UNIVERSITY OF ALBERTA

UNIQUENESS THEOREMS OF POSITIVE RADIAL SOLUTIONS FOR QUASILINEAR ELLIPTIC EQUATIONS

BY

MOXUN TANG ©

A thesis submitted to the Faculty of Graduate Studies and Research in partial fulfillment of the requirements for the degree of DOCTOR OF PHILOSOPHY.

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DEDICATED

TO MY FATHER

AND MY WIFE QUAN

Abstract

We establish the uniqueness of positive radial solutions to the quasilinear elliptic equation $\operatorname{div}(|\nabla u|^{m-2}\nabla u) + f(|x|, u) = 0$ in a ball or an annulus Ω in $\mathbb{R}^n$, $n \geq 3$, with the Dirichlet boundary condition on $\partial\Omega$, where $1 < m \leq n$, ∇u denotes the gradient of u, and f is radially symmetric in x and satisfies $f(|x|, 0) = 0$. We also establish the uniqueness of ground state solutions to the equation in $\mathbb{R}^n$. Our results applied to a wide class of nonlinearities, including some important model cases such as $f(|x|, u) = \mu u^p + u^q$, $u^p - u^q$, $m - 1 \leq p < q$, and $K(|x|)\gamma(u)$. In particular, we prove that there exists a unique positive solution to the semilinear Dirichlet problem $\Delta u + \mu u^p + u^q = 0$ in B with $u = 0$ on ∂B, where B is the unit ball in $\mathbb{R}^n$, $\mu > 0$ and $1 \leq p < q \leq (n+2)/(n-2)$, provided that $n \geq 6$. This result partially solves an open problem raised by Brezis and Nirenberg in 1983. It is very interesting in view of the well-known non-uniqueness proof in the case $n = 3$ by Atkinson and Peletier in 1986. The basic approach used in our proof is to make extensive use of a new Pohozaev-type identity which we develop in this thesis.

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Chapter 1
INTRODUCTION

Let Ω be a ball or an annulus in $\mathbb{R}^n$, $n \geq 3$. We are concerned with the problem of uniqueness of radial solutions of the quasilinear elliptic equation

$$\begin{aligned} &\text{div } (|\nabla u|^{m-2}\nabla u) + f(|x|, u) = 0 \quad \text{in} \quad \Omega, \\ &u > 0 \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{on} \quad \partial\Omega, \end{aligned} \tag{1.1_m}$$

where $1 < m \leq n$, ∇u denotes the gradient of u, and f is radially symmetric in x and satisfies $f(|x|, 0) = 0$. We shall also study the question of uniqueness of ground state solutions of the nonlinear problem

$$\begin{aligned} &\text{div } (|\nabla u|^{m-2}\nabla u) + f(|x|, u) = 0 \quad \text{in} \quad \mathbb{R}^n, \\ &u > 0 \quad \text{in} \quad \mathbb{R}^n, \quad u \to 0 \quad \text{as} \quad |x| \to \infty. \end{aligned} \tag{1.2_m}$$

A ground state solution is, roughly speaking, a solution of (1.2_m) vanishing at infinity and behaving like $|x|^{-\frac{n-m}{m-1}}$ for large $|x|$. It is also called a fast decaying solution (see the definition below).

When $m = 2$, the equation of (2.1_m) reduces to the particular interesting semilinear elliptic equation $\Delta u + f(|x|, u) = 0$, where Δ denotes the n-dimensional Laplacian. Now problems $(1.1_m) - (1.2_m)$ take the form

$$\begin{aligned} &\Delta u + f(|x|, u) = 0 \quad \text{in} \quad \Omega, \\ &u > 0 \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{on} \quad \partial\Omega, \end{aligned} \tag{1.1}$$

and

$$\begin{aligned} &\Delta u + f(|x|, u) = 0 \quad \text{in} \quad \mathbb{R}^n, \\ &u > 0 \quad \text{in} \quad \mathbb{R}^n, \quad u \to 0 \quad \text{as} \quad |x| \to \infty. \end{aligned} \tag{1.2}$$

The existence and uniqueness of nontrivial solutions to these problems, especially problems (1.1) and (1.2), have received extensive investigations in recent years. The simplest and best understood example of (1.1)-(1.2) seems to be the case when $f(|x|, u) = u^p, p > 1$. The semilinear equation

$$\Delta u + u^p = 0 \quad p > 1, \tag{1.3}$$

is known as the Lane-Emden equation in astrophysics. In this context, when $n = 3$, the function u represents the density of a single star. When $p = (n+2)/(n-2)$, equation (1.3) is also a special case of the Yamabe problem in differential geometry, and it is relevant to Yang-Mills equations for $n = 4$. In a celebrated paper of Gidas, Ni and Nirenberg [53], it is proved that when $1 < p < (n+2)/(n-2)$ and $\Omega = B_1$, the unit ball of $\mathbb{R}^n$, the problem

$$\begin{aligned} &\Delta u + u^p = 0 \quad \text{in} \quad B_1, \\ &u > 0 \quad \text{in} \quad B_1, \quad u = 0 \quad \text{on} \quad \partial B_1, \end{aligned} \tag{1.4}$$

has a unique solution and the solution is necessarily radial. On the other hand, a well-known result of the pioneering work of Fowler [47] in 1931 asserts that equation (1.3) has a positive radial solution in $\mathbb{R}^n$ if and only if $p \geq (n+2)/(n-2)$. More precisely, if $p \geq (n+2)/(n-2)$, then for any $\alpha > 0$, there is a unique radial solution u of (1.3) with $u(0) = \alpha$, and u is a ground state solution when $p = (n+2)/(n-2)$, a slowly decaying solution (behaving like $|x|^{-2/(p-1)}$ as $|x| \to \infty$, see definition below) when $p > (n+2)/(n-2)$. The exponent $p = (n+2)/(n-2)$ sets up a dividing number for the existence and nonexistence of solutions of problems (1.1)-(1.2). It is critical from the point of view of Sobolev embedding. Since $p+1 = 2n/(n-2)$ is the limiting Sobolev exponent for the embedding $H_0^1(\Omega) \subset L^{p+1}(\Omega)$. This embedding is not compact when $p \geq (n+2)/(n-2)$. A powerful identity of Pohozaev [101] demonstrates that problem (1.1) has no solution at all (radial or nonradial) when $f = u^p, p \geq \frac{n+2}{n-2}$ and Ω is a starshaped bounded domain (not necessarily a ball).

A typical example of the nonlinearities without a constant growth is $f(u) = u^p$, for $u > 1; = u^q$, for $0 \leq u \leq 1$. For this nonlinearity, Erbe and Tang [40] [42] investigated the uniqueness and the global structure of radial solutions of problems (1.1) and (1.2) in the range $1 < p < (n+2)/(n-2) < q$. They proved that problem (1.1) has a unique solution on a ball, and problem (1.2) has a unique ground state and infinitely many slowly decaying solutions. This result can be generalized. Namely, it holds if $f(u)$ is positive for $u > 0$, and the growth of f is a nonincreasing function of $u > 0$ and is subcritical for large u, while supercritical for small u.

The proof of [53] for the radial symmetry of solutions of (1.4) is very general. Namely, it was proved there that if the nonlinearity is independent of $|x|$ and Lipschitz continuous in u, and Ω is a ball, then any solution of (1.1) is radial. The situation is quite different when Ω is an annulus. At first, on an annular domain Ω, it was shown by Brezis and Nirenberg [14] that even for the simplest nonlinearity $f = u^p, p > 1$ problem (1.1) has both radial and nonradial solutions (see also Caffarelli and Friedman [17] for other nonlinearities). Secondly, Pohozaev's result does not apply since an annulus is not starshaped. Therefore problem (1.1) may admit a solution for a wider class of nonlinearities. For example, if $f = u^p$, then problem (1.1) has a solution in any finite annulus for all $p > 1$, and it was proved by Ni [88] that the solution is unique in the class of radial solutions. Ni and Nussbaum [90] continued the study of [88] and proved that if $f(u)$ is positive for $u > 0$ and has superlinear growth, then problem (1.1) possesses at most one radial solution provided that the growth of f is less than $n/(n-2)$.

When $f(u)$ has the property that $f(0) = 0, f'(0) < 0$ and $f(u)$ has a unique positive zero, the equation of (1.1) is sometimes called the Euclidean scalar field equation (see Anderson and Weinberger [8] and Berestycki and Lions [13]). The existence and uniqueness of ground state solutions of such an equation are relevant to the search for certain kinds of stationary states of nonlinear Klein-Gordon or Schrodinger equations and some reaction-diffusion equations which arise in population dynamics theory (see Fife [44]). The existence problem has been extensively studied by Berestycki and Lions [13]. A typical model of scalar field equations is $\Delta u - u + u^p = 0$, $p > 1$. It was shown that (see [13]) this equation has positive radial solutions in $\mathbb{R}^n$ if and only if p is subcritical. Gidas, Ni and Nirenberg [54] proved that such solutions are radially symmetric. The first uniqueness result for the solutions was proved by Coffman [22], who studied the case $n = p = 3$. His result was extended by MacLeod and Serrin [83], Peletier and Serrin [98] [99]. An important development was due to Kwong [67] who finally proved the uniqueness of solutions of (1.2) when $f = -u + u^p, n \geq 3$, and $1 < p < (n+2)/(n-2)$. His proof was simplified and generalized by MacLeod [82] and Kwong and Zhang [70].

When $f(|x|, u) = K(|x|)u^p, p > 1$, the equation of (1.1) arises both in physics and geometry. When $K(|x|) = 1/(1+|x|^2)$, and $n = 3$, the equation was proposed by Matukuma as a mathematical model to describe the dynamics of a globular cluster of stars. When p is the critical number, the equation is now known as a conformal scalar curvature equation in $\mathbb{R}^n$. The symmetricity and the existence and uniqueness of ground state solutions have been extensively investigated in an abundant list of literature (see, for example [29] [30] [74] [75] [87] and [110], etc.).

Brezis and Nirenberg [14] considered the existence of solutions of problem (1.1) when $f(|x|, u) = u^q + g(|x|, u)$ and Ω is a bounded domain in $\mathbb{R}^n$, where q is the critical Sobolev exponent and $g(|x|, u)$ is a lower-order perturbation of u^q in the sense that $\lim_{u \to \infty} g(|x|, u)/u^q = 0$. Some typical examples are $g(|x|, u) = \lambda u$ and $g(|x|, u) = \mu u^p, 1 < p < q$. As we have seen, if $f = u^q, q = (n+2)/(n-2)$, and Ω is starshaped, then problem (1.1) possesses no solution at all. But this situation can be reversed by a lower-term perturbation. Let us restrict $\Omega = B$, a finite ball in $\mathbb{R}^n$, and consider the problems

$$\begin{gathered} \Delta u + \lambda u + u^q = 0 \quad \text{in} \quad B, \\ u > 0 \quad \text{in} \quad B, \quad u = 0 \quad \text{on} \quad \partial B, \end{gathered} \tag{1.5}$$

and

$$\begin{gathered} \Delta u + \mu u^p + u^q = 0 \quad \text{in} \quad B, \\ u > 0 \quad \text{in} \quad B, \quad u = 0 \quad \text{on} \quad \partial B. \end{gathered} \tag{1.6}$$

Let $q = (n+2)/(n-2)$, and $1 < p < q$. Then the main results of [14] can be summarized as follows:

(i) When $n \geq 4$, problem (1.5) has a solution if and only if $\lambda \in (0, \lambda_1)$, where λ_1 denotes the first eigenvalue of $-\Delta$; while problem (1.6) has a solution for every $\mu > 0$.

(ii) When $n = 3$, problem (1.5) has a solution if and only if $\lambda \in (\lambda_1/4, \lambda_1)$; while problem (1.6) has a solution only for large values of $\mu > 0$.

Moreover, it is suggested by numerical computations that

(iii) when $n = 3$, $q = 5$, and $1 < p < 3$, there is some $\mu_0 > 0$ such that problem (1.6) has at least two solutions for $\mu > \mu_0$, a unique solution for $\mu = \mu_0$, and no solution for $\mu < \mu_0$.

Assertion (iii) was proved later by Atkinson and Peletier [10]. An interesting open problem raised by Brezis and Nirenberg was whether or not the solution of (1.5)-(1.6), whose existence is ensured in (i) and (ii), is unique (except the case $n = 3$, $q = 5$, and $1 < p < 3$ in (1.6)).

The uniqueness of solutions of problem (1.5) was proved by Kwong and Li [68] for the range $1 < q < (n+2)/(n-2)$, and by Zhang [113] for $1 < q \leq (n+2)/(n-2)$ simultaneously and independently. Some alternative proofs were provided by Srikanth [106] and Adimurthi and Yadava [4]. Their proofs made use of the linearity of the perturbation term λu and it may not be possible to extend their proofs to study the uniqueness problem of (1.6). In fact, in view of assertion (iii) above, one can not expect that the uniqueness of solutions of problem (1.6) holds for all $n \geq 3$ and $1 < p < q \leq (n+2)/(n-2)$. Very recently, Zhang [114] proved the uniqueness of (1.6) under the assumption $(q-1)/(p+1) \leq 2/n$. For instance, if $n = p = 3$, and $3 < q \leq 11/3$, then (1.6) has a unique solution.

Unfortunately, Zhang's result does not include the important case when q is the critical Sobolev exponent at all. Since once $q = (n+2)/(n-2)$, the condition $(q-1)/(p+1) \leq 2/n$ becomes $p \geq (n+2)/(n-2)$. In this thesis, we shall prove that *when $n \geq 6$, problem (1.6) has a unique solution for all $\mu > 0$ and all p, q satisfying* $1 < p < q \leq (n+2)/(n-2)$. Thus, the open problem concerning the uniqueness of (1.6) is completely solved for $n \geq 6$. Some detailed analyses will be given for the cases $n = 3, 4, 5$. For example, when $n = p = 3$, problem (1.6) has at most one solution if $3 < q < 4.7748332$. These results can be simply derived from one of the main results of this thesis.

This thesis is organized as follows. In Chapters 2-4, we shall limit ourselves to the case $f(|x|, u) = f(u), f(0) = 0$, and the finite domain Ω is restricted to be a finite ball. Our development is based on a shooting argument. Let $\alpha > 0$ and $u = u(t, \alpha), t = |x|$, be a radial solution of the quasilinear equation of (1.1_m) or (1.2_m) with $u(0, \alpha) = \alpha$. Then $u(t, \alpha)$ is the solution of an initial value problem of ordinary differential equations. In Chapter 2, we shall present some preliminary results on the properties of $u(t, \alpha)$. In particular, we shall show that $u(t, \alpha)$ is a strictly decreasing function of t whenever it is positive. We say that

(i) $u(t, \alpha)$ is a *crossing solution* if it vanishes at some $0 < t < \infty$;

(ii) $u(t, \alpha)$ is a *ground state solution* or a *fast decaying solution* if $u(t, \alpha) > 0$ on $[0, \infty)$ and $\lim_{t\to\infty} t^{(n-m)/(m-1)} u(t, \alpha)$ exists and is finite and positive.

(iii) $u(t, \alpha)$ is a *a slowly decaying solution* if $u(t, \alpha) > 0$ on $[0, \infty)$ and

$$\lim_{t\to\infty} t^{(n-m)/(m-1)} u(t, \alpha) = \infty.$$

If $f(s) > 0$ for $s > 0$, then $u(t, \alpha)$ can be classified into one of the above three types and it is either a solution of problem (1.1_m) for a finite ball or a solution of (1.2_m) vanishing at infinity. Due to the strict monotonicity of $u(t, \alpha)$, its inverse is well-defined in the interval $(0, \alpha)$. We shall establish a new Pohozaev-type identity involving the inverse of u. This identity plays essential roles in the proof of the main results. Under some natural restrictions on the nonlinearity, we are able to give a characteristic description for each type of solution by using the Pohozaev Identity. This result allows us to prove a preliminary result on the global structure of the set of solutions of $u(t, \alpha), \alpha > 0$.

For a given nonlinearity $f(s) \in C^1([0, \infty))$, let $F(s)$ and $H(s)$ be defined by

$F(s) := \int_0^s f(\tau)d\tau$ and

$$H(s) := [(n-m)sf(s) - nmF(s)]/f(s), \quad \text{for} \quad s > 0;$$
$$H(s) := 0, \quad \text{for} \quad s = 0.$$

Now the main results of Chapters 3-4 can be stated as follows:

(i) if $0 < (m-1)f(s) < sf'(s)$ and $H'(s) \leq 0$ for $s > 0$, then problem (1.1_m) admits at most one radial solution in any finite ball Ω (see Theorem 3.1).

(ii) If $f(s) > 0$ for $s > 0$, and

$$\limsup_{u \to 0+} \frac{F(u)}{u^{\epsilon_m}} = 0, \quad \text{where} \quad \epsilon_m = \frac{n(m-1)}{n-m} > 0,$$

then a necessary condition for the existence of a ground state solution of problem (1.2_m) is that $H'(s)$ assumes both positive and negative values in $s > 0$. Moreover, if there exists a number η, $0 < \eta < \infty$, such that $H'(s) \geq 0$ for $0 < s < \eta$, and $H'(s) \leq 0$ for $s \geq \eta$, and $H'(s)$ is not identically zero in any subinterval of $(0, \infty)$, then problem (1.2_m) admits at most one ground state solution (see Theorem 4.1).

We shall prove the first result, which is exactly the same as Theorem 3.1, in Chapter 3. The significance of Theorem 3.1 will be exhibited in various aspects. At first, its assumptions are natural. For instance, if we take $f = u^p$, $p > m-1$, then the conditions of Theorem 3.1 are fulfilled, that is, problem (1.1_m) has a uniqueness property, if and only if it has an existence property. Secondly, we shall give two examples to show that if the conditions are weakened, then problem (1.1_m) may have multiple solutions. Moreover, the theorem can be applied in studying the uniqueness problem of (1.1_m) with a large class of nonlinearities. In particular, the uniqueness property concerning problem (1.6) will be proved in Section 3.2.

The nonuniqueness problem is investigated in Section 3.3. We shall show that if $f(u)$ is given by

$$f(u) = u^p - u^q, \quad 1 < p < \frac{n+2}{n-2} \leq q,$$

and Ω is a finite ball with radius R sufficiently large, then problem (1.1) has at least two solutions.

The second result, which is essentially Theorem 4.1, will be proved in Chapter 4. As an application, we shall prove that problem (1.2_m) has at most one ground state solution if $f(u)$ is given by

$$f(u) = u^p - u^q, \quad \frac{nm-n+m}{n-m} < p < q,$$

or

$$f(u) = u^p, \quad u > 1; \quad = u^q, \quad u \leq 1, \quad m-1 < p < \frac{nm-n+m}{n-m} < q.$$

Note that the number $\frac{nm-n+m}{n-m}$ can be viewed as a critical exponent for the $m-$Laplacian operator, since it becomes the usual Sobolev critical exponent when $m = 2$.

In Chapter 5 we shall continue the study of the uniqueness problem by using a so-called *Kolodner-Coffman method* due to Kolodner [66] and Coffman [21]. We shall only consider semilinear elliptic equations, but the nonlinearity will be no longer independent of t, and the domain Ω may be a finite ball, a finite annulus, or the entire space $\mathbb{R}^n$. All our developments in this chapter are based on a principal lemma which will be given and proved in the first section. Some sufficient conditions for the uniqueness of positive radial solutions to problem (1.1) are given in Section 5.2. In Sections 5.3 and 5.4 we shall study the uniqueness of ground state solutions of problem (1.2) with f a separable function of t and u. Some examples such as *the generalized Matukuma's equation*, *Henon's equation*, and *the conformal scalar curvature equation* are examined in Section 5.5. The main body of Chapter 5 is an extension of the study of Erbe and Tang [40], [42], and [43].

In the final chapter, we shall give some further remarks and discussion.

The proofs of the main results are very involved. It is our intention to present them as transparently and elementarily as possible.

Chapter 2
PRELIMINARY RESULTS

In this chapter and the next two chapters, we shall be mainly concerned with the uniqueness of radial solutions of problems $(1.1_m)-(1.2_m)$. We shall limit ourselves to the cases when the nonlinearity is independent of t, i.e., $f(t,u) \equiv f(u)$, and the bounded domain Ω is a finite ball centered at the origin of $\mathbb{R}^n$.

Let $f(u)$ be defined and locally Lipschitz continuous on $[0,\infty)$ with $f(0) = 0$. Recall from the well-known result of Gidas, Ni and Nirenberg [53] that if $u(x)$ is a solution of the nonlinear Dirichlet problem

$$\begin{aligned} &\Delta u + f(u) = 0 \quad \text{in} \quad \Omega, \\ &u > 0 \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{on} \quad \partial\Omega, \end{aligned} \tag{2.1}$$

then $u(x)$ is radially symmetric, i.e., $u(x) = u(|x|)$. Concerning the radial symmetricity of solutions of the problem

$$\begin{aligned} &\Delta u + f(u) = 0 \quad \text{in} \quad \mathbb{R}^n, \\ &u > 0 \quad \text{in} \quad \mathbb{R}^n, \quad u(x) \to 0 \quad \text{as} \quad x \to \infty, \end{aligned} \tag{2.2}$$

it is not known yet whether or not every solution is radial. Nevertheless, it was shown in Gidas et al. [54] that for a large class of nonlinearities f, if $u(x)$ is a solution of (2.2) behaving like $|x|^l, l \leq 2-n$ for large $|x|$, then $u(x)$ is necessarily radial (see also Li and Ni [75] [76] for related results).

Thus in order to establish the uniqueness of solutions of the above problems, it is of primary interest to consider the corresponding radial problem

$$\begin{aligned} &u'' + \frac{n-1}{t}u' + f(u) = 0, \\ &u'(0) = 0, \quad u(t) > 0 \quad \text{in} \quad [0,b), \quad u(b) = 0, \end{aligned} \tag{2.3}$$

where u is now a function of the radial variable $t = |x|$ and the primes denote differentiation with respect to t, and $b > 0$ is the radius of Ω. We allow $b = \infty$ in (2.3) and by it we mean that $u(t) > 0$ in $[0,\infty)$, and $\lim_{t\to\infty} u(t) = 0$.

In order to establish the existence and uniqueness of solutions of problem (2.3), it has become standard to consider the initial value problem

$$\begin{aligned} &u'' + \frac{n-1}{t}u' + f(u) = 0, \\ &u(0) = \alpha > 0, \quad u'(0) = 0, \end{aligned} \tag{2.4}$$

where u extends maximally to the right with $u \geq 0$. Recall that (see Peletier and Serrin [98]) problem (2.4) has a unique solution if f is locally Lipschitz continuous on $[0, \infty)$.

More generally, let $1 < m \leq n$, we shall study the uniqueness of radial solutions of the following boundary value problems of quasilinear elliptic equations. Let $u(x)$ be a radial solution of the problem

$$\begin{aligned} &\operatorname{div}(|\nabla u|^{m-2}\nabla u) + f(u) = 0 \quad \text{in} \quad \Omega, \\ &u > 0 \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{on} \quad \partial\Omega \end{aligned} \tag{2.1_m}$$

or

$$\begin{aligned} &\operatorname{div}(|\nabla u|^{m-2}\nabla u) + f(u) = 0 \quad \text{in} \quad \mathbb{R}^n, \\ &u > 0 \quad \text{in} \quad \mathbb{R}^n, \quad u(x) \to 0 \quad \text{as} \quad |x| \to \infty. \end{aligned} \tag{2.2_m}$$

Then $u(t)$ is the solution of the corresponding radial problem

$$\begin{aligned} &[(m-1)u'' + \frac{n-1}{t}u']|u'|^{m-2} + f(u) = 0, \\ &u'(0) = 0, \quad u(t) > 0 \quad \text{in} \quad [0, b), \quad u(b) = 0, \end{aligned} \tag{2.3_m}$$

for some fixed $0 < b \leq \infty$. We shall also investigate the structure of solutions of the initial value problem

$$\begin{aligned} &[(m-1)u'' + \frac{n-1}{t}u']|u'|^{m-2} + f(u) = 0, \\ &u(0) = \alpha > 0, \quad u'(0) = 0, \end{aligned} \tag{2.4_m}$$

where u extends maximally to the right with $u \geq 0$. The uniqueness of solutions to problem (2.4_m) with some nonlinearities $f(u)$ has been observed by Knaap and Peletier [65], while a more general theorem has been proved by Franchi et al.[48]. If there exists a unique solution to problem (2.4) or (2.4_m), then we denote the solution by $u(t, \alpha)$. Note that if $m = 2$, then (2.1_m)-(2.4_m) reduce to (2.1)-(2.4), respectively.

We divide the rest of this chapter into three sections. In Section 2.1, we collect some fundamental properties of $u(t, \alpha)$. In Section 2.2, we shall establish a new Pohozaev-type identity which involves the inverse of $u(t, \alpha)$. This identity is crucial to the proof of the main results of this thesis. In Section 2.3, inspired by Kawano, Yanagida and Yotsutani [64], we characterize each type of solution $u(t, \alpha)$ by using the Pohozaev-type identity. This characterization allows us to give a preliminary result on the global structure of solutions of the initial value problem (2.4_m).

2.1 Properties of the Solution

First, we collect some hypotheses on f that may be assumed under various circumstances in the following development.

(F1) $f \in C^1([0,\infty))$, $f(0) = 0$ and $f(s) > 0$ for $s > 0$.

(F2) $0 < (m-1)f(s) < sf'(s)$, for $s > 0$.

(F2′) $0 < (m-1)f(s) < sf'(s)$, for $s > 0$ and $\lim_{u\to 0^+} \inf uf'(u)/f(u) > m-1$.

(F3) $\lim_{s\to 0^+} \sup \frac{F(s)}{s^{\epsilon_m}} = 0$, where $F(s) = \int_0^s f(\tau)d\tau$, and $\epsilon_m = \frac{n(m-1)}{n-m} > 0$.

Condition (F1) will be assumed throughout the remainder of this thesis, unless otherwise specified. (F2) can be viewed as a generalized superlinearity assumption on f. Since, when $m = 2$, (F2) implies that f is superlinear on $(0,\infty)$ in the usual sense.

Let $u(t,\alpha)$ be a solution of problem (2.4_m) with $\alpha > 0$. Define

$$b(\alpha) = \sup\{T : \ u(t,\alpha) \text{ is defined and } u(t,\alpha) > 0 \quad \text{in} \quad [0,T)\}. \tag{2.5}$$

Then one has $b(\alpha) > 0$.

In the following proposition we collect some fundamental properties of $u(t,\alpha)$ that are extensions of some well-known theorems in semilinear elliptic equations.

Proposition 2.1. *Let $u(t,\alpha)$ be a solution of problem (2.4_m) with $\alpha > 0$. Then we have*

(i) $u(b(\alpha),\alpha) = 0$, *if* $b(\alpha) < \infty$.

(ii) $u(t,\alpha) \in C^2(0,b(\alpha)] \cap C^1[0,b(\alpha)]$, *when* $b(\alpha) < \infty$, *or* $u(t,\alpha) \in C^2(0,\infty) \cap C^1[0,\infty)$ *when* $b(\alpha) = \infty$.

(iii) $u(t,\alpha)$ *is uniquely determined by* α. *Moreover, let* $u = u(t,\alpha)$, *and* $\hat{u} = u(t,\hat{\alpha})$ *with* $\alpha > 0$, $\hat{\alpha} > 0$, *be two solutions of* (2.4_m). *If there exists* $t_0 \in [0, \min\{b(\alpha), b(\hat{\alpha})\}]$, *or* $t_0 \in [0,\infty)$ *when* $b(\alpha) = b(\hat{\alpha}) = \infty$, *such that* $u(t_0) = \hat{u}(t_0)$ *and* $u'(t_0) = \hat{u}'(t_0)$, *then* $u \equiv \hat{u}$.

(iv) $u'(t,\alpha) < 0$ *in* $(0,b(\alpha)]$ *when* $b(\alpha) < \infty$, *or in* $(0,\infty)$ *otherwise.*

(v) If $b(\alpha) = \infty$, then $u(t,\alpha) \to 0$ and $u'(t,\alpha) \to 0$ as $t \to \infty$.

Proof. (i) follows from (ii) and (iv). (ii) was obtained by Ni and Serrin in the Appendix of [93], and (iii) was proved by Franchi, Lanconelli, and Serrin in the Appendix of [48]. We omit the proof of (i)-(iii), and give a detailed proof of (iv)-(v) here.

In the case $m = 2$, (iv) is well-known and easy to prove. In general, we rewrite the equation of (2.4_m) as

$$(t^{n-1}|u'|^{m-2}u')' = -t^{n-1}f(u). \tag{2.6}$$

If there were $t_1 \in (0, b(\alpha)]$ such that $u'(t_1) \geq 0$, then from (2.6) we would have, for any $t \in [0, t_1)$,

$$\begin{aligned} t^{n-1}|u'(t)|^{m-2}u'(t) &= t_1^{n-1}|u'(t_1)|^{m-2}u'(t_1) - \int_{t_1}^{t} s^{n-1}f(u(s))ds \\ &\geq \int_{t}^{t_1} s^{n-1}f(u(s))ds > 0. \end{aligned}$$

So then $u'(t) > 0$ in $[0, t_1)$, which contradicts $u'(0) = 0$. This proves (iv).

