University of Alberta
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Solution Techniques for Geometrically Nonlinear Structures

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NOTATION

a_1, a_2, a_3	-	local coordinate system
A	-	area of bar
b	-	horizontal projection of bar
c	-	vertical projection of bar
d	-	length of bar
E	-	vector of residual forces or error in equilibrium equations; modulus of elasticity.
$\mathcal{E}$	-	strain energy
f	-	equilibrium equation (function)
F	-	vector of equilibrium equations (functions); vector of total loads applied to augmented structure
$\tilde{K}$	-	nonlinear portion of stiffness matrix
K*	-	pseudo-load stiffness matrix
K_A	-	augmented stiffness matrix
K_r	-	stiffness matrix for r^{th} load increment
K_s	-	stiffness matrix of spring support system
P	-	vector of equivalent loads
ΔP	-	vector of equivalent load increments
$\hat{P}$	-	vector of equivalent loads on the unstiffened structure
p	-	load parameter
$\bar{p} = Rd^3/EAb^3$	-	nondimensional load parameter

Q	-	vector of pseudo-loads
r	-	load increment index
R	-	external truss load
S	-	external truss load
$\bar{s} = Sd^3/EAb^3$	-	nondimensional load parameter
s	-	iterate index
s_k	-	total number of iterations for load increment k
t_r	-	total number of iterations after (r-1) load increments
T	-	transformation matrix between local and global coordinates
u, v	-	displacements in ξ and η directions
$\bar{u}$, $\bar{v}$	-	nondimensional displacements in ξ and η directions
x	-	displacement coordinate
X	-	vector of displacement coordinates
x_i^*, X^*	-	reference configuration
α_i	-	over-relaxation factor
β	-	a relaxation parameter; nondimensional coordinate a_1/d
Δ	-	increment; deflection of piston
γ	-	stiffness of spring
$\mu = c/b$	-	geometric parameter for two-bar truss
θ	-	dummy variable for perturbation formulation; angle in radians

Solution Techniques for

Geometrically Nonlinear Structures

CHAPTER I INTRODUCTION

§1. *Scope of Report*

All ductile structures respond in a nonlinear manner immediately prior to and during collapse. In order to accurately predict maximum load carrying capacity it is therefore necessary to be able to assess the effects of this nonlinear response. In addition, in an indeterminate structure, individual components may have passed their maximum load carrying capacity prior to an overall collapse, and the behavior of these components in their unloading range may be critical to the overall behavior of the assemblage. Their capacity to deform in a ductile manner during collapse, without significant reduction in load carrying capacity, and their interaction with other structural components, therefore becomes a subject of interest to the structural analyst.

Nonlinearities in structural response may arise from material effects or from geometric effects. In general both types of nonlinearities are present during collapse. However, the true behavior of a structure beyond the range of linear response is often very difficult to predict. A prerequisite to solving the 'real' problem is the ability to predict the nonlinear geometric effects. This report is concerned with an assessment of the suitability of various techniques of nonlinear analysis to predict highly nonlinear geometric response, and in particular the unloading behavior. The response of simple pin-jointed structures is used to form a basis for this assessment.

Many techniques of nonlinear structural analysis have been proposed (2, 6, 8, 10, 14, 15, 17) and it is not possible to consider all of these in a single report. The report is therefore confined to an examination of the formulation and application of some of the most common techniques which appear to be in vogue in the recent literature.

Prior to applying the various techniques to some illustrative pin-jointed structures, an attempt is made to derive the basic equations of each technique in a unified manner, so that the inter-relationships between them can be examined.

§2. *Symbolic Statement of Problem*

Most general nonlinear analyses are based on computer oriented numerical solutions of the equilibrium equations, in which the dependent variables are a discrete set of displacement coordinates. Each equilibrium equation will therefore be assumed to be expressible as a function of the discrete set of displacement coordinates (x_1, x_2, ... x_n), and a loading parameter p. These equations will be written symbolically, as

$$\begin{aligned} f_1\ (x_1, x_2, \ldots, x_n, p) &= 0 \\ f_2\ (x_1, x_2, \ldots, x_n, p) &= 0 \\ &\cdot \\ &\cdot \\ &\cdot \\ f_n\ (x_1, x_2, \ldots, x_n, p) &= 0\ . \end{aligned} \qquad \text{(2-1a)}$$

Define the vector of displacement coordinates X, and the vector of equilibrium functions F, as

$$X^T = < x_1, x_2, \ldots, x_n >$$

and

$$F^T = < f_1, f_2, \ldots, f_n > .$$

Eqs. (2-1a) may then be represented as

$$F(X,p) = \{0\}\ . \qquad \text{(2-1b)}$$

A solution occurs when, for any specified value of the load parameter p, the vector X has been determined such that Eqs. (2-1) are satisfied.

If Eqs. (2-1) are not satisfied by a given (X,p), we may write

$$F(X,p) = -E \neq \{0\} \tag{2-2}$$

where E is the vector of 'residual force effects' (or 'unbalanced forces', or 'error in forces') for the given configuration. It is often useful to be able to visualize the effects of a particular procedure on these equations. A schematic representation may be obtained if the vector X is assumed to be composed of only a single displacement coordinate and the vector F is assumed to consist of only a single equilibrium function. This artifice will be used throughout the report. F(X,p) may then be represented as a surface in 3-space as shown in Fig. 1. The intersection of this surface with the coordinate plane represents a solution to the set of equilibrium equations (ie. $F(X,p) = \{0\}$). The value of the function $F(X,p) \neq \{0\}$ represents the unbalanced forces as indicated by Eq. (2-2).

In the particular case where loading and geometric effects can be separated, the functions of Eqs. (2-1) may be represented as

$$F(X,p) = G(X) - P(p) \,. \tag{2-3}$$

Using the artifice described above, the curve G(X) may be graphed in 2-space as shown in Fig. 2, and the point of intersection of the horizontal line for any P(p) with the curve G(X) represents the solution for that value of the load parameter. Note that in this case

$$\frac{\partial F}{\partial x_i} = \frac{\partial G}{\partial x_i} \qquad (2\text{-}4)$$

which is represented by the slope of the G(X) curve, and the unbalanced force -E is represented by the vertical difference between the magnitude of G(X) and the magnitude of P(p). (The negative sign on E disappears when P(p) exceeds G(X).)

CHAPTER II SOLUTION TECHNIQUES

§3. *Solution Techniques Arising Directly From Taylor's Series* (7)

Many of the commonest solution techniques arise in a natural manner from a Taylor's series expansion. For any specified values of X,p, say (X*,p*), the value of F(X,p) at a point sufficiently close to (X*,p*) may be determined from the Taylor's series expansion

$$\begin{aligned} F(X,p) = {} & F(X^*,p^*) + \left.\frac{\partial F}{\partial x_i}\right|_* (x_i - x_i^*) + \left.\frac{\partial F}{\partial p}\right|_* (p-p^*) \\ & + \frac{1}{2!}\left\{ \left.\frac{\partial^2 F}{\partial x_i \partial x_j}\right|_* (x_i - x_i^*)(x_j - x_j^*) + 2\left.\frac{\partial^2 F}{\partial x_i \partial p}\right|_* (x_i - x_i^*)(p-p^*) \right. \\ & \left. + \left.\frac{\partial^2 F}{\partial p^2}\right|_* (p-p^*)^2 \right\} + \frac{1}{3!}\left\{ \left.\frac{\partial^3 F}{\partial x_i \partial x_j \partial x_k}\right|_* (x_i - x_i^*)(x_j - x_j^*)(x_k - x_k^*) \right. \\ & + 3\left.\frac{\partial^3 F}{\partial x_i \partial x_j \partial p}\right|_* (x_i - x_i^*)(x_j - x_j^*)(p-p^*) + 3\left.\frac{\partial^3 F}{\partial x_i \partial p^2}\right|_* (x_i - x_i^*)(p-p^*)^2 \\ & \left. + \left.\frac{\partial^3 F}{\partial p^3}\right|_* (p-p^*)^3 \right\} + \ldots \end{aligned} \qquad (3\text{-}1)$$

where x_i is an element of the vector X and the summation convention has been adopted.

To find an equilibrium position it is necessary that F(X,p) be zero. Designating the point (X*, p*) by (X^r, p^r), an approximate equilibrium position (X^{r+1}, p^{r+1}) can be estimated from Eq. (3-1) by retaining only the linear terms on the right hand side and requiring that

$$F(X^{r+1},p^{r+1}) \doteq F(X^r,p^r) + \left.\frac{\partial F}{\partial x_i}\right|_r (x_i^{r+1}-x_i^r) + \left.\frac{\partial F}{\partial p}\right|_r (p^{r+1}-p^r) = \{0\} \qquad (3\text{-}2)$$

Using matrix notation and solving for X^{r+1} results in

$$X^{r+1} = X^r + \left[K\right]_r^{-1} \{E^r + \Delta P^{r+1}\} \qquad (3\text{-}3)$$

where $[K]_r = \left[\frac{\partial F}{\partial x_i}\right]_r$ is known as the stiffness matrix; $E^r = - F(X^r,p^r)$ is the vector of residual forces at (X^r, p^r); and $\Delta P^{r+1} = - \left.\frac{\partial F}{\partial p}\right|_r (p^{r+1}-p^r)$ is the change in the effective force vector.

$$P = - \int_0^p \frac{\partial F}{\partial p} \, d\, p . \qquad (3\text{-}4)$$

Eq. (3-3) is the basic equation for a *modified incremental analysis* as illustrated in Fig. 3b.

If it is assumed that the equilibrium equations are exactly satisfied at (X^r,p^r), then $E^r = \{0\}$, and the equation for *simple incremental analysis* results, namely

$$X^{r+1} = X^r + \left[K\right]_r^{-1} \{\Delta P\}^{r+1} . \qquad (3\text{-}5)$$

Solution by this equation is illustrated in Fig. 3a.

When the load is applied in a single increment, $p^{r+1} = p^r$ and $\Delta p^{r+1} = 0$ for $r > 0$. Eq. (3-3) then becomes the recursion equation for *Newton-Raphson iteration* which is expressed by the equations

$$X^1 = X^0 + \left[K\right]_0^{-1} \Delta P^1 \qquad \text{for} \quad r = 0 \tag{3-6a}$$

and

$$X^{r+1} = X^r + \left[K\right]_r^{-1} E^r \qquad \text{for} \quad r > 0 \ . \tag{3-6b}$$

This procedure is illustrated in Fig. 3c. Since Newton-Raphson iteration requires the evaluation and inversion of the matrix $\left[K\right]_r = \left[\frac{\partial F}{\partial x_i}\right]_r$ for every iterate, less computation may result if the matrix $[K]_0$ is used throughout. Eqs. (3-6) then become

$$X^1 = X^0 + \left[K\right]_0^{-1} \Delta P^1 \qquad \text{for} \quad r = 0 \tag{3-7a}$$

and

$$X^{r+1} = X^r + \left[K\right]_0^{-1} E^r \qquad \text{for} \quad r > 0 \ , \tag{3-7b}$$

which is the *modified Newton-Raphson* procedure illustrated in Fig. 3d.

For problems where nonlinearities are more severe, the above procedures may not perform satisfactorily and it may then be desirable to incorporate the iterative procedure of Eqs. (3-6) or (3-7) into the incremental loading procedure expressed in Eq. (3-3). Let r be the number of the load increment and s be the number of iterations from the beginning of that load increment. Let s_k be the total number of iterations occuring during load increment k and

$$t_r = \sum_{k=1}^{r-1} s_k$$

be the total number of iterations after r-1 load increments. Then Eq. (3-3) yields the equations for the *incremental Newton-Raphson procedure* which consists of

$$X^{t_r+1} = X^{t_r} + [K]^{-1}_{t_r} \left\{E^{t_r} + \Delta P^r\right\} \qquad \text{for} \quad s = 0 \qquad (3\text{-}8a)$$

and

$$X^{t_r+s+1} = X^{t_r+s} + [K]^{-1}_{t_r+s} \left\{E^{t_r+s}\right\} \qquad \text{for} \quad s > 0 \qquad (3\text{-}8b)$$

which are simply Eqs. (3-6) applied to each load step as illustrated in Fig. 3e. This technique suffers from the same deficiency as the Newton Raphson method and if the iteration is carried out with the matrix at the beginning of the load step the *modified incremental Newton-Raphson technique* results, as expressed by the equations

$$X^{t_r+1} = X^{t_r} + [K]^{-1}_{t_r} \left\{E^{t_r} + \Delta P^r\right\} \qquad \text{for} \quad s = 0 \qquad (3\text{-}9a)$$

and

$$X^{t_r+s+1} = X^{t_r+s} + [K]^{-1}_{t_r} \left\{E^{t_r+s}\right\} \qquad \text{for} \quad s > 0\,. \qquad (3\text{-}9b)$$

This procedure is illustrated in Fig. 3f.

All iterative techniques, such as those represented by Eqs. (3-6), (3-7), (3-8) and (3-9), assume that iteration will be continued until some measure of the displacement increments such as $\Delta X^T \cdot \Delta X$, or some measure of the unbalanced forces such as $E^T \cdot E$, becomes sufficiently small.

A *static perturbation technique* (14,17) may be derived directly from Eq. (5) by assuming that $x_i - x_i^*$ and $p - p^*$ are functions of a single parameter θ (with $\theta = 0$ at X^*, p^*), and expanding by Taylor's series in the forms

$$\Delta x_i = \Delta\dot{x}_i \, \theta + \frac{\Delta\ddot{x}_i}{2!} \theta^2 + \frac{\Delta\dddot{x}_i}{3!} \theta^3 + \ldots. \tag{3-10a}$$

$$\Delta p = \Delta\dot{p} \, \theta + \frac{\Delta\ddot{p}}{2!} \theta^2 + \frac{\Delta\dddot{p}}{3!} \theta^3 + \ldots. \tag{3-10b}$$

where $(\dot{\ }) = d/d\theta$ and $\theta < 1$. Substituting Eqs. (3-10) into Eq. (3-1), assuming $F(X,p) = F(X^*,p^*) = \{0\}$, and grouping terms with like powers of θ, requires that the following equations be satisfied.

$$\left.\frac{\partial F}{\partial x_i}\right|_* \Delta\dot{x} + \left.\frac{\partial F}{\partial p}\right|_* \Delta\dot{p} = \{0\} \tag{3-11a}$$

$$\left.\frac{\partial F}{\partial x_i}\right|_* \Delta\ddot{x}_i + \left.\frac{\partial F}{\partial p}\right|_* \Delta\ddot{p} + \left.\frac{\partial^2 F}{\partial x_i \partial x_j}\right|_* \Delta\dot{x}_i \, \Delta\dot{x}_j$$

$$+ \; 2 \left.\frac{\partial^2 F}{\partial x_i \partial p}\right|_* \Delta\dot{x}_i \, \Delta\dot{p} + \left.\frac{\partial^2 F}{\partial p^2}\right|_* (\Delta\dot{p})^2 = \{0\} \tag{3-11b}$$

$$\left.\frac{\partial F}{\partial x_i}\right|_* \dddot{\Delta x}_i + \left.\frac{\partial F}{\partial p}\right|_* \dddot{\Delta p} + 3 \left.\frac{\partial^2 F}{\partial x_i \partial x_j}\right|_* (\ddot{\Delta x}_i \dot{\Delta x}_j) + 3 \left.\frac{\partial^2 F}{\partial p \partial x_i}\right|_* (\ddot{\Delta x}_i \dot{\Delta p} + \dot{\Delta x}_i \ddot{\Delta p})$$

$$+ 3 \left.\frac{\partial^2 F}{\partial p^2}\right|_* \dot{\Delta p} \ddot{\Delta p} + \left.\frac{\partial^3 F}{\partial x_i \partial x_j \partial x_k}\right|_* \dot{\Delta x}_i \dot{\Delta x}_j \dot{\Delta x}_k + 3 \left.\frac{\partial^3 F}{\partial x_i \partial x_j \partial p}\right|_* \dot{\Delta x}_i \dot{\Delta x}_j \dot{\Delta p}$$

$$+ 3 \left.\frac{\partial^3 F}{\partial x_i \partial p^2}\right|_* \dot{\Delta x}_i (\dot{\Delta p})^2 + \left.\frac{\partial^3 F}{\partial p^3}\right|_* (\dot{\Delta p})^3 = \{0\} \quad \text{(3-11c)}$$

If θ is taken as the load parameter p, $\dot{\Delta p} = 1$, $\ddot{\Delta p} = \dddot{\Delta p} = \ldots = 0$, and Eq. (3-11a) becomes

$$\left.\frac{\partial F}{\partial x_i}\right|_* \dot{\Delta x}_i + \frac{\partial F}{\partial p} = \{0\} \quad \text{(3-12a)}$$

which is sufficient to determine $\dot{\Delta x}_i$ providing $\left.\frac{\partial F}{\partial x_i}\right|_*$ is nonsingular. Eq. (3-11b) becomes

$$\left.\frac{\partial F}{\partial x_i}\right|_* \ddot{\Delta x}_i = -\left[\left.\frac{\partial^2 F}{\partial x_i \partial x_j}\right|_* \dot{\Delta x}_i \dot{\Delta x}_j + 2 \left.\frac{\partial^2 F}{\partial x_i \partial p}\right|_* \dot{\Delta x}_i + \frac{\partial^2 F}{\partial p^2}\right] \quad \text{(3-12b)}$$

which is sufficient to determine $\ddot{\Delta x}_i$ once $\dot{\Delta x}_i$ is now known. Similarly Eq. (3-11c) may be solved for $\dddot{\Delta x}_i$. Only the matrix $\left.\frac{\partial F}{\partial x_i}\right|_*$ need be inverted in this procedure.

A similar procedure can be carried out if θ is taken as one of the displacement coordinates.

§4. *Relaxation Factors*

The rate of convergence of an iterative scheme may sometimes be improved by applying an arbitrary multiplier, usually called a relaxation factor, to either the displacement increment or the unbalanced forces. If a relaxation factor of β is applied to displacement increments, Eq. (3-3) becomes

$$X^{r+1} = X^r + \beta \left[K\right]_r^{-1} \left\{E^r + \Delta P^{r+1}\right\} \tag{4-1a}$$

and, if applied to the unbalanced forces, Eq. (3-3) becomes

$$X^{r+1} = X^r + \left[K\right]_r^{-1} \left\{\beta E^r + \Delta P^{r+1}\right\} . \tag{4-1b}$$

All of the iterative schemes of section 3 can be modified to include relaxation factors.

When convergence is monotonic the factor should be greater than one and is called an *over-relaxation factor*. When convergence is oscillatory the factor should be less than one and is called an *under-relaxation factor*.

§5. *Pseudo-Load Techniques* (2,3)

A common technique in nonlinear analysis is to retain only linear terms on the left hand side of the equation and transfer quadratic and higher order terms to the right hand side. In structural analysis these higher order terms may be interpreted as additional loads and hence may be referred to as pseudo-loads. To implement such an approach it is necessary that the linear terms of F(X,p) be separated from the higher order terms.

Assuming the restricted form of F(X,p) in Eq. (2-3), the Taylor's series expansion of Eq. (3-1) becomes

$$F(X,p) = F(X^*,p^*) + \left.\frac{\partial F}{\partial x_i}\right|_* (x_i - x_i^*) + \frac{1}{2!}\left.\frac{\partial^2 F}{\partial x_i \partial x_j}\right|_* (x_i - x_i^*)\,(x_j - x_j^*)$$

$$+ \frac{1}{3!}\left.\frac{\partial^3 F}{\partial x_i \partial x_j \partial x_k}\right|_* (x_i - x_i^*)\,(x_j - x_j^*)\,(x_k - x_k^*) + \ldots$$

$$+ \left.\frac{\partial F}{\partial p}\right|_* (p-p^*) + \frac{1}{2!}\left.\frac{\partial^2 F}{\partial p^2}\right|_* (p-p^*)^2 + \frac{1}{3!}\left.\frac{\partial^3 F}{\partial x^3}\right|_* (p-p^*)^3 + \ldots. \quad (5\text{-}1)$$

If it is further assumed that P is linear with respect to p, and using the notation $\Delta X = X - X^*$, Eq. (5-1) may be written in matrix notation as

$$F(X,p) = F(X^*,p^*) + \left[K^{(0)}\right]_* \Delta X + \left[K^{(1)}\right]_* \Delta X + \left[K^{(2)}\right]_* \Delta X + \ldots$$

$$- \Delta P \quad (5\text{-}2)$$

where the superscript on the K matrices indicates the degree of dependence of the elements of the matrices on ΔX.

Using Eq. (5-2) and assuming (X^*,p^*) as the origin, Eqs. (2-1) may be written as

$$\left[K^{(0)}\right] X + \left[\tilde{K}\right] X - P(p) = \{0\} \tag{5-3}$$

where $\left[\tilde{K}\right]$ includes the effects of all $\left[K^{(i)}\right]$ for $i > 0$. Designating the second term as the 'pseudo-load', $Q(X)$, Eq. (5-3) becomes

$$\left[K^{(0)}\right] X + Q(X) - P(p) = \{0\} \tag{5-4}$$

and the reason for the terminology becomes apparent when the last two terms are transferred to the right hand side. Pseudo-loads represent the departure of the force-deformation relationship from linearity as indicated schematically in Fig. 4.

The vector X which satisfies Eq. (5-3), for any specified value of p, represents an equilibrium configuration. However, since Q is a function of X, this vector is not known 'a priori', and Eq. (5-4) must be solved iteratively through the recurrence equation

$$X^{r+1} = \left[K^{(0)}\right]^{-1} \left\{P - Q^r\right\} \tag{5-5}$$

where $Q^r = Q(X^r)$. This *iterative pseudo-load technique* is illustrated in Fig. 4a. To relate the pseudo-load technique to the methods of section 3 we note that until convergence has been attained, after say m iterates, Eq. (5-4) is not completely satisfied and may therefore be written as

$$\left[K^{(0)}\right] X^r + Q^r - P = - E^r \tag{5-6}$$

where $-E^r$ is the unbalanced force vector at (X^r,p). However, Eq. (5-5) requires that

$$\left[K^{(0)}\right] X^r + Q^{r-1} - P = \{0\} \tag{5-7}$$

Subtracting Eq. (5-7) from Eq. (5-6) yields

$$Q^r = Q^{r-1} - E^r \tag{5-8}$$

and this relationship is apparent on Fig. 4a.

Since $Q^o = \{0\}$ we have

$$Q^r = \sum_{k=1}^{r} - E^k . \tag{5-9}$$

A comparison of Figs. 4a and 3d, and the corresponding equations, indicates that the *incremental pseudo-load* solution is essentially the same as a *modified Newton-Raphson* solution. Pseudo-loads often arise in a natural way from a consideration of nonlinear material response in the form of *initial strains*. They may be evaluated in total, if the nonlinear expressions are explicit, or accumulated through an equation such as Eq. (5-9).

If the load is incremented for every iterate the preceding equations (Eqs. (5-5) and (5-9)) become

$$Q^r = Q^{r-1} - E^r \qquad (r > 0) \tag{5-10a}$$

$$X^{r+1} = \left[K^{(0)}\right]^{-1} \left\{P^{r+1} - Q^r\right\} \qquad (r \geq 0) \tag{5-10b}$$

This *incremental pseudo-load procedure* is illustrated in Fig. 4b. A comparison of Fig. 4b with Fig. 3a indicates that these results are similar to those obtained from a simple incremental analysis but, since $K^{(0)}$ is retained throughout, they are not identical.

When the initial point (X*,p*) is not the origin, but is some point on the equilibrium path, so that F(X*,p*) = {0}, Eq. (5-2) yields

$$\left[K^{(0)}\right]_* \Delta X + \left[\tilde{K}\right]_* \Delta X - \Delta P = \{0\} \qquad (5\text{-}11a)$$

or

$$\left[K^{(0)}\right]_* \Delta X + Q(\Delta X) - \Delta P = \{0\} \qquad (5\text{-}11b)$$

where ΔX and ΔP are departures from point (X*,p*) and Q(ΔX) represents the deviation from the linear 'tangent plane' at (X*,p*) as shown in Fig. 4c. The equilibrium point for the load P* + ΔP may then be determined from the recursive equations

$$Q^s = Q^{s-1} - E^s \qquad (s > 0) \qquad (5\text{-}12a)$$

$$X^{s+1} = X^* + \left[K^{(0)}\right]_*^{-1} \left\{\Delta P - Q^s\right\} \qquad (s \geq 0) \qquad (5\text{-}12b)$$

as illustrated in Fig. 4c, where s is the iterate number from the time of application of the load increment ΔP.

