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EQUICONVERGENCE OF SOME INTERPOLATORY AND BEST APPROXIMATING PROCESSES

by

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The undersigned certify that they have read and recommend to the Faculty of Graduate Studies and Research for acceptance, a thesis entitled EQUICONVERGENCE OF SOME INTERPOLATORY AND BEST APPROXIMATING PROCESSES submitted by MUHAMMAD ASHFAQ BOKHARI in partial fulfilment of the requirements for the degree of Doctor of Philosophy.

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In the name of God, Most Gracious Most Merciful

Dedicated to
a street of Medina & a person from India,
I came across in my dream.

ABSTRACT

Let $L_{n-1}(z,f) \in \pi_{n-1}$ denote the Lagrange interpolant to f on the n^{th} roots of unity and let $p_{n-1}(z,f) \in \pi_{n-1}$ denote the Taylor polynomial of f where $f \in A_\rho$ (the class of functions analytic in $|z| < \rho$, but not in $|z| \leq \rho$). Although both $L_{n-1}(z,f)$ and $p_{n-1}(z,f)$ converge to f uniformly on compact subsets of $|z| < \rho$ as $n \to \infty$, Walsh showed [30] that their difference tends to zero uniformly on compact subsets of $|z| < \rho^2$ when $\rho > 1$. This dissertation begins with a recent extension of this theorem of Walsh by Saff and Sharma [18] using rational interpolants and is divided into six chapters.

Chapter I is historical and gives a brief survey of the main results related to Walsh equiconvergence. In Chapter II we prove an analogue of the Saff-Sharma theorem [18] by replacing the Lagrange interpolant with the Hermite interpolant.

In Chapter III we prove the equiconvergence of two sequences of rational functions $R^*_{n+m,n}(z,f)$ and $r_{n+m,n}(z,f)$ of the form $p_{n+m}(z)/(z^n-\sigma^n)$ where the numerator is a polynomial of degree $n+m$. The first one $R^*_{n+m,n}(z,f)$ minimizes $\sum_{k=0}^{qn-1} |f(\omega^k) - R(\omega^k,f)|^2$ over all rationals $R(z)$ of the above form where $\omega^{qn} = 1$, and the second $r_{n+m,n}(z,f)$ is the one considered by Saff and Sharma in [18].

The theme for Chapter IV is suggested by a paper of Szabados and Varga [23]. Here we examine the sensitivity of Walsh equiconvergence in the result of Saff-Sharma to a slight perturbation of the poles and nodes of interpolation.

Chapters V and VI are motivated by the results of V. Totik [25] and Szabados [22]. In Chapter V we obtain quantitative estimates for the difference $\Delta^{\sigma}_{\ell,n,m}(z,f)$ whose region of convergence was determined by Saff and Sharma [18]. This enables us to give an analogue of the so-called "pizza" theorem of Saff and Varga [19].

In Chapter VI we show that if f is analytic in $|z| < 1$ and continuous in $|z| \leq 1$, and if for some $\sigma > \rho^{\ell+1}$, $\rho > 1$, $\{\Delta^{\sigma}_{\ell,n,m}(z,f)\}_1^{\infty}$ is uniformly bounded on compact subsets of $|z| < \rho^{\ell+1}$, then f is analytic in $|z| < \rho$.

Finally in the Appendix we briefly outline a problem dealing with the asymptotic behaviour of zeros of Lagrange interpolants of a class of functions on the roots of unity suggested by the recent work of Edrei, Saff and Varga [8] on zeros of sections of power series.

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CHAPTER I

A HISTORICAL SURVEY

§1. INTRODUCTION. The present dissertation owes its motivation to a simple and elegant theorem of J.L. Walsh and some of its recent extensions. This theorem is of "such simplicity, directness and beauty that it is surprising that it has stood almost alone unnoticed for nearly half a century". These investigations of Walsh in turn can be traced for their roots, to the works of Meráy, Runge, Fejér, Kalmár and others. For all these references, we refer to the classic book of Walsh [28].

Let us denote by A_ρ, $\rho > 1$, the class of functions single-valued and analytic in $|z| < \rho$, but not single-valued and analytic on $|z| = \rho$. For $f(z) = \sum_{k=0}^{\infty} a_k z^k \in A_\rho$, let $p_{n-1}(z,f)$ denote its Taylor section of order $n-1$ and let $L_{n-1}(z,f)$ denote the Lagrange interpolant of f on the n^{th} roots of unity. It is known that both $L_{n-1}(z,f)$ and $p_{n-1}(z,f)$ converge to f uniformly on compact subsets of $|z| < \rho$ as $n \to \infty$. Walsh observed that the difference $\Delta_{1,n-1}(z,f) := L_{n-1}(z,f) - p_{n-1}(z,f)$, however, tends to zero for $|z| < \rho^2$. He showed ([28] p. 153) that the convergence is geometric and uniform on compact subsets of the region $|z| < \rho^2$. Moreover, the result is sharp in the sense that if $\hat{z} = \rho^2$, there is a function $\hat{f}(z) = (z-\rho)^{-1}$ for which $\{\Delta_{1,n-1}(\hat{z},\hat{f})\}_1^{\infty}$ does not tend to zero.

(Also see [30] p. 239, and [31] p. 718.)

In 1981, Cavaretta et al [5] extended Walsh's theorem in order to obtain a larger region of equiconvergence. They showed that if

$$(1.1)\qquad p_{n-1,j}(z,f) := \sum_{k=0}^{n-1} a_{k+jn} z^k \;, \quad (j = 0,1,2,\ldots) \;,$$

and if for any integer $\ell \geq 1$,

$$(1.2)\qquad \Delta_{\ell,n-1}(z,f) := L_{n-1}(z,f) - \sum_{j=0}^{\ell-1} p_{n-1,j}(z,f) \;, \quad f \in A_\rho \;,$$

then,

$$(1.3)\qquad \lim_{n\to\infty} \Delta_{\ell,n-1}(z,f) = 0 \;, \quad |z| < \rho^{\ell+1} \;,$$

and that this result is sharp in the same sense as that of Walsh.

An equivalent expression for $\Delta_{\ell,n-1}(z,f)$, viz.,

$$(1.4)\qquad \Delta_{\ell,n-1}(z,f) = \sum_{j=\ell}^{\infty} \sum_{k=0}^{n-1} a_{k+jn} z^k$$

has been used recently by J. Szabados [22] to give a very simple proof of (1.3). It is based on the observation that $f \in A_\rho$ implies that $\overline{\lim_{n\to\infty}} |a_n|^{1/n} = \rho^{-1}$.

§2. HERMITE INTERPOLATION. The first extension of (1.3) was to replace Lagrange by Hermite interpolation which in the notation of [5] can be called $(0,1,\ldots,r-1)$ interpolation. Let us denote by $h_{rn-1}(z,f)$ the Hermite interpolant of degree $rn-1$ to $f^{(\nu)}$,

$\nu = 0,1,2,\dots r-1$, on the n^{th} roots of unity. If $\beta_{j,r}(z)$ are polynomials given by

$$(2.1)\qquad \beta_{j,r}(z) = \sum_{k=0}^{r-1} \binom{r+j-1}{k}(z-1)^k, \quad (j = 1,2,\dots),$$

and if

$$(2.2)\qquad \begin{cases} H_{rn-1,0}(z,f) = \sum\limits_{k=0}^{rn-1} a_k z^k \\ H_{rn-1,j}(z,f) = \beta_{j,r}(z^n) \sum\limits_{k=0}^{n-1} a_{k+n(r+j-1)} z^k, \quad (j = 1,2,3,\dots) \end{cases}$$

then it was shown by Cavaretta et al [5] (or [9] p. 79) that

$$(2.3)\qquad \lim_{n\to\infty} \Delta_{\ell,rn-1}(z,f) = 0 \quad , \quad \text{for } |z| < \rho^{1+\frac{\ell}{r}} ,$$

where

$$(2.4)\qquad \Delta_{\ell,rn-1}(z,f) := h_{rn-1}(z,f) - \sum_{j=0}^{\ell-1} H_{rn-1,j}(z,f) .$$

Again, the convergence is uniform and geometric for all $|z| \le Z < \rho^{1+\frac{\ell}{r}}$, and the result (2.3) is best possible.

