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Renorming and Smoothness in Banach Spaces

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ABSTRACT

The following results concerning smoothness and the geometry of Banach spaces are established in this thesis.

A Banach space X admits C^1-smooth partitions of unity if X admits a locally uniformly rotund norm whose dual norm is also locally uniformly rotund.

There is a norm on a separable Hilbert space that cannot be approximated uniformly on bounded sets by functions whose second derivatives are uniformly continuous. On the other hand, on any separable space admitting a norm with modulus of smoothness of power type 2, every norm is a limit of uniformly rotund norms which are twice Gâteaux differentiable and whose first derivatives are Lipschitz.

If a Banach space admits a continuous bump function with pointwise directional Hölder derivative, then it is an Asplund space. A consequence of this is that X is isomorphic to a Hilbert space whenever X and X^* admit continuous twice Gâteaux differentiable bump functions.

Let X be a separable Banach space and L be a subspace of X with a countable algebraic basis. Then there is a locally uniformly rotund norm on X that is Fréchet differentiable at each point of L.

A characterization of Banach spaces which admit Markushevich bases is given. Some new results concerning the extension of Markushevich bases are obtained.

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NOTATION

B_r	$\equiv$	$\{x : \|x\| \leq r\}$
$B_r(x_0)$	$\equiv$	$\{x : \|x - x_0\| \leq r\}$
B_X	$\equiv$	$\{x : \|x\| \leq 1\}$
$\lvert S\rvert$	$\equiv$	the cardinality of the set S
$C_u^k(X)$	$\equiv$	real functions on X with uniformly continuous k-th derivatives
$\mathrm{conv}(S)$	$\equiv$	the convex hull of S
$\overline{\mathrm{conv}}(S)$	$\equiv$	the norm closed convex hull of S
$\overline{\mathrm{conv}}^{w^*}(S)$	$\equiv$	the w^*-closed convex hull of S
$\mathrm{dens}(X)$	$\equiv$	the smallest cardinality of a dense subset of X
$f \,\square\, g(x)$	$\equiv$	$\inf\{f(y) + g(x-y) : y \in X\}$
$K^{(\nu)}$	$\equiv$	the ν-th derived set of K
$l_\infty^c(\Gamma)$	$\equiv$	the countably supported elements of $l_\infty(\Gamma)$
LUR	$\equiv$	locally uniformly rotund
SC	$\equiv$	strictly convex
$\mathrm{span}(S)$	$\equiv$	the linear span of S
$\overline{\mathrm{span}}(S)$	$\equiv$	the norm closed linear span of S
$\overline{\mathrm{span}}^{w^*}(S)$	$\equiv$	the w^*-closed linear span of S
S_X	$\equiv$	$\{x : \|x\| = 1\}$
$\partial f(x_0)$	$\equiv$	$\{\Lambda \in X^* : \Lambda(x) - \Lambda(x_0) \leq f(x) - f(x_0) \text{ for all } x \in X\}$
$\partial_\epsilon f(x_0)$	$\equiv$	$\{\Lambda \in X^* : \Lambda(x) - \Lambda(x_0) \leq f(x) - f(x_0) + \epsilon \text{ for all } x \in X\}$
UR	$\equiv$	uniformly rotund
WCD	$\equiv$	weakly countably determined
WCG	$\equiv$	weakly compactly generated
WLD	$\equiv$	weakly Lindelöf determined

INTRODUCTION

This chapter will give a brief overview of the contents of this thesis. However, we have chosen not to include the definitions here but instead we have placed them in the chapters where they are used. It should be noted that all linear spaces considered in this thesis are over the real scalar field.

The question of smooth approximation in Banach spaces has been studied for some time, yet the fundamental question as to whether a Banach space with a C^k-smooth bump function admits C^k-smooth partitions of unity has not been answered. There have been several partial results, the most general to date seems to be: suppose X or X^* is weakly compactly generated and X admits a C^k-smooth bump function, then X admits C^k-smooth partitions of unity ([GTWZ$_1$], [Mc]). In Chapter One, we will improve this theorem in the case $k = 1$ but we do not study any higher order cases. In particular, in Theorem 1.2.1 it is shown that a Banach space X admits C^1-smooth partitions of unity whenever X admits a locally uniformly rotund norm whose dual norm is also locally uniformly rotund. Specific classes of spaces where this theorem applies are given in Section 1.3. The interesting feature of our methods is that C^1-smooth approximations are obtained without the use of Toruńczyk's fundamental characterization result in [To].

In Chapter Two we look at the question concerning the approximation of convex functions by smooth convex functions. Much less is known in this area than in the area of smooth partitions of unity. Indeed, it appears to be unknown whether every norm on a Hilbert space is a limit of C^k-smooth norms for $k \geq 2$. More generally, it is not known whether every norm on a Banach space is a limit of C^k-smooth norms provided the space itself admits a C^k-smooth norm. The analogous question for norms with uniformly continuous k-th derivatives fails in a very strong sense: namely, Section 2.2 uses the methods of Chapter 1 in conjunction with a result of

Nemirovskii and Semenov ([NS]) to show the existence of a norm on a separable Hilbert space that cannot be approximated uniformly on bounded sets by functions whose second derivatives are uniformly continuous. In contrast to this, we show in Section 2.3 that every norm can be approximated by twice Gâteaux differentiable norms with moduli of smoothness of power type 2 in a separable space that admits a norm with modulus of smoothness of power type 2. Similar results are also shown for convex functions. The last section of the second chapter contains a constuction of locally uniformly rotund norms that are limits of C^k-smooth norms in separable Banach spaces admitting C^k-smooth norms. A version of this is also shown for norms with uniformly continuous k-th derivatives. As already mentioned, in this case not every norm can be approximated by norms with uniformly continuous k-th derivatives so it seems to be unknown whether we can replace local uniform rotundity with uniform rotundity in the above result.

Structural properties possessed by Banach spaces that admit continuous bump functions with higher order directional smoothness are examined in the third chapter. Our starting point is [BN, Proposition 2.2] which shows that a point of twice Gâteaux differentiability of a continuous convex function is a point of Lipschitz smoothness. While there are (nonconvex) continuous functions on $\mathbb{R}^2$ that are twice Gâteaux differentiable yet not Fréchet differentiable, the proposition from [BN] still gives a good indication of the strength of higher order directional differentiability. In fact, with the help of techniques from [BN] and [DGZ_2], it is shown in Section 3.2 that any space with a continuous twice Gâteaux differentiable bump function is an Asplund space. Some applications of this theorem are given in Section 3.3. In particular, spaces isomorphic to Hilbert spaces are characterized as those spaces X for which X and X^* admit continuous twice Gâteaux differentiable bump functions.

The question of the existence of norms that are Fréchet differentiable on a normed linear space with a countable algebraic basis is studied in the fourth chapter.

Given an arbitrary countable set not containing the origin, we show that there are several norms that are Fréchet differentiable at each of those points. It turns out that since the set is countable, the Baire category theorem is nicely suited for proving this fact. However, given that the linear span of such a set is uncountable, a new method for constructing a norm that would be Fréchet differentiable on the span is needed. In Section 4.3, using results concerning spaces with Schauder basis ([JRZ]) and Kadets-Klee renorming techniques, it is shown that there is a norm that is Fréchet differentiable on a normed linear space with a countable algebraic basis. This is in contrast to the fact that the closure of this space might not admit a Fréchet differentiable norm (e.g. $l_1(\mathbb{N})$ admits no Fréchet differentiable norm). Some connections between this result and monotone Schauder bases are discussed.

In the final chapter we abandon our study of smoothness and focus on the relationship between some topological properties in Banach spaces and Markushevich bases. The existence of a Markushevich basis on a Banach space provides a useful tool for studying the structure of the space. For instance, a Markushevich basis can be used to construct an explicit linear injection into $c_0(\Gamma)$. The main goal of Chapter Five is to provide a characterization of spaces that admit Markushevich bases in terms of certain types of injections of the dual space into $l_\infty(\Gamma)$. The techniques we use are based on a paper of Plichko ([Pl_1]) and they enable us to prove that Markushevich bases can be extended in some very general situations.

Chapter One

ROTUNDITY AND SMOOTH PARTITIONS OF UNITY

1.1 Introduction

The main objective of this chapter is to prove that a Banach space X admits C^1-smooth partitions of unity when X admits a locally uniformly rotund (LUR) norm whose dual is also LUR. The study of smooth partitions of unity in Banach spaces is of interest as it provides a tool for approximating continuous functions by smooth functions (see [To], [BF]). In addition to this, smooth partitions of unity provide coordinatewise smooth homeomorphisms of a given Banach space into some $c_0(\Gamma)$ by the theorem in [To].

In 1966, Bonic and Frampton ([BF]) proved that a separable Banach space X admits C^k-smooth partitions of unity if and only if there is a C^k-smooth bump function on X, that is a function with bounded and nonempty support However, at that time it was unknown if a nonseparable Hilbert space admits C^1-smooth partitions of unity. This was answered affirmatively by Wells ([W_1], [W_2]) a few years later. Shortly thereafter in [To], Toruńczyk proved that a Banach space X admits C^k-smooth partitions of unity if and only if there is a coordinatewise C^k-smooth homeomorphism of X into $c_0(\Gamma)$. As an easy consequence of this fundamental theorem, Toruńczyk showed that a nonseparable Hilbert space admits C^∞-smooth partitions of unity.

Many of the subsequent results concerning smooth partitions of unity in Banach spaces have relied on the characterization of [To]. Usually a coordinatewise smooth embedding of the space into $c_0(\Gamma)$ is constructed with the aid of a linear injection of the given space into $c_0(\Gamma)$. Typical examples of this are the papers [$GTWZ_1$] and [Mc] which together show that X admits C^k-smooth partitions of unity whenever X has a C^k-smooth bump function and X or X^* is WCG. However, in Section 1.2,

C^1-smooth partitions of unity on X are obtained by the purely geometric condition that X admits an equivalent LUR norm whose dual is also LUR. Such a space need not linearly inject into any $c_0(\Gamma)$; notwithstanding, from [To] and our result stated above, there is a coordinatewise smooth embedding of such a space into some $c_0(\Gamma)$.

The third section of this chapter will give examples of spaces which satisfy the hypothesis of the theorem mentioned above. In particular, a new result on the existence of smooth partitions of unity on certain $C(K)$ spaces will be obtained. In recent years a lot of attention has been given to $C(K)$ spaces where K is a scattered compact set, that is, there exists an ordinal η such that $K^{(\eta)}$ (the η-th derived set of K) is empty. In [CP], Ciesielski and Pol constructed a $C(K)$ space such that $K^{(3)} = \emptyset$ while $C(K)$ does not linearly inject into any $c_0(\Gamma)$. Only recently it was shown in [DGZ$_1$] that $C(K)$ admits C^∞-smooth partitions of unity whenever $K^{(\omega_0)} = \emptyset$. In fact, until [DGZ$_1$], it had been unknown whether the $C(K)$ space of [CP] admits C^1-smooth partitions of unity. Using renormings of Deville ([De]) and Haydon and Rogers ([HR]), it follows from the result in Section 1.2 that $C(K)$ admits C^1-smooth partitions of unity whenever $K^{(\omega_1)} = \emptyset$. In contrast to this we should mention that in [H$_1$], Haydon constructed a $C(K)$ space such that $K^{(\omega_1)}$ is a singleton which nevertheless admits no equivalent Gâteaux smooth nor strictly convex norm. It is presently unknown whether this fundamentally important space admits C^1-smooth partitions of unity, although Haydon has recently shown ([H$_2$]) that there is a C^1-smooth bump function on it.

The notation and terminology used here should be quite standard.

1.1.1 DEFINITION. A norm is *locally uniformly rotund* (LUR) if $\|x_n - x\| \to 0$ whenever $\|x_n\| \to \|x\| = 1$ and $\|x_n + x\| \to 2$. A norm is *strictly convex* (SC) if $x = y$ whenever $\|x + y\| = \|x\| + \|y\|$. A *uniformly rotund* (UR) norm is a norm such that $\|x_n - y_n\| \to 0$ whenever $\|x_n\| \leq 1$, $\|y_n\| \leq 1$ and $\|x_n + y_n\| \to 2$.

As usual, C^k-smoothness is always meant in the continuous Fréchet sense. We call a function C^k*-smooth* if it is C^k-smooth on the whole space. Since norms are never differentiable at the origin, we will say a norm on X is C^k*-smooth* if it is C^k-smooth on $X\setminus\{0\}$; similarly for Gâteaux smoothness.

1.1.2 DEFINITION. A space is said to *admit* C^k*-smooth partitions of unity* (respectively *Gâteaux smooth partitions of unity*) if given any open cover there is a locally finite partition of unity consisting of C^k-smooth functions (respectively continuous Gâteaux differentiable functions) subordinate to this cover.

The notation $\partial f(x_0) = \{\Lambda \in X^* : \Lambda(x) - \Lambda(x_0) \le f(x) - f(x_0) \text{ for all } x \in X\}$, and for $\epsilon > 0$, $\partial_\epsilon f(x_0) = \{\Lambda \in X^* : \Lambda(x) - \Lambda(x_0) \le f(x) - f(x_0) + \epsilon \text{ for all } x \in X\}$ will be used. We also use the symbols $S_X = \{x : \|x\| = 1\}$, $B_r(x_0) = \{x : \|x - x_0\| \le r\}$, $B_r = B_r(0)$ and $B_X = B_1$.

For basic information concerning rotundity and smoothness one can consult the lecture notes [Di], [DGZ_3] and [Ph].

1.2. Rotundity and Smooth partitions of Unity

The goal of this section is to prove the following theorem.

1.2.1 THEOREM. *(a) If X has an LUR norm whose dual is also LUR, then X admits C^1-smooth partitions of unity.*

(b) If X has an LUR norm whose dual is SC, then X admits Gâteaux smooth partitions of unity.

Recall that from Asplund's averaging technique (see e.g. [FZZ, p. 351]), it follows that if X has an LUR norm and X^* has a dual LUR (respectively SC) norm, then X has an LUR norm whose dual norm is LUR (respectively SC). The proof of Theorem 2.1 will be separated into in several steps. The following lemma is a natural generalization of a result of Šmulyan ([Sm]). We include its proof for the reader's convenience.

1.2.2 LEMMA. *Let f be a continuous convex function on X. Then the following are equivalent:*

(a) f is Fréchet differentiable at x_0;

(b) $\|\Lambda_n - \Lambda\|^* \to 0$ whenever $\Lambda \in \partial f(x_0)$ and $\Lambda_n \in \partial_{\epsilon_n} f(x_0)$ where $\epsilon_n \downarrow 0$.

Proof. (a) $\Rightarrow$ (b): Suppose that (b) does not hold. Then there exists $\epsilon_n \downarrow 0$, $\Lambda_n \in \partial_{\epsilon_n} f(x_0)$, $\Lambda \in \partial f(x_0)$ and $\eta > 0$ such that

$$\|\Lambda_n - \Lambda\|^* \geq \eta \qquad \text{for all } n.$$

Let $t_n = \frac{4\epsilon_n}{\eta}$ and choose $h_n \in S_X$ so that

$$(\Lambda_n - \Lambda)(h_n) \geq \frac{\eta}{2}.$$

Thus,

$$\begin{aligned} f(x_0 + t_n h_n) - f(x_0) - \Lambda(t_n h_n) &\geq \Lambda_n(x_0 + t_n h_n) - \Lambda_n(x_0) - \epsilon_n - \Lambda(t_n h_n) \\ &= \Lambda_n(t_n h_n) - \Lambda(t_n h_n) - \epsilon_n \\ &\geq t_n \frac{\eta}{2} - \epsilon_n = t_n \frac{\eta}{4}. \end{aligned}$$

Therefore f is not Fréchet differentiable at x_0.

(b) $\Rightarrow$ (a): Suppose f is not Fréchet differentiable at x_0. Then there exists $t_n \downarrow 0$, $x_n \in S_X$ and $\epsilon > 0$ such that

$$f(x_0 + t_n x_n) - f(x_0) - \Lambda(t_n x_n) \geq \epsilon t_n \qquad \text{where } \Lambda \in \partial f(x_0).$$

Let $\Lambda_n \in \partial f(x_0 + t_n x_n)$. Since f is locally Lipschitzian (see e.g. [Ph, Proposition 1.6]), there exists $\delta > 0$ and $M > 0$ such that $|f(x) - f(y)| \leq M\|x - y\|$ whenever $x, y \in B_\delta(x_0)$. We may assume that $t_n < \delta$ for all n. Thus $\|\Lambda_n\|^* \leq M$ for all n; moreover,

$$\begin{aligned} \Lambda_n(y) - \Lambda_n(x_0) &= \Lambda_n(y) - \Lambda_n(x_0 + t_n x_n) + \Lambda_n(t_n x_n) \\ &\leq f(y) - f(x_0 + t_n x_n) + M t_n \\ &\leq f(y) - f(x_0) + 2M t_n. \end{aligned}$$

Whence, for all n, $\Lambda_n \in \partial_{\epsilon_n} f(x_0)$, where $\epsilon_n = 2Mt_n \downarrow 0$. However,

$$\Lambda_n(t_n x_n) \geq f(x_0 + t_n x_n) - f(x_0) \geq \Lambda(t_n x_n) + \epsilon t_n.$$

Therefore $\|\Lambda - \Lambda_n\|^* \geq \epsilon$ for all n. Thus (b) fails. □

Because of the importance of Šmulyan's characterization of differentiability of a norm ([Sm]), we record it here as

1.2.3 COROLLARY ([Sm]). *The norm $\|\cdot\|$ on X is Fréchet differentiable at $x_0 \neq 0$ if and only if $\|\Lambda - \Lambda_n\|^* \to 0$ whenever $\|\Lambda_n\|^* \leq \|\Lambda\|^* = 1$ and $\Lambda_n(x_0) \to \Lambda(x_0) = \|x_0\|$.*

Proof. This follows immediately from Lemma 1.2.2, because $\Lambda_n \in \partial_{\epsilon_n}\|\cdot\|(x_0)$ if and only if $\Lambda_n(x_0) \geq \|x_0\| - \epsilon_n$ and $\|\Lambda_n\|^* \leq 1$; and $\Lambda \in \partial\|\cdot\|(x_0)$ if and only if $\Lambda(x_0) = \|x_0\|$ and $\|\Lambda\|^* = 1$. □

An easy but useful application of Lemma 1.2.2 is contained in

1.2.4 LEMMA. *Let C be a closed convex subset of X, and $\rho(x) = \rho(x, C) = \inf\{\|x - y\| : y \in C\}$.*

(a) If $\|\cdot\|^*$ is LUR, then $\rho(x)$ is Fréchet differentiable at each $x \notin C$.

(b) If $\|\cdot\|^*$ is SC, then $\rho(x)$ is Gâteaux differentiable at each $x \notin C$.