If this 'tangent stiffness' concept is applied with an increment of loading occuring for every iterate, the *modified incremental pseudo-load technique* results, as illustrated in Fig. 4d. The recursive equations are

$$Q^r = Q^{r-1} - E^r \qquad (r > 0) \qquad (5\text{-}13a)$$

$$X^{r+1} = X^r + \left[K^{(0)}\right]_r^{-1} \left\{\Delta P^{r+1} - Q^r\right\} \qquad (r \geq 0) \qquad (5\text{-}13b)$$

A comparison of Fig. 4d with Fig. 3b indicates that the results from this analysis are essentially the same as those from the *modified incremental analysis* of section 3.

If, on the other hand, iteration is continued to convergence after each load increment, we may use the indices of section 3 to write

$$Q^s = Q^{s-1} - E^s \qquad (s > 0) \qquad (5\text{-}14a)$$

$$X^{t_r+s+1} = X^{t_r} + \left[K^{(0)}\right]_{t_r}^{-1} \left\{\Delta P^r - Q^s\right\} \qquad (s \geq 0) \qquad (5\text{-}14b)$$

which is the *iterative incremental pseudo-load procedure* illustrated in Fig. 4e. A comparison of this figure with Fig. 3f indicates the results are essentially the same as those from the *modified incremental Newton-Raphson* analysis.

Although it has been shown that pseudo-load and modified Newton-Raphson methods produce essentially the same results, there may be considerable difference in application. When closed form expressions are available to estimate E^r for any configuration, these differences are minor. However E^r may also be estimated as the departure from linearity in a small load increment. In this case the pseudo-load methods have no overall equilibrium check beyond the first load increment, since the convergence is checked against ΔP rather than P. They are, however, very useful for path dependent studies.

§6. *Initial Value Techniques* (2, 3, 11, 12)

Returning to Eqs. (2-1), another family of solution techniques may be derived by differentiating the equilibrium equations and applying numerical integration procedures. Assume the load parameter p, and the resulting displacements X, are continuous functions of some parameter θ. Also assume that the initial point (X*,p*) does not completely satisfy equilibrium so that Eq. (2-2) is in effect. Differentiating with respect to θ yields

$$\left.\frac{\partial F}{\partial x_i}\right|_* \dot{x}_i + \left.\frac{\partial F}{\partial p}\right|_* \dot{p} = -\dot{E} . \qquad (6\text{-}1)$$

Designating the initial point as (X^r,p^r), and using the notation of section 3, Eq. (6-1) becomes

$$\left[K\right]_r \dot{X} - \dot{P} \doteq -\dot{E} \qquad (6\text{-}2)$$

Any numerical integration technique applicable to *initial value* problems may now be applied.

In particular, applying the Euler approximation

$$\dot{X} = \frac{X^{r+1} - X^r}{\Delta\theta} \qquad (6\text{-}3)$$

a solution for X^{r+1} is obtained as

$$X^{r+1} = X^r + \left[K\right]_r^{-1} \left\{\dot{P} - \dot{E}\right\} \Delta\theta \qquad (6\text{-}4)$$

Since Δθ is the change in the parameter θ associated with the increment from configuration r to configuration (r+1), the value of $\dot{P}\Delta\theta$ is ΔP^{r+1}

and if it is desired to eliminate the unbalanced force at X^r, $\dot{E}\Delta\theta = - E^r$. Eq. (6-4) then becomes

$$X^{r+1} = X^r + \left[K\right]_r^{-1} \left\{\Delta P^{r+1} + E^r\right\} \tag{6-5}$$

which is identical to the modified incremental analysis of section 3. More sophisticated integration schemes, such as Runga-Kutta methods, may be applied to Eq. (6-2).

Haisler (2, 3, 11, 12) has applied a variety of initial value techniques to pseudo-load finite element formulations. The pseudo-load form of the equilibrium equations has been derived in this report from the Taylor's series expansion displayed in Eq. (5-1), and resulted in Eq. (5-4), However, if equilibrium is not completely satisfied at a configuration, Eq. (5-4) becomes

$$\left[K^{(0)}\right] X + Q(X) - P(p) = - E\,(X,p) \tag{6-6}$$

Proceeding in the same manner as above and differentiating this equation with respect to θ yields

$$\left[K^{(0)}\right] \dot{X} + \dot{Q} - \dot{P} = - \dot{E} \tag{6-7}$$

Assuming P is a linear function of θ and differentiating once more yields

$$\left[K^{(0)}\right] \ddot{X} + \ddot{Q} = - \ddot{E} \tag{6-8}$$

A variety of *pseudo-load initial value schemes* may now be obtained by using a variety of finite difference approximations of the derivatives in Eqs. (6-7) and (6-8). In particular Haisler has presented the following.

(a) An Initial Value Three-Point Difference Form

Assuming $E \equiv 0,\ \theta = p$,

$$\dot{X}^r = \frac{1}{\Delta p} (X^{r+1} - X^r) \tag{6-9a}$$

and

$$\dot{Q}^r = \frac{1}{\Delta p} (Q^r - Q^{r-1}) \tag{6-9b}$$

Eq. (6-7) yields the forward stepping equations

$$X^{r+1} = X^r + \left[K^{(0)}\right]^{-1} \left\{\Delta P^{r+1} - Q^r + Q^{r-1}\right\} \tag{6-9c}$$

This difference form becomes unstable for more than moderate nonlinearities. Eq. (6-9a) is a forward difference approximation while Eq. (6-9b) is a backward difference.

(b) An Initial Value Five-Point Difference Form

Assuming $E \equiv 0,\ \theta = p$,

$$\dot{X}^r = \frac{1}{2\Delta p} (3X^r - 4X^{r-1} + X^{r-2}) \tag{6-10a}$$

$$\dot{Q}^r = 2\dot{Q}^{r-1} - \dot{Q}^{r-2} \tag{6-10b}$$

which, if the $\dot{Q}$ are assumed of the same form as (6-10a), becomes

$$\dot{Q}^r = \frac{1}{2\Delta p} \left\{6Q^{r-1} - 11Q^{r-2} + 6Q^{r-3} - Q^{r-4}\right\} , \tag{6-10c}$$

then Eq. (6-7) becomes

$$X^r = \frac{4}{3} X^{r-1} - \frac{1}{3} X^{r-2} + \left[K^{(0)}\right]^{-1} \left\{\frac{2}{3} \Delta P^r - 2Q^{r-1} + \frac{11}{3} Q^{r-2} - 2Q^{r-3} + \frac{1}{3} Q^{r-4}\right\} \qquad (6\text{-}10d)$$

This difference form is apparently reasonably accurate and results from a linear extrapolation of $\dot{Q}$ (Eq. (6-10b)) and then the application of a 3 point backward difference form for first derivatives (Eq. (6-10a)).

If the error is not assumed to be zero, *'self-correcting' initial value* forms can be derived. By assuming

$$\dot{E} = - z E \qquad (6\text{-}11a)$$

where z is a scalar multiplying factor, and using Eq. (6-6) to evaluate E, Eq. (6-7) becomes

$$\left[K^{(0)}\right] \left\{\dot{X} + z X\right\} = \left\{\dot{P} + z P\right\} - \left\{\dot{Q} + z Q\right\} \; . \qquad (6\text{-}11b)$$

Using the chain rule of differentation $\dot{Q}$ may be expressed as

$$\dot{Q} = \dot{Q}(x) = \sum_i \frac{\partial Q}{\partial x_i} \dot{x}_i = \left[K^*\right] \dot{X} \qquad (6\text{-}11c)$$

which defines the matrix $\left[K^*\right]$. Substituting into Eq. (6-11b) yields

$$\left[\left[K^{(0)}\right] + \left[K^*\right]\right] \dot{X} + z \left[K^{(0)}\right] X = \dot{P} + z P - z Q \; . \qquad (6\text{-}11d)$$

Again various difference forms can be used.

Haisler (2, 3, 11, 12) also derives a "self correcting" initial value form from the second order differential equation, Eq. (6-8), by assuming

$$\ddot{E} = -(z\,E + c\,\dot{E}) \tag{6-12a}$$

where z and c are scalar multipliers. In this case Eq. (6-8) becomes

$$\left[K^{(0)}\right] \ddot{X} = -\ddot{Q} + z\,E + c\,\dot{E} \;.$$

Substituting from Eq. (6-6) for E, and from Eq. (6-7) for $\dot{E}$, Eq. (6-12b) becomes

$$\left[K^{(0)}\right]\left\{\ddot{X} + c\,\dot{X} + z\,X\right\} = \left\{z\,P + c\,\dot{P}\right\} - \left\{z\,Q + c\,\dot{Q} + \ddot{Q}\right\} \tag{6-12c}$$

(c) A Self-Correcting Initial Value Four Point Difference Form

Assume now that θ = p. Neglect $\ddot{Q}$ and assume the four point backward difference form

$$\ddot{X}^{r} = \frac{1}{(\Delta p)^2}\left\{2X^{r} - 5X^{r-1} + 4X^{r-2} - X^{r-3}\right\}. \tag{6-13a}$$

Express $\dot{X}^{r}$ by Eq. (6-10a) and Q^{r} by the form in Eq. (6-10b), that is

$$Q^{r} = 2Q^{r-1} - Q^{r-2} \tag{6-13b}$$

Express $\dot{Q}^{r}$ by Eq. (6-9b) and use Eq. (6-13b) to obtain

$$\dot{Q}^{r} = \frac{1}{\Delta p}\left(Q^{r-1} - Q^{r-2}\right). \tag{6-13c}$$

Then Eq. (6-12c) becomes

$$X^r = \left[(\Delta p)^2 \left[K^{(0)}\right]^{-1} \left\{z\, P^r + c\, \dot{P}^r - z\,(2Q^{r-1} - Q^{r-2}) - \frac{c}{\Delta p}\,(Q^{r-1} - Q^{r-2})\right\}\right.$$

$$\left. + \;(5 + 2\,c\,\Delta p)\, X^{r-1} - (4 + c\,\frac{\Delta p}{2})\, X^{r-2} + X^{r-3}\right] / \left(2 + \frac{3}{2}\,c\,\Delta p + z\,\Delta p^2\right) \quad \ldots \quad (6\text{-}13d)$$

Starting values for the solution may be obtained by one of the previously discussed iterative techniques. Haisler, from numerical experiments, recommended the values

$$z = 10 / (\Delta p \sqrt{\Delta p \cdot P}\,) \qquad (6\text{-}13e)$$

and

$$c = z^{0.2} / 2 \qquad (6\text{-}13f)$$

§7. *The Alpha-Constant Technique* (6)

The alpha-constant technique is a method of deriving a matrix of constants to estimate the displacement increments ΔX derived from a 'tangent stiffness' matrix, such as that in Eq. (3-6b), from the displacement increments ΔX derived from an 'initial tangent stiffness' matrix, such as that in Eq. (3-7b). To fix ideas, we may look at the α constants as 'over-relaxation factors' applied to modified Newton-Raphson displacement increments in an attempt to estimate the displacement increments that would occur in an unmodified Newton-Raphson process.

Eq. (3-6b) of the Newton-Raphson process may be written as

$$\Delta X^{r+1} = \left[K\right]_r^{-1} E^r \quad . \tag{7-1}$$

Expressing the tangent stiffness $[K]_r$ as

$$[K]_r = \left[K^{(0)}\right] + [\tilde{K}]_r \tag{7-2}$$

where $\left[K^{(0)}\right]$ is some reference stiffness and $[\tilde{K}]_r$ represents the departure of $[K]_r$ from this reference stiffness at X^r, Eq. (7-1) may be written as

$$\left[\left[K^{(0)}\right] + [\tilde{K}]_r\right] \Delta X^{r+1} = E^r \tag{7-3}$$

multiplying by $\left[K^{(0)}\right]^{-1}$ yields

$$\Delta X^{r+1} + \left[K^{(0)}\right]^{-1} [\tilde{K}]_r \, \Delta X^{r+1} = \left[K^{(0)}\right]^{-1} E^r = \hat{\Delta X}^{r+1} \tag{7-4}$$

where $\hat{\Delta X}^{r+1}$ is the displacement increment that would be predicted using only the reference matrix $[K^{(0)}]$. This may be computed immediately when E^r is known.

Now assume

$$\Delta X^{r+1} = [\alpha]^{r+1} \hat{\Delta X}^{r+1} \tag{7-5}$$

and we wish to determine the α for which this is true. If Eq. (7-5) is substituted into Eq. (7-4), but in the second term $[\alpha]^{r+1}$ is approximated by $[\alpha]^r$, Eq. (7-4) becomes

$$[\alpha]^{r+1} \hat{\Delta X}^{r+1} + [K^0]^{-1} [\tilde{K}]_r [\alpha]^r \hat{\Delta X}^{r+1} = \hat{\Delta X}^{r+1} \tag{7-6}$$

Since $\hat{\Delta X}^{r+1}$ and $[\alpha]^r$ are known, the second term of Eq. (7-6) may be evaluated. Representing this vector by ΔU^{r+1}, Eq. (7-6) yields

$$\alpha_j^{r+1} = 1 - \frac{\Delta u_j^{r+1}}{\hat{\Delta x}_j^{r+1}} \tag{7-7}$$

and $[\alpha]^{r+1}$ can be determined through this equation.

The iterative procedure is, therefore,

(a) Evaluate E^r

(b) Find $\hat{\Delta X}^{r+1} = \left[K^{(0)}\right]^{-1} E^r$ (7-8a)

(c) Evaluate $\Delta U^{r+1} = \left[K^{(0)}\right]^{-1} \left[\tilde{K}\right]_r [\alpha]^r \hat{\Delta X}^{r+1}$ (7-8b)

(d) Evaluate $[\alpha]^{r+1}$ from $\alpha_j^{r+1} = 1 - \dfrac{\Delta u_j^{r+1}}{\hat{\Delta x}_j^{r+1}}$ (7-8c)

(e) Find $X^{r+1} = X^r + [\alpha]^{r+1} \hat{\Delta X}^{r+1}$ (7-8d)

The application of this technique essentially converts a modified Newton-Raphson method to an approximate Newton-Raphson method. Similarly, it may be applied to convert the incremental modified Newton-Raphson method to an approximate incremental Newton-Raphson method (see Section 3). The advantage is the elimination of the need to invert the stiffness matrix at every iterate.

§8. *The Augmented Stiffness Technique* (10, 15)

Many structures respond in such a way that they become unstable after reaching a particular critical configuration. Such a 'critical point' is indicated as point A on the response curve G(X) in Fig. 5a. From 0 to A the 'tangent stiffness' matrix is positive definite and the structure is stable. The point A is characterized by the fact that the 'tangent stiffness' matrix becomes singular (ie. - det $[K_T] = 0$). If the structure becomes 'unstable' at this point, as indicated in Fig. 5a, there is a deformation path originating from point A along which increases of displacement must be accompanied by decreasing loads if equilibrium is to be maintained. Along such a path, indicated as A-B in Fig. 5a, the tangent stiffness matrix is negative definite. If the structure reaches a point, such as B in Fig. 5a, where increases of displacement must again be produced by positive load increments, the stiffness matrix again becomes positive definite.

A complete study of the behavior of such structures is rather complex and prediction of response is generally difficult. The critical point A is of the type occuring in 'snap-through' problems and is often referred to as a 'limit point'. On the other hand instability may occur in a precipitous manner associated with deformation components that are not induced by the loading in the initial response of the structure. Such a point, normally referred to as a 'bifurcation point', is shown as point A* in Fig. 5a, and the critical response in this case may be the curve A*-B*. There may be multiple equilibrium paths emanating from a bifurcation point.

The introduction of loads or initial deformations which induce displacements associated with the bifurcation behavior in the initial response converts bifurcation points to limit points. Therefore attention is focused on this type of behavior.

If the load is not decreased at point A, but increased, the only solution available on G(X) in Fig. 5a is a displacement along the curve C-E. In this case an analysis would predict the displacement, say D, and the unloading response of the structure along path A-B-C would be completely lost. On the other hand, an equilibrium position may exist for a negative displacement, say at D', and under monotonic loading the predicted structural response might be O-A-D'-E' rather than O-A-D-E. In addition, since the matrix at A is singular, the numerical method may simply fail to converge to any value at A (ie. - the solution 'diverges' or 'blows up').

Without attempting to overlook the difficulties inherent in such a problem, it can be said that if the response curve G(X) is locally unique one method of eliminating numerical problems at A is to artificially stiffen the structure. The stiffness of the basic structure is *augmented* by a series of elastic springs such that the stiffness of the augmented structure is always positive definite. A solution for this structure is then carried out and is represented by the curve O-A'-B'-C' in Fig. 5a. Once this solution has been obtained, the loads that the structure equilibrates may be determined by subtracting the 'spring forces' from the loads that the augmented structure equilibrates. This results in the response curve G(X). For such a technique to work the spring forces must be proportional to the specified applied forces. The detailed formulation follows.

Let P, $\hat{P}$ and F be the loads equilibrated by the basic structure, the augmenting springs, and the augmented structure, respectively. Then

$$F = P + \hat{P} . \tag{8-1}$$

Assume that the loads F are proportional to the specified forces P. It is therefore necessary that

$$\hat{P} = \alpha F \tag{8-2}$$

where α is a constant to be determined by the stiffness of the spring system.

One physical arrangement for achieving such a system of spring supports is shown in Fig. 5b. Assuming the ratio of the elements of F remain the same, restraining forces proportional to the applied forces are obtained from a closed hydraulic system in which the areas (A_i) of the pistons providing support are such that

$$A_i/A_j = \hat{p}_i/\hat{p}_j = f_i/f_j \tag{8-3}$$

where $\hat{p}_i$ and f_i are elements of $\hat{P}$ and F, respectively. Eq. (8-2) is then valid.

To determine the constant α, we note that by Eq. (8-2)

$$\sum_{i=1}^{n} \left| \hat{p}_i \right| = \sum_{i=1}^{n} \left| \alpha f_i \right| = \alpha \sum_{i=1}^{n} \left| f_i \right| \tag{8-4}$$

Assume the pressure in the hydraulic system is controlled by an elastic spring as indicated in Fig. 5b, such that

$$\sum_{i=1}^{n} \left| \hat{p}_i \right| = \gamma\Delta \tag{8-5}$$

where γ is the spring constant and Δ is the deflection of the piston.

Combining Eqs. (8-4) and (8-5) to evaluate α in terms of $\gamma\Delta$, and substituting into Eq. (8-2) yields

$$\hat{P} = \frac{\gamma\Delta}{\sum_{i=1}^{n} \left| f_i \right|} F . \tag{8-6}$$

Since the work done by the restraining forces is stored in the elastic spring we have, from Eq. (8-6),

$$\frac{1}{2} \hat{P}^T X = \frac{1}{2} \frac{\gamma\Delta}{\sum_{i=1}^{n} \left| f_i \right|} F^T X = \frac{1}{2} \gamma \Delta^2 \tag{8-7}$$

and the last equality requires that

$$\Delta = \frac{F^T X}{\sum_{i=1}^{n} \left| f_i \right|} \tag{8-8}$$

Substituting Eq. (8-8) into Eq. (8-6) yields

$$\hat{P} = \frac{\gamma F F^T}{(\Sigma |f_i|)^2} X \tag{8-9}$$

Denoting the stiffness matrix of the spring system by $[K_s]$, Eq. (8-9) yields

$$[K_s] = \frac{\gamma}{(\Sigma|f_i|)^2} F F^T , \tag{8-10}$$

and the stiffness of the spring system such that Eq. (8-2) is satisfied has been determined. The elements of $[K_s]$ are proportional to the spring stiffness γ but are independent of a scalar multiplication of F and only the ratios of the elements in this vector are significant.

Any of the previously discussed solution techniques may now be applied with the *augmented stiffness technique*. The technique is:
(a) to apply the loads F to a structure with an augmented stiffness matrix, defined by

$$[K_A] = [K] + [K_s] \tag{8-11}$$

where $[K_A]$ is the augmented stiffness matrix, [K] is the normal stiffness matrix and $[K_s]$ is the spring stiffness matrix of Eq. (8-10). (b) Solve for the displacements X required to equilibrate the loads F. (c) Determine the loads P from Eq. (8-1). This last operation is carried out by noting that Eq. (8-1) can be written as

$$P = F - \hat{P} \tag{8-12}$$

which, from Eq. (8-9) may be written as

$$P = F - \frac{\gamma\, F(F^T X)}{(\Sigma|f_i|)^2} = F - \frac{\gamma(F^T X)}{(\Sigma|f_i|)^2} F \tag{8-13}$$

or

$$P = \lambda F \tag{8-14}$$

where

$$\lambda = 1 - \frac{\gamma F^T X}{(\Sigma |f_i|)^2} . \tag{8-15}$$

The constant of proportionality, λ, is therefore, different for every loading condition.

The following points should be noted:

1. The spring stiffness γ must be carefully selected, generally to make $0 < \lambda < 0.5$.

2. Since K_s may be full, the banded nature of the stiffness matrix may be destroyed.

3. If the determinant of the augmented stiffness matrix becomes negative, the load should be decremented, which is also the case with unstiffened structures, as will become apparent in Chapter III.

CHAPTER III APPLICATIONS TO A TWO-BAR TRUSS

§9. *Introduction to Two-Bar Truss* (8, 9)

In this chapter the solution methods of Chapter II are applied to a simple two-bar truss in an attempt to assess their suitability for solving geometrically nonlinear problems. The problems are unrealistic in the sense that the material is assumed to be infinitely linear elastic, that changes in the bar areas do not occur (ie. $\nu = 0$), and that member instabilities do not occur. However, they form a convenient vehicle to study the ability of the solution technique to respond in highly nonlinear situations.

§9.1 *Formulation of Equilibrium Equations*

In large deformation problems an energy formulation is usually considerably simpler than an approach based on statics. Consider the truss in Fig. 6a. This truss has two displacement coordinates associated with joint B, which are designated as u and v. The deformed configuration is shown in Figs. 6b and 6c. The displaced configurations of the bars are shown in Figs. 6d and 6e.