§3. $(0,m_1,\dots,m_q)$ - INTERPOLATION. It is known that if $m_1 < m_2 < \dots < m_q$ are positive integers, and if $m_k \le kn$, $k = 1,2,\dots,q$, then the problem of $(0,m_1,\dots,m_q)$-interpolation is regular on the n^{th} roots of unity ([15], §11.2). Set $\underline{m} := (0,m_1,\dots,m_q)$. Let $b_{(q+1)n-1,\underline{m}}(z,f) \in \pi_{(q+1)n-1}$ denote the polynomial of $(0,m_1,\dots,m_q)$ interpolation to f on the n^{th} roots of unity. Set

$$B_{(q+1)n-1,\nu}(z,f) := \sum_{j=0}^{n-1} a_{j+(\nu+q)n}\, b_{(q+1)n-1,\underline{m}}(z,g) \;,$$

$(\nu = 1,2,3, \ldots)$, where $g(t) = t^{j+(\nu+q)n}$. It was conjectured in [5] and recently proved in [20] that

$$\lim_{n\to\infty} \{b_{(q+1)n-1,\underline{m}}(z,f) - \sum_{\nu=0}^{\ell-1} B_{(q+1)n-1,\nu}(z,f)\} = 0 \;,$$

for $|z| < \rho^{1+\frac{\ell}{q+1}}$, where $B_{(q+1)n-1,0}(z,f) = \sum_{\nu=0}^{(q+1)n-1} a_\nu z^\nu$.

Earlier, this was proved in [5] for the special cases of $(0,2)$, $(0,3)$ and $(0,1,\ldots,r-2,r)$ interpolation.

Remark. The above results in §1,2,3 led B.M. Baishanski [1] to ask why this phenomenon of equiconvergence was so closely related to interpolation at the origin and at the roots of unity. He considered two sets of nodes whose elementary symmetric functions are "close" in a certain sense and proved a corresponding equiconvergence result. His main theorem contains as a special case a weak form of Walsh's theorem. For fuller details see [1].

§4. ℓ_2- APPROXIMATION. Let $m = qn + c \in \mathbf{N}$, q,c being fixed positive integers. Let $f \in A_\rho$, $\rho > 1$ and let $\mathcal{P}_{n-1,m}(z,f)$ be the polynomial of degree $n-1$ which attains the minimum of the error $\sum_{k=0}^{m-1} |f(\omega^k) - p(\omega^k)|^2$ over all $p \in \pi_{n-1}$ where ω is an m^{th} primitive root of unity. T.J. Rivlin [17] showed that if $f \in A_\rho$, $\rho > 1$ then

$$(4.1) \qquad \lim_{n\to\infty} \{\mathcal{P}_{n-1,m}(z,f) - p_{n-1}(z,f)\} = 0 \;, \text{ for } |z| < \rho^{1+q} \;.$$

If $f(z) = \sum_{k=0}^{\infty} a_k z^k$, an easy generalization of (4.1) is that for any positive integer $\ell \geq 1$, we have

$$(4.2) \qquad \lim_{n\to\infty} \sum_{j=\ell}^{\infty} \sum_{k=0}^{n-1} a_{k+jm} z^k = 0 , \quad |z| < \rho^{1+\ell q} , \quad f \in A_\rho .$$

This result is best possible and gives (4.1) when $\ell = 1$.

Rivlin also extended the theorem of Walsh in another direction. Let E_ρ be an ellipse with foci -1, 1 which is obtained from the circle $|z| = \rho$ by the mapping $\omega = \frac{1}{2}(z + z^{-1})$. Let $A(E_\rho)$ denote the class of functions analytic in the interior of the ellipse E_ρ but not on the closure $\overline{E}_\rho$. If $f \in A(E_\rho)$, let $f(z) = \frac{1}{2} A_0 + \sum_{k=1}^{\infty} A_k T_k(z)$ be the Fourier-Chebyshev expansion of f on $[-1,1]$. Let $\{\xi_j^{(m)}\}_1^m$, $m = nq + c$, be the zeros of the Chebyshev polynomial $T_m(x)$ on $[-1,1]$. If $u_{n,m}(z,f)$ is the least-square approximation of degree n to $f(z)$ on $\{\xi_j^{(m)}\}_{j=1}^m$ then

$$(4.3) \qquad \lim_{n\to\infty} \{u_{n,m}(z,f) - s_n(z,f)\} = 0 ,$$

for z inside $E_{\rho^{2q-1}}$, where $s_n(z,f) := \frac{1}{2} A_0 + \sum_{k=1}^{n} A_k T_k(z)$ with

$$A_k := \frac{2}{\pi} \int_{-1}^{1} f(x)\, T_k(x) \frac{dx}{\sqrt{1-x^2}} , \quad (k = 0,1,\dots) .$$

The convergence is uniform and geometric inside and on E_τ for any $\tau < \rho^{2q-1}$.

§5. MIXED HERMITE AND ℓ_2- APPROXIMATION. Cavaretta et al [3] extended Rivlin's result (4.1) by combining Hermite interpolation with ℓ_2-approximation. Let $m = nq + c$ and let ω be a primitive m^{th} root of unity. Let $Q_{rm+n}(z,f)$ be a polynomial of degree $rm+n$ so that

(i) $Q^{(\nu)}_{rm+n}(\omega^k, f) = f^{(\nu)}(\omega^k)$, $(k = 0,1,\ldots,m-1);(\nu = 0,1,\ldots,r-1)$,

(ii) $Q^{(r)}_{rm+n}(z,f)$ minimizes $\sum_{k=0}^{m-1} |P^{(r)}_{rm+n}(\omega^k, f) - f^{(r)}(\omega^k)|^2$ over all polynomials of degree $\leq rm+n$ that satisfy (i) for $r \geq 1 \in N$.

By defining a sequence of polynomials $S_{rm+n,\ell}(z,f) \in \pi_{rm+n}$ related to the Taylor sections of f , it was shown that

$$(5.1) \qquad \lim_{n\to\infty}\{Q_{rm+n}(z,f) - S_{rm+n,\ell}(z,f)\} = 0 , \quad |z| < \rho^{1+\frac{\ell q}{1+rq}} ,$$

the convergence being uniform and geometric on any closed subset of $|z| < \rho^{1+\frac{\ell q}{1+rq}}$. (For explicit formulae for $S_{rm+n,\ell}(z,f)$ see [3].)

It may be noted that (5.1) reduces to (4.2) in case $r = 0$. For other variations of Rivlin's result see [2], [7] and [21].

§6 AVERAGE OF INTERPOLATING POLYNOMIALS. Let m and n be positive integers $\omega = \exp(2\pi i/mn)$. Set $f_k(z) = f(z\omega^k)$, $(k = 0,1,\ldots,m-1)$, and define the averages

$$A_{n-1}(z,f) := \frac{1}{m} \sum_{k=0}^{m-1} L_{n-1}(z\omega^{-k}, f_k) ,$$

and

$$A_{n-1,j}(z,f) := \frac{1}{m} \sum_{k=0}^{m-1} p_{n-1,j}(z\omega^{-k}, f_k), \quad j = 0,1,2,\ldots,$$

where $p_{n-1,j}(z,f)$ is given by (1.1). If β is the least positive integer such that $\beta m > \ell$ then Price [16] proved that

$$(6.1) \quad \lim_{n\to\infty}\{A_{n-1}(z,f) - \sum_{j=0}^{\ell} A_{n-1,j}(z,f)\} = 0, \qquad |z| < \rho^{1+\beta m}.$$

If $m = 1$ and $\ell \geq 0$, (6.1) yields (1.3), an extension of Walsh's theorem.

§7. SHARPNESS OF WALSH'S THEOREM.

Saff and Varga [19] were the first to raise the question whether the difference $\Delta_{\ell,n-1}(z,f)$ can be bounded at some points outside the circle $|z| = \rho^{1+\ell}$. They showed that this cannot happen at more than ℓ distinct points in $|z| > \rho^{\ell+1}$. Conversely, given any ℓ distinct points $\{z_j\}_{j=1}^{\ell}$ with $\rho^{\ell+1} < |z_j| < \rho^{\ell+2}$, $(j = 1,2,\ldots,\ell)$, they showed that there exists a function $\hat{f} \in A_\rho$ such that $\Delta_{\ell,n-1}(z_j,\hat{f})$ tends to zero as $n \to \infty$, $(j = 1,\ldots,\ell)$.

Later T. Hermann [10] removed the restriction $\rho^{\ell+1} < |z_j| < \rho^{\ell+2}$ and constructed a function $\hat{f} \in A_\rho$ corresponding to a given set $\{z_k\}_{k=1}^{s}$ of distinct points with $\rho^{\ell+1} < |z_j| < \rho^{L+1}$, $(j = 1,2,\ldots,s)$, $s \leq \ell < L$ for which $\lim_{n\to\infty} \Delta_{\ell,n-1}(z_k,\hat{f}) = 0$, $k = 1,\ldots,s$.

