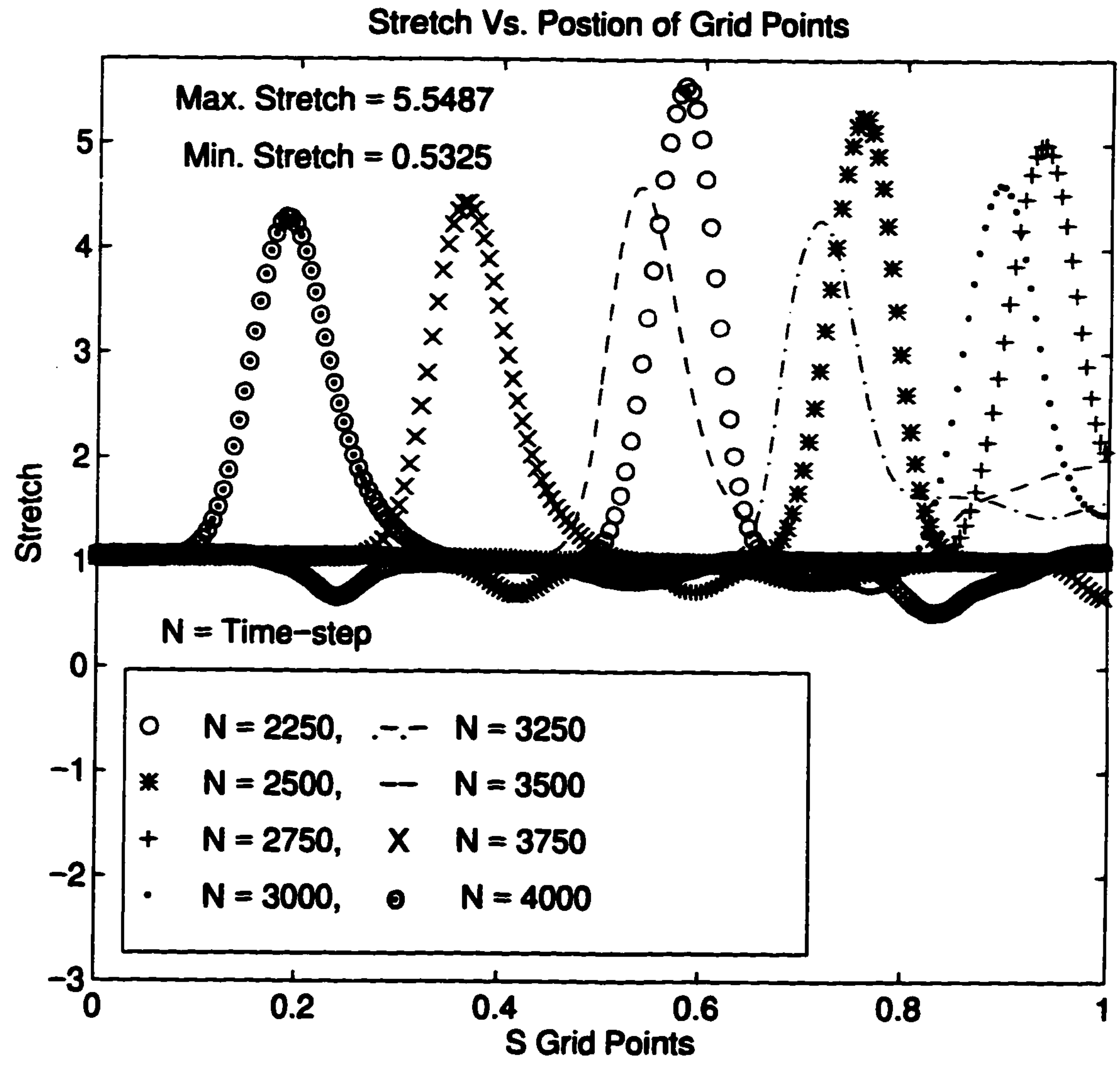


Fig. 6.17 Stretch Value from 2000 to 4000 Time-steps for Large Amplitude

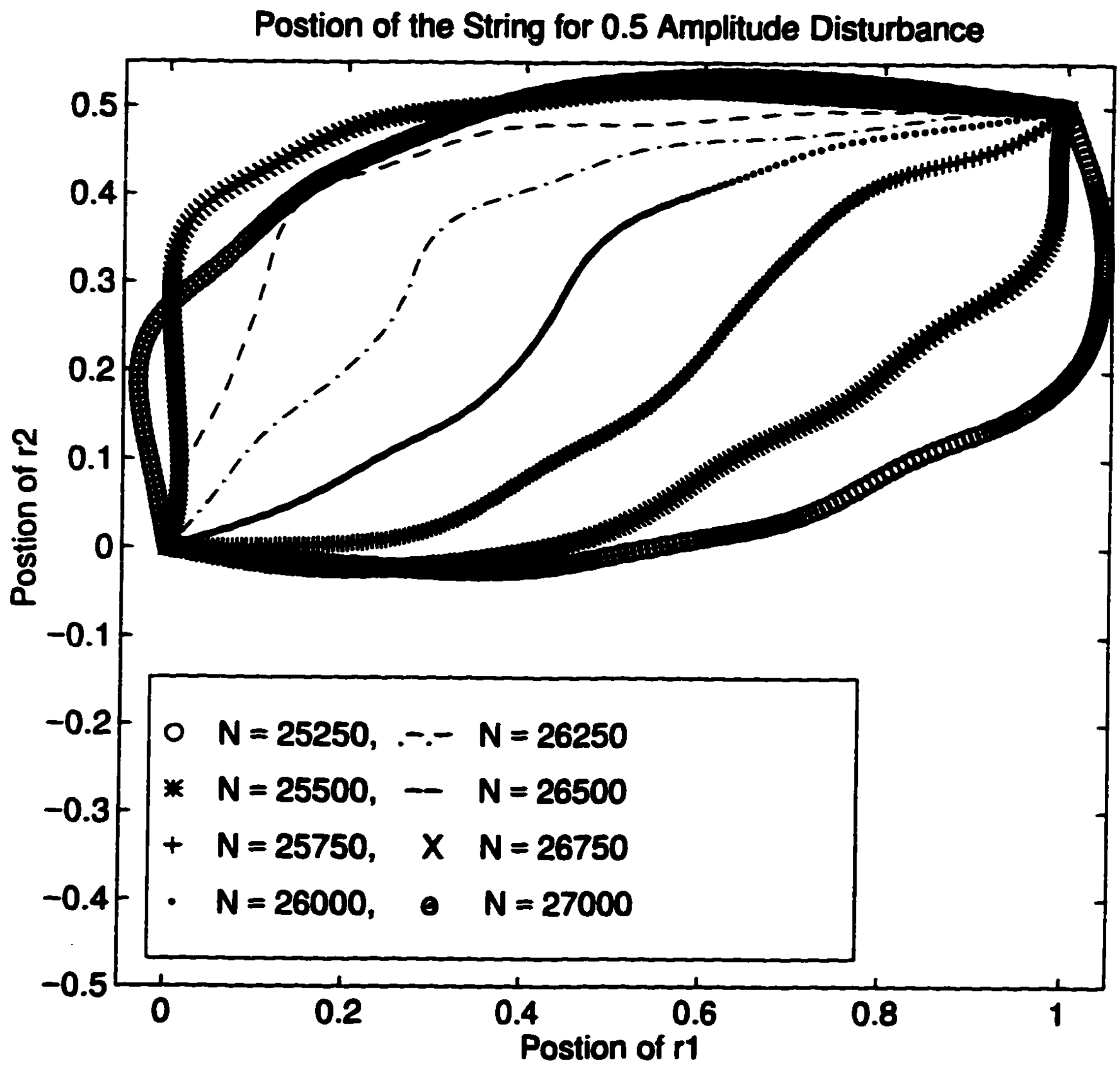


Fig. 6.18 Position of the String from 25250 to 27000 Time-steps for Large Amplitude

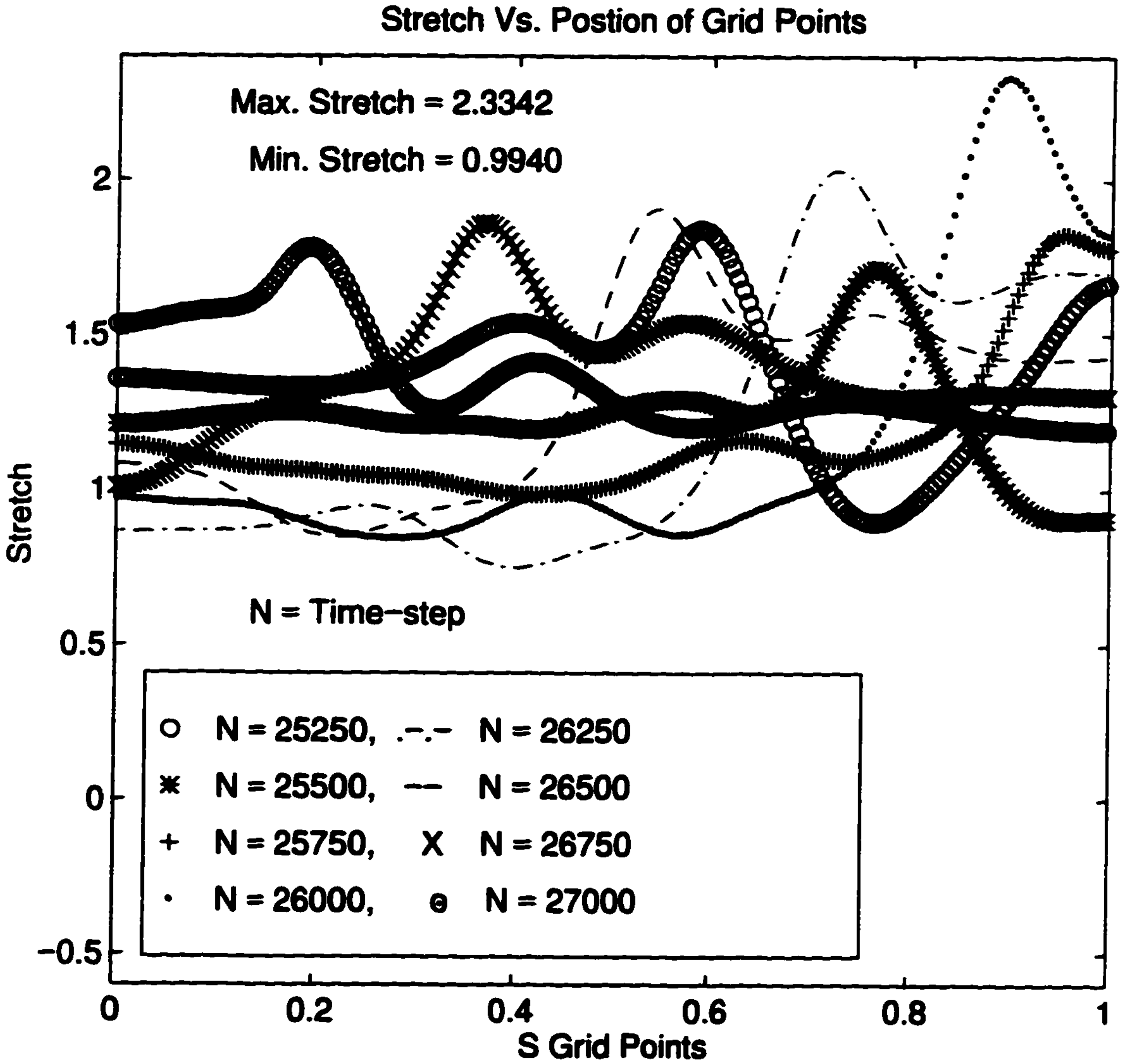


Fig. 6.19 Stretch Value from 25250 to 27000 Time-steps for Large Amplitude

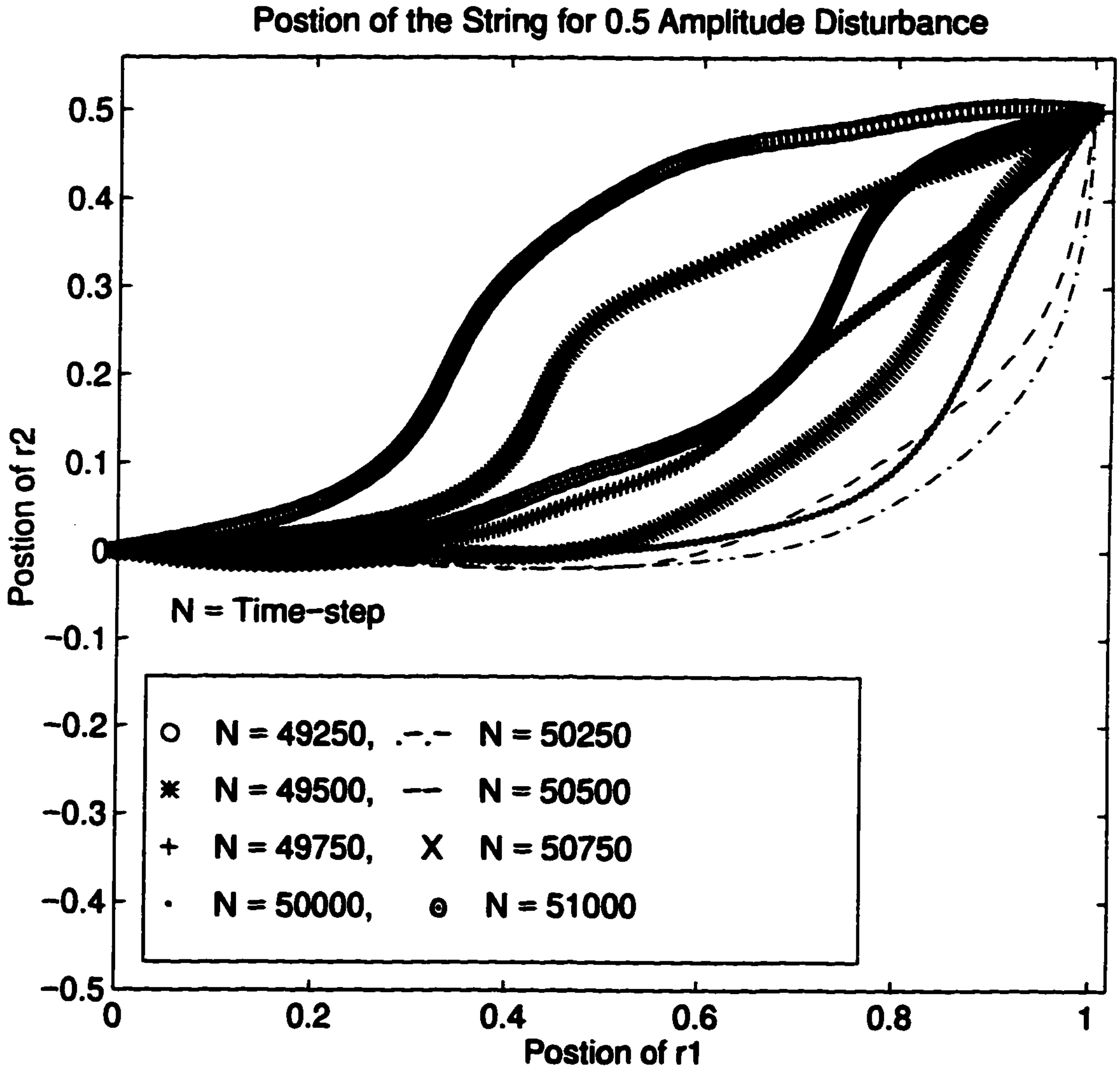


Fig. 6.20 Position of the String from 49250 to 51000 Time-steps for Large Amplitude

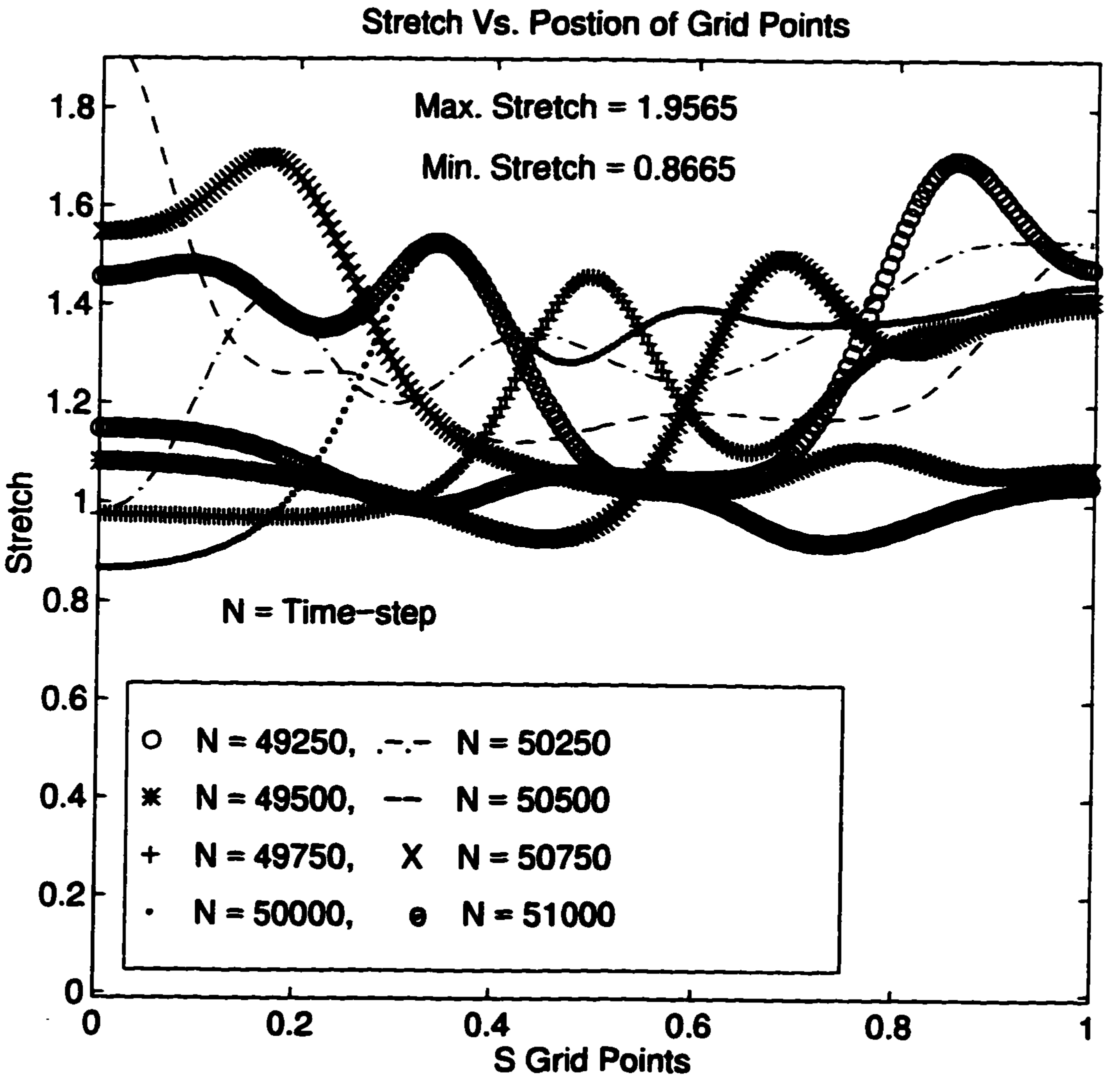


Fig. 6.21 Stretch Value from 49250 to 51000 Time-steps for Large Amplitude

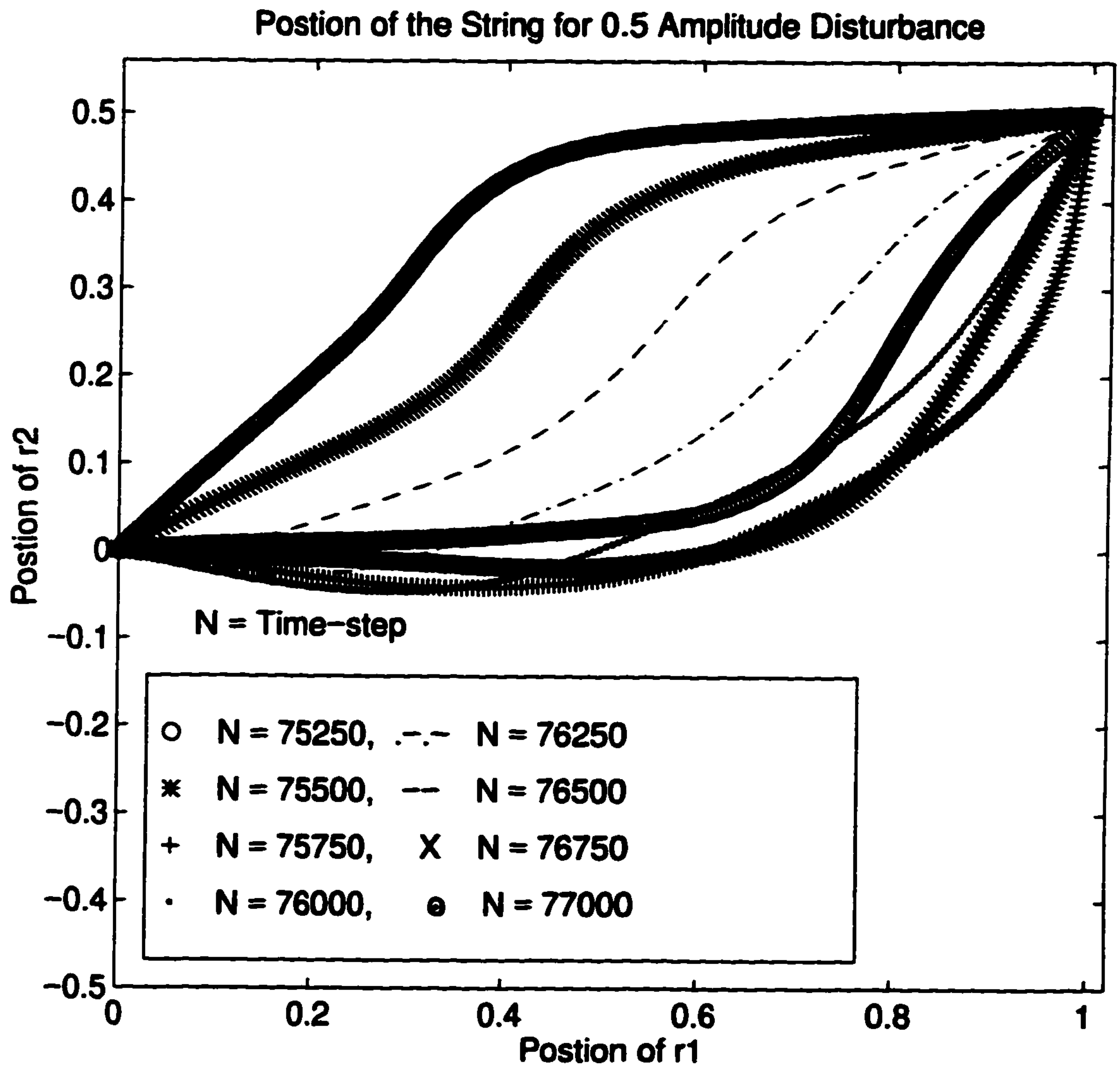


Fig. 6.22 Position of the String from 75250 to 77000 Time-steps for Large Amplitude