Let a_1, a_2 be the local coordinates of a point on the bar before deformation, and x_1, x_2 be the coordinates of the same point after deformation. Let $\tilde{u}_1$ and $\tilde{u}_2$ be the displacements in these local coordinate systems shown in Figs. 6d and 6e. Since the extensions of the bars are uniform we may write

(i) for bar $\overline{AB}$

$$\tilde{u}_1 = (- v \sin \theta + u \cos \theta)\, a_1 / d \tag{9-1a}$$

$$\tilde{u}_2 = (- v \cos \theta - u \sin \theta)\, a_1 / d \tag{9-1b}$$

(ii) for bar BC

$$\tilde{u}_1 = (+ v \sin \theta + u \cos \theta)\, a_1 / d \tag{9-1c}$$

$$\tilde{u}_2 = (- v \cos \theta + u \sin \theta)\, a_1 / d\,. \tag{9-1d}$$

Using Green's strain tensor the strain energy may be written as

$$\mathcal{E}_{AB} = \frac{1}{2} \int_{AB} E \left\{ \frac{\partial \tilde{u}_1}{\partial a_1} + \frac{1}{2}\left(\frac{\partial \tilde{u}_1}{\partial a_1}\right)^2 + \frac{1}{2}\left(\frac{\partial \tilde{u}_2}{\partial a_1}\right)^2 \right\} d\,V \tag{9-2}$$

with a similar expression for $\mathcal{E}_{BC}$. Differentiating Eqs. (9-1) with respect to a_1 simply removes a_1 from the expressions. Substituting the derivatives into Eq. (9-2) and simplifying results in

$$\begin{aligned}\mathcal{E} &= \mathcal{E}_{AB} + \mathcal{E}_{BC} \\ &= \frac{1}{2}\, E\, A\, d \left\{ \frac{- v \sin \theta + u \cos \theta}{d} + \frac{1}{2}\left(\frac{v^2}{d^2} + \frac{u^2}{d^2}\right)\right\}^2 \\ &+ \frac{1}{2}\, E\, A\, d \left\{ \frac{v \sin \theta + u \cos \theta}{d} + \frac{1}{2}\left(\frac{v^2}{d^2} + \frac{u^2}{d^2}\right)\right\}^2 \end{aligned} \tag{9-3}$$

Differentiating Eq. (9-3) with respect to u and v, and simplifying, results in

$$\frac{\partial \mathcal{E}}{\partial v} = \frac{EA}{d^3} \left\{ (v - c) \quad v^2 - 2vc + u^2 \right\} \tag{9-4a}$$

$$\frac{\partial \mathcal{E}}{\partial u} = \frac{EA}{d^3} \; u \left\{ v^2 + 2(b^2 - vc) + u^2 \right\} \tag{9-4b}$$

But, by Castigliano's Theorem, for an elastic structure in equilibrium

$$F_y^{\;B} = \frac{\partial \mathcal{E}}{\partial v} \tag{9-5a}$$

and

$$F_x^{\;B} = \frac{\partial \mathcal{E}}{\partial u} \tag{9-5b}$$

If we consider the load R to be conservative (it retains its line of action) while the load S is nonconservative (it always remains normal to bar AB'), we have, from Fig. 6b,

$$F_y^{\;B} = R + S \cos \theta' \tag{9-6a}$$

and

$$F_x^{\;B} = S \sin \theta' \tag{9-6b}$$

Evaluating cos θ' and sin θ' from Fig. 6d

$$\cos \theta' = (b + u) \,/\, d^* \tag{9-7a}$$

$$\sin \theta' = (c - v) \,/\, d^* \tag{9-7b}$$

where

$$d^* = \sqrt{(b + u)^2 + (c - v)^2} \tag{9-7c}$$

Combining Eqs. (9-4), (9-5), (9-6) and (9-7) yields the equilibrium equations

$$\frac{EA}{d^3} (v - c) \left\{v^2 - 2vc + u^2\right\} = R + S (b + u) / d^* \quad (9\text{-}8a)$$

$$\frac{EA}{d^3} u \left\{v^2 + 2(b^2 - vc) + u^2\right\} = S (c - v) / d^* \quad (9\text{-}8b)$$

These equations are more conveniently solved by putting them into non-dimensional form. Defining

$$\bar{v} = v/b \quad (9\text{-}9a)$$

$$\bar{u} = u/b \quad (9\text{-}9b)$$

$$\bar{r} = R\, d^3/EA\, b^3 \quad (9\text{-}9c)$$

$$\bar{s} = S\, d^3/EA\, b^3 \quad (9\text{-}9d)$$

$$\mu = c/b \quad (9\text{-}9e)$$

and

$$\bar{d} = \sqrt{(1 + \bar{u})^2 + (\mu - \bar{v})^2} \quad (9\text{-}9f)$$

Eqs. (9-8) become

$$(\bar{v} - \mu) (\bar{v}^2 - 2\mu\bar{v} + \bar{u}^2) = \bar{r} + \bar{s} (1 + \bar{u})/\bar{d} \quad (9\text{-}10a)$$

$$\bar{u} (\bar{v}^2 + 2 (1 - \mu\bar{v}) + \bar{u}^2) = \bar{s} (\mu - \bar{v})/\bar{d}. \quad (9\text{-}10b)$$

Let us now consider some special cases.

(i) <u>*CASE A: Load* S = 0</u>

For this condition Eqs. (9-10) become

$$(\bar{v} - \mu) (\bar{v}^2 - 2\mu\bar{v} + \bar{u}^2) = \bar{r} \quad (9\text{-}11a)$$

$$\bar{u}\,(\bar{v}^2 + 2\,(1 - \mu\bar{v}) + \bar{u}^2) \;=\; 0 \qquad \text{(9-11b)}$$

(ii) CASE B: Load $S = 0,\ \mu = 1$ *and* $u = 0$

Constraining the u displacement results in a symmetric structure at all times. Eq. (9-10b) is identically satisfied and Eq. (9-10a) becomes (for $b = c$, or $\mu = 1$)

$$\bar{v}\,(\bar{v} - 1)(\bar{v} - 2) \;=\; \bar{r} \qquad \text{(9-12)}$$

§9.2 *Solutions for Special Case B (Vertical Load and Symmetric Deformation Only)*

In this case we may identify the variables in Eq. (9-12) with the notation of Chapter II as follows. Let us denote $\bar{r}$ in Eq. (9-12) as $\bar{p}$ to avoid confusion with the iteration index.

$$X \;=\; \bar{v} \qquad \text{(9-14a)}$$

$$P \;=\; P(R) \;=\; \bar{p} \qquad \text{(9-14b)}$$

$$F(X,p) \;=\; F(\bar{v},R) \;=\; \bar{v}(\bar{v} - 1)\,(\bar{v} - 2) - \bar{p} \qquad \text{(9-14c)}$$

$$G(X) \;=\; G(\bar{v}) \;=\; \bar{v}(\bar{v} - 1)\,(\bar{v} - 2) \qquad \text{(9-14d)}$$

$$P(p) \;=\; P(R) \;=\; \bar{p} \;=\; R\,d^3/EA\,b^3 \qquad \text{(9-14e)}$$

$$[K] \;=\; \frac{\partial F}{\partial \bar{v}} \;=\; 2 - 6\bar{v} + 3\bar{v}^2 \qquad \text{(9-14f)}$$

The tangent stiffness matrix [K] will be singular when

$$3\bar{v}^2 - 6\bar{v} + 2 \;=\; 0 \qquad \text{(9-15a)}$$

or when

$$\bar{v} = \frac{6 \pm \sqrt{36 - 24}}{6} = 1 \pm \frac{1}{\sqrt{3}} \quad . \tag{9-15b}$$

Eqs. (9-15b) and (9-14c) yield critical points (corresponding to points A and B of Fig. 5a) of (0.365, 0.379) and (1.635, - 0.379).

The solution for this structure is shown in Fig. 7. This solution may be obtained by substituting values for $\bar{v}$ into Eq. 9-12 and computing the corresponding value of $\bar{p}$.

§9.2(a) *Simple Incremental Analysis*

Using the correspondence between variables established in Eqs. (9-14), Eq. (3-5) becomes

$$\bar{v}^{r+1} = \bar{v}^r + \left\{2 - 6\bar{v}^r + 3(\bar{v}^r)^2\right\}^{-1} (\Delta\bar{p}^{r+1}) \quad . \tag{9-16}$$

Solution by this technique is shown in Fig. 8. The increment in $\bar{p}$ for this solution was 0.05. The solution is adequate for the 'prebuckling' region but under monotonic loading predicts a response of the type shown by O-A-D'-E' in Fig. 5a.

§9.2(b) *Modified Incremental Analysis*

The modified incremental analysis adds the force unbalance from step r to the load increment for step r+1. Eq. (3-3) becomes

$$\bar{v}^{r+1} = \bar{v}^r + \left\{2 - 6\bar{v}^r + 3(\bar{v}^r)^2\right\}^{-1} \left\{\Delta\bar{p}^{r+1} + E^r\right\} \tag{9-17a}$$

where E^r is evaluated from Eq. (9-14c) as

$$E^r = \bar{p}^r - \bar{v}^r (\bar{v}^r - 1)(\bar{v}^r - 2) \qquad (9\text{-}17b)$$

The load displacement curve is shown in Fig. 8. Under monotonic loading this method attempted to follow a curve such as O-A-D-E of Fig. 5a. Again the prebuckling curve can be obtained reasonably well but the snap through phenomenon is missed.

§9.2(c) *Incremental Newton-Raphson Analysis*

Eqs. (3-8) become

$$\bar{v}^{t_r+1} = \bar{v}^{t_r+1} + \left\{2 - 6\bar{v}^{t_r} + 3(\bar{v}^{t_r})^2\right\}^{-1} \left\{\Delta\bar{p}^r + E^{t_r}\right\} \qquad (9\text{-}18a)$$

$$\bar{v}^{t_r+s+1} = \bar{v}^{t_r+s+1} + \left\{2 - 6\bar{v}^{t_r+s} + 3(\bar{v}^{t_r+s})^2\right\}^{-1} E^{t_r+s} \qquad (9\text{-}18b)$$

where

$$E^{t_r+s} = \bar{p}^r - \bar{v}^{t_r+s} (\bar{v}^{t_r+s} - 1)(\bar{v}^{t_r+s} - 2) \qquad (9\text{-}18c)$$

After convergence occurs for iterate s_r of load increment r,

$$t_r = t_{r-1} + s_r \qquad (9\text{-}18d)$$

and

$$s = 0 . \qquad (9\text{-}18e)$$

The load increment was again set to 0.05. As indicated on Fig. 8, this procedure predicts the prebuckling range and jumps to point D of Fig. 5a without the disturbance observed in the modified incremental procedure.

However, it is also inadequate to detect the snap-through phenomenon since the load increments are monotonic.

§9.2(d) *Modified Incremental Newton-Raphson Analysis*

Eqs. (3-9) yield recursive equations identical to Eqs. (9-18) except for Eq. (9-18b) which becomes

$$\bar{v}^{t_r+s+1} = \bar{v}^{t_r+s} + \left\{2 - 6\bar{v}^{t_r} + 3(\bar{v}^{t_r})^2\right\}^{-1} E^{t_r+s} . \qquad (9\text{-}19)$$

Results correspond to those of the incremental Newton-Raphson method for prebuckling response. However, the procedure 'blows up' at a loading of 0.4 (the first load above the critical load of 0.379).

§9.2(e) *Incremental Newton-Raphson with Gradient Test*

This procedure uses the recursive equations previously presented as Eqs. (9-18). However the determinant of $[K]_{t_r+s}$ is checked for sign. (In this problem the determinant of $[K]_{t_r+s}$ is equal to $\left\{2 - 6\bar{v}^{t_r+s} + 3(\bar{v}^{t_r+s})^2\right\}$). When this sign is negative the load is decremented. When the sign is positive the load is incremented. The test for sign is called a 'gradient test'.

The results are shown on Fig. 8 and indicate that unloading and postbuckling behavior can be readily determined with this type of procedure.

§9.2(f) *Combined Newton-Raphson and Modified Newton-Raphson with Gradient Test*

The computational inefficiency of the Newton-Raphson method was mentioned in section 3, namely, it is necessary to reform and invert [K] for every iterate. This suggests that the following procedure would be more efficient than that above.

(i) The modified Newton-Raphson technique (Equations (19-18) with Eq. (9-19) replacing Eq. (9-18b)) is used as long as convergence for a load increment is obtained.

(ii) If for load increment m the method begins to diverge, return to displacements $\bar{v}^{m-1}$ and initiate the Newton-Raphson technique with a gradient test as discussed above until the first point beyond the critical point is determined.

(iii) Switch back to the modified Newton-Raphson technique and continue to decrement the load until the method again begins to diverge, at which time return to (ii).

The load deflection curve obtained by this procedure is the same as in (e) and is shown on Fig. 8.

§9.2(g) *Initial Value Five-Point Difference Form*

The initial value five-point difference form of section 6 is a pseudo-load procedure. To put Eq. (9-14c) into pseudo-load form the equation must be written so the terms correspond with those of Eqs. (5-3) and (5-4). Thus Eq. (9-14c) is

$$F(\bar{v},p) = 2\bar{v} - 3\bar{v}^2 + \bar{v}^3 - \bar{p} = 0 \tag{9-20a}$$

which, in the notation of Eq. (5-3) is

$$[K^{(0)}]\,\bar{v} + [\tilde{K}]\,\bar{v} - \bar{p} = 0 \tag{9-20b}$$

where

$$[K^{(0)}] = 2 \tag{9-20c}$$

and

$$[\tilde{K}] = \bar{v}^2 - 3\bar{v} \tag{9-20d}$$

Then, identifying the second term as Q,

$$Q = [\tilde{K}]\,\bar{v} = \bar{v}^2\,(\bar{v} - 3) \tag{9-20e}$$

The initial-value procedure of Eqs. (6-10) then becomes

$$\bar{v}^r = \frac{4}{3}\,\bar{v}^{r-1} - \frac{1}{3}\,\bar{v}^{r-2} + \frac{1}{2}\left\{\frac{2}{3}\,\overline{\Delta p}^r - 2Q^{r-1} + \frac{11}{3}\,Q^{r-2} - 2Q^{r-3} + \frac{1}{3}\,Q^{r-4}\right\} \tag{9-21}$$

where Q is evaluated from Eq. (9-20e). Initial values for the first three points were obtained by iteration and Eq. (9-21) was solved for $\overline{\Delta p} = 0.01$ and $\overline{\Delta p} = 0.005$. Results are shown on Fig. 9. Prebuckling is predicted reasonably well but postbuckling predictions are erratic.

§9.2(h) *Self Correcting Initial Value Four Point Difference Form*

This procedure, summarized in Eqs. (6-13), is based on the second order differential equation presented in section 6. Eq. 6-13d becomes

$$\bar{v}^r = \left[(\overline{\Delta p})^2 \cdot \frac{1}{2}\left\{c + z\,\bar{p}^r - 2z\,Q^{r-1} + z\,Q^{r-2} - \frac{c}{\overline{\Delta p}}\left(Q^{r-1} - Q^{r-2}\right)\right\} + (5 + 2\;c\overline{\Delta p})\;\bar{v}^{r-1} - \left(4 + \frac{c\Delta p}{2}\right)\bar{v}^{r-2} + \bar{v}^{r-3}\right] / \left(2 + \frac{3}{2}\,c\overline{\Delta p} + z\,(\Delta\bar{p}^2)\right) \qquad \ldots (9\text{-}22)$$

where Q is given by (9-20e), and Eqs. (6-13e) and (6-13f) define z and c. Results are shown on Fig. 9 for a load increment of 0.005. Starting values were again determined by iteration. This method predicts the post-buckling region more accurately than the five point form used above but cannot detect the snap through behavior.

§9.2(i) *The Alpha-Constant Technique* (6)

The alpha constant technique has been tested with the modified incremental Newton-Raphson technique. Eq. 7-3 becomes

$$\left[[K^{(0)}] + [\tilde{K}]_r\right] \overline{\Delta v}^{r+1} = E^r \qquad (9\text{-}23)$$

where $[K^{(0)}]$ is given by Eq. (9-20c), $[\tilde{K}]$ is given by Eq. (9-20d) and E^r is obtained from Eq. (9-17b). The procedure follows Eqs. (7-8) and is:

1. Set $\bar{v}^0 = 0$ and $\alpha^0 = 1$.

2. Evaluate E^r from (9-17b)

$$E^r = \bar{p}^r - \bar{v}^r (\bar{v}^r - 1)(\bar{v}^r - 2) \quad (9\text{-}24a)$$

3. Find $\hat{\Delta\bar{v}}$ from Eq. (7-8a)

$$\hat{\Delta\bar{v}}^r = \frac{1}{2} E^r . \quad (9\text{-}24b)$$

4. Compute Δu from Eq. (7-8b)

$$\Delta u^r = \frac{1}{2}\left\{3(\bar{v}^r)^2 - 6\bar{v}^r\right\} \alpha^{r-1} \hat{\Delta\bar{v}}^r \quad (9\text{-}24c)$$

5. Compute $\alpha^r = 1 - \Delta u^r / \hat{\Delta\bar{v}}^r$ (9-24d)

6. Find $\bar{v}^r = \bar{v}^{r-1} + \alpha^r \hat{\Delta\bar{v}}^r$ (9-24d)

7. Return to step 2.

The alpha constant technique applied to the modified incremental Newton-Raphson method reduces the number of iterations required for each load increment as shown in Table 1. The solution is the same as that for the modified incremental Newton-Raphson method discussed in (b) above.

§*9.2(j)* *The Augmented Stiffness Technique* (10,15)

As described in Section 8, the augmented stiffness technique consists of applying the forces F to a stiffened (or augmented) structure for which the stiffening system is such that its resistance is always proportional to the applied loads. The stiffness of the augmented structure is expressed by Eqs. (8-11) where $[K_s]$ is given by Eq. (8-10).

For the present problem [K] is given by Eq. (9-14f). The technique may be applied with any of the previously presented solution techniques. It is applied here with the modified incremental technique of section 9.2(b). Eq. (3.3) becomes

$$\bar{v}^{r+1} = \bar{v}^{r} + [K_A]_r^{-1} \left\{E^r + \Delta\bar{f}^{r+1}\right\} \tag{9-25}$$

where, from Eqs. 9-14,

$$\bar{f} = \frac{F\,d^3}{EA\,b^3} \tag{9-26a}$$

From Eq. (8-10)

$$[K_s] = \frac{\gamma\,F\,F^T}{(\Sigma|f_i|)^2} = \gamma\ . \tag{9-26b}$$

The matrix $[K_A]$ then becomes

$$[K_A]_r = 2 - 6\bar{v}^r + 3\,(\bar{v}^r)^2 + \gamma \tag{9-26c}$$

and Eq. (9-14c) yields

$$E^r = \bar{f}^r - \bar{v}^r\,(\bar{v}^r - 1)\,(\bar{v}^r - 2) - \gamma\bar{v}^r \tag{9-26d}$$

The loads on the basic structure are given by Eqs. (8-14) and (8-15) which become

$$\bar{p}^r = \lambda^r\,\bar{f}^r \tag{9-26e}$$

where

$$\lambda^r = 1 - \frac{\gamma \bar{v}^r}{\bar{f}^r} . \tag{9-26f}$$

An initial value of γ to make $\lambda = 0.4$ was selected. The load-deflection curve is shown in Fig. 10c and 10b. The determinant of the incremental stiffness matrix remains positive and the snap-through phenomenon is detected without any special precautions.

§9.3 *Solutions for Special Case A (Vertical Load Only)*

The equilibrium equations for this condition are Eqs. (9-11). The previous solutions have assumed that the response of the truss was symmetric under vertical load. Eq. (9-11b) indicates that $\bar{u} = 0$ is certainly a condition under which equilibrium is maintained and is therefore a valid equilibrium solution. However the truss may respond in an unsymmetric manner, even with symmetric loading. The conditions under which this may occur are when the term in parentheses in Eq. (9-11b) is equal to zero. That is, when

$$\bar{v}^2 + 2 (1 - \mu\bar{v}) + \bar{u}^2 = 0 \tag{9-27a}$$

For a real nonzero value of $\bar{u}$ to occur, Eq. (9-27a) can be satisfied only if

$$\bar{v}^2 + 2 (1 - \mu\bar{v}) < 0 \tag{9-27b}$$

or

$$\bar{v}^2 - 2 \mu\bar{v} + 2 < 0 \tag{9-27c}$$

This equation can be factored, if μ is set to 2.25, and yields

$$(\bar{v} - 4)\ (\bar{v} - 0.5) \ < \ 0 \ . \tag{9-27d}$$

Eq. (9-27d) is satisfied only if

$$0.5 \ < \ \bar{v} \ < \ 4 \ . \tag{9-27e}$$

Therefore, for $\mu = 2.25$, nonzero $\bar{u}$ can occur (ie. an unsymmetric equilibrium configuration is possible) when $\bar{v}$ is in the range expressed by Eq. (9-27e). The objective of the following example is to determine if the more successful techniques of section 9.2 can detect this behavior.

Setting $\mu = 2.25$, and again designating $\bar{r}$ by $\bar{p}$, Eqs. (9-11) may be identified with the variables of Chapter II as

$$X = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} \bar{v} \\ \bar{u} \end{Bmatrix} \tag{9-28a}$$

$$P(p) = P(R) = \begin{Bmatrix} p_1 \\ p_2 \end{Bmatrix} = \begin{Bmatrix} \bar{p} \\ 0 \end{Bmatrix} = \begin{Bmatrix} R\ d^3/EA\ b^3 \\ 0 \end{Bmatrix} \tag{9-28b}$$

$$F(X,p) = F(X,R) = \begin{Bmatrix} (\bar{v} - 2.25)\ (\bar{v}^2 - 4.5\ \bar{v} + \bar{u}^2) - \bar{p} \\ \bar{u}\ (\bar{v}^2 - 4.5\ \bar{v} + \bar{u}^2 + 2) \end{Bmatrix} \tag{9-28c}$$

$$G(X) = \begin{Bmatrix} (\bar{v} - 2.25)\ (\bar{v}^2 - 4.5\ \bar{v} + \bar{u}^2) \\ \bar{u}\ (\bar{v}^2 - 4.5\ \bar{v} + \bar{u}^2 + 2) \end{Bmatrix} \tag{9-28d}$$

$$[K] = \left[\frac{\partial F}{\partial x_i}\right] = \begin{bmatrix} (3\bar{v}^2 - 13.5\bar{v} + \bar{u}^2 + 10.125) & 2\bar{u}\,(\bar{v} - 2.25) \\ 2\bar{u}\,(\bar{v} - 2.25) & (\bar{v}^2 - 4.5\bar{v} + 3\bar{u}^2 + 2) \end{bmatrix} \tag{9-28e}$$

Note that for $\bar{v} = 0.5$ and $\bar{u} = 0$, the first equilibrium equation from Eqs. (9-28c) predicts a load of $\bar{p} = 3.5$. This load therefore corresponds to the lower limit at which $\bar{u}$ displacements can be initiated as indicated by inequality 9-27e.

The problem was solved with the aid of the augmented stiffness technique. The spring stiffness matrix for the load vector of Eq. (9-28b) is, from Eq. (8-10)

$$[K_s] = \frac{\gamma \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} < 1 \quad 0 >}{(\Sigma|1|)^2} = \begin{bmatrix} \gamma & 0 \\ 0 & 0 \end{bmatrix} \tag{9-29}$$

and Eqs. (9-28c) and (9-28e) become

$$F_A(X,p) = \begin{Bmatrix} (\bar{v} - 2.25)\,(\bar{v}^2 - 4.5\bar{v} + \bar{u}^2) + \gamma\bar{v} - \bar{f} \\ \bar{u}\,(\bar{v}^2 - 4.5\bar{v} + \bar{u}^2 + 4.5) \end{Bmatrix} \tag{9-30a}$$

and

$$[K]_A = \begin{bmatrix} (3\bar{v}^2 - 13.5\bar{v} + \bar{u}^2 + 10.125 + \gamma) & 2\bar{u}\,(\bar{v} - 2.25) \\ 2\bar{u}\,(\bar{v} - 2.25) & (\bar{v}^2 - 4.5\bar{v} + 3\bar{u}^2 + 2) \end{bmatrix}$$

... (9-30b)

From Eqs. (8-12) and (8-15)

$$\begin{Bmatrix} \bar{p} \\ \\ 0 \end{Bmatrix} = \begin{Bmatrix} \bar{f} \\ \\ 0 \end{Bmatrix} - \begin{Bmatrix} \hat{\bar{p}} \\ \\ 0 \end{Bmatrix} = \lambda \begin{Bmatrix} \bar{f} \\ \\ 0 \end{Bmatrix} \tag{9-30c}$$

where

$$\lambda = 1 - \frac{\gamma < \bar{f} \ 0 >}{(|f|)^2} \begin{Bmatrix} \bar{v} \\ \bar{u} \end{Bmatrix} = 1 - \frac{\gamma \bar{f} \bar{v}}{(|\bar{f}|)^2} \tag{9-30d}$$

The solution of the augmented structural equations, Eqs. (9-30), without special precautions, was attempted using the Newton-Raphson procedure. The results, shown in Fig. 11, predict a response with $\bar{u} = 0$ and $\bar{v}$ identical to that obtained for one degree of freedom. This is not the correct solution and a direct application of the augmented stiffness technique therefore fails. If, however, a nonzero $\bar{u}$ displacement is artificially introduced into the solution at the beginning of the load increment for which nonzero $\bar{u}$ can occur, the load on the augmented structure continues to increase while the correct loading path is followed. This result is also shown in Fig. 11.

For a more general problem, the load at which an artificial displacement should be introduced is unknown, and an attempt was therefore made to follow the correct loading path by maintaining a small lateral load associated with the $\bar{u}$ displacement. Results for a lateral load of $\bar{f}_x = 0.0001$ are shown in Fig. 12. Although nonzero $\bar{u}$ displacements occur, this is not the proper solution, and the vertical response again follows a curve of the type for $\bar{u} = 0$ (see also the solid line of Fig. 13a). However, when a lateral load of $\bar{f}_x = 0.01$ is applied, the augmented structure response is that shown by the dashed lines of Fig. 13a and Fig. 13b. This is the desired solution. The response of the augmented structure, predicted by the Newton-Raphson technique, is therefore profoundly influenced by the magnitude of the disturbing force.

The observation that the determinant of the augmented structure remains positive for a disturbing force of 0.01 which follows the correct response, and is negative for a disturbing force of 0.0001 where incorrect response is obtained, prompted the use of a gradient check with the augmented stiffness method. When this technique is applied with $\bar{f}_x = 0.0001$ the correct response of Fig. 14 is obtained. It should be noted, however, that in all of the applications discussed in this section, the load on the augmented structure continues to increase.