Saff and Varga [19] also proved a corresponding result for Hermite interpolation.

§8. CONVERSE RESULTS. Recently, J. Szabados [22] gave a sort of converse result. Using the formula (1.4) and a simple interesting identity he proved:

If f is analytic in $|z| < 1$ and continuous in $|z| \leq 1$, and if the sequence $\{\Delta_{\ell,n-1}(z,f)\}_{n=1}^{\infty}$ is uniformly bounded on every closed subset of $|z| < \rho^{\ell+1}$, then f is analytic in $|z| < \rho$.

Cavaretta et al [4] proved the following somewhat similar result for the sequence $\{\Delta_{\ell,rn-1}(z,f)\}$ in (2.4):

Let f be analytic in $|z| < 1$ and let $f, f', \ldots, f^{(r-1)}$ be all continuous on $|z| = 1$. If the sequence $\{\Delta_{\ell,rn-1}(z,f)\}_{n=1}^{\infty}$ is uniformly bounded on every closed subset of $|z| < \rho^{1+(\ell/r)}$, then f is analytic in $|z| < \rho$.

§9. CONTRIBUTIONS OF V. TOTIK. The formula (1.4) suprisingly helps to break new ground in the theory of equiconvergence. Some elegant results by V. Totik [25], which also supplement the sharpness result of Saff and Varga (cf. §7), are based on this formula. If we set

$$(9.1)\quad \begin{cases} K_\ell(z,\rho) := \begin{cases} |z|\rho^{-\ell-1}, & |z| \geq \rho \\ \rho^{-\ell}, & 0 \leq |z| < \rho \end{cases} \\ B_\ell(z,f) := \overline{\lim_{n\to\infty}} |\Delta_{\ell,n-1}(z,f)|^{1/n} \\ \delta_{\ell,\rho}(f) := \{z: B_\ell(z,f) < K_\ell(z,\rho)\}, \end{cases}$$

then Totik [25] proved that

$$(9.2)\qquad \overline{\lim_{n\to\infty}}\left(\max_{|z|=R} |\Delta_{\ell,n-1}(z,f)|\right)^{1/n} = K_\ell(R,\rho);$$

$$(9.3)\qquad \begin{cases} |\delta_{\ell,\rho}(f) \cap \{z: |z| > \rho\}| \le \ell \ , \\ \\ |\delta_{\ell,\rho}(f) \cap \{z: 0 < |z| < \rho\}| \le \ell-1 \ , \end{cases}$$

where $|F|$ denotes the cardinality of the set F. He also proved the following: For any ℓ distinct points $\{z_j\}_1^\ell$ with moduli $> \rho$ (or $\ell-1$ points $\{z_j\}_1^{\ell-1}$ with $0 < |z_j| \lesssim \rho$), there exists an $f_1 \in A_\rho$ (or $f_2 \in A_\rho$) such that $z_j \in \delta_{\ell,\rho}(f_1)$, $j = 1,2,\ldots,\ell$ (or $z_j \in \delta_{\ell,\rho}(f_2)$, $j = 1,\ldots,\ell-1$).

Totik [26] also solved three problems raised by Szabados and Varga [24] concerning the exact domain of equiconvergence for certain sequences of polynomial interpolants (see §11).

§10. FOLLOW-UP OF TOTIK'S RESULTS. From the results of Totik it is not clear whether the ℓ points in $\delta_{\ell,\rho}(f) \cap \{z: |z| > \rho\}$ and $\ell-1$ points in $\delta_{\ell,\rho}(f) \cap \{z: 0 < |z| < \rho\}$ can exist at the same time. Ivanov and Sharma, in a forthcoming paper [11], introduced the notion of <u>(ℓ,ρ)-distinguished set</u>. A set Z is (ℓ,ρ)-distinguished if there exists an $f \in A_\rho$ so that $B_\ell(z,f) < K_\ell(z,\rho)$, $z \in Z$. If $|z_k| < \rho$, $k = 1,\ldots,\mu$ and if $|z_k| > \rho$, $k = \mu+1,\ldots,m$, let X and Y be the matrices given by

$$X = (z_i^j)_{i=1,\ j=0}^{\mu\quad \ell-1} \quad , \quad Y = (z_i^j)_{i=\mu+1,\ j=0}^{m\quad \ell} .$$

Set $M(X,Y) := \begin{pmatrix} X & & & \\ & X & & () \\ () & & \ddots & \\ & & & X \\ Y & & & \\ & Y & & () \\ () & & \ddots & \\ & & & Y \end{pmatrix}$ where X and Y are repeated $\ell+1$ and ℓ times respectively. Two principal results of Ivanov and Sharma [11] are:

I. The set $Z = \{z_k\}_{k=1}^m$ is (ℓ,ρ)-distinguished if and only if Rank $M < \ell(\ell+1)$.

II. Any set of $\ell+1$ distinct points on the circle $|z| = \rho$ is (ℓ,ρ)-distinguished.

They also give analogues of the above results in [13] for Hermite interpolation and for ℓ_2-approximation.

§11 SAFF-VARGA CONJECTURE AND SUBSEQUENT WORK. Let $Z = [z_{k,n}]_{k=1,n=1}^{n\quad \infty}$ be an infinite triangular matrix of nodes whose entries satisfy

$$(11.1) \qquad 1 \leq |z_{k,n}| < \rho \quad , \quad k = 1,2,\dots,n; \quad n = 1,2,\dots$$

For any $f \in A_\rho$, let $L^*_{n-1}(z,Z,f)$ denote the Lagrange interpolant to f in the n points $\{z_{k,n}\}_{k=1}^n$ of the n^{th} row of Z. In particular, if $z_{k,n} = \exp(2\pi ki/n)$, we denote the corresponding matrix by E. Now Walsh's theorem can be stated as

$$(11.2) \qquad \lim_{n\to\infty} \left(L^*_{n-1}(z,E,f) - p_{n-1}(z,f)\right) = 0 \ , \quad |z| < \rho^2 \ , \quad f \in A_\rho .$$

It was conjectured by Saff and Varga [27] that the quantity ρ^2

in (11.2) is maximal for any interpolating matrix Z satisfying (11.1). Szabados and Varga [23] proved that the Saff-Varga conjecture is valid in the more general setting of the formula (1.3) (with $\rho^{\ell+1}$ instead of ρ^2 in (11.2)), and under a weaker hypothesis than (11.1). In [24] they proposed to determine the precise domains in $\mathbb{C}$ for which the sequence (1.1) of [24] converges geometrically to zero for all $f \in A_\rho$. This problem was settled by V. Totik [26] who proved that the precise domain is circular with centre at the origin. For details of the other problems settled by Totik see [26].

§12. RATIONAL INTERPOLANTS AND WALSH'S THEOREM. It is well-known that the Taylor polynomial $p_{n-1}(z,f)$ is the best L_2-approximant to $f(z)$ on $|z| = 1$ among all polynomials in π_{n-1}.

The following theorem being a direct consequence of Walsh's result ([28] p. 224) generalizes this property of Taylor polynomials:

Theorem A. Let the function $f(z)$ be analytic on and within C: $|z| = 1$. Let $\{\sigma_k\}_1^n$ with $|\sigma_k| > 1$ be n given numbers. If $m \geq -1$ is a fixed integer, then the unique function of the form

$$r_{n+m}(z) = \frac{b_0 z^{n+m} + b_1 z^{n+m-1} + \dots + b_{n+m}}{(z-\sigma_1)(z-\sigma_2)\dots(z-\sigma_n)} \tag{12.1}$$

of best approximation to $f(z)$ on C in the sense of least squares is the unique function of the form (12.1) which interpolates $f(z)$ in the zeros of $z^{m+1}\prod_{k=1}^{n}(z-\bar{\sigma}_k^{-1})$.

Remark 12.1. Theorem A remains valid even if $C: |z| = 1$ is replaced by $C_\delta : |z| = \delta$, where $\max_k |\frac{1}{\sigma_k}| < \delta < 1$.

