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The Dimensional-Reduction Anomaly

by

Patrick James Sutton ©

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Patrick James Sutton

Patrick James Sutton
Physics Department
University of Alberta
Edmonton, Alberta
T6G 2J1

20 September 2000

University of Alberta

Faculty of Graduate Studies and Research

The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research for acceptance, a thesis entitled **The Dimensional-Reduction Anomaly** submitted by **Patrick James Sutton** in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

Prof. V. P. Frolov (Supervisor)

Don N. Page

Prof. D. N. Page (Chair)

Robert Mann

Prof. R. B. Mann (External)

B. A. Campbell

Prof. B. A. Campbell

M. Razavy

Prof. M. Razavy

Prof. H. Kunzle

DATE: September 8, 2000

Abstract

In D-dimensional spacetimes which can be foliated by n-dimensional homogeneous subspaces, a quantum field can be decomposed in terms of modes on the subspaces. Substitution into the field equations reduces the free quantum field in D dimensions to a collection of quantum fields in $D-n$ dimensions. This dimensional reduction establishes a formal relationship whereby objects in the physical theory, such as the Green function and the effective action, can be written as sums of the corresponding objects in the dimensionally reduced theories.

In this thesis we examine dimensional reduction in the context of renormalization. Quantities of physical interest in quantum field theory, such as the expectation value of the stress tensor, are divergent and must be renormalized. Though the equivalence of the original and dimensionally-reduced theories is easily established at the bare level, we demonstrate that the divergent terms which must be subtracted to renormalize the higher- and lower-dimensional theories are not related by the mode decomposition. As a result, renormalized expectation values in D dimensions can be obtained by summing over their renormalized $(D-n)$-dimensional counterparts only if the

contribution of each mode is modified by an extra anomalous contribution. We call this effect the dimensional-reduction anomaly.

We explicitly calculate the dimensional-reduction anomaly in the field fluctuations and the stress tensor for several classes of spacetimes of physical interest, with particular emphasis on spherical and static spaces. In each case, the anomaly is shown to produce significant mode-by-mode corrections to renormalized expectation values in the dimensionally reduced theories. For spherical geometries we investigate the relevance of the anomaly to recent attempts to calculate the stress tensor and Hawking radiation from Schwarzschild black holes using two-dimensional dilaton-gravity models. For static spaces we find an intriguing relationship between the anomaly and a new, general approximation scheme for renormalized expectation values of quantum fields.

To

Catherine

Preface

The research presented here was performed by the author, partially in collaboration with V. P. Frolov and A. I. Zelnikov, at the University of Alberta between November 1998 and July 2000. Unless otherwise noted, all of the material in this report is original. Much of it has been published in the following articles:

- P. Sutton, "Dimensional-reduction anomaly in spherically symmetric spacetimes". Physical Review **D 62** 044033 (2000).
- V. Frolov, P. Sutton, and A. Zelnikov, "Dimensional-reduction anomaly". Physical Review **D 61** 024021 (2000).
- P. J. Sutton, "Dimensional Reduction of the Effective Action and the Multiplicative Anomaly". Proceedings of the Eighth Canadian Conference on General Relativity and Relativistic Astrophysics, pp. 257-261 (American Institute of Physics, New York, 1999).

These results have also been presented at the Second International School on Field Theory and Gravitation (April 25-28, 2000, Vitoria, Brasil) and at the Fradkin Memorial meeting (June 5-10, 2000, Moscow, Russia).

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Most of all, I thank my wife Catherine. She has been my light and my love, a pillar of faith, and an unending source of support and encouragement throughout these many years. All that I have and all that I am I owe to her.

Table of Contents

	D dimensions	**D − n dimensions**	**n dimensions**
indices	$\mu, \nu, \ldots$	$a, b, c, \ldots$	$i, j, k, \ldots$
coordinates	X^{μ}	x^{a}	y^{i}
metric	$g_{\mu\nu}$	h_{ab}	Ω_{ij}
d'Alembertian	$\Box$	Δ_{h}	Δ_{Ω}
covariant derivative	$\nabla_{\mu}, ()_{;\mu}$	$\nabla_{a}, \nabla, ()_{\|a}$	$\nabla_{i}, ()_{:i}$
field objects	$G, W, \ldots$	$\mathcal{G}, \mathcal{W}, \ldots$	
curvatures	${}^{D}R = R[g]$	$R = R[h]$	$R[\Omega]$

Table 1: Conventions on Notation

Chapter 1

Introduction

In the absence of a full quantum theory of gravitation, attempts to model the behaviour of quantized matter incorporating gravitational interactions are forced to rely on the semiclassical approximation. In this approach, the gravitational field is treated as a classical background on which the quantized matter fields propagate. Quantum amplitudes may then be calculated perturbatively to the desired order.[1]

Even at the semiclassical level, vacuum polarization effects in quantized matter are notoriously difficult to compute. As a result, much effort has been devoted to calculating vacuum polarization and particle production in privileged spacetimes which possess a high degree of symmetry; these include homogeneous cosmological models [1–4], and eternal black hole spacetimes [5–19]. In these systems, one can take advantage of the continuous symmetries of the geometry to simplify the calculation of field-dependent functions. By selecting a coordinate system based on the symmetries and using separation of variables, the quantum field may be decomposed in terms of modes in the symmetry directions to effec-

[1]Small gravitational fluctuations about the classical background may be treated as a separate matter field of gravitons and quantized to the one-loop level, the non-renormalizability of gravity preventing a consistent extension to higher orders.

tively reduce the dimensionality of the system. For example, in a D-dimensional static spacetime one can decompose the quantum field in terms of Fourier time modes $e^{i\omega t}$. Substituting this Fourier transform into the field equation yields a new field equation in $(D-1)$ dimensions with an ω-dependent potential. Quantities of interest in the physical theory can then be obtained by solving for the corresponding quantities in the dimensionally reduced theory and summing over all ω (performing the inverse Fourier transform).

This technique is standard in the solution of partial differential equations for classical fields. There is, however, a difficulty that one encounters in applying dimensional reduction to quantum field theories. Specifically, quantities of physical interest in quantum field theories, such as expectation values, typically are divergent and must be renormalized to yield a finite, meaningful result. This raises the question of how renormalization affects the relationship between the dimensionally reduced theory and the original physical theory. In the language of operators, we ask whether dimensional reduction and renormalization commute. In this report, we find that they do not. This may be understood from the fact that renormalization is a purely local operation, depending only on the short-distance or high-frequency behaviour of the theory. Mode decomposition, however, is sensitive to global characteristics such as topology, and so probes the long-distance or low-frequency behaviour. Integrating out some portion of the manifold can change the global properties of the system, a change to which the renormalization is insensitive. This leads to different results when renormalization is carried out before versus after the mode decomposition. In subsequent chapters we will see that further discrepancies between renormalization in the two theories arise due to local effects depending on the curvature and field potential. We call the resulting non-commutability of dimensional reduction and renormalization the **dimensional-reduction anomaly**.

The problem of dimensional reduction and renormalization may be of relevance to various issues that have appeared in the literature in recent years. One of the most notable of these is the attempt to calculate the stress tensor and Hawking radiation of Schwarzschild black holes using two-dimensional dilaton-gravity models [20–31]. In this case, a massless, minimally coupled scalar field in a four-dimensional spherically symmetric spacetime (such as that of a Schwarzschild black hole) is decomposed into spherical harmonics, reducing the field to one propagating in two dimensions. Using the conformal properties of two-dimensional spaces, the calculation of the effective action is greatly simplified. One can then obtain the contribution of the $\ell = 0$ spherical mode to the stress tensor and Hawking radiation in the original, physical spacetime. Various attempts along these lines [32–39] have been rewarded with unsettling results [40], including in some cases a negative Hawking flux, and in all cases two-dimensional stress tensors that are qualitatively different from the expected four-dimensional one near the black hole. We show that the dimensional-reduction anomaly supplies a state-independent contribution to the stress tensor which appears to correct the predictions of the dimensionally reduced theories near the event horizon, while leaving the asymptotic behaviour (including the Hawking radiation) unaffected.

A second system that we shall consider is that of a scalar field in static space at finite temperature. In this case we demonstrate that the dimensional-reduction anomaly can be used to derive a new general analytic approximation for the renormalized physical theory which is based on the high-frequency behaviour of the field. This technique can be viewed as an extension of previous approximations for conformal fields by Page, Brown, and Ottewill [8, 9] and by Frolov and Zelnikov [10], as well as the approximation of Anderson, Hiscock, and Samuel [11, 12] for fields in static spherically symmetric spacetimes.

This thesis is organized as follows. In Chapter 2 we review some of the basic formalism of free quantum fields in curved space. We then examine the dimensional reduction of a quantum field in a spacetime with a homogeneous subspace. We indicate how the dimensional-reduction anomaly arises and relate it to the more general multiplicative anomaly. Chapter 3 is devoted to explicit calculations of the dimensional-reduction anomaly for the simple case of spherical decompositions in flat space. We show how the anomaly arises and why it must be included for the dimensional reduction to yield standard results for flat spacetime. In Chapter 4 we extend our considerations to general four-dimensional spherically symmetric spaces, with particular emphasis on the Schwarzschild geometry. We calculate explicitly the anomalies in $\langle\hat{\Phi}^2\rangle$, the effective action, and the stress tensor. Comparisons to known results for renormalized quantum fields in two and four dimensions demonstrate the significance of the anomaly to dilaton-gravity models. In Chapter 5 we examine dimensional reduction in four-dimensional static spaces and derive the corresponding anomaly. For the zero-temperature case, we demonstrate in Chapter 6 how this anomaly may be used to obtain a new approximation scheme for quantum fields in static spaces. We conclude in Chapter 7 with a summary of our results and point out some prospects for future work.

Chapter 2

Quantum Field Theory in Curved Spacetime and Dimensional Reduction

Our main subject is special properties of quantum fields propagating in curved spacetimes. In this chapter we collect some general results which will be useful later. We begin in Section 2.1 with a brief review of the quantization of a free scalar field in a gravitational background, with emphasis on such topics as Green functions, the effective action, the heat kernel, and the choice of quantum state. In Section 2.2 we examine the dimensional reduction of such a theory when the spacetime contains a homogeneous subspace, showing how it may be rewritten as a collection of fields in a lower-dimensional spacetime. In Section 2.3 we indicate how this dimensional reduction breaks down under renormalization, producing the dimensional-reduction anomaly.

2.1 Quantum Field Theory in Curved Spacetime

Let us review the general scheme for the quantization of a free field in a gravitational background. It will be sufficient for our purposes to consider real scalar fields only, though the following formalism is easily extended to more general bosonic or fermionic fields. For more general and detailed accounts see, for example, DeWitt [41, 42], Berezin [43], Birrell and Davies [44], and Wald [45].

2.1.1 Classical Field

A self-consistent description of a classical scalar field Φ propagating in a curved spacetime with metric $g_{\mu\nu}$, where the gravitational interaction of the matter is taken into account, may be formulated in terms of the actions for the scalar and gravitational fields. The classical action for Φ is taken to be

$$S = \frac{1}{2}\int dx\sqrt{|g|}\,\Phi F \Phi\,, \tag{2.1}$$

where F is a self-adjoint operator.[1] Combining the action (2.1) with the classical Einstein-Hilbert action for general relativity,[2]

$$S_{\text{grav}} = -\frac{1}{16\pi}\int dx\sqrt{|g|}\,\left[R - 2\Lambda\right]\,, \tag{2.2}$$

and setting the variation of the total action $S + S_{\text{grav}}$ with respect to the metric equal to zero yields the Einstein equations,

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu}\,. \tag{2.3}$$

---

[1] Our F is self-adjoint if, for any smooth complex functions ψ_1, ψ_2 having compact support in an open spacetime region of interest,

$$\int dx\sqrt{|g|}\,\left[\overline{\psi}_1(F\psi_2) - (\overline{F\psi_1})\psi_2\right] = 0\,,$$

where the bar denotes the complex conjugate.

[2] We ignore the surface term that should be included for rigour [46].

Here $T_{\mu\nu}$ is the stress tensor for the field Φ, and is defined by

$$T_{\mu\nu} \equiv \frac{2}{\sqrt{|g|}} \frac{\delta S}{\delta g^{\mu\nu}} . \tag{2.4}$$

In this thesis we will consider the gravitational background as a fixed external field and ignore the back-reaction of the (quantized) matter on the geometry, focusing rather on the vacuum polarization of the scalar field.

For a fixed background geometry the field equation satisfied by Φ is found by setting to zero the variation of the action (2.1) with respect to the field, and is

$$F\Phi = 0 . \tag{2.5}$$

The self-adjointness of the field operator F can be used to show that there exists an invariant "inner product" for solutions of the field equation which is bilinear in the fields and independent of the Cauchy surface on which it is evaluated. Specifically, if Φ_1, Φ_2 are any two complex solutions of (2.5) and Σ is a complete Cauchy hypersurface, one can show that

$$(\Phi_1, \Phi_2) \equiv -i \int_\Sigma d\Sigma^\mu \, \overline{\Phi}_1 \overleftrightarrow{f}_\mu \Phi_2$$

is invariant under smooth deformations and displacements of Σ. (If the space is noncompact, one assumes that the solutions fall off sufficiently rapidly at infinity.) Here $\overleftrightarrow{f}_\mu$ is related to F via

$$\int_\Omega dx \sqrt{|g|} \left[\overline{\psi}_1 (F\psi_2) - (\overline{F\psi_1})\psi_2 \right] = \int_{\partial\Omega} d\Sigma^\mu \, \overline{\psi}_1 \overleftrightarrow{f}_\mu \psi_2 ,$$

where Ω is any compact region of spacetime with smooth boundary $\partial\Omega$, $d\Sigma^\mu$ is the outward-directed surface element of $\partial\Omega$, and ψ_1, ψ_2 are any smooth complex functions defined on an open region containing Ω. Using this inner product, one can introduce a complete, orthonormal set of solutions $\{\Phi_i, \overline{\Phi}_i\}$ (the bar denoting the complex conjugate) such that

$$(\Phi_i, \Phi_j) = -(\overline{\Phi}_i, \overline{\Phi}_j) = \delta_{ij} , \qquad (\overline{\Phi}_i, \Phi_j) = (\Phi_i, \overline{\Phi}_j) = 0 . \tag{2.6}$$

Since the set $\{\Phi_i, \overline{\Phi}_i\}$ is complete, we can decompose the field Φ as

$$\Phi = \sum_i \left[a_i \Phi_i + \overline{a}_i \overline{\Phi}_i\right] , \tag{2.7}$$

where a_i represents the constant amplitude of the field in the mode Φ_i. Note that the set $\{\Phi_i, \overline{\Phi}_i\}$ is not unique; this will be of some importance later.

2.1.2 Canonical Quantization

Assuming that the spacetime is globally hyperbolic, so that it may be foliated by spacelike Cauchy hypersurfaces, we can define a canonical momentum Π conjugate to Φ via

$$\Pi \equiv \frac{\delta S}{\delta \Phi_{,0}} . \tag{2.8}$$

Here the x^0 coordinate enumerates the hypersurfaces.

A standard procedure for quantizing the theory (2.1) is to elevate Φ, Π to operators and impose on each slice the standard canonical commutation relations

$$\left[\hat{\Phi}(x^0, \mathbf{x}), \hat{\Phi}(x^0, \mathbf{x}')\right] = 0 , \tag{2.9}$$

$$\left[\hat{\Pi}(x^0, \mathbf{x}), \hat{\Pi}(x^0, \mathbf{x}')\right] = 0 , \tag{2.10}$$

$$\left[\hat{\Phi}(x^0, \mathbf{x}), \hat{\Pi}(x^0, \mathbf{x}')\right] = i\,\delta(\mathbf{x} - \mathbf{x}') , \tag{2.11}$$

where the $\mathbf{x}$ are the coordinates on the hypersurfaces.

In the decomposition (2.7) the a_i, $\overline{a}_i$ become operators:

$$\hat{\Phi} = \sum_i \left[\hat{a}_i \Phi_i + \hat{a}_i^\dagger \overline{\Phi}_i\right] , \tag{2.12}$$

where the dagger ($\dagger$) denotes the hermitian conjugate. The commutation relations (2.9)–(2.11) then imply that the $\hat{a}_i$, $\hat{a}_i^\dagger$ act respectively as the annihilation and creation operators for quanta in the mode Φ_i, as they satisfy the commutation relations

$$[\hat{a}_i, \hat{a}_j] = [\hat{a}_i^\dagger, \hat{a}_j^\dagger] = 0 , \qquad [\hat{a}_i, \hat{a}_j^\dagger] = \delta_{ij} . \tag{2.13}$$

The free quantum field thus decomposes to a collection of independent quantum harmonic oscillators, and we can construct a Fock basis for the Hilbert space just as one would do for the quantum oscillator. In particular, the vacuum state $|0\rangle$ is defined as that state annihilated by all $\hat{a}_i$:

$$\hat{a}_i|0\rangle = 0\,. \tag{2.14}$$

Multiparticle states can then be built up by repeated applications of the creation operator $\hat{a}_i^\dagger$.

2.1.3 Vacuum States

It should be noted that the physical nature of the vacuum state (2.14) depends on the choice of basis modes $\{\Phi_i, \overline{\Phi}_i\}$. One of the most interesting features of quantum field theory in curved spacetime is that in general there is no set of modes which is "preferred" by the geometry. As a result, there is no unique vacuum state.

To see the consequences of this, consider two complete, orthonormal sets of states,

$$\{u_i, \overline{u}_i\}\,, \qquad\qquad \{v_i, \overline{v}_i\}\,, \tag{2.15}$$

with associated creation and annihilation operators

$$\{\hat{a}_i, \hat{a}_i^\dagger\}\,, \qquad\qquad \{\hat{b}_i, \hat{b}_i^\dagger\}\,, \tag{2.16}$$

and vacua $|0\rangle_u$, $|0\rangle_v$ such that

$$\hat{a}_i|0\rangle_u = 0\,, \qquad\qquad \hat{b}_i|0\rangle_v = 0\,. \tag{2.17}$$

Since both sets are complete, we must be able to write the $\{v_i, \overline{v}_i\}$ as linear combinations of the $\{u_i, \overline{u}_i\}$. For example,

$$v_i = \sum_k [\alpha_{ik} u_k + \beta_{ik} \overline{u}_k]\,. \tag{2.18}$$

This is known as a Bogoliubov transformation. The orthonormality and completeness of both sets of modes dictates that the matrices α and β must satisfy

$$\alpha\alpha^\dagger - \beta\beta^\dagger = I\,, \tag{2.19}$$

$$\overline{\alpha}\beta^\dagger - \overline{\beta}\alpha^\dagger = 0\,, \tag{2.20}$$

where I is the identity matrix.

The operators $\{\hat{b}_i, \hat{b}_i^\dagger\}$ can also be written in terms of the $\{\hat{a}_i, \hat{a}_i^\dagger\}$. For example,

$$\hat{b}_i = \sum_k [\overline{\alpha}_{ik}\hat{a}_k - \overline{\beta}_{ik}\hat{a}_k^\dagger]\,. \tag{2.21}$$

If $\beta \neq 0$, the annihilation and creation operators are mixed in the Bogoliubov transformation. As a result, the vacua $|0\rangle_u$ and $|0\rangle_v$ are inequivalent. In particular, the number of particles in the mode v_i contained in the $|0\rangle_u$ vacuum is

$$_u\langle 0|\hat{b}_i^\dagger \hat{b}_i|0\rangle_u = \sum_k |\beta_{ki}|^2\,. \tag{2.22}$$

If the gravitational field vanishes (or becomes static) at very early or very late times, the vacuum state in these asymptotic regions can be chosen naturally. To do so, one identifies the basis modes Φ_i with positive-frequency solutions ($\Phi \propto e^{-i\omega t}$, $\omega > 0$). For such a definition the corresponding vacuum state is the state of lowest possible energy. For $x^0 \to -\infty$ it is known as the "in" vacuum and denoted by $|0;\text{in}\rangle$, while for $x^0 \to +\infty$ it is the "out" vacuum $|0;\text{out}\rangle$. In the presence of both "in" and "out" asymptotic regions one has at least two generically different privileged vacuum states. If at intermediate times the gravitational field is non-trivial, a given "in" mode Φ_i^{in} will typically scatter into a linear combination of the late-time modes $\{\Phi_j^{\text{out}}, \overline{\Phi}_j^{\text{out}}\}$. In particular, if the decomposition of Φ_i^{in} into "out" modes contains any of the $\overline{\Phi}_j^{\text{out}}$ (i.e., if $\beta \neq 0$) then the vacuum states $|0;\text{in}\rangle$ and $|0;\text{out}\rangle$ are not equivalent. In this case a field which begins in the state $|0;\text{in}\rangle$ at early times will contain "out" particles at late times. The interpretation is that the nontrivial gravitational field has produced particles of the quantum field.

2.1.4 Green Functions

The various Green functions associated with the quantized theory may be defined as expectation values or matrix elements of products of the field operators for a given quantum state. Of particular interest to us are the Hadamard Green function $G^{(1)}(x,x')$ and the Feynman Green function $G_F(x,x')$ for some vacuum state $|0\rangle$:

$$G^{(1)}(x,x') \equiv \langle 0|\{\hat{\Phi}(x),\hat{\Phi}(x')\}|0\rangle\,, \tag{2.23}$$

$$G_F(x,x') \equiv i\langle 0|T(\hat{\Phi}(x),\hat{\Phi}(x'))|0\rangle\,. \tag{2.24}$$

Here $\{,\}$ denotes the anticommutator and $T()$ the time-ordered product,

$$T(\hat{\Phi}(x),\hat{\Phi}(x')) = \theta(x,x')\hat{\Phi}(x)\hat{\Phi}(x') + \theta(x',x)\hat{\Phi}(x')\hat{\Phi}(x)\,,$$

where $\theta(x,x') = 1$ if x lies to the future of a spacelike hypersurface through x', and vanishes otherwise. The choice of hypersurface is arbitrary, as the field operators commute for spacelike separations.

Using the commutation relations (2.9)–(2.11), one can show that the Green functions obey the differential equations

$$F\,G^{(1)}(x,x') = 0\,, \tag{2.25}$$

$$F\,G_F(x,x') = -\delta(x,x') \equiv -\frac{\delta(x-x')}{\sqrt{|g|}}\,. \tag{2.26}$$

Note that the boundary conditions obeyed by the Green functions are contained implicitly by the choice of modes $\{\Phi_i,\overline{\Phi}_i\}$ used to define the vacuum state.

For a given mode set $\{\Phi_i,\overline{\Phi}_i\}$, the Green functions can be found by substituting the mode decomposition (2.12) into (2.23) or (2.24). For example, for the

Hadamard function we find

$$G^{(1)}(x,x') = \sum_i [\Phi_i(x)\overline{\Phi}_i(x') + \overline{\Phi}_i(x)\Phi_i(x')] \,. \tag{2.27}$$

The stress tensor for the quantized theory may also be expressed as a sum over modes by taking the quadratic-in-Φ expression resulting from the differentiation of the action with respect to the metric in (2.4), making the replacement $\Phi \to \hat{\Phi}$, and taking the expectation value. For the vacuum state we obtain

$$\langle 0|\hat{T}_{\mu\nu}|0\rangle = \sum_i T_{\mu\nu}(\Phi_i, \overline{\Phi}_i) \,, \tag{2.28}$$

where $T_{\mu\nu}(\Phi_i, \overline{\Phi}_i)$ is the expression resulting from (2.4); it is symmetric in x, x' and in Φ_i, $\overline{\Phi}_i$. Equivalently, we will see in Section 2.2 that the stress tensor can be written as the coincidence limit $x' \to x$ of a differential operator acting on the Hadamard Green function $G^{(1)}(x,x')$. Using the mode-decomposed form (2.27) for $G^{(1)}(x,x')$ then yields (2.28).

2.1.5 Effective Action

We will be interested in calculating the expectation value of the stress tensor for quantized scalar fields. While the mode-sum technique is physically transparent, the required mode set is typically extremely difficult to find. An alternative approach which is both elegant and enlightening is to define a quantized version of the action (2.1), called the effective action and denoted W, from which the expectation value of the stress tensor for a given state can be obtained by functional differentiation in analogy to (2.4):

$$\langle \hat{T}_{\mu\nu}\rangle \equiv \frac{2}{\sqrt{|g|}}\frac{\delta W}{\delta g^{\mu\nu}} \,. \tag{2.29}$$

Using the 'in-out' formalism, it is not difficult to show that W may be obtained from the vacuum-to-vacuum transition amplitude $\langle 0;\mathrm{out}|0;\mathrm{in}\rangle$. With hindsight we define

$$e^{iW} \equiv \langle 0;\mathrm{out}|0;\mathrm{in}\rangle = \int \mathcal{D}[\Phi]\, e^{iS[\Phi]}\,, \tag{2.30}$$

where we use the path integral representation for $\langle 0;\mathrm{out}|0;\mathrm{in}\rangle$ (see, e.g., [44, 47]). Under a change δS in the classical action, the corresponding variation in W is given by

$$\delta W = e^{-iW} \int \mathcal{D}[\Phi]\, \delta S\, e^{iS[\phi]} \tag{2.31}$$

$$= \frac{\langle 0;\mathrm{out}|\delta S|0;\mathrm{in}\rangle}{\langle 0;\mathrm{out}|0;\mathrm{in}\rangle}\,. \tag{2.32}$$

Equation (2.32) is a statement of the well-known "Schwinger variational principle" [48]. For the special case where the variation δS is due to a variation $\delta g^{\mu\nu}$ in the metric, we have by definition of the stress tensor

$$\delta S = \frac{1}{2}\int dx \sqrt{|g|}\, \delta g^{\mu\nu}\, T_{\mu\nu}\,. \tag{2.33}$$

Substituting (2.33) into (2.32) it clearly follows that

$$\frac{\langle 0;\mathrm{out}|\hat{T}_{\mu\nu}|0;\mathrm{in}\rangle}{\langle 0;\mathrm{out}|0;\mathrm{in}\rangle} = \frac{2}{\sqrt{|g|}}\frac{\delta W}{\delta g^{\mu\nu}}\,. \tag{2.34}$$

Other expectation values such as $\langle 0;\mathrm{in}|\hat{T}_{\mu\nu}|0;\mathrm{in}\rangle$ differ from (2.34) by finite, well-defined amounts [42].

The next task is to find a convenient method for calculating W, which is typically a difficult problem. Returning to (2.30) and inserting the expression (2.1) for the classical field action, we obtain a Gaussian path integral for the free field. This evaluates to [44, 47]

$$e^{iW} \propto [\det(F)]^{-\frac{1}{2}}\,, \tag{2.35}$$

where we treat the field operator F formally as a matrix. From this we determine

$$W = \frac{i}{2}\ln\det(F) = \frac{i}{2}\mathrm{Tr}\ln(F)\,. \tag{2.36}$$

In writing (2.36) we have dropped a metric-independent additive constant, as it will not contribute to the stress tensor.

2.1.6 Heat Kernel

The result (2.36) for the effective action can be reformulated in a more convenient manner through the introduction of a quantity known as the heat kernel. Let us formally consider G_F as an operator on the space of vectors $|x\rangle$ normalized by

$$\langle x|x'\rangle = \delta(x, x'), \tag{2.37}$$

such that

$$G_F(x, x') = \langle x|G_F|x'\rangle. \tag{2.38}$$

The differential equation for G_F then becomes $G_F = -F^{-1}$, and hence one can write

$$G_F = -F^{-1} = i\int_0^\infty ds\, e^{iFs}, \tag{2.39}$$

$$\ln(F) = -\int_0^\infty \frac{ds}{s} e^{iFs}, \tag{2.40}$$

where in the second equation we have discarded a metric-independent infinite constant. For rigour, we should add a small positive imaginary part to F to ensure that the integrals over s converge in the $s \to \infty$ limit; for a massive field this is equivalent to setting $m^2 \to m^2 - i\epsilon$. This procedure also assures that (2.39) evaluates to the Feynman Green function rather than one of the other Green functions [49].

Taking matrix elements, we find

$$G_F(x, x') = i\int_0^\infty ds\, K(x, x'|s), \tag{2.41}$$

$$W = -\frac{i}{2}\int_0^\infty \frac{ds}{s}\int dx\sqrt{|g|}\, K(x, x|s), \tag{2.42}$$

where the function $K(x, x'|s) = \langle x|e^{iFs}|x'\rangle$ is known as the heat kernel for the operator F. It is clearly a solution of the Schrödinger-like equation

$$\left(i\frac{\partial}{\partial s} + F\right) K(x, x'|s) = 0 \tag{2.43}$$

with initial condition

$$K(x, x'|s = 0) = \delta(x, x') . \tag{2.44}$$

These results can also be obtained through alternative means which avoid the formal constructions (2.37)–(2.40) [50].

The heat kernel formulation has several virtues. It provides a unified means of studying both the Green function and the effective action. From a conceptual standpoint, it is interesting because it gives a quantum-mechanical solution[3] to a field theory problem. Most important for our purposes, the heat kernel formulation is convenient for renormalizing quantum field theories in curved spacetime, which we consider next.

2.1.7 Renormalization

One of the principle features of quantum field theory is that quantities of physical interest contain infinities. For example, the expectation value of the square of the field operator, which is a measure of the quantum fluctuations in the field, can be obtained from the coincidence limit of the Hadamard Green function (2.23):

$$\langle \hat{\Phi}^2(x) \rangle = \lim_{x' \to x} \frac{1}{2} G^{(1)}(x, x') . \tag{2.45}$$

However, this Green function diverges in the coincidence limit, so that $\langle \hat{\Phi}^2(x) \rangle$ is not well-defined. Similar infinities plague the stress tensor and the effective action. The procedure by which these infinities are eliminated to yield finite, physically meaningful results is known as renormalization.

The heat kernel formalism just discussed provides a powerful method for renormalizing quantum field theories in curved spacetimes. In (2.41)–(2.42), the divergences in both the Green function and the effective action come from

[3]Note that the heat kernel is formally identical to the amplitude for a quantum-mechanical particle to propagate from x' to x in "time" s under the "Hamiltonian" $-F$.

the $s \to 0$ limit of the s integral (the $s \to \infty$ limit is well-behaved due to the $i\epsilon$ prescription). The advantage of the heat kernel formulation is that the small-s behaviour of the heat kernel is known for arbitrary curved spaces of any dimension, in the form of the Schwinger-DeWitt expansion [41, 42, 51]. We postpone a detailed examination of this expansion until Section 4.1; suffice it to say that the divergences in the effective action (Green function) in D-dimensional spacetime are contained in the first N_D terms ($N_D - 1$ terms) of the Schwinger-DeWitt expansion of the heat kernel, where

$$N_D = \begin{cases} \frac{D}{2} + 1 & \text{for D even,} \\ \frac{D+1}{2} & \text{for D odd.} \end{cases} \tag{2.46}$$

Moreover, the coefficients in the Schwinger-DeWitt expansion are purely local functions of the background fields appearing in the classical gravitational and matter actions: the curvature, the field potential, and their covariant derivatives. (This is because the divergences come about in the $x' \to x$ limit, and so must be independent of boundary conditions and the global nature of the spacetime.) Because of their purely local form, the divergences can be absorbed into the classical actions (2.1), (2.2) by redefinitions of the Newtonian gravitational constant, the cosmological constant, the coupling in the potential, and the other parameters in these actions. From a practical standpoint, that means that we can renormalize a given expectation value by simply subtracting the contribution from the first few terms of the Schwinger-DeWitt expansion.

2.1.8 Euclidean Approach

Until now we have been considering quantum field theory on Lorentzian manifolds; i.e., on spacetimes with indefinite metric. An alternative approach which is frequently used for quantum field theory calculations is to define and compute all quantities of interest on a Euclidean manifold; i.e., on a space with positive-definite metric [52]. For static spaces, this is equivalent to moving from real to imaginary time via the 'Wick rotation' $t \to -it$. Results for the physical spacetime can be obtained by analytic continuation from Euclidean space at the end of the calculation.

The Euclidean approach has several advantages. Foremost among these is that the differential operator F, which is hyperbolic in Lorentzian manifolds, becomes elliptic in Euclidean manifolds. For a wide class of Euclidean spaces and operators, its inverse (the Green function) is then well-defined and unique [52]. This Green function G obeys the differential equation

$$FG(X,X') = \delta^D(X,X') \equiv \frac{\delta^D(X-X')}{\sqrt{g}}, \tag{2.47}$$

and vanishes as the separation distance goes to infinity in noncompact manifolds.

One can also obtain the Euclidean Green function from the heat kernel, which is so named because in Euclidean signature it obeys the heat equation

$$\left(-\frac{\partial}{\partial s} + F\right) K(x,x'|s) = 0 \tag{2.48}$$

with initial condition

$$K(x,x'|s=0) = \delta(x,x'). \tag{2.49}$$

The heat equation is obtained from (2.43) by the replacement $s \to -is$, $t \to -it$. Similarly, the Euclidean Green function and effective action are related to the Euclidean heat kernel via

$$G(x,x') = \int_0^\infty ds\, K(x,x'|s), \tag{2.50}$$

$$W = -\frac{1}{2}\int_0^\infty \frac{ds}{s} \int dx\sqrt{g}\, K(x,x|s). \tag{2.51}$$

Note that for massive fields these integrals are well-defined in the $s \to \infty$ limit because the Euclidean heat kernel is exponentially damped for large s; no $i\epsilon$ prescription is needed.

Because of these properties, it is often more convenient to work in Euclidean space and continue the final results back to Lorentzian spacetime. For a static space, the inverse Wick rotation gives

$$\begin{array}{rcl} \text{Lorentzian} & & \text{Euclidean} \\ G_F & = & i\,G\,\big|_{t\to it}\,, \\ W & = & i\,W\,\big|_{t\to it}\,. \end{array} \tag{2.52}$$

Note that the Green function that results from the continuation of this G back to real time is the Feynman function G_F [49, 52].

One other useful characteristic of Euclidean spaces is that they are very convenient for considering quantum fields at nonzero temperatures [53]. In particular, one can show that expectation values for a quantum field at temperature $t = \beta^{-1}$ can be obtained by treating the field as propagating on a manifold which is periodic in imaginary time with period β [54, 55]. From this it follows that the Euclidean Green function and the Lorentzian Hadamard function are both periodic in Euclidean time (antiperiodic for fermions) [56, 57]. These properties will be useful when we consider quantum fields at finite temperatures.

One drawback of the Euclidean approach is that many spacetimes of interest are not sections of a complex manifold which also contains a unique Euclidean section. However, this formulation is applicable to the spacetimes we shall be most interested in: flat spacetime, Schwarzschild spacetime, and general static spacetimes. In future chapters our calculations of the dimensional-reduction anomaly will be done for Euclidean manifolds.

2.2 Dimensional Reduction

In this section we consider the dimensional reduction of a scalar field in a formal manner. We demonstrate how field objects such as the effective action and stress tensor may be related to the corresponding objects in a dimensionally reduced theory. In the following section we will indicate how this relationship breaks down under renormalization to give rise to the dimensional-reduction anomaly. Henceforth we work in Euclidean space unless stated otherwise.

2.2.1 Spacetime Metric and Notations

Consider a D-dimensional space with a line element of the form

$$ds^2 = g_{\mu\nu}(X^\tau)dX^\mu dX^\nu = h_{ab}(x^c)dx^a dx^b + \mathrm{e}^{-4\phi(x^c)/n}\Omega_{ij}(y^k)dy^i dy^j . \qquad (2.53)$$

Here $X^\alpha = (x^a, y^i)$, Ω_{ij} is the metric of an n-dimensional homogeneous space called the *internal space*, and h_{ab} is an arbitrary $(D-n)$-dimensional Euclidean metric. The function ϕ is known as the *dilaton*, and sets the scale of the internal space. The normalization of the dilaton field is a question of convenience; we set it by requiring that $\sqrt{g}$ for the metric (2.53) be proportional to $\mathrm{e}^{-2\phi}$ for any number of internal dimensions n. Well-known examples of metrics of the form (2.53) are those of spherical spacetimes, and metrics connected with a dimensional reduction in Kaluza-Klein theories.

At this point some conventions on notation are in order. We shall need to be able to distinguish quantities like Green functions defined in different dimensions. "Ordinary" letters such as G, W are used for the original D-dimensional theory, while calligraphic letters such as $\mathcal{G}$, $\mathcal{W}$ refer to dimensionally reduced quantities. All curvatures will be with respect to h unless explicitly labelled otherwise; for example, $R = R[h]$ and ${}^D R = R[g]$. As for differential operators, we shall understand $\Box$ and $()_{;\mu}$, ∇_μ to represent the d'Alembertian and covariant derivative using the metric g, while Δ_h, $()_{|a}$, ∇_a are the d'Alembertian and covariant derivative using the metric h. We also use the shorthand $(\nabla U)^2 \equiv h^{ab}U_{|a}U_{|b}$, $\nabla U \cdot \nabla S \equiv h^{ab}U_{|a}S_{|b}$. For the dilaton ϕ we shall understand ϕ_a, ϕ_{ab}, etc. to denote multiple $(D-n)$-dimensional covariant derivatives of ϕ calculated using the metric h. In the rare cases where we need to take derivatives in the internal homogeneous space, we use the notation Δ_Ω, $()_{:i}$. These conventions are repeated in Table 1, which is located at the beginning of the thesis for easy reference.

2.2.2 Quantum Field in D Dimensions

Let $\hat{\Phi}$ be a scalar field propagating in the spacetime (2.53) and described by the classical action

$$S = \frac{1}{2}\int dX\sqrt{g}\left[g^{\alpha\beta}\Phi_{;\alpha}\Phi_{;\beta} + (m^2+V)\,\Phi^2\right], \tag{2.54}$$

variation of which with respect to Φ yields the field equation

$$F\Phi \equiv \left(\Box - m^2 - V\right)\Phi = 0\,. \tag{2.55}$$

Note that we explicitly separate the mass term m^2 from the potential V. The latter may contain an interaction with the curvature, $\xi R[g]$, for a non-minimally coupled field, but is not fixed at the moment. We only assume that when calculated on the background (2.53), the potential V is independent of the y^i coordinates.

The stress tensor for this theory is obtained by varying the action (2.54) with respect to the metric $g_{\mu\nu}$ as in (2.4). Since we need to know the behaviour of the potential under variations of the metric to compute the stress tensor, we calculate $T_{\mu\nu}$ for the physically interesting case $V=\xi R[g]$. We find

$$T_{\mu\nu} \equiv \frac{2}{\sqrt{g}}\frac{\delta S}{\delta g^{\mu\nu}} \tag{2.56}$$

$$= (1-2\xi)\Phi_{;\mu}\Phi_{;\nu} - 2\xi\Phi_{;\mu\nu}\Phi + \xi R_{\mu\nu}[g]\Phi^2 + (2\xi-\frac{1}{2})g_{\mu\nu}\left[g^{\alpha\beta}\Phi_{;\alpha}\Phi_{;\beta} + (m^2+\xi R[g])\Phi^2\right], \tag{2.57}$$

where we have used the field equation (2.55) to remove $\Box\Phi$ terms. Note that this classical stress tensor is traceless for conformally invariant fields, i.e., for $m^2 = 0$, $\xi = \frac{1}{4}\frac{D-2}{D-1}$.

In the quantum theory, the field Φ is elevated to an operator. Quantities of physical interest, such as $\langle\hat{\Phi}^2\rangle$ and $\langle\hat{T}_{\mu\nu}\rangle$, can be written in terms of the Euclidean Green function,

$$G(X, X') = \langle\hat{\Phi}(X)\hat{\Phi}(X')\rangle\,, \tag{2.58}$$

which satisfies

$$F\,G(X, X') = -\delta(X, X') \equiv -\frac{\delta(X - X')}{\sqrt{g}}\,. \tag{2.59}$$

Clearly,

$$\langle\hat{\Phi}^2(X)\rangle = \lim_{X'\to X} G(X, X')\,. \tag{2.60}$$

Using (2.57), the expectation value of the quantum stress tensor can be written in terms of a differential operator acting on $G(X, X')$ as follows:

$$\langle\hat{T}_{\mu\nu}\rangle = \lim_{X'\to X} D_{\mu\nu}\, G(X, X')\,, \tag{2.61}$$

where for $V = \xi R[g]$

$$\begin{aligned} D_{\mu\nu} &= (\frac{1}{2} - \xi)\left(g_\mu^{\alpha'}\nabla_{\alpha'}\nabla_\nu + g_\nu^{\alpha'}\nabla_\mu\nabla_{\alpha'}\right) - \xi\left(\nabla_\mu\nabla_\nu + g_\mu^{\alpha'} g_\nu^{\beta'}\nabla_{\alpha'}\nabla_{\beta'}\right) \\ &\quad +(2\xi - \frac{1}{2})g_{\mu\nu}\left(g^{\alpha\beta'}\nabla_\alpha\nabla_{\beta'} + m^2 + \xi R[g]\right) + \xi R_{\mu\nu}[g]\,. \end{aligned} \tag{2.62}$$

Here $g_\mu^{\alpha'}$ is the bivector of parallel transport, and ∇_α ($\nabla_{\alpha'}$) is the covariant derivative at X (X') using the metric g. If the Green function has been 'renormalized' so that it and its first two derivatives are finite as $X' \to X$ then one may set $g_\mu^{\alpha'} \to g_\mu^\alpha$.

2.2.3 Mode Decompositions

Let us now consider what happens when we take advantage of the homogeneity of the internal space $\Omega_{ij}dy^i dy^j$. Using the line element (2.53), the operator $\Box$

becomes

$$\Box = \Delta_h - 2\,\nabla\phi\cdot\nabla + e^{4\phi/n}\Delta_\Omega\,, \tag{2.63}$$

where Δ_h, Δ_Ω are the d'Alembertians corresponding to the metrics h_{ab}, Ω_{ij} respectively, and ∇ is understood to denote the covariant derivative with respect to the metric h_{ab}.

Considerable simplification of the problem in spacetime (2.53) is connected with the fact that for a wide class of homogeneous metrics the eigenvalue problem

$$\Delta_\Omega Y(y) = -\lambda Y(y) \tag{2.64}$$

is well-studied [58]. We denote by $Y_{\lambda\varrho}$ the harmonics or eigenfunctions of (2.64), and use a collective index ϱ to distinguish between different solutions of (2.64) for the same λ. These modes may be chosen to obey standard orthonormality and completeness conditions,

$$\int dy\,\sqrt{\Omega}\,Y_{\lambda\varrho}(y)\overline{Y}_{\lambda'\varrho'}(y) = \delta_{\lambda\lambda'}\delta_{\varrho\varrho'}\,, \tag{2.65}$$

$$\sum_{\lambda\varrho} Y_{\lambda\varrho}(y)\overline{Y}_{\lambda\varrho}(y') = \delta(y,y') \equiv \frac{\delta^n(y-y')}{\sqrt{\Omega}}\,. \tag{2.66}$$

We write summation over indices assuming that the spectrum is discrete (or equivalently, that the internal space is compact). For a continuous spectrum one must replace summation by integration over the spectrum. In what follows we shall assume that this rule is automatically applied.