§9.4 *Solutions for Nonconservative Loading*

The equilibrium equations, Eqs. (9-10) contain the nonconservative load $\bar{s}$. In this example the load $\bar{r}$ is set to zero and only the nonconservative load is considered. This example differs from the previous examples since it is impossible to write F(X,p) in the special form indicated in Eq. (2-3). However, none of the techniques contained in this report require the special form of Eq. (2-3), even though it has often been assumed for convenience. Using the notation of section 3, Eqs. (9-10) with $\bar{r} = 0$, and denoting now $\bar{s}$ by $\bar{p}$, we have

$$F(X,p) = F(X,S) = \begin{Bmatrix} (\bar{v} - \mu)\,(\bar{v}^2 - 2\mu\bar{v} + \bar{u}^2) - \bar{p}\,(1 + \bar{u}) / \bar{d} \\ \bar{u}\,(\bar{v}^2 + 2\,(1 - \mu\bar{v}) + \bar{u}^2) - \bar{p}\,(\mu - \bar{v}) / \bar{d} \end{Bmatrix} \qquad \text{(9-31a)}$$

Now $[K] = \left[\dfrac{\partial F}{\partial x_i}\right]$, where $x_1 = \bar{v}$, $x_2 = \bar{u}$, becomes

$$[K] = \begin{bmatrix} 3\bar{v}^2 - 4\mu\bar{v} + \bar{u}^2 + \mu^2 + \dfrac{\bar{p}}{\bar{d}^3}\,(1 + \bar{u})\,(\bar{v} - \mu) \;; & 2\bar{u}(\bar{v} - \mu) - \dfrac{\bar{p}}{\bar{d}} + \dfrac{\bar{p}(1 + \bar{u})^2}{\bar{d}^3} \\ 2\bar{u}\,(\bar{v} - \mu) + \dfrac{\bar{p}}{\bar{d}} - \dfrac{\bar{p}\,(\bar{v} - \mu)^2}{\bar{d}^3} \;; & \bar{v}^2 + 2(1 - \mu\bar{v}) + 3\bar{u}^2 - \dfrac{\bar{p}(1 + \bar{u})(\bar{v} - \mu)}{\bar{d}^3} \end{bmatrix}$$

... (9-31b)

and

$$- \left\{\frac{\partial F}{\partial \bar{p}}\right\} = \begin{Bmatrix} (1 + \bar{u}) / \bar{d} \\ (\mu - \bar{v}) / \bar{d} \end{Bmatrix} \qquad \text{(9-31c)}$$

The incremental Newton-Raphson equations, Eqs. (3-8), may now be used with the above expressions for [K], $\Delta P = -\frac{\partial F}{\partial \bar{p}}\ \overline{\Delta p}$, and $-\ E^r$ evaluated from Eqs. (9-31a). Eqs. (9-31) may also be used with any of the other techniques of section 3.

Displaced configurations for zero applied load may be determined by setting F(X,p) of Eqs. (9-31a) equal to zero. This requires the simultaneous satisfaction of the equations

$$(\bar{v} - \mu)\ (\bar{v}^2 - 2\mu\bar{v} + \bar{u}^2) \ = \ 0 \qquad (9\text{-}32a)$$

$$\bar{u}\ (\bar{v}^2 + 2\ (1 - \mu\bar{v}) + \bar{u}^2) \ = \ 0 \qquad (9\text{-}32b)$$

Eq. (9-32b) is satisfied for $\bar{u} = 0$, and in this case Eq. (9-32a) becomes

$$\bar{v}\ (\bar{v} - \mu)\ (\bar{v} - 2\mu) \ = \ 0\ . \qquad (9\text{-}32c)$$

Selecting simple geometry, let $\mu = 2$, and the equilibrium positions $(\bar{v}, \bar{u})$ available for $\bar{u} = 0$ are then (0,0), (0,2) and (0,4). For $\bar{u} \neq 0$, Eqs. (9-32) require the simultaneous satisfaction of

$$\bar{v}^2 - 4\bar{v} + \bar{u}^2 \ = \ 0 \qquad (9\text{-}33a)$$

$$\bar{v}^2 - 4\bar{v} + u^2 \ = \ -\ 2 \qquad (9\text{-}33b)$$

providing $\bar{v} \neq 2$. Eqs. (9-33) are obviously inconsistent and possess no solutions. Therefore it is sufficient to look for additional solutions only for $\bar{v} = 2$, in which case (9-32a) is identically satisfied and Eq. (9-32c) requires

$$\bar{u}^2 = -4 - 2(1 - 4) = 2$$

The additional solutions therefore are (+ $\sqrt{2}$, 2) and (- $\sqrt{2}$, 2). In summary, the equilibrium positions possible with zero load are (0,0), (0,2), (0,4), (1.414, 2), (-1.414, 2). These solutions may be contained on possible equilibrium paths of the structure.

The load deflection curve obtained by the combined Newton-Raphson, modified Newton-Raphson technique, with gradient test is shown in Fig. 15. Fig. 16 shows the displaced configurations of the truss along the loading paths and the displacements for configurations A, B, C and D are indicated on Fig. 15. The ability of this technique to follow complex load-deformation response is apparent.

The solution by the self-correcting initial value four point difference form is also indicated on Fig. 15 for $\overline{\Delta p} = 0.05$. This solution oscillates after the limit load is reached and cannot be used to determine the post-buckling region.

Fig. 17 shows the load-deflection curve obtained by the augmented stiffness approach. This solution diverged at the load of 2.0 between configurations C and D of Fig. 16.

§9.5 *Discussion of Results*

The applications of the techniques of Chapter 2 to the problems associated with a two-bar truss, considered in this section indicate that the following conclusions can be drawn.

(1) In order to follow an unloading path it is necessary to include a gradient check in the analysis. This follows from the results of §9.2 where the only successful techniques, other than the augmented stiffness technique, were those using a gradient test. However the augmented stiffness technique failed to detect a bifurcation point in §9.3 when no gradient test was used (on the augmented stiffness matrix). Therefore the conclusion is a general one.

(2) Initial value and numerical integration techniques do not seem to be suitable for structures which unload.

(3) In order to detect a bifurcation point, it is necessary that a nonzero load be associated with every degree of freedom. This follows from the first analysis of §9.3. These 'disturbing forces' may be small compared to those in the normal applied load vector. This in itself is insufficient and a gradient test must also be included. (Fig. 12).

(4) The modified Newton-Raphson, Newton-Raphson, technique with a gradient test (§9.2f) appears to be the most versatile and reliable technique. This may be applied to either the primary or the augmented structure. However, an augmented structure technique, by itself is less versatile (see the last result in §9.4). The alpha-constant method may be included in this technique to reduce the number of iterations.

CHAPTER IV APPLICATION TO A SPACE TRUSS

§10. *Introduction to Space Truss Problem*

The examples of Chapter III were simple ones with one or two degrees of freedom. In this chapter the more general problem of formulation and solution of geometrically nonlinear space trusses is considered. The formulation is based on the same simplifying assumptions as were employed in Chapter III.

Shallow reticulated shell structures often display instability phenomena of the snap-through or bifurcation type. This is illustrated schematically in Fig. 5a where the response curve exhibits a limit point at A and a bifurcation point at A*. Hangai and Kawamata (17, 18, 19) have solved for the nonlinear response for a reticulated shell structure by static perturbation methods. The augmented stiffness approach of Chapter III is applied to this type of problem in this chapter.

§11. *Formulation of Equilibrium Equations*

A pin-jointed bar is shown in an arbitrary spatial orientation in Fig. 18a, with ends designated as joints i and j. Adopting the notation of Chapter II, let a_1, a_2 and a_3 be a local coordinate system with origin at joint i, as shown in Fig. 18a. Let $\tilde{u}$, $\tilde{v}$ and $\tilde{w}$ be the displacements of an arbitrary point on the bar in these local coordinate directions. The bar has six displacement degrees of freedom designated as u_1, v_1 and w_1, at joint i, and u_2, v_2 and w_2, at joint j, as shown in Fig. 18b. For uniform extension of the bar

$$\tilde{u} = u_1 (1 - \beta) + u_2 \beta \qquad (11\text{-}1a)$$

$$\tilde{v} = v_1 (1 - \beta) + v_2 \beta \qquad (11\text{-}1b)$$

$$\tilde{w} = w_1 (1 - \beta) + w_2 \beta \qquad (11\text{-}1c)$$

where β is the nondimensional coordinate a_1/ℓ.

But as in Eq. (9-2), the strain energy of the bar is

$$\mathcal{E} = \frac{1}{2} EA\ell \, \varepsilon^2 \qquad (11\text{-}2)$$

where

$$\varepsilon = \frac{\partial \tilde{u}}{\partial a_1} + \frac{1}{2} \left\{ \left(\frac{\partial \tilde{u}}{\partial a_1}\right)^2 + \left(\frac{\partial \tilde{v}}{\partial a_1}\right)^2 + \left(\frac{\partial \tilde{w}}{\partial a_1}\right)^2 \right\} \qquad (11\text{-}3a)$$

Substituting Eqs. (11-1) into Eq. (11-3a) yields

$$\varepsilon = \frac{u_2 - u_1}{\ell} + \frac{1}{2\ell^2} \left((u_2 - u_1)^2 + (v_2 - v_1)^2 + (w_2 - w_1)^2 \right) \qquad \ldots (11\text{-}3b)$$

which may be written in the symbolic form

$$\varepsilon = \varepsilon_L + \varepsilon_{NL} \tag{11-3c}$$

where ε_L and ε_{NL} are identified as the corresponding terms in Eq. (11-3b) and are the linear and nonlinear components of strain, respectively.

Using the notation of Chapters I and II, we identify the displacement coordinates for a single bar in the local coordinate system as

$$X^T = \langle x_1, x_2, x_3, x_4, x_5, x_6 \rangle = \langle u_1, v_1, w_1, u_2, v_2, w_2 \rangle \tag{11-4}$$

Then the components of strain in Eq. (11-3c), may be obtained by substituting Eqs. (11-1) into Eq. (11-3b) and result in

$$\varepsilon_L = L^T X \tag{11-5a}$$

where L is the vector defined as

$$L^T = \frac{1}{\ell} \langle -1 \quad . \quad . \quad 1 \quad . \quad . \rangle \tag{11-5b}$$

and

$$\varepsilon_{NL} = \frac{1}{2} X^T H X \tag{11-6a}$$

where

$$H = \frac{1}{\ell^2} \begin{bmatrix} 1 & . & . & -1 & . & . \\ . & 1 & . & . & -1 & . \\ . & . & 1 & . & . & -1 \\ -1 & . & . & 1 & . & . \\ . & -1 & . & . & 1 & . \\ . & . & -1 & . & . & 1 \end{bmatrix} . \tag{11-6b}$$

Equation (11-2) now can be introduced into the potential energy expression for the bar and results in

$$\pi^e = \mathcal{E} - P^T X \tag{11-7a}$$

Where P is the vector of element nodal forces in the (a_1, a_2, a_3) coordinate system. Substituting Eq. (11-3c) into the expression for strain energy, Eq. (11-2), and using Eqs. (11-5) yields

$$\pi^e = \frac{EA\ell}{2} (X^T L + \frac{X^T H X}{2}) (L^T X + \frac{X^T H X}{2}) - P^T X \tag{11-8a}$$

$$= \frac{EA\ell}{2} \left(X^T L L^T X + \frac{1}{2} X^T (LX^T H + HXL^T) X + \frac{X^T H X X^T H X}{4} \right) - X^T P$$

or

$$\pi^e = \frac{EA\ell}{2} \left(X^T L L^T X + \frac{1}{3} X^T (LX^T H + L^T X H + HXL^T) X + \frac{1}{6} X^T (\frac{X^T H X H}{2} + H X X^T H) X - X^T P \right. \tag{11-8b}$$

The energy expression may be written in many different ways such that the first, second and third terms on the right hand side are symmetric.

The equilibrium equation for the element is then obtained from Eqs. (11-8b) or (11-8c) by differentiating with respect to the displacement coordinates. In the notation of Chapter II

$$F^e\ (X,p) = \left\{\frac{\partial \pi^e}{\partial x_k}\right\} = \frac{EA\ell}{2}\left\{2\ L\ L^T\ X + (L\ X^T\ H + L^T\ X\ H + H\ X\ L^T)\ X\right.$$

$$+ (H\ X\ X^T\ H)\ X - \{P\} = \{0\} \qquad (11\text{-}9a)$$

or

$$F^e\ (X,p) = \left\{\frac{\partial \pi^e}{\partial x_k}\right\} = EA\ell\left\{L\ L^T\ X + \frac{1}{2}(L\ X^T\ H + L^T\ X\ H + H\ X\ L^T)\ X\right.$$

$$+ \frac{1}{3}\ \left(\frac{X^T\ H\ X\ H}{2} + H\ X\ X^T\ H\right)\ X - \{P\} = \{0\} \qquad (11\text{-}9b)$$

Eqs. (11-9a) and (11-9b) are the equilibrium equations obtained by differentiating the energy expressions of Eqs. (11-8a) and (11-8b) respectively. Eq. (11-9a) and Eq. (11-9b) represent identically the same equilibrium equations but they are written in two different symmetric ways. In obtaining Eq. (11-9) the symmetry of H has been recognized and it is also recognized that scalar products and quadratic forms can be commuted as normal scalars. Eq. (11-9) is the set of equilibrium equations for a single bar and corresponds to Eq. (2-1b). When equilibrium is not completely satisfied it may be used for the evaluation of residual forces as indicated in Eqs. (2-2).

Mallet and Marcal (5) have introduced special notation for Eq. (11-9) and written the form

$$\left[[K]^e + \frac{1}{2} [N_1]^e + \frac{1}{3} [N_2]^e \right] X^e - p^e = \{0\} \quad . \qquad (11\text{-}10)$$

where

$$[K]^e = EA\ell \; L \; L^T$$

$$[N_1]^e = EA\ell \; [L \; X^T \; H + L^T \; X \; H + H \; X \; L^T]$$

and

$$[N_2]^e = EA\ell \left[\frac{X^T \; H \; X \; H}{2} + H \; X \; X^T \; H \right] .$$

To write an approximate incremental equation such as Eq. (3-2), using the Taylor series expansion, it is necessary to evaluate

$$\left[\frac{\partial F^e \; (X,p)}{\partial x_i} \right] \Delta X$$

Differentiating either Eq. (11-9a) or Eq. (11-9b) results in

$$\left[\frac{\partial F^e \; (X,p)}{\partial_{xn}} \right] = EA\ell \left[L \; L^T + (L \; X^T \; H + L^T \; X \; H + H \; X \; L^T) + (\frac{X^T \; H \; X \; H}{2} + H \; X \; X^T \; H) \right] \qquad (11\text{-}11a)$$

Identifying the matrices in this equation with those in Eqs. (11-10) we may write

$$\left[\frac{\partial F^e\ (X,p)}{\partial x_n}\right] = [K]^e + [N_1]^e + [N_2]^e \tag{11-11b}$$

The neat relationship of Mallet and Marcal such as

$$\pi^e = X^T\left[\frac{1}{2}[K]^e + \frac{1}{6}[N_1]^e + \frac{1}{12}[N_2]^e\right]X - X^T P$$

$$\left[[K]^e + \frac{1}{2}[N_1]^e + \frac{1}{3}[N_2]^e\right]X = P \tag{11-12a}$$

$$\left[[K]^e + [N_1]^e + [N_2]^e\right]\Delta X = \Delta P \tag{11-12b}$$

will exist only if the energy expression is written as Eq. (11-8b). Writing the energy expression having proper symmetry of the resultant matrices above will not insure that the above neat relationship is valid at all times.

Upon expansion, using the definitions of X, L and H in Eqs. (11-4), (11-5) and (11-6) the matrices in Eqs. (11-10) and (11-11) are

$$[K]^e = \frac{EA}{\ell}\begin{bmatrix} 1 & \cdot & \cdot & -1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -1 & \cdot & \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \tag{11-12a}$$

$$[N_1]^e = \frac{EA}{\ell^2} \begin{bmatrix} 3U_{21} & V_{21} & W_{21} & -3U_{21} & -V_{21} & -W_{21} \\ & U_{21} & \cdot & -V_{21} & -U_{21} & \cdot \\ & & U_{21} & -W_{21} & \cdot & -U_{21} \\ & & & 3U_{21} & V_{21} & W_{21} \\ & \text{SYM.} & & & U_{21} & \cdot \\ & & & & & U_{21} \end{bmatrix} \qquad (11\text{-}12b)$$

Where the notation $u_2 - u_1 = U_{21}$, etc has been used. Continuing

$$[N_2]^e = \frac{EA}{\ell^3} \begin{bmatrix} U_{21}^2 + \frac{1}{2}\Delta & U_{21}V_{21} & U_{21}W_{21} & -U_{21}^2 - \frac{1}{2}\Delta & -U_{21}V_{21} & -U_{21}W_{21} \\ & V_{21}^2 + \frac{1}{2}\Delta & V_{21}W_{21} & -V_{21}U_{21} & -V_{21}^2 - \frac{1}{2}\Delta & -V_{21}W_{21} \\ & & W_{21}^2 + \frac{1}{2}\Delta & -W_{21}U_{21} & -W_{21}U_{21} & -W_{21}^2 - \frac{1}{2}\Delta \\ & & & U_{21}^2 + \frac{1}{2}\Delta & U_{21}V_{21} & U_{21}W_{21} \\ & & & & V_{21}^2 + \frac{1}{2}\Delta & V_{21}W_{21} \\ & & & & & W_{21}^2 + \frac{1}{2}\Delta \end{bmatrix}$$

... (11-12c)

Where $\Delta = U_{21}^2 + V_{21}^2 + W_{21}^2$

All the matrices associated with a truss element for a total or linear incremental solution have now been determined. However, it remains to assemble the equilibrium equations for the structure.

The above equations are in the local coordinate system. If ξ, η, ζ, represent the global axes in Fig. 18a, the coordinate transformation from the (a_1, a_2, a_3) axes to the local axes may be written as

$$\begin{Bmatrix} \xi \\ \eta \\ \zeta \end{Bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = [A] \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} \qquad (11\text{-}13)$$

Where a_{ij} are the direction cosines between the axis a_j and the axis ξ_i where $(\xi_1, \xi_2, \xi_3) \rightarrow (\xi, \eta, \zeta)$. The force transformation to the global axes at the node i of a bar is then (Fig. 18c).

$$\begin{Bmatrix} U_\xi \\ V_\eta \\ W_\zeta \end{Bmatrix}^e = [A] \begin{Bmatrix} U \\ V \\ W \end{Bmatrix}^e_i \qquad (11\text{-}14a)$$

and the displacement transformation is

$$\begin{Bmatrix} u \\ v \\ w \end{Bmatrix}_i = [A]^{-1} \begin{Bmatrix} u_\xi \\ v_\eta \\ w_\zeta \end{Bmatrix}_i = [A]^T \begin{Bmatrix} u_\xi \\ v_\eta \\ w_\zeta \end{Bmatrix} \qquad (11\text{-}14b)$$

Transforming all elements to the global system of coordinates, and assembling, Eqs. (11-10a) and (11-11d) become

$$F(X,p) = [K + \frac{1}{2} N_1 + \frac{1}{3} N_2] X - P = \{0\} \qquad (11\text{-}15a)$$

$$F(X^{r+1}, p^{r+1}) = F(X^r, p^r) + [K_r + N_{1,r} + N_{2,r}] \Delta X - \Delta P$$

$$= \{0\} \qquad (11\text{-}15b)$$

Where X is the asembled vector of global displacement coordinates, P is the assembled vector of global forces, and

$$K = \sum_{m=1}^{N} [T] \; [K^e] \; [T]^T \qquad (11\text{-}16a)$$

$$N_1 = \sum_{m=1}^{N} [T] \; [N_1{}^e] \; [T]^T \qquad (11\text{-}16b)$$

$$N_2 = \sum_{m=1}^{N} [T] \; [N_2{}^e] \; [T]^T \qquad (11\text{-}16c)$$

and

$$[T] = \begin{bmatrix} [A] & \cdot \\ \cdot & [A] \end{bmatrix} \qquad (11\text{-}16d)$$

A variety of solution procedures may be used to solve these equations.

§12. *Solution of Reticulated Shell Structure*

The reticulated shell structure solved by Hangai and Kawamata (18) is shown in Fig. 19. Results for the vertical displacement of joint 1, obtained by the augmented stiffness modified Newton-Raphson, Newton-Raphson, technique with a gradient test, are also compared to those of reference 19 in Fig. 19. The displacements of joint 2 are plotted in Fig. 20.

It is apparent that the technique employed is capable of producing results for reasonably complex problems. The results of Ref. 18 were obtained by a static perturbation technique. The discrepancy of results between the two approaches can be attributed to two causes: (a) Ref. 18 employed a different strain measure in the constitutive relation, and (b) the static perturbation technique does not have an overall equilibrium check.

SUMMARY

The formulation of a number of different techniques for carrying out the solution of geometrically nonlinear structural problems has been presented. These techniques have been implemented and results compared for a number of simple structures exhibiting pronounced geometrically nonlinear response.

