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THE EFFECTIVE ACTION AT ZERO AND NON-ZERO TEMPERATURE

by

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research for acceptance, a thesis entitled "The Effective Action at Zero and Non-zero Temperature" submitted by Robert Ivars Griganis in partial fulfilment of the requirements for the degree of Doctor of Philosophy in Theoretical Physics.

Supervisor

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Abstract

The work presented herein is essentially divided into two parts. The first deals with computing the effective action for a scalar field model, to the two loop level. This has been discussed at great length elsewhere. However, the approach used here provides, one hopes, a more coherent picture of the renormalization procedure. Thus its value is perhaps largely pedagogical.

The remainder of the work is concerned with aspects of a formalism, developed by various people over the last twelve years, which admits the incorporation of temperature into quantum field theory. This is achieved in such a way as to allow the calculation of thermodynamical quantities using the Feynman rules and diagrams already familiar to particle physicists. The quantity considered here is the free energy, which is shown to be related to a generalization of the effective potential of Chapter One. Two models are examined; a single real scalar field with ϕ^4 interaction, and a scalar field interacting with a fermion field.

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Chapter 1 An Approach To Calculating the Effective Action

1. Introduction

The effective action has great importance in quantum field theory[1,2]. The functions it generates are intimately related to the scattering amplitudes of the particular theory it represents. Furthermore, it embodies the symmetries of the theory, allowing them to be expressed in a compact way via the Ward-Takahashi identities. The effective potential, which is the first term in the momentum expansion of the effective action, is a useful tool in studying symmetry breaking.

This chapter presents a straightforward method for calculating perturbatively the coefficients of the momentum expansion of the effective action. It should be pointed out that more sophisticated methods are available for these calculations[3-5], but they don't seem to have the generality of the method presented here. For example, in the review article by Iliopoulos et al.[4], the two loop calculation of $Z(\phi)$ requires deft manipulation of operators in a way that is not at all transparent.

To begin with, the formalism is reviewed. Although only a single real scalar field appears explicitly, the generalization to more complicated models is straightforward.

The functions $V(\phi)$ and $Z(\phi)$ are then computed to two loops in a model with a single, real self-interacting scalar field. These are compared with previous calculations, and a discrepancy between the results for $Z(\phi)$ is discussed.

To demonstrate how simply the method lends itself to higher orders in the momentum expansion, the order p^4 functions are computed to one loop.

Finally, it is pointed out that this method provides an efficient way of determining the coefficents of the renormalization group equations.

2. Review of the formalism

The generating functional of renormalized, configuration-space Green's functions of a scalar field theory has the path-integral representation

$$Z[j] = \frac{\int [d\hat{\phi}] \exp i \int d^4x (L(\hat{\phi}) + j\hat{\phi})}{\int [d\hat{\phi}] \exp i \int d^4x \, L(\hat{\phi})} \qquad 1\text{-}1$$

there, $L(\hat{\phi})$ is the renormalized Lagrange density (i.e. with counterterms included). The Green's functions generated by $Z[j]$ are

$$\begin{aligned} G_n(x_1,\ldots,x_n) &= (-i)^n \frac{\delta^n Z}{\delta j(x_1)\ldots\delta j(x_n)} \Big|_{j=0} \\ &= \langle 0|T(\Phi(x_1)\ldots\Phi(x_n)|0\rangle \end{aligned} \qquad 1\text{-}2$$

The latter equality expresses the link between the path-integral and the operator formalisms. $\Phi(x)$ is the scalar field operator, and T is the time-ordering operator. $G_n(x_i)$ is the sum of all amplitudes of n particles created at space-time points x_i. This can be decomposed into a sum of products of connected Green's functions, which are generated by

$$G_c[j] = \ln Z[j] \qquad 1\text{-}3$$

Thus, the n-point connected Green's function is

$$G_{cn}(x_i) = (-i)^n \frac{\delta^n G_c[j]}{\delta j(x_1)\ldots\delta j(x_n)}\Big|_{j=0} \qquad 1\text{-}4$$

The connected n-point Green's function can itself be decomposed, by first factoring out the external legs (two-point functions) and expressing the remainder as a sum of products of proper functions. These are represented by Feynman graphs that cannot be disconnected by the removal of one internal line, hence their alternate designation of "one particle irreducible" (1PI) functions.

The proper functions are generated by the Legendre transform of $G_c[j]$ with respect to the functional variable $\phi(x)$ defined by

$$\phi(x) = -i \frac{\delta G_c[j]}{\delta j(x)} \qquad 1\text{-}5$$

Thus

$$\Gamma[\phi] \equiv i \int d^4x\, j(x)\phi(x) - G_c[j] \qquad 1\text{-}6$$

generates the n-point proper functions

$$\Gamma_n(x_i) = \frac{\delta^n \Gamma[\phi]}{\delta\phi(x_1)\ldots\delta\phi(x_n)}\Big|_{\phi=0} \qquad 1\text{-}7$$

$\Gamma[\phi]$ is the object which is called the effective action. By translation invariance, $\Gamma[\phi]$ has the momentum space expansion

$$\Gamma[\phi] = \sum_{n=0}^{\infty} \frac{1}{n!}\int d^4p_1\ldots d^4p_n (2\pi)^4\delta^{(4)}(\sum_i p_i)\Gamma_n(p_1,\ldots,p_n)\tilde{\phi}(p_1)\ldots\tilde{\phi}(p_n) \qquad 1\text{-}8$$

Here, $\tilde{\phi}(p)$ is the Fourier transform of $\phi(x)$, and $\Gamma_n(p_i)$ is the renormalized n-point proper function in momentum space. This function is calculated perturbatively to a given order (number of loops) by summing all proper n-point Feynman graphs up to that order, each graph being evaluated using the Feynman rules and the counterterms contained in $L(\phi)$. The counterterms depend on what interpretation is given to the field, mass and coupling parameters occurring in $L(\phi)$. These parameters can be fixed by imposing normalization conditions on the functions $\Gamma_n(p_i)$. For example, in a scalar theory with ϕ^4 interaction, there are three renormalizations; mass, coupling constant and wavefunction. So there are three conditions to be imposed. Conventionally, one specifies the values of Γ_2, $\partial\Gamma_2/\partial p^2$ and Γ_4 at some choice of external momenta.

Now, translation invariance also implies that the effective action has the expansion

$$\Gamma[\phi] = -i\int d^4x\{V(\phi) - \frac{1}{2} Z(\phi)(\partial\phi)^2 + O(\partial^4)\} \qquad \text{1-9}$$

This is essentially an expansion about zero momentum, so in calculating the functions on the right hand side, it is convenient to specify normalization conditions at zero momentum. Thus for the ϕ^4 model, we might require

$$\Gamma_2(p=0) = -im_0^2 \qquad \text{1-10a}$$

$$\left.\frac{\partial\Gamma_2(p)}{\partial p^2}\right|_{p^2=0} = i \qquad \text{1-10b}$$

$$\Gamma_4(p_i=0) = -i\lambda \qquad \text{1-10c}$$

These are the tree-level values for a model with Lagrangian

$$L = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2} m_0^2 \phi^2 - \frac{\lambda}{4!}\phi^4 + (\text{counterterms of } O(\hbar)) \qquad \text{1-11}$$

Proceeding to higher orders in $\hbar$ (the number of loops), the normalization conditions 1-10 determine the counterterms uniquely. In four dimensions, these counterterms are of course infinite, but with a suitable regularization scheme, the (in principle measurable) functions $\Gamma_n(p_i)$ are finite and well-defined.

3. The Shifted-field method

The first step in calculating the functions on the right-hand side of equation 1-9 is to observe that the path integral is invariant under a translation of the integrated field; $\hat{\phi}(x) \to \hat{\phi}(x) + a$, where a is a constant. Performing this translation in equation 1-1 gives

$$Z[j] = Z_a[j]\exp[i \int d^4x \; a \; j(x)] \qquad 1\text{-}12$$

where $Z_a[j]$ is the generating functional for a theory with Lagrangian density $L_a(\hat{\phi}) = L(\hat{\phi}+a)$. Equation 1-3 then implies

$$G_c[j] = ia \int d^4x j(x) + G_{ca}[j] \qquad 1\text{-}13$$

where $G_{ca}[j]$ generates connected Green's functions in the shifted theory. Now, defining a field $\phi_a(x)$ by

$$\phi_a(x) \equiv -i \frac{\delta G_{ca}[j]}{\delta j(x)} = \phi(x) - a \qquad 1\text{-}14$$

the effective action in the shifted theory is

$$\Gamma_a[\phi_a] = i\int d^4x \; j(x)\phi_a(x) - G_{ca}[j] = \Gamma[\phi] \qquad 1\text{-}15$$

where use was made of 1-7, 1-13 and 1-14. Thus,

$$\Gamma_a[\phi] = \Gamma[\phi+a] \tag{1-16}$$

Now, the left-hand side is expanded according to equation 1-8, and the right-hand is expanded according to equation 1-9, about $\phi = 0$. In the first case, we have

$$\Gamma_a[\phi] = \sum_{n=0}^{\infty} \frac{1}{n!} \int d^4p_1 \ldots d^4p_n (2\pi)^4 \delta^{(4)}(\sum_1 p_i) \Gamma_{na}(p_j) \tilde{\phi}(p_1) \ldots \tilde{\phi}(p_n) \tag{1-17}$$

where $\Gamma_{na}(p_j)$ is the n-point proper function in momentum space calculated with the Feynman rules and counterterms of the shifted Lagrangian $L(\hat{\phi}+a)$. On the other hand, we have

$$\Gamma[\phi+a] = -i\int d^4x \{ \sum_{n=0}^{\infty} \frac{1}{n!} V^{(n)}(a)\phi^n(x) - \frac{1}{2} \sum_{n=0}^{\infty} \frac{1}{n!} Z^{(n)}(a)\phi^n(x)(\partial\phi)^2 + O(\partial^4)\} \tag{1-18}$$

The order ∂^4 terms will be considered separately later. If we now Fourier transform the fields in 1-18, and perform the x-integration we obtain

$$\Gamma[\phi+a] = -i \sum_{n=0}^{\infty} \frac{1}{n!} V^{(n)}(a) \int d^4p_1 \cdots d^4p_n (2\pi)^4 \delta^{(4)}\left(\sum_i p_i\right)\tilde{\phi}(p_1)\cdots\tilde{\phi}(p_n)$$

$$- i \sum_{n=0}^{\infty} \frac{1}{(n+2)!} Z^{(n)}(a) \int d^4p_1 \cdots d^4p_n (2\pi)^4 \delta^{(4)}\left(\sum_i p_i\right)\tilde{\phi}(p_1)\cdots\tilde{\phi}(p_{n+2}) \sum_{i<j} p_i \cdot p_j$$

$$+ O(p^4) \qquad 1\text{-}19$$

comparing to equation 1-8, we see that

$$\Gamma_{na}(p_1,\ldots,p_n) = -iV^{(n)}(a) - i\, Z^{(n-1)}(a) \sum_{i<j} p_i \cdot p_j + O(p^4) \qquad 1\text{-}20$$

In particular,

$$\Gamma_{0a} = -iV(a) \qquad 1\text{-}21$$

$$\left.\frac{\partial \Gamma_{2a}(p)}{\partial p^2}\right|_{p^2=0} = i\, Z(a) \qquad 1\text{-}22$$

The left side of 1-21 is simply the sum of the vacuum diagrams in the shifted theory. The left side of 1-22 is the coefficient of p^2 in the proper two-point function in the shifted theory.

4. $V(\phi)$, $Z(\phi)$ to two loops

We now turn to the computation of the effective potential $V(\phi)$ and the function $Z(\phi)$, for a real scalar field ϕ with a ϕ^4 interaction. Dimensional regularization is used (for specific notation, see Appendix I). The Lagrangian density in $D(=4-\varepsilon)$ dimensions is

$$L(\phi) = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m_0^2\phi^2 - \mu^\varepsilon\frac{\lambda}{4!}\phi^4 + \frac{1}{2}A(\partial\phi)^2 - \frac{1}{2}Bm_0^2\phi^2 - \mu^\varepsilon C\frac{\lambda}{4!}\phi^4 \tag{1-23}$$

μ is an arbritary mass scale entering at $D \neq 4$ dimensions, used to keep the coupling λ dimensionless. A, B, and C are counterterms of order one loop and higher, determined loop by loop by the normalization conditions 1-10, which, using 1-21 and 1-22, can also be written

$$V''(0) = m_0^2 \tag{1-24a}$$

$$V^{(iv)}(0) = \lambda \tag{1-24b}$$

$$Z(0) = 1 \tag{1-24c}$$

The shifted Lagrangian is

$$L_a(\phi) = L(\phi+a)$$

$$= -\frac{1}{2} m_0^2 a^2 - \mu^\epsilon \frac{\lambda}{4!} a^4 - \frac{1}{2} B m_0^2 a^2 - \mu^\epsilon C \frac{\lambda}{4!} a^4$$

$$-(m_0^2 a^2 + \mu^\epsilon \frac{\lambda}{3!} a^3 + B m_0^2 a^2 + \mu^\epsilon C \frac{\lambda}{3!} a^3)\phi$$

$$+ \frac{1}{2} (\partial\phi)^2 - \frac{1}{2} (m_0^2 a^2 + \mu^\epsilon \frac{\lambda a^2}{2})\phi^2 - \mu^\epsilon \frac{a\lambda}{3!} \phi^3 - \mu^\epsilon \frac{\lambda}{4!} \phi^4$$

$$+ \frac{1}{2} A(\partial\phi)^2 - \frac{1}{2} (B m_0^2 + \mu^\epsilon C \frac{\lambda a^2}{2})\phi^2 - \mu^\epsilon C \frac{a\lambda}{3!} \phi^3 - \mu^\epsilon \frac{C\lambda}{4!} \phi^4 \qquad 1\text{-}25$$

The relevant Feynman rules are listed in Fig. 1-1, with the more compact notation $x = \mu^\epsilon \frac{\lambda a^2}{2m_0^2}$. The tree level effective potential $V^{(0)}(\phi)$ is simply the vacuum "vertex"

$$V^{(0)}(\phi) = \frac{m_0^4 \mu^{-\epsilon}}{\lambda} (x + \frac{1}{6} x^2) \qquad 1\text{-}26$$

and, of course

$$Z^{(0)}(\phi) = 1 \qquad 1\text{-}27$$

(i) The one loop functions

The one-loop contribution to the vacuum proper function is simply the bubble diagram (Fig. 1-2(a)). To evaluate it, we first differentiate it by m^2 , using

$$\frac{\partial}{\partial m^2}\frac{i}{p^2-m^2} = \frac{i}{p^2-m^2}\,(-1)\,\frac{i}{p^2-m^2} \qquad 1\text{-}28$$

This gives

$$\frac{\partial}{\partial m^2}\bigcirc = -\frac{i}{2}\int dk\,\frac{i}{k^2-m^2}$$

$$= -\frac{i}{2(4\pi)^2}\,\Gamma\left(\frac{\varepsilon}{2}-1\right)(m^2)^{1-\frac{\varepsilon}{2}}$$

so that

$$\bigcirc = -\frac{i}{2(4\pi)^2}\,\frac{\Gamma\left(\frac{\varepsilon}{2}-1\right)}{2-\frac{\varepsilon}{2}}\,(m^2)^{2-\frac{\varepsilon}{2}}$$

$$= \frac{i(m_0^2)^{2-\frac{\varepsilon}{2}}}{4(4\pi)^2}\,(1+x)^2\left(\frac{2}{\varepsilon}-\gamma+\frac{3}{2}-\ln(1+x)\right) \qquad 1\text{-}29$$

where terms of order ε have been dropped. Now, the one loop vacuum counterterm

$$-i\,\frac{m_0^4\mu^{-\epsilon}}{\lambda}\,(B_1 x + \frac{1}{6}\,C_1 x^2) \qquad 1\text{-}30$$

serves the role of subracting terms of order a^2, $a^4(x,x^2)$ in 1-29, so as to maintain the normalization conditions 1-24. This determines

$$B_1 = +\,\frac{\lambda}{2(4\pi)^2}\,(\frac{\mu}{m_0})^{\epsilon}(\frac{2}{\epsilon} - \gamma+1) \qquad 1\text{-}31a$$

$$C_1 = +\,\frac{3\lambda}{2(4\pi)^2}\,(\frac{\mu}{m_0})^{\epsilon}(\frac{2}{\epsilon} - \gamma) \qquad 1\text{-}31b$$

Finally, adding 1-29 and 1-30 gives the one loop effective action, and

$$V_1(a) = \frac{m_0^4}{4(4\pi)^2}\,[(1+x)^2\ln(1+x) - x - \frac{3}{2}\,x^2] \qquad 1\text{-}32$$

The one-loop wavefunction counterterm A_1 is zero, because at one loop in the unshifted theory there are no p^2-dependent diagrams. So the entire contribution to $Z_1(a)$ comes from Fig. 1-2(b) which is

$$\frac{i a^2\lambda^2\mu^{2\epsilon}}{2(4\pi)^2}\,\Gamma(\frac{\epsilon}{2})\int_0^1 dy[m^2-y(1-y)p^2]^{-\frac{\epsilon}{2}} \qquad 1\text{-}33$$

the part of this proportional to p^2 gives

$$Z_1(a) = \frac{\lambda}{6(4\pi)^2}\frac{x}{1+x} \qquad \text{1-34}$$

(ii) The two loop functions

The two loop contribution to the effective action is given by the diagrams in figure 1-3. In fig. 1-3(c), the dot represents the one loop, two-point counterterm, given by figure 1-1 and equations 1-31. There is, in addition, an overall two loop vacuum counterterm, which merely serves to subtract terms of order x, x^2 from the sum of these diagrams. The results are (see Appendix I for details of calculations)

$$(a) = -\frac{i\lambda\mu^{\varepsilon}}{(4\pi)^4}(m_0^2)^{2-\varepsilon}\, x(1+x)\left(\frac{1}{\varepsilon^2} - \frac{\gamma}{\varepsilon} + \frac{3}{2\varepsilon}\right)(1+x)^{-\varepsilon} \qquad \text{1-35}$$

$$(b) = -\frac{i\lambda\mu^{\varepsilon}}{(4\pi)^4}(m_0^2)^{2-\varepsilon}\,\frac{1}{2}(1+x)^2\left(\frac{1}{\varepsilon^2} - \frac{\gamma}{\varepsilon} + \frac{1}{\varepsilon}\right)(1+x)^{-\varepsilon} \qquad \text{1-36}$$

$$(c) = \frac{i\lambda\mu^{\varepsilon}}{(4\pi)^4}(m_0^2)^{2-\varepsilon}\,(1+x)\left[(1+3x)\left(\frac{1}{\varepsilon^2} - \frac{\gamma}{\varepsilon}\right) + \left(1+\frac{3}{2}x\right) + \frac{1}{\varepsilon}\right](1+x)^{-\varepsilon}$$

1-37

Note that only pole terms $\left(\frac{1}{\varepsilon^2}, \frac{1}{\varepsilon}\right)$ have been retained. The omitted finite terms will only contribute terms of order $1, x, x^2$, which are to be subtracted in the end anyway. They would have to be included, however, if one were to proceed to a three-loop calculation.