Proof. (a) Let $x_0 \notin C$ and $\Lambda_n \in \partial_{\epsilon_n}\rho(x_0)$ for each n, where $\epsilon_n \downarrow 0$. Since $|\rho(x) - \rho(y)| \leq \|x - y\|$, it follows that $\|\Lambda_n\|^* \leq 1$. Thus $\limsup_n \|\Lambda_n\|^* \leq 1$. On the other hand, choosing $x_n \in C$ such that $\|x_0 - x_n\| \to \rho(x_0) > 0$, one has

$$\frac{\Lambda_n(x_0 - x_n)}{\|x_0 - x_n\|} \geq \frac{\rho(x_0) - \rho(x_n) - \epsilon_n}{\|x_0 - x_n\|} \longrightarrow 1.$$

Therefore, $\liminf_n \|\Lambda_n\|^* \geq 1$. In particular, $\|\Lambda\|^* = 1$ whenever $\Lambda \in \partial\rho(x_0)$. Also, $\frac{\Lambda + \Lambda_n}{2} \in \partial_{\epsilon_n}\rho(x_0)$, for Λ, Λ_n and ϵ_n as chosen above. Now $\|\Lambda_n\|^* \to \|\Lambda\|^* = 1$, $\|\frac{\Lambda + \Lambda_n}{2}\|^* \to 1$ and thus $\|\Lambda - \Lambda_n\|^* \to 0$, because $\|\cdot\|^*$ is LUR. By Lemma 1.2.2, ρ is Fréchet differentiable at x_0.

(b) Let $x_0 \notin C$ and $\Lambda_1, \Lambda_2 \in \partial\rho(x_0)$. From the proof of (a) it follows that $\|\Lambda_1\|^* = \|\Lambda_2\|^* = \|\frac{\Lambda_1+\Lambda_2}{2}\|^* = 1$. Since $\|\cdot\|^*$ is SC, $\Lambda_1 = \Lambda_2$. Therefore ρ is Gâteaux differentiable at x_0; see e.g. [Ph, Proposition 1.8]. □

1.2.5 REMARK. (a) Observe that $\rho(x, C)$ need not even be Gâteaux differentiable if the norm on X is Fréchet differentiable but the dual norm is not LUR. This follows, for example, because $X = C[0, \omega_1]$ admits a Fréchet differentiable norm while X^* has no dual SC norm ([Ta, Theorems 3 and 4]). Thus given any norm on X, X^* has a two dimensional subspace H on which its dual norm is not SC. Let L be a closed subspace of X such that $(X/L)^* = H$. On finite dimensional subspaces, a norm is Gâteaux differentiable if and only if its dual norm is SC. Therefore, given any norm on X, there is a two dimensional quotient space X/L such that the quotient norm of X/L is not Gâteaux differentiable.

(b) For any nonempty set F, $\rho^2(x, F)$ is Fréchet differentiable at each point of F. In particular, $\rho^2(x, C)$ is C^1-smooth whenever C is a closed and convex set and $\|\cdot\|^*$ is LUR (because Fréchet differentiable convex functions are C^1-smooth; see e.g. [DGZ, Chapter I] or [Ph, p. 20]).

(c) Choosing $C = \{0\}$, we have the well-known fact that the norm on X is Fréchet differentiable if its dual norm is LUR.

The next lemma uses ideas from Theorem 3.2 of [BF] and Theorem 1.1 of [NS].

1.2.6 LEMMA. *Suppose X has an LUR norm $\|\cdot\|$ whose dual is also LUR. Let f be a bounded continuous function on B_X, then given $\epsilon > 0$ there exists a C^1-smooth function g such that $|g(x) - f(x)| < \epsilon$ for all $x \in S_X$.*

Proof. One may assume $-1 \leq f(x) \leq 1$ for $x \in B_X$. Choose N such that $\frac{7}{2N} < \epsilon$. Let

$$\Delta_i = \left[\frac{i-1-N}{N}, \frac{i-N}{N}\right] \quad \text{for } i = 1, \ldots, 2N.$$

Let $Q_0 = Q_{2N+1} = \emptyset$ and for $i = 1, \ldots, 2N$, define

$$Q_i = f^{-1}(\Delta_i) \quad \text{and} \quad \widetilde{Q}_i = \overline{\text{conv}}(B_X \backslash (Q_{i-1} \cup Q_i \cup Q_{i+1})).$$

For each i, it is assumed that $\widetilde{Q}_i \neq \emptyset$, since in case $\widetilde{Q}_i = \emptyset$, $g(x) = \frac{i-1}{N}$ would work. Now let $r_i(x) = \rho^2(x, \widetilde{Q}_i)$ for $i = 1, \ldots, 2N$. By Remark 1.2.5(b), r_i is a C^1-smooth function for each i. Let $x_0 \in S_X$ be fixed. Certainly $x_0 \in Q_{i_0}$ for some i_0. For some $\delta > 0$, one has $|f(x_0) - f(y)| < \frac{1}{N}$ for all $y \in B_X$ such that $\|x_0 - y\| < \delta$. It follows that $y \notin B_X \backslash (Q_{i_0-1} \cup Q_{i_0} \cup Q_{i_0+1})$ whenever $\|x_0 - y\| < \delta$. Since $\|\cdot\|$ is LUR, there exists $\Lambda \in S_{X^*}$ with $\Lambda(x_0) = 1$ such that $\|y - x_0\| < \delta$ whenever $y \in B_X$ and $\Lambda(y) \geq 1 - \alpha$ for some $\alpha > 0$. Hence $\overline{\text{conv}}(B_X \backslash (Q_{i_0-1} \cup Q_{i_0} \cup Q_{i_0+1})) \subset \{y : \Lambda(y) \leq 1 - \alpha\}$. Therefore $r_{i_0}(x_0) \geq \alpha^2 > 0$. Since x_0 was arbitrary, this shows that $\sum_{i=1}^{2N} r_i(x) > 0$ for each $x \in S_X$. Let $r(x) = (1 - \|x\|^2)^2$. Notice that $r(x) = 0$ for all $x \in S_X$, $r(x) > 0$ for $x \notin S_X$ and r is C^1-smooth; cf. Remark 1.2.5(b), (c). Therefore

$$h_i(x) = \frac{r(x) + r_i(x)}{r(x) + \sum_{i=1}^{2N} r_i(x)}$$

is C^1-smooth. Let $\alpha_i =$ midpoint of Δ_i and

$$g(x) = \sum_{i=1}^{2N} \alpha_i h_i(x).$$

Certainly g is C^1-smooth. Finally, for $x_0 \in S_X$, choose i_0 such that $x_0 \in Q_{i_0}$ and use the fact that $r_i(x_0) = 0$ for $i \notin \{i_0 - 1, i_0, i_0 + 1\}$ to estimate

$$\begin{aligned}
|g(x_0) - f(x_0)| &= \left| \frac{\sum_{i=1}^{2N} \alpha_i r_i(x_0)}{\sum_{i=1}^{2N} r_i(x_0)} - f(x_0) \frac{\sum_{i=1}^{2N} r_i(x_0)}{\sum_{i=1}^{2N} r_i(x_0)} \right| \\
&\leq \sum_{i=1}^{2N} |\alpha_i - f(x_0)| \frac{r_i(x_0)}{\sum_{i=1}^{2N} r_i(x_0)} \\
&\leq |\alpha_{i_0-1} - f(x_0)| + |\alpha_{i_0} - f(x_0)| + |\alpha_{i_0+1} - f(x_0)| \\
&\leq \frac{7}{2N} < \epsilon.
\end{aligned}$$

□

1.2.7 PROPOSITION. *Suppose X has an LUR norm $\|\cdot\|$ whose dual is also LUR, then every bounded continuous function can be approximated uniformly on bounded sets by C^1-smooth functions.*

Proof. Consider $Y = X \oplus \mathbb{R}$ with norm $\|(x,r)\|_Y = (\|x\|^2 + |r|^2)^{\frac{1}{2}}$. Certainly $\|\cdot\|_Y$ and its dual norm are LUR. Let $x_0 = (0,1) \in S_Y$. Now, $\Lambda \in S_{Y^*}$ defined by $\Lambda(x,r) = r$ is the supporting functional at x_0 and $H = \{(x,r) : \Lambda(x,r) = 1\} = \{(x,1) : x \in X\}$ is the supporting hyperplane. Let f be a bounded continuous function on H and C be a bounded subset of H. Thus for some $m > 0$, $C \subset \{(x,1) : \|x\| \le m\} \equiv F$. Define $p : Y\backslash\{0\} \to S_Y$ by $p(y) = \frac{y}{\|y\|_Y}$; let $p_1 = p\,|_H$. For $p_1^{-1}(y) \in F$, set $f_1(y) = f(p_1^{-1}(y))$. Extend f_1 to a bounded continuous function on Y and denote it again by f_1. By Lemma 1.2.6, choose a C^1-smooth function g_1 such that $|g_1(y) - f_1(y)| < \epsilon$ for an arbitrary fixed $\epsilon > 0$ and all $y \in S_Y$. For $y \in Y$ define $g(y) = g_1(p(y))$. By Remark 1.2.5(c), $\|\cdot\|$ is Fréchet differentiable, and thus g is C^1-smooth on $Y\backslash\{0\}$. In particular, g is C^1-smooth on H. For $h \in F$, one computes

$$|g(h) - f(h)| = |g_1(p(h)) - f(p_1^{-1}(p(h)))| = |g_1(p(h)) - f_1(p(h))| < \epsilon.$$

Since H is a translate of X, the proposition is proved. □

The following proposition is more general than what is required to complete the proof of Theorem 1.2.1, nevertheless we are including it here, because it seems to have some merit on its own.

1.2.8 PROPOSITION. *For a Banach space X the following are equivalent.*

(a) Every continuous function can be approximated uniformly by C^k-smooth functions.

(b) Every Lipschitz function can be approximated uniformly on bounded sets by C^k-smooth functions.

(c) X admits C^k-smooth partitions of unity.

Proof. We will only prove (b) $\Rightarrow$ (c), because (a) $\Rightarrow$ (b) is obvious and (c) $\Rightarrow$ (a) is well-known (see [BF], [To]).

Let S be the set of real-valued C^k-smooth functions on X and $\mathcal{U}_S$ be the family of sets $\{f^{-1}(0,\infty) : f \in S \text{ and } f : X \to [0,1]\}$. By Lemma 1 of [To] it suffices to show that $\mathcal{U}_S$ contains a σ-locally finite base for the norm topology on X. To accomplish this, we use a technique from [Su, Lemma 6]. Let O be a bounded open subset of X and let $F = X \backslash O$. Choose $r > 0$ such that $O \subset B_r$. Let $\phi_n : \mathbb{R} \to [0,1]$ be a C^k-smooth function such that

$$\phi_n(t) = \begin{cases} 0 & \text{if } t \leq \frac{1}{2n} \\ > 0 & \text{otherwise.} \end{cases}$$

By the hypothesis, one can choose a C^k-smooth function h_n such that $|h_n(x) - \rho(x,F)| < \frac{1}{2n}$ for $x \in B_{r+3}$. Now define

$$g_n(x) = \begin{cases} \phi_n(h_n(x)) & \text{if } x \in B_{r+3} \\ 0 & \text{otherwise.} \end{cases}$$

Clearly g_n is C^k-smooth on B_{r+2} as it is a composition of C^k-smooth functions there. Also, g_n is C^k-smooth on $X \backslash B_{r+1}$ since $g_n(x) = 0$ for all $x \in X \backslash B_r$. Therefore, g_n is C^k-smooth on X. Let $G_n = \{x : g_n(x) > 0\}$. Observe that $\{x : \rho(x,F) > \frac{1}{n}\} \subset G_n \subset O$ and thus $\bigcup_{n=1}^{\infty} G_n = O$. Because X is a metric space, X has a σ-locally finite base $\mathcal{V} = \bigcup_{n=1}^{\infty} \mathcal{V}_n$ where each $\mathcal{V}_n$ is locally finite and consists of bounded sets. By the above argument, for each $V \in \mathcal{V}$, choose a fixed sequence $\{G_{V,k}\}_k \subset \mathcal{U}_S$ such that $V = \bigcup_{k=1}^{\infty} G_{V,k}$. Let $\mathcal{G}_{n,k} = \{G_{V,k} : V \in \mathcal{V}_n\}$. Certainly each $\mathcal{G}_{n,k}$ is locally finite. Therefore $\mathcal{G} = \bigcup_{n,k} \mathcal{G}_{n,k} \subset \mathcal{U}_S$ is a σ-locally finite base for X. □

Proof of Theorem 1.2.1. (a) This follows immediately from Proposition 1.2.7 and Proposition 1.2.8.

(b) If, in Lemma 1.2.6, the dual norm is SC instead of LUR, then the function g constructed there is Gâteaux smooth by Lemma 1.2.4(b). Moreover, note that g is

Fréchet differentiable on finite dimensional subspaces of X (since distance functions and norms are convex). For p and g_1 in the proof of Proposition 1.2.7 (with g_1 constructed as g is in Lemma 1.2.6), $g_1 \circ p$ is Gâteaux differentiable everywhere except the origin, since for x_0 and x_1 fixed, $p(x_0 + tx_1) \in L$ for all $t \in \mathbb{R}$ where L is the linear subspace generated by x_0 and x_1. Hence Proposition 1.2.7 holds with SC replacing LUR in the dual norm and Gâteaux smoothness replacing C^1-smoothness in the conclusion. To complete the proof of (b) one need only observe that Proposition 1.2.8 is true when C^k-smoothness is replaced by Gâteaux smoothness. □

1.2.9 REMARK. Recall that a point x_0 of a convex set C is a *strongly exposed point of* C if there exists $\Lambda \in X^*$ such that $\Lambda(x) \leq \Lambda(x_0)$ for all $x \in C$ and $\|x_n - x_0\| \to 0$ whenever $\{x_n\}_{n=1}^{\infty} \subset C$ and $\Lambda(x_n) \to \Lambda(x_0)$.

If $\|\cdot\|$ is LUR and $x \in S_X$, then x is a strongly exposed point of B_X. On the other hand, there are non-LUR norms for which every point on the unit sphere is strongly exposed. To see this, recall that Yost ([Y, Theorem 2.3]) proved that there is a C^1-smooth norm $|||\cdot|||$ on a separable reflexive space X such that $|||\cdot|||^*$ is not LUR. Let $f \in S_{X^*}$ and choose $x \in B_{X^{**}} = B_X$ such that $f(x) = 1$. By Šmulyan's criterion (see Corollary 1.2.3), $|||x_n - x||| \to 0$ whenever $x_n \in B_X$ and $f(x_n) \to 1$. Therefore f is strongly exposed.

Now observe that the full strength of the LUR norm on X was not used in the proof of Lemma 1.2.6 or elsewhere in the proof of Theorem 1.2.1. That is, only the strictly weaker condition that every point of S_X is strongly exposed was used. This observation yields a result which is, at least formally, better than Theorem 1.2.1.

1.3 Smooth Partitions of Unity in Nonseparable Spaces

Using some deep renorming theorems we will list some concrete examples of spaces for which Theorem 1.2.1 yields the existence of C^1-smooth partitions of unity.

1.3.1 THEOREM. *If $K^{(\omega_1)} = \emptyset$, then $C(K)$ admits C^1-smooth partitions of unity.*

Proof. From the renorming theorems [De, Theorem 2.1] and [HR] it follows that $C(K)$ admits an LUR norm whose dual is LUR whenever $K^{(\omega_1)} = \emptyset$. □

1.3.2 THEOREM. *If there is a w^*-compact $K \subset X^*$ such that $(K, w^*)^{(\omega_1)} = \emptyset$ and the linear span of K is norm dense in X^*, then X admits C^1-smooth partitions of unity.*

Proof. Transfer techniques as used in [De], [$GTWZ_1$] and [$GTWZ_2$] along with the fact that $C(K)$ admits an LUR norm whose dual is LUR show that X admits an LUR norm whose dual is LUR. □

For further details on the renormings used in this section one can consult [DGZ_3, Chapter VII]. Before presenting the next result we need a definition.

1.3.3 DEFINITION. A Banach space X is said to be *weakly countably determined* (WCD) if there exists a countable collection K_n of w^*-compact subsets of X^{**} such that for every $x \in X$ and every $u \in X^{**}$, there exists an n_0 such that $x \in K_{n_0}$ and $u \notin K_{n_0}$. A Banach space X is said to be *weakly compactly generated* (WCG) if there is a weakly compact subset K of X such that the linear span of K is norm dense in X.

1.3.4 THEOREM. *If X^* is WCD, then X admits C^1-smooth partitions of unity.*

Proof. From [FT, Theorem 4] and [F_4, Theorem 3] it follows that such an X admits an LUR norm whose dual is LUR. □

We also point out that Theorem 1.2.1 provides alternate methods for proving some known results on smooth partitions of unity.

1.3.5 THEOREM ([JZ_3], [Vas]). *If X is WCD, then X admits Gâteaux smooth partitions of unity.*

Proof. It is shown in [Mer, Theorem 4.8] that such an X admits an LUR norm whose dual is SC. □

We also obtain the results of Godefroy *et al.* ([GTWZ_1]) and McLaughlin ([Mc]) in the C^1-smooth case.

1.3.6 THEOREM ([GTWZ_1], [Mc]). *If either X or X^* is WCG and X admits a C^1-smooth bump function, then X admits C^1-smooth partitions of unity.*

Proof. If X^* is WCG, then X^* is WCD so this follows from Theorem 1.3.4. If X is WCG and admits a C^1-smooth bump function, then from [JZ_1, Theorem 1] and [Tr, Theorem 1] it follows that X admits an LUR norm whose dual is also LUR. □

Chapter Two

SMOOTH CONVEX APPROXIMATION IN SEPARABLE BANACH SPACES

2.1 Introduction

The purpose of this chapter is to investigate when continuous convex functions can be approximated by functions of higher order smoothness in either the Gâteaux or Fréchet sense.

In the paper [NS], Nemirovskii and Semenov proved that there is a uniformly continuous function on a separable Hilbert space which cannot be approximated by functions with uniformly continuous second derivatives. In the first section we show, using the method from Section 2.2 and the above result of [NS], that there is a norm on a separable Hilbert space which is not a limit of functions with uniformly continuous second derivatives.

As mentioned in Section 1.1, the question of approximating continuous functions on Hilbert spaces by C^k-smooth functions has been settled for a long time. However, as pointed out by Borwein and Noll in [BN], very little is known about approximating convex functions by smooth convex functions. In [BN] it is shown that on a Hilbert space every convex function which is bounded on bounded sets can be approximated uniformly on bounded sets by convex functions with Lipschitz derivatives. In the third section we will show that if a separable Banach space admits a norm with modulus of smoothness of power type 2 (see Definition 2.1.1), then every convex function which is bounded on bounded sets can be approximated uniformly on bounded sets by convex functions which are twice Gâteaux differentiable with Lipschitz first derivatives. This is done by extending the result from [BN] mentioned above to the setting of spaces which admit norms with moduli of smoothness of power type 2 and using the techniques of Fabian *et al.* in [FWZ, Theorem 3.1].

We also show that in such spaces there are many norms which are UR and twice Gâteaux differentiable. It is not known if this result is valid in nonseparable Hilbert spaces.

In the fourth section, it is proved that if a separable Banach space admits a C^k-smooth norm, then there is an LUR norm which is a limit of C^k-smooth norms on bounded sets. Such approximations are useful, for instance, in constructing smooth homeomorphic maps of spaces into c_0 or l_2; see [DGZ$_3$, Chapter V] for further details.

Recall that LUR and UR norms were defined in Definition 1.1.1.

Much of this chapter will deal with spaces that admit norms with moduli of smoothness of power type 2.