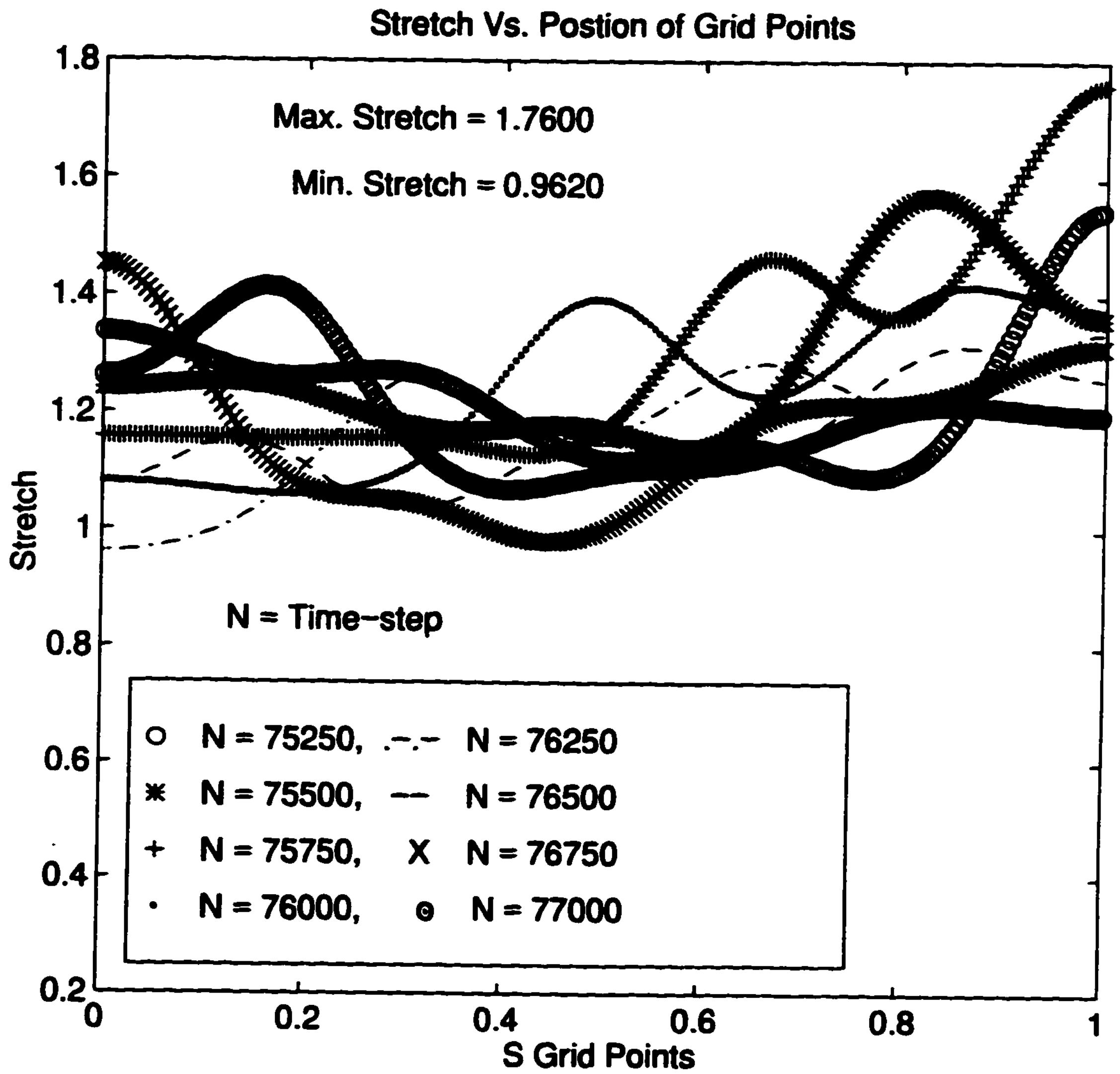


Fig. 6.23 Stretch Value from 75250 to 77000 Time-steps for Large Amplitude

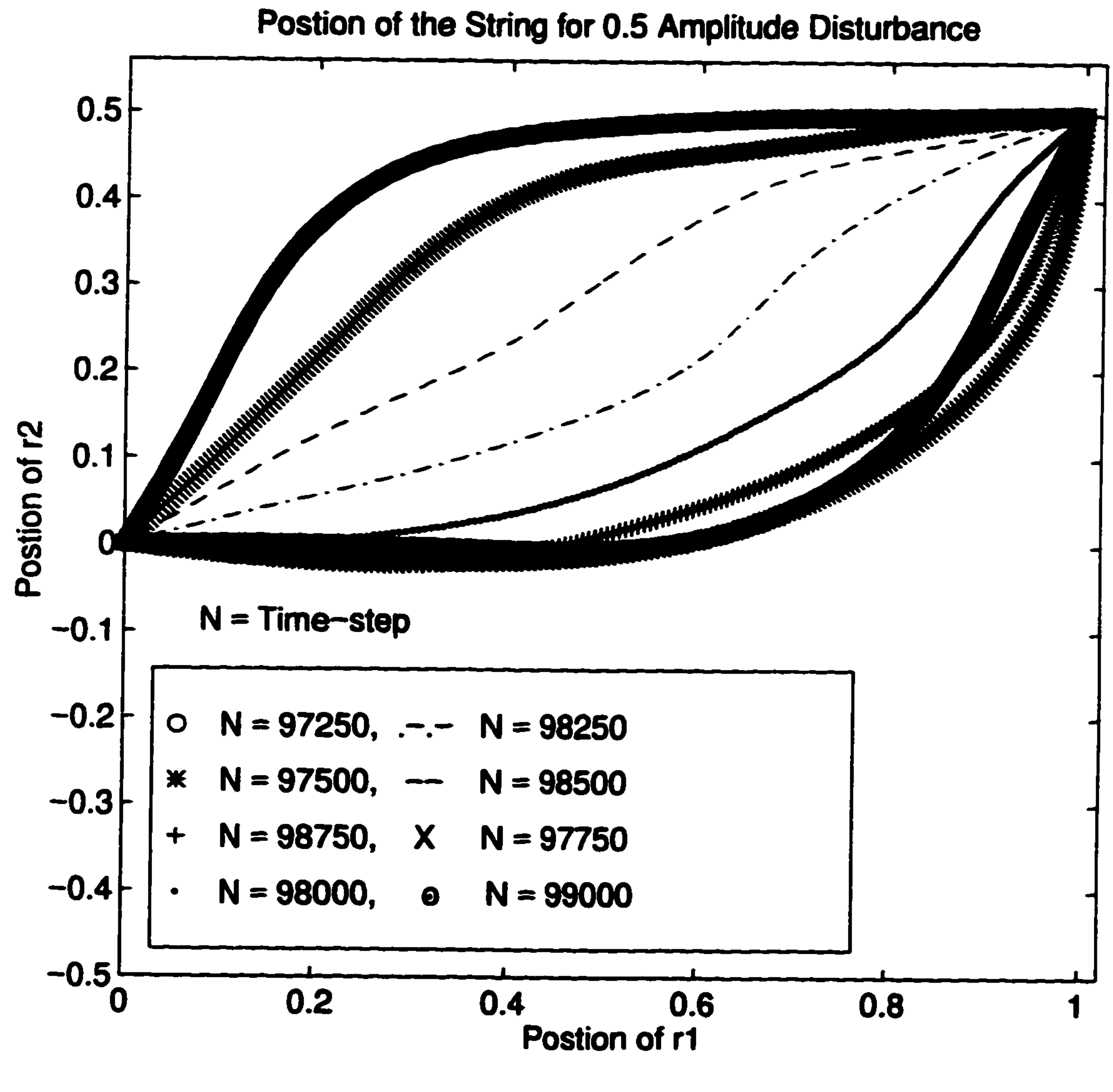


Fig. 6.24 Position of the String from 97250 to 99000 Time-steps for Large Amplitude

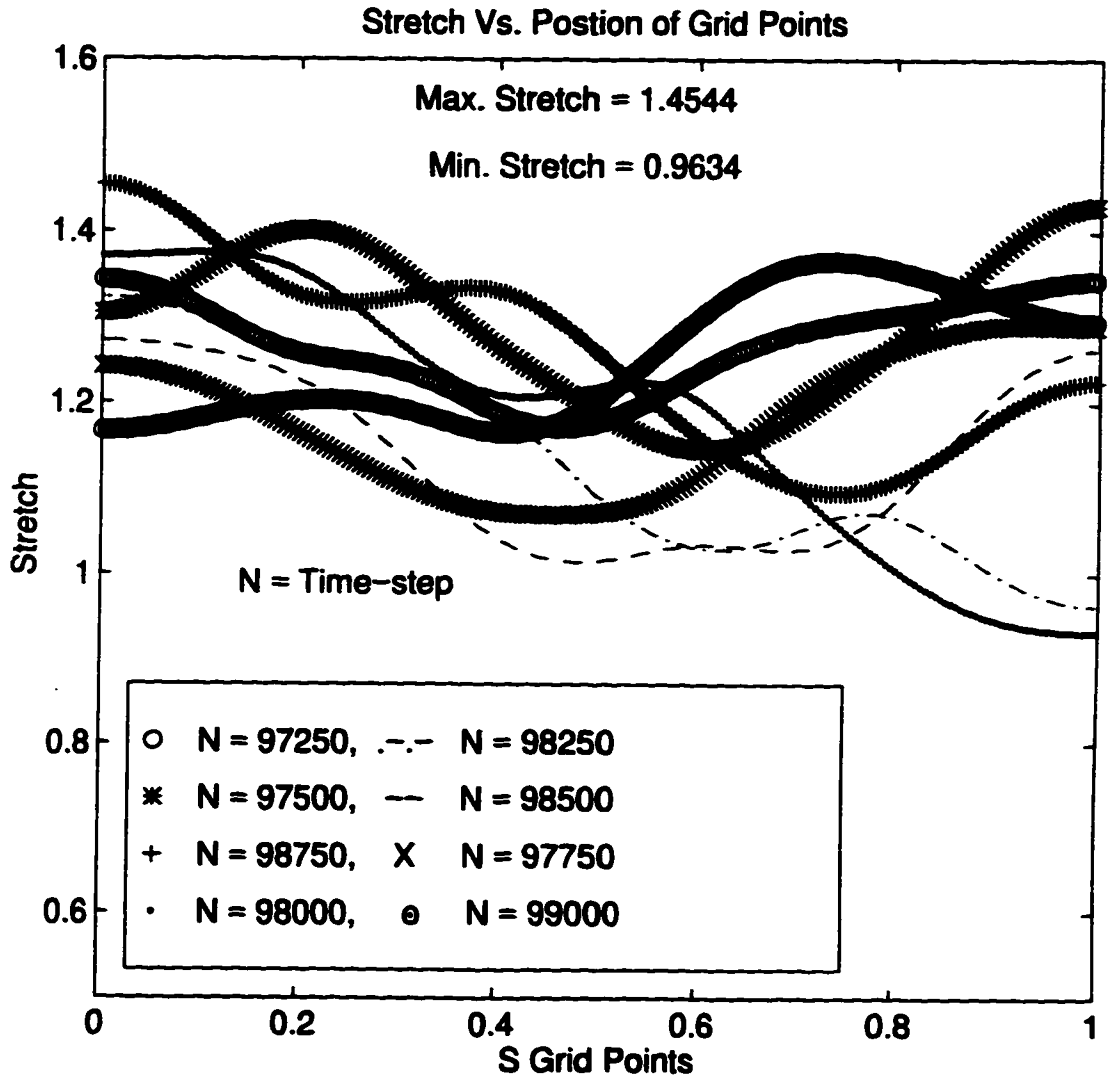


Fig. 6.25 Stretch Value from 97250 to 99000 Time-steps for Large Amplitude

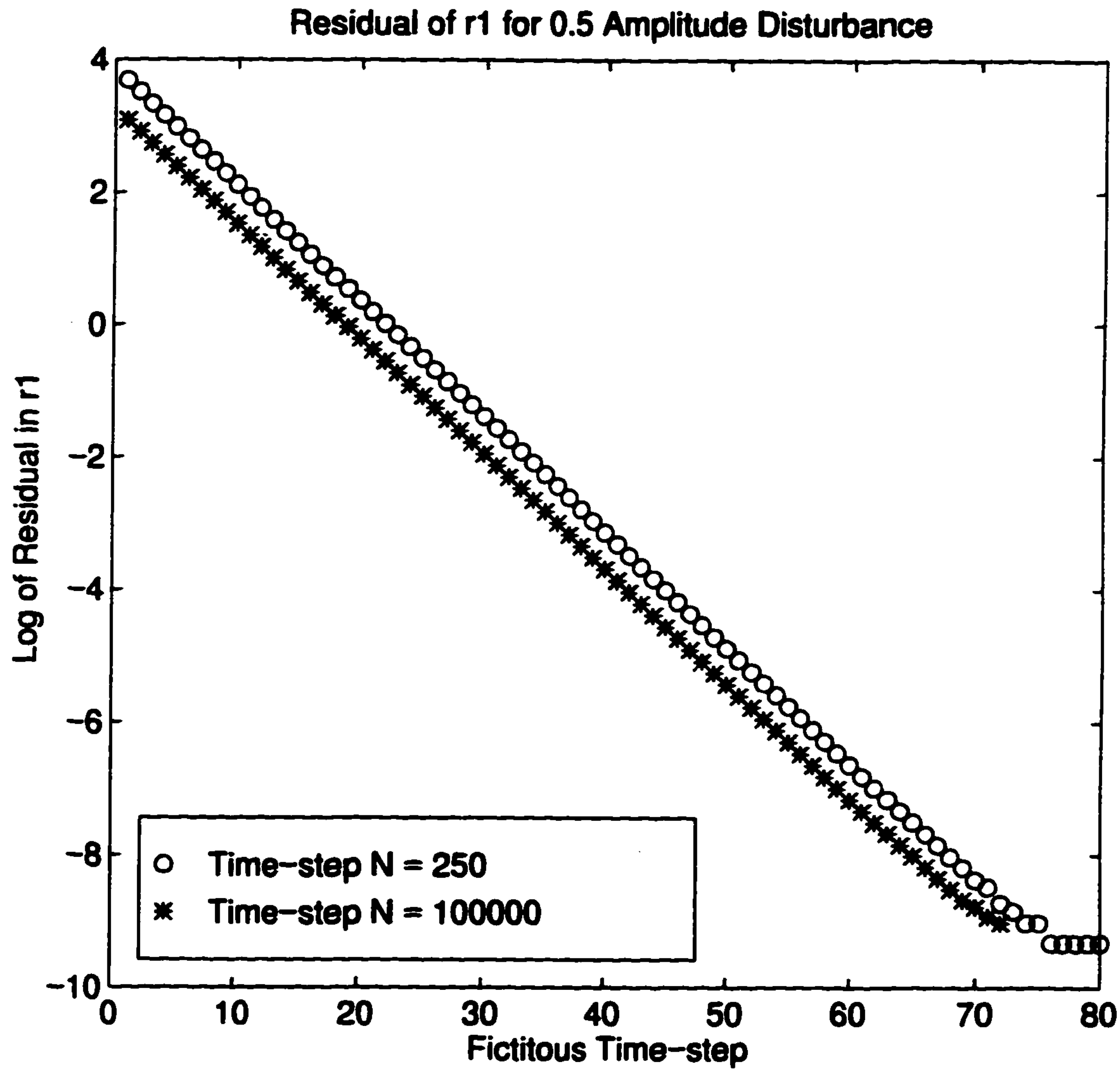


Fig. 6.26 Residual of $\mathbf{r}_1$ for Large Amplitude Case

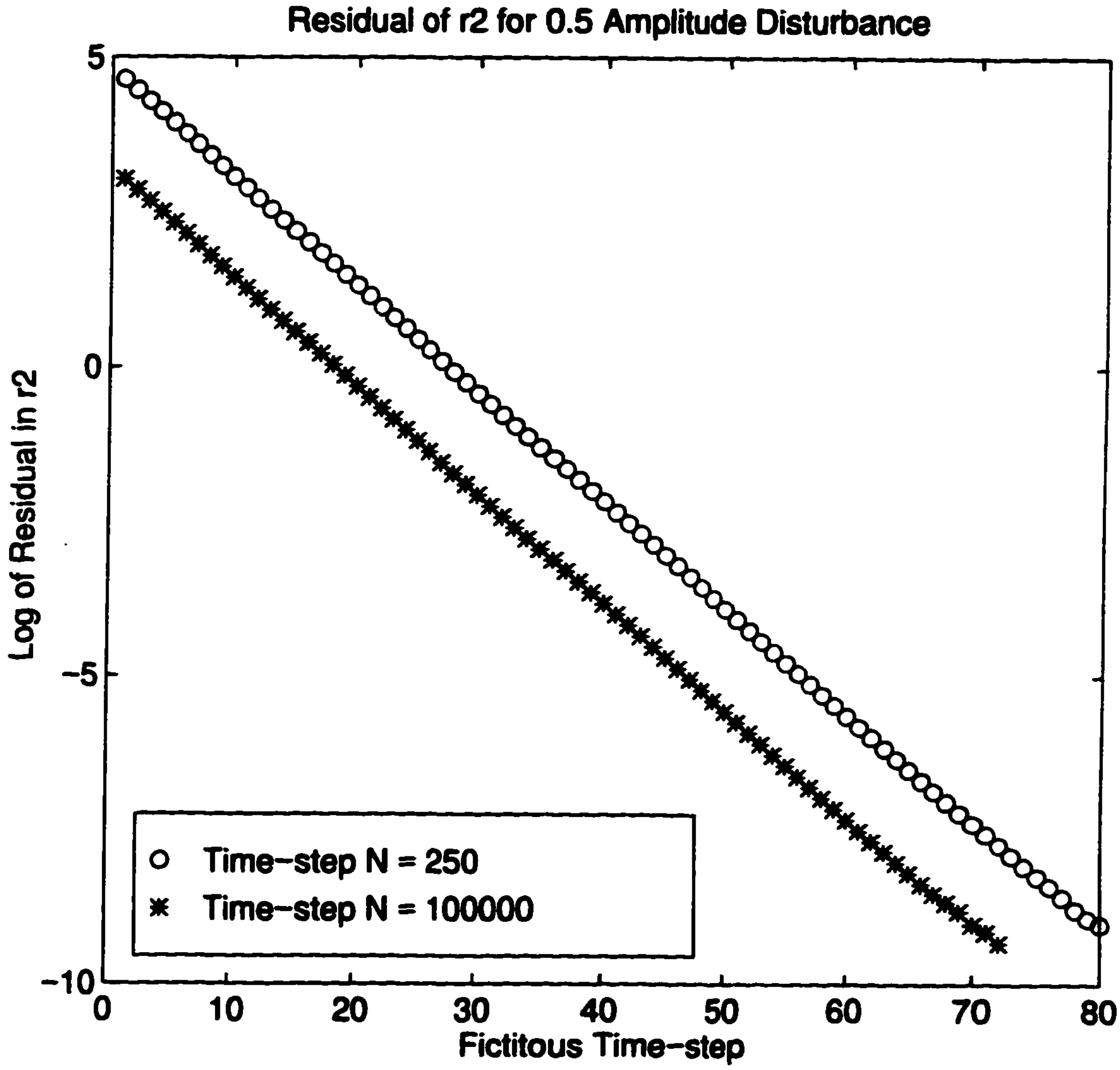


Fig. 6.27 Residual of $\mathbf{r}_2$ for Large Amplitude Case

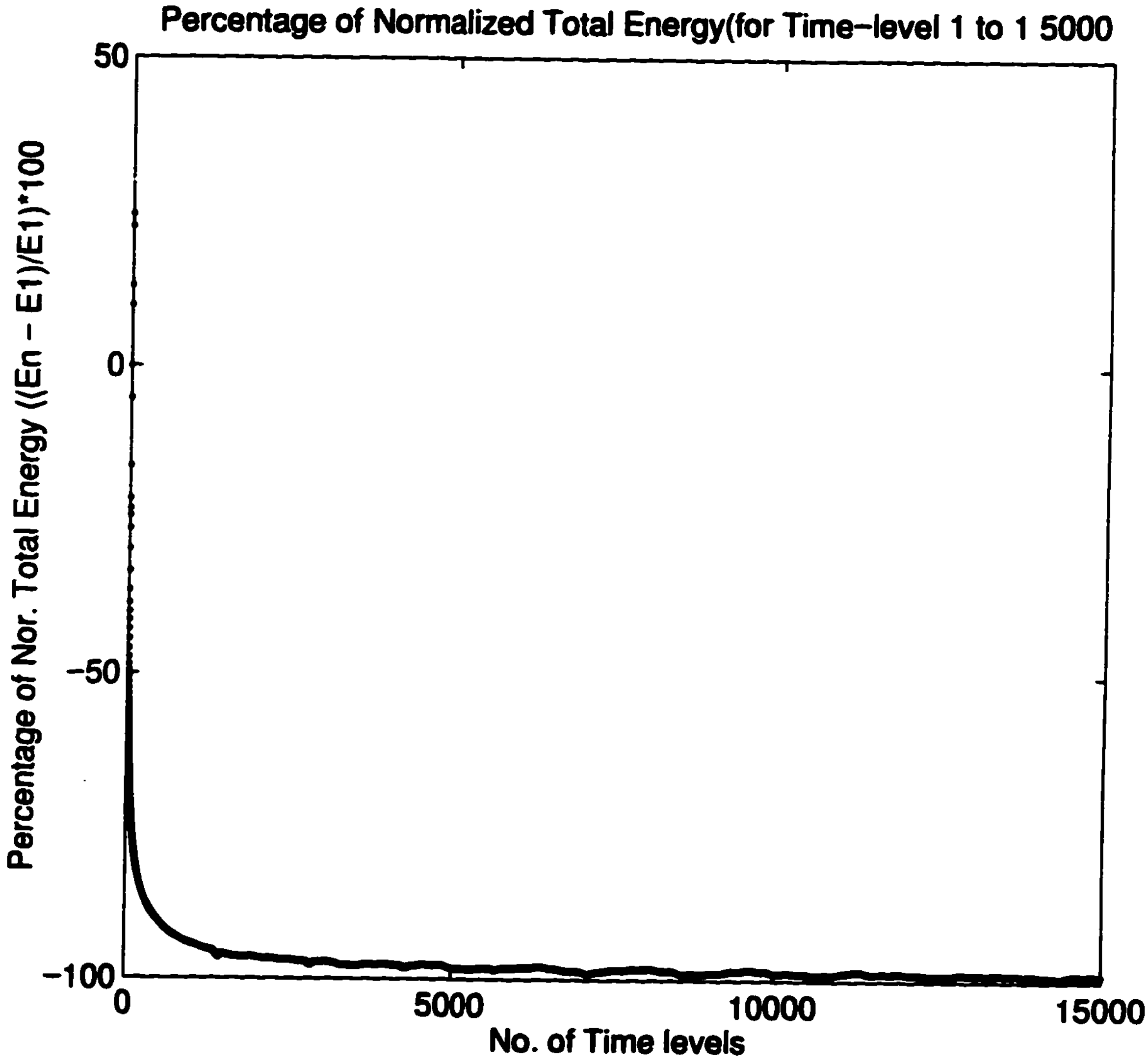


Fig. 6.28 Percentage of Normalized Total Energy in the First 15000 Time-steps for Large Amplitude case

