UNIVERSITY OF ALBERTA

Smoothness and Convexity in Banach Spaces

WEE-KEE TANG

A thesis submitted to the Faculty of Graduate Studies and Research in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Mathematics.

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ABSTRACT

Equivalent conditions for the generic Fréchet differentiability of a given convex Lipschitz function f defined on a separable Banach space are established. The conditions are in terms of a majorization of f by a C^1-smooth function, separability of the boundary for f or an approximation of f by Fréchet smooth convex functions.

Approximation by smooth convex functions and questions on the Smooth Variational Principle for a continuous convex function defined on a WCG space are also studied.

A class of continuous convex functions defined on a Banach space (not necessarily an Asplund space) is defined. This class of functions is found to exhibit similar properties possessed by the norm of an Asplund space.

We also construct a uniformly smooth norm on a separable Banach space X that contains an isomorphic copy of ℓ_1 such that this norm, when extended canonically to X^{**}, is nowhere differentiable at the points of X.

An extension of norms from a closed subspace of a Banach space to the whole space that preserves various types of rotundity possessed by the subspace norms is constructed. We also construct a strictly convex norm such that a prescribed set of points lies on the unit sphere of this norm.

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Introduction

This dissertation is divided into two main parts. The first part, which consists of chapters one to three, deals with the analysis of convex functions in Banach spaces. The second part, chapters four and five, is devoted to the study of extensions of norms.

One of the important results in classical analysis is Rademacher's Theorem, which says that every locally Lipschitzian function from $\mathbb{R}^n$ to $\mathbb{R}^m$ is Fréchet differentiable almost everywhere. In particular, every convex continuous function defined on $\mathbb{R}^n$ is differentiable on a dense set. In the case of infinite dimensional spaces, the scenario is more complex. For instance, the norm $\|\cdot\|_1$ of l_1 is nowhere Fréchet differentiable, whereas every continuous convex function on l_2 (or any separable reflexive Banach space) is Fréchet differentiable on a dense G_δ set, according to a result of J. Lindenstrauss. In 1968, E. Asplund discovered more general spaces in which the Fréchet differentiability conclusion holds. This class of spaces, now called *Asplund spaces* (An Asplund space is a Banach space on which every convex continuous function is Fréchet differentiable on a dense set), received much attention in the subsequent decades. It was first found by E. Asplund that if a Banach space has separable dual, then it must be an Asplund space. In fact the converse is also valid.

In 1975, I. Namioka and R. R. Phelps discovered that the dual of an Asplund space has the Radon-Nikodym property, i.e. every bounded set admits weak* slices of arbitrarily small diameter. The converse was proved by C. Stegall in 1978.

J. E. Jayne and C. A. Rogers characterized an Asplund space X as one which on the duality mapping of X has a selector that is the pointwise limit of a sequence of norm to norm continuous mappings from X into X^*.

In this dissertation, we generalize the above results. We translate the above classifications of Asplund spaces into classifications of a certain class of convex functions. We show that this class of functions, which we call Asplund functions, exhibit similar behavior as Asplund spaces. For instance, when restricted to a separable subspace, the image of the subdifferential map of an Asplund function is separable. We also see that subdifferential map of an Asplund function admits a Jayne-Rogers type of selector.

In Chapter One, we consider convex Lipschitzian functions on a separable Banach space. We prove some equivalent conditions for which the image of the subdifferential map of a Lipschitzian convex function f is separable. One of the equivalent conditions is that every convex function majorized by f can be approximated uniformly by a Fréchet differentiable convex function.

In Chapter Two, we study convex functions on a WCG space, which is more general than a separable space. We see that it is still possible to approximate every convex function majorized by an Asplund function by smooth functions. We also show the existence of smooth points of higher order for a convex function on certain uniformly smooth spaces.

Chapter Three deals with more general spaces. Smooth approximation is no longer possible in general, however, some other characterizations still hold for Asplund functions on a general Banach space.

The second part of this dissertation is devoted to the study of extensions of norms in a Banach space. In Chapter Four, we study the notion of preserved smooth points. A point x on the sphere is said to be smooth when the norm is Gâteaux differentiable at x. A smooth point is said to be a preserved smooth point if it is still a smooth point with respect to the canonical extension of the norm into the second dual. We give a sufficient condition for a space to admit a smooth norm with rough canonical extension.

In Chapter Five, we study the extensions of norms that preserve rotundity. K. John and V. Zizler had shown that if Y is a closed subspace of a separable Banach space X and if Y admits a LUR norm, then this LUR norm on Y can be extended to a LUR norm on X. Recently, it was shown by M. Fabian that if Y is reflexive, and if the norms of X and Y possess some kind of rotundity, then the norm on Y can be extended to a norm on X with the same rotundity. We give a construction that does not require the subspace Y to be reflexive. The extension given in this chapter seems to preserve most notions of rotundity. We conclude the chapter by looking at some natural geometrical questions regarding rotund norms.

Chapter One

Fréchet differentiability of convex functions on separable Banach spaces.

Introduction. It is known that on a given separable Banach space X all continuous convex functions are generically Fréchet differentiable if and only if X^* is separable, and if and only if X admits a C^1-smooth bump function. In this case, every equivalent norm in X can be uniformly approximated by Fréchet smooth equivalent norms on bounded sets.

The purpose of this chapter is to generalize these results. We give some equivalent conditions for the generic Fréchet differentiability of a given Lipschitz convex function defined on a separable Banach space in terms of the properties of the function f rather than that of X. In this setting, we cover some continuous convex functions defined on separable non-Asplund spaces. For instance if $\|\cdot\|$ denotes the Hilbertian norm on l_2 and T is a continuous linear map of a separable Banach space X into l_2, then any Lipschitz convex function f defined on X such that $f(x) \leq \|T(x)\|^2$ for $x \in X$ satisfies the assumptions in Theorem 1.6 below. At the end of this chapter, we show how the methods from variational principles can be applied to find a sufficient condition for the w^*−lower semicontinuity of convex functions.

Definition 1.1. Let f be a convex continuous function defined on a Banach space X. We say that f is Gâteaux differentiable at a point $x \in X$ if for every $h \in S_X$,

$$f'(x)(h) = \lim_{t \to 0} \frac{f(x+th) - f(x)}{t}$$

exists. The functional is then called the Gâteaux derivative of f at $x \in X$. If in addition, the above limit is uniform in $h \in S_X$, we say that f is Fréchet differentiable at x, and the functional is called the Fréchet derivative of f at x. A convex continuous function is said to be generically Fréchet differentiable if it is Fréchet differentiable on a dense G_δ set.

Definition 1.2. Let f be a continuous convex function defined on a Banach space X, the subdifferential of f at $x \in X$ is the set

$$\partial f(x) = \{x^* \in X^* : \langle x^*, y - x\rangle \leq f(y) - f(x) \text{ for all } y \in X\}.$$

We write $\partial f(X) = \cup_{x \in X} \partial f(x)$.

Definition 1.3. Let (Z, τ) be a topological space and g be a real valued function on Z. We say that g is τ-lower semicontinuous if $g^{-1}(-\infty, r]$ is τ-closed for every $r \in I\!R$.

Definition 1.4. Let f be as above, the Fenchel dual (or conjugate) of f, is an extended real valued function on X^* defined by

$$f^*(x^*) = \sup\{\langle x^*, x\rangle - f(x) : x \in X\}, \text{ for } x^* \in X^*.$$

It is clear that f^* is a w^*-lower semicontinuous convex function on X^*.

Definition 1.5. Given two continuous convex functions f and g defined on X, the infimal convolution $f \Box g$ of f and g is defined by

$$f \Box g(x) = \inf\{f(y) + g(y - x) : y \in X\}.$$

A subset $B \subset \partial f(X)$ is called a boundary for f if B intersects $\partial f(x)$ for each $x \in X$ (see e.g. [G]). By a selector for ∂f we mean a single-valued mapping $s : X \to X^*$ such that $s(x) \in \partial f(x)$ for every $x \in X$. Unless stated otherwise, all topological terms in dual Banach spaces refer to the norm topology of these spaces.

We refer to [Ph] and [D-G-Z] for some unexplained notions and results used in this chapter.

A main result in this chapter is the following statement.

Theorem 1.6. *Let X be a separable Banach space and f be a Lipschitz convex function defined on X. The following are equivalent.*

(1) The set $\partial f(X)$ is separable.

(2) There is a selector s for ∂f such that $s(X) = \{s(x) : x \in X\}$ is separable.

(3) There is a continuously Fréchet differentiable function ϕ such that $\phi \geq f$ on X.

(4) f can be approximated uniformly on X by Fréchet differentiable convex functions.

(5) If h is a convex function on X such that $h \leq f$ on X, then h is generically Fréchet differentiable on X.

Proof. Clearly (1) $\implies$ (2). We shall show (2) $\implies$ (1) using Simons' lemma ([S]). Put $B = s(X)$ and let $\gamma = \inf\{f^*(y^*) : y^* \in B\}$. Clearly, $\gamma < \infty$ as B is nonempty. We show that $C := dom f^* \subset \overline{conv} B$. If this is not so, pick $y_o^* \in C \setminus \overline{conv} B$. By separation theorem, there exist $z \in X^{**}$ and $\alpha, \beta \in I\!R$ such that $z(y_o^*) > \beta > \alpha > z(y^*)$ for each $y^* \in B$. By scaling the functional z, we may assume that $\frac{\beta - \alpha}{2} > f^*(y_o^*) - \gamma$. For every $x \in X$, define a function $h_x \in l^\infty(B)$ by

$$h_x(x^*) = (x^*, x) - f^*(x^*).$$

Let $E = \{x \in X : \|x\| \leq \|z\|,\ x(y_o^*) > \beta\}$. Since B is separable, there exists a s quence $\{x_n\}$ in E such that x_n converges to z in the topology of pointwise convergence on B. Define a sequence $h_n \in l^\infty(C)$ by $h_n(x^*) = h_{x_n}(x^*)$ for $x^* \in C$. Note that for any $x = \sum_{k=1}^{\infty} \lambda_k x_k$, where $\lambda_k \geq 0$ and $\sum_{k=1}^{\infty} \lambda_k = 1$, we have $s(x) \in B$ and

$$\sum \lambda_k h_k(s(x)) = h_x(s(x)) = f(x) = \sup\{(x^*, x) - f^*(x^*) : x^* \in C\}$$

$$= \sup\{(x^*, \sum \lambda_k x_k) - f^*(x^*) : x^* \in C\} = \sup\{\sum \lambda_k h_k(x^*) : x^* \in C\}.$$

Since $z(y^*) < \alpha$, we have $\limsup h_n(y^*) \leq \alpha - f^*(y^*)$ for all $y^* \in B$. Consequently $\sup\{\limsup h_n(y^*) : y^* \in B\} \leq \alpha - \gamma$. By Simons' lemma there is $g \in conv\{h_n\}$, $g = \sum_{k=1}^{N} \rho_k h_k$, $\rho_k \geq 0$, $\sum_{k=1}^{N} \rho_k = 1$ such that

$$\sup\{g(x^*) : x^* \in C\} \quad \leq \quad \frac{\alpha + \beta}{2} - \gamma.$$

On the other hand, $g(y_o^*) = \sum_{k=1}^{N} \rho_k h_k(y_o^*) = (y_o^*, \sum_{1}^{N} \rho_k x_k) - f^*(y_o^*) > \beta - f^*(y_o^*)$ and thus $\beta - f^*(y_o^*) < \frac{\alpha+\beta}{2} - \gamma$. Therefore $\frac{\beta-\alpha}{2} < f^*(y_o^*) - \gamma$. This contradiction shows that (2) implies (1).