To prove (v), we let $\alpha > 0$ be such that $u(t,\alpha) > 0$ in $[0, \infty)$. By (iv) it follows that there is some u_∞ such that $0 \leq u_\infty < \infty$ and

$$\lim_{t\to\infty} u(t,\alpha) = u_\infty.$$

We shall show that $u'(t,\alpha)$ is also convergent as $t \to \infty$. For this purpose, we introduce an energy function defined by

$$E(t) = E(t,\alpha) := \frac{m-1}{m}|u'|^m + F(u). \tag{2.7}$$

If $u = u(t,\alpha)$, then one can easily verify that

$$\frac{dE(t)}{dt} = -\frac{n-1}{t}|u'|^m. \tag{2.8}$$

Therefore $E(t)$ is decreasing whenever $u(t)$ is defined. In particular, $E(t,\alpha)$ tends to a nonnegative constant as $t \to \infty$. Because of the convergence of $u(t,\alpha)$ and the continuity of $F(u)$, we conclude from (2.7) that $|u'(t,\alpha)|^m$ is convergent, so is $u'(t,\alpha)$. Due to the fact that $u(t,\alpha) > 0$ on $[0,\infty)$, one must have

$$\lim_{t\to\infty} u'(t,\alpha) = 0.$$

It remains to show that $u_\infty = 0$. Suppose to the contrary that $u_\infty \neq 0$. Then $u_\infty > 0$. Recall that

$$[(m-1)u''(t,\alpha) + \frac{n-1}{t}u'(t,\alpha)]|u'(t,\alpha|^{m-2} = -f(u(t,\alpha)).$$

Passing t to ∞ in this identity we have

$$\lim_{t\to\infty} u''(t,\alpha)|u'(t,\alpha)|^{m-2} = -f(u_\infty)/(m-1) < 0, \tag{2.9}$$

because $m > 1$ and $\lim_{t\to\infty} u'(t,\bar{\alpha}) = 0$. It is readily seen that (2.9) implies

$$\lim_{t\to\infty} u''(t,\alpha) = -\infty \quad \text{as} \quad m > 2, \quad \text{and} \quad \lim_{t\to\infty} u''(t,\alpha) = -f(u_\infty) < 0 \quad \text{as} \quad m = 2.$$

They are not compatible with $\lim_{t\to\infty} u'(t,\bar{\alpha}) = 0$. In the case $1 < m < 2$, it follows from (2.9) that there exist some real numbers $\iota > 0$ and $T_\iota > 0$ such that if $t > T_\iota$, then

$$u''(t,\alpha)|u'(t,\alpha)|^{-1} < -\iota,$$

or equivalently,

$$u''(t,\alpha)/u'(t,\alpha) > \iota.$$

Integrating both sides of the latter inequality over $[T_\iota, t)$ and letting $t \to \infty$ yield

$$\lim_{t\to\infty} |u'(t,\alpha)| = \infty,$$

which contradicts $\lim_{t\to\infty} u'(t,\alpha) = 0$. Thus we obtain $\lim_{t\to\infty} u'(t,\alpha) = 0$ as desired. □

Definition 2.2. A solution $u(t,\alpha)$ is called a *crossing solution* if $b(\alpha) < \infty$, $u(t,\alpha) > 0$ in $[0, b(\alpha))$, and $u(b(\alpha),\alpha) = 0$. It is called a *decaying solution* if $u(t,\alpha) > 0$ in $[0,\infty)$ and $\lim_{t\to\infty} u(t,\alpha) = 0$.

It follows from (i) and (v) of the last proposition that any solution $u(t,\alpha)$ is either a crossing solution or a decaying solution.

Before we start investigating the asymptotic behavior of decaying solutions, we digress for a moment to the special case $n = m$. We shall show that, in this case, every solution $u(t,\alpha), \alpha > 0$ is necessarily a crossing solution, and therefore no decaying solutions exist. To the best of our knowledge, this interesting result has not been observed in the current literature.

Proposition 2.3. *If $n = m$, then (2.4_m) possesses no decaying solution. Consequently, there is no radial solution to problem (2.2_m).*

Proof. Let $n = m$, and suppose for contradiction that $u = u(t, \alpha)$, $\alpha > 0$ is a decaying solution of (2.4_m). Then $u' < 0$ in $(0, \infty)$ and

$$\lim_{t \to \infty} tu' = 0. \tag{2.10}$$

To prove (2.10), note that from the equation of (2.4_m) we can deduce

$$(m-1)u'' + \frac{n-1}{t}u' < 0, \quad t \in (0, \infty),$$

which in turn implies,

$$(tu')' < 0, \quad t \in (0, \infty),$$

since $n = m$. Thus tu' is strictly decreasing in $(0, \infty)$, and there is some number $\iota \le 0$ such that $\lim_{t \to \infty} tu' = \iota$. If $\iota < 0$, then one can easily demonstrate that $\lim_{t \to \infty} u = -\infty$ to yield a contradiction. Hence $\iota = 0$ and (2.10) is proved.

Now, combining (2.10) and the fact that tu' is decreasing in $(0, \infty)$, we conclude that $tu' > 0$ in $(0, \infty)$. But this is impossible since $u' < 0$. The proof is completed. $\square$

Proposition 2.4. *Let $1 < m < n$, and let $u(t, \alpha)$ be a decaying solution of (2.4_m), then $t^{(n-m)/(m-1)}u(t, \alpha)$ is strictly increasing in $[0, \infty)$.*

Proof. Let $1 < m < n$, and $u = u(t, \alpha)$ be a decaying solution of (2.4_m). Then a straightforward calculation yields

$$(t^{(n-m)/(m-1)}u)' = t^{((n-m)/(m-1))-1}\left(\frac{n-m}{m-1}u + tu'\right), \quad t > 0.$$

It suffices to show that

$$\frac{n-m}{m-1}u + tu' > 0 \quad \text{for} \quad t > 0. \tag{2.11}$$

Note that

$$\left(\frac{n-m}{m-1}u + tu'\right)' = \frac{t}{m-1}((m-1)u'' + \frac{n-1}{t}u') < 0.$$

Hence $\frac{n-m}{m-1}u + tu'$ is decreasing on $(0, \infty)$ and $\lim_{t\to\infty}(\frac{n-m}{m-1}u + tu')$ exists. In fact,

$$\lim_{t\to\infty}(\frac{n-m}{m-1}u + tu') = 0,$$

because of $\lim_{t\to\infty} u = 0$. Thus (2.11) follows and the proof is completed. □

Definition 2.5. We call a decaying solution $u(t, \alpha)$ of (2.4_m) a *fast decaying solution* or a *ground state solution* if

$$\lim_{t\to\infty} t^{(n-m)/(m-1)}u(t,\alpha) \quad \text{exists and is finite,} \tag{2.12}$$

and a *slowly decaying solution* if

$$\lim_{t\to\infty} t^{(n-m)/(m-1)}u(t,\alpha) = \infty. \tag{2.13}$$

2.2. Pohozaev-type Variational Identities

First we recall in the next lemma a generalized Pohozaev-type variational identity due to Ni and Serrin [92] [93] and Pucci and Serrin [102]. For completeness, we give a proof of the identity following the lemma.

Lemma 2.6. *Let $u = u(t, \alpha)$ be a solution of (2.4_m). Then*

$$\begin{aligned}\int_0^t & \left\{auf(u) - nF(u) + |u'|^m\left(\frac{n}{m} - a - 1\right)\right\}\tau^{n-1}\,d\tau \\ &= -au'(t)|u'(t)|^{m-2}u(t)t^{n-1} - \left(1 - \frac{1}{m}\right)|u'(t)|^m t^n - F(u(t))t^n,\end{aligned} \tag{2.14}$$

where a is any real number.

Proof. Recall that u satisfies

$$(\tau^{n-1}|u'(\tau)|^{m-2}u'(\tau))' = -\tau^{n-1}f(u(\tau)). \tag{2.15}$$

First, multiplying (2.15) by $au(\tau)$ and integrating the resultant identity over $[0, t]$, we obtain

$$\int_0^t [auf(u) - a|u'|^m]\tau^{n-1}\,d\tau = -au'|u'|^{m-2}ut^{n-1}. \tag{2.16}$$

Next, multiply (2.15) by $\tau u'(\tau)$ and integrate the resultant identity over $[0,t]$. Making use of integration by parts and observing that

$$|u'|^{m-2}u'' = -\frac{f(u)}{m-1} - \frac{n-1}{m-1}\frac{u'}{\tau}|u'|^{m-2},$$

we have

$$\int_0^t [-nF(u) + \frac{n-m}{m}|u'|^m]\tau^{n-1}\, d\tau = -\frac{m-1}{m}t^n|u'|^m - F(u)t^n. \tag{2.17}$$

Combining (2.16) and (2.17), we obtain (2.14). $\square$

Let $a = \frac{n}{m} - 1$ in (2.14). Then we have

Lemma 2.7. *Let $u = u(t,\alpha)$ be a solution of (2.4_m). Then*

$$\bar{P}(t) = \int_0^t \bar{H}(u(\tau))\tau^{n-1}\, d\tau, \tag{2.18}$$

where

$$\begin{aligned}\bar{P}(t) &= \bar{P}(t,\alpha) = \bar{P}(t,\alpha,u(t,\alpha))\\ &:= -(n-m)u'(t)|u'(t)|^{m-2}u(t)t^{n-1} - (m-1)|u'(t)|^m t^n - mF(u(t))t^n,\end{aligned} \tag{2.19}$$

and

$$\bar{H}(u) := (n-m)uf(u) - nmF(u). \tag{2.20}$$

In the rest of this section, we concentrate on developing a Pohozaev-type variational identity involving the inverse of $u(t,\alpha)$. This identity is important and unusual. It plays a crucial role in proving the main results of this thesis.

Recall from (iv) of Proposition 2.1 that any solution $u(t,\alpha)$ of (2.4_m) is strictly decreasing in $(0,b(\alpha))$. Thus the inverse of $u(t,\alpha)$, denoted by $t = t(u,\alpha)$, is well-defined and is also strictly decreasing in $(0,\alpha)$. We have

$$u_t = 1/t_u, \quad u_{tt} = -t_{uu}/t_u^3.$$

Hence $t = t(u,\alpha)$ satisfies the equation

$$(m-1)t'' = \frac{n-1}{t}t'^2 + f(u)|t'|^m t', \tag{2.21}$$

where $' = \frac{d}{du}$.

Lemma 2.8. *Let $u = u(t,\alpha)$ be a solution of (2.4_m) and $t = t(u,\alpha)$ be its inverse in $(0,\alpha]$. We have, for any $u \in (0,\alpha)$,*

$$P(u) = \int_\alpha^u H'(s)t^{n-1}/(t'|t'|^{m-2})\, ds, \tag{2.22}$$

where $' = \frac{d}{du}$,

$$\begin{aligned} P(u) &= P(u,\alpha) = P(u,\alpha,t(u,\alpha)) \\ &:= [H(u) - (n-m)u]\frac{t^{n-1}}{t'|t'|^{m-2}} - (m-1)\frac{t^n}{|t'|^m} - mF(u)t^n, \end{aligned} \tag{2.23}$$

and

$$H(u) := \begin{cases} [(n-m)uf(u) - nmF(u)]/f(u), & u > 0, \\ 0, & u = 0. \end{cases} \tag{2.24}$$

Proof. A straightforward way to verify (2.22) is to differentiate both sides of the identity with respect to u. We prefer to give a proof by starting with (2.18). By using the fact that $t_u = 1/u_t$, we have

$$\begin{aligned} &-(n-m)\frac{ut^{n-1}}{t'|t'|^{m-2}} - (m-1)\frac{t^n}{|t'|^m} - mF(u)t^n \\ &= \int_\alpha^u [(n-m)sf(s) - mnF(s)]t^{n-1}t'\, ds. \end{aligned} \tag{2.25}$$

Recall from (2.21) that

$$f(u)t' = ((m-1)t'' - \frac{n-1}{t}t'^2)/|t'|^m.$$

We can evaluate the right side of (2.25), using integration by parts, as follows:

$$\begin{aligned} &\int_\alpha^u [(n-m)sf(s) - mnF(s)]t^{n-1}t'\, ds \\ &= \int_\alpha^u H(s)f(s)t^{n-1}t'\, ds \\ &= \int_\alpha^u H(s)t^{n-1}((m-1)t'' - \frac{n-1}{t}t'^2)/|t'|^m\, ds \\ &= \int_\alpha^u H(s)t^{n-1}\frac{(m-1)t''}{|t'|^m}\, ds - (n-1)\int_\alpha^u H(s)t^{n-2}/|t'|^{m-2}\, ds \\ &= -\int_\alpha^u H(s)t^{n-1}\, d\left(\frac{1}{t'|t'|^{m-2}}\right) - (n-1)\int_\alpha^u H(s)t^{n-2}/|t'|^{m-2}\, ds \\ &= -H(u)t^{n-1}/(t'|t'|^{n-2}) + \int_\alpha^u H'(s)t^{n-1}/(t'|t'|^{m-2})\, ds \\ &\quad + (n-1)\int_\alpha^u H(s)t^{n-2}/|t'|^{m-2}\, ds - (n-1)\int_\alpha^u H(s)t^{n-2}/|t'|^{m-2}\, ds \\ &= -H(u)t^{n-1}/(t'|t'|^{n-2}) + \int_\alpha^u H'(s)t^{n-1}/(t'|t'|^{m-2})\, ds. \end{aligned}$$

Hence (2.22) follows. □

2.3. Characterization of Positive Radial solutions

It follows from Proposition 2.1 (i) and (v), and Proposition 2.4 that ever solution $u(t, \alpha)$ is classified into one of three types: a crossing solution, a slowl decaying solution, or a ground state solution (a fast decaying solution). In thi section we shall give a characteristic description for each type of solutions of (2.4_m) As a simple application, we can present a structure theorem on the set of solution $u(t, \alpha), \alpha > 0$. In view of Proposition 2.3, we are only interested in the case

$$1 < m < n. \tag{2.26}$$

Recall that $b(\alpha)$ is defined to be the first zero of $u(t, \alpha)$ in $t > 0$, and $b(\alpha) = \infty$ when $u(t, \alpha)$ is positive in $t > 0$. Let $\bar{P}(t, \alpha)$ be as in (2.19). Define

$$\bar{P}_\alpha := \limsup_{t \to b(\alpha)^-} P(t, \alpha). \tag{2.27}$$

Lemma 2.9. *Let $\bar{P}_\alpha$ be as in (2.27). We have*

(i). If $u(t, \alpha)$ is a crossing solution, then

$$\bar{P}_\alpha = \bar{P}(b(\alpha), \alpha) < 0. \tag{2.28}$$

(ii). If $u(t, \alpha)$ is a ground solution, and $f(u)$ satisfies (F3), then

$$\bar{P}_\alpha = \lim_{t \to \infty} \bar{P}(t, \alpha) = 0. \tag{2.29}$$

(iii). If $u(t, \alpha)$ is a slowly decaying solution, and $f(u)$ satisfies (F2′), then

$$\bar{P}_\alpha \geq 0, \tag{2.30}$$

and for any $T' > 0$, there exists $T'' > T'$ such that

$$\bar{P}(T'', \alpha) > 0. \tag{2.31}$$

Proof. (i). The proof of (2.28) is trivial, we omit it.

(ii). Let $u(t,\alpha)$ be a ground state solution. Then there exists a number c_α such that

$$0 < c_\alpha < \infty, \tag{2.32}$$

and

$$\lim_{t\to\infty} t^{(n-m)/(m-1)} u(t,\alpha) = c_\alpha. \tag{2.33}$$

Observe that

$$\begin{aligned}
[t^{(n-1)/(m-1)} u'(t,\alpha)]' &= \frac{1}{m-1} t^{(n-1)/(m-1)}((m-1)u''(t,\alpha) + \frac{n-1}{t} u'(t,\alpha)) \\
&= -\frac{t^{(n-1)/(m-1)} f(u)}{(m-1)|u'(t,\alpha)|^{m-2}} \\
&< 0,
\end{aligned} \tag{2.34}$$

for all $t > 0$. This shows that $\lim_{t\to\infty} t^{(n-1)/(m-1)} u'(t,\alpha)$ exists, and we can evaluate it by L'Hospital's rule to give

$$\begin{aligned}
c_\alpha &= \lim_{t\to\infty} t^{(n-m)/(m-1)} u(t,\alpha) \\
&= \lim_{t\to\infty} \frac{u(t,\alpha)}{t^{-(n-m)/(m-1)}} = \lim_{t\to\infty} \frac{u'(t,\alpha)}{\frac{n-m}{m-1} t^{-(n-1)/(m-1)}} \\
&= \lim_{t\to\infty} -\frac{m-1}{n-m} t^{(n-1)/(m-1)} u'(t,\alpha).
\end{aligned}$$

Hence we obtain that

$$\lim_{t\to\infty} t^{(n-1)/(m-1)} u'(t,\alpha) = -\frac{n-m}{m-1} c_\alpha,$$

and

$$\lim_{t\to\infty} t^{(n-1)/(m-1)} |u'(t,\alpha)| = \frac{n-m}{m-1} c_\alpha. \tag{2.35}$$

By our assumption $n > m$ it follows that $(n-1)/(m-1) > 1$, which in turn, combining with (2.35), implies

$$\lim_{t\to\infty} t u'(t,\alpha) = 0. \tag{2.36}$$

Thus

$$\begin{aligned}
\bar{P}_\alpha &= \lim_{t\to\infty} \bar{P}(t,\alpha) \\
&= \lim_{t\to\infty} [-(n-m)t^{n-1}u(t,\alpha)u'(t,\alpha)|u'(t,\alpha)|^{m-2} \\
&\quad - (m-1)t^n|u'(t,\alpha)|^m - mt^nF(u(t,\alpha))] \\
&= -\lim_{t\to\infty}(n-m)t^{n-1}uu'|u'|^{m-2} - (m-1)\lim_{t\to\infty} t^n|u'|^m - m\lim_{t\to\infty} t^nF(u) \\
&=(n-m)\lim_{t\to\infty} u \lim_{t\to\infty}(t^{(n-1)/(m-1)}|u'|)^{m-1} \\
&\quad - m\lim_{t\to\infty}\frac{F(u)}{u^{\epsilon_m}}\lim_{t\to\infty}(t^{(n-m)/(m-1)}u)^{\epsilon_m} \\
&\quad + (m-1)\lim_{t\to\infty}(tu')\lim_{t\to\infty}(t^{(n-1)/(m-1)}|u'|)^{m-1} \\
&=0,
\end{aligned}$$

because of (2.33), (2.35), (2.36) and (F3).

(iii). Let $u(t,\alpha)$ be a slowly decaying solution. In view of assumption (F2$'$) we may pick some number $\bar{m}$, $\bar{m} > m$ so that

$$\frac{sf'(s)}{f(s)} > \bar{m} - 1, \quad 0 < s < \alpha. \tag{2.37}$$

We can use this inequality to estimate $\frac{F(u)}{uf(u)}$. Note that when $0 < u < \alpha$,

$$\begin{aligned}
\frac{F(u)}{uf(u)} &= \frac{\int_0^u f(s)ds}{uf(u)} \\
&= \frac{uf(u) - \int_0^u sf'(s)ds}{uf(u)} = 1 - \frac{\int_0^u sf'(s)ds}{uf(u)} \\
&< 1 - (\bar{m}-1)\frac{\int_0^u f(s)ds}{uf(u)} = 1 - (\bar{m}-1)\frac{F(u)}{uf(u)}.
\end{aligned}$$

This gives

$$\frac{F(u(t,\alpha))}{u(t,\alpha)f(u(t,\alpha))} < \frac{1}{\bar{m}}. \tag{2.38}$$

For simplicity of notations, let $\xi = \frac{m}{\bar{m}}$. Then

$$0 < \xi < 1, \quad mF(u(t,\alpha)) < \xi u(t,\alpha)f(u(t,\alpha)), \quad t > 0. \tag{2.39}$$

Now, let $u = u(t,\alpha)$ and $t > 0$, we have

$$\begin{aligned}
\bar{P}(t,\alpha) &= -(n-m)t^{n-1}uu'|u'|^{m-2} - (m-1)t^n|u'|^m - mt^nF(u) \\
&\geq -(n-m)t^{n-1}uu'|u'|^{m-2} - (m-1)t^n|u'|^m - \xi t^n uf(u) \\
&= -(m-1)t^{n-\frac{n-m}{m-1}}u'|u'|^{m-2}\left[\frac{n-m}{m-1}t^{\frac{n-m}{m-1}-1}u + t^{\frac{n-m}{m-1}}u'\right] \\
&\quad + \xi tu(t^{n-1}u'|u'|^{m-2})' \quad \text{(see (2.6))} \\
&= -[(m-1)t^{n-\frac{n-m}{m-1}}u'|u'|^{m-2}(t^{\frac{n-m}{m-1}}u)' + \xi tu(t^{n-1}|u'|^{m-1})']
\end{aligned}$$

$$
\begin{aligned}
&= t^n u|u'|^{m-1}\left[(m-1)\frac{(t^{\frac{n-m}{m-1}}u)'}{t^{\frac{n-m}{m-1}}u} - \xi\frac{(t^{n-1}|u'|^{m-1})'}{t^{n-1}|u'|^{m-1}}\right] \\
&= t^n u|u'|^{m-1}[(m-1)\ln(t^{\frac{n-m}{m-1}}u) - \xi\ln(t^{n-1}|u'|^{m-1})]' \\
&= t^n u|u'|^{m-1}[(m-1)\ln(t^{\frac{n-m}{m-1}}u) - (m-1)\xi\ln(t^{\frac{n-m}{m-1}}u) \\
&\quad + (m-1)\xi\ln(t^{\frac{n-m}{m-1}}u) - \xi\ln(t^{n-1}|u'|^{m-1})]' \\
&= t^n u|u'|^{m-1}[(m-1)\ln(t^{\frac{n-m}{m-1}}u) - (m-1)\xi\ln(t^{\frac{n-m}{m-1}}u) \\
&\quad + \xi\ln(t^{\frac{n-m}{m-1}}u)^{m-1} - \xi\ln(t^{\frac{n-1}{m-1}}|u'|)^{m-1}]' \\
&= t^n u|u'|^{m-1}[(m-1)(1-\xi)\ln(t^{\frac{n-m}{m-1}}u) + \xi\ln((t^{\frac{n-m}{m-1}}u)/(t^{\frac{n-1}{m-1}}|u'|))^{m-1}]' \\
&= t^n u|u'|^{m-1}[(m-1)(1-\xi)\ln(t^{\frac{n-m}{m-1}}u) + \xi\ln(u/(t|u'|))^{m-1}]'.
\end{aligned}
$$

Recall that $\lim_{t\to\infty} t^{\frac{m-1}{n-m}}u = +\infty$, and

$$
u/(t|u'|) \geq \frac{m-1}{n-m}, \quad \text{(by (2.11))}.
$$

We have

$$
\lim_{t\to\infty}[(m-1)(1-\xi)\ln(t^{\frac{n-m}{m-1}}u) + \xi\ln(u/(t|u'|))^{m-1}] = +\infty.
$$

Thus for any $T' > 0$, there exists a $T'' > T'$ such that

$$
[(m-1)(1-\xi)\ln(t^{\frac{n-m}{m-1}}u) + \xi\ln(u/(t|u'|))^{m-1}]'|_{t=T''} > 0.
$$

Now (2.30) and (2.31) follow immediately. The proof is completed. □

The next theorem in which we present some criteria for the existence and nonexistence of radial solutions to problems (2.1_m) or (2.2_m), is of fundamental importance. Its proof is based on the Pohozaev-type Identity (2.18) and the last lemma.

Theorem 2.10. *Suppose that (F2′) and (F3) hold. Let $\bar{H}(u)$ be defined as in (2.20). We have*

(i) if there exists some $0 < \eta_H \leq \infty$ such that $\bar{H}(u) \equiv 0$ in $[0, \eta_H)$, then every solution $u(t,\alpha)$, $0 < \alpha \leq \eta_H$ is a ground state solution.

(ii) If there exists an $\epsilon_H > 0$ such that $\bar{H}(u)$ is not identically zero in any subinterval of $[0, \epsilon_H)$, and

$$
\bar{H}(u) \geq 0 \quad in \quad [0, \infty), \tag{2.40}
$$

then every solution $u(t,\alpha)$, $\alpha > 0$ is a slowly decaying solution. Consequently, problem (2.1_m) admits no radial solutions in any finite ball.

(iii) *If there exists an $\epsilon_H > 0$ such that $\bar{H}(u)$ is not identically zero in any subinterval of $[0, \epsilon_H)$, and*

$$\bar{H}(u) \leq 0 \quad in \quad [0, \infty). \tag{2.41}$$

Then every solution $u(t,\alpha)$, $\alpha > 0$ is a crossing solution. Consequently, problem (2.2_m) admits no radial solutions.

Proof. (i). Let $u(t,\alpha)$ be a solution with $0 < \alpha \leq \eta_H$. Then $u(t,\alpha) < \eta_H$ when $0 < t < b(\alpha)$. Now by the assumptions of (i) and identity (2.18) we obtain

$$\bar{P}(t,\alpha) \equiv 0, \quad 0 \leq t < b(\alpha).$$

Thus $b(\alpha) = \infty$ and $u(t,\alpha)$ is not a crossing solution. By (2.31) one sees that $u(t,\alpha)$ is not a slowly decaying solution. Therefore $u(t,\alpha)$ must be a ground state solution.

(ii). Let $u(t,\alpha)$ be a solution with $\alpha > 0$. In this case, by using (2.18) and (2.40) we see that

$$\bar{P}(t,\alpha) \geq 0, \quad 0 \leq t < b(\alpha).$$

Therefore once again it follows that $b(\alpha) = \infty$ and $u(t,\alpha)$ is not a crossing solution. We see that $u(t,\alpha)$ is a decaying solution and there is some $t = T_\epsilon \geq 0$ such that

$$0 < u(t,\alpha) < \epsilon_H, \quad \text{for} \quad t > T_\epsilon,$$

which implies $\bar{P}_\alpha > 0$. Hence $u(t,\alpha)$ is a slowly decaying solution.

(iii). The proof is similar to that of (ii), and so we omit it. □

As in [64], [111] and [112], we introduce the following definition.

Definition 2.11. The structure of positive solutions of the initial value problem (2.4) or (2.4_m) is of

Type C : if $u(t,\alpha)$ is a crossing solution for every $\alpha > 0$;

Type F : if $u(t,\alpha)$ is a fast decaying solution (or a ground state solution) for every $\alpha > 0$;

Type S : if $u(t,\alpha)$ is a slowly decaying solution for every $\alpha > 0$;

Type M : if there exists a unique positive number $\alpha^* > 0$ such that $u(t,\alpha^*)$ is a ground state solution, and $u(t,\alpha)$ is a crossing solution for every $\alpha \in (\alpha^*,\infty)$, a slowly decaying solution for every $\alpha \in (0,\alpha^*)$.

The next result, which is merely a restatement of Theorem 2.10, describes the structures of solution sets of (2.4_m) in some special cases.

Proposition 2.12. *Suppose that $(F2')$ and $(F3)$ hold. Let $\bar{H}(u)$ be defined as in(2.20). Then we have*

(i) If $\bar{H}(u) \equiv 0$ in $[0,\infty)$, then the structure of positive solutions of (2.4_m) is of Type F.

(ii) If there exists an $\epsilon_H > 0$ such that $H'(u)$ is not identically zero in any subinterval of $[0,\epsilon_H)$, and (2.40) holds. Then the structure of positive solutions of (2.4_m) is of Type S.

(ii) If there exists an $\epsilon_H > 0$ such that $H'(u)$ is not identically zero in any subinterval of $[0,\epsilon_H)$, and (2.41) holds. Then the structure of positive solutions of (2.4_m) is of Type C.

If we repeat the above argument of this section by employing identity (2.22) rather than (2.18), then we can establish some analogous results with Lemma 2.9, Theorem 2.10 and Proposition 2.12. We state the results in the next proposition. Their proofs are omitted.

Proposition 2.13. *Let $P(u,\alpha)$ be defined as in (2.23), and $H(u)$ be defined as in (2.24). Define*

$$P_\alpha = \limsup_{u\to 0^+} P(u,\alpha). \tag{2.42}$$

Suppose that $(F2')$ holds. Then we have *(i) If $u(t,\alpha)$ is a crossing solution, then*

$$P_\alpha = P(0,\alpha) < 0. \tag{2.43}$$

If $u(t,\alpha)$ is a ground solution, and $f(u)$ satisfies (F3), then

$$P_\alpha = \lim_{u\to 0+} P(u,\alpha) = 0. \tag{2.44}$$

(ii) If there exists an $\epsilon_H > 0$ such that $H'(u)$ is not identically zero in any subinterval of $[0,\epsilon_H)$, and

$$H'(u) \geq 0 \quad in \quad [0,\infty). \tag{2.45}$$

Then every solution $u(t,\alpha)$, $\alpha > 0$ is a slowly decaying solution, and the structure of positive solutions of (2.4_m) is of Type S. Consequently, problem (2.1_m) admits no radial solutions in any finite ball.

(iii) If there exists an $\epsilon_H > 0$ such that $H'(u)$ is not identically zero in any subinterval of $[0,\epsilon_H)$. Then a necessary condition for the existence of a solution of problem (2.1_m) is that $H'(u_0) < 0$ for some $u_0 > 0$, and a necessary condition for the existence of a ground state solution of problem (2.2_m) is that $H'(u_1) > 0$ and $H'(u_2) < 0$ for some $u_1 > 0$ and $u_2 > 0$.

Note that we are unable to establish an analogue of (iii) of Lemma 9.

From (iii) of Theorem 2.10 we have seen that if (2.41) is strengthened to

$$\bar{H}(u) < 0 \quad \text{in} \quad [0,\infty), \tag{2.46}$$

then every solution $u(t,\alpha)$, $\alpha > 0$ is a crossing solution. It is natural to ask the question: is (2.46) sufficient to ensure the uniqueness of solutions of problem (2.1_m) in any given finite ball Ω? This question will be answered negatively at the beginning of the next chapter. More interestingly, we shall show there that a slightly stronger condition than (2.46) is sufficient.