Adding 1-35, 1-36 and 1-37, one sees that terms of the type $\frac{\ln(1+x)}{\varepsilon}$ cancel. If they did not, the theory would be unrenormalizable, as it would require non-polynomial counterterms to be rendered finite. The sum of the three expressions is

$$-\frac{i\lambda\mu^{\varepsilon}}{8(4\pi)^4}(m_0^2)^{2-\varepsilon}\{(1+4x+3x^2)\ln^2(1+x) - 10x(1+x)\ln(1+x) - 4(1+4x+3x^2)(\frac{1}{\varepsilon^2} - \frac{\gamma}{\varepsilon}) - 4(1-x^2)\frac{1}{\varepsilon}\} \qquad 1\text{-}38$$

It follows that the necessary two loop counterterms are

$$B_2 = \frac{2\lambda^2}{(4\pi)^4}\left(\frac{\mu}{m_0}\right)^{2\varepsilon}\left(\frac{1}{\varepsilon^2} - \frac{\gamma}{\varepsilon} + O(1)\right) \qquad 1\text{-}39$$

$$C_2 = \frac{9\lambda^2}{2(4\pi)^4}\left(\frac{\mu}{m_0}\right)^{2\varepsilon}\left(\frac{1}{\varepsilon^2} - \frac{\gamma}{\varepsilon} - \frac{1}{3\varepsilon} + O(1)\right) \qquad 1\text{-}40$$

and the two loop contribution to the effective potential is

$$V_2(\phi) = \frac{\lambda m_0^2}{8(4\pi)^4}\left[(1+4x+3x^2)\ln^2(1+x) - 10x(1+x)\ln(1+x) + 9x^2\right] \qquad 1\text{-}41$$

which agrees with previous calculations.[4]

The two loop contributions to $iZ(\phi)$ are given by the diagrams of

Figure 1-4, apart from an overall counterterm $-iA_2$, which is independent of $x(=\mu^{\varepsilon}\frac{\lambda\phi^2}{2m_0^2})$. Figures 1-4(a) and 1-4(b) contain, respectively, the one loop two-point and three-point counterterms. The results are (upon differentiating the diagrams by p^2, and setting $p^2 = 0$);

$$(a) = -\frac{i\lambda^2}{6(4\pi)^4}\left(\frac{\mu}{m_0}\right)^{2\varepsilon}\left[\frac{x}{1+x}\left(\frac{1}{\varepsilon}-\gamma+1\right)+\frac{2x^2}{(1+x)^2}\left(\frac{1}{\varepsilon}-\gamma+\frac{1}{4}\right)\right](1+x)^{-\varepsilon/2} \tag{1-42}$$

$$(b) = \frac{i\lambda^2}{(4\pi)^4}\left(\frac{\mu}{m_0}\right)^{2\varepsilon}\frac{x}{1+x}\left(\frac{1}{\varepsilon}-\gamma\right)(1+x)^{-\varepsilon/2} \tag{1-43}$$

$$(c) = \frac{2i\lambda^2}{3(4\pi)^4}\frac{x}{(1+x)^2}\left(\frac{1}{9}J+\frac{2}{9}\right) \tag{1-44}$$

$$(d) = \frac{i\lambda^2}{3(4\pi)^4}\left(\frac{\mu}{m_0}\right)^{2\varepsilon}\frac{x}{(1+x)^2}\left(\frac{1}{\varepsilon}-\gamma-\frac{8}{9}J+\frac{53}{36}\right)(1+x)^{-\varepsilon} \tag{1-45}$$

$$(e) = -\frac{2i\lambda^2}{3(4\pi)^4}\left(\frac{\mu}{m_0}\right)^{2\varepsilon}\frac{x}{1+x}\left(\frac{1}{e}-\gamma-\frac{2}{3}J+\frac{2}{3}\right)(1+x)^{-\varepsilon} \tag{1-46}$$

$$(f) = \frac{i\lambda^2}{12(4\pi)^4}\left(\frac{\mu}{m_0}\right)^{2\varepsilon}\left(\frac{1}{\varepsilon}-\gamma\right)(1+x)^{-\varepsilon} \tag{1-47}$$

$$(g) = -\frac{i\lambda^2}{3(4\pi)^4}\left(\frac{\mu}{m_0}\right)^{2\varepsilon}\frac{x}{1+x}\left(\frac{1}{\varepsilon}-\gamma\right)(1+x)^{-\varepsilon} \tag{1-48}$$

$$(h) = \frac{i\lambda^2}{6(4\pi)^4}\left(\frac{\mu}{m_0}\right)^{2\varepsilon}\frac{x}{1+x}\left(\frac{1}{\varepsilon}-\gamma+1\right)(1+x)^{-\varepsilon} \tag{1-49}$$

The number J occurring in 1-44, 1-45 and 1-46 is defined by

$$J = -\int_0^1 dx\,\frac{\ln x}{1-x+x^2} \tag{1-50}$$

Again, some details of the calculations can be found in Appendix I.

Now, adding the above contributions together, all non-polynomial poles $\left(\frac{1}{\varepsilon}\frac{x}{1+x}, \frac{1}{\varepsilon}\frac{x^2}{(1+x)^2}\right)$ cancel, leaving only a constant term to be subtracted by the two loop wavefunction renormalization counterterm;

$$A_2 = -\frac{\lambda^2}{12(4\pi)^4}\left(\frac{\mu}{m_0}\right)^{2\varepsilon}\left(\frac{1}{\varepsilon}-\gamma+O(\varepsilon)\right) \tag{1-51}$$

The remainder is simply $iZ_2(\phi)$; thus

$$Z_2(\phi) = \frac{\lambda^2}{12(4\pi)^4}\left\{\left[-1+\frac{5x}{1+x}+\frac{2x^2}{(1+x)^2}\right]\ln(1+x)+\frac{\xi_1 x}{1+x}+\frac{\xi_2 x^2}{(1+x)^2}\right\} \tag{1-52}$$

where $\xi_1 = \frac{16}{3}J - \frac{16}{3}$ and $\xi_2 = -\frac{8}{3}J + \frac{20}{3}$. We note here a discrepancy with the results obtained by Iliopoulos et al.[4] They obtained

$$\xi_1 = \frac{16}{3} J - 2 \quad \text{and} \quad \xi_2 = -\frac{8}{3} J + \frac{19}{3}$$

Now, it is always possible to change the non-logarithmic parts of Z_2 by altering the finite parts of the one loop counterterms. In general, this would alter the $\ell n(1+x)$ term in V_2. However, even if one could adjust the finite parts of A_1, B_1, C_1 so as to leave V_2 unchanged, there would still be a change in the normalization conditions at the one loop level

$$Z(0) = 1 + A_1 \ ; \quad V''(0) = m_0^2(1+\delta B_1) \ ; \quad V^{(iv)}(0) = \lambda(1+\delta C_1)$$

where A_1 , δB_1 , δC_1 denote the finite changes in the counterterms. So, if one requries any kind of consistency in maintaining the normalization conditions, one is tempted to conclude that the result of Iliopoulos et al. was in error. In case there is any doubt about the validity of the shifted field method in the calculation of Z_2 , there is a way to check the coefficients ξ_1 and ξ_2 , independently of this method. The check was made for ξ_1 , and began with the observation that

$$\frac{\partial^2 Z}{\partial \phi^2}(\phi = 0) \propto \frac{\partial Z}{\partial x}(x = 0)$$

$= \text{coefficient of } \sum_{i<j} p_i \cdot p_j \text{ in } \Gamma_4(p_k)$ in the unshifted theory.

This follows directly from equations 1-8 and 1-9, and does not depend on shifting the field. Upon computing the relevant part of the four-point function in the unshifted theory to two loops, the result for ξ_1 was verified.

5. Fourth order functions in the momentum expansion

All functions occurring at fourth order in the momentum expansion (order ∂^4 or p^4) will be denoted by a square bracket with subscript 4. Thus, the relevant part of the effective action (first term excluded in equation 1-9) is

$$[\Gamma[\phi]]_4 = -i\int d^4x\{b(\phi)(\partial\phi)^2(\partial\phi)^2 + c(\phi)(\partial\phi)^2\partial^2\phi + d(\phi)\partial_\mu\partial_\nu\phi\partial^\mu\partial^\nu\phi + e(\phi)\partial^2\phi\partial^2\phi + f(\phi)\partial^\mu\phi\partial_\mu\partial^2\phi + g(\phi)\partial_\mu\partial_\nu\phi\partial^\mu\partial^\nu\phi + h(\phi)(\partial^2)^2\phi\} \qquad 1\text{-}53$$

We first note that the function $h(\phi)$ can be absorbed into $f(\phi)$ by an integration by parts. Expanding, as before, about $\phi = a$, and taking Fourier Transforms, one can obtain the $O(p^4)$ part of the n-point proper function in the shifted theory. For $n \geq 4$,

$$i[\Gamma_{na}(p_1,\ldots,p_n)]_4 = \sum_{i=j}^{n} (p_i^2)^2[-f^{(n-2)}+d^{(n-3)}+c^{(n-3)}-3b^{(n-4)}]$$

$$+ 2\sum_{i<j} p_i^2 p_j^2 [e^{(n-2)}-c^{(n-3)}+b^{(n-4)}]$$

$$+ 2\sum_{i<j} (p_i\cdot p_j)^2 \; [g^{(n-2)}-d^{(n-3)}+2b^{(n-4)}] \qquad 1\text{-}54$$

The superscripts denote differentiation with respect to a. It is convenient for later calculations to express the momentum functions in terms of the scalars $p_i\cdot p_j\, i \neq j$ which will be denoted S_α (α = $1,2,\ldots, \frac{1}{2}\, n(n-1)$). For $n \geq 4$, it can be shown that

$$\sum_i (p_i^2)^2 = 2\sum_\alpha S_\alpha^2 + 2\sum_{\alpha<\beta}{}' S_\alpha S_\beta \qquad 1\text{-}55$$

$$\sum_{i<j} p_i^2 p_j^2 = \sum_\alpha S_\alpha^2 + 3\sum_{\alpha<\beta}{}' S_\alpha S_\beta + 4\sum_{\alpha<\beta}{}'' S_\alpha S_\beta \qquad 1\text{-}56$$

$$\sum_{i<j} (p_i\cdot p_j)^2 = \sum_\alpha S_\alpha^2 \qquad 1\text{-}57$$

$\sum'$ denotes a sum over products in which S_α , S_β have one momentum in common (e.g. $-(p_1\cdot p_2)(p_1\cdot p_3)$). $\sum''$ denotes a sum over products in which all four momenta in S_α, S_β are different (e.g.

$-(p_1\cdot p_2)(p_3\cdot p_4))$. So one can re-write equation 1-54 as

$$
\begin{aligned}
i[\Gamma_{na}(p_1,\dots,p_n)]_4 = {} & 2\sum_\alpha s_\alpha^2(e^{(n-2)}-f^{(n-2)}+g^{(n-2)}) \\
& + 2\sum_{\alpha<\beta}{}' \, s_\alpha s_\beta(3e^{(n-2)}-f^{(n-2)}+d^{(n-3)}-2c^{(n-3)}) \\
& + 8\sum_{\alpha<\beta}{}'' \, s_\alpha s_\beta(e^{(n-2)}-c^{(n-3)}+b^{(n-4)})
\end{aligned}
\qquad \text{1-58}
$$

Two things are obvious from equation 1-54 or 1-58. Firstly, all the information about the functions $b,\dots,g$ is contained in the two-, three-, and four-point functions in the shifted theory. Higher values of n simply have derivatives (with respect to a) of the same combinations of these functions. Secondly, there are only three equations determining the six functions $b,\dots,g$. Some arbitrariness is to be expected, as they can be "mixed up" by integrating by parts in equation 1-53. This gives one a certain freedom in choosing the solution. One possibility is $f = c = d = 0$. From equation 1-58, the two-, three-, and four-point functions are, respectively;

$$i[\Gamma_{2a}(r)]_4 = 2r^2(e+g) \qquad \text{1-59}$$

where $r \equiv p^2 = -p_1\cdot p_2$

$$i[\Gamma_{3a}(\sigma,\tau)]_4 = 2(\sigma^2+\tau^2)(e'+g') + 2\sigma\tau(3e') \qquad 1\text{-}60$$

where $\sigma = p_1 \cdot p_2$, $\tau = p_1 \cdot p_3$, $p_1 \cdot p_3 = 0$

$$i[\Gamma_{4a}(s,t)]_4 = 2(s^2+t^2)(e''+g'') + 8st(e''+b) \qquad 1\text{-}61$$

where $s = p_1 \cdot p_2$, $t = p_3 \cdot p_4$, $p_1 \cdot p_3 = p_1 \cdot p_4 = p_1 \cdot p_3 = p_1 \cdot p_4 = 0$. So, one now can write

$$e + g = \frac{1}{4}\frac{\partial^2}{\partial r^2}\Gamma_{2a}(r)\Big|_{r=0} \qquad 1\text{-}62$$

$$e' = \frac{1}{6}\frac{\partial^2}{\partial\sigma\partial\tau}\Gamma_{3a}(\sigma,\tau)\Big|_{\sigma=\tau=0} \qquad 1\text{-}63$$

$$b = \frac{1}{8}\frac{\partial^2}{\partial s\partial t}\Gamma_{4a}(s,t)\Big|_{s=t=0} - e'' \qquad 1\text{-}64$$

To calculate the functions to one loop order, ~~the~~ relevant diagrams are given in figure 1-5. The calculations are done with the external momenta given above. The results are (again expressed in terms of $x = \lambda\phi^2/2m_0^2$)

$$e(\phi) = \frac{-\lambda}{1080(4\pi)^2 m_0^2}\left(\frac{7}{1+x} - \frac{5}{(1+x)^2}\right) \qquad 1\text{-}65$$

$$g(\phi) = \frac{-\lambda}{1080(4\pi)^2 m_0^2}\left(\frac{2}{1+x} - \frac{4}{(1+x)^2}\right) \qquad 1\text{-}66$$

$$b(\phi) = \frac{\lambda^2}{720(4\pi)^2 m_0^4}\left(\frac{1}{(1+x)^2} - \frac{12}{(1+x)^2} + \frac{10}{(1+x)^4}\right) \qquad 1\text{-}67$$

6. The Renormalization Group

The essence of the renormalization group is contained in the behaviour of the path integral 1-1 under scaling of the field and dynamical parameters occurring in the Lagrangian. For example, in a $\lambda\phi^4$ theory with mass parameter m^2, coupling parameter λ and specific counterterms, one can define new scaled parameters (with ϕ being the field integrated over in the path integral)

$$\phi = \xi\hat{\phi} \; ; \quad m^2 = \zeta\hat{m}^2 \; ; \quad \lambda = \eta\hat{\lambda} \qquad 1\text{-}68$$

It then follows readily that

$$\hat{\Gamma}[\hat{\phi};\hat{m}^2,\hat{\lambda}] = \Gamma[\phi;m^2,\lambda] = \Gamma[\xi\hat{\phi};\zeta\hat{m}^2,\eta\hat{\lambda}] \qquad 1\text{-}69$$

where the left-hand side is the effective action for a theory with Lagrangian

$$\hat{L}(\hat{\phi};\hat{m}^2,\hat{\lambda}) = L(\xi\hat{\phi};\zeta\hat{m}^2,\eta\hat{\lambda}) \qquad 1\text{-}70$$

From 1-69, one infers that the n-point proper functions satisfy

$$\hat{\Gamma}_n(p_i;\hat{m}^2,\hat{\lambda}) = \xi^n \Gamma_n(p_i;m^2,\lambda) \qquad \text{1-71}$$

Another way of expressing this invariance is to start with unspecified parameters $\hat{m}^2$, $\hat{\lambda}$, and determine the proper functions $\hat{\Gamma}_n$ by defining the normalization conditions (i.e. the counterterms) with respect to some mass scale μ^2 . This could be a characteristic external momentum squared, or a parameter entering via regularization. Then the proper functions are completely determined as functions of p_i, $\hat{m}^2$, $\hat{\lambda}$ and μ^2. The requirement that these functions describe the same physical theory as the original functions $\Gamma_n(p_i;m^2,\lambda)$ is then expressed as

$$\hat{\Gamma}_n(p_i;\hat{m}^2,\hat{\lambda},\mu^2) = \xi^n \Gamma_n(p_i;m^2,\lambda) \qquad \text{1-72}$$

The three normalization conditions on the Γ_n's then determine the parameters $\hat{m}^2$, $\hat{\lambda}$ and ξ as functions of μ^2 , m^2 and λ. In terms of the effective action, 1-72 reads

$$\hat{\Gamma}[\phi;\hat{m}^2,\hat{\lambda},\mu^2] = \Gamma[\xi\phi;m^2,\lambda] \qquad \text{1-73}$$

The expansion 1-9 also yields

$$\hat{V}(\hat{\phi};\hat{m}^2,\hat{\lambda},\mu^2) = V(\xi\hat{\phi};m^2,\lambda) \qquad 1\text{-}74$$

$$\hat{Z}(\hat{\phi};\hat{m}^2,\hat{\lambda},\mu^2) = \xi^2 Z(\xi\hat{\phi};m^2,\lambda) \qquad 1\text{-}75$$

Differentiating these expressions with respect to μ, keeping m^2 and λ fixed, gives us the renormalization group equations

$$\left[\mu\frac{\partial}{\partial\mu} + \beta\frac{\partial}{\partial\hat{\lambda}} + \hat{m}\gamma_{\hat{m}}\frac{\partial}{\partial\hat{m}} - \gamma_\xi\int d^4x\hat{\phi}(x)\frac{\delta}{\delta\hat{\phi}(x)}\right]\hat{\Gamma}(\hat{\phi};\hat{m}^2,\hat{\lambda},\mu^2) = 0 \qquad 1\text{-}76$$

where

$$\beta = \mu\frac{\partial\hat{\lambda}}{\partial\mu} \qquad 1\text{-}77a$$

$$\gamma_{\hat{m}} = \frac{\mu}{\hat{m}}\frac{\partial\hat{\lambda}}{\partial\mu} \qquad 1\text{-}77b$$

$$\gamma_\xi = \mu\frac{\partial}{\partial\mu}\ln\xi \qquad 1\text{-}77c$$

The value of the renormalization group equations is the following: When the coefficients 1-77 are computed to some order, the solutions of 1-76 are in general valid over a larger range (of the field ϕ in the case 1-76 is applied to the effective potential; of the external momenta when it is applied to the proper functions $\Gamma_n(p_i))$, than if one were to simply compute the effective action to the same order. Hence the phrase "renormalization group-improved". We now turn to the computation of these coefficients in a $\lambda\phi^4$ theory. It will be seen that the method expounded in the previous sections provides an efficient way of determining them. Henceforth the caret notation will be dropped, and it is to be understood that the parameters m^2 and λ now depend on the mass scale μ.