(3) $\implies$ (1) We follow the idea in [F]. Let $x \in X$, $q \in \partial f(x)$ and $\epsilon > 0$ be given. The function $\phi - q$ is a bounded below continuous function on X. By Ekeland's variational principle, there is a x_q such that for each $h \in X$ and $t > 0$,

$$(\phi - q)(x_q + th) \geq (\phi - q)(x_q) - \epsilon\|h\|t.$$

Hence,

$$\|\phi'(x_q) - q\|^* \leq \epsilon.$$

Therefore $\partial f(X) \subset \overline{\{\phi'(x) : x \in X\}}$. Since ϕ' is continuous and X is separable, the set $\overline{\{\phi'(x) : x \in X\}}$ and thus also $\partial f(X)$ are separable.

(1) $\implies$ (5) By using the above argument for the functions h and f, we see that $\partial h(X) \subset \overline{\partial f(X)}$. Therefore $\partial h(X)$ is also separable and the statement follows immediately from the proof of Theorem 1 in [Pr-Z] (see also [Ph, Theorem 2.11]).

(5) $\implies$ (1) Since $\partial f(X) \subset \mathrm{dom} f^*$, it suffices to show that $dom f^*$ is separable. We split $\mathrm{dom} f^*$ into w^*-compact sets C_n and show that all C_n are norm separable. We put $C_n = \{x^* \in X^* : f^*(x^*) \leq n\}$, and note that $\mathrm{dom} f^* = \cup_{n=1}^{\infty} C_n$.

Assume for some $n \in \mathbb{N}$, the set C_n is not norm separable. Since C_n is compact and metrizable in the w^*-topology, we find a w^*-compact subset $A \subset C_N$ and $\epsilon > 0$ such that every w^*-slice has diameter greater than $\epsilon > 0$ (see the proof of [Ph, Theorem 2.19]). Define $h(x) = \sup\langle A, x\rangle - N, x \in X$. Then $h(x) \leq \sup\{\langle x^*, x\rangle - f^*(x^*) : x^* \in C_N\}$, and the function h is nowhere Fréchet differentiable (see the proof of [Ph, Lemma 2.18]). As $h \leq f$ on X, we obtain a contradiction.

(1) $\implies$ (4) Let $Y = \overline{span}\{\partial f(x) : x \in X\}$. Since Y is norm separable, there is an equivalent norm $\|\cdot\|$ on X such that its dual norm $\|\cdot\|^*$ on X^* is locally uniformly rotund at points of domf^*. In other words, if $y \in$ domf^*, $y_k \in X^*$, and $\lim(\frac{\|y_k\|^{*2}+\|y\|^{*2}}{2} - \|\frac{y_k+y}{2}\|^{*2}) = 0$, then $\lim \|y_k - y\|^* = 0$ (see e.g. the proof of [D-G-Z, Prop.IV.5.2]).

Now, define a sequence of functions $\{h_n\}$ on X^* by $h_n(x^*) = f^*(x^*) + \frac{1}{4n^4}\|x^*\|^{*2}$. Clearly dom $h_n =$ domf^*. Note that for any $n \in \mathbb{N}$, $\lim(\dfrac{h_n(y)}{2} + \dfrac{h_n(y_k)}{2} - h_n(\dfrac{y+y_k}{2})) = 0$, implies $\lim \|y_k - y\|^* = 0$. Define $g_n := f \square n^4\|\cdot\|^2$, the infimal convolution of f and $n^4\|\cdot\|^2$. Note that the function g_n is a convex continuous function on X for all n and $g_n{}^* = h_n$.

Given $n \in \mathbb{N}$, $x \in X$ and $y \in \partial g_n(x)$, note that h_n is rotund at y with respect to x in the sense of [A-R], i.e., for every $\epsilon > 0$, there exists $\delta > 0$ such that

$$\{v : h_n(y+v) - h_n(y) - (x, v) \leq \delta\} \subset \epsilon B_{X^*}.$$

Indeed, if this is not so, there exists an $\epsilon > 0$ such that for all $k \in \mathbb{N}$, there is a v_k, with $\|v_k\| > \epsilon$ and

$$\frac{1}{2}h_n(y+v_k) - \frac{1}{2}h_n(y) - (x, \frac{v_k}{2}) \leq \frac{1}{2k}.$$

Since $h_n = g_n{}^*$, we have $x \in \partial h_n(y)$, and thus

$$(x, \frac{v_k}{2}) \leq h_n(y + \frac{v_k}{2}) - h_n(y)$$

Putting these two inequalities together, we obtain for every $k \in I\!N$,

$$\frac{h_n(y) + h_n(y + v_k)}{2} - h_n(y + \frac{v_k}{2}) \leq \frac{1}{2k}.$$

From the local uniform convexity of h_n, we have $\lim \|v_k\| = 0$, a contradiction. By [A-R, Proposition 4], g_n is Fréchet differentiable at x with the derivative y. By the proof of Lemma 2.4 in [MPVZ], one can show that $\lim g_n = f$ uniformly on X.

(4) $\Longrightarrow$ (3). By (4), there exists a Fréchet differentiable convex function ψ such that $|\psi(x) - f(x)| \leq \frac{1}{2}$ for every $x \in X$. Then $\psi + 1$ is the desired function.

This completes the proof of Theorem 1.6. □

Note that in Theorem 1.6, the implications (3) $\Longrightarrow$ (1) $\Longrightarrow$ (5) are still valid without requiring f to be Lipschitz. The assumption of separability of X in the statement of Theorem 1.6 cannot be dropped in general. Indeed, Haydon constructed a nonseparable space X where all convex continuous functions are generically Fréchet differentiable and yet no equivalent norm can be approximated uniformly on bounded sets by Fréchet differentiable convex functions (see e.g. [D-G-Z]).

Note also that in Theorem 1.6, it is crucial that the function f be defined on the whole of X, as there may exist nowhere differentiable norms bounded on the open ball by constant functions.

The following statement shows how Ekeland's variational principle can be used in questions on w^*-lower-continuity of convex functions.

Theorem 1.7. *Let X be a Banach space and f be a w^*-lower semicontinuous Fréchet differentiable function on X^*. Then every norm-lower semicontinuous convex function g on X^* such that $g \leq f$ on X^* is w^*-lower semicontinuous on X^*.*

Proof. We first note that $f'(X^*) := \cup\{f'(y) : y \in X^*\} \subset X$. Indeed, for any $y \in X^*$, $f'(y)$ is w^*-lower semicontinuous on B_{X^*}, as it is a uniform limit of w^*-lower semicontinuous functions on B_{X^*}. Since $f'(y)$ is linear, $f'(y)$ is w^*-continuous on B_{X^*}. By Banach-Dieudonné Theorem, $f'(y)$ is w^*-continuous on X^*. Hence $f'(y) \in X$ for any $y \in X^*$.

We claim that $\text{dom}g^* \subset X$. Indeed, for any $h \in \text{dom}g^*$, we have $\sup\{h(x^*) - g(x^*) : x^* \in X^*\} < \infty$. This implies that $f - h$ is bounded below. As in the proof of (3) $\implies$ (1), we can show $h \in \overline{f'(X^*)}$. Therefore $h \in X$.

Since g is norm-lower semicontinuous, we have $g = g^{**}{\restriction_{X^*}}$. However, $g^{**} = (g^*)^* = (g^*{\restriction_{\text{dom}g^*}})^* = (g^*{\restriction_X})^*$. Hence g is a dual to a function defined on X, therefore g is w^*-lower semicontinuous. □

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Chapter Two

Sets of differentials and smoothness of convex functions.

Introduction. It is known that a weakly compactly generated (WCG) Banach space admits an equivalent Fréchet differentiable norm if it admits a Fréchet differentiable bump function (cf. e.g. [J-Z]). However, there are nonseparable spaces that admit Fréchet differentiable bump functions but admit no equivalent Fréchet differentiable norm (cf. e.g. [D-G-Z, Chapter VII]). If the space X admits an equivalent norm with modulus of smoothness of power type 2, then every convex continuous function on X has points of Lipschitz smoothness (cf. e.g. [D-G-Z, Chapter IV]). The purpose of this chapter is to localize these results. We prove that any convex Lipschitz function f that is defined on a WCG Banach space X can be uniformly approximated by Fréchet differentiable convex functions if f is majorized on X by a Fréchet smooth convex function. If, moreover, $\overline{span}^{\|\cdot\|}\{\partial f(x) : x \in X\}$ is a subspace of X^* that admits a norm with modulus of rotundity of power type 2, then there is a convex function ψ with ψ' Lipschitz on X such that $\psi \geq f$ on X and $\psi(x) = f(x)$ for some $x \in X$. Thus in particular, f has points of Lipschitz smoothness.

Definition 2.1. A Banach space X is said to be weakly compactly generated (WCG) if there exists a weakly compact set W of X that spans a dense linear subspace in X.

Definition 2.2. Let X be a Banach space. We denote by μ the smallest ordinal such that its cardinality $|\mu| = \text{dens(X)}$. A projectional resolution of identity (PRI) on X is a collection $\{P_\alpha : \omega_0 \leq \alpha \leq \mu\}$ of projections of X onto X that satisfies, for every α, $\omega_0 \leq \alpha \leq \mu$, the following conditions.

(i) $\|P_\alpha\| = 1$.

(ii) $P_\alpha P_\beta = P_\alpha$ if $\omega_0 \le \alpha \le \beta \le \mu$.

(iii) $\text{dens}(P_\alpha(X)) \le |\alpha|$.

(iv) $\bigcup\{P_{\beta+1}(X) : \beta < \alpha\}$ is norm dense in $P_\alpha(X)$.

(v) $P_\mu = Id_X$.

It is well known that every WCG space admits a PRI (see e.g. [D-G-Z, Chapter VI]). A main result of this chapter is the following theorem.

Theorem 2.3. *Let f be a convex Lipschitz function defined on a WCG Banach space X. Then the following are equivalent.*

(1) The function f can be uniformly approximated on X by a Fréchet differentiable convex function.