Owing to (2.65), (2.66), the field Φ can be decomposed in terms of the harmonics $Y_{\lambda\varrho}(y)$:

$$\Phi(X) = e^{\phi(x)}\sum_{\lambda,\varrho} c_{\lambda\varrho}\,\varphi_\lambda(x)\,Y_{\lambda\varrho}(y)\,, \tag{2.67}$$

where $c_{\lambda\varrho}$ are constants. Substituting this mode decomposition into the classical action (2.54) and using (2.64), (2.65) gives

$$S = \frac{1}{2}\int dx\sqrt{h}\int dy\sqrt{\Omega}\,e^{-2\phi}\left[e^{4\phi/n}\Omega^{ij}\Phi_{;i}\overline{\Phi}_{;j} + h^{ab}\Phi_{;a}\overline{\Phi}_{;b} + (m^2+V)\Phi\overline{\Phi}\right]$$

$$
\begin{aligned}
&= \sum_{\lambda,\varrho}\sum_{\lambda',\varrho'} c_{\lambda\varrho}c_{\lambda'\varrho'}\,\frac{1}{2}\int dx\sqrt{h}\int dy\sqrt{\Omega}\,\Big[\mathrm{e}^{4\phi/n}\varphi_\lambda\varphi_{\lambda'}\Omega^{ij}Y_{\lambda\varrho:i}\overline{Y}_{\lambda'\varrho':j} \\
&\qquad + h^{ab}(\phi_a\varphi_\lambda+\varphi_{\lambda|a})(\phi_b\varphi_{\lambda'}+\varphi_{\lambda'|b})Y_{\lambda\varrho}\overline{Y}_{\lambda'\varrho'} + (m^2+V)\varphi_\lambda\varphi_{\lambda'}Y_{\lambda\varrho}\overline{Y}_{\lambda'\varrho'}\Big] \\
&= \sum_{\lambda,\varrho}|c_{\lambda\varrho}|^2\,\frac{1}{2}\int dx\sqrt{h}\,\big[h^{ab}\varphi_{\lambda|a}\varphi_{\lambda|b} + \big(m^2+V+\phi^a\phi_a-\phi^a_a+\lambda\mathrm{e}^{4\phi/n}\big)\,\varphi_\lambda^2\big]\;.
\end{aligned}
\tag{2.68}
$$

We use complex notation in (2.68) in case the modes $Y_{\lambda\varrho}$ are complex.[4]

The classical action S for the D-dimensional theory can thus be written as the sum of $(D-n)$-dimensional actions $\mathcal{S}_\lambda$,

$$
S = \sum_{\lambda,\varrho}|c_{\lambda\varrho}|^2\,\mathcal{S}_\lambda\,, \tag{2.69}
$$

where

$$
\mathcal{S}_\lambda \;=\; \frac{1}{2}\int dx\sqrt{h}\,\big[h^{ab}\nabla_a\varphi_\lambda\nabla_b\varphi_\lambda + \big(m^2+V_\lambda\big)\,\varphi_\lambda^2\big]\;, \tag{2.70}
$$

and the 'induced potential' V_λ is

$$
V_\lambda \equiv V - \Delta_h\phi + (\nabla\phi)^2 + \lambda\mathrm{e}^{4\phi/n}\,. \tag{2.71}
$$

In other words, by expanding the field in modes we effectively reduce the original D-dimensional system to a similar system in $(D-n)$-dimensional space with an effective potential V_λ depending on the dilaton field ϕ.

An important feature of this dimensional reduction is that the $(D-n)$-dimensional action $\mathcal{S}_\lambda$ is independent of ϱ. As a result, for a given λ each ϱ makes an equal contribution to the classical action (2.68) up to the choice of the constants $c_{\lambda\varrho}$. In the quantum theory, the $c_{\lambda\varrho}$ will be replaced by creation and annihilation operators, and each ϱ contributes equally to the quantum action and other expectation values. The total action due to all modes ϱ for a given λ

[4] One can always choose real $Y_{\lambda\varrho}$. In practice, however, it is often more convenient to work with complex modes, such as the Fourier modes $\mathrm{e}^{\pm i\omega t}$ or the spherical harmonics $Y_{\ell m}$.

will then be simply the action for any one ϱ multiplied by the degeneracy of the eigenvalue λ.

The $(D-n)$-dimensional field $\varphi_\lambda(x)$ obeys the equation

$$\mathcal{F}_\lambda \varphi_\lambda \equiv \left(\Delta_h - m^2 - V_\lambda\right) \varphi_\lambda = 0\,, \tag{2.72}$$

which may be derived from the variation of the reduced action (2.70) or by direct substitution of the mode decomposition (2.67) into the D-dimensional field equation (2.55) and using (2.63). The special case $V = \xi R[g]$ is of particular interest. Substituting for $R[g]$ in terms of $R[h]$, $R[\Omega]$, and the dilaton (see, for example, Appendix B.1), we have

$$V_\lambda = \xi R[h] + (1 - 4\frac{n+1}{n}\xi)(\nabla\phi)^2 + (4\xi - 1)\Delta_h\phi + (\lambda + \xi R[\Omega])\mathrm{e}^{4\phi/n}\,. \tag{2.73}$$

Note that $R[\Omega]$ is a constant, since Ω_{ij} describes a homogeneous space.

The stress tensor $\mathcal{T}^{(\lambda)}_{ab}$ for our dimensionally reduced theory is obtained from the variation of (2.70) with respect to the $(D-n)$-dimensional metric h^{ab}. For the potential (2.73), we find

$$\begin{aligned}
\mathcal{T}^{(\lambda)}_{ab} &\equiv \frac{2}{\sqrt{h}}\frac{\delta S_\lambda}{\delta h^{ab}} &&(2.74)\\
&= (1-2\xi)\nabla_a\varphi_\lambda\nabla_b\varphi_\lambda - 2\xi\varphi_\lambda\nabla_a\nabla_b\varphi_\lambda + (2\xi - \tfrac{1}{2})h_{ab}(\nabla\varphi_\lambda)^2 + \xi R_{ab}[h]\varphi_\lambda^2 \\
&\quad + (2\xi - \tfrac{1}{2})h_{ab}\left[m^2 + \xi R[h] + (4\xi-1)\Delta_h\phi + (1 - 4\frac{n+1}{n}\xi)(\nabla\phi)^2\right. \\
&\quad \left. + \mathrm{e}^{4\phi/n}(\xi R[\Omega] + \lambda)\right]\varphi_\lambda^2 + (1 - 4\frac{n+1}{n}\xi)\phi_a\phi_b\varphi_\lambda^2 \\
&\quad + (1-4\xi)\varphi_\lambda\left[\phi_a\nabla_b + \phi_b\nabla_a - \frac{1}{2}h_{ab}\Delta_h\phi - h_{ab}\nabla\phi\cdot\nabla\right]\varphi_\lambda\,, &&(2.75)
\end{aligned}$$

where we use the field equation (2.72) to remove $\Delta_h\varphi_\lambda$ terms. It is interesting that this dimensionally reduced theory is only conformally invariant for $D - n = 2$ (i.e., when the reduction is to a two-dimensional theory), $\lambda = 0$, $m = 0$, and $\xi = 0$. In this case the classical stress tensor (2.75) will be traceless, while the trace of its quantum counterpart will be determined entirely by the conformal trace anomaly. This useful property, combined with the fact that any two-dimensional space is conformally related to flat space, is at the heart of most efforts involving dilaton gravity models (see Section 2.2.7).

Strictly speaking, the dimensionally reduced stress tensor does not have any components in the y^i sector. It is easily demonstrated, however, that one can write the D-dimensional $T_i{}^i$ in terms of the functional derivative of the action (2.54) with respect to the dilaton. Defining the effective pressure $P \equiv T_i{}^i/n$, one finds

$$P \equiv \frac{1}{n} T_i{}^i = \frac{1}{2\sqrt{g}} \frac{\delta S}{\delta \phi}. \tag{2.76}$$

For the dimensionally reduced theory we thus define analogously

$$\mathcal{P}^{(\lambda)} \equiv \frac{1}{2\sqrt{h}} \frac{\delta S_\lambda}{\delta \phi} \tag{2.77}$$

$$\begin{aligned} &= (2\xi - \frac{1}{2})(\nabla\varphi_\lambda)^2 + (2\xi - \frac{1}{2}) \Big[m^2 + \xi R[h] + (4\xi - 1)\Delta_h \phi \\ &\quad + (1 - 4\frac{n+1}{n}\xi)(\nabla\phi)^2 + \mathrm{e}^{4\phi/n}(\xi R[\Omega] + \lambda) \Big] \varphi_\lambda^2 \\ &\quad - (1 - 4\frac{n+1}{n}\xi)\, \varphi_\lambda \left[\nabla\phi \cdot \nabla + \frac{1}{2}\Delta_h \phi \right] \varphi_\lambda \\ &\quad + \frac{1}{n} \mathrm{e}^{4\phi/n}(\xi R[\Omega] + \lambda)\, \varphi_\lambda^2 . \end{aligned} \tag{2.78}$$

For the quantized theory, the Euclidean Green function is defined by

$$\mathcal{G}_\lambda(x, x') = \langle \hat{\varphi}_\lambda(x) \hat{\varphi}_\lambda(x') \rangle , \tag{2.79}$$

and satisfies

$$\mathcal{F}_\lambda\, \mathcal{G}_\lambda(x, x') = -\delta(x, x') \equiv -\frac{\delta(x - x')}{\sqrt{h}}. \tag{2.80}$$

In analogy to (2.60) we have

$$\langle \hat{\varphi}_\lambda^2(x) \rangle = \lim_{x' \to x} \mathcal{G}_\lambda(x, x') . \tag{2.81}$$

Replacing φ_λ by its operator equivalent and taking the expectation value of both sides of (2.75), one finds that the quantum stress tensor may be calculated using

$$\langle \hat{\mathcal{T}}_{ab}^{(\lambda)} \rangle = \lim_{x' \to x} \mathcal{D}_{ab}^{(\lambda)} \, \mathcal{G}_\lambda(x, x') , \tag{2.82}$$

where

$$\begin{aligned} \mathcal{D}_{ab}^{(\lambda)} \equiv & \; (\frac{1}{2} - \xi) \left(h_a^{c'} \nabla_{c'} \nabla_b + h_b^{c'} \nabla_a \nabla_{c'} \right) - \xi \left(\nabla_a \nabla_b + h_a^{c'} h_b^{d'} \nabla_{c'} \nabla_{d'} \right) \\ & + (2\xi - \frac{1}{2}) h_{ab} h^{cd'} \nabla_c \nabla_{d'} + \xi R_{ab}[h] + (2\xi - \frac{1}{2}) h_{ab} \Big[m^2 + \xi R[h] \\ & + 4\xi \Delta_h \phi + (1 - 4\frac{n+1}{n}\xi)(\nabla\phi)^2 + \mathrm{e}^{4\phi/n}(\xi R[\Omega] + \lambda) \Big] \\ & + (1 - 4\frac{n+1}{n}\xi)\phi_a \phi_b + (2\xi - \frac{1}{2}) h_{ab} \phi^c \left(\nabla_c + h_c^{d'} \nabla_{d'} \right) \\ & - (2\xi - \frac{1}{2}) \Big[\phi_a (\nabla_b + h_b^{d'} \nabla_{d'}) + \phi_b (\nabla_a + h_a^{d'} \nabla_{d'}) \Big] . \end{aligned} \tag{2.83}$$

Similarly, the effective pressure $\mathcal{P}^{(\lambda)}$ may be calculated using

$$\langle \hat{\mathcal{P}}^{(\lambda)} \rangle = \lim_{x' \to x} \mathcal{D}_{\mathcal{P}}^{(\lambda)} \, \mathcal{G}_\lambda(x, x') , \tag{2.84}$$

where

$$\begin{aligned} \mathcal{D}_{\mathcal{P}}^{(\lambda)} \equiv & \; (2\xi - \frac{1}{2}) h^{cd'} \nabla_c \nabla_{d'} - \frac{1}{2}(1 - 4\frac{n+1}{n}\xi) \left[\phi^c \left(\nabla_c + h_c^{d'} \nabla_{d'} \right) + \Delta_h \phi \right] \\ & + (2\xi - \frac{1}{2}) h_{ab} \Big[m^2 + \xi R[h] + (4\xi - 1)\Delta_h \phi + (1 - 4\frac{n+1}{n}\xi)(\nabla\phi)^2 \\ & + \mathrm{e}^{4\phi/n}(\xi R[\Omega] + \lambda) \Big] + \frac{1}{n} \mathrm{e}^{4\phi/n}(\xi R[\Omega] + \lambda) . \end{aligned} \tag{2.85}$$

2.2.4 Dimensional Reduction of $\langle \hat{\mathbf{\Phi}}^2 \rangle$ and $\langle \hat{\mathbf{T}}_{\mu\nu} \rangle$

Mode-decomposition relations such as (2.67) for the field and (2.69) for the action can be extended to expectation values in the quantized theories. For example, by considering the differential equations (2.59), (2.80) for the Green functions and using (2.63) and the properties (2.64) and (2.66) of the $Y_{\lambda\varrho}$, one can easily verify that G and $\mathcal{G}_\lambda$ are related by

$$G(X,X') = \sum_{\lambda,\varrho} e^{\phi(x)+\phi(x')} Y_{\lambda\varrho}(y)\overline{Y}_{\lambda\varrho}(y')\, \mathcal{G}_\lambda(x,x')\,. \tag{2.86}$$

Let us assume for convenience that the homogeneous space is compact (similar results will hold for the noncompact case). Then one can show that

$$\lim_{y'\to y} \sum_{\varrho} Y_{\lambda\varrho}(y)\overline{Y}_{\lambda\varrho}(y') = \frac{\mathcal{N}_\lambda}{V_\Omega}\,, \tag{2.87}$$

where V_Ω is the volume of the homogeneous space, and $\mathcal{N}_\lambda$ is the degeneracy[5] of the eigenvalue λ:

$$\mathcal{N}_\lambda = \sum_{\varrho}\,. \tag{2.88}$$

Taking the coincidence limit of both sides of (2.86) then yields

$$\langle\hat{\Phi}^2\rangle = \sum_{\lambda} \frac{\mathcal{N}_\lambda}{V_\Omega} e^{2\phi}\, \langle\hat{\varphi}_\lambda^2\rangle\,. \tag{2.89}$$

Similarly, inserting the decomposition (2.86) into the expression (2.61) for the D-dimensional stress tensor, applying the operator D_{ab}, and comparing to (2.82), (2.83), one finds

$$\begin{aligned}
\langle\hat{T}_{ab}\rangle &= \lim_{X'\to X} D_{ab} \sum_{\lambda,\varrho} e^{\phi(x)+\phi(x')} Y_{\lambda\varrho}(y)\overline{Y}_{\lambda\varrho}(y')\mathcal{G}_\lambda(x,x') \\
&= \lim_{x'\to x} \sum_{\lambda,\varrho} e^{\phi(x)+\phi(x')} Y_{\lambda\varrho}(y)\overline{Y}_{\lambda\varrho}(y)\, \mathcal{D}^{(\lambda)}_{ab}\mathcal{G}_\lambda(x,x') \\
&= \sum_{\lambda} \frac{\mathcal{N}_\lambda}{V_\Omega} e^{2\phi}\, \langle\hat{\mathcal{T}}^{(\lambda)}_{ab}\rangle\,.
\end{aligned} \tag{2.90}$$

This demonstrates formally that the x^a sector of the D-dimensional stress tensor can be obtained as the sum of the stress tensors for the dimensionally reduced

[5]For discrete spectra $\mathcal{N}_\lambda$ is the dimension of the space of eigenfunctions $Y_{\lambda\varrho}$ with fixed λ. It is finite [60].

theories. A similar procedure can be carried out for the components of the stress tensor in the y^i sector (the pressures), yielding an analogous result:

$$\langle \hat{P} \rangle = \sum_{\lambda} \frac{\mathcal{N}_\lambda}{V_\Omega} e^{2\phi} \langle \hat{\mathcal{P}}^{(\lambda)} \rangle . \tag{2.91}$$

Another elegant means to obtain these dimensional-reduction results is by employing the heat kernel. In Section 2.1 we were introduced to the heat kernel as a standard approach to renormalization in curved space. For a field satisfying the differential equation (2.55) in Euclidean space, the heat kernel $K(X, X'|s)$ is a solution of of the heat equation (2.48) with initial condition (2.49); from it the Euclidean Green function and effective action may be obtained using (2.50), (2.51). Analogous formulae also hold for the dimensionally reduced theory with operator $\mathcal{F}_\lambda$, heat kernel $\mathcal{K}_\lambda(x, x'|s)$, Green function $\mathcal{G}_\lambda(x, x')$, and effective action $\mathcal{W}_\lambda$.

Using the field equations (2.55), (2.72) and the properties (2.65), (2.66) of the $Y_{\lambda\varrho}$, one sees that the heat kernels $K(X, X'|s)$, $\mathcal{K}_\lambda(x, x'|s)$ obey a mode-decomposition relation exactly analogous to that for the Green functions, (2.86):

$$K(X, X'|s) = \sum_{\lambda,\varrho} e^{\phi(x)+\phi(x')} Y_{\lambda\varrho}(y) \overline{Y}_{\lambda\varrho}(y') \mathcal{K}_\lambda(x, x'|s) . \tag{2.92}$$

This relation provides an easy way to produce the mode-decomposition relations for the (unrenormalized) Green function and the effective action. For example, in the compact case, integrating both sides of (2.92) over s as in (2.42) yields the mode decomposition of the quantized effective action,

$$W = \sum_{\lambda} \mathcal{N}_\lambda \mathcal{W}_\lambda , \tag{2.93}$$

in analogy to the result (2.69) for the classical actions. The mode decompositions of the stress tensor and pressure can then be derived from (2.93) by functional differentiation.

For noncompact internal spaces, the decompositions (2.86) and (2.92) for the Green function and heat kernel still hold, but $V_\Omega \to \infty$ and (2.87) is no longer valid. We must therefore go back to (2.89)–(2.91) and replace $\mathcal{N}_\lambda/V_\Omega \to \infty$ by $\sum_\varrho Y_{\lambda\varrho}(y)\overline{Y}_{\lambda\varrho}(y)$, which is easily shown to be y-independent. We will see explicit examples of this case in Chapters 5 and 6.

2.2.5 Dimensional Reduction of the Conservation Equation

The D-dimensional action S is invariant under the variation

$$\begin{aligned} \delta g_{\mu\nu} &= \eta_{\mu;\nu} + \eta_{\nu;\mu}\,, \\ \delta\Phi &= \eta^\nu \Phi_\nu\,, \end{aligned} \tag{2.94}$$

where η is an arbitrary smooth vector field. This invariance implies that the D-dimensional stress tensor obeys the conservation equation

$$T_{\mu\nu}{}^{;\nu} = 0\,. \tag{2.95}$$

This invariance survives quantum corrections (see Section 2.3), and so applies to the quantum stress tensor as well:

$$\langle \hat{T}_{\mu\nu}\rangle^{;\nu} = 0\,. \tag{2.96}$$

Substituting the mode-decomposed form for $\langle \hat{T}_{\mu\nu}\rangle$ yields the corresponding (non)-conservation equation for the dimensionally reduced stress tensor,

$$\nabla^b \langle \hat{\mathcal{T}}^{(\lambda)}_{ab}\rangle = -2\phi_a \langle \hat{\mathcal{P}}^{(\lambda)}\rangle\,. \tag{2.97}$$

The same result could also be derived directly in $(D-n)$ dimensions by noting the invariance of $\mathcal{S}_\lambda$ under the transformation

$$\delta h_{ab} = \eta_{a|b} + \eta_{b|a}\,,$$

$$\begin{aligned} \delta\varphi_\lambda &= \eta^a \varphi_{\lambda|a}\,, \\ \delta\Phi &= \eta^a \Phi_a\,. \end{aligned} \tag{2.98}$$

This invariance also survives quantum corrections.

2.2.6 Induced Boundary Conditions

A final point regarding general dimensional reductions which will be very important for us later concerns global properties of the space and induced boundary conditions.

Let us assume that there exist points $x^a = x_0$ in the space (2.53) such that $e^{-\phi(x_0)} = 0$; i.e., at which the volume of the internal space vanishes. If the D-dimensional manifold is regular at these points, then for a regular quantum state the (renormalized) expectation values of the field in that state will also be finite there. Considering the mode decomposition formulae (2.67) and (2.86), it is clear that the dimensionally reduced field $\hat{\varphi}_\lambda$ and Green function $\mathcal{G}_\lambda$ must vanish at these points. We call this effect "induced boundary conditions". It will be very important in future chapters.

Note also that the induced potential (2.71) diverges at x_0 if $\lambda \neq 0$.

2.2.7 Dilaton-Gravity Models

Classical Background

One of the interesting examples of dimensional reduction appearing in the literature in recent years is the spherical reduction of Einsteinian gravity in $D \geq 4$ dimensions to two dimensions [20]. As an example, consider the total (matter plus gravitational) action for a minimally coupled massless scalar field in a four-dimensional spacetime with zero cosmological constant:

$$S_{\text{grav}} + S = -\frac{1}{2}\int d^4X\sqrt{|g|}\left[\frac{1}{8\pi}R[g] + (\nabla\Phi)^2\right]. \tag{2.99}$$

Under the assumption that both the spacetime and field configuration are spherically symmetric (i.e., keeping only the $\ell = 0$ mode of the field in an expansion in spherical harmonics), this action reduces to

$$\begin{aligned} S_{\mathrm{grav}} + S &= -\frac{1}{2}\int d^2x\sqrt{|h|}\left[\frac{1}{2}\mathrm{e}^{-2\phi}\left(R[h] + 4\Delta_h\phi - 6(\nabla\phi)^2\right) + \frac{1}{2}R[\Omega]\right. \\ &\qquad \left. + (\nabla\varphi_{\ell=0})^2 + \left((\nabla\phi)^2 - \Delta_h\phi\right)\varphi_{\ell=0}^2\right], \end{aligned} \tag{2.100}$$

where we write

$$\Phi(x) = \frac{\mathrm{e}^{\phi(x)}\varphi_{\ell=0}(x)}{\sqrt{4\pi}}, \tag{2.101}$$

and $R[\Omega] = 2$ is the curvature of the unit two-sphere.

The first line of (2.100) describes two-dimensional gravity interacting with a dilaton background field ϕ. This theory can be generalized from Einsteinian gravity in $D \geq 4$ dimensions to include models inspired by string theory [21] and other sources [22–28]; the resulting class of systems is known as *dilaton gravity*. All classical solutions for dilaton gravity containing one event horizon and one singularity in the absence of matter are known [22, 25, 26], and their global properties and other characteristics have been studied [29–31]. The second line of (2.100) contains the classical action for a massless, minimally coupled two-dimensional scalar field propagating in the dilaton-dependent potential $(\nabla\phi)^2 - \Delta_h\phi$. This corresponds to the s mode (the $\ell = 0$ harmonic) of the four-dimensional scalar field Φ. The quantization of this sector of the theory has been the subject of much interest in recent years, and we will discuss it momentarily.

Quantum Fields: Vacuum Polarization and Particle Creation

The interest in dilaton-gravity models stems from the fact that semiclassical [32–39] and quantum-gravitational [61–63] calculations seem to be much easier to perform in two dimensions than in higher-dimensional spacetimes. For example, the Riemann tensor in two dimensions has only one independent component, and two-dimensional spaces are conformally flat. As well, for conformally invariant fields in two dimensions the quantum stress tensor is largely fixed by knowledge of its trace. These properties make it possible to perform some vacuum polarization and other calculations for arbitrary metrics, making the study of the back-reaction of quantized matter on the gravitational background possible [64]. This leads to the hope of being able to use dilaton-gravity models to obtain information on the quantum creation and evaporation of genuine four-dimensional Schwarzschild black holes, including insight into the information-loss puzzle and black-hole thermodynamics [65].

If dilaton-gravity models are to shed light on the nature of black-hole evaporation and quantum gravity, it is clear that effects which are already known must be reproduced by them. The most interesting of these is Hawking radiation [66–68]. The first attempt to calculate the Hawking radiation in the $\ell = 0$ mode for a four-dimensional black hole using a two-dimensional model was made by Mukhanov, Wipf, and Zelnikov [32]. They used the conformal anomaly in two dimensions to calculate the effective action for a massless scalar field coupled minimally to gravity but nonminimally to the dilaton background. This technique leaves undetermined a conformally invariant term. The authors developed a perturbative technique to compute this conformally invariant part but were unable to calculate it completely. Various attempts since then by a number of other authors [33–39] have led to different results for the Hawking radiation and stress tensor for dilaton-gravity, and disagreement on the correct approach to computing them. (Some authors [36] have stated that the Hawking radiation should be derived by integrating the (non)-conservation equation (2.97) for the two-dimensional stress tensor, while others [37] maintain that the effective action formalism is sufficient.) In addition to questions concerning their asymptotic behaviour, comparison of the stress tensors obtained from dilaton gravity to the four-dimensional stress tensor for a massless scalar field near the event horizon also shows qualitatively different behaviour. These problems call into question the usefulness of dilaton-gravity models for understanding the physics of four-dimensional black holes, and at present there does not appear to be a consensus on the issue of the quantum effective action for dilaton gravity.

Part of this thesis (Chapter 4) is devoted to an examination of this issue, and the connection between spherically reduced quantum fields and physics in four dimensions.

2.3 The Dimensional-Reduction Anomaly

In the preceding section we saw that in a space (2.53), the main objects which characterize a free quantum field, such as the Green function, the square of the field operator, and the stress tensor, can all be written as formal sums of the corresponding objects in dimensionally reduced theories. These arguments have been applied to the field theory at the classical or bare level, and have commonly been assumed to hold for the renormalized theories as well. A central point of this thesis is to demonstrate that relationships like (2.89)–(2.91) in fact break down under renormalization.

The need for renormalization is apparent when one considers, for example, the expressions (2.60) and (2.81) for the expectation value of the square of the field operator. The Green function is divergent in the coincidence limit. As a result, our mode-decomposition expressions like (2.89) are purely formal; both sides contain divergences which must be removed to yield physically meaningful results. This raises the question of whether the renormalization in D dimensions is equivalent to renormalization in $(D-n)$ dimensions. Specifically, is the sum over λ of the divergent terms in $(D-n)$ dimensions equal to the divergent terms in D dimensions?

As discussed in Section 2.1, the divergences in the effective action W in D-dimensional spacetime can be removed by subtracting the first N_D terms of the Schwinger-DeWitt expansion of the heat kernel K for the operator F, where N_D is given by (2.46). By contrast, the dimensionally reduced action $\mathcal{W}_\lambda$ is renormalized by subtracting the first N_{D-n} terms of the Schwinger-DeWitt expansion of the heat kernel $\mathcal{K}_\lambda$ for the operator $\mathcal{F}_\lambda$. In subsequent chapters we will calculate the "divergent" parts of these heat kernels for various spaces and demonstrate explicitly that they do not satisfy a mode-decomposition relation like (2.92). As a result, the renormalization procedure destroys the formal representation (2.93) for the effective actions, so that after renormalization one gets

$$W_{\text{ren}} = \sum_\lambda \mathcal{N}_\lambda \left[\mathcal{W}_{\lambda|\text{ren}} + \Delta\mathcal{W}_\lambda\right] . \tag{2.102}$$

We call the additional contribution $\Delta\mathcal{W}_\lambda$ the *dimensional-reduction anomaly.* Similar anomalies occur in the other expectation values of physical interest:

$$G_{\text{ren}}(X,X') = \sum_{\lambda,\varrho} \mathrm{e}^{\phi(x)+\phi(x')} Y_{\lambda\varrho}(y)\overline{Y}_{\lambda\varrho}(y') \left[\mathcal{G}_{\lambda|\text{ren}}(x,x') + \Delta\mathcal{G}_\lambda(x,x')\right] , \tag{2.103}$$

$$\langle\hat{\Phi}^2\rangle_{\text{ren}} = \sum_\lambda \frac{\mathcal{N}_\lambda}{V_\Omega}\mathrm{e}^{2\phi}\left[\langle\hat{\varphi}_\lambda^2\rangle_{\text{ren}} + \Delta\langle\hat{\varphi}_\lambda^2\rangle\right] , \tag{2.104}$$

$$\langle\hat{T}_{ab}\rangle_{\text{ren}} = \sum_\lambda \frac{\mathcal{N}_\lambda}{V_\Omega}\mathrm{e}^{2\phi}\left[\langle\hat{\mathcal{T}}_{ab}^{(\lambda)}\rangle_{\text{ren}} + \Delta\langle\hat{\mathcal{T}}_{ab}^{(\lambda)}\rangle\right] , \tag{2.105}$$

$$\langle\hat{P}\rangle_{\text{ren}} = \sum_\lambda \frac{\mathcal{N}_\lambda}{V_\Omega}\mathrm{e}^{2\phi}\left[\langle\hat{\mathcal{P}}^{(\lambda)}\rangle_{\text{ren}} + \Delta\langle\hat{\mathcal{P}}^{(\lambda)}\rangle\right] . \tag{2.106}$$

In each case the anomaly is found as the difference between the renormalization terms for the $(D-n)$-dimensional theory and the mode-decomposed renormalization terms from D dimensions. For example, using the orthonormality condition (2.65) for the $Y_{\lambda\varrho}$, one can show that for compact internal spaces the anomaly in the Green function is

$$\Delta\mathcal{G}_\lambda(x,x') = \mathcal{G}_{\lambda|\text{div}}(x,x') - \mathrm{e}^{-(\phi(x)+\phi(x'))}\int dy\sqrt{\Omega}\int dy'\sqrt{\Omega'}\,\overline{Y}_{\lambda\varrho}(y)\,Y_{\lambda\varrho}(y')\,G_{\text{div}}(X,X') , \tag{2.107}$$

where the subscript 'div' denotes the divergent part obtained from the Schwinger-DeWitt expansion. In subsequent chapters it will be demonstrated that this anomaly is generally nonvanishing.

One might observe that there exists a relationship between the dimensional-reduction anomaly and the so-called *multiplicative anomaly* [69–71]. Formally, one can write

$$F = \prod_{\lambda,\varrho} \mathcal{F}_\lambda , \tag{2.108}$$

$$-\frac{1}{2}\log\det F = -\frac{1}{2}\sum_\lambda \mathcal{N}_\lambda \log\det \mathcal{F}_\lambda . \tag{2.109}$$

The latter relation is nothing but (2.93) for the quantum action. The violation of the formal relation (2.109) for products of operators after renormalization is known as the multiplicative anomaly.

In the following chapters we discuss special examples of the dimensional-reduction anomaly. For simplicity, we restrict ourselves to the physically interesting case where the number of spacetime dimensions D is 4, and the number of dimensions of the "internal" homogeneous space is 1 or 2. Unless specified otherwise, we work with manifolds of Euclidean signature. When we do consider Lorentzian spacetimes, our metric signature is always $(-+++)$. We use the conventions of Misner, Thorne, and Wheeler [72] for the definition of the curvature.

Chapter 3

Spherical Decompositions in Flat Space

We begin our study of the dimensional-reduction anomaly with an examination of one of the simplest and most familiar examples of dimensional reduction: the decomposition of a scalar field in flat space into spherical harmonics. This surprisingly nontrivial example illustrates most aspects of the dimensional-reduction anomaly, and allows their direct physical interpretation. At the same time, it is simple enough for exact results to be obtained, and will serve as a check on later calculations for the anomaly in curved spherically symmetric spaces.

We will be particularly interested in the role of the s mode (the $\ell = 0$ spherical harmonic) for quantum fields at nonzero temperatures. Accordingly, we begin in Section 3.2 with a direct calculation of the s-mode contribution to $\langle\hat{\Phi}^2\rangle$ and $\langle\hat{T}_{\mu\nu}\rangle$ at finite temperature in four dimensions. We then consider in Section 3.3 how to solve the same problem via dimensional reduction to a two-dimensional theory. We will demonstrate that renormalization of the dimensionally reduced theory fails to yield the correct expectation values, and explain why. We will then show how the dimensional-reduction anomaly corrects the predictions of the dimensionally reduced theory to reproduce the expected results for four dimensions.

3.1 Spherical Dimensional Reductions

In section 2.2 we examined in a formal manner the dimensional reduction of a scalar field without specifying the nature of the mode decomposition. As preparation for Sections 3.2 and 3.3 and for Chapter 4, we rewrite the more important formulae from Section 2.2 for the special case of spherical decompositions in a four-dimensional space.

Choosing standard[1] angular coordinates (θ, η), the line element for a four-dimensional spherically symmetric space may be written as

$$ds^2 = g_{\mu\nu}(X^\tau)dX^\mu dX^\nu = h_{ab}(x^c)dx^a dx^b + \rho^2 e^{-2\phi(x^c)}\left(d\theta^2 + \sin^2\theta\, d\eta^2\right) . \quad (3.1)$$

Here ρ is a constant with the dimensions of length. The radius of a two-sphere of fixed x^a is given by $r = \rho\, e^{-\phi(x^a)}$. In the general case h_{ab} is an arbitrary two-dimensional metric; in this chapter we consider only flat two-dimensional space.

Consider a massive scalar field propagating on the space (3.1) and obeying the field equation (2.55), where the potential V is also spherically symmetric. A natural procedure is to decompose the field into spherical harmonics $Y_{\ell m}(\theta, \eta)$,

$$\Phi(X) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} c_{\ell m}\, \varphi_\ell(x^a) \frac{Y_{\ell m}(\theta, \eta)}{r} , \quad (3.2)$$

where the $c_{\ell m}$ are constants. Using the orthogonality and completeness of the $Y_{\ell m}(\theta, \eta)$ on the two-sphere, one easily verifies that the dimensional-reduction

[1] We denote the azimuthal coordinate by η rather than ϕ to avoid confusion with the dilaton.

formulae from the previous chapter can be carried over directly for this decomposition using $D = 4$, $n = 2$, $R[\Omega] = 2$, and making the substitutions

$$Y_{\lambda\varrho}(y) \to Y_{\ell m}(\theta, \eta)\,, \tag{3.3}$$

$$\lambda \to \ell(\ell+1)\,, \tag{3.4}$$

$$\mathcal{N}_\lambda \to 2\ell+1\,, \tag{3.5}$$

$$V_\Omega \to 4\pi\,, \tag{3.6}$$

$$\sum_\lambda \to \sum_{\ell=0}^{\infty}\,. \tag{3.7}$$

In addition,

$$\sum_{m=-\ell}^{\ell} Y_{\ell m}(\theta,\eta)\overline{Y}_{\ell m}(\theta',\eta') = \frac{2\ell+1}{4\pi} P_\ell(\cos\lambda)\,. \tag{3.8}$$

Here P_ℓ is a Legendre polynomial and we now use λ to represent the angular separation of the points X, X', given implicitly by

$$\cos\lambda = \cos\theta\,\cos\theta' + \sin\theta\,\sin\theta'\,\cos(\eta-\eta')\,. \tag{3.9}$$

Following the analysis of Section 2.2, we see that φ_ℓ behaves as a field propagating in the two-dimensional space with metric h_{ab}. It satisfies the field equation (2.72) with induced potential

$$V_\ell = V + \frac{\ell(\ell+1)}{r^2} - \Delta\phi + (\nabla\phi)^2\,, \tag{3.10}$$

and obeys Dirichlet boundary conditions wherever $r = 0$ (provided the manifold is regular there). The corresponding two-dimensional Green functions $\mathcal{G}_\ell$ are related to the four-dimensional G via

$$G(X,X') = \sum_{\ell=0}^{\infty} \frac{(2\ell+1)}{4\pi r r'} P_\ell(\cos\lambda)\mathcal{G}_\ell(x,x')\,, \tag{3.11}$$

while the renormalized quantities (2.102)–(2.106) take the following explicit form:

$$G_{\text{ren}}(X,X') = \sum_{\ell=0}^{\infty} \frac{(2\ell+1)}{4\pi r r'} P_\ell(\cos\lambda)\left[\mathcal{G}_{\ell|\text{ren}}(x,x') + \Delta\mathcal{G}_\ell(x,x')\right]\,, \tag{3.12}$$

$$\langle\hat{\Phi}^2\rangle_{\text{ren}} = \sum_{\ell=0}^{\infty} \frac{(2\ell+1)}{4\pi r^2} \left[\langle\hat{\varphi}_\ell^2\rangle_{\text{ren}} + \Delta\langle\hat{\varphi}_\ell^2\rangle\right] , \tag{3.13}$$

$$W_{\text{ren}} = \sum_{\ell=0}^{\infty} (2\ell+1) \left[\mathcal{W}_{\ell|\text{ren}} + \Delta\mathcal{W}_\ell\right] , \tag{3.14}$$

$$\langle\hat{T}_{\mu\nu}\rangle_{\text{ren}} = \sum_{\ell=0}^{\infty} \frac{(2\ell+1)}{4\pi r^2} \left[\langle\hat{\mathcal{T}}_{\mu\nu}^{(\ell)}\rangle_{\text{ren}} + \Delta\langle\hat{\mathcal{T}}_{\mu\nu}^{(\ell)}\rangle\right] . \tag{3.15}$$

$$\langle\hat{P}\rangle_{\text{ren}} = \sum_{\ell=0}^{\infty} \frac{(2\ell+1)}{4\pi r^2} \left[\langle\hat{\mathcal{P}}^{(\ell)}\rangle_{\text{ren}} + \Delta\langle\hat{\mathcal{P}}^{(\ell)}\rangle\right] . \tag{3.16}$$

In each case the anomaly is the difference between the divergent subtraction terms for the dimensionally reduced theory and the mode-decomposed subtraction terms from four dimensions. For example, in analogy to (2.107) we find

$$\Delta\mathcal{G}_\ell(x,x') = \mathcal{G}_{\ell|\text{div}}(x,x') - 2\pi r r' \int_{-1}^{1} d(\cos\lambda) P_\ell(\cos\lambda) G_{\text{div}}(X,X') . \tag{3.17}$$

The remainder of this chapter is devoted to explicit calculations of these anomalies and elucidation of their importance for dimensional reductions in flat space. General spherically symmetric geometries are considered in Chapter 4.

3.2 s-mode in Flat Space at Finite Temperature

To better understand the role played by individual angular modes in spherical decompositions, we calculate the contribution made by the s modes to the square of the field operator and the stress tensor of a scalar field in flat four-dimensional spacetime at finite temperature. These follow the calculations of Balbinot, Fabbri, and Frolov for $\langle\hat{T}_{\mu\nu}\rangle$ in the massless case [73]. We use the metric signature $(-+++)$.

3.2.1 Formalism

In Section 2.2 we saw explicitly how the expectation values of the square of the field operator and the stress tensor for a scalar field in Euclidean space can be obtained from the Euclidean Green function. Formulae (2.60)–(2.62) hold in Lorentzian geometries as well with the replacement $G \to \frac{1}{2}G^{(1)}$, where $G^{(1)}$ is the Hadamard function for the quantum state of interest. Thus,

$$\langle\hat{\Phi}^2\rangle = \lim_{X'\to X} \frac{1}{2} G^{(1)}(X,X'), \tag{3.18}$$

$$\begin{aligned}\langle\hat{T}_{\mu\nu}\rangle &= \lim_{X'\to X} \frac{1}{2}(\frac{1}{2}-\xi)\left(G^{(1)}_{\mu'\nu}+G^{(1)}_{\mu\nu'}\right) - \frac{1}{2}\xi\left(G^{(1)}_{\mu\nu}+G^{(1)}_{\mu'\nu'}\right) + \frac{1}{2}\xi R_{\mu\nu}[g]G^{(1)} \\ &\quad + (\xi-\frac{1}{4})g_{\mu\nu}\left[g^{\rho\sigma'}G^{(1)}_{\rho\sigma'} + (m^2+\xi R[g])G^{(1)}\right], \end{aligned}\tag{3.19}$$

where $G^{(1)}_{\mu\nu'} \equiv g^{\rho'}_{\nu}\nabla'_{\rho}\nabla_{\mu}G^{(1)}(X,X')$, etc. Let us decompose the field operator $\hat{\Phi}$ in terms of a complete orthonormal set of modes $\{\Phi_{k\ell m}, \overline{\Phi}_{k\ell m}\}$,

$$\hat{\Phi}(X) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\int dk\left[\hat{a}_{k\ell m}\Phi_{k\ell m}(X) + \hat{a}^{\dagger}_{k\ell m}\overline{\Phi}_{k\ell m}(X)\right], \tag{3.20}$$

where

$$\Phi_{k\ell m} = \varphi_{k\ell}(t,r)\,\frac{Y_{\ell m}(\theta,\eta)}{r}. \tag{3.21}$$

Under this spherical decomposition the four-dimensional field equation

$$\left(\Box - m^2\right)\Phi_{k\ell m} = 0 \tag{3.22}$$

reduces to

$$\left(-\partial_t^2 + \partial_r^2 - m^2 - \frac{\ell(\ell+1)}{r^2}\right)\varphi_{k\ell} = 0. \tag{3.23}$$

In particular, for the $\ell = 0$ mode the φ_{k0} are just plane waves. However, to account for the fact that $r \in [0,\infty)$, we must impose Dirichlet boundary conditions on the $\varphi_{k\ell}$ at $r = 0$. The correctly normalized[2] s modes are then

$$\Phi_{k00} = \frac{-i}{2\pi r\sqrt{\omega_k}} \mathrm{e}^{-i\omega_k t} \sin(kr) , \tag{3.24}$$

where $k \in (0, \infty)$, and $\omega_k \equiv \sqrt{k^2 + m^2}$ is the energy eigenvalue of the mode $\Phi_{k\ell m}$.

It can be shown that choosing the time dependence of the modes such that $\Phi_{k\ell m} \propto \mathrm{e}^{-i\omega_k t}$ guarantees that the vacuum state associated with these modes will be the ordinary Minkowski (zero-temperature) vacuum. The zero-temperature Hadamard function is then given by (2.27):

$$G^{(1)}_{\beta\to\infty}(X, X') = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \int_0^\infty dk \left(\Phi_{k\ell m} \overline{\Phi}'_{k\ell m} + \overline{\Phi}_{k\ell m} \Phi'_{k\ell m} \right) , \tag{3.25}$$

with $\Phi' \equiv \Phi(X')$. Meanwhile, the Hadamard function at temperature $T = \beta^{-1}$ can be shown to be [44]

$$G^{(1)}_{\beta}(X, X') = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \int_0^\infty dk \left(\Phi_{k\ell m} \overline{\Phi}'_{k\ell m} + \overline{\Phi}_{k\ell m} \Phi'_{k\ell m} \right) \frac{1 + \mathrm{e}^{-\beta\omega_k}}{1 - \mathrm{e}^{-\beta\omega_k}} . \tag{3.26}$$

In flat space with vanishing potential, a free quantum field is renormalized by subtracting from the bare quantity the corresponding result for the Minkowski vacuum state. Taking the difference of (3.26) and (3.25), one finds

$$\langle \hat{\Phi}^2 \rangle^\beta = \sum_{\ell=0}^{\infty} (2\ell + 1)\, f^{(\ell)} , \qquad \langle \hat{T}_{\mu\nu} \rangle^\beta = \sum_{\ell=0}^{\infty} (2\ell + 1)\, t^{(\ell)}_{\mu\nu} , \tag{3.27}$$

where the contributions to $\langle \hat{\Phi}^2 \rangle^\beta$ and $\langle \hat{T}_{\mu\nu} \rangle^\beta$ due to modes of fixed ℓ with $m = 0$ are

$$f^{(\ell)} = \int_0^\infty dk \, \frac{2}{\mathrm{e}^{\beta\omega_k} - 1} \, \Phi_{k\ell 0} \, \overline{\Phi}_{k\ell 0} , \tag{3.28}$$

[2]The normalized s-modes are those satisfying

$$(\Phi_{k_1 00}, \Phi_{k_2 00}) \equiv -i4\pi \int_0^\infty dr\, r^2 (\Phi_{k_1 00} \partial_t \overline{\Phi}_{k_2 00} - \partial_t \Phi_{k_1 00} \overline{\Phi}_{k_2 00})_{t=t'} = \delta(k_1 - k_2) .$$

$$\begin{aligned} t^{(\ell)}_{\mu\nu} &= \int_0^\infty dk \, \frac{2}{e^{\beta\omega_k}-1} \Big[(\frac{1}{2}-\xi) \left(\partial_\mu \Phi_{k\ell 0} \partial_\nu \overline{\Phi}_{k\ell 0} + \partial_\nu \Phi_{k\ell 0} \partial_\mu \overline{\Phi}_{k\ell 0}\right) \\ &\quad + (2\xi - \frac{1}{2}) g_{\mu\nu} \left(g^{\rho\sigma} \partial_\rho \Phi_{k\ell 0} \partial_\sigma \overline{\Phi}_{k\ell 0} + m^2 \Phi_{k\ell 0} \overline{\Phi}_{k\ell 0}\right) \\ &\quad - \xi \left(\Phi_{k\ell 0} (\nabla_\mu \nabla_\nu \overline{\Phi}_{k\ell 0}) + (\nabla_\mu \nabla_\nu \Phi_{k\ell 0}) \overline{\Phi}_{k\ell 0}\right) \Big] . \end{aligned} \tag{3.29}$$

Since each value of the azimuthal number m for a given ℓ gives the same contribution to $\langle\hat{\Phi}^2\rangle$ and $\langle\hat{T}_{\mu\nu}\rangle$, (3.28) and (3.29) are written for the $m = 0$ mode alone, and the degeneracy $\mathcal{N}_\ell = 2\ell + 1$ is accounted for in (3.27).