Using dimensional regularization, at $D(=4-\varepsilon)$ dimensions the four-point coupling acquires a dimension $(\text{mass})^\varepsilon$, and this can be used as the renormalization scale μ. The D-dimensional Lagrangian is

$$L = \frac{1}{2}(1+A)(\partial\phi)^2 - \frac{1}{2}m^2(1+B)\phi^2 - \frac{1}{4!}\lambda\mu^\varepsilon(1+C)\phi^4 \qquad \text{1-78}$$

where A, B and C are counterterms. The corresponding bare Lagrangian is

$$L = \frac{1}{2}(\partial\phi_B)^2 - \frac{1}{2}m_B^2\phi_B^2 - \frac{1}{4!}\lambda_B\phi_B^4 \qquad \text{1-79}$$

which implies that

$$\phi_B^2 = \phi^2(1+A) = \phi^2(1+a_j\varepsilon^{-j}) \qquad \text{1-80a}$$

$$m_B^2 = m^2\left(\frac{1+B}{1+A}\right) = m^2(1+b_j\varepsilon^{-j}) \qquad \text{1-80b}$$

$$\lambda_B = \mu^\varepsilon\lambda\,\frac{(1+C)}{(1+A)^2} = \mu^\varepsilon\lambda(1+c_j\varepsilon^{-j}) \qquad \text{1-80c}$$

Here, the renormalization constants have been expanded in powers of $1/\varepsilon$. The index j is summed over from $j = 0$ to $j = \infty$. The dimensionless coefficients a_j, b_j, c_j are in general functions of λ and μ^2/m^2. Now the bare parameters are to be understood to be independent of any particular regularization scheme, and are thus independent of the scale μ^2. So, differentiating equations 1-80 by μ will yield information about the functions β, γ_m, γ_ξ. If the counterterms A, B, and C are chosen to be independent of m^2 and μ^2, these functions will only depend on λ. This is assumed in the following. Differentiating equations 1-80, and equating powers of ε, one obtains

$$\beta = \mu\frac{\partial\lambda}{\partial\mu} = \frac{\lambda^2}{[1+(\lambda c_0)']^2}\left[c_1'(1+c_0) - c_1c_0'\right] \qquad \text{1-81a}$$

$$2\gamma_\xi(1+a_0) = \beta a_0' + \frac{\lambda(1+c_0)}{1+(\lambda c_0)'}\left[\frac{a_1a_0'}{1+a_0} - a_1'\right] \qquad \text{1-81b}$$

$$2\gamma_m(1+b_0) = -\beta b_0' + \frac{\lambda(1+c_0)}{1+(\lambda c_0)'}\left[\frac{b_1b_0'}{1+b_0} - b_1'\right] \qquad \text{1-81c}$$

Here the prime denotes differentiation with respect to λ. In the minimal subtraction scheme, where only pole terms are subtracted, one has $a_0 = b_0 = c_0 = 0$, and equations 1-81 simplify to

$$\beta = \lambda^2 c_1' = \lambda^2 (C_1' - 2A_1') \qquad \text{1-82a}$$

$$\gamma_\xi = -\frac{1}{2}\lambda^2 a_1' = -\frac{1}{2}\lambda A_1' \qquad \text{1-82b}$$

$$\gamma_m = \frac{1}{2}\lambda b_1' = \frac{1}{2}\lambda (B_1' - A_1') \qquad \text{1-82c}$$

The subscript 1 here denotes the single-pole coefficient of the counterterm, to all orders in the perturbation expansion. This is not to be confused with the previous notation, which signified the one loop counterterm. The loop order will now be denoted by a superscript in brackets.

The method outlined in the previous sections provides one with an efficient way of evaluating the counterterms A, B and C, and thus the coefficients β, γ_ξ and γ_m. The only modifications to the counterterms already evaluated (equations 1-31, 1-39, 1-10 and 1-51) are to have them in the minimal subtraction scheme. In terms of the single pole $(\frac{1}{\varepsilon})$ parts, this means only a re-evaluation of the diagram of Fig. 1-3(c), which involves the one loop counterterm. This is readily done, yielding

$$B_1^{(1)} = \frac{\lambda}{(4\pi)^2} \qquad C_1^{(1)} = \frac{3\lambda}{(4\pi)^2} \qquad A_1^{(1)} = 0 \qquad \text{1-83}$$

$$B_1^{(2)} = -\frac{\lambda^2}{2(4\pi)^4} \qquad C_1^{(1)} = -\frac{3\lambda^2}{(4\pi)^4} \qquad A_1^{(1)} = -\frac{\lambda^2}{12(4\pi)^4} \qquad \text{1-84}$$

And so, we have from equations 1-82, with $\alpha \equiv \lambda/(4\pi)^2$

$$\beta = \lambda(3\alpha - \frac{17}{3}\alpha^2 + O(\alpha^3)) \qquad \text{1-85a}$$

$$\gamma_\xi = \frac{\alpha^2}{12} + O(\alpha^3) \qquad \text{1-85b}$$

$$\gamma_m = \frac{1}{2}(\alpha - \frac{5}{6}\alpha^2 + O(\alpha^3)) \qquad \text{1-85c}$$

Fig. 1-1. Feynman rules and counterterms in the shifted theory (with $x = \mu^{\varepsilon}\lambda\alpha^2/2m_0^2$)

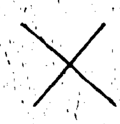
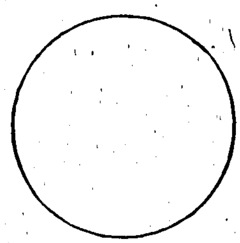
zero-point vertex	$-\frac{im_0^4\mu^{-\varepsilon}}{\lambda}(x + \frac{1}{6}x^2)$	
three-point vetex	$-i\alpha\lambda\mu^{\varepsilon}$	
four-point vertex	$-i\lambda\mu^{\varepsilon}$	
propagator	$\frac{i}{p^2-m^2}$ $(m^2 = m_0^2(1+x))$	
zero-point counterterm	$-\frac{im_0^4\mu^{-\varepsilon}}{\lambda}(Bx + \frac{1}{6}Cx^2)$	
two-point counterterm	$iAp^2 - im_0^2(B+Cx)$	
three-point counterterm	$-iC\alpha\lambda\mu^{\varepsilon}$	
four-point counterterm	$-iC\lambda\mu^{\varepsilon}$	

Fig. 1-2. one loop bare diagrams

(a) contribution to effective potential



(b) Contribution to $Z(\phi)$

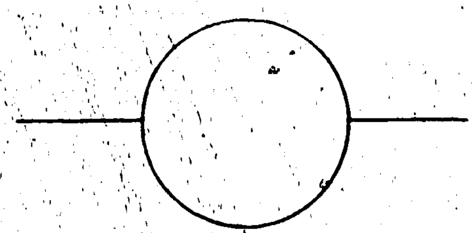


Fig. 1-3. Feynman diagrams contributing to the effective potential

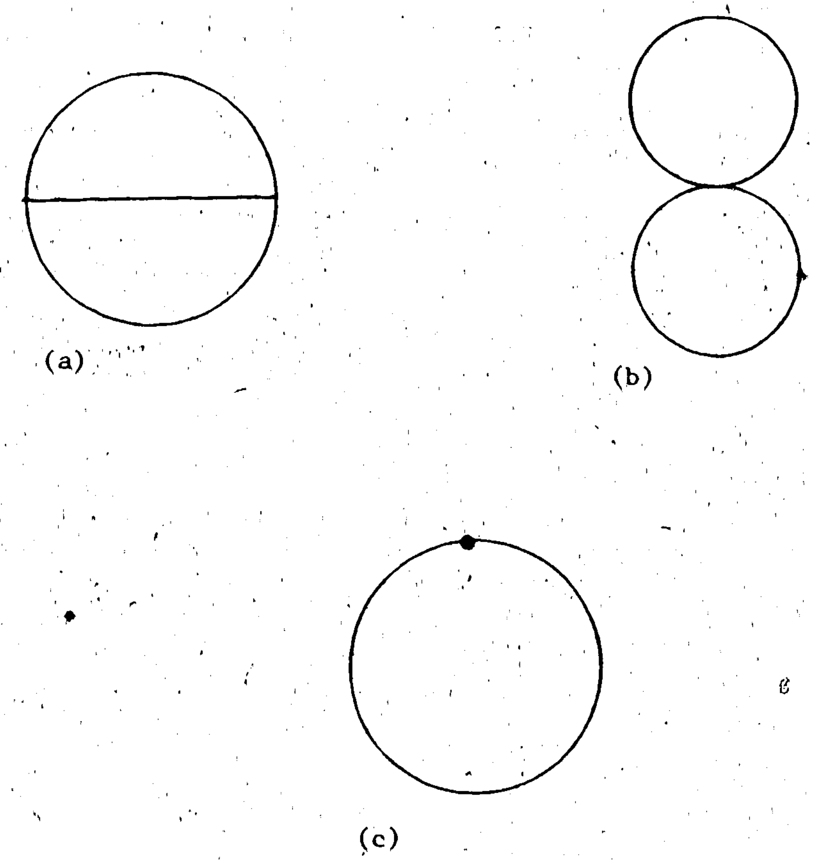


Fig. 1-4. Feynman diagrams contributing to the function $Z(\phi)$ at two loops.

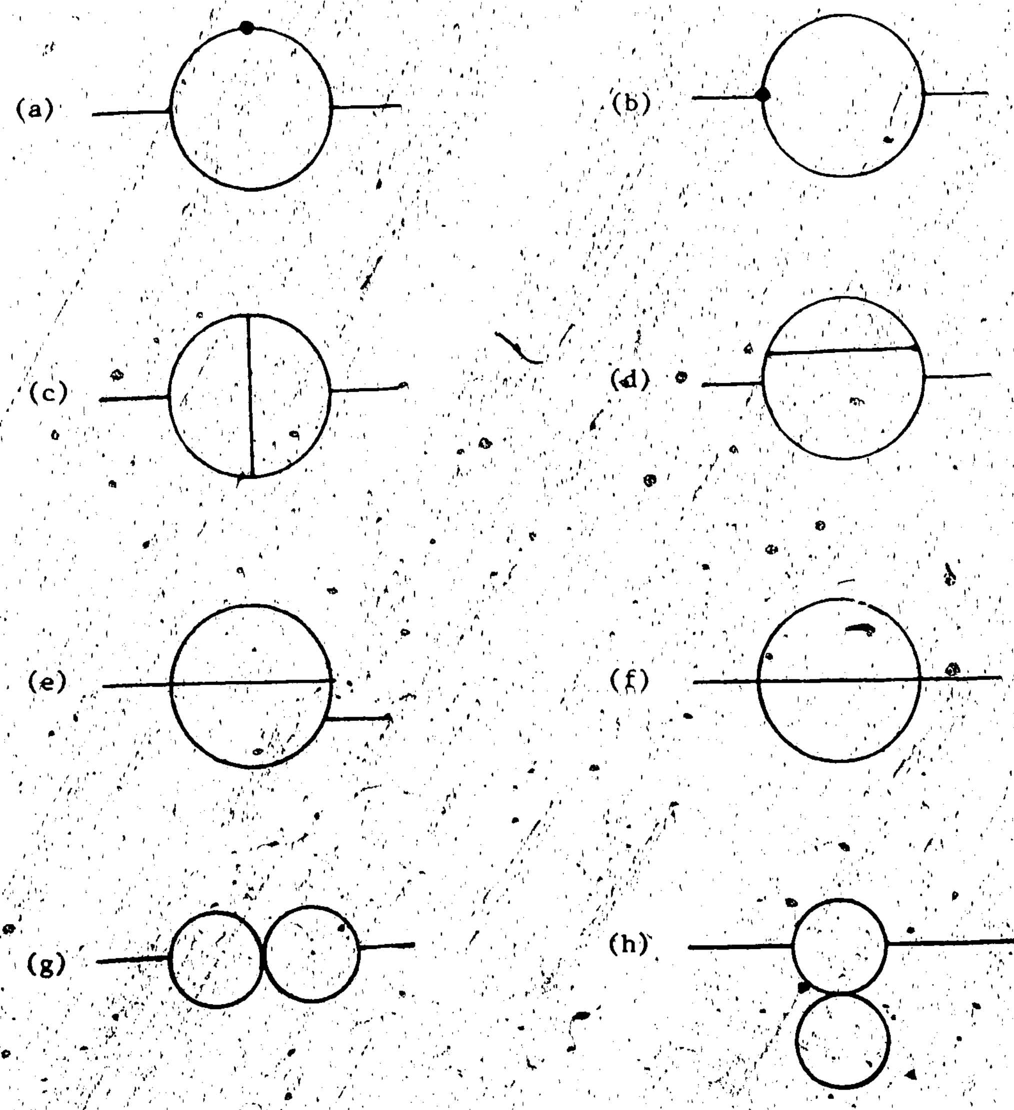
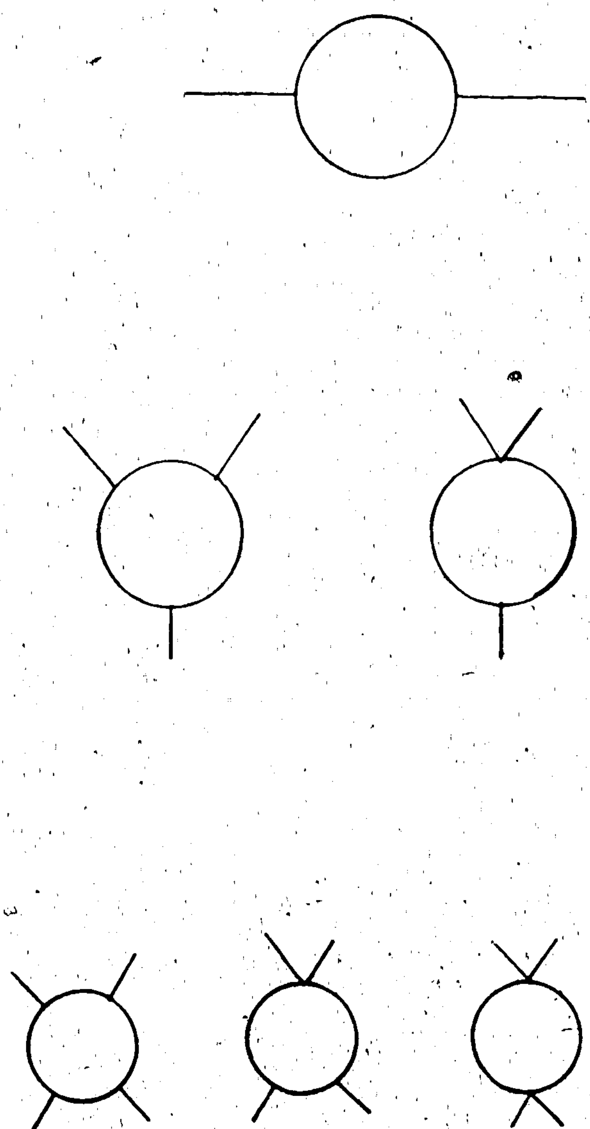


Fig. 1-5. Feynman diagrams contributing to fourth order in the momentum expansion of the effective action at the one loop level



(a) Two-point function

(b) Three-point function

(c) Four-point function

Chapter Two A Brief Survey of Field Theory at Non-Zero Temperature.