(2) There exists a Fréchet differentiable convex function ϕ defined on X such that $\phi \ge f$ on X.

Proof. Clearly (1) $\implies$ (2). The proof (2) $\implies$ (1) is divided into a few steps.

Proposition 2.4. *Let X be a WCG Banach space, ϕ be a Fréchet differentiable convex function defined on X and let $Y := \overline{span}^{\|\cdot\|}\{\phi'(x) : x \in X\}$. Then there exists a projectional resolution of identity (PRI) $\{P_\alpha : \omega_0 \le \alpha \le \mu\}$ on X such that*

(i) $P_\mu^* = I$, $\|P_\alpha^*\| = 1$ for all α.

(ii) $P_\alpha^* P_\beta^* = P_\beta^* P_\alpha^* = P_{min(\alpha,\beta)}^*$.

(iii) $P_\alpha^* Y \subset Y$ for all α.

(iv) $dens(P_\alpha^* Y) \le |\alpha|$ for all $\alpha \le \mu$.

(v) $P_\alpha^* Y = \overline{span}^{\|\cdot\|} \bigcup_{\beta<\alpha} P_{\beta+1}^* Y$ for all $\alpha \le \mu$.

Proof. Using standard techniques for constructing projectional resolutions of identity (see e.g. [D-Z-G, Chapter VI]), we only need to show $P_\alpha^* Y \subset Y$

and $P_\alpha^* Y = \overline{span}^{\|\cdot\|} \bigcup_{\beta<\alpha} P_{\beta+1}^* Y$ for all α. The proof of this is contained in Lemmas 2.5 to 2.7.

Lemma 2.5. *In the notation as above, we can construct a PRI $\{P_\alpha : \omega_0 \leq \alpha \leq \mu\}$ so that $P_\alpha^* \phi'(y) = \phi'(y)$ for all $y \in P_\alpha X$.*

Proof. See Lemma 5 in [J-Z]. □

Lemma 2.6. *With the notation as above, $P_\alpha^* Y = \overline{span}^{\|\cdot\|}\{\phi'(x) : x \in P_\alpha X\}$.*

Proof. To see $P_\alpha^* Y \supset \overline{span}^{\|\cdot\|}\{\phi'(x) : x \in P_\alpha X\}$, we let $x \in P_\alpha X$, and show that $\phi'(x) \in P_\alpha^* Y$. Since ϕ is C^1-smooth, given $\epsilon > 0$, there exists an $x_\beta \in P_{\beta+1} X$ for some $\beta < \alpha$, such that $\|\phi'(x) - \phi'(x_\beta)\| < \epsilon$. By Lemma 2.5, $\phi'(x_\beta) = P_{\beta+1}^* \phi'(x_\beta)$. Therefore $\phi'(x_\beta) \in P_{\beta+1}^* Y \subset P_\alpha^* Y$. As $P_\alpha^* Y$ is closed, $\phi'(x) \in P_\alpha^* Y$. For the converse inclusion, we follow the idea in [F]. Let $\phi'(x) \in Y$. Clearly $g(\cdot) = \phi(\cdot) - \phi'(x)(\cdot)$ is a continuous bounded below function on X. Hence its restriction $g_{|P_\alpha X}$ is also continuous and bounded below. By Ekeland's variational principle, given $\epsilon > 0$, there exists $x_\alpha \in P_\alpha X$ such that for every $w \in B_{P_\alpha X}$, $t > 0$, we have $g(x_\alpha + tw) \geq g(x_\alpha) - \epsilon t$, thus, $\phi'(x)(w) \leq (\phi(x_\alpha + tw) - \phi(x_\alpha))/t + \epsilon$. Hence, by taking limits, we have $\phi'(x)(w) - \phi'(x_\alpha)(w) \leq \epsilon$. Therefore $\sup\{|\phi'(x)(v) - \phi'(x_\alpha)(v)| : v \in B_{P_\alpha X}\} \leq \epsilon$. Given any $h \in B_X$, we have $(h, P_\alpha^* \phi'(x) - \phi'(x_\alpha)) = (h, P_\alpha^* \phi'(x) - P_\alpha^* \phi'(x_\alpha)) = (P_\alpha h, \phi'(x) - \phi'(x_\alpha)) \leq \epsilon$. Therefore $\|P_\alpha^* \phi'(x) - \phi'(x_\alpha)\| \leq \epsilon$. Finally, since Y is the closed linear span of the derivatives of ϕ and P_α is bounded, the assertion follows. □

Lemma 2.7. *$P_\alpha^* Y = \overline{span}^{\|\cdot\|} \bigcup_{\beta<\alpha} P_{\beta+1}^* Y$ for every $\alpha \leq \mu$.*

Proof. Clearly $P_\alpha^* Y \supset \overline{span}^{\|\cdot\|} \bigcup_{\beta<\alpha} P_{\beta+1}^* Y$. The converse inclusion follows from Lemma 2.6 and the continuity of ϕ'. □

Proof of Theorem 2.3. Since $f \leq \phi$, using Ekeland's variational principle as in Lemma 2.6 we show that dom $f^* \subset Y$. Using Proposition 2.4, and the classical Troyanski-Zizler construction (see e.g. [D-G-Z, Chapter VII]) we obtain a dual norm $\|\cdot\|^*$ in X^* such that its restriction on Y is locally uniformly rotund (LUR). Define a sequence of functions $\{h_n\}$ on X^* by $h_n(x^*) = f^*(x^*) + \frac{1}{4n^4}\|x^*\|^{*2}$. Clearly, dom h_n =dom f^*. Define $g_n := f \square n^4\|\cdot\|^2$, where $\square$ denotes the infimal convolution. Note that g_n is convex and continuous on X and $g_n^* = h_n$ for all n. Given $n \in I\!N$, $x \in X$ and $y \in \partial g_n(x)$, note that h_n is rotund at y with respect to x in the sense of [A-R], i.e., for every $\epsilon > 0$, there exist $\delta > 0$ such that $\{v : h_n(y + v) - h_n(y) - (x, v) \leq \delta\} \subset \epsilon B_{X^*}$ (cf. Chapter 1). By [A-R, Proposition 4], g_n is Fréchet differentiable at x with the derivative y. One can also show that $\lim g_n = f$ uniformly on X (see e.g. [M-P-V-Z, Lemma 2.4]). $\square$

Since the function f can be quite "flat" in Theorem 2.3, there is a difficulty in applying the techniques of Smooth Variational Principles (cf. [D-G-Z, Chapter I]) in this situation. However, under more restrictive assumptions we can use the Stegall-Fabian variational principle and obtain our variational result by duality. We will say that $x \in X$ is a point of Lipschitz smoothness of a convex function f if $f(x + h) + f(x - h) - 2f(x) = O(\|h\|^2)$. Given a bounded set $A \subset X^*$, the indicator function $\delta_A(\cdot)$ is a convex function on X^* that takes value zero in A and ∞ elsewhere.

Lemma 2.8. *Let f be a convex continuous function on a Banach Space X and g be its dual function. Suppose there exists a constant C such that for any $x \in X$, $y \in \partial f(x)$, and for any $\epsilon > 0$, we have*

$$\{v : g(y + v) - g(y) - (x, v) \leq C\epsilon^2\} \subset \epsilon B_{X^*}.$$

Then f is Fréchet differentiable and f' is Lipschitz on X.

Proof. By taking polars, we have $\epsilon^{-1}B_X \subset \{v : g(y+v) - g(y) - (x,v) \leq C\epsilon^2\}^0$. According to Proposition 3 of [A-R], $\{v : g(y+v) - g(y) - (x,v) \leq C\epsilon^2\}^0 \subset C^{-1}\epsilon^{-2}\{u : f(x+u) - f(x) - (y,u) \leq C\epsilon^2\}$. Therefore, $\epsilon C B_X \subset \{u : f(x+u) - f(x) - (y,u) \leq C\epsilon^2\}$, i.e. for any $u \in \epsilon C B_X, f(x+u) + f(x-u) - 2f(x) \leq \frac{2}{C}(\epsilon C)^2$. Thus f' exists at x and we have that f' is Lipschitz on X (cf. e.g. [D-G-Z, Lemma V.3.5]). □

Theorem 2.9. *Let f be a Lipschitz convex function on a Banach space X and $Y = \overline{span}^{\|\cdot\|}\{\partial f(x) : x \in X\}$. Suppose that Y admits an equivalent norm with modulus of convexity of power type 2. Then f can be majorized by a convex function ψ that has Lipschitz derivative and $\psi(x) = f(x)$ for some $x \in X$. In particular, f has points of Lipschitz smoothness.*

Proof. Let $\|\cdot\|$ be an equivalent norm on X^* such that its restriction on Y has modulus of convexity of power type 2 (cf. e.g. [D-G-Z, Lemma II.8.1]). We note that Y is w^*-closed. Indeed, since Y is reflexive, B_Y is compact in the weak topology of X^* and thus B_Y is w^*-compact in X^*. By the Banach-Dieudonné theorem, Y is w^*-closed. Assume that $f(0) = 0$, and thus we have $f^* \geq 0$ on X^*. Let

$$h(x^*) = \begin{cases} \frac{1}{2}\|x^*\|^2 - \frac{1}{2}m^2 \text{ if } x^* \in Y \\ \infty \text{ otherwise,} \end{cases}$$

where $m = Lip(f)$. Since Y is w^*-closed, h is w^*-lower semicontinuous and $h = (h^*_{\restriction X})^*$. We show that h satisfies the condition on the function g given in Lemma 2.8. Indeed, by the modulus of rotundity of $\|\cdot\|$, there exists $L > 0$ such that for any $y_1, y_2 \in Y$, we have

$$\frac{1}{2}\{\|y_1\|^2 + \|y_2\|^2\} - \|\frac{y_1 + y_2}{2}\|^2 \geq L\|y_1 - y_2\|^2 \qquad (*)$$

(cf. e.g. [B, Lemma 5.I.4]). Assume that for every $k \in \mathbb{N}$ there exist $\epsilon_k > 0$, $x_k \in X$, $y_k \in \partial h^*_{\restriction X}(x_k)$ and $v_k \in X^*$, $\|v_k\| > \epsilon_k$, such that $h(y_k+v_k) - h(y_k) -$