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T A B L E S   A N D   F I G U R E S

Load	Modified Newton-Raphson	Alpha-Constant Stiffness Method
0.05	6	4
0.10	6	4
0.15	7	5
0.20	7	5
0.25	8	6
0.30	10	9

TABLE 1 Comparison of Alpha-Constant Stiffness and Modified Newton-Raphson Technique

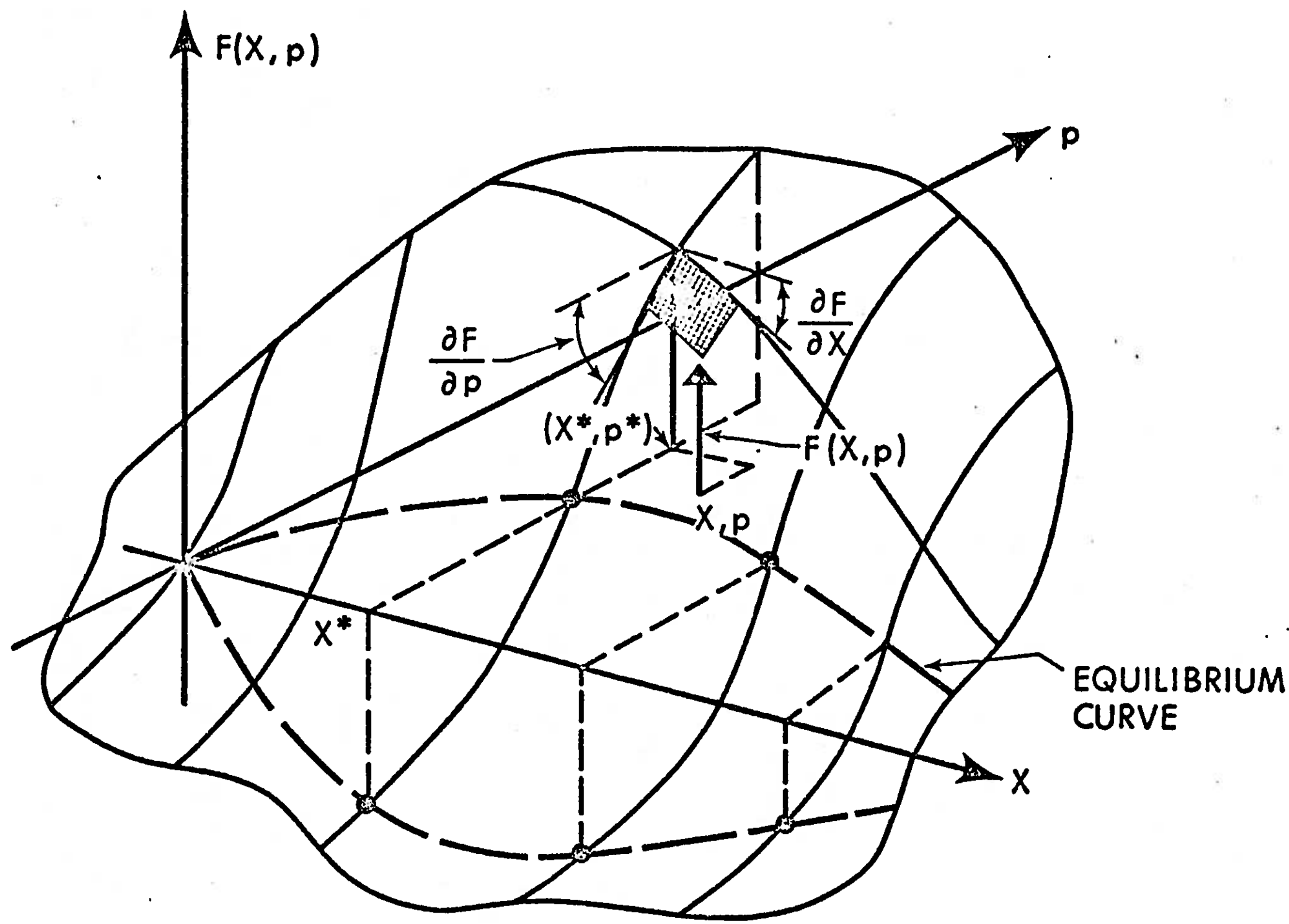


FIG. 1 - THREE DIMENSIONAL REPRESENTATION OF EQUATIONS

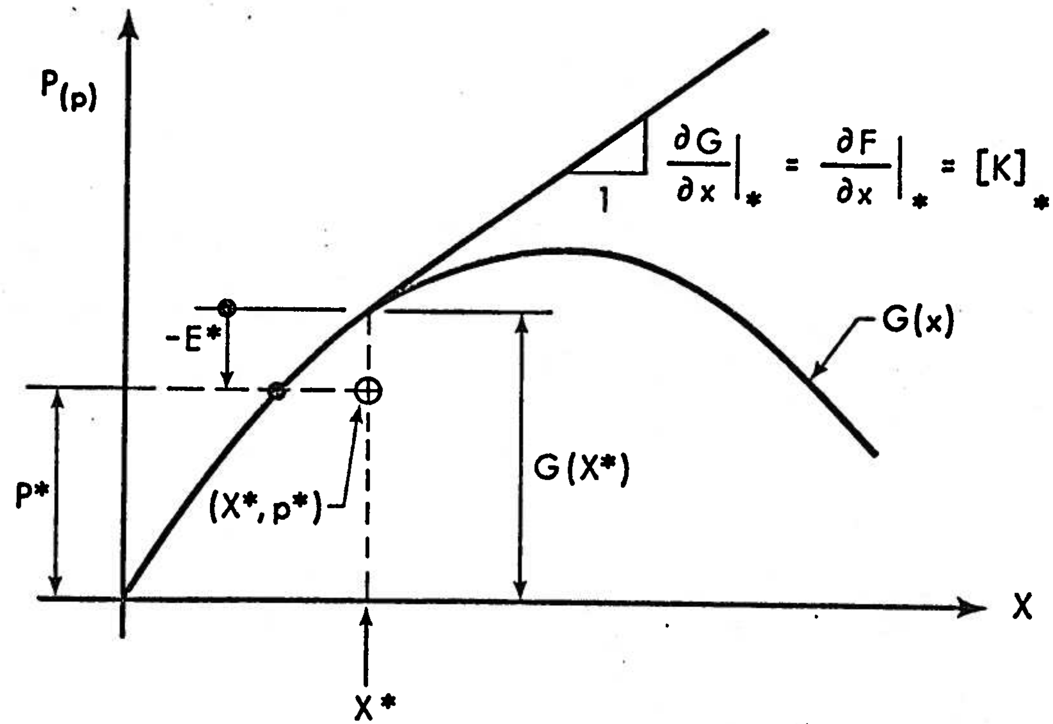
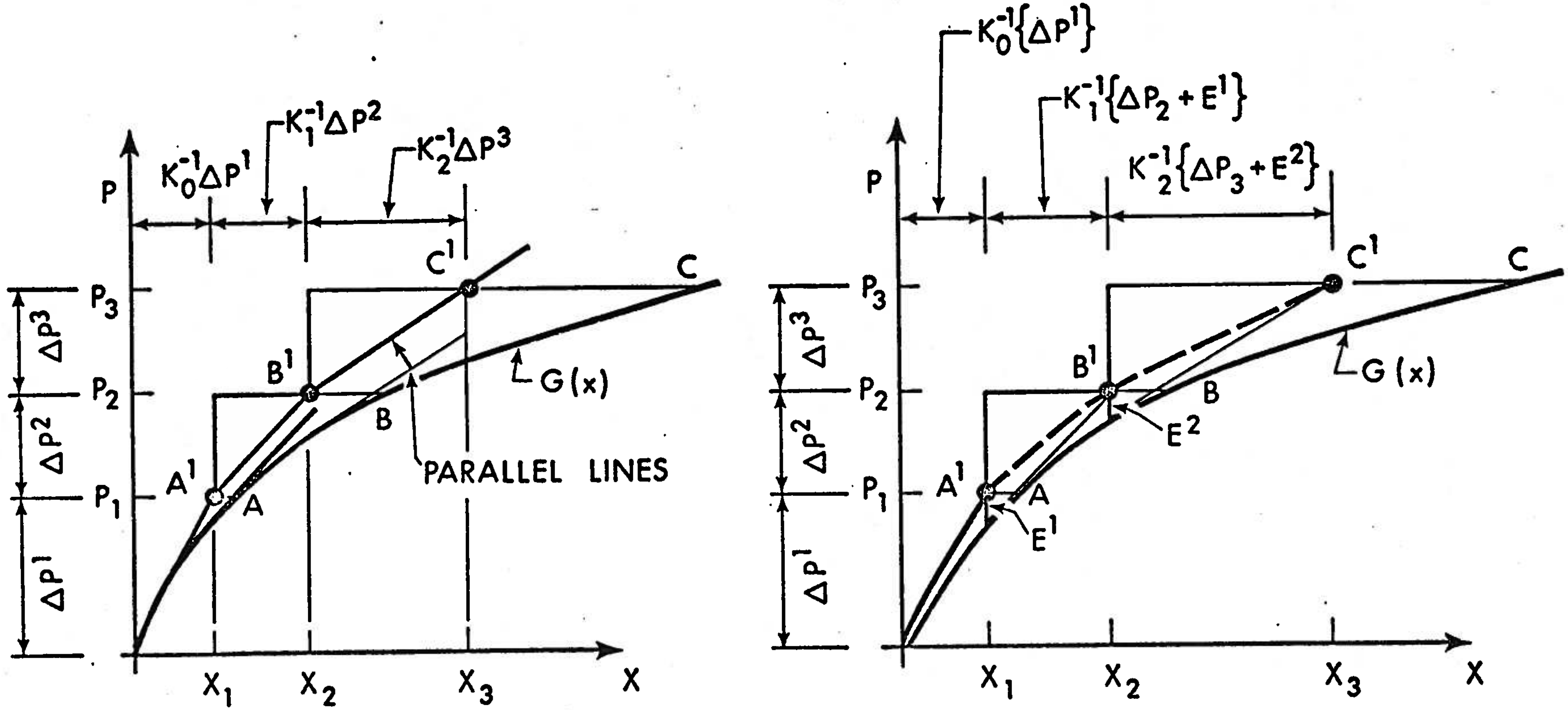


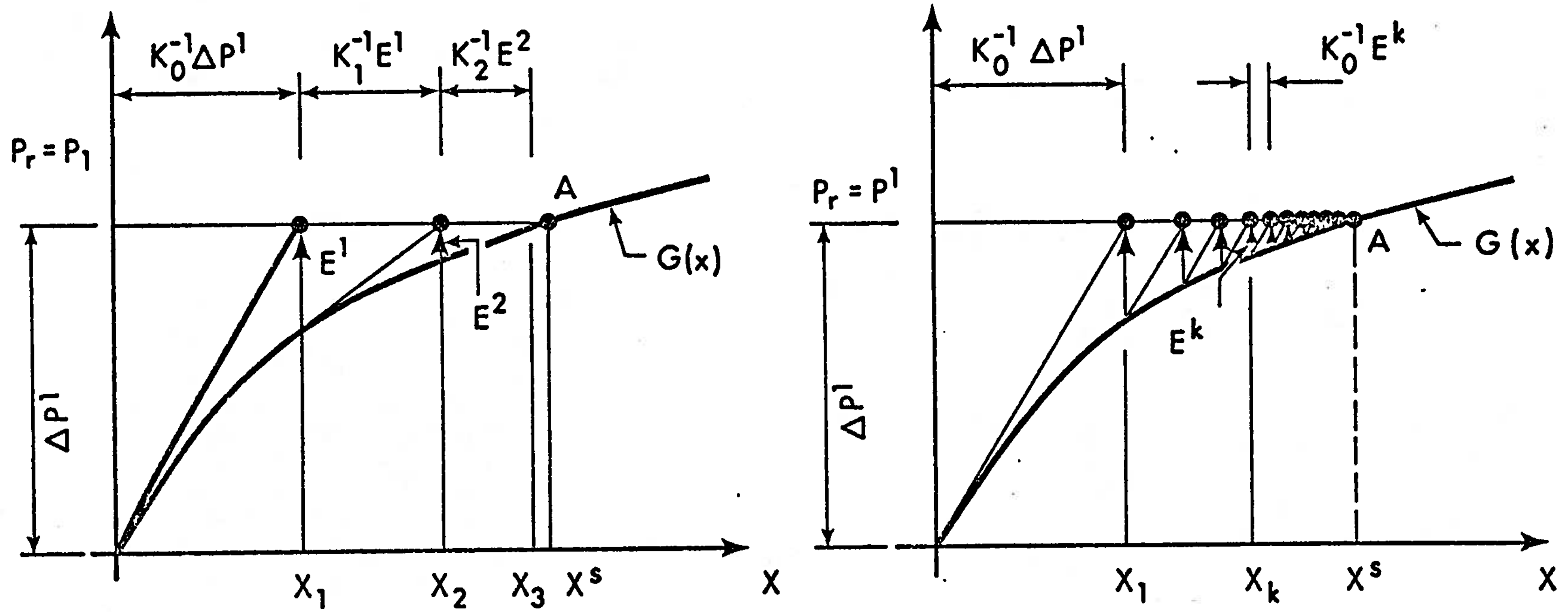
FIG. 2 - TWO DIMENSIONAL REPRESENTATION OF EQUATIONS



a) SIMPLE INCREMENTAL

b) MODIFIED INCREMENTAL

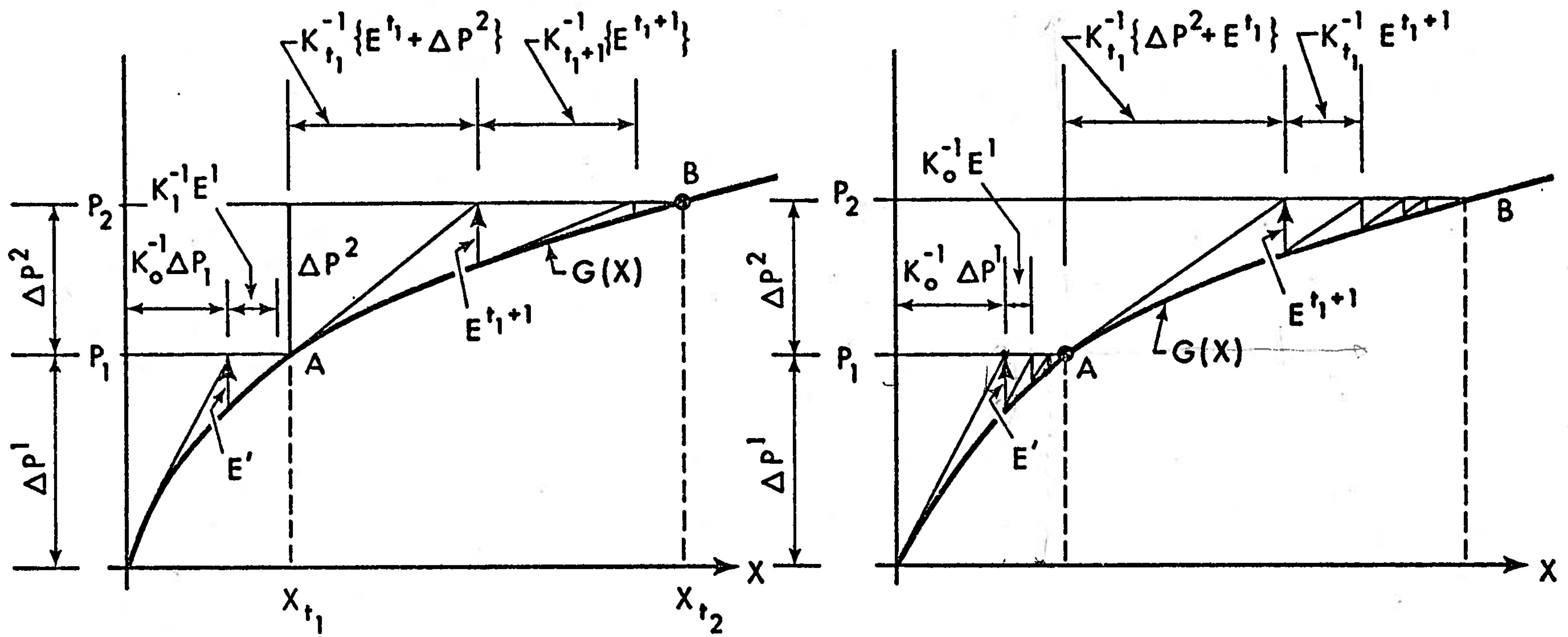
FIG. 3 - SCHEMATICS OF SOLUTION PROCEDURES



c) NEWTON - RAPHSON

d) MODIFIED NEWTON-RAPHSON

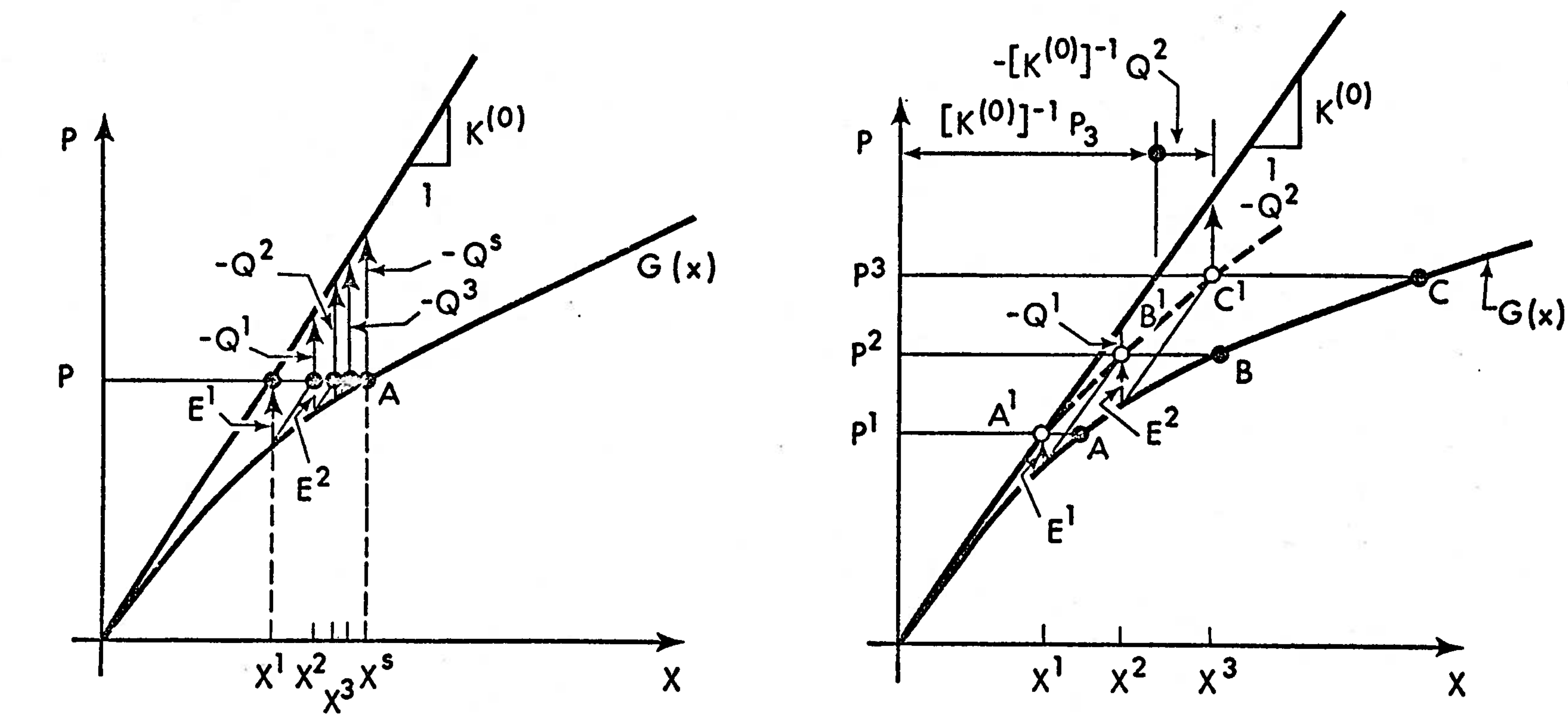
FIG. 3 - SCHEMATICS OF SOLUTION PROCEDURES (CONTINUATION 1)



(e) INCREMENTAL NEWTON-RAPHSON

(f) MODIFIED INCREMENTAL NEWTON-RAPHSON

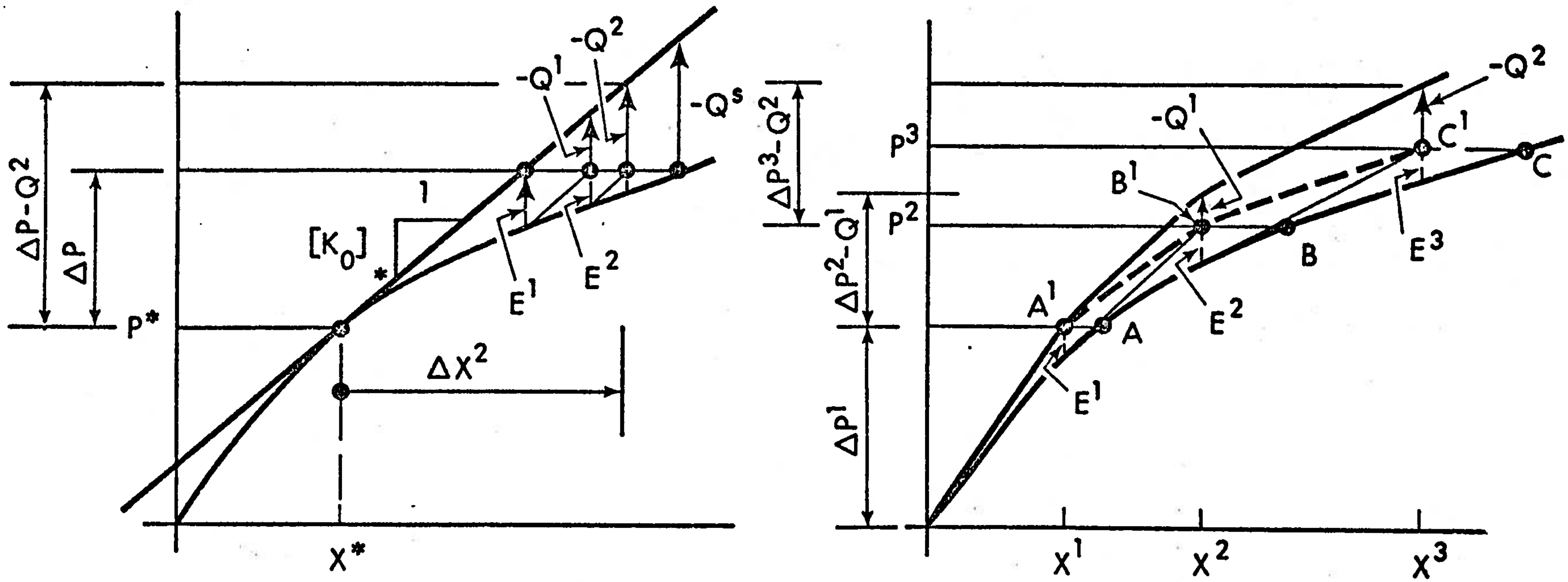
FIG. 3 - SCHEMATICS OF SOLUTION PROCEDURES (CONTINUATION 2)



a) ITERATIVE PSEUDO-LOAD

b) INCREMENTAL PSEUDO-LOAD

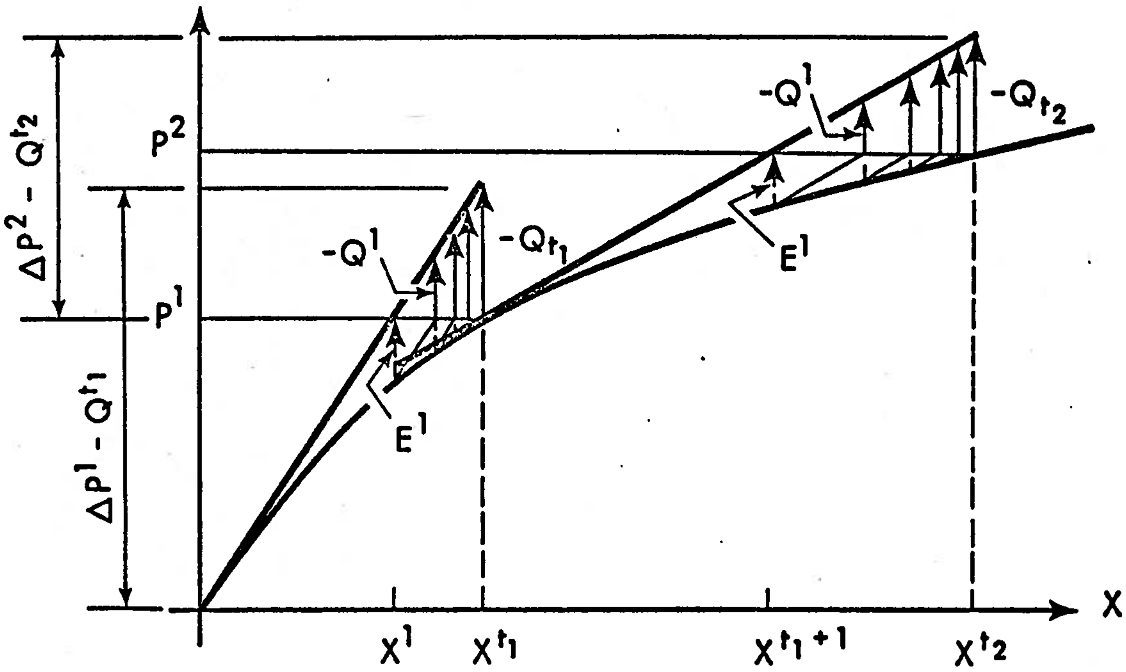
FIG. 4 - SCHEMATICS OF PSEUDO-LOAD SOLUTION PROCEDURES



c) UNBALANCED FORCES AND PSEUDO-LOADS

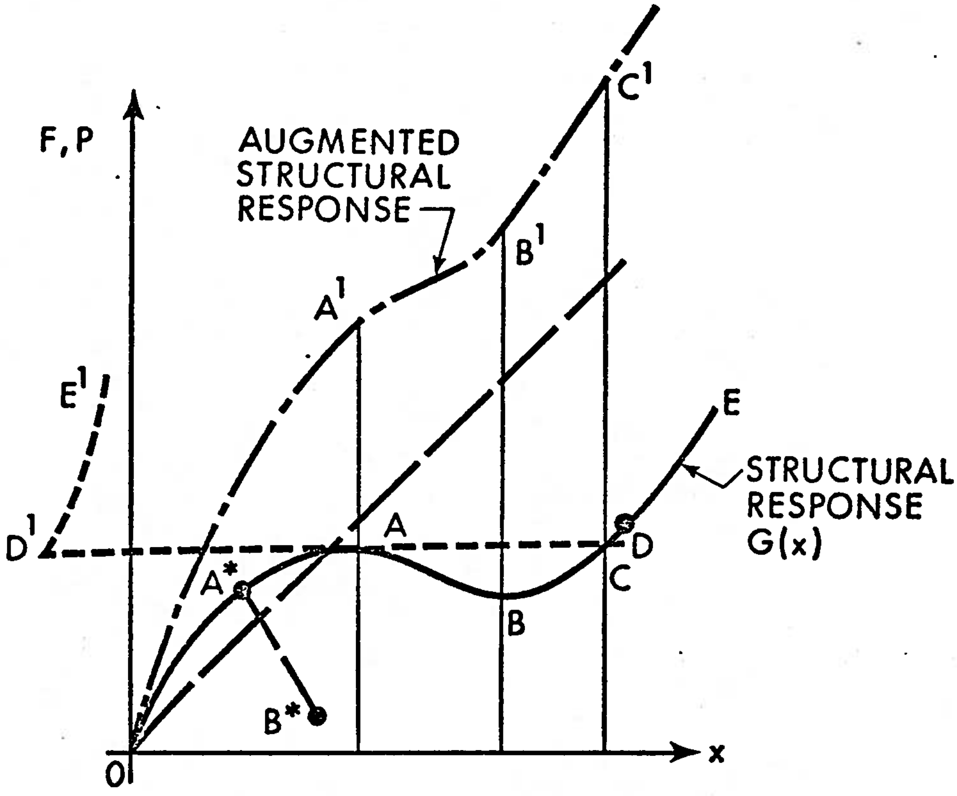
d) MODIFIED INCREMENTAL PSEUDO-LOADS

FIG. 4 - SCHEMATICS OF PSEUDO-LOAD SOLUTION PROCEDURES (CONTINUATION 1)