In dealing with the uniqueness of ground state solutions to problem (2.2_m), we may only consider the case when $H'(u)$ switches signs in $[0,\infty)$ in view of (iii) of Proposition 2.13. In general, the uniqueness problem of ground states is extremely difficult to study even for the semilinear problem (2.2), and we have not seen any result for the general quasilinear problem (2.2_m). In Chapter 4, we shall show that if $H'(u)$ is positive for small u and negative for large u, and $H'(u)$ changes signs only once, then one has the uniqueness for the ground states of problem (2.2_m).

Chapter 3
UNIQUENESS IN A FINITE BALL

In this chapter, we are concerned with the problem of the uniqueness of solutions of $(2.1)_m$ in a finite ball Ω of $\mathbb{R}^n$, in which $n \geq 3$ and $1 < m < n$.

First, let us consider an example. Consider the semilinear Dirichlet problem

$$\begin{aligned} &\Delta u + \mu u^p + u^5 = 0 \quad \text{in} \quad \Omega, \Omega \subset \mathbb{R}^3, \\ &u > 0 \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{on} \quad \partial\Omega, \end{aligned} \tag{3.1}$$

in which $\mu > 0$ is a real number. (3.1) is a special case of (2.1). Let $\bar{H}(u)$ be defined as in (2.20). Since $n = 3, m = 2$ and $f(u) = \mu u^p + u^5$, $\mu > 0$, we have

$$\bar{H}(u) = \left(1 - \frac{6}{p+1}\right) \mu u^{p+1}.$$

It follows that $\bar{H}(u) < 0$ for $u > 0$ when $1 < p < 5$. On the other hand, as we have mentioned, Atkinson and Peletier [10] proved that problem (3.1) may have at least two solutions for some $\mu > 0$ and $1 < p < 3$. This shows that (2.46) is not a sufficient condition for the uniqueness of radial solutions of problem (2.1) or (2.1_m). Nevertheless, we shall show that if (2.46) is strengthened by replacing $\bar{H}(u)$ by $H'(u)$, then one obtains the uniqueness.

We state the main result of this chapter as follows.

Theorem 3.1. *Problem* $(2.1)_m$ *admits at most one radial solution in any finite ball* Ω *provided that* $f(u)$ *satisfies* $(F2)$ *and*

$$H'(u) \leq 0 \quad in \quad (0, \infty). \tag{3.2}$$

Theorem 3.1 will be proved in Section 3.1. The key ingredient in our proof is making extensive use of the Pohozaev-type identity (2.22) we established in the previous chapter. In Section 3.2 we apply Theorem 3.1 to study the important model case when $f(u) = \lambda u^p + u^q$, in which $p < q$ and q is a subcritical or a critical exponent. Some interesting results will be derived, especially, an open problem which arose in Brezis and Nirenberg [14] is solved in higher dimensions when $n \geq 6$. In Section 3.3 we discuss the nonuniqueness problem. When condition (F2) is removed, we shall show that the assertion of Theorem 3.1 is no longer true.

We shall examine an example when $f(u) = u^p - u^q, p < q$, where p is subcritical, while q is supercritical.

3.1. Proof of Theorem 3.1

In this section, we shall complete the proof of Theorem 3.1. For a given $\alpha > 0$, we always let $u = u(t, \alpha)$ denote the unique solution of (2.4_m) or (2.4), and $t = t(u, \alpha)$ be its inverse defined in $(0, \alpha]$. The proof of Theorem 3.1 is based on the following two technical lemmas.

Lemma 3.2. *Suppose that (3.2) holds. Let $0 < \alpha_1 < \alpha_2$ and $t_1 = t(u, \alpha_1)$, $t_2 = t(u, \alpha_2)$ be the inverses of $u(t, \alpha_1)$, and $u(t, \alpha_2)$ respectively. Then*

$$t_1'(u) < t_2'(u), \quad \textit{for all} \quad u \in [0, \alpha_1). \tag{3.3}$$

Lemma 3.3. *Assume that $f(u)$ satisfies (F2). If $u_1 = u(t, \alpha_1)$ and $u_2 = u(t, \alpha_2)$ are two crossing solutions of problem (2.4_m) with $b(\alpha_1) = b(\alpha_2)$ and $u_1 \leq u_2$ in $[0, b(\alpha_1)]$, then $u_1 \equiv u_2$.*

Accepting Lemma 3.2 and Lemma 3.3 for the moment, we can readily complete the proof of Theorem 3.1. Let Ω be a finite ball in $\mathbb{R}^n$, and let b be the radius of Ω with $0 < b < \infty$. Suppose to the contrary that problem $(2.1)_m$ has more than one radial solution. Then we may assume that $u_1 = u(t, \alpha_1)$ and $u_2 = u(t, \alpha_2)$ where $0 < \alpha_1 < \alpha_2$ are two solutions of $(2.1)_m$.

Let $t_1 = t(u, \alpha_1)$, $t_2 = t(u, \alpha_2)$ be the inverses of u_1 and u_2, respectively. From (F2) and Lemma 3.3 it follows that there is some $u_I \in (0, \alpha_1)$ such that

$$t_1(u_I) = t_2(u_I), t_1'(u_I) < t_2'(u_I), \quad \text{and} \quad t_1(u) < t_2(u) \quad \text{in} \quad (u_I, \alpha_1).$$

But

$$t_1(0) = t_2(0) = b.$$

There is a point $u_{II} \in [0, u_I)$ such that

$$t_1(u_{II}) = t_2(u_{II}), \quad \text{and} \quad t_1(u) > t_2(u) \quad \text{in} \quad (u_{II}, u_I).$$

In particular, one has

$$t_1'(u_{II}) \geq t_2'(u_{II}).$$

But this contradicts (3.3). The proof is completed.

It remains to prove Lemmas 3.2-3.3. We provide a lemma before proving Lemma 3.2.

Lemma 3.4. *Let $t_1(u)$ and $t_2(u)$ be as in Lemma 3.2. Define*

$$S_{12}(u) := \frac{t_1^{n-1}}{t_1'|t_1'|^{m-2}} \Big/ \frac{t_2^{n-1}}{t_2'|t_2'|^{m-2}}. \tag{3.4}$$

Then

$$S_{12}'(u) > 0 \quad \textit{if and only if} \quad t_1'(u) > t_2'(u), \quad u \in (0, \alpha_1). \tag{3.5}$$

Proof. We have

$$\begin{aligned}(t_1^{1-n}t_1'|t_1'|^{m-2})' &= (1-n)t_1^{-n}t_1'^2|t_1'|^{m-2} + t_1^{1-n}(m-1)|t_1'|^{m-2}t_1'' \\ &= t_1^{1-n}|t_1'|^{m-2}\left(\frac{1-n}{t_1}t_1'^2 + (m-1)t_1''\right) \\ &= t_1^{1-n}t_1'|t_1'|^{2m-2}f(u),\end{aligned}$$

and a similar identity holds for t_2. Hence

$$\begin{aligned}\frac{dS_{12}(u)}{du} =& (t_1^{1-n}t_1'|t_1'|^{m-2})^{-2}\left[t_2^{1-n}t_2'|t_2'|^{2m-2}f(u)t_1^{1-n}t_1'|t_1'|^{m-2}\right. \\ &\left.- t_1^{1-n}t_1'|t_1'|^{2m-2}f(u)t_2^{1-n}t_2'|t_2'|^{m-2}\right] \\ =& (t_1^{1-n}t_1'|t_1'|^{m-2})^{-2}t_1^{1-n}t_2^{1-n}t_1'|t_1'|^{m-2}t_2'|t_2'|^{m-2}f(u)(|t_2'|^m - |t_1'|^m).\end{aligned}$$

Now (3.5) readily follows. □

Proof of Lemma 3.2. Observe that both $u(t, \alpha_1)$ and $u(t, \alpha_2)$ are crossing solutions. Thus t_i, $i = 1, 2$ are defined in $[0, \alpha_i]$. Because

$$\lim_{u \to \alpha_1^-} t_1'(u) = -\infty,$$

we see that (3.3) holds in a left neighborhood of α_1.

(i). First we consider the case

$$t_1(u) < t_2(u), \quad u \in (0, \alpha_1). \tag{3.6}$$

Note that (3.6) is possible since in this lemma we do not assume (F2). Suppose to the contrary that there is some $u \in [0, \alpha)$ such that $t_1'(u) \geq t_2'(u)$. Then we can find a $u_c \in [0, \alpha)$ such that

$$t_1'(u_c) = t_2'(u_c), \quad \text{and} \quad t_1'(u) < t_2'(u) \quad \text{for} \quad u \in (u_c, \alpha_1). \tag{3.7}$$

Thus $t_1(u_c) < t_2(u_c)$ and

$$\begin{aligned}
0 &\geq t_1''(u_c) - t_2''(u_c) \\
&= \frac{1}{m-1}\left(\frac{n-1}{t_1}t_1'^2 + f(u)|t_1'|^{m-1}t_1' - \frac{n-1}{t_2}t_2'^2 - f(u)|t_2'|^{m-1}t_2'\right)\Big|_{u=u_c} \\
&= \frac{n-1}{m-1}\left(\frac{1}{t_1(u_c)} - \frac{1}{t_2(u_c)}\right)t_1'^2(u_c) \\
&> 0.
\end{aligned}$$

This is a contradiction.

(ii). Next, we consider the case when the graphs of t_1 and t_2 intersect in $(0, \alpha_1)$, i.e., there is some $u_I \in (0, \alpha_1)$ such that

$$t_1(u_I) = t_2(u_I), t_1'(u_I) < t_2'(u_I), \quad \text{and} \quad t_1(u) < t_2(u) \quad \text{in} \quad (u_I, \alpha_1). \tag{3.8}$$

If (3.3) does not hold in this case, then we can find a point $u_c \in [0, \alpha_1)$ as above such that (3.7) holds. From the proof above it follows that

$$u_c < u_I. \tag{3.9}$$

In view of (iii) of Proposition 2.1, the intersection points of t_1 and t_2 are isolated, in the sense that they intersect just once in some neighborhood of an intersection point. Now, if u_I is the only intersection point, then

$$t_1(u_c) > t_2(u_c). \tag{3.10}$$

If t_1 and t_2 intersect in $[0, u_I)$, say at $u = \bar{u_I} < u_I$, and $t_1 > t_2$ in $(\bar{u_I}, u_I)$, then we must have $u_c > \bar{u_I}$ because of (3.7). Therefore (3.10) is also fulfilled.

Let $S_c = S_{12}(u_c)$, where $S_{12}(u)$ is defined in (3.4), then

$$1 < S_c = \frac{t_1^{n-1}(u_c)}{t_2^{n-1}(u_c)} < \frac{t_1^n(u_c)}{t_2^n(u_c)}. \tag{3.11}$$

Let $P_i(u)$, $i = 1, 2$, denote the corresponding functions of (2.23) when t and t' in (2.23) are replaced by t_i, t_i'. Then

$$\begin{aligned}
&P_1(u_c) - S_c P_2(u_c) \\
&= [H(u_c) - (n-m)u_c]\left(\frac{t_1^{n-1}(u_c)}{t_1'(u_c)|t_1'(u_c)|^{m-2}} - S_c \frac{t_2^{n-1}(u_c)}{t_2'(u_c)|t_2'(u_c)|^{m-2}}\right) \\
&\quad - (m-1)\left[\frac{t_1^n(u_c)}{|t_1'(u_c)|^m} - S_c \frac{t_2^n(u_c)}{|t_2'(u_c)|^m}\right] - mF(u_c)(t_1^n(u_c) - S_c t_2^n(u_c)) \qquad (3.12) \\
&= \left[-\frac{m-1}{|t_1'(u_c)|^m} - mF(u_c)\right][t_1^n(u_c) - S_c t_2^n(u_c)] \\
&< 0.
\end{aligned}$$

Since $m > 1$, $F(u_c) > 0$, and (3.11) is satisfied, by using identity (2.22), we have

$$\begin{aligned}
P_1(u_c) - S_c P_2(u_c) &= \int_{\alpha_1}^{u_c} H'(\tau)t_1^{n-1}(\tau)/(t_1'(\tau)|t_1'(\tau)|^{m-2})\, d\tau \\
&\quad - S_c \int_{\alpha_2}^{u_c} H'(\tau)t_2^{n-1}(\tau)/(t_2'(\tau)|t_2'(\tau)|^{m-2})\, d\tau \\
&= \int_{\alpha_1}^{u_c} H'(\tau)\,[t_1^{n-1}(\tau)/(t_1'(\tau)|t_1'(\tau)|^{m-2}) \qquad (3.13) \\
&\quad - S_c t_2^{n-1}(\tau)/(t_2'(\tau)|t_2'(\tau)|^{m-2})]\, d\tau \\
&\quad - S_c \int_{\alpha_2}^{\alpha_1} H'(\tau)t_2^{n-1}(\tau)/(t_2'(\tau)|t_2'(\tau)|^{m-2})\, d\tau \\
&=: I_1 - I_2.
\end{aligned}$$

Now, since $\alpha_1 < \alpha_2$, $H'(\tau) \leq 0$ in (α_1, α_2), and $t_2'(u) < 0$ in (α_1, α_2), we have

$$I_2 \leq 0. \qquad (3.14)$$

By (3.8) and Lemma 3.4, we know that $S_{12}(u)$ is strictly decreasing in (u_c, α_1). Hence

$$S_{12}(u) < S_c, \quad \text{for} \quad u \in (u_c, \alpha_1).$$

That is,

$$\frac{t_1^{n-1}(u)}{t_1'(u)|t_1'(u)|^{m-2}} - S_c \frac{t_2^{n-1}(u)}{t_2'(u)|t_2'(u)|^{m-2}} > 0, \quad u \in (u_c, \alpha_1). \qquad (3.15)$$

Combining (3.15), and the fact that $u_c < \alpha_1$ and $H'(\tau) \leq 0$, we get

$$I_1 \geq 0. \qquad (3.16)$$

Therefore,

$$P_1(u_c) - S_c P_2(u_c) \geq 0,$$

which contradicts (3.12). The proof is completed. □

As an immediate consequence of Lemma 3.2, we have:

Corollary 3.5. *Suppose that (3.2) holds. Let* $u_1 = u(t, \alpha_1)$ *and* $u_2 = u(t, \alpha_2)$ *with* $0 < \alpha_1 < \alpha_2$. *We have*

(i) *The graphs of* u_1 *and* u_2 *intersect at most once in* $[0, \min\{b(\alpha_1), b(\alpha_2)\})$.

(ii) *If* $b(\alpha_1) = b(\alpha_2) < \infty$, *then* $u_1 < u_2$ *for* $t \in [0, b(\alpha_1))$.

The proof of Lemma 3.3 is standard, it involves the study of some eigenvalue problems.

Proof of Lemma 3.3. For simplicity of notations, let $\bar{b} = b(\alpha_1) = b(\alpha_2)$. It follows for (F2) that the function

$$\bar{f}(u) := f(u)/u^{m-1}$$

is strictly increasing for $u > 0$, and $\bar{f}_0 := \lim_{u\to 0^+} \bar{f}(u)$ exists and is nonnegative. Let

$$\rho_i(t) = f(u_i(t))/u_i^{m-1}(t), \quad 0 \le t < \bar{b}, \quad \rho_i(\bar{b}) = \bar{f}_0, \quad i = 1, 2.$$

Then $\rho_i(t)$, $i = 1, 2$ are continuous and nonnegative in $[0, \bar{b}]$. It follows from the strict monotonicity of $\bar{f}$ that

$$\rho_1(t) = \rho_2(t) \Leftrightarrow u_1(t) = u_2(t), \quad t \in [0, \bar{b}]. \tag{3.17}$$

Since $u_1 \le u_2$ in $[0, \bar{b}]$, we have

$$\rho_1(t) \le \rho_2(t), \quad t \in [0, \bar{b}]. \tag{3.18}$$

Note that as u_i is a solution of (2.4_m), $(1, u_i)$ is an eigenpair of the eigenvalue problem

$$\begin{aligned} &-(t^{n-1}|\phi'|^{m-2}\phi')' = \lambda \rho_i(t)|\phi|^{m-2}\phi t^{n-1}, \quad t \in [0, \bar{b}], \\ &\phi'(0) = \phi(\bar{b}) = 0. \end{aligned} \tag{3.19}$$

As is well known (see Rabinowitz [103]), the eigenvalue of (3.19) can be characterized variationally. In particular, if we define

$$E_{\bar{b}} = \{u \in C^1[0, \bar{b}] \ \ |u'(0) = u(\bar{b}) = 0\},$$

then $u_i \in E_{\bar{b}}$, $i = 1, 2$, and

$$\begin{aligned} 1 &= \min_{\phi \in E_{\bar{b}}} \frac{\int_0^{\bar{b}} |\phi'|^m t^{n-1}\, dt}{\int_0^{\bar{b}} \rho_i(t)|\phi|^m t^{n-1}\, dt} \\ &= \frac{\int_0^{\bar{b}} |u_i'|^m t^{n-1}\, dt}{\int_0^{\bar{b}} \rho_i(t)|u_i|^m t^{n-1}\, dt}. \end{aligned}$$

Thus,

$$1 = \frac{\int_0^{\bar{b}} |u_1'|^m t^{n-1}\, dt}{\int_0^{\bar{b}} \rho_1(t)|u_1|^m t^{n-1}\, dt}$$
$$= \frac{\int_0^{\bar{b}} |u_2'|^m t^{n-1}\, dt}{\int_0^{\bar{b}} \rho_2(t)|u_2|^m t^{n-1}\, dt}$$
$$\leq \frac{\int_0^{\bar{b}} |u_1'|^m t^{n-1}\, dt}{\int_0^{\bar{b}} \rho_2(t)|u_1|^m t^{n-1}\, dt},$$

therefore

$$\int_0^{\bar{b}} \rho_1(t)|u_1|^m t^{n-1}\, dt \geq \int_0^{\bar{b}} \rho_2(t)|u_1|^m t^{n-1}\, dt. \tag{3.20}$$

Since $u_1 > 0$ in $[0, \bar{b})$, (3.20) and (3.18) imply $\rho_1(t) \equiv \rho_2(t)$, $t \in [0, \bar{b})$, i.e., $u_1 \equiv u_2$. The proof is completed. □

Before closing this section, we present an example to give a simple application of Theorem 3.1. Some more examples will be provided in following sections.

Example 3.6. $f(u) = u^p$, $p > 0$ is a constant.

It is seen that (F2) is fulfilled if $p > m - 1$, and

$$H(u) = \left(n - m - \frac{nm}{p+1}\right) u.$$

Thus $H'(u) = n - m - \frac{nm}{p+1}$ and

$$H'(u) < 0 \quad \text{if and only if} \quad p < \frac{nm - n + m}{n - m}. \tag{3.21}$$

The following result follows easily from Theorem 3.1 and Theorem 2.10.

Corollary 3.7. *Suppose that $f(u) = u^p$, where $p > m - 1$. We have*

(i) if p is supercritical, i.e.,

$$p > \frac{nm - n + m}{n - m}.$$

then problem (2.1_m) has no solution. Moreover, every solution $u(t, \alpha)$, $\alpha > 0$ is a slowly decaying solution.

(ii) If p is critical, i.e.,

$$p = \frac{nm - n + m}{n - m},$$

then problem (2.1_m) *has no solution. Moreover, every solution* $u(t, \alpha)$, $\alpha > 0$ *is a ground state solution.*

(iii) If p is subcritical, i.e.,

$$p < \frac{nm - n + m}{n - m},$$

then every solution $u(t, \alpha)$, $\alpha > 0$ *is a crossing solution. and problem* $(2.1)_m$ *admits at most one radial solution.*

Remark 3.8. We believe that Corollary 3.7 can be proved by other methods, but we have not found any paper in which the assertions of Corollary 3.7 were mentioned or proved. We believe that in the case (iii), problem $(2.1)_m$ possesses a unique positive radial solution.

3.2. Applications of Theorem 3.1

We shall apply Theorem 3.1 to study the problem of uniqueness of radial solutions u satisfying

$$\begin{aligned} &\operatorname{div}(|\nabla u|^{m-2}\nabla u) + \lambda u^p + u^q = 0 \quad \text{in} \quad \Omega, \\ &u > 0 \quad \text{in} \quad \Omega, \qquad u = 0 \quad \text{on} \quad \partial\Omega, \end{aligned} \tag{3.22}$$

where $\lambda > 0$ is a constant, $\Omega \subset \mathbb{R}^n$ is a ball, and $1 < m < n$. When $m = 2$, (3.22) is reduced to the semilinear elliptic Dirichlet problem

$$\begin{aligned} &\Delta u + \lambda u^p + u^q = 0 \quad \text{in} \quad \Omega, \\ &u > 0 \quad \text{in} \quad \Omega, \qquad u = 0 \quad \text{on} \quad \partial\Omega. \end{aligned} \tag{3.23}$$

We are only concerned here with the case when the nonlinearity is superlinear and subcritical (or critical). We may assume, corresponding to (3.23),

$$1 \le p < q \le \frac{n+2}{n-2}. \tag{3.24}$$

In general, corresponding to the m-Laplacian equation (3.22), we assume

$$m - 1 \le p < q \le \frac{nm - n + m}{n - m}. \tag{3.25}$$

Note that if $1 < m < n$, then we always have

$$m - 1 < \frac{nm - n + m}{n - m},$$

to assure that (3.25) is feasible.

Let $f(u) = \lambda u^p + u^q$. Then

$$F(u) = \frac{\lambda}{p+1} u^{p+1} + \frac{1}{q+1} u^{q+1},$$

and

$$\begin{aligned} H(u) &= [(n-m)uf(u) - nmF(u)]/f(u) \\ &= \left[\lambda(n-m)u^{p+1} + (n-m)u^{q+1} - \frac{\lambda nm}{p+1}u^{p+1} - \frac{nm}{q+1}u^{q+1}\right] / (\lambda u^p + u^q) \\ &= \left[\lambda\left(n - m - \frac{nm}{p+1}\right)u^{p+1} + \left(n - m - \frac{nm}{q+1}\right)u^{q+1}\right] / (\lambda u^p + u^q). \\ &= \left[\lambda\left(n - m - \frac{nm}{p+1}\right)u + \left(n - m - \frac{nm}{q+1}\right)u^{q-p+1}\right] / (\lambda + u^{q-p}). \end{aligned}$$

For simplicity of notation, we let

$$\nu = -\left(n - m - \frac{nm}{p+1}\right), \quad \xi = -\left(n - m - \frac{nm}{q+1}\right). \tag{3.26}$$

Then

$$H(u) = -\frac{\lambda \nu u + \xi u^{q-p+1}}{\lambda + u^{q-p}}. \tag{3.27}$$

It can be easily verified that when (3.25) is satisfied, $\nu < 0$, $\xi \leq 0$ and $H(u) < 0$, for $u > 0$. Thus every solution $u(t, \alpha)$, $\alpha > 0$, has a finite zero. Differentiating (3.27) with respect to u yields

$$H'(u) = -\frac{1}{(\lambda + u^{q-p})^2}\left[\lambda^2 \nu + \lambda[\nu(p-q+1) + \xi(q-p+1)]v + \xi v^2\right],$$

where $v = u^{q-p}$. Let

$$\sigma = \nu(p-q+1) + \xi(q-p+1), \tag{3.28}$$

and

$$D(x) := D(x; p, q, \lambda) := \xi x^2 + \lambda \sigma x + \lambda^2 \nu. \tag{3.29}$$

Then

$$H'(u) = -\frac{1}{(\lambda + u^{q-p})^2} \cdot D(v). \tag{3.30}$$

Applying Theorem 3.1, we have:

Theorem 3.9. *Suppose that $\lambda > 0$, p and q satisfy (3.25). Let $D(x)$ be the quadratic form of (3.29). Then problem (3.22) has at most one radial solution if*

$$D(x) > 0, \quad \text{for} \quad x > 0, \tag{3.31}$$

or equivalently, if

$$\sigma \geq 0 \quad \text{or} \quad \sigma^2 \leq 4\nu\xi, \tag{3.32}$$

where ν, ξ and σ are defined in (3.26) and (3.28).

Since $\nu > 0$, $\xi \geq 0$ and $p < q$, we can give a simple condition which is sufficient for $\sigma \geq 0$. It is $p - q + 1 \geq 0$, or

$$q - p \leq 1. \tag{3.33}$$

Corollary 3.10. *Suppose that $\lambda > 0$, p and q satisfy (3.25). Then problem (3.22) possesses at most one radial solution if $q - p \leq 1$.*

Corollary 3.11. *Suppose that $\lambda > 0$, p and q satisfy (3.25). Then problem (3.22) possesses at most one radial solution if $n \geq m + m^2$.*

Proof. It suffices to show that $n \geq m + m^2$ implies $q - p \leq 1$ under (3.25). In fact, by (3.25) we obtain

$$\begin{aligned} q - p &\leq \frac{nm - n + m}{n - m} - (m - 1) \\ &= \frac{m^2}{n - m} \leq \frac{m^2}{m + m^2 - m} = 1. \end{aligned}$$

□

The existence and uniqueness of positive radial solutions to the semilinear elliptic problem (3.23) is of particular interest. In fact, to study the classical solutions of (3.23) is essentially the same as to study the radial ones, since a solution u of (3.23) is necessarily radially symmetric. It is well-known that the existence of positive solutions of problem (3.23) holds for $1 < p < q < \frac{n+2}{n-2}$. But the situation is changed drastically and becames very delicate when the Sobolev critical exponent is involved, i.e., $q = \frac{n+2}{n-2}$ (see Brezis and Nirenberg [14]).

Let $p = 1$, and $1 < q \leq \frac{n+2}{n-2}$. We are led to a semilinear Dirichlet problem

$$\begin{aligned} &\Delta u + \lambda u + u^q = 0 \quad \text{in} \quad B_1, \\ &u > 0 \quad \text{in} \quad B_1, \quad u = 0 \quad \text{on} \quad \partial B_1, \end{aligned} \tag{3.34}$$

where B_1 is the unit ball in $\mathbb{R}^n$. The uniqueness of solutions of (3.34) was studied independently and almost simultaneously by Kwong and Li [68], Zhang [113], and later by Srikanth [106] using different approaches. They proved that (3.34) has at most one solution when $n \geq 3$, and $1 < q \leq \frac{n+2}{n-2}$. Later on, a simpler and elementary proof was given by Adimurthi [4]. In particular, when $q = \frac{n+2}{n-2}$, the nonlinearity of (3.34) is a linear perturbation of the critical term $u^{(n+2)/(n-2)}$. Their results solve an open problem which arose in [14] concerning the uniqueness of solutions of (3.23) when $p = 1$.

Let us replace the linear term λu of (3.34) by a superlinear and subcritical term, and consider

$$\begin{aligned} &\Delta u + \lambda u^p + u^q = 0 \quad \text{in} \quad B_1, \\ &u > 0 \quad \text{in} \quad B_1, \quad u = 0 \quad \text{on} \quad \partial B_1, \end{aligned} \tag{3.35}$$

with $1 < p < q \leq (n+2)/(n-2)$. It is natural to ask whether or not the solution to this problem, if it exists, is unique. In general, the answer is "no". As we have mentioned, Atkinson and Peletier [10] proved that when $n = 3$, $1 < p < 3$, and $q = 5$, problem (3.35) has at least two solutions for λ sufficiently large. Therefore, at least for $n = 3$, the uniqueness to (3.35) is lost for some numbers p,q and λ.

Very recently, Zhang [114] proved that (3.35) admits at most one solution if

$$(q-1)/(p+1) \leq 2/n. \tag{3.36}$$

For example, when $n = 3$ and $p = 3$, uniqueness of positive solutions of (3.35) holds for $3 < q \leq 11/3$. In a more general setting, Ni and Nussbaum [90] studied the uniqueness of positive solutions for a general nonlinearity $f(u)$. For (3.35), their results reveal that problem (3.35) has a unique solution if

$$1 < p < q < n/(n-2). \tag{3.37}$$

Note that (3.36) and (3.37) cover only a part of the range $1 < p < q \leq (n+2)/(n-2)$, since on this range one has

$$(q-1)/(p+1) < \frac{2}{n-2}.$$

In particular, the most important case when $q = (n+2)/(n-2)$ is not included.

Surprisingly, as an important application of Theorem 3.1, we have:

Theorem 3.12. *Let $\lambda > 0$, $1 \leq p < q \leq \frac{n+2}{n-2}$ and B_1 be the unit ball in $\mathbb{R}^n$, $n \geq 3$. The Dirichlet problem*

$$\Delta u + \lambda u^p + u^q = 0 \quad in \quad B_1,$$
$$u > 0 \quad in \quad B_1, \quad u = 0 \quad on \quad \partial B_1,$$

admits at most one solution when $n \geq 6$.

It remains interesting to deal with the lower-dimension cases $n = 3, 4, 5$.

Remark 3.13. When $n = 3$, we have

$$\nu = \frac{5-p}{p+1}, \quad \xi = \frac{5-q}{q+1},$$

and

$$\sigma = \frac{2}{(p+1)(q+1)}(5pq + 2p + 2q - 3p^2 - 3q^2 + 5).$$

Condition (3.32) becomes

$$5pq + 2p + 2q - 3p^2 - 3q^2 + 5 \geq 0, \tag{3.38}$$

or

$$(5pq + 2p + 2q - 3p^2 - 3q^2 + 5)^2 \leq (5-p)(5-q)(p+1)(q+1). \tag{3.39}$$

It follows from Theorem 3.9 that if either (3.43) or (3.44) holds, then the nonlinear problem (3.35) admits at most one solution.

Some similar inequalities to (3.38) and (3.39) can be established when $n = 4$ or $n = 5$.

Example 3.14. The case $n = 3$ and $p = 3$: Inequalities (3.38) and (3.39) can be simplified to

$$3q^2 - 17q + 16 \leq 0 \tag{3.40}$$

and

$$9q^4 - 102q^3 + 393q^2 - 576q + 216 \leq 0, \tag{3.41}$$

respectively. By a numerical computation, (3.41) is satisfied when $3 < p < 4.7748332$. Therefore, *the semilinear Dirichlet problem*

$$\begin{aligned} &\Delta u + \lambda u^3 + u^q = 0 \quad \text{on} \quad B_1, \\ &u > 0 \quad \text{on} \quad B_1, \quad u = 0 \quad \text{on} \quad \partial B_1. \end{aligned} \tag{3.42}$$

has a unique solution when $3 < q < 4.7748332$.