Saff and Sharma [18] used Theorem A in the special case when the σ_k's are the zeros of $z^n - \sigma^n$. For $\sigma > 1$, and a fixed integer $m \geq -1$, let

(12.2) $\quad R_{n+m,n}(z,f) := B_{n+m,n}(z,f)/(z^n-\sigma^n), \; (B_{n+m,n}(z,f) \in \pi_{n+m}),$

interpolate $f(z)$ in the $(n+m+1)^{th}$ roots of unity. Let

(12.3) $\quad r_{n+m,n}(z,f) := P_{n+m,n}(z,f)/(z^n-\sigma^n), \; (P_{n+m,n}(z,f) \in \pi_{n+m})$

be the best L_2-rational approximant to $f(z)$ of the form $P(z)/(z^n-\sigma^n)$, $(P(z) \in \pi_{n+m})$, on $|z| = 1$. From Theorem A, we note that $P_{n+m,n}(z,f)$ interpolates $(z^n-\sigma^n)f(z)$ in the $(n+m+1)$ zeros of $z^{m+1}(z^n-\sigma^{-n})$. Saff and Sharma showed that if $f \in A_\rho$, $\rho > 1$ and if $\sigma > 1$, then

$$(12.4) \quad \lim_{n\to\infty} \Delta^{\sigma}_{1,n,m}(z,f) = 0 \begin{cases} \text{for } |z| < \rho^2, & \text{if } \sigma \geq \rho^2, \\ \text{for } |z| \neq \sigma, & \text{if } \sigma < \rho^2, \end{cases}$$

where $\Delta^{\sigma}_{1,n,m}(z,f) := R_{n+m,n}(z,f) - r_{n+m,n}(z,f)$. The result (12.4) is best possible in the sense that the region of convergence cannot be improved.

Remark 12.2. Walsh's result ([28] p. 153) can be derived from (12.4) on letting $\sigma \to \infty$. Also (12.4) asserts that equiconvergence holds at all points of the plane excluding those on the circle $|z| = \sigma$, when $\sigma < \rho^2$. This is a new phenomenon which does not arise in Walsh's theorem. (See also [14].)

Next, set

(12.5) $$\alpha_{n,m}(z) = 1 - z^{m+1}\sigma^{-n}, \quad \beta_{n,m}(z) = z^{m+1}(z^{n}-\sigma^{-n}).$$

If $f(z)$ is an analytic function in $|z| \leq 1$, it is proved in [18] that

(12.6) $$(z^{n}-\sigma^{n})f(z) = \sum_{\nu=0}^{\infty} \left\{\frac{\beta_{n,m}(z)}{\alpha_{n,m}(z)}\right\}^{\nu} P_{n+m,n}(z,f,\nu),$$

where $P_{n+m,n}(z,f,\nu) \in \pi_{n+m}$, $(\nu = 0,1,2,\ldots)$, with $P_{n+m,n}(z,f,0) := P_{n+m}(z,f)$, (cf. (12.3)). An integral representation of $P_{n+m}(z,f,\nu)$, $(\nu \geq 1)$, is given by

(12.7) $$P_{n+m,n}(z,f,\nu) = \int_{|t|=1} \left(\frac{(\alpha_{n,m}(z))^{\nu-1}}{(\beta_{n,m}(z))^{\nu+1}}\right)(t^{n}-\sigma^{n})K_{n,m}(t,z)f(t)dt$$

with

(12.8) $$K_{n,m}(t,z) := \frac{\alpha_{n,m}(z)\beta_{n,m}(t) - \alpha_{n,m}(t)\beta_{n,m}(z)}{t-z}.$$

For a fixed positive integer ℓ, if we set

(12.9) $$\Delta^{\sigma}_{\ell,n,m}(z,f) := R_{n+m,n}(z,f) - \sum_{\nu=0}^{\ell-1} r_{n+m,n}(z,f,\nu)$$

where,

$$(12.10) \qquad r_{n+m,n}(z,f,\nu) := P_{n+m,n}(z,f,\nu)/(z^n-\sigma^n) \quad , \quad \nu \geq 0,$$

then ([18], Theorem 3.3)

$$(12.11) \qquad \lim_{n\to\infty} \Delta^{\sigma}_{\ell,n,m}(z,f) = 0 \begin{cases} \text{for } |z| < \rho^{1+\ell} , & \text{if } \sigma \geq \rho^{1+\ell} , \\ \text{for } |z| \neq \sigma , & \text{if } \sigma < \rho^{1+\ell} . \end{cases}$$

It may be noted that (12.11) is an obvious extension of (12.4).

Remark 12.3. In case $m < -1$, Saff and Sharma showed that the L_2-minimization polynomial $P_{n+m,n}(z,f)$ (cf. (12.3)) is obtained by truncating the polynomial $P_{n-1,n}(z,f)$ up to degree $n+m$. Notice that $P_{n-1,n}(z,f)$ interpolates $(z^n-\sigma^n)f(z)$ in the zeros of $z^n - \sigma^{-n}$. The results given by (12.4) and (12.11) still remain valid for $m < -1$. The corresponding help polynomials (cf. (12.7)) in this case have a suitable integral representation ([18], formulae (4.19)).

CHAPTER II

EQUICONVERGENCE OF CERTAIN RATIONAL INTERPOLANTS

(HERMITE CASE)

§1. INTRODUCTION. In his classic book on interpolation and approximation, Walsh [28] has shown that approximation in the sense of least squares by polynomials is intimately connected with Taylor series and he suggested that "approximation in the sense of least squares by more general rational functions may also be connected with interpolation in points related to the poles of the rational functions". A number of his theorems ([28], Chapter IX) justify this assertion. Recently, Saff and Sharma [18] took the cue and proved a theorem which further supplements the above statements of Walsh. A detailed description of their result is given in [illegible], §12).

This chapter deals with a generalization of the Saff-Sharma result using Hermite interpolation. It turns out that the involvement of derivatives in the interpolatory processes gives a smaller region of equiconvergence (cf. [18], Theorem 3.3). §2 contains the statement of the results. The "help functions" used in the extension of our first result are described in §3. Finally in §4 we obtain some estimates and prove the results stated in §2.

§ 2. STATEMENT OF RESULTS. Throughout the following discussion we shall regard m, r and s as fixed integers with $m \geq -1$ and $1 \leq r \leq s$. For the sake of simplicity, we set

(2.1) $$N := n+m+1 \quad , \quad N(s) := s(n+m+1)-1 \; .$$

Let

(2.2) $$Q_{N(s),n}(z,f,r) := B_{N(s),n}(z,f)/(z^n-\sigma^n)^r \;,\; B_{N(s),n}(z,f) \in \pi_{N(s)} \;,$$

be the rational function such that

$$Q^{(j)}_{N(s),n}(\omega^k,f,r) = f^{(j)}(\omega^k) \quad , \quad (j = 0,1,\ldots,s-1) \;,$$

$(k = 0,1,2,\ldots,n+m)$, where ω is the N^{th} root of unity. Let

(2.3) $$q_{N(s),n}(z,f,r) := P_{N(s),n}(z,f)/(z^n-\sigma^n)^r \;,\; P_{N(s),n}(z,f) \in \pi_{N(s)} \;,$$

be the best L_2 approximant of the above form to $f(z)$ on $|z| = 1$.

Remark 1.1. As a consequence of a Theorem of Walsh ([28] p. 224), $q_{N(s),n}(z,f,r)$ interpolates $f(z)$ in the Hermite sense in the sN zeros of $z^{sN-rn}(z^n-\sigma^{-n})^r$ if $\sigma > 1$. The following result gives (I, (12.4)) as a special case when $r = s = 1$.

Theorem 2.1. For any $f \in A_\rho$ with $\rho > 1$ and $\sigma > 1$, and for any fixed integers m,r,s with $m \geq -1$ and $1 \leq r \leq s$,

$$\lim_{n\to\infty}\left\{Q_{N(s),n}(z,f,r)-q_{N(s),n}(z,f,r)\right\} = 0 \begin{cases} |z| < \rho^{1+\frac{1}{s}}, & \text{if } \sigma \geq \rho^{1+\frac{1}{s}}, \\ |z| \neq \sigma, & \text{if } \sigma < \rho^{1+\frac{1}{s}}, \end{cases}$$

the convergence being uniform and geometric in any compact subset of the regions described above. Moreover, the result is best possible in the sense that the region of convergence can not be improved.

We shall omit the proof of Theorem 2.1 since it is a special case of Theorem 2.2.