Since in the following we deal solely with the s mode, we drop the superscript (ℓ) and understand f, $t_{\mu\nu}$ as referring to the $\ell = 0$ mode only.

3.2.2 s-mode Contribution to $\langle\hat{\Phi}^2\rangle^\beta$

Computing (3.28) for the $\ell = 0$ modes (3.24) we find

$$f = \frac{2}{(2\pi r)^2} \int_0^\infty dk \, \frac{1}{e^{\beta\sqrt{k^2+m^2}}-1} \, \frac{\sin^2(kr)}{\sqrt{k^2+m^2}} . \tag{3.30}$$

This can be expressed in terms of the known integral

$$\int_0^\infty dk \, \frac{e^{-\beta\sqrt{k^2+m^2}}}{\sqrt{k^2+m^2}} \cos(2kr) = K_0(m\sqrt{\beta^2+4r^2}) \tag{3.31}$$

by rewriting the β-dependent factor in (3.30) as a geometric series in $e^{-\beta\omega_k}$. The result is

$$f = \frac{1}{(2\pi r)^2} \sum_{n=1}^\infty \left[K_0(nm\beta) - K_0(m\sqrt{(n\beta)^2+4r^2}) \right] . \tag{3.32}$$

Massless limit

Taking the limit $m \to 0$ in (3.32) reduces the K_0 Bessel function to a logarithm:

$$\begin{aligned} f_{m=0} &= \frac{1}{(2\pi r)^2} \frac{1}{2} \sum_{n=1}^\infty \ln\left[1 + \left(\frac{2r}{n\beta}\right)^2\right] \\ &= \frac{1}{(2\pi r)^2} \frac{1}{2} \ln \frac{\sinh(2\pi r/\beta)}{(2\pi r/\beta)} . \end{aligned} \tag{3.33}$$

The same result could also be obtained by taking the $m \to 0$ limit in (3.30) before evaluating the integral. As expected, $f_{m=0}$ is finite at $r = 0$,

$$f_{m=0} \stackrel{r=0}{=} \frac{1}{12\beta^2}, \tag{3.34}$$

while for large r it decreases as

$$f_{m=0} \stackrel{r\to\infty}{=} \frac{1}{4\pi\beta r}. \tag{3.35}$$

Low-temperature limit of massive theory

For $m\beta \gg 1$, the $n > 1$ terms in (3.32) are exponentially damped with respect to the $n = 1$ term, so we may neglect them to obtain

$$f_{m\beta\gg1} = \frac{1}{(2\pi r)^2}\left[K_0(m\beta) - K_0(m\sqrt{\beta^2+4r^2})\right]. \tag{3.36}$$

Here the limiting values for large and small r are

$$f_{m\beta\gg1} \stackrel{r=0}{=} \frac{1}{2\pi^2}\frac{m}{\beta}K_1(m\beta), \tag{3.37}$$

$$f_{m\beta\gg1} \stackrel{r\to\infty}{=} \frac{K_0(m\beta)}{(2\pi r)^2}. \tag{3.38}$$

High-temperature limit of massive theory

The evaluation of the sum (3.32) in the previous cases has relied upon the asymptotic behaviour of the Bessel function K_0 for large and small values of its argument. However, in the high temperature limit, $m\beta \ll 1$, these arguments are neither always large nor always small, making the sums difficult to evaluate for general r. In spite of this, for very large and small values of mr we can still obtain f by making use of the sums

$$\sum_{n=1}^{\infty}\frac{K_1(nb)}{nb} = \frac{\pi^2}{6b^2} - \frac{\pi}{2b} - \frac{1}{4}\left(\gamma + \ln\frac{b}{4\pi}\right) + \frac{1}{8} + O(b^2), \tag{3.39}$$

$$\sum_{n=1}^{\infty} K_0(nb) = \frac{\pi}{2b} + \frac{1}{2}\left(\gamma + \ln\frac{b}{4\pi}\right) + O(b^2), \tag{3.40}$$

$$\sum_{n=1}^{\infty}\left[\frac{K_0(nb)}{(nb)^2} + 2\frac{K_1(nb)}{(nb)^3}\right] = \frac{\pi^4}{45b^4} + \frac{\pi^2}{12b^2} - \frac{\pi}{6b} + \frac{1}{16}\left(\gamma + \ln\frac{b}{4\pi}\right) - \frac{3}{64} + O(b^2) \tag{3.41}$$

(see, e.g., 5.9.1.4 of [74]).

For $r \to 0$ a Taylor series expansion for $K_0(m\sqrt{(n\beta)^2 + 4r^2})$ yields

$$\begin{aligned} f_{m\beta \ll 1} &\overset{r=0}{=} \frac{m^2}{2\pi^2}\sum_{n=1}^{\infty}\frac{K_1(nm\beta)}{nm\beta} \\ &= \frac{1}{12\beta^2} - \frac{m}{4\pi\beta} - \frac{m^2}{8\pi^2}\left(\gamma + \ln\frac{m\beta}{4\pi}\right) + \frac{m^2}{16\pi^2} + O(\beta^2). \end{aligned} \tag{3.42}$$

Note that taking the $m \to 0$ limit returns the expected result (3.34) for a massless field.

For $mr \gg 1$, we can ignore the $K_0(m\sqrt{(n\beta)^2 + 4r^2})$ terms as exponentially damped with respect to the $K_0(nm\beta)$. We then have

$$\begin{aligned} f_{m\beta \ll 1} &\overset{r\to\infty}{=} \frac{1}{(2\pi r)^2}\sum_{n=1}^{\infty} K_0(nm\beta) \\ &= \frac{1}{4\pi r^2}\left[\frac{1}{2m\beta} + \frac{1}{2\pi}\left(\gamma + \ln\frac{m\beta}{4\pi}\right) + O(\beta^2)\right]. \end{aligned} \tag{3.43}$$

Note that the $m \to 0$ limit is ill-defined. This is not surprising, since the assumption $mr \gg 1$ used to derive (3.43) excludes $m = 0$.

It should be noted that our results (3.34), (3.37), and (3.42) for the s-mode contribution to $\langle\hat{\Phi}^2\rangle$ at $r = 0$ exactly reproduce the value obtained by summing over all modes [see (A.5), (A.7), (A.8) in Appendix A]. This occurs because as $r \to 0$, $\Phi_{klm} \propto r^{\ell}$, so only the s-mode makes a nonvanishing contribution to $\langle\hat{\Phi}^2\rangle$ at the origin. Meanwhile, the $r \to \infty$ limit taken in (3.38), (3.43) equals $(4\pi r^2)^{-1}$ times the corresponding results for $\langle\hat{\Phi}^2\rangle^{\beta}$ in two dimensions [see (A.17), (A.18)]. Similar phenomena will be seen with the stress tensor.

3.2.3 s-mode Contribution to $\langle \hat{\mathbf{T}}_{\mu\nu} \rangle^\beta$

Using (3.29), one finds that the stress tensor due to the s-modes has only diagonal components,

$$\begin{aligned} t_\mu{}^\nu &= \frac{1}{(2\pi r)^2}\int_0^\infty dk\, \frac{2}{e^{\beta\omega_k}-1}\frac{1}{\omega_k}\left[\frac{1}{2}k^2 T^{(1)\,\nu}_{\mu} + \frac{1}{2}k^2\cos(2kr)T^{(2)\,\nu}_{\mu}\right. \\ &+ \frac{k}{2r}\sin(2kr)T^{(3)\,\nu}_{\mu} + \frac{1}{4r^2}\left(1-\cos(2kr)\right)T^{(4)\,\nu}_{\mu} \\ &\left. + \frac{m^2}{2}\left(1-\cos(2kr)\right)T^{(5)\,\nu}_{\mu}\right], \end{aligned} \tag{3.44}$$

where the constant matrices $T^{(i)\,\nu}_{\mu}$ are

$$T^{(1)\,\nu}_{\mu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{3.45}$$

$$T^{(2)\,\nu}_{\mu} = \begin{pmatrix} 4\xi & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 4\xi-1 & 0 \\ 0 & 0 & 0 & 4\xi-1 \end{pmatrix}, \tag{3.46}$$

$$T^{(3)\,\nu}_{\mu} = \begin{pmatrix} -4\xi+1 & 0 & 0 & 0 \\ 0 & 4\xi-1 & 0 & 0 \\ 0 & 0 & -6\xi+1 & 0 \\ 0 & 0 & 0 & -6\xi+1 \end{pmatrix}, \tag{3.47}$$

$$T^{(4)\,\nu}_{\mu} = \begin{pmatrix} 4\xi-1 & 0 & 0 & 0 \\ 0 & -8\xi+1 & 0 & 0 \\ 0 & 0 & 8\xi-1 & 0 \\ 0 & 0 & 0 & 8\xi-1 \end{pmatrix}, \tag{3.48}$$

$$T^{(5)\,\nu}_{\mu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3.49}$$

It is easy to verify that all of the integrals in (3.44) can be written as derivatives with respect to r of the integral (3.31) after expanding the β-dependent part as a geometric series in $e^{-\beta\omega_k}$. Defining $\mu_n \equiv \sqrt{(n\beta)^2 + 4r^2}$, we find

$$\begin{aligned}(2\pi r)^2\, t_\mu{}^\nu &= \sum_{n=1}^{\infty} \frac{m}{n\beta} K_1(nm\beta)\; T^{(1)\,\nu}_{\mu} \\ &+ \sum_{n=1}^{\infty} \left[\frac{m}{\mu_n}\left(1 - \frac{8r^2}{\mu_n^2}\right) K_1(m\mu_n) - \frac{4m^2r^2}{\mu_n^2} K_0(m\mu_n)\right]\; T^{(2)\,\nu}_{\mu} \\ &+ \sum_{n=1}^{\infty} \frac{2m}{\mu_n} K_1(m\mu_n)\; T^{(3)\,\nu}_{\mu} \\ &+ \frac{1}{2r^2} \sum_{n=1}^{\infty} \left[K_0(nm\beta) - K_0(m\mu_n)\right]\; T^{(4)\,\nu}_{\mu} \\ &+ m^2 \sum_{n=1}^{\infty} \left[K_0(nm\beta) - K_0(m\mu_n)\right]\; T^{(5)\,\nu}_{\mu}. \end{aligned} \tag{3.50}$$

These sums can be evaluated for the massless, low-temperature, and high-temperature limits just as was done for the s-mode contribution to $\langle\hat{\Phi}^2\rangle^\beta$. Moreover, it is easily verified that this stress tensor is conserved for all ξ at all r, and that it is traceless for $\xi = \frac{1}{6}$ and $m = 0$.

Massless limit

Taking $m \to 0$ in (3.50) replaces the Bessel functions by their small-argument limits. Using the sums

$$\sum_{n=1}^{\infty} [n^2 + x^2]^{-1} = \frac{\pi}{2x}\left[\coth(\pi x) - \frac{1}{\pi x}\right], \tag{3.51}$$

$$\sum_{n=1}^{\infty} \ln\left[1 + \left(\frac{x}{n}\right)^2\right] = \ln \frac{\sinh \pi x}{\pi x}, \tag{3.52}$$

one finds

$$\begin{aligned}\left(t_\mu{}^\nu\right)_{m=0} &= \frac{1}{4\pi r^2}\frac{\pi}{6\beta^2}\,T^{(1)\,\nu}_{\ \mu} \\ &+ \left(\frac{1}{32\pi^2 r^4} - \frac{1}{8\beta^2 r^2\sinh^2(2\pi r/\beta)}\right) T^{(2)\,\nu}_{\ \mu} \\ &+ \left(\frac{1}{8\pi\beta r^3}\coth(2\pi r/\beta) - \frac{1}{16\pi^2 r^4}\right) T^{(3)\,\nu}_{\ \mu} \\ &+ \frac{1}{16\pi^2 r^4}\ln\left\{\frac{\sinh(2\pi r/\beta)}{2\pi r/\beta}\right\} T^{(4)\,\nu}_{\ \mu}\,. \end{aligned} \tag{3.53}$$

The same result is obtained by taking the $m \to 0$ limit before evaluating the integral over k in (3.44).

This stress tensor is finite at $r = 0$:

$$\left(t_\mu{}^\nu\right)_{m=0} \overset{r=0}{=} -\frac{\pi^2}{180\beta^4}\left(6T^{(2)\,\nu}_{\ \mu} + 4T^{(3)\,\nu}_{\ \mu} + T^{(4)\,\nu}_{\ \mu}\right)\,. \tag{3.54}$$

In particular, for $\xi = \frac{1}{6}$,

$$\left(t_\mu{}^\nu\right)_{m=0} \overset{r=0}{=} \frac{\pi^2}{36\beta^4}\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{3}\end{pmatrix}\,. \tag{3.55}$$

Comparing to (A.11), we conclude that at the origin $\frac{5}{6}$ of the total stress tensor (i.e., that obtained by summing over all ℓ) is due to the s mode. The difference is made up by the $\ell = 1$ mode, which can make a nonvanishing contribution at the origin due to the fact that the expression (3.29) for the stress tensor contains up to two derivatives per term quadratic in the field.

Finally, at large r we have

$$\left(t_\mu{}^\nu\right)_{m=0} \overset{r\to\infty}{=} \frac{1}{4\pi r^2}\frac{\pi}{6\beta^2}\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0\end{pmatrix} + O\left(\frac{1}{\beta r^3}\right)\,. \tag{3.56}$$

Note that the leading order term is just $(4\pi r^2)^{-1}$ times the stress tensor for a massless scalar field in two dimensions, (A.19) .

Low-temperature limit of massive theory

For low temperatures, $m\beta \gg 1$, the arguments $nm\beta$, $m\sqrt{(n\beta)^2+4r^2}$ of the Bessel functions in (3.50) are large for all n and so the Bessel functions are exponentially damped. The leading contribution to the sums then comes from the $n=1$ term alone, as we saw earlier for $\langle\hat{\Phi}^2\rangle^\beta$. Recalling that $\mu_1 = \sqrt{\beta^2+4r^2}$, we find

$$\begin{aligned}(2\pi r)^2 \left(t_\mu{}^\nu\right)_{m\beta\gg 1} &= \frac{m}{\beta}K_1(m\beta)\, T^{(1)}_\mu{}^\nu \\ &+ \left[\frac{m}{\mu_1}\left(1-\frac{8r^2}{\mu_1^2}\right)K_1(m\mu_1) - \frac{4m^2r^2}{\mu_1^2}K_0(m\mu_1)\right]\, T^{(2)}_\mu{}^\nu \\ &+ \frac{2m}{\mu_1}K_1(m\mu_1)\, T^{(3)}_\mu{}^\nu \\ &+ \frac{1}{2r^2}\left[K_0(m\beta)-K_0(m\mu_1)\right]\, T^{(4)}_\mu{}^\nu \\ &+ m^2\left[K_0(m\beta)-K_0(m\mu_1)\right]\, T^{(5)}_\mu{}^\nu . \end{aligned} \tag{3.57}$$

As in the massless case, the stress tensor is finite at $r=0$:

$$\begin{aligned}\left(t_\mu{}^\nu\right)_{m\beta\gg 1} &\overset{r=0}{=} -\left(6T^{(2)}_\mu{}^\nu + 4T^{(3)}_\mu{}^\nu + T^{(4)}_\mu{}^\nu\right)\frac{1}{(2\pi)^2}\left[\frac{m^2}{\beta^2}K_0(m\beta) + \frac{2m}{\beta^3}K_1(m\beta)\right] \\ &+ T^{(5)}_\mu{}^\nu \frac{1}{(2\pi)^2}\frac{2m^3}{\beta}K_1(m\beta) . \end{aligned} \tag{3.58}$$

In particular, for $\xi = \frac{1}{6}$,

$$\left(t_\mu{}^\nu\right)_{m\beta\gg 1} \overset{r\to 0}{=} 5\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{3}\end{pmatrix}\frac{1}{(2\pi)^2}\left[\frac{m^2}{\beta^2}K_0(m\beta) + \frac{2m}{\beta^3}K_1(m\beta)\right]$$

$$+\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \frac{1}{(2\pi)^2}\frac{2m^3}{\beta}K_1(m\beta)\,. \tag{3.59}$$

Let us compare this result to the total stress tensor (due to all ℓ) for a massive field at low temperature, (A.14). We see that the second term in (A.14), which vanishes in the massless case, is due entirely to the s mode, while $\frac{5}{6}$ of the first term is due to the s mode.

Meanwhile, as $r \to \infty$ we find

$$\left(t_\mu{}^\nu\right)_{m\beta\gg 1} \overset{r\to\infty}{=} \frac{m^2}{(2\pi r)^2}\left[K_0(m\beta)\, T^{(5)}{}_\mu{}^\nu - \frac{K_1(m\beta)}{m\beta}\, T^{(1)}{}_\mu{}^\nu\right] + O\left(\frac{1}{r^4}\right)\,. \tag{3.60}$$

By comparing to (A.20) we see that the leading-order terms are just $(4\pi r^2)^{-1}$ times the low-temperature stress tensor for a scalar field in two dimensions.

High-temperature limit of massive theory

For $m\beta \ll 1$ we can evaluate the stress tensor for $r \to 0$ and $r \to \infty$ only.

Let us consider the case $r \to 0$ first. By expanding the r-dependent Bessel functions in (3.50) for small r and making use of the sums (3.39)–(3.41) one finds

$$\begin{aligned}\left(t_\mu{}^\nu\right)_{m\beta\ll 1} \overset{r=0}{=}\; & -\left(6T^{(2)}{}_\mu{}^\nu + 4T^{(3)}{}_\mu{}^\nu + T^{(4)}{}_\mu{}^\nu\right)\left[\frac{\pi^2}{180\beta^4} + \frac{m^2}{48\beta^2} - \frac{m^3}{24\pi\beta}\right.\\ & \left. + \frac{m^4}{64\pi^2}\left(\gamma + \ln\frac{b}{4\pi}\right) - \frac{3m^4}{256\pi^2}\right]\\ & + T^{(5)}{}_\mu{}^\nu\left[\frac{m^2}{12\beta^2} - \frac{m^3}{4\pi\beta} - \frac{m^4}{8\pi^2}\left(\gamma + \ln\frac{b}{4\pi}\right) + \frac{m^4}{16\pi^2}\right]\,. \end{aligned} \tag{3.61}$$

Note that taking the $m \to 0$ limit returns the massless result (3.54). Also, if we take $\xi = \frac{1}{6}$ and compare to the total stress tensor for a high-temperature field, (A.15), we see that the second term in (A.15) is entirely due to the s mode, while $\frac{5}{6}$ of the first term is due to the s mode.

For $mr \gg 1$, all of the Bessel functions $K_j(m\mu_n)$ will be exponentially small compared to the $K_j(mn\beta)$, and may be ignored. We then find

$$\left(t_\mu{}^\nu\right)_{m\beta\ll 1} \overset{mr\to\infty}{=} \frac{1}{(2\pi r)^2}\left[\frac{\pi^2}{6\beta^2} - \frac{\pi m}{2\beta} - \frac{m^2}{4}\left(\gamma + \ln\frac{m\beta}{4\pi}\right) + \frac{m^2}{8}\right] T^{(1)}{}_\mu{}^\nu + \frac{1}{(2\pi r)^2}\left[\frac{\pi m}{2\beta} + \frac{m^2}{2}\left(\gamma + \ln\frac{m\beta}{4\pi}\right)\right] T^{(5)}{}_\mu{}^\nu + O\left(\frac{1}{r^4}\right). \tag{3.62}$$

Again, the leading-order terms are just $(4\pi r^2)^{-1}$ times the corresponding result for a scalar field in two dimensions, (A.21). As well, for $m \to 0$, this leading part reproduces the $1/r^2$ part of the stress tensor from the massless case, (3.56).

3.3 The Dimensional-Reduction Anomaly in Flat Space

In the previous section, we calculated directly the s-mode contribution to the expectation values $\langle\hat{\Phi}^2\rangle$ and $\langle\hat{T}_{\mu\nu}\rangle$ at finite temperature in four dimensions. In this section we attempt to solve the same problem using dimensional reduction. We show that naïve dimensional reduction fails to produce the correct expectation values for flat space. We then examine the dimensionally reduced theory in detail and show how the anomaly corrects its predictions to yield expectation values in agreement with those of the previous section.

Consider a quantum field $\hat{\Phi}$ in four-dimensional flat space. For convenience, we will calculate the anomaly in Euclidean space and continue the final results back to Lorentzian signature. We assume that the field satisfies the field equation (2.55), where the potential V vanishes inside the region of interest and is spherically symmetric outside. In this case, the Green function for a given state is renormalized by subtracting the Green function for the Euclidean vacuum. In four dimensions the latter is[3]

$$G_{\rm div}(X,X') = \frac{m}{4\pi^2\sqrt{2\sigma}}K_1(m\sqrt{2\sigma})\,, \tag{3.63}$$

where σ is one-half the square of the geodesic distance between X and X', and K_1 is a modified Bessel function. In spherical coordinates $X^\mu = (t, r, \theta, \eta)$ the line element is

$$ds^2 = dt^2 + dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\eta^2\right)\,, \tag{3.64}$$

and

$$2\sigma = (t-t')^2 + (r-r')^2 + 2rr'\,(1-\cos\lambda)\,, \tag{3.65}$$

where λ, the angle between X and X', is given by (3.9).

If there exists a boundary Σ surrounding the region $\mathcal{M}$ under consideration and the field obeys a non-trivial boundary condition at Σ, or equation (2.55) includes a non-vanishing potential V which vanishes in $\mathcal{M}$, then the Green function $G(X,X')$ will differ from $G_{\rm div}$. The renormalized Green function is then

$$G_{\rm ren}(X,X') = G(X,X') - G_{\rm div}(X,X')\,. \tag{3.66}$$

It is evident that $\langle\hat\Phi^2\rangle_{\rm ren}$ and $\langle\hat T_{\mu\nu}\rangle_{\rm ren}$ vanish in the absence of the external field V and boundaries.

If before renormalizing we first decompose the field $\hat\Phi$ into spherical harmonics as in (3.2), we will be left with an effective field $\hat\varphi_\ell$ propagating on the two-dimensional space with line element $ds^2 = dt^2 + dr^2$, where $t \in (-\infty,\infty)$ and $r \in [0,\infty)$. The geodesic distance function σ is simply $\frac{1}{2}[(t-t')^2 + (r-r')^2]$, and from (3.10) the induced potential is

$$V_\ell = \frac{\ell(\ell+1)}{r^2}\,, \tag{3.67}$$

[3]Eq. (3.63) may be obtained using the Schwinger-DeWitt expansion, as described in Section 4.1.

as $\Delta\phi - (\nabla\phi)^2 = 0$ for the line element (3.64). Though this induced potential is nonvanishing, it does not influence the divergences in two dimensions, as we shall see later. As a result, the theory is renormalized by subtracting the two-dimensional vacuum Green function[4]

$$\mathcal{G}_{\ell|\mathrm{div}}(x,x') = \frac{1}{2\pi} K_0\left(m\sqrt{(t-t')^2 + (r-r')^2}\right). \tag{3.68}$$

By this point one might have noticed a curious feature of the dimensional reduction. For the original theory in four dimensions, one typically expects the renormalized stress tensor and field fluctuations to be finite everywhere (such as for a finite-temperature state). In particular, for the Euclidean vacuum state, all expectation values should vanish identically. By (3.2) and (3.11), finiteness at $r = 0$ in four dimensions implies that the dimensionally reduced field and Green function must vanish there. On general grounds, however, one expects that this boundary condition will produce infinite vacuum polarization which causes the dimensionally reduced stress tensor to diverge as $r \to 0$ [75]. Thus, if one attempts to calculate $\langle \hat{T}_{\mu\nu} \rangle_{\mathrm{ren}}$ by naïvely summing over the $\langle \hat{T}^{(\ell)}_{\mu\nu} \rangle_{\mathrm{ren}}$, the resulting four-dimensional stress tensor will diverge[5] at $r = 0$, even for the vacuum state! A key result of this section is that the dimensional-reduction anomaly cancels these divergences in the two-dimensional theory to yield finite results in four dimensions.

The dimensional-reduction anomaly arises because of the inequivalence of the divergent subtraction terms for the higher- and lower-dimensional theories. To

[4] Eq. (3.68) can be obtained using the Schwinger-DeWitt expansion in Section 4.1.

[5] In fact, one can show that the sum over ℓ will diverge at *all* r. For example, the anomaly $\Delta\langle\hat{\varphi}^2_\ell\rangle$ varies as ℓ^0 as $\ell \to \infty$, and hence sums like

$$\sum_{\ell=0}^{\infty} \frac{(2\ell+1)}{4\pi r^2} \Delta\langle\hat{\varphi}^2_\ell\rangle, \qquad \sum_{\ell=0}^{\infty} \frac{(2\ell+1)}{4\pi r^2} \langle\hat{\varphi}^2_\ell\rangle_{\mathrm{ren}}$$

are divergent. Similar effects were noticed in [6, 11, 12].

illustrate this, we decompose $G_{\rm div}$ into spherical harmonics and compare to $\mathcal{G}_{\ell|\rm div}$. Defining the mode decomposition by

$$G_{\rm div}(X,X') = \sum_{\ell=0}^{\infty} \frac{(2\ell+1)}{4\pi r r'} P_\ell(\cos\lambda) G_{{\rm div}|\ell}(x,x') \tag{3.69}$$

in accordance with (3.11), we have

$$G_{{\rm div}|\ell}(x,x') = 2\pi r r' \int_{-1}^{1} d(\cos\lambda) P_\ell(\cos\lambda) G_{\rm div}(X,X')\,. \tag{3.70}$$

Inserting (3.63) into (3.70) and using the well-known integral representation for K_ν,

$$\int_0^\infty dx\, x^{-1-\nu} \exp\left\{-x - \frac{\alpha^2}{4x}\right\} = 2\left(\frac{2}{\alpha}\right)^\nu K_\nu(\alpha) \tag{3.71}$$

(see Appendix C.3.1), the integral

$$\int_{-1}^{1} dz\; P_\ell(z)\; e^{p(1-z)} = (-1)^\ell e^p \sqrt{\frac{2\pi}{p}} I_{\ell+1/2}(p)\,, \tag{3.72}$$

where $I_{\ell+1/2}$ is a modified Bessel function (see e.g. [74], vol.2, eq.2.17.5.2), and the representation

$$I_{\ell+1/2}(p) = \frac{1}{\sqrt{2\pi p}} \sum_{k=0}^{\ell} \frac{(\ell+k)!}{k!(\ell-k)!} \frac{1}{(2p)^k} \left[(-1)^k e^p - (-1)^\ell e^{-p}\right]\,, \tag{3.73}$$

(see, for example, 8.467 of [76]), we obtain

$$\begin{aligned} G_{{\rm div}|\ell}(x,x') \;=\; & \frac{1}{2\pi} \sum_{k=0}^{\ell} \frac{(\ell+k)!}{k!(\ell-k)!} \left[(-1)^k \frac{[2\sigma^-]^{k/2}}{(2mrr')^k} K_k(m\sqrt{2\sigma^-}) \right. \\ & \left. - (-1)^\ell \frac{[2\sigma^+]^{k/2}}{(2mrr')^k} K_k(m\sqrt{2\sigma^+}) \right]\,, \end{aligned} \tag{3.74}$$

where

$$\sigma^\pm \equiv \frac{1}{2}[(t-t')^2 + (r \pm r')^2]\,. \tag{3.75}$$

Let us compare this result for the mode-decomposed subtraction terms from four dimensions with the subtraction term for the two-dimensional theory, (3.68).

While $\mathcal{G}_{\ell|\text{div}}$ is the free-field Green function in two dimensions, it is not difficult to verify that $G_{\text{div}|\ell}$ is the Green function for a field propagating in the centrifugal barrier potential (3.67) and obeying Dirichlet boundary conditions[6] at $r = 0$. The anomaly in the Green function, given by

$$\Delta\mathcal{G}_\ell(x, x') = \mathcal{G}_{\ell|\text{div}}(x, x') - G_{\text{div}|\ell}(x, x'), \tag{3.76}$$

is then simply the negative of the renormalized Green function for a field propagating in the centrifugal barrier potential (3.67) in two dimensions and obeying Dirichlet boundary conditions at $r = 0$. The infinite vacuum polarization effects in the effective two-dimensional theory due to the boundary at $r = 0$ will thus be exactly canceled by the dimensional-reduction anomaly. In particular, for the four-dimensional vacuum state the anomaly exactly cancels the renormalized two-dimensional Green function, yielding vanishing expectation values in four dimensions. Failure to include the anomaly will leave one with divergent results for four-dimensional expectation values, even for the vacuum state.

The anomaly in the Green function can be used to calculate the anomalies in $\langle\hat{\Phi}^2\rangle$ and $\langle\hat{T}_{\mu\nu}\rangle$. For $\langle\hat{\Phi}^2\rangle$ we find

$$\begin{aligned} \Delta\langle\hat{\varphi}_\ell^2\rangle &= \lim_{x'\to x}\left[\mathcal{G}_{\ell|\text{div}}(x, x') - G_{\text{div}|\ell}(x, x')\right] & (3.77)\\ &= \frac{1}{4\pi}\sum_{k=1}^{\ell}\frac{(\ell+k)!}{(\ell-k)!}\frac{1}{k}\frac{(-1)^{k+1}}{(mr)^{2k}} + \frac{(-1)^\ell}{2\pi}\sum_{k=0}^{\ell}\frac{(\ell+k)!}{k!(\ell-k)!}\frac{K_k(2mr)}{(mr)^k}. & (3.78)\end{aligned}$$

For example, for the first three modes the anomalies are

$$\begin{aligned} \Delta\langle\hat{\varphi}_{\ell=0}^2\rangle &= \frac{1}{2\pi}K_0(2mr), & (3.79)\\ \Delta\langle\hat{\varphi}_{\ell=1}^2\rangle &= \frac{1}{2\pi}\left[\frac{1}{(mr)^2} - K_0(2mr) - \frac{2}{(mr)}K_1(2mr)\right], & (3.80)\\ \Delta\langle\hat{\varphi}_{\ell=2}^2\rangle &= \frac{1}{2\pi}\left[\frac{3}{(mr)^2} - \frac{6}{(mr)^4} + K_0(2mr) + \frac{6}{(mr)}K_1(2mr)\right.\end{aligned}$$

[6]Since (3.72) is finite for $p = 0$, $G_{\text{div}|\ell}$ obeys Dirichlet boundary conditions at $r = 0$, $r' = 0$ due to (3.70).

$$+\frac{12}{(mr)^2}K_2(2mr)\Bigg] . \tag{3.81}$$

Note that the anomaly diverges logarithmically as $r \to 0$, and vanishes as $r \to \infty$. The divergence at $r = 0$ precisely cancels the corresponding divergence in the renormalized two-dimensional theory due to the boundary condition.

The expressions for the anomaly in the stress tensor are more complicated, so we show the results for the $\ell = 0$ mode only. Substituting (3.76) into (2.82), (2.83) we obtain

$$\begin{aligned}
\Delta\langle\hat{\mathcal{T}}^{(0)}{}_t{}^t\rangle &= \frac{m^2}{2\pi}\left[(4\xi-1)K_0(2mr)+(12\xi-2)\frac{K_1(2mr)}{(2mr)}\right. \\
&\left. \quad +(8\xi-2)\frac{K_0(2mr)}{(2mr)^2}\right] , && (3.82) \\
\Delta\langle\hat{\mathcal{T}}^{(0)}{}_r{}^r\rangle &= \frac{m^2}{2\pi}\left[(2-8\xi)\frac{K_1(2mr)}{(2mr)}+(2-16\xi)\frac{K_0(2mr)}{(2mr)^2}\right] , && (3.83) \\
\Delta\langle\hat{\mathcal{P}}^{(0)}\rangle &= \frac{m^2}{2\pi}\left[(4\xi-1)K_0(2mr)+(16\xi-3)\frac{K_1(2mr)}{(2mr)}\right. \\
&\left. \quad +(16\xi-2)\frac{K_0(2mr)}{(2mr)^2}\right] . && (3.84)
\end{aligned}$$

In the $m \to 0$ limit the anomalous stress tensor becomes

$$\begin{aligned}
\Delta\langle\hat{\mathcal{T}}^{(0)}{}_t{}^t\rangle &= \frac{1}{4\pi r^2}\left[(1-4\xi)\{\gamma+\ln mr\}+(6\xi-1)\right] , && (3.85) \\
\Delta\langle\hat{\mathcal{T}}^{(0)}{}_r{}^r\rangle &= \frac{1}{4\pi r^2}\left[(8\xi-1)\{\gamma+\ln mr\}+(1-4\xi)\right] , && (3.86) \\
\Delta\langle\hat{\mathcal{P}}^{(0)}\rangle &= \frac{1}{8\pi r^2}\left[(2-16\xi)\{\gamma+\ln mr\}+(16\xi-3)\right] , && (3.87)
\end{aligned}$$

where the m in the logarithms must now be considered an arbitrary parameter. Note that the anomalous stress tensor diverges as $\ln(r)/r^2$ as $r \to 0$, and vanishes as $r \to \infty$. As was the case for $\Delta\langle\hat{\varphi}_\ell^2\rangle$, the divergence at $r = 0$ precisely cancels the corresponding divergence in the renormalized two-dimensional theory due to the boundary condition, leaving a finite stress tensor in four dimensions.

3.3.1 Scalar Field at Finite Temperature

Let us return to the spherical decomposition of a scalar field at temperature $T = \beta^{-1}$ in four-dimensional flat space, and use this case as an illustration of the role of the dimensional-reduction anomaly. From the preceding discussion, the dimensional reduction of a free scalar field with vanishing potential produces a dimensionally reduced field $\hat{\varphi}_\ell$ propagating in the centrifugal barrier potential $\ell(\ell+1)/r^2$ and satisfying Dirichlet boundary conditions at $r = 0$. The bare zero-temperature Green function for this theory is just (3.74). The finite-temperature Green function is most conveniently obtained by recalling that in Euclidean space the temperature dependence can be realized by requiring the field to be periodic in (imaginary) time with period β. Thus, the finite-temperature Green function in two dimensions can be obtained from the zero-temperature function (3.74) by the replacement $(t-t') \to (t-t'+n\beta)$ and summing over all integers n:

$$\begin{aligned} \mathcal{G}_\ell^\beta(x,x') &= \sum_{n=-\infty}^{\infty} \frac{1}{2\pi} \sum_{k=0}^{\ell} \frac{(\ell+k)!}{k!(\ell-k)!} \left[(-1)^k \frac{[2\sigma_n^-]^{k/2}}{(2mrr')^k} K_k(m\sqrt{2\sigma_n^-}) \right. \\ &\quad \left. - (-1)^\ell \frac{[2\sigma_n^+]^{k/2}}{(2mrr')^k} K_k(m\sqrt{2\sigma_n^+}) \right], \end{aligned} \tag{3.88}$$

where

$$\sigma_n^\pm \equiv \frac{1}{2}[(t-t'+n\beta)^2 + (r\pm r')^2]. \tag{3.89}$$

The zero-temperature result is the $n = 0$ term, $G_{\mathrm{div}|\ell}(x,x')$. The renormalized value of $\langle\hat{\varphi}_\ell^2\rangle$ for the dimensionally reduced field is then

$$\langle\hat{\varphi}_\ell^2\rangle_{\mathrm{ren}}^\beta = \lim_{x'\to x}\left[\mathcal{G}_\ell^\beta(x,x') - \mathcal{G}_{\ell|\mathrm{div}}(x,x')\right] \tag{3.90}$$

$$\begin{aligned} &= 2\sum_{n=1}^{\infty} \frac{1}{2\pi} \sum_{k=0}^{\ell} \frac{(\ell+k)!}{k!(\ell-k)!} \left[(-1)^k \left(\frac{n\beta}{2mr^2}\right)^k K_k(nm\beta) \right. \\ &\quad \left. - (-1)^\ell \left(\frac{\sqrt{(n\beta)^2+(2r)^2}}{2mr^2}\right)^k K_k(m\sqrt{(n\beta)^2+(2r)^2}) \right] \\ &\quad + \lim_{x'\to x}\left[G_{\mathrm{div}|\ell}(x,x') - \mathcal{G}_{\ell|\mathrm{div}}(x,x')\right] \end{aligned} \tag{3.91}$$

Note that we have separated the $n = 0$ term and grouped it with the two-dimensional renormalization term, $\mathcal{G}_{\ell|_{\rm div}}$. This last line is precisely the negative of the anomaly $\Delta\langle\varphi_\ell^2\rangle$, (3.77). As we have seen, this last part will produce a contribution to $\langle\varphi_\ell^2\rangle$ which diverges as $r \to 0$. However, in calculating the renormalized $\langle\hat{\Phi}^2\rangle^\beta$ in four dimensions this term is precisely cancelled by the dimensional-reduction anomaly:

$$\langle\hat{\Phi}^2\rangle^\beta_{\rm ren} = \sum_{\ell=0}^{\infty}\frac{2\ell+1}{4\pi r^2}\left[\langle\varphi_\ell^2\rangle^\beta_{\rm ren} + \Delta\langle\varphi_\ell^2\rangle\right] \tag{3.92}$$

$$\begin{aligned} &= \sum_{\ell=0}^{\infty}\frac{2\ell+1}{4\pi r^2}2\sum_{n=1}^{\infty}\frac{1}{2\pi}\sum_{k=0}^{\ell}\frac{(\ell+k)!}{k!(\ell-k)!}\left[(-1)^k\left(\frac{n\beta}{2mr^2}\right)^k K_k(nm\beta)\right.\\ &\left.\quad -(-1)^\ell\left(\frac{\sqrt{(n\beta)^2+(2r)^2}}{2mr^2}\right)^k K_k(m\sqrt{(n\beta)^2+(2r)^2})\right] . \end{aligned} \tag{3.93}$$

The remaining terms are the finite, temperature-dependent contributions to $\langle\hat{\Phi}^2\rangle^\beta_{\rm ren}$ in four dimensions. For example, the $\ell = 0$ mode alone makes a contribution of

$$f^{(\ell=0)} = \frac{1}{4\pi^2 r^2}\sum_{n=1}^{\infty}\left[K_0(nm\beta) - K_k(m\sqrt{(n\beta)^2+(2r)^2})\right] . \tag{3.94}$$

This is precisely the result (3.32) obtained in the previous section by working directly in four dimensions.

To summarize, we have demonstrated that expectation values such as $\langle\hat{\Phi}^2\rangle_{\rm ren}$ and $\langle\hat{T}_{\mu\nu}\rangle_{\rm ren}$ can be expressed as sums of the corresponding renormalized quantities from the two-dimensional theory using relations like (3.12)–(3.16). However, to obtain the correct result we see that each term in the decomposition must be modified by adding a state-independent[7] anomaly term, such as (3.78), (3.82)–(3.84). Failure to account for these extra terms would result, for example, in nonzero expectation values in the Minkowski state. We conclude that one can

[7]See also the discussion at the end of Chapter 5.

model correctly quantum fields using lower-dimensional theories, but only if the dimensional-reduction anomaly is accounted for.

Chapter 4

Spherical Decompositions in Curved Space

In the previous chapter we demonstrated the existence of the dimensional-reduction anomaly using the simple example of spherical decompositions in flat space. We saw that the anomaly has to be accounted for if the dimensionally reduced theory is to yield correct expectation values for the physical theory in four dimensions. In this chapter we extend our analysis to the more general case of spherical decompositions in *curved* spherically symmetric spaces. These include the very interesting example of the Schwarzschild black hole, which we shall consider at some length.

We begin in Section 4.1 with a detailed derivation of the dimensional-reduction anomaly in $\langle\hat{\Phi}^2\rangle$. The techniques developed there are then applied in Section 4.2 to the calculation of the anomaly in the effective action. In Section 4.3 we derive the anomalous stress tensor for a general spherical decomposition, and in Section 4.4 apply these results to the interesting case of the s-mode in Schwarzschild spacetime.

4.1 The Dimensional-Reduction Anomaly in $\langle\hat{\Phi}^2\rangle$

While our chief aim is the study of the dimensional-reduction anomaly in the effective action, in order to illustrate more clearly the effect of curvature on local reduction anomalies we begin with the simpler case of the anomaly for $\langle\hat{\Phi}^2\rangle_{\text{ren}}$.

4.1.1 Structure of Divergences

We take as our system a massive scalar field with arbitrary coupling to the four-dimensional scalar curvature, described by the field equation (2.55) with

$$V = \xi\, {}^4R[g]\,. \tag{4.1}$$

We assume that the space of interest is Euclidean and described by the line element (3.1).

As we saw in Section 3.1, under the dimensional reduction (3.2) the quantum field $\hat{\Phi}$ reduces to a collection of effective fields $\hat{\varphi}_\ell$ in the two-dimensional space with metric h_{ab}. These dimensionally reduced fields propagate in the induced potential (3.10),

$$\begin{aligned} V_\ell &= \xi\, {}^4R[g] + \frac{\ell(\ell+1)}{r^2} - \Delta\phi + (\nabla\phi)^2 \\ &= \xi R[h] + [\ell(\ell+1) + 2\xi]\,\rho^{-2}\mathrm{e}^{2\phi} + (4\xi - 1)\Delta_h\phi + (1 - 6\xi)(\nabla\phi)^2\,, \end{aligned} \tag{4.2}$$

where $r = \rho\,\mathrm{e}^{-\phi}$, and vanish at any regular points of the manifold where $r = 0$. We wish to compute the anomaly associated with renormalizing this dimensionally reduced theory versus (2.55).