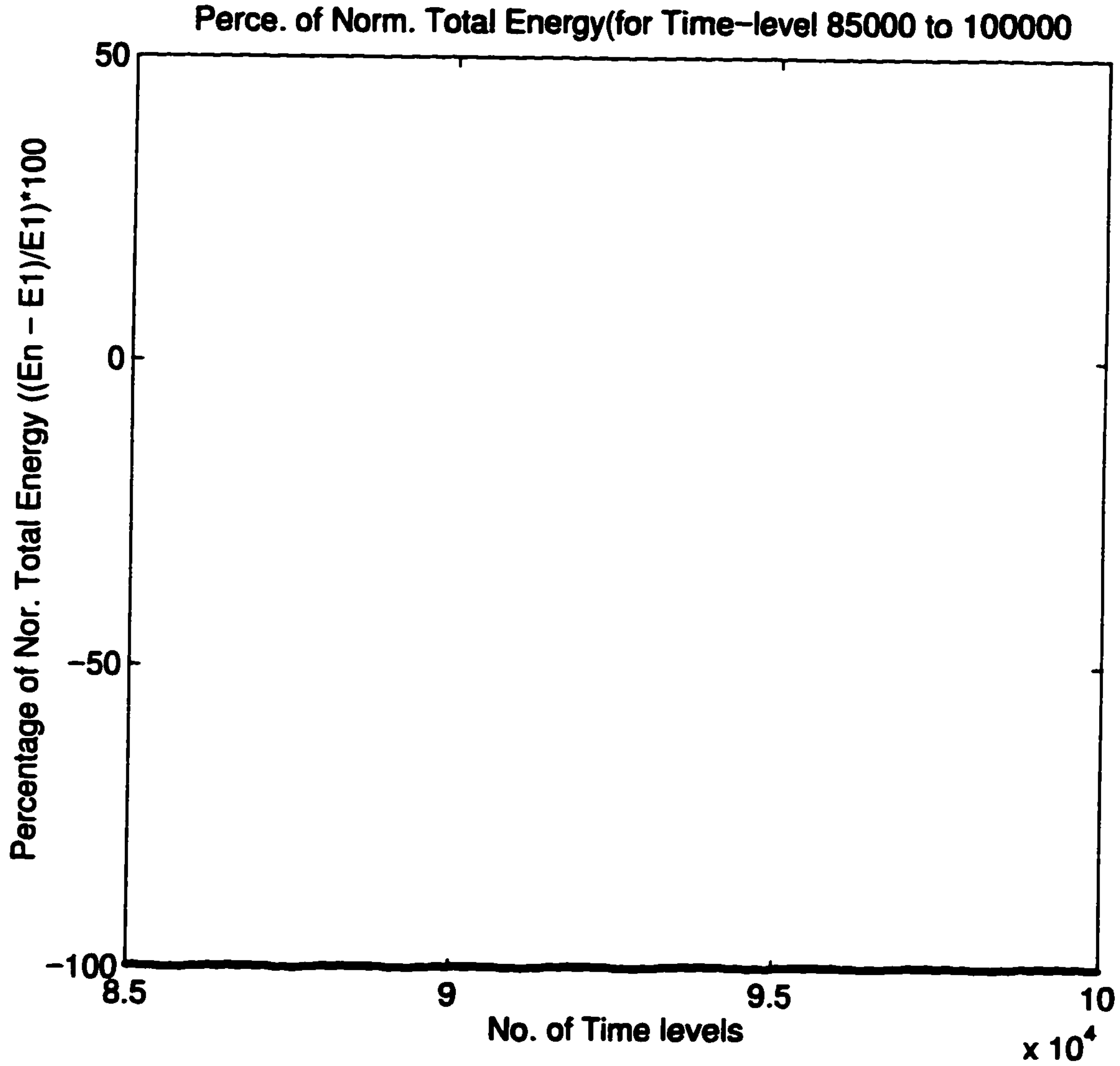


Fig. 6.29 Percentage of Normalized Total Energy from 85000 to 100000 Time-steps for Large Amplitude case

6-4 Small Amplitude Disturbance by Low Pressure

In this test case a low intensity instantaneous pressure applied on five grid points only at the first time step. This low pressure initiates a small amplitude wave. The plots of Fig 6.13 to Fig.6.21 shows the position of the wave at different time levels, the initial stage before reflection, after reflection and a comparison of two waves at varies time levels. Since the amplitude is very small, the wave propagate as a linear wave. Thus the waves initiated at the middle of the string progresses to the right and left with constant speed as expected for the linear range.This property is preserved for the tested length of time, for fifty reflections.Smoothing property observed at the corners of the wave front and 3% wave shift at the center of the string.

From the plot it can be seen that the numerical method preserved the symmetry of the problem for all the tested length of time.The plot for the residual of the fictitious first order partial differential equation in the Runge-Kutta method vs. the time steps required to reach this equation to steady state are shown on Fig. 6.22 and Fig.6.23.The percentage of the normalized energy vs. time step is plotted from Fig. 6.10 to Fig. 6.12 to show the stability of the scheme and the numerical does not violet the conservation of the total energy. For clarity, the plots for positions of the string are done at an interval of three data points.

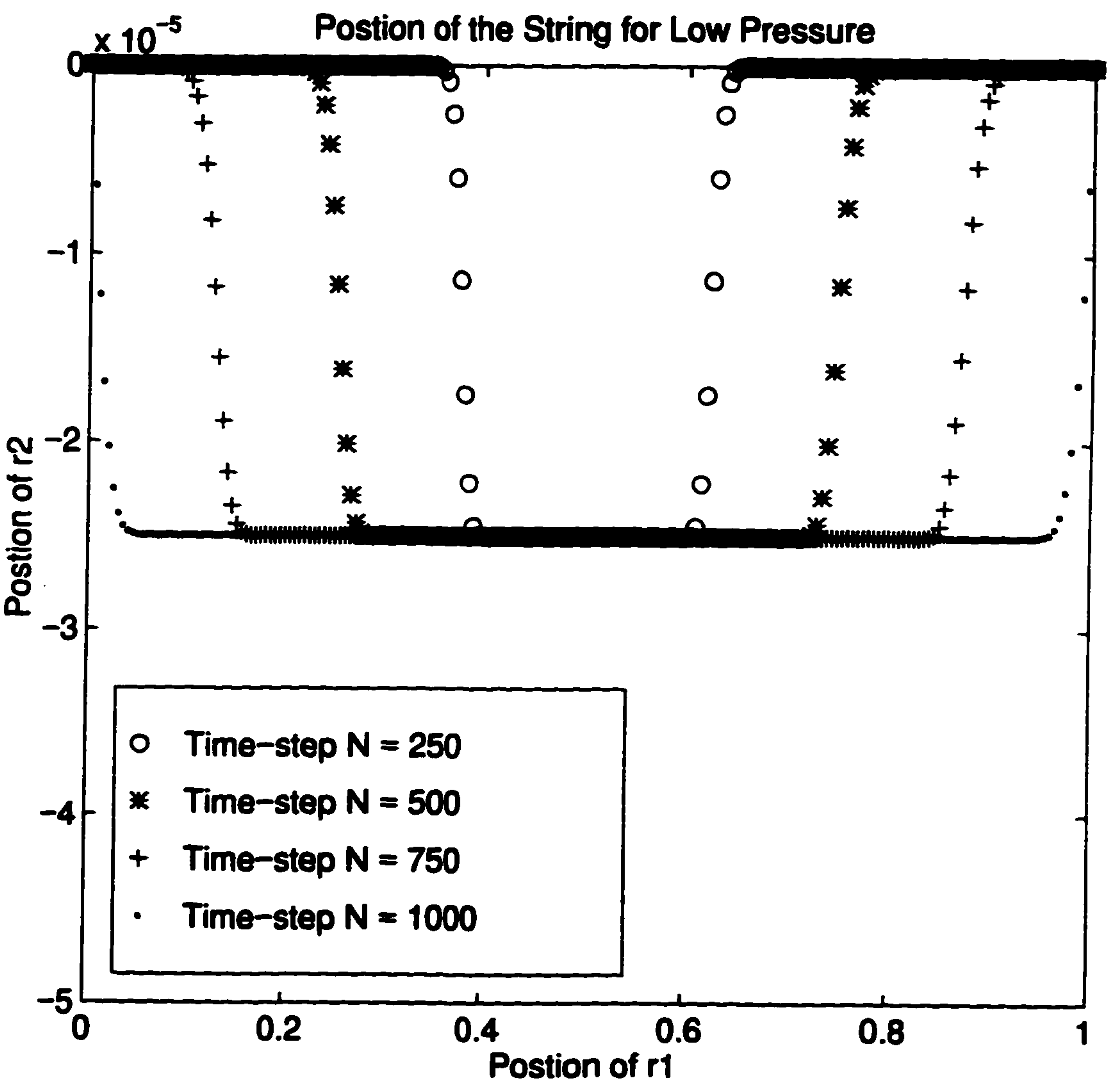


Fig. 6.30 Positions of the String for First 1000 Time-steps for Low Pressure Case

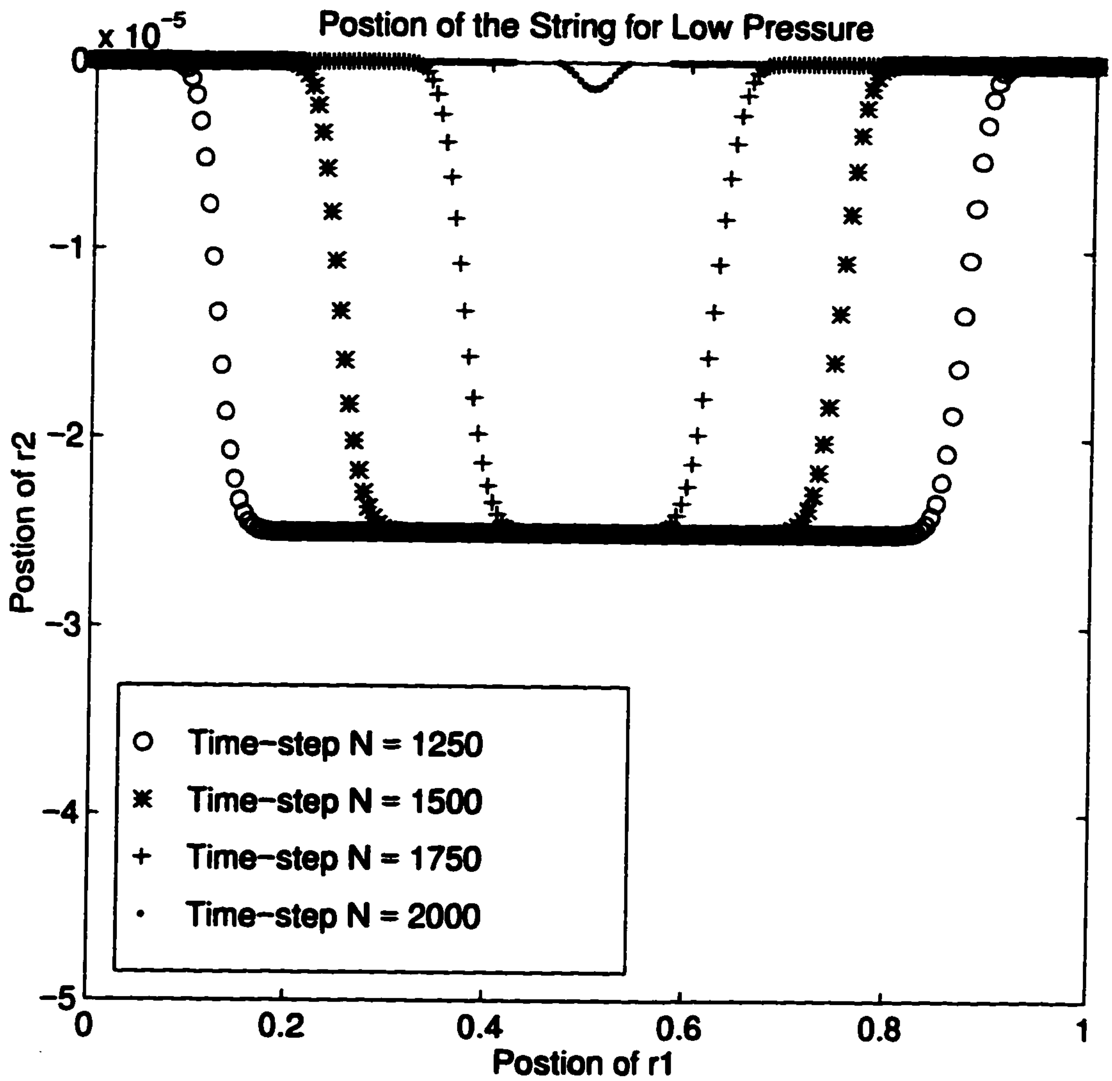


Fig. 6.31 Positions of the String form 1250 to 2000 Time-steps for Low Pressure Case

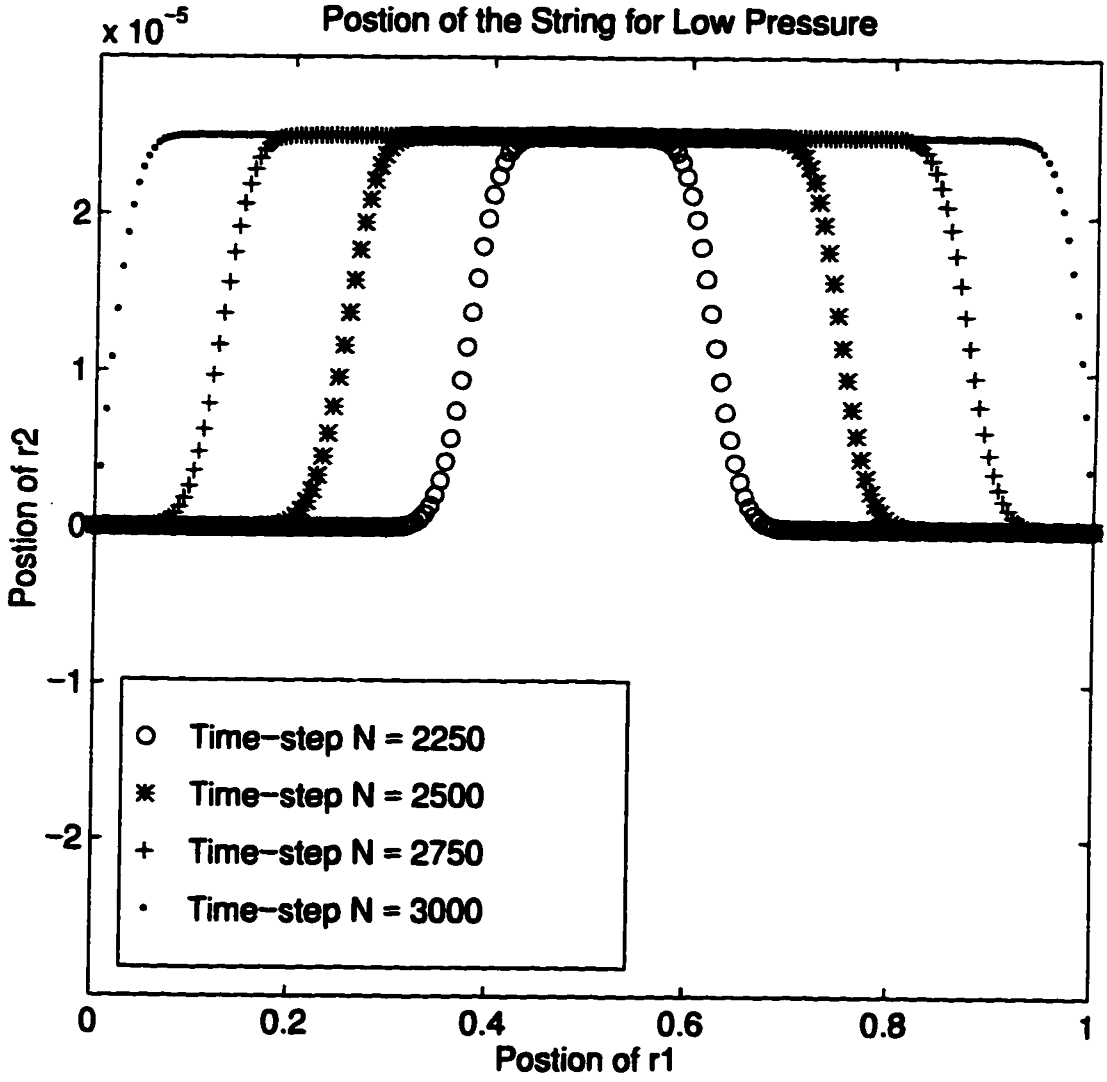


Fig. 6.32 Positions of the String form 2250 to 3000 Time-steps for Low Pressure Case

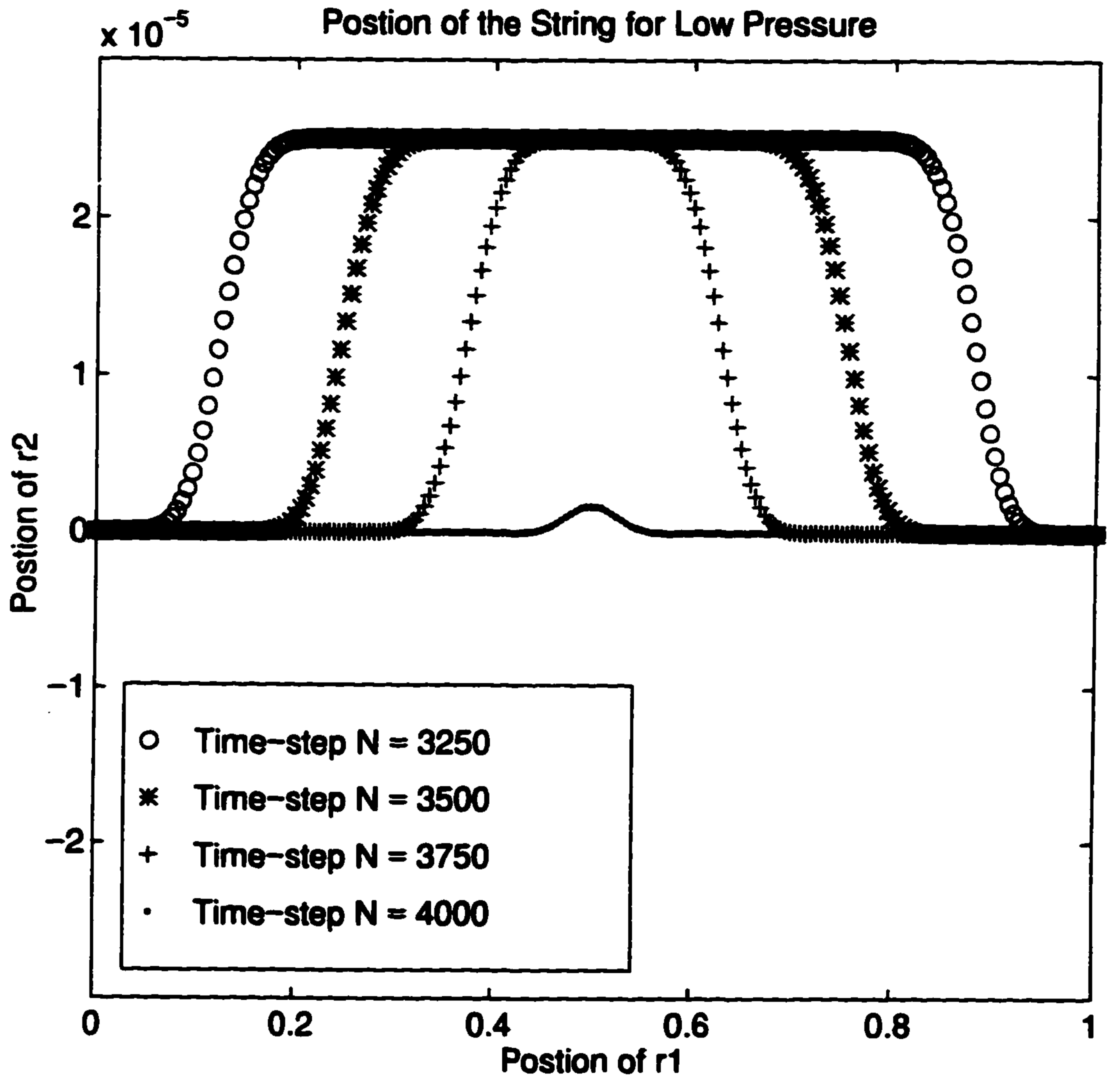


Fig. 6.33 Positions of the String form 3250 to 4000 Time-steps for Low Pressure Case