2.1.1 DEFINITION. The *modulus of smoothness* $\rho_X(\tau)$ on $(X, \|\cdot\|)$ is defined by

$$\rho_X(\tau) = \sup\{\frac{1}{2}(\|x+y\| + \|x-y\| - 2) : \|x\| = 1, \|y\| \leq \tau\}$$

and $\rho_X(\tau)$ is said to be of *power type p* for $1 < p \leq 2$ if there exists $C > 0$ such that $\rho_X(\tau) \leq C\tau^p$.

If X admits a norm which has modulus of smoothness of power type p, then X is superreflexive and thus the UR norms are dense among all norms on X (see e.g. [B]). In addition, the proof of [FWZ, Lemma 2.4] shows that $\|\cdot\|$ has modulus of smoothness of power type 2 if and only if $\|\cdot\|$ has Lipschitz derivative on its sphere. Using this, a direct computation shows that $\|\cdot\|^2$ has Lipschitz derivative on all of X. See [DGZ$_3$, Chapter V] for further details on this and for the proof of the following proposition.

2.1.2 PROPOSITION. *Let f be a continuous convex function. Then the following are equivalent:*

(a) there exists $C > 0$ such that $f(x+y) + f(x-y) - 2f(x) \leq C\|y\|^2$ for all $y \in X$;

(b) f' is Lipschitz.

2.2 Approximation in Hilbert Spaces

The basic methods of Section 1.2 are used to show that if every norm on a Hilbert space X can be approximated uniformly on bounded sets by functions in $C_u^2(X)$ (the space of real-valued functions with uniformly continuous second derivatives on X), then the same is true for every uniformly continuous function. In particular, using a proposition of [NS] which shows that there is a function on a separable Hilbert space X which has Lipschitz derivative but cannot be approximated uniformly on bounded sets by functions in $C_u^2(X)$ we obtain

2.2.1 PROPOSITION. *There is a norm with modulus of smoothness of power type 2 on a Hilbert space X which is not a uniform limit on B_X of functions in $C_u^2(X)$.*

Two lemmas will be used to prove Proposition 2.2.1. The first is analogous to Lemma 1.2.6. In the sequel a function f will be called *even* if $f(x) = f(-x)$ for all $x \in X$.

2.2.2 LEMMA. *Suppose every norm on a Hilbert space X can be approximated uniformly on bounded sets by functions in $C_u^2(X)$. Then given $\epsilon > 0$ and a uniformly continuous even function f on B_X, there is a $g \in C_u^2(X)$ such that $|g(x) - f(x)| < \epsilon$ for all $x \in S_X$.*

Proof. Note that the usual Hilbert norm $\|\cdot\|$ is UR and $\|\cdot\|^2 \in C_u^2(X)$. Without loss of generality assume that $f(x) \in [-1, 1]$ for all $x \in B_X$. Let $\epsilon > 0$ be fixed and choose N so that $\frac{7}{2N} < \epsilon$. Let

$$\Delta_i = \left[\frac{i-1-N}{N}, \frac{i-N}{N}\right] \quad \text{for } i = 1, \ldots, 2N.$$

Let $Q_0 = Q_{2N+1} = \emptyset$ and for $i = 1, \ldots, 2N$, define

$$Q_i = f^{-1}(\Delta_i) \quad \text{and} \quad \widetilde{Q}_i = \overline{\text{conv}}\Big(\big(B_X \backslash (Q_{i-1} \cup Q_i \cup Q_{i+1})\big) \cup B_{\frac{1}{2}}\Big).$$

Since f is even, it follows that $\tilde{Q}_i$ is the unit ball of an equivalent norm $\|\cdot\|_i$ on X. Let $\nu_i(x) = \|x\|_i$. By uniform rotundity and the uniform continuity of f, it follows that there exists $\alpha > 0$ such that $\nu_i(x) \geq 1 + \alpha$ for every $x \in Q_i \cap S_X$ and for all i. Choose $h_i \in C_u^2(X)$ such that $|h_i(x) - \nu_i(x)| < \frac{\alpha}{4}$ for $x \in B_3$. Construct a function $\phi : [0, \infty) \to [0, 1]$ with uniformly continuous second derivative which moreover satisfies $\phi(t) = 1$ if $t \geq 1 + \frac{3}{4}\alpha$ and $\phi(t) = 0$ if $t \leq 1 + \frac{1}{4}\alpha$. Set $r_i(x) = \phi(h_i(x))$. Observe that $\sum_{i=1}^{2N} r_i(x) \geq 1$ for $1 \leq \|x\| \leq 3$. Since $\sum_{i=1}^{2N} r_i(x)$ is uniformly continuous, given $F = \{x \in B_X : \sum_{i=1}^{2N} r_i(x) \leq \frac{1}{2}\}$, the distance from F to S_X is greater than 0. The functions r and θ which are used below can be expressed as composites of appropriate functions on $\mathbb{R}$ with the norm. Pick $r \in C_u^2(X)$ such that $r(x) = 1$ for all $x \in F$ and $r(x) = 0$ for $\|x\| \geq 1$. Let $\alpha_i =$ midpoint of Δ_i and for $x \in B_3$ define

$$h(x) = \frac{r(x) + \sum_{i=1}^{2N} \alpha_i r_i(x)}{r(x) + \sum_{i=1}^{2N} r_i(x)}.$$

Since h is not necessarily defined on all of X, we extend h to a continuous function on X. Now construct $g \in C_u^2(X)$ such that $g(x) = h(x)$ for all $x \in B_X$ as follows. Let $g(x) = h(x)\theta(x)$ where $\theta \in C_u^2(X)$ satisfies $\theta : X \to [0, 1]$, $\theta(x) = 1$ for $x \in B_1$ and $\theta(x) = 0$ for $x \notin B_2$. As in the proof of Lemma 1.2.6, $|g(x) - f(x)| < \epsilon$ for all $x \in S_X$. □

2.2.3 Lemma. *Let X be a Hilbert space. If every norm on X can be approximated uniformly on bounded sets by functions in $C_u^2(X)$, then every real-valued uniformly continuous function can be approximated uniformly on bounded sets by functions in $C_u^2(X)$.*

Proof. The basic proof of Proposition 1.2.7 works. Let the notation be as in the proof of Proposition 1.2.7 and let f be uniformly continuous on the bounded subset F of the hyperplane H. Note that $Y = X \oplus \mathbb{R}$ is a Hilbert space; and that $p_1^{-1}(y) \in F$ implies $\Lambda(y) \geq \alpha$ for some fixed $\alpha > 0$. Since $\alpha > 0$, one can extend f_1

to a uniformly continuous even function on $\mathcal{O}_Y$. Apply Lemma 2.2.2 and proceed as in the proof of Proposition 1.2.7 observing that p is a C_u^2-smooth mapping outside any neighborhood of the origin in Y. □

Proof of Proposition 2.2.1. It is known that the norms with moduli of smoothness of power type 2 are dense among all norms on a Hilbert space (see [B] or Corollary 2.3.4 for a proof). Hence, if every norm with modulus of smoothness of power type 2 can be approximated uniformly on bounded sets by functions in $C_u^2(X)$, then so can every norm. Thus by Lemma 2.2.3 every uniformly continuous function can be approximated uniformly on B_X by functions in $C_u^2(X)$. This contradicts the results of Proposition 3, Section 7 in [NS] and the remark following it which show that there is a uniformly continuous function on a separable Hilbert space which cannot be approximated uniformly on bounded sets by functions with uniformly continuous second derivatives. □

2.2.4 Remark. The proof of Proposition 2.2.1 shows that given any class of norms which are dense among all norms on a Hilbert space (e.g. the UR or LUR norms), then at least one of them cannot be approximated uniformly on bounded sets by functions with uniformly continuous second derivatives.

2.3 Second Order Gâteaux Smooth Convex Approximation

Recall that a function $\phi : X \to \mathbb{R}$ is *twice Gâteaux differentiable* at $x \in X$ provided that $\phi'(y)$ (the Gâteaux derivative of ϕ) exists for y in a neighborhood of x, the limit

$$\phi''(x)(h,k) = \lim_{t \downarrow 0} \frac{1}{t}\big(\phi'(x+th) - \phi'(x)\big)(k)$$

exists for each $h, k \in X$ and $\phi''(\cdot,\cdot)$ is a continuous symmetric bilinear form.

Much of this section will be devoted to proving the following theorem.

2.3.1 THEOREM. *Let X be a separable Banach space that admits a norm with modulus of smoothness of power type 2.*

(a) Every convex function which is bounded on bounded subsets of X can be approximated uniformly on bounded sets by twice Gâteaux differentiable convex functions whose first derivatives are also Lipschitz.

(b) Every norm on X is a limit of UR norms which are twice Gâteaux differentiable and have moduli of smoothness of power type 2.

Some preliminary results will be given before proceeding to the proof of Theorem 2.3.1. In what follows, $f \,\square\, g(x) = \inf\{f(y) + g(x-y) : y \in X\}$ and is called the *infimal convolution* of the convex functions f and g on a Banach space X.

2.3.2 LEMMA. *Suppose X is a Banach space and let f be a convex function on X which is bounded on bounded sets. If $\{g_k\}_{k=1}^{\infty}$ is a sequence of convex functions such that $g_k(0) \leq \frac{1}{k}$ and $g_k(x) \geq k\|x\| - \frac{1}{k}$ for all $x \in X$, then $f \,\square\, g_k \to f$ uniformly on bounded subsets of X.*

Proof. Let $r > 0$ and let K be the Lipschitz constant of f on B_{r+1}. For $x_0 \in B_r$ fixed and for each k we can choose y_k so that $f \,\square\, g_k(x_0) \geq f(y_k) + g_k(x_0 - y_k) - \frac{1}{k}$. For any $k \geq K+1$ with $k \geq 3$ we have

$$
\begin{aligned}
f(x_0) + \frac{1}{k} \geq f(x_0) + g_k(0) &\geq f \,\square\, g_k(x_0) \\
&\geq f(y_k) + g_k(x_0 - y_k) - \frac{1}{k} \\
&\geq f(y_k) + k\|x_0 - y_k\| - \frac{2}{k}. \qquad (1)
\end{aligned}
$$

Let $\Lambda_0 \in \partial f(x_0)$, then $\|\Lambda_0\|^* \leq K$, since f has Lipschitz constant K on B_{r+1}. Because $f(y_k) - f(x_0) \geq \Lambda_0(y_k) - \Lambda_0(x_0)$, we have

$$f(x_0) - f(y_k) \leq \|\Lambda_0\|^* \|y_k - x_0\| \leq K\|y_k - x_0\|.$$

Thus it follows from (1) that

$$K\|y_k - x_0\| + \frac{3}{k} \geq k\|x_0 - y_k\|.$$

In other words,

$$\|x_0 - y_k\| \leq \frac{3}{k(k-K)}.$$

In particular, $y_k \in B_{r+1}$ and so $|f(y_k) - f(x_0)| \leq K\|y_k - x_0\|$. From this we obtain

$$\begin{aligned} f(y_k) + k\|x_0 - y_k\| - \frac{2}{k} &\geq f(x_0) - K\|x_0 - y_k\| + k\|x_0 - y_k\| - \frac{2}{k} \\ &\geq f(x_0) - \frac{2}{k}. \end{aligned} \tag{2}$$

Clearly the lemma follows from (1) and (2). □

In the following proposition, part (a) generalizes [BN, Theorem 5.2(1)] while part (b) is well-known (see e.g. [B]) and is given here for the reader's convenience.

2.3.3 PROPOSITION. *Let X be a Banach space which has a norm with modulus of smoothness of power type 2.*

(a) Any convex function f which is bounded on bounded subsets of X can be approximated uniformly on bounded sets by convex functions with Lipschitz derivatives.

(b) Every norm on X can be approximated by norms with moduli of smoothness of power type 2.

Proof. Let $\|\cdot\|$ have modulus of smoothness of power type 2. Then $\|\cdot\|^2$ has Lipschitz derivative on all of X (see [DGZ$_3$, Chapter 5]); hence so does g_k where $g_k(x) = k^4\|x\|^2$. Easily $g_k(x) \geq k\|x\| - \frac{1}{k}$ for all k and $g_k(0) = 0$, therefore $f \square g_k \to f$ uniformly on bounded sets by Lemma 2.3.2.

To see that $f_k = f \square g_k$ has Lipschitz derivative for each k we use Proposition 2.1.2 to choose a $C_k > 0$ such that

$$g_k(x+h) + g_k(x-h) - 2g_k(x) \leq C_k\|h\|^2 \tag{1}$$

for all $x, h \in X$. Fix arbitrary $x_0, h_0 \in X$ and choose y_k so that

$$f_k(x_0) \geq f(y_k) + g_k(x_0 - y_k) - \|h_0\|^2.$$

Then, using (1), we have

$$\begin{aligned} f_k(x_0+h_0) + f_k(x_0-h_0) - 2f_k(x_0) &\leq f(y_k) + g_k(x_0 + h_0 - y_k) + \\ &\quad + f(y_k) + g_k(x_0 - h_0 - y_k) - \\ &\quad - 2(f(y_k) + g_k(x_0 - y_k) - \|h_0\|^2) \\ &= g_k(x_0 - y_k + h_0) + g_k(x_0 - y_k - h_0) - \\ &\quad - 2g_k(x_0 - y_k) + 2\|h_0\|^2 \\ &\leq (C_k + 2)\|h_0\|^2. \end{aligned}$$

Since C_k does not depend on x_0 or h_0, it follows from Proposition 2.1.2 that f_k' is Lipschitz. This proves (a).

To see (b), for a given norm $|\cdot|$ let $f = |\cdot|^2$. Then by (a) the norms $|\cdot|_k = (f \,\square\, g_k)^{\frac{1}{2}}$ have moduli of smoothness of power type 2 and converge to $|\cdot|$ uniformly on bounded sets. □

To obtain approximating functions which are twice Gâteaux differentiable, we need a lemma whose proof is almost identical to the proof of [FWZ, Theorem 3.1].

2.3.4 LEMMA. *Let X be a separable Banach space and let $\epsilon > 0$ and $r > 0$ be given.*

(a) If f is a convex function whose first derivative is Lipschitz, then there is a convex function g such that $|g(x) - f(x)| < \epsilon$ for all $x \in B_r$ and g is twice Gâteaux differentiable with Lipschitz first derivative.

(b) If $\|\cdot\|$ is a norm with modulus of smoothness of power type 2, then there is a norm $\|\cdot\|_1$ such that $(1-\epsilon)\|x\| \leq \|x\|_1 \leq (1+\epsilon)\|x\|$ for all x and $\|\cdot\|_1$ is twice Gâteaux differentiable with modulus of smoothness of power type 2.

Proof. To begin the proof, we fix $\epsilon > 0$ and $r > 0$. Let $C \in \mathbb{R}$ be such that f is Lipschitz with constant C on B_{r+1}, and select a set $\{h_i\}_{i=0}^{\infty}$ dense in S_X. Next, fix a C^∞-smooth function $\phi_0 : \mathbb{R} \to \mathbb{R}$ such that ϕ_0 is nonnegative and even, vanishes outside $[\frac{-\epsilon}{2C}, \frac{\epsilon}{2C}]$ and satisfies $\int_{\mathbb{R}} \phi_0 = 1$. Setting $f_0 = f$ and $\phi_n = 2^n\phi_0(2^n t)$ for $t \in \mathbb{R}$, $n \geq 1$, we define a sequence of functions $\{f_n : X \to \mathbb{R}\}_{n=1}^{\infty}$ by

$$f_n(x) = \int_{\mathbb{R}^{n+1}} f_0(x - \sum_{i=0}^{n} t_i h_i) \prod_{i=0}^{n} \phi_i(t_i) dt_0 \cdots dt_n.$$

As in the proof of [FWZ, Theorem 3.1], there is a function $g : X \to \mathbb{R}$ such that $f_n \to g$ uniformly on bounded sets and g is twice Gâteaux differentiable with Lipschitz first derivative.

Moreover, for $x \in B_r$, we have

$$\begin{aligned} |f(x) - g(x)| &= \lim_n \Big| \int_{\mathbb{R}^{n+1}} \Big[f_0(x) - f_0(x - \sum_{i=0}^{n} t_i h_i) \Big] \prod_{i=0}^{n} \phi_i(t_i) dt_0 \cdots dt_m \Big| \\ &\leq \int_{|t_i| \leq \epsilon/(2^{i+1}C)} C \| \sum_{i=0}^{n} t_i h_i \| \prod_{i=0}^{n} \phi_i(t_i) dt_0 \cdots dt_m \\ &\leq C \frac{\epsilon}{C} = \epsilon. \end{aligned}$$

In case (b) where the function f is a norm, we set $f_0(x) = \|x\|^2$. It follows that the function g as obtained above is convex and even. By (a), choose g so that $g(x) \in [\|x\|^2 - \epsilon, \|x\|^2 + \epsilon]$ whenever $\|x\| \leq 5$. If we set $B = \{x \in X : g(x) \leq 16\}$, then as in [FWZ] the Minkowski functional of B is an equivalent norm $|\cdot|$ which is twice Gâteaux differentiable and has modulus of smoothness of power type 2. Let $\|\cdot\|_1 = 4|\cdot|$. Now $\|x\|_1 = 4$ if and only if $g(x) = 16$ which occurs if and only if $16 - \epsilon \leq \|x\|^2 \leq 16 + \epsilon$. Therefore, $(1-\epsilon)\|x\| \leq \|x\|_1 \leq (1+\epsilon)\|x\|$ for all $x \in X$. □

Proof of Theorem 2.3.1. From Proposition 2.3.3(a) and Lemma 2.3.4(a) one immediately sees that (a) is true. We now prove (b).

Step 1. We first show that every UR norm is a limit of UR norms with moduli of smoothness of power type 2.

Let $|\cdot|$ be UR and let $\epsilon > 0$. By Proposition 2.3.3(b), choose norms $|\cdot|_n$ with moduli of smoothness of power type 2 so that $(1-\epsilon)|x| \leq |x|_n \leq |x|$ and $|\cdot|_n \to |\cdot|$.

Choose $C_n \geq 2$ so that $|x+h|_n^2 + |x-h|_n^2 - 2|x|_n^2 \leq C_n |h|_n^2$ for all $x, h \in X$ and define $|||\cdot|||$ by

$$|||x||| = \left(|x|_1^2 + \epsilon \sum_{n=1}^{\infty} \frac{2^{-n}}{C_n} |x|_n^2 \right)^{\frac{1}{2}}.$$

Easily $|||x+h|||^2 + |||x-h|||^2 - 2|||x|||^2 \leq (C_1+1)|||h|||^2$ for all $x, h \in X$ and $(1-\epsilon)|x| \leq |||x||| \leq (1+\epsilon)|x|$. To see that $|||\cdot|||$ is UR, suppose that

$$2|||x_k|||^2 + 2|||y_k|||^2 - |||x_k + y_k|||^2 \to 0.$$

From this it follows, for each n, that

$$2|x_k|_n^2 + 2|y_k|_n^2 - |x_k + y_k|_n^2 \to 0.$$

Thus,

$$2|x_k|^2 + 2|y_k|^2 - |x_k + y_k|^2 \to 0.$$

Therefore $|x_k - y_k| \to 0$; this implies that $|||x_k - y_k||| \to 0$. That is, $|||\cdot|||$ is UR.