Remark 3.15. Let B_1 be the unit ball in $\mathbb{R}^3$. Consider the problem

$$\begin{aligned} &\Delta u + \lambda u^p + u^5 = 0 \quad \text{on} \quad B_1, \\ &u > 0 \quad \text{on} \quad B_1, \quad u = 0 \quad \text{on} \quad \partial B_1. \end{aligned} \tag{3.43}$$

Recall from [10] and [14] that if $1 < p \leq 3$, $\lambda > \lambda_0$ for some positive constant λ_0, then (3.43) has at least two solutions. On the other hand, it follows from Corollary 3.10 that if $4 \leq p < 5$, then (3.43) has a unique solution for all $\lambda > 0$. It is unclear what happens for the case $3 < p < 4$.

It is also interesting to see that problem (3.43) exhibits a bifurcation phenomenon with respect to the parameter p, in the sense that there is a number $p = p^*$ with $3 \leq p^* \leq 4$ such that if λ is sufficiently large, then the uniqueness of solutions of (3.48) holds for $p^* < p < 5$, and the uniqueness is lost if $1 < p < p^*$.

Remark 3.16. In the situation where supercritical growth is involved in (3.23), one may not expect to have uniqueness of solutions. Budd and Norbary [16] considered problem (3.23) and proved that, when Ω is the unit ball in $\mathbb{R}^3$, $p = 1$ and $q > 5$, there is a critical value $\lambda = \lambda_c(q)$ at which the problem has an infinity of positive C^2 solutions.

Remark 3.17. In general, consider problem (3.23) when p and q satisfy (3.24). The examples of the problem we have seen in the literature which admits more than one solution for some λ are only for the case that $n = 3$ and the critical growth term u^5 is involved. We conjecture that the uniqueness of solutions of problem (3.23) holds if (3.24) is satisfied and $n \geq 4$. Even in the case $n = 3$, we conjecture that the uniqueness still holds if $1 < p < q < (n+2)/(n-2)$.

Remark 3.18. The breakdown of uniqueness for lower dimension cases is interesting. A similar phenomenon occurs for the existence of positive solutions or nodal solutions to the Dirichlet problem in the unit ball B_1 of $\mathbb{R}^n$, $n \geq 3$,

$$\begin{aligned} &\Delta u + \lambda u + u|u|^{\frac{4}{n-2}} = 0 \quad \text{on} \quad B_1, \\ &u(0) \neq 0, \quad u = 0 \quad \text{on} \quad \partial B_1. \end{aligned} \tag{3.44}$$

A nodal solution of (3.44) is a nontrivial solution which changes sign in B_1. In the celebrated paper of Atkinson, Brezis and Peletier [9], it is proved that problem (3.44) admits no nodal solutions when $\lambda > 0$ is sufficiently small and $n = 3, 4, 5, 6$. An elementary proof of this result was given in [1]. On the other hand, Cerami, Solimini and Struve [20] have proved that if $n \geq 7$, then problem (3.43) admits infinitely many nodal solutions for every $\lambda > 0$. Recently, Filippicci et al. [45-46] have shown that the same phenomenon occurs for a much wider class of equations, including those arising from the m-Laplace operator and the mean curvature operator.

3.3. A Nonuniqueness Example – Multiple Ordered Solutions

First let us take a look at condition (F2) which we assumed in Theorem 3.1. In the case $m = 2$, (F2) is simply reduced to the requirement that $f(u)$ is superlinear, i.e., $f(u)/u$ is an increasing function of u when $u > 0$. When $m = 2$, Theorem 3.5 can be stated as: the semilinear elliptic problem (2.1) has at most one solution in a finite ball of $\mathbb{R}^n$, $n \geq 3$ if f is superlinear and $H'(u) < 0$ in $u > 0$.

If we reverse the inequality of (F2) to get, for $m = 2$,

$$uf'(u) < f(u) \quad \text{for} \quad u > 0, \tag{3.45}$$

i.e., f is sublinear, and we can simply demonstrate that the assertion of Theorem 3.1 remains valid even without condition (3.2). In fact, if this assertion is false, then one can find some $0 < b < \infty$, and $0 < \alpha_1 < \alpha_2$ such that $b(\alpha_1) = b(\alpha_2) = b$. Observe that both $u_1 = u(t, \alpha_1)$ and $u_2 = u(t, \alpha_2)$ satisfy the equation of (2.4). Multiplying the equation of u_1 (or u_2) of (2.4) by u_2 (or u_1), and then subtracting the resultant equations, we have

$$[t^{n-1}(u_1'u_2 - u_1u_2')]' = -t^{n-1}(u_2f(u_1) - u_1f(u_2)). \tag{3.46}$$

Since $\alpha_1 < \alpha_2$, $u_1 < u_2$ in a right neighborhood of $t = 0$. Let $t = t_b$ be the first point at which $u_1(t_b) = u_2(t_b)$. Then

$$u_1(t) < u_2(t) \quad \text{in} \quad (0, t_b), \quad \text{and} \quad u_1'(t_b) > u_2'(t_b). \tag{3.47}$$

Note that such a point t_b does exist, and $0 < t_b \leq b$ because of $b(\alpha_1) = b(\alpha_2)$. Integrate both sides of (3.46) over $[0, t_b)$ to obtain

$$t^{n-1}(u_1'u_2 - u_1u_2')|_{t=t_0} = -\int_0^{t_b} t^{n-1}u_1u_2\left(\frac{f(u_1)}{u_1} - \frac{f(u_2)}{u_2}\right)\, dt.$$

The left side of this equality is positive by (3.47), while the right side is negative since $u_1 < u_2$ in $[0, t_b)$ and $f(u)/u$ is decreasing in $u > 0$, giving a contradiction.

The above arguments show that one has uniqueness in problem (2.1) whenever f is sublinear, or f is superlinear and $H'(u) < 0$. In this section, we shall show that, if both sublinearity and superlinearity are involved, then one may have nonuniqueness even though $H'(u) < 0$. Nevertheless, recall from Corollary 3.5 that if $H'(u) < 0$ for $u > 0$ and $u_1 = u(t, \alpha_1)$, $u_2 = u(t, \alpha_2)$, $0 < \alpha_1 < \alpha_2$ are two distinct solutions of the semilinear Dirichlet problem (2.1), then they are strictly ordered in the sense that

$$u_1(x) < u_2(x), \quad x \in \Omega. \tag{3.48}$$

We shall be particularly concerned with the existence of multiple solutions of problem (2.1) with

$$f(u) = u^p - u^q, \quad 1 < p < \frac{n+2}{n-2} \leq q. \tag{3.49}$$

Proposition 3.19. *Let f be defined in (3.49). Let $u(t, \alpha)$ be a solution of (2.4). We have*

(i) for $\alpha > 1$, $u(t, \alpha)$ is positive and strictly increasing in $(0, \zeta(\alpha))$, where $0 < \zeta(\alpha) \leq \infty$, and $\lim_{t \to \zeta(\alpha)} u(t, \alpha) = \infty$ when $\zeta(\alpha) < \infty$.

(ii) $u(t, 1) \equiv 1$.

(iii) for $0 < \alpha < 1$, $u(t, \alpha)$ is a crossing solution and is strictly decreasing before it vanishes.

Proof. (i). As $\alpha > 1$, $u(t, \alpha) > 1$ in a right neighborhood of $t = 0$. Let $t = \bar{t} > 0$ be a point such that $u(t, \alpha) > 1$ in $[0, \bar{t}]$. It suffices to prove that $u'(\bar{t}, \alpha) > 0$. In fact, from the equation of (2.4), one simply has that

$$\bar{t}^{n-1} u'(\bar{t}, \alpha) = -\int_0^{\bar{t}} t^{n-1} f(u)\, dt > 0,$$

since $f(u(t, \alpha)) < 0$ in $[0, \bar{t}]$. Thus $u'(\bar{t}, \alpha) > 0$ and the proof for this case is completed.

(ii). This follows from the fact that $f(1) = 0$.

(iii). Observe that $f(u) > 0$ when $0 < u < 1$. Thus if $0 < \alpha < 1$, then $u(t, \alpha)$ is strictly decreasing whenever it is positive. We only need to prove that $u(t, \alpha)$ has a finite zero. Since, for the function $f(u)$ in (3.49),

$$H(u) = \frac{1}{f(u)} \left[\left(n - 2 - \frac{2n}{p+1} \right) u^{p+1} - \left(n - 2 - \frac{2n}{q+1} \right) u^{q+1} \right], \tag{3.50}$$

we see that $H(u) < 0$ when $0 < u < 1$. It follows from Theorem 2.10 that $u(t, \alpha)$ is a crossing solution as $0 < \alpha < 1$. The proof is completed. □

When $\alpha \geq 1$, this proposition shows that $u(t, \alpha)$ is neither a crossing nor a decaying solution. Thus our consideration below shall be restricted in $0 < u < 1$. The main result of this section is:

Theorem 3.20. *Let $f(u)$ be as in (3.49). Let $R > 0$ be the radius of Ω. Then there exists $\bar{R} > 0$ such that*

i) if $R < \bar{R}$, then problem (2.1) has no solutions;

ii) if $R > \bar{R}$, then problem (2.1) admits at least two solutions. And any two distinct solutions are strictly ordered.

Proof. First we prove the second part of ii). In view of Corollary 3.5 and Proposition 3.19, we only need to show $H'(u) < 0$ in $0 < u < 1$. By (3.50) we obtain

$$H'(u) = \frac{u^{2p}}{f^2(u)} \cdot T(v), \tag{3.51}$$

where $v = u^{q-p}$, and

$$T(v) = \left(n - 2 - \frac{2n}{q+1} \right) v^2 + \left[2n + 4 - 2n \left(\frac{p}{q+1} + \frac{q}{p+1} \right) \right] v + \left(n - 2 - \frac{2n}{p+1} \right).$$

Since $T(0) = n - 2 - \frac{2n}{p+1} < 0$, $T''(v) \equiv 2\left(n - 2 - \frac{2n}{q+1}\right) \geq 0$, and

$$T(1) = -\frac{2n}{(p+1)(q+1)}(p-q)^2 < 0.$$

We conclude that $T(v) < 0$ when $0 \le v \le 1$. Hence

$$H'(u) < 0 \quad \text{in} \quad 0 < u < 1, \tag{3.52}$$

as desired.

It follows from Proposition 3.19 that when $0 < \alpha < 1$, $b(\alpha)$ is defined and $b(\alpha) < \infty$. The existence and uniqueness theorem for initial value problems for ordinary differential equations implies that $u'(b(\alpha), \alpha) < 0$. Thus, combining the fact that $u(t, \alpha)$ is C^2 in t and C^1 in α, we see that $b(\alpha)$ is continuous. Since $u(t, 1) \equiv 1$. The continuity of $b(\alpha)$ implies that $b(\alpha) \to +\infty$ as $\alpha \to 1^-$.

We claim that if $\alpha \to 0^+$, then it also holds that $b(\alpha) \to +\infty$. Assuming the claim for the moment, we can readily complete the proof of this theorem. In fact, if $b(\alpha) \to +\infty$ as $\alpha \to 0^+$ or $\alpha \to 1^-$, then $b(\alpha)$ attains its absolute minimum at some points, say $\alpha = \bar{\alpha}$, with $0 < \bar{\alpha} < 1$. That is

$$b(\bar{\alpha}) = \inf_{0<\alpha<1} \{b(\alpha)\}, \quad 0 < \bar{\alpha} < 1.$$

Obviously, $b(\bar{\alpha}) > 0$. Let $\bar{R} = b(\bar{\alpha})$. Then problem (2.1) has no solution if $R < \bar{R}$. Thus i) is proved. The first part of ii) follows from the continuity of $b(\alpha)$ and the mean value theorem.

Now we turn out to the proof of the claim. In order to investigate the behavior of solutions with sufficiently small initial data, we use a standard scaling argument. The proof we present next is essentially due to Ni and Yotsutani [95].

Corresponding to the solution $u = u(t, \alpha)$, let $w = w(t, \alpha) = \frac{1}{2} u(\frac{t}{\beta}, \alpha)$, then

$$w(0) = 1, \quad w'(0) = 0, \tag{3.53}$$

and

$$w'' + \frac{n-1}{t} w' + \frac{\alpha^{p-1}}{\beta^2} w^p - \frac{\alpha^{q-1}}{\beta^2} w^q = 0.$$

If $\beta = \alpha^{(p-1)/2}$, then

$$w'' + \frac{n-1}{t} w' + w^p - \alpha^{q-p} w^q = 0. \tag{3.54}$$

Let $v(t)$ be the unique solution of problem

$$\begin{aligned} &v'' + \frac{n-1}{t} v' + v^p = 0, \\ &v(0) = 1, \quad v'(0) = 0. \end{aligned} \tag{3.55}$$

Then $v'(t) < 0$ when $v(t) > 0$. Let t_v be the unique point at which $v(t_v) = 1/2$, then

$$v(t) \geq \frac{1}{2} \quad \text{in} \quad [0, t_v].$$

Note that t_v does exist in view of (v) of Proposition 2.1. We claim that

$$\lim_{\alpha \to 0+} \left[\sup_{0 \leq t \leq t_v} |w - v| \right] = 0. \tag{3.56}$$

If the claim is true, then for α sufficiently close to 0, one has

$$w(t, \alpha) = \frac{1}{2} u(\frac{t}{\beta}, \alpha) > 0, \quad t \in [0, t_v].$$

Therefore, $b(\alpha) > t_v/\beta$, which implies that $b(\alpha) \to +\infty$ as $\alpha \to 0^+$ as desired.

It remains to prove (3.56). It can be easily verified that (see [95])

$$v(t) = 1 - \frac{1}{n-2} \int_0^t \left[1 - \left(\frac{s}{t}\right)^{n-2} \right] s v^p(s) \, ds,$$
$$w(t) = 1 - \frac{1}{n-2} \int_0^t \left[1 - \left(\frac{s}{t}\right)^{n-2} \right] [w^p(s) - \alpha^{q-p} w^q(s)] \, ds.$$

Taking the difference of $v(t)$ and $w(t)$. we obtain

$$\begin{aligned}(n-2)|w(t) - v(t)| &= |\int_0^t \left[1 - \left(\frac{s}{t}\right)^{n-2} \right] s[v^p - w^p + \alpha^{q-p} w^q] \, ds| \\ &\leq \int_0^t s|v^p - w^p| \, dt + \alpha^{q-p} \int_0^t s w^q \, ds.\end{aligned}$$

Thus, for $0 \leq t \leq t_v$, one has $|v(t)| \leq 1$, $|w(t)| \leq 1$, and

$$|w(t) - v(t)| \leq \frac{1}{n-2} \cdot \alpha^{q-p} \cdot \frac{t_v^2}{2} + \frac{p}{n-2} \cdot \int_0^t s|v - w| \, dt,$$

By Gronwall's inequality (see Hartman [57]), we obtain

$$\sup_{0 \leq t \leq t_v} |w(t) - v(t)| \leq \frac{1}{n-2} \cdot \alpha^{q-p} \cdot \frac{t_v^2}{2} \cdot \exp\left(\frac{p}{n-2} \cdot \frac{t_v^2}{2} \right),$$

which implies $\lim_{\alpha \to 0+} \sup_{0 \leq t \leq t_v} |w(t) - v(t)| = 0$. The proof is completed. □

Theorem 3.20 can be easily extended to some more general nonlinearities $f(u)$. But a more interesting problem is to obtain the exact multiplicity of solutions whose existence is ensured by Theorem 3.20. We conjecture that the number of critical points of $b(\alpha)$ in $(0,1)$ is one, and there exist exactly two solutions when $R > \bar{R}$. For the study of exact multiplicity problem, there are few results in the literature. Ouyang [96] considered the structure of positive solutions of semilinear equations $\Delta u + \lambda u + hu^p = 0$ on compact manifolds, where $\lambda > 0$, $p > 1$ are real numbers, and $h = h(x)$ is a function. Under some natural conditions, he proved that the Dirichlet problem has exactly two positive solutions. But his idea seems not applicable to problem (2.1) when f is defined in (3.49). Note that

$$f(u) = u^p - u^q, \quad 1 < p < \frac{n+2}{n-2} \leq q.$$

contains both supercritical (or critical) and subcritical terms. It has a superlinear growth for $0 < u < ((p-1)/(q-1))^{1/(q-p)}$, and a sublinear growth for $((p-1)/(q-1))^{1/(q-p)} < u < 1$.

Chapter 4
UNIQUENESS OF GROUND STATE SOLUTIONS

Recall that a ground state or fast decaying solution $u(x)$ of problem (2.2_m) is a positive radial solution in $\mathbb{R}^n$ satisfying

$$\lim_{t\to\infty} t^{(n-m)/(m-1)}u(t) = c, \quad 0 < c < \infty, \tag{4.1}$$

or equivalently,

$$\lim_{t\to\infty} t^{(n-1)/(m-1)}u'(t) = -\frac{(n-m)c}{m-1}. \tag{4.2}$$

Under assumption (F3), that is,

$$\limsup_{u\to 0+} \frac{F(u)}{u^{\epsilon_m}} = 0, \quad \text{where} \quad \epsilon_m = \frac{n(m-1)}{n-m} > 0,$$

we have given a characteristic description for ground state solutions. More precisely, if $P(u,\alpha)$ is defined as in (2.23), and $P_\alpha := \limsup_{u\to 0+} P(u,\alpha)$, then u is a ground state solution only if $P_\alpha = 0$. By using this characterization, it has been shown that a necessary condition for the existence of a ground state solution is that $H'(s)$ changes signs in $s > 0$, where $\bar{H}(s)$ is defined in (2.24).

The main purpose of this chapter is to prove a theorem on the uniqueness of ground state solutions to problem (2.2_m). In addition to basic conditions (F1) and (F3), we shall need

(H_η) there exists $0 < \eta < \infty$ such that $H'(s) \geq 0$ for $0 < s < \eta$, and $H'(s) \leq 0$ for $s \geq \eta$, and $H'(s)$ is not identically zero in any subinterval of $(0,\infty)$. There exists a $\bar{\eta} > 0$ such that $f'(s) \geq 0$ for $0 < s < \bar{\eta}$.

Note that when (F1) is assumed, the second part of condition (H_η) is very mild. Recall that $\bar{H}(s) = f(s)H(s)$, and then $\bar{H}'(s) = f'(s)H(s) + f(s)H'(s)$. Under conditions (F1) and (H_η), it holds that $\bar{H}'(s) \geq 0$ and not identically zero in any subinterval of $(0, \min\{\eta,\bar{\eta}\})$. Hence $\bar{H}(s) > 0$ in $(0, \min\{\eta,\bar{\eta}\})$.

Under assumptions (F1), (F3) and H_η), we shall show that problem (2.2_m) admits at most one ground state solution. It is worth mentioning that we do not need (F2) here.

The remainder of this chapter is divided into five sections. In Section 4.1, we state the uniqueness theorem, whose proof is based on two technical lemmas which we shall prove in Sections 4.3-4.4. The significance of the theorem is exhibited by some examples in Section 4.2. We shall also investigate the global structure of solutions to the initial value problem (2.4_m) in Section 4.5.

4.1. Main Result

We state the main result of this chapter as follows.

Theorem 4.1. *Suppose that (F1) (F3) and (H_η) hold. Then problem (2.2_m) admits at most one ground state solution.*

The proof of Theorem 4.1 relies on the following two technical lemmas. As before, we let $u_i(t) = u(t, \alpha_i)$, $i = 1, 2$ be two solutions of (2.4_m). We are going to investigate the intersection behavior of these two solutions.

Lemma 4.2. *Suppose thst (F1) and (F3) hold. If $u_1 = u(t, \alpha_1)$, $u_2 = u(t, \alpha_2)$ are two distinct ground state solutions of problem (2.2_m), then u_1 and u_2 must intersect in $(0, \infty)$. That is, there is some $t_I \in (0, \infty)$ such that*

$$u_1(t_I) = u_2(t_I), \quad \textit{and} \quad u_1'(t_I) \neq u_2'(t_I), \tag{4.3}$$

Lemma 4.3. *Suppose thst (F1), (F3) and (H_η) hold. Then u_1 and u_2 can have at most a finite number of intersection points.*

The proof of both lemmas will be postponed to Sections 4.3-4.4. Before their proof, we show here how they are used to prove Theorem 4.1.

Suppose for contradiction that $u_1 = u(t, \alpha_1)$, $u_2 = u(t, \alpha_2)$ are two different ground state solutions of (2.2_m). Then both u_1 and u_2 are strictly decreasing in $(0, \infty)$, and thus their inverses are well-defined. Let $t_i = t(u, \alpha_i)$, $0 < u \leq \alpha_i$, $i = 1, 2$, be the inverses of u_i. Without loss of generality, we assume

$$0 < \alpha_1 < \alpha_2. \tag{4.4}$$

We claim that

$$\alpha_1 > \eta, \tag{4.5}$$

where η is as in assumption H_η). In fact, if $\alpha_1 \leq \eta$, then $u_1 = u(t, \alpha_1) < \eta$ in $(0, \infty)$. But then one has $H'(u_1) \geq 0$, which implies, by Proposition 2.13, that u_1 is a slowly decaying solution. This is a contradiction.

From Lemmas 4.2-4.3 we see that there is a unique $u_I \in (0, \alpha_1)$ such that either

$$t_1(u) > t_2(u) \quad \text{in} \quad (0, u_I), \quad t_1(u_I) = t_2(u_I), \tag{4.6}$$

or

$$t_1(u) < t_2(u) \quad \text{in} \quad (0, u_I), \quad t_1(u_I) = t_2(u_I). \tag{4.7}$$

We suppose that the first case occurs. The second case can be similarly handled.

Step 1. Suppose that (4.6) holds. We claim that

$$t_1'(u) < t_2'(u) \quad \text{in} \quad (0, u_I]. \tag{4.8}$$

It is easily seen that (4.8) holds at $u = u_I$ because of (4.6) and (iii) of Proposition 2.1. If (4.8) is false, then we can find a point, say $u = u_J$, $0 < u_J < u_I$, at which

$$t_1'(u_J) = t_2'(u_J), \quad \text{and} \quad t_1(u_J) > t_2(u_J). \tag{4.9}$$

By using identity (2.21), we have

$$(m-1)(t_1'' - t_2'')(u_J) = (m-1)t_1'^2(u_J)\left(\frac{1}{t_1(u_J)} - \frac{1}{t_2(u_J)}\right) < 0.$$

Thus the function $t_1(u) - t_2(u)$ attains a strict maximum at u_J, which implies that u_J is the unique critical point of $t_1 - t_2$ in $(0, u_I)$, and

$$t_1'(u) > t_2'(u) \quad \text{in} \quad (0, u_J). \tag{4.10}$$

Now we make use of the Pohozaev-type identity (2.22) to derive a contradiction. We have

$$\begin{aligned} P_i(u) &= [H(u) - (n-m)u]\frac{t_i^{n-1}}{t_i'|t_i'|^{m-2}} - (m-1)\frac{t_i^n}{|t_i'|^m} - mF(u)t_i^n \\ &= \int_{\alpha_i}^{u} H'(\tau)t_i^{n-1}/(t_i'|t_i'|^{m-2})\, d\tau, \quad i = 1, 2. \end{aligned} \tag{4.11}$$

and

$$\lim_{u \to 0+} P_i(u) = 0, \quad i = 1, 2. \tag{4.12}$$

It follows from Lemma 3.2 that

$$\eta > u_J. \tag{4.13}$$

Since otherwise, if $\eta \le u_J$, then we can repeat the second part of the proof of Lemma 3.2 to conclude that $t_1'(u) < t_2'(u)$ in $[u_J, \alpha_1)$, which contradicts (4.9). Now, (4.13) implies

$$H'(\tau) \ge 0 \quad \text{in} \quad (0, u_J]. \tag{4.14}$$

Let the function $S_{12}(u)$ be defined as in (3.4). Then Lemma 3.4 and (4.10) imply that

$$S_{12}'(u) > 0 \quad \text{in} \quad (0, u_J). \tag{4.15}$$

Define

$$S_J = S_{12}(u_J) = t_i^{n-1}(u_J)/t_2^{n-1}(u_J). \tag{4.16}$$

Then

$$S_J > 1, \quad \text{and} \quad S_{12}(u) < S_J \quad \text{in} \quad (0, u_J). \tag{4.17}$$

Therefore,

$$\begin{aligned} P_1(u) - S_J P_2(u_J) &= \left[\frac{(m-1)}{|t_1'(u_J)|^m} + mF(u_J)\right][S_J t_2^n(u_J) - t_1^n(u_J)] \\ &= \left[\frac{(m-1)}{|t_1'(u_J)|^m} + mF(u_J)\right] t_1^{n-1}(u_J)[t_2(u_J) - t_1(u_J)] \\ &< 0. \end{aligned} \tag{4.18}$$

We also have, by using identity (4.11),

$$\begin{aligned} &P_1(u) - S_J P_2(u_J) \\ &= \int_{\alpha_1}^{u_J} H'(\tau) t_1^{n-1}/(t_1'|t_1'|^{m-2})\, d\tau - S_J \int_{\alpha_2}^{u_J} H'(\tau) t_2^{n-1}/(t_2'|t_2'|^{m-2})\, d\tau \\ &= \int_0^{u_J} H'(\tau) t_1^{n-1}/(t_1'|t_1'|^{m-2})\, d\tau - S_J \int_0^{u_J} H'(\tau) t_2^{n-1}/(t_2'|t_2'|^{m-2})\, d\tau \\ &= \int_0^{u_J} H'(\tau)[S_{12}(\tau) - S_J] t_2^{n-1}/(t_2'|t_2'|^{m-2})\, d\tau \\ &\ge 0, \end{aligned} \tag{4.19}$$

because of (4.14), (4.17) and the fact that $t_2' < 0$ in $(0, \alpha_2)$. Thus we get a contradiction.

Step 2. We claim that u_I cannot be the unique intersection point of t_1 and t_2 in $(0, \alpha_1)$. Suppose to the contrary that t_1 and t_2 only intersect at u_I. Then, in addition to (4.6) and (4.8), we have

$$t_1(u) < t_2(u) \quad \text{in} \quad (u_I, \alpha_1). \tag{4.20}$$

And we can use the argument in the first part of the proof of Lemma 3.2 to demonstrate that

$$t_1'(u) < t_2'(u) \quad \text{in} \quad (u_I, \alpha_1).$$

Combining this inequality with (4.8) we have

$$t_1'(u) < t_2'(u) \quad \text{in} \quad (0, \alpha_1). \tag{4.21}$$

At this case, we define

$$S_\eta = S_{12}(\eta).$$

Thus, by Lemma 3.4 again,

$$S_{12}(u) > S_\eta \quad \text{if} \quad 0 < u < \eta, \quad \text{and} \quad S_{12}(u) < S_\eta \quad \text{if} \quad \eta < u < \alpha_1. \tag{4.22}$$

As above, we calculate $P_1(\eta) - S_\eta P_2(\eta)$ by two different ways and then to get a contradiction. Similar to the proof of (4.19), we have

$$P_1(\eta) - S_\eta P_2(\eta) = \int_0^\eta H'(\tau)[S_{12}(\tau) - S_\eta] t_2^{n-1}/(t_2'|t_2'|^{m-2})\, d\tau < 0,$$

and

$$\begin{aligned} &P_1(\eta) - S_\eta P_2(\eta) \\ &= \int_{\alpha_1}^{\eta} H'(\tau)[S_{12}(\tau) - S_\eta] t_2^{n-1}/(t_2'|t_2'|^{m-2})\, d\tau \\ &\quad - S_\eta \int_{\alpha_2}^{\alpha_1} H'(\tau) t_2^{n-1}/(t_2'|t_2'|^{m-2})\, d\tau \\ &\geq \int_{\alpha_1}^{\eta} H'(\tau)[S_{12}(\tau) - S_\eta] t_2^{n-1}/(t_2'|t_2'|^{m-2})\, d\tau \\ &> 0, \end{aligned}$$

because of H_η), (4.22) and the fact that $0 < \eta < \alpha_1 < \alpha_2$. We again arrive at a contradiction.

Step 3. As is the last step to complete the proof of Theorem 4.1, we claim that u_I must be the unique intersection point of t_1 and t_2 in $(0, \alpha_1)$. If this claim is proved, then Theorem 4.1 readily follows.

Suppose to the contrary that there are at least two intersection points. Then we can find a $u = u_{II}$ which is the second intersection point of t_1 and t_2 in the sense that

$$0 < u_I < u_{II} < \alpha_1, \quad t_2(u) > t_1(u) \quad \text{in} \quad (u_I, u_{II}). \tag{4.23}$$

Then there is some $u = u_d$ such that

$$u_I < u_d < u_{II}, \quad t_1'(u_d) = t_2'(u_d), \quad \text{and} \quad t_1'(u) < t_2'(u) \quad \text{in} \quad (0, u_d). \tag{4.24}$$

At this case, we define

$$S_d = S_{12}(u_d). \tag{4.25}$$

Making use of Lemma 3.4 again, we have

$$S_{12}(u) > S_d \quad \text{in} \quad (0, u_d), \tag{4.26}$$

and by Lemma 3.2,

$$u_d < \eta. \tag{4.27}$$

As in the proof of (4.18), we obtain

$$\begin{aligned} &P_1(u_d) - S_d P_2(u_d) \\ &= \left[\frac{(m-1)}{|t_1'(u_d)|^m} + mF(u_d)\right] [S_d t_2^n(u_d) - t_1^n(u_d)] \\ &= \left[\frac{(m-1)}{|t_1'(u_d)|^m} + mF(u_d)\right] t_1^{n-1}(u_d)[t_2(u_d) - t_1(u_d)] \\ &> 0 \end{aligned} \tag{4.28}$$

due to (4.23) and (4.24).