The imaginary time thermal field theory formalism (ITF) was introduced by Matsubara[6]. Kirzhnitz was the first to apply the formalism to the discussion of gauge theories and phase transitions at high temperature[7], followed by other authors.[8] The starting point is the statistical average of a physical quantity F, at temperature $T = 1/\beta$;

$$\langle F \rangle_\beta = \frac{\sum_n \langle n | e^{-\beta H} F | n \rangle}{\sum_n \langle n | e^{-\beta H} | n \rangle} \qquad 2\text{-}1$$

Now, the time development of the states is given by

$$|n;t\rangle = e^{-iHt}|n\rangle \qquad 2\text{-}2$$

Thus, if the time argument is continued into the complex plane, the statistical average above can be written, at least formally;

$$\langle F \rangle_\beta = \frac{\sum_n \langle n; t-i\beta | F | n;t \rangle}{\sum_n \langle n; t-i\beta | n;t \rangle} \qquad 2\text{-}3$$

The numerator of this expression can now be cast into a path integral, defined on a contour extending from t to $t-i\beta$ in the complex time plane. Extending these arguments to a relativistic field theory, in a manner analogous to the generating functional at zero temperature, the result is a formalism which allows the computation of statistical quantities by Feynman rules and diagrams. For a scalar field, one obtains the propagator;

$$\frac{1}{(2\pi n/\beta)^2+k^2+m^2} \qquad 2\text{-}3$$

and the usual momentum integration changes

$$\int \frac{d^4k}{(2\pi)^4} \to \frac{1}{\beta} \int \frac{d^3k}{(2\pi)^3} \sum_n \qquad 2\text{-}4$$

Another approach to equation 2-1 was invented by Takahashi and Umezawa(9), and developed by Matsumoto and others(10-17). The resulting formalism was denoted thermo field theory (TFT). In describing this approach, the example of a one-dimensional harmonic oscillator will be used. Then, equation 2-1 becomes

$$\langle F\rangle_\beta = \left(1- e^{-\beta\omega}\right) \sum_n e^{-n\beta\omega} \langle n|F|n\rangle \qquad 2\text{-}5$$

This can be written as an amplitude in a single, temperature dependent state if one introduces a "fictitious" copy of the original state space, with states denoted $|\tilde{n}\rangle$, whose creation and annihilation operators commute with those of the original states.

$$\langle F\rangle_\beta = \langle 0(\beta)|F|0(\beta)\rangle \qquad 2\text{-}6$$

$$|0(\beta)\rangle = \left(1-e^{-\beta\omega}\right)^{\frac{1}{2}} \sum_n e^{-\frac{n\beta\omega}{2}} |n,\tilde{n}\rangle \qquad 2\text{-}7$$

Further, it can be easily demonstrated, by differentiating with respect to β, that this state is related to the vacuum by a Bogoliubov transformation;

$$|0(\beta)\rangle = e^{G(\beta)}|0,\tilde{0}\rangle \qquad 2\text{-}8$$

where

$$G(\beta) = \theta(\beta)\left(a^{+}\tilde{a}^{+} - a\tilde{a}\right) \qquad 2\text{-}9$$

and

$$\theta(\beta) = \frac{1}{2}\ln \coth\left(\frac{\beta\omega}{4}\right) \qquad 2\text{-}10$$

or

$$\cosh^2\theta = \frac{1}{1-e^{-\beta\omega}} \qquad \sinh^2\theta = \frac{e^{-\beta\omega}}{1-e^{-\beta\omega}} \qquad 2\text{-}11$$

In equation 2-9, a, a^{+} $(\tilde{a}, \tilde{a}^{+})$ are the annihilation and creation operators of the states $|n\rangle\ (|\tilde{n}\rangle)$, satisfying

$$\left[a, a^{+}\right] = \left[\tilde{a}, \tilde{a}^{+}\right] = 1$$
$$\left[a, a^{+}\right] = \left[\tilde{a}, \tilde{a}^{+}\right] = 0 \qquad 2\text{-}12$$

and

$$|n\rangle = \frac{(a^{+})^{n}}{\sqrt{n!}}\,|0\rangle\ ; \quad |\tilde{n}\rangle = \frac{(\tilde{a}^{+})^{n}}{\sqrt{n!}}\,|\tilde{0}\rangle \qquad 2\text{-}13$$

The identification of $|0(\beta)\rangle$ as a "thermal vacuum" is more complete if one demands that it have zero energy. This requires the total Hamiltonian to be

$$H_{T} = \omega\left(a^{+}a - \tilde{a}^{+}\tilde{a}\right) \qquad 2\text{-}14$$

Note, however, that the physical energy is $\omega a^{+}a$.

One can now define temperature-dependent creation and annihilation operators

$$a_{\beta}^{+} = e^{G(\beta)}a^{+}e^{-G(\beta)} = a^{+}\cosh\theta - \tilde{a}\ \text{sihn}\ \theta \qquad 2\text{-}15$$
$$\tilde{a}_{\beta}^{+} = e^{G(\beta)}\tilde{a}^{+}e^{-G(\beta)} = \tilde{a}^{+}\cosh\theta - a\ \text{sihn}\ \theta$$

and so forth. Extending these arguments to, for example, a scalar field theory, one now has two fields ϕ, $\tilde{\phi}$ (denoted ϕ_1, ϕ_2 in ensuing chapters). The creation operators $\tilde{a}^{+}(k)$ of the field $\tilde{\phi}$

create states of momentum $(-k^{\mu})$. Expanding the fields in terms of the creation and annihilation operators, one can readily compute the propagators from the time-ordered products;

$$\begin{aligned}
\Delta_{11}(x-y) &= \langle 0(\beta)|T(\phi(x)\phi(y))|0(\beta)\rangle \\
\Delta_{12}(x-y) &= \langle 0(\beta)|T(\phi(x)\tilde{\phi}(y))|0(\beta)\rangle \qquad \text{2-16} \\
\Delta_{22}(x-y) &= \langle 0(\beta)|T(\tilde{\phi}(x)\tilde{\phi}(y))|0(\beta)\rangle
\end{aligned}$$

The results are just those obtained in Chapter Three. The other Feynman rules are obtained essentially by generalizing the expression 2-12 for the Hamiltonian

$$L(\phi,\tilde{\phi}) = L(\phi) - L(\tilde{\phi}) \qquad \text{2-17}$$

The corresponding path-integral approach to relativistic thermal field theory was formulated by Niemi and Semenoff. They showed that the real-time formalism outlined above, with some differences, arises from a path integral defined on a contour in the complex time plane extending along the real-time axis, and doubling back along the line $t - \frac{\beta}{2}$. (See Figure 3-2)

Until recently, the applications of thermal quantum field theory have been largely in solid state physics(11). The development of a real-time formalism has made it more palatable to particle physicists, with a resulting increase in interest.

Chapter 3 Finite Temperature Propagators and Vertices.

1. Introduction

The tools required for thermo field theoretic calculations are presented here. The Feynman rules as derived from a path integral approach have been discussed elsewhere[14,15]. However, the conventions used here are somewhat different, so derivations of the thermal propagators for scalar, spinor and massless vector fields are given. A degree of arbitrariness in the Feynman rules, arising from the presence of an unphysical degree of freedom, is discussed. The starting point assumes familiarity with path integral formalism in general, and with the initial applications to thermo field theory by Niemi and Semenoff[14,15], in particular.

2. The Thermal Generating Functional

The contour in the complex time plane relevant to thermal quantum field theory, is one joining an initial real time t to a final "time" $t-i\beta$. The imaginary time formalism is derived using the contour in Fig (3-1). Thermo field theory arises from the contour in Fig (3-2), which is characterized by variation in the real-time direction. This doubling up of the contour results in the appearance of a "fictitious" field for each physical degree of freedom. There is, of course, no reason why one could not consider a contour as a Fig (3-3), which zig-zags $2n$ times. The resulting model will then contain $2n-1$ "fictitious" fields for each degree of freedom. However, henceforth only the contour in Fig (3-2) will be considered.

Now consider a model with one scalar field ϕ , one spinor field ψ and its conjugate $\bar{\psi}$, and a massless vector field A_μ. Their respective generating currents will be denoted $j(x)$, $\bar{\xi}(x)$, $\eta(x)$, and $J^\mu(k)$ ($\bar{\xi}$, η being Grossman-valued spinors). The interaction potential is $V(\phi,\psi,\bar{\psi},A)$. Then the generator of real-time thermal Green's functions in this model is

$$\begin{aligned} Z_\beta = N \exp[-i \int d^4x(V_1-V_2)] \\ \times \exp i \int d^4y \int d^4z[\tfrac{1}{2} j_i(y)\Delta_{ik}(y-x)j_k(z) \\ - \tfrac{1}{2} J_{i\mu}(y)D^{\mu\nu}_{ik}(y-x)J_{k\nu}(z) \\ - \bar{\xi}_{i\alpha}(y)\, S^{\alpha\gamma}_{ik}(y-x)\eta_{k\gamma}(z)] \end{aligned} \qquad 3\text{-}1$$

Here, the indices i, $k \in (1,2)$, where $i=1$ corresponds to the physical field, and $i=2$ corresponds to the "fictitios" field. Δ, D and S are the thermal propagators for the scalar, vector and spinor fields respectively. Finally,

$$V_1 \equiv V(\frac{1}{i}\frac{\delta}{\delta j_1}, \frac{1}{i}\frac{\delta}{\delta\bar{\xi}_1}, \frac{1}{i}\frac{\delta}{\delta\eta_1}, -\frac{1}{i}\frac{\delta}{\delta J_1}) \qquad 3\text{-}2$$

$$V_2 \equiv V(-\frac{1}{i}\frac{\delta}{\delta j_2}, -\frac{1}{i}\frac{\delta}{\delta\bar{\xi}_2}, \frac{1}{i}\frac{\delta}{\delta\eta_2}, \frac{1}{i}\frac{\delta}{\delta J_2}) \qquad 3\text{-}3$$

Now, any physical quantity generated by Z_β is obtained by functional differentiation with respect to the type-1 currents. The type-2 currents are thus dummy variables, and can be re-defined arbitrarily, with resulting changes in the vertices of V_2 and the components of the propagators. The consequences of the current conventions implicit in equations 3-1, 3-2 and 3-3 will now be outlined.

Denote the generic propagator (scalar, spinor or vector) on the contour C in Fig (3-2) by $\Delta(\tau)$. Spatial as well as internal indices have been suppressed, and τ is the ordered time parameter on the contour. $\Delta(\tau)$ satisfies the equation

$$\mathcal{D}_c(\partial)\Delta(\tau) = \delta_c^4(\bar{x},\tau) \tag{3-4}$$

$\mathcal{D}_c$ is the relevant differential operator for the particular field in question, modified to the contour C. Thus, time differentiation becomes differentiation with respect to the parameter τ. Likewise, δ_c^4 is the four-dimensional delta function defined on C. One next writes

$$\Delta(\tau) = \theta_c(\tau)\Delta_>(\tau) + \theta_c(-\tau)\Delta_<(\tau) \tag{3-5}$$

with θ_c being the step function defined on C. The trace nature of Z_β implies the boundary condition

$$\Delta_<(\tau) = \pm\, \Delta_>(\tau-i\beta) \tag{3-6}$$

where the negative sign is for fermions. Now, with the conventions for the type-2 currents used in equations 3-1, 3-2 and 3-3, the components of the thermal propagators appearing in 3-1 satisfy the following relations (with t being real time);

$$\Delta_{11}(t) = \theta_c(t)\Delta_>(t) + \theta_c(-t)\Delta_<(t) \qquad 3\text{-}7$$

$$\Delta_{12}(t) = \Delta_{21}(t) = \Delta_>\left(t - i\,\frac{\beta}{2}\right) \qquad 3\text{-}8$$

$$\Delta_{22}(t) = \pm\left[\theta(t)\Delta_<(t) + \theta(-t)\Delta_>(t)\right] \qquad 3\text{-}9$$

In equation 3-9, the negative sign is for fermions. Some possible vertices arising from these conventions are given in Table 3-1

	Type 1	Type 2
(i) Scalar	$\frac{g\phi_1^3}{3!}$	$\frac{g\phi_2^3}{3!}$
	$\frac{\lambda\phi_1^4}{4!}$	$-\frac{\lambda\phi_2^4}{4!}$
(ii) Yukawa	$h\bar{\psi}_1\psi_1\phi_1$	$-h\bar{\psi}_2\psi_2\phi_2$
(iii) Spinor-Vector	$e\bar{\psi}_1 A\!\!\!/_1\psi_1$	$-e\bar{\psi}_2 A\!\!\!/_2\psi_2$

Table 3-1 Vertices

The momentum-space thermal propagators will now be computed, using the equations of motion 3-4, and equations 3-5 to 3-9. Unlike the zero-temperature case, one cannot simply perform a Fourier transform on the configuration-space equations of motion, since the time derivative is defined on the contour C. First, a spatial Fourier transform is performed, and the resulting equations are solved for the relevant Green's function $\Delta(\tau, k_i)$ (k_i being the space components of the momentum). The components of the propagator are then identified according to equations 3-7 to 3-9. Finally, the momentum-space propagators are obtained by Fourier-transforming over the real time t, regulated by a "switching-off" term $e^{-\epsilon|t|}$.

3. Scalar Propagator

The equation of motion is

$$(\partial^2 + m^2)_c \Delta(x) = \delta_c^4(x) \qquad 3\text{-}10$$

Defining

$$\Delta(\tau, k_i) = \int d^3x \, e^{-i\bar{k}\cdot\bar{x}} \Delta(x) \qquad 3\text{-}11$$

and

$$\omega = (\bar{k}^2 + m^2)^{\frac{1}{2}} \qquad 3\text{-}12$$

one obtains

$$(\partial_\tau^2 + \omega^2)\Delta(\tau,k_i) = \delta_c(\tau) \qquad 3\text{-}13$$

The general homogeneous solution of 3-13 is

$$\Delta_h = Ae^{-i\omega\tau} + Be^{i\omega t} \qquad 3\text{-}14$$

so one writes

$$\begin{aligned}\Delta(\tau,k_i) &= \theta_c(\tau)[Ae^{-i\omega\tau} + Be^{i\omega\tau}] \\ &\quad + \theta_c(-\tau)[Ce^{-i\omega\tau} + De^{i\omega\tau}]\end{aligned} \qquad 3\text{-}15$$

the boundary condition 3-6 implies

$$C = Ae^{-\beta\omega} \qquad B = De^{-\beta\omega} \qquad 3\text{-}16$$

Integrating equation 3-13 from 0 to 0_+ yields

$$\partial_\tau \Delta(\tau,k_i)\Big|_-^+ = 1 \qquad 3\text{-}17$$

To completely specify Δ, there remains continuity at $\tau = 0$;

$$\Delta_<(0) = \Delta_>(0) \qquad 3\text{-}18$$

Combining 3-16 to 3-18, one obtains

$$\Delta(\tau,k_i) = \frac{i}{2\omega(1-e^{-\beta\omega})}\{\theta_c(\tau)(e^{-i\omega\tau} + e^{-\beta\omega}e^{i\omega\tau}) + \theta_c(-\tau)(e^{-\beta\omega}e^{-i\omega\tau} + e^{i\omega\tau})\} \tag{3-19}$$

the components of the propagator are identified from equations 3-7 to 3-9;

$$\Delta_{11}(t,k_i) = \frac{i}{2\omega(1-e^{-\beta\omega})}\{\theta(t)(e^{-i\omega_- t} + e^{-\beta\omega}e^{i\omega_+ t}) + \theta(-t)(e^{-\beta\omega}e^{-i\omega_+ t} + e^{i\omega_- t})\} \tag{3-20}$$

$$\Delta_{22}(t,k_i) = -\Delta^*_{11}(t,k_i) \tag{3-21}$$

$$\Delta_{12}(t,k_i) = \Delta_{21}(t,k_i) = \frac{ie^{-\beta\frac{\omega}{2}}}{2\omega(1-e^{-\beta\omega})}\{\theta(t)(e^{-i\omega_- t} + e^{i\omega_+ t}) + \theta(-t)(e^{-i\omega_+ t} + e^{i\omega_- t})\} \tag{3-22}$$

In the above, the regulating term $e^{-\varepsilon|t|}$ has been included, and absorbed into

$$\omega_\pm \equiv \omega \pm i\varepsilon \tag{3-23}$$

It only remains now to perform the integration $\int_{-\infty}^{+\infty} dt\, e^{ik_\circ t}$ to obtain the momentum-space propagator. The integrations are elementary, and they yield

$$\Delta_{11}(k) = \frac{-1}{k_\circ^2 - \omega^2} + \frac{i}{e^{\beta\omega}-1}\,\frac{k_\circ^2 + \omega^2}{\omega^2}\,\frac{2\varepsilon\omega}{(k_\circ^2-\omega^2)^2+4\varepsilon^2\omega^2} \qquad \text{3-24}$$

$$\Delta_{22}(k) = -\Delta_{11}^*(k) \qquad \text{3-25}$$

$$\Delta_{12}(k) = \Delta_{21}(k) = \frac{ie^{\beta\frac{\omega}{2}}}{e^{\beta\omega}-1}\,\frac{k_\circ^2 + \omega^2}{\omega^2}\,\frac{2\varepsilon\omega}{(k_\circ^2-\omega^2)^2+4\varepsilon^2\omega^2} \qquad \text{3-26}$$

Using the identification

$$\lim_{\varepsilon\to 0} \frac{\varepsilon}{x^2+\varepsilon^2} = \pi\delta(x)$$

these can be written

$$\Delta_{11}(k) = \frac{-1}{k_\circ^2-\omega^2} + \frac{2\pi i\delta(k_\circ^2-\omega^2)}{e^{\beta\omega}-1} \qquad \text{3-27}$$

$$\Delta_{22}(k) = -\Delta_{11}^*(k) \qquad \text{3-28}$$

$$\Delta_{12}(k) = \Delta_{21}(k) = \frac{2\pi i e^{\frac{\beta\omega}{2}} \delta(k_o^2 - \omega^2)}{e^{\beta\omega} - 1} \qquad 3\text{-}29$$

4. Fermion Propagator

The equation of motion is

$$(i\not\partial - m)_{\alpha\gamma} S_{\gamma\delta} = \delta_{\alpha\delta} \delta_c^4 \qquad 3\text{-}30$$

A spatial Fourier transform gives

$$(i\gamma_o \partial_\tau + k_i \gamma^i - m)S = \delta_c(\tau) \qquad 3\text{-}31$$

which can be written

$$\partial_\tau S - i\gamma^\circ (k_i \gamma^i - m) S = -i\gamma^\circ \delta_c(T) \qquad 3\text{-}32$$

The homogeneous solution of 3-32 is

$$\begin{aligned} S_h &= \left[\cos \omega\tau + i\gamma^\circ \frac{(k_i \gamma^i - m)}{\omega} \sin \omega\tau\right] S_o \qquad 3\text{-}33 \\ &\equiv A(\tau) S_o \end{aligned}$$

Thus

$$S = \theta_c(\tau)A(\tau)S_\circ + \theta_c(-\tau)A(\tau)S'_\circ \qquad 3\text{-}34$$

The trace condition for fermions is

$$S_<(\tau) = -S_>(\tau-i\beta) \qquad 3\text{-}35$$

and equation 3-32 when integrated gives

$$S_>(0) - S_<(0) = -i\gamma^\circ \qquad 3\text{-}36$$

The unique solution is

$$S_\circ = -\frac{1}{2}\left[\gamma_\circ + \frac{(k_i\gamma^i+m)}{\omega}\left(1-\frac{2}{e^{\beta\omega}+1}\right)\right] \qquad 3\text{-}37$$

$$S'_\circ = -\frac{1}{2}\left[-\gamma_\circ + \frac{(k_i\gamma^i+m)}{\omega}\left(1-\frac{2}{e^{\beta\omega}+1}\right)\right] \qquad 3\text{-}38$$