Step 2. If the initial norm in Lemma 2.3.4(b) is UR, then so is the norm in the conclusion.

Let $|\cdot|$ be UR with modulus of smoothness of power type 2. By uniform rotundity, for a fixed $r > 0$, given $\delta > 0$ there exists $\epsilon > 0$ so that $2|x|^2 + 2|y|^2 \geq |x+y|^2 + 4\epsilon$ whenever $|x| \leq r+1, |y| \leq r+1$ and $|x-y| \geq \delta$. Hence using $f_0(\cdot) = |\cdot|^2$ to construct the functions f_n as in the proof of Lemma 2.3.4, for $|x-y| \geq \delta$ and $|x| \leq r$, $|y| \leq r$, we have

$$\begin{aligned} f_n\left(\frac{x+y}{2}\right) &= \int_{\mathbb{R}^{n+1}} \left| \frac{x+y}{2} - \sum_{i=0}^{n} t_i h_i \right|^2 \prod_{i=0}^{n} \phi(t_i) dt_0 \ldots dt_n \\ &= \int_{\mathbb{R}^{n+1}} \left| \frac{x - \sum_{i=0}^{n} t_i h_i}{2} + \frac{y - \sum_{i=0}^{n} t_i h_i}{2} \right|^2 \prod_{i=0}^{n} \phi(t_i) dt_0 \ldots dt_n \\ &\leq \int_{\mathbb{R}^{n+1}} \left(\frac{1}{2} \left| x - \sum_{i=0}^{n} t_i h_i \right|^2 + \frac{1}{2} \left| y - \sum_{i=0}^{n} t_i h_i \right|^2 - \epsilon \right) \prod_{i=0}^{n} \phi(t_i) dt_0 \ldots dt_n \\ &= \frac{1}{2} f_n(x) + \frac{1}{2} f_n(y) - \epsilon. \end{aligned}$$

Now $f_n \to f$ uniformly on bounded sets for some f, therefore

$$f\Big(\frac{x+y}{2}\Big) \le \frac{1}{2}f(x) + \frac{1}{2}f(y) - \epsilon.$$

Let $B = \{x : f(x) \le M\}$ be the unit ball of some norm $\|\cdot\|$. We will show that $\|\cdot\|$ is UR. Now $B \subset \{x : |x| \le r\}$ for some $r > 0$. Given $\delta > 0$, there exists an $\epsilon > 0$ so that

$$f\Big(\frac{x+y}{2}\Big) \le \frac{1}{2}f(x) + \frac{1}{2}f(y) - \epsilon \le M - \epsilon$$

whenever $\|x - y\| \ge \delta$ and $x, y \in B$. Since f is convex and bounded on bounded sets, it is certainly uniformly continuous on B. Thus there is an $\eta > 0$ such that $\|u - v\| \le \eta$ and $u, v \in B$ imply $|f(u) - f(v)| \le \epsilon$. Hence

$$f\Big((1+\eta)\big(\frac{x+y}{2}\big)\Big) \le f\Big(\frac{x+y}{2}\Big) + \epsilon \le M.$$

This implies

$$\Big\|\frac{x+y}{2}\Big\| \le \frac{1}{1+\eta}$$

whenever $x, y \in B$ and $\|x - y\| \ge \delta$. Thus $\|\cdot\|$ is UR. This finishes Step 2.

Finally, since the UR norms are dense among all norms in X, Step 1, Step 2 and Lemma 2.3.4(b) show that (b) is true. □

2.3.5 Remark. (a) If f is globally Lipschitz with Lipschitz constant K and g_k is as in Lemma 2.3.2, then arguing as in inequalities (1) and (2) in the proof of Lemma 2.3.2, for any $x_0 \in X$ we have

$$f(x_0) + \frac{1}{k} \ge f \,\square\, g(x_0) \ge f(x_0) + (k - K)\|x_0 - y_k\| - \frac{2}{k}.$$

Thus the approximation is uniform on all of X. Moreover in Lemma 2.3.4(a) the approximation is uniform on all of X provided the given function f is globally Lipschitz. Therefore the approximation in Theorem 2.3.1(a) will be uniform on all of X provided the initial function is globally Lipschitz.

(b) Variants of Proposition 2.3.3 are also valid, for example, in spaces which admit uniformly smooth norms or norms with moduli of smoothness of power type $1+\alpha$ for $0<\alpha<1$ and for C^1-smoothness in reflexive spaces.

2.4 Approximating LUR Norms by C^k-Smooth Norms

It seems to be unknown whether the existence of a C^k-smooth norm on X implies that every norm on X can be approximated C^k-smooth norms. The next proposition shows one can construct an LUR norm on a separable Banach space X that is a limit of C^k-smooth norms provided X admits a C^k-smooth norm; see [PWZ, Proposition 2] for a similar construction on $c_0(\Gamma)$.

2.4.1 PROPOSITION. *Let X be a separable Banach space which admits a C^k-smooth norm for some $k \in \mathbb{N} \cup \{\infty\}$. Then there is an LUR norm on X which is C^1-smooth and is limit of C^k-smooth norms.*

Proof. Let the norm $\|\cdot\|$ be C^k-smooth and $\{h_n\}_{n=1}^{\infty}$ be dense in S_X. Choose $f_n \in S_{X^*}$ such that $f_n(h_n) = 1$ and define the projections P_n by $P_n x = f_n(x)h_n$. For $m = 1, 2, \ldots$ let ϕ_m be even, convex and C^∞-smooth functions on $\mathbb{R}$ such that $\phi_m(t) = 0$ if $|t| \le \frac{1}{m}$ and $\phi_m(t) > 0$ if $|t| > \frac{1}{m}$; suppose also that $\phi_m(2) \le \frac{1}{2}$ for all m. Now set

$$\theta_{n,m}(x) = \phi_m(\|x\|) + \phi_m(\|x - P_n x\|).$$

Observe that $\theta_{n,m}$ is C^k-smooth, even, convex and uniformly continuous on bounded subsets of X. If $V_{n,m} = \{x : \theta_{n,m}(x) \le 1\}$, then $V_{n,m}$ is the unit ball of an equivalent norm $\|\cdot\|_{n,m}$. Because $\theta_{n,m}(x) \le 1$ whenever $\|x\| \le 1$, one has $\|\cdot\|_{n,m} \le \|\cdot\|$. Moreover, $\theta_{n,m}(0) = 0$ and $\theta_{n,m}(x) = 1$ whenever $\|x\|_{n,m} = 1$; this implies $\theta'_{n,m}(x)(x) \ge 1$. According to the implicit function theorem, $\|\cdot\|_{n,m}$ is C^k-smooth on $X \backslash \{0\}$.

Consider the norm $|||\cdot|||$ defined by

$$|||x||| = (\|x\|^2 + \sum_{n,m} \frac{1}{2^{n+m}}\|x\|^2_{n,m} + \sum_{n=1}^{\infty} \frac{1}{2^n} f_n^2(x))^{\frac{1}{2}}.$$

Notice that $|||\cdot|||$ is C^1-smooth, the norms

$$|||x|||_j = (\|x\|^2 + \sum_{n,m\leq j} \frac{1}{2^{n+m}}\|x\|^2_{n,m} + \sum_{n=1}^{j} \frac{1}{2^n} f_n^2(x))^{\frac{1}{2}}$$

are C^k-smooth and $|||\cdot|||_j \to |||\cdot|||$.

We now show that $|||\cdot|||$ is LUR. Suppose that $|||x||| = 1$ and

$$(1) \qquad 2|||x|||^2 + 2|||x_i|||^2 - |||x + x_i|||^2 \to 0.$$

We now show, for every n, that $\|x_i - P_n x_i\| \to \|x - P_n x\|$. To do this we first assume that $\limsup_i \|x_i - P_n x_i\| > \|x - P_n x\|$ for some n. Thus there is a subsequence $\{x_{i_j}\}$ such that $\|x_{i_j} - P_n x_{i_j}\| \geq \|x - P_n x\| + \delta$ for some $\delta > 0$ and for all j. Choose m so that $\frac{1}{m} \leq \frac{\delta}{2}$, and since $\|x\|_{n,m} \leq \|x\| \leq |||x||| = 1$, we choose $\alpha \geq 1$ so that

$$\phi_m(\alpha\|x\|) + \phi_m(\alpha\|x - P_n x\|) = 1.$$

Because of (1) and the definition of $|||\cdot|||$, $\|x_i\| \to \|x\|$ and therefore

$$\liminf_j \phi_m(\alpha\|x_{i_j}\|) + \phi_m(\alpha\|x_{i_j} - P_n x_{i_j}\|) \geq \phi_m(\alpha\|x\|) + \phi_m(\alpha\|x - P_n x\| + \delta).$$

Since ϕ_m is convex and increasing on $(\frac{1}{m}, \infty)$ and $\delta \geq \frac{2}{m}$ it follows that

$$\begin{aligned}\phi_m(\alpha\|x\|) + \phi_m(\alpha\|x - P_n x\| + \delta) &\geq \phi_m(\alpha\|x\|) + \phi_m(\alpha\|x - P_n x\|) + \\ &\quad + \phi_m(\alpha\|x - P_n x\| + \frac{2}{m}) - \phi_m(\alpha\|x - P_n x\|) \\ &\geq \phi_m(\alpha\|x\|) + \phi_m(\alpha\|x - P_n x\|) + \lambda \\ &= 1 + \lambda\end{aligned}$$

where $\lambda = \phi_m(\frac{2}{m}) - \phi_m(\frac{1}{m}) > 0$. Now $\phi_m(\alpha\|x_{i_j}\|) + \phi_m(\alpha\|x_{i_j} - P_m x_{i_j}\|) \geq 1 + \frac{\lambda}{2}$ for $j \geq j_0$. Since ϕ_m is uniformly continuous on bounded sets, there is an $\epsilon > 0$ so that

$$\phi_m\big((1-\epsilon)\alpha\|x_{i_j}\|\big) + \phi_m\big((1-\epsilon)\alpha\|x_{i_j} - P_m x_{i_j}\|\big) \geq 1$$

for $j \geq j_0$. Hence $\liminf_j (1-\epsilon)\|x_{i_j}\|_{n,m} \geq \|x\|_{n,m}$. However, this leads to a contradiction, since (1) and the definition of $|||\cdot|||$ imply that $\|x_i\|_{n,m} \to \|x\|_{n,m}$. Therefore, $\limsup_i \|x_i - P_n x_i\| \leq \|x - P_n x\|$ for each n.

Similarly we see that $\liminf_i \|x_i - P_n x_i\| \geq \|x - P_n x\|$ for each n. Therefore,

$$\|x_i - P_n x_i\| \to \|x - P_n x\| \quad \text{for each} \quad n. \tag{2}$$

We now argue as in [JZ$_2$] to show that $|||\cdot|||$ is LUR. Let $\epsilon > 0$ and recall that $P_n v = f_n(v)h_n$ where $\|f_n\|^* = \|h_n\| = f_n(h_n) = 1$. Since $\{h_n\}_{n=1}^{\infty}$ is dense in S_X, we choose and fix n such that

$$\|x - P_n x\| < \epsilon. \tag{3}$$

According to (2) and (3) there is an i_0 such that

$$\|x_i - P_n x_i\| < \epsilon \quad \text{for all} \quad i \geq i_0. \tag{4}$$

Because of (1) and the definition of $|||\cdot|||$, it follows that $\lim_i f_n(x_i) = f_n(x)$. Thus replacing i_0 by a larger number if necessary we also have:

$$|f_n(x_i) - f_n(x)| < \epsilon \quad \text{for all} \quad i \geq i_0. \tag{5}$$

Finally, for $i \geq i_0$ (3), (4) and (5) imply

$$\begin{aligned}\|x - x_i\| &\leq \|x_i - P_n x_i\| + \|P_n x_i - P_n x\| + \|P_n x - x\| \\ &= \|x_i - P_n x_i\| + \|(f_n(x_i) - f_n(x))h_n\| + \|P_n x - x\| \\ &< 3\epsilon.\end{aligned}$$

Since $|||\cdot|||$ is equivalent to $\|\cdot\|$, $|||x - x_i||| \to 0$. Therefore $|||\cdot|||$ is LUR. □

In a similar manner one can also prove:

2.4.2 PROPOSITION. *If X is a separable Banach space with a norm whose k-th Fréchet derivative is uniformly continuous on its sphere for some $k \in \mathbb{N}$, then X admits an LUR norm which has uniformly continuous k-th Fréchet derivative on its sphere.*

Proof. Essentially the same proof as in Proposition 2.4.1 works. As before, for $\|x\|_{n,m} = 1$, we have $\theta'_{n,m}(x)(x) \geq 1$, thus the implicit function theorem asserts that $\|\cdot\|_{n,m}$ has uniformly continuous k-th derivative, since $\theta_{n,m}$ has uniformly continuous k-th derivative. Now define the norm $|||\cdot|||$ by

$$|||x||| = \Big(\|x\|^2 + \sum_{n,m} \frac{1}{C_{n,m}2^{n+m}}\|x\|^2_{n,m} + \sum_{n=1}^{\infty} \frac{1}{2^n} f_n^2(x)\Big)^{\frac{1}{2}}$$

where $C_{n,m} \geq 1$ is chosen so that the k-th derivative of $\frac{1}{C_{n,m}}\|\cdot\|^2_{n,m}$ has norm ≤ 1 on $X\backslash\{0\}$. The rules for differentiating an infinite sum show that $|||\cdot|||$ has uniformly continuous k-th derivative on its sphere. $\square$

In the following proposition, part (a) is similar to [DGZ$_2$, Corollary III.8] but we give a simpler proof here using the methods of [FWZ, Theorem 3.3] and the Baire category theorem; part (b) is an easy consequence of the methods of [FWZ, Theorem 3.3] but was not included in that paper. Before we state the proposition, recall that a norm is *Lipschitz smooth* at $x \neq 0$ if and only if there exists a $C > 0$ so that $\|x+h\| + \|x-h\| - 2\|x\| \leq C\|h\|^2$ for all $h \in X$; see [FWZ, Lemma 2.4].

2.4.3 PROPOSITION. *(a) Suppose X admits an LUR norm which is Lipschitz smooth at each point of a dense G_δ subset of X. Then X admits a norm with modulus of smoothness of power type 2.*

(b) Suppose X does not contain a subspace isomorphic to $c_0(\mathbb{N})$. If X admits a norm whose k-th derivative is locally uniformly continuous on $X\backslash\{0\}$, then X admits a norm with uniformly continuous k-th derivative on its sphere.

Proof. (a) Let $\|\cdot\|$ be an LUR norm on X which is Lipschitz smooth at each point of Ω a dense G_δ subset of X. Define

$$F_n = \{x : \|x+h\| + \|x-h\| - 2\|x\| \leq n\|h\|^2 \quad \text{for all} \quad h \in X\}.$$

Then F_n is closed and $\Omega \subset \cup F_n$. By the Baire category theorem, for some n_0, F_{n_0} has nonempty interior, say, $B(x_0, 2\epsilon) \subset F_{n_0}$ for some $\epsilon > 0$ and $x_0 \in X$. Therefore, $\|\cdot\|'$ is Lipschitz on $B(x_0, \epsilon)$.

We now proceed as in the proof of [FWZ, Theorem 3.3]. Let $H = \{h \in X : \|x_0\|'(h) = 0\}$. Since $\|\cdot\|$ is LUR, there is a $\delta > 0$ such that for $h \in H$ and $\|h\| > \epsilon$, we have $\|x_0 + h\| \geq \|x_0\| + \delta$. For $h \in H$, let $\phi(h) = \|x_0 + h\| + \|x_0 - h\| - 2\|x_0\|$. Set $Q = \{h \in H : \phi(h) \leq \frac{\delta}{2}\}$. Let q be the Minkowski functional of Q. The implicit function theorem asserts that as an equivalent norm on H, q has Lipschitz derivative on its sphere. Thus there is such a norm on X.

(b) By [FWZ, Theorem 3.3(i)], X is superreflexive. Therefore there exists a strongly exposed point on B_X. Choose $\epsilon > 0$ so that the k-th derivative of $\|\cdot\|$ is uniformly continuous on $B(x_0, \epsilon)$ and proceed as in part (a). □

The following corollary is an immediate consequence of Theorem 2.3.1(b), Proposition 2.4.3(a) and the result that a point of twice Gateaux differentiability of a convex function is a point of Lipschitz smoothness ([BN, Proposition 2.2]).

2.4.4 Corollary. *For a separable Banach space X, the following are equivalent.*

(a) X admits an LUR norm which is twice Gâteaux differentiable on a dense G_δ set.

(b) X admits an LUR norm which is Lipschitz smooth on a dense G_δ set.

(c) X admits a twice Gâteaux differentiable UR norm with modulus of smoothness of power type 2.

In particular, from Corollary 2.4.4 we see that the LUR norm in Proposition 2.4.1 cannot generally be twice Gâteaux differentiable on a dense G_δ subset of X. We conclude this section with a dichotomy on averaging processes.

2.4.5 Corollary. *Suppose that X is separable and admits a C^{k+1}-smooth norm for some $k \geq 1$.*

(a) If X does not contain a subspace isomorphic to $c_0(\mathbb{N})$, then X admits an LUR norm which has uniformly continuous k-th derivative on S_X. Moreover X admits a twice Gâteaux differentiable UR norm.

(b) If X contains a subspace isomorphic to $c_0(\mathbb{N})$, then there is no LUR norm on X which is twice Gâteaux differentiable on a dense G_δ subset of X.

Proof. (a) By Propositions 2.4.2 and 2.4.3(b) X admits an LUR norm which has uniformly continuous k-th derivative on its sphere. In particular, X is superreflexive. Moreover, a norm which is C^{k+1}-smooth on $X\backslash\{0\}$ has a locally Lipschitz derivative on $X\backslash\{0\}$, therefore [FWZ, Theorem 3.4] shows that X admits a norm with modulus of smoothness of power type 2. Applying Theorem 2.3.1 shows that (a) is true in this case.

(b) Because X contains a subspace isomorphic to $c_0(\mathbb{N})$, X is not superreflexive; in particular, X admits no UR norm. The proof is completed by applying Corollary 2.4.4. □

Chapter Three

POINTWISE DIRECTIONAL HOLDER DIFFERENTIABLE BUMP FUNCTIONS

3.1 Introduction

The main goal of this chapter is to show that if a Banach space X and its dual space admit continuous bump functions with pointwise directional Lipschitz derivatives, then X is isomorphic to a Hilbert space. In proving this, we will develop results that are valid for bump functions with directional Hölder derivatives which, it seems, should be presented in the more general Hölder case.

This chapter is organized as follows. In this section, we introduce the basic notions which will be used throughout the chapter.

The second section will focus on Asplund spaces. Recall that a Banach space X is an *Asplund space* if every continuous convex function is Fréchet differentiable on a dense G_δ subset of X. It is well-known that a Banach space is an Asplund space if and only if every separable subspace has a separable dual; see e.g. [Ph, Theorem 2.34]. We will show that a Banach space is an Asplund space if it admits a continuous bump function with pointwise directional Hölder derivative. In particular, any Banach space which admits a continuous twice Gâteaux differentiable bump function is an Asplund space. Recall that any separable Banach space admits a continuous Gâteaux differentiable bump function (see e.g. [DGZ$_3$]) while not every separable Banach space is an Asplund space; for example $l_1(\mathbb{N})$ has a nonseparable dual.