On the other hand, as is similar to the proof of (4.19), we can show that $P_1(u_d) - S_d P_2(u_d) < 0$, which gives a contradiction. The proof is completed.

4.2. Applications of Theorem 4.1

As applications of Theorem 4.1, some examples are presented in this section to exhibit the significance of the result.

(I). $f(s) = s^p - s^q, \quad \frac{nm-n+m}{n-m} < p < q.$

In the case $m = 2$, Kwong et al. [69] studied the existence and uniqueness of the ground state solutions $u(x)$ of the problem

$$\begin{aligned} &\Delta u + u^p - u^q = 0 \quad \text{in} \quad \mathbb{R}^n, \\ &u > 0 \quad \text{in} \quad \mathbb{R}^n, \quad u(|x|) \to 0 \quad \text{as} \quad |x| \to \infty, \end{aligned} \tag{4.29}$$

in which $n \geq 3$ and

$$\frac{n+2}{n-2} < p < q. \tag{4.30}$$

The motivation for their study came from the investigation of the asymptotic behavior of positive solutions of the Dirichlet problem

$$\begin{aligned} &\Delta u + u^p - \epsilon u^q = 0 \quad \text{in} \quad B_1, \\ &u > 0 \quad \text{in} \quad B_1, \quad u = 0 \quad \text{on} \quad \partial B_1, \end{aligned} \tag{4.31}$$

where $\epsilon > 0$ is a small constant, B_1 is the unit ball in $\mathbb{R}^n$. When $\epsilon = 0$, (4.31) has no solution at all. On the other hand, Merle and Peletier [85] showed that for $\epsilon > 0$ and small, problem (4.31) has at least two ordered solutions. Thus, an interesting and natural question to ask is what happens to solutions of (4.31) as $\epsilon \to 0$. In [85], it was proved that the larger of the solution of (4.31) becomes unbounded at every $x \in B_1$. While the smaller one, denoted by u_ϵ, "concentrates" at the origin, i.e., $u_\epsilon(0) \to \infty$ and $u_\epsilon(x) \to 0$ when $x \neq 0$. Near the origin u_ϵ approaches the ground state solution of problem (4.29). More precisely, setting $\alpha_\epsilon = u_\epsilon(0)$ and $\alpha_g = u_g(0)$, where u_g is the ground state solution of (4.29), then in terms of scaled variables

$$y = \alpha_g^{-(p-1)/2} \alpha_\epsilon^{(p-1)/2} x, \quad v_\epsilon(y) = (\alpha_g/\alpha_\epsilon) u_\epsilon(x),$$

it holds that

$$v_\epsilon(y) \to u(y) \quad \text{as} \quad \epsilon \to 0.$$

By a suitable change of variables, Kwong et al. [69] proved the existence of a ground state solution u_g of (4.29), and

$$\left(\frac{c(p,n)}{c(q,n)}\right)^{1/(q-p)} < \alpha_g < 1,$$

where

$$c(s,n) = \frac{(n-2)s - (n+2)}{2(s+1)}.$$

The hardest part of their work was to prove the uniqueness of u_g. By employing a method developed in Kwong [67], they were able to obtain some subtle properties of the variational functions of the solutions $u(t, \alpha)$. Their Proof for the uniqueness is ingenious, but very complicated. It seems that their method is difficult to generalize to the quasilinear problem, or even to semilinear problem with a different nonlinearity.

In this section, as a simple application of Theorem 4.1, we shall give an alternative proof for the uniqueness of u_g. Our proof seems simpler and more straightforward. In fact, we shall prove the uniqueness of ground state solutions for the general quasilinear problem

$$\begin{aligned}&\operatorname{div}(|\nabla u|^{m-2}\nabla u) + u^p - u^q = 0 \quad \text{in} \quad \mathbb{R}^n,\\ &u > 0 \quad \text{in} \quad \mathbb{R}^n, \quad u(|x|) \to 0 \quad \text{as} \quad |x| \to \infty.\end{aligned} \tag{4.32}$$

Theorem 4.4. *Problem (4.32) possesses at most one ground state solution if*

$$\frac{nm - n + m}{n - m} < p < q. \tag{4.33}$$

Proof. As before, we let $u = u(t, \alpha)$ be the unique solution of the problem

$$\begin{aligned}&\left[(m-1)u'' + \frac{n-1}{t}u'\right]|u'|^{m-2} + u^p - u^q = 0,\\ &u(0) = \alpha > 0, \quad u'(0) = 0.\end{aligned}$$

Let $f(s) = s^p - s^q$, in which p and q satisfy (4.33). Then $f(s) < 0$ for all $s > 1$. By a similar argument to the proof of Proposition 3.19, we can easily prove that $u(t, \alpha)$ is strictly decreasing (or increasing) until to $t = \infty$ or a point at which it vanishes (or blows up) if $\alpha < 1$ (or $\alpha > 1$). Thus any ground state solution u_g, if it does exist, satisfies

$$0 < u_g(t) < 1, \quad t \geq 0.$$

Thus, in order to establish the uniqueness of ground state solutions, we are restricted to the region $0 < u < 1$.

Obviously, conditions (F1) and (F3) are satisfied. In view of Theorem 4.1, it suffices to show that (H_η) is fulfilled by f to complete the proof of this theorem. In fact, a routine calculation yields

$$H(s) = \frac{1}{f(s)}\left[\left(n - m - \frac{nm}{p+1}\right)u^{p+1} - \left(n - m - \frac{nm}{q+1}\right)u^{q+1}\right].$$

Hence

$$H'(s) = \frac{u^{2p}}{f^2(s)}T(v), \tag{4.34}$$

where $v = u^{q-p}$ and

$$\begin{aligned}T(v) = &\left(n - m - \frac{nm}{q+1}\right)v^2 + \left[2nm + 2m - 2n - nm\left(\frac{p}{q+1} + \frac{q}{p+1}\right)\right]v\\ &+ \left(n - m - \frac{nm}{p+1}\right).\end{aligned} \tag{4.35}$$

Note that (4.33) implies

$$n - m - \frac{nm}{p+1} > 0, \quad \text{and} \quad n - m - \frac{nm}{q+1} > 0. \tag{4.36}$$

For the quadratic form $T(v)$, we have

$$T(0) = n - m - \frac{nm}{p+1} > 0,$$

and

$$T(1) = -\frac{nm}{(p+1)(q+1)}(p-q)^2 < 0.$$

Thus there is a unique $0 < v_\eta < 1$ such that

$$H'(v) > 0 \quad \text{in} \quad (0, v_\eta), \quad \text{and} \quad H'(v) < 0 \quad \text{in} \quad (v_\eta, 1).$$

If we take $\eta = v_\eta^{1/(q-p)}$, then $H_\eta)$ is satisfied and the proof is completed. □

(II).

$$f(s) = \begin{cases} s^p, & s > 1, \\ s^q, & s \leq 1, \end{cases} \qquad m - 1 < p < \frac{nm - n + m}{n - m} < q.$$

As we mentioned earlier, the structure of positive radial solutions to the quasilinear problem (2.4_m) is well understood when $f(s) = u^p$, $p > m-1$. Let $u(t,\alpha)$ be a solution of the corresponding initial value problem (2.4_m). Then $u(t,\alpha)$ is (uniformly for all $\alpha > 0$) a slowly decaying solution, a ground state solution, or a crossing solution when p is supercritical, critical or subcritical, respectively. A natural question one may ask is what happens if $f(s)$ varies the growth order in $s > 0$. When $m = 2$, this problem has been studied recently by several authors. Kajikiya ([61-62]) has extensively studied the existence of ground state solutions of problem (2.2) and the structure of solutions of (2.4) for a class of nonlinearities which have a supercritical growth for small s and a subcritical growth for large s. A typical and important model case is

$$f(s) = \begin{cases} s^p, & s > 1, \\ s^q, & s \leq 1, \end{cases} \tag{4.37}$$

with $1 < p < \frac{n+2}{n-2} < q$. The uniqueness of ground state solutions of problem (2.2) with f defined in (4.37) was proved by Erbe and Tang [40]. They also proved that there is a unique $\alpha^* > 0$ such that $u(t,\alpha)$ is a crossing solution for $\alpha > \alpha^*$, a slowly decaying solution for $\alpha < \alpha^*$, and a ground state solution for $\alpha = \alpha^*$.

By using Theorem 4.1, we generalize a uniqueness result of [40] as follows.

Theorem 4.5. *Let $f(s)$ be defined in (4.37), and suppose that*

$$m - 1 < p < \frac{nm - n + m}{n - m} < q.$$

Then problem (2.2_m) *has at most one ground state solution.*

Proof. Although f is not C^1 in $s > 0$ due to the nondifferentiability at $s = 1$, it does not essentially affect the uniqueness of radial solution. Since f is locally Lipschitz continuous. (H_η) is satisfied if we take $\eta = 1$, (F1) and (F3) are obviously fulfilled. The theorem follows readily from Theorem 4.1. □

4.3. Proof of Lemma 4.2

We need three lemmas to complete the proof of Lemma 4.2.

Lemma 4.6. *Suppose that (F1) holds. Let $u_1 = u(t, \alpha_1)$, and $u_2 = u(t, \alpha_2)$, $0 < \alpha_1 < \alpha_2$, be two decaying solutions of* (2.4_m). *If*

$$u_1(t) < u_2(t) \quad in \quad [0, \infty), \tag{4.38}$$

and $t_1 = t(u, \alpha_1)$, $t_2 = t_2(u, \alpha_2)$ are the inverses of u_1 and u_2 respectively, then $t_1'(u) < t_2'(u)$ in $(0, \alpha_1)$.

Proof. Note that $\lim_{u \to \alpha_1^-} (t_2' - t_1')(u) = +\infty$, and $t_2(u) > t_1(u) > 0$ for $u \in (0, \alpha_1)$. If there exists $u \in (0, \alpha_1)$ such that $(t_2' - t_1')(u) \leq 0$, then we can find a $u_c \in (0, \alpha_1)$ such that

$$(t_2' - t_1')(u_c) = 0, \quad \text{and} \quad (t_2' - t_1')(u) > 0 \quad \text{if} \quad u_c < u < \alpha_1.$$

Therefore,

$$(t_2'' - t_1'')(u_c) \geq 0. \tag{4.39}$$

On the other hand, by (2.21) we can deduce

$$(m - 1)(t_2''(u_c) - t_1''(u_c)) = (n - 1)(t_2'(u_c))^2 \left(\frac{1}{t_2(u_c)} - \frac{1}{t_1(u_c)} \right) < 0,$$

which contradicts (4.39), and the lemma is proved. □

Lemma 4.7. *Suppose that (F1) holds. If u_1, u_2 are two ground state solutions of* (2.4_m) *satisfying (4.38), then*

$$\lim_{u\to 0+} \Theta(u) = 0, \tag{4.40}$$

where

$$\Theta(u) := \frac{m-1}{m} t_2^\beta \left(\frac{1}{|t_2'|^m} - \frac{1}{|t_1'|^m}\right), \quad \beta = \frac{m(n-1)}{m-1}.$$

Proof. Suppose that

$$\lim_{t\to\infty} t^{(n-m)/(m-1)} u(t, \alpha_i) = c_i \geq 0,$$

then both c_1 and c_2 are finite, and $c_1 \leq c_2$. We show that

$$c_1 = c_2. \tag{4.41}$$

Suppose (4.41) is not true, then we should have $c_1 < c_2$. But

$$t_i(u) \sim c_i^{(m-1)/(n-m)} u^{-(m-1)/(n-m)}, \quad \text{as} \quad u \to 0^+, \quad i = 1, 2.$$

Hence

$$\lim_{u\to 0+} (t_2(u) - t_1(u)) = \infty. \tag{4.42}$$

On the other hand, by the proof of Lemma 4.6 we see that $t_2(u) - t_1(u)$ is strictly increasing in $(0, \alpha_1)$, which contradicts (4.42). This proves (4.41).

Now we have

$$\begin{aligned}
\lim_{u\to 0+} \Theta(u) &= \frac{m-1}{m} \left(\lim_{u\to 0+} \frac{t_2^\beta}{|t_2'|^m} - \lim_{u\to 0+} \frac{t_2^\beta}{|t_1'|^m} \right) \\
&= \frac{m-1}{m} \left(\lim_{t\to\infty} t^\beta |u_2'|^m - \lim_{u\to 0+} \frac{t_2^\beta}{t_1^\beta} \cdot \lim_{t\to\infty} t^\beta |u_1'|^m \right) \\
&= \frac{m-1}{m} \left(\lim_{t\to\infty} (t^{(n-1)/(m-1)} |u_2'|)^m - \lim_{t\to\infty} (t^{(n-1)/(m-1)} |u_1'|)^m \right) \\
&= 0.
\end{aligned}$$

Note that the last equality follows from (4.2). □

Lemma 4.8. *If $t = t(u)$ is a solution of (2.21), then*

$$\frac{d}{du}\left[\frac{m-1}{m}\frac{1}{|t'|^m} + F(u)\right] = \frac{n-1}{t}\frac{1}{|t'|^{m-1}}. \tag{4.43}$$

This lemma can be verified by straightforward computation, and so we omit the proof.

Proof of Lemma 4.2. Suppose to the contrary that u_1 and u_2 are two distinct ground state solutions of (2.4_m) satisfying (4.38), and t_1, t_2 and $\Theta(u)$ are defined as above. Then $t_1 < t_2$ and $t_1' < t_2'$ in $(0, \alpha_1)$ and $\lim_{u\to 0^+}\Theta(u) = 0$. With the aid of identity (4.43), we have, for $u \in (0, \alpha_1)$,

$$\begin{aligned}
\frac{d\Theta}{du} &= \frac{(m-1)}{m}\cdot \beta t_2^{\beta-1} t_2' \left(\frac{1}{|t_2'|^m} - \frac{1}{|t_1'|^m}\right) + t_2^\beta \left[\frac{m-1}{m}\left(\frac{1}{|t_2'|^m} - \frac{1}{|t_1'|^m}\right)\right]' \\
&= (n-1)t_2^{\beta-1}\left(\frac{t_2'}{|t_2'|^m} - \frac{t_2'}{|t_1'|^m}\right) + (n-1)t_2^\beta\left(\frac{1}{t_2|t_2'|^{m-1}} - \frac{1}{t_1|t_1'|^{m-1}}\right) \\
&= (n-1)t_2^{\beta-1}\left(\frac{t_2'}{|t_2'|^m} + \frac{1}{|t_2'|^{m-1}} - \frac{t_2'}{|t_1'|^m} - \frac{t_2}{t_1|t_1'|^{m-1}}\right) \\
&= -(n-1)(t_2^{\beta-1}/|t_1'|^{m-1})\left(\frac{t_2'}{|t_1'|} + \frac{t_2}{t_1}\right).
\end{aligned}$$

Since $0 < t_1(u) < t_2(u)$, $t_1'(u) < t_2'(u) < 0$ in $(0, \alpha_1)$,

$$\frac{t_2'}{|t_1'|} + \frac{t_2}{t_1} > 0. \tag{4.44}$$

Thus, $\frac{d\Theta}{du} < 0$ in $(0, \alpha_1)$. Therefore it follows from Lemma 4.7 that $\Theta < 0$ in $(0, \alpha_1)$. But by the definition of $\Theta(u)$ and the fact that $t_1'(u) < t_2'(u)$, or equivalent, $|t_1'(u)| > |t_2'(u)|$, we obtain $\Theta > 0$ in $(0, \alpha_1)$, which leads to a contradiction. □

4.4. Proof of Lemma 4.3

Let $u_1 = u(t, \alpha_1)$ and $u_2 = u(t, \alpha_2)$ be two distinct ground state solutions of Problem (2.2_m) with $0 < \alpha_1 < \alpha_2$. Suppose $u_1(\xi) = u_2(\xi)$ at some $\xi > 0$. We have seen that $u_1'(\xi) \neq u_2'(\xi)$. Thus the intersection points of u_1 and u_2 are isolated. Suppose for contradiction that they intersect infinitely many times. Then the intersection points can be enumerated as $0 < \xi_1 < \xi_2 < \cdots$, and $\xi_i \to \infty$ as

$i \to \infty$. Observe that $0 < \alpha_1 < \alpha_2$ and thus $u_1 < u_2$ in $[0, \xi_1)$, and $u_1'(\xi_1) > u_2'(\xi_1)$. Generally, let k be a positive integer. Then one has

$$u_1'(\xi_{2k-1}) > u_2'(\xi_{2k-1}), \quad u_1'(\xi_{2k}) < u_2'(\xi_{2k}), \tag{4.45}$$

and

$$u_1 > u_2 \quad \text{in} \quad (\xi_{2k-1}, \xi_{2k}), \quad u_1 < u_2 \quad \text{in} \quad (\xi_{2k}, \xi_{2k+1}). \tag{4.46}$$

For $i = 1, 2$, let

$$\bar{P}_i(t) = -(n-m)u_i'|u_i'|^{m-2}u_i t^{n-1} - (m-1)|u_i'|^m t^n - mF(u_i)t^n. \tag{4.47}$$

It follows from the generalized Pohozaev-type identity (2.18) that

$$\bar{P}_i(t) = \int_0^t \bar{H}(u_i(\tau))\tau^{n-1} d\tau, \tag{4.48}$$

where $\bar{H}(s) = (n-m)sf(s) - nmF(s)$. Since both u_1 and u_2 approach zero as $t \to \infty$. We can find some integer $K > 0$ such that $u_1(\xi_{2k-1}) = u_2(\xi_{2k-1}) < \bar{\eta}$, where $\bar{\eta}$ is determined according to assumption (H_η). By the strict monotonicity of u_1 and u_2 we have

$$u_i(\xi_{2k}) < u_i(\xi_{2k-1}) < \bar{\eta}, \quad i = 1, 2, \quad \text{for all} \quad k > K. \tag{4.49}$$

Thus, condition (H_η) implies that

$$\bar{H}'(s) \geq 0 \quad \text{for} \quad 0 \leq s < u_i(\xi_{2k-1}), \quad i = 1, 2, \quad k > K. \tag{4.50}$$

Now, suppose for some $k > K$, it holds that $\bar{P}_1(\xi_{2k-1}) - \bar{P}_2(\xi_{2k-1}) \geq 0$. Then identity (4.48) yields

$$\begin{aligned}
\bar{P}_1(\xi_{2k}) - \bar{P}_2(\xi_{2k}) &= \bar{P}_1(\xi_{2k-1}) - \bar{P}_2(\xi_{2k-1}) \\
&\quad + \int_{\xi_{2k-1}}^{\xi_{2k}} [\bar{H}(u_1(\tau)) - \bar{H}(u_2(\tau))]\tau^{n-1}\, d\tau \\
&\geq \int_{\xi_{2k-1}}^{\xi_{2k}} [\bar{H}(u_1(\tau)) - \bar{H}(u_2(\tau))]\tau^{n-1}\, d\tau \\
&> 0,
\end{aligned}$$

due to (4.46) and (4.50). Thus, we conclude that for any $k > K$, it happens that either

$$\bar{P}_1(\xi_{2k-1}) < \bar{P}_2(\xi_{2k-1}), \tag{4.51}$$

or

$$\bar{P}_1(\xi_{2k}) > \bar{P}_2(\xi_{2k}). \tag{4.52}$$

Suppose at the moment that (4.51) holds for all $k > K$. Then the definition of $\bar{P}_i(t)$ yields, at $t = \xi_{2k-1}$,

$$\begin{aligned}
&-(n-m)u_1'|u_1'|^{m-2}u_1\xi_{2k-1}^{n-1} - (m-1)|u_1'|^m\xi_{2k-1}^n \\
<&-(n-m)u_2'|u_2'|^{m-2}u_2\xi_{2k-1}^{n-1} - (m-1)|u_2'|^m\xi_{2k-1}^n,
\end{aligned}$$

or

$$(n-m)\xi_{2k-1}^{n-2}u_1(u_2'|u_2'|^{m-2} - u_1'|u_1'|^{m-2}) + (m-1)\xi_{2k-1}^{n-1}(|u_2'|^m - |u_1'|^m) < 0.$$

Observe that (4.45) implies

$$u_2'|u_2'|^{m-2} - u_1'|u_1'|^{m-2} < 0, \quad \text{at} \quad t = \xi_{2k-1}.$$

Thus

$$(n-m)\xi_{2k-1}^{n-2}u_1 + (m-1)\xi_{2k-1}^{n-1} \cdot \frac{|u_2'|^m - |u_1'|^m}{u_2'|u_2'|^{m-2} - u_1'|u_1'|^{m-2}} > 0. \tag{4.53}$$

Let $b_{2k-1} = \frac{|u_2'(\xi_{2k-1})|}{|u_1'(\xi_{2k-1})|} > 1$. Then (4.53) becomes

$$(n-m)\xi_{2k-1}^{n-2}u_1 + (m-1)\xi_{2k-1}^{n-1}u_1' \cdot \frac{b_{2k-1}^m - 1}{b_{2k-1}^{m-1} - 1} > 0. \tag{4.54}$$

If we define a function $b(x) = (x^m - 1)/(x^{m-1} - 1)$. Then it can be easily verified that $b'(x) > 0$ when $m > 1$ and $x > 1$. But

$$\lim_{x \to 1+} b(x) = \frac{m}{m-1}.$$

So we have

$$\frac{b_{2k-1}^m - 1}{b_{2k-1}^{m-1} - 1} > \frac{m}{m-1}.$$

Combining this inequality and (4.54) we obtain

$$(n-m)\xi_{2k-1}^{n-2}u_1 + m\xi_{2k-1}^{n-1}u_1' > 0,$$

or equivalently,

$$(n-m)\xi_{2k-1}^{n-2}u_1|u_1'|^{m-2} + m\xi_{2k-1}^{n-1}u_1'|v_1'|^{m-2} > 0. \tag{4.55}$$

. Hence

$$\limsup_{t\to\infty}[(n-m)t^{n-2}u_1|u_1'|^{m-2}+mt^{n-1}u_1'|u_1'|^{m-2}]\geq 0. \tag{4.56}$$

On the other hand, if for any given $\bar{k}>K$, one can find $k>\bar{k}$ such that (4.51) fails, then there is a subsequence of $\{\xi_{2k}\}$ such that (4.52) always holds. For simplicity of notations, denote this subsequence again by $\{\xi_{2k}\}$. Then we can use a similar argument as above to obtain

$$(n-m)\xi_{2k}^{n-2}u_2|u_2'|^{m-2}+m\xi_{2k}^{n-1}u_2'|u_2'|^{m-2}>0.$$

Thus,

$$\limsup_{t\to\infty}\{(n-m)t^{n-2}u_2|u_2'|^{m-2}+mt^{n-1}u_2'|u_2'|^{m-2}\}\geq 0. \tag{4.57}$$

It remains to show that both (4.56) and (4.57) are impossible. In fact, let

$$\lim_{t\to\infty}t^{(n-m)/(m-1)}u_i=c_i>0,\quad i=1,2. \tag{4.58}$$

Then

$$\lim_{t\to\infty}t^{(n-1)/(m-1)}u_i'=-\frac{n-m}{m-1}c_i<0,\quad i=1,2. \tag{4.59}$$

Therefore, for $i=1,2$,

$$\begin{aligned}
&\lim_{t\to\infty}[(n-m)t^{n-2}u_i|u_i'|^{m-2}+mt^{n-1}u_i'|u_i'|^{m-2}]\\
=&\lim_{t\to\infty}[(n-m)t^{(n-m)/(m-1)}u_i|t^{(n-1)/(m-1)}u_i'|^{m-2}-m|t^{(n-1)/(m-1)}u_i'|^{m-1}]\\
=&(n-m)c_i\left(\frac{n-m}{m-1}\right)^{m-2}c_i^{m-2}-m\left(\frac{n-m}{m-1}\right)^{m-1}c_i^{m-1}\\
=&-\left(\frac{n-m}{m-1}\right)^{m-1}c_i^{m-1}<0,
\end{aligned}$$

which contradicts (4.56) and (4.57). The proof is completed.

Remark 4.8. When $f(s)<0$ or $F(s)<0$ in $0<s<\epsilon$ for some small $\epsilon>0$. Peletier and Serrin [98] [99] have proved the same result as Lemma 4.3 in the special case $m=2$. They made use of an energy function argument and showed essentially that if u_1 and u_2 are two distinct decaying solutions, then they do not intersect in $0<u_1<\epsilon$.

In our situation where $f(s)>0$ for $s>0$, the proof is different and much more complicated. The delicate asymptotic estimates (4.55)-(4.56) play key roles in our proof. In the study of the global structure of positive radial solutions to a class of semilinear elliptic equations, Erbe and Tang [42] have detected an interesting intersection phenomenon for slowly decaying solutions. More precisely, let $f(s) = s^q$, $q > \frac{n+2}{n-2}$ in a small right neighborhood of $s = 0$, $3 \leq n \leq 10$, and $\bar{u}(t)$ be a slowly decaying solution with $\bar{u}(0) > 0$ sufficiently small. Then to each integer $k > 0$, one can find another slowly decaying solution $\bar{u_k}$ such that $\bar{u_k}$ intersects $\bar{u}$ more than k times in $t > 0$.

Remark 4.9. Lemma 4.3 and its proof are of intrinsic interest. The lemma can be extended in various circumstances. Since what we essentially need in the proof is the asymptotic estimates of ground state solutions and the requirement that $\bar{H}(s)$ is increasing for small $s > 0$.

4.5. A Global Structure Theorem

Let $u(t, \alpha)$, $\alpha > 0$ be the solution of problem (2.4_m). Let us define some subsets of $(0, \infty)$ as follows,

$$C := \{\alpha | \alpha > 0, \quad u(t, \alpha) \text{ is a crossing solution }\},$$
$$D_s := \{\alpha | \alpha > 0, \quad u(t, \alpha) \text{ is a slowly decaying solution}\},$$
$$D_f := \{\alpha | \alpha > 0, \quad u(t, \alpha) \text{ is a fast decaying or ground state solution}\}.$$

Proposition 4.10. *Suppose that (F1) holds, and there exists some $\eta^* > 0$ such that $\bar{H}(s) > 0$ in $(0, \eta^*)$. Then*

(i) $(0, \infty) = C \cup D_s \cup D_f$.

(ii) The sets C and D_s are open sets, and D_f is closed.

Proof. (i) follows from Propositions 2.1 and 2.4. We prove (ii). That the set C is open follows from the continuous dependence of solutions of (2.4_m) on the initial data α. It remains to show that D_s is open. Let $\alpha_s \in D_s$. We need to show that if α is sufficiently close to α_s, then $\alpha \in D_s$. Let $u_s = u(t, \alpha_s)$, and

$$\bar{P}_s(t) = -(n-m)u_s'|u_s'|^{m-2}u_s t^{n-1} - (m-1)|u_s'|^m t^n - mF(u_s)t^n.$$

Then the characteristic property of slowly decaying solutions implies that there is a large $t = T_s$ such that

$$\bar{P}_s(T_s) > 0,$$

and

$$0 < u_s(T_s) < \eta^*.$$

Now, if α is chosen to be sufficiently close to α_s, then we can use the continuous dependence of solutions as well as the first derivatives of solutions of (2.4_m) on the initial data to conclude that

$$0 < u(T_s) < \eta^* \quad \text{and} \quad \bar{P}(T_s) > 0. \tag{4.60}$$

Thus, for any $t > T_s$, whenever $u(t) > 0$, identity (2.18) yields

$$\bar{P}(t) = \bar{P}(T_s) + \int_{T_s}^{t} \bar{H}(u(\tau))\tau^{n-1}d\tau.$$

By (4.60) and the strict monotonicity of $u(t)$, this identity implies

$$\bar{P}(t) \geq \bar{P}(T_s) > 0, \quad \text{in} \quad t \geq T_s \quad \text{and} \quad u(t) > 0. \tag{4.61}$$

Note that if there was any point $t = T_0 > T_s$ such that $u(T_0) = 0$, then the definition of $\bar{P}(t)$ would imply

$$\bar{P}(T_0) = -(m-1)|u'(T_0)|^m T_0^n < 0,$$

which contradicts (4.61). So $u(t)$ must be a decaying solution. Letting $t \to \infty$ in (4.61), we obtain

$$\lim_{t\to\infty} \sup \bar{P}(t) \geq \bar{P}(T_s) > 0.$$

It follows again from the characteristic property of solutions of (2.4_m) that $u(t)$ must be a slowly decaying solution. The proof is completed. □

As a simple consequence of this proposition, we have

Proposition 4.11. *Under the assumption of Proposition 4.10, the existence of a crossing solution of* (2.4_m) *implies the existence of a ground state solution.*

Proof. For any $0 < \alpha < \eta^*$, $u(t, \alpha)$ must be a slowly decaying solution. If there is some α, say, $\alpha = \alpha_c$, such that $u(t, \alpha_c)$ is a crossing solution, then $\alpha_c > \eta^*$. By the openness of the sets C and D_s and the first assertion of Proposition 4.10 it follows that $D_f \cap (\eta^*, \alpha_c) \neq \emptyset$, the empty set. This completes the proof. □

Note that under the assumptions of Proposition 4.10, if $u(t, \alpha^*)$ is a ground state solution, then $\alpha^* > \eta^*$. Also note that the existence of η^* is ensured by (H_η). Now we state a theorem on the structure of solutions of the initial value problem (2.4_m). The proof of this result follows easily from Propositions 4.10-4.11, and so we omit it.