$v_k(x_k) \leq \frac{1}{k}\epsilon_k{}^2$. Then $\frac{1}{2}\|y_k+v_k\|^2 - \frac{1}{2}\|y_k\|^2 - (x_k, v_k) \leq \frac{1}{k}\epsilon_k{}^2$ for all k. From the definition of a subdifferential, we have $-(x_k, v_k) \geq \|y_k\|^2 - \|y_k + \frac{v_k}{2}\|^2$. Therefore, $\frac{\|y_k\|^2+\|y_k+v_k\|^2}{2} - \|y_k + \frac{v_k}{2}\|^2 \leq \frac{1}{k}\epsilon_k{}^2 \leq \frac{1}{k}\|v_k\|^2$, which contradicts (*). Now, for each $x^* \in \text{dom } f^* \subset mB_{X^*}$, we have $h(x^*) \leq 0 \leq f^*(x^*)$. Therefore $f^* - h$ is a lower semicontinuous convex function on dom f^* that is bounded below. Note that $f^* - h \geq \|\cdot\| - m$. By the Stegall-Fabian result (cf. e.g. [Ph, Corollary 5.22]), there exists $\hat{x} \in Y^*$ such that $f^* - h - \hat{x}$ attains its minimum in dom f^*, i.e. there is a $x^* \in \text{dom } f^*$ such that $f^*(x^*) - h(x^*) - \hat{x}(x^*) = \alpha \leq f^*(y^*) - h(y^*) - \hat{x}(y^*)$ for all $y^* \in \text{dom } f^*$. Therefore we have $h(\cdot)+\hat{x}(\cdot)+\alpha \leq f^*(\cdot)$ on dom f^* and the equality holds at x^*. Since Y is reflexive, there exists $x \in X$ such that $y^*(x) = \hat{x}(y^*)$ for each $y^* \in Y$. Let $k : X^* \to I\!R$ be a function defined by $k(\cdot) = h(\cdot)+x(\cdot)+\alpha$. Then k is a convex function such that $k \leq f^*$ and $k(x^*) = f^*(x^*)$. Put $l = k_{\restriction Y}$. The function l is continuous and convex on Y. Let $\hat{y} \in \partial l(x^*) \subset Y^*$. As Y is reflexive, there exists $y \in X$ such that $\hat{y}(y^*) = y^*(y)$ for each $y^* \in Y$. We claim that $y \in \partial k(x^*)$. Indeed, let $z^* \in X^*$. If $z^* \in Y$, $y(z^* - x^*) = \hat{y}(z^* - x^*) \leq k(z^*) - k(x^*)$. If $z^* \notin Y$, then $y(z^* - x^*) < k(z^*) - k(x^*) = \infty$. Hence $y \in \partial k(x^*)$. Since $k(x^*) = f^*(x^*)$, we have $y \in \partial f^*(x^*)$. Thus $k^*(y) + k(x^*) = (x^*, y) = f^*(x^*) + f(y)$. Therefore $f(y) = k^*(y)$. Since $f^* \geq k$, we have $k^* \geq f$. Put $\psi = k^*_{\restriction X}$. The function ψ has Lipschitz derivative and is our required function. Indeed, $k^* = (h(\cdot) + x(\cdot) + \alpha)^* = (h+x)^* - \alpha = h^*(\cdot)\Box\delta_x(\cdot) - \alpha = h^*(\cdot - x) - \alpha$ (where δ_x is the indicator function of the singleton $\{x\}$) and h^* has the desired differentiability by Lemma 2.8. Finally, since $f(y) = k^*(y) = \psi(y)$ and $f \leq \psi$ on X, we have $f(y+v)+f(y-v)-2f(y) \leq \psi(y+v)+\psi(y-v)-2\psi(y) \leq C\|v\|^2$, for some constant C. Therefore the function f is Fréchet differentiable at y and f' is Lipschitz at y. □

Similarly, using Troyanski's result that reflexive spaces admit equivalent LUR norms (see e.g. [D-G-Z, Chapter VII]), we can show the following result.

Corollary 2.10. *Let f be a Lipschitz convex function on a Banach space X and $Y = \overline{span}^{\|\cdot\|}\{\partial f(x) : x \in X\}$. If Y is reflexive, then f can be majorized on X by a convex function ϕ that is Fréchet differentiable and $\phi(x) = f(x)$ for some $x \in X$.*

Under the assumptions in Theorem 2.9, techniques in Theorem 2.3 may be applied to obtain approximation by functions with Lipschitz derivatives.

Theorem 2.11. *Let X,Y and f be as in Theorem 2.9. Then f can be uniformly approximated on X by convex functions that have Lipschitz derivative.*

Proof. As in the proof of Theorem 2.9, let $\|\cdot\|$ be an equivalent norm of X^* such that its restriction on Y is LUR. Let $h = \frac{1}{2}\|\cdot\|^2$ and $g := h + f^*$ on X^*. The function g is w^*-lower semicontinuous on X^*. Let k be a convex function on X such that $k^* = g$. We claim that there exists a constant C such that for any $\epsilon > 0$, $x \in X$ and $y \in \partial k(x)$, we have $\{v : g(v+y)-g(y)-(x,v) \leq C\epsilon^2\} \subset \epsilon B_{X^*}$. Since $g(u) = \infty$ whenever $u \notin Y$, we only need to consider points in Y. Let $v \in Y$, then $\frac{g(y)+g(y+v)}{2} - g(\frac{2y+v}{2}) \geq \frac{h(y)+h(y+v)}{2} - h(\frac{2y+v}{2})$ for any $y \in Y$. Using (*), we have $\frac{g(y)+g(y+v)}{2} - g(\frac{2y+v}{2}) \geq L\|v\|^2$ for any $v \in Y$ and for any $y \in Y$. Following the same idea as in the proof of Theorem 2.7, we complete the proof of the claim. By Lemma 2.8, k is Fréchet differentiable and k' is Lipschitz. For each $n \in I\!N$ define $g_n := f^* + \frac{1}{2n^4}h$ and k_n such that $k_n^* = g$. By the above argument, the function k_n is Fréchet differentiable and k_n' is Lipschitz for each $n \in I\!N$. By [M-P-V-Z, Lemma 2.1], $\lim g_n = f$ uniformly on X. □

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Chapter Three

Asplund Functions

Introduction. It is known that a Banach space X is an Asplund space if only if its dual X^* has the RNP, and if and only if every separable subspace Y of X has separable dual Y^*. The above conditions are also equivalent to the existence of a Jayne-Rogers selector for the duality map of a norm on X. As in the preceding two chapters, we study an analogue of these equivalent conditions in the sense of subdifferentials in a certain class of functions which may be defined on a non-Asplund space.

We recall that given a bounded set $A \subset X^*$, the indicator function $\delta_A(\cdot)$ of A is a convex function that takes value zero in A and ∞ elsewhere. The function $\delta_A(\cdot)$ is w^*-lower semicontinuous if and only if A is w^*-closed.

Definition 3.1. Let X be a Banach space and A be a convex subset of X. A slice of A is a set of the form

$$S(A, x^*, \alpha) = \{x \in A : x^*(x) > \sup x^*(A) - \alpha\},$$

for some $x^* \in X^*$ and $\alpha \in \mathbb{R}$.

Definition 3.2. Let X and A be as above, we say that A is dentable if given any $\epsilon > 0$, there is a slice of A of diameter less than ϵ. We say that A is subset dentable if every closed bounded subset of A is dentable. A Banach space is said to have RNP if X is subset dentable.

Definition 3.3. Let X^* be a dual Banach space and A be a convex subset of X^*. A w^*-slice of A is a set of the form

$$S(A, x^*, \alpha) = \{x \in A : x^*(x) > \sup x^*(A) - \alpha\},$$

for some $x^* \in X$ and $\alpha \in \mathbb{R}$. We say that A is w^*-dentable if A admits arbitrarily small w^*-slice, and we say that A is subset w^*-dentable if every closed bounded subset of A is w^*-dentable.

Definition 3.4. Let Z be a topological space and ρ a metric on Z that is not necessarily related to the topology of Z. For $\epsilon > 0$, the space Z is said to be fragmented down to ϵ when each non-empty subset of Z contains a non-empty open subset of ρ-diameter less than ϵ. The space Z is said to be fragmentable if it is fragmented down to ϵ for each $\epsilon > 0$. The space Z is said to be σ-fragmentable if, for each $\epsilon > 0$, Z can be written as a countable union of sets $\{Z_i\}$ with each Z_i fragmented down to ϵ.

Definition 3.5. A Banach space X is called an Asplund space if every continuous convex function on X is generically Fréchet differentiable.

It is well known that a Banach space X is an Asplund space if and only if X^* is w^*-subset dentable, and equivalently, the topological space (B_{X^*}, w^*) is fragmentable by the dual norm (cf. [D-G-Z]).

The main result in this chapter is the following:

Theorem 3.6. *Let f be a continuous convex function defined on a Banach space X. The following are equivalent.*

(1) If h is a continuous convex function on X such that $h \leq f$ on X, then h is generically Fréchet differentiable on X.

(2) For each positive integer n, the set $C_n := \{x^* \in X^* : f^*(x^*) \leq n\}$ is subset w^*-dentable.

(3) For each separable subspace Y of X, the set $\partial f_{|Y}(Y)$ is separable.

(4) For each Baire space Y and each minimal upper semi-continuous set-valued map Φ on Y, taking non-empty compact values in $(\overline{\partial f(X)}^{\|\cdot\|}, w^*)$ (usco), there is a dense G_δ-subset D of Y such that Φ is single valued at each point of D and Φ, when regarded as a map from Y to $\overline{\partial f(X)}^{\|\cdot\|}$ with the norm topology, is upper semi-continuous as each point of D.

(5) Every w^*-compact subset of $\overline{\partial f(X)}^{\|\cdot\|}$ is fragmentable by the norm.

If moreover, f is a Lipschitz function, then the above conditions are also *equivalent to:*

(6) For each separable subspace Y *of* X*, there is a selector* s *for* $\partial f_{\restriction Y}$ *such that* $s(Y) = \{s(y) : y \in Y\}$ *is separable.*

Proof. (1) $\implies$ (2). Indeed, for otherwise, there exists a bounded w^*-closed subset A of C_n that is not dentable. The function $((\delta_A(\cdot)+n)^*)_{\restriction X}$ is nowhere differentiable and is less than f.

(2) $\implies$ (3). Let Y be a separable subspace of X. Let $R : X^* \to Y^*$ be the restriction map. The map R is w^* to w^* continuous, and by the Hahn-Banach theorem, we have $R\partial f(y) = \partial f_{\restriction Y}(y)$, for all $y \in Y$. For each positive integer n, we define the sets C_n^Y and H_n as follows:

$$C_n^Y = \{y^* \in Y^* : f^*_{\restriction Y}(y^*) \leq n\} \text{ , and}$$

$$H_n = \partial f_{\restriction Y}(Y) \cap C_n^Y.$$

We note that

$$\bigcup_{n=1}^{\infty} H_n = \bigcup_{n=1}^{\infty} (\partial f_{\restriction Y}(Y) \cap C_n^Y) = \partial f_{\restriction Y}(Y) \cap (\bigcup_{n=1}^{\infty} C_n^Y)$$
$$= \partial f_{\restriction Y}(Y) \cap \text{dom} f^*_{\restriction Y} = \partial f_{\restriction Y}(Y).$$

We claim that for each n, $H_n \subset R(C_n)$. Indeed, let $y^* \in H_n$. Then $y^* \in \partial f_{\restriction Y}(y)$ for some $y \in Y$. Let $\hat{y}^* \in \partial f(y)$ such that $R(\hat{y}^*) = y^*$. We have $f^*(\hat{y}^*) = (\hat{y}^*, y) - f(y) = (y^*, y) - f_{\restriction Y}(y) = f^*_{\restriction Y}(y^*) \leq n$. Therefore $\hat{y}^* \in C_n$. Hence $y^* = R(\hat{y}^*) \in R(C_n)$.