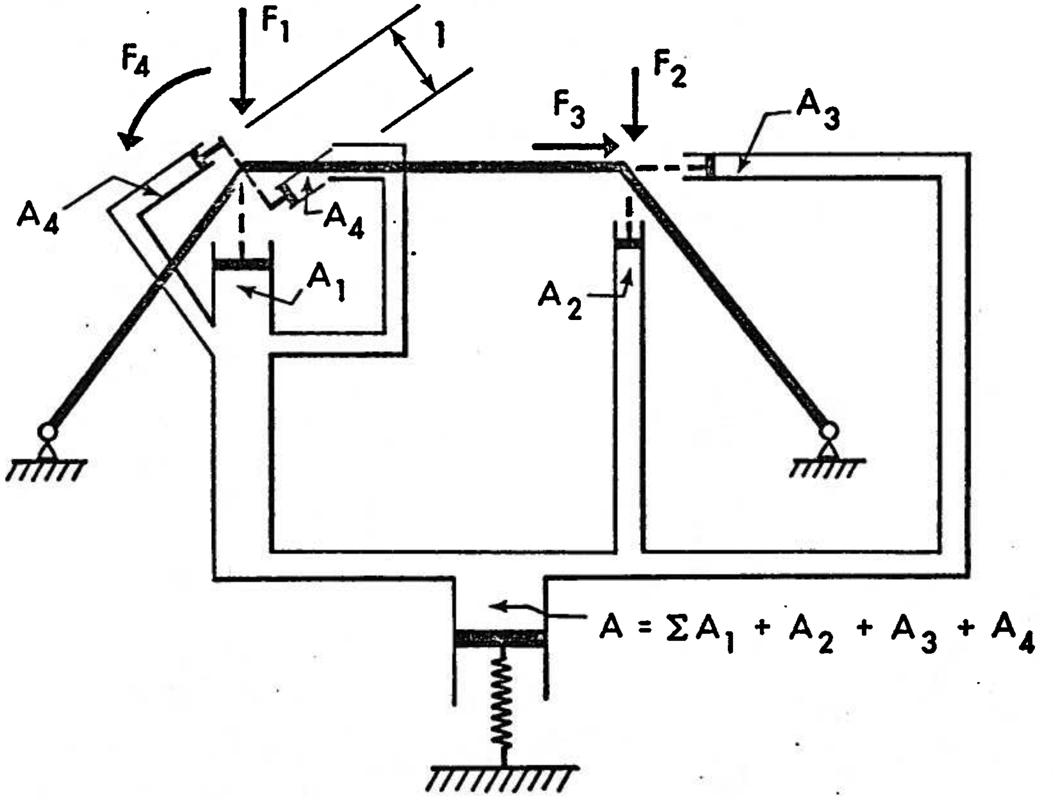
e) ITERATIVE INCREMENTAL PSEUDO - LOAD

FIG. 4 - SCHEMATICS OF PSEUDO-LOAD SOLUTION PROCEDURES (CONTINUATION 2)



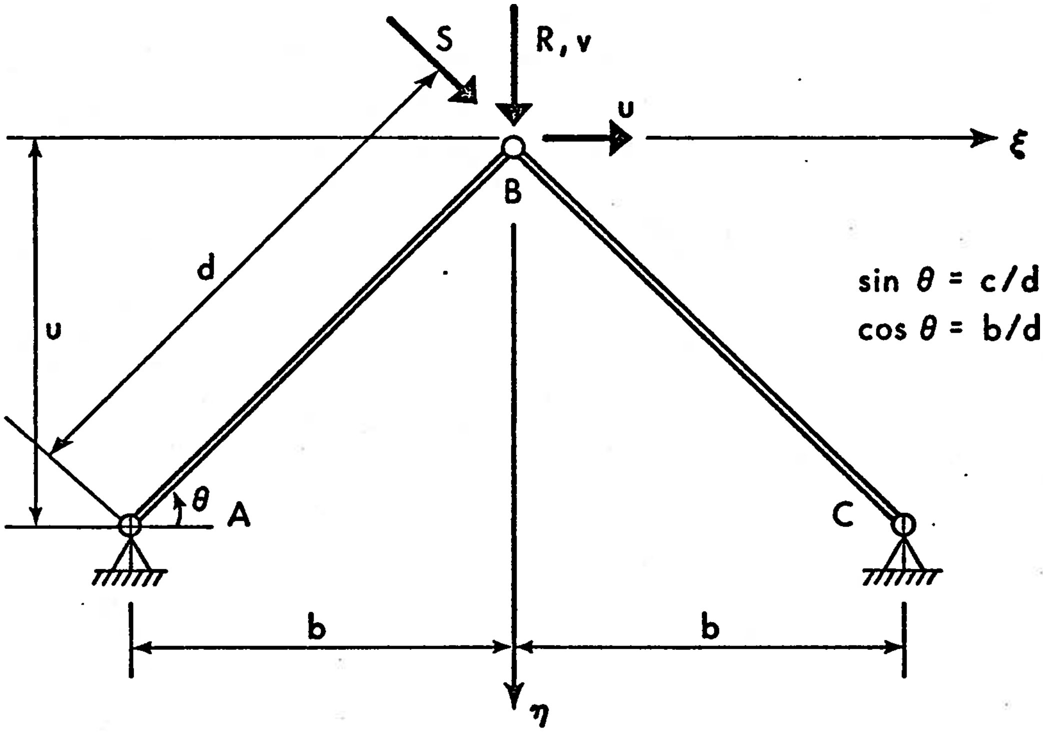
a) RESPONSE OF AUGMENTED STRUCTURE

FIG. 5 - AUGMENTED STIFFNESS ANALYSIS



b) AUGMENTED STRUCTURE

FIG. 5 - AUGMENTED STIFFNESS ANALYSIS (CONTINUATION)



(a)

FIG. 6 - TWO BAR TRUSS

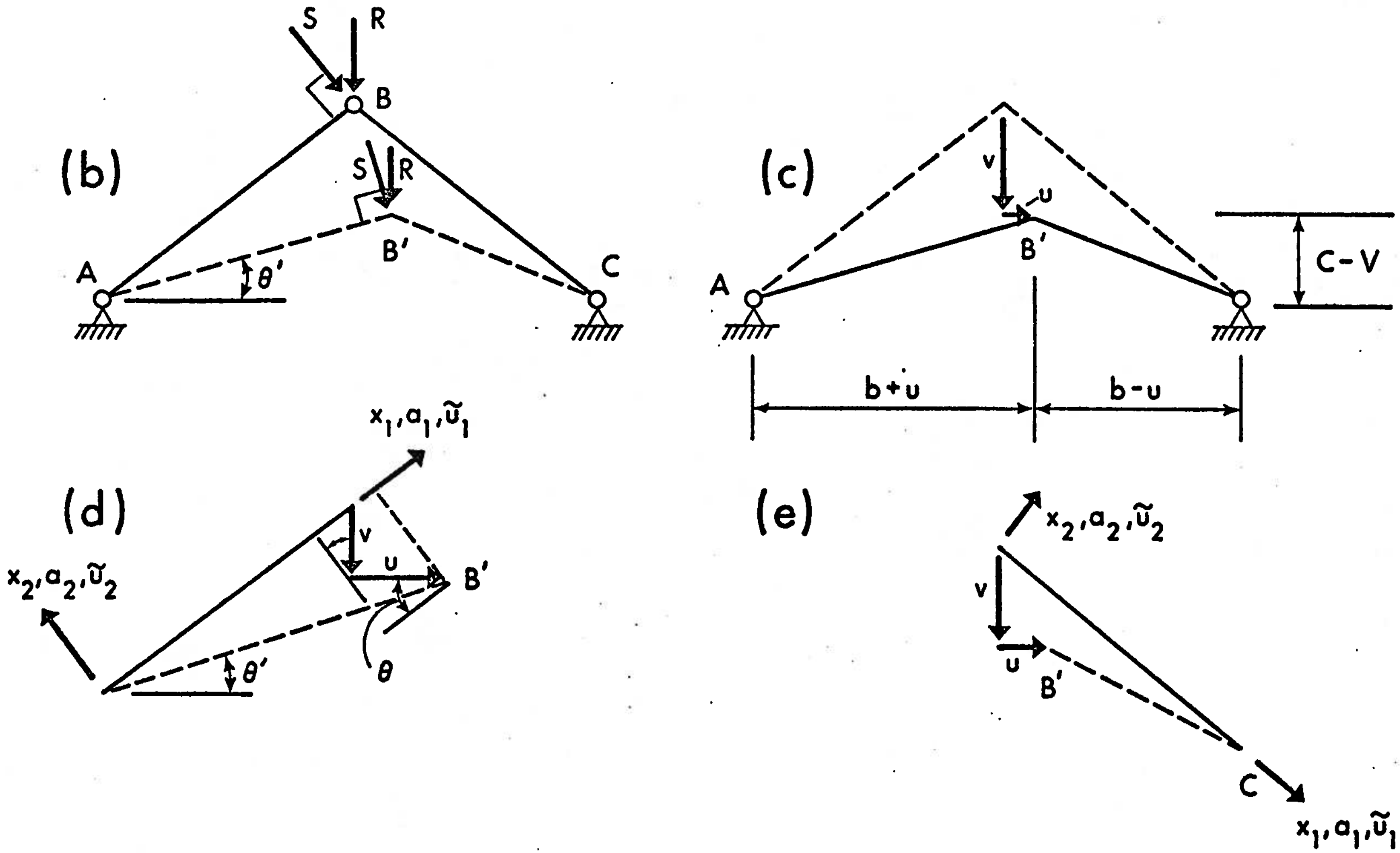


FIG. 6 - TWO BAR TRUSS (CONTINUATION)

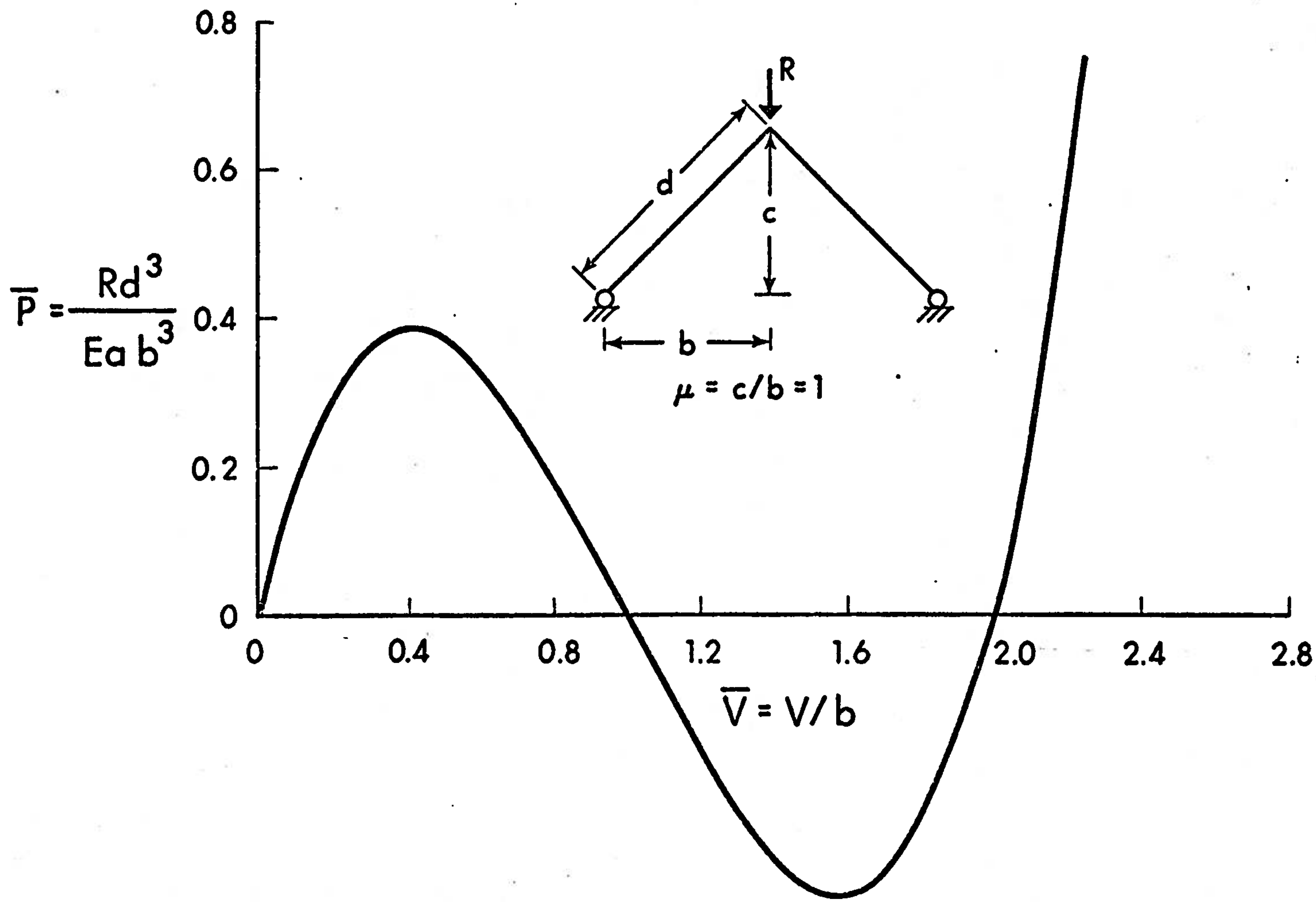


FIG. 7 - ANALYTIC SOLUTION FOR TWO BAR TRUSS ($\mu = 1$)

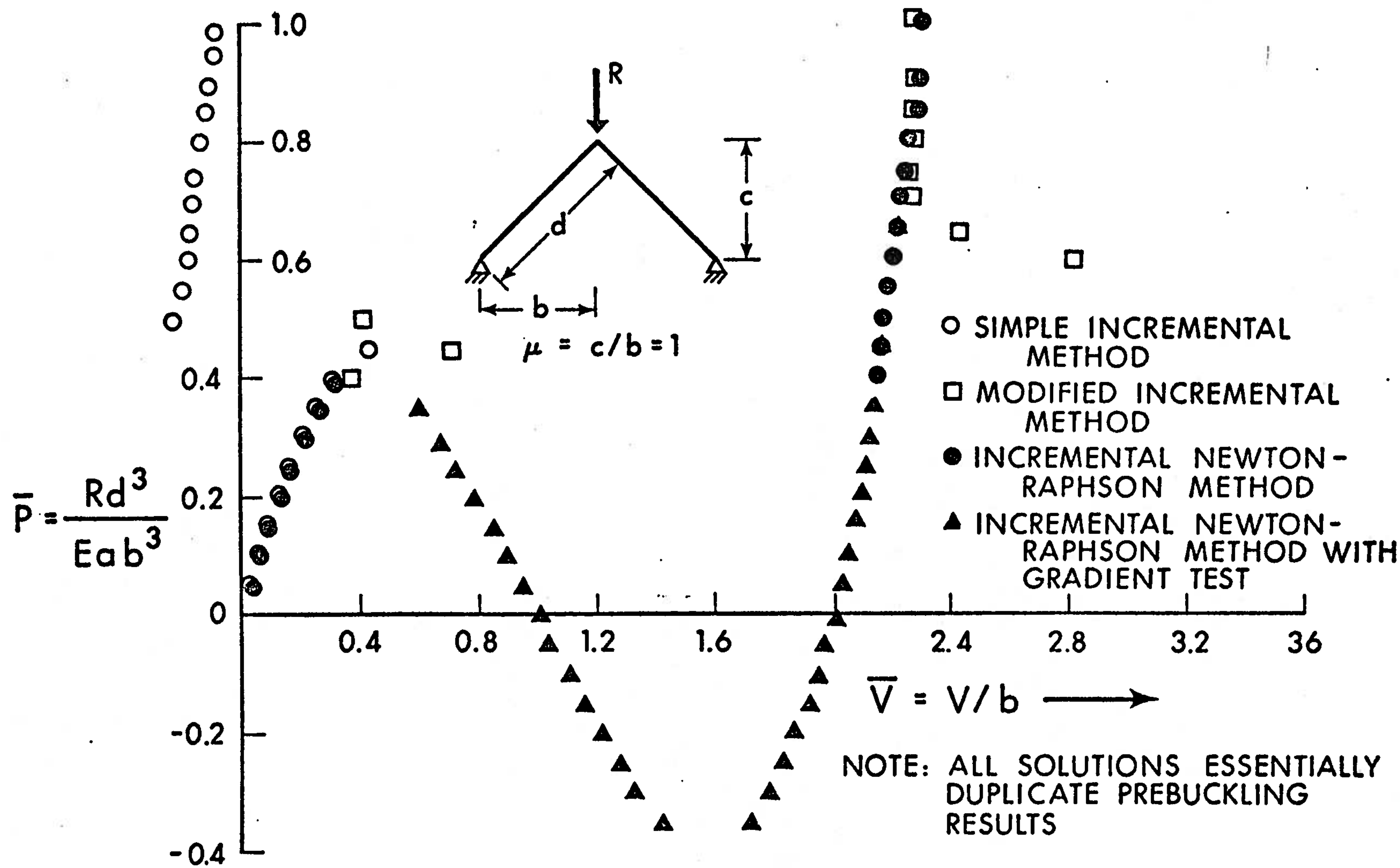


FIG. 8 - ANALYTIC SOLUTION FOR TWO BAR TRUSS (μ = 1)

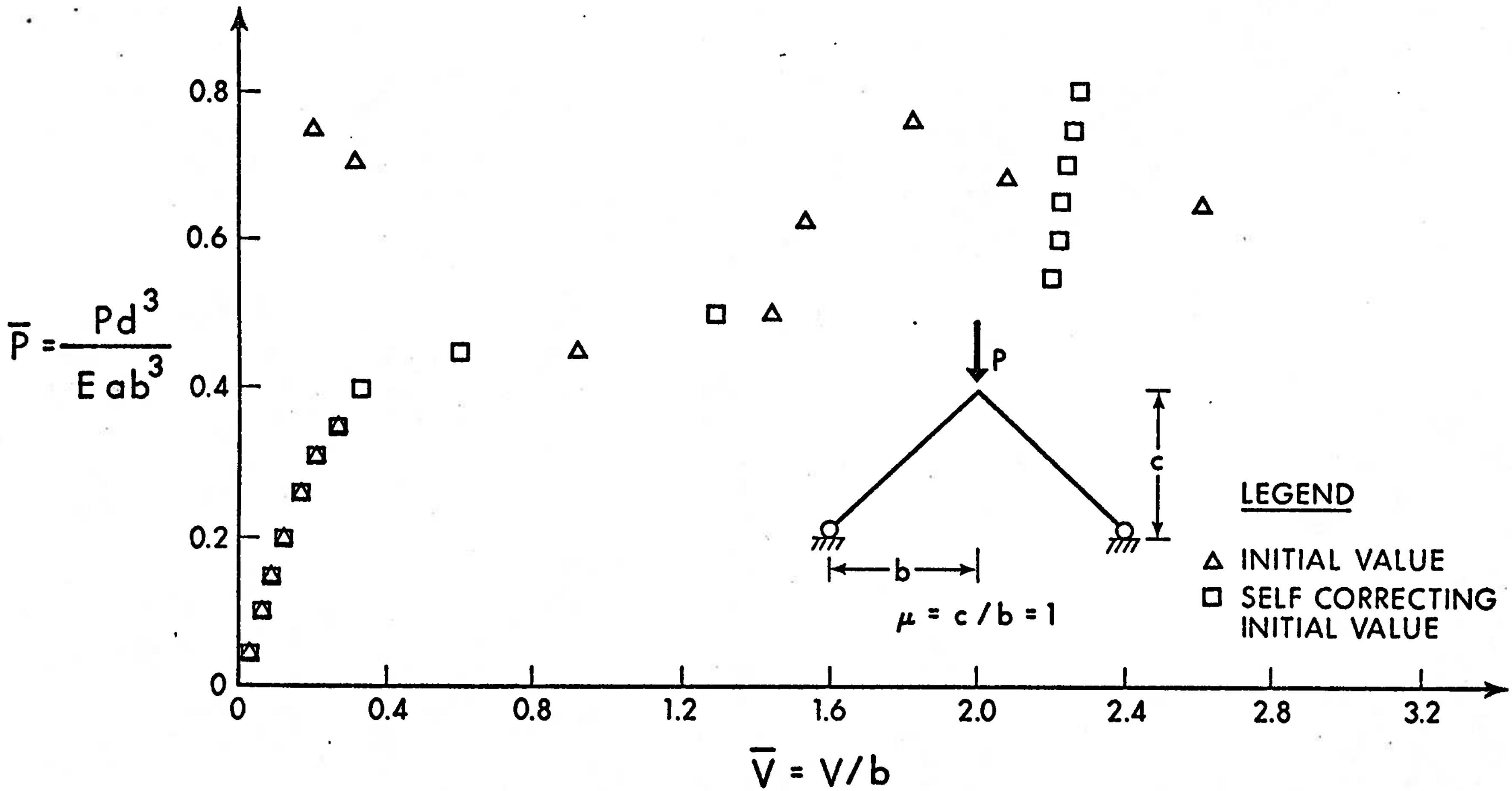


FIG. 9 - INITIAL VALUE SOLUTIONS FOR TWO BAR TRUSS (μ = 1)

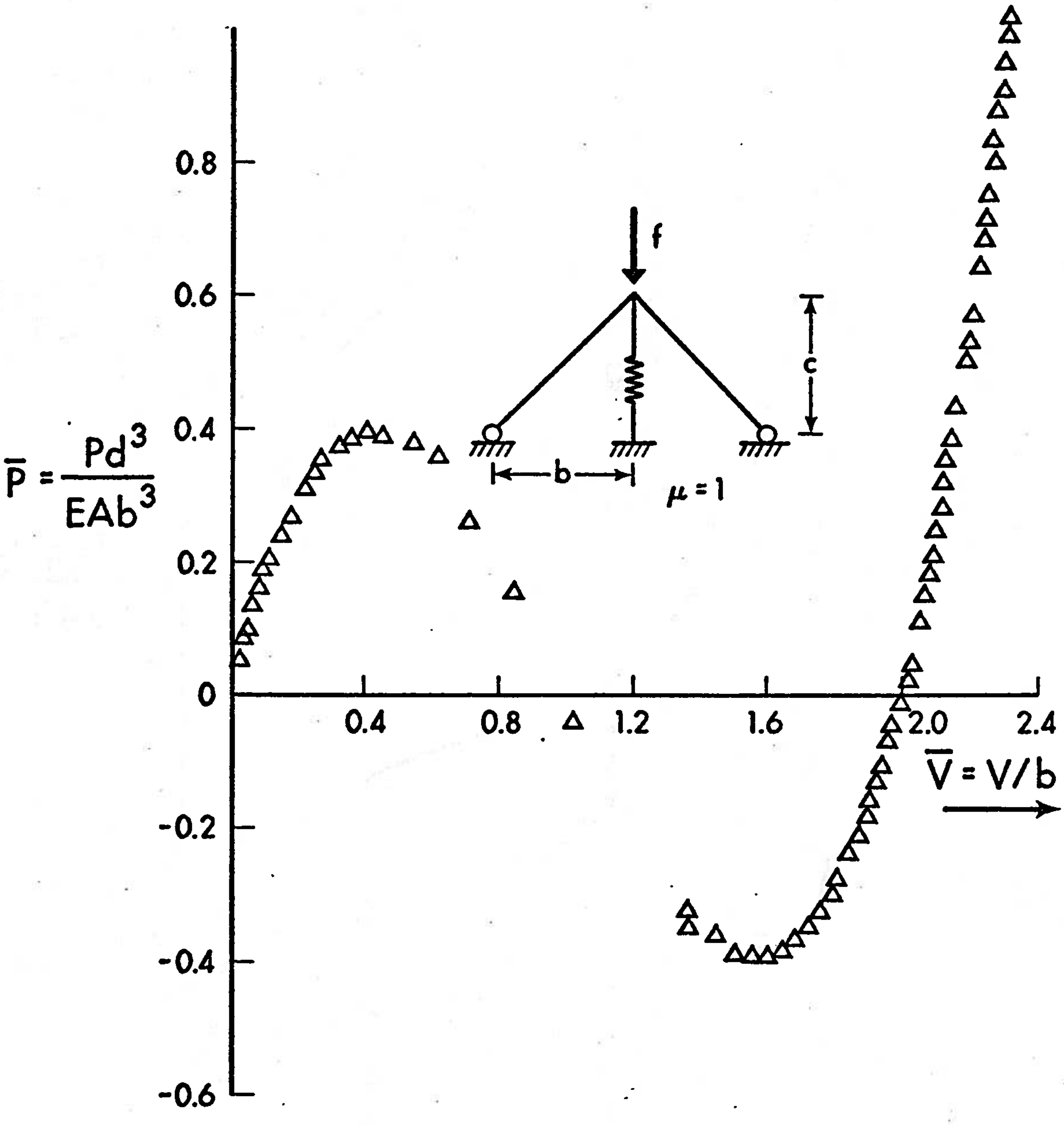


FIG. 10a - AUGMENTED STIFFNESS SOLUTION FOR TWO BAR TRUSS ($\mu = 1$)