Theorem 4.12. *Under the assumptions of Theorem 4.1, the structure of* (2.4_m) *is one of the following types:*

i) type S: every solution $u(t,\alpha)$, $\alpha > 0$ *is a slowly decaying solution;*

ii) type M: there exists an $\alpha = \alpha^*$, $\alpha^* > \eta^*$ *such that* $u(t,\alpha^*)$ *is a ground state solution;* $u(t,\alpha)$ *is a slowly decaying solution if* $0 < \alpha < \alpha^*$, *a crossing solution if* $\alpha > \alpha^*$;

iii) type FS: there exists a unique $\alpha = \alpha^*$, $\alpha^* > \eta^*$ *such that* $u(t,\alpha)$ *is a ground state solution if* $\alpha = \alpha^*$, *and a slowly decaying solution otherwise.*

Remark 4.13. We conjecture that, under the assumptions of Theorem 4.12, type FS is impossible. That is, the existence of crossing solutions is equivalent to the existence of a ground state solution.

Remark 4.14. Type S is possible since in Theorem 4.12 we do not require that $\bar{H}(s)$ becomes negative for large s. A necessary condition for the existence of a crossing solution or a ground state solution is that $\bar{H}(s)$ assumes negative values for some $s > 0$. In addition to the conditions of Theorem 4.12, if we assume further that there is some $\xi > 0$ such that $\bar{H}(s) < 0$ for $s > \xi$, then it is still unclear whether or not the existence of crossing solutions is ensured.

Chapter 5
A VARIATIONAL FUNCTIONAL APPROACH

In the previous chapters, by using various types of Pohozaev Identities, we have successfully established some criteria for the uniqueness of positive radial solutions to the Dirichlet problems for quasilinear and semilinear elliptic equations. In particular, Identity (2.22) we developed in Chapter 2 is an effective tool in the study of the uniqueness problem. But we have to mention that our previous approaches to the uniqueness problem can not be successfully adapted to study the nonlinear Dirichlet problem when Ω is an annulus. Furthermore, all the Pohozaev-type identities we have employed so far in this work are only related to the nonlinearity f which is independent on the radius t.

In this chapter we shall continue the study of the uniqueness problem by using a so-called Kolodner-Coffman method due to Kolodner [66] and Coffman [21]. We shall limit ourselves here to semilinear elliptic equations. But the nonlinearity is no longer independent of t, and the domain Ω may be a finite ball or a finite annulus. We shall also consider the uniqueness of positive radial solutions in the whole space R^n for some special cases. More precisely, we are interested in the uniqueness of radial solutions of the problems

$$\begin{aligned} &\Delta u + f(t,u) = 0, \quad \text{in} \ \ \Omega \\ &u > 0 \quad \text{in} \quad \Omega, \ \ u = 0 \quad \text{on} \quad \partial\Omega, \\ &\Omega = \{x : \ x \in R^n, \ |x| < b\}, \\ \textit{or} \quad &\Omega = \{x : \ x \in R^n, \ a < |x| < b\}, \quad 0 < a < b < \infty, \end{aligned} \tag{5.1}$$

and

$$\begin{aligned} &\Delta u + f(t,u) = 0, \quad \text{in} \ \ R^n \\ &u > 0 \quad \text{in} \quad R^n, \quad u \to 0 \quad \text{as} \quad t \to \infty. \end{aligned} \tag{5.2}$$

We assume again $n \geq 3$. To use a shooting argument, we consider the following initial value problem

$$\begin{aligned} &u'' + \frac{n-1}{t}u' + f(t,u) = 0, \quad t > 0, \\ &u(0) = \alpha > 0, \quad u'(0) = 0, \end{aligned} \tag{5.3}$$

or

$$\begin{aligned} &u'' + \frac{n-1}{t}u' + f(t,u) = 0, \quad t \geq a, \quad a > 0, \\ &u(a) = 0, \quad u'(a) = \alpha > 0. \end{aligned} \tag{5.4}$$

For the sake of simplicity, we assume in the whole chapter that the function $f(t,u)$ is continuously differentiable with respect to both $t > 0$ and $u \geq 0$, and that problems (5.3) and (5.4) have a unique solution for every $\alpha > 0$. Obviously, this assumption can be weakened in various circumstances. As before, we let $u(t,\alpha)$ denote the unique solution of (5.3) or (5.4). The variational function of $u(t,\alpha)$ with respect to $\alpha > 0$ is defined by

$$\phi(t,\alpha) = \frac{\partial u(t,\alpha)}{\partial \alpha}.$$

It follows that $\phi(t,\alpha)$ is the solution of the following variational problem

$$\begin{aligned} &\phi'' + \frac{n-1}{t}\phi' + f_u(t,u)\phi = 0, \\ &\phi(0) = 1, \quad \phi'(0) = 0, \end{aligned} \tag{5.5}$$

or

$$\begin{aligned} &\phi'' + \frac{n-1}{t}\phi' + f_u(t,u)\phi = 0, \\ &\phi(a) = 0, \quad \phi'(a) = 1, \quad a > 0. \end{aligned} \tag{5.6}$$

When $u(t,\alpha)$ is the unique solution of (5.3) and it vanishes in $t > 0$, we denote again by $b(\alpha)$ its first zero. Similarly, if $u(t,\alpha)$ is the unique solution of (5.4) and it vanishes in $t > a$, then $b(\alpha)$ denotes its first zero in $t > a$. We allow $b(\alpha) = \infty$ referring to the case when $u(t,\alpha)$ is defined and positive in $t > 0$ or $t > a$. Thus, when $b(\alpha) < \infty$, it holds that

$$u(b(\alpha),\alpha) = 0.$$

If we differentiate both sides of the identity with respect to α we have

$$u'(b(\alpha),\alpha)b'(\alpha) + \phi(b(\alpha),\alpha) = 0.$$

If $f(t,0) \equiv 0$ for all $t \geq 0$, then $u \equiv 0$, $t \geq 0$ is the unique solution of the equation of (5.3) or (5.4) satisfying $u(t_0) = u'(t_0) = 0$ for any $t = t_0 > 0$. Therefore, the uniqueness of solutions of initial value problems of ordinary differential equations implies that $u'(b(\alpha),\alpha) < 0$. Moreover, in view of the last identity, it is evident that $b'(\alpha) < 0 (> 0)$ if and only if $\phi(b(\alpha),\alpha) < 0 (> 0)$.

Thus, in order to study the uniqueness of positive radial solutions to problems (5.1)-(5.2), it is an effective way to investigate the oscillatory behavior of the variational functions. In fact, the essence of the Kolodner-Coffman method is to show that every solution $u(t,\alpha)$ is nondegenerate in the sense that $\phi(t,\alpha)$ does not vanish at $t = b(\alpha)$ once $b(\alpha)$ is defined and finite. This method has been employeed by several authors, including Ni and Nussbaum [90], Kwong [67], and McLeod [82], etc. A common approach in their proof was to use some Sturm-type Comparison Theorems to show that $\phi(t, \alpha)$ has exactly one zero between the initial state of t and $b(\alpha)$, and more importantly, $\phi(b(\alpha), \alpha) < 0$. Recently, Erbe and Tang [42-43] have developed a modified approach and given some easily verified conditions on $f(t, u)$ that are sufficient for $\phi(b(\alpha), \alpha) \neq 0$ as $b(\alpha) < \infty$.

The remainder of this chapter is organized as follows. We present a principal lemma and give some preliminary results on the properties of $u(t, \alpha)$ and $\phi(t, \alpha)$ in Section 5.1. Some conditions on f which guarantee the uniqueness of positive radial solutions to problem (5.1) are given in Section 5.2. In Sections 5.3-5.4 we study the uniqueness of ground state solutions of problem (5.2) when f is a separable function of t and u, i.e., $f(t, u) \equiv K(t)\gamma(u)$. Some examples are examined in Section 5.5. The main body of this chapter is an extension of the studies of Erbe and Tang [40, 42-43].

5.1. Preliminary Results

In the beginning of this section, we present and prove a principal lemma. It is fundamental in our approach to the uniqueness problem of radial solutions of (5.1) in a finite ball or annulus. Throughout the rest of this chapter, we assume that

$$f(t, 0) \equiv 0 \quad \text{for all} \quad t > 0.$$

Lemma 5.1. *Problem (5.1) has at most one positive radial solution provided that*

(H1) $\phi(b(\alpha), \alpha) \neq 0$ *whenever* $u(t, \alpha)$ *has a finite zero* $b(\alpha)$.

(H2) For any given $0 < \alpha_1 < \alpha_2$, *if at least one of* $u(t, \alpha_1)$ *and* $u(t, \alpha_2)$ *has a finite zero, then they intersect at least once before one of them vanishes.*

Proof. As in Chapter 4, we let N denote the set of $\alpha > 0$ for which the solution $u(t, \alpha)$ of (5.3) or (5.4) has a finite zero $b(\alpha)$. Obviously, we are only interested in the case when N is a nonempty set. N is an open set, and $u'(b(\alpha), \alpha) < 0$ for all $\alpha \in N$. Furthermore, if $0 < \bar{\alpha} < \infty$, $\bar{\alpha} \notin N$, and if $\alpha_n \in N$ and $\alpha_n \to \bar{\alpha}$ as $n \to \infty$, then $b(\alpha_n) \to +\infty$.

As we have seen, condition $(H1)$ implies that $b'(\alpha) \neq 0$ for all $\alpha \in N$. Suppose that J is a component of N, that is, J is an open interval in N whose end points are not in N, then $b'(\alpha)$ does not change sign in J. Thus $b(\alpha)$ is either strictly decreasing or increasing in J. For any given $b > 0$, there is at most one $\alpha \in J$ so that $b(\alpha) = b$. It is readily seen that the proof will be completed if it can be shown that N has only one component.

Suppose to the contrary that N has more than one component. Let $J_1 = (j_0, j_1)$, $J_2 = (j_2, j_3)$ be two disjoint components of N with $0 < j_1 \leq j_2 < \infty$. Then

$$\lim_{\alpha \to j_1^-} b(\alpha) = \lim_{\alpha \to j_2^+} b(\alpha) = +\infty.$$

Since $b(\alpha)$ is strictly monotone in each component of N, we see that $b'(\alpha) > 0$ in J_1, and $b'(\alpha) < 0$ in J_2. Hence,

$$\lim_{\alpha \to j_0^+} b(\alpha) < +\infty, \quad \lim_{\alpha \to j_3^-} b(\alpha) < +\infty.$$

It follows that $j_0 = 0$ and $j_3 = +\infty$. Moreover, $N = (0, j_1) \cup (j_2, \infty)$. We shall show that this violates (H2).

Now we have $J_1 = (0, j_1)$ and $b'(\alpha) > 0$ in J_1. Let $b_0 = \lim_{\alpha \to 0^+} b(\alpha)$, then $0 \leq b_0 < \infty$. If Ω is a finite ball, then $u(t, \alpha)$ is a solution of (5.3). Pick an $\epsilon > 0$ sufficiently small, and define

$$\tilde{U} := \inf\{u(t, j_1) |\ 0 \leq t < b_0 + \epsilon\}.$$

Then $\tilde{U} > 0$, since $u(t, j_1) > 0$ for all $t > 0$. On the other hand, we can find an $\alpha_0 > 0$ sufficiently small so that $b(\alpha_0) < b_0 + \epsilon$ and

$$\bar{U} := \sup\{u(t, \alpha_0) | 0 \leq t \leq b(\alpha_0)\} < \tilde{U}/2.$$

Thus $u(t, \alpha_0)$ does not intersect $u(t, j_1)$ before $u(t, \alpha_0)$ vanishes, which contradicts $(H2)$.

On the other hand, if Ω is an annulus, then $u(t, \alpha)$ solves problem (5.4). Thus $u(a, \alpha) = 0$, $u'(a, \alpha) = \alpha$, and $b_0 \geq a$. Pick an $\epsilon > 0$ sufficiently small so that $u'(t, j_1) > \frac{j_1}{2}$ for $a \leq t \leq a + \epsilon$. Define

$$\tilde{U} := \inf\{u(t, j_1) \mid a + \epsilon \leq t \leq b_0 + \epsilon\}.$$

Then again $\tilde{U} > 0$. If $\alpha_0 > 0$ is sufficiently small, then $b(\alpha_0) < b_0 + \epsilon$ and $u'(t, \alpha_0) < \frac{j_1}{2}$ for $a \leq t \leq \min\{a + \epsilon, b(\alpha_0)\}$, and when $b(\alpha_0) \geq a + \epsilon$,

$$\bar{U} = \sup\{u(t, \alpha_0) \mid a + \epsilon \leq t \leq b(\alpha_0)\} < \tilde{U}/2.$$

Thus $u(t, \alpha_0)$ does not intersect $u(t, j_1)$ at all. (H2) is again violated, and the proof is completed. □

Remark 5.2. From the proof of Theorem 5.1 it follows that if condition $(H2)$ is not satisfied, then problem (5.2) has at most two radial solutions.

Now we prepare here a series of preliminary results. Note that in what follows $u(t, \alpha)$ refers to the solution of either (5.3) or (5.4) if no further comments are made. To simplify our statements, we let $a(\alpha)$ denote the initial state of $u(t, \alpha)$. That is, $a(\alpha) = 0$ in (5.3) and $a(\alpha) = a > 0$ in (5.4).

Proposition 5.3. *Suppose that $f(t, u) > 0$ when $t > 0$ and $u > 0$. Then we have*

(i) if $u = u(t, \alpha), \alpha > 0$, is the solution of problem (5.3), then $u'(t, \alpha) < 0$ for all $t \in (0, b(\alpha))$,

(ii) if $u = u(t, \alpha), \alpha > 0$, is the solution of the initial value problem (5.4) such that $b(\alpha) < \infty$, then there exists a unique $t = c(\alpha)$ such that $a < c(\alpha) < b(\alpha)$ and

$$u'(t, \alpha) > 0, \text{ for } a < t < c(\alpha), \text{ and } u'(t, \alpha) < 0 \text{ for } c(\alpha) < t < b(\alpha). \quad (5.7)$$

Proof. The proof of (i) is the same as that of (iv) of Proposition 2.1. We prove (ii) here. From the positivity of u and the boundary conditions we see that u has at least one critical point in $(a, b(\alpha))$. At any such critical point, say $t = t_0$, one has $u''(t_0) = -f(t_0, u) < 0$. Thus u assumes a strict maximum value at $t = t_0$. It follows that the critical point is unique, and (ii) is proved. □

Proposition 5.4. *Suppose that f satisfies*

(FT1) $0 < f(t, u) < u f_u(t, u)$ *for* $t > 0$ *and* $u > 0$.

Let $u = u(t, \alpha), \alpha > 0$, be the solution of problem (5.3) or (5.4) with $b(\alpha) < \infty$, and let $\phi = \phi(t, \alpha)$ be its variational function. Then we have

(i) $\phi(t, \alpha)$ vanishes in $(a(\alpha), b(\alpha))$,

(ii) if $\alpha_0 > \alpha$, then $u_0 = u(t, \alpha_0)$ intersects u at some point in $(a(\alpha), b(\alpha))$.

Proof. Rewrite the equations of u and ϕ as

$$(t^{n-1}u')' = -t^{n-1}f(t, u), \tag{5.8}$$

and

$$(t^{n-1}\phi')' = -t^{n-1}f_u(t, u)\phi. \tag{5.9}$$

Multiply both sides of (5.8) by ϕ,and (5.9) by u, then subtract the resulting identities to obtain:

$$[t^{n-1}(u'\phi - u\phi')]' = -t^{n-1}[f(t, u) - uf_u(t, u)]\phi. \tag{5.10}$$

By the initial conditions of u and ϕ, we have

$$t^{n-1}(u'\phi - u\phi') = 0, \qquad \text{at } t = a(\alpha).$$

Suppose to the contrary that ϕ does not vanish in $(a(\alpha), b(\alpha))$. Then $\phi(t, \alpha) > 0$ for all $t \in (a(\alpha), b(\alpha))$. It follows from (5.10) and (FT1) that

$$t^{n-1}(u'\phi - u\phi') > 0, \qquad \text{at } t = b(\alpha).$$

Thus,

$$b(\alpha)^{n-1}u'(b(\alpha), \alpha)\phi(b(\alpha), \alpha) > 0.$$

But this is impossible because $u'(b(\alpha)) < 0$ and $\phi(b(\alpha)) \geq 0$, and we obtain a contradiction. Therefore (i) is proved. In order to prove (ii), we can use a similar argument as above. In this case, the argument is based on the identity

$$[t^{n-1}(u'u_0 - uu_0')]' = -t^{n-1}[f(t, u)u_0 - f(t, u_0)u], \tag{5.11}$$

rather than (5.10). We omit the details. □

Proposition (5.4) can also be simply proved by using the well-known Sturm Comparison Principle (see Hartman [57]). As another simple application of this principle, we have

Proposition 5.5. *For any given $\alpha > 0$ with $b(\alpha) < \infty$, $\phi(t, \alpha)$ has at most a finite number of zeros in $[a(\alpha), b(\alpha)]$.*

Proof. If $b(\alpha) < \infty$, then $u(t, \alpha)$ is bounded in $[a(\alpha), b(\alpha)]$. Let $0 < M < \infty$ be such that

$$|f_u(t, u(t, \alpha))| \leq M \quad \text{in} \quad [a(\alpha), b(\alpha)].$$

Let $v(t)$ be an arbitrary solution of the linear equation

$$v'' + \frac{n-1}{t}v' + Mv = 0.$$

If $v(t)$ is defined in $[a(\alpha), b(\alpha)]$, then it can have at most a finite number of zeros in this range. But the Sturm Comparison Principle implies that $\phi(t, \alpha)$ can not have two consecutive zeros within any two consecutive zeros of v. Thus $\phi(t, \alpha)$ has at most a finite number of zeros in $[a(\alpha), b(\alpha)]$, and the proof is completed.

For any real numbers $h \geq 1$ and $\alpha > 0$, we introduce a function

$$G_h(t) = G_h(t, u, \alpha) := u(t, \alpha) + \frac{h-1}{2}tu'(t, \alpha). \tag{5.12}$$

This function is well-defined in $[a(\alpha), b(\alpha)]$. It is useful in investigating the oscillatory behavior of $\phi(t, \alpha)$. Corresponding to a solution $u = u(t, \alpha)$, let L_u denote the linear operator defined as

$$L_u(\phi) := \frac{d^2\phi}{dt^2} + \frac{n-1}{t}\frac{d\phi}{dt} + f_u(t, u)\phi. \tag{5.13}$$

Then it is straightforward to verify

$$L_u(G_h(t)) = uf_u(t, u) - hf(t, u) - \frac{h-1}{2}tf_t(t, u). \tag{5.14}$$

Note that if $h = 1$, then $G_1(t) = u(t)$, and

$$L_u(u(t)) = uf_u(t, u) - f(t, u).$$

To close this section, we state two propositions on $G_h(t)$. In the first one we establish an identity which is similar to (5.10). The proofs of both propositions are straightforward, and we omit them.

Proposition 5.6. *For any given $h \geq 1$ and $\alpha > 0$, we have*

$$[t^{n-1}(G_h'(t)\phi(t,\alpha) - G_h(t)\phi'(t,\alpha))]' = t^{n-1}\phi(t,\alpha)L_u(G_h(t)). \tag{5.15}$$

Proposition 5.7. *Suppose that $h \geq 1$ and $\alpha > 0$. We have*

(i) if $u = u(t,\alpha)$ is the solution of problem (5.3) with $b(\alpha) < \infty$, and $u'(t,\alpha) < 0$ for $0 < t \leq b(\alpha)$, then $G_h(t)$ has at least one zero in $0 < t \leq b(\alpha)$. Denote the first zero of $G_h(t)$, $h \geq 1$, by τ_h. Then τ_h is a strictly decreasing function of h, and

$$\lim_{h\to 1+} \tau_h = b(\alpha), \quad \lim_{h\to\infty} \tau_h = 0; \tag{5.16}$$

(ii) if $u = u(t,\alpha)$ is the solution of problem (5.4) with $b(\alpha) < \infty$, and $u(t,\alpha)$ has a unique strict maximum at $c(\alpha)$ over $(a, b(\alpha)$, then $G_h(t)$ has at least one zero in $c(\alpha) < t \leq b(\alpha)$, and τ_h is again a strictly decreasing function of h, and

$$\lim_{h\to 1+} \tau_h = b(\alpha), \quad \lim_{h\to\infty} \tau_h = c(\alpha). \tag{5.17}$$

5.2. Uniqueness of Radial Solutions of (5.1)

In this section we are concerned with the uniqueness of radial solutions of problem (5.1). Our discussion is basically based on Lemma 5.1. From Lemma 5.4 it is seen that condition (FT1) implies (H2). As we shall see, (H1) is satisfied by a various class of nonlinearities. On the other hand, it is usually extremely difficult to show that the variational function ϕ is nondegenerate, i.e., (H2) is satisfied. The main purpose of this section is to find some easily verified conditions on f so that (H1) is fulfilled.

The following assumption plays an essential role in our discussion.

(FTH) Let $h \geq 1$, $\alpha > 0$, and $u = u(t,\alpha)$. The function

$$\psi_h(t) = \psi_h(t,\alpha,u,h) := uf_u(t,u) - hf(t,u) - \frac{h-1}{2}tf_t(t,u)$$

does not change signs from positive to negative as t increases from $a(\alpha)$ to $b(\alpha)$.

Theorem 5.8. *Problem (5.1) has at most one positive radial solution provided that*

(i) $f(t,u)$ satisfies (FT1) and (FTH), in the case that Ω is a finite ball, or,

(ii) $f(t,u)$ satisfies (FT1), (FTH) and

(FT2) $f(t,u) + \frac{1}{2} t f_t(t,u) > 0, \quad$ *for all* $\quad t > 0, \quad u > 0,$

in the case that Ω is a finite annulus.

Proof. Let α be such that $b(\alpha) < \infty$. In view of Theorem 5.1 and Lemma 5.4, we only need to show that $\phi(b(\alpha), \alpha) \neq 0$.

Suppose for contradiction that $\phi(b(\alpha), \alpha) = 0$. By Lemmas 5.4-5.5, ϕ vanishcs at least once and at most finitely many times in $a(\alpha) < t < b(\alpha)$. Thus we can find a point $t = \tau(\alpha)$ such that

$$a(\alpha) < \tau(\alpha) < b(\alpha), \quad \phi(\tau(\alpha), \alpha) = 0, \tag{5.18}$$

and

$$\phi(\tau(\alpha), \alpha) \neq 0, \quad \text{in} \quad (\tau(\alpha), b(\alpha)). \tag{5.19}$$

That is, $\tau(\alpha)$ is the last zero of ϕ in $t < b(\alpha)$.

(i). In the case that Ω is a finite ball, u and ϕ are the solutions of problems (5.3) and (5.5), respectively. $G_h(t, \alpha)$ satisfies

$$\begin{aligned} &L_u(G_h(t,\alpha)) = G_h'' + \frac{n-1}{t} G_h' + f_u(t,u) G_h = \psi_h(t) \\ &G_h(0) = \alpha, \quad G_h'(0) = 0. \end{aligned} \tag{5.20}$$

It follows from Proposition 5.3 that $u'(t,\alpha) < 0$ for $0 < t < b(\alpha)$. By using Proposition (5.7), the first zero of $G_h(t)$, denoted by τ_h, is defined for $h \geq 1$. τ_h is a strictly decreasing function of h and satisfies (5.16). Thus there is a unique $1 < \bar{h} < \infty$ such that $\tau_{\bar{h}} = \tau(\alpha)$.

We claim that $\psi_{\bar{h}}(\tau_{\bar{h}}) \geq 0$. In fact, if $\psi_{\bar{h}}(\tau_{\bar{h}}) < 0$, then condition (FTH) implies that $\psi_{\bar{h}}(t) < 0$ for all $t \in [0, \tau(\alpha)]$. By the equation of ϕ in (5.5), identities in (5.20) and a similar argument as in the proof of Proposition 5.4, we can show that $G_{\bar{h}}(t)$ must have a zero in $0 < t < \tau(\alpha)$, which contradicts the definition of $\tau_{\bar{h}}$. This claim can also be proved by using the Sturm Comparison Principle. Actually, if $\psi_{\bar{h}}(t) < 0$ for all $t \in [0, \tau(\alpha)]$, then $G_{\bar{h}}(t)$ oscillates faster than ϕ over $[0, \tau(\alpha)]$ whenever $G_{\bar{h}}(t)$ is positive. We again obtain a contradiction.

It follows from the above claim and (FTH) that $\psi_{\bar{h}}(t) \geq 0$ for all $t \in [\tau(\alpha), b(\alpha)]$. Moreover, it holds that

$$\psi_{\bar{h}}(t) \geq 0, \quad \psi_{\bar{h}}(t) \text{ is not identically zero for } t \in [\tau(\alpha), b(\alpha)]. \tag{5.21}$$

Since if $\psi_{\bar{h}}(t) \equiv 0$ in $[\tau(\alpha), b(\alpha)]$, then both $G_{\bar{h}}(t)$ and ϕ are solutions of the same homogeneous linear differential equation in $[\tau(\alpha), b(\alpha)]$. Because

$$G_{\bar{h}}(\tau(\alpha)) = \phi(\tau(\alpha), \alpha) = 0,$$

$G_{\bar{h}}$ and ϕ are linearly dependent over $[\tau(\alpha), b(\alpha)]$. Therefore

$$G_{\bar{h}}(b(\alpha)) = \phi(b(\alpha), \alpha) = 0.$$

But, on the other hand,

$$G_{\bar{h}}(b(\alpha)) = \frac{\bar{h}-1}{2} b(\alpha) u'(b(\alpha), \alpha) < 0,$$

yielding a contradiction.

It is easy to show that $G_{\bar{h}}(t)$ is not identically zero in any right neighborhood of $\tau(\alpha)$. The behavior of $G_{\bar{h}}(t)$ in $[\tau(\alpha), b(\alpha)]$ is then classified into three cases. We shall show that each case leads to a contradiction.

Case 1. There exists a $\delta > 0$ such that

$$G_{\bar{h}}(t) < 0 \quad \text{in} \quad (\tau(\alpha), \tau(\alpha)+\delta), \quad \text{and} \quad \tau(\alpha)+\delta \leq b(\alpha). \tag{5.22}$$

First we show that

$$\tau(\alpha) + \delta < b(\alpha). \tag{5.23}$$

If not, then $\tau(\alpha) + \delta = b(\alpha)$, i.e., $G_{\bar{h}}(t) < 0$ for all $t \in (\tau(\alpha), b(\alpha))$. Integrating both sides of (5.15) from $\tau(\alpha)$ to $b(\alpha)$ yields

$$-b^{n-1}(\alpha) G_{\bar{h}}(b(\alpha)) \phi'(b(\alpha), \alpha) = \int_{\tau(\alpha)}^{b(\alpha)} t^{n-1} \phi(t, \alpha) \psi_{\bar{h}}(t) dt. \tag{5.24}$$

If $\phi(t, \alpha) < 0$ in $(\tau(\alpha), b(\alpha))$, then the right side of (5.24) is negative. Thus the left side of (5.24) is also negative, which implies that $\phi'(b(\alpha), \alpha) < 0$, contradicting

$\phi(b(\alpha),\alpha) = 0$. On the other hand, if $\phi(t,\alpha) > 0$ in $(\tau(\alpha), b(\alpha))$, then a similar argument using (5.24) again implies $\phi'(b(\alpha),\alpha) > 0$, also giving a contradiction. This proves (5.23).

Observe that (5.23) implies that $G_{\bar{h}}$ vanishes within $(\tau(\alpha), b(\alpha))$. Let $t = \tau_1(\alpha)$ be such that $\tau(\alpha) < \tau_1(\alpha) < b(\alpha)$ and

$$G_{\bar{h}}(\tau_1(\alpha)) = 0, \quad G_{\bar{h}}(t) < 0 \text{ in } (\tau(\alpha), \tau_1(\alpha)). \tag{5.25}$$

By identity (5.15), it is easy to show that $G'_{\bar{h}}(t) \neq 0$ at $t = \tau_1(\alpha)$. Hence $G'_{\bar{h}}(\tau_1(\alpha)) > 0$. There is a right neighborhood of $\tau_1(\alpha)$ in which $G_{\bar{h}}(t)$ is positive. But $G_{\bar{h}}(b(\alpha)) < 0$, and hence we can find a number $t = \tau_2(\alpha)$ such that

$$\tau_1(\alpha) < \tau_2(\alpha) < b(\alpha)$$

and

$$G_{\bar{h}}(\tau_2(\alpha)) = 0, \quad G_{\bar{h}}(t) > 0, \quad \text{in } (\tau_1(\alpha), \tau_2(\alpha)). \tag{5.26}$$

Note that in $(\tau_1(\alpha), \tau_2(\alpha))$, $G_{\bar{h}}(t) > 0$, and $L(G_{\bar{h}}(t)) = \psi_{\bar{h}}(t) \geq 0$. By using identity (5.15) we can show that $G_{\bar{h}}(t)$ oscillates more slowly than $\phi(t,\alpha)$ in this interval, i.e., $\phi(t,\alpha)$ has at least one zero in $[\tau_1(\alpha), \tau_2(\alpha)]$. But this contradicts the fact that $\tau(\alpha)$ is the last zero of ϕ in $t < b(\alpha)$.

Case 2. $G_{\bar{h}}(t) > 0$ in a right neighborhood of $\tau(\alpha)$, corresponding to the case when $G_{\bar{h}}(t)$ has a local minimum at $\tau(\alpha)$. If this happens, then we can find a point $t = \tau_3(\alpha)$ such that

$$\tau(\alpha) < \tau_3(\alpha) < b(\alpha)$$

and

$$G_{\bar{h}}(\tau_3(\alpha)) = 0, \quad G_{\bar{h}}(t) > 0 \text{ in } (\tau(\alpha), \tau_3(\alpha)).$$

A similar argument as in Case 1 will show that $\phi(t,\alpha)$ has at least one zero in $(\tau(\alpha), \tau_3(\alpha))$, again yielding a contradiction.