In Section 3, the methods of Deville *et al.* ([DGZ$_2$]) in conjunction with a technique of Borwein and Noll ([BN]) are used to show that if a Banach space has the Radon-Nikodym property (RNP) and has a continuous bump function with pointwise directional α-Hölder derivative, then it has a norm with modulus of smoothness of power type $1+\alpha$; see [Bou] for further details on the RNP. Applying this result to those in the second section will yield new isomorphic characterizations of Hilbert and superreflexive spaces. We now introduce some terminology.

3.1.1 DEFINITION. A function ϕ on X is said to have a *directional Hölder derivative at x_0* if ϕ' (Gâteaux derivative) exists in a neighborbood of x_0 and for each $h \in S_X$ there exist $K_h \geq 2$, $\delta_h > 0$ and $\alpha_h > 0$ such that

$$\left|\left(\phi'(x_0+th)-\phi'(x_0)\right)(h)\right| \leq K_h t^{\alpha_h}$$

for all $0 \leq t \leq \delta_h$. In case $\alpha_h \geq \alpha > 0$ for all $h \in B_X$, ϕ' is said to be *directionally α-Hölder at x_0*. We will say that ϕ has *pointwise directional Hölder derivative on X*, if ϕ has directional Hölder derviative at each $x \in X$.

3.1.2 REMARK. If ϕ is continuous at x_0 and if ϕ' is directionally Hölder at x_0, then by the mean value theorem and the fact ϕ is bounded on a neighborhood of x_0, there is a $\delta > 0$ so that for each $h \in B_X$, there are $C_h \geq 2$ and $\alpha_h > 0$ such that

$$|\phi(x_0+th)-\phi(x_0)-\phi'(x_0)(th)| \leq C_h\|th\|^{1+\alpha_h}$$

for all $0 \leq t \leq \delta$. This is the property of ϕ which we will work with in this chapter.

3.2 Pointwise Directional Hölder Differentiability and Asplund Spaces

Before stating the main result of this section (Theorem 3.2.3) we will prove two lemmas. The first lemma is due to Fabian ([F_2]) and we include it here for completeness.

3.2.1 LEMMA ([F_2, Lemma 0]). *Let ϕ be a continuous and Gâteaux differentiable bump function on a Banach space X and let $S = \{x \in X : \phi(x) \neq 0\}$. Consider the mapping $\Phi : S \to X^*$ defined by*

$$\Phi(x) = \left(\phi^{-2}\right)'(x).$$

Then $\Phi(S)$ is norm dense in X^*.

Proof. We begin by defining a function ψ on X by $\psi(x) = \phi^{-2}(x)$ if $\phi(x) \neq 0$ and $\psi(x) = \infty$ if $\phi(x) = 0$. Let $f \in X^*$ and $\epsilon > 0$ be given. Since ϕ is continuous, the function $\psi - f$ is lower semicontinuous on X. Moreover $\psi \geq 0$ on X and $\psi = \infty$ outside a bounded set in X. Therefore $\psi - f$ is bounded below on X. According to Ekeland's variational principle (see e.g. [Ph, Lemma 3.13]), there is an $x_0 \in X$ such that $\psi(x_0) < \infty$, that is $\phi(x_0) \neq 0$, and for every $h \in X$ and $t > 0$ we have

$$\psi(x_0 + th) - f(x_0 + th) \geq \psi(x_0) - f(x_0) - \epsilon f\|h\|.$$

Hence for $h \in X$ and $t > 0$,

$$\frac{\psi(x_0 + th) - \psi(x_0)}{t} \geq \frac{f(x_0 + th) - f(x_0)}{t} = f(h) - \epsilon\|h\|.$$

Since ψ is Gâteaux differentiable at x_0, it follows for $h \in X$ that

$$\psi'(x_0)(h) = \lim_{t \downarrow 0} \frac{\psi(x_0 + th) - \psi(x_0)}{t} \geq f(h) - \epsilon\|h\|.$$

Therefore $\|\psi'(x_0) - f\|^* \leq \epsilon$. □

The next lemma is quite similar to [BN, Proposition 2.2].

3.2.2 LEMMA. *Suppose ϕ has directional Hölder derivative at x_0 and that ϕ is continuous on a neighborhood of x_0. Then there exists $\delta > 0$, $\alpha > 0$, $K \geq 2$ and an open set $U \subset B_X$ such that*

$$|\phi(x_0 + th) - \phi(x_0) - \phi'(x_0)(th)| \leq K\|th\|^{1+\alpha}$$

for all $h \in U$, and $0 \leq t \leq \delta$.

Proof. We use the assumptions to choose $\delta > 0$ so that ϕ is continuous at $x_0 + th$ for $h \in B_X$ and $0 \leq t \leq \delta$; moreover δ is chosen so that ϕ is bounded on $B_\delta(x_0)$. Hence for ach $h \in B_X$ there exists $C_h \geq 2$ and $\alpha_h > 0$ such that

$$|\phi(x_0 + th) - \phi(x_0) - \phi'(x_0)(th)| \leq C_h\|th\|^{1+\alpha_h} \quad \text{for all} \quad 0 \leq t \leq \delta.$$

Motivated by the elegant proof of [BN, Proposition 2.2], we define

$$F_{m,n} = \{h \in B_X : |\phi(x_0 + th) - \phi(x_0) - \phi'(x_0)(th)| \le n\|th\|^{1+\frac{1}{m}} \text{ for all } 0 \le t \le \delta\}.$$

Now each $F_{m,n}$ is closed and $\cup F_{m,n} = B_X$. According to the Baire category theorem, for some m_0 and n_0, there is an open set U such that $U \subset F_{m_0,n_0}$. Setting $K = n_0$ and $\alpha = \frac{1}{m_0}$ completes the proof. □

We are now ready for the main result of this section. The strategy used to prove the following theorem is similar to that of the proof of [DGZ$_2$, Lemma III.6].

3.2.3 THEOREM. *Suppose X admits a continuous bump function with pointwise directional Hölder derivative on X, then dens(Y) = dens(Y^*) for any subspace Y of X, where dens(X) is the smallest cardinality of a dense subset in X. In particular, X is an Asplund space.*

Proof. Suppose that ϕ is a continuous bump function on X with pointwise directional Hölder derivative. As in the proof of Lemma 3.2.1, let $\psi(x) = \phi^{-2}(x)$ if $\phi(x) \neq 0$ and $\psi(x) = \infty$ if $\phi(x) = 0$. Notice that ψ' is pointwise directionally Hölder whenever $\phi(x) \neq 0$. Let $D = \{x_j\}_{j \in J}$ be a dense set in X with $|J| = \text{dens}(X)$.

Let $f \in X^*$ and $0 < \epsilon \le 1$ be given. Using Lemma 3.2.1, choose x_0 so that $\phi(x_0) \neq 0$ and $\|\psi'(x_0) - f\| < \epsilon$. According to Lemma 3.2.2, there are $\delta > 0, \alpha > 0, K \ge 2$, $r > 0$ and $U = y_0 + B_r \subset B_X$ such that

$$(1) \quad |\psi(x_0 + th) - \psi(x_0) - \psi'(x_0)(th)| \le K\|th\|^{1+\alpha} \quad \text{if} \quad h \in U \text{ and } 0 \le t \le \delta.$$

Now let $C = \{th : h \in U, 0 \le t \le \delta\}$ and $C_1 = \{th : h \in y_0 + B_{\frac{r}{2}}, 0 \le t \le \delta\}$. From (1) and the definition of C we have

$$(2) \qquad |\psi(x_0 + z) - \psi(x_0) - \psi'(x_0)(z)| \le K\|z\|^{1+\alpha} \quad \text{for all } z \in C.$$

We will also need following elementary fact whose obvious proof is omitted.

Claim (i). If $\|u\| > 0$ and $u \in C_1$, then $u + \|u\|\frac{r}{2}v \in C$ for all $v \in B_X$.

Now choose $x_1 \in D$ so that $x_1 - x_0 \in C_1$ and $0 < \|x_1 - x_0\|^{\frac{\alpha}{2}} \le \min\{\frac{\epsilon}{10K}, \frac{r}{2}\}$. We also choose $n \in \mathbb{N}$ such that $\frac{1}{2}\|x_1 - x_0\|^{1+\frac{\alpha}{2}} \le \frac{1}{n} \le \|x_1 - x_0\|^{1+\frac{\alpha}{2}}$. In particular, $\frac{1}{n} \le \|x_1 - x_0\| \cdot \|x_1 - x_0\|^{\frac{\alpha}{2}} \le \|x_1 - x_0\|\frac{r}{2}$. According to Claim (i) we have

$$x_1 - x_0 + \frac{1}{n}y \in C \quad \text{for all } y \in B_X. \tag{3}$$

Now, for $y \in B_X$, defining $\rho(h) = \psi(x_0 + h) - \psi(x_0) - \psi'(x_0)(h)$ we have

$$\begin{aligned}\psi(x_1 + \frac{1}{n}y) - \psi(x_1) &= \psi\big(x_0 + (x_1 - x_0) + \frac{1}{n}y\big) - \psi(x_0) - \big(\psi(x_1) - \psi(x_0)\big)\\ &= \psi'(x_0)(x_1 - x_0) + \frac{1}{n}\psi'(x_0)(y) + \rho(x_1 - x_0 + \frac{1}{n}y) -\\ &\quad - \psi'(x_0)(x_1 - x_0) - \rho(x_1 - x_0).\end{aligned}$$

Hence,

$$\psi(x_1 + \frac{1}{n}y) - \psi(x_1) - \frac{1}{n}\psi'(x_0)(y) = \rho(x_1 - x_0 + \frac{1}{n}y) - \rho(x_1 - x_0).$$

Using this with (2) and (3) yields:

$$\begin{aligned}\big|\psi(x_1 + \frac{1}{n}y) - \psi(x_1) - \frac{1}{n}\psi'(x_0)(y)\big| &\le \big|\rho(x_1 - x_0 + \frac{1}{n}y)\big| + \big|\rho(x_1 - x_0)\big|\\ &\le K\|x_1 - x_0 + \frac{1}{n}y\|^{1+\alpha} + K\|x_1 - x_0\|^{1+\alpha}\\ &\le 2^{1+\alpha}K\|x_1 - x_0\|^{1+\alpha} + K\|x_1 - x_0\|^{1+\alpha}.\end{aligned}$$

This implies that

$$\begin{aligned}\big|n\big(\psi(x_1 + \frac{1}{n}y) - \psi(x_1)\big) - \psi'(x_0)(y)\big| &\le 5K\|x_1 - x_0\|^{\frac{\alpha}{2}}n\|x_1 - x_0\|^{1+\frac{\alpha}{2}}\\ &\le 5K \cdot \frac{\epsilon}{10K} \cdot n \cdot \frac{2}{n} = \epsilon.\end{aligned}$$

Since $\|\psi'(x_0) - f\| < \epsilon$, it follows, for $y \in B_X$, that

$$\big|n\big(\psi(x_1 + \frac{1}{n}y) - \psi(x_1)\big) - f(y)\big| \le 2\epsilon.$$

Consider the Banach space $l_\infty(B_X)$ in its canonical supremum norm and let

$$S = \{n\big(\psi(x_j + \frac{1}{n}y) - \psi(x_j)\}_{(j,n)}$$

where (j, n) are all indices in $J \times \mathbb{N}$ such that $\phi(x_j + \frac{1}{n}y) \neq 0$ for all $y \in B_X$. Since $|S| \leq \text{dens}(X)$ and X^* is in the norm closure of S as subsets of $l_\infty(B_X)$ it follows that $\text{dens}(X) = \text{dens}(X^*)$.

If Y is a closed subspace of X, then Y admits a continuous bump function with pointwise directional Hölder derivative. Therefore the above argument shows that $\text{dens}(Y) = \text{dens}(Y^*)$. In particular, X is an Asplund space (see e.g. [Ph, Theorem 2.34]). □

3.2.4 EXAMPLE. Since $l_1(\mathbb{N})$ is separable, it follows that $l_1(\mathbb{N})$ has a Gâteaux differentiable norm and hence a continuous Gâteaux differentiable bump function; see e.g. [DGZ_3]. On the other hand, by Theorem 3.2.3, it cannot admit any continuous bump function with pointwise directional Hölder derivative, since $l_\infty(\mathbb{N})$ is nonseparable.

Because a function which is twice Gâteaux differentiable at a point certainly has directional Hölder derivative at that point, Theorem 3.2.3 immediately yields

3.2.5 COROLLARY. *If a Banach space X admits a continuous twice Gâteaux differentiable bump function, then X is an Asplund space.*

In [BN, Proposition 2.2] it is shown that every norm which is twice Gâteaux differentiable on $X \setminus \{0\}$ is also Lipschitz smooth at each point of $X \setminus \{0\}$. However, it is not difficult to construct an example of a continuous function on $\mathbb{R}^2$ which is twice Gâteaux differentiable everywhere but there are points at which it is not Fréchet differentiable. Therefore the bump function in Theorem 3.2.3 need not be Fréchet differentiable everywhere. Because of this, it is natural to ask whether the existence of, say, a continuous twice Gâteaux differentiable bump function on X implies the existence of a Fréchet differentiable bump function on X. Although we do not know the answer to this in general, for a wide class of spaces a stronger result is true.

3.2.6 COROLLARY. *Suppose X is WCD and admits a continuous bump function with pointwise directional Hölder derivative, then X admits an LUR norm whose dual is also LUR.*

Proof. By Theorem 3.2.3, X is an Asplund space. The corollary thus follows from [F_3, Theorem 1] which shows that a WCD Asplund space admits an LUR norm whose dual is also LUR. □

Combining Corollary 3.2.6 and Theorem 1.2.1(a) we obtain

3.2.7 COROLLARY. *If X is WCD and admits a continuous bump function with pointwise directional Hölder derivative, then X admits C^1-smooth partitions of unity.*

3.3 Characterizations of Hilbert and Superreflexive Spaces

We begin by giving a sufficient condition for a Banach space to be superreflexive. It should be noted that Proposition 3.3.1 is an improvement of [DGZ_2, Theorem III.1] which was observed by R. Poliquin and V. Zizler. We include their proof for the reader's convenience.

3.3.1 PROPOSITION. *Suppose X is a Banach space with the RNP. If there is a continuous bump function ϕ on X with pointwise directional Hölder derivative, then X is superreflexive. Moreover, if ϕ' is pointwise directionally α-Hölder for some $\alpha > 0$, then X admits a norm with modulus of smoothness of power type $1 + \alpha$.*

Proof. We essentially follow the proof of [DGZ_2, Theorem III.1].

First define $\psi : X \to \mathbb{R} \cup \{\infty\}$ by $\psi(x) = \phi^{-2}(x)$ if $\phi(x) \neq 0$ and $\psi(x) = \infty$ if $\phi(x) = 0$. Let ψ^* be the Fenchel conjugate function of ψ i.e. for $y \in X^*$

$$\psi^*(y) = \sup\{\langle y, x\rangle - \psi(x) : x \in X\}.$$

Because $\psi(x) = \infty$ outside a bounded set, the function ψ^* is finite, convex and w^*-lower semicontinuous on X^*. Because X has RNP, the function ψ^* is Fréchet differentiable at each point of a norm dense G_δ subset Ω of X^* (cf. [Col]) with derivative in X (x^* is in the subdifferential of ψ at x if and only if x is in the subdifferential of ψ^* at x^*; see [ET]). Let $\widetilde{\psi}$ denote the Fenchel conjugate of ψ^* on X. It is shown in the proof of [DGZ$_2$, Theorem III.1] that if $y_0 \in \Omega$ and $x_0 = (\psi^*)'(y_0)$ then $(x_0, \widetilde{\psi}(x_0))$ is a strongly exposed point of the epigraph of $\widetilde{\psi}$—exposed by $(y_0, -1)$. Because of strong exposedness, the point $(x_0, \widetilde{\psi}(x_0))$ actually belongs to the epigraph of ψ and this means that $\psi(x_0) = \widetilde{\psi}(x_0) < \infty$. Therefore it follows that ψ has directional Hölder derivative and is continuous at x_0. Because $\widetilde{\psi}$ is convex, majorized by ψ and agrees with ψ at x_0, it is straightforward to verify that there exists $\delta > 0$ so that for each $h \in B_X$ there is a $C_h > 0$ and an $\alpha_h > 0$ such that

$$|\widetilde{\psi}(x_0 + th) - \widetilde{\psi}(x_0) - \widetilde{\psi}'(x_0)(th)| \leq C_h \|th\|^{1+\alpha_h}.$$

Hence Lemma 3.2.2 shows that there exists $\delta_1 > 0$, $\alpha_1 > 0$, $K \geq 2$ and an open set $U \subset B_X$ such that

$$|\widetilde{\psi}(x_0 + th) - \widetilde{\psi}(x_0) - \widetilde{\psi}'(x_0)(th)| \leq K\|th\|^{1+\alpha_1}$$

for all $0 \leq t \leq \delta$. Since $\widetilde{\psi}$ is convex and bounded above on a neighborhood of x_0 it follows, as in the proof of [BN, Proposition 2.2] that there exists $\delta_2 > 0$ and $K_1 > 0$ so that

$$|\widetilde{\psi}(x_0 + h) - \widetilde{\psi}(x_0) - \widetilde{\psi}'(x_0)(h)| \leq K_1\|h\|^{1+\alpha_1}$$

whenever $\|h\| \leq \delta_2$. From here, the argument is exactly the same as the proof of [DGZ$_2$, Theorem III.1] starting from equation (4). □

Applying Proposition 3.3.1 in the case $\alpha = 1$, that is ϕ' is *pointwise directionally Lipschitz*, to the results in Section 2.3 we obtain the following theorem.

3.3.2 THEOREM. *For a separable Banach space X, the following are equivalent.*

(a) X has RNP and admits a continuous bump function with pointwise directional Lipschitz derivative.

(b) X admits a norm with modulus of smoothness of power type 2.

(c) Every norm on X is a limit of twice Gâteaux differentiable UR norms with moduli of smoothness of power type 2.

(d) X admits an LUR norm which is twice Gâteaux differentiable on a dense G_δ set.

(e) Every convex function which is bounded on bounded sets can be approximated uniformly on bounded subsets of X by twice Gâteaux differentiable convex functions whose first derivatives are also Lipschitz.

Proof. Proposition 3.3.1 shows that (a) $\Rightarrow$ (b). From Theorem 2.3.1(b), we have (b) $\Rightarrow$ (c); (c) $\Rightarrow$ (d) is obvious and (d) $\Rightarrow$ (e) is a consequence of Corollary 2.4.4 and Theorem 2.3.1(a).

To prove (e) $\Rightarrow$ (a), notice that (e) easily implies that X admits a continuous twice Gâteaux differentiable bump function with Lipschitz derivative and therefore is superreflexive by [FWZ, Theorem 3.2] (it is also easy to directly construct a norm with modulus of smoothness of power type 2 using (e) and the implicit function theorem). □

As a consequence of what has been done in this chapter, one obtains

3.3.3 THEOREM. *For a Banach space X the following are equivalent.*

(a) X is isomorphic to a Hilbert space.

(b) Both X and X^* admit continuous twice Gâteaux differentiable bump functions.

(c) Both X and X^* admit continuous bump functions whose derivatives are pointwise directionally Lipschitz.