Next, we extend Theorem 2.1 in the spirit of Theorem 3.3 ([18]) and Theorem 3 ([5]). For this purpose we shall define a sequence $\{q_{N(s),n}(z,f,r;\nu)\}_{\nu=0}^{\infty}$ of "help functions" which arise from some interpolatory schemes related to $f(z)$ (see §3). For any fixed integer $\ell \geq 1$, if we set

$$W_{N(s),n,\ell}(z,f,r) = \sum_{\nu=0}^{\ell-1} q_{N(s),n}(z,f,r;\nu) , \tag{2.4}$$

where $q_{N(s),n}(z,f,r;\nu)$, $(\nu = 0,1,\ldots)$, are given by (3.20), then we can prove

<u>Theorem 2.2</u>: <u>Let</u> $\rho > 1$, $\sigma > 1$ <u>and</u> $f \in A_\rho$. <u>If</u> ℓ, m, r, s <u>are fixed integers with</u> $\ell \geq 1$, $m \geq -1$, $1 \leq r \leq s$ <u>then</u>

$$\lim_{n\to\infty}\left\{Q_{N(s),n}(z,f,r)-W_{N(s),n,\ell}(z,f,r)\right\} = 0 \begin{cases} |z| < \rho^{1+\frac{\ell}{s}} & \text{if } \sigma \geq \rho^{1+\frac{\ell}{s}}, \\ |z| \neq \sigma & \text{if } \sigma < \rho^{1+\frac{\ell}{s}}, \end{cases}$$

<u>the convergence being uniform and geometric on any compact subset of the regions described above. Moreover, the result is best possible.</u>

<u>Remark 2.1</u>. Letting σ tend to infinity and taking $r = s$ in Theorem 2.2, we obtain Theorem 3 ([5]).

Remark 2.2. For $r = s = 1$, Theorem 2.2 reduces to Theorem 3.3 ([18]) (See I, (12.11)).

§3. CONSTRUCTION OF HELP FUNCTIONS. The help functions $q_{N(s),n}(z,f,r;\nu)$, $(\nu = 0,1,\dots)$, are connected with an expansion of the kernel

$$K(t,z) := \left(\frac{z}{t}\right)^{sN-rn}\left(\frac{z^n-\sigma^{-n}}{t^n-\sigma^{-n}}\right)^r - \left(\frac{z^N-1}{t^N-1}\right)^s \tag{3.1}$$

which appears in the integral representation of the difference (cf. Theorem 2.1)

$$Q_{N(s),n}(z,f,r) - q_{N(s),n}(z,f,r) . \tag{3.2}$$

The following lemmas play a key role in defining these functions precisely.

Lemma 3.1 : For fixed integers $m \geq -1$, $n \geq 1$, $s \geq r \geq 1$, set $M(k,s) := (k+s)N-1$, $(k = 0,1,2,\dots)$, $N = n+m+1$, and put

$$\alpha_{n,m}(z) = 1-z^{m+1}\sigma^{-n} , \quad \beta_{n,m}(z) = z^{m+1}(z^n-\sigma^{-n}) , \quad \sigma > 1 . \tag{3.3}$$

Let $S_{M(k,s)}(z,f,r) \in \pi_{M(k,s)}$ interpolate $\{\alpha_{n,m}(z)\}^k(z^n-\sigma^n)^r f(z)$ in $(k+s)N$ zeros of $z^{(s-r)N}\{\beta_{n,m}(z)\}^{k+r}$ in the Hermite sense. If $f(z)$ is analytic in $|z| \leq 1$, then

$$S_{M(k,s)}(z,f,r) - \alpha_{n,m}(z)S_{M(k-1,s)}(z,f,r)$$

(3.4)

$$= z^{(s-r)N}\{\beta_{n,m}(z)\}^{k+r-1}\; P_{n+m}(z,f;k)$$

where $P_{n+m}(z,f;k) \in \pi_{n+m}$ $(k = 1,2,3,\ldots)$. Moreover, for each n sufficiently large

(3.5)
$$\lim_{k\to\infty} \frac{S_{M(k,s)}(z,f,r)}{\{\alpha_{n,m}(z)\}^k} = (z^n-\sigma^n)^r f(z)$$

uniformly on $|z| \leq 1$. Consequently, for $|z| \leq 1$,

(3.6)
$$(z^n-\sigma^n)^r f(z) - S_{M(0,s)}(z,f,r)$$

$$= z^{(s-r)N}\{\beta_{n,m}(z)\}^{r-1} \sum_{k=1}^{\infty} \left(\frac{\beta_{n,m}(z)}{\alpha_{n,m}(z)}\right)^k P_{n+m}(z,f;k)$$

Proof. The formula (3.4) follows from the intepolatory properties of the polynomial $S_{M(k,s)}(z,f,r)$ and the fact that $z^{(s-r)N}\{\beta_{n,m}(z)\}^{k+r-1}$ divides the polynomial

$$S_{M(k,s)}(z,f,r) - \alpha_{n,m}(z)S_{M(k-1,s)}(z,f,r).$$

In order to prove (3.5) we note that

$$G_k(z) := (z^n-\sigma^n)^r f(z) - \frac{S_{M(k,s)}(z,f,r)}{\{\alpha_{n,m}(z)\}^k}$$

$$= (z^n-\sigma^n)^r f(z) - \frac{1}{2\pi i}\int_\Gamma \left\{\frac{\alpha_{n,m}(t)}{\alpha_{n,m}(z)}\right\}^k \frac{(t^n-\sigma^n)^r f(t)}{t-z} \times$$

$$\times \frac{t^{(s-r)N}[\beta_{n,m}(t)]^{k+r} - z^{(s-r)N}[\beta_{n,m}(z)]^{k+r}}{t^{(s-r)N}[\beta_{n,m}(t)]^{k+r}}\, dt$$

where Γ is the circle $|t| = R$ in which $f(z)$ is analytic. The integral involved in the above equality can be written as the sum of two integrals one of which, according to Cauchy integral formula, is exactly $(z^n-\sigma^n)^r f(z)$. Therefore,

$$G_k(z) = \frac{1}{2\pi i}\int_\Gamma \left(\frac{z}{t}\right)^{(s-r)N}\left(\frac{\alpha_{n,m}(t)}{\alpha_{n,m}(z)}\right)^k\left(\frac{\beta_{n,m}(z)}{\beta_{n,m}(t)}\right)^{k+r}\frac{(t^n-\sigma^n)^r}{t-z}\,f(t)dt.$$

A straightforward estimate now yields $\overline{\lim_{k\to\infty}}\left(\max_{|z|\le 1}|G_k(z)|\right)^{1/k} < 1$ for $n > n_0$ where n_0 depends on m, r, s, R and σ. This proves (3.5). (3.6) follows from (3.4) and (3.5) by direct computation. □

<u>Remark 3.1</u>. The polynomials $P_{n+m}(z,f;k)$ in the formula (3.4) are associated with the function $f(z)$ and have the integral representation

$$(3.7)\qquad \frac{1}{2\pi i}\int_\Gamma \frac{(t^n-\sigma^n)^r f(t)}{t-z}\left(\frac{A(t,z)}{t^{(s-r)N}\alpha_{n,m}(t)[\beta_{n,m}(t)]^r}\right)\left(\frac{\alpha_{n,m}(t)}{\beta_{n,m}(t)}\right)^k dt$$

where

$$(3.8)\qquad A(t,z) := \alpha_{n,m}(z)\beta_{n,m}(t) - \alpha_{n,m}(t)\beta_{n,m}(z)$$

is a polynomial in z of degree $N = n+m+1$.

The next lemma is a direct consequence of the following identity due to Cavaretta, Sharma and Varga ([5], Lemma 1):

For all t with $|t| > 1$, we have

$$(3.9)\qquad S_{1,r}(t,z) := \left(\frac{z}{t}\right)^{rN} - \left(\frac{z^N-1}{t^N-1}\right)^r = \frac{t^N - z^N}{t^{rN}} \sum_{j=1}^{\infty} \frac{B_{j,r}(z)}{t^{jN}}$$

where

$$(3.10)\qquad B_{j,r}(z) := \sum_{k=0}^{r-1} \binom{r+j-1}{k}(z^N-1)^k \quad , \quad (j = 1,2,\ldots),$$

is a polynomial of degree $\leq (r-1)N$.

Lemma 3.2. For all t with $|t| > 1$, the following identities hold:

(i) If $A(t,z)$ is given by (3.8), then

$$(3.11)\qquad S_{2,r}(t,z) := \left(\frac{\beta_{n,m}(z)}{\beta_{n,m}(t)}\right)^r - \left(\frac{z^N-1}{t^N-1}\right)^r$$

$$= \frac{A(t,z)}{[\beta_{n,m}(t)]^{r+1}} \sum_{j=1}^{\infty} \left(\frac{\alpha_{n,m}(t)}{\beta_{n,m}(t)}\right)^{j-1} \gamma_{j,r}(z)$$

where $\gamma_{j,r}(z)$ is a polynomial of degree $\leq (r-1)N$ given by

$$(3.12)\qquad \gamma_{j,r}(z) = \sum_{k=0}^{r-1} \binom{r+j-1}{k}[\alpha_{n,m}(z)]^{r-k-1}(z^N-1)^k .$$

(ii) If $K(t,z)$ is given by (3.1), then we have

$$(3.13)\qquad \begin{aligned} K(t,z) &= \left(\frac{z}{t}\right)^{(s-r)N} S_{2,r}(t,z) + \left(\frac{z}{t}\right)^{rN} S_{1,s-r}(t,z) \\ &\quad - S_{1,r}(t,z)S_{1,s-r}(t,z) \end{aligned}$$

where $S_{1,r}(t,z)$ and $S_{2,r}(t,z)$ are given by (3.9) and (3.11) respectively.