Since the dimensional-reduction anomaly is connected with the divergences, let us discuss first the general structure of these divergences. The most convenient way for our purposes is to use the so-called Schwinger-DeWitt proper-time formalism [51], [41], [42]. In this approach, one can write the Green function $G(X, X')$ and effective action W in the form

$$G(X,X') = \int_0^\infty ds\, K(X, X'|s)\,, \tag{4.3}$$

$$W = -\frac{1}{2}\int_0^\infty \frac{ds}{s} \int d^4X \sqrt{g}\, K(X, X|s)\,, \tag{4.4}$$

where $K(X, X'|s)$ is the heat kernel for the operator F, defined by (2.48), (2.49). Analogous formulae also hold for the dimensionally reduced theory with operator $\mathcal{F}_\ell$, heat kernel $\mathcal{K}_\ell(x, x'|s)$, Green function $\mathcal{G}_\ell(x, x')$ and effective action $\mathcal{W}_\ell$.

The divergences in both the Green function and the effective action come from the $s \to 0$ limit of the s integral. The advantage of the heat kernel formulation in curved space is that the small-s behaviour of the heat kernel is known for arbitrary curved spaces of any dimension, in the form of the Schwinger-DeWitt expansion [41, 42, 51]. In four dimensions this is

$$K(X, X'|s) = \frac{D^{\frac{1}{2}}(X, X')}{(4\pi s)^2} \exp\left\{-m^2 s - \frac{\sigma(X, X')}{2s}\right\} \sum_{i=0}^{\infty} a_i(X, X') s^i\,. \tag{4.5}$$

Here again $\sigma(X, X')$ is one-half of the square of the geodesic distance between the points X and X' (sometimes referred to as the Synge world function [77]), while $D(X, X')$ is the Van Vleck-Morette determinant [78, 79],

$$D(X, X') = \frac{1}{\sqrt{g(X)}\sqrt{g(X')}} \det\left[-\frac{\partial}{\partial X^\mu}\frac{\partial}{\partial X'^\nu}\sigma(X, X')\right]\,. \tag{4.6}$$

The a_n are the Schwinger-DeWitt coefficients for the operator F of (2.55) with potential (4.1). In the coincidence limit $X' \to X$ the first few of these are [41]

$$a_0^{\Box-\xi\,{}^4R} = 1\,, \tag{4.7}$$

$$a_1^{\Box-\xi\,{}^4R} = \left(\frac{1}{6}-\xi\right) {}^4R\,, \tag{4.8}$$

$$\begin{aligned} a_2^{\Box-\xi\,{}^4R} &= \frac{1}{180}\left[{}^4R_{\alpha\beta\gamma\delta}\,{}^4R^{\alpha\beta\gamma\delta} - {}^4R_{\alpha\beta}\,{}^4R^{\alpha\beta} + \Box\,{}^4R\right] \\ &\quad + \frac{1}{6}\left(\frac{1}{6}-\xi\right)\Box\,{}^4R + \frac{1}{2}\left(\frac{1}{6}-\xi\right)^2 ({}^4R)^2\,. \end{aligned} \tag{4.9}$$

Note that, as promised, the a_n are purely local functions of the curvature, the potential, and their covariant derivatives. They are independent of the field mass, as this is accounted for separately in (4.5).

For the two-dimensional operator $\mathcal{F}_\ell$ of (2.72) with induced potential (4.2) we shall need the Schwinger-DeWitt expansion of the heat kernel only in the coincidence limit. This is

$$\mathcal{K}_\ell(x,x|s) = \frac{1}{4\pi s}\exp\left\{-m^2 s\right\}\left[1 + s\left(\frac{1}{6}R - V_\ell\right) + \cdots\right] . \tag{4.10}$$

Considering (4.3), (4.4), it is clear that in four dimensions the divergences in G (W) arise from the first two[1] (three) terms in the Schwinger-DeWitt expansion for K, while in two dimensions we need consider only the first term (first two terms) in $\mathcal{K}_\ell$. The anomaly in $\langle\hat{\Phi}^2\rangle$ or W can then be calculated by mode-decomposing the appropriate terms from K, comparing to the heat kernel $\mathcal{K}_\ell$ for the dimensionally reduced theory, and finally integrating the difference over s according to (4.3) or (4.4).

4.1.2 Calculation of the Anomaly

Let us begin with the anomaly in $\langle\hat{\Phi}^2\rangle$. The divergent part of the Green function in four dimensions is given by

$$G_{\rm div}(X,X') = \int_0^\infty ds\, K_{\rm div}(X,X'|s)\,, \tag{4.11}$$

where $K_{\rm div}$ consists of the first two terms of (4.5):

$$K_{\rm div}(X,X'|s) = \frac{1}{(4\pi s)^2}\exp\left\{-m^2 s - \frac{\sigma}{2s}\right\}\left[\Re_0^{\Box-\xi\,{}^4\!R}(X,X') + s\Re_1^{\Box-\xi\,{}^4\!R}(X,X')\right] . \tag{4.12}$$

[1]Comparing to (3.63), one sees that the integral representation used for $G_{\rm div}$ in Section 3.3 was just the heat kernel representation (4.3) with (4.5)–(4.8).

Here we use the convenient notation

$$\mathfrak{R}_n^{\Box-\xi\,{}^4R}(X,X') \equiv D^{\frac{1}{2}}(X,X')\, a_n^{\Box-\xi\,{}^4R}(X,X')\,. \tag{4.13}$$

In principle, the anomaly in $\langle\hat{\Phi}^2\rangle$ is straightforward to calculate. We mode-decompose $K_{\rm div}$ in terms of Legendre polynomials in the usual manner:

$$K_{\rm div}(X,X'|s) \;=\; \sum_{\ell=0}^{\infty} \frac{(2\ell+1)}{4\pi r r'}\, P_\ell(\cos\lambda)\, K_{{\rm div}|\ell}(x,x'|s)\,; \tag{4.14}$$

$$K_{{\rm div}|\ell}(x,x'|s) \;=\; 2\pi r r' \int_{-1}^{1} d(\cos\lambda)\, P_\ell(\cos\lambda)\, K_{\rm div}(X,X'|s)\,. \tag{4.15}$$

The anomaly in $\langle\hat{\Phi}^2\rangle$ is then just the coincidence limit of the difference between the subtraction terms in two dimensions and those mode-decomposed from four dimensions, integrated over s:

$$\Delta\langle\hat{\varphi}^2(x)\rangle = \int_0^\infty ds\, \left[K_{\ell|{\rm div}}(x,x|s) - K_{{\rm div}|\ell}(x,x|s)\right]\,. \tag{4.16}$$

We encounter a difficulty, however, when we try to perform the mode decomposition. For a general space, σ and the $a_n^{\Box-\xi\,{}^4R}$ are known only for infinitesimal separations[2] of X and X', while evaluation of the mode-decomposition integral (4.15) requires knowing σ and the $a_n^{\Box-\xi\,{}^4R}$ for finite separations of X, X' on the two-sphere. We proceed by determining an approximate $K_{\rm div}$ for finite separation based on the following criteria:

1. Our approximate $K_{\rm div}$ must reduce to the known value in the flat-space limit.

2. Our approximate $K_{\rm div}$ must respect the periodicity of the two-spheres (i.e., it must be periodic in the angular separation λ with period 2π).

[2]In terms of momentum integrals, finite separations correspond to the low-frequency regime, where the renormalization terms are not fixed by the divergences in the theory.

The first criterion may be met in the following manner. We split the points X and X' in the angular direction only. Using the well-known short-distance expansions obtained in [80, 81], σ and the $\Re_n^{\Box-\xi\,{}^4R}$ may be expanded in powers of the angular separation λ, which is equivalent to expanding in powers of the curvature. These expansions are then substituted into K_{div} and, assuming small curvatures, truncated at first order in the curvature for $\Delta\langle\hat{\varphi}_\ell^2\rangle$ and at second order for $\Delta\mathcal{W}_\ell$. To take account of the periodicity of the two-spheres, the resulting expansions in powers of λ^2 are then converted into expansions in $(1-\cos\lambda)$. Defining $z=\cos\lambda$, we have

$$\lambda^2 = 2(1-z) + \frac{1}{3}(1-z)^2 + \frac{4}{45}(1-z)^3 + \cdots . \tag{4.17}$$

We then substitute (4.17) for each λ^2, truncating at the lowest order in $(1-z)$ which will yield the correct flat-space limit. This replacement of λ^2 by a finite series in $(1-z)$ means that our expansions are only modified for large angular separations, where the renormalization terms are inherently ambiguous. Our choice simply corresponds to a natural extension of the flat-space heat kernel which respects the periodicity of the two-spheres for large angular separations. The details of this procedure are presented in Appendix B.2.

Following this procedure, one finds that to first order in the curvature

$$\begin{aligned} 2\sigma &= 2r^2(1-z) + \frac{r^2}{3}[1 - r^2(\nabla\phi)^2](1-z)^2 , && (4.18)\\ \Re_0^{\Box-\xi\,{}^4R} &= 1 + \frac{1}{6}\,{}^4R_{\theta\theta}(1-z) , && (4.19)\\ \Re_1^{\Box-\xi\,{}^4R} &= \left(\frac{1}{6}-\xi\right){}^4R . && (4.20) \end{aligned}$$

Inserting these expansions into (4.5) yields our approximation for the "divergent" part of the four-dimensional heat kernel,

$$\begin{aligned} K_{\text{div}}(X,X'|s) &= \frac{1}{(4\pi s)^2}\exp\left\{-m^2 s - \frac{r^2}{2s}(1-z)\right\}\left[1 + s\left(\frac{1}{6}-\xi\right){}^4R \right.\\ &\quad \left. + \frac{1}{6}\,{}^4R_{\theta\theta}(1-z) - \frac{r^2}{12s}[1-r^2(\nabla\phi)^2](1-z)^2\right] . \end{aligned} \tag{4.21}$$

The mode decomposition (4.15) of this K_{div} then boils down to evaluating the integral

$$J_{\ell n} \equiv -p \int_{-1}^{1} dz\, P_\ell(z)\, e^{p(1-z)}\, (1-z)^n\,, \tag{4.22}$$

where $p \equiv -r^2/2s$ is a dimensionless parameter and n is an integer. The integrals for $n \neq 0$ can be obtained from the $n=0$ result (3.72), (3.73) used in the flat-space case by differentiating with respect to p, yielding

$$J_{\ell n} = \sum_{k=0}^{\ell} \frac{(\ell+k)!}{k!(\ell-k)!} \left[\frac{(-1)^k 2^n}{(-2p)^{n+k}} \frac{(k+n)!}{k!} - (-1)^\ell e^{2p} \sum_{\alpha=0}^{n} \frac{2^n}{(-2p)^{\alpha+k}} \frac{(k+\alpha)!n!}{k!\alpha!(n-\alpha)!} \right]. \tag{4.23}$$

The mode-decomposed heat kernel subtraction terms for a general four-dimensional spherically symmetric spacetime are then

$$K_{\text{div}|\ell}(x,x|s) = \frac{e^{-m^2 s}}{4\pi s} \left[J_{\ell 0} + s\left(\frac{1}{6} - \xi\right) {}^4\!R\, J_{\ell 0} + \frac{1}{6} {}^4\!R_{\theta\theta} J_{\ell 1} - \frac{r^2}{12 s} [1 - r^2 (\nabla\phi)^2] J_{\ell 2} \right]. \tag{4.24}$$

The first term in (4.24) is the mode decomposition for flat space, while the other terms carry the contributions due to the curvature. Meanwhile, the various parts of the $J_{\ell n}$ fulfill several roles. First, the $k \neq 0$ terms in (4.23) are associated with the centrifugal potential $\ell(\ell+1)/r^2$ induced by the the mode decomposition. This potential is ignored in the renormalization in two dimensions, since only the first (potential-independent) term in the Schwinger-DeWitt expansion of the heat kernel contributes to the divergences of the two-dimensional Green function. Second, the terms in (4.23) proportional to $e^{2p} = e^{-r^2/s}$ enforce the Dirichlet boundary condition at $r=0$ (if this point belongs to the four-dimensional manifold), which is required if the four-dimensional subtraction term is to be finite there [see (4.14)].

These results are to be compared with the subtraction term in two dimensions, which consists of the first term of (4.10):

$$K_{\ell|\text{div}}(x,x|s) = \frac{e^{-m^2 s}}{4\pi s}. \tag{4.25}$$

In contrast to $K_{\text{div}|\ell}$, $\mathcal{K}_{\ell|\text{div}}$ is independent of both the position, the two-metric h_{ab}, and the mode number ℓ. It matches just the first term in the $k = 0$ contribution to the flat space part of $K_{\text{div}|\ell}$.

The anomaly in the heat kernel is found by subtracting (4.24) from (4.25). It is clear that part of this anomaly is due to the different number of terms in the Schwinger-DeWitt expansion used to renormalize the four- and two-dimensional theories. One might hope that the anomaly could be eliminated by keeping the second term in the Schwinger-DeWitt expansion in two dimensions, (4.10). It is easily verified, however, that doing so only cancels the first term in the $k = 1$ contribution to the flat-space part of $K_{\text{div}|\ell}$, and the first term in the $k = 0$ contribution to the curvature-dependent parts of $K_{\text{div}|\ell}$. Furthermore, there is no hope that renormalization, a purely local procedure, could cancel the terms responsible for the induced boundary condition at $r = 0$, which is essentially nonlocal.

The anomaly in $\langle\hat{\Phi}^2\rangle$ is now obtained by integrating the difference of $\mathcal{K}_{\ell|\text{div}}$ and $K_{\text{div}|\ell}$ as in (4.16). We find

$$\begin{aligned}\Delta\langle\hat{\varphi}_\ell^2\rangle &= \frac{1}{2\pi}\Bigg[-I[m^2sJ_{\ell0}] - \frac{1}{m^2}\left(\frac{1}{6}-\xi\right){}^4R\, I[(m^2s)^2J_{\ell0}] - \frac{1}{6}{}^4R_{\theta\theta}\, I[m^2sJ_{\ell1}] \\ &\qquad + \frac{(mr)^2}{12}[1-r^2(\nabla\phi)^2]\, I[J_{\ell2}]\Bigg], \end{aligned} \tag{4.26}$$

where

$$\begin{aligned} I[(m^2s)^tJ_{\ell n}] &\equiv \sum_{k=2-t-n}^{\ell}\frac{(\ell+k)!}{k!(\ell-k)!}2^n\left[\frac{1}{2}\frac{(-1)^k}{(mr)^{2n+2k}}\frac{(k+n)!}{k!}(k+t+n-2)!\right] \\ &\quad -(-1)^\ell\sum_{k=0}^{\ell}\frac{(\ell+k)!}{k!(\ell-k)!}2^n\left[\sum_{\alpha=0}^{n}\frac{(k+\alpha)!n!}{k!\alpha!(n-\alpha)!}\frac{K_{k+t+\alpha-1}(2mr)}{(mr)^{k-t+\alpha+1}}\right]. \end{aligned} \tag{4.27}$$

(The $I[(m^2s)^tJ_{\ell n}]$ result from integrating terms of the form $(m^2s)^{t-2}J_{\ell n}$ over s.) Note that the anomaly in $\langle\hat{\Phi}^2\rangle$ generally diverges at any point x^a such that

$r(x^a) = 0$, while for asymptotically flat spaces it vanishes as $r \to \infty$. From the discussion of Section 3.3, this behaviour is to be expected.

4.1.3 Spherical-Reduction Anomaly in Schwarzschild Spacetime

As an example, let us consider a quantum field in Euclidean Schwarzschild space, with line element

$$ds^2 = \left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\eta^2\right) , \tag{4.28}$$

where M is the black-hole mass. For this geometry ${}^4R = 0$, ${}^4R_{\mu\nu} = 0$, and

$$[1 - r^2(\nabla\phi)^2] = \frac{2M}{r} . \tag{4.29}$$

Figure 4.1 shows plots of $K_{\mathrm{div}|\ell=0}(x, x|s)$ for fixed s and various values of $M/\sqrt{s}$. Note that large values of M cause the mode-decomposed subtraction terms to become negative.

The anomaly in $\langle\hat{\Phi}^2\rangle$ for the $\ell = 0$ mode is

$$\Delta\langle\hat{\varphi}^2_{\ell=0}\rangle = \frac{K_0(2mr)}{2\pi} + \frac{M}{3\pi r}\left[\frac{1}{(mr)^2} - \frac{2}{mr}K_1(2mr) - 2K_0(2mr) - mrK_1(2mr)\right] . \tag{4.30}$$

Plots of $\Delta\langle\hat{\varphi}^2_{\ell=0}\rangle$ for various values of mM are shown in Figure 4.2. Note that in the $M \to 0$ limit we recover the exact anomaly in flat space, (3.79).

4.2 The Dimensional-Reduction Anomaly in the Effective Action

In the previous section we calculated the dimensional-reduction anomaly in $\langle\hat{\Phi}^2\rangle$ for a general four-dimensional spherically symmetric space. We now use the same

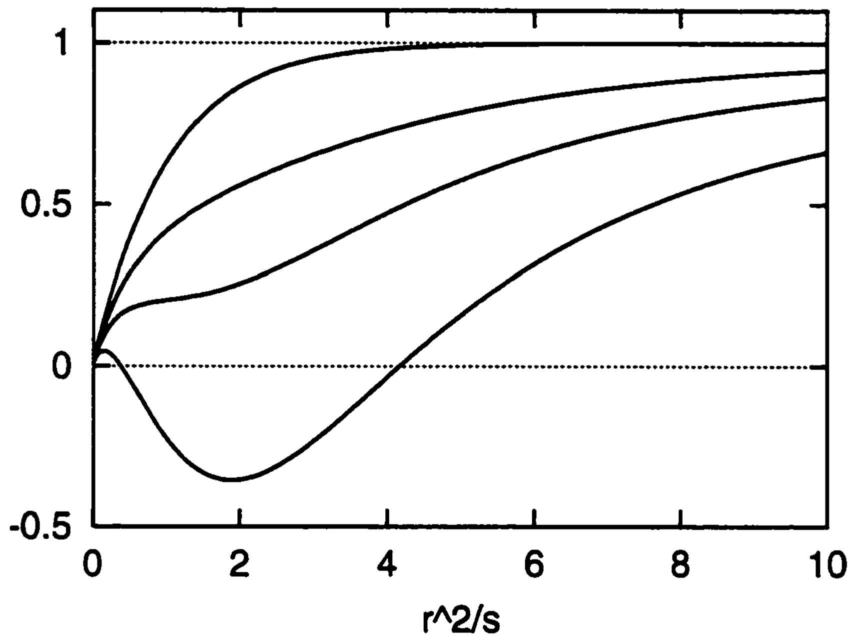

Figure 4.1: $K_{\text{div}|\ell=0}(x,x|s)$ in Schwarzschild space for fixed s and $M/\sqrt{s} = 0, 2, 4, 8$ from top to bottom. The common factor $e^{-m^2 s}/(4\pi s)$ has been removed. The two-dimensional subtraction term $\mathcal{K}_{\ell|\text{div}}$ would be a horizontal line at 1 on this plot.

procedure to determine the anomaly in the effective action, denoted by $\Delta\mathcal{W}_\ell$ in (3.14). Functional differentiation of $\Delta\mathcal{W}_\ell$ with respect to the metric h_{ab} and the dilaton ϕ will then give the corresponding anomalies in the stress tensor and pressure in (3.15), (3.16).

The divergences in the four-dimensional effective action (4.4) come from the first three terms of the heat kernel expansion (4.5):

$$K_{\text{div}}(X,X'|s) = \frac{1}{(4\pi s)^2}\exp\left\{-m^2 s - \frac{\sigma}{2s}\right\}\left[\mathfrak{R}_0^{\Box-\xi\,{}^4\!R} + s\mathfrak{R}_1^{\Box-\xi\,{}^4\!R} + s^2\mathfrak{R}_2^{\Box-\xi\,{}^4\!R}\right]. \tag{4.31}$$

As in the previous section, we split the points X, X' in the angular direction

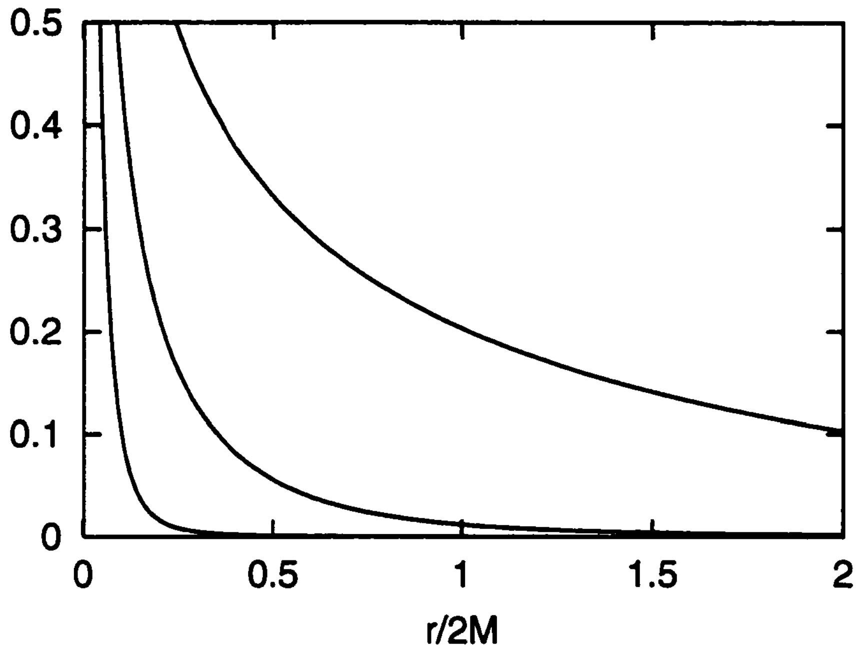

Figure 4.2: $\Delta\langle\hat{\varphi}^2_{\ell=0}\rangle$ in Schwarzschild space for $mM = 0.1, 1, 10$, from top to bottom.

only. Then one can write

$$2\sigma = 2r^2\left[(1-z) + u(1-z)^2 + v(1-z)^3 + \cdots\right], \tag{4.32}$$

$$\Re_n^{\Box-\xi^4R} = \Re_{n(0)}^{\Box-\xi^4R} + \Re_{n(1)}^{\Box-\xi^4R}(1-z) + \Re_{n(2)}^{\Box-\xi^4R}(1-z)^2 + \cdots, \tag{4.33}$$

where $z = \cos\lambda$. From the calculations for the anomaly in $\langle\hat{\Phi}^2\rangle$ we have seen that $u = \frac{1}{6}[1 - r^2(\nabla\phi)^2]$, $\Re_{0(0)}^{\Box-\xi^4R} = 1$, $\Re_{0(1)}^{\Box-\xi^4R} = \frac{1}{6}\,{}^4R_{\theta\theta}$, and $\Re_{1(0)}^{\Box-\xi^4R} = \left(\frac{1}{6} - \xi\right){}^4R$. The other $\Re_{n(k)}^{\Box-\xi^4R}$ and v are found in Appendix B.2. Inserting these expansions into (4.31) and truncating at second order in the curvature, we find

$$\begin{aligned} K_{\text{div}}(X, X'|s) = {} & \frac{1}{(4\pi s)^2}\exp\left\{-m^2 s - \frac{r^2}{2s}(1-z)\right\}\left[1 + s\Re_{1(0)}^{\Box-\xi^4R}\right. \\ & + s^2\Re_{2(0)}^{\Box-\xi^4R} + \left(\Re_{0(1)}^{\Box-\xi^4R} + s\Re_{1(1)}^{\Box-\xi^4R}\right)(1-z) \\ & + \left(\Re_{0(2)}^{\Box-\xi^4R} - \frac{r^2u}{2s} - \frac{r^2u}{2}\Re_{1(0)}^{\Box-\xi^4R}\right)(1-z)^2 \end{aligned}$$

$$+\left(-\frac{r^2u}{2s}\Re_{0(1)}^{\Box-\xi^4R}-\frac{r^2v}{2s}\right)(1-z)^3+\frac{r^4u^2}{8s^2}(1-z)^4\Bigg]. \quad (4.34)$$

The decomposition of the heat kernel subtraction terms (4.34) is done in the same manner as in the previous section. Employing the definition (4.15) of the spherical decomposition and using the functions $J_{\ell n}$ of (4.22), (4.23), we obtain

$$\begin{aligned}
K_{\mathrm{div}|\ell}(x,x|s) &= \lim_{x'\to x} 2\pi r^2 \int_{-1}^{1} d(\cos\lambda)\, P_\ell(\cos\lambda)\, K_{\mathrm{div}}(X,X'|s) \\
&= \frac{\mathrm{e}^{-m^2s}}{4\pi s}\Bigg[\left(1+s\Re_{1(0)}^{\Box-\xi^4R}+s^2\Re_{2(0)}^{\Box-\xi^4R}\right)J_{\ell 0} \\
&\quad + \left(\Re_{0(1)}^{\Box-\xi^4R}+s\Re_{1(1)}^{\Box-\xi^4R}\right)J_{\ell 1} \\
&\quad + \left(\Re_{0(2)}^{\Box-\xi^4R}-\frac{r^2u}{2s}-\frac{r^2u}{2}\Re_{1(0)}^{\Box-\xi^4R}\right)J_{\ell 2} \\
&\quad + \left(-\frac{r^2v}{2s}-\frac{r^2u}{2s}\Re_{0(1)}^{\Box-\xi^4R}\right)J_{\ell 3}+\frac{r^4u^2}{8s^2}J_{\ell 4}\Bigg]. \quad (4.35)
\end{aligned}$$

Meanwhile, the divergences in the effective action for the two-dimensional theory with the potential V_ℓ of (4.2) arise from the first two terms of (4.10):

$$\mathcal{K}_{\ell|\mathrm{div}}(x,x|s)=\frac{\mathrm{e}^{-m^2s}}{4\pi s}\left[1+s\left(\frac{1}{6}R-V_\ell\right)\right]. \quad (4.36)$$

The anomaly in the effective action is found by integrating the difference of (4.35), (4.36) over s as in (4.4):

$$\begin{aligned}
\Delta\mathcal{W}_\ell &= -\frac{1}{2}\int d^2x\sqrt{h}\int_0^\infty \frac{ds}{s}\left[\mathcal{K}_{\ell|\mathrm{div}}(x,x|s)-K_{\mathrm{div}|\ell}(x,x|s)\right] \\
&= \frac{m^2}{4\pi}\int d^2x\sqrt{h}\Bigg[I[J_{\ell 0}]+\frac{1}{m^2}\Re_{1(0)}^{\Box-\xi^4R}I[m^2sJ_{\ell 0}]+\frac{1}{m^4}\Re_{2(0)}^{\Box-\xi^4R}I[m^4s^2J_{\ell 0}] \\
&\quad +\Re_{0(1)}^{\Box-\xi^4R}I[J_{\ell 1}]+\frac{1}{m^2}\Re_{1(1)}^{\Box-\xi^4R}I[m^2sJ_{\ell 1}]+(\Re_{0(2)}^{\Box-\xi^4R}-\frac{r^2u}{2}\Re_{1(0)}^{\Box-\xi^4R})I[J_{\ell 2}] \\
&\quad -\frac{(mr)^2u}{2}I[\frac{1}{m^2s}J_{\ell 2}]-\frac{(mr)^2}{2}(u\Re_{0(1)}^{\Box-\xi^4R}+v)I[\frac{1}{m^2s}J_{\ell 3}] \\
&\quad +\frac{(mr)^4u^2}{8}I[\frac{1}{m^4s^2}J_{\ell 4}]\Bigg]. \quad (4.37)
\end{aligned}$$

The $I[(m^2s)^tJ_{\ell n}]$ are given by (4.27). Using (4.37) and the values of u, v, and the $\Re_{n(k)}^{\Box-\xi^4R}$ given in Appendix B.2, one can compute the anomalous contribution to the stress tensor. This is done in the next section.

4.3 The Dimensional-Reduction Anomaly in the Stress Tensor

The stress-tensor induced by a given effective action $\mathcal{W}$ in a space with metric h_{ab} may be calculated using

$$\langle \hat{\mathcal{T}}_{ab} \rangle = \frac{2}{\sqrt{h}} \frac{\delta \mathcal{W}}{\delta h^{ab}} . \tag{4.38}$$

In the previous section we found the anomaly in the effective action for the decomposition of a general four-dimensional spherically symmetric space, (4.37). We now calculate the anomalous contribution to the stress tensor due to this action. We do this by substituting the values of u, v, and the $\Re^{\Box-\xi^4 R}_{n(k)}$ given in Appendix B.2 into (4.37), using integration by parts to cast the action into a form suitable for performing the variations, and then calculating $\Delta\langle \hat{\mathcal{T}}^{(\ell)}_{ab} \rangle$ using (4.38). The anomaly in the pressure, $\Delta\langle \hat{\mathcal{P}}^{(\ell)} \rangle$, is most easily computed once $\Delta\langle \hat{\mathcal{T}}^{(\ell)}_{ab} \rangle$ is known by using the dimensionally reduced conservation equation (2.97), which implies

$$\langle \hat{\mathcal{P}}^{(\ell)} \rangle = -\frac{1}{2(\nabla\phi)^2} \phi^a \nabla^b \langle \hat{\mathcal{T}}^{(\ell)}_{ab} \rangle . \tag{4.39}$$

This procedure is straightforward but tedious, so we skip most of the details.

There are only a limited number of invariants dependent on h^{ab} (such as $\Delta_h \phi$, $(\nabla\phi)^2$, and $R[h]$) which enter $\Delta \mathcal{W}_\ell$. For this reason it will be convenient to define the following functions which result from the variation of these objects. Assuming S and N are scalar functions, and that $\delta N = 0$ (but not δS) under a variation δh^{ab}, one can show that

$$\int d^2x \sqrt{h}\, \delta([\nabla\phi]^2) S = \int d^2x \sqrt{h}\, \delta h^{ab}\, F^0_{ab}(S) , \tag{4.40}$$

$$\int d^2x \sqrt{h}\, \delta([\nabla\phi]^2) S = \int d^2x \sqrt{h}\, \delta h^{ab}\, F^0_{ab}(S) , \tag{4.41}$$

$$\int d^2x \sqrt{h}\, \delta(\Delta_h N) S = \int d^2x \sqrt{h}\, \delta h^{ab}\, F^1_{ab}(N, S) , \tag{4.42}$$

$$\int d^2x \sqrt{h}\, \delta(R) S = \int d^2x \sqrt{h}\, \delta h^{ab}\, F^2_{ab}(R, S) ,$$

where

$$
\begin{aligned}
F^0_{ab}(S) &= \phi_a\phi_b S\,, && (4.43)\\
F^1_{ab}(N,S) &= \frac{1}{2}\left[-N_a\partial_b - N_b\partial_a + h_{ab}\Delta_h N + h_{ab}\nabla N\cdot\nabla\right] S\,, && (4.44)\\
F^2_{ab}(R,S) &= \frac{1}{2}\left[-\nabla_a\nabla_b + h_{ab}\Delta_h + \frac{1}{2}h_{ab}R\right] S\,. && (4.45)
\end{aligned}
$$

The other variations which we shall require can be written in terms of these functions. For example,

$$
\int d^2x\sqrt{h}\,\delta({}^4R)S = \int d^2x\sqrt{h}\,\delta h^{ab}\left[F^2_{ab}(R,S) + 4F^1_{ab}(\phi,S) - 6F^0_{ab}(S)\right]\,, \quad (4.46)
$$

$$
\begin{aligned}
\int d^2x\sqrt{h}\,\delta(\Delta_h{}^4R)S &= \int d^2x\sqrt{h}\,\delta h^{ab}\left[F^2_{ab}(R,\Delta_h S) + F^1_{ab}({}^4R,S)\right.\\
&\qquad \left. + 4F^1_{ab}(\phi,\Delta_h S) - 6F^0_{ab}(\Delta_h S)\right]\,. && (4.47)
\end{aligned}
$$

We also have

$$
\int d^2x\delta(\sqrt{h})S = \int d^2x\sqrt{h}\,\delta h^{ab}\left[-\frac{1}{2}h_{ab}S\right]\,. \quad (4.48)
$$

For convenience, we divide our effective action (4.37) up into 9 separate parts,

$$
\Delta W_\ell = \frac{m^2}{4\pi}\sum_{i=\mathrm{I}}^{\mathrm{IX}} W_i\,. \quad (4.49)
$$

These parts and their associated stress tensors for the case $V = \xi\,{}^4R$ are listed below. The second form shown for each W_i is obtained using integration by parts and is used for the calculation of the $\langle T_{ab}\rangle_i$. Note that each W_i is constructed using a different $I[(m^2s)^t J_{\ell n}]$. For convenience, we write the $I[(m^2s)^t J_{\ell n}]$ as simply I on all but the first line of each result.

$$
W_{\mathrm{I}} = \int d^2x\sqrt{h}\,I[J_{\ell 0}]\,, \quad (4.50)
$$

$$
\langle T_{ab}\rangle_{\mathrm{I}} = -h_{ab}I\,, \quad (4.51)
$$

$$
\begin{aligned}
W_{\text{II}} &= \int d^2x\sqrt{h}\,\frac{1}{m^2}\,\Re_{1(0)}^{\Box-\xi\,{}^4R}\,I[m^2 s J_{\ell 0}] \\
&= \frac{(\frac{1}{6}-\xi)}{m^2}\int d^2x\sqrt{h}\;{}^4R\,I\,,
\end{aligned}
\tag{4.52}
$$

$$
\langle T_{ab}\rangle_{\text{II}} = \frac{2\,(\frac{1}{6}-\xi)}{m^2}\left[-\frac{1}{2}h_{ab}\;{}^4R\,I - 6F^0_{ab}(I) + 4F^1_{ab}(\phi, I) + F^2_{ab}(R, I)\right]\,, \tag{4.53}
$$

$$
\begin{aligned}
W_{\text{III}} &= \int d^2x\sqrt{h}\,\frac{1}{m^4}\Re_{2(0)}^{\Box-\xi\,{}^4R} I[m^4 s^2 J_{\ell 0}] \\
&= \frac{(\frac{1}{6}-\xi)^2}{2m^4}\int d^2x\sqrt{h}\,({}^4R)^2 I \\
&\quad + \frac{(\frac{1}{6}-\xi)}{6m^4}\int d^2x\sqrt{h}\;\left[\Delta_h{}^4R\,I + 2\,{}^4R\,\Delta_h\phi\,I + 2\,{}^4R\nabla\phi\cdot\nabla I\right] \\
&\quad + \frac{1}{180m^4}\int d^2x\sqrt{h}\;\left[\left(\Delta_h{}^4R + \frac{1}{2}R^2 + 10(\Delta_h\phi)^2 + \frac{2}{r^4}\right) I\right. \\
&\qquad + \left(2R + 12\Delta_h\phi - 8(\nabla\phi)^2 + \frac{4}{r^2}\right)\nabla\phi\cdot\nabla I \\
&\qquad \left. + 2(\nabla\phi)^2\Delta_h I\right]\,,
\end{aligned}
\tag{4.54}
$$

$$
\begin{aligned}
\langle T_{ab}\rangle_{\text{III}} &= \frac{(\frac{1}{6}-\xi)^2}{m^4}\left[-\frac{1}{2}h_{ab}({}^4R)^2 I - 12F^0_{ab}({}^4RI) + 8F^1_{ab}(\phi,\,{}^4RI)\right. \\
&\qquad \left. + 2F^2_{ab}(R,\,{}^4RI)\right] \\
&\quad + \frac{(\frac{1}{6}-\xi)}{3m^4}\left[-\frac{1}{2}h_{ab}\left(\Delta_h{}^4R\,I + 2\,{}^4R\,\Delta_h\phi\,I + 2\,{}^4R\nabla\phi\cdot\nabla I\right)\right. \\
&\qquad + 2\,{}^4R\,\phi_{(a}I_{b)} - 6F^0_{ab}(2\Delta_h\phi I + 2\nabla\phi\cdot\nabla I + \Delta_h I) \\
&\qquad + F^1_{ab}(\phi, 2\,{}^4RI + 8\Delta_h\phi I + 8\nabla\phi\cdot\nabla I + 4\Delta_h I) \\
&\qquad \left. + F^1_{ab}({}^4R, I) + F^2_{ab}(R, 2\Delta_h\phi I + 2\nabla\phi\cdot\nabla I + \Delta_h I)\right] \\
&\quad + \frac{1}{90m^4}\left[-\frac{1}{2}h_{ab}\left((\Delta_h{}^4R + \frac{1}{2}R^2 + 10(\Delta_h\phi)^2 + \frac{2}{r^4})I\right.\right.
\end{aligned}
$$

$$
\begin{aligned}
&+ (2R + 12\Delta_h\phi - 8(\nabla\phi)^2 + \frac{4}{r^2})\, \nabla\phi\cdot\nabla I + 2(\nabla\phi)^2\Delta_h I\Big) \\
&+ \left(2R + 12\Delta_h\phi - 8(\nabla\phi)^2 + \frac{4}{r^2}\right)\, \phi_{(a}I_{b)} \\
&+ F^0_{ab}(-8\nabla\phi\cdot\nabla I - 4\Delta_h I) \\
&+ F^1_{ab}(\phi, 20\Delta_h\phi I + 12\nabla\phi\cdot\nabla I + 4\Delta_h I) \\
&+ F^1_{ab}(I, 2(\nabla\phi)^2) + F^1_{ab}({}^4R, I) \\
&+ F^2_{ab}(R, RI + 2\nabla\phi\cdot\nabla I + \Delta_h I)\Big]\,, \qquad (4.55)
\end{aligned}
$$

$$
\begin{aligned}
W_{\rm IV} &= \int d^2x\sqrt{h}\,\Re^{\Box-\xi\,{}^4R}_{0(1)} I[J_{\ell 1}] \\
&= \int d^2x\sqrt{h}\, \frac{1}{6}(1 + r^2 B)\, I\,, \qquad (4.56)
\end{aligned}
$$

$$
\langle T_{ab}\rangle_{\rm IV} = \frac{1}{3}\left[-\frac{1}{2}h_{ab}\,(1 + r^2 B)\, I - 2F^0_{ab}(r^2 I) + F^1_{ab}(\phi, r^2 I)\right]\,, \qquad (4.57)
$$

$$
\begin{aligned}
W_{\rm V} &= \int d^2x\sqrt{h}\,\frac{1}{m^2}\Re^{\Box-\xi\,{}^4R}_{1(1)} I[m^2 s J_{\ell 1}] \\
&= \frac{(\frac{1}{6}-\xi)}{6m^2}\int d^2x\sqrt{h}\,\left[\,{}^4R(I - r^2\nabla\phi\cdot\nabla I)\,\right] \\
&+ \frac{1}{180m^2}\int d^2x\sqrt{h}\,\Big[\frac{2}{r^2}I + (-R - 2\Delta_h\phi + 4(\nabla\phi)^2)r^2\nabla\phi\cdot\nabla I \\
&\quad + (3\Delta_h\phi - 2\nabla\phi^2)r^2\Delta_h I\Big]\,, \qquad (4.58)
\end{aligned}
$$

$$
\begin{aligned}
\langle T_{ab}\rangle_{\rm V} = \frac{(\frac{1}{6}-\xi)}{3m^2}\Big[&-\frac{1}{2}h_{ab}\,{}^4R(I - r^2\nabla\phi\cdot\nabla I) - {}^4Rr^2\phi_{(a}I_{b)} \\
&- 6F^0_{ab}(I - r^2\nabla\phi\cdot\nabla I) + 4F^1_{ab}(\phi, I - r^2\nabla\phi\cdot\nabla I) \\
&+ F^2_{ab}(R, I - r^2\nabla\phi\cdot\nabla I)\Big]
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{90m^2}\left[-\frac{1}{2}h_{ab}\left(\frac{2}{r^2}I+(-R-2\Delta_h\phi+4(\nabla\phi)^2)r^2\nabla\phi\cdot\nabla I\right.\right.\\
&\qquad\left.+(3\Delta_h\phi-2\nabla\phi^2)r^2\Delta_h I\right)\\
&\qquad+(-R-2\Delta_h\phi+4(\nabla\phi)^2)r^2\phi_{(a}I_{b)}\\
&\qquad+F^0_{ab}(4r^2\nabla\phi\cdot\nabla I-2r^2\Delta_h I)\\
&\qquad+F^1_{ab}(\phi,-2r^2\nabla\phi\cdot\nabla I+3r^2\Delta_h I)\\
&\qquad+F^1_{ab}(I,(3\Delta_h\phi-2\nabla\phi^2)r^2)\\
&\qquad\left.+F^2_{ab}(R,-r^2\nabla\phi\cdot\nabla I)\right]\,,
\end{aligned}
\tag{4.59}
$$

$$
\begin{aligned}
W_{\mathrm{VI}} &= \int d^2x\sqrt{h}\left(\Re^{\Box-\xi{}^4R}_{0(2)}-\frac{r^2u}{2}\Re^{\Box-\xi{}^4R}_{1(0)}\right)I[J_{\ell 2}]\\
&= -\frac{(\frac{1}{6}-\xi)}{12}\int d^2x\sqrt{h}\left[(1-r^2(\nabla\phi)^2)\,{}^4R\,r^2I\right]\\
&\quad+\frac{1}{360}\int d^2x\sqrt{h}\left[19I+(20\Delta_h\phi-40\nabla\phi^2)r^2I\right.\\
&\qquad+(R\nabla\phi^2-3(\Delta_h\phi)^2+10\Delta_h\phi\nabla\phi^2-8\nabla\phi^4)r^4I\\
&\qquad\left.+(-8\Delta_h\phi+6\nabla\phi^2)r^4\nabla\phi\cdot\nabla I+2\nabla\phi^2r^4\Delta_h I\right]\,,
\end{aligned}
\tag{4.60}
$$

$$
\begin{aligned}
\langle T_{ab}\rangle_{\mathrm{VI}} &= -\frac{(\frac{1}{6}-\xi)}{6}\left[-\frac{1}{2}h_{ab}(1-r^2\nabla\phi^2)\,{}^4R\,r^2I\right.\\
&\qquad+F^0_{ab}(-{}^4Rr^4I-6(1-r^2\nabla\phi^2)r^2I)\\
&\qquad\left.+4F^1_{ab}(\phi,(1-r^2\nabla\phi^2)r^2I)+F^2_{ab}(R,(1-r^2\nabla\phi^2)r^2I)\right]\\
&\quad+\frac{1}{180}\left[-\frac{1}{2}h_{ab}\left(19I+(20\Delta_h\phi-40\nabla\phi^2)r^2I\right.\right.\\
&\qquad+(R\nabla\phi^2-3(\Delta_h\phi)^2+10\Delta_h\phi\nabla\phi^2-8\nabla\phi^4)r^4I\\
&\qquad\left.+(-8\Delta_h\phi+6\nabla\phi^2)r^4\nabla\phi\cdot\nabla I+2\nabla\phi^2r^4\Delta_h I\right)\\
&\qquad+(-8\Delta_h\phi+6\nabla\phi^2)r^4\phi_{(a}I_{b)}\\
&\qquad+F^0_{ab}(-40r^2I+[R+10\Delta_h\phi-16(\nabla\phi)^2]r^4I+6r^4\nabla\phi\cdot\nabla I
\end{aligned}
$$

$$
\begin{aligned}
&+ 2r^4\Delta_h I) \\
&+ F^1_{ab}(\phi, 20r^2 I + [-6\Delta_h\phi + 10(\nabla\phi)^2]r^4 I - 8r^4\nabla\phi\cdot\nabla I) \\
&+ F^1_{ab}(I, 2(\nabla\phi)^2 r^4) \\
&+ F^2_{ab}(R, (\nabla\phi)^2 r^4 I)\Big] \,, \qquad (4.61)
\end{aligned}
$$

$$
\begin{aligned}
W_{\text{VII}} &= -\int d^2x\sqrt{h}\left[\frac{(mr)^2 u}{2} I[\frac{1}{m^2 s}J_{\ell 2}]\right] \\
&= -\frac{m^2}{12}\int d^2x\sqrt{h}\,\left[(1 - r^2\nabla\phi^2)\, r^2 I\right] \,, \qquad (4.62)
\end{aligned}
$$

$$
\langle T_{ab}\rangle_{\text{VII}} = -\frac{m^2}{6}\left[-\frac{1}{2}h_{ab}\,(1 - r^2\nabla\phi^2) - r^2\phi_a\phi_b\right] r^2 I \,, \qquad (4.63)
$$

$$
\begin{aligned}
W_{\text{VIII}} &= -\int d^2x\sqrt{h}\left[\frac{(mr)^2}{2}(u\Re^{\Box-\xi^4 R}_{0(1)} + v) I[\frac{1}{m^2 s}J_{\ell 3}]\right] \\
&= -\frac{m^2}{360}\int d^2x\sqrt{h}\,\Big[13r^2 I + (5\Delta_h\phi - 25\nabla\phi^2)r^4 I \\
&\qquad + (-2\Delta_h\phi\nabla\phi^2)r^6 I + 3r^6\nabla\phi^2\nabla\phi\cdot\nabla I\Big] \,, \qquad (4.64)
\end{aligned}
$$

$$
\begin{aligned}
\langle T_{ab}\rangle_{\text{VIII}} &= \frac{m^2}{360}\Big[h_{ab}\,\big(13r^2 I + (5\Delta_h\phi - 25\nabla\phi^2)r^4 I \\
&\qquad + (-2\Delta_h\phi\nabla\phi^2)r^6 I + 3r^6\nabla\phi^2\nabla\phi\cdot\nabla I\big) \\
&\qquad - 6\nabla\phi^2 r^6\phi_{(a}I_{b)} \\
&\qquad + F^0_{ab}(50r^4 I + 4\Delta_h\phi r^6 I - 6r^6\nabla\phi\cdot\nabla I) \\
&\qquad + F^1_{ab}(\phi, -10r^4 I + 4\nabla\phi^2 r^6 I)\Big] \,, \qquad (4.65)
\end{aligned}
$$

$$\begin{aligned} W_{\text{IX}} &= \int d^2x\sqrt{h}\,\frac{(mr)^4u^2}{8}I[\frac{1}{m^4s^2}J_{\ell 4}] \\ &= \frac{m^4}{288}\int d^2x\sqrt{h}\,\left[(1-r^2\nabla\phi^2)^2r^4I\right]\,, \end{aligned} \tag{4.66}$$

$$\langle T_{ab}\rangle_{\text{IX}} = -\frac{m^4}{288}\left[h_{ab}\,(1-r^2\nabla\phi^2)+4r^2\phi_a\phi_b\right](1-r^2\nabla\phi^2)r^4I\,. \tag{4.67}$$

The total anomalous stress tensor in the x^a sector is

$$\Delta\langle\hat{T}^{(\ell)}_{ab}\rangle = \frac{m^2}{4\pi}\sum_{i=\text{I}}^{\text{IX}}\langle T_{ab}\rangle_i\,. \tag{4.68}$$

The anomalous pressure is most easily calculated for a given geometry using (4.39). In the next section, both $\Delta\langle\hat{T}^{(\ell)}_{ab}\rangle$ and $\Delta\langle\hat{\mathcal{P}}^{(\ell)}\rangle$ are computed explicitly for the $\ell=0$ mode in the Schwarzschild geometry.