Case 3. $\tau(\alpha)$ is a cluster point of the zeros of $G_{\bar{h}}(t)$. In this case, an easy argument by using the Sturm Comparison Principle will show that $\phi(t,\alpha)$ has to have infinitely many zeros near $\tau(\alpha)$. This contradicts Lemma 5.5.

The proof is completed when Ω is a finite ball.

(ii) In the case that Ω is a finite annulus, u and ϕ are the solutions of problems (5.4) and (5.6), respectively. It follows from (FT1) and Proposition 5.3

that $u(t,\alpha)$ has a unique critical point, denoted by $c(\alpha)$, in $(a, b(\alpha))$, satisfying (5.7). Since $u(t,\alpha) > 0$ in $(a, b(\alpha))$, it follows that $G_{\bar{h}}(t) > 0$ in $(a, c(\alpha)]$ for all $h \geq 1$. We show that assumption conditions (FTH) and (FT2) imply

$$\phi(t,\alpha) > 0 \ \text{ in}(a, c(\alpha)]. \tag{5.27}$$

In fact, if (FT2) holds, then we can find a sufficiently large $h = \hat{h}$ such that

$$\psi_{\hat{h}}(c(\alpha)) = uf_u(t,u) + \frac{t}{2}f_t(t,u) - \hat{h}\left(f(t,u) + \frac{1}{2}tf_t(t,u)\right)\Big|_{t=c(\alpha)} < 0.$$

Then, by assumption (FTH), we have

$$\psi_{\hat{h}}(t) < 0, \quad t \in (a, c(\alpha)]. \tag{5.28}$$

Since $G_h(t) > 0$ in $(a, c(\alpha)]$, we can use a similar argument to that in the proof of Proposition 5.4 or use the Sturm Comparison Principle to show that $\phi(t,\alpha)$ oscillates more slowly than $G_{\hat{h}}(t)$ in this interval. Thus $\phi(t,\alpha)$ can not have a zero in $(a, c(\alpha)]$, and (5.27) is proved. The remaining part of the proof in this case is the same as the case when Ω is a ball. We can simply repeat the proof there, replacing $a(\alpha)$ by $c(\alpha)$, and omit the details. The proof is completed. □

5.3. Uniqueness of Ground State Solutions of (5.2)

In this section and the next one we study the uniqueness of ground state solutions to problem (5.2). This problem has been studied in Chapter 4 for the nonlinearity independent of t. But the situation becames very complicated in the general case. We can prove a uniqueness theorem only for the case when f is a separable function of t and u. Even for this specialized nonlinearity, our assumptions are restrictive. It is worthwhile to mention that the techniques we develope below may not be extendable to more general cases. Our discussion is based on a Pohozaev-type Identity and some detailed investigations for the oscillatory and asymptotic behavior of the solutions and their variational functions.

When $f(t,u) = K(t)u^p, p > 1$, the semilinear equation arose from differential geometry and physics (see Ni and Yotsutani [95]), and the problem of the existence and uniqueness of ground state solutions has been a subject of extensive studies since the first general and systematic study of Ni [87]. See also [29-30] [64] [75-76] [89] and [110-112].

In this section we investigate the general behavior of solutions of initial value problem (5.3) by a shooting method, and state the main results at the end. Their proof will be given in the next section. Analogously, we say that

(i) $u(t, \alpha)$ is a *crossing solution* if $b(\alpha) < \infty$,

(ii) $u(t, \alpha)$ is a *slowly decaying solution* if $u(t, \alpha) > 0$ in $[0, \infty)$ and $\lim_{t \to \infty} t^{n-2} u(t, \alpha) = \infty$,

(iii) $u(t, \alpha)$ is a *ground state solution* or a *fast decaying solution* if $u(t, \alpha) > 0$ in $[0, \infty)$, $\lim_{t \to \infty} t^{n-2}\ u(t, \alpha)$ exists, and is finite and positive.

Note that if u is a ground state solution, then it necessarily holds that $\lim_{t \to \infty} u = 0$. But the definition of a slowly decaying solution here is slightly different from that of Chapter 3, since it is not required now that u really "decay", i.e., it may or may not hold that $\lim_{t \to \infty} u = 0$.

Proposition 5.9. *Suppose that $f(t, u) > 0$ when $t > 0$ and $u > 0$. Then every solution $u(t, \alpha), \alpha > 0$ is classified into one of the above three types.*

Proof. By Proposition 5.3, $u(t, \alpha)$ is strictly decreasing in $t > 0$ when it is positive. Thus we have either $b(\alpha) < \infty$, or $u(t, \alpha) > 0$ in $[0, \infty)$. It remains to show that in the second case $\lim_{t \to \infty} t^{n-2}\ u(t, \alpha)$ does exist and the limit is either a positive finite number or ∞. We do this by showing that the function $t^{n-2} u(t, \alpha)$ is increasing in $t > 0$.

Suppose $u(t) = u(t, \alpha) > 0$ in $[0, \infty)$. Let $v(s) = t^{n-2} u(t)$, $s = t^{n-2}$. Then $v(s) > 0$ for all $s > 0$. By a routine calculation we obtain

$$\frac{dv}{ds} = u(t) + \frac{1}{n-2} t u',$$

where $' = \frac{d}{dt}$, and

$$\begin{aligned} \frac{d^2 v}{ds^2} &= \left[\frac{1}{n-2} t u'' + \frac{n-1}{n-2} u' \right] \cdot \frac{1}{n-2} \cdot t^{3-n} \\ &= -\frac{1}{(n-2)^2} t^{4-n} f(t, u). \end{aligned}$$

Thus

$$\frac{d^2 v}{ds^2} < 0, \quad \text{for } \ s > 0.$$

In view of $v(s) > 0$ in $s > 0$, we must have

$$\frac{dv}{ds} > 0, \quad \text{for} \quad s > 0. \tag{5.29}$$

Observing that $\frac{dv}{dt} = (n-2)t^{n-3}\frac{dv}{ds}$, (5.29) implies $\frac{dv}{dt} > 0$. This completes the proof. □

In the rest of this section, we will restrict ourselves to the special case

$$f(t,u) = K(t)\gamma(u). \tag{5.30}$$

Some assumptions we shall impose on K and γ are

(K1) $K(t) \in C^1(0,\infty)$, $K(t) > 0$ in $(0,\infty)$,

(K2) $K'(t) \leq 0$ in $(0,\infty)$,

(Γ1) $\gamma(s) \in C^1(0,\infty)$, $\gamma(s) > 0$ in $(0,\infty)$,

(Γ2) $\gamma(s)$ is superlinear. For all $s > 0$, $u(s)\gamma'(s) > \gamma(s)$,

(Γ3) there exists an $\eta_\gamma > 0$ such that if $0 < s < \eta_\gamma$, then $(n-2)s\gamma(s) - 2n\Gamma(s) \geq 0$, in which $\Gamma(u) = \int_0^u \gamma(\tau)d\tau$.

Proposition 5.10. *Let $f(t,u) = K(t)\gamma(u)$. Let $u = u(t,\alpha)$ be the solution of problem (5.3). Define*

$$Q(t) := Q(t,u,\alpha) := -t^n u'^2 - (n-2)t^{n-1}uu' - 2t^n K(t)\Gamma(u). \tag{5.31}$$

Then

$$Q(t) = \int_0^t \tau^{n-1}K(\tau)[(n-2)u\gamma(u) - 2n\Gamma(u)]d\tau - 2\int_0^t \tau^n K'(\tau)\Gamma(u)d\tau. \tag{5.32}$$

Proof. From the equation for u we obtain

$$(t^{n-1}u')' = -t^{n-1}K(t)\gamma(u),$$
$$[tu' + (n-2)u]' = -tK(t)\gamma(u).$$

Hence

$$\frac{d}{dt}\left[(t^{n-1}u')(tu' + (n-2)u)\right]$$
$$= - t^{n-1}K(t)\gamma(u)(tu' + (n-2)u) - t^{n-1}u'tK(t)\gamma(u)$$
$$= - 2t^n K(t)u'\gamma(u) - (n-2)t^{n-1}K(t)u\gamma(u).$$

Integrating this identity over $[0,t]$, and using integration by parts yields

$$t^{n-1}u'[tu' + (n-2)u]$$
$$= - \int_0^t 2\tau^n K(\tau)\gamma(u)u'd\tau - (n-2)\int_0^t \tau^{n-1}K(\tau)u\gamma(u)d\tau$$
$$= - \int_0^t 2\tau^n K(\tau)\left[\frac{d}{d\tau}\Gamma(u)\right]d\tau - (n-2)\int_0^t \tau^{n-1}K(\tau)u\gamma(u)d\tau$$
$$= - 2t^n K(t)\Gamma(u) + 2\int_0^t \Gamma(u)d(\tau^n K(\tau)) - (n-2)\int_0^t \tau^{n-1}K(\tau)u\gamma(u)d\tau.$$

Thus,

$$Q(t) = - t^n u'^2 - (n-2)t^{n-1}uu' - 2t^n K(t)\Gamma(u)$$
$$= - \int_0^t 2n\tau^{n-1}K(\tau)\Gamma(u)d\tau - 2\int_0^t \tau^n K'(\tau)\Gamma(u)d\tau$$
$$+ (n-2)\int_0^t \tau^{n-1}K(\tau)u\gamma(u)d\tau$$
$$= \int_0^t \tau^{n-1}K(\tau)[(n-2)u\gamma(u) - 2n\Gamma(u)]d\tau - 2\int_0^t \tau^n K'(\tau)\Gamma(u)d\tau.$$

□

The next lemma is an analogue of Lemma 3.2, giving characteristic properties of each type of solution.

Proposition 5.11. *Suppose that* $(K1), (\Gamma1)$ *and* $(\Gamma2)$ *hold. We have*

(i) if $u(t,\alpha)$ *is a crossing solution, then* $Q(b(\alpha)) < 0$,

(ii) if $u(t,\alpha)$ *is a slowly decaying solution, then for any* $t = T > 0$, *there is a* $T_1 > T$ *such that* $Q(T_1) > 0$,

(iii) if $u(t,\alpha)$ *is a ground state solution, then* $\lim_{t\to\infty} Q(t) = 0$ *provided that*

$$\lim_{s\to\infty} s^2 K(s)\gamma(s^{2-n}) = 0. \tag{5.33}$$

Proof. The proof of (i) trivial. Observe that ($\Gamma 2$) implies $\Gamma(u) < \frac{1}{2}u\gamma(u)$. Therefore the proof of (ii) is the same as that of Lemma 3.2, except we take $m = 2$ and replace $f(u)$ with $K(t)\gamma(u)$ there. As before, if u is a ground state solution, then

$$\lim_{t\to\infty}[-t^n u'^2 - (n-2)t^{n-1}uu'] = 0.$$

On the other hand,

$$\lim_{t\to\infty} -2t^n K(t)\Gamma(u) = 0$$

follows from (5.33). This proves (iii). □

Remark 5.12. Condition (5.33) is very mild. It reduces to ($\Gamma 2$) if $n \geq 4$. In fact, (5.33) is always satisfied for all $n \geq 3$ if ($\Gamma 2$) holds and $\lim_{t\to\infty} tK(t) = 0$. In any case, (5.33) is fulfilled if $(K1), (K2), (\Gamma 1)$ and $(\Gamma 3)$ hold.

Remark 5.13. Suppose that $(K1 - K2)$ and $(\Gamma 1 - \Gamma 3)$ hold. If it happens that

$$K'(t) \equiv 0 \text{ in } (0,\infty), \quad \text{and} \quad (n-2)s\gamma(s) - 2n\Gamma(s) \equiv 0, \quad \text{for } 0 < s < \eta_\gamma, \tag{5.34}$$

then for any $0 < \alpha < \eta_\gamma$, one has $Q(t) \equiv 0$ in $(0,\infty)$. It follows from Proposition 5.11 that $u(t,\alpha)$ is a ground state solution. Therefore Problem (5.2) possesses infinitely many ground state solutions in this case.

Since we are particularly interested in the uniqueness of ground state solutions, we shall assume that (5.34) does not hold. More precisely, we need to assume that

($K\Gamma$) if $(n-2)s\gamma(s) - 2n\Gamma(s) \equiv 0$, for $0 < s < \eta_\gamma$, then $K'(t) < 0$ for some $t \in (0,\infty)$.

We can make an argument similar to that of Remark 5.13 to establish the following

Proposition 5.14. *Suppose that $(K1 - K2)$, $(\Gamma 1 - \Gamma 3)$ and $(K\Gamma)$ hold. If $0 < \alpha < \eta_\gamma$, then $u(t,\alpha)$ is a slowly decaying solution.*

Next, we will present two interesting propositions concerning the asymptotic behavior of slowly decaying solutions.

Proposition 5.15. *Suppose that* $(K1)$ *and* $(\Gamma 1)$ *hold. Then any slowly decaying solution tends to a positive number if there exists a* $t = T_K > 0$ *such that*

$$tK'(t) + (2n-2)K(t) \leq 0, \quad \text{if } \ t > T_K. \tag{5.35}$$

Proof. Let $u = u(t, \alpha)$ be a slowly decaying solution. Suppose to the contrary that $\lim_{t\to\infty} u = 0$. Then by using L'Hospital's rule we have

$$\lim_{t\to\infty} t^{(n-1)}u' = -(n-2) \lim_{t\to\infty} t^{(n-2)}u = -\infty. \tag{5.36}$$

Since $t^{(n-1)}u'$ is decreasing and $\lim_{t\to\infty} t^{(n-2)}u = \infty$. Define

$$M(t) := M(t, u, \alpha) = t^{2n-2}[u'(t,\alpha)^2/2 + K(t)\Gamma(u(t,\alpha))]. \tag{5.37}$$

Then

$$\lim_{t\to\infty} M(t) = \infty. \tag{5.38}$$

It is easy to verify that

$$M'(t, \alpha) = t^{2n-3}\Gamma(u)[(2n-2)K(t) + tK'(t)]. \tag{5.39}$$

Thus $M' \leq 0$ for $t > T_K$ because of (5.35) and $(\Gamma 1)$. But this contradicts (5.38). The proof is completed. □

Proposition 5.16. *Suppose that* $(K1)$ *and* $(\Gamma 1)$ *hold. Then any slowly decaying solution tends to zero if*

$$\int^{\infty} tK(t)dt = \infty.$$

Proof. Let $u = u(t, \alpha)$ be a slowly decaying solution. Let

$$w(t, \alpha) := (n-2)u + tu'.$$

It follows from the proof of Proposition 5.9 that $w > 0$ in $t > 0$. By the equation for u we have

$$w'(t) = -tK(t)\Gamma(u).$$

Now, suppose to the contrary that there is a $u_\infty > 0$ such that

$$\lim_{t\to\infty} u = u_\infty.$$

Then we can find constants $c > 0$ and t_Γ such that

$$w'(t) \le -ctK(t), \quad \text{if} \quad t > t_\Gamma$$

Integrating both sides of this inequality from t_Γ to $T > t_\Gamma$ and letting T tend to ∞, we obtain

$$\lim_{t\to\infty} w = -\infty.$$

Hence

$$\lim_{t\to\infty} tu' = -\infty.$$

But the last identity would imply $\lim_{t\to\infty} u = -\infty$. We obtain a contradiction. □

For technical reasons, we need one more condition on K.

(K3) There is a $t = \bar{T}_K > 0$ such that

$$tK'(t) + (2n-2)K(t) \ge 0, \quad \text{if} \quad t > \bar{T}_K.$$

and

$$\int^{\infty} tK(t)dt = \infty.$$

Similar to condition (FTH), we assume

$(K\Gamma H)$ let $h \ge 1$, $\alpha > 0$, and $u = u(t, \alpha)$. The function

$$\psi_h(t) = K(t)(u\gamma'(u) - h\gamma(u)) - \frac{h-1}{2}t\gamma(u)K'(t). \tag{5.40}$$

does not change signs from positive to negative as t increases from 0 to $b(\alpha)$, and $\psi_h(t)$ is not identically zero in (Ψ, ∞) for any $\Psi > 0$.

Now we are in the position to state the main results on this subject.

Theorem 5.17. *Suppose that $(K1 - K3)$,$(\Gamma 1 - \Gamma 3)$, $(K\Gamma)$ and $(K\Gamma H)$ hold. Then Problem (5.2) has at most one ground state solution, but infinitely many slowly decaying solutions. Moreover, problem (5.1) has at most one positive radial solution in any finite ball Ω.*

On the global structure of solutions of problem (5.3), we have

Theorem 5.18. *Under the same assumptions of Theorem 5.17, the structure of positive solutions of (5.3) is of either Type S or Type M. More precisely, the structure is classified into one of the following types:*

(i) every solution $u(t,\alpha)$ is a slowly decaying solution,

(ii) there is a unique $\alpha^* > \eta_\gamma$ such that $u(t,\alpha)$ is a slowly decaying solution for $0 < \alpha < \alpha^*$, $u(t,\alpha^*)$ is a ground state solution, and $u(t,\alpha)$ is a crossing solution for $\alpha > \alpha^*$.

5.4. Proof of Theorems 5.17-5.18

Defining the subsets N, D_s and D_f of $(0,\infty)$ as those in Chapter 4, we have:

Lemma 5.19. *Under the assumptions of Theorem 5.17*

(i) $N \cup D_s \cup D_f = (0,\infty)$.

(ii) D_s *is nonempty, and* $(0,\eta_\gamma] \subset D_s$.

iii) Both N *and* D_s *are open sets,* D_f *is a closed set.*

Proof. (i) follows from Proposition 5.9. (ii) follows from Proposition 5.14. The continuous dependence of solutions of (5.3) on initial data implies that N is an open set. In view of (i), we only need to show that D_s is an open set to complete the proof of (iii). Let $\bar{u} = u(t,\bar{\alpha})$ be a slowly decaying solution. By Propositions (5.11) and (5.16) we can find a $t = T_\eta$ sufficiently large such that

$$u(T_\eta,\bar{\alpha}) < \eta_\gamma/2, \quad \text{and} \quad Q(T_\eta,\bar{\alpha}) > 0. \tag{5.41}$$

Thus, if α is sufficiently close to $\bar{\alpha}$, then

$$0 < u(T_\eta,\alpha) < \eta_\gamma/2, \quad \text{and} \quad Q(T_\eta\alpha) > 0. \tag{5.42}$$

Since $u(t,\alpha)$ is decreasing whenever it is positive, we have

$$u(t,\alpha) < \eta_\gamma/2, \quad \text{if} \quad t > T_\eta, \quad \text{and} \quad u(t,\alpha) > 0. \tag{5.43}$$

By Identity (5.32) and conditions $(K2), (\Gamma 3)$ we obtain

$$Q(t,\alpha) > Q(T_\eta,\alpha), \quad \text{if} \quad t > T_\eta, \quad \text{and} \quad u(t,\alpha) > 0, \tag{5.44}$$

which implies that $u(t,\alpha) > 0$ for all $t > T_\eta$. Since otherwise, say, at $t = T_0 > T_\eta$, one has $u(T_0,\alpha) = 0$. Then $Q(T_0,\alpha) < 0$, contradicts (5.44). By using (5.44) again, we have

$$\lim_{t\to\infty} Q(t,\alpha) > Q(T_\eta,\alpha).$$

Therefore $u(t,\alpha)$ must be a slowly decaying solution in view of Proposition (5.11). The proof is completed. □

Now we can give an outline of the proof of Theorems 5.17-5.18. Since $(0,\eta_\gamma) \subset D_s$, we can define

$$\alpha^* := \sup\{\alpha > 0 : \ \alpha' \in D_s \quad \text{if } 0 < \alpha' < \alpha.\} \tag{5.45}$$

If $\alpha^* = \infty$, then the structure of solutions of (5.3) is of Type S, and every solution $u(t,\alpha)$ is a slowly decaying solution. If

$$\alpha^* < \infty, \tag{5.46}$$

then $u(t,\alpha^*)$ is a ground state solution because D_s and N are open sets. In this case, we shall show that every solution $u(t,\alpha)$ with $\alpha > \alpha^*$ is a crossing solution. Once this assertion is proved, the uniqueness of ground state solutions readily follows.

To complete the proof of the main results, we shall apply the theory of linear second order ordinary differential equations to analyze the oscillatory and asymptotic behavior of $\phi(t,\alpha^*)$. In what follows, we assume that (5.46) holds. For simplicity of notations, we let $u^* = u(t,\alpha^*)$ and $\phi^* = \phi(t,\alpha^*)$. The following two technical lemmas are crucial in our proof.

Lemma 5.20. *Suppose that* $(K1)-(K2),(\Gamma 1-3)$ *and* $(K\Gamma)$ *hold. Then* ϕ^* *vanishes exactly once in* $(0,\infty)$.

Lemma 5.21. *Under the assumptions of Theorem 5.17. There exists a constant* $\phi_0^* > 0$ *such that*

$$\lim_{t\to\infty} \phi^* = -\phi_0^*. \tag{5.47}$$

We divide the rest of this section into three parts. We shall prove Lemma 5.20 and 5.21 in 5.4.1 and 5.4.2, respectively. The proof of Theorems 5.17-5.18 is completed in 5.4.3.

5.4.1. Proof of Lemma 5.20

In what follows, we shall show that ϕ^* vanishes at least once in $(0,\infty)$ in Lemma 5.22, and that it vanishes at most once in $(0,\infty)$ in Lemma 5.23. Once Lemmas 5.22-5.23 are established, Lemma 5.20 is proved.

Lemma 5.22. *Suppose that* $(K1)$ *and* $(\Gamma 1-2)$ *hold. Then* ϕ^* *vanishes at least once in* $(0,\infty)$.

Proof. From (5.10) we have

$$[t^{n-1}(u^{*'}\phi^* - u^*\phi^{*'})]' = -t^{n-1}K(t)[\gamma(u^*) - u^*\gamma'(u^*)]\phi^*. \tag{5.48}$$

Integrating both sides of (5.48) from 0 to $t > 0$ we get

$$t^{n-1}(u^{*'}\phi^* - u^*\phi^{*'}) = \int_0^t \tau^{n-1}K(\tau)\phi^*(u^*\gamma'(u^*) - \gamma(u^*))d\tau. \tag{5.49}$$

Suppose to the contrary that $\phi^* > 0$ for all $t \in (0,\infty)$. Then $t^{n-1}(u^{*'}\phi^* - u^*\phi^{*'}) > 0$ in $(0,\infty)$, which implies that u^*/ϕ^* is strictly increasing in $(0,\infty)$ and so is $(t^{n-2}u^*)/(t^{n-2}\phi^*)$. Since $\lim_{t\to\infty} t^{n-2}u^*$ exists and is finite, there is a number $0 \le d^* < \infty$ such that

$$\lim_{t\to\infty} t^{n-2}\phi^* = d^*. \tag{5.50}$$

By L'Hospital's rule one has

$$\lim_{t\to\infty} t^{n-1}\phi^{*'} = (2-n)d^* \le 0. \tag{5.51}$$

Note that (5.50) implies that $\lim_{t\to\infty}\phi^* = 0$. By (5.50) (5.51) and the fact that u^* is a ground state solution, we obtain

$$\lim_{t\to\infty} t^{n-1}(u^{*'}\phi^* - u^*\phi^{*'}) = 0.$$

On the other hand, letting t tend to ∞ in (5.49), in view of $(K1)$ and $(\Gamma 2)$, we have

$$\lim_{t\to\infty} t^{n-1}(u^{*'}\phi^* - u^*\phi^{*'}) > 0.$$

We get a contradiction, and the lemma is proved.

Lemma 5.23. *Suppose that* $(K1)-(K2), (\Gamma1)-(Gamma3)$ *and* $(K\Gamma)$ *hold. Then* ϕ^* *vanishes at most once in* $(0, \infty)$.

Proof. It suffices to show that u^* intersects every solution $u(t, \alpha), 0 < \alpha < \alpha^*$ exactly once in $t \in (0, \infty)$. Recall that $u(t, \alpha)$ is a slowly decaying solution when $0 < \alpha < \alpha^*$. Hence it must intersect u^* at least once in $t \in (0, \infty)$.

At first we claim that there is a $0 < \epsilon < \alpha^*$ such that u^* intersects $u(t, \alpha)$ at most once when $0 < \alpha < \epsilon$.

Suppose no such an ϵ exists. Then we can find a sequence $\{\alpha_i\}_{i=1}^{\infty}$ such that $\lim_{t\to\infty} \alpha_i = 0$, and every $u(t, \alpha_i)$ intersects u^* at least twice. Denote the second intersection point by a_i. Since $\alpha_i \to 0$ as $i \to \infty$, and $u'(t, \alpha_i) < 0$ in $(0, \infty)$, we have

$$\lim_{i\to\infty} a_i = \infty. \tag{5.52}$$

Recall that $\alpha_i < \alpha^*$ and a_i is the second intersection point, we have

$$u(a_i, \alpha_i) = u^*(a_i), \text{ and } u'(a_i, \alpha_i) < {u^*}'(a_i) < 0, \quad i = 1, 2, \cdots . \tag{5.53}$$

Without loss of generality, we assume

$$\alpha_i < \eta_\gamma/2, i = 1, 2, \cdots .$$

Recall the Pohozaev-type Identity (5.32), conditions $(K2), (\Gamma3)$ and $(K\Gamma)$, and let $u_i = u(t, \alpha_i)$ we get

$$\begin{aligned} Q_i(a_i) &:= Q(a_i, \alpha_i) \\ &= \int_0^{a_i} \tau^{n-1} K(\tau)[(n-2)u_i\gamma(u_i) \\ &\quad - 2n\Gamma(u_i)]d\tau - 2\int_0^{a_i} \tau^n K'(\tau)\Gamma(u_i)d\tau \\ &> 0. \end{aligned} \tag{5.54}$$

Let $Q^*(t) = Q(t, \alpha^*)$. Recall the characteristic property of u^* that $\lim_{t\to\infty} Q^*(t) = 0$. Using identity (5.32) again yields

$$\begin{aligned} Q^*(a_i) = &-\int_{a_i}^{\infty} \tau^{n-1} K(\tau)[(n-2)u^*\gamma(u^*) - 2n\Gamma(u^*)]d\tau \\ &+ 2\int_{a_i}^{\infty} \tau^n K'(\tau)\Gamma(u^*)d\tau \\ &< 0. \end{aligned} \tag{5.55}$$

(5.54) and (5.55) lead to

$$Q_i(a_i) > Q^*(a_i). \tag{5.56}$$

Combining this inequality with (5.31) and (5.53) we obtain

$$(n-2)[u^{*'}(a_i) - u_i'(a_i)]u^*(a_i)a_i^{n-1} + [u^{*'}(a_i)^2 - u_i'(a_i)^2]a_i^n > 0. \tag{5.57}$$

Factoring $[u^{*'}(a_i) - u_i'(a_i)]$ out, and using (5.53) again, one has

$$(n-2)u^*(a_i)a_i^{n-2} + 2u^{*'}(a_i)a_i^{n-1} > 0. \tag{5.58}$$

But then we have a contradiction. Since u^* is a ground state solution, it has been proved in Chapter 4 that

$$\lim_{t\to\infty}[(n-2)t^{n-2}u^*(t) + 2t^{n-1}u^{*'}(t)] < 0.$$

The claim is proved.

Let $\bar{\epsilon}$ be the largest number in $(0, \alpha^*)$ so that the claim is valid. It remains to prove $\bar{\epsilon} = \alpha^*$. Suppose that $\bar{\epsilon} < \alpha^*$, then there is a sequence $\{\beta_j\}_{j=1}^{\infty}$ such that $\bar{\epsilon} < \beta_j < \alpha^*$, $\lim_{j\to\infty}\beta_j = \bar{\epsilon}$, and $u(t,\beta_j)$ crosses u^* at least twice. Since $\beta_j < \alpha^*$ and $u^* < u(t,\beta_j)$ for large t, $u(t,\beta_j)$ and u^* must intersect a third time, say, at $t = c_j$. Then $\lim_{j\to\infty} c_j = \infty$, and

$$u^*(c_j) = u(c_j,\beta_j), \quad u^{*'}(c_j) < u'(c_j,\beta_j) < 0, \quad j = 1, 2, \cdots. \tag{5.59}$$

For simplicity of notation, let $u_j = u(t,\beta_j)$, $M^*(t) = M(t, u^*, \alpha^*)$ and $M_j(t) = M(t, u_j, \beta_j)$. Where M is defined in (5.37). Thus

$$M^*(c_j) > M_j(c_j). \tag{5.60}$$

Recall that $\lim_{t\to\infty} M(t) = \infty$ when u is a slowly decaying solution. While

$$\lim_{t\to\infty} M^*(t) < \infty,$$

since $\lim_{t\to\infty} t^{n-1}u^{*'}$ is finite, and $\lim_{t\to\infty} t^{2n-2}\Gamma(u^*(t)) = 0$ by (Γ3). Denote $M_* = \lim_{t\to\infty} M^*(t) < \infty$. Choose $T_M > \bar{T}_K$ sufficiently large that $M(T_M, \bar{\epsilon}) > 4M_*$, where $\bar{T}_K$ is as in (K3). Without loss of generality, we can suppose $c_j > T_M$ and $M_j(T_M) > 2M_*$ for all $j = 1, 2, \cdots$. Since M_j is increasing in $(\bar{T}_K, \infty)$ by (K3), we must have $M_j(c_j) > 2M_*$, which is impossible in view of (5.60). The proof is completed.

5.4.2. Proof of Lemma 5.21

Consider a linear second order equation

$$(t^{n-1}v')' + t^{n-1}K(t)\gamma'(u^*(t)) = 0. \tag{5.61}$$

We say that equation (5.61) is *nonoscillatory* on $(0,\infty)$ if it has a solution vanishing at most finitely many times on $(0,\infty)$. Since ϕ^* is a solution of (5.61) with $v(0) = 1$, $v'(0) = 0$, and ϕ^* has exactly one zero in $t > 0$ as we proved in Lemma 5.20, equation (5.61) is nonoscillatory on $(0,\infty)$.