Proof: (i) Since $z^N - 1 = \beta_{n,m}(z) - \alpha_{n,m}(z)$, the left side of (3.11) can be factorized as

$$(3.14)\qquad \left(\frac{\alpha_{n,m}(z)}{\alpha_{n,m}(t)}\right)^r \left\{\frac{\left(\frac{\beta_{n,m}(z)}{\alpha_{n,m}(z)}\right)^r - \left(\frac{\beta_{n,m}(z)}{\alpha_{n,m}(z)}-1\right)^r}{\left(\frac{\beta_{n,m}(t)}{\alpha_{n,m}(t)}\right)^r - \left(\frac{\beta_{n,m}(t)}{\alpha_{n,m}(t)}-1\right)^r}\right\}.$$

If we substitute T and Z for $\left(\frac{\beta_{n,m}(t)}{\alpha_{n,m}(t)}\right)^{1/n}$ and $\left(\frac{\beta_{n,m}(z)}{\alpha_{n,m}(z)}\right)^{1/n}$ respectively, the second factor in (3.14) becomes the left side of (3.9) in the variables T and Z. An application of (3.9) to this factor and some straightforward computations lead to the identity (3.11).

(ii) The kernel $K(t,z)$ in (3.1) can be rewritten as follows on using (3.3):

$$K(t,z) = \left(\frac{z}{t}\right)^{(s-r)N}\left(\frac{\beta_{n,m}(z)}{\beta_{n,m}(t)}\right)^r - \left(\frac{z^N-1}{t^N-1}\right)^s$$

$$= \left(\frac{z}{t}\right)^{(s-r)N}\left\{\left(\frac{\beta_{n,m}(z)}{\beta_{n,m}(t)}\right)^r - \left(\frac{z^N-1}{t^N-1}\right)^r\right\} + \left(\frac{z}{t}\right)^{rN}\left\{\left(\frac{z}{t}\right)^{(s-r)N} - \left(\frac{z^N-1}{t^N-1}\right)^{s-r}\right\}$$

$$+ \left\{\left(\frac{z^N-1}{t^N-1}\right)^r - \left(\frac{z}{t}\right)^{rN}\right\}\left\{\left(\frac{z}{t}\right)^{(s-r)N} - \left(\frac{z^N-1}{t^N-1}\right)^{s-r}\right\}.$$

Now the identity (3.13) follows from (3.9) and (3.11). □

Remark 3.2. The product $S_{1,r}(t,z)S_{1,s-r}(t,z)$ appearing in (3.13) can be explicitly written down by using Cauchy product of two series. We then have

$$(3.15)\qquad S_{1,r}(t,z)S_{1,s-r}(t,z) = \frac{(t^N-z^N)^2}{t^{sN}} \sum_{k=1}^{\infty} \frac{C_{k,s,r}(z)}{t^{(k+1)N}}$$

where $C_{k,s,r}(z)$ is a polynomial of degree $(s-2)N$ given by

$$(3.16)\qquad C_{k,s,r}(z) = \sum_{j=1}^{k} \sum_{u=0}^{r-1} \sum_{\nu=0}^{s-r-1} \binom{r+j-1}{u}\binom{s+k+j-r}{\nu}(z^N-1)^{u+\nu} .$$

If $f(z) = \sum_{j=0}^{\infty} a_j z^j$, we define three polynomials which depend on f(z) in some sense. Indeed, we set

$$(3.17)\qquad V_1(z,k) := z^{(s-r)N}\, \gamma_{k,r}(z) P_{n+m}(z,f;k) ,$$

$$(3.18)\quad V_2(z,k) := B_{k,s-r}(z) \sum_{j=0}^{r} (-1)^r \binom{r}{j} \sigma^{(r-j)n} \sum_{u=0}^{N-1} a_{(s+k-1)N-jn+u}\, z^{rN+u},$$

where $B_{k,s-r}(z)$ is given by (3.10), and

$$(3.19)\quad V_3(z,k) := C_{k,s,r}(z) \left\{ \sum_{j=0}^{r} \sum_{u=0}^{N-1} (-1)^j \binom{r}{j} \sigma^{(r-j)u} \right.$$

$$\left. \times \left\{ a_{(u+k)N-jn+u}\, z^{N+u} - a_{(s+k-1)N-jN+u}\, z^u \right\} \right\} .$$

This enables us to assert that the rational function $q_{N(s),n}$ mentioned in (2.4) is given by

$$(3.20)\quad q_{N(s),n}(z,f,r;k) := P_{N(s),n}(z,f,k)/(z^n-\sigma^n)^r, \ (k = 0,1,2,3,\ldots),$$

where

$$(3.21)\qquad P_{N(s),n}(z,f,k) := \sum_{j=1}^{3} V_j(z,k) \quad , k = 1,2,3,4,\ldots,$$

and for $k = 0$,

(3.22) $$P_{N(s),n}(z,f,0) := S_{M(0,s)}(z,f,r) \ , \quad M(0,s) = Ns-1 \ .$$

Remark 3.3. Using integral representations for each of the polynomials $V_j(z,k)$, $j = 1,2,3$, the help functions $q_{N(s),n}(z,f,r;k)$, $k \geq 1$, can be written as

$$\frac{1}{2\pi i}\int_{\Gamma} \left(\frac{t^n-\sigma^n}{z^n-\sigma^n}\right)^r \frac{K_1(t,z;k)}{t-z} f(t)dt$$

where Γ is the circle $|t| = R$, $1 < R < \rho$, and

(3.23) $$K_1(t,z;k) := A(t,z)\left(\frac{\alpha_{n,m}(t)}{\beta_{n,m}(t)}\right)^{k-1} \frac{\gamma_{k,r}(z)}{[\beta_{n,m}(t)]^{r+1}} \left(\frac{z}{t}\right)^{(s-r)N} + \left(\frac{t^N - z^N}{t^{(s+k)N}}\right)B_{k,s-r}(z)z^{rN} - \left(\frac{(t^N-z^N)^2}{t^{(s+k+1)N}}\right)C_{k,s,r}(z) \ , \quad (k=1,2,3,\ldots).$$

§4. SOME ESTIMATES AND PROOF OF THEOREM 2.2. We shall require the following estimates in the proof of Theorem 2.2:

i) For all δ and t with $1 < \delta < |t|$, there is an integer n_0 such that

(4.1) $$\left|\frac{\alpha_{n,m}(t)}{\beta_{n,m}(t)}\right| < \frac{\delta}{|t|^N} \ , \qquad n \geq n_0 \ .$$

(This follows from the fact that for $\varepsilon > 0$, there exists n_0 such that $|1-t^{m+1}\sigma^{-n}| < 1+\varepsilon$ and $|1-(t\sigma)^{-n}| \geq 1-\varepsilon$, for $n \geq n_0$. Choosing ε such that $\delta := \frac{1+\varepsilon}{1-\varepsilon} < |t|$, we get (4.1).)

ii) From (3.8), recalling the definition

$$A(t,z) := \alpha_{n,m}(z)\beta_{n,m}(t) - \alpha_{n,m}(t)\beta_{n,m}(z),$$

it follows that

(4.2) $$|A(t,z)| \le C_1(|t|^N+|z|^N) \quad , \text{ for all sufficiently large } n,$$

and for some constant C_1 independent of n.