To obtain the 11-component of the momentum-space propagator, one simply puts $\tau = t$ in equation 3-34, includes the regulating term $e^{-\varepsilon|t|}$, and integrates;

$$S_{11}(k) = \int_{-\infty}^{+\infty} dt\; e^{ik_\circ t}\, S(t,k_i)e^{-\varepsilon|t|} \qquad 3\text{-}39$$

$$= (\not{k} + m)\left[\frac{1}{k_o^2 - \omega_-^2} + \frac{2\pi i\delta(k_o^2 - \omega^2)}{e^{\beta\omega} + 1}\right]$$

The 22-component is obtained from equation 3-9. The result is

$$S_{22}(k) = (\not{k} + m)\left[\frac{1}{k_o^2 - \omega_+^2} + \frac{2\pi i\delta(k_o^2 - \omega^2)}{e^{\beta\omega} + 1}\right] \qquad 3\text{-}40$$

Then 12- and 21- components follow from equation 3-8;

$$S_{12}(k) = S_{21}(k) = -i\epsilon(k_o)2\pi\delta(k_o^2 - \omega^2)\ \frac{e^{\frac{\beta\omega}{2}}}{e^{\beta\omega} + 1} \qquad 3\text{-}41$$

where, at the risk of confusing symbols,

$$\epsilon(k_o) = \begin{cases} +1, & k_o > 0 \\ -1, & k_o < 0 \end{cases}$$

5. Massless vector propagator

The equation of motion is

$$[\partial^2 g_{\mu\nu} - (1 - \tfrac{1}{\alpha})\partial_\mu \partial_\nu]D^\nu_\lambda = g_{\mu\lambda}\delta^4_c \qquad 3\text{-}42$$

A spatial fourier transform gives $\left(\omega = \left(\vec{k}^2\right)^{\frac{1}{2}}\right)$

$$\left(\partial_\tau^2 + \omega^2\right) D_{o\lambda} - \left(1 - \frac{1}{\alpha}\right)\partial_\tau\left(\partial_\tau D^{o}_{\lambda} - ik_j D^{j}_{\lambda}\right) = g_{o\lambda}\delta_c(\tau) \qquad 3\text{-}43$$

$$\left(\partial_\tau^2 + \omega^2\right) D_{i\lambda} + \left(1 - \frac{1}{\alpha}\right) ik_i\left(\partial_\tau D^{o}_{\lambda} - ik_j D^{j}_{\lambda}\right) = g_{i\lambda}\delta_c(\tau) \qquad 3\text{-}44$$

Furthermore, differentiating equation 3-42 gives

$$\partial^2 \partial_\nu D^{\nu}_{\lambda} = \alpha \partial_\lambda \delta^4_c \qquad 3\text{-}45$$

and its spatial Fourier transform

$$\left(\partial_\tau^2 + \omega^2\right)\left(\partial_\tau D^{o}_{\lambda} - ik_j D^{j}_{\lambda}\right) = \alpha \begin{cases} \alpha_\tau \delta_c(\tau) & , \ \lambda = 0 \\ -ik_\ell \delta_c(\tau) & , \ \lambda = \ell \end{cases} \qquad 3\text{-}46$$

The homogeneous solution to equation 3-46 is

$$\partial_\tau (D_h)^{o}_{\lambda} - ik_j (D_h)^{j}_{\lambda} = A_\lambda e^{-i\omega\tau} + B_\lambda e^{i\omega\tau} \qquad 3\text{-}47$$

where A_λ and B_λ are constant four-vectors to be determined. Now the homogeneous form of equations 3-43, 3-44 can be written

$$\left(\partial_\tau^2 + \omega^2\right)(D_h)_{o\lambda} + \left(1 + \frac{1}{\alpha}\right) i\omega\left(A_\lambda e^{-i\omega\tau} - B_\lambda e^{i\omega\tau}\right) = 0 \qquad 3\text{-}48$$

$$\left(\partial_\tau^2+\omega^2\right)(D_h)_{i\lambda} + \left(1-\frac{1}{\alpha}\right) i k_i \left(A_\lambda e^{-i\omega\tau} + B_\lambda e^{i\omega\tau}\right) = 0 \qquad 3\text{-}49$$

Defining $(k_\pm)^\mu = (\omega, \pm\bar{k})$ these can be combined;

$$\left(\partial_\tau^2+\omega^2\right)(D_h)_{\mu\lambda} + i\left(1-\frac{1}{\alpha}\right)\left[(k_+)_\mu A_\lambda e^{-i\omega\tau} - (k_-)_\mu B_\lambda e^{i\omega\tau}\right] = 0 \qquad 3\text{-}50$$

The general solution to this is

$$\begin{aligned}(D_h)_{\mu\lambda} &= \left[\frac{\tau}{2\omega}\left(1-\frac{1}{\alpha}\right)(k_+)_\mu A_\lambda + a_{\mu\lambda}\right]e^{-i\omega\tau}\\ &+ \left[\frac{\tau}{2\omega}\left(1-\frac{1}{\alpha}\right)(k_-)_\mu B_\lambda + b_{\mu\lambda}\right]e^{i\omega\tau}\end{aligned} \qquad 3\text{-}51$$

where $a_{\mu\lambda}$ and $b_{\mu\lambda}$ are constant tensors to be determined. Consistency with equation 3-47 requires

$$A_\lambda = -\frac{2i\alpha}{1+\alpha}(k_+)^\mu a_{\mu\lambda} \qquad B_\lambda = \frac{2i\alpha}{1+\alpha}(k_-)^\mu b_{\mu\lambda} \qquad 3\text{-}52$$

The solution to the homogeneous equation 3-50 is therefore

$$\begin{aligned}(D_h)_{\mu\lambda} &= \left[g_\mu^\nu - \frac{i\tau}{\omega}\frac{\alpha-1}{\alpha+1}(k_+)_\mu(k_+)^\nu\right]a_{\nu\lambda}e^{-i\omega\tau}\\ &+ \left[g_\mu^\nu + \frac{i\tau}{\omega}\frac{\alpha-1}{\alpha+1}(k_-)_\mu(k_-)^\nu\right]b_{\nu\lambda}e^{i\omega\tau}\\ &\equiv P_+(\tau)_\mu^\nu a_{\nu\lambda}e^{-\omega\tau} + P_-(\tau)_\mu^\nu b_{\nu\lambda}e^{i\omega\tau}\end{aligned} \qquad 3\text{-}53$$

Using the trace condition $D^<_{\mu\nu}(\tau) = D^>_{\mu\nu}(\tau-i\beta)$, the solution to equations 3-43, 3-44 is

$$D^{>}_{\mu\nu} = P_+(\tau)^{\nu}_{\mu} a_{\nu\lambda} e^{-i\omega\tau} + P_-(\tau+i\beta)^{\nu}_{\mu} b_{\nu\lambda} e^{-\beta\omega} e^{i\omega\tau} \qquad 3\text{-}54$$

$$D^{<}_{\mu\nu} = P_+(\tau-i\beta)^{\nu}_{\mu} a_{\nu\lambda} e^{-\beta\omega} e^{i\omega\tau} + P_-(\tau)^{\nu}_{\mu} b_{\nu\lambda} e^{i\omega\tau} \qquad 3\text{-}55$$

The tensors $a_{\mu\nu}$, $b_{\mu\nu}$ are determined by two conditions at $\tau = 0$; continuity:

$$D^{>}_{\mu\nu}(0) = D^{<}_{\mu\nu}(0) \qquad 3\text{-}56$$

and the integrated equation of motion;

$$\partial_\tau D^{>}_{\mu\nu}(0) - \partial_\tau D^{<}_{\mu\nu}(0) = g_{\mu\nu} + (\alpha-1) g_{\mu o} g_{\nu o} \qquad 3\text{-}57$$

The rather tedious details of the arithmetic are omitted. In presenting the result, Lorentz indices are suppressed. Thus, $g_{\mu\nu}$ is written simply g, $(k_+)_\mu (k_+)_\nu$ is written k_+k_+ , and so on.

$$a = \frac{1}{2\omega(1-e^{-\beta\omega})}\left[g + \frac{\beta(1-\alpha)}{2\omega(e^{\beta\omega}-1)} k_+k_+ - \frac{1-\alpha}{4\omega^2}(k_+k_- + k_-k_+)\right] \qquad 3\text{-}58$$

and b is this same expression, but with k_+ and k_- interchanged.

Substituting into equations 3-54 and 3-55 gives, with

$$Q \equiv g - \frac{(1-\alpha)}{4\omega^2}(k_+k_- + k_-k_+) \ ,$$

$$D^> = \frac{i}{2\omega(1-e^{-\beta\omega})}\Big\{Q(e^{-i\omega\tau} + e^{-\beta\omega}e^{i\omega\tau}) + \frac{(1-\alpha)}{2\omega}\frac{\beta}{e^{\beta\omega}-1}\left(k_+k_+e^{i\omega\tau} + k_-k_-e^{i\omega\tau}\right) + \frac{(1-\alpha)}{2\omega}\, i\tau\left(k_+k_+e^{-i\omega\tau} - k_-k_-e^{-\beta\omega}e^{i\omega\tau}\right)\Big\} \qquad 3\text{-}59$$

$$D^< = \frac{i}{2\omega(1-e^{-\beta\omega})}\Big\{Q(e^{-\beta\omega}e^{-i\omega\tau} + e^{i\omega\tau}) + \frac{(1-\alpha)}{2\omega}\frac{\beta}{e^{\beta\omega}-1}\left(k_+k_+e^{-i\omega\tau} + k_-k_-e^{i\omega\tau}\right) + \frac{(1-\alpha)}{2\omega}\, i\tau\left(k_+k_+e^{-\beta\omega}e^{-i\omega\tau} - k_-k_-e^{i\omega\tau}\right)\Big\} \qquad 3\text{-}60$$

The 11 component of the thermal propagator in momentum space is (Lorentz indices suppressed);

$$D_{11}(k) = \int_{-\infty}^{+\infty} dt\, e^{ik_\circ t}e^{-\epsilon|t|}\left[D^>(t)\theta(t) + D^<(t)\theta(-t)\right] \qquad 3\text{-}61$$

Again omitting the elementary integrations and subsequent arithmetic, the result is

$$(D_{\mu\nu}(k))_{11} = [g_{\mu\nu} - (1-\alpha)k_\mu k_\nu \frac{\partial}{\partial\omega^2}]\Delta_{11}(k) \qquad 3\text{-}62$$

where $\Delta_{11}(k)$ is the 11 component of the massless scalar boson propagator. It turns out that the other components follow the same pattern, so that

$$(D_{\mu\nu}(k))_{ij} = [g_{\mu\nu} - (1-\alpha)k_\mu k_\nu \frac{\partial}{\partial\omega^2}]\Delta_{ij}(k) \qquad 3\text{-}63$$

There is one caveat that should be mentioned. The temperature-dependent exponential occurring in these equations is $\exp(\beta\omega)$, not $\exp(\beta|k_o|)$ as occurs in other conventions. This would not be significant were it not for the derivative $\partial/\partial\omega^2$ occurring in the vector propagator. Using $\exp(\beta|k_o|)$ in the above expressions would give a different distribution.

6. Summary

The thermal propagators for scalar, fermion and massless vector fields have been computed. They are presented here together with conventional factors of i included.

(i) Scalar of mass m ;

$$\Delta(k) = \begin{vmatrix} \Delta_{11}(k) & \Delta_{12}(k) \\ \Delta_{21}(k) & \Delta_{22}(k) \end{vmatrix} = \begin{vmatrix} \Delta_B(k) + \Delta_\beta(k) & \tilde{\Delta}_\beta(k) \\ \tilde{\Delta}_\beta(k) & -\tilde{\Delta}_B(k) + \Delta_\beta(k) \end{vmatrix} \quad 3\text{-}64$$

where

$$\Delta_B(k) = \frac{i}{k^2-m^2+i\varepsilon} \;, \; \tilde{\Delta}_B(k) = \frac{i}{k^2-m^2-i\varepsilon}$$
$$\Delta_\beta(k) = 2\pi\delta(k^2-m^2)f_B(k)$$
$$\tilde{\Delta}_\beta(k) = 2\pi\delta(k^2-m^2)g_B(k) \quad 3\text{-}65$$
$$f_B(k) = \frac{1}{e^{\beta\omega}-1} \;, \; g_B(k) = \frac{e^{\frac{\beta\omega}{2}}}{e^{\beta\omega}-1}$$
$$\omega = (\vec{k}^2 + m^2)^{\frac{1}{2}}$$

(ii) Fermion of mass m

$$S(k) = (\not{k} + m) \times \begin{vmatrix} \Delta_B(k) - 2\pi\delta(k^2-m^2)f_F(k) & \varepsilon(k_o)2\pi\delta(k^2-m^2)g_F(k) \\ \varepsilon(k_o)2\pi\delta(k^2-m^2)g_F(k) & \tilde{\Delta}_B(k) + 2\pi\delta(k^2-m^2)f_F(k) \end{vmatrix} \quad 3\text{-}66$$

where

$$\varepsilon(k_o) = \begin{cases} +1 & (k_o > 0) \\ -1 & (k_o < 0) \end{cases} \quad 3\text{-}67$$

$$f_F(k) = \frac{1}{e^{\beta\omega}+1} \;, \; g_F(k) = \frac{e^{\frac{\beta\omega}{2}}}{e^{\beta\omega}+1}$$

(iii) Massless vector boson

$$- D_{\mu\nu}(k) = - [g_{\mu\nu} - (1-\alpha) k_\mu k_\nu \frac{\partial}{\partial \omega^2}] \Delta(k) \qquad 3\text{-}68$$

where $\Delta(k)$ is the thermal propagator for a massless scalar boson, as in (i) with $m = 0$.

Figure 3-1: The contour of the imaginary time formalism (ITF).

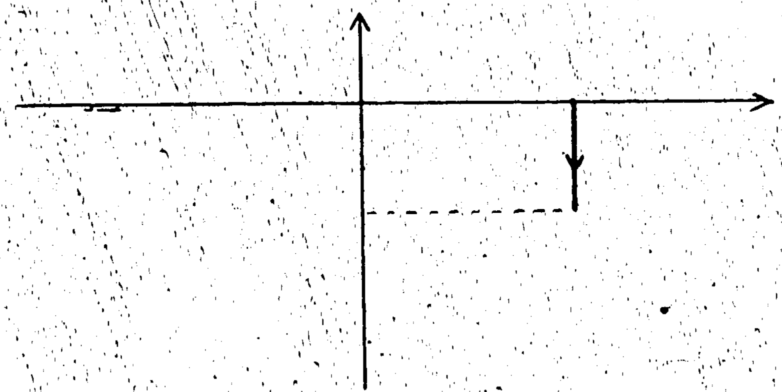


Figure 3-2; The contour of the real-time formalism (TFT).

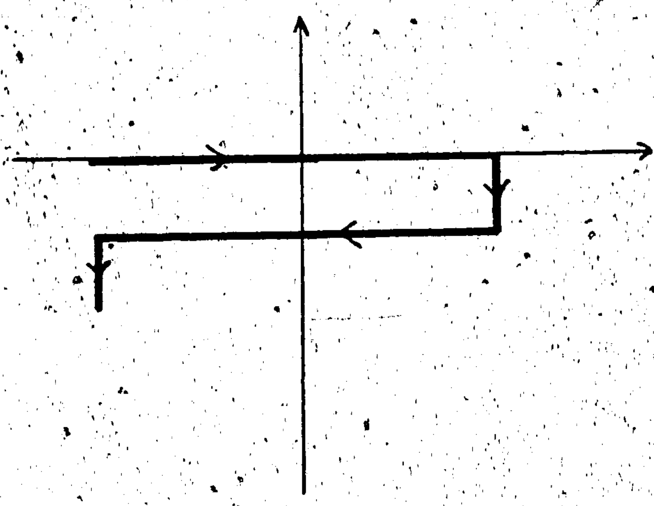
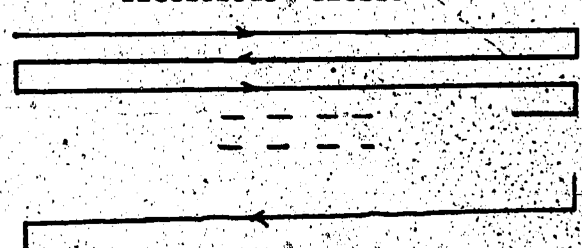


Figure 3-3; The contour of real-time formalism with 2n-1 "fictitious" fields



Chapter 4 The Free Energy and the Finite Temperature Effective Potential: Scalar Field Theory.

1. Introduction

The link between thermodynamics and quantum field theory, via the formalism of Thermo Field Theory, has been discussed elsewhere[15]. An important expression of this connection is the equation

$$f(\phi) = i \int^{\phi} da\, \Gamma_{1,0}(\phi_1,\phi_2)|_{\phi_1=\phi_2=a} \qquad 4\text{-}1$$

$f(\phi)$ is the free energy of the vacuum at some non-zero temperature in a model with a scalar field ϕ. $\Gamma_{1,0}$ is the one-point thermal proper function, with the external field being the physical one (ϕ_1). This is calculable perturbatively, using the Feynman rules discussed previously. Before more is said about these calculations, perhaps a few remarks should be made about the physical significance of the free energy.

There has been much interest in the last decade or so in grand unified theories of matter, for which various models have been proposed. Any realistic model must include a mechanism for symmetry-breaking, to account for the asymmetries of the observed particle spectrum and its interactions. Having adopted a particular model, one can examine its predictions about the early universe by looking at the behaviour of the scattering amplitudes as interaction energies increase. The standard picture is that, as the disparate particles' energies increase, they and their interactions "converge", until a point is reached where full symmetry is restored.