4.4 Anomalous Stress Tensor in Schwarzschild Space

In this section we calculate the anomalous contribution to the stress tensor for the $\ell=0$ mode in the Schwarzschild geometry. This example is of particular significance to attempts to calculate the stress tensor of four-dimensional black holes using dimensionally reduced dilaton-gravity theories [32–39], discussed briefly in Section 2.2.7.

Using (4.68) and expressions (4.50)–(4.67), we can calculate the dimensional-reduction anomaly in a spacetime with given metric. In particular, for the $\ell=0$ mode in the Euclidean Schwarzschild geometry (4.28), we find

$$\begin{aligned} \Delta\langle\hat{T}^{(0)\,t}{}_{t}\rangle &= \frac{1}{180\pi m^2r^6}\Big[-28rM+54M^2+\big(-15m^2r^4-30m^4r^6+56m^2r^3M \\ &\quad +82m^4r^5M-108m^2r^2M^2-96m^4r^4M^2\big)\,K_0(2mr) \\ &\quad +\big(56mr^2M+110m^3r^4M+20m^5r^6M-108mrM^2 \end{aligned}$$

$$
\begin{aligned}
&\quad - 162m^3r^3M^2 - 30m^5r^5M^2\big) K_1(2mr)\big] \\
&+ \frac{(\xi - \frac{1}{6})}{3\pi m^2 r^6}\big[20rM - 45M^2 + \big(3m^2r^4 + 6m^4r^6 - 49m^2r^3M \\
&\quad - 26m^4r^5M + 90m^2r^2M^2 + 30m^4r^4M^2\big) K_0(2mr) \\
&\quad + \big(9m^3r^5 - 40mr^2M - 55m^3r^4M - 4m^5r^6M \\
&\quad + 90mrM^2 + 75m^3r^3M^2 + 8m^5r^5M^2\big) K_1(2mr)\big], \qquad (4.69)
\end{aligned}
$$

$$
\begin{aligned}
\Delta\langle \hat{\mathcal{T}}^{(0)}{}_r{}^r\rangle &= \frac{1}{180\pi m^2 r^6}\big[20rM - 42M^2 + \big(-15m^2r^4 - 40m^2r^3M \\
&\quad - 38m^4r^5M + 84m^2r^2M^2 + 24m^4r^4M^2\big) K_0(2mr) \\
&\quad + \big(30m^3r^5 - 40mr^2M - 58m^3r^4M + 84mrM^2 \\
&\quad + 54m^3r^3M^2 + 10m^5r^5M^2\big) K_1(2mr)\big] \\
&+ \frac{(\xi - \frac{1}{6})}{3\pi m^2 r^6}\big[-10rM + 15M^2 + \big(-6m^2r^4 + 29m^2r^3M \\
&\quad + 4m^4r^5M - 30m^2r^2M^2 - 6m^4r^4M^2\big) K_0(2mr) \\
&\quad + \big(-6m^3r^5 + 20mr^2M + 23m^3r^4M - 30mrM^2 \\
&\quad - 21m^3r^3M^2\big) K_1(2mr)\big], \qquad (4.70)
\end{aligned}
$$

$$
\begin{aligned}
\Delta\langle \hat{\mathcal{P}}^{(0)}\rangle &= \frac{1}{180\pi m^2 r^6}\big[-50rM + 150M^2 + \big(15m^2r^4 - 30m^4r^6 \\
&\quad + 100m^2r^3M + 92m^4r^5M - 300m^2r^2M^2 \\
&\quad - 108m^4r^4M^2 - 10m^6r^6M^2\big) K_0(2mr) \\
&\quad + \big(-15m^3r^5 + 100mr^2M + 142m^3r^4M + 38m^5r^6M \\
&\quad - 300mrM^2 - 246m^3r^3M^2 - 44m^5r^5M^2\big) K_1(2mr)\big] \\
&+ \frac{(\xi - \frac{1}{6})}{3\pi m^2 r^6}\big[25rM - 60M^2 + \big(6m^2r^4 + 6m^4r^6 - 68m^2r^3M \\
&\quad - 28m^4r^5M + 120m^2r^2M^2 + 36m^4r^4M^2\big) K_0(2mr) \\
&\quad + \big(12m^3r^5 - 50mr^2M - 71m^3r^4M - 4m^5r^6M \\
&\quad + 120mrM^2 + 96m^3r^3M^2 \\
&\quad + 8m^5r^5M^2\big) K_1(2mr)\big]. \qquad (4.71)
\end{aligned}
$$

While these stress tensors were calculated for Euclidean Schwarzschild space, they are also valid for Lorentzian signature.[3] It is easily verified that the anomalous stress tensor is finite on both the past and future event horizons. In the $M \to 0$ limit we recover the exact results obtained earlier for the spherical decomposition of flat space, (3.82)–(3.84). Other limits of interest are $m \to 0$ and $mr \gg 1$. For massless fields we find

$$\begin{aligned}
\Delta\langle \hat{T}^{(0)}{}_t{}^t \rangle &= \frac{1}{60\pi r^2}\left[5\,(\gamma+\ln\mu r)+9\frac{M}{r}-9\frac{M^2}{r^2}\right] \\
&\quad + \frac{(\xi-\frac{1}{6})}{4\pi r^2}\left[\left(-4+12\frac{M}{r}\right)(\gamma+\ln\mu r)+6-10\frac{M}{r}-10\frac{M^2}{r^2}\right],
\end{aligned} \tag{4.72}$$

$$\begin{aligned}
\Delta\langle \hat{T}^{(0)}{}_r{}^r \rangle &= \frac{1}{60\pi r^2}\left[5\,(\gamma+\ln\mu r)+5-3\frac{M}{r}-5\frac{M^2}{r^2}\right] \\
&\quad + \frac{(\xi-\frac{1}{6})}{4\pi r^2}\left[\left(8-12\frac{M}{r}\right)(\gamma+\ln\mu r)-4+2\frac{M}{r}+6\frac{M^2}{r^2}\right],
\end{aligned} \tag{4.73}$$

$$\begin{aligned}
\Delta\langle \hat{\mathcal{P}}^{(0)} \rangle &= \frac{1}{120\pi r^2}\left[-10\,(\gamma+\ln\mu r)-5+14\frac{M}{r}+18\frac{M^2}{r^2}\right] \\
&\quad + \frac{(\xi-\frac{1}{6})}{8\pi r^2}\left[\left(-16+48\frac{M}{r}\right)(\gamma+\ln\mu r)+16-28\frac{M}{r}-32\frac{M^2}{r^2}\right],
\end{aligned} \tag{4.74}$$

where the μ appearing in the logarithms is an arbitrary parameter with units of mass. In the large mr limit, the Bessel functions in (4.69)–(4.71) are exponentially damped and may be dropped, leaving

$$\Delta\langle \hat{T}^{(0)}{}_t{}^t \rangle = \frac{1}{180\pi m^2 r^4}\left[-28\frac{M}{r}+54\frac{M^2}{r^2}+60\left(\xi-\frac{1}{6}\right)(20\frac{M}{r}-45\frac{M^2}{r^2})\right], \tag{4.75}$$

$$\Delta\langle \hat{T}^{(0)}{}_r{}^r \rangle = \frac{1}{180\pi m^2 r^4}\left[20\frac{M}{r}-42\frac{M^2}{r^2}+60\left(\xi-\frac{1}{6}\right)(-10\frac{M}{r}+15\frac{M^2}{r^2})\right], \tag{4.76}$$

[3]One converts from Euclidean to Lorentzian signature by changing the sign of $\langle \hat{T}_{tt} \rangle$ and $\langle \hat{T}^{tt} \rangle$, but leaving $\langle \hat{T}_t{}^t \rangle$ and the other diagonal components of the stress tensor unchanged.

$$\Delta\langle\hat{\mathcal{P}}^{(0)}\rangle = \frac{1}{360\pi m^2 r^4}\left[-100\frac{M}{r} + 300\frac{M^2}{r^2} + 60\left(\xi - \frac{1}{6}\right)\left(50\frac{M}{r} - 120\frac{M^2}{r^2}\right)\right]. \tag{4.77}$$

Plots of (4.75)–(4.77) versus the full expression (4.69)–(4.71) indicate that the asymptotic form is valid for all $r \geq 2M$ for approximately $mM > 3$.

To appreciate the significance of the anomaly, it will be convenient to analyze the massless and massive cases separately.

4.4.1 Massless Field

The calculation of the vacuum expectation value of the stress tensor for a massless scalar field in the Schwarzschild geometry has long been of interest [5–15]. In particular, in recent years numerous attempts have been made to obtain the stress tensor due to the $\ell = 0$ mode using dimensionally reduced models [32–39].

For an analysis of the dimensional reduction, let us consider the stress tensor for a field in the Hartle-Hawking state $|H\rangle$ [82]. This is the state for a black-hole spacetime which is regular on the past and future event horizons and which reduces at infinity to thermal radiation at temperature $T_H = (8\pi M)^{-1}$. It represents a black hole in (unstable) thermal equilibrium with radiation; i.e., it most closely corresponds to the thermal states considered in the previous chapter. As well, it is the state obtained by continuation from the regular Euclidean Schwarzschild geometry [82].

Anderson [11] and Anderson, Hiscock, and Samuel [12] (henceforth AHS) have developed an approximation scheme for $\langle\hat{\Phi}^2\rangle$ and $\langle\hat{T}_\mu{}^\nu\rangle$ for scalar fields in four-dimensional static, spherically symmetric spacetimes. Their approach is to write the Green function as a mode sum, as discussed in Section 2.1.4, where the time dependence is of the form $e^{i\omega t}$ and the angular dependence is in terms of spherical harmonics $Y_{\ell m}(\theta, \eta)$. A WKB (high-frequency) approximation is then used to obtain the radial modes. The mode sum is evaluated for large ω and ℓ, and renormalized using point splitting [80,81]. The resulting stress tensor has been verified numerically for massless scalar fields with arbitrary coupling ξ in the Schwarzschild geometry [12]. For a field at temperature $T = \beta^{-1}$ they find

$$\begin{aligned}
\langle \hat{T}_t{}^t \rangle_{\text{AHS}} &= \frac{1}{2880\pi^2 r^4 f^2}\left[-96\pi^4\frac{r^4}{\beta^4} + 60\frac{M^2}{r^2} - 216\frac{M^3}{r^3} + 198\frac{M^4}{r^4}\right] \\
&+ \frac{(\xi - \frac{1}{6})}{16\pi^2 r^4 f^2}\left[8\pi^2\frac{M^2}{\beta^2} - 4\frac{M^2}{r^2} + 12\frac{M^3}{r^3} - 10\frac{M^4}{r^4}\right],
\end{aligned} \tag{4.78}$$

$$\begin{aligned}
\langle \hat{T}_r{}^r \rangle_{\text{AHS}} &= \frac{1}{2880\pi^2 r^4 f^2}\left[32\pi^4\frac{r^4}{\beta^4} + 4\frac{M^2}{r^2} - 24\frac{M^3}{r^3} + 30\frac{M^4}{r^4}\right] \\
&+ \frac{(\xi - \frac{1}{6})}{48\pi^2 r^4 f^2}\left[-16\pi^2\frac{Mr}{\beta^2} + 24\pi^2\frac{M^2}{\beta^2} + 8\frac{M^2}{r^2} - 24\frac{M^3}{r^3} + 18\frac{M^4}{r^4}\right],
\end{aligned} \tag{4.79}$$

$$\begin{aligned}
\langle \hat{T}_\theta{}^\theta \rangle_{\text{AHS}} &= \frac{1}{2880\pi^2 r^4 f^2}\left[32\pi^4\frac{r^4}{\beta^4} - 8\frac{M^2}{r^2} + 24\frac{M^3}{r^3} - 18\frac{M^4}{r^4}\right] \\
&+ \frac{(\xi - \frac{1}{6})}{48\pi^2 r^4 f^2}\left[8\pi^2\frac{Mr}{\beta^2} - 16\frac{M^2}{r^2} + 54\frac{M^3}{r^3} - 48\frac{M^4}{r^4}\right],
\end{aligned} \tag{4.80}$$

where $f \equiv (1 - 2M/r)$. (In a spherical geometry the effective pressure P is equal to $\langle \hat{T}_\theta{}^\theta \rangle$.) The Hartle-Hawking state corresponds to the choice $\beta = 8\pi M$. Note that (4.78)–(4.80) is the stress tensor due to *all* modes ($\ell = 0, 1, 2, \ldots$) of the field.

Recently, Balbinot, Fabbri, and Nicolini (henceforth BFN) have used the AHS technique to calculate the stress tensor for the $\ell = 0$ mode of the dimensionally reduced massless scalar field with $\xi = 0$ in the Schwarzschild geometry [83]. For the Hartle-Hawking state they find

$$\langle \hat{\mathcal{T}}_t{}^t \rangle_{\text{BFN}} = -\frac{1}{384\pi M^2}\left[1 + 2\frac{M}{r} + 4\frac{M^2}{r^2} + 40\frac{M^3}{r^3}\right], \tag{4.81}$$

$$\langle \hat{\mathcal{T}}_r{}^r \rangle_{\text{BFN}} = \frac{1}{384\pi M^2}\left[1 + 2\frac{M}{r} + 4\frac{M^2}{r^2} - 88\frac{M^3}{r^3}\right], \tag{4.82}$$

$$\langle \hat{\mathcal{P}} \rangle_{\text{BFN}} = \frac{1}{384\pi M^2}\left[144\frac{M^3}{r^3}\right]. \tag{4.83}$$

The reader is reminded of our convention of using Latin letters to denote four-dimensional quantities (e.g., $\langle \hat{T}_t{}^t \rangle$) and calligraphic letters for two-dimensional quantities (e.g., $\langle \hat{\mathcal{T}}_t{}^t \rangle$).

In Sections 3.2 and 3.3 we saw that in flat space the s-mode makes the dominant contribution to the stress tensor near the origin. However, the stress tensor calculated from the dimensionally reduced theory for the $\ell = 0$ mode is larger than the expected result in four dimensions for approximately $r < \beta$, and in this region the contribution of the dimensional-reduction anomaly must be taken into account to obtain the correct stress tensor for a given mode. It is interesting to analyze the situation for the Schwarzschild geometry. Does the $\ell = 0$ stress tensor $\langle \hat{\mathcal{T}}_a{}^b \rangle_{\text{BFN}}$ serve as an approximation for the full stress tensor $\langle \hat{T}_a{}^b \rangle_{\text{AHS}}$ in the immediate vicinity of the black hole, and is the dimensional-reduction anomaly significant?

To test these ideas, we multiply the two-dimensional stress tensor of BFN by $(4\pi r^2)^{-1}$ as in (3.15) and compare to the stress tensor of AHS in four dimensions. The results are displayed in Figures 4.3 and 4.4. Near $r = 2M$, the two-dimensional stress tensor of BFN (from the $\ell = 0$ mode alone) is approximately an order of magnitude larger than and of opposite sign to the expected four-dimensional stress tensor of AHS (due to all modes). Clearly, the $\ell = 0$ contribution alone cannot serve as a useful approximation for the full stress tensor; furthermore, it seems unlikely that $\ell > 0$ modes should cancel this $\ell = 0$ result to a sufficiently high degree to restore agreement with the four-dimensional stress tensor. This indicates a dismal failure of the dimensional reduction.

The resolution of this dilemma may be that we have not taken into account the dimensional-reduction anomaly. The anomalous stress tensor is shown in Figures 4.5 and 4.6; it is of the same order as but of opposite sign to the predictions of BFN near the horizon. If the anomaly is added to the two-dimensional stress tensor and the total compared to $\langle \hat{T}_a{}^b \rangle_{\text{AHS}}$, we see that the agreement between

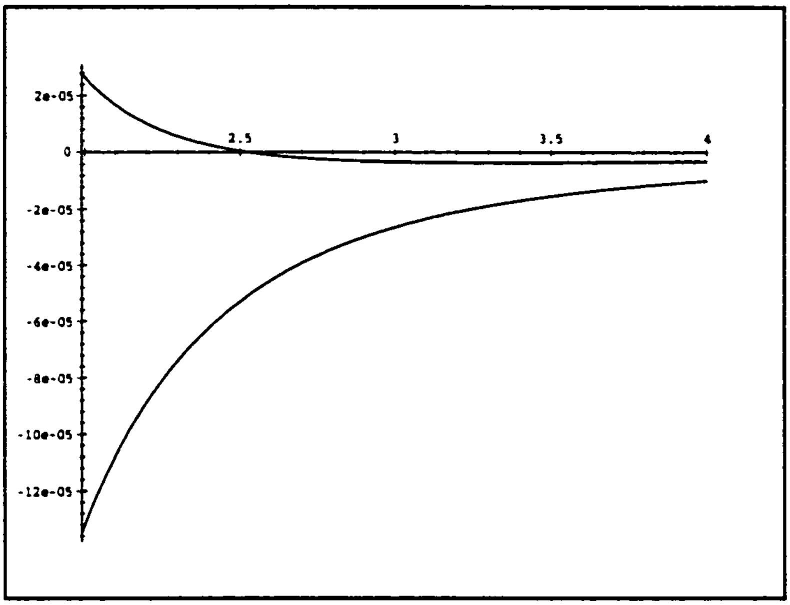

Figure 4.3: Comparison of the two-dimensional stress tensor of BFN for the $\ell = 0$ mode to the four-dimensional result of AHS for a massless, minimally coupled scalar field. The upper curve is $\langle \hat{T}_t^t \rangle_{\text{AHS}}$; the lower curve is $(4\pi r^2)^{-1} \langle \hat{T}_t^t \rangle_{\text{BFN}}$. This plot uses $\mu = M^{-1}$. The horizontal axis is r/M; the vertical axis is $M^4 \times$ [stress tensor].

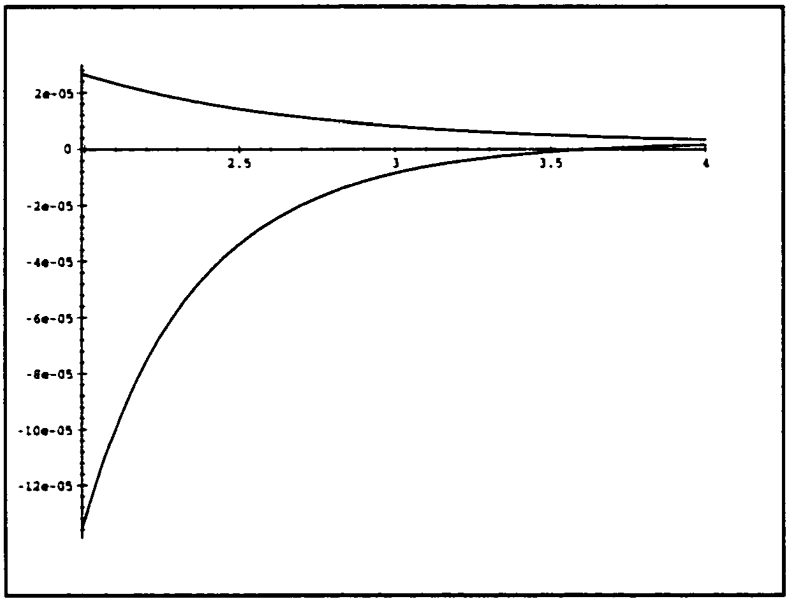

Figure 4.4: Comparison of the two-dimensional stress tensor of BFN for the $\ell = 0$ mode to the four-dimensional result of AHS for a massless, minimally coupled scalar field. The upper curve is $\langle \hat{T}_r^r \rangle_{\mathrm{AHS}}$; the lower curve is $(4\pi r^2)^{-1} \langle \hat{T}_r^r \rangle_{\mathrm{BFN}}$. This plot uses $\mu = M^{-1}$. The horizontal axis is r/M; the vertical axis is $M^4 \times$ [stress tensor].

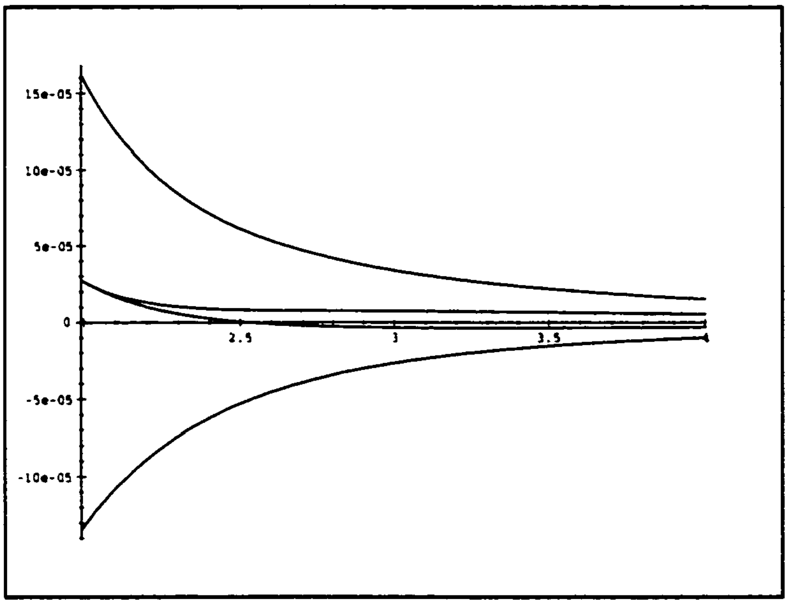

Figure 4.5: Comparison of the two-dimensional stress tensor of BFN for the $\ell = 0$ mode to the four-dimensional result of AHS when the dimensional-reduction anomaly is taken into account. From top to bottom near the horizon, the curves are $(4\pi r^2)^{-1}\Delta\langle\hat{T}_t^t\rangle$, $(4\pi r^2)^{-1}[\langle\hat{T}_t^t\rangle_{\mathrm{BFN}}+\Delta\langle\hat{T}_t^t\rangle]$, $\langle\hat{T}_t^t\rangle_{\mathrm{AHS}}$, $(4\pi r^2)^{-1}\langle\hat{T}_t^t\rangle_{\mathrm{BFN}}$. This plot uses $\mu = M^{-1}$. The horizontal axis is r/M; the vertical axis is $M^4 \times$ [stress tensor].

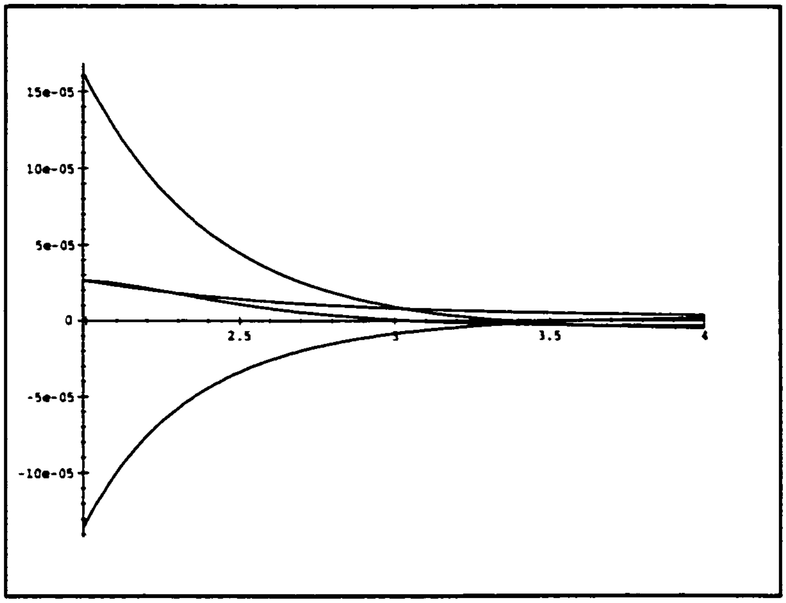

Figure 4.6: Comparison of the two-dimensional stress tensor of BFN for the $\ell = 0$ mode to the four-dimensional result of AHS when the dimensional-reduction anomaly is taken into account. From top to bottom near the horizon, the curves are $(4\pi r^2)^{-1}\Delta\langle\hat{T}_r^r\rangle$, $(4\pi r^2)^{-1}[\langle\hat{T}_r^r\rangle_{\text{BFN}} + \Delta\langle\hat{T}_r^r\rangle]$, $\langle\hat{T}_r^r\rangle_{\text{AHS}}$, $(4\pi r^2)^{-1}\langle\hat{T}_r^r\rangle_{\text{BFN}}$. This plot uses $\mu = M^{-1}$. The horizontal axis is r/M; the vertical axis is $M^4 \times$ [stress tensor].

the two- and four-dimensional stress tensors is much improved near the event horizon. In fact, we find the remarkable result that at $r = 2M$ the anomaly-modified stress tensor agrees *exactly* with the four-dimensional approximation of AHS.

As we move away from the horizon towards $r = 3M$, the match between the anomaly-corrected stress tensor and the AHS result is worse. This is likely due to the fact that the stress tensor of AHS includes the $\ell > 0$ modes, which become more significant at larger radii. It is also possible that the potential barrier, which is largely ignored by the WKB approximations of AHS and BFN, may be of some relevance. (Note that the high frequency modes used in the WKB approximation penetrate the potential barrier with little reflection, and so it has little influence on these approximations.)

Finally, at very large distances from the black hole ($r \approx 10^2 M$ or greater), the anomaly is vanishingly small compared to the two-dimensional stress tensor, and so does not influence the asymptotic behaviour of the dimensionally reduced theory. At the same time, the geometry becomes flat and $\langle \hat{T}_a{}^b \rangle_{\text{BFN}}$ and $\langle \hat{T}_a{}^b \rangle_{\text{AHS}}$ reduce to the stress tensors for a massless scalar field at the Hartle-Hawking temperature in two and four dimensions. The results of Section 3.2 then assure us that in this regime the s-mode contribution to $\langle \hat{T}_a{}^b \rangle_{\text{AHS}}$ is precisely $(4\pi r^2)^{-1}$ times the dimensionally reduced stress tensor $\langle \hat{T}_a{}^b \rangle_{\text{BFN}}$.

This analysis indicates that dilaton-gravity models may indeed be used to model vacuum polarization and particle creation effects for massless fields in higher-dimensional black-hole spacetimes, but only if the dimensional-reduction anomaly is taken into account. Specifically, it appears that in order to obtain the correct expectation values in the vicinity of the event horizon from a reduced theory one must include the contributions of the dimensional-reduction anomaly. It should be noted, however, that while the improvements in the dimensionally reduced stress tensor due to the anomaly are impressive, they are still tentative. For example, the anomaly for the $\ell = 0$ mode cannot be fixed exactly due to the inherent ambiguity in low-frequency structure of the divergences for a quantum field. Furthermore, the anomaly for the massless field contains an arbitrary infrared cutoff parameter μ which has a strong influence on the anomalous stress tensor. The value $\mu \approx 1$ which was used in Figures 4.5 and 4.6 is merely that which gives the closest match between the dimensionally reduced theory and the full four-dimensional stress tensor near the horizon (though the exact agreement at $r = 2M$ occurs for all μ). Finally, the accuracy of the two-dimensional stress tensor of BFN has not been determined; indeed, one would expect from general principles that it should also contain the cutoff parameter μ.

These difficulties could be reduced and our conclusions strengthened if the dimensionally reduced stress tensors for $\ell \geq 0$ could be calculated accurately. In this case, by combining the stress tensors for each ℓ with the corresponding anomaly and summing over even the first few ℓ a much better match with the four-dimensional stress tensor may be obtained in the vicinity of the horizon and the potential barrier. In particular, the effect of the low-frequency ambiguity in the anomaly would be lessened since the sum of the anomaly over all modes is well-defined. In any case, it is clear that the dimensional-reduction anomaly induces very significant alterations in the predictions of dilaton-gravity models for local observables in the vicinity of a black hole, and likely must be included for a proper comparison to physics in higher dimensions.

4.4.2 Large-Mass Case

For $mM \gg 1$, the Compton wavelength m^{-1} of the quantum field is small compared to the length scale M over which the geometry changes. In this case one can use the Schwinger-DeWitt expansion for the heat kernel directly as an approximate solution for the system. Taking the coincidence limit of (4.5) and integrating over s results in an expansion for the effective action in terms of $a_n/m^{2n} \approx (R/m^2)^n \approx (mM)^{-2n}$. For large mM, one can obtain an approximate effective action for the field by simply discarding the divergent terms in the Schwinger-DeWitt expansion and keeping the first finite term (the a_3 term in four dimensions, or the a_2 term in two dimensions). The stress tensor for large field mass can then be calculated from this approximate effective action.

For four-dimensional black-hole spacetimes the Schwinger-DeWitt approximation has been calculated by Frolov and Zelnikov [16]. They found

$$\langle \hat{T}_t{}^t \rangle_{\rm DS} = A \left[\left(-285\frac{M^2}{r^2} + 626\frac{M^3}{r^3} \right) + 14\left(\xi - \frac{1}{6}\right)\left(360\frac{M^2}{r^2} - 792\frac{M^3}{r^3}\right) \right], \tag{4.84}$$

$$\langle \hat{T}_r{}^r \rangle_{\rm DS} = A \left[\left(105\frac{M^2}{r^2} - 154\frac{M^3}{r^3} \right) + 14\left(\xi - \frac{1}{6}\right)\left(-144\frac{M^2}{r^2} + 216\frac{M^3}{r^3}\right) \right], \tag{4.85}$$

$$\langle \hat{T}_\theta{}^\theta \rangle_{\rm DS} = A \left[\left(-315\frac{M^2}{r^2} + 734\frac{M^3}{r^3} \right) + 14\left(\xi - \frac{1}{6}\right)\left(432\frac{M^2}{r^2} - 1008\frac{M^3}{r^3}\right) \right], \tag{4.86}$$

where $A \equiv (10080\pi^2 m^2 r^6)^{-1}$. Anderson, Hiscock, and Samuel [12] have verified numerically that (4.84)–(4.86) is a good approximation to the total stress tensor for the Hartle-Hawking state near the event horizon in four dimensions for $mM > 2$.

In the dimensionally reduced Schwarzschild geometry the Schwinger-DeWitt approximation for the $\ell = 0$ mode is easily calculated, and is precisely the negative of the large mr limit of the anomaly, (4.75)–(4.77). We denote this stress tensor by $\langle \hat{\mathcal{T}}_a{}^b \rangle_{\rm DS}$.

One can compare the Schwinger-DeWitt approximations in two and four dimensions just as we compared the stress tensors for the massless field; the results are shown in Figure 4.7. As in the massless case, we see that the stress tensor predicted by dimensional reduction for one mode alone is much larger than the full stress tensor in four dimensions. Again, the cause (and cure) are clear. In four dimensions $W_{\rm DS}$ is constructed starting with the a_3 Schwinger-DeWitt coefficient, which is $O(R^3)$. In two dimensions the first contributions to $\mathcal{W}_{\rm DS}$ come from the a_2 coefficient, which is $O(R^2)$. To reproduce the four-dimensional results, this $O(R^2)$ contribution must be removed somehow. The dimensional-reduction anomaly is the difference between the mode-decomposed subtraction terms from four dimensions (the a_0, a_1, and a_2 terms) and those from two dimensions (the a_0 and a_1 terms); i.e., it is essentially the negative of the mode-decomposed a_2 from four dimensions. This is why adding the anomaly to the stress tensor in the two-dimensional Schwinger-DeWitt approximation exactly cancels the leading (a_2) contributions. Thus, by taking the dimensional-reduction anomaly into account, the leading mode-by-mode contributions to the stress tensor in four dimensions will be $O(R^3)$, as expected.

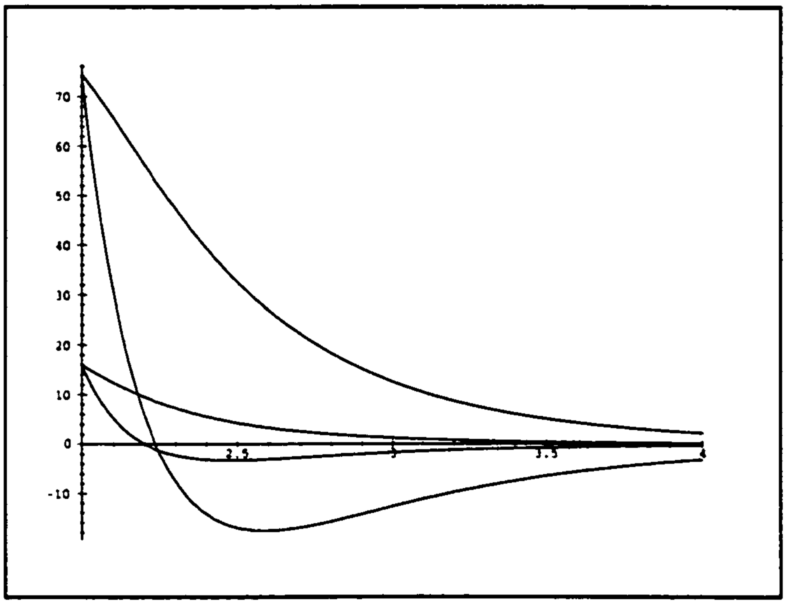

Figure 4.7: Comparison of the stress tensors from the Schwinger-DeWitt approximations for the $\ell = 0$ mode in two dimensions and the total stress tensor in four dimensions. The plot uses $\xi = 0$, $mM = 1$. From top to bottom near the event horizon, the curves are $(4\pi r^2)^{-1}\langle \hat{T}_r^r \rangle_{\rm DS}$, $(4\pi r^2)^{-1}\langle \hat{T}_t^t \rangle_{\rm DS}$, $\langle \hat{T}_r^r \rangle_{\rm DS}$, $\langle \hat{T}_t^t \rangle_{\rm DS}$, The horizontal axis is r/M; the vertical axis is $90\pi^2(8M)^4 \times$ [stress tensor].

From these examples it is clear that the dimensional-reduction anomaly induces very significant corrections to the contribution of modes of fixed angular momentum to the stress tensor when calculated from spherically reduced models. It is thus vital to account for the anomaly if one is to obtain the correct stress tensor for a quantum field in the Schwarzschild spacetime from dimensional reduction. (In fact, it can be shown that the sum of the anomaly over all modes is divergent, so that the total error in the four-dimensional stress tensor from ignoring the anomaly is infinite.) Note, however, that the anomalous stress tensor does not contribute to the Hawking radiation, as the flux terms $\Delta\langle\hat{\mathcal{T}}_{tr}\rangle$ vanish. More generally, we see that the anomaly vanishes at large r in the dimensionally reduced spacetime, and so cannot affect the asymptotic behaviour of the dimensionally reduced system. This indicates that the dimensional-reduction anomaly is not associated with the negative Hawking flux from spherically reduced gravity obtained by some authors [37].

Chapter 5

Static Decompositions

In this chapter we consider the dimensional reduction of a scalar field when the internal space is flat and of dimension $n = 1$ or $n = 2$. If one of the internal-space coordinates is the Euclidean "time," and if the internal space metric is independent of this coordinate, then these spaces are static. As we saw in Chapter 2, static spaces are physically very important for their close connection to quantum field theory at finite temperature. Specifically, a quantum scalar field which is periodic in Euclidean time with period β corresponds to a thermal state at temperature β^{-1}.

In the following sections we calculate the dimensional-reduction anomaly in static spaces. For a finite-temperature field ($0 < \beta < \infty$) we derive the anomaly in $\langle\hat{\Phi}^2\rangle$ for a one-dimensional internal space. For zero-temperature fields ($\beta \to \infty$) we calculate the anomalies in $\langle\hat{\Phi}^2\rangle$ and W for both $n = 1$ and $n = 2$. We begin in Section 5.1 with the anomaly for the decomposition of flat space in Rindler coordinates, where the internal space is compact ($t \in [0, 2\pi]$) with $n = 1$. Sections 5.2 and 5.3 deal with noncompact ($\beta \to \infty$) internal spaces. Finally, in Section 5.4 we derive the dimensional-reduction anomaly for the case $n = 1$ when the time coordinate has arbitrary periodicity β.

5.1 Rindler space

In our consideration of dimensional reductions in spherically symmetric spaces, we began with the simplest example of flat space. This system had the advantage of being exactly solvable, which aided in the physical interpretation of the dimensional-reduction anomaly.

To begin our study of dimensional reductions in static spaces, we shall again begin with the simplest nontrivial example: quantum theory in Rindler space [84]. Because Rindler space is simply flat space in non-inertial coordinates, we can again perform mode-decomposition calculations exactly. We will find results closely analogous to those for spherical decompositions in flat space.

Consider a scalar field obeying the field equation (2.55) in Euclidean Rindler space,

$$ds^2 = z^2 dt^2 + dz^2 + dx^2 + dy^2 \,, \tag{5.1}$$

where $t \in [0, 2\pi]$, $z \in [0, \infty)$, $x, y \in (-\infty, \infty)$, and the points $t = 0$, $t = 2\pi$ are identified. This corresponds to the special case of the line element (2.53) where $x^a = (x, y, z)$, $y^i = t$, $h_{ab} = \delta_{ab}$, $\Omega_{ij} = 1$, and $\rho\, e^{-2\phi} = z$. The Rindler-space line element may be obtained from the standard flat-space line element

$$ds^2 = dT^2 + dX^2 + dY^2 + dZ^2 \tag{5.2}$$

by the coordinate transformation

$$\begin{aligned} T &= z\sin(t)\,, \quad X = x\,, \\ Z &= z\cos(t)\,, \quad Y = y\,, \end{aligned} \tag{5.3}$$

and is clearly just flat space in polar coordinates.

Since Euclidean Rindler space is not periodic in the usual Minkowski time coordinate T of (5.3), an inertial observer will see a zero-temperature state. However, a line of fixed Rindler coordinates (x, y, z) is periodic in Rindler time with period $2\pi z$; hence a Rindler observer at fixed z will find the quantum field to be in a thermal state with temperature $(2\pi z)^{-1}$. This is a statement of the well-known *Unruh effect*: for two systems in relative accelerated motion, the vacuum of one system appears relative to the other as a state with thermally distributed particles at a temperature which is proportional to the acceleration [67, 85].

Because the line element (5.1) is static, a quantum theory on this space may be dimensionally reduced by decomposing the field in terms of Rindler "time" modes $\cos(k[t-t'])/\sqrt{2\pi}$. We wish to calculate the anomaly associated with this dimensional reduction.