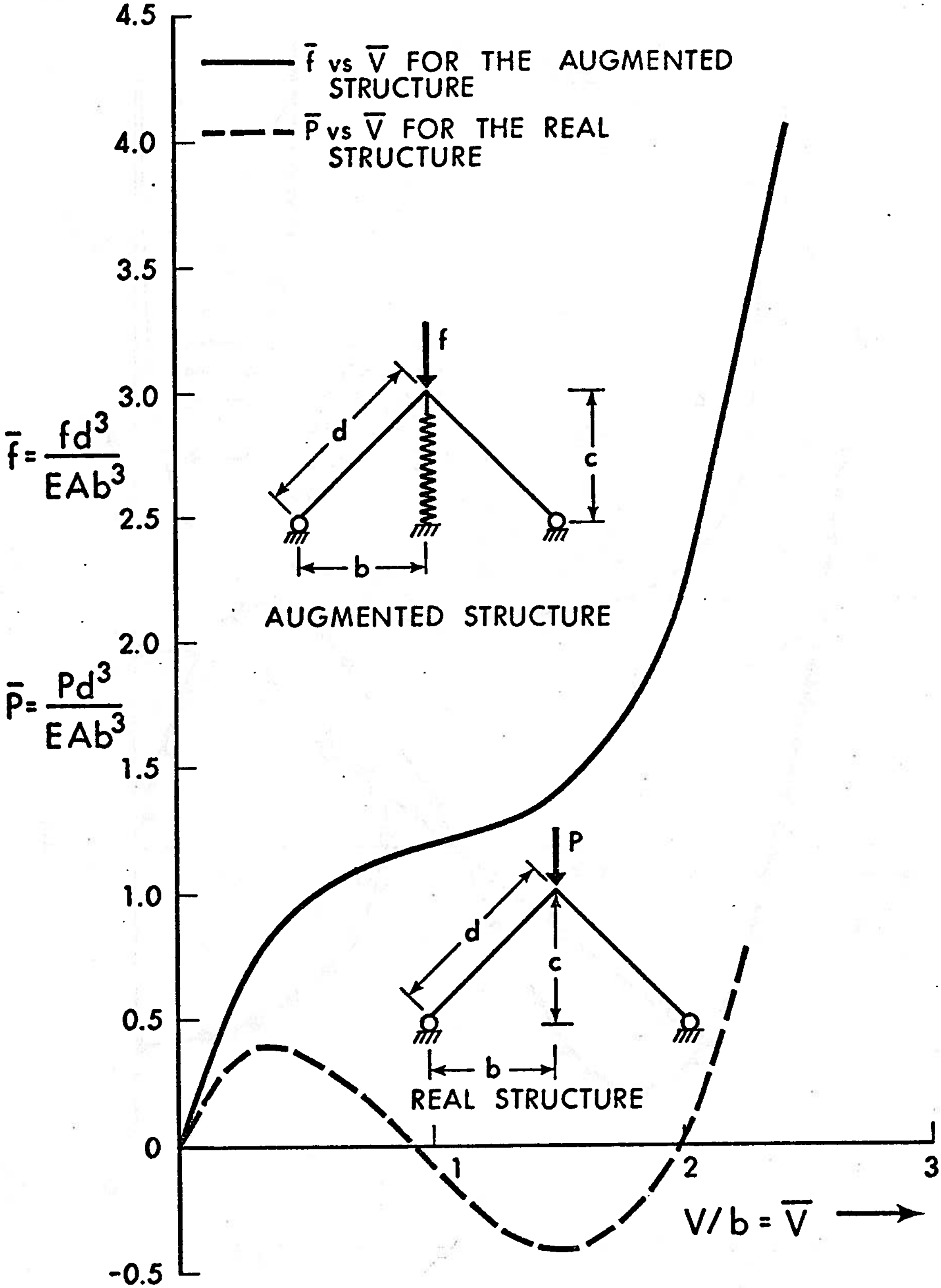


FIG. 10b - LOAD - DEFLECTION CURVES FOR AUGMENTED AND REAL STRUCTURES

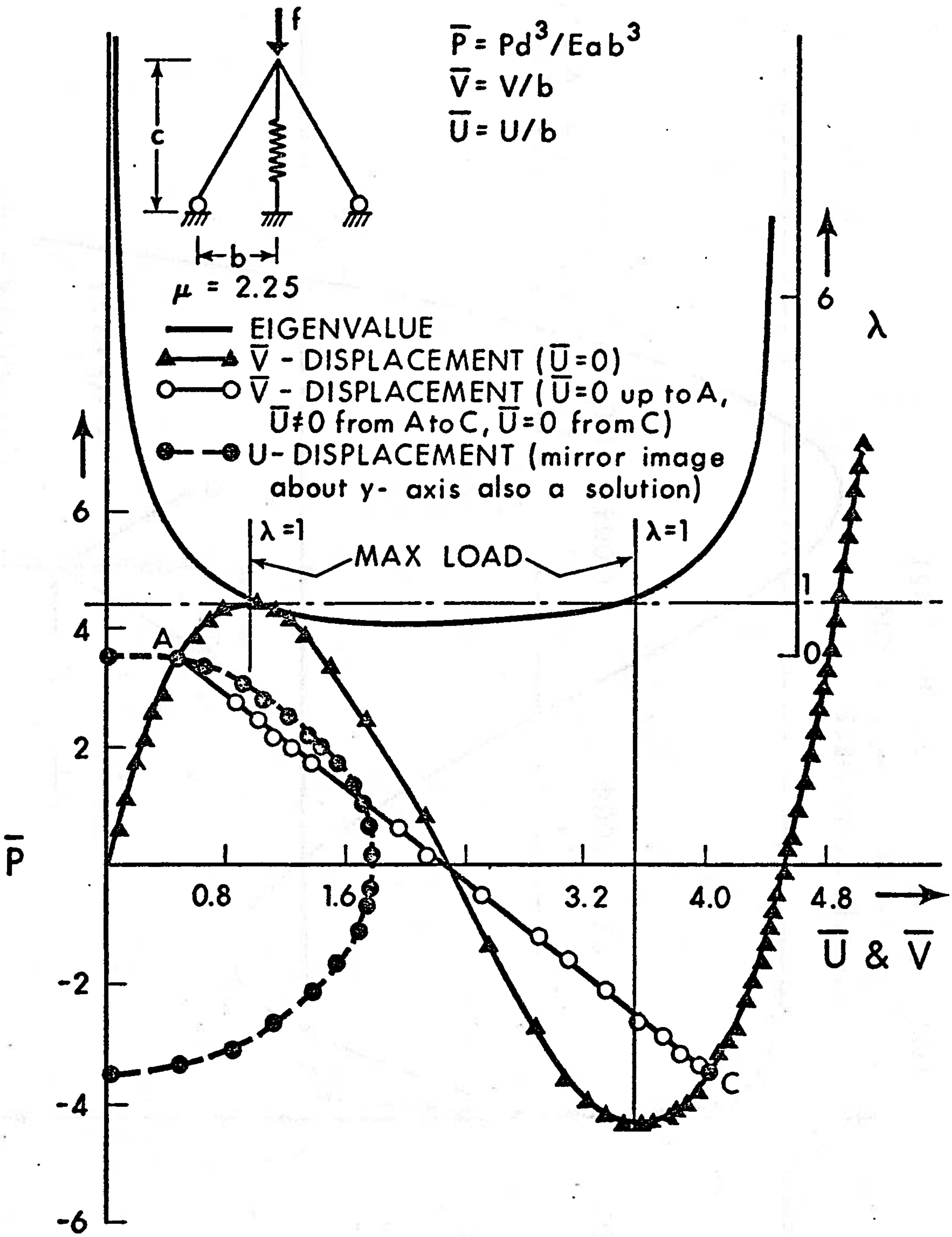
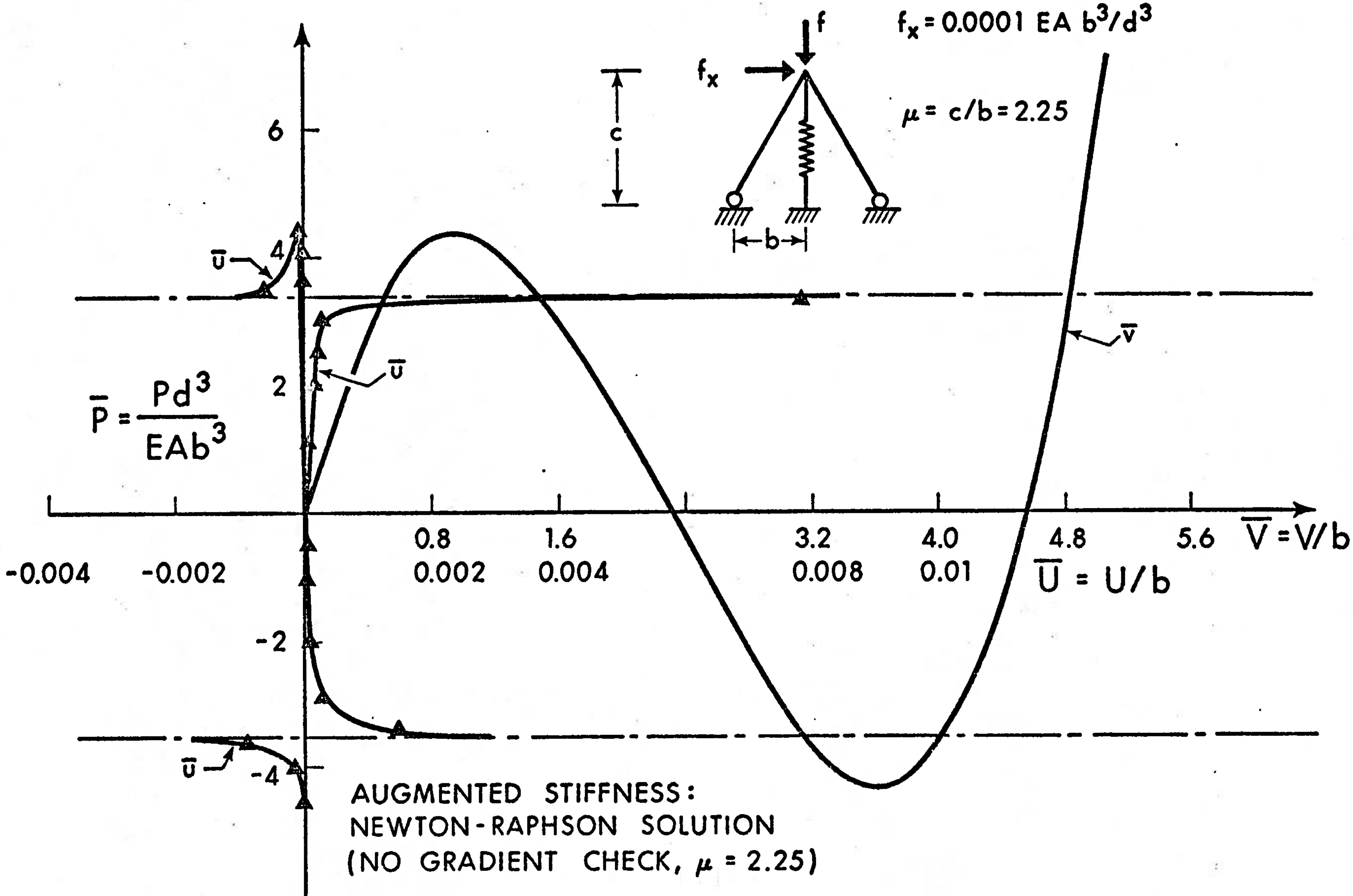


FIG. 11 - SOLUTIONS FOR TWO BAR TRUSS (μ = 2.25)



AUGMENTED STIFFNESS:
NEWTON-RAPHSON SOLUTION
(NO GRADIENT CHECK, μ = 2.25)

FIG. 12

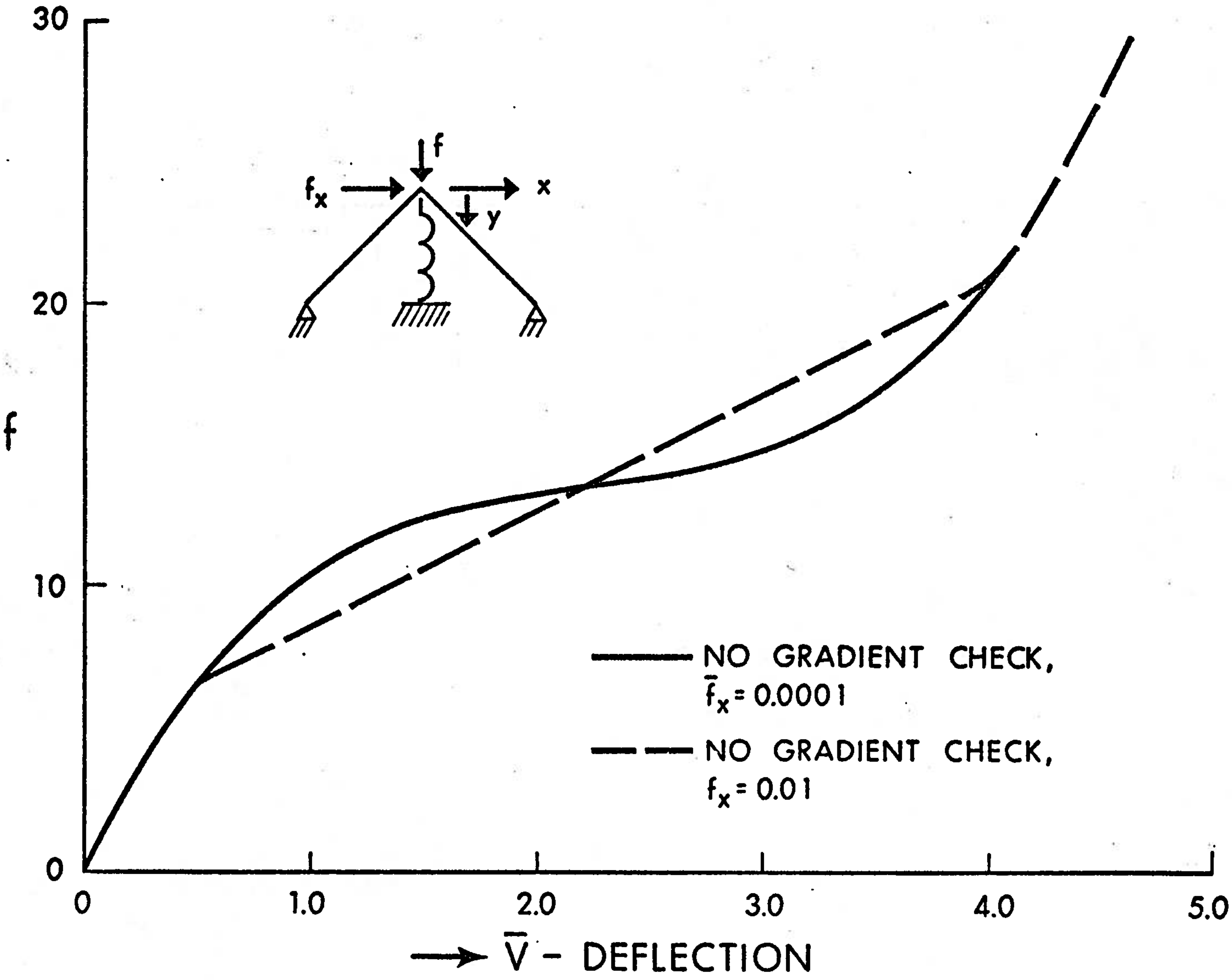


FIG. 13a - AUGMENTED STIFFNESS $\overline{V}$ DISPLACEMENT WITH DISTRUBING FORCES ($\mu = 2.25$)

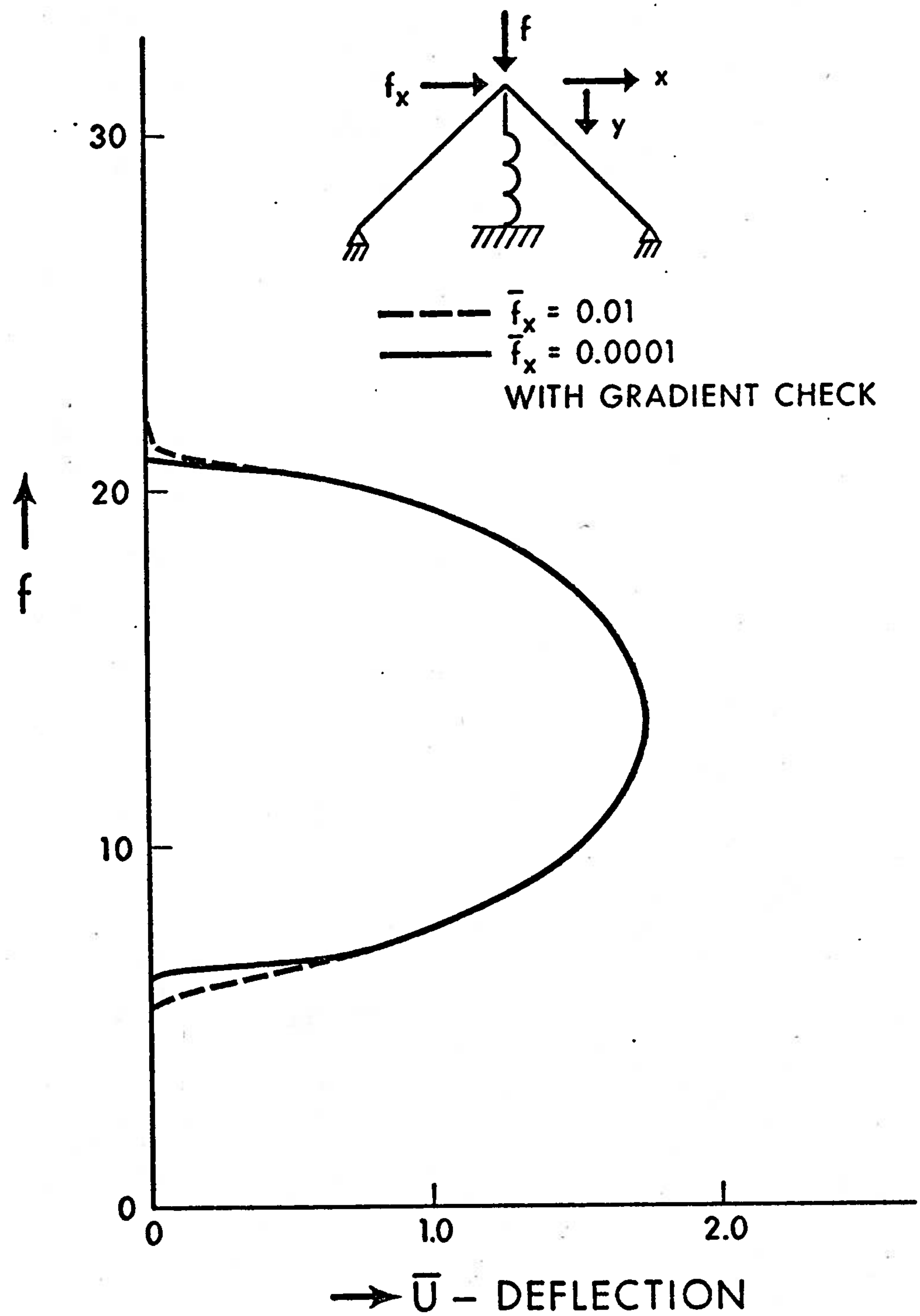


FIG. 13b - AUGMENTED STIFFNESS DISPLACEMENT WITH DISTURBING FORCES ($\mu = 2.25$)

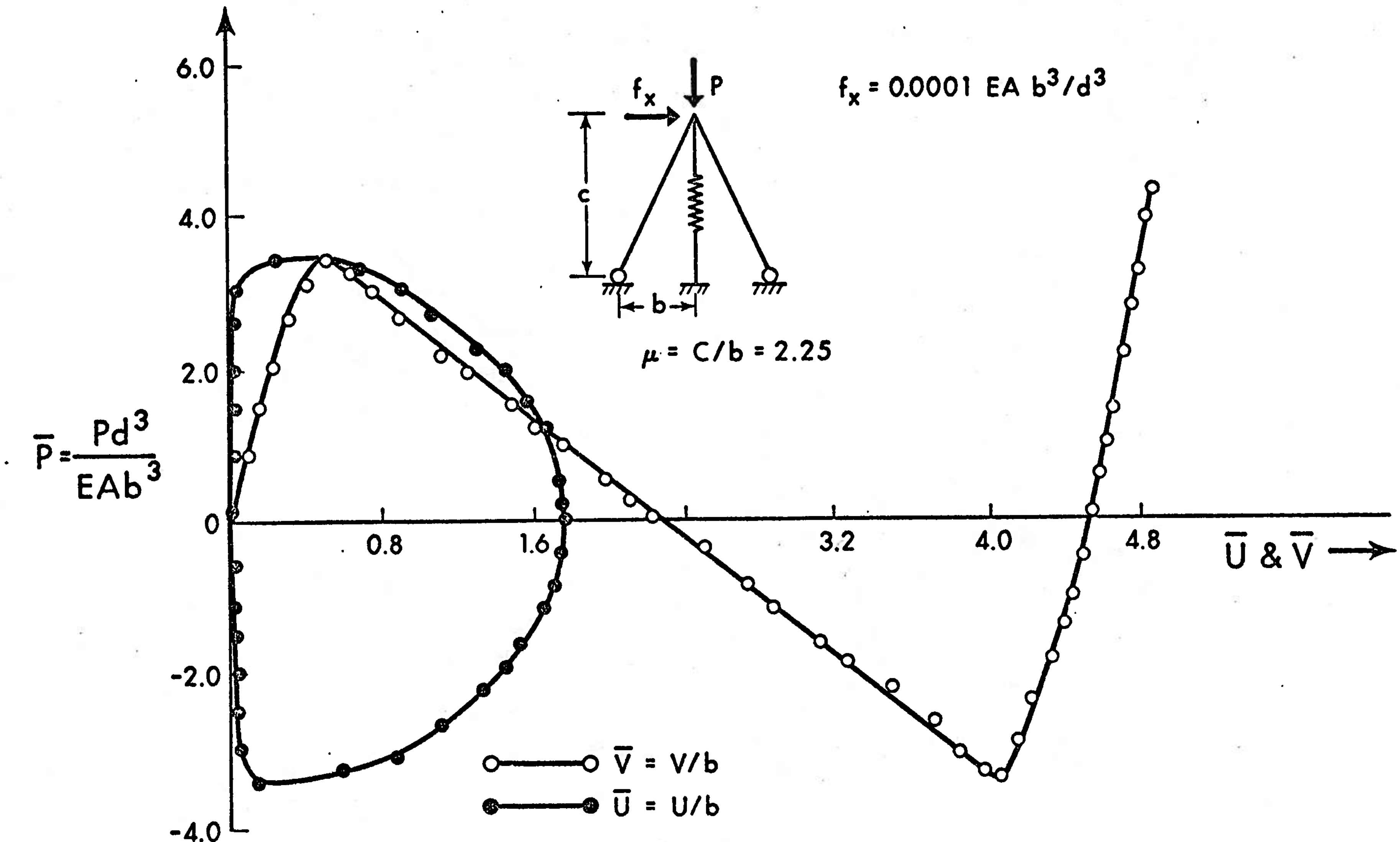


FIG. 14 - AUGMENTED STIFFNESS SOLUTION WITH DISTURBING FORCE AND GRADIENT CHECK OF AUGMENTED STRUCTURE ($\mu = 2.25$)