Another approach could be to view the various particles as interacting in a thermal background. Thermo field theory provides the formalism with which to embark on this approach. The physical parameters of a model would now develop a temperature dependence, which would determine in particular the behaviour of the model in the early universe (i.e. at extremely high temperatures). The restoration of symmetry with increasing temperature would now be expressed as a phase transition occurring at some critical temperature T_c. This critical temperature would in principle be calculable in terms of the observed parameters appearing in the model (masses and coupling constants measured in the laboratory). It is the free energy, as defined in equation 4-1, which allows one to examine the behaviour of the effective, temperature-dependent parameters. For example, in a model of a single, real scalar field ϕ with ϕ^4 interaction, the effective parameters would be the mass squared;

$$m^2(T) = \frac{d^2 f(\phi)}{d\phi^2}\Big|_{\phi=0}$$

and the coupling 4-2

$$\lambda(T) = \frac{d^4 f(\phi)}{d\phi^4}\Big|_{\phi=0}$$

The right-hand sides of these equations, as computed using equation

4-1, would depend on the physical parameters and the temperature T.

The remainder of this chapter will be concerned with the calculation of $f(\phi)$ in the $\lambda\phi^4$ model. The elucidation of the techniques employed would, it is hoped, facilitate their application to more realistic models.

After establishing a derivative formula for the thermal propagators of Chapter Three, this is used in an examination of the two-and three-loop contributions to the one-point function $\Gamma_{1;0}(a,a)$. It is demonstrated to this order that, at least in the simple model considered here, the free energy is identical to another, more easily computed function, which shall be called the finite temperature effective potential (FTEP). It is given this designation in recognition[1] of its similarity to the effective potential of Chapter One.

The appearance of imaginary contributions to the free energy in more complicated models is discussed.

Next, a two-loop computation is done explicitly in both the Thermo Field and the Imaginary Time formalisms, and the results are shown to agree, apart from an imaginary term in the Thermo Field calculation.

Finally, a complete two-loop calculation of the FTEP is performed.

2. The Finite Temperature Effective Potential

To begin with, it is fairly straightforward to establish that the propagators of Chapter Three have the following mass derivative formulae (with the change of $e^{\beta\omega_k} \to e^{\beta|k_0|}$);

(i) Scalar Boson

$$\frac{1}{n!}\left(i\frac{\partial}{\partial m^2}\right)^n \Delta(k)\tau = \left[\Delta(k)\tau\right]^{n+1} \qquad 4\text{-}3$$

where

$$\tau = \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} \qquad 4\text{-}4$$

(ii) Fermion

$$\frac{1}{n!}\left(i\frac{\partial}{\partial m}\right)^n S(k) = \left[S(k)\right]^{n+1} \qquad 4\text{-}5$$

These formulae will be useful in discussing the "taming" of serious singularities that appear in individual Feynman diagrams. They will also help in the formulation of the FTEP.

Apart from the propagators, the Feynman rules for the type-1 (physical) fields are identical to those in the model considered in Chapter One (Fig-2-1). In addition, there is a corresponding set of vertices and counterterms for the type-2 field, differing from the type-1 only in sign. The thermal propagators in the shifted theory ($\phi_1 \to \phi_1 + a$; $\phi_2 \to \phi_2 + a$) are just those of Chapter Three, with $m^2 = m_0^2 + \lambda a^2/2$.

The Feynman diagrams contributing to two loops, to the one-point function $\Gamma_{1,0}(a,a)$ are shown in Figures 4-1, 4-2 and 4-3, apart from an overall two-loop counterterm. The external leg is understood to be always of type 1. The vertices and counterterms are labelled according to their type (1 or 2). The propagators are determined by the types of the vertices that they join. Thus, the propagator carrying momentum k, and joining a type-i vertex to a type-j vertex, would be $\Delta_{ij}(k)$.

For the purpose of defining the FI[illegible] only the two- and three-loop diagrams of 4-2 to 4-5 will be considered for the time being.

The two-loop contributions to the one-point function are shown in Figure 4-2. Those of Figures 4-2(a) and (b) are

$$-\frac{\lambda^2 a}{4}\int\frac{d^Dk}{(2\pi)^D}\frac{d^Dp}{(2\pi)^D}\left[\Delta_{11}^2(k)\Delta_{11}(p)-\Delta_{12}^2(k)\Delta_{22}(p)\right] \qquad 4\text{-}6$$

Figures 4-2(c) and (d) give

$$-\frac{\lambda^2 a}{6}\int\frac{d^Dk}{(2\pi)^D}\frac{d^Dp}{(2\pi)^D}\left[\Delta_{11}(k)\Delta_{11}(p)\Delta_{11}(k+p)-\Delta_{12}(k)\Delta_{12}(p)\Delta_{12}(k+p)\right] \qquad 4\text{-}7$$

(e), (f) and (g) contribute

$$-\frac{i\lambda^3 a^3}{4}\int\frac{d^Dk}{(2\pi)^D}\frac{d^Dp}{(2\pi)^D}\left[\Delta_{11}^2(k)\Delta_{11}(p)\Delta_{11}(k+p) - 2\Delta_{11}(k)\Delta_{12}(k)\Delta_{12}(p)\Delta_{12}(k+p)+\Delta_{12}^2(k)\Delta_{22}(p)\Delta_{22}(k+p)\right] \qquad 4\text{-}8$$

The contributions involving one-loop counterterms are; from (h) and (i):

$$-\frac{i\lambda a}{2}\int\frac{d^Dk}{(2\pi)^D}\left[iAk^2-im_0^2(B+C\lambda a^2/2m_0^2)\right]\left[\Delta_{11}^2(k)-\Delta_{12}^2(k)\right] \quad 4\text{-}9$$

And from (j)

$$-i\frac{C\lambda a}{2}\int\frac{d^Dk}{(2\pi)^D}\Delta_{11}(k) \quad 4\text{-}10$$

In 4-6, one can use the identity

$$\int\frac{d^Dk}{(2\pi)^D}\Delta_{22}(k)=\int\frac{d^Dk}{(2\pi)^D}\Delta_{11}(k) \quad 4\text{-}11$$

Then, using the derivative formula equation 4-3, this implies that

$$\begin{aligned}[\text{expression 4-6}] &= -\frac{i\lambda^2 a}{2}\frac{\partial}{\partial m^2}\int\frac{d^Dk}{(2\pi)^D}\frac{d^Dp}{(2\pi)^D}\Delta_{11}(k)\Delta_{11}(p)\\ &= -\frac{i\lambda}{8}\frac{\partial}{\partial a}\int\frac{d^Dk}{(2\pi)^D}\frac{d^Dp}{(2\pi)^D}\Delta_{11}(k)\Delta_{11}(p)\\ &= \frac{\partial}{\partial a}[\text{Fig 4-4(a)}]\end{aligned} \quad 4\text{-}12$$

So, it is now apparent how the individual terms in 4-6, which contain the dangerous singular term $\delta^2(k^2-m^2)$, combine to give a well-defined distribution. It is simply by the derivative formula 4-3.

Here, use was made of the relation $a\lambda\partial/\partial m^2 = \partial/\partial a$. In 4-7, the second term in square brackets vanishes because of the three delta functions involved. The arguments of these delta functions cannot simultaneously vanish. So, one can write

$$[\text{expression 4-7}] = \frac{\partial}{\partial a}\left(-\frac{\lambda^2 a^2}{12}\right) \int \frac{d^D k}{(2\pi)^D} \frac{d^D p}{(2\pi)^D} \Delta_{11}(k)\Delta_{11}(p)\Delta_{11}(k+p) \qquad \text{4-13}$$

With some manipulations, the expression 4-8 can be written

$$-\frac{i\lambda^3 a^3}{4} \int \frac{d^D k}{(2\pi)^D} \frac{d^D p}{(2\pi)^D} [\Delta_{11}^2(k) - \Delta_{12}^2(k)]\Delta_{11}(p)\Delta_{11}(k+p) + R \qquad \text{4-14}$$

R is a term that contains the factor

$$\delta^2(k^2-m^2)\delta(p^2-m^2)\delta((k+p)^2-m^2) \qquad \text{4-15}$$

This is a troublesome distribution, in that a naive evaluation of R yields a finite result which depends on the prescription used to define the delta function. An argument for discarding it is presented by Niemi and Semenoff[15]. Discarding R enables one to combine 4-13 and

4-14 as

$$\frac{\partial}{\partial a}\left[-\frac{i\lambda^2 a^2}{12}\int \frac{d^Dk}{(2\pi)^D}\frac{d^Dp}{(2\pi)^D}\Delta_{11}(k)\Delta_{11}(p)\Delta_{11}(k+p)\right]$$
$$=\frac{\partial}{\partial a}\left[\text{Fig 4-4(b)}\right] \qquad 4\text{-}16$$

Finally, the expressions 4-9, 4-10 containing one-loop counterterms can be combined, and written

[expression 4-9+4-10] =

$$=\frac{\partial}{\partial a}\left\{\frac{1}{2}\int \frac{d^Dk}{(2\pi)^D}\left[iAk^2-im_0^2(B + C\lambda a^2/2m_0^2)\right]\Delta_{11}(k)\right\}$$
$$=\frac{\partial}{\partial a}\left[\text{Fig. 4-4(c)}\right] \qquad 4\text{-}17$$

So, it has been shown that the two-loop contribution to the one-point function is just the derivative of the two-loop contribution to the "effective potential" calculated with only type 1 vertices and propagators. One may then naively equate the free energy and this effective potential. However, this association breaks down at the three-loop level, where diagrams like those of Fig. 4-5 contain a singular term $\Delta_{11}^2(k)$. Instead, one considers the one-point diagrams of Fig. 4-3, which combine to give

[Fig. 4-3] =

$$=\frac{\partial}{\partial a}\left\{-\frac{\lambda^2}{16}\int \frac{d^Dk}{(2\pi)^D}\frac{d^Dp}{(2\pi)^D}\frac{d^Dq}{(2\pi)^D}\Delta_{11}(k)\left[i\frac{\partial}{\partial m^2}\Delta_{11}^2(p)\right]\Delta_{11}(q)\right\} \qquad 4\text{-}18$$

The expression in curly brackets can be represented by the vacuum diagram of Fig. 4-5, if one makes the substitution

$$\Delta_{11}^{2}(k) \rightarrow i \frac{\partial}{\partial m^{2}} \Delta_{11}(k) \qquad 4\text{-}19$$

One may now postulate the following prescription for computing the free energy;

1. Draw all diagrams contributing to the effective potential just as in Chapter One.

2. Use the temperature-dependent propagator $\Delta_{11}(k)$ in place of the zero-temperature one $i/(k^2 - m^2 + i\epsilon)$.

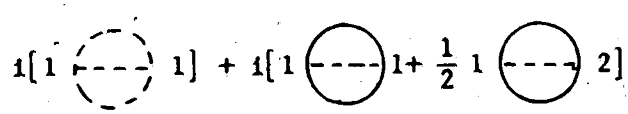
3. Whenever a product of propagators carrying the same momentum occurs, make the substitution

$$[\Delta_{11}(k)]^{n} \rightarrow \frac{1}{(n-1)!} \left(i \frac{\partial}{\partial m^{2}}\right)^{n-1} \Delta_{11}(k) \qquad 4\text{-}20$$

The resulting function will be called the finite temperature effective potential, and is postulated to be equal to the free energy. The economy of this identification is apparent from the drastic reduction in the number of diagrams to be evaluated.

3. Imaginary contribution to $f(\phi)$

A complication arises when other fields are present. In this case, diagrams of the type 4-2(d) do not vanish, if the propagators are those of different fields. For example, consider a model with two interacting scalar fields ϕ and Φ, with masses m and M respectively. If one calculates the two-loop contribution to the free energy $f(\phi)$ from overlapping diagrams like those of Fig. 4.2(c) to (g), the result is (having shifted the field $\phi\to\phi+a$);



4-21

Here, the solid lines are Φ propagators, the dashed lines are ϕ propagators, and the vertices are labelled according to their thermal type. In this case, the prescription given in Section Two fails because of the contribution of the second diagram in the second term of 4-21. Note, however, that this contribution is manifestly imaginary, due to the real nature of the 12 components of the propagators. It has been demonstrated(19) that the imaginary part of the free energy must vanish. Thus, the imaginary contribution of the last diagram is cancelled by imaginary contributions from the second diagram. The only modification to the prescription given in section two, therefore, is to neglect any imaginary contributions arising from vacuum graphs.

4. Comparison with the Imaginary Time Formalism [18]

The Imaginary Time Formalism (ITF) also admits the use of conventional Feynman rules in the computation of thermal Green's functions. One striking difference between ITF and Thermo Field Theory (TFT) is the absence of the time co-ordinate in the former, due to the nature of the analytic continuation in time (see Chapter Three). In practical calculations, ITF presents further difficulties. Firstly, the separation between temperature-dependent and temperature-independent terms is not apparent, because of the nature of the propagator; for a scalar field, this is

$$\frac{i}{(2\pi n/\beta)^2 + \bar{k}^2 + m^2} \qquad 4\text{-}25$$

Secondly, at orders higher than one loop, manipulation of the summations in the propagators becomes unwieldy.

The purpose of this section is to perform a calculation of the two-loop diagram Fig. 4-4(b) in both formalisms, and compare the results. It will be assumed that the masses occurring in the propagators are all different. This comparison will demonstrate the difficulty of doing higher order calculations in ITF.

With the relevant masses being m_a, m_b, m_c, the integral corresponding to Fig. 4-4(b) in TFT is

$$I = \int \frac{d^Dk}{(2\pi)^D} \frac{d^D\ell}{(2\pi)^D} \left\{ \frac{i}{k^2-m_a^2+i\varepsilon} \, \frac{i}{(k-\ell)^2-m_b^2+i\varepsilon} \, \frac{}{\ell^2-m_c^2+i\varepsilon} \right. \qquad 4\text{-}26$$

$$+\frac{i}{k^2-m_a^2+i\varepsilon}\,\frac{i}{(k-\ell)^2-m_b^2+i\varepsilon}\,\frac{2\pi\delta(\ell^2-m_c^2)}{e^{\beta|\ell_0|}-1}$$

$$+\frac{i}{k^2-m_a^2+i\varepsilon}\,\frac{i}{\ell^2-m_c^2+i\varepsilon}\,\frac{2\pi\delta((k-\ell)^2-m_b^2)}{e^{\beta|k_0-\ell_0|}-1}$$

$$+\frac{i}{\ell^2-m_c^2+i\varepsilon}\,\frac{i}{(k-\ell)^2-m_b^2+i\varepsilon}\,\frac{2\pi\delta(k^2-m_a^2)}{e^{\beta|\ell_0|}-1} \qquad 4\text{-}27$$

$$+\frac{i}{k^2-m_a^2+i\varepsilon}\,\frac{2\pi\delta((k-\ell)^2-m_b^2)}{e^{\beta|k_0-\ell_0|}-1}\,\frac{2\pi\delta(\ell^2-m_c^2)}{e^{\beta|\ell_0|}-1}$$

$$+\frac{i}{(k-\ell)^2-m_b^2+i\varepsilon}\,\frac{2\pi\delta(k^2-m_a^2)}{e^{\beta|\ell_0|}-1}\,\frac{2\pi\delta(\ell^2-m_c^2)}{e^{\beta|\ell_0|}-1}$$

$$+\frac{i}{\ell^2-m_c^2+i\varepsilon}\,\frac{2\pi\delta(k^2-m_a^2)}{e^{\beta|k_0|}-1}\,\frac{2\pi\delta((k-\ell)^2-m_b^2)}{e^{\beta|k_0-\ell_0|}-1} \qquad 4\text{-}28$$

$$+\frac{2\pi\delta(k^2-m_a^2)}{e^{\beta|k_0|}-1}\,\frac{2\pi\delta((k-\ell)^2-m_b^2)}{e^{\beta|k_0-\ell_0|}-1}\,\frac{2\pi\delta(\ell^2-m_c^2)}{e^{\beta|\ell_0|}-1}\Bigg\} \qquad 4\text{-}29$$

Here, the contributions involving zero, one, two and three statistical factors have been numbered individually. The last term (4-29) is precisely the one discussed in section three, giving an imaginary contribution to the free energy. The corresponding expression in the IFT formalism can be written

$$I = \frac{i^2}{\beta^2} \sum_{n=-\infty}^{+\infty} \sum_{m=-\infty}^{+\infty} \int \frac{d^3k}{(2\pi)^3} \frac{d^3\ell}{(2\pi)^3} \frac{i}{x^2+a^2} \frac{i}{(x-y)^2+b^2} \frac{i}{y^2+c^2} \qquad 4\text{-}30$$

where

$$x = 2\pi n/\beta, \; y = 2\pi m/\beta, \; a^2 = \bar{k}^2 + m_a^2, \; b^2 = (\bar{k} - \bar{\ell})^2 + m_b^2, \; c^2 = \bar{\ell}^2 + m_c^2 \qquad 4\text{-}31$$

One approach to evaluating 4-30 is to use the formula;

$$\frac{1}{\beta} \sum_{n=-\infty}^{+\infty} f(\omega_n^{\pm}) = \frac{1}{4\pi} \int_C dz \, \frac{f(z)\left(e^{i\beta z/2} \pm e^{-i\beta z/2}\right)}{e^{i\beta z/2} \mp e^{-\beta z/2}}$$

$$= \frac{1}{4\pi} \left(\int_{-i\varepsilon-\infty}^{-i\varepsilon+\infty} dz + \int_{i\varepsilon+\infty}^{i\varepsilon-\infty} dz \right) \frac{f(z)\left(e^{i\beta z/2} \pm e^{-\beta z/2}\right)}{e^{i\beta z/2} \mp e^{-\beta z/2}} \qquad 4\text{-}32$$

where

$$\omega_n^{\pm} = \begin{cases} 2\pi n/\beta \;, & \text{bosons} \\ (2n+1)\pi/\beta \;, & \text{fermions} \end{cases}$$

and, as indicated in the second line of equation 4-32, the contour C encloses the real axis. Making this substitution in 4-30, for the summation over m, and performing the y-integration, yields (the integrals over the spatial components of k and ℓ are dropped)

$$I = \frac{1}{2\beta} \sum_{n=-\infty}^{+\infty} \left[\frac{1}{2c} \frac{1}{x^2+a^2} \left(\frac{1}{(x-ic)^2+b^2} + \frac{1}{(x-ic)^2+b^2} \right) f(c) + \frac{1}{2b} \frac{1}{x^2+a^2} \left(\frac{1}{(x-ib)^2+c^2} + \frac{1}{(x-ib)^2+c^2} \right) f(b) \right] \qquad 4\text{-}33$$

where

$$f(c) = -\coth(\beta c/2) \qquad 4\text{-}34$$

Re-arranging terms in 4-33, using 4-32 again, and finally re-arranging the result, one obtains

$$I = \frac{1}{4} \left(\frac{1}{ac} \frac{b^2-(a^2+c^2)}{[b^2-(a+c)^2][b^2-(a-c)^2]} + \frac{1}{ab} \frac{c^2-(a^2+b^2)}{[c^2-(a+b)^2][c^2-(a-b)^2]} f(a)f(b) + \frac{1}{bc} \frac{a^2-(b^2+c^2)}{[a^2-(b+c)^2][a^2-(b-c)^2]} f(b)f(c) + \frac{2}{[a^2-(b+c)^2][a^2-(b-c)^2]} \right) \qquad 4\text{-}35$$

Using $f(a) = -[1+ (e^{\beta a}-1)^{-1}]$, one can re-write this as a sum of; a temperature-independent part, a term with one statistical factor

$(e^{\beta\omega}-1)^{-1}$, and a term with two statistical factors:

$$I_0 = \frac{1}{abc}\,\frac{1}{a+b+c} \qquad 4\text{-}36$$

$$I_1 = -\frac{2}{abc}\,\frac{1}{A}\left\{\frac{(b+c)[a^2-(b-c)^2]}{e^{\beta a}-1} + \frac{(c+a)[b^2-(c-a)^2]}{e^{\beta b}-1} + \frac{(a+b)[c^2-(a-b)^2]}{e^{\beta c}-1}\right. \qquad 4\text{-}37$$

$$I_2 = \frac{4}{A}\left\{\frac{c^2-(a^2+b^2)}{ab(e^{\beta a}-1)(e^{\beta b}-1)} + \frac{a^2-(b^2+c^2)}{bc(e^{\beta a}-1)(e^{\beta c}-1)} + \frac{b^2-(c^2+a^2)}{cb(e^{\beta c}-1)(e^{\beta a}-1)}\right. \qquad 4\text{-}38$$

where

$$\begin{aligned} A &= [b^2-(a+c)^2][b^2-(c-a)^2] \\ &= [c^2-(a+c)^2][c^2-(a-b)^2] \\ &= [a^2-(b+c)^2][a^2-(b-c)^2] \\ &= a^4+b^4+c^4-2(a^2b^2+a^2c^2+b^2c^2) \end{aligned} \qquad 4\text{-}39$$

Equations 4-36 to 4-38 are formally equivalent to what is obtained by performing the k_0 and ℓ_0 integrations in equations 4-26 to 4-28. Note that the imaginary contribution (the real part of I, 4-29) does not appear in the ITF calculation. This appears to reflect an inequivalence between the two formalisms.