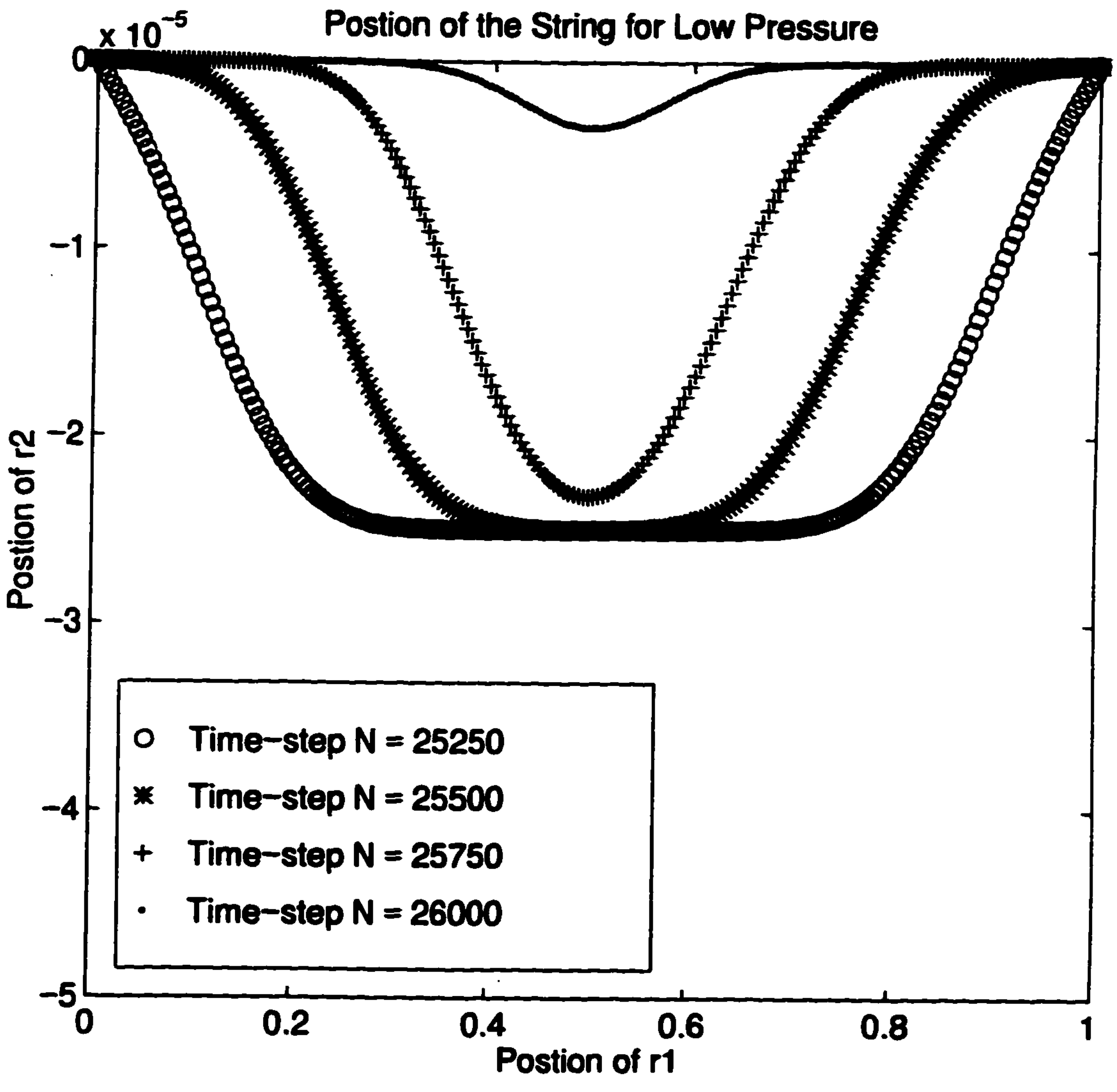


Fig. 6.34 Positions of the String form 25250 to 26000 Time-steps for Low Pressure Case

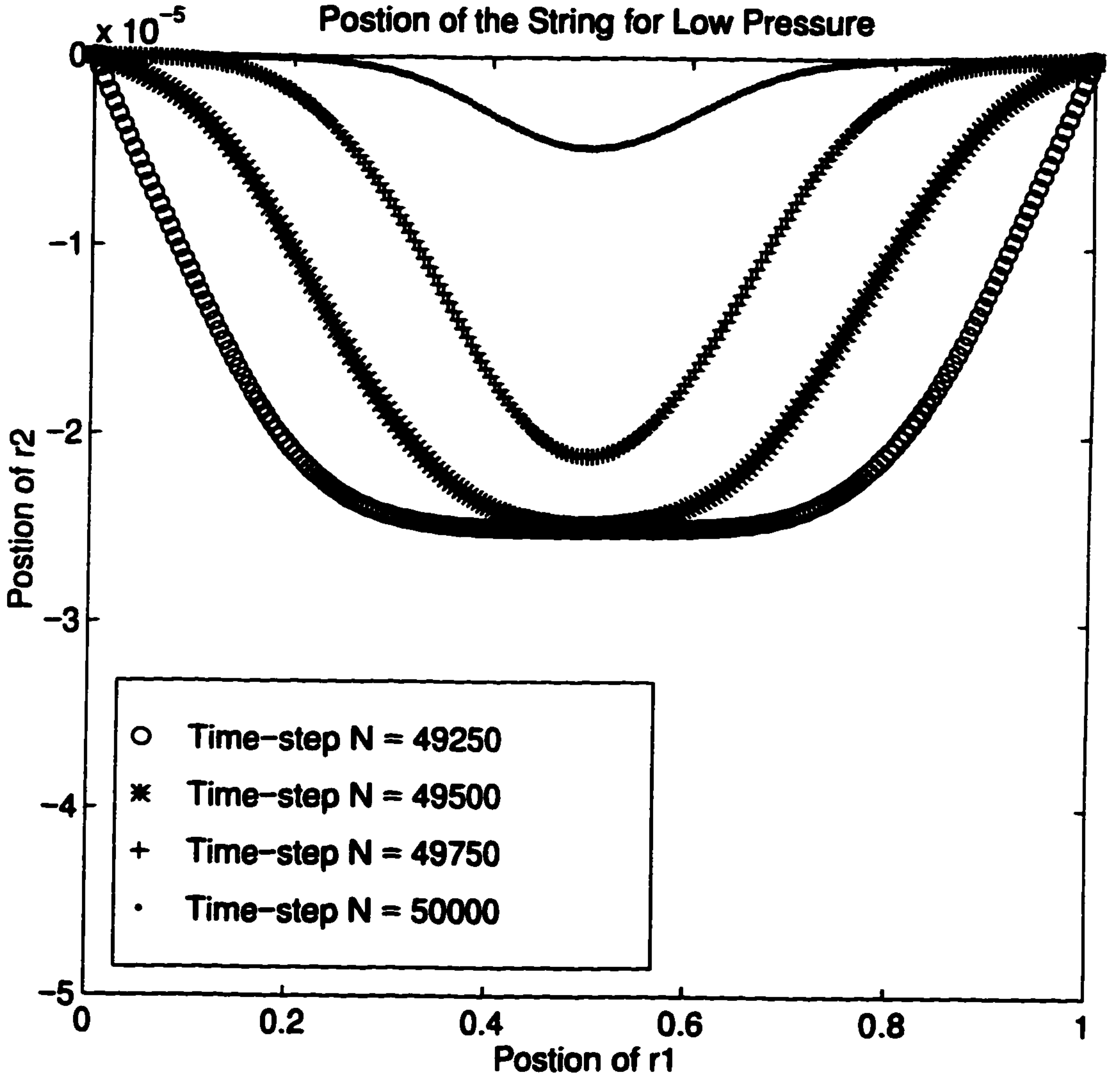


Fig. 6.35 Positions of the String form 49250 to 50000 Time-steps for Low Pressure Case

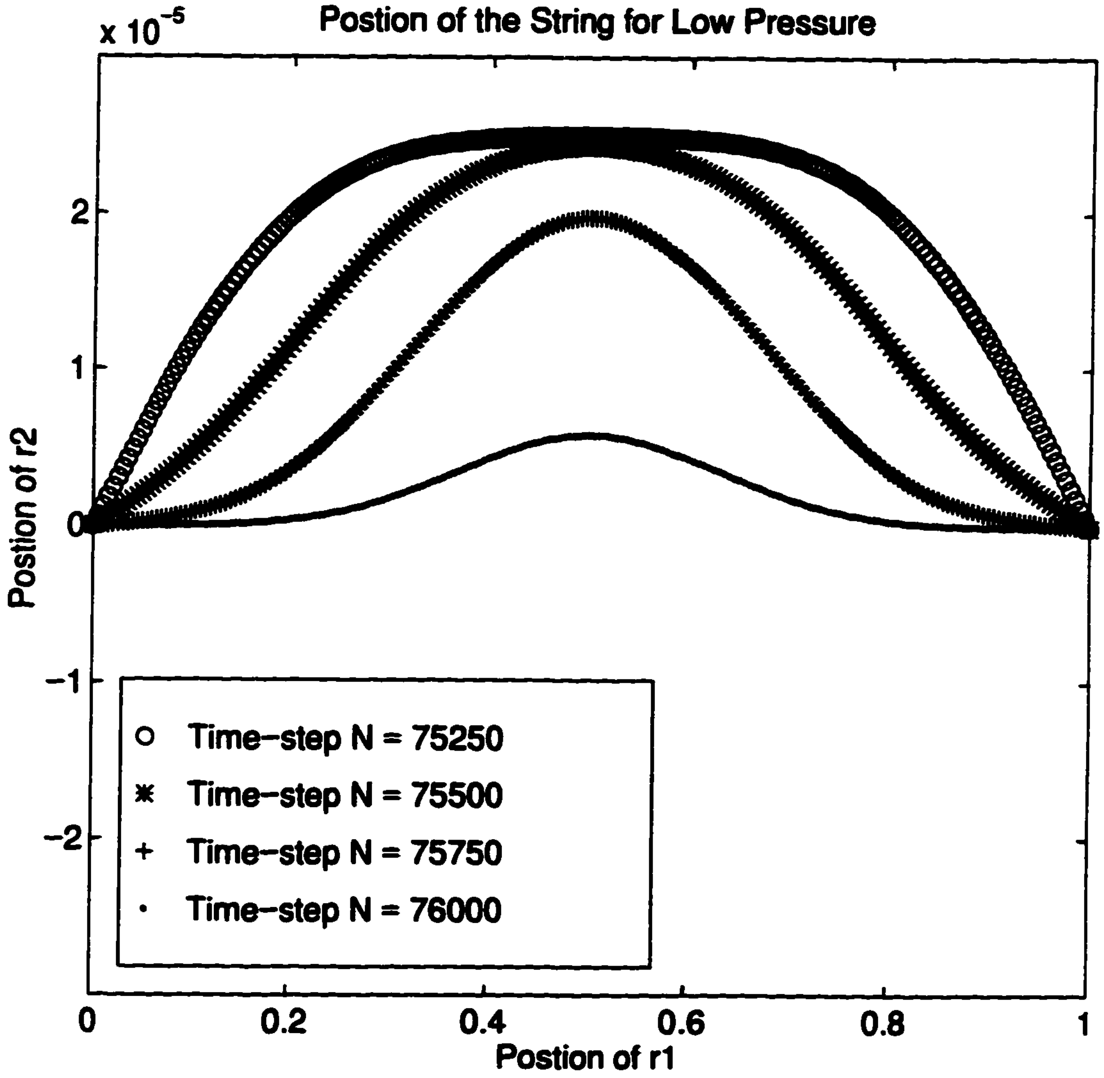


Fig. 6.36 Positions of the String form 75250 to 76000 Time-steps for Low Pressure Case

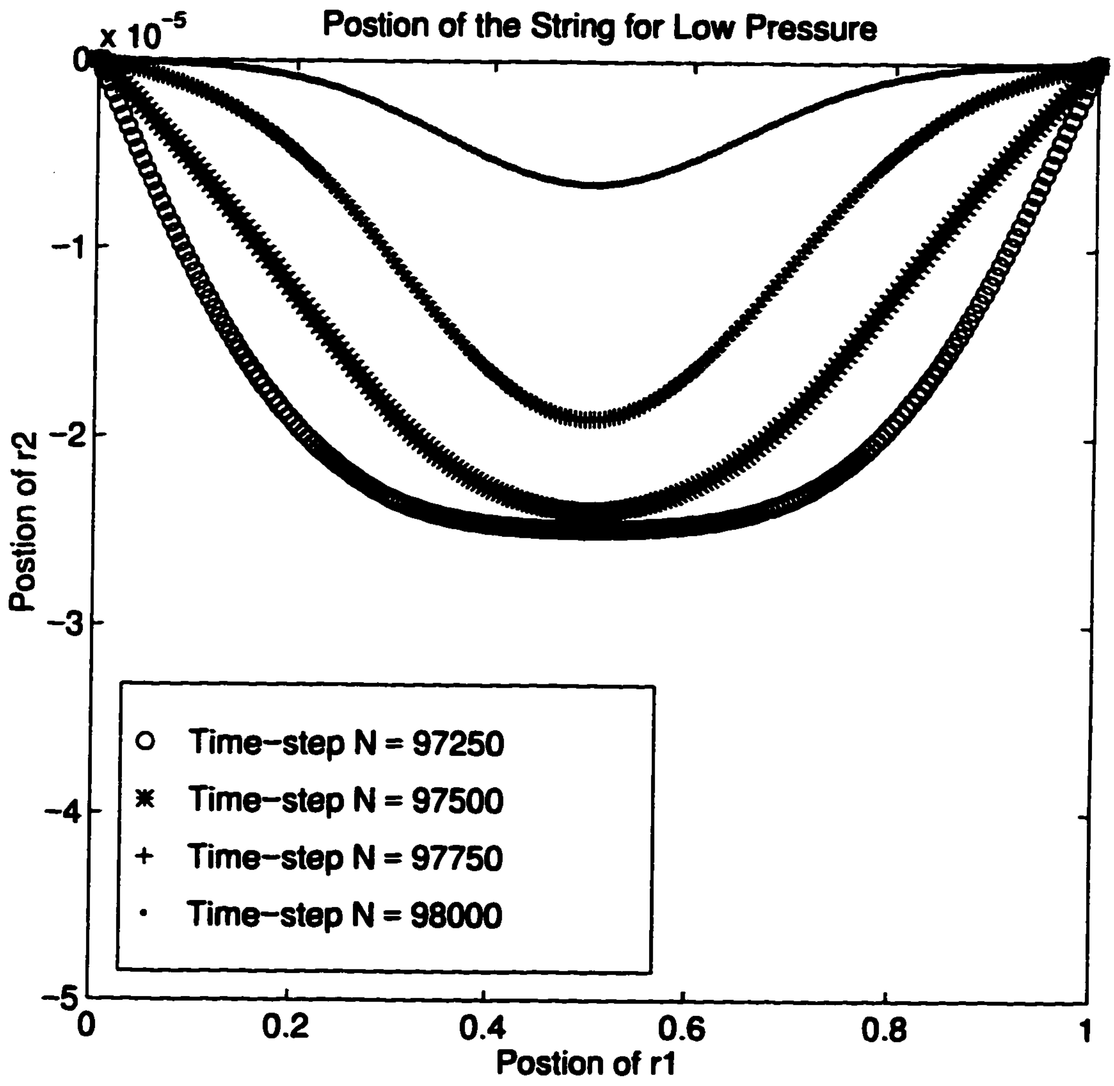


Fig. 6.37 Positions of the String form 97250 to 98000 Time-steps for Low Pressure Case