From the claim and the observation above we have $\partial f_{\restriction Y}(Y) \subset \bigcup_{n=1}^{\infty} R(C_n)$. Suppose $\partial f_{\restriction Y}(Y)$ is not separable, then there exists an integer N such that $R(C_N)$ is not separable. Therefore there exist $k \in \mathbb{N}$ such that $R(C_N \cap kB_{X^*})$ is not separable. For simplicity, denote $C'_N = C_N \cap kB_{X^*}$.

Note that $R(C'_N)$ is a w^*-compact subset of Y^*. By employing the technique in [Ph, 2.19], we obtain a w^*-compact set $A \subset R(C'_N)$ and $\epsilon > 0$ such that any non-empty w^*-open subset of A contains two distinct points x^* and y^* such that $\|x^* - y^*\| > \epsilon$.

Now we follow the arguments in [Ph, 5.4], let $A_1 \subset C'_N$ be a minimal w^*-compact set such that $R(A_1) = A$. If U is a non-empty relatively w^*-open subset of A_1, then $A_1 \setminus U$ is compact and $A_2 = R(A_1 \setminus U)$ is a proper compact subset of A (since A_1 is minimal). Thus $A \setminus A_2$ is a non-empty w^*-open subset of A and it contains two distinct points with distance at least ϵ apart. Therefore there exist x^* and y^* in U such that $\|x^* - y^*\| > \epsilon$, contradicting the assumption that C_N is subset w^*-dentable.

(3) $\Longrightarrow$ (1). According to [Pr-Z], $f_{\restriction Y}$ is generically differentiable for each separable subspace $Y \subset X$. By the separable reduction theorem in [Pr], f is generically differentiable.

(2) $\Longrightarrow$ (4). We first recall that $\partial f(X) \subset dom f^* \subset \overline{\partial f(X)}^{\|\cdot\|}$. We also note that $dom f^* = \bigcup_{n,k} C_{n,k}$. where $C_{n,k} = kB_{X^*} \cap C_n$. For each n and $k \in I\!N$, the set $C_{n,k}$ is w^*-compact and subset w^*-dentable, and thus a w^*-compact norm fragmentable subset. Therefore $\overline{\partial f(X)}^{\|\cdot\|}$ is the closure of a countable union of compact norm fragmentable subsets, hence $\overline{\partial f(X)}^{\|\cdot\|}$ is σ-fragmentable by the norm (cf. [J-N-R, Lemma 2.3]). By [J-N-R, Theorem 3.2], given any usco map $\Phi : Y \to 2^{\overline{\partial f(X)}^{\|\cdot\|}}$ from a Baire space Y, there is a dense G_δ-subset D of Y that satisfies our requirements.

(4) $\Longrightarrow$ (1). Let h be a convex function bounded above by f. Then the subdifferentiable map ∂h is a usco map on X, taking set values in $\partial h(X) \subset \overline{\partial f(X)}^{\|\cdot\|}$. Let Φ be a minimal usco map such that $\Phi \subset \partial h$. Then by assumption, Φ (and also ∂h) has a selector which is single valued and norm to norm continuous on a dense G_δ set D of X. Therefore h is Fréchet differentiable on D (cf. eg. [Ph, 2.8]).

(4) $\Longrightarrow$ (5). This follows from (b) $\Longrightarrow$ (d) of [J-N-R, Theorem 3.1].

(5) $\implies$ (4). As in the proof of (2) $\implies$ (4), $\overline{\partial f(X)}^{\|\cdot\|}$ is the closure of a countable union of compact subsets $C_{n,k}$. By the hypothesis, $C_{n,k}$ is fragmentable by the norm for each n and k. Again by [J-N-R, Lemma 2.3], $\overline{\partial f(X)}^{\|\cdot\|}$ is σ-fragmentable by the norm and (4) follows from [J-N-R, Theorem 3.2]. Finally, suppose f is a Lipschitz convex function, then (3) $\iff$ (6) follows from Theorem 1.6. □□

Definition 3.7. Let f be a convex function on a Banach space X, we say that f is an *Asplund function* if f satisfies any of conditions (1) to (6).

Remarks.

1. Note that suppose the Asplund function f is a norm on a Banach space X, then $B_{X^*}(= C_1 = dom f^* = \overline{\partial f(X)}^{\|\cdot\|})$ is fragmentable by the dual norm, as B_{X^*} is w^*-compact. Consequently, X is an Asplund space (cf. eg. [D-G-Z]). Furthermore, given any separable subspace Y of X, $B_{Y^*}(= \partial f_{|Y}(Y))$ is separable. Therefore, the above theorem yields a notion that is more general than that of Asplund spaces, it also summarizes several important developments in the theory of Asplund spaces.

2. We note that a convex function f is Lipschitzian if and only if $\partial f(X)$ (and thus also $dom f^*$) is bounded. Therefore for a non-Lipschitzian function f, the techniques in the proof of Theorem 1.6 (2) $\implies$ (1) cannot be used to show (6) $\implies$ (3).

The following proposition is immediate.

Proposition 3.8. *Every continuous convex function defined on an Asplund space is an Asplund function.*

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Chapter Four

Preserved Smoothness

Introduction. Let X be a Banach space equipped with a norm $\|\cdot\|$. We say that $\|\cdot\|$ is Gâteaux differentiable at x if for every $h \in S_X(\|\cdot\|)$,

$$\lim_{t\to 0} \frac{\|x+th\| - \|x\|}{t} \tag{*}$$

exists. We say that the norm $\|\cdot\|$ is Gâteaux differentiable if $\|\cdot\|$ is Gâteaux differentiable at all $x \in S_X(\|\cdot\|)$. Suppose the limit in (*) exists uniformly in $x \in S_X(\|\cdot\|)$ for every $h \in S_X(\|\cdot\|)$, we say that $\|\cdot\|$ is uniformly Gâteaux differentiable (UG for short). A point $x \in S_X(\|\cdot\|)$ is said to be a smooth point if the norm $\|\cdot\|$ is Gâteaux differentiable at x. A smooth point x is said to be a preserved smooth point if the bidual norm is also Gâteaux differentiable at x. The norm $\|\cdot\|$ is said to be octahedral if there exists a $u \in X^{**} \setminus \{0\}$ such that $\|x+u\| = \|x\| + \|u\|$ for all $x \in X$.

B. V. Godun [Go1] has shown that a separable Banach space is reflexive if and only if each smooth point is preserved in each equivalent norm. On the other hand, the dual version of the above result has also been obtained in another paper of B. V. Godun (cf., [Go2]), which says that a Banach space is reflexive if and only if for each equivalent norm the extreme points of the unit ball are preserved. P. Morris [Mo] has shown that a separable Banach space has a subspace isomorphic to c_0 if and only it admits an equivalent strictly convex norm in which no element on the sphere is a preserved extreme point. Some other related results on extreme points are also contained in [Ma].

It is clear that if a Gâteaux differentiable norm $\|\cdot\|$ is octahedral, then every point on the sphere $S_X(\|\cdot\|)$ is not a preserved smooth point. It is unknown whether a separable space that contains l_1 necessarily admits a Gâteaux differentiable octahedral norm. However, in this chapter, we show the following:

Theorem 4.1. *Let X be a separable Banach space containing an isomorphic copy of l_1. Then X admits a uniformly Gâteaux differentiable LUR norm such that its bidual norm is nowhere Gâteaux differentiable at points of X.*

At the end of the chapter, we show by using elementary methods (without using octahedrality of norms) that l_1 has also such a property. We refer the readers to [D-G-Z] for some unexplained notions and results used in this note. We also refer to [G] for more related results.

Proof of Theorem 4.1. Since X contains l_1, it admits an octahedral norm $\|\cdot\|$ (cf. [G]). Let $\{x_n\}$ be a countable dense set of $S_X(\|\cdot\|)$. Define an equivalent dual norm $|\cdot|$ on X^* by $|f| = (\|f\|^2 + p^2(f))^{1/2}$, where $p(f) = (\sum_{i=0}^{\infty} \frac{f(x_i)^2}{2^i})^{1/2}$. The norm $|\cdot|$ is w^*-lower semicontinuous and weak* uniformly rotund (W^*UR). Therefore its predual is uniformly Gâteaux differentiable (cf. [D-G-Z, II.6.7]).

Let $x \in S_X(|\cdot|)$, $f = |\cdot|'(x) \in S_{X^*}(|\cdot|)$. By octahedrality of $\|\cdot\|$ there exists a $u \in X^{**}$ such that u has no point of continuity on $(\|f\|B_{X^*}(\|\cdot\|), w^*)$ (cf. [D-G-Z, III.2.4]). Therefore, there exists a sequence $\{f_n\} \subset \|f\|B_{X^*}(\|\cdot\|)$ such that $f_n \longrightarrow f$ in the w^*-topology but $u(f_n)$ does not converge to $u(f)$. We note that $|f_n| \longrightarrow |f|$. Indeed, since p is w^*-continuous, $p(f_n) \longrightarrow p(f)$; furthermore $\|f_n\| \longrightarrow \|f\|$ as $\|\cdot\|$ is w^*-lower semicontinuous. Hence, according to the Šmulyan's lemma (cf. [D-G-Z, I.1.4]), the dual norm of $|\cdot|$ is not Gâteaux differentiable at x.

Finally, let $|\cdot|_1$ be an equivalent locally uniformly rotund (LUR), uniformly Gâteaux differentiable norm on X. The norm $|||\cdot||| = [|\cdot|_1^2 + |\cdot|^2]^{\frac{1}{2}}$ is LUR and uniformly Gâteaux differentiable on X but its bidual is not differentiable at points of X. $\square$

Example 4.2. Using a different method, we show that l_1 admits a uniformly Gâteaux differentiable norm such that its bidual norm is nowhere differentiable at points of l_1

Proof. We show that the norm in [Ph, p.86] is a required norm. Define $\|\cdot\|$ on l_∞ by $\|y\| = (\|y\|_\infty^2 + p(y)^2)^{1/2}$ where $p(y) = (\sum \frac{y_i^2}{2^i})^{\frac{1}{2}}$. The norm $\|\cdot\|$ is W^*UR, thus its predual is uniformly Gâteaux differentiable. Let $x \in S_{l_1}(\|\cdot\|)$ and $y = \|\cdot\|'(x) \in S_{l_\infty}(\|\cdot\|)$. We shall construct a sequence y^k such that $\|y^k\| \longrightarrow \|y\|$, $y^k(x) \longrightarrow y(x)$, but y^k does not converge to y weakly. We may write $y = (y_1, y_2, y_3, ...)$ and consider two cases:

Case I If $y_n \longrightarrow 0$.