We assume that the potential V is t-independent and vanishes in the region of interest. Hence, the Green function for a given state in four dimensions is renormalized by subtracting the Euclidean vacuum Green function (3.63), where

$$2\sigma = (x-x')^2 + (y-y')^2 + (z-z')^2 + 2zz'\left[1-\cos(t-t')\right] . \tag{5.4}$$

Since any physical quantity must be periodic in t, we may decompose this Green function in terms of the Rindler time as an ordinary Fourier cosine series,[1]

$$G_{\text{div}}(X,X') = \frac{1}{2\pi\sqrt{zz'}} \sum_{k=-\infty}^{\infty} \cos(k[t-t'])G_{\text{div}|k}(\vec{x},\vec{x}') , \tag{5.5}$$

$$G_{\text{div}|k}(\vec{x},\vec{x}') = \sqrt{zz'}\int_0^{2\pi} d(t-t')\cos(k[t-t'])G_{\text{div}}(X,X') , \tag{5.6}$$

where $\vec{x} = (x,y,z)$; note that $G_{\text{div}|-k} = G_{\text{div}|k}$. One may verify that $G_{\text{div}|k}(\vec{x},\vec{x}')$ is a solution of the three-dimensional Green-function equation

$$\mathcal{F}_k\, \mathcal{G}(\vec{x},\vec{x}') \equiv (\Delta - m^2 - V_k)\mathcal{G}_k(\vec{x},\vec{x}') = -\delta(x-x')\,\delta(y-y')\,\delta(z-z') , \tag{5.7}$$

where $\Delta = \partial_x^2 + \partial_y^2 + \partial_z^2$ and

$$V_k(z) = \frac{4k^2-1}{4z^2} , \tag{5.8}$$

[1] This is a special case of the decomposition for arbitrary β discussed in more detail in Section 5.4.

and which vanishes at $z = \rho\, e^{-2\phi} = 0$, $z' = 0$. [The potential (5.8) follows from (2.71) with $\lambda \to k^2$, where V is assumed to vanish in the region of interest.] We may obtain an explicit expression for $G_{\mathrm{div}|k}(\vec{x},\vec{x}')$ by decomposing the four-dimensional Green function as in (5.6). Using the integral representation for K_1 from Appendix C.3.1 (i.e., the heat kernel representation for G_{div}), combined with the integrals

$$\int_0^{2\pi} dx \cos(kx) e^{p\cos(x)} = 2\pi I_k(p) \tag{5.9}$$

and

$$\begin{aligned}\int_0^{\infty} dx \frac{1}{x} e^{-ax-\frac{b}{x}} I_k(cx) &= 2 I_k(\sqrt{b(a+c)} - \sqrt{b(a-c)}) \\ &\quad \times K_k(\sqrt{b(a+c)} + \sqrt{b(a-c)})\end{aligned} \tag{5.10}$$

(see, for example, 5.1.937.2 of [76] and 2.15.6.4 of [74] respectively), one can show that

$$\begin{aligned}G_{\mathrm{div}|k}(\vec{x},\vec{x}') &= -\frac{m^2}{4\pi}\sqrt{zz'}\left[\left(I_{k+1}(\alpha_-) + \frac{k}{\alpha_-} I_k(\alpha_-)\right) K_k(\alpha_+)\left(\frac{1}{md_+} - \frac{1}{md_-}\right)\right. \\ &\quad \left. - I_k(\alpha_-)\left(K_{k+1}(\alpha_+) - \frac{k}{\alpha_+}K_k(\alpha_+)\right)\left(\frac{1}{md_+} + \frac{1}{md_-}\right)\right],\end{aligned} \tag{5.11}$$

where we have defined

$$\alpha_\pm = \frac{m}{2}(d_+ \pm d_-)\,, \qquad d_\pm = \sqrt{(x-x')^2 + (y-y')^2 + (z\pm z')^2}\,. \tag{5.12}$$

Note the similarity of $d_\pm$ to $\sigma^\pm$ of (3.75).

The physical observable of interest, $\langle\hat{\Phi}^2\rangle_{\mathrm{ren}}$, can be calculated from the full Green function $G(X,X')$ with a t-independent boundary or potential outside the region of interest using

$$\langle\hat{\Phi}^2\rangle_{\mathrm{ren}} = \lim_{X'\to X}\left[G(X,X') - G_{\mathrm{div}}(X,X')\right]. \tag{5.13}$$

Decomposing G in the same manner as G_{div} then allows us to write a decomposed form for $\langle\hat{\Phi}^2\rangle_{\mathrm{ren}}$:

$$\langle\hat{\Phi}^2(\vec{x})\rangle_{\mathrm{ren}} = \frac{1}{2\pi z}\lim_{\vec{x}'\to\vec{x}}\left\{\sum_{k=-\infty}^{\infty}\left[\mathcal{G}_k(\vec{x},\vec{x}') - G_{\mathrm{div}|k}(\vec{x},\vec{x}')\right]\right\}. \tag{5.14}$$

On the other hand, the renormalized value of $\langle\hat{\varphi}_k^2\rangle$ for the three-dimensional operator $\mathcal{F}_k$ in (5.7) is obtained by subtracting from the full three-dimensional Green function $\mathcal{G}_k(\vec{x},\vec{x}')$ not $G_{\text{div}|k}(\vec{x},\vec{x}')$, but rather the first term in the Schwinger-DeWitt expansion for $\mathcal{F}_k$, denoted $\mathcal{G}_{k|\text{div}}(\vec{x},\vec{x}')$. Specifically, for each k we have

$$\langle\hat{\varphi}_k^2(\vec{x})\rangle_{\text{ren}} = \lim_{\vec{x}'\to\vec{x}} \left[\mathcal{G}_k(\vec{x},\vec{x}') - \mathcal{G}_{k|\text{div}}(\vec{x},\vec{x}')\right] , \tag{5.15}$$

where[2]

$$\mathcal{G}_k^{\text{div}}(\vec{x},\vec{x}') = \left(\frac{m}{4\pi^2\sqrt{2\sigma}}\right)^{\frac{1}{2}} K_{\frac{1}{2}}(m\sqrt{2\sigma}) . \tag{5.16}$$

Comparing (5.14) to (5.15) we find

$$\langle\hat{\Phi}^2(\vec{x})\rangle_{\text{ren}} = \frac{1}{2\pi z}\sum_{k=-\infty}^{\infty}\left[\langle\hat{\varphi}_k^2(\vec{x})\rangle_{\text{ren}} + \Delta\langle\hat{\varphi}_k^2(\vec{x})\rangle\right] , \tag{5.17}$$

where for each k the anomaly is

$$\begin{aligned}
\Delta\langle\hat{\varphi}_k^2(\vec{x})\rangle &= \lim_{\vec{x}'\to\vec{x}}\left[\mathcal{G}_{k|\text{div}}(\vec{x},\vec{x}') - G_{\text{div}|k}(\vec{x},\vec{x}')\right] &(5.18)\\
&= \frac{m}{4\pi}\Big[-1 + mz\Big(I_{k+1}(mz)K_{k+1}(mz) + I_k(mz)K_k(mz)\Big)\\
&\qquad + k\Big(I_k(mz)K_{k+1}(mz) - I_{k+1}(mz)K_k(mz)\Big)\Big] . &(5.19)
\end{aligned}$$

Once again, the anomaly is seen as the difference in the subtraction terms used to renormalize the higher- and lower-dimensional theories. It is easily shown that (5.19) falls off as $O(z^{-1})$ for large mz, and diverges as $O(z^{-1})$ when $mz \to 0$ for $k \neq 0$. For $k = 0$ the anomaly is finite as $mz \to 0$.

The divergence of the anomaly as $z \to 0$ is familiar from the spherical decompositions, and arises for the same reason. As in the spherical decomposition of flat space, the dimensionally reduced theory obeys Dirichlet boundary conditions (i.e. the Green function vanishes) at $z = \rho e^{-2\phi} = 0$. We therefore expect divergences in the three-dimensional renormalized expectation values at $z = 0$

[2]Equation (5.16) can be obtained from the Schwinger-DeWitt expansion in three dimensions [41].

due to the vacuum polarization produced by this boundary. These are cancelled by the dimensional-reduction anomaly, leaving finite expectation values in four dimensions, as expected for flat space.

These results are qualitatively very similar to those from the spherical decomposition of flat spacetime in Section 3.3. This should not be surprising, considering that the Euclidean Rindler space (5.1) is simply flat space in polar coordinates. As a result, the calculations of this section amount to a one-dimensional "spherical" decomposition of flat space.

To be precise, recall that mode decompositions take advantage of a continuous symmetry of the manifold. For flat space, there are three distinct sets of continuous symmetries (the Poincare group): translations, rotations, and boosts. The translations are fairly trivial; one can get the dimensional-reduction anomaly for these by taking $\phi \to 0$ in the results of the next section. Rotations were covered in Chapter 3. Translations in Rindler time are the boosts, which mix the Cartesian time and space coordinates. Of course, in Euclidean space they are just another set of rotations. As a result, the decompositions of this section are the one-dimensional counterparts of the two-dimensional decompositions of Section 3.3.

5.2 The Dimensional-Reduction Anomaly in $\langle\hat{\Phi}^2\rangle$

In the previous section we examined the dimensional-reduction anomaly for the decomposition of a field in flat space into Rindler time modes. There the internal space was compact and of dimension 1. In the next two sections we consider general static spaces with internal spaces which are noncompact and of dimension 1 or 2. While qualitatively different from Rindler space, we shall see in Section 5.4

how the anomalies in these two examples may be related.

5.2.1 (1+3) Reduction

We begin with the case $n = 1$ and write the metric (2.53) in the form

$$ds^2 = g_{\mu\nu}\, dX^\mu\, dX^\nu = \mathrm{e}^{-4\phi(x)} dt^2 + h_{ab}(x) dx^a dx^b\,, \tag{5.20}$$

where $a, b \in \{1, 2, 3\}$. We assume that $t \in (-\infty, \infty)$, corresponding to a zero-temperature state. The scalar field operator is again taken to be $F = \Box - m^2 - V$, where the potential V is independent of Euclidean "time" t. It is easy to see that the operator Δ_Ω for the metric (5.20) is $\partial^2/\partial t^2$. Hence, the mode decomposition in terms of its eigenvalues is simply the standard Fourier transform with

$$Y_\omega(t) = \frac{\exp(-i\omega t)}{\sqrt{2\pi}}\,. \tag{5.21}$$

In the language of Section 2.2 we have

$$\lambda = \omega^2\,, \qquad \sum_{\lambda, \varrho} = \int_{-\infty}^{\infty} d\omega\,. \tag{5.22}$$

For example, the orthogonality and completeness relations become

$$\int_{-\infty}^{\infty} \frac{dt}{2\pi} \mathrm{e}^{-i(\omega-\omega')t} = \delta(\omega - \omega')\,, \tag{5.23}$$

$$\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \mathrm{e}^{-i\omega(t-t')} = \delta(t - t')\,. \tag{5.24}$$

The bare $\langle\hat{\Phi}^2\rangle$ is obtained from the coincidence limit of the Green function,

$$\langle\hat{\Phi}^2(X)\rangle = \lim_{X' \to X} G(X, X')\,. \tag{5.25}$$

We saw in the previous chapter that for a general four-dimensional space the divergences of the Green function in the coincidence limit come from the first two terms of the Schwinger-DeWitt expansion of the heat kernel. From (4.11) and (4.12),

$$G_{\rm div}(t,x;t',x') = \int_0^\infty ds\, \frac{1}{(4\pi s)^2} \exp\left\{-m^2 s - \frac{2\sigma+\epsilon^2}{4s}\right\} \left[\Re_0^{\Box-V} + s\Re_1^{\Box-V}\right] . \tag{5.26}$$

Here $\sigma = \sigma(X,X')$ is one-half of the square of the geodesic distance between points $X = (t,\vec{x})$ and $X' = (t',\vec{x}')$, $\Re_n^{\Box-V}$ is defined as in (4.13), $D(X,X')$ is the Van Vleck determinant (4.6), and the first few Schwinger-DeWitt coefficients for arbitrary potential V in the coincidence limit $X' \to X$ are

$$a_0^{\Box-V} = 1\,, \tag{5.27}$$

$$a_1^{\Box-V} = \frac{1}{6}\,{}^4R - V\,, \tag{5.28}$$

$$\begin{aligned} a_2^{\Box-V} = {} & \frac{1}{180}\left[{}^4R_{\alpha\beta\gamma\delta}\,{}^4R^{\alpha\beta\gamma\delta} - {}^4R_{\alpha\beta}\,{}^4R^{\alpha\beta}\right] \\ & + \frac{1}{2}\left(\frac{1}{6}\,{}^4R - V\right)^2 + \frac{1}{30}\Box\,{}^4R - \frac{1}{6}\Box V\,. \end{aligned} \tag{5.29}$$

Expansions of σ and the $\Re_n^{\Box-V}$ for the metric (5.20) with $x = x'$ and $e^{-2\phi}(t-t')$ small are given in Appendix C.1. We also include a cut-off parameter ϵ in the exponent of the integrand in (5.26) which ensures convergence of the integral for small s with σ vanishing. The anomaly is independent of this regularization parameter, and it is useful for displaying intermediate results. (This is the way the calculations of the anomaly were done in Chapter 4; ϵ did not appear explicitly since only final results were displayed.)

Our purpose is to compare the divergences of four- and three-dimensional theories related by a Fourier time transform. Unfortunately, it is not possible to evaluate the Fourier transform of (5.26) exactly for general h_{ab}, as σ and the $\Re_n^{\Box-V}$ are known in the general case only for infinitesimal point separations. Our response is the same as that in Chapter 4: we make point splitting in the t-direction, expand all t-dependent quantities in powers of the curvature, and truncate all expressions at first order in the curvature (two derivatives of the dilaton or metric). Denoting $\tau \equiv \mathrm{e}^{-2\phi}(t-t')$ and putting $x = x'$ we have up to first order in the curvature

$$2\sigma(t,x;t',x) = \tau^2 - \frac{1}{3}(\nabla\phi)^2\tau^4, \tag{5.30}$$

$$\Re_0^{\Box-V} = 1 + \frac{1}{6}\Box\phi\,\tau^2, \tag{5.31}$$

$$\Re_1^{\Box-V} = \frac{1}{6}{}^4R - V. \tag{5.32}$$

See Appendix C.1 for details. Note that in this case the internal space is not compact, and so there is no need to make our expansions periodic in $(t-t')$.

Substituting (5.30) into (5.26), expanding the exponent, and keeping in the exponent only terms which are quadratic in τ, we get

$$G_{\mathrm{div}}(t,x;t',x) = \int_0^\infty ds\, \frac{\mathrm{e}^{-m^2 s-(\tau^2+\epsilon^2)/4s}}{(4\pi s)^2}\left[\left(1+\frac{(\nabla\phi)^2}{12s}\tau^4\right)\Re_0^{\Box-V} + s\Re_1^{\Box-V}\right]. \tag{5.33}$$

The integral over the parameter s can be taken with the following result (see Appendix C.3.1):

$$G_{\mathrm{div}}(t,x;t',x) = \frac{1}{8\pi^2}\left[\left(\frac{1}{6}{}^4R - V\right)\mathbf{K}_0(z) + m^2\left(1+\frac{1}{6}\Box\phi\,\tau^2\right)\mathbf{K}_1(z) + m^4\frac{(\nabla\phi)^2}{12}\tau^4\,\mathbf{K}_2(z)\right]. \tag{5.34}$$

Here $z \equiv m\sqrt{\tau^2+\epsilon^2}$,

$$\mathbf{K}_\nu(z) = \left(\frac{2}{z}\right)^\nu K_\nu(z), \tag{5.35}$$

and the K_ν are modified Bessel functions. For example, putting $\epsilon = 0$ and expanding G^F_{div} in a Laurent series in τ we get

$$G_{\mathrm{div}}(t,x;t',x) = \frac{1}{4\pi^2\tau^2} - \frac{1}{8\pi^2}\left[\frac{1}{6}{}^4R - V - m^2\right]\left\{\gamma + \frac{1}{2}\ln\left(\frac{m^2\tau^2}{4}\right)\right\} - \frac{m^2}{16\pi^2} + \frac{(\nabla\phi)^2}{12\pi^2} + \frac{\Box\phi}{24\pi^2} + \dots. \tag{5.36}$$

Here γ is the Euler constant, and the dots denote terms of higher order in τ. The terms displayed are just the usual DeWitt-Schwinger expansion for the divergent parts of G for the metric (5.20).

The renormalized value of $\langle\hat{\Phi}^2\rangle$ can be written in the form

$$\langle\hat{\Phi}^2(t,x)\rangle_{\rm ren} = \lim_{\epsilon\to 0}\lim_{t-t'\to 0}\left[G(t,x;t',x) - G_{\rm div}(t,x;t',x)\right]. \tag{5.37}$$

Taking the limit $\epsilon \to 0$ in this expression is a trivial operation since the difference in the square brackets is already a finite quantity.

Let us now analyze what happens when we mode decompose $G_{\rm div}$ and compare to the corresponding divergent terms from the three-dimensional theory. Following (2.86) and (5.21)–(5.24), the Fourier time-transform pair is defined as

$$G_{\rm div}(x;x'|\omega) = \mathrm{e}^{-(\phi+\phi')}\int_{-\infty}^{\infty} d(t-t')\,\mathrm{e}^{i\omega(t-t')}\,G_{\rm div}(t,x;t',x'), \tag{5.38}$$

$$G_{\rm div}(t,x;t',x') = \mathrm{e}^{(\phi+\phi')}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\,\mathrm{e}^{-i\omega(t-t')}\,G_{\rm div}(x;x'|\omega). \tag{5.39}$$

Since $G_{\rm div}(t,x;t',x')$ depends only on the difference $t-t'$, the function $G_{\rm div}(x;x'|\omega)$ does not depend on t and t'. Calculating the integral in (5.38) using (5.34) and (C.57) we obtain

$$G_{\rm div}(x;x|\omega) = \frac{1}{4\pi}\left[\frac{1}{\epsilon} - \mu + \frac{1}{2\mu}\left(\frac{1}{6}{}^4R - V\right) + \frac{m^2\Box\phi}{6\mu^3} + \frac{m^4(\nabla\phi)^2}{2\mu^5} + O(\epsilon)\right], \tag{5.40}$$

where $\mu \equiv \sqrt{m^2 + \mathrm{e}^{4\phi}\omega^2}$. Meanwhile, the operator $\mathcal{F}_\omega$ which determines the reduced equation of motion (2.72) is

$$\mathcal{F}_\omega = \Delta_h - m^2 - V_\omega[\phi], \tag{5.41}$$

where

$$V_\omega[\phi] = V + \omega^2\mathrm{e}^{4\phi} + (\nabla\phi)^2 - \Delta_h\phi. \tag{5.42}$$

The Schwinger-DeWitt expansion of the heat kernel [41] for the operator $\mathcal{F}_\omega$ in three dimensions is

$$\mathcal{K}_\omega(x,x|s) = \frac{1}{(4\pi s)^{\frac{3}{2}}}\exp\left\{-m^2 s\right\}\left[1 + s\left(\frac{1}{6}R - V_\omega\right) + \cdots\right]. \tag{5.43}$$

The divergent part of the Green function for $\mathcal{F}_\omega$ is generated by the first term in this expansion,

$$\mathcal{G}_{\omega|\mathrm{div}}(x;x) = \int_0^\infty ds\, \frac{1}{(4\pi s)^{\frac{3}{2}}} \exp\left\{-m^2 s - \frac{\epsilon^2}{4s}\right\} = \frac{1}{4\pi}\left[\frac{1}{\epsilon} - m + O(\epsilon)\right]. \quad (5.44)$$

Hence, if we start with a three-dimensional theory with the field equation

$$\mathcal{F}_\omega\, \hat{\varphi}_\omega(x) = 0\,, \quad (5.45)$$

we will obtain for the renormalized value of $\langle\hat{\varphi}_\omega^2\rangle$ the representation

$$\langle\hat{\varphi}_\omega^2(x)\rangle_{\mathrm{ren}} = \lim_{\epsilon\to 0}\left[\mathcal{G}_\omega(x;x) - \mathcal{G}_{\omega|\mathrm{div}}(x;x)\right]. \quad (5.46)$$

By comparing (5.37) with (5.46) we can get the following relation between $\langle\hat{\Phi}^2\rangle$ in the four- and three-dimensional theories:

$$\langle\hat{\Phi}^2\rangle_{\mathrm{ren}} = \mathrm{e}^{2\phi}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\left[\langle\hat{\varphi}_\omega^2\rangle_{\mathrm{ren}} + \Delta\langle\hat{\varphi}_\omega^2\rangle\right], \quad (5.47)$$

where the anomalous term is

$$\begin{aligned}\Delta\langle\hat{\varphi}_\omega^2\rangle &= \lim_{\epsilon\to 0}\left[\mathcal{G}_{\omega|\mathrm{div}}(x;x) - G_{\mathrm{div}}(x;x|\omega)\right] & (5.48)\\ &= \frac{1}{4\pi}\left[\mu - m - \frac{1}{2\mu}\left(\frac{1}{6}{}^4\!R - V\right) - \frac{m^2\,\Box\phi}{6\mu^3} - \frac{m^4(\nabla\phi)^2}{2\mu^5}\right]. & (5.49)\end{aligned}$$

Since $\Delta\langle\hat{\varphi}_\omega^2\rangle$ does not vanish we have another example of the dimensional-reduction anomaly.

It is useful to note that for large frequencies $\Delta\langle\hat{\varphi}_\omega^2\rangle$ is proportional to ω. Thus, the integral over ω of the anomaly diverges as $\omega\to\pm\infty$. Since the renormalized $\langle\hat{\Phi}^2\rangle$ in four dimensions should be finite, this implies that the integral of $\langle\hat{\varphi}_\omega^2\rangle_{\mathrm{ren}}$ over ω also diverges. Thus, if one attempted to calculate $\langle\hat{\Phi}^2\rangle_{\mathrm{ren}}$ in four dimensions by summing over the $\langle\hat{\varphi}_\omega^2\rangle_{\mathrm{ren}}$, the result would diverge. Similar results held for the spherical reductions considered in Chapters 3 and 4.

5.2.2 (2+2) Reduction

Let us discuss now the dimensional-reduction anomaly for $\langle\hat{\Phi}^2\rangle$ for the case when the metric of the internal space is flat and two-dimensional; that is, the spacetime metric is of the form

$$ds^2 = \mathrm{e}^{-2\phi(x)}(dt_0^2 + dt_1^2) + h_{AB}(x)dx^A dx^B\,, \tag{5.50}$$

where $A, B \in \{2,3\}$. First of all, it should be noticed that this metric is a special case of the metric (5.20) when the dilaton field does not depend on one of the coordinates x^a. Equation (5.50) can be obtained from (5.20) by rescaling the dilaton field $\phi \to \phi/2$ and putting

$$h_{ab}dx^a dx^b = \mathrm{e}^{-2\phi(x)}\, dt_1^2 + h_{AB}(x)dx^A dx^B\,. \tag{5.51}$$

For the divergent part of the four-dimensional Green function expanded to first order in the curvature we have an expression similar to (5.34),

$$\begin{aligned} G_{\mathrm{div}}(\mathrm{t},x;\mathrm{t}',x) &= \frac{1}{8\pi^2}\left[m^2\,\mathrm{K}_1(z) + \frac{(\nabla\phi)^2}{48}m^4\tau^4\,\mathrm{K}_2(z) + \frac{\Box\phi}{12}m^2\tau^2\,\mathrm{K}_1(z)\right.\\ &\qquad\left. + \left(\frac{1}{6}{}^4R - V\right)\mathrm{K}_0(z)\right]\,. \end{aligned} \tag{5.52}$$

See Appendix C.2. Here ${}^4R$ refers to the curvature of the full four-dimensional space with metric $g_{\mu\nu}$ of (5.50), and we define

$$z = m\sqrt{\tau^2+\epsilon^2}\,, \qquad \tau^2 = \mathrm{e}^{-2\phi}\mathrm{t}^2 \equiv \mathrm{e}^{-2\phi}(t_0^2 + t_1^2)\,. \tag{5.53}$$

To obtain the mode decomposition of the divergent part of the Green function we make a Fourier transform similar to (5.38)

$$G_{\mathrm{div}}(x;x'|p) = \mathrm{e}^{-(\phi+\phi')}\int_{-\infty}^{\infty} d(\mathrm{t}-\mathrm{t}')\,\mathrm{e}^{\mathrm{i}\mathrm{p}(\mathrm{t}-\mathrm{t}')}\,G_{\mathrm{div}}(\mathrm{t},x;\mathrm{t}',x')\,, \tag{5.54}$$

$$G_{\mathrm{div}}(\mathrm{t},x;\mathrm{t}',x') = \mathrm{e}^{(\phi+\phi')}\int_{-\infty}^{\infty}\frac{d\mathrm{p}}{(2\pi)^2}\,\mathrm{e}^{-\mathrm{i}\mathrm{p}(\mathrm{t}-\mathrm{t}')}\,G_{\mathrm{div}}(x;x'|p)\,. \tag{5.55}$$

Here we use the vector notations $\mathbf{p} = (p_0, p_1)$ and $\mathbf{pt} = p_0t_0+p_1t_1$. We also denote $p^2 = \mathbf{p}^2$. Since the function $G_{\text{div}}(\mathbf{t}, x; \mathbf{t}', x')$ depends only on the difference $\mathbf{t} - \mathbf{t}'$, its Fourier transform depends only on p. Calculating the integrals we obtain

$$\begin{aligned} G_{\text{div}}(x;x'|p) &= \frac{1}{2\pi}\left[-\left\{\gamma + \frac{1}{2}\ln\left(\frac{\mu^2\epsilon^2}{4}\right)\right\} + \frac{1}{2\mu^2}\left(\frac{1}{6}{}^4R - V\right)\right. \\ &\quad \left. + \frac{m^2}{6\mu^4}\Box\phi + \frac{m^4}{3\mu^6}(\nabla\phi)^2\right] + O(\epsilon)\,, \end{aligned} \tag{5.56}$$

where

$$\mu = \sqrt{m^2 + p^2 e^{2\phi}}\,. \tag{5.57}$$

The adopted mode expansion into plane waves $\exp(i\mathbf{pt})/2\pi$ reduces the initial system for the four-dimensional operator $F = \Box - m^2 - V$ a two-dimensional system with a dilaton-dependent potential. The corresponding wave operator $\mathcal{F}_p$ is

$$\mathcal{F}_p = \Delta_h - m^2 - V_p[\phi]\,, \tag{5.58}$$

where

$$V_p[\phi] = V + p^2 e^{2\phi} + (\nabla\phi)^2 - \Delta_h\phi\,. \tag{5.59}$$

The divergent part of the two-dimensional Green function for the operator $\mathcal{F}_p$ can be obtained from the first term of the Schwinger-DeWitt expansion for this operator, (4.10) with $V_\ell \to V_p$. With the regularization parameter ϵ inserted,

$$\mathcal{G}_{p|\text{div}}(x;x) = \int_0^\infty ds\,\frac{1}{4\pi s}\exp\left\{-m^2 s - \frac{\epsilon^2}{4s}\right\} = -\frac{1}{2\pi}\left\{\gamma + \frac{1}{2}\ln\left(\frac{m^2\epsilon^2}{4}\right)\right\} + O(\epsilon)\,. \tag{5.60}$$

We define the four- and two-dimensional renormalized values $\langle\hat{\Phi}^2(x)\rangle_{\text{ren}}$ and $\langle\hat{\varphi}_p^2(x)\rangle_{\text{ren}}$ by expressions similar to (5.37) and (5.46) respectively. By comparing these definitions, and using relations (5.56) and (5.60), we obtain the representation

$$\langle\hat{\Phi}^2\rangle_{\text{ren}} = e^{2\phi}\int\frac{d\mathbf{p}}{(2\pi)^2}\left[\langle\hat{\varphi}_p^2\rangle_{\text{ren}} + \Delta\langle\hat{\varphi}_p^2\rangle\right]\,, \tag{5.61}$$

where the dimensional-reduction anomaly $\Delta\langle\hat{\varphi}_p^2\rangle$ is

$$\Delta\langle\hat{\varphi}_p^2\rangle = \frac{1}{4\pi}\left[\ln\left(\frac{\mu^2}{m^2}\right) - \frac{1}{\mu^2}\left(\frac{1}{6}{}^4R - V\right) - \frac{m^2}{3\mu^4}\Box\phi - \frac{2m^4}{3\mu^6}(\nabla\phi)^2\right]. \quad (5.62)$$

Compare this to the anomaly for the $(1+3)$ reduction, (5.49).

5.3 The Dimensional-Reduction Anomaly in the Effective Action

5.3.1 (1+3) Reduction

For the static spacetime (5.20), the calculation of the anomaly in the effective action W, where

$$W[g] = \int dX\sqrt{g}\,L\,, \quad (5.63)$$

proceeds analogously to the calculation of the anomaly in $\langle\hat{\Phi}^2\rangle$. To analyze the divergent part of the effective action we introduce first a point-split version of the effective Lagrangian[3] L. Using the Schwinger-DeWitt expansion for the heat kernel, we have for the divergent part of L

$$L_{\text{div}}(t,x;t',x) = -\frac{1}{2}\int_0^\infty \frac{ds}{s}\,\frac{e^{-m^2s-(\tau^2+\epsilon^2)/4s}}{(4\pi s)^2}\left[\Re_0^{\Box-V} + s\Re_1^{\Box-V} + s^2\Re_2^{\Box-V}\right]. \quad (5.64)$$

As earlier, the points are split in the t direction. Since the internal space is homogeneous, the point-split Lagrangian depends on t and t' only in the combination $t-t'$.

Faced with the same problem as before, we expand σ and the $\Re_n^{\Box-V}$ in terms of $\tau \equiv e^{-2\phi}(t-t')$ for $x=x'$, this time truncating at second order in the curvature

[3]It is more convenient to work with L than W for noncompact internal spaces, as the spacetime volume integral (5.63) contains an infinite volume factor $V_\Omega \to \infty$.

(four derivatives of the metric or dilaton). Writing

$$\begin{aligned} 2\sigma(t,x;t',x) &= \tau^2 + u\tau^4 + v\tau^6 + \cdots , \\ \Re_n^{\Box-V}(t,x;t',x) &= \Re_{n(0)}^{\Box-V} + \Re_{n(2)}^{\Box-V}\tau^2 + \Re_{n(4)}^{\Box-V}\tau^4 + \cdots , \end{aligned} \tag{5.65}$$

we have $\Re_{0(0)}^{\Box-V} = 1$, while u, v, and the other $\Re_{n(k)}^{\Box-V}$ may be found in Appendix C.1. Inserting and truncating at $O(R^2)$ gives

$$\begin{aligned} L_{\text{div}}(t,x;t',x) &= -\frac{1}{(4\pi)^2}\Bigg[m^4\mathbf{K}_2(z) - \frac{1}{4}(u\tau^4+v\tau^6)m^6\mathbf{K}_3(z) \\ &\quad + \frac{1}{32}u^2\tau^8m^8\mathbf{K}_4(z) + \Re_{0(2)}^{\Box-V}\tau^2m^4\left(\mathbf{K}_2(z) - \frac{1}{4}u\tau^4m^2\mathbf{K}_3(z)\right) \\ &\quad + \Re_{0(4)}^{\Box-V}\tau^4m^4\mathbf{K}_2(z) + \Re_{1(0)}^{\Box-V}m^2\left(\mathbf{K}_1(z) - \frac{1}{4}u\tau^4m^2\mathbf{K}_2(z)\right) \\ &\quad + \Re_{1(2)}^{\Box-V}\tau^2m^2\mathbf{K}_1(z) + \Re_{2(0)}^{\Box-V}\mathbf{K}_0(z)\Bigg] . \end{aligned} \tag{5.66}$$

The function $\mathbf{K}_\nu(z)$ is defined by (5.35).

We define the Fourier transform of L_{div} as in (5.38). (This follows from using the heat-kernel decomposition formula (2.92) and the mode (5.21) from the previous section.) Evaluating the transform as before yields

$$\begin{aligned} L_{\text{div}}(x|\omega) &= -\frac{1}{4\pi}\Bigg\{\frac{1}{\epsilon^3} + \frac{1}{2\epsilon}\left[-3u + 2\Re_{0(2)}^{\Box-V} + \Re_{1(0)}^{\Box-V} - \mu^2\right] \\ &\quad + \frac{1}{3}\mu^3 - \frac{1}{2}\mu\left(-3u + 2\Re_{0(2)}^{\Box-V} + \Re_{1(0)}^{\Box-V}\right) \\ &\quad + \frac{5u\omega^2e^{4\phi}}{2\mu} + \frac{m^2u\omega^2e^{4\phi}}{2\mu^3} - \frac{15m^6v}{2\mu^7} + \frac{105m^8u^2}{8\mu^9} \\ &\quad + \Re_{0(2)}^{\Box-V}\left[-\frac{\omega^2e^{4\phi}}{\mu} - \frac{15m^6u}{2\mu^7}\right] + \Re_{0(4)}^{\Box-V}\left[\frac{3m^4}{\mu^5}\right] \\ &\quad + \Re_{1(0)}^{\Box-V}\left[-\frac{3m^4u}{4\mu^5}\right] + \Re_{1(2)}^{\Box-V}\left[\frac{m^2}{2\mu^3}\right] + \Re_{2(0)}^{\Box-V}\left[\frac{1}{4\mu}\right] + O(\epsilon)\Bigg\} . \end{aligned} \tag{5.67}$$

Meanwhile, for the three-dimensional theory with the field operator $\mathcal{F}_\omega$ given by (5.41) we have

$$\mathcal{L}_{\omega|\text{div}}(x) = -\frac{1}{2}\int_0^\infty \frac{ds}{s}\frac{1}{(4\pi s)^{\frac{3}{2}}}\exp\left\{-m^2s - \frac{\epsilon^2}{4s}\right\}\left[1 + s\left(\frac{1}{6}R - V_\omega\right)\right]$$

$$= -\frac{1}{4\pi}\left[\frac{1}{\epsilon^3}+\frac{1}{2\epsilon}\left(\frac{1}{6}R-V_\omega-m^2\right)+\frac{m^3}{3}-\frac{m}{2}\left(\frac{1}{6}R-V_\omega\right)\right]+O(\epsilon)\,. \tag{5.68}$$

Here V_ω is the effective potential of the three-dimensional system, given by (5.42), and R is the scalar curvature calculated from the three-metric h_{ab}. It is related to the four-dimensional curvature ${}^4R$ via

$${}^4R[g] = R[h] + 4\Box\phi = R[h] + 4\Delta_h\phi - 8(\nabla\phi)^2\,. \tag{5.69}$$

The renormalized effective Lagrangians in four and three dimensions are obtained by subtracting from the exact effective action its divergent part, as given by (5.66) and (5.68) respectively. By comparing these divergent parts using (5.38), (5.39) and (5.67) we can write

$$L_{\rm ren}(x) = e^{2\phi}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\left[\mathcal{L}_{\omega|\rm ren}(x)+\Delta\mathcal{L}_\omega(x)\right]\,, \tag{5.70}$$

where $\Delta\mathcal{L}_\omega$ is the term representing the dimensional-reduction anomaly of the renormalized effective Lagrangian,

$$\begin{aligned}\Delta\mathcal{L}_\omega = \frac{1}{4\pi}\Bigg\{&\frac{1}{3}(\mu^3-m^3)-\frac{1}{2}(\mu-m)\left(\frac{1}{6}R-V-(\nabla\phi)^2+\Delta_h\phi\right)-\frac{1}{2}m\omega^2e^{4\phi}\\
&+\mu\left(\frac{5}{2}u-\Re^{\Box-V}_{0(2)}\right)+\frac{1}{\mu}\left(-2m^2u+m^2\Re^{\Box-V}_{0(2)}+\frac{1}{4}\Re^{\Box-V}_{2(0)}\right)\\
&+\frac{m^2}{\mu^3}\left(-\frac{m^2}{2}u+\frac{1}{2}\Re^{\Box-V}_{1(2)}\right)+\frac{m^4}{\mu^5}\left(3\Re^{\Box-V}_{0(4)}-\frac{3}{4}u\Re^{\Box-V}_{1(0)}\right)\\
&+\frac{m^6}{\mu^7}\left(-\frac{15}{2}v-\frac{15}{2}u\Re^{\Box-V}_{0(2)}\right)+\frac{m^8}{\mu^9}\left(\frac{105}{8}u^2\right)\Bigg\}\,.\end{aligned} \tag{5.71}$$

5.3.2 (2+2) Reduction

As we already mentioned, the $(2+2)$ reduction is a special case of the "static" spacetime reduction. The calculation of the dimensional-reduction anomaly is very similar to the calculations of the previous subsection – straightforward but quite involved. We do not reproduce the details of these calculations here but simply give the final results.

The mode decomposition of the renormalized effective Lagrangian for the operator $F = \Box - m^2 - V$ has the form

$$L_{\text{ren}} = e^{2\phi} \int \frac{d\mathbf{p}}{(2\pi)^2} \left[\mathcal{L}_{\mathbf{p}|\text{ren}} + \Delta\mathcal{L}_p\right] , \tag{5.72}$$

where the dimensional-reduction anomaly $\Delta\mathcal{L}_p$ is

$$\begin{aligned}
\Delta\mathcal{L}_p = \; = \; & \frac{1}{8\pi}\Bigg[-p^2 e^{2\phi} + \left(\frac{1}{6}R - V + \Delta\phi - (\nabla\phi)^2 - m^2 - p^2 e^{2\phi}\right)\ln\left(\frac{m^2}{\mu^2}\right) \\
& + \left(12u - 4\Re^{\Box-V}_{0(2)}\right) + \frac{1}{\mu^2}\left(\Re^{\Box-V}_{2(0)} - 8m^2u + 4m^2\Re^{\Box-V}_{0(2)}\right) \\
& + \frac{m^2}{\mu^4}\left(-4m^2u + 4\Re^{\Box-V}_{1(2)}\right) + \frac{m^4}{\mu^6}\left(-8u\Re^{\Box-V}_{1(0)} + 32\Re^{\Box-V}_{0(4)}\right) \\
& + \frac{m^6}{\mu^8}\left(-96v - 96u\Re^{\Box-V}_{0(2)}\right) + \frac{m^8}{\mu^{10}}\left(192u^2\right)\Bigg] .
\end{aligned} \tag{5.73}$$

Recall that μ is given by relation (5.57).

In the next chapter we will use these anomalies to derive analytic approximations for $\langle\hat{\Phi}^2\rangle$ and W at zero temperature. First, however, we will consider an approach to mode decompositions in static spaces with arbitrary periodicity β in an attempt to unify the Rindler-space results of Section 5.1 with these zero-temperature anomalies.

5.4 The Dimensional-Reduction Anomaly in $\langle\hat{\Phi}^2\rangle$ at Finite Temperature

In this section we determine the dimensional-reduction anomaly in $\langle\hat{\Phi}^2\rangle$ for the decomposition of a scalar field in a static space where the Euclidean time coordinate has an arbitrary period β. In the context of thermal quantum field theory, this corresponds to a field at temperature $T = \beta^{-1}$. Although we consider only the anomaly in $\langle\hat{\Phi}^2\rangle$, the techniques used here are readily applied to the calculation of the anomaly in W.

Consider the line element (5.20),

$$ds^2 = \mathrm{e}^{-4\phi(x^c)}dt^2 + h_{ab}(x^c)dx^a dx^b\,, \tag{5.74}$$

where we now choose $t \in [0, \beta]$ with $t = 0$ and $t = \beta$ identified. Rindler space (5.1) corresponds to $\beta = 2\pi$, $\mathrm{e}^{-2\phi} = x^3 = z$ with $h_{ab} = \delta_{ab}$, while the zero-temperature case (5.20) corresponds to $\beta \to \infty$.

Following the steps of the previous sections, we split the points in the t-direction only, and expand all geometric quantities in powers of $(t-t')^2$. Since the internal space of (5.74) is periodic, as for the spherical decompositions, we follow the technique of Section 4.1 and make our expansions periodic as well. Guided by the Rindler space example, in which the t-dependence of σ was in the form $(1-\cos(t-t'))$ with $t \in [0, 2\pi]$, we define an effective angular coordinate λ by

$$\lambda \equiv 2\pi\frac{(t-t')}{\beta} \in [0, 2\pi]\,. \tag{5.75}$$

We then replace all $(t-t')^2$ by

$$(t-t')^2 = \left(\frac{\beta}{2\pi}\right)^2\left[2(1-\cos\lambda) + \frac{1}{3}(1-\cos\lambda)^2 + \frac{4}{45}(1-\cos\lambda)^3 + \cdots\right], \tag{5.76}$$

truncating at the lowest order that gives the correct flat-space limit. With this technique our expansions (5.30)–(5.32) for σ and the $\Re_n^{\Box-V}$ become

$$2\sigma = 2\mathrm{L}^2(1-\cos\lambda) + \mathrm{L}^2\left[\frac{1}{3} - \frac{4}{3}(\nabla\phi)^2\mathrm{L}^2\right](1-\cos\lambda)^2\,, \tag{5.77}$$

$$\Re_0^{\Box-V} = 1 + \frac{\mathrm{L}^2}{3}\Box\phi\,(1-\cos\lambda)\,, \tag{5.78}$$

$$\Re_1^{\Box-V} = \frac{1}{6}{}^4R - V\,, \tag{5.79}$$

where for convenience we define

$$\mathrm{L} \equiv \frac{\beta e^{-2\phi}}{2\pi}, \tag{5.80}$$

which is the effective radius of a line of constant t. The local temperature is then given by $T_{\rm loc} = (2\pi \mathrm{L})^{-1}$. It is easily verified that (5.77)–(5.79) reduce to the previously derived results for both Rindler space and zero-temperature static space.

The mode decomposition is similar to that for the Rindler-space example:

$$K(t,x;t',x'|s) = \frac{e^{\phi+\phi'}}{\beta} \sum_{n=-\infty}^{\infty} e^{-in\frac{2\pi}{\beta}(t-t')} K_n(x,x'|s), \tag{5.81}$$

$$K_n(x,x'|s) = e^{-(\phi+\phi')} \int_0^\beta d(t-t')\, e^{in\frac{2\pi}{\beta}(t-t')} K(t,x;t',x'|s), \tag{5.82}$$

where we use

$$Y_{\lambda\varrho}(y) \to \frac{e^{-in\frac{2\pi}{\beta}t}}{\sqrt{\beta}}, \qquad \lambda \to \left(n\frac{2\pi}{\beta}\right)^2, \qquad \sum_{\lambda,\varrho} \to \sum_{n=-\infty}^{\infty}. \tag{5.83}$$

The renormalized values of the square of the field operator will then be related via

$$\langle \hat{\Phi}^2 \rangle_{\rm ren} = \frac{e^{2\phi}}{\beta} \sum_{n=-\infty}^{\infty} \left[\langle \hat{\varphi}_n^2 \rangle_{\rm ren} + \Delta \langle \hat{\varphi}_n^2 \rangle \right]. \tag{5.84}$$

Note also that we can return to the zero-temperature ($\beta \to \infty$) formalism with the replacements

$$n\frac{2\pi}{\beta} \to \omega, \qquad \frac{1}{\beta}\sum_{n=-\infty}^{\infty} \to \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega, \tag{5.85}$$

$$\int_0^\beta d(t-t') \to \int_{-\infty}^{\infty} d(t-t'). \tag{5.86}$$

By inserting the expansions (5.77)–(5.79) into the expression (4.12) for the divergent part of the heat kernel in four dimensions and performing the mode decomposition (5.82), we obtain

$$\begin{aligned} K_{\mathrm{div}|n}(x,x|s) &= Ł\,\frac{e^{-m^2 s}}{(4\pi s)^2}\left[J_{n0} + s\left(\frac{1}{6}{}^4R - V\right)J_{n0} + \frac{Ł^2}{3}\Box\phi J_{n1}\right. \\ &\quad \left. - \frac{Ł^2}{12s}(1 - 4(\nabla\phi)^2Ł^2)J_{n2}\right], \end{aligned} \tag{5.87}$$

where

$$J_{nq} \equiv \int_0^{2\pi} d\lambda\, e^{in\lambda}\, e^{p(1-\cos\lambda)}(1-\cos\lambda)^q, \tag{5.88}$$

and $p \equiv -Ł^2/2s$. These integrals are evaluated by taking derivatives with respect to p of the $q=0$ result (5.9) used in the mode decomposition of Rindler space:

$$J_{n0} = 2\pi(-1)^n e^p I_n(p), \tag{5.89}$$

$$J_{nq} = \frac{d^q}{dp^q}J_{n0}. \tag{5.90}$$

Equation (5.87) is to be compared to the subtraction term for the dimensionally reduced theory, consisting of the first term of the Schwinger-DeWitt expansion (5.43) in three dimensions:

$$\mathcal{K}_{n|\mathrm{div}} = \frac{e^{-m^2 s}}{(4\pi s)^{\frac{3}{2}}}. \tag{5.91}$$

The anomaly in $\langle\hat{\Phi}^2\rangle$ can now be obtained by integrating the difference of (5.91) and (5.87) over s:

$$\Delta\langle\hat{\varphi}_n^2\rangle = \int_0^\infty ds\,\left[\mathcal{K}_{n|\mathrm{div}}(x,x|s) - K_{\mathrm{div}|n}(x,x|s)\right]. \tag{5.92}$$

Employing the formula (5.10) from the Rindler example, one can show with much tedious algebra that the anomaly is given by

$$\begin{aligned} \Delta\langle\hat{\varphi}_n^2\rangle &= \frac{m}{4\pi}\Big[-1 + n\,(I_n(mŁ)K_{n+1}(mŁ) - I_{n+1}(mŁ)K_n(mŁ)) \\ &\quad + mŁ\,(I_n(mŁ)K_n(mŁ) + I_{n+1}(mŁ)K_{n+1}(mŁ))\Big] \\ &\quad - \frac{Ł}{4\pi}\left(\frac{1}{6}{}^4R - V\right)I_n(mŁ)K_n(mŁ) \end{aligned}$$

$$
\begin{aligned}
&+\frac{\mathrm{L}}{6\pi}\Box\phi\left[nI_n(m\mathrm{L})K_n(m\mathrm{L})-\frac{1}{2}m\mathrm{L}\left(I_n(m\mathrm{L})K_{n+1}(m\mathrm{L})\right.\right.\\
&\left.\left.-I_{n+1}(m\mathrm{L})K_n(m\mathrm{L})\right)\right]\\
&+\frac{1}{12\pi\mathrm{L}}(1-4(\nabla\phi)^2\mathrm{L}^2)\left[(n^2-n)I_n(m\mathrm{L})K_n(m\mathrm{L})\right.\\
&+\frac{n-1}{2}m\mathrm{L}\left(I_{n+1}(m\mathrm{L})K_n(m\mathrm{L})-I_n(m\mathrm{L})K_{n+1}(m\mathrm{L})\right)\\
&\left.+\frac{1}{2}(m\mathrm{L})^2\left(I_n(m\mathrm{L})K_n(m\mathrm{L})-I_{n+1}(m\mathrm{L})K_{n+1}(m\mathrm{L})\right)\right]. \quad (5.93)
\end{aligned}
$$

Using uniform asymptotic expansions of the Bessel functions, one can verify that in the $\mathrm{L} \to \infty$ ($\beta \to \infty$, $T \to 0$) limit (5.93) reproduces the zero-temperature result (5.49).[4] As well, the special case of $\beta = 2\pi$ with line element (5.1) reproduces the exact Rindler-space anomaly, (5.19).