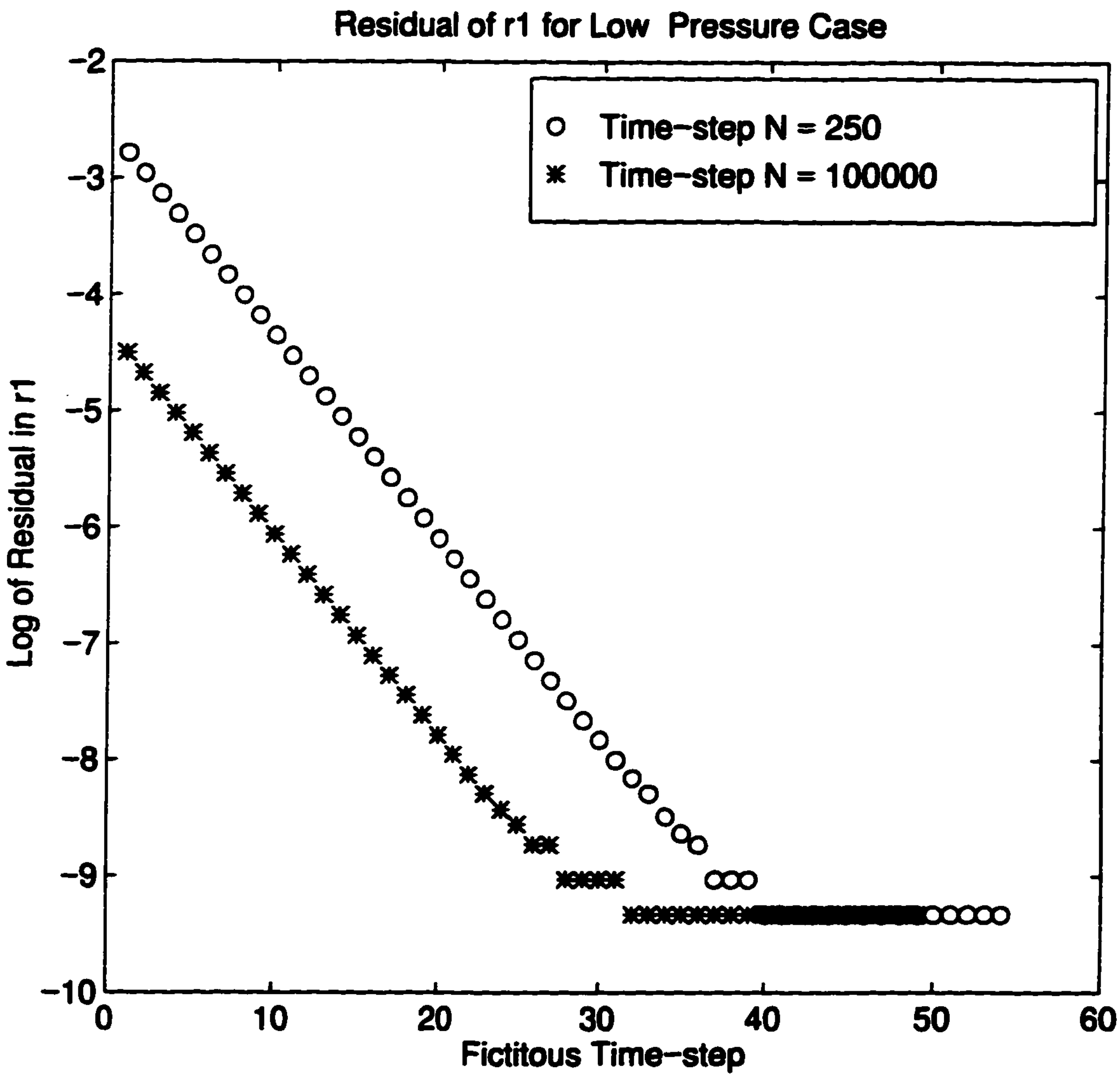


Fig. 6.38 Residual of $\mathbf{r}_1$ for the Low Pressure Case

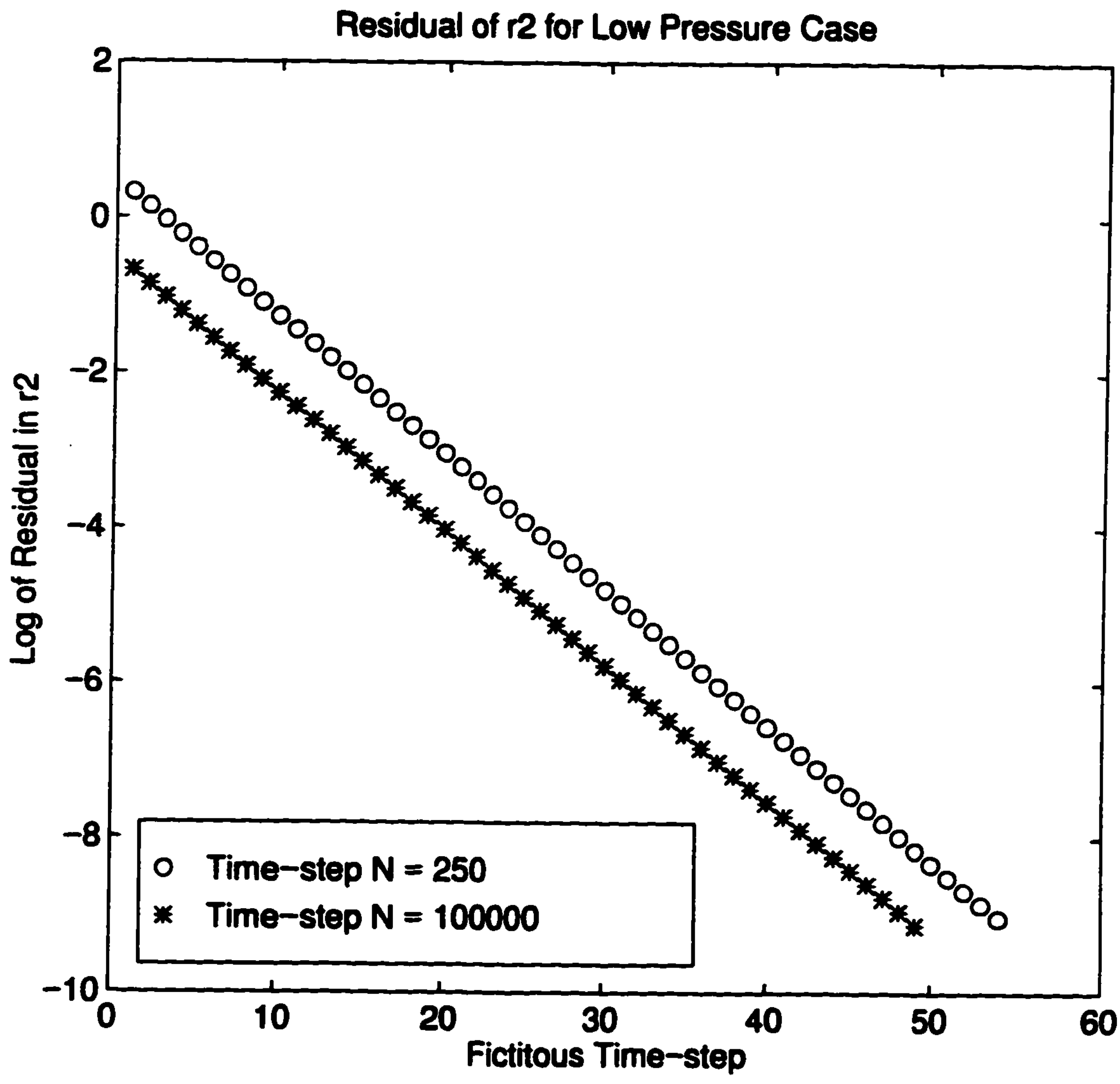


Fig. 6.39 Residual of $\mathbf{r}_2$ for the Low Pressure Case

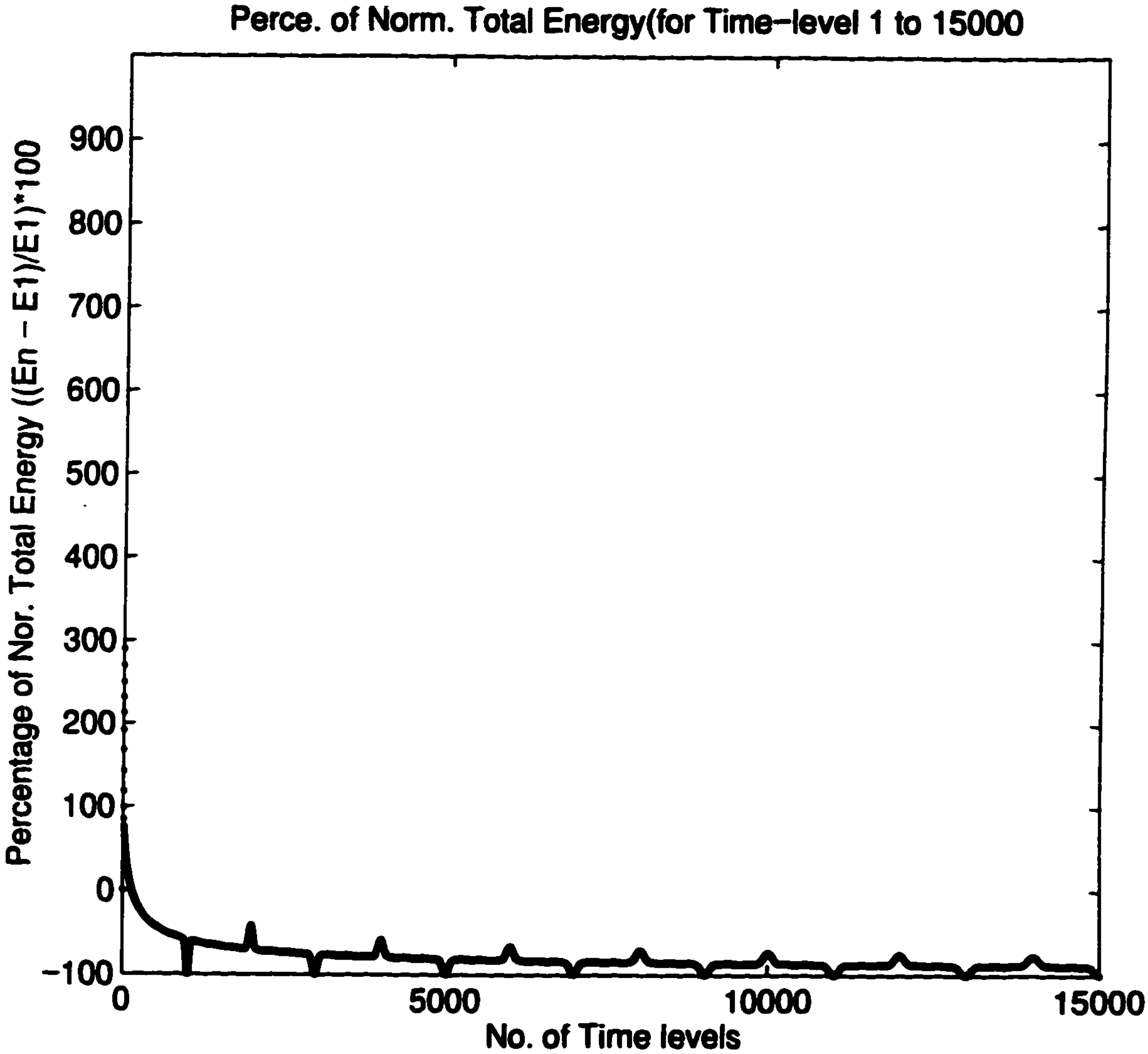


Fig. 6.40 Percentage of Normalized Total Energy in the First 15000 Time-steps for Low Pressure Case

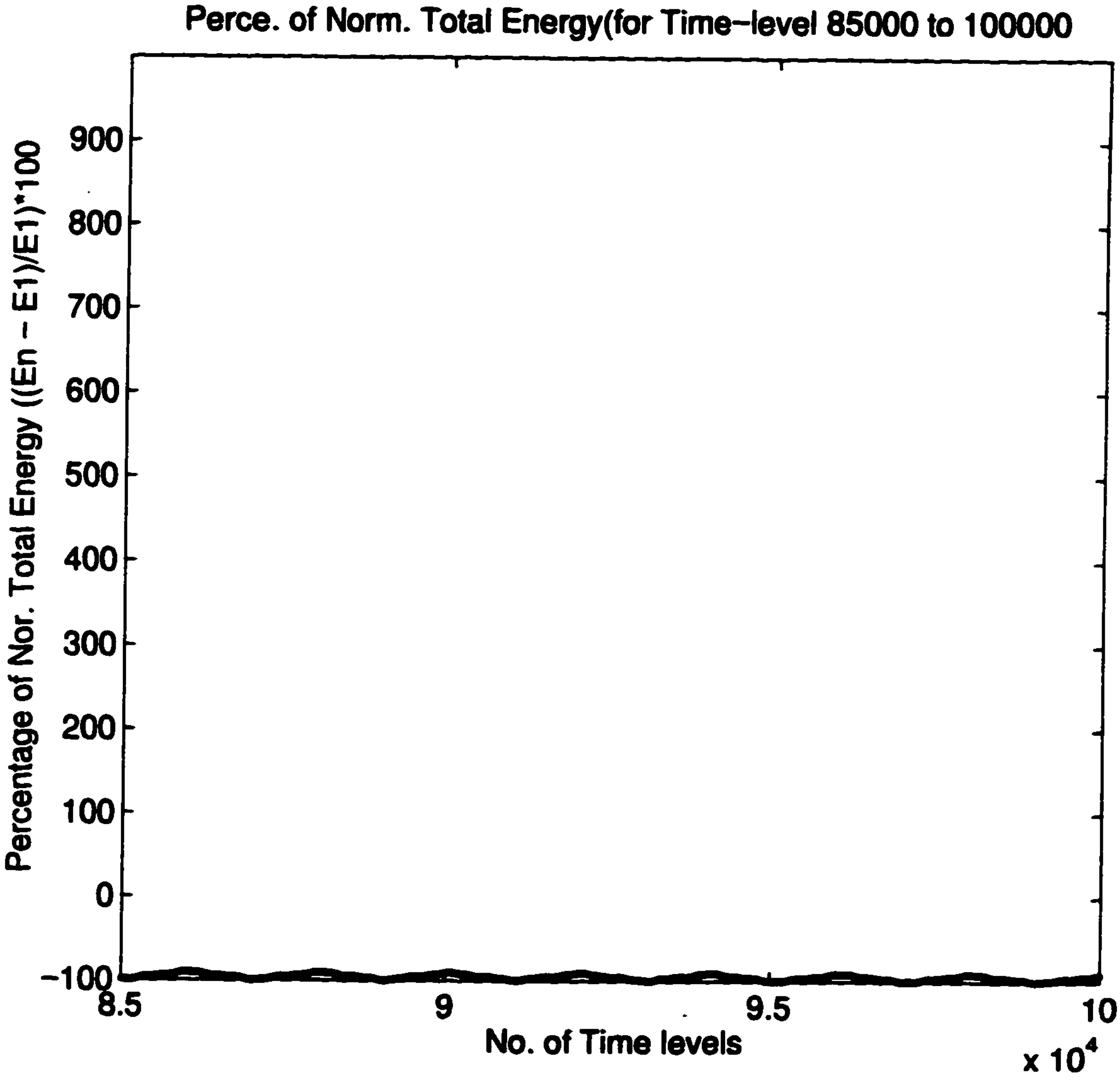


Fig. 6.41 Percentage of Normalized Total Energy from 85000 to 100000 Time-steps for Low Pressure Case

6-5 Large Amplitude Disturbance by High Pressure

In this test case a high intensity instantaneous pressure applied on five grid points only at the first time step. This high pressure initiates a large amplitude wave. The plots of Fig 6.13 to Fig.6.21 shows the position of the wave at different time levels, the initial stage before reflection, after reflection and a comparison of two waves at varies time levels. The stretch vs. grid poison are plotted next to each plot of the string position. From the plots of the position of the string it can be seen that the string motion is nonlinear. When the stretch value goes well bellow one the shape of the string corresponds to this low stretch value is different for other regions.But the value of the stretch below one is observed on a short length of the string for short period of time.

From the plot it can be seen that the numerical method preserved the symmetry of the problem for all the tested length of time.The plot for the residual of the fictitious first order partial differential equation in the Runge-Kutta method vs. the time steps required to reach this equation to steady state are shown on Fig. 6.22 and Fig.6.23.The percentage of the normalized energy vs. time step is plotted from Fig. 6.10 to Fig. 6.12 to show the stability of the scheme and the numerical does not violet the conservation of the total energy. For clarity, the plots for positions of the string are done at an interval of three data points.

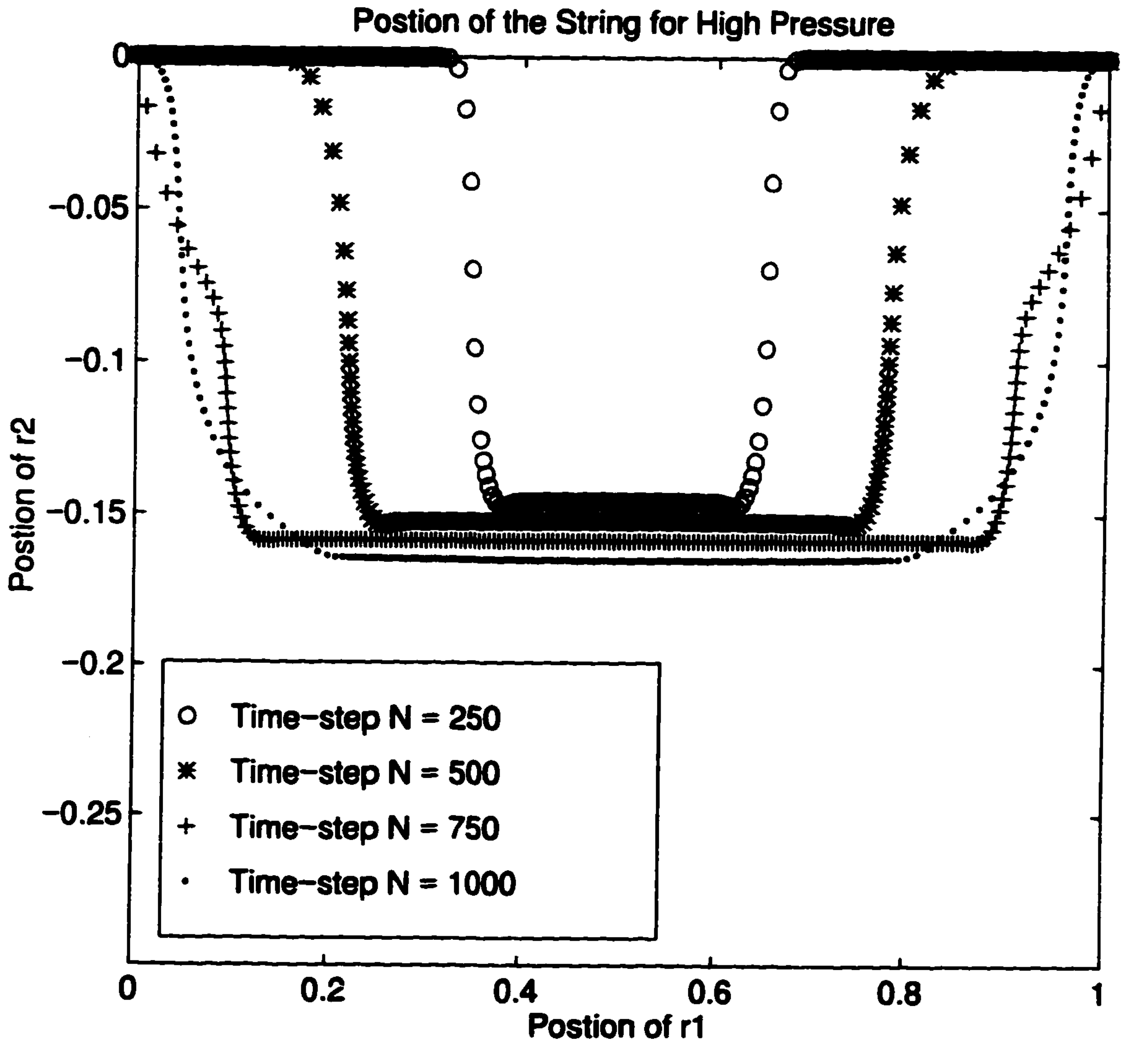


Fig. 6.42 Positions of the String for First 1000 Time-steps for High Pressure Case

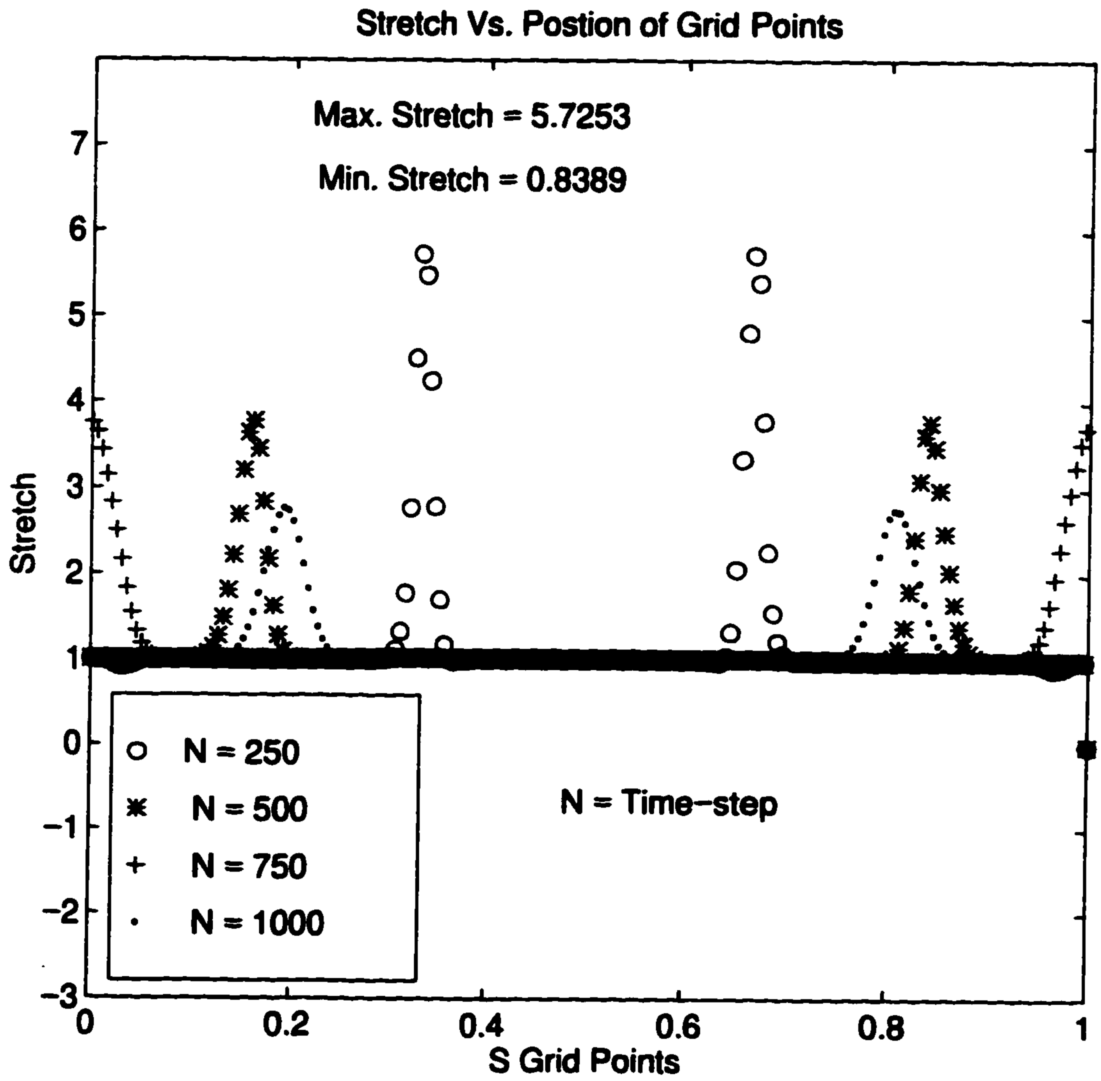


Fig. 6.43 Stretch Value for First 1000 Time-steps for High Pressure Case

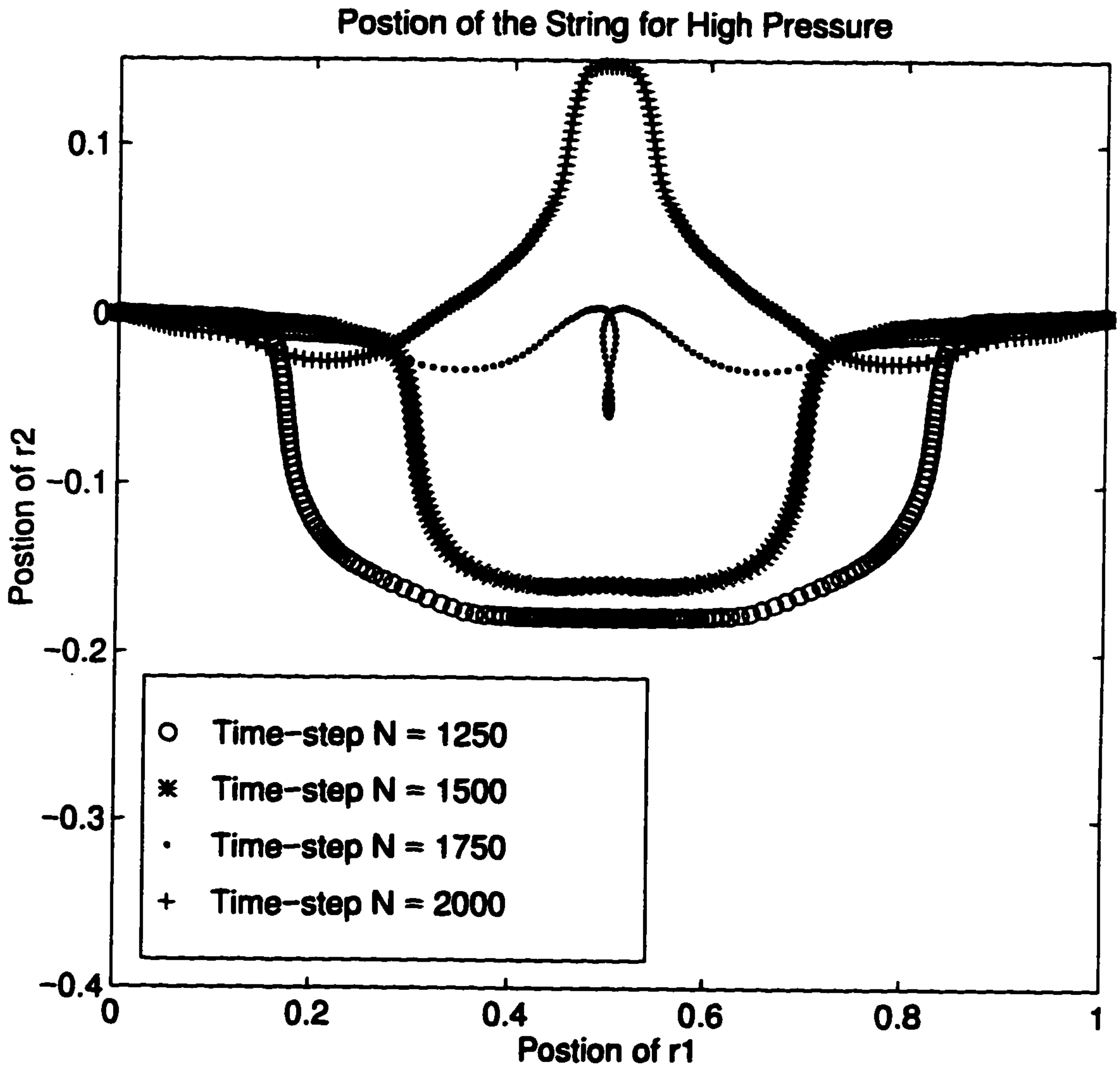


Fig. 6.44 Positions of the String form 1250 to 2000 Time-steps for High Pressure Case