Proof. Both (a) $\Rightarrow$ (b) and (b) $\Rightarrow$ (c) are obvious. Thus only (c) $\Rightarrow$ (a) needs proving. By Theorem 3.2.3, X and X^* are Asplund spaces, therefore X and X^*

have the RNP (see e.g. [Bou, Theorem 5.2.12]). According to Propostion 3.3.1, X and X^* admit norms with moduli of smoothness of power type 2. Therefore X has type 2 and cotype 2 (see e.g. [B, Propositions 2 and 3, pp. 309–311]) and hence is isomorphic to a Hilbert space by Kwapien's theorem in [Kw]. □

We close this chapter with a characterization of superreflexive spaces.

3.3.4 PROPOSITION. *For a Banach space X the following are equivalent.*

(a) X and X^* admit continuous bump functions with pointwise directional Hölder derivatives.

(b) X is superreflexive.

Proof. According to Theorem 3.2.3, X^* is an Asplund space, therefore X has the RNP (see e.g. [Bou, Theorem 5.2.12]). Invoking Proposition 3.3.1 shows that X is superreflexive. This proves (a) $\Rightarrow$ (b). Of course, (b) $\Rightarrow$ (a) follows from a deep theorem of Pisier ([Pi, Theorem 3.2]). □

It seems even to obtain (a) from superreflexivity one has to use Pisier's theorem, although the existence of a bump function with pointwise directional Hölder derivative is much weaker than the existence of a norm with a modulus of smoothness of power type. For instance, $c_0(\mathbb{N})$ admits a bump function with locally Lipschitz derivative while $c_0(\mathbb{N})$ is not superreflexive.

Chapter Four

FRECHET SMOOTH NORMS ON SPACES OF COUNTABLE DIMENSION

4.1 Introduction

If a separable Banach space X has a Fréchet differentiable norm, then X^* is also separable (see e.g. [DGZ3, Chapter II] or the proof of Corollary 4.3.3). Therefore certain separable Banach spaces (e.g. $l_1(\mathbb{N})$, $C[0,1]$ and $L_1[0,1]$) do not admit Fréchet differentiable norms. In this chapter we will investigate the existence of norms which are Fréchet differentiable on certain dense subsets of a separable Banach space.

In Section 4.2 it is shown that there is an abundance of norms which are Fréchet differentiable at each point of a prescribed countable set not containing the origin. In addition, many of these norms can be chosen to also be LUR whenever the space itself admits an LUR norm (which is the case for separable spaces). Moreover, if every norm on a separable space X is Fréchet differentiable on dense set, then X is an Asplund space and thus X^* is separable (see e.g. [Ph, Theorem 2.34 and Corollary 2.35]). Therefore, even in separable Banach spaces we cannot hope to have all norms Fréchet differentiable at each point of some fixed countable set of nonzero elements.

The third section considers smoothness on (noncomplete) normed linear spaces with countable algebraic bases. It is shown if a subspace L of a separable Banach space X has dimension $\aleph_0$, then there is an LUR norm which is Fréchet differentiable at each nonzero element of L (Theorem 4.3.1). Notice Šmulyan's criterion implies that if a norm is Fréchet differentiable and LUR at a point x, then the dual norm is LUR at the support functional of x (see Remark 4.3.2). Therefore, in order to construct a norm which is Fréchet differentiable and LUR at each nonzero element of a subspace L, one must construct a dual norm which is LUR at each support functional to nonzero elements of L. We will construct such a norm using Kadets-Klee renorming techniques (see [DJ], [Di], [K_1] and [K_2]) in conjunction with a result of Johnson *et al.* from [JRZ]. The result of [JRZ] is crucial in our construction, because it enables us to ensure that support functionals to elements of the given $\aleph_0$-dimensional subspace are in a fixed separable subspace of the dual under certain renormings. The techniques of [JZ_4] will be used to combine smoothness and rotundity. Some connections between Theorem 4.3.1, monotone Schauder bases and support functionals will also be discussed.

We will use the following notions concerning Schauder bases.

4.1.1 DEFINITION. A Schauder basis $\{e_k\}_{k=1}^{\infty}$ on a Banach space X with norm $\|\cdot\|$ is said to be *monotone* if for each n, $\|P_n\| = 1$ where $P_n(\sum_{k=1}^{\infty} a_k e_k) = \sum_{k=1}^{n} a_k e_k$ is the *natural projection* on the basis $\{e_k\}_{k=1}^{\infty}$.

4.2 Fréchet Smooth Norms on Arbitrary Countable Sets

In this chapter it will be necessary to recognize certain norms as being LUR. These norms are described in the following known lemma (see [DJ], [K_1], [K_2]). For the reader's convenience, we have included a proof of this lemma which is very similar to the last part of the proof of Proposition 2.4.1.

4.2.1 LEMMA. *Suppose $(X, \|\cdot\|)$ is a Banach space and $\{P_n\}_{n=1}^{\infty}$ be a sequence of finite rank projections with $\{\|P_n\|\}_{n=1}^{\infty}$ bounded. Let $\{f_n\}_{n=1}^{\infty} \subset X^*\backslash\{0\}$ be chosen so that we can write*

$$P_n(x) = \sum_{k\in I_n} f_k(x) h_{n(k)} \quad \text{where} \quad I_n \subset \mathbb{N},\ |I_n| < \infty \quad \text{and} \quad h_n(k) \in X.$$

If $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty} \subset (0,\infty)$ are chosen so that $\sum_{n=1}^{\infty} \alpha_n$ and $\sum_{n=1}^{\infty} \beta_n(\|f_n\|^*)^2$ converge, then $|||\cdot|||$ is LUR at each point of $\overline{\cup_{n=1}^{\infty} P_n X}$ where $|||\cdot|||$ is defined by

$$|||x||| = \left(\|x\|^2 + \sum_{n=1}^{\infty} \alpha_n \|P_n x - x\|^2 + \sum_{n=1}^{\infty} \beta_n f_n^2(x)\right)^{\frac{1}{2}}.$$

Proof. The choice of $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ guarantee that $|||\cdot|||$ is an equivalent norm. Let $\epsilon > 0$ and let $x \in \overline{\cup_{n=1}^{\infty} P_n X}$ and $x \neq 0$. Suppose that

$$2|||x|||^2 + |||x_n|||^2 - |||x + x_n|||^2 \to 0.$$

Then by the convexity of the terms in the definition of $|||\cdot|||$ it follows that

$$\text{(1)} \qquad \lim_{j\to\infty} \|P_n x_j - x_j\| = \|P_n x - x\| \quad \text{for each} \quad n$$

and

$$\text{(2)} \qquad \lim_{j\to\infty} f_n(x_j) = f_n(x) \quad \text{for each} \quad n.$$

Because $x \in \overline{\cup_{n=1}^{\infty} P_n X}$ and $\{\|P_n\|\}_{n=1}^{\infty}$ is bounded one can choose n so that

$$\text{(3)} \qquad \|P_n x - x\| < \epsilon.$$

Using (1) and (3), for some j_0 we have

$$\text{(4)} \qquad \|P_n x_j - x_j\| < \epsilon \quad \text{for} \quad j \geq j_0.$$

Let $m = |I_n|$; using (2) and replacing j_0 by a larger number if necessary, we have

$$\text{(5)} \qquad \|h_{n(k)}\| \cdot |f_k(x_j) - f_k(x)| < \frac{\epsilon}{m} \quad \text{for } k \in I_n,\ j \geq j_0.$$

Finally, for $j \geq j_0$, from (3), (4) and (5) it follows that

$$\begin{aligned}\|x_j - x\| &\leq \|x_j - P_n x_j\| + \|P_n x_j - P_n x\| + \|P_n x - x\| \\ &\leq \|x_j - P_n x_j\| + \sum_{k\in I_n} |f_k(x_j) - f_k(x)|\|h_{n(k)}\| + \|P_n x - x\| \\ &< 3\epsilon.\end{aligned}$$

Therefore $|||x_j - x||| \to 0$. □

Since we refer to the following well-known theorem in this thesis, we will include it here as an easy consequence of the above lemma.

4.2.2 COROLLARY. (a) *If X is separable, then X admits an LUR norm.*

(b) *If X^* is separable, then X^* admits a dual LUR norm.*

Proof. (a) Let $\{h_n\}_{n=1}^{\infty}$ be dense in S_X and $f_n \in S_{X^*}$ satisfy $f_n(h_n) = 1$. Set $\alpha_n = \beta_n = 2^{-n}$ and $P_n x = f_n(x)h_n$, then $\overline{\cup_{n=1}^{\infty} P_n X} = X$. Thus $|||\cdot|||$ defined in Lemma 4.2.1 is LUR on X.

(b) Let $\{f_n\}_{n=1}^{\infty}$ be dense in S_{X^*} with $\|\cdot\|^*$ a dual norm on X^*. Choose $x_n \in X$ with $\|x_n\| \leq 2$ so that $f_n(x_n) = 1$. Hence $P_n f = f(x_n)f_n$ is a dual projection. Moreover $\cup_{n=1}^{\infty} P_n X^*$ is norm dense in X^* and $\|P_n\| \leq 2$ for each n. Define a norm $|||\cdot|||$ by

$$|||f||| = ((\|f\|^*)^2 + \sum_{n=1}^{\infty} \frac{1}{2^n}(\|f - P_n f\|^*)2 + \sum_{n=1}^{\infty} \frac{1}{2^n} f^2(x_n))^{\frac{1}{2}}.$$

Since all terms involved are w^*-lower semicontinuous it follows that $|||\cdot|||$ is a dual norm; Lemma 4.2.1 ensures that $|||\cdot|||$ is LUR. □

This concludes our digression on LUR renormings in separable spaces. The next proposition concerns differentiability of norms on arbitrary countable sets.

We will use the following notation from [FZZ]. The set P will denote the set of all equivalent norms on X and B_1 will denote the unit ball of a fixed norm $\|\cdot\|$ on X. It follows that (P, ρ) is a Baire space with $\rho(\nu, \mu) = \sup\{|\nu(x) - \mu(x)| : x \in B_1\}$; see [FZZ].

4.2.3 PROPOSITION. *Let $S = \{x_k\}_{k=1}^{\infty} \subset X\backslash\{0\}$. Then the set of equivalent norms which are Fréchet differentiable at each point of S is residual. In particular, if X is separable, then the set of LUR norms which are Fréchet differentiable at each point of S is residual in P.*

Proof. Let B_1 denote the unit ball of a fixed norm $\|\cdot\|$ on X. Let $O_{n,k}$ be the the set of all norms $\nu \in P$ for which there exists a $\delta_\nu > 0$ such that:

$$\sup_{y \in B_1} \{\nu(x_k + \delta_\nu y) + \nu(x_k - \delta_\nu y) - 2\nu(x_k)\} < \frac{\delta_\nu}{n}. \qquad (1)$$

The following routine argument shows that $O_{n,k}$ is open. Let $\nu \in O_{n,k}$ and choose $\delta > 0$ so that

$$\nu(x_k + \delta y) + \nu(x_k - \delta y) - 2\nu(x_k) \leq a < \frac{\delta}{n} \quad \text{for all} \quad y \in B_1.$$

Let $K = \sup_{y \in B_1}\{\|x_k + \delta y\|, \|x_k - \delta y\|, 2\|x_k\|\}$. If $\rho(\mu, \nu) < \frac{1}{4K}(\frac{\delta}{n} - a)$, then

$$\begin{aligned} \mu(x_k + \delta y) + \mu(x_k - \delta y) - 2\mu(x_k) &\leq \nu(x_k + \delta y) + \nu(x_k - \delta y) - 2\nu(x_k) + \\ &\quad + \frac{1}{4K}(\frac{\delta}{n} - a)3K \\ &\leq a + \frac{3}{4}(\frac{\delta}{n} - a) < \frac{\delta}{n}. \end{aligned}$$

Hence $\mu \in O_{n,k}$. This shows that $O_{n,k}$ is open.

If ν is Fréchet differentiable at x_k, it follows from the definition of Fréchet differentiability that $\nu \in O_{n,k}$. To show that $O_{n,k}$ is dense, we will show that the collection of norms that are Fréchet differentiable at x_k is dense in P. Fix x_k, $\epsilon > 0$ and let $|\cdot|$ be an arbitrary norm in P. Define a norm μ^* by

$$\mu^*(f) = \left((|f|^*)^2 + \epsilon f^2(x_k)\right)^{\frac{1}{2}}.$$

It follows that μ^* is an equivalent dual norm on X^*, since it is w^*-lower semicontinuous. Choose $g \in X^*$ such that $g(x_k) = \mu^*(g)\mu(x_k) = 1$. Now define a dual norm ν^* by

$$\nu^*(f) = \left((|f|^*)^2 + \epsilon f^2(x_k) + \epsilon(|f - f(x_k)g|^*)^2\right)^{\frac{1}{2}}.$$

According to Lemma 4.2.1, ν^* is LUR on span($\{g\}$). Now observe that $\nu^*(g) = \mu^*(g)$. Moreover, $\mu^* \leq \nu^*$ and thus $\nu(x_k) \leq \mu(x_k)$. Whence

$$g(x_k) = \mu^*(g)\mu(x_k) \geq \mu^*(g)\nu(x_k) = \nu^*(g)\nu(x_k).$$

This shows that $\frac{1}{\nu^*(g)}g$ supports x_k with respect to ν. Because ν^* is LUR at $\frac{1}{\nu^*(g)}g$, Šmulyan's criterion (see Corollary 1.2.3) implies that ν is Fréchet differentiable at

x_k. Since $\epsilon > 0$, was arbitrary this shows that the norms that are differentiable at x_k are dense in P. Thus $O_{n,k}$ is dense for each n and k.

Now consider the following residual set $O \subset P$:

$$O = \bigcap_{n,k} O_{n,k}$$

We now show that each $\mu \in O$ is Fréchet differentiable at each x_k. It is a well-known fact that norms satisfying (1) for all n are Fréchet differentiable at x_k; nevertheless, a proof using Šmulyan's criterion is included here. Let $\mu \in O$ and fix x_k. Suppose $\mu^*(g) = 1$, $\mu^*(f_n) \leq 1$ and $\lim_n f_n(x_k) = g(x_k) = \mu(x_k)$. Since $\mu \in O_{n,k}$ for each n, choose $\delta_n > 0$ so that

$$(2) \qquad \mu(x_k + \delta_n y) + \mu(x_k - \delta_n y) - 2\mu(x_k) < \frac{\delta_n}{n} \quad \text{for all} \quad y \in B_1.$$

Choose m_n so that

$$(3) \qquad f_m(x_k) \geq g(x_k) - \frac{\delta_n}{n} \quad \text{for all} \quad m \geq m_n.$$

According to (2) and (3) for $y \in B_1$ and $m \geq m_n$, one has

$$\begin{aligned} f_m(\delta_n y) - g(\delta_n y) &\leq f_m(x_k + \delta_n y) + g(x_k - \delta_n y) - 2g(x_k) + \frac{\delta_n}{n} \\ &\leq \mu(x_k + \delta_n y) + \mu(x_k - \delta_n y) - 2\mu(x_k) + \frac{\delta_n}{n} \\ &\leq \frac{2\delta_n}{n}. \end{aligned}$$

This shows that $\|f_m - g\|^* \leq \frac{2}{n}$ for $m \geq m_n$; hence $\mu^*(f_n - g) \to 0$, since μ is an equivalent norm. By Šmulyan's criterion (Corollary 1.2.3), μ is Fréchet differentiable at x_k.

In particular, if X is separable, then by Kadets' theorem ([K$_1$]; see Corollary 4.2.2(a)) X admits an LUR norm and thus by [FZZ, Theorem 1] the set of LUR norms is residual in P. □

4.3 Fréchet Smooth Norms on Spaces of Dimension $\aleph_0$

The main result of this chapter is the following theorem. As in the proof of Proposition 4.2.3, we will use a dual argument to obtain points of Fréchet differentiability. This time, since the span of a countable set is uncountable, the Baire category argument as used before does not work. To overcome this, we take advantage of a result of Johnson *et al.* ([JRZ, Lemma 4.2]) that provides us with a nice collection of norm one finite rank projections which still have norm one under certain renormings. As will be seen in the proof, the key is that these projections allow us to maintain control on the support functionals of the union of their ranges.

4.3.1 THEOREM. *If X is a separable Banach space and L is a subspace of dimension $\aleph_0$, then X admits an equivalent LUR norm which is Fréchet differentiable on $L\backslash\{0\}$. In particular, any normed linear space of dimension $\aleph_0$ admits a Fréchet differentiable norm.*

Proof. Let $\{x_k\}_{k=1}^{\infty}$ be an algebraic basis of L. By the Banach-Mazur theorem X embeds isometrically into $C[0,1]$. Thus considering $\text{span}(\{x_k : k \in \mathbb{N}\}) = L$ as a subspace of $C[0,1]$, we will show that there is an equivalent LUR norm on $C[0,1]$ which is Fréchet differentiable at all points of $L\backslash\{0\}$. First let $\{x_k\}_{k=1}^{\infty} \subset D$ where D is a countable dense subset of $C[0,1]$, and let $\{b_k\}_{k=1}^{\infty}$ be a countable algebraic basis of $\text{span}(D)$. Fix a Schauder basis $\{e_k\}_{k=1}^{\infty}$ of $C[0,1]$ and let $K = \sup\{\|P_n\| : n \in \mathbb{N}\}$ where $\|\cdot\|$ is the supremum norm on $C[0,1]$ and P_n is the natural projection on the basis $\{e_k\}_{k=1}^{\infty}$ for each n. For the rest of the proof, let $E = C[0,1]$. We now inductively choose finite dimensional subspaces $E_1 \subset E_2 \subset E_3 \subset \dots$ and projections $Q_k : E \to E_k$ as follows. First assume that $E_0 = \{0\}$ and $Q_0 : E \to \{0\}$. Let $Y = \overline{\text{span}}(\{e_k^* : k \in \mathbb{N}\})$ where $\{e_k^*\}_{k=1}^{\infty}$ are the biorthogonal functionals to $\{e_k\}_{k=1}^{\infty}$. Suppose that E_{k-1} and Q_{k-1} have been chosen and $Q_{k-1}^* E^* \subset Y$. By Lemma 4.2 of [JRZ], there is a projection Q_k such that $\text{span}(E_{k-1} \cup \{b_k\}) \subset Q_k E$,

$\|Q_k\| \le 4K + 4K^2$, $Q_{k-1}^* E^* \subset Q_k^* E^* \subset Y$ and $Q_k E$ is $\frac{1}{k}$-close to $P_{n(k)}E$ for some $n(k)$ (that is, there exists an invertible operator T_k mapping $P_{n(k)}E$ onto $Q_k E$ such that $\|T_k x - x\| \le \frac{1}{k}\|x\|$ for all $x \in P_{n(k)}E$). Now set $E_k = Q_k E$. This completes the inductive procedure.