To end this section, the result of integration 4-26 to 4-29 is presented, with $m_a = m_b = m_c = m$ (thus causing the last term to vanish).

$$I = \frac{3im^2}{4(4\pi)^4}\left[-\frac{4}{\varepsilon}\ln m^2 + 2\ln^2 m^2 + (4\gamma - 6)\ln m^2\right]$$
$$+ \frac{3im^2}{32\pi^4}\left[\frac{2}{\varepsilon} - \gamma - \ln m^2 + \left(2 - \pi/\sqrt{3}\right)\right]\int_1^\infty dx\,\frac{(x^2-1)^{\frac{1}{2}}}{e^{\beta mx}-1}$$
$$+ \frac{3im^2}{32\pi^4}\int_1^\infty dx\int_1^\infty dy\,\frac{\ln(X_+/X_-)}{(e^{\beta mx}-1)(e^{\beta my}-1)} \qquad 4\text{-}40$$

where

$$X_\pm = \left[1 \pm 2(x^2-1)^{\frac{1}{2}}(y^2-1)^{\frac{1}{2}}\right]^2 - 4x^2y^2 \qquad 4\text{-}41$$

5. The FTEP to two loops in scalar ϕ^4 theory[16]

Since the effective potential (FTEP at T=0) has been calculated in Chapter One, only the temperature-dependent contributions will be considered here. The relevant Feynman diagrams are those of Figures 1-2(a), and 1-3(a),(b),(c), with the replacement of the zero-temperature propagators by the thermal propagator $\Delta_{11}(k)$. Recall that the mass in the shifted theory is

$$m^2 = m_0^2 + \lambda a^2/2 \qquad 4\text{-}42$$

The one-loop contribution is well-known, and is

$$V_\beta^{1\text{-loop}} = -\frac{m^4}{6\pi^2}\int_1^\infty dx(x^2-1)^{\frac{3}{2}}\frac{1}{e^{\beta mx}-1} \qquad 4\text{-}43$$

At the two-loop level, one has to include the one-loop counterterms, and so renormalization-scheme dependence enters the temperature contributions. In particular, any temperature-dependent divergences must be cancelled by the effect of these counterterms, otherwise new, temperature-dependent counterterms would have to be introduced.

Fig. 1-3(a) has been presented in the last section; including coupling constants and the symmetry factor, the temperature-dependent part is

$$I_A = -\frac{i\lambda^2 a^2 m^2}{128\pi^4}\int_1^\infty dx\int_1^\infty dy\,\frac{\ln(X_+/X_-)}{(e^{\beta mx}-1)(e^{\beta my}-1)} + \frac{i\lambda^2 a^2 m^2}{128\pi^4}\left(\frac{2}{\varepsilon}-\gamma-\ln m^2+\left(2-\frac{\pi}{\sqrt{3}}\right)\right)F_1(\beta m) \qquad 4\text{-}44$$

where

$$F_1(\beta m) = \int_1^\infty dx\,(x^2-1)^{\frac{1}{2}}\,\frac{1}{e^{\beta mx}-1} \qquad 4\text{-}45$$

Fig. 1-3(b) contributes

$$I_B = -\frac{i\lambda m^4}{32\pi^4}\left[F_1(\beta m)\right]^2 + \frac{i\lambda m^4}{128\pi^4}\left(\frac{2}{\varepsilon} - \gamma + 1 - \ln m^2\right)F_1(\beta m) \qquad 4\text{-}46$$

Using the two-point counterterm determined in Chapter One, the final contribution, from Fig. 1-3(c), is

$$I_C = -\frac{i\lambda m^2}{128\pi^4}\left[\left(m^2+\lambda a^2\right)\left(\frac{2}{\varepsilon} - \gamma\right) + m_0^2\right]F_1(\beta m) \qquad 4\text{-}47$$

Adding I_A, I_B, I_C, the cancellation of temperature-dependent divergences is evident[13-16].

Figure 4-1 One loop contribution to the one-point function

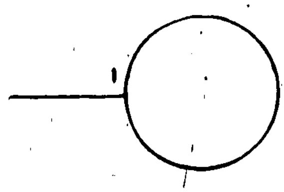


Figure 4-2 Two loop contribution to the one-point function

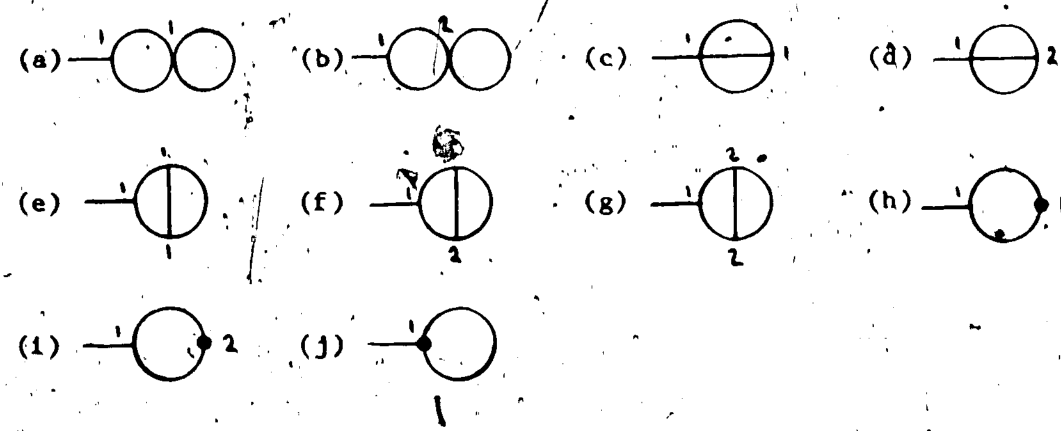


Figure 4-3 Some three loop contributions to the one-point function

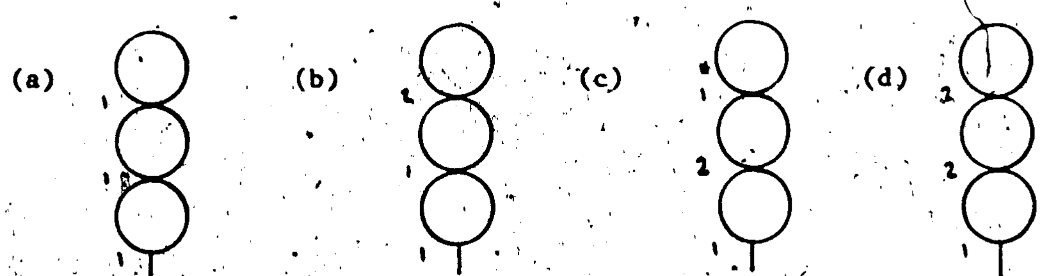


Figure 4-4 Two loop contributions to FTEP

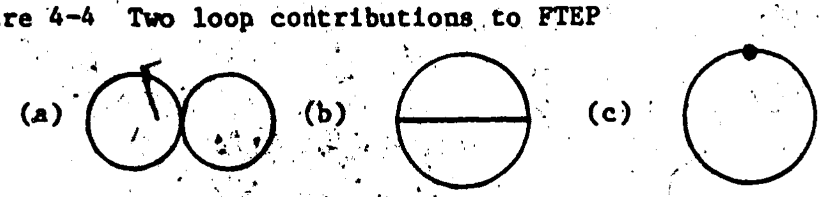
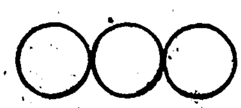


Figure 4-5 A three loop contribution to FTEP



Chapter 5 Two-loop FTEP with fermions(17)

1. Introduction.

A step up in complexity is now taken, by the addition of an interacting fermion field into the previous simple scalar model. In this way, one can gain experience in performing calculations in a more realistic model.

The (zero-temperature) Lagrangian density considered here will be

$$L(\phi,\psi) = L(\phi) + i\bar{\psi}\partial\!\!\!/\psi - M_0\bar{\psi}\psi - f\bar{\psi}\psi\phi + \text{(Additional counterterms)} \qquad 5\text{-}1$$

where $L(\phi)$ is the scalar Lagrangian used thus far (see equation 1-23). The additional counterterms are those arising from the fermion field.

This chapter will present the contribution to two loops from the fermion, to the FTEP.

2. Zero Temperature

The three new parameters occurring in the Lagrangian (ψ, M_0, f) require three normalization conditions. The one involving the fermion-fermion-scalar vertex is not relevant to the calculation done here, as the corresponding counterterms do not contribute up to and inc luding the two-loop level. The normalization conditions adopted for the fermion two-point function will be

$$S^{-1}(k=0) = iM_0$$

$$\frac{\partial S^{-1}}{\partial k^\mu}(k=0) = -i\gamma^\mu \qquad 5\text{-}2$$

Upon shifting the field $\phi \to \phi + a$ in equation 5-1, the effective fermion mass becomes

$$M \equiv M_0 + af \ . \qquad 5\text{-}3$$

Now, in the two-loop calculation one will require the counterterms for the two-point functions arising from the presence of the fermion field. These are necessitated by the diagrams of figure 5-2. If the counterterms for these diagrams are denoted

$$\delta Z_F \not{k} + (\delta M_F)M \quad \text{(Fermion)}$$

$$\delta Z_B k^2 + (\delta M_B)m^2 \quad \text{(Scalar)} \qquad 5\text{-}4$$

Then satisfying the normalization conditions requires

$$\delta Z_B = -\frac{i}{(4\pi)^2}\, 2f^2\left(\frac{2}{\varepsilon} - \gamma - \frac{2}{3} - \ln M_0^2\right)$$

$$\delta Z_F = -\frac{1}{(4\pi)^2} f^2\left(\frac{1}{\varepsilon} - \frac{\gamma}{2} - C_1\right)$$

$$\delta M_B = -\frac{1}{(4\pi)^2} (12f^2M^2)\left(\frac{2}{\varepsilon} - \gamma + \frac{1}{3} - \ell n\ M_0^2\right)$$

$$\delta M_F = -\frac{1}{(4\pi)^2} f^2\left(\frac{2}{\varepsilon} - \gamma - C_2\right) \qquad 5\text{-}5$$

where

$$C_1 = \frac{1}{2}\ell n\ m_0^2 + \frac{1}{2}\left(\frac{M_0^2}{m_0^2 - M_0^2}\right)^2 \ell n\ \frac{M_0^2}{m_0^2} - \frac{1}{4} + \frac{M_0^2}{2(m_0^2 - M_0^2)}$$

$$C_2 = \frac{1}{m_0^2 - M_0^2}\left(m_0^2\ \ell n\ m_0^2 - M_0^2\ \ell n\ M_0^2\right) - 1 \qquad 5\text{-}6$$

3. Non-zero temperature

The fermionic contribution to the FTEP will be computed using the prescription of chapter four, section two. The relevant fermion propagator is the 11-component of the matrix in equation 3-66, with mass M. With a slight change of notation to avoid confusion with the boson propagator, this will be written;

$$S_{11}(k) = (\not{k} + M)\left[\Delta_F(k) - 2\pi\delta(k^2 - M^2)f_F(k)\right] \qquad 5\text{-}7$$

The one-loop fermionic contribution to the FTEP comes from the diagram of Figure 5-1 and its counterterm. This is readily evaluated to be

$$V^f_{1\text{-loop}} = -\frac{1}{16\pi^2} M^4 \ln \frac{M^2}{M_0^2} - \Delta V^f_{1\text{-loop}}$$

$$- \frac{2M^4}{3\pi^2} \int_1^\infty dx \, \frac{(x^2-1)^{3/2}}{e^{\beta m x} + 1} \qquad 5\text{-}8$$

where

$$\Delta V^f_{1\text{-loop}} = \frac{1}{64\pi^2} \left(8afM_0^3 + 7a^2f^2M_0^2 + \frac{104}{3} a^3f^3M_0 + \frac{62}{3} a^4f^4\right) \qquad 5\text{-}9$$

The two-loop fermionic contributions come from the diagrams of Figure 5-3. The counterterms in Figure 5-3 (b) and (c) are, respectively, the scalar and fermion two-point counterterms 5-4. Each contribution will be separated into terms with zero, one and two statistical factors. The term with three statistical factors in Figure 5-3(a) will be neglected, as its contribution to the FTEP is imaginary.

Figure 5-3(a) contributes

$$I(A) = \frac{1}{2} (-if)^2 (-1) \mathrm{Tr} \int \frac{d^D k}{(2\pi)^D} \frac{d^D \ell}{(2\pi)^D} (\not{k} + M)(\not{\ell} + M)$$

$$\times \left[\Delta_F(k) - 2\pi\delta(k^2 - M^2) f_F(k)\right]$$

$$\times \left[\Delta_B(k-\ell) + 2\pi\delta\left((k-\ell)^2 - m^2\right) f_B(k-\ell)\right]$$

$$\times \left[\Delta_F(\ell) - 2\pi\delta(\ell^2 - M^2) f_F(\ell)\right]$$

$$= I_0(A) + I_{\beta-1}(A) + I_{\beta-2}(A) \qquad 5\text{-}10$$

Figure 5-3 (b) gives

$$I(B) = \frac{1}{2}\int \frac{d^Dk}{(2\pi)^D} (\delta Z_B k^2 + \delta M_B m^2)\left[\Delta_B(k) + 2\pi\delta(k^2 - m^2) f_B(k)\right]$$

$$= I_0(B) + I_{\beta-1}(B) \qquad 5\text{-}11$$

Figure 5-3(c) contributes

$$I(C) = -\mathrm{Tr}\int \frac{d^Dk}{(2\pi)^D} (\delta Z_F \not{k} + \delta M_F M)\left[\Delta_F(k) - 2\pi\delta(k^2 - M^2) f_F(k)\right]$$

$$= I_0(C) + I_{\beta-1}(C) \qquad 5\text{-}12$$

The evaluation of $I_0(A)$, the zero temperature part of Figure 5-3(a), is instructive in the handling of Feynman integrals. Details of the calculation are given in Appendix II. In presenting the results, polynomials in m, M are omitted; these are subtracted by a two-loop counterterm anyway. The contributions are;

$$I_0(A) = -\frac{2if^2}{(4\pi)^4}\left(\frac{1}{\epsilon} - \gamma + 1\right)(6M^4 \ln M^2 + m^2 \ln m^2)$$

$$+ \frac{if^2}{(4\pi)^4}\{(4M^2 - m^2)(-2M^2 \ln M^2 - m^2 \ln m^2 + 2M^2 \ln^2 M^2$$

$$+ m^2 \ln^2 m^2 - m^2 g(\alpha)) + m^2 M^2 \ln^2 (m^2 M^2) - 2M^4 \ln^2 M^2\} \qquad 5\text{-}13$$