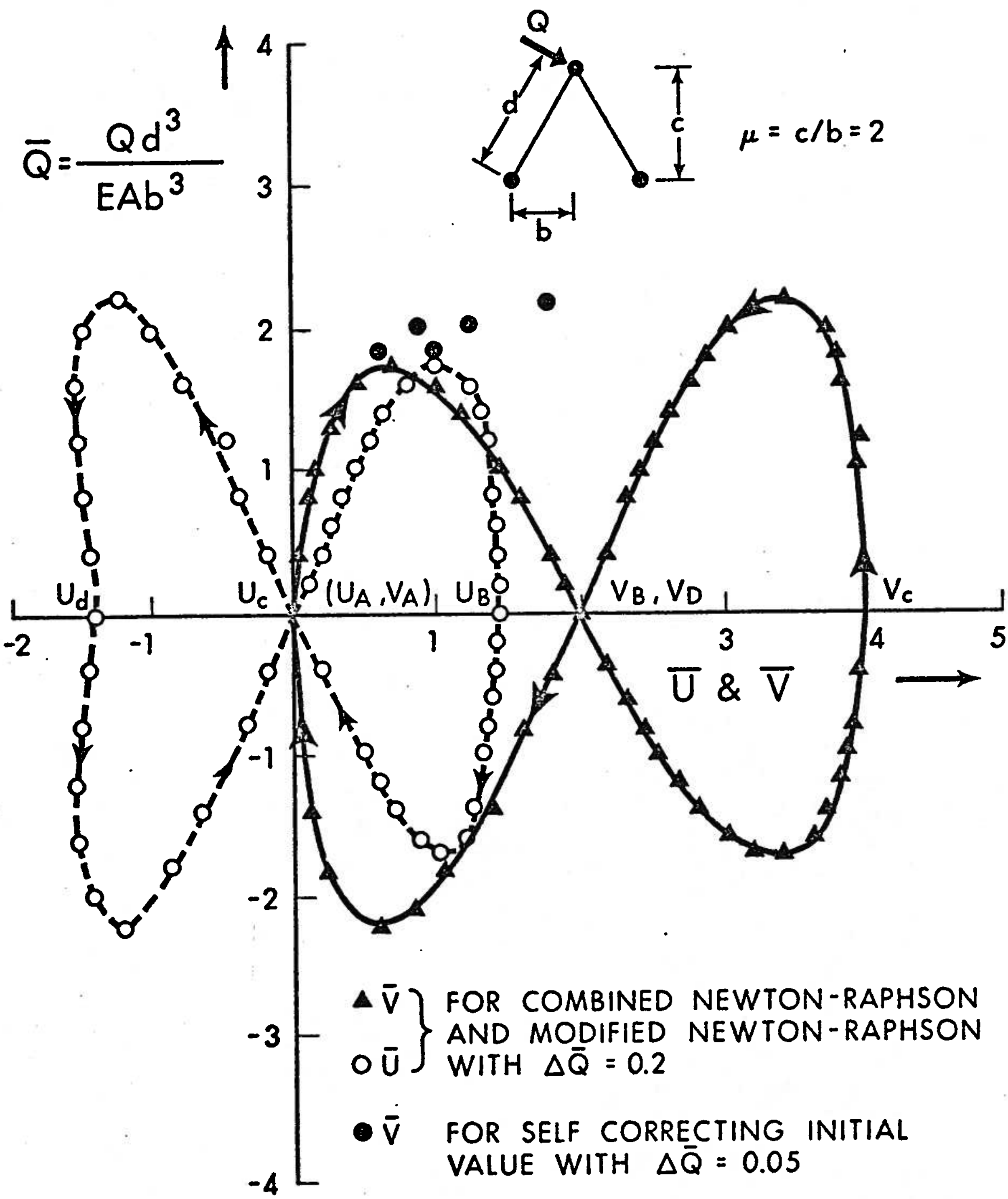


FIG. 15 - SOLUTIONS FOR TWO BAR TRUSS (μ=2.0) WITH NON-CONSERVATIVE LOADING

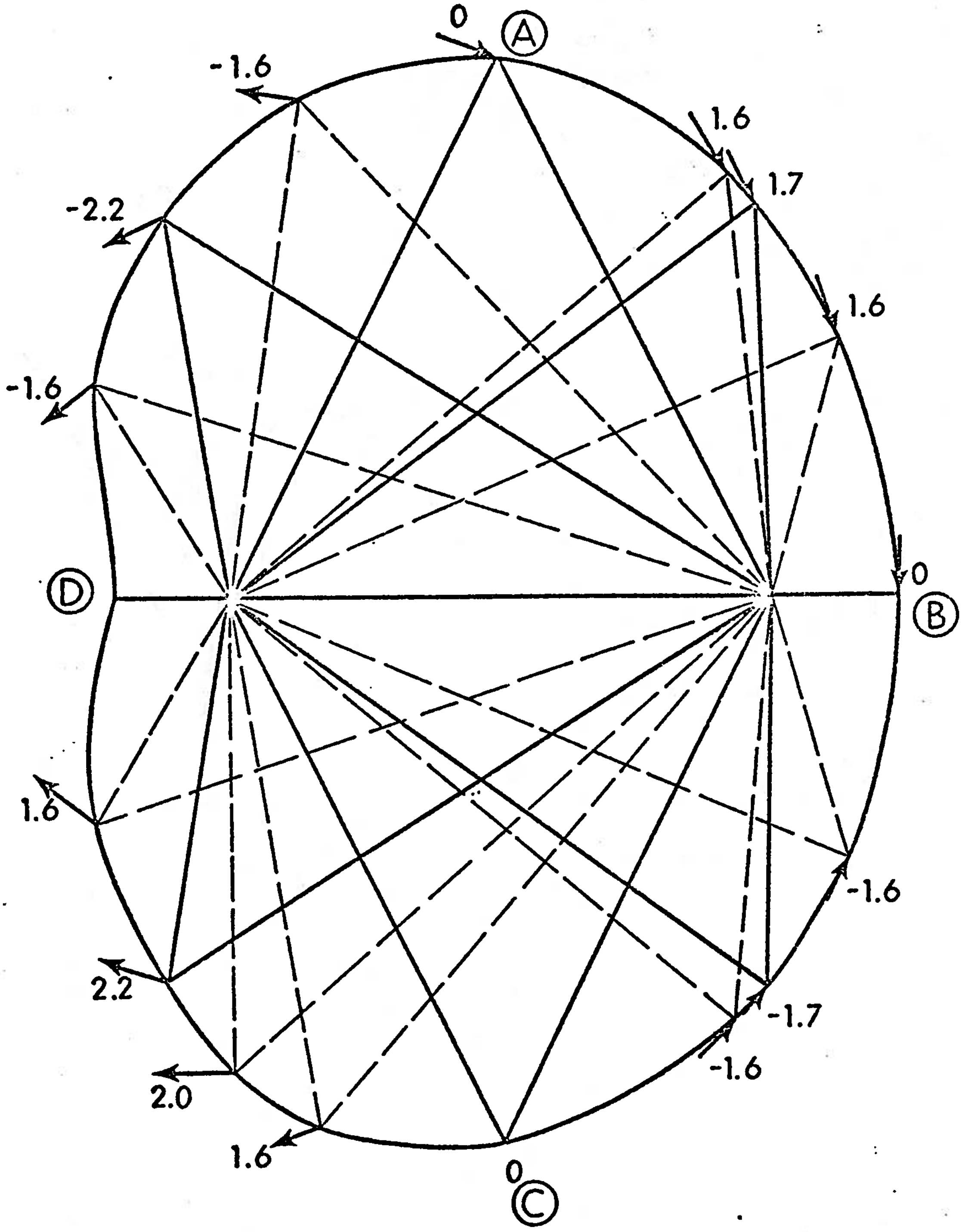


FIG. 16 - DISPLACED CONFIGURATIONS OF TWO BAR TRUSS (μ = 2.0) WITH NONCONSERVATIVE LOADING

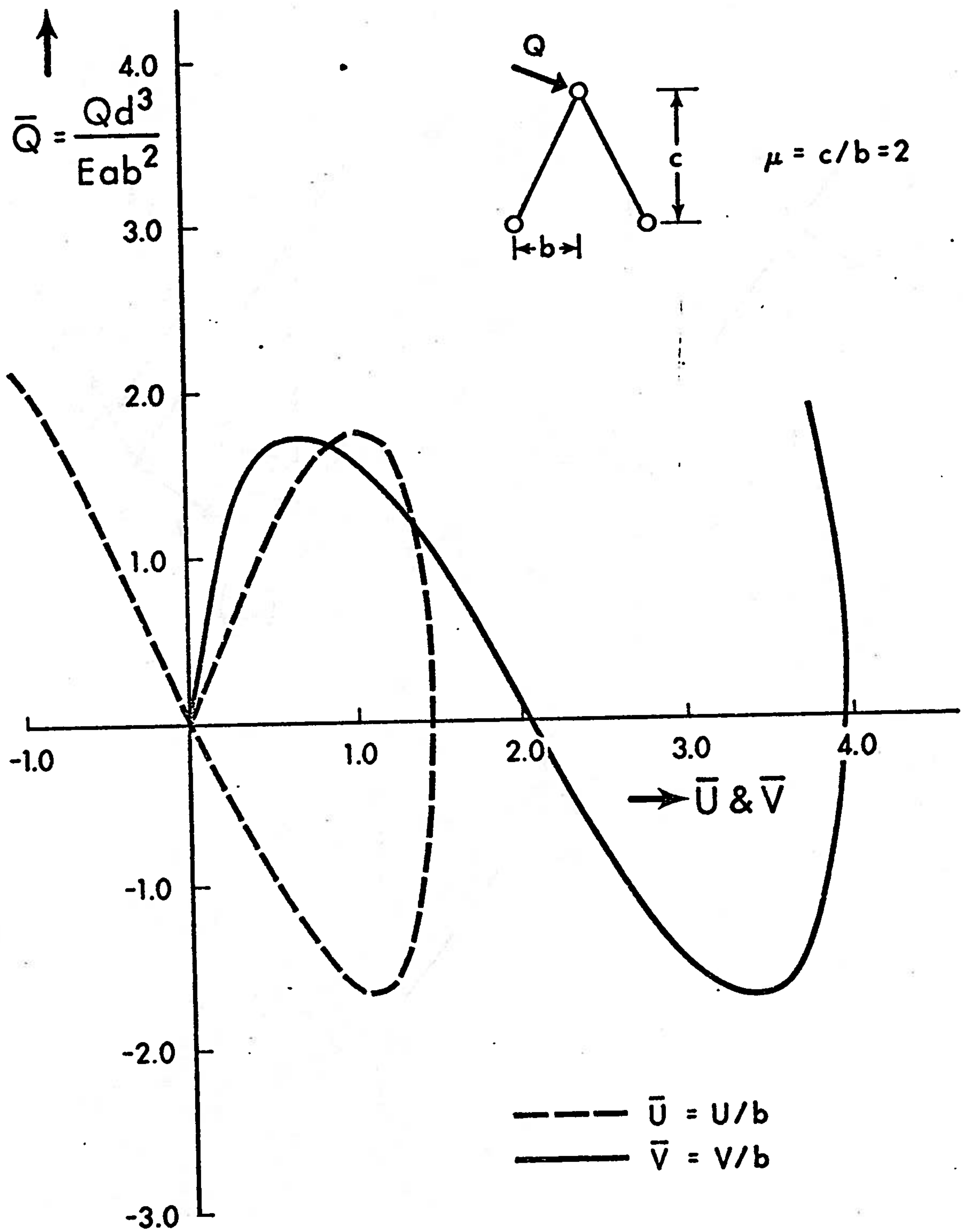


FIG. 17 - AUGMENTED STIFFNESS SOLUTION FOR TWO BAR TRUSS (μ=2.0) WITH NONCONSERVATIVE LOADING

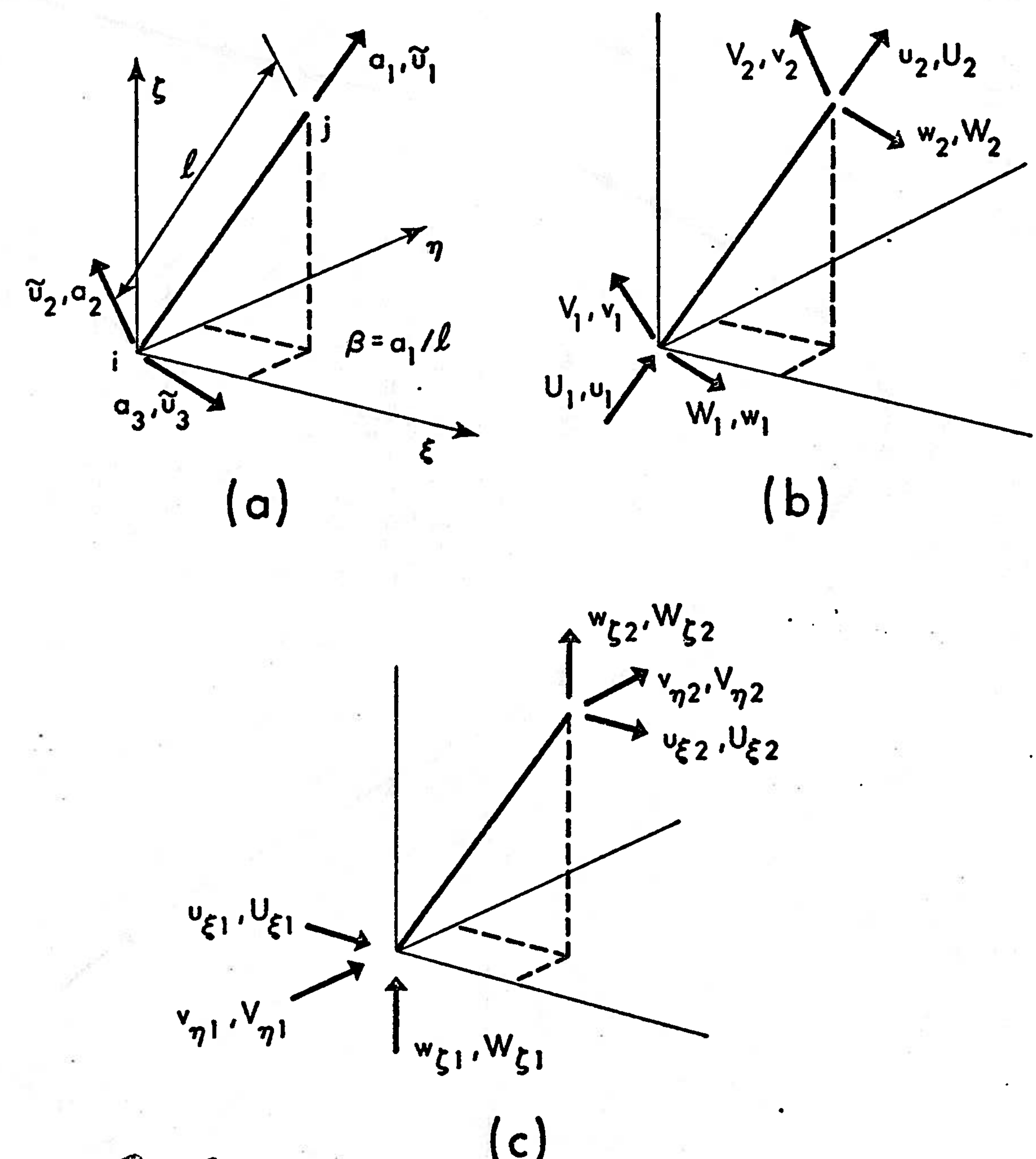


FIG. 18 - COORDINATES AND NODAL DISPLACEMENTS FOR SPACE TRUSS MEMBER

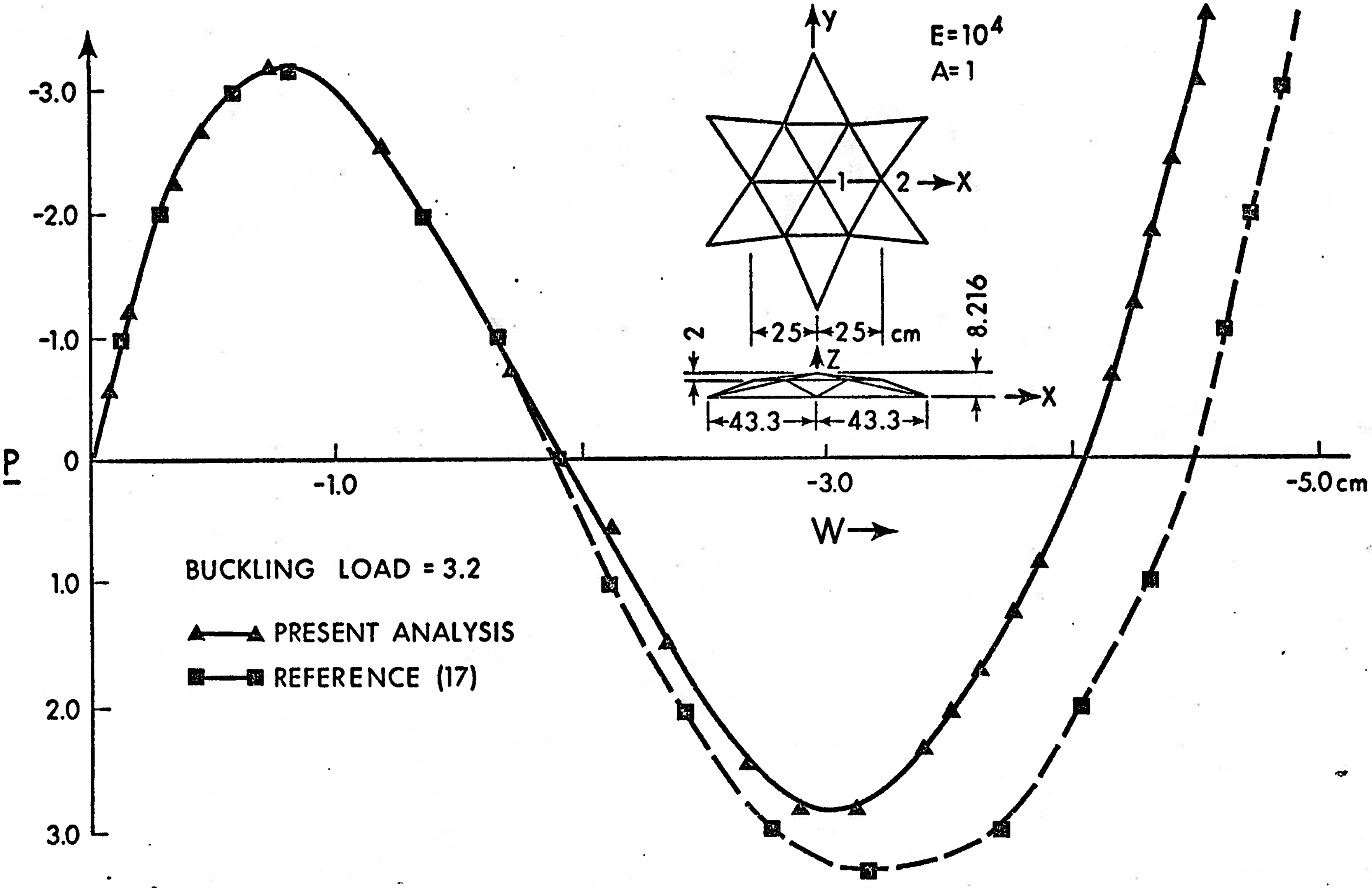


FIG. 19 - VERTICAL DISPLACEMENT OF JOINT 1 OF RETICULATED SHELL

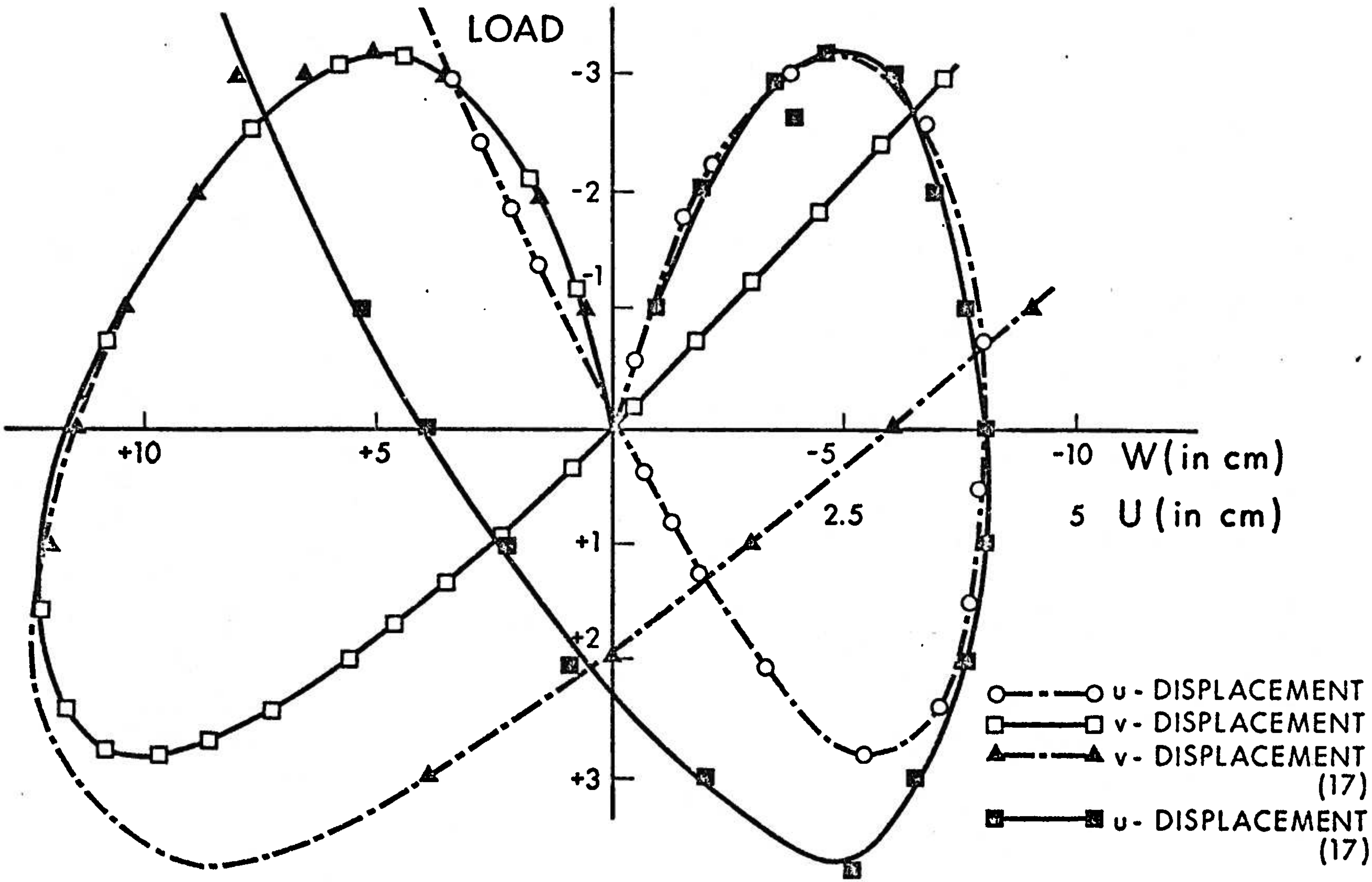


FIG. 20 - DISPLACEMENTS OF JOINT 2 OF RETICULATED SHELL