Choose an algebraic basis $\{v_k\}_{k=1}^\infty$ of $\bigcup_{k=1}^\infty E_k$ as follows. Fix a basis of E_1 and denote it by $\{v_1, \dots, v_{n(1)}\}$. Supposing $\{v_1, \dots, v_{n(k)}\}$ a basis of E_k has been chosen, extend it to a basis $\{v_1, \dots, v_{n(k)}, y_{n(k)+1}, \dots, y_{n(k+1)}\}$ of E_{k+1}. For $n(k)+1 \le i \le n(k+1)$ choose $v_i \in \ker Q_k$ such that $y_i = v_i + w_i$ for some $w_i \in E_k$. We may assume that $\{v_i\}$ is normalized by replacing v_i with $\|v_i\|^{-1} v_i$ if necessary. From the above process one has $\{b_i\}_{i=1}^k \subset E_k$ and $\|Q_k\| \le 4K + 4K^2$ for all $k \in \mathbb{N}$. Also $Q_j Q_k = Q_k Q_j = Q_{\min(j,k)}$ for all $j, k \in \mathbb{N}$, since $Q_k^* E^* \subset Q_{k+1}^* E^*$ and thus $\ker Q_{k+1} \subset \ker Q_k$. Whence $Q_k(v_i) = 0$ if $v_i \notin E_k$. Letting $T_k : P_{n(k)}E \to E_k$ satisfy $\|T_k x - x\| \le \frac{1}{k}\|x\|$ for all $x \in P_{n(k)}E$, we have

$$\begin{aligned}\|P_{n(k)}x - Q_k(P_{n(k)}x)\| &= \|P_{n(k)}x - Q_k\big(P_{n(k)}x - T_k(P_{n(k)}x)\big) - T_k(P_{n(k)}x)\| \\ &\le \|P_{n(k)}x - T_k(P_{n(k)}x)\| + \|Q_k\big(P_{n(k)}x - T_k(P_{n(k)}x)\big)\| \\ &\le \frac{\|P_{n(k)}x\|}{k}[1 + 4K + 4K^2] \quad \le \quad \frac{K\,[\text{illegible}] + 4K^3}{k}.\end{aligned}$$

Consequently $\lim_{k\to\infty} \|Q_k x - x\| = 0$ for all $x \in E$, since $\lim_{k\to\infty} \|P_{n(k)}x - x\| = 0$ for all $x \in E$. Thus, defining $|||x||| = \sup\{\|Q_k x\| : k \in \mathbb{N}\}$ we see that $\|x\| \le |||x||| \le (4K + 4K^2)\|x\|$ for each $x \in E$. Therefore $|||\cdot|||$ is an equivalent norm on E for which $|||Q_k||| = 1$ for each $k \in \mathbb{N}$. One can now define an equivalent norm $|\cdot|^*$ on E^* by

$$|f|^* = \Big[(|||f|||^*)^2 + \sum_{k=1}^\infty 2^{-k}(|||f - Q_k^* f|||^*)^2 + \sum_{k=1}^\infty 2^{-k} f^2(v_k)\Big]^{\frac{1}{2}}.$$

Since all the terms involved in the definition of $|\cdot|^*$ are w^*-lower semicontinuous, it is easy to see that $|\cdot|^*$ is an equivalent dual norm on E^*.

Notice that $|||v_k||| = 1$, $|||Q_k||| = 1$ and since $Q_k X = \operatorname{span}(\{v_j : 1 \le j \le n(k)\})$, it follows that we can write $Q_k^* f = \sum_{j=1}^{n(k)} v_j(f) f_j$ where $f_j \in X^*$ and $v_j \in X \subset X^{**}$. Thus, by Lemma 4.2.1, $|\cdot|^*$ is LUR at all points of $H = \overline{\bigcup_{k=1}^\infty Q_k^* E^*}$.

Moreover, because $Q_kQ_j = Q_jQ_k$ and $Q_kv_i = v_i$ or $Q_kv_i = 0$, the following is satisfied for each n.

$$\begin{aligned}|Q_n^*f|^* &= [(|||Q_n^*f|||^*)^2 + \sum_{k=1}^{\infty} 2^{-k}(|||Q_n^*f - Q_k^*Q_n^*f|||^*)^2 + \sum_{k=1}^{\infty} 2^{-k}f^2(Q_nv_k)]^{\frac{1}{2}} \\ &\leq [(|||Q_n^*f|||^*)^2 + \sum_{k=1}^{\infty} 2^{-k}(|||Q_n^*|||^*|||f - Q_k^*f|||^*)^2 \ + \sum_{k=1}^{\infty} 2^{-k}f^2(Q_nv_k)]^{\frac{1}{2}} \\ &\leq |f|^*.\end{aligned}$$

Thus $|Q_n| \leq 1$ for all n where $|\cdot|$ is the predual norm on E to $|\cdot|^*$.

Let $x \in E_k\backslash\{0\}$ for some k and choose $f \in E^*$ which satisfies $|f|^* = 1$ and $f(x) = |x|$. Now $|Q_k^*f| \leq 1$ while $Q_k^*f(x) = f(Q_kx) = f(x) = |x|$. By Šmulyan's criterion (see Corollary 1.2.3), $|\cdot|$ is Fréchet differentiable at x, since $|\cdot|^*$ is LUR at its support functional $Q_k^*f \in H$. That is, $|\cdot|$ is Fréchet differentiable on $L \subset \bigcup_{k=1}^{\infty} E_k$.

Now let $|\cdot|$ be any norm on E such that $|Q_k| = 1$ for all $k \in \mathbb{N}$. Let $\{f_k\}_{k=1}^{\infty}$ be the biorthogonal functionals to $\{v_k\}_{k=1}^{\infty}$. That is, for each k, by the Hahn-Banach theorem select a continuous functional $\tilde{f}_k$ satisfying $\tilde{f}_k(v_i) = 0$ for $i \neq k$, $i \leq n(k)$ and $\tilde{f}_k(v_k) = 1$, then set $f_k = Q_k^*\tilde{f}_k$. Now define $\|\cdot\|_1$ on E by

$$\|x\|_1 = [|x|^2 + \sum_{k=1}^{\infty} 2^{-k}|x - Q_kx|^2 + \sum_{k=1}^{\infty} \frac{2^{-k}}{(|f_k|^*)^2} f_k^2(x)]^{\frac{1}{2}}.$$

It follows from Lemma 4.2.1 that $\|\cdot\|_1$ is LUR. Moreover, since $Q_jQ_k = Q_kQ_j$ and $Q_k^*f_j = f_j$ or $Q_k^*f_j = 0$, as above one has $\|Q_k\|_1 = 1$ for all k.

Because $\|Q_k\|_1 = 1$ for all $k \in \mathbb{N}$, it follows as above that $|\cdot|_n$ defined on E by

$$|f|_n^* = [(\|f\|_1^*)^2 + \frac{1}{n}\sum_{k=1}^{\infty} 2^{-k}(\|f - Q_k^*f\|_1^*)^2 + \frac{1}{n}\sum_{k=1}^{\infty} 2^{-k}f^2(v_k)]^{\frac{1}{2}}$$

is Fréchet differentiable at each point of $L\backslash\{0\}$. Furthermore, $|\cdot|_n \to \|\cdot\|_1$. Using this and the fact that $\|\cdot\|_1$ is LUR it follows as in [JZ$_4$] that $\|\cdot\|_X$ defined on X by

$$\|x\|_X = (\sum_{n=1}^{\infty} 2^{-n}|x|_n^2)^{\frac{1}{2}}$$

is LUR and is Fréchet differentiable at each point of $L\backslash\{0\}$. □

Because there are spaces with C^1-smooth norms whose duals do not admit dual LUR norms (e.g. $C[0,\omega_1]$; [Ta]), the natural question is whether the dual arguments used in the proofs of Proposition 4.2.3 and Theorem 4.3.1 are necessary. However, since we obtain norms that are both LUR and Fréchet smooth at a fixed set of points, the following remark somewhat justifies our construction.

4.3.2 REMARK. Let $x \in X$ and $\|x\| = 1$. If x is a strongly exposed point of B_X (in particular if $\|\cdot\|$ is LUR at x) and if $\|\cdot\|$ is Fréchet differentiable at x, then $\|\cdot\|^*$ is LUR at $\Lambda \in S_{X^*}$ where $\Lambda(x) = \|x\|$. To see this, suppose that $\Lambda_n \in B_{X^*}$ and $\|\Lambda_n + \Lambda\|^* \to 2$. Choosing $x_n \in B_X$ so that $(\Lambda_n + \Lambda)(x_n) \to 2$, we have $\Lambda(x_n) \to 1$. Consequently $\|x - x_n\| \to 0$, since Λ is the strongly exposing functional of x. Combining this with the fact that $\Lambda_n(x_n) \to 1$, shows that $\Lambda_n(x) \to 1$. It now follows from Šmulyan's criterion (see Corollary 1.2.3) that $\|\Lambda_n - \Lambda\|^* \to 0$. That is, $\|\cdot\|^*$ is LUR at Λ.

It is also worth noting that we don't know if the set of norms satisyfing the conclusion of Theorem 4.3.1 are residual or even dense among all norms on a separable Banach space. It is clear that the construction we used cannot be done densely. Notice that it is also unknown if the C^1-smooth norms on $C[0,\omega_1]$ are dense.

The following corollary shows a relationship between Theorem 4.3.1 and the failure of the Bishop-Phelps theorem in noncomplete spaces. Recall that the Bishop-Phelps theorem asserts that the support functionals to the unit ball B_X of a Banach space are dense in X^* (see e.g. [Ph, Theorem 3.20]).

4.3.3 COROLLARY. *If L is a normed linear space of countable dimension with nonseparable dual space L^*, then L can be renormed so that the support functionals of B_L are not dense in L^*.*

Proof. A careful examination of the proof of Theorem 4.3.1 shows that with the norm constructed there the support functionals of B_L are in a separable subspace of L^*. However, we can also derive this from the statement of Theorem 4.3.1.

Let $X = \widetilde{L}$ be the completion of L. Using Theorem 4.3.1, let $\|\cdot\|$ be a norm which is Fréchet differentiable at each point of S_L. Let $D = \{d_n\}_{n=1}^{\infty}$ be dense in S_L and fix $\Lambda_n \in S_{L^*}$ such that $\Lambda_n(d_n) = 1$. For $\Lambda \in S_{L^*}$ such that Λ supports B_L, choose $x \in B_L$ such that $\Lambda(x) = 1$. Let $\{d_{n(k)}\} \subset D$ be such that $d_{n(k)} \to x$. Then $\Lambda_{n(k)}(x) \to 1$ which means, by Šmulyan's criterion (see Corollary 1.2.3), that $\|\Lambda_{n(k)} - \Lambda\|^* \to 0$. Therefore the support functionals to B_L are in the separable subspace $\overline{\text{span}}(\{\Lambda_n : d_n \in D\})$. □

From the proof Theorem 4.3.1 we also obtain

4.3.4 PROPOSITION. *Let X be a Banach space with Schauder basis $\{e_k\}_{k=1}^{\infty}$ and biorthogonal functionals $\{e_k^*\}_{k=1}^{\infty}$. Then there is an equivalent norm $|\cdot|$ on X such that:*

(a) $\{e_k\}_{k=1}^{\infty}$ is monotone with respect to $|\cdot|$;

(b) $|\cdot|$ is LUR;

(c) $|\cdot|$ is Fréchet differentiable on span($\{e_k : k \in \mathbb{N}\}$)$\backslash\{0\}$;

(d) $\lim_\nu |x - x_\nu| = 0$ whenever $\lim_\nu |x_\nu| = |x|$ and $\lim_\nu e_k^*(x_\nu) = e_k^*(x)$ for all $k \in \mathbb{N}$.

Proof. Mimicking the proof of Theorem 4.3.1, let $v_n = e_n$ and $Q_n = P_n$, where P_n is the natural projection on the basis $\{e_k\}_{k=1}^{\infty}$. The existence of $|||\cdot|||$ satisfying $|||P_n||| \leq 1$ for all n (see e.g. [LT]) and the proof of Theorem 4.3.1 show that X admits an equivalent norm $|\cdot|$ satisfying (a), (b) and (c). To see (d), check that $|x + x_\nu| \geq |P_n(x + x_\nu)|$ for all n, ν, $\lim_\nu P_n(x + x_\nu) = 2P_n x$ for each n and $\lim_{n\to\infty} P_n x = x$. Therefore, under the assumptions of (d), one has $\lim_\nu |x + x_\nu| = 2|x|$. From (b) we conclude, $\lim_\nu |x - x_\nu| = 0$. □

In [K$_2$] Kadets constructed a norm on a space with Schauder basis satisfying (a), (b) and (d) to prove that all infinite dimensional separable Banach spaces are homeomorphic.

4.3.5 REMARK. Let $\{e_k\}_{k=1}^{\infty}$ be a Schauder basis of X with biorthogonal functionals $\{e_k^*\}_{k=1}^{\infty}$. If $\{e_k\}_{k=1}^{\infty}$ is monotone with respect to $\|\cdot\|$ and $\|\cdot\|$ is Fréchet differentiable on $X\backslash\{0\}$, then $\{e_k^*\}_{k=1}^{\infty}$ forms a basis for X^*. To prove this, let $x \in X\backslash\{0\}$ and choose $f \in X^*$ a support functional to x. Now $\|P_n^* f\|^* \leq \|f\|^*$ and $\lim_{n\to\infty} P_n^* f(x) = f(x)$, hence by Šmulyan's criterion (see Corollary 1.2.3) one has $\lim_{n\to\infty} \|P_n^* f - f\|^* = 0$. By the Bishop-Phelps theorem $\lim_{n\to\infty} \|P_n^* \Lambda - \Lambda\|^* = 0$ for all $\Lambda \in X^*$. Therefore $\{e_k^*\}_{k=1}^{\infty}$ is a basis for X^*, since it is always a weak-star basis. In particular for X^* separable such that $\{e_k^*\}_{k=1}^{\infty}$ is not a basis of X^* (for example $X = J^*$ the dual of the original James space; see [LT, p. 25]) there is no Fréchet differentiable norm $|||\cdot|||$ for which $\{e_k\}_{k=1}^{\infty}$ is monotone although the set of Fréchet differentiable norms on X is residual (because the dual LUR norms on X^* are residual; use [FZZ, Theorem 2] with e.g. Corollary 4.2.2(b)).

Chapter Five

BANACH SPACES WHICH ADMIT MARKUSHEVICH BASES

5.1 Introduction

Thus far, most of the results proved in this thesis have dealt with smoothness. However, in this chapter we will focus on some topological properties in Banach spaces. Whereas Chapter Four studied some connections between Schauder bases and smoothness, this chapter looks at some relationships between Markushevich bases and notions related to Corson compacta. Moreover, the techniques used in this chapter are similar to those in Chapter Four in that many of the arguments are accomplished by working on algebraic spans of a given set of elements in a Banach space.

The main goal of this chapter is to give a characterization of Banach spaces which admit M-bases and to give conditions under which M-bases can be extended. This will be done in Section 5.2.

In this section we list the definitions of some types of compacta and indicate their relationships to M-bases in Banach spaces.

5.1.1 DEFINITION. A subspace $Y \subset X^*$ is called *λ-norming* if $\sup\{f(x) : f \in Y \cap B_{X^*}\} \geq \lambda\|x\|$; if the above holds for some $\lambda > 0$, then we may refer to Y as *norming*.

Suppose that Y is a norming subspace of X^*. Defining $|||x||| = \sup\{f(x) : f \in Y \cap B_{X^*}\}$ we see that $|||\cdot|||$ is an equivalent norm on X and that Y is 1-norming on $(X, |||\cdot|||)$.

5.1.2 DEFINITION. A system $\{x_i, f_i\}_{i\in I} \subset X \times X^*$ is called a ***Markushevich basis*** (M-basis) if $f_i(x_j) = \delta_{ij}$ (the Kronecker delta), $\overline{\text{span}}(\{x_i : i \in I\}) = X$ and $\overline{\text{span}}^{w^*}(\{f_i : i \in I\}) = X^*$. An M-basis $\{x_i, f_i\}_{i\in I}$ is said to be ***countably norming*** (*countably λ-norming*) if there is a norming (λ-norming) subspace $Y \subset X^*$ such that $|\{i : f(x_i) \neq 0\}| \leq \aleph_0$ for each $f \in Y$.

A topological concept that has been extensively studied in recent years is that of a Corson compact space.

5.1.3 DEFINITION. A compact topological space is said to be *Corson compact* if it is homeomorphic to a subset K of a cube $[0,1]^\Gamma$ such that each element of K has at most countably many nonzero coordinates.

If X is a WCD Banach space, then (B_X, w^*) is Corson compact (see e.g. [DGZ$_3$, Chapter VI]). The following linearized version of (B_X, w^*) being Corson compact was introduced and studied by Argyros and Mercourakis in [AM].

5.1.4 DEFINITION. A Banach space X is said to be *weakly Lindelöf determined* (WLD) if there is a one-to-one bounded linear operator $T : X^* \to l_\infty^c(\Gamma)$ which is w^* to pointwise continuous where $l_\infty^c(\Gamma)$ denotes the subspace of countably supported elements in $l_\infty(\Gamma)$.

The last more or less standard definition that we introduce is that of a projectional resolution of identity. Note that projectional resolutions of identity are a very useful tool for proving the existence of nice norms on certain classes of spaces by transfinite induction (see e.g. [DGZ$_3$, Chapter VII]), thus a lot of work has gone into showing various spaces admit projectional resolutions of identity.

5.1.5 DEFINITION. A *projectional resolution of identity* (PRI) is a "long sequence" of projections $P_\beta : X \to X$, $\|P_\beta\| \leq 1$ for $\omega_0 \leq \beta \leq \mu$ where μ is the first ordinal of cardinality dens(X) satisfying $P_\beta P_\gamma = P_\gamma P_\beta = P_{\min(\gamma,\beta)}$, dens$(P_\beta X) \leq |\beta|$, $P_\beta X = \overline{\cup\{P_{\gamma+1} X : \gamma < \beta\}}$, and $P_\mu = I$.

The following known theorem is the starting point for the work we will do in the next section. This is because it provides a relationship between M-bases and spaces whose duals have nice injections into some $l_\infty(\Gamma)$. For the reader's convenience we have outlined a proof of this theorem which is based on PRI's and is indicative of the type of arguments we will use in the next section.

5.1.6 THEOREM. *For a Banach space X the following are equivalent.*

(a) X is WLD.

(b) (B_{X^*}, w^*) is Corson compact.

(c) X has an M-basis. Moreover every M-basis $\{x_\gamma, f_\gamma\}_{\gamma\in\Gamma}$ satisfies that $|\{\gamma : f(x_\gamma) \neq 0\}| \leq \aleph_0$ for each $f \in X^*$.

Sketch of Proof. (a) $\Rightarrow$ (b): This follows immediately from the definitions.

(b) $\Rightarrow$ (c): By [V₁, Theorem 1] it follows that X admits a PRI say $\{P_\alpha\}$. We may assume $P_{\omega_0} = 0$ and let $X_\alpha = (P_{\alpha+1} - P_\alpha)X$. Since $B_{X_\alpha^*}$ is isomorphic to $(P^*_{\alpha+1} - P^*_\alpha)B_{X^*} \subset B_{X^*}$, it follows that $(B_{X_\alpha^*}, w^*)$ is Corson compact. Using a transfinite induction argument on the density character of X, starting with the fact that separable spaces have M-bases ([LT, Proposition 1.f.3]), one can show that X has an M-basis provided (B_{X^*}, w^*) is Corson compact.