Then there exists an integer N such that $|y_n| < \frac{1}{4}$ for all $n > N$. We define y^k for $k \geq N$:

$$y_n^k = \begin{cases} \frac{1}{2} & \text{if } n > k \\ y_n & \text{otherwise,} \end{cases}$$

We note the following:

(1) $(y^k, x) = (y, x) - \sum_{n=k}^{\infty} x_n(y_n - \frac{1}{2}) \longrightarrow (y, x) = 1$ as $k \longrightarrow \infty$.

(2) $p^2(y^k) = p^2(y) - \sum_{n=k}^{\infty}(\frac{y_n^2 - 1/4}{2^n}) \longrightarrow p^2(y)$ as $k \longrightarrow \infty$.

(3) $\|y^k\|_\infty = \|y\|_\infty$ as $1 = \|y\| \leq 2\|y\|_\infty$.

(2) and (3) imply that $\|y^k\| \longrightarrow \|y\|$.

However, y^k does not converge weakly to y, since any convex combination of $\{y^k\}$ has distance at least $\frac{1}{4}$ from y. Therefore by the Šmulyan's lemma, $\|\cdot\|$ is not differentiable in l_1^{**} at x.

Case II If y_n does not converge to zero.

Then there exists $\epsilon > 0$ and a subsequence $\{y_{nk}\}$ such that $|y_{nk}| > 2\epsilon$ for all k. Define y^m as follows:

$$y_n^m = \begin{cases} 0 & \text{if } n > k \\ y_n & \text{otherwise,} \end{cases}$$

As in case I, $\|y^m\| \longrightarrow \|y\|$ as $m \longrightarrow \infty$, but y^m does not coverge to y weakly.□

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Chapter Five

Extensions and Geometry of Rotund Spheres

Introduction. It is shown in [J-Z_1] (see also [D-G-Z, II.8] and [J-Z_2] for a more general result) that if Y is a closed subspace of a separable Banach space X and if Y admits a LUR norm, then this LUR norm on Y can be extended to a LUR norm on X. In [F], it is shown that if Y is reflexive, and if the norms of X and Y possess some kind of rotundity, then the norm on Y can be extended to a norm on X with the same rotundity. It is also shown in [F] that Fabián's extension preserves moduli of convexity of power type. In section one of this chapter, we give an extension which does not require the subspace Y to be reflexive. Our construction, although unnatural at first sight, turns out to be very useful in various situations. We also show that our method of extension preserves convexity of power type. However, the modulus of convexity of this extended norm may be of larger order as compared to the original moduli.

Section Two of this chapter is devoted to a natural question: given an arbitrary set G in a Banach space X that admits a strictly convex norm, can we construct a strictly convex norm which unit sphere contains the set G? A partial solution is given in this section.

Let $(X, \|\cdot\|)$ be a real Banach space. We say that $\|\cdot\|$ is *strictly convex* (R) if $x = y$ whenever $x, y \in S_X(\|\cdot\|)$ and $\frac{x+y}{2} \in S_X(\|\cdot\|)$. The norm $\|\cdot\|$ is called *locally uniformly rotund* (LUR) if $x_n, x \in X$ and if $\frac{1}{2}\|x_n\|^2 + \frac{1}{2}\|x\|^2 - \|\frac{x_n+x}{2}\|^2 \longrightarrow 0$, then $\|x_n - x\| \longrightarrow 0$. The norm $\|\cdot\|$ is *uniformly rotund* (UR) if given bounded sequences $\{x_n\}$, $\{y_n\} \subset X$ and if $\frac{1}{2}\|x_n\|^2 + \frac{1}{2}\|y_n\|^2 - \|\frac{x_n+y_n}{2}\|^2 \longrightarrow 0$, then $\|x_n - y_n\| \longrightarrow 0$; if $x_n - y_n \longrightarrow 0$ weakly instead of in the norm topology, we say that $\|\cdot\|$ is *weakly uniformly rotund* (WUR). The norm $\|\cdot\|$ is said to be *uniformly rotund in every direction* (URED) if given bounded sequences $\{x_n\}$,$\{y_n\} \subset X$ such that $x_n - y_n \in$ span$z$, for some $z \in X$, and if $\frac{1}{2}\|x_n\|^2 + \frac{1}{2}\|y_n\|^2 - \|\frac{x_n+y_n}{2}\|^2 \longrightarrow 0$, then $\|x_n - y_n\| \longrightarrow 0$. The norm $\|\cdot\|$ is said to have the *Kadeč-Klee property* (KKP) if given $x, x_n \in X$ such that $\|x_n\| \longrightarrow \|x\|$ and $x_n \longrightarrow x$ weakly, then $\|x_n - x\| \longrightarrow 0$. The modulus of convexity of a norm $\|\cdot\|$ is defined as:

$$\delta_{\|\cdot\|}(\epsilon) = \inf\{1 - \|\frac{x+y}{2}\| : \|x\| = \|y\| = 1 \ , \ \|x-y\| \geq \epsilon\}.$$

The norm $\|\cdot\|$ is said to have modulus of convexity of power type r if $\delta_{\|\cdot\|}(\epsilon) \geq K\epsilon^r$, for some constant $K > 0$ and for all small $\epsilon > 0$. It is known that a norm $\|\cdot\|$ has modulus of convexity of power type r if and ond if $\frac{1}{2}\|x\|^r + \frac{1}{2}\|y\|^r - \|\frac{x+y}{2}\|^r \geq L\|x-y\|^r$(cf. [B, V.3.2]). This inequality turns out to be very useful in the computation of modulus of convexity in Theorem 5.1.3.

A set $\{x_\alpha; f_\alpha\}_{\alpha\in\Gamma} \subset X \times X^*$ is a bounded biorthogonal system if $f_\alpha(x_\beta) = \delta_{\alpha\beta}$ and $\sup_{\alpha\in\Gamma} \|x_\alpha\|\|f_\alpha\| < \infty$. A set of functionals $\{g_\alpha\} \subset X^*$ is said to be total on a subspace $Y \subset X$ if $\{g_a\}$ separates the points of Y.

Section One : Extension of Rotund Norms.

A main result of this section is the following theorem.

Theorem 5.1.1. *Let Y be a closed subspace of a Banach space X. Suppose X and Y both admit UR (LUR, WUR, URED, R or KKP) norms say $\|\cdot\|$ and $|\cdot|_Y$ respectively, then $|\cdot|_Y$ can be extended to a UR (respectively LUR, WUR, URED, R or KKP) norm on X.*

We recall a useful fact (cf. [D-G-Z, II.2.3]):

Lemma 5.1.2. *If p is a positive convex function on a bounded convex set $C \subset X$ and x, $y \in C$, then (1) $\implies$ (2) $\implies$ (3), where*

(1) $\frac{1}{2}p^2(x) + \frac{1}{2}p^2(y) - p^2(\frac{x+y}{2}) \leq \epsilon$, for some $\epsilon > 0$.

(2) $|p(x) - p(y)| \leq K_1\sqrt{\epsilon}$ and $|p(\frac{x+y}{2}) - p(x)| \leq K_2\sqrt{\epsilon}$, for some constants K_1 and K_2.

(3) $\frac{1}{2}p^2(x) + \frac{1}{2}p^2(y) - p^2(\frac{x+y}{2}) \leq K_3\sqrt{\epsilon}$, for some constant K_3.

Proof. To see that (1) $\implies$ (2), we note that $p(\frac{x+y}{2}) \leq \frac{p(x)+p(y)}{2}$. Therefore we have

$$(\frac{p(x)-p(y)}{2})^2 \leq \frac{1}{2}p^2(x) + \frac{1}{2}p^2(y) - p^2(\frac{x+y}{2}) \leq \epsilon.$$

Proof of Theorem 5.1.1. First we prove the theorem for the case when $\|\cdot\|$ and $|\cdot|_Y$ are both UR, the proofs for the cases of LUR, WUR and R are similar. Let $|\cdot|$ be any extension of $|\cdot|_Y$. Without loss of any generality, we may assume that $\|\cdot\| \leq \frac{1}{\sqrt{2}}|\cdot|$. Define a function $p(\cdot)$ on $B_X(|\cdot|)$ by

$$p^2(x) = |x|^2 + q(x) + \operatorname{dist}(x, Y)^2,$$

Where $q(x) = \operatorname{dist}(x, Y)^2 e^{\|x\|^2}$. We note that $q(x)$ is a convex function on $B_X(\|\cdot\|)$. To see this, we compute the Hessian of the function $f(x, y) = y^2 e^{x^2}$, where $(x, y) \in I\!R^2$ and observe that the Hessian is positive definite on $(0, \frac{1}{\sqrt{2}}) \times (0, \frac{1}{\sqrt{2}})$, we leave the details to the readers. Therefore p is a convex symmetric function. Define $B = \{x \in X : p(x) \leq 1\}$. The set B is bounded as $p(\cdot) \geq |\cdot|$. On the other hand, $p^2(x) \leq (2+e)|x|^2$ for all $x \in B$. Therefore B is an equivalent ball on X. Let $|\!|\!|\cdot|\!|\!|$ denote the corresponding norm defined by B.

To see that $|\!|\!|\cdot|\!|\!|$ is UR, let $x_n, y_n \in S_X(|\!|\!|\cdot|\!|\!|)$ such that $|\!|\!|\frac{y_n+x_n}{2}|\!|\!| \to 1$. Therefore we have $\frac{1}{2}p^2(x_n) + \frac{1}{2}p^2(y_n) - p^2(\frac{x_n+y_n}{2}) \longrightarrow 0$. Then by convexity, we have

$$\frac{1}{2}|x_n|^2 + \frac{1}{2}|y_n|^2 - |\frac{x_n + y_n}{2}|^2 \to 0, \tag{1.1}$$

$$\frac{1}{2}q(x_n) + \frac{1}{2}q(y_n) - q(\frac{x_n + y_n}{2}) \to 0, \text{ and} \tag{1.2}$$

$$\frac{1}{2}\operatorname{dist}(x_n, Y)^2 + \frac{1}{2}\operatorname{dist}(y_n, Y)^2 - \operatorname{dist}(\frac{x_n + y_n}{2}, Y)^2 \to 0 \tag{1.3}$$

Applying Lemma 5.1.2 to (1.3) we have the following:

$$\lim(\operatorname{dist}(x_n, Y) - \operatorname{dist}(y_n, Y)) = 0, \text{ and} \tag{1.4}$$

$$\lim(\operatorname{dist}(\frac{x_n + y_n}{2}, Y) - \operatorname{dist}(x_n, Y)) = 0. \tag{1.5}$$

Assume that $|||x_n - y_n|||$ does not converge to zero, then there exists a subsequence of $\{x_n - y_n\}$ which we label again as $\{x_n - y_n\}$ that is bounded from zero, i.e., there exists an $\epsilon > 0$ such that

$$|||x_n - y_n||| > \epsilon \text{ for each } n \in I\!N. \tag{1.6}$$

Now we consider two cases:

Case I: Suppose $\lim \operatorname{dist}(x_n, Y) = 0$. Then by (1.4), there exists two sequences $\{x'_n\}, \{y'_n\} \subset Y$ such that $|x_n - x'_n|, |y_n - y'_n| \longrightarrow 0$. Since $\{x_n\}$ and $\{y_n\}$ are bounded, we have

$$\frac{1}{2}|x'_n|^2 + \frac{1}{2}|y'_n|^2 - |\frac{x'_n + y'_n}{2}|^2 \to 0.$$

by the uniform rotundity of $|\cdot|$ on Y, we have $|x'_n - y'_n| \longrightarrow 0$ and thus $|x_n - y_n| \longrightarrow 0$, contradicting (1.6).