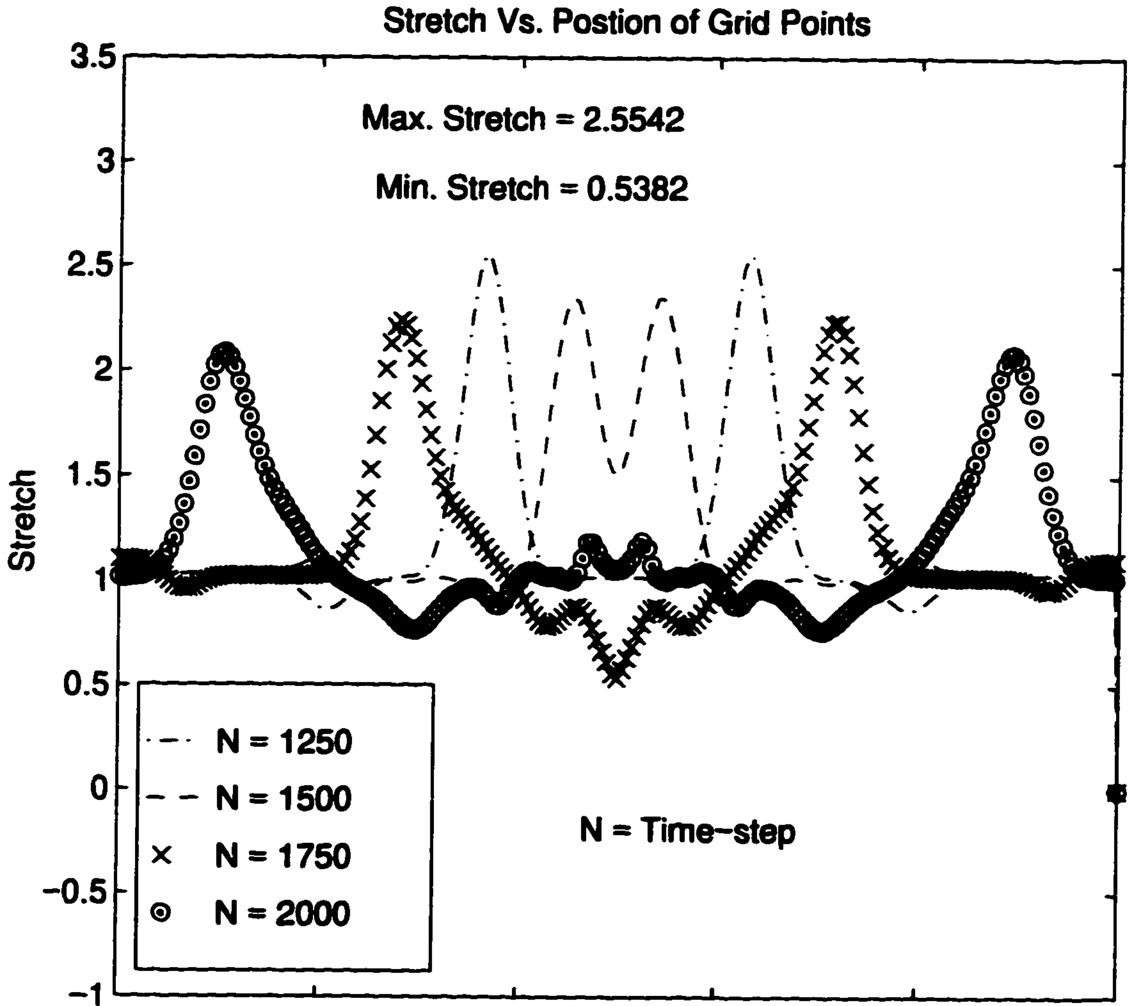


Fig. 6.45 Stretch Value form 1250 to 2000 Time-steps for High Pressure Case

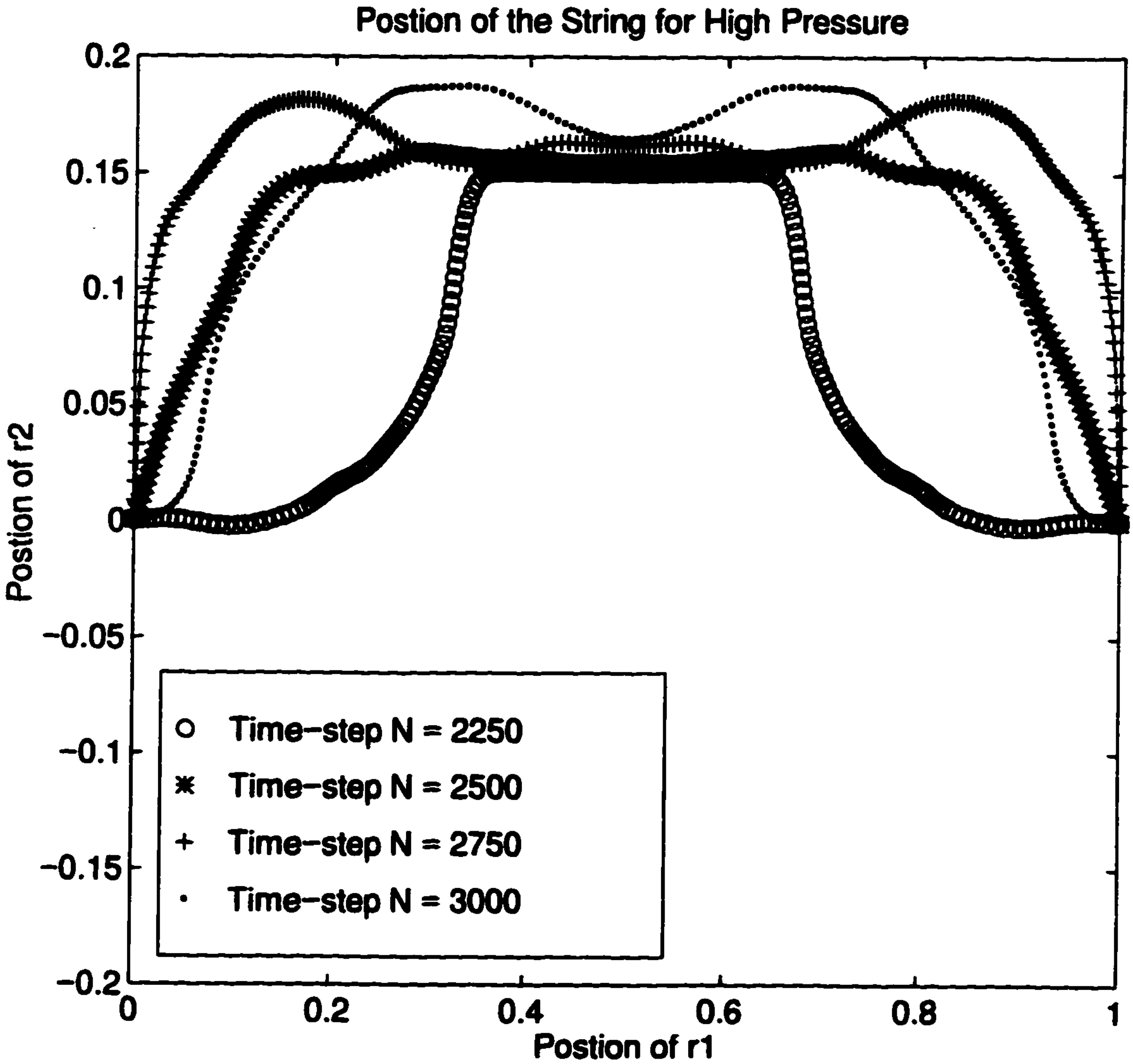


Fig. 6.46 Positions of the String form 2250 to 3000 Time-steps for High Pressure Case

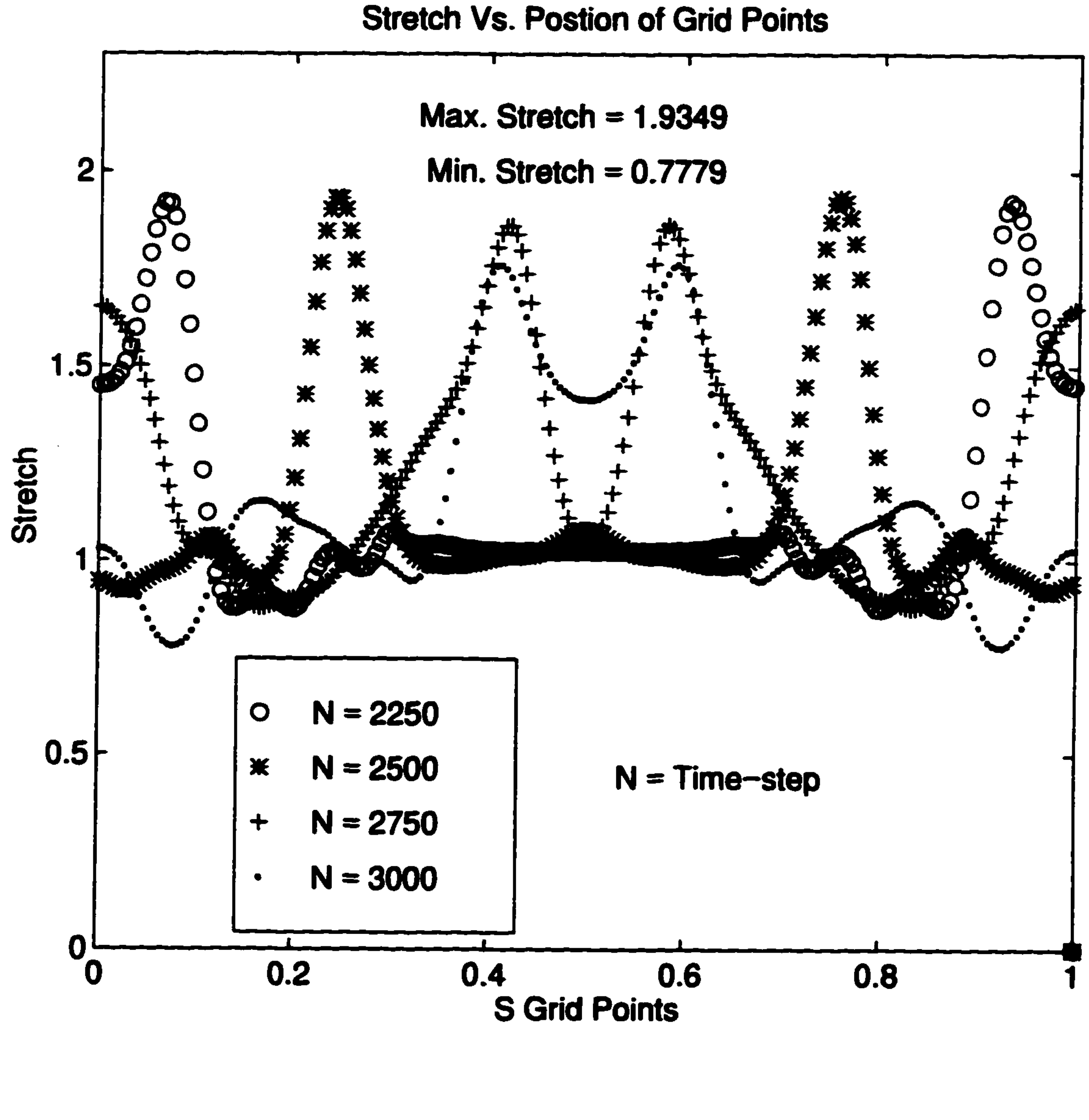


Fig. 6.47 Stretch Value form 2250 to 3000 Time-steps for High Pressure Case

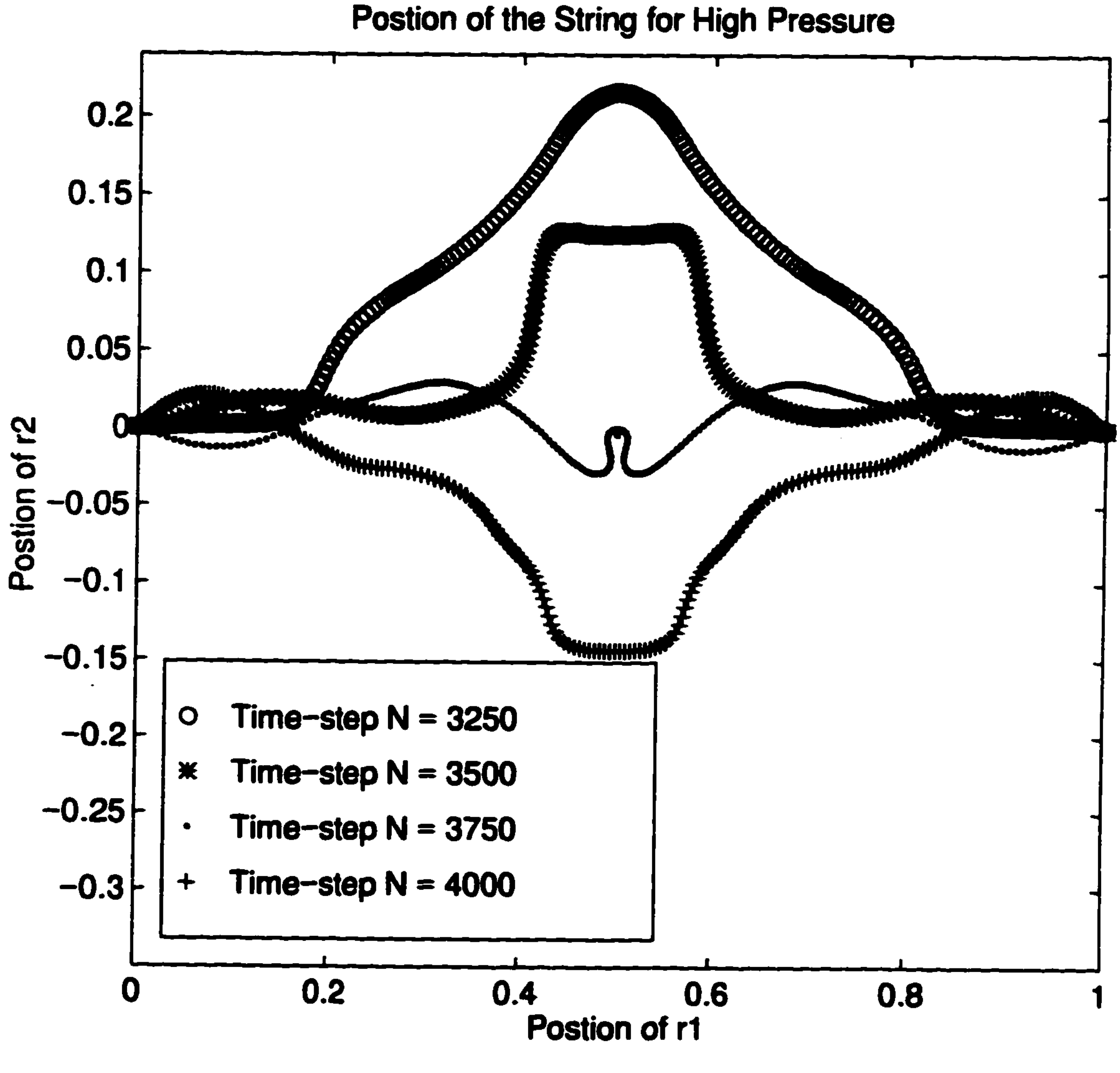


Fig. 6.48 Positions of the String form 3250 to 4000 Time-steps for High Pressure Case

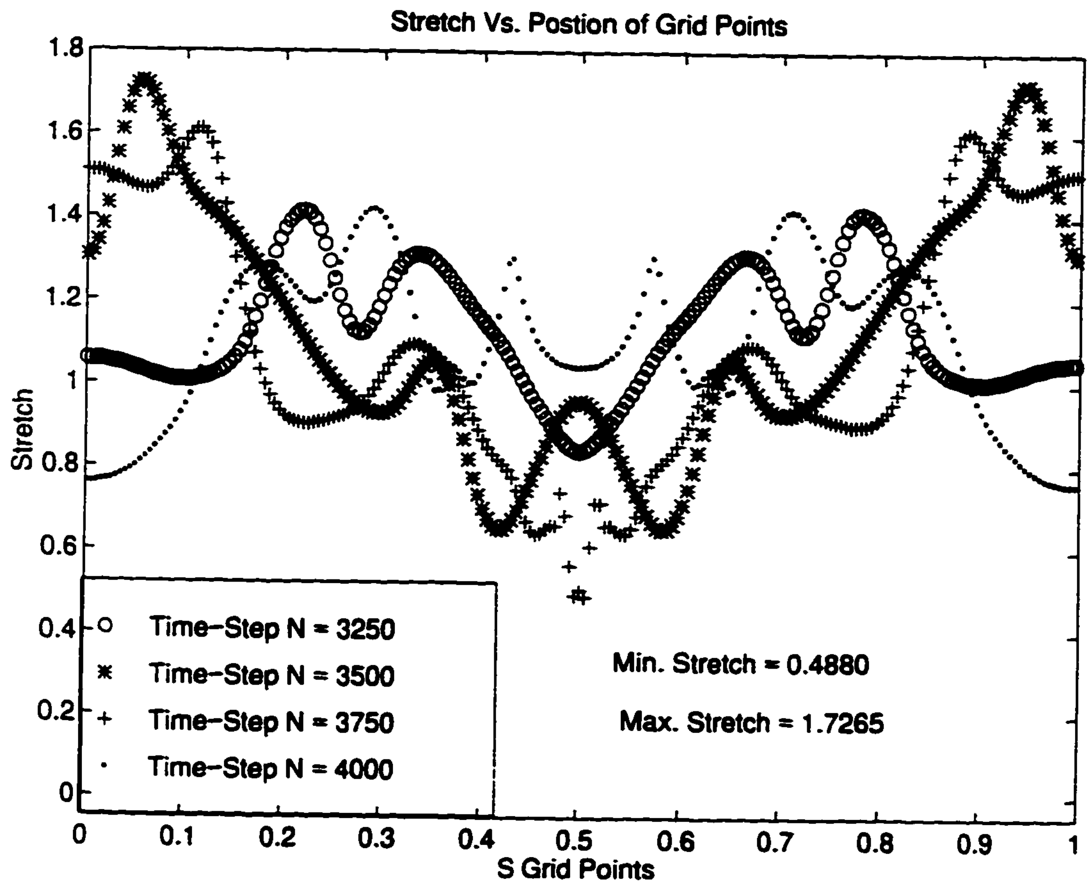


Fig. 6.49 Stretch Value form 3250 to 4000 Time-steps for High Pressure Case

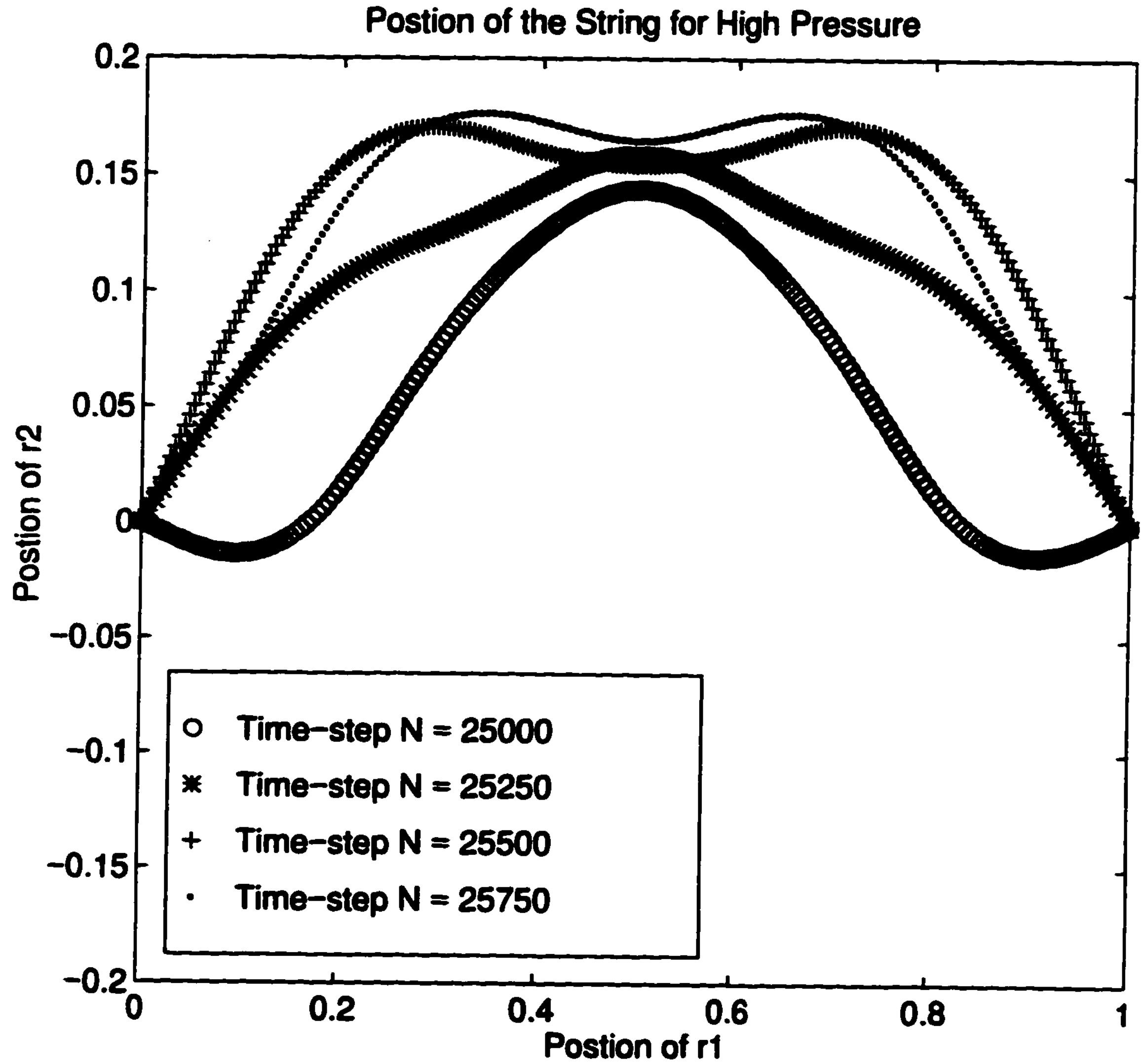


Fig. 6.50 Positions of the String form 25000 to 25750 Time-steps for High Pressure Case