Now let $\{x_\gamma, f_\gamma\}_{\gamma\in\Gamma}$ be an M-basis of X. We argue as in [Pl₁] to show that $|\{\gamma : f(x_\gamma) \neq 0\}| \leq \aleph_0$ for each $f \in X^*$. Indeed, let $F = \{f \in X^* : |\{\gamma : f(x_\gamma) \neq 0\}| \leq \aleph_0\}$. Because Corson compact spaces are angelic (sequential closure of subsets coincides with closure), it follows that $F \cap B_{X^*}$ is w^*-closed. Therefore F is w^*-closed and thus $F = X^*$.

(c) $\Rightarrow$ (a): The map $T : X^* \to l^c_\infty(\Gamma)$ defined by $Tf = \{f(x_\gamma)\}_{\gamma\in\Gamma}$ is w^* to pointwise continuous. □

For further results related to Theorem 5.1.6 and alternative methods of proving it, one can consult: [C], [OSV], [Po], and [V₃]. In fact, using different arguments, the equivalence of (a) and (b) is shown in [OSV, Proposition 4.1] while the equivalence of (b) and (c) follows from in [V3, Theorem 2 and Corollary 3.1]

5.2 Spaces With and Extensions of Markushevich Bases

As was just mentioned in Theorem 5.1.6, WLD spaces can be characterized in terms of M-bases on which continuous linear functionals are countably supported.

What we will do now, is weaken the definition of a WLD space and use this definition to characterize spaces that admit M-bases. To obtain this characterization, we will use an elegant method of Plichko ([Pl_1, Theorem 1]) and in the process obtain some new results on the extension of M-bases.

5.2.1 DEFINITION. Suppose there is a bounded linear one-to-one operator $T : X^* \to l_\infty(\Gamma)$ which is w^* to pointwise continuous such that $TY \subset l^c_\infty(\Gamma)$ for a certain subspace Y of X^*. If $\overline{Y}^{w^*} = X$ (respectively Y is norming) we say that X has the ***total property*** (respectively ***norming property***).

We will abbreviate the above properties as TP and NP respectively. The following implications are clear: X is WLD $\Rightarrow$ X has the NP $\Rightarrow$ X has the TP. However, the reverse implications do not hold as will be seen in the next example.

5.2.2 EXAMPLE. The space $l_1(\Gamma)$ has the NP, since $c_0(\Gamma) \subset l_\infty(\Gamma)$ is norming. However, $l_1(\Gamma)$ is not WLD, since its dual unit ball is not Corson compact.

(b) There is a Banach space X which does not have a locally uniformly rotund norm such that X has an M-basis and a complemented subspace which has no M-basis ([Pl_2, Theorem 3]). Therefore, from [V_2] it follows that (B_{X^*}, w^*), for any equivalent norm on X, is not homeomorphic to a subset K of a cube $[0,1]^\Gamma$ such that the elements in K which have at most countably many nonzero coordinates are dense in K. In particular, X does not have the NP. However, it is easy to see that any space with an M-basis has the TP (see Theorem 5.2.3), thus X has the TP. Note that the TP is not hereditary, since the complemented subspace of X without an M-basis does not have the TP by Theorem 5.2.3.

In the paper [V_3], Valdivia showed, among other things, that X has the NP if and only if X has a countably norming M-basis. We now give a similar characterization of the TP.

5.2.3 THEOREM. *For a Banach space X, the following are equivalent.*

(a) X admits an M-basis.

(b) X has the TP.

(c) There is a subset $\{x_\alpha\}_{\alpha\in A}$ of X and a subspace $Y \subset X^*$ such that $\overline{Y}^{w^*} = X^*$, $\overline{span}(\{x_\alpha : \alpha \in A\}) = X$ and $|\{\alpha : f(x_\alpha) \neq 0\}| \leq \aleph_0$ for each $f \in Y$.

Before proving Theorem 5.2.3, we need to make some preliminary observations.

5.2.4 REMARK. Given $\{x_\gamma\}_{\gamma\in\Gamma} \subset X$, if there is norming subspace Y of X^* and $|\{\gamma : f(x_\gamma) \neq 0\}| \leq \aleph_0$ for each $f \in Y$, then $|\Gamma| \leq \text{dens}(X)$. Indeed let S be dense in X and Y be λ-norming. For each $s \in S$, choose $f_s \in B_{X^*} \cap Y$ such that $f_s(s) \geq \frac{\lambda}{2}\|s\|$. The result follows because the map $s \to \{\gamma : f_s(x_\gamma) \neq 0\}$ is countably valued and onto Γ.

From the above remark, and the proof of [Pl_1, Theorem 1] one easily obtains the following result.

5.2.5 THEOREM (Pl_1). *Let $\{x_i\}_{i\in I} \subset X$ and $Y \subset X^*$ be 1-norming. If $|\{i : f(x_i) \neq 0\}| \leq \aleph_0$ for each $f \in Y$ and $\overline{span}(\{x_i : i \in I\}) = X$, then there is a PRI $\{P_\beta\}_{\omega_0\leq\beta\leq\mu}$ such that for each β, $P_\beta X = \overline{span}(\{x_i : i \in J_\beta \subset I\})$, $|J_\beta| \leq |\beta|$ and $kerP_\beta = \overline{span}(\{x_i : i \in I\backslash J_\beta\})$.*

After a preliminary version of this chapter was written, we were informed that Valdivia has already obtained Proposition 5.2.6 ([V_4, Theorem 2]). However, our proof based on Plichko's techniques is different and useful for our extension results.

5.2.6 PROPOSITION. *Let X be a Banach space and Y a norming subspace of X^*. If $\{x_\alpha\}_{\alpha\in A}$ is such that $\overline{span}(\{x_\alpha : \alpha \in A\}) = X$ and $|\{\alpha : f(x_\alpha) \neq 0\}| \leq \aleph_0$ for each $f \in Y$, then X admits an M-basis $\{e_\gamma, f_\gamma\}_{\gamma\in\Gamma}$ such that span$(\{e_\gamma : \gamma \in \Gamma\}) =$ span$(\{x_\alpha : \alpha \in A\})$ and $|\{\gamma : f(e_\gamma) \neq 0\}| \leq \aleph_0$ for each $f \in Y$.*

Proof. We may assume without loss of generality that Y is 1-norming; see comment following Definition 5.1.1. If X is finite dimensional then this is trivial.

In the case X is separable, this result follows from the proof of Proposition 1.f.3 in [LT], because Remark 5.2.4 guarantees that $|A| \leq \aleph_0$.

Suppose that $\text{dens}(X) = |\mu|$ and that the result holds for all E with $\text{dens}(E) < |\mu|$ where μ is an ordinal. By Theorem 5.2.5, there is a PRI $\{P_\beta\}_{\omega_0 \leq \beta \leq \mu}$ such that $P_\beta X = \overline{\text{span}}(\{x_\alpha : \alpha \in J_\beta \subset A\})$ and $\ker P_\beta = \overline{\text{span}}(\{x_\alpha : \alpha \in A \backslash J_\beta\})$ where $|J_\beta| \leq |\beta|$. We may assume that $P_{\omega_0} = 0$. We set $X_\beta = (P_{\beta+1} - P_\beta)X$ for $\omega_0 \leq \beta < \mu$. Thus $X_\beta = \overline{\text{span}}(\{x_i : i \in J_{\beta+1} \backslash J_\beta\})$. Let $Y_\beta = \{f|_{X_\beta} : f \in Y\}$. Now $\{x_i : i \in J_{\beta+1} \backslash J_\beta\}$ and Y_β satisfy the induction hypothesis on X_β, so there is an M-basis $\{e_{i,\beta}, f_{i,\beta}\}$ of X_β such that

$$\text{span}(\{e_{i,\beta}\}) = \text{span}(\{x_i : i \in J_{\beta+1} \backslash J_\beta\})$$

and $|\{i : f(e_{i,\beta}) \neq 0\}| \leq \aleph_0$ for each $f \in Y_\beta$. Define

$$\{e_\gamma, f_\gamma\}_{\gamma \in \Gamma} = \bigcup \{e_{i,\beta}, (P^*_{\beta+1} - P^*_\beta) f_{i,\beta} : \omega_0 \leq \beta < \mu\}.$$

This is an M-basis on X. If $f \in Y$, then $f(x_\alpha) = 0$ except for countably many α; hence $f|_{X_\beta} = 0$ for all but countably many β. But for each such β, f is countably supported on $\{e_{i,\beta}\}$. Therefore $|\{\gamma : f(e_\gamma) \neq 0\}| \leq \aleph_0$ for each $f \in Y$. □

Proof of Theorem 5.2.3. (a) $\Rightarrow$ (b): Let $T : X^* \to l_\infty(\Gamma)$ be defined by $Tf = \{f(x_\gamma)\}_{\gamma \in \Gamma}$ where $\{x_\gamma, f_\gamma\}_{\gamma \in \Gamma}$ is an M-basis of X. We see that this T with $Y = \overline{\text{span}}(\{f_\gamma : \gamma \in \Gamma\})$ satisfy the conditions in the definition of the TP.

(b) $\Rightarrow$ (c): Let T and Y be as in the defintion of the TP, and let $e_\gamma = \pi_\gamma \circ T$ where π_γ is the projection of $l_\infty(\Gamma)$ onto its γ-th coordinate. Clearly $e_\gamma \in X$, $\overline{\text{span}}(\{e_\gamma : \gamma \in \Gamma\}) = X$ (by the Hahn-Banach theorem, since T is one-to-one) and $|\{\gamma : f(e_\gamma) \neq 0\}| \leq \aleph_0$ for each $f \in Y$.

(c) $\Rightarrow$ (a): Define a new norm $|\cdot|$ on X by

$$|x| = \sup\{f(x) : f \in Y \cap B_{X^*}\}.$$

Let $(\widetilde{X}, |\cdot|)$ be the completion of $(X, |\cdot|)$ and $\widetilde{Y}$ be all Hahn-Banach extensions of Y on $\widetilde{X}$. Observe that $\overline{\text{span}}^{|\cdot|}(\{x_\alpha : \alpha \in A\}) = \widetilde{X}$ and $\widetilde{Y}$ is 1-norming on $(\widetilde{X}, |\cdot|)$. Therefore $(\widetilde{X}, |\cdot|)$ has the NP. By Proposition 5.2.6 there is a countably 1-norming M-basis $\{z_i, f_i\}_{i\in I}$ of $(\widetilde{X}, |\cdot|)$ such that $\text{span}(\{z_i : i \in I\}) = \text{span}(\{x_\alpha : \alpha \in A\})$. Since $|\cdot| \leq \|\cdot\|$ where $\|\cdot\|$ is the original norm on X it follows that $\hat{f}_i = f_i|X \in (X, \|\cdot\|)^*$. Therefore $\{z_i, \hat{f}_i\}$ is an M-basis of $(X, \|\cdot\|)$. □

From Proposition 5.2.6 and Theorem 5.2.3 we obtain:

5.2.7 Corollary. *(a) If Z has a countably norming M-basis and X/Z is separable, then X admits a countably norming M-basis.*

(b) If Z has an M-basis and X/Z is separable, then X has an M-basis.

Proof. To prove (a), let $\{z_\alpha, f_\alpha\}_{\alpha\in A}$ be an M-basis of Z and $Y \subset Z^*$ be λ-norming on Z. Choose $\{\hat{x}_n\}_{n=1}^\infty$ dense in X/Z. For each n, let $x_n \in \hat{x}_n$. Set $\widetilde{Y} = \{f \in X^* : f|_Z \in Y\}$. We will show that $\widetilde{Y}$ is $\frac{\lambda}{4}$-norming on X. Let $x \in X$, if $\rho(x, Z) < \frac{\lambda}{4}$ where $\rho(x, Z) = \inf\{\|x - z\| : z \in Z\}$, choose $z \in Z$ such that $\|x - z\| \leq \frac{\lambda}{4}$ and $f \in \widetilde{Y} \cap B_{X^*}$ such that $f(z) \geq \frac{3\lambda}{4}$. Certainly $f(x) \geq \frac{\lambda}{2}$. On the other hand, if $\rho(x, Z) \geq \frac{\lambda}{4}$, then there is an $f \in B_{X^*}$ such that $f(x) = \frac{\lambda}{4}$ and $f(z) = 0$ for all $z \in Z$ and hence $f \in \widetilde{Y}$. This shows that $\widetilde{Y}$ is $\frac{\lambda}{4}$-norming on X and each of its elements is countably supported on $\{z_\alpha\}_{\alpha\in A} \cup \{x_n\}_n^\infty$. Therefore it follows from Proposition 5.2.6 that X admits a countably norming M-basis. Notice that (b) is proved similarly using Theorem 5.2.3. □

Under the stronger assumption that (B_{Z^*}, w^*) is Corson compact, it is quite easy to prove the statement analogous to Corollary 5.2.7.

5.2.8 Remark. If X/Z is separable and (B_{Z^*}, w^*) is Corson compact, then it is easy to see that (B_{X^*}, w^*) is Corson compact. Simply take $d_n \in \hat{d}_n$, $d_n \in B_X$ where $\{\hat{d}_n\}_{n=1}^\infty$ is dense in $\frac{1}{2}B_{X/Z}$ and $\{z_i, f_i\}_{i\in I}$ an M-basis of Z with $z_i \in B_X$ for each i and define $T : B_{X^*} \to [-1, 1]^\Gamma$ by $Tf = \{f(x_\gamma)\}_{\gamma\in\Gamma}$ where $\{x_\gamma\}_{\gamma\in\Gamma} = \{z_i\}_{i\in I} \cup$

$\{d_n\}_{n=1}^{\infty}$. Since (B_{Z^*}, w^*) is Corson compact it follows that $|\{i : f(z_i) \neq 0\}| \leq \aleph_0$ for each $f \in X^*$ ([Pl$_1$]; see the proof of Theorem 5.1.6), therefore $|\{\gamma : f(x_\gamma) \neq 0\}| \leq \aleph_0$.

We outline how the methods used in the proof of Theorem 5.2.3 can be used to obtain

5.2.9 PROPOSITION. *For a Banach space $(X, \|\cdot\|)$, the following are equivalent.*

(a) $(X, \|\cdot\|)$ continuously linearly injects into $c_0(\Gamma)$ for some Γ.

(b) There is a subspace $Y \subset X^*$ such that $\overline{Y}^{w^*} = X^*$ and a convex symmetric $K \subset Y$ which is Corson compact in its w^*-topology such that $span(K) = Y$.

Proof. (b) $\Rightarrow$ (a): Consider $(X, |\cdot|)$ where $|x| = \sup\{f(x) : f \in K\}$, and let $(\widetilde{X}, |\cdot|)$ be the completion of $(X, |\cdot|)$. Then $\widetilde{K} = \{\tilde{f} \in (\widetilde{X}, |\cdot|)^* : \tilde{f}|_X \in K\}$ is the unit ball of $(\widetilde{X}, |\cdot|)^*$ and is Corson compact in its w^*-topology since K is. Hence $(\widetilde{X}, |\cdot|)$ has an M-basis and thus there is a continuous linear injection $T : (\widetilde{X}, |\cdot|) \to c_0(\Gamma)$. Certainly T is continuous on $(X, \|\cdot\|)$.

(a) $\Rightarrow$ (b): Let T be a bounded linear injection of X into $c_0(\Gamma)$. Set $K = T^* B_{c_0^*(\Gamma)}$ and $Y = \text{span}(K)$; K is Corson (even Eberlein—see e.g. [Day]) compact in its w^*-topology and $\overline{Y}^{w^*} = X^*$, since T is one-to-one. □

We conclude this thesis with a result concerning the extension of M-bases. Notice that there is a Banach space X with an M-basis and a complemented subspace Z of X such that Z has an M-basis yet no M-basis on Z can be extended to an M-basis on X ([Pl$_3$, Proposition 1]). However, using the techniques of [Pl$_1$], we are able to obtain extensions in quite general situations.

5.2.10 THEOREM. *Let $Z \subset X$, $\{z_j, f_j\}_{j \in J}$ be an M-basis of Z, and $\{x_i\}_{i \in I}$ be such that $\overline{span}(\{x_i : i \in I\}) = X$. If $|\{i : f(x_i) \neq 0\} \cup \{j : f(z_j) \neq 0\}| \leq \aleph_0$ for* *each $f \in Y \subset X^*$ and Y is norming on X, then $\{z_j, f_j\}_{j\in J}$ can be extended to an M-basis of X such that the functionals in Y are countably supported on it.*

Proof. Assume without loss of generality that Y is 1-norming. Using Theorem there is a PRI $\{P_\beta\}_{\omega_0 \le \beta \le \mu}$ on X such that

$$P_\beta X = \overline{\text{span}}(\{\{x_i\} \cup \{z_j\} : i \in I_\beta \in I \ , \ j \in J_\beta \in J\})$$

and

$$\ker P_\beta = \overline{\text{span}}(\{\{x_i\} \cup \{z_j\} : i \in I \backslash I_\beta \ , \ j \in J \backslash J_\beta\})$$

where $|I_\beta \cup J_\beta| \le |\beta|$. Thus $P_\beta z_j = z_j$ or $P_\beta z_j = 0$ for all j and β. In particular, $P_\beta Z \subset Z$. The proof now reduces to a standard transfinite induction argument.

For X separable, the result is always true by [GK]. Suppose that dens$(X) = |\mu|$ and that the result holds for any Banach space E with dens$(E) < |\mu|$. For convenience we assume that $P_{\omega_0} = 0$ and let $X_\beta = (P_{\beta+1} - P_\beta)X$. Letting $J_\beta^1 = \{j : (P_{\beta+1} - P_\beta)z_j = z_j\}$ and $I_\beta^1 = \{i : (P_{\beta+1} - P_\beta)x_i = x_i\}$ we see that

$$X_\beta = \overline{\text{span}}(\{\{z_j\} \cup \{x_i\} \ . \ j \in J_\beta^1 \ , \ i \in I_\beta^1\}).$$

Thus X_β and $Y_\beta = \{f|_{X_\beta} : f \in Y\}$ satisfy the induction hypothesis. Therefore $\{z_j, f_j\}_{j \in J_\beta}$ can be extended to an M-basis, say, $\{z_\gamma^\beta, f_\gamma^\beta\}_{\gamma \in \Gamma_\beta}$ of X_β such that $|\{\gamma : f(z_\gamma^\beta) \ne 0\}| \le \aleph_0$ for each $f \in Y_\beta$. Now let

$$\{e_\alpha, f_\alpha\}_{\alpha \in A} = \bigcup_\beta \{z_\gamma^\beta, (P_{\beta+1}^* - P_\beta^*) f_\gamma^\beta\};$$

as in the proof of Proposition 5.2.6, this is an M-basis of X such that $|\{\alpha : f(e_\alpha) \ne 0\}| \le \aleph_0$ for each $f \in Y$. □

An interesting consequence of Theorem 5.2.10 is

5.2.11 COROLLARY. *If X admits a countably norming M-basis and $Z \subset X$ is such that (B_{Z^*}, w^*) is Corson compact, then any M-basis on Z can be extended to a countably norming M-basis on X.*

Proof. The corollary follows from the preceding theorem and the argument of Plichko ([Pl$_1$]) which shows all the elements of Z^* are countably supported on any M-basis of Z, since (B_{Z^*}, w^*) is Corson compact; see the proof of Theorem 5.1.6.

□

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