Let us introduce another equation

$$(t^{n-1}w')' = 0, \tag{5.62}$$

which has two linearly independent solutions $w_1(t) \equiv 1$, $w_2(t) \equiv t^{2-n}$. Let $v_1(t), v_2(t)$ be two independent solutions of (5.61). Then every solution of (5.61), particularly, ϕ^*, is a linear combination of $v_1(t), v_2(t)$. At first, we shall show that ultimately v_i behaves like w_i, $i = 1, 2$.

We state Theorem 9.1 of Hartman (page 379 of [57]) in a simplified version as follows.

Lemma 5.24. *$v_i \sim w_i, i = 1, 2$ as $t \to \infty$, in other words, $\lim_{t\to\infty} v_i/w_i$ is a nonzero finite constant, provided that*

$$\int^{\infty} w_1 w_2 t^{n-1} K(t)\gamma'(u^*(t))dt < \infty. \tag{5.63}$$

To see that (5.63) is valid, recall that $K(t)$ is bounded above, $u^* \sim t^{2-n}$, and ($\Gamma3$) implies that the growth of γ' is greater than $4/(n-2)$. Thus

$$w_1 w_2 t^{n-1} K(t)\gamma'(u^*(t)) = o(t^{(2-n)+(n-1)-4}) = o(t^{-3}),$$

and (5.63) follows. Which in turn, implies that

$$v_i(t) \sim w_i(t), \quad i = 1, 2. \tag{5.64}$$

Proof of Lemma 5.21. Let ν_1, ν_2 be two constants such that

$$\phi^* = \nu_1 v_1(t) + \nu_2 v_2(t). \tag{5.65}$$

It suffices to show that $\nu_1 \neq 0$.

Suppose to the contrary that $\nu_1 = 0$, then $\phi^* \sim t^{2-n}$ as $t \to \infty$. Let $G_h^*(t) = G_h(t, \alpha^*)$, it holds for all $h \geq 1$ that

$$\lim_{t\to\infty} t^{n-1}(G_h^{*'}(t)\phi^*(t) - G_h^*(t)\phi^{*'}(t)) = 0. \tag{5.66}$$

Let $\tau(\alpha^*)$ be the unique zero of ϕ^*. Let ζ_h be the first zero of $G_h^*(t)$, if $G_h^*(t)$ does vanish in $(0, \infty)$. Note that ζ_h may not be defined for all $h > 1$. By a similar argument to that in the proof of (i) of Theorem 5.8, we can show that there is a number $\bar{h}$ such that

$$\tau(\alpha^*) = \zeta_{\bar{h}},$$

and

$$\psi_{\bar{h}}(\tau(\alpha^*)) = \psi_{\bar{h}}(\zeta_{\bar{h}}) \geq 0. \tag{5.67}$$

Therefore,

$$t^{n-1}(G_h^{'*}(t)\phi^*(t) - G_h^*(t)\phi^{*'}(t)) = 0, \quad \text{at} \quad t = \tau(\alpha^*). \tag{5.68}$$

In view of (5.67) and $(K\Gamma H)$, we have

$$\psi_{\bar{h}}(t) \geq 0, \quad \text{and} \quad \psi_{\bar{h}}(t) \quad \text{is not identically zero in} (\tau(\alpha^*), \infty). \tag{5.69}$$

Combining (5.68), (5.69) and identity (5.15), we obtain

$$\begin{aligned} &\lim_{t\to\infty} t^{n-1}(G_h^{'*}(t)\phi^*(t) - G_h^*(t)\phi^{*'}(t)) \\ &= \int_{\tau(\alpha^*)}^{\infty} t^{n-1}\phi^*(t)\psi_{\bar{h}}(t)dt \\ &< 0. \end{aligned}$$

But this contradicts (5.66). The proof is completed.

5.4.3. Proof of Main Results

Lemma 5.25. *There exists a $\rho > 0$ such that $u(t, \alpha)$ is a crossing solution if $\alpha^* < \alpha < \alpha^* + \rho$.*

Proof. For any given $0 < \lambda < n - 2$, define

$$z(t) := z_\lambda(t, \alpha) = t^\lambda(u(t, \alpha^*) - u(t, \alpha)). \tag{5.70}$$

Then $z(t)$ satisfies

$$z'' + (n-1-2\lambda)\frac{z'}{t} + \lambda(\lambda+2-n)\frac{z}{t^2} + t^\lambda K(t)[\gamma(u^*) - \gamma(u)] = 0. \tag{5.71}$$

If there are $\alpha > 0$ and $t_0 > 0$ such that

$$z'(t_0) = 0, \quad \text{and} \quad 0 < u(t_0) < u^*(t_0), \tag{5.72}$$

then

$$\begin{aligned} z''(t_0) &= \{\lambda(n-2-\lambda)/t_0^2 - K(t_0)[\gamma(u^*(t_0)) - \gamma(u(t_0))]/[u^*(t_0) - u(t)]\}z \\ &= [\lambda(n-2-\lambda)/t_0^2 - K(t_0)\gamma'(\theta(t_0))]z, \end{aligned} \tag{5.73}$$

where $u^*(t_0) \le \theta(t_0) \le u(t_0)$. We recall that $\gamma'(u^*(t)) = o(t^{-4})$ when $t \to \infty$. It follows that if t_0 is sufficiently large and (5.72) is valid, then

$$z''(t_0) > 0. \tag{5.74}$$

Recalling that ϕ^* has a unique zero $t = \tau(\alpha^*)$ and behaves like a negative constant for t large, we have $t^\lambda\phi^*(t) \to -\infty$ as $t \to \infty$, $\lambda > 0$. It follows that when $t = T^* > \tau(\alpha^*)$ is sufficiently large,

$$t^\lambda\phi^*(t) < 0, \quad (t^\lambda\phi^*(t))' < 0, \text{ if } t = T^*. \tag{5.75}$$

Note that

$$t^\lambda\phi^*(t) = \lim_{\alpha\to\alpha^*} z(t)/(\alpha^* - \alpha). \tag{5.76}$$

Suppose to the contrary that $\alpha > \alpha^*$ is sufficiently close to α^* and $u(t,\alpha)$ is not a crossing solution, then

$$z(T^*) > 0, \quad z'(T^*) > 0. \tag{5.77}$$

Therefore, $z(t)$ increases in a right neighborhood of T^*. If $z(t)$ decreases for some $t > T^*$, then there exists some $t_0 > T^*$ such that

$$z(t_0) > 0, \quad z'(t_0) = 0, \text{ and } z(t) \text{ has a local maximum at } t_0.$$

But this contradicts (5.72) and (5.74). If $z(t)$ is increasing for all $t > T^*$, then

$$z(t) > z(T^*) > 0 \ \text{ for } \ t > T^*, \lim_{t\to\infty} z(t) > z(T^*) > 0,$$

which is not compatible with the assumption that u is not a crossing solution. The lemma is proved.

Proof of Theorem 5.17. As we have seen, if α^* defined in (5.45) is finite, then $u(t,\alpha^*)$ is a ground state solution, u is a slowly decaying solution if $0<\alpha<\alpha^*$, and u is a crossing solution if $\alpha^*<\alpha<\alpha^*+\rho$. Thus, for $\alpha^*<\alpha<\alpha^*+\rho$, the first zero $b(\alpha)$ of u is well defined, and

$$\lim_{\alpha\to\alpha^*+} b(\alpha)=\infty.$$

Therefore, $b(\alpha)$ is decreasing in a right neighborhood of α^*. In the proof of Theorem 5.8 we have shown that $b'(\alpha)\neq 0$. Thus $b'(\alpha)<0$ whenever $\alpha>\alpha^*$ and $b(\alpha)$ is defined.

Define

$$\alpha_\infty := \sup\{\alpha>\alpha^*:\quad \alpha'\in N \quad \text{if } \alpha^*<\alpha'<\alpha.\} \tag{5.78}$$

Then $b(\alpha)$ is defined and strictly decreasing in (α^*,α_∞). Therefore, $\lim_{\alpha\to\alpha_\infty^-} b(\alpha) <\infty$, which in turn implies

$$\alpha_\infty=\infty,$$

since otherwise, one should have $\lim_{\alpha\to\alpha_\infty^-} b(\alpha)=\infty$.

In summary, we have shown that

$$N=(\alpha^*,\infty),\quad b'(\alpha)<0,\quad \text{if}\quad \alpha\in N, D_f=\alpha^*, D_s=(0,\alpha^*). \tag{5.79}$$

This proves Theorem 5.17. □

It is evident that Theorem 5.18 follows from (5.79).

5.5. Examples

5.5.1. $f(t,u)=K(t)u^p, p>1$.

The following mathematical model, now called *Matukuma's equation*, was proposed by Matukuma in 1930 to describe the dynamics of a globular cluster of stars in astrophysics,

$$\Delta u+\frac{1}{1+|x|^2}u^p=0,\quad x\in R^3.$$

This equation and its generalization

$$\Delta u+K(|x|)u^p=0,\quad x\in R^n. \tag{5.80}$$

have been extensively studied by many authors. See for example [29],[75],[87] and [110], just to mention a few. When $p = \frac{n+2}{n-2}$, problem (5.80) is also called a *conformal scalar curvature problem.* It arises from the problem of finding conformal Riemannian metrics with prescribed scalar curvature K.

It has been observed that if $K(t) \equiv 1$, and $u(t,\alpha)$ is a positive radial solution of the equation

$$\Delta u + K(|x|)u^{\frac{n+2}{n-2}} = 0, \quad x \in R^n, \tag{5.81}$$

with $u(0,\alpha) = \alpha$, then it is a ground state solution. Let $\kappa(t)$ be a nonnegative, nonincreasing, and nonconstant function defined in $[0,\infty)$ with $\kappa(0) < 1$. It was proved in Ding and Ni [29] that

(i) if $K(t) = 1 + \kappa(t)$, then $u(t,\alpha)$ is a slowly decaying solution behaving like $t^{\frac{2-n}{2}}$ as $t \to \infty$,

(ii) if $K(t) = 1 - \kappa(t)$, then $u(t,\alpha)$ vanishes at some finite t, i.e., it is a crossing solution.

Note that $\kappa(t)$ can be taken arbitrarily small and with compact support. This example shows that the structure of solutions of (5.81) is quite delicate and very sensitive to small perturbations. More generally, we can prove:

Proposition 5.26. *Suppose that $K(t) \in C^1((0,\infty))$ is a positive, and nonconstant function. Let $u(t,\alpha), \alpha > 0$ be a radial solution of (5.81). We have*

(i) if $K'(t) \leq 0$, then $u(t,\alpha)$ is a slowly decaying solution,

(ii) if $K'(t) \geq 0$, then $u(t,\alpha)$ is a crossing solution.

In other words, when $K(t) \in C^1((0,\infty))$ is positive, the structure of radial solutions of problem (5.81) is of Type F if K is a constant; it is of Type S if K is decreasing (and not a constant); and it is of Type C if K is increasing (and not a constant).

We sketch only the outline of a simple proof for this proposition. Let $Q(t)$ be defined as in (5.31). Substituting $\gamma(u)$ by $u^{\frac{n+2}{n-2}}$ in (5.32) yields

$$Q(t) = -\frac{4n}{n-2}\int_0^t \tau^n K'(\tau)u(\tau)^{\frac{n+2}{n-2}}d\tau.$$

In the first case, it holds that $Q(t) \geq 0$ if $u(t,\alpha) > 0$, implying that u is not a crossing solution. Moreover, $\lim_{t\to\infty} Q(t)$ exists and is positive. Thus u is a slowly decaying solution. The second case can be similarly handled.

As an application of Theorem 5.8, we give some uniqueness results to the positive radial solutions of equation (5.80).

Proposition 5.27. *Let $f(t,u) = K(t)u^p, p > 1$. Then Problem (5.1) has at most one positive radial solution provided that one of the following conditions is fulfilled,*

(i) $K(t) = t^l, l > -2$,

(ii) $K(t) = \frac{1}{1+t^2}$,

(iii) $K(t) = \frac{1}{1+t^r}, r \geq 2$, *and Ω is a finite ball.*

In the first case, the equation of (5.1) is proposed by Henon [58] as a model in astrophysics to study "rotating stellar systems". When K is as in (iii), the corresponding equation of (5.1) is sometimes called the *generalized Matukuma's equation.*

Proof. It is easily seen that condition (FT1) is satisfied because $p > 1$. In order to check (FTH), we write $\psi_h(t)$ as

$$\psi_h(t) = u^p((p-h)K(t) - \frac{h-1}{2}tK'(t)) \tag{5.82}$$

Let the functions $J(t)$ and $J_h(t)$ be defined by

$$\begin{aligned} J(t) &= K(t) + \frac{1}{2}tK'(t), \\ J_h(t) &= (p-h)K(t) - \frac{h-1}{2}tK'(t), \quad h \geq 1. \end{aligned} \tag{5.83}$$

Then $\psi_h(t) = u^p J_h(t)$, and condition $(FT2)$ is fulfilled if $J(t) > 0$ for $t > 0$. We give in the following the proof of (i) and (iii) separately, the proof of (ii) follows from that of (iii).

(i). In this case, we have $J(t) = t^l(1 + \frac{l}{2})$, and $J_h(t) = t^l(p - h - \frac{h-1}{2}l)$. Thus (FTH) is obviously satisfied for all $l \in R$, and $(FT2)$ holds for $l > -2$. The assertion follows from Theorem 5.8.

(iii). It is straightforward to verify that

$$J(t) = \frac{1}{(1+t^\tau)^2}(t^\tau(1-\frac{\tau}{2})+1),$$

$$J_h(t) = \frac{1}{(1+t^\tau)^2}[(p-h)+(p-h+\frac{\tau(h-1)}{2})t^\tau].$$

Thus $J(t) > 0$ in $t > 0$ if $\tau \le 2$, and (FTH) holds if

$$(p-h+\frac{\tau(h-1)}{2})\tau \ge 0 \text{ for all } h > 1,$$

which is satisfied, for example, when $\tau \ge 2$. The remaining argument is trivial, and so we omit it. □

5.5.2. $f(t,u) = t^l\gamma(u)$.

Theorem 5.28. *Let* $f(t,u) = t^l\gamma(u), l > -2$. *Suppose that* $\gamma(u) \in C^1([0,\infty))$, *and* $0 < \gamma(u) < u\gamma'(u)$. *Let* $\bar{\gamma}(u) = u\gamma'(u)/\gamma(u)$. *We have*

- *(i) problem (5.1) has at most one positive radial solution provided that* $\bar{\gamma}(u)$ *is nonincreasing in* u.
- *(ii) Problem (5.2) has at most one ground state solution provided that* $\bar{\gamma}(u)$ *is nonincreasing in* u, *in addition,* $l \le 0$ *and* $(\Gamma 3), (K\Gamma)$ *are satisfied.*

Proof. Observe that $\psi_h(t)$ can be simplified to

$$\psi_h(t) = t^l\gamma(u)(\gamma_\psi(u) - h - \frac{h-1}{2}l) \tag{5.84}$$

Since $u(t,\alpha)$ is strictly decreasing in $[c(\alpha), b(\alpha)]$, we see that (FTH) is satisfied if $\bar{\gamma}(u)$ is nonincreasing in u. On the other hand, $(FT2)$ is fulfilled if

$$t^l\gamma(u)(1+l/2) > 0,$$

which is valid if $l > -2$. Thus (i) follows from Theorem 5.8. The proof of (ii) can be similarly completed by using Theorem 5.17. □

Corollary 5.29. *Let* $f(t,u) = t^lu^p$ *for* $u \ge 1$; $f(t,u) = t^lu^q$ *for* $0 \le u < 1$. *Suppose that* $-2 < l \le 0$, $1 < p < \frac{n+2}{n-2} < q$. *We have*

- *(i) problem (5.1) has at most one radial solution,*
- *(ii) problem (5.2) has at most one ground state solution.*

Chapter 6
FURTHER REMARKS AND DISCUSSION

6.1. Existence Results

The existence of positive radial solutions for semilinear elliptic equations in annular domains subject to various boundary conditions has been extensively studied in recent years. Let $0 < R_1 < R_0 < \infty$, and Ω be an annulus in $\mathbb{R}^n$ defined by $\Omega = \{x : \; 0 < R_1 < t = |x| < R_0\}$, $n \geq 3$. Let $f \in C^1([0,\infty))$, $f(0) = 0$, $f(s) > 0$ for $s > 0$. Consider the problem of the semilinear equation

$$\Delta u + f(u) = 0 \quad \text{in} \quad \Omega, \tag{6.1}$$

subject to one of the following sets of boundary conditions:

$$u = 0 \quad \text{on} \quad t = R_1, \quad \text{and} \quad t = R_0, \tag{6.2$_a$}$$

$$u = 0 \quad \text{on} \quad t = R_1, \quad \text{and} \quad \frac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad t = R_0, \tag{6.2$_b$}$$

$$\frac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad t = R_1, \quad \text{and} \quad u = 0 \quad \text{on} \quad t = R_0. \tag{6.2$_c$}$$

Bandle, Coffman and Marcus [11] have proved that if $f(s)$ is superlinear at $s = 0$ and $s = \infty$, i.e., $\lim_{s\to 0^+} f(s)/s = 0$, and $\lim_{s\to\infty} f(s)/s = \infty$, and if f is non-decreasing in $(0,\infty)$, then problem (6.1)-(6.2) possesses a positive radial solution. Their result was extended by Lin [77] to the equation

$$\Delta u + g(|x|)f(u) = 0 \quad \text{in} \quad \Omega \tag{6.3}$$

with one of the boundary conditions of (6.2). Very recently, Lee and Lin [72] proved the existence of positive radial solutions to equation (6.1) subject to some non-homogeneous Dirichlet boundary conditions.

The result of Bandle et al. [11] has been generalized in Erbe, Hu and Wang [34], Erbe and Wang [37-38], Erbe and Tang [39] [41] and Hu [59]. In particular, Erbe and Tang [41] have established the existence of multiple radial solutions to problem (6.1)-(6.2).

For results on the existence of positive radial or nonradial solutions to various types of nonlinear boundary value problems of semilinear elliptic equations, quasilinear elliptic equations and second order ordinary differential equations, see

[25-27] [31-33] [35-36] [49] [52] [63] [79-80] [104] and [109] and the references cited therein.

6.2. Nodal Solutions

Let $f \in C^1([0,\infty))$, $f(0) = 0$. Let $u(t,\alpha)$ be a crossing solution to the initial value problem (2.4). If we extend the domain of f in such a way that $f(-s) = -f(s)$, then $f \in C^1((-\infty,\infty))$ and is an odd function. Correspondingly, $u(t,\alpha)$ can be extended after it reaches the first zero $b(\alpha)$. In this context, it is of particular interest to study the existence and uniqueness of solutions which change signs in $t > 0$, to the problem

$$\begin{aligned} &\Delta u + f(u) = 0, \\ &u(0) > 0, \quad u = 0 \quad \text{on} \quad \Omega, \end{aligned} \tag{6.4}$$

and

$$\begin{aligned} &\Delta u + f(u) = 0, \\ &u(0) > 0, \quad u \to 0 \quad \text{as} \quad t \to \infty. \end{aligned} \tag{6.5}$$

By a nodal solution we mean a solution of (6.4) or (6.5) which has k simple zeros, where $k > 0$ is an integer. Note that when $f(0) = 0$, any zero point of a nontrivial solution is simple and isolated. Namely, if $u(t_0) = 0$ at some $t_0 > 0$, then it necessarily holds that $u'(t_0) \neq 0$. The existence of nodal solutions, especially infinitely many nodal solutions of problems (6.4) or (6.5) has been studied with various nonlinearities. An important model case is the so-called scalar field equation

$$\begin{aligned} &\Delta u - u + |u|^{p-1}u = 0, \\ &u(t) \to 0 \quad \text{as} \quad t \to \infty, \end{aligned} \tag{6.6}$$

where $1 < p < (n+2)/(n-2)$. Berestycki and Lions [13] have proved the existence of infinitely many radially symmetric nodal solutions of (6.6) by variational methods. But they did not give detailed information on the shape of solutions, and in particular, it was an open problem for some years as to whether solutions exist with a prescribed numbers of zeros. This question was answered in the affirmative by Jones and Kupper [60] by using a dynamical system approach and an application of the theory of the Conley index. A simple proof of Jones-Kupper's result was given by McLeod et al. [84] using ordinary differential equations theory
.

It was conjectured by Berestycki and Lions [13] that for any given $k > 0$, the nontrivial solutions of (6.6) with k zeros are unique. This problem still remains open. It seems that the uniqueness of nodal solutions of (6.4) or (6.5) is extremely difficult to study. The only exceptions are the cases when $f(u)$ has some homogeneous property or Ω is a sufficiently "thin" annulus. For instance, if $f(u) = u^p$, $p > 1$, and Ω is an annulus, then one may transform the equation of (6.4) into an autonomous planar system by making a change of variables. A simple phase plane analysis for the autonomous system may lead to the desired uniqueness result (see [24], [86] and [90] for details). On the other hand, if Ω is a thin annulus in the sense that the ratio of the outer radius and the inner radius is sufficiently small, and if f is superlinear, then problem (6.4) has at most one nodal solution with any prescribed number of zeros (see Hale and Raugel [55-56] and Ni and Nussbaum [90]).

For other results on the existence of infinitely many radial solutions or solutions with a prescribed number of zeros, see [12] [19] [61-62] [81] and [107-108].

6.3. Nonradial Solutions

Let $f \in C^1([0,\infty))$ and $f(0) = 0$. Let B be a finite ball in $\mathbb{R}^n$. As we have mentioned earlier, any solution of the Dirichlet problem

$$\begin{aligned} &\Delta u + f(u) = 0 \quad \text{in} \quad B, \\ &u > 0 \quad \text{in} \quad B, \quad u = 0 \quad \text{on} \quad \partial B, \end{aligned} \tag{6.7}$$

is radially symmetric according to [53]. A natural question to ask is if we replace B by an annulus in (6.7), is it still true that any solution of the new problem has to be radially symmetric? In the usual case, the answer is "No", even if $f(u)$ is of a simple form. The existence of nonradial solutions has been investigated by several authors in recent years. In [23], Coffman considered the following problem

$$\begin{aligned} &\Delta u - u + u^p = 0 \quad \text{in} \quad D(r,d) \subset \mathbb{R}^n, \\ &u > 0 \quad \text{in} \quad D(r,d), \quad u = 0 \quad \text{on} \quad \partial D(r,d), \end{aligned} \tag{6.8}$$

where $p > 1$ if $n = 2$, and $1 < p < (n+2)/(n-2)$ if $n \geq 3$. Both r and d are positive numbers, and

$$D(r,d) := \{x \in \mathbb{R}^n : \quad r^2 \leq |x|^2 \leq (r+d)^2\}.$$

It is proved in [23] that there could be many nonrotationally equivalent nonradial solutions to problem (6.8). More precisely, the number of nonrotationally equivalent nonradial solutions of problem (6.8) tends to $+\infty$ as $r \to +\infty$, in the case of

$n = 2$, $p > 1$, or $n \geq 3$ being even and $1 < p < n/(n-2)$. This result was improved by Li [73] who filled up the gap for $n \geq 4$ and $n/(n-2) \leq p < (n+2)/(n-2)$. It remains open if the same result is true for the case $n = 3$.

It is also not clear whether or not all positive solutions of the problem

$$\begin{aligned} &\Delta u + f(u) = 0 \quad \text{in} \quad \mathbb{R}^n, \\ &u \to 0 \quad \text{at} \quad \infty, \end{aligned} \tag{6.9}$$

are necessary radial (see the survey paper of Ni [89]). If we drop the positivity requirement of (6.9), then Ding [28] proved that the problem

$$\begin{aligned} &\Delta u + |u|^{\frac{4}{n-2}} u = 0 \quad \text{in} \quad \mathbb{R}^n, \\ &u \to 0 \quad \text{at} \quad \infty, \end{aligned}$$

has infinitely many distinct solutions which change signs on $\mathbb{R}^n$. Moreover, none of these solutions are radial.

6.4. Singular Solutions

Let Ω be a finite ball in $\mathbb{R}^n$ or $\Omega = \mathbb{R}^n$, $n \geq 3$. Consider the following problem

$$\begin{aligned} &\Delta u + f(u) = 0 \quad \text{in} \quad \Omega - \{0\}, \\ &\lim_{x \to 0} u(x) = \infty. \end{aligned} \tag{6.10}$$

A positive solution $u(x)$ of (6.10) is called a singular solution. It is called a singular ground state solution if $\Omega = \mathbb{R}^n$ and $\lim_{|x| \to \infty} u(x) = 0$.

When $f(0) = 0$ and f is subcritical, the existence of positive singular solutions of (6.10) on finite balls has been shown by Ni and Sacks [91], and Lin [78]. While the nonexistence of positive singular solutions of (6.10) on finite balls has been shown by Ni and Serrin [94] for the case when f is supercritical. Recently, Pan [97] studied the existence of singular positive ground state solutions of problem (6.10) for $f(u) = u^p + u^{(n+2)/(n-2)}$. He proved that Problem (6.10) possesses a singular ground state solution when $n/(n-2) < p < (n+2)/(n-2)$. Note that a singular ground state solution does not exist if $1 < p \leq n/(n-2)$.

6.5. Singular Equations

When a nonlinearity $f(t, u)$ is assumed to be singular in the second variable, i.e., $\lim_{u\to 0^+} f(t,u) = \infty$ for each fixed $t \in (0,1]$, the boundary value problem

$$\Delta u + f(t,u) = 0,$$
$$x > 0 \quad \text{in} \quad B, \quad x = 0 \quad \text{on} \quad \partial B,$$

was studied by Gatica, Hernandez and Waltman [50] with a view to obtaining the existence of classical solutions. A model case for such singular nonlinearities is $f(t,u) = a(t)u^{-p}$, $p > 0$. Other relevant results can be found in [51] and [71].

6.6. Other Quasilinear Equations

Our study for the m-Laplace equations can be extended to other quasilinear equations. In particular, the uniqueness of positive radial solutions to the mean curvature equation

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) + f(u) = 0$$

can be similarly investigated.

We mention here a quasilinear equation studied by Serrin and Zou [105] which takes the form

$$\Delta u + u^p - |\nabla u|^q = 0. \tag{6.11}$$

They were interested in the question of existence and non-existence of radial ground states of (6.11). It was proved that problem (6.11) has a ground state solution if either

$$\text{i) } p > \frac{n+2}{n-2}, \quad \text{ii) } p = \frac{n+2}{n-2}, q < p, \quad \text{or iii) } p < \frac{n+2}{n-2}, \quad q < \frac{2p}{p+1}.$$

and (6.11) does not have a ground state solution if either

$$\text{i) } q \geq \frac{2p}{p+1}, \quad 0 < p < 1, \quad \text{or ii) } q > \frac{2p}{p+1}, \quad 1 \leq p \leq \frac{n}{n-2}.$$

The uniqueness and asymptotic behavior of ground states of (6.11) was studied in Peletier, Serrin and Zou [100] for the case $n = 1$.

6.7. Semipositone Problems

All our studies for the existence and uniqueness problems of (2.1)-(2.2) are restricted to the case $f(t,0) \equiv 0$. Some other cases have been studied recently.

For instance, Castro et al. [18] considered the radial solutions to the problem

$$\begin{aligned} &\Delta u + \lambda f(u(x)) = 0 \quad \text{and} \quad x \in B_1, \\ &u(x) = 0 \qquad\qquad\quad\ \text{for} \quad x \in \partial B_1 \end{aligned}$$

where B_1 denotes the unit ball in $\mathbb{R}^n$, $n > 1$, centered at origin and $\lambda > 0$. They assumed that $f: \mathbb{R} \to \mathbb{R}$ is monotonically increasing, superlinear with subcritical growth on $[0, \infty)$. In particular, they were concerned with the case $f(0) < 0$, i.e., f is semipositone. They established the structure of radial solution branches for the above problem. It was also proved that if f is convex and $f(s)/(sf'(s) - f(s))$ is a nondecreasing function, then for each $\lambda > 0$ there exists at most one positive solution u such that (λ, u) belongs to the unbounded branch of positive solutions. The model case $f(s) = s^p - k$, $k > 0$, $1 < p < (n+2)/(n-2)$ was studied. See Allegretto et al. [6-7] for other studies on the positive solutions to semipositone problems.

6.8. Effect of the Dimension

As we have seen, a striking feature of the elliptic problems is that the existence of nontrivial solutions is sometimes dependent on the dimensions of the space $\mathbb{R}^n$, especially when the Sobolev critical exponent is involved. It is worth mentioning that this interesting phenomenon also occurs in the study of Neumann problems. Let $\lambda > 0$ and consider the following problem

$$\begin{aligned} &\Delta u - \lambda u + u^{(n+2)/(n-2)} = 0 \quad \text{in} \quad B_1, \\ &u > 0 \quad \text{in} B_1, \quad \frac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad \partial B_1, \end{aligned} \tag{6.12}$$

where ν denotes the outer normal vector on ∂B_1. Problem (6.12) always admits a constant solution $u_0 = \lambda^{(n-2)/4}$. The existence of a nonconstant solution to this problem has been considered by Adimurthi [2-3] and Budd, Knaap and Peletier [15]. They proved that there exists a constant $\lambda_0 > 0$ such that if $0 < \lambda < \lambda_0$, then (6.12) admits a nonconstant solution when $n = 4, 5, 6$, while no solution exists when $n = 3$. Recently, Adimurthi [5] proved that problem (6.12) possesses no solution for $0 < \lambda < \lambda_0$ when $n \geq 7$. Thus, if we restrict n to be an integer and $n \geq 3$, then problem (6.12) has a nonconstant solution for small λ if and only if $n = 4, 5, 6$.

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