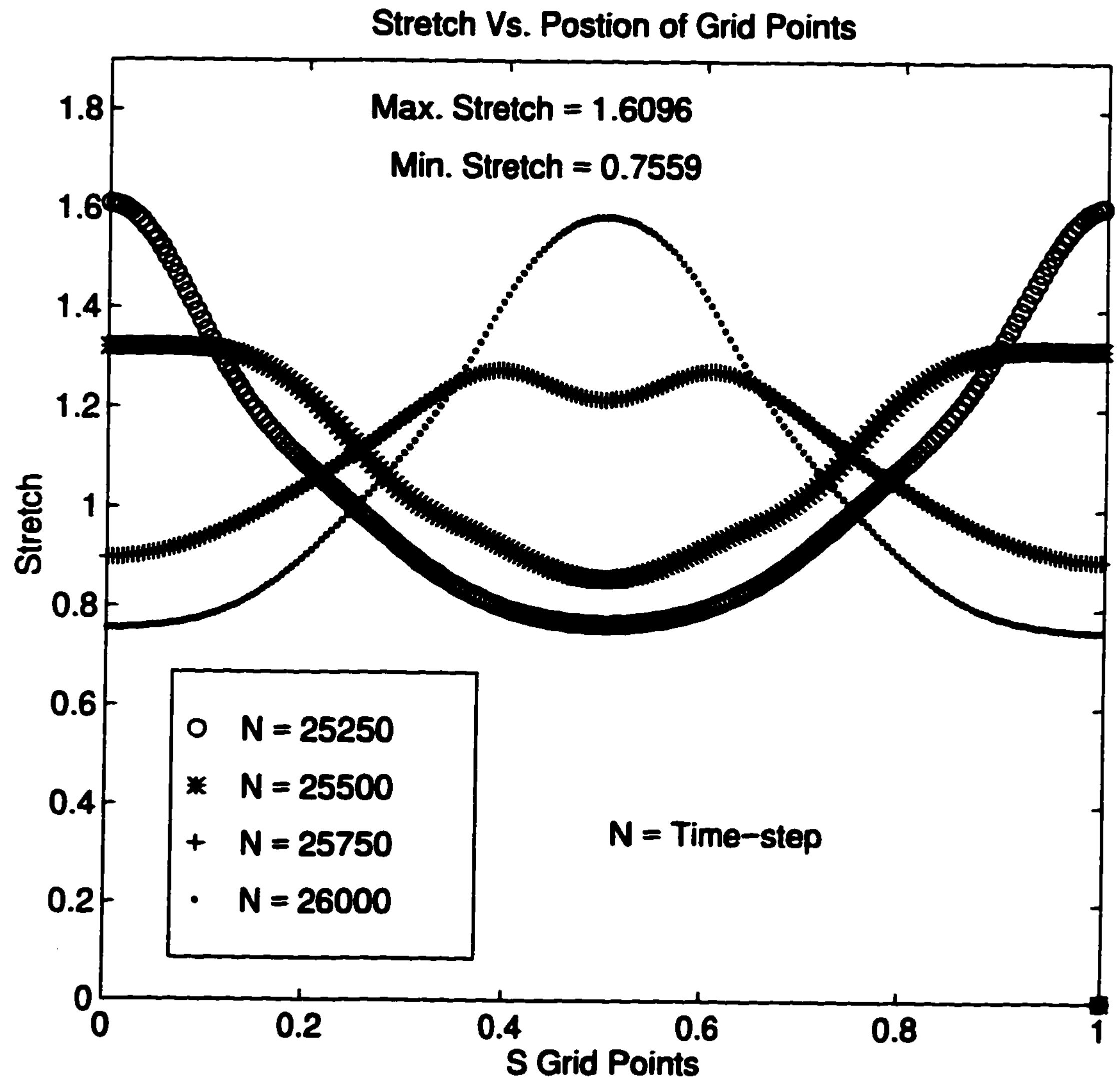


Fig. 6.51 Stretch Value form 25250 to 26000 Time-steps for High Pressure Case

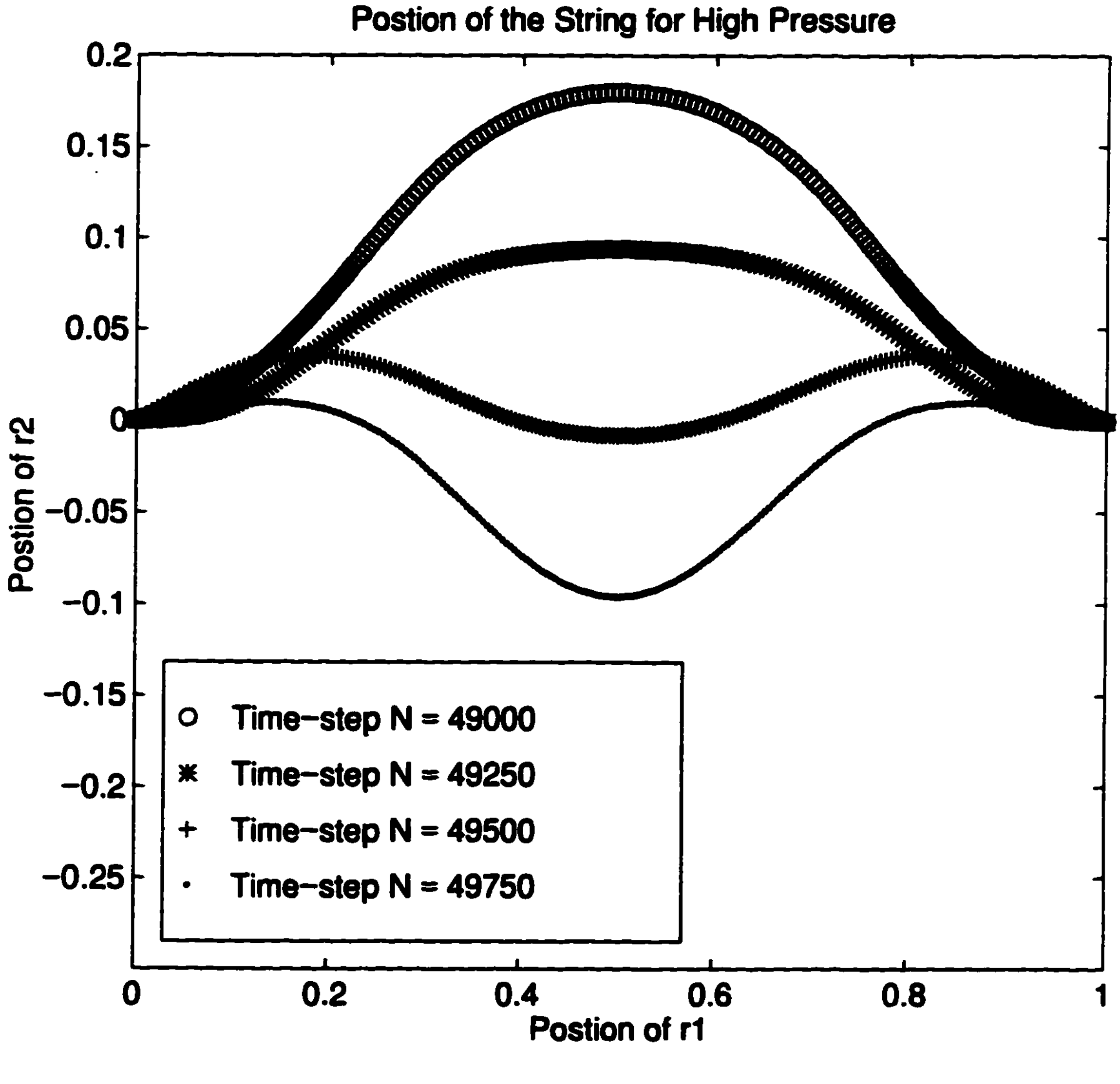


Fig. 6.52 Positions of the String form 49000 to 49750 Time-steps for High Pressure Case

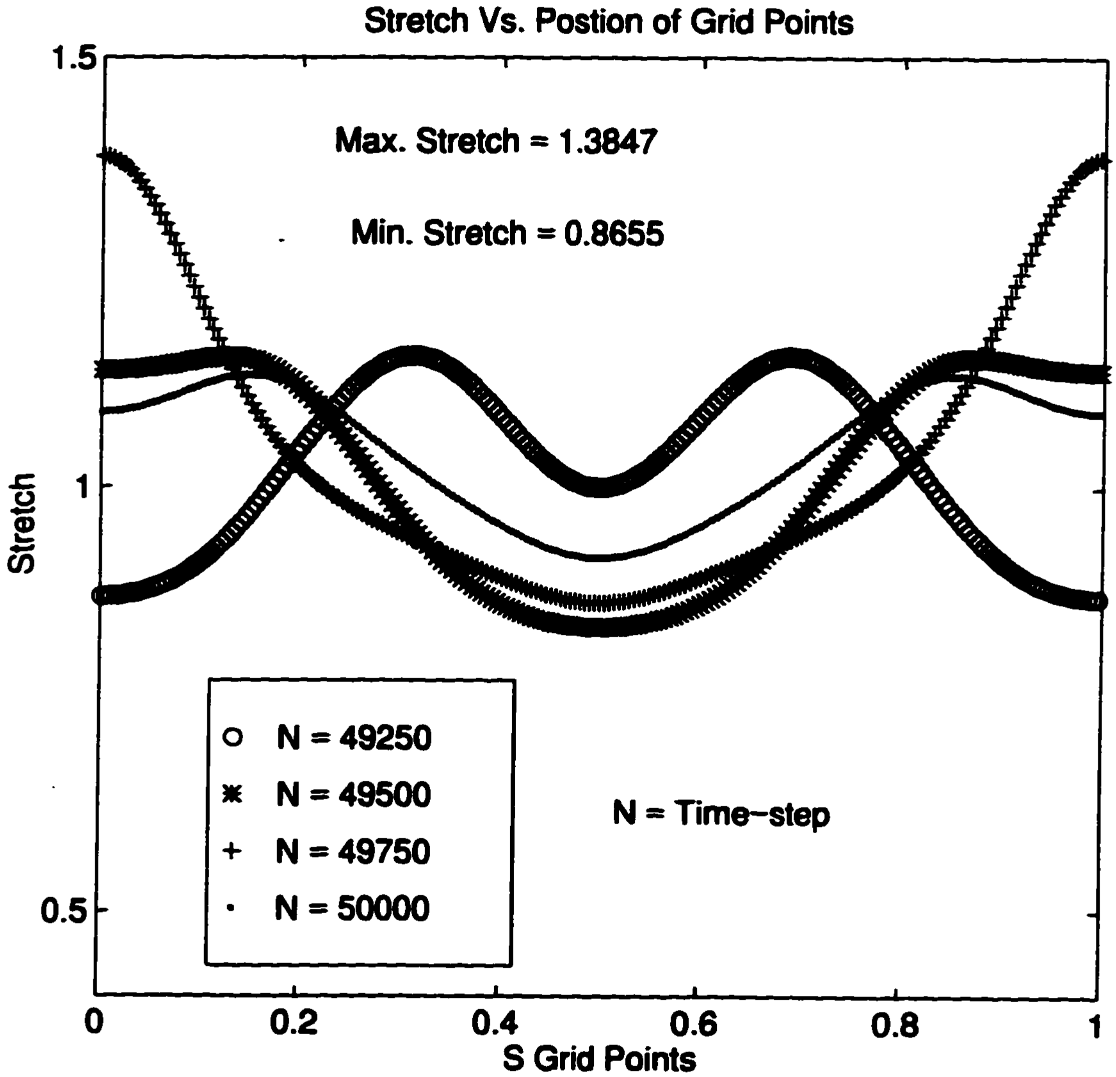


Fig. 6.53 Stretch Value form 49250 to 50000 Time-steps for High Pressure Case

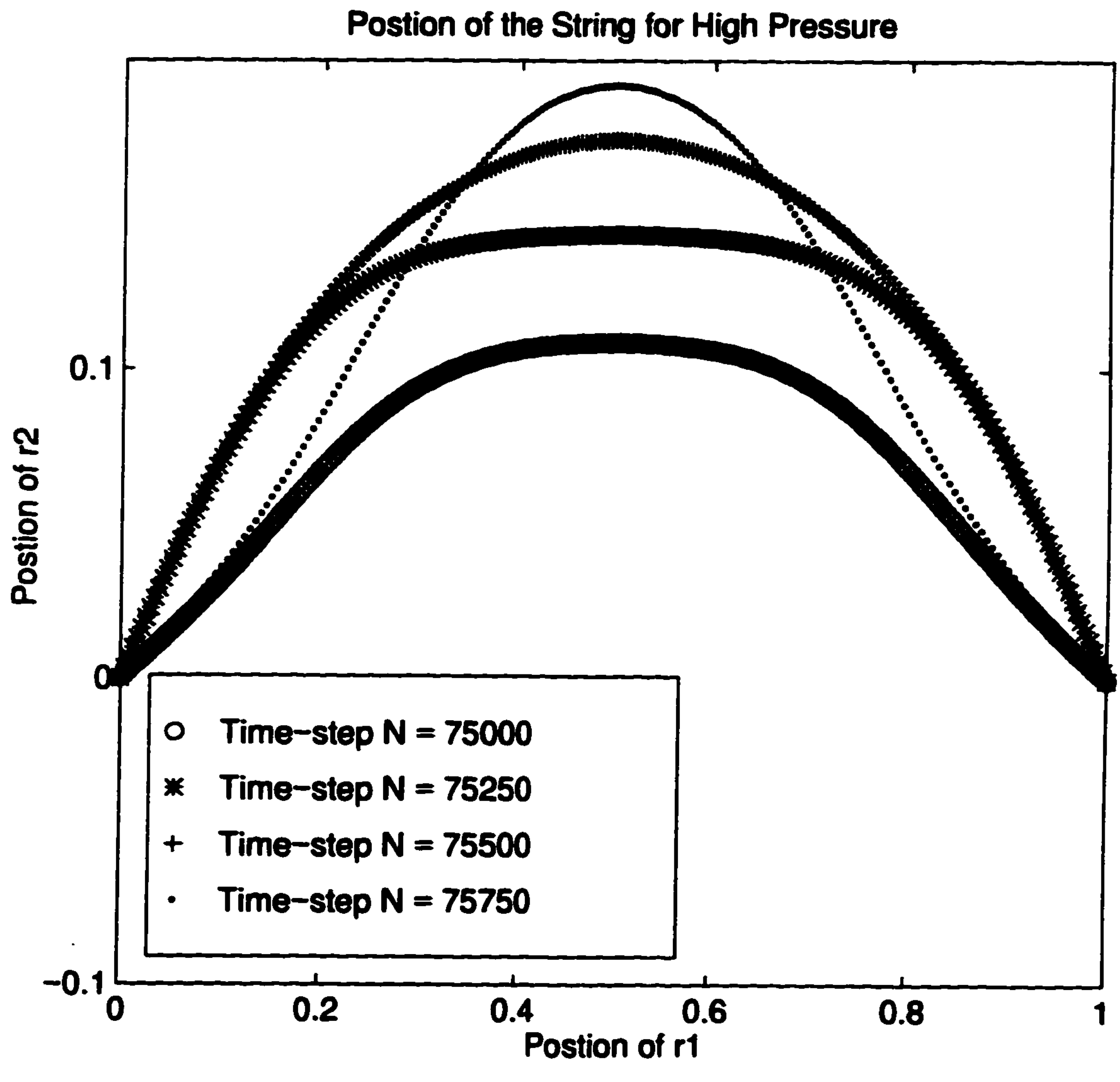


Fig. 6.54 Positions of the String form 75000 to 75750 Time-steps for High Pressure Case

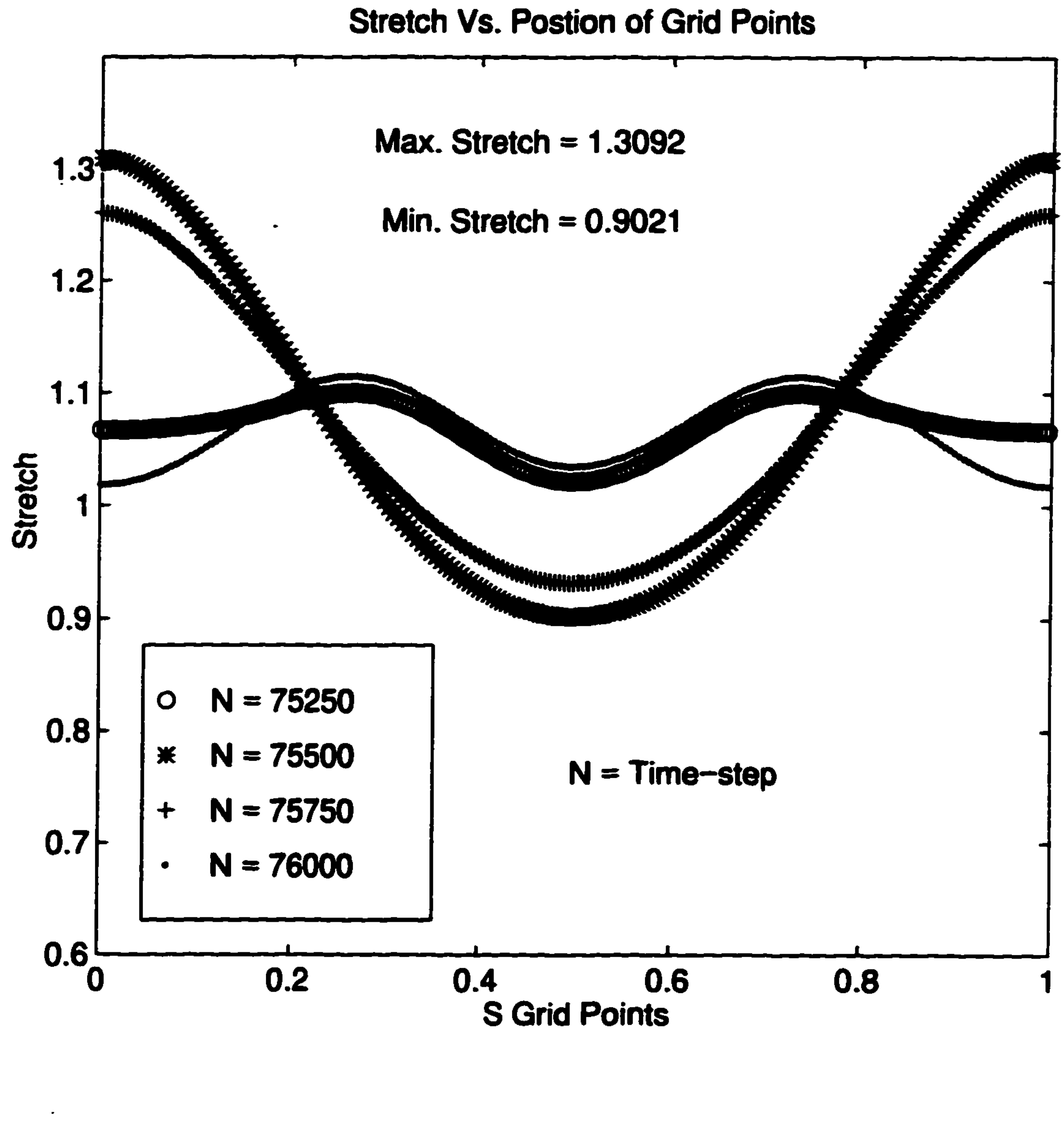


Fig. 6.55 Stretch Value form 75250 to 76000 Time-steps for High Pressure Case

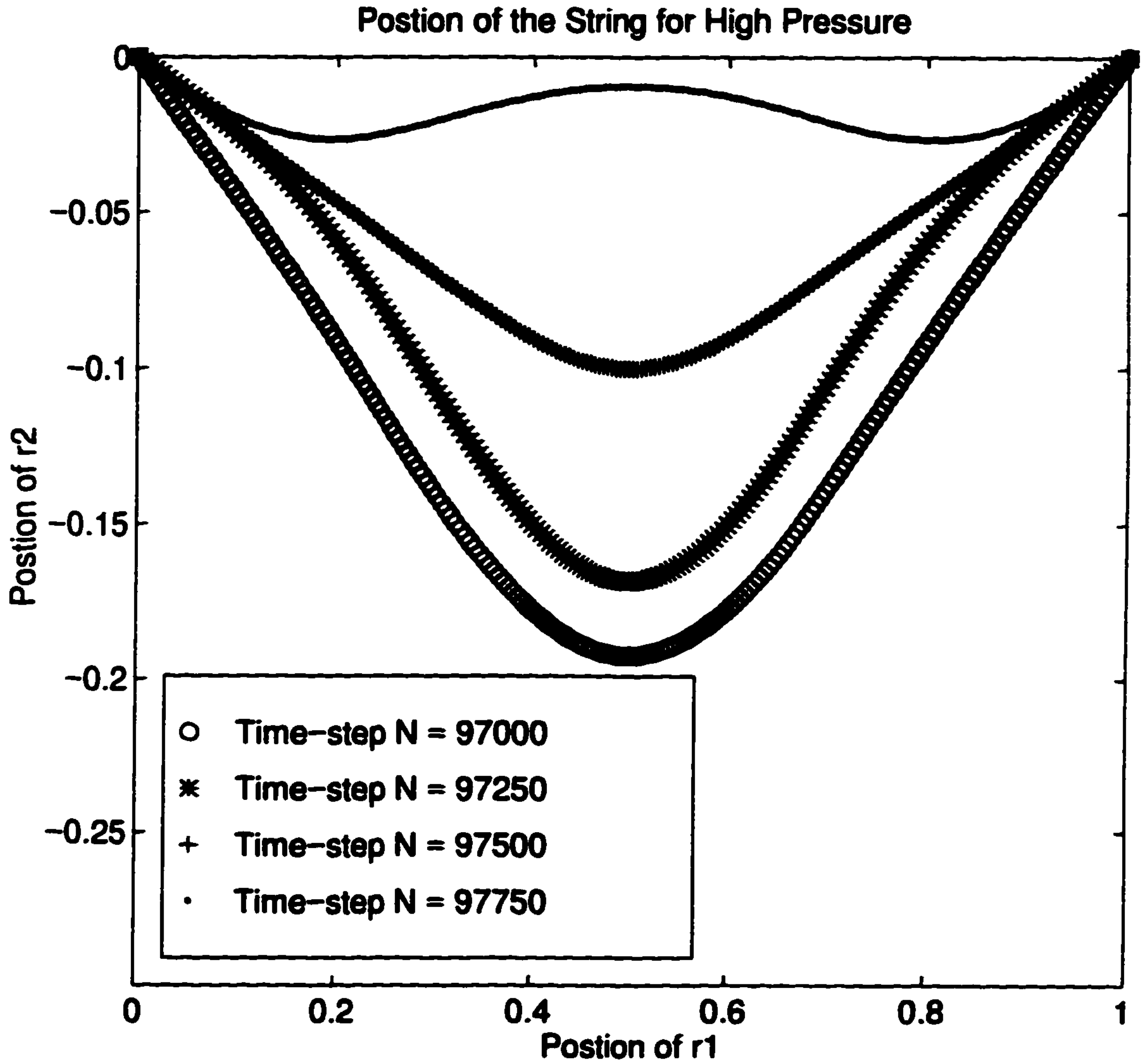


Fig. 6.56 Positions of the String form 97000 to 97750 Time-steps for High Pressure Case