Here,

$$g(\alpha) = \int_0^1 dx \int_0^1 \partial y \, \frac{1}{y} \ln\left[1 + y\left(\frac{1}{\alpha x(1-x)} - 1\right)\right]$$

$$\alpha = \frac{m^2}{M^2} \qquad 5\text{-}14$$

$$I_0(B) + I_0(C) = \frac{if^2}{(4\pi)^4}\{2\left(\frac{1}{\epsilon} - \gamma + 1\right)(6m^2M^2 - m^4)\ln m^2$$

$$+ \left[-\left(\frac{1}{3} - \ln M_0^2\right)m^4 + 6\left(\frac{4}{3} - \ln M_0^2\right)m^2M^2\right]\ln m^2$$

$$+ (-3m^2M^2 + \frac{1}{2} m^4)\ln^2 m^2 + 2\left(\frac{1}{\epsilon} - \gamma + 1\right)6M^4 \ln M^2$$

$$+ (6 - 4C_1 - 4C_2)M^4 \ln M^2 - 3M^4 \ln^2 M^2\} \qquad 5\text{-}15$$

$$I_{\beta-1}(A) = \frac{if^2}{32\pi^4}\{6M^4(\frac{2}{\varepsilon} - \gamma) + 2(4M^2 - m^4)M^2P + 2M^4\ell n\, M^2$$

$$+ 2M^2m^2(1 - \ell n\, m^2)\}\tilde{F}_1(\beta M)$$

$$+ \frac{if^2}{32\pi^4}\{(-6M^2m^2 + m^4)(\frac{2}{\varepsilon} - \gamma) + (4M^2 - m^2)m^2Q$$

$$-2M^2m^2(1 - \ell n\, M^2)\}F_1(\beta m) \qquad 5\text{-}16$$

Here,

$$P = \int_0^1 dx\, \ell n[(1-x)^2M^2 + xm^2]$$

$$Q = \int_0^1 dx\, \ell n[x(1-x)m^2 - M^2]$$

$$\tilde{F}_1(\beta M) = \int_1^\infty dx\, \frac{(x^2-1)^{1/2}}{e^{\beta Mx} + 1} \qquad 5\text{-}17$$

$$F_1(\beta m) = \int_1^\infty dx\, \frac{(x^2-1)^{1/2}}{e^{\beta mx} - 1} \qquad 5\text{-}18$$

$$I_{\beta-1}(B) = -\frac{if^2}{32\pi^4}\{(-6M^2m^2 + m^4)(\frac{2}{\varepsilon} - \gamma) - 6(\frac{1}{3} - \ln M_0^2)m^2M^2 \qquad \text{5-19}$$

$$- (\frac{2}{3} - \ln M_0^2)m^4\}F_1(\beta m)$$

$$I_{\beta-1}(C) = -\frac{if^2}{32\pi^4}\{6M^4(\frac{2}{\varepsilon} - \gamma) + 4M^4(4C_1 + C_2)\}\tilde{F}_1(\beta M) \qquad \text{5-20}$$

And finally

$$I_{\beta-2}(A) = -\frac{if^2M^2}{4\pi^4}\{M^2\,\tilde{F}_1^2(\beta M) + 2m^2\tilde{F}_1(\beta M)F_1(\beta M)\}$$

$$-\frac{if^2}{32\pi^4}(4M^2 - m^2)\{2Mm\int_1^\infty dx\int_1^\infty dy\,\frac{F_2(x,y)}{(e^{\beta mx}-1)(e^{\beta My}+1)}$$

$$+ M^2\int_1^\infty dx\int_1^\infty dy\,\frac{F_3(x,y)}{(e^{\beta Mx}+1)(e^{\beta My}+1)}\} \qquad \text{5-21}$$

Here,

$$F_2(x,y) = F(x,y,m_1,m_2;m_3)\big|_{m_1=m_3=M,\,m_2=m}$$

$$F_3(x,y) = F(x,y,m_1,m_2;m_3)\big|_{m_1=m_2=M,\,m_3=m}$$

$$F(x,y,m_1,m_2;m_3) = \ln \frac{X_+}{X_-}$$

$$X_\pm = [A \pm (x^2 - 1)^{1/2}(y^2-1)^{1/2}]^2 - x^2y^2$$

$$A = \frac{m_1^2 + m_2^2 - m_3^2}{2m_1m_2} \qquad 5\text{-}22$$

As in the purely scalar case, it is observed that temperature-dependent divergent terms cancel.

Figure 5-1 Fermion contribution to one-loop effective potential (the dashed line is the fermion propagator)

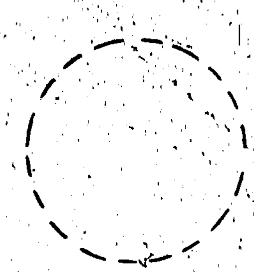


Figure 5-2 Fermion contribution to two-point functions

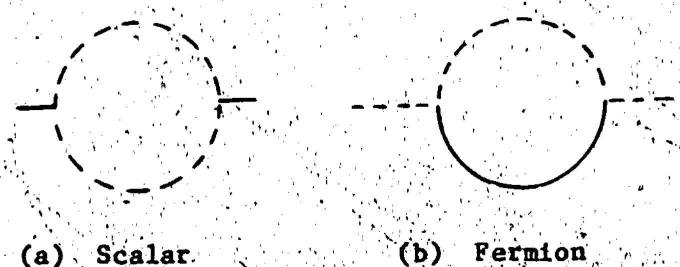
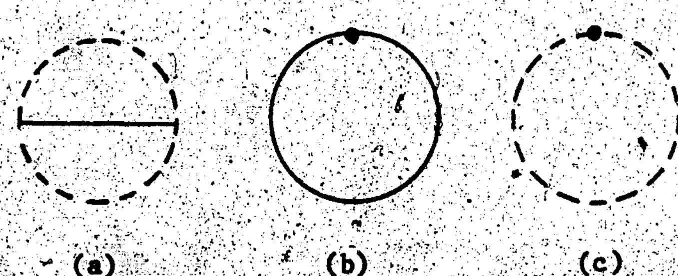


Figure 5-3 Fermion contributions to the two-loop effective potential



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Appendix I A Two-Loop Contribution to $Z(\phi)$

Details are given of the calculation of the two-loop diagram of Fig. 1-4(d), contributing to the function $Z(\phi)$. This is probably the most involved of the calculations, and the others present no additional difficulties. First, some preliminary remarks.

The D-dimensional momentum-space (Minkowskian) integral of a product of propagators is;

$$\int \frac{d^D k}{(2\pi)^D} \frac{1}{(k^2-m^2)^n} = \frac{(-1)^{n_1}}{(4\pi)^2} \frac{\Gamma(n-\frac{D}{2})}{\Gamma(n)} (m^2)^{\frac{D}{2}-n} \qquad \text{I-1}$$

Liberal use is also made of the relation;

$$\frac{1}{A_1^{n_1} \cdots A_k^{n_k}} = \frac{\Gamma(n_1+\cdots+n_k)}{\Gamma(n_1)\cdots\Gamma(n_k)} \int_0^1 dx_1 \cdots \int_0^1 dx_k \, \delta(\sum_j x_j - 1) [A_1 x_1 + \cdots + A_k x_k]^{-\sum_j n_j} \qquad \text{I-2}$$

This is used in combining products of propagators.

Finally, mention should be made of certain integrals that arise in the finite parts of some two-loop diagrams. These are of two kinds;

$$I_n \equiv \int_0^1 dx \frac{\ln x}{(x^2-x+1)^n} \qquad J_n \equiv \int_0^1 dx \frac{1}{(x^2-x+1)^n} \qquad \text{I-3}$$

Defining

$$I_n(a) = \int_0^1 dx \frac{\ln x}{(x^2-x+1)^n} \qquad J_n(a) = \int_0^1 dx \frac{1}{(x^2-x+a)^n} \qquad \text{I-4}$$

It is clear that

$$I_n = \frac{(-1)^n}{(n-1)!} \frac{\partial^n I_1(a)}{\partial a^n}\Big|_{a=1} \qquad J_n = \frac{(-1)^n}{(n-1)!} \frac{\partial^n J_1(a)}{\partial a^n}\Big|_{a=1} \qquad \text{I-5}$$

It can be shown that

$$\frac{\partial I_1(a)}{\partial a} = -\frac{1}{2}\left(a-\frac{1}{4}\right)^{-1} I_1(a) + \frac{1}{2a} J_1(a) \qquad \text{I-6}$$

$$\frac{\partial J_1(a)}{\partial a} = -\frac{1}{2}\left(a-\frac{1}{4}\right)^{-1}\left[J_1(a) + \frac{1}{a}\right] \qquad \text{I-7}$$

Thus, given I_1 and J_1, all I_n and J_n can be generated. Some are listed below;

$$I_2 = \frac{2}{3} I_1 - \frac{\pi}{3\sqrt{3}} \qquad J_1 = \frac{2\pi}{3\sqrt{3}}$$

$$I_3 = \frac{2}{3} I_1 - \frac{7\pi}{18\sqrt{3}} - \frac{1}{6} \qquad J_2 = \frac{4\pi}{9\sqrt{3}} + \frac{2}{3} \qquad \text{I-8}$$

$$I_4 = \frac{20}{27} I_1 - \frac{4\pi}{9\sqrt{3}} - \frac{17}{54} \qquad J_3 = \frac{4\pi}{9\sqrt{3}} + 1$$

The ground is now prepared for the evaluation of diagram 4-1(d). The Feynman rules give;

[Fig 4-1(d)] =

$$-\frac{i\lambda^4 a^4}{2} \int \frac{d^D k}{(2\pi)^D} \frac{d^D q}{(2\pi)^D} \frac{1}{(k^2-m^2)^2((k+p)^2-m^2)(q^2-m^2)((q+k)^2-m^2)} \qquad \text{I-9}$$

Here, p^μ is the external momentum. Use is now made of relation I-2 to combine the propagators which depend on q^r. Equation I-1 is used to perform the q-integration. Then this procedure is repeated to do the k-integration. The result is

$$-\frac{i\lambda^4 a^4}{2} \Gamma(1+\varepsilon) \int_0^1 dx[x(1-x)]^{1+\frac{\varepsilon}{2}} \int_0^1 dy\, y^{\frac{\varepsilon}{2}-1} \int_0^{1-y} dz(1-y-z)$$
$$\times \left[ym^2 + (1-y)x(1-x)m^2 - x(1-x)z(1-z)p^2 \right]^{-(1+\varepsilon)} \qquad \text{I-10}$$

To obtain the contribution to $Z(\phi)$, one must differentiate this with respect to p^2, and set $p^2 = 0$. This yields

$$-\frac{i a^4 \lambda^4}{24^4 (4\pi)^4} \Gamma(2+\epsilon)(m^2)^{-(2+\epsilon)} \int_0^1 dx\left[x(1-x)\right]^{2+\frac{\epsilon}{2}} \sim$$

$$\int_0^1 dy\, y^{\frac{\epsilon}{2}-1} \frac{(1-y)^3(1+y)}{[y+(1-y)x(1-x)]^{2+\epsilon}} \qquad \text{I-11}$$

where the trivial z-integration has been done. In the limit as $\epsilon \to 0$ $(D \to 4)$, this expression diverges due to the $y^{\frac{\epsilon}{2}-1}$ term in the integrand. Since the divergence arise from $y = 0$, it can be extracted by putting $y = 0$ in all other terms. This gives the integral

$$\int_0^1 dx\left[x(1-x)\right]^{-\frac{\epsilon}{2}} \int_0^1 dy\, y^{\frac{\epsilon}{2}-1} = \frac{2}{\epsilon}\left(1+\epsilon+O(\epsilon^2)\right) \qquad \text{I-12}$$

To obtain the finite remainder, one simply subtracts the integrand of I-12 from that of I-11, and puts $\epsilon = 0$. The resulting finite remainder of the parameter integration gives

$$12I_2 - 16I_3 + 6I_4 - \frac{1}{2} + 3J_1 - \frac{13}{2} J_2 + 3J_3 \qquad \text{I-13}$$

Using the relations I-8, this can be reduced to

$$\frac{16}{9} I_1 - \frac{19}{18} \qquad \text{I-14}$$

Adding this to the divergent part I-12 and including the multiplier of the parameter integration gives the contribution of Fig. 4-1(d) to $Z(\phi)$;

$$-\frac{ia^4\lambda^4}{24(4\pi)^4}\,\Gamma(2+\epsilon)(m^2)^{-(2+\epsilon)}\left(\frac{2}{\epsilon}+\frac{16}{9}I_1+\frac{17}{18}\right)$$

$$=-\frac{ia^4\lambda^4m^{-4}}{12(4\pi)^4}\left(\frac{1}{\epsilon}-\ln m^2-\gamma+\frac{8}{9}I_1+\frac{53}{36}\right)$$

where the expansion $\Gamma(2+\epsilon) = 1+\epsilon-\epsilon\gamma$ was used, and γ is Euler's constant.

Appendix II-CALCULATION OF $I_0(A)$

Details of the calculation of the zero-temperature part $I_0(A)$ of Figure 5-3(a) are presented. Use is made of the relations set down in Appendix I.

To begin with

$$I_0(A) = -\frac{if^2}{2}\int \frac{d^Dk}{(2\pi)^D}\frac{d^D\ell}{(2\pi)^D}\frac{Tr(\not k + M)(\not \ell + M)}{(k^2-M^2+i\varepsilon)(\ell^2-M^2+i\varepsilon)((k-\ell)^2-m^2+i\varepsilon)} \qquad \text{II-1}$$

The trace term is

$$Tr(\not k + M)(\not \ell + m) = 4(k\cdot\ell + M^2)$$

$$= 2\{-((k-\ell)^2-m^2) + (k^2-M^2) + (\ell^2-M^2) + 4M^2 - m^2\} \qquad \text{II-2}$$

Putting this into equation II-1 yields

$$I_0(A) = -if^2[-J^2(M) + 2J(m)J(M) + (4M^2-m^2)K(m,M,M)] \qquad \text{II-3}$$

where

$$J(\mu) = \int \frac{d^Dk}{(2\pi)^D}\frac{1}{k^2-\mu^2+i\varepsilon}$$

$$= -\frac{1}{(4\pi)^2}\Gamma\left(\frac{\varepsilon}{2}-1\right)(\mu^2)^{1-\frac{\varepsilon}{2}} \qquad \text{II-3}$$

(Note: the ε occurring in the denominator of the propagator is not to be confused with the dimensional $\varepsilon = 4-D$. The former will only appear in explicitly written propagators).

And

$$K(m_1,m_2,m_3) = \int \frac{d^Dk}{(2\pi)^D}\frac{d^D\ell}{(2\pi)^D}\frac{1}{(k^2-m_1^2+i\varepsilon)(\ell^2-m_2^2+i\varepsilon)((k-\ell)^2-m_3^2+i\varepsilon)} \qquad \text{II-5}$$

The first two terms on the right-hand side of II-3 are easily expanded;

$$2J(m)J(M) = -\frac{2m^2M^2}{(4\pi)^4}\{(\frac{2}{\varepsilon}-\gamma+1)^2 - 2(\frac{1}{\varepsilon}-\gamma+1)(\ln m^2+\ln M^2) + \frac{1}{2}\ln^2(m^2M^2)\} \qquad \text{II-6}$$

$$-J^2(M) = \frac{M^4}{(4\pi)^4}\{(\frac{2}{\varepsilon}-\gamma+1)^2 - 4(\frac{1}{\varepsilon}-\gamma+1)\ln M^2 + 2\ln^2M^2\} \qquad \text{II-7}$$

The last term requries considerably more effort. Using the relations in Appendix I, it can be integrated to the form;

$$K(m,M,M) = \frac{1}{(4\pi)^4}\Gamma(\varepsilon-1)\int_0^1 dx[x(1-x)]^{\frac{\varepsilon}{2}-1}\int_0^1 dy\, y^{\frac{\varepsilon}{2}-1} \times [(1-y)x(1-x)m^2 + yM^2]^{1-\varepsilon} \qquad \text{II-8}$$

The pole contributions of the parameter integration are now isolated.

They come from the limits $x \to 0$, $x \to 1$, and $y \to 0$. The following limiting forms of the integrand are chosen to extract the poles (the choice is not unique);

$$x^{\frac{\varepsilon}{2}-1}\, y^{-\frac{\varepsilon}{2}}\, (M^2)^{1-\varepsilon}\,, \quad x \to 0$$

$$(1-x)^{\frac{\varepsilon}{2}-1}\, y^{-\frac{\varepsilon}{2}}\, (M^2)^{1-\varepsilon}\,, \quad x \to 1 \qquad \text{II.9}$$

$$[x(1-x)]^{-\varepsilon/2}\, y^{\frac{\varepsilon}{2}-1}\, (m^2)^{1-\varepsilon}\,, \quad y \to 0$$

Integrating these gives, respectively

$$\frac{2}{\varepsilon}\,\frac{1}{1-\frac{\varepsilon}{2}}\,(M^2)^{1-\varepsilon}$$

$$\frac{2}{\varepsilon}\,\frac{1}{1-\frac{\varepsilon}{2}}\,(M^2)^{1-\varepsilon} \qquad \text{II-10}$$

$$\frac{2}{\varepsilon}\,(1+\varepsilon+O(\varepsilon^2))(m^2)^{1-\varepsilon}$$

Having isolated the pole terms, one now subtracts the integrands in II-9 from the parameter integration in II-8. Unlike the example considered in Appendix I however, one is not free to then set $\varepsilon = 0$. This is because there is another pole, in the form of $\Gamma(\varepsilon-1)$, sitting in front of the integral. This means that the integrand has to be expanded to order ε, to ensure that all finite contributions are taken into account. When the expansion has been made, terms can be collected in such a way that they can be fairly easily integrated, with one exception. The result of the subtracted, expanded parameter integration is

$$-m^2 + \varepsilon\left[\frac{1}{2} m^2(2\ln m^2 - 3) - \frac{2\pi^2}{3} M^2 - m^2 g(\alpha)\right] \qquad \text{II-11}$$

where

$$g(\alpha) = \int_0^1 dx \int_0^1 dy \, \frac{1}{y} \ln\left[1 + y\left(\frac{1}{\alpha x(1-x)} - 1\right)\right]$$

$$\alpha = \frac{m^2}{M^2} \qquad \text{II-12}$$

When II-11 is added to the pole contributions of II-10, and the sum is multiplied by $\Gamma(\varepsilon-1)/(4\pi)^4$, the computation of $K(m,M,M)$ is completed. Expanding the result in ε, and discarding polynomials in m and M (which are irrelevant unless one wishes to proceed to a three-loop calculation), the outcome is

$$K(m,M,M) = \frac{1}{(4\pi)^4}\Big\{\left(\frac{1}{\varepsilon} - \gamma\right)(4M^2\ln M^2 + 2m^2\ln m^2) + 6M^2\ln M^2$$

$$+ 3m^2\ln m^2 - 2M^2\ln^2 M^2 - m^2\ln^2 m^2 + m^2 g(\alpha)\Big\} \qquad \text{II-13}$$

Inserting II-13, II-6 and II-7 into equation II-3 finally give $I_0(A)$, as written in equation 5-13.

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