In the following chapter we turn our attention to making "practical" use of these dimensional-reduction anomaly results. First, however, a few comments regarding the dimensional-reduction anomaly and the choice of quantum state are in order. The anomaly is determined by the divergent subtraction terms of the D- and $(D-n)$-dimensional theories, and the mode decomposition. The divergences are local, depending on the geometry and potential, and so are state-independent. The mode decomposition, however, is sensitive to the global behaviour of the internal space (at least for low frequencies), and so may be affected by the choice of state. In particular, if the time direction lies in the internal space, the mode decomposition will be sensitive to the temperature; this means that the dimensional-reduction anomaly for static spaces will be state-dependent. By contrast, for the spherical reductions considered previously, the decomposition was over spatial directions, and so the dimensional-reduction anomalies of Chapters 3 and 4 are state-independent.

[4]In taking the zero-temperature limit it is important to hold the frequency fixed. One does this by replacing n by $\omega\beta/2\pi$ and holding ω constant as $\beta \to \infty$.

In either case, it is important to recognize the distinction between the low- and high-frequency parts of the dimensional-reduction anomaly. The high-frequency part of the anomaly is responsible for the divergences in the D-dimensional theory. It is purely local, and so is universal and unambiguous. By contrast, the low-frequency part of the anomaly is sensitive to the global behaviour of the system. As a result, it is inherently somewhat ambiguous in the sense that different assumptions for the large-distance behaviour of the system will alter the anomaly at low frequencies. In particular, the assumption of periodicity used in the spherical decompositions of Chapter 4 and in the static decompositions of this section do influence the anomaly. While they are perhaps the most natural choices for determining the anomaly for compact internal spaces, they are not unique.

Chapter 6

Analytic Approximations for $\langle\hat{\Phi}^2\rangle$ and W

In the preceding chapter we examined mode decompositions and the dimensional-reduction anomaly in static spaces. In this chapter we will see how these results may be used to derive a new type of analytic approximation for renormalized expectation values in static spaces. At the present time, this scheme has only been applied to zero-temperature fields; its generalization to fields at arbitrary temperature is still underway.

6.1 Analytic Approximation for $\langle\hat{\Phi}^2\rangle$

In Section 5.2.1 we derived the dimensional-reduction anomaly in $\langle\hat{\Phi}^2\rangle$ for a field at zero-temperature in the static space (5.20),

$$\Delta\langle\hat{\varphi}^2_\omega\rangle = \frac{1}{4\pi}\left[\mu - m - \frac{1}{2\mu}\left(\frac{1}{6}{}^4R - V\right) - \frac{m^2\,\Box\phi}{6\mu^3} - \frac{m^4(\nabla\phi)^2}{2\mu^5}\right], \tag{6.1}$$

where the four- and three-dimensional theories are related via

$$\langle\hat{\Phi}^2\rangle_{\text{ren}} = e^{2\phi}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\left[\langle\hat{\varphi}^2_\omega\rangle_{\text{ren}} + \Delta\langle\hat{\varphi}^2_\omega\rangle\right]. \tag{6.2}$$

The anomalous term $\Delta\langle\hat{\varphi}^2_\omega\rangle$ is finite and can be written in the form

$$\Delta\langle\hat{\varphi}^2_\omega\rangle = \Delta\langle\hat{\varphi}^2_\omega\rangle^\sharp + \Delta\langle\hat{\varphi}^2_\omega\rangle^\flat , \tag{6.3}$$

where[1]

$$\Delta\langle\hat{\varphi}^2_\omega\rangle^\sharp = \frac{1}{4\pi}\left[\bar{\omega} + \frac{m^2}{2\bar{\omega}} - m - \frac{1}{2\bar{\omega}}\left(\frac{1}{6}{}^4R - V\right)\right], \tag{6.4}$$

$$\Delta\langle\hat{\varphi}^2_\omega\rangle^\flat = \frac{1}{4\pi}\left[\mu - \bar{\omega} - \frac{m^2}{2\bar{\omega}} - \frac{1}{2}\left(\frac{1}{6}{}^4R - V\right)\left(\frac{1}{\mu} - \frac{1}{\bar{\omega}}\right) - \frac{m^2\,\Box\phi}{6\mu^3} - \frac{m^4(\nabla\phi)^2}{2\mu^5}\right]. \tag{6.5}$$

Here and later we use the notation $\bar{\omega} = e^{2\phi}\omega$. The quantity $\Delta\langle\hat{\varphi}^2_\omega\rangle^\sharp$ is that part of the anomaly which dominates at high frequencies ($\omega \to \infty$); it consists of all terms of $O(\omega^{-1})$ and higher in the large-ω expansion of $\Delta\langle\hat{\varphi}^2_\omega\rangle$. These are the terms which diverge in the integration over ω, and hence which lead to the divergences in the four-dimensional Green function as $t - t' \to 0$. The part of the anomaly that remains when $\Delta\langle\hat{\varphi}^2_\omega\rangle^\sharp$ is subtracted off is denoted by $\Delta\langle\hat{\varphi}^2_\omega\rangle^\flat$. It is of $O(\omega^{-2})$ for high frequencies, so the inverse Fourier transform of $\Delta\langle\hat{\varphi}^2_\omega\rangle^\flat$ is finite as $t - t' \to 0$.

Let us examine the inverse Fourier transform of $\Delta\langle\hat{\varphi}^2_\omega\rangle^\flat$. Defining

$$\langle\hat{\Phi}^2\rangle_{\text{approx}} = \int_0^\infty \frac{d\bar{\omega}}{\pi}\,\Delta\langle\hat{\varphi}^2_\omega\rangle^\flat , \tag{6.6}$$

and performing the integration (see Appendix C.3.2), we obtain

$$\langle\hat{\Phi}^2\rangle_{\text{approx}} = \frac{m^2}{16\pi^2} + \frac{1}{16\pi^2}\left(\frac{1}{6}{}^4R - V - m^2\right)\ln\frac{m^2 e^{-4\phi}}{4\eta^2} - \frac{\Box\phi}{24\pi^2} - \frac{(\nabla\phi)^2}{12\pi^2}\,. \tag{6.7}$$

The parameter η is a low-frequency cut-off which is required to make the integral convergent. It corresponds to a well-known ambiguity in the renormalization prescription. This ambiguity is absent for a conformally invariant theory, when $m = 0$ and $V = \frac{1}{6}{}^4R$.

[1]The symbol $\sharp$ (*sharp*) is borrowed from musical notation to denote the *high-frequency* part of the anomaly; the remainder is labelled with $\flat$ (*flat*).

The reason that (6.7) is of interest (and the justification for the name $\langle\hat{\Phi}^2\rangle_{\mathrm{approx}}$) is that in certain special cases it reproduces previously known approximation schemes for quantum fields in curved spacetimes. In particular, for a massless, conformally coupled field (6.7) reduces to the zero-temperature limit of the Killing approximation of Frolov and Zelnikov [10] for static spacetimes. This scheme is based on the assumption that one can approximate the expectation value of interest ($\langle\hat{\Phi}^2\rangle$ or $\langle\hat{T}_{\mu\nu}\rangle$) by a tensor which is a local function of the curvatures, the Killing vector for the static spacetime, and their covariant derivatives. Imposing physical properties, such as the covariant conservation of $\langle\hat{T}_{\mu\nu}\rangle$, then uniquely determines the approximate tensor up to a few arbitrary constants which depend on the choice of state. For the further specialization to static Einstein spaces the Killing approximation contains the approximations of Page [8], and Page, Brown, and Ottewill [9], and so our result (6.7) is equivalent to the zero-temperature limit of these approximations as well. Finally, for the special case of static spherically symmetric spacetimes (6.7) also reduces to the zero-temperature limit of the WKB approximation of Anderson [11]. (The corresponding result for the stress tensor, derived by Anderson, Hiscock, and Samuel [12], was discussed in Section 4.4, where it was denoted $\langle\hat{T}_{\mu\nu}\rangle_{\mathrm{AHS}}$.)

The agreement between our $\langle\hat{\Phi}^2\rangle_{\mathrm{approx}}$ and the results of Anderson is particularly interesting when we recall that (6.7) is constructed using the low-frequency part of the dimensional-reduction anomaly, $\Delta\langle\hat{\varphi}^2_\omega\rangle^\flat$. We are thus led to the seemingly paradoxical conclusion that there is a connection between the low-frequency part of the anomaly and the high-frequency behaviour of the field.

A more careful examination of how $\langle\hat{\Phi}^2\rangle_{\mathrm{approx}}$ is constructed resolves this mystery. Note that the renormalization terms $\mathcal{G}_{\omega|\mathrm{div}}$ (5.44) from the dimensionally reduced theory do not contribute[2] to $\Delta\langle\hat{\varphi}^2_\omega\rangle^\flat$. The analytic approximation is

[2]This is because of their dependence on the frequency, ω. Specifically, the renormalization terms from the dimensionally reduced theory are polynomials in ω^2 [ω only enters $\mathcal{K}_{\omega|\mathrm{div}}$

therefore constructed using only the renormalization terms $G_{\mathrm{div}|\omega}$ (5.40) from four dimensions. Schematically, we have

$$\begin{aligned}\langle\hat{\Phi}^2(x)\rangle_{\mathrm{approx}} &= \int_0^\infty \frac{d\bar{\omega}}{\pi}\left[\Delta\langle\hat{\varphi}_\omega^2(x)\rangle - \Delta\langle\hat{\varphi}_\omega^2(x)\rangle^\sharp\right] \\ &= \int_0^\infty \frac{d\bar{\omega}}{\pi}\left[G^\sharp(x;x|\omega) - G_{\mathrm{div}}(x;x|\omega)\right] \qquad (6.8)\\ &= \lim_{t'\to t}\left[G^\sharp(t,x;t',x) - G_{\mathrm{div}}(t,x;t',x)\right]. \qquad (6.9)\end{aligned}$$

Here $G^\sharp(t,x;t',x)$ denotes the high-frequency part of the Green function, obtained by Fourier transforming the Green function, keeping only the terms of $O(\omega^{-1})$ and higher in the large-ω limit, and then inverting the Fourier transform. Viewed in this manner, we see that the analytic approximation is simply the expression resulting from using a high-frequency approximation for the Green function, renormalized in the standard manner in four dimensions (using (5.36)). This is why our technique reproduces the WKB approximation of Anderson.

There are several points to note regarding our approximation. First, it is unambiguous since it is constructed using the high-frequency behaviour of the Green function, which is determined by the ultraviolet divergences of the theory. In particular, $G^\sharp$ is completely independent of the particular ansatz used for the Green function for large point separations in the Fourier transform. Different choices change only the low-frequency behaviour of the Green function, which is discarded anyway in the high-frequency limit. This means that we can construct $G^\sharp$ using any ansatz for the Green function which contains all of its divergences for the given point-splitting; the Schwinger-DeWitt expansion is merely the most convenient and general such ansatz.

Similarly, the low-frequency part of G_{div} used for renormalization in four dimensions is also unimportant, as it is not affected by the subtraction in (6.8) and

through the induced potential V_ω, given by (5.42)]. Thus, they are unaltered in the high-frequency expansions and so are removed identically when we subtract the high-frequency limit, $\Delta\langle\hat{\varphi}_\omega^2\rangle^\sharp$.

is just returned identically in four dimensions when the mode sum is done. For example, note that the terms in (5.36) which are finite as $\tau \to 0$ all appear identically in the analytic approximation (6.7) [except for a sign reversal, since (5.36) is subtracted from $G^\sharp$ in the renormalization]. As a result, these finite terms need not be mode-decomposed in the first place. We could just as easily have begun with the Laurent series (5.36) for G, and only Fourier-transformed those terms which diverge as $t \to 0$. Removing the high-frequency limit and performing the inverse Fourier transform would then yield our analytic approximation (6.7).

The only knowledge used to construct $\langle\hat{\Phi}^2\rangle_{\rm approx}$ is the structure of the divergences of the physical theory and the static nature of the space. Since the divergences are known for arbitrary spaces and field parameters via the Schwinger-DeWitt expansion, this method can (in theory) be applied to any spacetime with a continuous symmetry. This is a distinct advantage over the WKB approximations used by Anderson, Hiscock, and Samuel and by Balbinot, Fabbri, and Nicolini, which require explicit mode solutions for the entire field $\hat{\Phi}$, not just for the temporal behaviour. Our approach does not presume detailed knowledge of the spacetime; in particular, we do not need to assume any symmetries other than being static. Furthermore, since our approximation is constructed (in the massless case) using only covariant objects from the static space (i.e., the curvature, the Killing vector, and their covariant derivatives), it is clear that it must reduce to the Killing approximation of Frolov and Zelnikov and the approximations of Page, Brown, and Ottewill for massless conformally coupled fields. Our high-frequency approximation is not restricted to conformal fields, however, and hence relation (6.7) can be considered as an extension of these approximations to the general case when the spacetime is static, but not necessary spherically symmetric, and the field equation includes an arbitrary mass and potential V. This approximation should be valid whenever the high-frequency limit of the

Green function is appropriate for modelling the full Green function.

Finally, we note that our approximation for $\langle\hat{\Phi}^2\rangle$ is the same if we were to use the anomaly from the $(2+2)$ reduction in Section 5.2.2. In this case, the square of the field operators for a $(2+2)$ decomposition of the space (5.50) were shown to be related by (5.61),

$$\langle\hat{\Phi}^2\rangle_{\text{ren}} = e^{2\phi} \int \frac{d\mathrm{p}}{(2\pi)^2} \left[\langle\hat{\varphi}_p^2\rangle_{\text{ren}} + \Delta\langle\hat{\varphi}_p^2\rangle\right] , \tag{6.10}$$

where the dimensional-reduction anomaly $\Delta\langle\hat{\varphi}_p^2\rangle$ is

$$\Delta\langle\hat{\varphi}_p^2\rangle = \frac{1}{4\pi}\left[\ln\left(\frac{\mu^2}{m^2}\right) - \frac{1}{\mu^2}\left(\frac{1}{6}{}^4R - V\right) - \frac{m^2}{3\mu^4}\Box\phi - \frac{2m^4}{3\mu^6}(\nabla\phi)^2\right] . \tag{6.11}$$

The part of $\Delta\langle\hat{\varphi}_p^2\rangle$ which dominates at large "momentum" p and which is responsible for the divergences of the four-dimensional Green function in the coincidence limit $\mathbf{t} - \mathbf{t}' \to 0$ is

$$\Delta\langle\hat{\varphi}_p^2\rangle^\sharp = \frac{1}{4\pi}\left[\ln\frac{\bar{p}^2}{m^2} - \frac{1}{\bar{p}^2}\left(\frac{1}{6}{}^4R - V - m^2\right)\right] , \tag{6.12}$$

where we define $\bar{p} = pe^\phi$. Defining the sub-leading part $\Delta\langle\hat{\varphi}_p^2\rangle^\flat$ of the anomaly and $\langle\hat{\Phi}^2\rangle_{\text{approx}}$ by relations similar to (6.3) and (6.6), respectively, we obtain an expression for $\langle\hat{\Phi}^2\rangle_{\text{approx}}$ which is identical to (6.7); see Appendix C.3.3. One can expect this result, since the $(2+2)$ reduction may be considered as a special case of the $(1+3)$ reduction.

Thus confident as to the physical meaning of our new approximation scheme, let us apply it to the effective action and the stress tensor.

6.2 Analytic Approximation for the Effective Action

Our approximation procedure is easily applied to the effective action. From (5.70), the effective Lagrangians in four and three dimensions are related by

$$L_{\rm ren}(x) = e^{2\phi}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\left[\mathcal{L}_{\omega|\rm ren}(x) + \Delta\mathcal{L}_\omega(x)\right] , \tag{6.13}$$

where $\Delta\mathcal{L}_\omega$ is the dimensional-reduction anomaly (5.71),

$$\begin{aligned}\Delta\mathcal{L}_\omega \;=\; & \frac{1}{4\pi}\left\{\frac{1}{3}(\mu^3 - m^3) - \frac{1}{2}(\mu - m)\left(\frac{1}{6}R - V - (\nabla\phi)^2 + \Delta_h\phi\right) - \frac{1}{2}m\bar{\omega}^2\right.\\ & + \mu\left(\frac{5}{2}u - \Re^{\Box-V}_{0(2)}\right) + \frac{1}{\mu}\left(-2m^2u + m^2\Re^{\Box-V}_{0(2)} + \frac{1}{4}\Re^{\Box-V}_{2(0)}\right)\\ & + \frac{m^2}{\mu^3}\left(-\frac{m^2}{2}u + \frac{1}{2}\Re^{\Box-V}_{1(2)}\right) + \frac{m^4}{\mu^5}\left(3\Re^{\Box-V}_{0(4)} - \frac{3}{4}u\Re^{\Box-V}_{1(0)}\right)\\ & \left. + \frac{m^6}{\mu^7}\left(-\frac{15}{2}v - \frac{15}{2}u\Re^{\Box-V}_{0(2)}\right) + \frac{m^8}{\mu^9}\left(\frac{105}{8}u^2\right)\right\} .\end{aligned} \tag{6.14}$$

As earlier, we write

$$\Delta\mathcal{L}_\omega = \Delta\mathcal{L}^\sharp_\omega + \Delta\mathcal{L}^\flat_\omega , \tag{6.15}$$

where $\Delta\mathcal{L}^\sharp_\omega$ is the part of the anomaly which dominates at high frequencies ($\omega \to \infty$),

$$\begin{aligned}\Delta\mathcal{L}^\sharp_\omega \;=\; & \frac{1}{4\pi}\left\{\frac{1}{3}\bar{\omega}^3 - \frac{1}{2}m\bar{\omega}^2 + \bar{\omega}\left[\frac{m^2}{2} - \frac{1}{2}\left(\frac{1}{6}R - V - (\nabla\phi)^2 + \Delta_h\phi\right)\right.\right.\\ & \left. + \frac{5u}{2} - \Re^{\Box-V}_{0(2)}\right] + \left[\frac{m}{2}\left(\frac{1}{6}R - V - (\nabla\phi)^2 + \Delta_h\phi\right) - \frac{m^3}{3}\right]\\ & + \frac{1}{\bar{\omega}}\left[\frac{3m^4}{8} - \frac{m^2}{4}\left(\frac{1}{6}R - V - (\nabla\phi)^2 + \Delta_h\phi + 3u - 2\Re^{\Box-V}_{0(2)}\right)\right.\\ & \left.\left. + \frac{1}{4}\Re^{\Box-V}_{2(0)}\right]\right\} .\end{aligned} \tag{6.16}$$

By subtracting this large-ω limit from the anomaly (6.14) and making the inverse Fourier transform, we can construct an approximate effective Lagrangian for the four-dimensional theory:

$$L_{\rm approx} = \int_0^\infty \frac{d\bar{\omega}}{\pi} \Delta\mathcal{L}^{\flat}_{\omega} . \qquad (6.17)$$

Performing the ω-integration (see Appendix C.3.2) gives

$$\begin{aligned} L_{\rm approx} &= \frac{3m^4}{128\pi^2} + \frac{m^2}{32\pi^2}\left[-\frac{1}{6}R + V + (\nabla\phi)^2 - \Delta_h\phi + u - 2\Re^{\Box-V}_{0(2)}\right] \\ &+ \frac{1}{8\pi^2}\left[\Re^{\Box-V}_{1(2)} + 4\Re^{\Box-V}_{0(4)} - u\Re^{\Box-V}_{1(0)} - 8u\Re^{\Box-V}_{0(2)} - 8v + 12u^2\right] \\ &+ \frac{1}{32\pi^2}\left[-\frac{m^4}{2} + m^2\left(\frac{1}{6}R - V - (\nabla\phi)^2 + \Delta_h\phi + 3u - 2\Re^{\Box-V}_{0(2)}\right)\right. \\ &\left. - \Re^{\Box-V}_{2(0)}\right] \ln\left(\frac{m^2 e^{-4\phi}}{4\eta^2}\right) . \end{aligned} \qquad (6.18)$$

The effective action corresponding to this Lagrangian may be simplified considerably using integration by parts. Substituting for the u, v, and $\Re^{\Box-V}_{n(k)}$ from Appendix C.1 and neglecting surface terms, one can show that the effective action for the important case $V = \xi\,{}^4R$ may be written as

$$\begin{aligned} W_{\rm approx} &= \int d^4x\sqrt{g}\left\{-\frac{1}{64\pi^2}\ln\left(\frac{m^2e^{-4\phi}}{4\eta^2}\right)\left[m^4 + \frac{1}{90}\left({}^4R_{\alpha\beta\gamma\delta}\,{}^4R^{\alpha\beta\gamma\delta}\right.\right.\right. \\ &\left.\left. - {}^4R_{\alpha\beta}\,{}^4R^{\alpha\beta} + \Box\,{}^4R\right)\right] + \frac{3m^4}{128\pi^2} - \frac{m^2(\nabla\phi)^2}{24\pi^2} \\ &+ \frac{1}{360\pi^2}\left[{}^4R_{\alpha\beta}\phi^\alpha\phi^\beta - \frac{3}{2}(\Box\phi)^2 - 4\Box\phi(\nabla\phi)^2 - 4(\nabla\phi)^4\right] \\ &+ \frac{(\xi-\frac{1}{6})}{32\pi^2}\left[m^2\,{}^4R - \frac{4}{3}{}^4R(\nabla\phi)^2 + \ln\left(\frac{m^2e^{-4\phi}}{4\eta^2}\right)\left(-m^2\,{}^4R + \frac{1}{6}\Box\,{}^4R\right)\right] \\ &\left. - \frac{(\xi-\frac{1}{6})^2}{64\pi^2}\left[\ln\left(\frac{m^2e^{-4\phi}}{4\eta^2}\right)({}^4R)^2\right]\right\} . \end{aligned} \qquad (6.19)$$

Note that (6.19) has been written entirely in terms of four-dimensional quantities. The stress tensor $\langle\hat{T}_{\mu\nu}\rangle_{\rm approx}$ resulting from the variation of this action with respect to the metric is displayed in Section 6.3. It can be shown that in the special case of a static, spherically symmetric spacetime $\langle\hat{T}_{\mu\nu}\rangle_{\rm approx}$ coincides with the analytic approximation of Anderson, Hiscock, and Samuel [12] for the zero-temperature case. Furthermore, in the massless, conformally coupled limit (6.19) coincides with the zero-temperature Killing approximation [10] and the zero-temperature Page-Brown-Ottewill approximation [8, 9].

6.3 Analytic Approximation for $\langle\hat{\mathbf{T}}_{\mu\nu}\rangle$

Using the result (6.19) for the approximate effective action for a scalar field in a static four-dimensional spacetime with potential $V = \xi\,{}^4R$, one can calculate the resulting stress tensor. For convenience, we split the action into pieces according to its dependence on the field mass m and coupling ξ as follows:

$$W_{\rm approx} = W^0_{m^4} + W^0_{m^2} + W^0_{m^0} + \left(\xi - \frac{1}{6}\right)\left[W^1_{m^2} + W^1_{m^0}\right] + \left(\xi - \frac{1}{6}\right)^2 W^2_{m^0}\ ; \quad (6.20)$$

where

$$W^0_{m^4} = \frac{m^4}{32\pi^2}\int d^4x\sqrt{g}\left\{\frac{3}{4} - \frac{1}{2}\ln\left(\frac{m^2\chi^2}{4\eta^2}\right)\right\}, \quad (6.21)$$

$$W^0_{m^2} = \frac{m^2}{32\pi^2}\int d^4x\sqrt{g}\left\{-\frac{4}{3}(\nabla\phi)^2\right\}, \quad (6.22)$$

$$\begin{aligned} W^0_{m^0} = \frac{1}{720\pi^2}\int d^4x\sqrt{g}\Big\{&\left[2R_{\alpha\beta}\phi^\alpha\phi^\beta - 3(\Box\phi)^2 - 8\Box\phi(\nabla\phi)^2 - 8(\nabla\phi)^4\right] \\ &- \frac{1}{8}\ln\left(\frac{m^2\chi^2}{4\eta^2}\right)\left(R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} - R_{\alpha\beta}R_{\alpha\beta} + \Box R\right)\Big\}, \end{aligned} \quad (6.23)$$

$$W^1_{m^2} = \frac{m^2}{32\pi^2}\int d^4x\sqrt{g}\left\{R - R\ln\left(\frac{m^2\chi^2}{4\eta^2}\right)\right\}, \quad (6.24)$$

$$W^1_{m^0} = \frac{1}{32\pi^2}\int d^4x\sqrt{g}\left\{-\frac{4}{3}R(\nabla\phi)^2 + \frac{1}{6}\ln\left(\frac{m^2\chi^2}{4\eta^2}\right)\Box R\right\}, \quad (6.25)$$

$$W^2_{m^0} = \frac{1}{32\pi^2}\int d^4x\sqrt{g}\left\{-\frac{1}{2}R^2\ln\left(\frac{m^2\chi^2}{4\eta^2}\right)\right\}. \quad (6.26)$$

Here and for the remainder of this section we assume all curvatures and derivatives to be four-dimensional; i.e., calculated using the metric $g_{\mu\nu}$. For example, multiple four-dimensional covariant derivatives of ϕ are represented by $\phi_{\alpha\beta} \equiv \phi_{;\alpha\beta}$, etc. Also, χ^μ is the Killing vector of the space (5.20), with $\chi^2 = e^{-4\phi}$.

Using
$$\langle\hat{T}_{\mu\nu}\rangle = \frac{2}{\sqrt{g}}\frac{\delta W}{\delta g^{\mu\nu}}\,, \tag{6.27}$$
the stress tensor due to each part of the effective action is found to be (in Euclidean signature)
$$\langle\hat{T}_{\mu\nu}\rangle^{0}_{m^4} = \frac{m^4}{32\pi^2}\left\{\frac{\chi_\mu\chi_\nu}{\chi^2} + g_{\mu\nu}\left[\frac{1}{2}\ln\left(\frac{m^2 e^{-4\phi}}{4\eta^2}\right) - \frac{3}{4}\right]\right\}, \tag{6.28}$$

$$\langle\hat{T}_{\mu\nu}\rangle^{0}_{m^2} = \frac{m^4}{24\pi^2}\left\{\frac{\chi_\mu\chi_\nu}{\chi^2}\Box\phi + g_{\mu\nu}(\nabla\phi)^2 - 2\phi_\mu\phi_\nu\right\}, \tag{6.29}$$

$$\begin{aligned}
\langle\hat{T}_{\mu\nu}\rangle^{0}_{m^0} = & -\frac{1}{2880\pi^2}\frac{1}{2}\ln\left(\frac{m^2\chi^2}{4\eta^2}\right)\Bigg[12(R_{\mu\alpha\nu\beta} - \frac{1}{4}g_{\mu\nu}R_{\alpha\beta})R^{\alpha\beta} \\
& - 4R(R_{\mu\nu} - \frac{1}{4}g_{\mu\nu}R) + 6\Box R_{\mu\nu} - 2R_{;\mu\nu} - g_{\mu\nu}\Box R\Bigg] \\
& + \frac{1}{2880\pi^2}\frac{\chi_\mu\chi_\nu}{\chi^2}\Bigg[\left(R^{\alpha\beta\gamma\delta}R_{\alpha\beta\gamma\delta} - R^{\alpha\beta}R_{\alpha\beta} + \Box R\right) \\
& + 4\left(-R^{;\alpha}\phi_\alpha - 2R^{\alpha\beta}\phi_{\alpha\beta} - 3\Box^2\phi - 4\Box((\nabla\phi)^2) + 8(\Box\phi)^2\right. \\
& \left. + 8\nabla\phi\cdot\nabla(\Box\phi) + 16\Box\phi(\nabla\phi)^2 + 16\nabla\phi\cdot\nabla((\nabla\phi)^2)\right)\Bigg] \\
& + \frac{1}{720\pi^2}g_{\mu\nu}\Bigg[-R^{\alpha\beta}\phi_{\alpha\beta} - \frac{1}{2}R^{;\alpha}\phi_\alpha + \Box^2\phi - 2R^{\alpha\beta}\phi_\alpha\phi_\beta \\
& + 2(\phi^\alpha\phi^\beta)_{;\alpha\beta} - 3(\Box\phi)^2 - 6\nabla\phi\cdot\nabla(\Box\phi) - 8\nabla\phi\cdot\nabla((\nabla\phi)^2) \\
& + 8(\nabla\phi)^4\Bigg] \\
& + \frac{1}{720\pi^2}\Bigg[4R_{\mu\alpha\nu\beta}\phi^{\alpha\beta} + \frac{1}{2}\left(R_{\alpha\mu}\phi_\nu{}^\alpha + R_{\alpha\nu}\phi_\mu{}^\alpha\right) \\
& + 3\left(R_{\mu\nu;\alpha} - R_{\mu\alpha;\nu} - R_{\nu\alpha;\mu}\right)\phi^\alpha \\
& - (\Box\phi)_{;\mu\nu} + 4\left(R_{\alpha\mu}\phi_\nu + R_{\alpha\nu}\phi_\mu\right)\phi^\alpha + 2\Box(\phi_\mu\phi_\nu) \\
& - 2\left((\phi_\mu\phi_\alpha)_\nu{}^\alpha + (\phi_\nu\phi_\alpha)_\mu{}^\alpha\right) + 6\left(\phi_\mu(\Box\phi)_{;\nu} + \phi_\nu(\Box\phi)_{;\mu}\right) \\
& + 8\left(\phi_\mu((\nabla\phi)^2)_{;\nu} + \phi_\nu((\nabla\phi)^2)_{;\mu}\right) - 16(\phi_\mu\phi_\nu)\Box\phi
\end{aligned}$$

$$- 32(\phi_\mu\phi_\nu)(\nabla\phi)^2\Big] \, , \tag{6.30}$$

$$\begin{aligned}\langle\hat{T}_{\mu\nu}\rangle^1_{m^2} &= \frac{m^2}{16\pi^2}\Bigg\{\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right)\left[1 - \ln\left(\frac{m^2\chi^2}{4\eta^2}\right)\right] + \frac{\chi_\mu\chi_\nu}{\chi^2}R \\ &\quad + 4g_{\mu\nu}\Box\phi - 4\phi_{\mu\nu}\Bigg\} \, ,\end{aligned} \tag{6.31}$$

$$\begin{aligned}\langle\hat{T}_{\mu\nu}\rangle^1_{m^0} &= \frac{1}{24\pi^2}\Bigg\{\frac{\chi_\mu\chi_\nu}{\chi^2}\left[R^{;\alpha}\phi_\alpha + R\Box\phi - \frac{1}{4}\Box R\right] - 2\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right)(\nabla\phi)^2 \\ &\quad - R_{\mu\nu}\Box\phi - g_{\mu\nu}\Box\left(\Box\phi + 2(\nabla\phi)^2\right) + \left(\Box\phi + 2(\nabla\phi)^2\right)_{;\mu\nu} \\ &\quad - 2R\phi_\mu\phi_\nu + \frac{1}{2}\left(R_{;\mu}\phi_\nu + R_{;\nu}\phi_\mu - g_{\mu\nu}R^{;\alpha}\phi_\alpha\right)\Bigg\} \, ,\end{aligned} \tag{6.32}$$

$$\begin{aligned}\langle\hat{T}_{\mu\nu}\rangle^2_{m^0} &= \frac{1}{32\pi^2}\Bigg\{\ln\left(\frac{m^2\chi^2}{4\eta^2}\right)\left[\frac{1}{2}g_{\mu\nu}R^2 + 2R_{;\mu\nu} - 2g_{\mu\nu}\Box R - 2R_{\mu\nu}R\right] \\ &\quad + \frac{\chi_\mu\chi_\nu}{\chi^2}R^2 - 8\left(R_{;\mu}\phi_\nu + R_{;\nu}\phi_\mu + R\phi_{\mu\nu}\right) \\ &\quad + 8g_{\mu\nu}\left(R\Box\phi + R^{;\alpha}\phi_\alpha\right)\Bigg\} \, .\end{aligned} \tag{6.33}$$

As noted before, in the special case of a static, spherically symmetric spacetime this stress-energy tensor coincides with the analytic approximation of Anderson *et al* [12] for the zero-temperature case. Furthermore, in the massless, conformally coupled limit, it coincides with the zero-temperature Killing approximation [10], and the zero-temperature Page-Brown-Ottewill approximation [8, 9]. Our high-frequency approximation can in fact be considered as an extension of these schemes to the general case when the spacetime is static, but not necessary spherically symmetric, and the field equation includes an arbitrary mass and potential V.

Chapter 7

Discussion and Future Work

In the presence of a continuous spacetime symmetry, the field equation can be solved by decomposition of the field into harmonics. This effectively replaces the single field in the physical theory by a collection of lower-dimensional fields. One can then express quantities of interest, such as the Green function and the effective action, as sums of the corresponding objects in a dimensionally reduced theory. Due to the presence of ultraviolet divergences, however, these decompositions have only formal meaning, as the renormalization violates the exact form of such representations. As a result, the expression for the renormalized expectation value of the object in the physical spacetime can be obtained by summing the contributions of corresponding lower-dimensional quantities only if additional anomalous terms are added to each of the modes. We call this effect the *dimensional-reduction anomaly.*

The anomaly may have several sources. The dimensional reduction may change the global properties of the spacetime, inducing additional boundary conditions which the dimensionally reduced field must satisfy. This effect was seen in the decomposition of flat space into spherical modes (Section 3.3), and into Rindler-time modes (Section 5.1). Because of its nonlocal nature, this effect cannot be eliminated by additional finite renormalization of the dimensionally reduced theory. In addition to this "global" contribution to the dimensional-reduction anomaly, there also exist "local" contributions due to the curvature and the potential induced by the dimensional reduction. The corresponding anomalous terms are local invariants constructed from the curvature, the dilaton field, and their covariant derivatives. Comparison to the Schwinger-DeWitt expansion shows that these additional anomalous terms also cannot be eliminated by additional finite renormalizations. The dimensional-reduction anomaly is therefore not merely an artifact of renormalization ambiguity, and cannot be ignored in the analysis of quantum fields via dimensional reduction. Failure to account for the anomaly will lead to incorrect predictions for the contributions of fixed modes to expectation values in the physical theory. Furthermore, when summed over all modes, the renormalized expectation values of the dimensionally reduced theory will in general diverge if the anomaly is not included. Naïve dimensional reduction and renormalization (ignoring the anomaly) cannot yield finite results for the physical theory.

We have explicitly demonstrated the importance of the dimensional-reduction anomaly for the study of the stress tensor of a quantum field in a black-hole spacetime using dimensional reduction. The stress tensors predicted by two-dimensional dilaton-gravity models for the $\ell = 0$ mode are qualitatively very different near the event horizon from the stress tensor in four dimensions. The anomaly appears to correct the predictions of the dimensionally reduced theory at the event horizon, while leaving the asymptotic behaviour from individual modes unaffected.

We have also demonstrated how mode decompositions can be used to obtain a very simple and general high-frequency approximation scheme for renormalized expectation values of quantum fields in static spaces. This scheme requires only the structure of the divergences in the theory and minimal assumptions about the spacetime (the existence of a Killing vector). It can therefore be applied to fields with arbitrary parameters in a wide class of spacetimes.

There are a number of topics presented in this thesis which merit further investigation. In dilaton-gravity theories, the contribution of the $\ell > 0$ modes is of interest, particularly near the potential barrier at $r \approx 3M$. The high-frequency approximation scheme developed in Chapter 6 may be useful here for confirming the WKB approximation of Balbinot *et al* for the $\ell = 0$ mode, and for deriving the dimensionally reduced stress tensors for $\ell > 0$. Better approximations for the stress tensor in black-hole spacetimes would aid in the accurate modelling of the back-reaction on the geometry and the subsequent evaporation process.

The dimensional-reduction anomaly could also be of relevance to Kaluza-Klein theories [86, 87], in which the assumption of extra 'hidden' dimensions is used in attempts to explain the physical properties of our four-dimensional universe in a more natural manner. Such theories arise, for example, as a natural consequence of the low-energy limit of string theory [88]. The dimensional-reduction anomaly implies that one might expect additional quantitative and even qualitative differences between the higher-dimensional quantum field theory and the effective four-dimensional behaviour of matter in our universe, and this possibility has yet to be fully explored [89].

The generalization of our approximation scheme to include finite-temperature effects would be of great value. In that case, one could regard our scheme as a proper extension of the Killing and WKB techniques to general static spaces with nonconformal fields.

Also important is the testing of our approximation scheme to establish its range of validity. While in the special cases of conformally invariant fields and Schwarzschild spacetimes it reduces to known approximation methods (Killing and WKB respectively), one does not expect this high-frequency approximation to be useful in all cases, such as for fields of large mass. A determination of the criteria for the validity of this scheme is required if it is to be applied to other systems.

Appendix A

Quantum Fields in Flat Spacetime at Finite Temperature

As an exercise we derive the renormalized expectation values of the square of the field operator and the stress tensor for a scalar field in four- and two-dimensional flat spacetimes at finite temperature. These are useful for interpreting the contribution of s mode alone, derived in Section 3.2.

The total renormalized values of $\langle\hat{\Phi}^2\rangle^\beta$ and $\langle\hat{T}_{\mu\nu}\rangle^\beta$ for a scalar field in flat spacetime are easily calculated using (3.18), (3.19) of Section 3.2. Decomposing the field $\hat{\Phi}$ in terms of the standard plane-wave modes

$$\Phi_{\vec{k}} \equiv \frac{e^{ik_\alpha x^\alpha}}{\sqrt{2\omega(2\pi)^{n-1}}}, \tag{A.1}$$

where $k^0 = \omega_{\vec{k}} \equiv \sqrt{k^2+m^2}$, $k \equiv |\vec{k}|$, and each of the components of the vector $\vec{k}$ range over $(-\infty,\infty)$, we find the renormalized Hadamard Green function to be

$$G^{(1)}_\beta(X,X') = \int_{-\infty}^{\infty} d\vec{k}\, \frac{2}{e^{\beta\omega_{\vec{k}}}-1}\left(\Phi_{\vec{k}}\overline{\Phi}'_{\vec{k}} + \overline{\Phi}_{\vec{k}}\Phi'_{\vec{k}}\right). \tag{A.2}$$

A.1 Four dimensions

In four dimensions we have

$$\int_{-\infty}^{\infty} d\vec{k} = 4\pi \int_0^{\infty} dk\, k^2 . \tag{A.3}$$

Using the modes (A.1) with (3.18), we obtain

$$\langle\hat{\Phi}^2\rangle^\beta = \frac{1}{2\pi^2}\int_0^\infty dk\, \frac{1}{e^{\beta\sqrt{k^2+m^2}}-1}\frac{k^2}{\sqrt{k^2+m^2}} . \tag{A.4}$$

Note that this is equal to the expression for the contribution of the s-mode alone, (3.30), evaluated at $r = 0$. For a massless field we find

$$\langle\hat{\Phi}^2\rangle^\beta_{m=0} = \frac{1}{2\pi^2}\int_0^\infty d\omega\, \frac{\omega}{e^{\beta\omega}-1} = \frac{1}{12\beta^2} , \tag{A.5}$$

while in the massive case we have

$$\begin{aligned}
\langle\hat{\Phi}^2\rangle^\beta &= \frac{m^2}{2\pi^2}\sum_{n=1}^{\infty}\frac{K_1(nm\beta)}{nm\beta} && \text{(A.6)}\\
&\overset{m\beta\gg1}{=} \frac{1}{2\pi^2}\frac{m}{\beta}K_1(m\beta) && \text{(A.7)}\\
&\overset{m\beta\ll1}{=} \frac{1}{12\beta^2} - \frac{m}{4\pi\beta} + \frac{m^2}{16\pi^2} - \frac{m^2}{8\pi^2}\left(\gamma + \ln\frac{m\beta}{4\pi}\right) . && \text{(A.8)}
\end{aligned}$$

Similarly, for the total stress tensor we have

$$\langle\hat{T}_\mu{}^\nu\rangle^\beta = \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty} d\vec{k}\, \frac{1}{e^{\beta\sqrt{k^2+m^2}}-1}\frac{k_\mu k^\nu}{\sqrt{k^2+m^2}} . \tag{A.9}$$

Note that $\langle\hat{T}_\mu{}^\nu\rangle^\beta$ is diagonal, independent of ξ, and also traceless for $m = 0$. In the massless case we need only compute the time-time component,

$$\langle\hat{T}_{tt}\rangle^\beta = \frac{1}{2\pi^2}\int_0^\infty d\omega\, \frac{\omega^3}{e^{\beta\omega}-1} = \frac{\pi^2}{30}T^4 , \tag{A.10}$$

and the total stress tensor is then[1]

$$\langle \hat{T}_\mu{}^\nu \rangle^\beta = \frac{\pi^2}{30} T^4 \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{3} \end{pmatrix} . \tag{A.11}$$

In the massive case let us consider first the low temperature limit, $m\beta \gg 1$. The trace of the stress tensor is given by

$$\begin{aligned} \langle \hat{T}_\nu{}^\nu \rangle_{m\beta \gg 1} &= \frac{-m^2}{2\pi^2} \int_0^\infty dk \frac{k^2}{\sqrt{k^2+m^2}} \mathrm{e}^{-\beta\sqrt{k^2+m^2}} \\ &= \frac{1}{(2\pi)^2} \left[-\frac{2m^3}{\beta} K_1(m\beta) \right] . \end{aligned} \tag{A.12}$$

Combining this with the time-time component

$$\begin{aligned} \langle \hat{T}_{tt} \rangle_{m\beta \gg 1} &= \frac{1}{2\pi^2} \int_0^\infty dk\, k^2 \sqrt{k^2+m^2}\, \mathrm{e}^{-\beta\sqrt{k^2+m^2}} \\ &= \frac{1}{(2\pi)^2} \left[\frac{2m^3}{\beta} K_1(m\beta) + \frac{6m^2}{\beta^2} K_0(m\beta) + \frac{12m}{\beta^3} K_1(m\beta) \right] \end{aligned} \tag{A.13}$$

yields the total stress tensor

$$\begin{aligned} \langle \hat{T}_\mu{}^\nu \rangle_{m\beta \gg 1} &= \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{3} \end{pmatrix} \frac{1}{(2\pi)^2} \left[\frac{6m^2}{\beta^2} K_0(m\beta) + \frac{12m}{\beta^3} K_1(m\beta) \right] \\ &+ \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \frac{1}{(2\pi)^2} \frac{2m^3}{\beta} K_1(m\beta) . \end{aligned} \tag{A.14}$$

[1] These calculations have been performed in Cartesian coordinates; however, in transforming to spherical coordinates any second-rank tensor of the form $T_\mu^\nu = \mathrm{diag}(\mathrm{a}, \mathrm{b}, \mathrm{b}, \mathrm{b})$ has the same components in both spherical and Cartesian coordinates.