If for a fixed integer $p \ge 1$, $h(x) \in \pi_p$, then $\sum_{k=0}^{\infty} h(k)z^k$ is analytic in $|z| < 1$. Applying this fact to the special cases when

$$h(k) = \binom{r+k+\ell-1}{u} \text{ or } \sum_{j=0}^{k+\ell-1} \binom{r+j}{u}\binom{s+k+\ell+j-r-1}{\nu} ,$$

we can easily prove (4.3) and (4.4) in (iii):

iii) For any integers ℓ, k, s, r and any integers u, ν with $0 \le u \le r-1$, $0 \le \nu \le s-r-1$, and any $|t| > 1$, there exists a constant $c_0 = c_0(|t|,\ell,k,r,s)$ such that

(4.3) $$\left| \sum_{k=0}^{\infty} \sum_{j=0}^{k+\ell-1} \binom{r+j}{u}\binom{s+k+\ell+j-r-1}{\nu} t^{-kN} \right| \le c_0 , \quad n \ge 1,$$

and

(4.4) $$\left| \sum_{k=0}^{\infty} \binom{r+k+\ell-1}{u}\left[\frac{\alpha_{n,m}(t)}{\beta_{n,m}(t)}\right]^k \right| \le c_0 ,$$

for all sufficiently large n. (For (4.4), apply (4.1)).

<u>Proof of Theorem 2.2.</u> From (2.4) and the representation (3.1), we

can write

$$Q_{N(s),n}(z,f,r) - W_{N(s),n,\ell}(z,f,r) = \frac{1}{2\pi i}\int_\Gamma \left(\frac{t^n-\sigma^n}{z^n-\sigma^n}\right)^r \frac{f(t)}{t-z} K_{2,\ell}(t,z)dt$$

with

$$K_{2,\ell}(t,z) = K(t,z) - \sum_{k=1}^{\ell-1} K_1(t,z;k) \tag{4.5}$$

where $K_1(t,z;k)$ is given by (3.23). Using (3.13) and (3.23), we obtain after some algebra

$$K_{2,\ell}(t,z) = \left(\frac{A(t,z)}{[\beta_{n,m}(t)]^{r+1}}\right)\left(\frac{z}{t}\right)^{(s-r)N} \sum_{k=0}^{\infty}\left(\frac{\alpha_{n,m}(t)}{\beta_{n,m}(t)}\right)^{k+\ell-1} \gamma_{k+\ell,r}(z) +$$

$$+ \frac{(t^N-z^N)z^{rN}}{t^{sN}} \sum_{k=0}^{\infty} \frac{B_{k+\ell,s-r}(z)}{t^{(k+\ell)N}} - \frac{(t^N-z^N)^2}{t^{(s+1)N}} \sum_{k=0}^{\infty} \frac{C_{k+\ell,s,r}(z)}{t^{(k+\ell)N}} .$$

Since $|t| = R > 1$ on Γ, it follows from the estimates (4.1) - (4.4) that for sufficiently large n

$$|Q_{N(s),n}(z,f,r) - W_{N(s),n,\ell}(z,f,r)|$$

$$\le \left|\frac{R^n+\sigma^n}{\sigma^n-|z|^n}\right|^r M.C^* \left\{\frac{R^N+|z|^N}{R^{N(q+\ell)}}\left(\frac{|z|}{R}\right)^{(s-r)N}(|z|^N+1)^{r-1}\right. \tag{4.6}$$

$$\left. + \frac{R^N+|z|^N}{R^{(s+\ell)N}}|z|^{rN}(|z|^N+1)^{s-r-1} + \frac{(R^N+|z|^N)^2}{R^{N(s+\ell+1)}}(|z|^N+1)^{s-2}\right\}$$

where $|f(t)| \le M$ for $|t| < \rho$ and C^* is a positive constant independent of n. A straightforward analysis of (4.6) then yields the region of convergence for the sequence of the functions on the left side of (4.6).

Finally, direct computation with the function $\hat{f}(z) := (z-\rho)^{-1}$ shows that the result is best possible. □

CHAPTER III

ON EQUICONVERGENCE OF CERTAIN SEQUENCES OF L_2 AND ℓ_2 RATIONAL APPROXIMANTS

§1. INTRODUCTION. A recent paper of T.J. Rivlin [17] extends Walsh's theorem on equiconvergence of two sequences of interpolating polynomials. His main idea is to replace the Lagrange interpolating polynomial in the Walsh's theorem by the polynomial of minimization in the ℓ_2-norm. For fuller detail we refer to (I, §4).

In the present chapter we find an analogue of Rivlin's result for certain rational sequences of the form $p(z)/(z^n-\sigma^n)$, $(p(z) \in \pi_{n+m})$, discussed by Saff and Sharma (I, §12). §2 deals with the ℓ_2-minimization problem on the set $\{\omega^k\}_0^{qn-1}$, where ω is a primitive $(qn)^{th}$ root of unity. Its solution will replace the polynomial $\mathcal{P}_{n-1,m}(z,f)$ in (I, (4.1)) as stated in our main result (see Theorem 2.1). Similarly, in place of the Taylor polynomial $p_{n-1}(z,f)$ in (I, (4.1)), we shall consider the unique rational function $r_{n+m,n}(z,f)$ given by (I, (12.3)). The ℓ_2-minimization problem will be solved in §3. We shall prove our main result in §4. The last section §5 is devoted to extensions of Theorem 2.1.

§2. A MINIMIZATION PROBLEM & STATEMENT OF RESULT. Consider the following problem:

(P1) Let $m \geq -1$ and $q \geq 2$ be fixed integers and let

$\omega = \exp(2\pi i/qn)$. For $f \in A_\rho$, we want to minimize

$$(2.1) \qquad \sum_{k=0}^{qn-1} |f(\omega^k) - R(\omega^k, f)|^2$$

over all rational functions of the form

$$R(z,f) = \frac{p(z)}{z^n - \sigma^n}, \quad p(z) \in \pi_{n+m}.$$

If the solution of the problem (P1) is given by

$$(2.2) \qquad R^*_{n+m,n}(z,f) := \frac{P^*_{n+m,n}(z,f)}{z^n - \sigma^n}, \quad P^*_{n+m,n}(z,f) \in \pi_{n+m},$$

then we can state our main result as

Theorem 2.1. Let $m \geq -1$ and $q \geq 2$ be two fixed positive integers and let $\sigma > 1$. If $f \in A_\rho$, $1 < \rho < \infty$, then

$$(2.3) \quad \lim_{n\to\infty}\{R^*_{n+m,n}(z,f) - r_{n+m,n}(z,f)\} = 0 \begin{cases} |z| < \rho^{1+q} & \text{if } \sigma \geq \rho^{1+q}, \\ |z| \neq \sigma & \text{if } \sigma < \rho^{1+q}, \end{cases}$$

the convergence being uniform and geometric in any compact subset of the regions described above. Moreover, the result is sharp in the sense that for each $|z| = \rho^{1+q}$ with $\sigma \geq \rho^{1+q}$, there is an $\hat{f} \in A_\rho$ for which (2.3) does not hold.

The proof of the preceding theorem will be given in §4. First we turn to the reformulation of the minimization problem (P1) which requires the following lemma:

Lemma 2.1. Let $m \geq -1$ and $q \geq 2$ be fixed integers and let

$g(z) := P(z)/(z^n - \sigma^n)$, <u>where</u> $P(z) = \sum_{j=0}^{n+m} c_j z^j \in \pi_{n+m}$ <u>is given</u>. <u>Then the Lagrange interpolant of</u> $g(z)$ <u>on the</u> $(qn)^{th}$ <u>roots of unity is given by</u>

$$L_{qn-1}(z,g) := \sum_{\nu=0}^{qn-1} A_\nu z^\nu$$

<u>where</u>

(2.4) $$A_{\nu n+j} = \begin{cases} \lambda_1 c_j + \lambda_q c_{j+n} , & \nu = 0 , \quad 0 \le j \le m , \\ \lambda_{\nu+1} c_j , & 0 \le \nu \le q-1 , \quad m+1 \le j \le n-1 , \\ \lambda_{\nu+1} c_j + \lambda_\nu c_{j+n} , & 1 \le \nu \le q-1 , \quad 0 \le j \le m , \end{cases}$$

<u>with</u>

(2.5) $$\lambda_\nu = \sigma^{(q-\nu)n}/(1-\sigma^{qn}) \quad , \quad \nu = 1,2,\dots,q .$$

<u>Proof</u>: The Lagrange interpolant of the function $(z^n - \sigma^n)^{-1}$ on the $(qn)^{th}$ roots of unity is given by

$$L_{qn-1}(z,(z^n-\sigma^n)^{-1}) = \frac{z^{qn}-\sigma^{qn}}{(z^n-\sigma^n)(1-\sigma^{qn})} = \sum_{\nu=1}^{q} \lambda_\nu z^{(\nu-1)n} .$$

For $z = \omega$, we have

(2.6) $$(\omega^{kn} - \sigma^n)^{-1} = \sum_{\nu=1}^{q} \lambda_\nu \omega^{(\nu-1)nk} , \quad k = 0,1,2,\dots,qn-1 .$$