Case II: Suppose $\overline{\lim} \operatorname{dist}(x_n, Y) = d > 0$. Then there exists a subsequence $\{x_{nk}\}$ of $\{x_n\}$ such that $\lim \operatorname{dist}(x_{nk}, Y) = d$, by relabeling the sequence, we may assume that $\lim \operatorname{dist}(x_n, Y) = d$.

Then by (1.2),

$$\lim \frac{1}{2}\operatorname{dist}(x_n, Y)^2 e^{\|x_n\|^2} + \frac{1}{2}\operatorname{dist}(y_n, Y)^2 e^{\|y_n\|^2} - \operatorname{dist}(\frac{x_n+y_n}{2}, Y)^2 e^{\|\frac{x_n+y_n}{2}\|^2} = 0,$$

which implies

$$\lim(\frac{1}{2}e^{\|x_n\|^2} + \frac{1}{2}e^{\|y_n\|^2} - e^{\|\frac{x_n+y_n}{2}\|^2}) = 0. \tag{1.7}$$

Hence,

$$\sum_{k=1}^{\infty} \frac{1}{k!}(\frac{1}{2}\|x_n\|^{2k} + \frac{1}{2}\|y_n\|^{2k} - \|\frac{x_n+y_n}{2}\|^{2k}) \longrightarrow 0, \text{ as } n \longrightarrow \infty. \tag{1.8}$$

Consequently, we have

$$\lim(\frac{1}{2}\|x_n\|^2 + \frac{1}{2}\|y_n\|^2 - \|\frac{x_n+y_n}{2}\|^2) = 0. \tag{1.9}$$

The rotundity of $\|\cdot\|$ together with (1.9) imply that $\|x_n - y_n\| \longrightarrow 0$, contradicting (1.6). Hence proved.

For the case of URED norms, we use the same extension as above. To see that $|||\cdot|||$ is URED, let $x_n, y_n, z \in S_X(|||\cdot|||)$ such that $|||\frac{y_n+x_n}{2}||| \to 1$ and $x_n - y_n = \lambda_n z$ for some $\lambda_n \in I\!R$ for each $n \in I\!N$.
Assume that $|\lambda_n| = |||x_n - y_n|||$ does not converge to zero, then there exists a subsequence of $\{x_n - y_n\}$ which we label again as $\{x_n - y_n\}$ that is bounded from zero, i.e., there exists an $\epsilon > 0$ such that

$$|\lambda_n| = |||x_n - y_n||| > \epsilon. \tag{1.6'}$$

Again we consider two cases:

Case I: Suppose that $\lim \operatorname{dist}(x_n, Y) = 0$. Suppose that $\operatorname{dist}(z, Y) > 0$ then $|\lambda_n| \operatorname{dist}(z, Y) = \operatorname{dist}(x_n - y_n, Y) \leq \operatorname{dist}(x_n, Y) + \operatorname{dist}(y_n, Y) \longrightarrow 0$. Thus $|\lambda_n| = |||x_n - y_n||| \longrightarrow 0$, contradicting (1.6'). Therefore we assume $z \in Y$.

Since $\text{dist}(x_n, Y) \longrightarrow 0$, there exists $\{x'_n\} \subset Y$ such that $|x_n - x'_n| \longrightarrow 0$. Let $y'_n = x'_n - \lambda_n z$. Then $y'_n \in Y$ and $x'_n - y'_n = \lambda_n z$. Furthermore, $|y'_n - y_n| = |x'_n - x_n| \longrightarrow 0$. Since $\{x_n\}$ is bounded, using (1.1) we have $\frac{1}{2}|x'_n|^2 + \frac{1}{2}|y'_n|^2 - |\frac{x'_n + y'_n}{2}|^2 \to 0$, as the norm $|\cdot|$ is URED on Y, $|x'_n - y'_n| \longrightarrow 0$ and thus $|x_n - y_n| \longrightarrow 0$, contradicting (1.6′).

Case II: Suppose $\overline{\lim}\, \text{dist}(x_n, Y) = d > 0$. Then we follow the same argument as in the case of UR norms.

Suppose now $|\cdot|_Y$ and $\|\cdot\|$ have KKP, let $x \in S_X(|||\cdot|||)$ and $x_n \in X$ such that $x_n \longrightarrow x$ weakly and $|||x_n||| \longrightarrow |||x|||$. Therefore we have $|x_n| \longrightarrow |x|$, $q(x_n) \longrightarrow q(x)$ and $\text{dist}(x_n, Y) \longrightarrow \text{dist}(x, Y)$, as these functions are weakly lower-semicontinuous. Suppose $\text{dist}(x, Y) > 0$, then $q(x_n) \longrightarrow q(x)$ would imply $\|x_n\| \longrightarrow \|x\|$, consequently $\|x_n - x\| \longrightarrow 0$ by the KKP of $\|\cdot\|$. If $\text{dist}(x, Y) = 0$, then we may find $x'_n \in Y$ such that $|x'_n - x_n| \longrightarrow 0$. It is clear that $|x'_n| \longrightarrow |x|$ and $x'_n \longrightarrow x$ weakly. Therefore we have $|x'_n - x| \longrightarrow 0$ and thus $|x_n - x| \longrightarrow 0$. Hence $|||\cdot|||$ is a norm with the Kadeč-Klee property. The proof is complete. □

A quantitative version of Theorem 5.1.1 is as follows:

Theorem 5.1.3. *Let Y be a closed subspace of a Banach space X. Suppose X and Y both admit norms say $\|\cdot\|$ and $|\cdot|_Y$ with modulus of convexity of power type $r \geq 2$, then for any $r' > r$, $|\cdot|_Y$ can be extended to a norm on X with modulus of convexity of power type $4r'$, .*

Proof. Let $\alpha > 0$ be given and $|\cdot|$ be any extension of $|\cdot|_Y$ to the whole Banach space X. Without any loss of generality, we may assume that $\|\cdot\| \leq (\frac{\alpha(r-1)}{r})^{\frac{1}{r}}|\cdot|$. We define a function $p(\cdot)$ on $B_X(|\cdot|)$ by

$$p^r(x) = |x|^r + q(x) + \text{dist}(x, Y)^r,$$

Where $q(x) = \text{dist}(x, Y)^{1+\alpha} e^{\|x\|^r}$. As in Theorem 5.1.1, $q(x)$ can be shown to be convex and symmetric on $B_X(|\cdot|)$, and the norm $|||\cdot|||$ defined by the convex symmetric set $B := \{x \in X : p(x) \leq 1\}$ is an equivalent norm that extends $|\cdot|_Y$. We show that $|||\cdot|||$ has modulus of convexity of power type $4(1+2\alpha)r$. To this end, let $\epsilon > 0$ be given. Let $x, y \in S_X(|||\cdot|||)$ such that

$$1 - |||\frac{x+y}{2}||| < \frac{\epsilon}{2r}.$$

Therefore we have

$$\frac{1}{2}p^r(x) + \frac{1}{2}p^r(y) - p^r(\frac{x+y}{2}) < \epsilon.$$

Consequently,

$$\frac{1}{2}|x|^r + \frac{1}{2}|y|^r - |\frac{x+y}{2}|^r < \epsilon, \quad (2.1)$$

$$\frac{1}{2}q(x) + \frac{1}{2}q(y) - q(\frac{x+y}{2}) < \epsilon, \text{ and} \quad (2.2)$$

$$\frac{1}{2}\text{dist}(x, Y)^r + \frac{1}{2}\text{dist}(y, Y)^r - \text{dist}(\frac{x+y}{2}, Y)^r < \epsilon. \quad (2.3)$$

Inequality (2.3) and a variation of Lemma 5.1.2 yield

$$|\text{dist}(x, Y) - \text{dist}(y, Y)| < C\sqrt{\epsilon} \text{ and } |\text{dist}(x, Y) - \text{dist}(\frac{x+y}{2})| < C\sqrt{\epsilon} \quad (2.4)$$

for some constant $C := C(r)$. We consider two cases:

Case I: If $\text{dist}(x, Y) < \epsilon^{\frac{1}{4}}$. Then $\text{dist}(y, Y) < C'\epsilon^{\frac{1}{4}}$. For some constant $C' > 1$. Let $x', y' \in Y$ such that $|x - x'| < 3C'\epsilon^{\frac{1}{4}}$ and $|y - y'| < 3C'\epsilon^{\frac{1}{4}}$. Therefore we have

$$\frac{1}{2}|x|^r + \frac{1}{2}|y|^r - |\frac{x+y}{2}|^r$$
$$< \frac{1}{2}|x'|^r + \frac{1}{2}|y'|^r - |\frac{x'+y'}{2}|^r + C''\epsilon^{\frac{1}{4}},$$

for some constant C''. By the modulus of convexity of $|\cdot|$ on Y, we have

$$L|x-y|^r \leq \frac{1}{2}|x|^r + \frac{1}{2}|y|^r - |\frac{x+y}{2}|^r \leq \epsilon + C''\epsilon^{\frac{1}{4}},$$

for some constant L. Consequently we have

$$|x-y|^r \leq K_1\epsilon^{\frac{1}{4}}, \text{ for some constant } K_1. \tag{2.5}$$