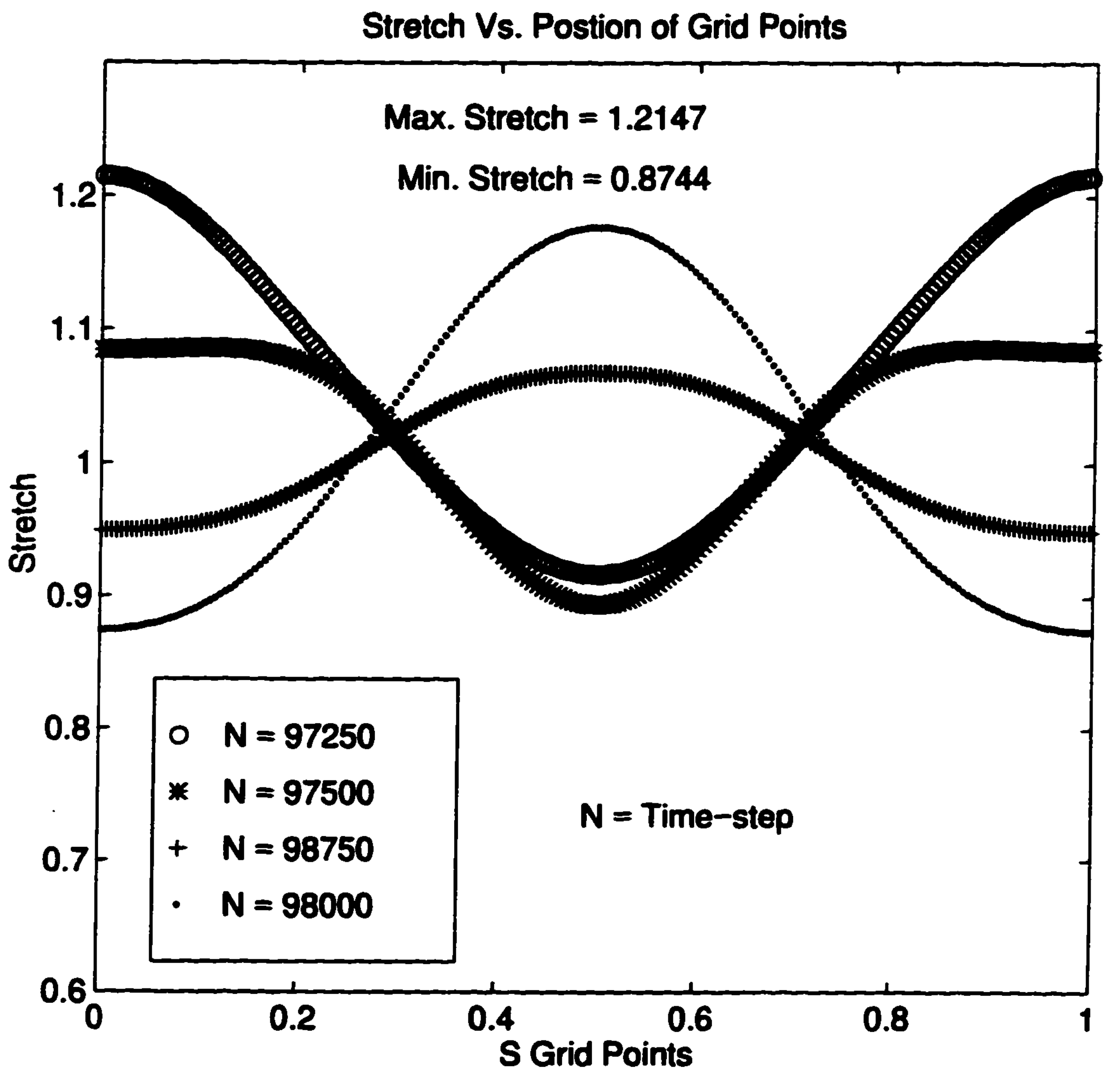


Fig. 6.57 Stretch Value form 97250 to 98000 Time-steps for High Pressure Case

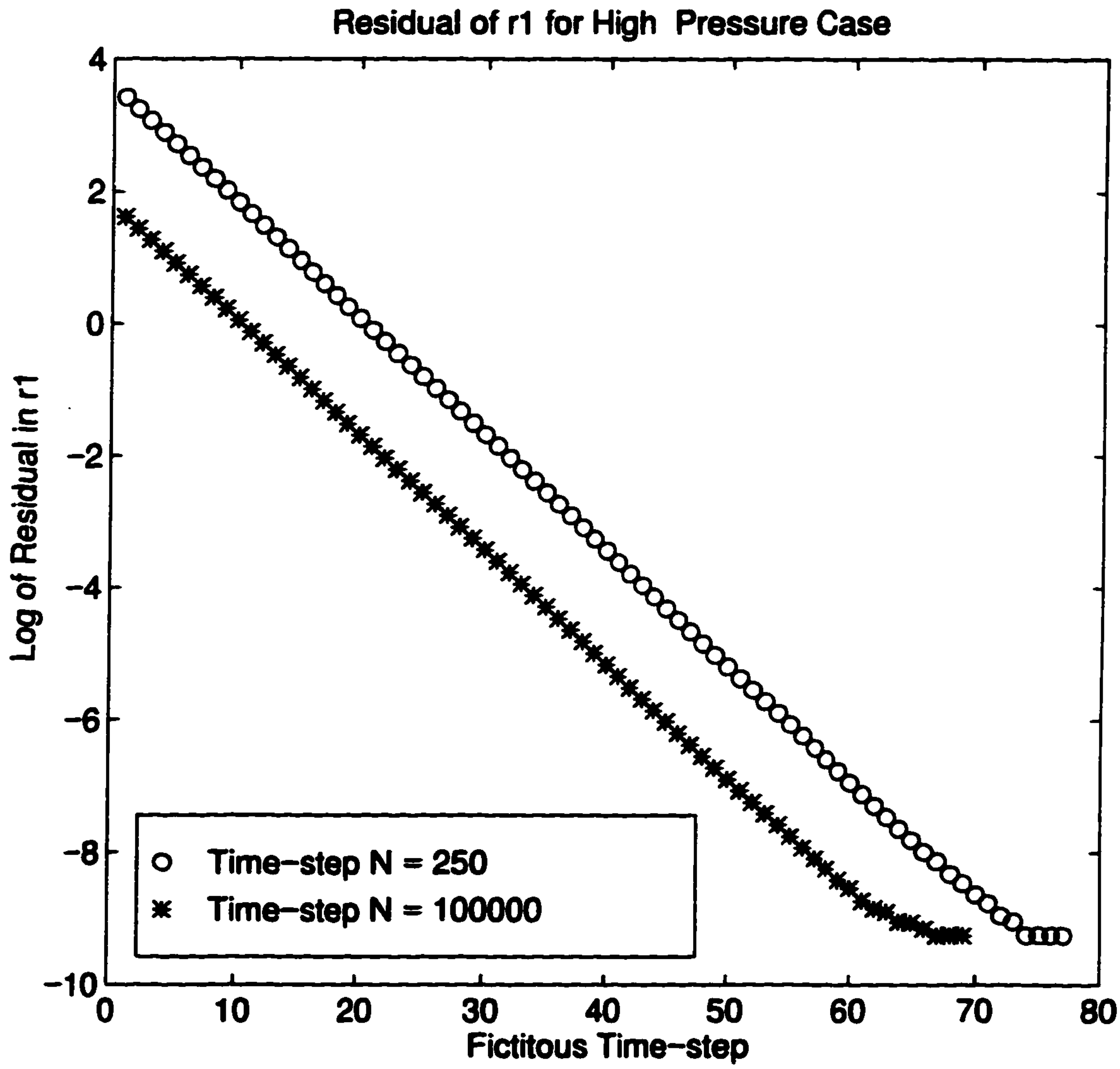


Fig. 6.58 Residual of $\mathbf{r}_1$ for the High Pressure Case

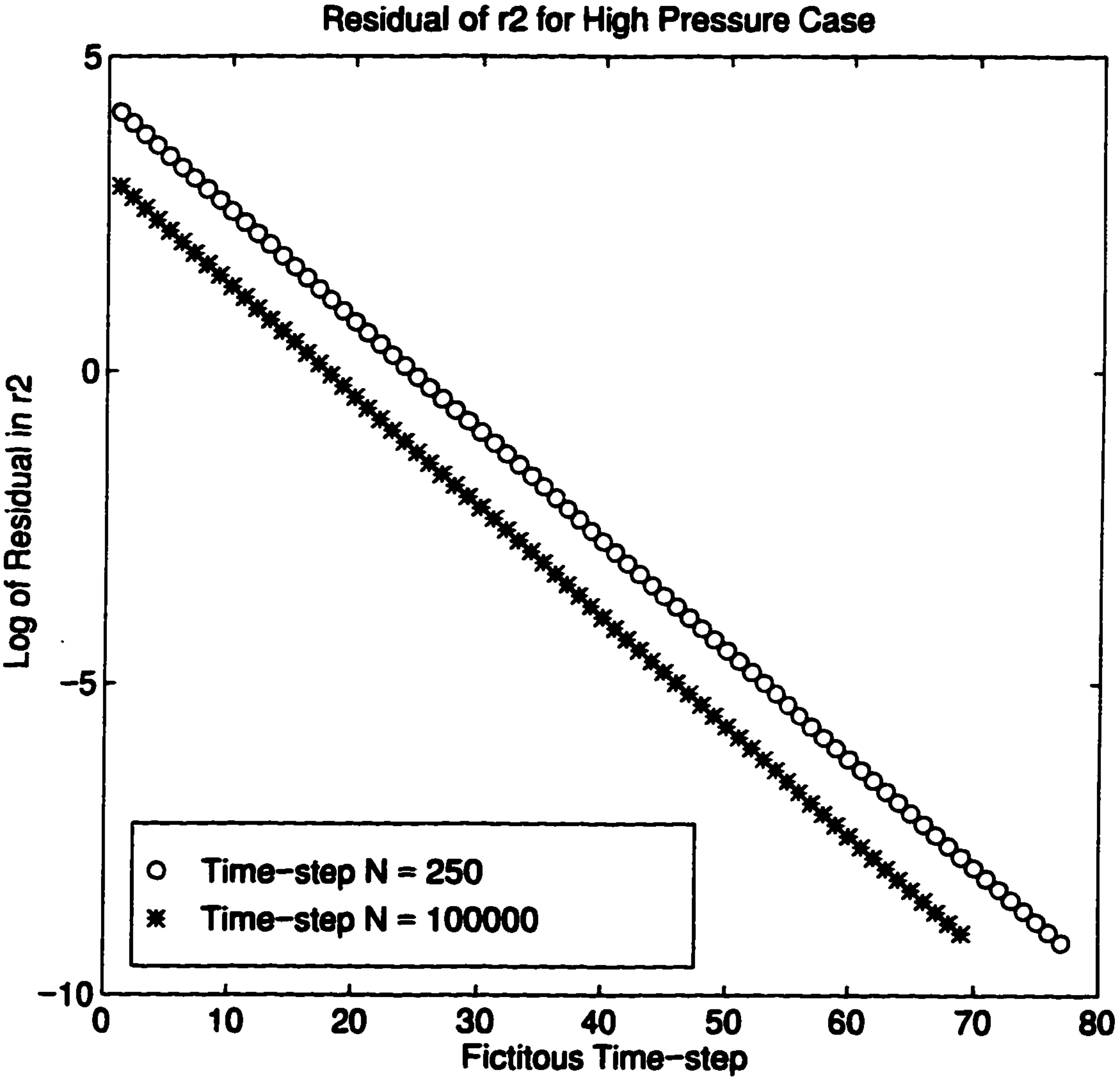


Fig. 6.59 Residual of $\mathbf{r}_2$ for the High Pressure Case

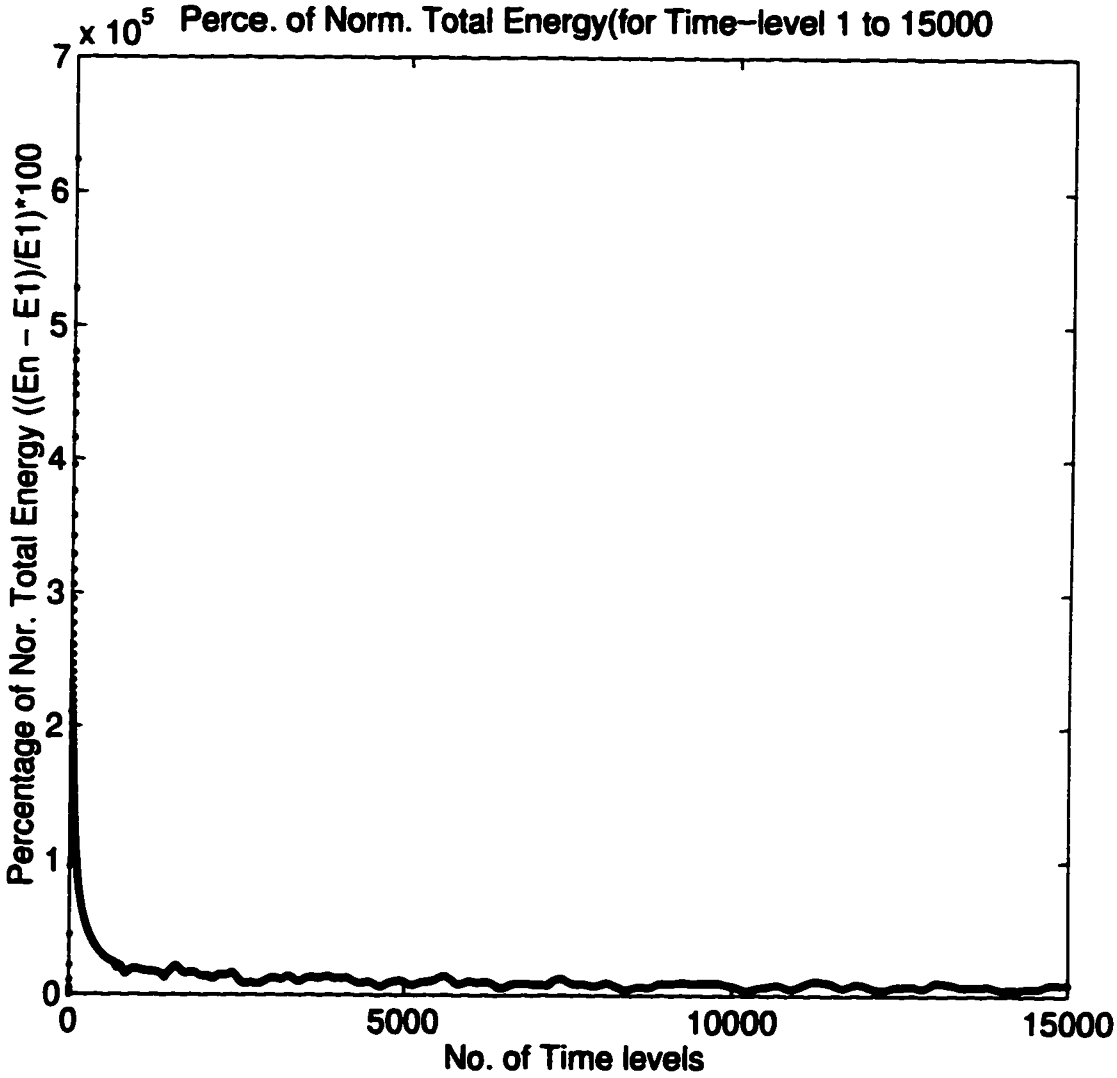


Fig. 6.60 Percentage of Normalized Total Energy in the First 15000 Time-steps for High Pressure Case

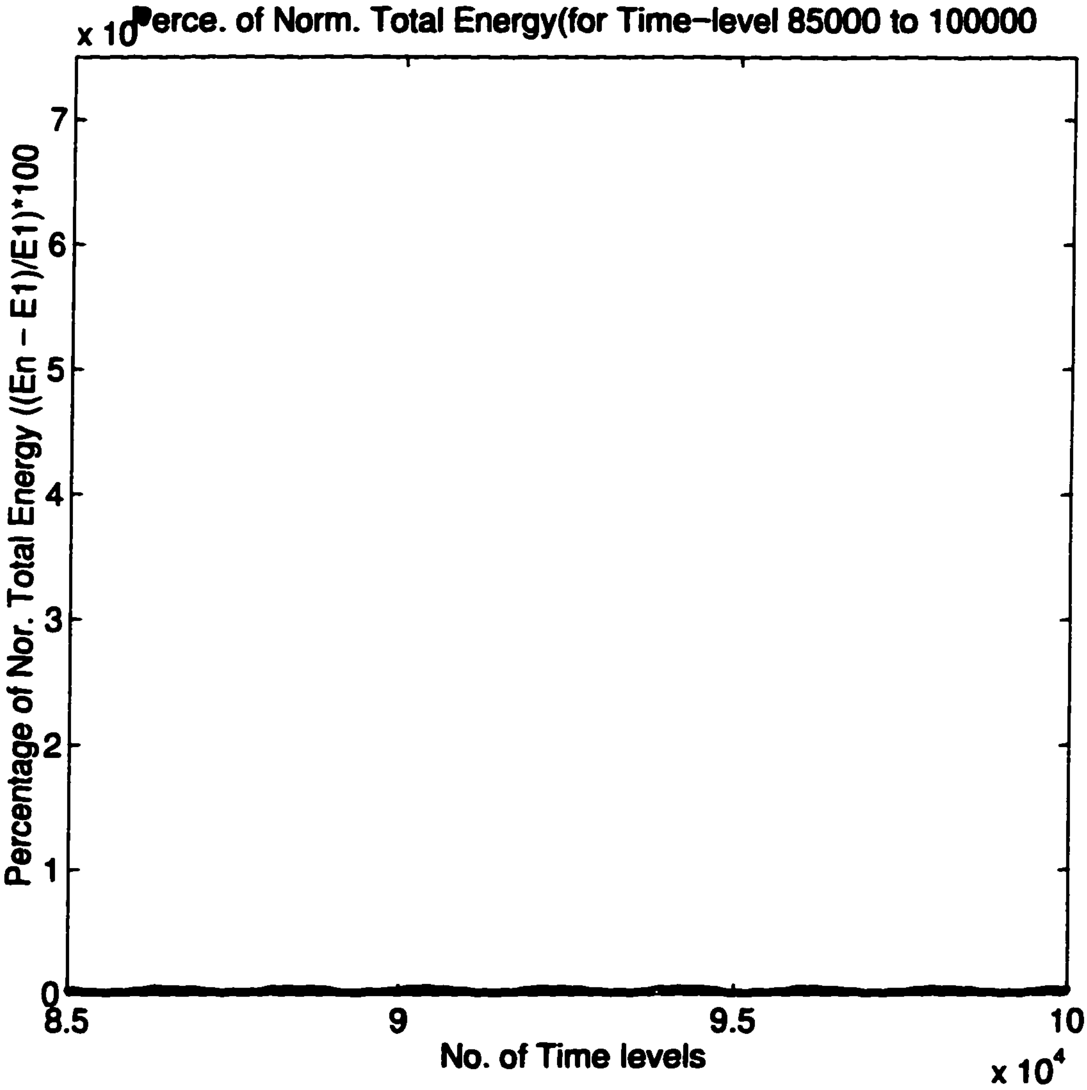


Fig. 6.61 Percentage of Normalized Total Energy from 85000 to 100000 Time-steps for High Pressure Case

CHAPTER SEVEN

CONCLUSION

The numerical method developed in this thesis is based on the governing equation of motion for an elastic string. The partial differential equation of string motion is constructed and discretized in conservative form. This makes the numerical scheme to obtain the solution even when there is a discontinuity in the first order term of the partial differential equation.

A linear stability analysis is used as a guide in the choice of the type of finite difference technique, the grid size and the time steps to implement in order to produce a stable numerical solution.The form of discretization found from the stability analysis is a central finite differencing in time and space, which approximate the flux at two time levels with implicitness control parameter.

The method of solution developed to solve the implicit discretized form of the governing equation of the string motion is by defining a fictitious first order partial differential equation, in such a way that the steady state solution of this first order equation is equivalent to one time step advancement of the solution of the string equation of motion.

To march the defined first order partial differential equation to steady state, a modified, explicit, four stage Runge-Kutta method is used. Two of the stage coefficients of Runge-Kutta method are specified to get a second order accuracy and the rest two are determined from stability analysis.

Five test cases are conducted on the developed method of solution, to investigate its validity, accuracy, response to small and large amplitude waves.Also an investigation is conducted on the conservation of the total energy of the system and for small amplitude waves the maintenance of the propagation speed for number of reflections. For small amplitude waves the speed of propagation for 100,000 time steps equivalent to fifty reflections was constant as the pre determined calculated value.

In the first test case comparison of analytical and numerical solution is done. This comparison done using a constitutive relation of Mooney-Rivlin. The numerical solution showed in the impact angle an average error of 1.65% at the point of discontinuity of θ. The result from the comparison gives as a confidence on the validity and the accuracy of the scheme.

The conservative equation of string motion which requires a constant total energy is used to check the obtained numerical solution for the maintenance of the conservation of the total energy. From the conservation test the total energy remain constant for 100,000 time steps equivalent to fifty reflections after experiencing a big jump initially since the initial condition is not the solution of the equation of the string motion.

The method of solution developed is not based on the characteristic equation of an elastic string. Thus it can be extended to solve fluid structural problems.

BIBLIOGRAPHY

[1] C.Truesdell, The Rational Mechanics of Flexible or Elastic Bodies, 1638 -1788, in L.Euleri Opera Ominia II.

[2] J.B.Keller, Large Amplitude Motion of a String, Amer. Jour. of Phys. 1959 27, 584 - 586.

[3] J.Von Neumann, Proposal and Analysis of a New Numerical Method for the Treatment of Hydrodynamical Shock Problems, 1944, Collected Works, Vol VI, 361 - 379.

[4] S.S. Antman, Non-linear Problems of Elasticity, 1994, New York, Springer-Verlag.

[5] C.Hirsch, Numerical Computation of Internal and External Flows, 1988, John Wiley & Sons, Vol. I.

[6] R.W.Dickey, A Quasilinear Evolution Equation and The Method of Galerkin, 1973, Proc. Amer. Math. Soc. 37, 149 -156.

[7] J.Wegner,J.B.Haddow & R.J.Tait, Finite Amplitude Wave Propagation in a Stretched Elastic String, 1989, Elastic Wave Propagation, North - Holland.

[8] K.Abdella, Application of Goundnov's Scheme to The Propagation of waves in non-linear elastic string, 1989, Msc. Thesis, U of A.

[9] J.L.Zhong, Impact Problems for Non-linear Elastic String and Membranes with Variable Boundary Conditions, 1994, Ph.D. Thesis, U of A.

[10] R.J.Tait, K.Abdella & D.B.Duncan, Approximate Riemann Solvers and Waves on a Non-linear Elastic String, 1991, Computers Math. Appl. Vol. 21, No.4, 77 - 89.

[11] R.D.Richtmyer & K.W.Morton, Difference Methods for initial-Value Problems, 1967, John Wiley & Sons. Inc.

[12] A.Jameson, Time Dependant Calculations Using Multigrid, With Applications to Unsteady Flows Past Airfoils and Wings, 1991, AIAA 10th Comp. Fluid Dyn.conf.

[13] P.J.Chen, Selected Topics in Wave Propagation, 1976, Noordhoff international publishing Leyden.

[14] R.Courant & K.O.Friedrichs, Supersonic Flow and Shock Waves, 1948, New York, Intersince Publishers, Inc.

[15] T.C.Truesdell, The Non-linear Field Theories of Mechanics, Encyclopedia of Physics, Vol. 3,1965, New York, Springer-Verlag.

[16] P.Lax & B.Wendroff, Systems of Conservation Laws, 1960, Comm. on Pure & App. Math. Vol. XIII, 217 - 237.

[17] L.M.Brekhovskh & V.Gonchorov, Mechanics of Continua and wave Dynamics, 1994, New York, Springer-Verlag.

[18] M.F.Beatty & J.B.Haddow, Transverse impact of a Hyperelastic stretched string, 1985, Tran. ASME Jour. of App. Mechanics Vol 32, 137 -143

[19] J.D.Achenbach, Wave Propagation in Elastic Solids, 1973, North-Holland Publishing

[20] K.Washizu, Variational Methods in Elasticity & Plasticity, 1982, Pergamon Press

[21] A.A.Atai & D.J.Steigamann, On the Nonlinear Mechanics of Discrete Networks, 1996 to be published

[22] G.A.Sod, Numerical Methods in Fluid Dynamics, 1985, Cambridge University Press.

[23] P.Radinowitz, Numerical Methods for Nonlinear Algebraic Equations, 1970, Gordon and Breach Science Publishers.

[24] W.A.Smith, Multigrid solution of the Euler Equations, 1987, Cornell University, Ph.D. Thesis.

[25] J.L.Wegener,Some Problems of Hyperbolic Wave Propagation 1988, Ph.D. Thesis, U of A