The high-temperature limit can be obtained from the low-temperature result by substituting $\beta \to n\beta$ and summing over n from 1 to ∞. Using the sums (3.39)–(3.41), we obtain

$$\langle \hat{T}_{\mu}{}^{\nu} \rangle_{m\beta \ll 1} =$$
$$6 \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{3} \end{pmatrix} \left[\frac{\pi^2}{180\beta^4} + \frac{m^2}{48\beta^2} - \frac{m^3}{24\pi\beta} + \frac{m^4}{64\pi^2} \left(\gamma + \ln \frac{b}{4\pi} \right) - \frac{3m^4}{256\pi^2} \right]$$
$$+ \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \left[\frac{m^2}{12\beta^2} - \frac{m^3}{4\pi\beta} - \frac{m^4}{8\pi^2} \left(\gamma + \ln \frac{b}{4\pi} \right) + \frac{m^4}{16\pi^2} \right] . \quad \text{(A.15)}$$

A.2 Two dimensions

Repeating these calculations for two-dimensional space, where

$$\int_{-\infty}^{\infty} d\vec{k} = 2 \int_{0}^{\infty} dk \,, \quad \text{(A.16)}$$

we find that $\langle \hat{\Phi}^2 \rangle^{\beta}$ is divergent for a massless field, while in the massive case we have

$$\langle \hat{\Phi}^2 \rangle^{\beta} \overset{m\beta \gg 1}{=} \frac{1}{\pi} K_0(m\beta) \,, \quad \text{(A.17)}$$
$$\overset{m\beta \ll 1}{=} \frac{1}{2m\beta} + \frac{1}{2\pi} \left(\gamma + \ln \frac{m\beta}{4\pi} \right) . \quad \text{(A.18)}$$

The stress tensor is

$$\langle \hat{T}_{\mu}{}^{\nu} \rangle^{\beta} \overset{m=0}{=} \frac{\pi^2}{6\beta^2} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} , \quad \text{(A.19)}$$

$$\overset{m\beta\gg 1}{=} \frac{m^2}{\pi}\left[\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\frac{K_1(m\beta)}{m\beta} + \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} K_0(m\beta)\right], \tag{A.20}$$

$$\begin{aligned}\overset{m\beta\ll 1}{=} & \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\left\{\frac{\pi}{6\beta^2} - \frac{m}{2\beta} - \frac{m^2}{4\pi}\left(\gamma + \ln\frac{m\beta}{4\pi}\right) + \frac{m^2}{8\pi}\right\} \\ & + \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}\left\{\frac{m}{2\beta} + \frac{m^2}{2\pi}\left(\gamma + \ln\frac{m\beta}{4\pi}\right)\right\}.\end{aligned} \tag{A.21}$$

Note that the $r \to \infty$ limit of the contribution of the s mode alone in four dimensions, derived in Section 3.2, are $(4\pi r^2)^{-1}$ times these.

Appendix B

Spherical Decompositions

B.1 Spherical Decomposition of Curvatures

Consider a line element of the form

$$ds^2 = g_{\mu\nu} dX^\mu dX^\nu = h_{ab} dx^a dx^b + \rho^2 e^{-2\phi} \omega_{ij} dy^i dy^j \ , \tag{B.1}$$

where $h_{ab} = h_{ab}(x^c)$ is an arbitrary two-dimensional metric and $\omega_{ij} = \omega_{ij}(y^k)$ is the metric of a two-sphere. The dilaton ϕ is a function of the x^a only, and ρ is a constant with the dimensions of length. The radius of a two-sphere of fixed x^a is $r = \rho e^{-\phi}$.

We wish to decompose our field theory in terms of modes on the two-sphere. This requires rewriting four-dimensional geometric quantities like the curvatures in terms of the corresponding curvatures for the metric h.

Our notational conventions are as follows: four-dimensional covariant derivatives are denoted by $()_{;a}$, while $\Box$ is understood to represent the d'Alembertian with respect to g. Meanwhile, ∇, $()_{|a}$ and Δ_h are the two-dimensional covariant derivatives and d'Alembertian calculated using the metric h_{ab}. For the dilaton ϕ we shall understand ϕ_a, ϕ_{ab}, etc. to denote multiple two-dimensional covariant derivatives of ϕ. For example, the four-dimensional d'Alembertian of an angle-independent scalar S decomposes to

$$\Box S = \Delta_h S - 2\nabla\phi\cdot\nabla S\,. \tag{B.2}$$

In particular,

$$\Box\phi = \Delta_h\phi - 2(\nabla\phi)^2\,. \tag{B.3}$$

For the given line element, the nonvanishing Christoffel symbols are

$$
\begin{aligned}
{}^4\Gamma^a_{bc}[g] &= {}^2\Gamma^a_{bc}[h]\,, && (B.4)\\
{}^4\Gamma^a_{ij}[g] &= \phi^a g_{ij}\,, && (B.5)\\
{}^4\Gamma^i_{ja}[g] &= -\phi_a\delta^i_j\,, && (B.6)\\
{}^4\Gamma^k_{ij}[g] &= {}^2\Gamma^k_{ij}[\omega]\,. && (B.7)
\end{aligned}
$$

Selecting coordinates (θ,η) on the two-spheres, where

$$\omega_{ij}dy^i dy^j = d\theta^2 + \sin^2\theta\, d\eta^2\,, \tag{B.8}$$

one finds

$${}^4\Gamma^\theta_{\eta\eta}[g] = -\sin\theta\cos\theta\,, \qquad {}^4\Gamma^\eta_{\eta\theta}[g] = \frac{\cos\theta}{\sin\theta}\,. \tag{B.9}$$

For convenience, we define the following commonly occurring functions of the dilaton field:

$$
\begin{aligned}
A &= 1 - r^2(\nabla\phi)^2\,, && (B.10)\\
B &= \Delta_h\phi - 2(\nabla\phi)^2\,, && (B.11)\\
T_{ab} &= \phi_{ab} - \phi_a\phi_b\,, && (B.12)\\
T &= h^{ab}T_{ab} = \Delta_h\phi - (\nabla\phi)^2\,. && (B.13)
\end{aligned}
$$

Since the two-sphere metric has constant curvature, ${}^2R[\omega] = 2$, explicit reference to it may be dropped. Henceforth we shall assume all curvatures to be with respect to the two-dimensional metric h_{ab} unless explicitly labelled otherwise.

Using this notation, one can show that the only nonvanishing components of the four-dimensional curvatures are

$$\begin{aligned} {}^4R_{abcd}[g] &= \frac{1}{2}R\,(h_{ac}h_{bd} - h_{ad}h_{bc})\,, && \text{(B.14)} \\ {}^4R_{aibj}[g] &= g_{ij}T_{ab}\,, && \text{(B.15)} \\ {}^4R_{ijkm}[g] &= \frac{A}{r^2}(g_{ik}g_{jm} - g_{im}g_{jk})\,, && \text{(B.16)} \end{aligned}$$

$$\begin{aligned} {}^4R_{ab}[g] &= \frac{1}{2}R\,h_{ab} + 2T_{ab}\,, && \text{(B.17)} \\ {}^4R_{ij}[g] &= g_{ij}\left[\frac{1}{r^2} + B\right]\,, && \text{(B.18)} \end{aligned}$$

$$ {}^4R[g] = R + 4\Delta_h\phi - 6(\nabla\phi)^2 + \frac{2}{r^2}\,, \qquad \text{(B.19)}$$

while the only nonvanishing ${}^4R_{\alpha\beta;\gamma}$ are

$$\begin{aligned} {}^4R_{ab;c}[g] &= \frac{1}{2}h_{ab}\,R_{|c} + 2T_{ab|c}\,, && \text{(B.20)} \\ {}^4R_{am;n}[g] &= g_{mn}\left[\left(-\frac{1}{2}R + \frac{1}{r^2} + B\right)\phi_a - 2T_{ab}\phi^b\right]\,, && \text{(B.21)} \\ {}^4R_{mn;a}[g] &= g_{mn}\left(\frac{1}{r^2} + B\right)_{|a}\,. && \text{(B.22)} \end{aligned}$$

Also,

$$\begin{aligned} {}^4R_{mn;ab}[g] &= g_{mn}\left(\frac{1}{r^2} + B\right)_{|ab}\,, && \text{(B.23)} \\ {}^4R_{mn;jk}[g] &= -\left(g_{km}g_{nj} + g_{kn}g_{mj}\right)\left[\left(-\frac{1}{2}R + \frac{1}{r^2} + B\right)(\nabla\phi)^2 - 2T_{ab}\phi^a\phi^b\right] \\ &\quad - g_{jk}g_{mn}\left(\frac{1}{r^2} + B\right)_{|a}\phi^a\,, && \text{(B.24)} \\ \Box\,{}^4R_{mn}[g] &= g_{mn}\left\{[\Delta_h - 2\nabla\phi\cdot\nabla]\left(\frac{1}{r^2} + B\right) + R(\nabla\phi)^2 - 2\left(\frac{1}{r^2} + B\right)(\nabla\phi)^2 \right. \\ &\quad \left. + 4T_{ab}\phi^a\phi^b\right\}\,, && \text{(B.25)} \end{aligned}$$

$$^4R_{;a}[g] = \left(R + 4\Delta_h\phi - 6(\nabla\phi)^2 + \frac{2}{r^2}\right)_{|a}, \tag{B.26}$$

$$^4R_{;ab}[g] = \left(R + 4\Delta_h\phi - 6(\nabla\phi)^2 + \frac{2}{r^2}\right)_{|ab}, \tag{B.27}$$

$$^4R_{;mn}[g] = -g_{mn}\left(R + 4\Delta_h\phi - 6(\nabla\phi)^2 + \frac{2}{r^2}\right)_{|a}\phi^a, \tag{B.28}$$

$$\Box\,^4R[g] = [\Delta_h - 2\nabla\phi\cdot\nabla]\left(R + 4\Delta_h\phi - 6(\nabla\phi)^2 + \frac{2}{r^2}\right). \tag{B.29}$$

B.2 Small-Distance Expansions

To perform the decomposition into spherical harmonics it is necessary to know the behaviour of σ and the $D^{\frac{1}{2}}a_n$ for X and X' separated along the two-spheres. Without loss of generality we take the points to be split in the θ direction only, with angular separation $\lambda = \theta - \theta'$. Our procedure will be to calculate the desired quantities first as expansions in powers of λ^2, and then to convert them to expansions in powers of $(1-\cos\lambda)$ for use in the mode-decomposition calculations.

We take as our ansatz for the geodetic interval σ

$$2\sigma(x,y;x',y') = (\bar{r}\lambda)^2 + U(\bar{x})(\bar{r}\lambda)^4 + V(\bar{x})(\bar{r}\lambda)^6 + \cdots, \tag{B.30}$$

where $\bar{x} \equiv \frac{1}{2}(x + x')$. Taking the derivative of $\tilde{\sigma}$ with respect to each of the coordinates and requiring $\sigma = \frac{1}{2}g^{\alpha\beta}\sigma_\alpha\sigma_\beta$ in the coincidence limit, one can show that

$$U(x) = -\frac{1}{12}(\nabla\phi)^2, \tag{B.31}$$

$$V(x) = \frac{1}{90}(\nabla\phi)^4 - \frac{1}{120}\phi^a\phi^b\phi_{ab}, \tag{B.32}$$

and

$$(\sigma^\theta)^2 = \lambda^2\left[1 - \frac{1}{3}r^2(\nabla\phi)^2\lambda^2 + r^4\left(\frac{17}{180}(\nabla\phi)^4 - \frac{1}{20}\phi^a\phi^b\phi_{ab}\right)\lambda^4 + \cdots\right], \tag{B.33}$$

$$\begin{aligned}
\sigma^{\eta} &= 0\,, &\text{(B.34)}\\
\sigma^{a} &= -\frac{1}{2}\phi^{a}(r\lambda)^{2} + \left[-\frac{1}{24}\phi^{ab}\phi_{b} + \frac{1}{12}(\nabla\phi)^{2}\phi^{a}\right](r\lambda)^{4} + \cdots\,. &\text{(B.35)}
\end{aligned}$$

The expansion (B.30) for σ can be converted into one in terms of $(1-\cos\lambda)$ using

$$\lambda^{2} = 2(1-z) + \frac{1}{3}(1-z)^{2} + \frac{4}{45}(1-z)^{3} + \cdots\,, \tag{B.36}$$

where $z \equiv \cos\lambda$. Defining the functions $u(x)$, $v(x)$ by

$$2\sigma(x,y;x,y') = 2r^{2}\left[(1-z) + u(x)(1-z)^{2} + v(x)(1-z)^{3} + \cdots\right]\,, \tag{B.37}$$

we obtain

$$\begin{aligned}
u(x) &= \frac{1}{6}[1 - r^{2}(\nabla\phi)^{2}]\,, &\text{(B.38)}\\
v(x) &= \frac{2}{45}\left[1 - \frac{5}{4}r^{2}(\nabla\phi)^{2} + r^{4}(\nabla\phi)^{4} - \frac{3}{8}r^{4}\nabla\phi\cdot\nabla[(\nabla\phi)^{2}]\right]\,. &\text{(B.39)}
\end{aligned}$$

Combining (B.33–B.36) with the results of Appendix B.1 and the short-distance expansions of [80, 81], one can derive expansions for the $D^{\frac{1}{2}}a_n$ in powers of $(1-z)$. Writing

$$D^{\frac{1}{2}}a_{n}^{\square-\xi^{4}R} = \Re_{n}^{\square-\xi^{4}R} = \Re_{n(0)}^{\square-\xi^{4}R} + \Re_{n(1)}^{\square-\xi^{4}R}(1-z) + \Re_{n(2)}^{\square-\xi^{4}R}(1-z)^{2} + \cdots\,, \tag{B.40}$$

it can be shown that

$$\Re_{0(0)}^{\square-\xi^{4}R} = 1\,, \tag{B.41}$$

$$\Re_{0(1)}^{\square-\xi^{4}R} = \frac{1}{6}(1 + r^{2}B)\,, \tag{B.42}$$

$$\begin{aligned}
\Re_{0(2)}^{\square-\xi^{4}R} &= \frac{1}{90}A^{2} + \frac{1}{72}(1+r^{2}B)^{2} + \frac{1}{36}(1+r^{2}B)(1-4r^{2}(\nabla\phi)^{2})\\
&\quad + \frac{r^{4}}{180}\left[\frac{3}{2}R(\nabla\phi)^{2} + 6T_{ab}\phi^{a}\phi^{b} + 2T_{ab}T^{ab}\right.\\
&\quad \left. + 12\left(\frac{1}{r^{2}} + B\right)(\nabla\phi)^{2} + 6\left(\frac{1}{r^{2}} + B\right)_{|a}\phi^{a}\right]\,, &\text{(B.43)}
\end{aligned}$$

$$\mathfrak{R}_{1(0)}^{\Box-\xi^4 R} = \left(\frac{1}{6}-\xi\right)\left(R+4\Delta_h\phi-6(\nabla\phi)^2+\frac{2}{r^2}\right), \tag{B.44}$$

$$\begin{aligned}\mathfrak{R}_{1(1)}^{\Box-\xi^4 R} &= \frac{1}{6}\left(\frac{1}{6}-\xi\right)\left[(1+r^2B)+r^2\nabla\phi\cdot\nabla\right]\left(R+4\Delta_h\phi-6(\nabla\phi)^2+\frac{2}{r^2}\right)\\ &+\frac{r^2}{180}\Bigg[RT+3R(\nabla\phi)^2+8T_{ab}T^{ab}+12T_{ab}\phi^a\phi^b\\ &+\left(R+4\Delta_h\phi-6(\nabla\phi)^2+\frac{2}{r^2}\right)_{|a}\phi^a\\ &+3[\Delta_h-2\nabla\phi\cdot\nabla]\left(\frac{1}{r^2}+B\right)-6\left(\frac{1}{r^2}+B\right)(\nabla\phi)^2\Bigg]\\ &+\frac{1}{90r^2}\left(2A^2+A(1+r^2B)-2(1+r^2B)^2\right),\end{aligned} \tag{B.45}$$

$$\begin{aligned}\mathfrak{R}_{2(0)}^{\Box-\xi^4 R} &= \frac{1}{2}\left(\frac{1}{6}-\xi\right)^2\left(R+4\Delta_h\phi-6(\nabla\phi)^2+\frac{2}{r^2}\right)^2\\ &+\frac{1}{6}\left(\frac{1}{6}-\xi\right)[\Delta_h-2\nabla\phi\cdot\nabla]\left(R+4\Delta_h\phi-6(\nabla\phi)^2+\frac{2}{r^2}\right)\\ &+\frac{1}{180}\Bigg[[\Delta_h-2\nabla\phi\cdot\nabla]\left(R+4\Delta_h\phi-6(\nabla\phi)^2+\frac{2}{r^2}\right)\\ &+\frac{1}{2}R^2-2RT+4T_{ab}T^{ab}+\frac{4}{r^4}A^2-\frac{2}{r^4}(1+r^2B)^2\Bigg].\end{aligned} \tag{B.46}$$

It is easily verified that for flat spacetime each of the $\mathfrak{R}_{n(k)}^{\Box-\xi^4 R}$ vanishes, except for $\mathfrak{R}_{0(0)}^{\Box-\xi^4 R}$.

Appendix C

Static Decompositions

These appendices contain formulae required for the calculations in Chapters 5 and 6, which deal with mode decompositions in static spaces.

C.1 Small-Distance Expansions for (1+3) Reductions

The line element for a static space may be written in the form

$$ds^2 = e^{-4\phi(x)}dt^2 + h_{ab}(x)dx^a dx^b \ . \tag{C.1}$$

For this static metric h_{ab} is an induced 3-metric and $n^\alpha = e^{2\phi(x)}\delta^\alpha_t$ is a unit vector normal to the surfaces of constant t. The extrinsic curvature K_{ab} on these surfaces vanishes. The nonvanishing Christoffel symbols are

$${}^4\Gamma^a_{bc}[g] = \Gamma^a_{bc}[h]\,, \quad {}^4\Gamma^a_{00} = 2e^{-4\phi}\phi^{;a}\,, \quad {}^4\Gamma^0_{0a} = -2\phi_{;a} \ . \tag{C.2}$$

Because we will be using some quantities defined in terms of the full four-dimensional metric g and others defined in terms of the three-dimensional metric h, some conventions on notation are in order. Henceforth four-dimensional curvatures and covariant derivatives will be denoted by ${}^4R_{\dots}$ and $()_{;\alpha}$ respectively, while $\Box$ is understood to represent the d'Alembertian with respect to g. All other curvatures and covariant derivatives are understood to be calculated using the three-metric h. In particular, ∇ and $()_{|a}$ are three-dimensional covariant derivatives, and Δ_h is the three-dimensional d'Alembertian. For the dilaton ϕ we shall understand ϕ_a, ϕ_{ab}, etc. to denote multiple three-dimensional covariant derivatives of ϕ.

With these conventions we have, for example,

$$\Box\phi = \Delta_h\phi - 2(\nabla\phi)^2 \,. \tag{C.3}$$

It is convenient to define the following three-dimensional tensor which occurs naturally in the 1 + 3 reduction:

$$T_{ab} \;=\; 2\left[\phi_{ab} - 2\phi_a\phi_b\right] , \tag{C.4}$$

$$T \;=\; T^a_a \;=\; 2\left[\Delta_h\phi - 2(\nabla\phi)^2\right] \;=\; 2\Box\phi \,. \tag{C.5}$$

In terms of T_{ab} the only nonvanishing components of the four-dimensional curvatures are

$$^4R_{abcd} = R_{abcd} \,, \quad {}^4R_{0a0b} = e^{-4\phi}T_{ab} \,, \tag{C.6}$$

$$^4R_{ab} = R_{ab} + T_{ab} \,, \quad {}^4R_{00} = e^{-4\phi}T \,, \quad {}^4R = R + 2T \,. \tag{C.7}$$

We shall also need the following expressions for $^4R_{\alpha\beta;\gamma}$, $^4R_{\alpha\beta;\gamma\delta}$, $^4R_{;\alpha}$, and $^4R_{;\alpha\beta}$:

$$^4R_{ab;c} \;=\; (R_{ab} + T_{ab})_{|c} \,, \tag{C.8}$$

$$^4R_{a0;0} \;=\; -2e^{-4\phi}\left[(R_{ab} + T_{ab})\,\phi^b - T\phi_a\right] , \tag{C.9}$$

$$^4R_{00;c} \;=\; e^{-4\phi}T_{|c} \,, \tag{C.10}$$

$$^4R_{ab;cd} \;=\; (R_{ab} + T_{ab})_{|cd} \,, \tag{C.11}$$

$$\begin{aligned} {}^4R_{ab;00} \;=\; & -2e^{-4\phi}\Big[(R_{ab} + T_{ab})_{|c}\,\phi^c - 4\phi_a\phi_b T + 2\phi_a\,(R_{bc} + T_{bc})\,\phi^c \\ & + 2\phi_b\,(R_{ac} + T_{ac})\,\phi^c\Big] \,, \end{aligned} \tag{C.12}$$

$$
\begin{aligned}
{}^4R_{a0;c0} &= -2e^{-4\phi}\Big[(R_{ab}+T_{ab})_{|c}\phi^b + 2\phi^b(R_{ab}+T_{ab})\phi_c \\
&\quad - 2\phi_a\phi_c T - \phi_a T_{|c}\Big]\,, && \text{(C.13)} \\
{}^4R_{a0;0d} &= -2e^{-4\phi}\left[(R_{ab}+T_{ab})\phi^b - T\phi_a\right]_{|d}\,, && \text{(C.14)} \\
{}^4R_{00;cd} &= e^{-4\phi}T_{|cd}\,, && \text{(C.15)} \\
{}^4R_{00;00} &= 2e^{-8\phi}\left[4(R_{ab}+T_{ab})\phi^a\phi^b - 4T\phi^a\phi_a - T_{|a}\phi^a\right]\,, && \text{(C.16)}
\end{aligned}
$$

$$
\begin{aligned}
{}^4R_{;a} &= (R+2T)_{|a}\,, && \text{(C.17)} \\
{}^4R_{;ab} &= (R+2T)_{|ab}\,, && \text{(C.18)} \\
{}^4R_{;00} &= -2e^{-4\phi}(R+2T)_{|a}\phi^a\,. && \text{(C.19)}
\end{aligned}
$$

For the mode decomposition (Fourier time transform) we need to know the behaviour of the two-point functions σ and $D^{1/2}a_n^{\square-V}$ for points $X^\alpha = (t,x)$ and $X'^\alpha = (t',x)$ where $\tau \equiv e^{-2\phi}(t-t')$ is small. Using the procedure of Appendix B.2 one can easily show that

$$
\begin{aligned}
2\sigma(t,x;t',x) &= \tau^2 - \frac{1}{3}\phi^a\phi_a\tau^4 + \frac{1}{45}\left[8(\phi^a\phi_a)^2 - 3\phi^a\phi^b\phi_{ab}\right]\tau^6 + \cdots\,, && \text{(C.20)} \\
\sigma_a(t,x;t',x) &= -\phi_a\tau^2 + \frac{1}{6}\left[\phi^b\phi_b\phi_a - \phi^b\phi_{ba}\right]\tau^4 + \cdots\,, && \text{(C.21)} \\
\sigma_t(t,x;t',x) &= e^{-2\phi}\tau\left[1 - \frac{2}{3}\phi^a\phi_a\tau^2 + \frac{1}{15}\left[8(\phi^a\phi_a)^2 - 3\phi^a\phi^b\phi_{ab}\right]\tau^4\right] \\
&\quad + \cdots\,. && \text{(C.22)}
\end{aligned}
$$

Combining the above expressions with the results of [80, 81] for small-σ^α expansions of the $D^{1/2}a_n$, it is easily shown that for the operator $\square - V$, where the potential function V is independent of t, the first three $D^{1/2}a_n^{\square-V}$ to $O(R^2)$ are

$$
\begin{aligned}
D^{1/2}a_0^{\square-V} &= 1 + \frac{1}{12}T\tau^2 + \frac{1}{360}\Big[6(R_{ab}+T_{ab})\phi^a\phi^b - 16T\phi^a\phi_a + 6T_{|a}\phi^a \\
&\quad + \frac{5}{4}T^2 + T^{ab}T_{ab}\Big]\tau^4\,, && \text{(C.23)}
\end{aligned}
$$

$$\begin{aligned} D^{1/2}a_1^{\Box-V} &= \left(\frac{1}{6}R - V + \frac{1}{3}T\right) + \left\{\frac{1}{2}\left(\frac{1}{6}R - V + \frac{1}{3}T\right)_{|a}\phi^a \right. \\ &+ \left(\frac{1}{3}V_{|a}\phi^a - \frac{1}{12}VT\right) + \frac{1}{360}\Big[3\Delta_h T + 6T^2 + 6T^{ab}T_{ab} \\ &+ 5RT + 2R^{ab}T_{ab} + 24\left(R_{ab} + T_{ab}\right)\phi^a\phi^b - 24T\phi^a\phi_a \\ &\left. - 18R_{|a}\phi^a - 42T_{|a}\phi^a\Big]\right\}\tau^2 , \end{aligned} \tag{C.24}$$

$$\begin{aligned} D^{1/2}a_2^{\Box-V} &= \frac{1}{2}\left(\frac{1}{6}R - V + \frac{1}{3}T\right)^2 + \frac{1}{30}\left[\Delta_h(R+2T) - 2(R+2T)_{|a}\phi^a\right] \\ &+ \frac{1}{180}\left[R_{abcd}R^{abcd} + 4T^{ab}T_{ab} - \left(R^{ab} + T^{ab}\right)\left(R_{ab} + T_{ab}\right) - T^2\right] \\ &- \frac{1}{6}\left[\Delta_h V - 2V_{|a}\phi^a\right] . \end{aligned} \tag{C.25}$$

C.2 Small-Distance Expansions for (2+2) Reductions

Consider a spacetime with the line element

$$ds^2 = \mathrm{e}^{-2\phi(x)}(dt_0^2 + dt_1^2) + h_{AB}(x)dx^A dx^B . \tag{C.26}$$

In this case we will be using some quantities defined in terms of the full four-dimensional metric $g_{\mu\nu}$, and others in terms of the two-dimensional metric h_{AB} $(A, B = 2, 3)$.[1] In analogy to the $(1+3)$-splitting case, four-dimensional curvatures and covariant derivatives will be denoted by ${}^4R_{\dots}$ and $()_{;a}$ respectively, while $\Box$ is understood to represent the d'Alembertian with respect to $g_{\mu\nu}$. All other curvatures and covariant derivatives are understood to be calculated using the two-metric h_{AB}. In particular, ∇ and $()_{|A}$ are two-dimensional covariant

[1]We use uppercase Latin indices for the $n = 2$ decomposition to avoid confusion with the $n = 1$ case.

derivatives, and Δ_h is the two-dimensional d'Alembertian. For the dilaton ϕ we shall understand ϕ_A, ϕ_{AB}, etc. to denote multiple two-dimensional covariant derivatives of ϕ. For example, with these conventions the four-dimensional d'Alembertian of a y-independent scalar S decomposes to

$$\Box S = \Delta_h S - 2\nabla\phi\cdot\nabla S \,. \tag{C.27}$$

In particular,

$$\Box\phi = \Delta_h\phi - 2(\nabla\phi)^2 = -\frac{1}{2}e^{2\phi}\Delta_h e^{-2\phi} \,. \tag{C.28}$$

For the given line element, the nonvanishing Christoffel symbols are $(i, j = 0, 1)$

$$\begin{aligned}
\Gamma^A_{BC}[g] &= \Gamma^A_{BC}[h] = \frac{1}{2}h^{AD}(h_{DB,C} + h_{CD,B} - h_{BC,D})\,, && \text{(C.29)}\\
\Gamma^A_{ij}[g] &= \phi^A e^{-2\phi}\eta_{ij} = \phi^A g_{ij}\,, && \text{(C.30)}\\
\Gamma^i_{jA}[g] &= -\phi_A\delta^i_j = -\phi_A g^i_j\,, && \text{(C.31)}\\
\Gamma_{Aij}[g] &= -\Gamma_{ijA}[g] = \phi_A e^{-2\phi}\eta_{ij} = \phi_A g_{ij}\,. && \text{(C.32)}
\end{aligned}$$

Meanwhile, the only nonvanishing components of the four-dimensional curvatures are

$$\begin{aligned}
{}^4R_{ABCD}[g] &= \frac{1}{2}R\,(g_{AC}g_{BD} - g_{AD}g_{BC})\,, && \text{(C.33)}\\
{}^4R_{AmBn}[g] &= g_{mn}[\phi_{AB} - \phi_A\phi_B]\,, && \text{(C.34)}\\
{}^4R_{ijkm}[g] &= -\phi_A\phi^A(g_{ik}g_{jm} - g_{im}g_{jk})\,, && \text{(C.35)}
\end{aligned}$$

$$\begin{aligned}
{}^4R_{AB}[g] &= \frac{1}{2}R\,g_{AB} + 2[\phi_{AB} - \phi_A\phi_B]\,, && \text{(C.36)}\\
{}^4R_{mn}[g] &= g_{mn}[\phi^A_A - 2\phi^A\phi_A]\,, && \text{(C.37)}
\end{aligned}$$

$${}^4R[g] = R + 4\phi^A_A - 6\phi^A\phi_A\,. \tag{C.38}$$

We shall also need the following components of ${}^4R_{\alpha\beta;\gamma}$, ${}^4R_{\alpha\beta;\gamma\delta}$, ${}^4R_{;\alpha}$, and ${}^4R_{;\alpha\beta}$:

$$\begin{aligned} {}^4R_{AB;C}[g] &= \frac{1}{2}g_{AB}R_{,C} + 2[\phi_{AB} - \phi_A\phi_B]_{|C}\,, & \text{(C.39)}\\ {}^4R_{Am;n}[g] &= g_{mn}\Big[-\frac{1}{2}R\phi_A + \phi_A\Delta_h\phi - 2\phi^B\phi_{BA}\Big]\,, & \text{(C.40)}\\ {}^4R_{mn;A}[g] &= g_{mn}[\Delta_h\phi - 2(\nabla\phi)^2]_{|A}\,, & \text{(C.41)} \end{aligned}$$

$$\begin{aligned} {}^4R_{mn;AB}[g] &= g_{mn}[\Delta_h\phi - 2(\nabla\phi)^2]_{|AB}\,, & \text{(C.42)}\\ {}^4R_{ij;km}[g] &= (g_{ik}g_{jm} + g_{im}g_{jk})\Big[\frac{1}{2}R(\nabla\phi)^2 - (\nabla\phi)^2\Delta_h\phi + 2\phi^A\phi^B\phi_{AB}\Big]\\ &\quad - (g_{ij}g_{km})\left[\Delta_h\phi - 2(\nabla\phi)^2\right]_{|A}\phi^A\,, & \text{(C.43)} \end{aligned}$$

$$ {}^4R_{;A}[g] = \left[R + 4\Delta_h\phi - 6(\nabla\phi)^2\right]_{,A}\,, \qquad \text{(C.44)}$$

$$\begin{aligned} {}^4R_{;AB}[g] &= \left[R + 4\Delta_h\phi - 6(\nabla\phi)^2\right]_{|AB}\,, & \text{(C.45)}\\ {}^4R_{;mn}[g] &= -g_{mn}\left[R + 4\Delta_h\phi - 6(\nabla\phi)^2\right]_{|A}\phi^A\,. & \text{(C.46)} \end{aligned}$$

Note again that operators and curvatures are with respect to the two-dimensional metric h_{AB} unless explicitly labelled otherwise.

We now write out the expansions of $\sigma(\mathbf{t}, x; \mathbf{t}', x)$ and the $D^{1/2}a_n(\mathbf{t}, x; \mathbf{t}', x)$ for small separations. Defining $\tau^i \equiv \mathrm{e}^{-\phi}(t^i - t'^i)$, we have to second order in the curvature

$$\begin{aligned} 2\sigma(\mathbf{t}, x; \mathbf{t}', x) &= \tau^2 - \frac{1}{12}(\nabla\phi)^2\tau^4 + \frac{1}{360}\left[4(\nabla\phi)^4 - 3\phi^A\phi^B\phi_{AB}\right]\tau^6\,, & \text{(C.47)}\\ \sigma^i(\mathbf{t}, x; \mathbf{t}', x) &= \mathrm{e}^{\phi}\tau^i\left[1 - \frac{1}{6}\phi_A\phi^A\tau^2 + \frac{1}{120}\left[4(\phi_A\phi^A)^2 - 3\phi^A\phi^B\phi_{AB}\right]\tau^4\right]\,, & \text{(C.48)}\\ \sigma^a(\mathbf{t}, x; \mathbf{t}', x) &= -\frac{1}{2}\phi^A\tau^2 - \frac{1}{24}\phi_B\left(\phi^{BA} - 2\phi^B\phi^A\right)\tau^4\\ &\quad + \frac{1}{720}\Big[-12(\phi^B\phi_B)^2\phi^A + 8\phi^B\phi_B\phi_C\phi^{CA} + 9\phi^A\phi^B\phi^C\phi_{BC}\\ &\quad - 3\phi^B\phi_{BC}\phi^{CA} - \frac{3}{2}\phi_B\phi_C\phi^{BCA}\Big]\tau^6\,. & \text{(C.49)} \end{aligned}$$

Combining these expressions with the results of [80, 81], it is easily shown that for the operator $\Box - V$, where the potential function V is independent of t^i, the first three $D^{1/2}a_n^{\Box-V}$ are

$$\begin{aligned}
D^{1/2}a_0^{\Box-V} &= 1 + \frac{1}{12}\left[\Delta_h\phi - 2(\nabla\phi)^2\right]\tau^2 + \frac{1}{1440}\Big[3R(\nabla\phi)^2 + 48(\nabla\phi)^4 \\
&\quad - 36(\nabla\phi)^2\Delta_h\phi - 44\phi^A\phi^B\phi_{AB} + 5(\Delta_h\phi)^2 + 4\phi^{AB}\phi_{AB} \\
&\quad + 12\phi^A(\Delta_h\phi)_A\Big]\tau^4 + O(\tau^6)\,, \qquad \text{(C.50)}
\end{aligned}$$

$$\begin{aligned}
D^{1/2}a_1^{\Box-V} &= \left[\frac{1}{6}R - V + \frac{2}{3}\phi^A_A - \phi^A\phi_A\right] + \frac{1}{2}\Big[-\frac{1}{6}V_A\phi^A - \frac{1}{6}V[\Delta_h\phi - 2(\nabla\phi)^2] \\
&\quad + \frac{1}{30}R_{,A}\phi^A + R[\frac{1}{30}\Delta_h\phi - \frac{2}{45}(\nabla\phi)^2] + \frac{1}{180}\big(60(\nabla\phi)^4 \\
&\quad - 62(\nabla\phi)^2\Delta_h\phi - 52\phi^A\phi^B\phi_{AB} + 16(\Delta_h\phi)^2 - 4\phi^{AB}\phi_{AB} \\
&\quad + 18\phi^A(\Delta_h\phi)_A - 12\phi^A\Delta_h(\phi_A) + 3\Delta_h^2\phi\big)\Big]\tau^2 + O(\tau^4)\,, \qquad \text{(C.51)}
\end{aligned}$$

$$\begin{aligned}
D^{1/2}a_2^{\Box-V} &= \frac{1}{2}\left[\frac{1}{6}R - V + \frac{2}{3}\phi^A_A - \phi^A\phi_A\right]^2 - \frac{1}{6}\Delta_h V + \frac{1}{3}V_{,A}\phi^A + \frac{1}{30}\Delta_h R \\
&\quad - \frac{1}{15}R_{,A}\phi^A + \frac{1}{180}\Big[\frac{1}{2}R^2 - 2R[\Delta_h\phi - (\nabla\phi)^2] + 8(\nabla\phi)^2\Delta_h\phi \\
&\quad + 136\phi^A\phi^B\phi_{AB} - 2(\Delta_h\phi)^2 - 68\phi^{AB}\phi_{AB} - 72\phi^A\Delta_h(\phi_A) \\
&\quad - 48\phi^A(\Delta_h\phi)_A + 24\Delta_h^2\phi\Big] + O(\tau^2)\,. \qquad \text{(C.52)}
\end{aligned}$$

C.3 Useful Formulae

C.3.1 The Modified Bessel Function K_ν

The modified Bessel functions $K_\nu(z)$ may be defined via the integral

$$\int_0^\infty dx\, x^{-1-\nu}\exp\left\{-x - \frac{z^2}{4x}\right\} = 2\left(\frac{2}{z}\right)^\nu K_\nu(z)\,. \qquad \text{(C.53)}$$

It may be shown that $K_{-\nu}(z) = K_\nu(z)$. Furthermore, for $\nu > 0$ the $K_\nu(z)$ obey the differential relation

$$\left(-\frac{1}{z}\frac{d}{dz}\right)^n z^\nu K_\nu(z) = z^{\nu-n} K_{\nu-n}(z) \tag{C.54}$$

In particular, for $z = m\sqrt{\tau^2+\epsilon^2}$ one can easily show that

$$\frac{1}{z^n}K_n(z) = \left(-\frac{1}{m^2\epsilon}\frac{d}{d\epsilon}\right)^n K_0(z) \tag{C.55}$$

Combining (C.55) with integral (6.677) of [76],

$$\int_{-\infty}^{\infty} d\tau\ \cos(\bar{\omega}\tau)\ K_0(m\sqrt{\epsilon^2+\tau^2}) = \frac{\pi}{\sqrt{m^2+\bar{\omega}^2}} \exp(-\epsilon\sqrt{m^2+\bar{\omega}^2})\ , \tag{C.56}$$

allows us to evaluate the $(1+3)$-splitting Fourier transforms of Sections 5.2.1, 5.3.1 as follows:

$$\begin{aligned}\int_{-\infty}^{\infty} d\tau\ \cos(\bar{\omega}\tau)\ \tau^{2k}\ \frac{1}{(m\sqrt{\tau^2+\epsilon^2})^n} K_n(m\sqrt{\epsilon^2+\tau^2}) \ &= \\ = (-1)^k \frac{d^{(2k)}}{d\bar{\omega}^{(2k)}} \left(-\frac{1}{m^2\epsilon}\frac{d}{d\epsilon}\right)^n \frac{\pi}{\sqrt{m^2+\bar{\omega}^2}} \exp(-\epsilon\sqrt{m^2+\bar{\omega}^2})\,. \end{aligned}\tag{C.57}$$

For convenience, we have used the notation $\bar{\omega} = e^{2\phi}\omega$ introduced in Section 6.1.

C.3.2 Integrals of μ^n for the (1+3) Reduction

For $\mu = \sqrt{m^2+x^2}$ it is easily shown that for large $\bar{\omega}$

$$\int_0^{\bar{\omega}} dx\mu^3 \ = \ \frac{1}{4}\bar{\omega}^4 + \frac{3}{4}m^2\bar{\omega}^2 + \frac{9}{32}m^4 + \frac{3}{8}m^4 \ln\frac{2\bar{\omega}}{m}\ , \tag{C.58}$$

$$\int_0^{\bar{\omega}} dx\mu \ = \ \frac{1}{2}\bar{\omega}^2 + \frac{1}{4}m^2 + \frac{1}{2}m^2 \ln\frac{2\bar{\omega}}{m}\ , \tag{C.59}$$

$$\int_0^{\bar{\omega}} dx\frac{1}{\mu} \ = \ \ln\frac{2\bar{\omega}}{m}\ . \tag{C.60}$$

In addition, for $n \geq 1$,

$$\int_0^{\infty} dx\mu^{-(2n+1)} = \frac{1}{m^{2n}} \frac{2^{n-1}\,(n-1)!}{(2n-1)!!}\,. \tag{C.61}$$

See for example (2.271) of [76]. These results are sufficient to perform the sums over modes in Sections 6.1 and 6.2.

C.3.3 Formulae for the (2+2) Reduction

The Fourier transforms of Sections 5.2.2 and 5.3.2 were computed before performing the s-integration by expanding all z-dependent quantities for small curvatures and using

$$\int_{-\infty}^{\infty} d^2z \exp\left\{-\frac{1}{4s}z^2 + i\bar{p}z\right\} z^{2n} = (4\pi s)\mathrm{e}^{-\bar{p}^2 s}(4s)^n I_n \,, \tag{C.62}$$

where $\bar{p} = \mathrm{e}^{\phi}p$ and

$$\begin{aligned}
I_0 &= 1\,, && \text{(C.63)}\\
I_1 &= 1 - \bar{p}^2 s\,, && \text{(C.64)}\\
I_2 &= 2 - 4\bar{p}^2 s + \bar{p}^4 s^2\,, && \text{(C.65)}\\
I_3 &= 6 - 18\bar{p}^2 s + 9\bar{p}^4 s^2 - \bar{p}^6 s^3\,, && \text{(C.66)}\\
I_4 &= 24 - 96\bar{p}^2 s + 72\bar{p}^4 s^2 - 16\bar{p}^6 s^3 + \bar{p}^8 s^4\,. && \text{(C.67)}
\end{aligned}$$

For the summation over modes referred to in Sections 6.1 and 6.2 one may use the integrals

$$\begin{aligned}
\int_0^{\bar{p}} dx\, x \ln\frac{\mu^2}{m^2} &= \frac{1}{2}(\bar{p}^2 + m^2)\ln\frac{\bar{p}^2+m^2}{m^2} - \frac{1}{2}\bar{p}^2\,, && \text{(C.68)}\\
\int_0^{\bar{p}} dx\, x\, \mu^{-2} &= \frac{1}{2}\ln\frac{\bar{p}^2+m^2}{m^2}\,, && \text{(C.69)}
\end{aligned}$$

and for $n > 1$,

$$\int_0^{\infty} dx\, x\, \mu^{-2n} = \frac{1}{2(n-1)m^{2(n-1)}}\,, \tag{C.70}$$

where $\mu = \sqrt{m^2 + x^2}$.

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