Case II: If $\operatorname{dist}(x,Y) > \epsilon^{\frac{1}{4}}$, by (2.2), we have

$$\begin{aligned}
&\frac{1}{2}\operatorname{dist}(x,Y)^{1+\alpha}e^{\|x\|^r} + \frac{1}{2}\operatorname{dist}(x,Y)^{1+\alpha}e^{\|y\|^r} - \operatorname{dist}(x,Y)^{1+\alpha}e^{\|\frac{x+y}{2}\|^r}\\
= &\frac{1}{2}\operatorname{dist}(x,Y)^{1+\alpha}e^{\|x\|^r} + \frac{1}{2}|\operatorname{dist}(y,Y)^{1+\alpha}e^{\|y\|^r} - \operatorname{dist}(\frac{x+y}{2},Y)^{1+\alpha}e^{\|\frac{x+y}{2}\|^r}\\
&+\frac{1}{2}(\operatorname{dist}(x,Y)^{1+\alpha} - \operatorname{dist}(y,Y)^{1+\alpha})e^{\|y\|^r} - (\operatorname{dist}(x,Y)^{1+\alpha}\\
&-\operatorname{dist}(\frac{x+y}{2},Y)^{1+\alpha})e^{\|\frac{x+y}{2}\|^r} \leq \epsilon + D\sqrt{\epsilon} \leq E\sqrt{\epsilon},
\end{aligned}$$

for some constants D and E. Consequently we have,

$$\frac{1}{2}e^{\|x\|^r} + \frac{1}{2}e^{\|y\|^r} - e^{\|\frac{x+y}{2}\|^r} \leq E\epsilon^{\frac{1-\alpha}{4}},$$

which is equivalent to

$$\sum_{k=1}^{\infty}\frac{1}{k!}(\frac{1}{2}\|x\|^{kr} + \frac{1}{2}\|y\|^{kr} - \|\frac{x+y}{2}\|^{kr}) \leq E\epsilon^{\frac{1-\alpha}{4}}.$$

By convexity, each term of the series is non-negative, hence we have,

$$\frac{1}{2}\|x\|^r + \frac{1}{2}\|y\|^r - \|\frac{x+y}{2}\|^r \leq E\epsilon^{\frac{1-\alpha}{4}}. \tag{2.6}$$

Therefore as before, by the modulus of convexity of $\|\cdot\|$,

$$\|x-y\|^r < K_2\epsilon^{\frac{1-\alpha}{4}}, \text{ for some constant } K_2 \tag{2.7}$$

Finally, inequalities (2.5) and (2.7) imply that $1 - |||\frac{x+y}{2}||| > K_3\epsilon^{4r(1+2\alpha)}$, for some constant K_3 whenever we have $|||x-y||| \geq \epsilon$. Now if we are given

$r' > r$, we can find an $\alpha > 0$ such that $r(1+2\alpha) > r'$. Then the norm $|||\cdot|||$ we constructed above with the corresponding α is our required norm. $\square$.

Remarks

1. We are convinced that the extension in Theorem 5.1.1 preserves all types of convexity.

2. We do not know if there is a similar extension that admits modulus of convexity of power type r' for any $r' > r$.

3. As noted in [F], we do not know if we could extend smooth norms from a subspace to the whole space (see also [Z]).

Section Two : Prescribed Extreme Points

In this section, we consider the following problem:

Problem Let X be a Banach space that admits a strictly convex norm and G be a subset of the unit sphere of an equivalent norm (not necessary strictly convex), is there an equivalent strictly convex norm $\|\cdot\|$ such that $G \subset S_X(\|\cdot\|)$?

Clearly the answer to the problem would be negative if G contains collinear points. However, with some restrictions to the set G, we are able to construct such a norm.

Theorem 5.2.1. *Let X be a Banach space that admits a strictly convex norm $\|\cdot\|$ and G be a subset of the unit sphere of some equivalent norm $|\cdot|$. Suppose there exists an $\epsilon > 0$ such that*

$$\text{dist}(x, G_x) > \epsilon, \textit{ for all } x \in G,$$

where $G_x = \overline{conv}^{|\cdot|}(G \setminus \{x\})$, then X admits an equivalent strictly convex norm $|||\cdot|||$ such that $G \subset S_X(|||\cdot|||)$.

Proof. Without loss of generality, we may assume that $\|\cdot\| < |\cdot|$ and $\epsilon < 1$. For each $x \in G$, there exists $f_x \in MB_{X^*}$ such that

$$f_x(x) > 1 > 1 - \frac{\epsilon}{2} > \sup f_x(G_x),$$

for some $M \geq \frac{2}{\epsilon}$. Define a set of convex functions $\{r_x; x \in G\}$ on X by

$$r_x(\cdot) = f_x^2(\cdot) + (\frac{\epsilon}{3})^2\|\cdot\|^2.$$

We note that $M + 1 \geq r_x(x) > 1$ and $r_x(y) < 1$, whenever $y \in G \setminus \{x\}$. Let $c_x = \frac{1}{r_x(x)}$. Note that $c_x < 1$, for all $x \in G$. Define a set of functions $\{p_x; x \in G\}$ on X by $p_x(\cdot) = c_x r_x(\cdot)$. Finally, let

$$p(\cdot) = \sup_{x \in G} p_x(\cdot).$$

Clearly, the function p is convex. We note that $p_x(x) = 1$ for all $x \in G$ and $p_y(x) < 1$, whenever $y \in G \setminus \{x\}$. Therefore $p(x) = 1$, for all $x \in G$. We claim that p is strictly convex on Y. Indeed, let $u, v \in Y$ such that

$$\frac{1}{2}p(u) + \frac{1}{2}p(v) - p(\frac{u+v}{2}) = 0.$$

Let $\eta > 0$ be given, then there exists $x \in G$ such that $p_x(\frac{u+v}{2}) + \eta \geq p(\frac{u+v}{2})$. Thus,

$$\begin{aligned}
&\frac{1}{2}p(u) + \frac{1}{2}p(y) - p(\frac{u+v}{2}) \\
&\geq \frac{1}{2}p_x(u) + \frac{1}{2}p_x(v) - p_x(\frac{u+v}{2}) - \eta \\
&\geq \frac{\epsilon^2 c_x}{18}\|u\|^2 + \frac{\epsilon^2 c_x}{18}\|v\|^2 - \frac{\epsilon^2 c_x}{9}\|\frac{u+v}{2}\|^2 - \eta
\end{aligned}$$

Consequently,

$$\frac{1}{2}\|u\|^2 + \frac{1}{2}\|v\|^2 - \|\frac{u+v}{2}\|^2 \leq \frac{9(M+1)\eta}{\epsilon^2},$$

for any $\eta > 0$. Therefore

$$\frac{1}{2}\|u\|^2 + \frac{1}{2}\|v\|^2 - \|\frac{u+v}{2}\|^2 = 0,$$

and thus $u = v$, as $\|\cdot\|$ is strictly convex. Define a new norm $|||\cdot|||$ by $|||\cdot|||^2 = \frac{|\cdot|^2 + p(\cdot)}{2}$. The norm $|||\cdot|||$ is strictly convex on X and $|||x||| = 1$ for all $x \in G$. □

Theorem 5.2.2. *Let X be a Banach space that admits a UR norm $\|\cdot\|$ with modulus of convexity of power type 2 and G be as in Theorem 5.2.1, then X admits an equivalent UR norm $|||\cdot|||$ with modulus of convexity of power type 2 such that $|||x||| = 1$ for each $x \in G$.*

Proof. As in the proof of Theorem 5.2.1, we let $p(\cdot) = \sup_{x \in G} c_x(f_x^2(\cdot) + (\frac{\epsilon}{3})^2\|\cdot\|^2)$, where $\|\cdot\|$ is the uniformly rotund norm. It suffices to show that the norm $\sqrt{p(\cdot)}$ has modulus of convexity of power type 2. Given $u, v \in X$, let $x \in G$ such that

$$p(\frac{u+v}{2}) \leq c_x(f_x^2(\frac{u+v}{2}) + (\frac{\epsilon}{3})^2\|\frac{u+v}{2}\|^2) + \frac{1}{2}p(u) + \frac{1}{2}p(v) - p(\frac{u+v}{2}).$$

Therefore

$$\begin{aligned} &\frac{1}{2}p(u) + \frac{1}{2}p(v) - p(\frac{u+v}{2}) \\ \geq\ &\frac{\epsilon^2 c_x}{18}(\frac{1}{2}\|u\|^2 + \frac{1}{2}\|v\|^2 - \|\frac{u+v}{2}\|^2) \\ \geq\ &L\|u-v\|^2, \end{aligned}$$

for some constant L. Consequently, we have

$$\frac{1}{2}p(u) + \frac{1}{2}p(v) - p(\frac{u+v}{2}) \geq Kp(u-v),$$

for some constant $K > 0$. □

Suppose a stronger condition is assumed on the set G, we could obtain a required norm in a simpler manner.

Theorem 5.2.3. *Let X be a Banach space that admits a strictly convex (rotund) norm and $\{x_n; n \in \mathbb{N}\}$ be a subset of the unit sphere of some equivalent norm $|\cdot|$. Suppose there exists an $\epsilon > 0$ such that*

$$\operatorname{dist}(x_n, Y_n) > \epsilon, \text{ for all } n \in \mathbb{N},$$

where $Y_n = \overline{span}^{|\cdot|}\{x_m; m \in \mathbb{N} \setminus \{n\}\}$, then X admits an equivalent rotund norm $|||\cdot|||$ such that $|||x_n||| = 1$ for each $n \in \mathbb{N}$.

Proof. Let $n \in \mathbb{N}$, there exists $f_n \in \frac{1}{\epsilon}B_{X^*}$ such that

$$f_n(x_n) = 1 \text{ and } f_n(Y_n) = \{0\}.$$

Let p be a convex function defined by

$$p(x) = \sup_{\pi} \sum_{n=1}^{\infty} \frac{f^2_{\pi(n)}(x)}{2^n},$$

where the supremum is taken over the set of all possible permutations π of $\mathbb{N}$. Clearly, $p(x_n) = 1$ for all $n \in \mathbb{N}$. It can be verified easily that the function p is strictly convex on $Y = \overline{span}^{|\cdot|}\{x_n; n \in \mathbb{N}\}$. Define a new norm $|||\cdot|||$ by $|||\cdot|||^2 = \frac{1}{2}(p(\cdot) + |\cdot|^2)$. The norm $|||\cdot|||$ is strictly convex on Y and $|||x_n||| = 1$ for all $n \in \mathbb{N}$. By Theorem 5.1.1, there exists an equivalent strictly convex norm on X that coincides with $|||\cdot|||$ on Y. □

Corollary 5.2.4. *Let $\{x_n, f_n\} \subset S_X \times X^*$ be a bounded biorthogonal system. Then there exists an equivalent strictly convex norm $|||\cdot|||$ such that $|||x_n||| = 1$ for each $n \in \mathbb{N}$.*

Corollary 5.2.5. *Let X be a Banach space that admits a strictly convex norm and $\{x_n\} \subset X$ be a normalized monotone basic sequence with respect to an equivalent norm $|\cdot|$. Then there exists an equivalent strictly convex norm $|||\cdot|||$ such that $|||x_n||| = 1$ for all $n \in \mathbb{N}$ and $\{x_n\}$ is monotone with respect to $|||\cdot|||$.*

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