If we multiply (2.6) by $P(\omega^k) = \sum_{j=0}^{n+m} c_j \omega^{kj}$ then a simple calculation leads to

$$\frac{P(\omega^k)}{\omega^{kn}-\sigma^n} = \lambda_1 \sum_{j=0}^{n+m} c_j \omega^{kj} + \lambda_2 \sum_{j=n}^{2n+m} c_{j-n} \omega^{kj} + \dots + \lambda_q \sum_{j=(q-1)n}^{qn+m} c_{j-(q-1)n} \omega^{kj} .$$

Notice that in the above, each of the q summations has $m+1$ distinct powers of ω^k which appear in its preceding sum. If we group the terms involving identical powers of ω^k in separate summations and then rearrange them, we obtain for each $k = 0,1,2,\ldots,qn-1$,

$$\frac{P(\omega^k)}{\omega^{kn}-\sigma^n} = \sum_{j=0}^{m} [\lambda_1 c_j + \lambda_q c_{j+n}]\omega^{jk} + \sum_{\nu=0}^{q-1} \sum_{j=m+1}^{n-1} \lambda_{\nu+1} c_j \omega^{(\nu n+j)k} +$$

$$+ \sum_{\nu=1}^{q-1} \sum_{j=0}^{m} [\lambda_{\nu+1} c_j + \lambda_\nu c_{j+n}]\omega^{(\nu n+j)k}$$

which reduces to $L_{qn-1}(\omega^k,g) = \sum_{j=0}^{qn-1} A_j \omega^{jk}$ on recalling that ω is a primitive $(qn)^{th}$ root of unity, where A_j's are given by (2.4). □

Remark 2.1. If the a_ν's are given complex numbers, then from the properties of the roots of unity, it follows that

$$(2.7) \qquad \sum_{k=0}^{qn-1} \Big| \sum_{\nu=0}^{qn-1} a_\nu \omega^{k\nu} \Big|^2 = qn \sum_{\nu=0}^{qn-1} |a_\nu|^2 .$$

Remark 2.2. If $L_{qn-1}(z,f) := \sum_{j=0}^{qn-1} b_j z^j$ is the Lagrange interpolant of degree $qn-1$ to $f(z) \in A_\rho$ in the $(qn)^{th}$ roots of unity, then we recall that

$$(2.8) \qquad b_j = \frac{1}{2\pi i} \int_\Gamma \frac{f(t)t^{qn-j-1}}{t^{qn}-1} dt , \quad j = 0,1,2,\ldots,qn-1 ,$$

where Γ is the circle $|t| = R$, $1 < R < \rho$.

Remark 2.3. Since $L_{qn-1}(\omega^k,f) = \sum_{\nu=0}^{qn-1} b_\nu \omega^{k\nu} = f(\omega^k)$ and $L_{qn-1}(\omega^k,g) = \sum_{\nu=0}^{qn-1} A_\nu \omega^{k\nu} = \frac{P(\omega^k)}{\omega^{kn}-\sigma^n}$, $k = 0,1,2,\ldots,qn-1$, it follows from (2.7) that the minimization problem (P1) is equivalent to the problem (P2):

(P2) Minimize

$$G := \sum_{\nu=0}^{qn-1} |b_\nu - A_\nu|^2 \tag{2.9}$$

over c_j's , $j = 0,1,2,\ldots,n+m$, where A_ν's are given by (2.4).

It may be noted that b_ν's are well-defined in (2.8) and that A_ν's are functions of c_j's .

§3 SOLUTION OF MINIMIZATION PROBLEM (P1). The following proposition determines the solution of the problem (P1):

Proposition 3.1. The polynomial $P^*_{n+m,n}(z,f)$ (cf.(2.2)) which solves the minimization problem (P1) is given by

$$P^*_{n+m,n}(z,f) = \sum_{\nu=0}^{n+m} p_\nu z^\nu$$

where

$$p_\nu := \begin{cases} -b_\nu \sigma^n + \dfrac{\sigma^{(q-1)n}(1-\sigma^{2n})}{1-\sigma^{(q-1)2n}} \displaystyle\sum_{j=1}^{q-1} \sigma^{-jn} b_{\nu+jn} , & 0 \le \nu \le m , \\ \dfrac{\sigma^{(q-1)n}(1-\sigma^{2n})}{1+\sigma^{qn}} \displaystyle\sum_{j=0}^{q-1} \sigma^{-jn} b_{\nu+jn} , & m+1 \le \nu \le n-1, \\ b_{\nu-n} - \dfrac{\sigma^{2(q-1)n}(1-\sigma^{2n})}{1-\sigma^{(q-1)2n}} \displaystyle\sum_{j=1}^{q-1} \sigma^{-jn} b_{\nu+(j-1)n} , & n \le \nu \le n+m, \end{cases} \tag{3.1}$$

and the b_k , $k = 0,1,\ldots,qn-1$ are given by (2.8).

Proof. Since (P1) is equivalent to (P2), it is sufficient to solve the system of equations $\frac{\partial G}{\partial \bar{c}_\nu} = 0$, $\nu = 0,1,2,\ldots,n+m$, in terms of λ_ν's and b_j's. Using (2.4), we can rewrite (2.9) as:

$$G = \sum_{j=0}^{m} |b_j - \lambda_1 c_j - \lambda_q c_{j+n}|^2 + \sum_{\nu=0}^{q-1} \sum_{j=m+1}^{n-1} |b_{j+\nu n} - \lambda_{\nu+1} c_j|^2 + \sum_{\nu=1}^{q-1} \sum_{j=0}^{m} |b_{j+\nu n} - \lambda_{\nu+1} c_j - \lambda_\nu c_{j+n}|^2 .$$

Therefore, for $0 \le j \le m$, we have

$$(3.2) \quad \frac{\partial G}{\partial \bar{c}_j} = -(b_j - \lambda_1 c_j - \lambda_q c_{j+n})\lambda_1 - \sum_{\nu=1}^{q-1} (b_{j+\nu n} - \lambda_{\nu+1} c_j - \lambda_\nu c_{j+n})\lambda_{\nu+1}$$

$$= \sum_{\nu=1}^{q} \lambda_\nu^2 c_j + (\lambda_q \lambda_1 + \lambda_1 \lambda_2 + \ldots + \lambda_{q-1}\lambda_q) c_{j+n} - \sum_{\nu=0}^{q-1} \lambda_{\nu+1} b_{j+\nu n} ,$$

and

$$(3.3) \quad \frac{\partial G}{\partial \bar{c}_{n+j}} = -(b_j - \lambda_1 c_j - \lambda_q c_{j+n})\lambda_q - \sum_{\nu=1}^{q-1} (b_{j+\nu n} - \lambda_{\nu+1} c_j - \lambda_\nu c_{j+n})\lambda_\nu$$

$$= \sum_{\nu=1}^{q} \lambda_\nu^2 c_{j+n} + (\lambda_q \lambda_1 + \lambda_1 \lambda_2 + \ldots + \lambda_{q-1}\lambda_q) c_j - \lambda_q b_j - \sum_{\nu=1}^{q-1} \lambda_\nu b_{j+\nu n} .$$

Also, for $m+1 \le j \le n-1$, we have

$$(3.4) \quad \frac{\partial G}{\partial \bar{c}_j} = - \sum_{\nu=0}^{q-1} (b_{j+\nu n} - \lambda_{\nu+1} c_j)\lambda_{\nu+1} = \sum_{\nu=0}^{q-1} \lambda_{\nu+1}^2 c_j - \sum_{\nu=0}^{q-1} \lambda_{\nu+1} b_{j+\nu n} .$$

For the sake of simplicity, we set

$$(3.5) \quad \begin{cases} \alpha := \lambda_1^2 + \lambda_2^2 + \ldots + \lambda_q^2 = \left(\frac{1+\sigma^{qn}}{1-\sigma^{2n}}\right)\lambda_q , \\ \beta = \lambda_q \lambda_1 + \lambda_1 \lambda_2 + \ldots + \lambda_{q-1}\lambda_q = \left(\frac{\sigma^n + \sigma^{(q-1)n}}{1-\sigma^{2n}}\right)\lambda_q . \end{cases}$$

Then from (3.2)-(3.4) we obtain, for $j = 0,1,\ldots,m$:

$$(3.6)\qquad \begin{cases} \dfrac{\partial G}{\partial \bar{c}_j} = \alpha c_j + \beta c_{n+j} - \sum\limits_{\nu=0}^{q-1} \lambda_{\nu+1} b_{\nu n+j}\,, \\ \dfrac{\partial G}{\partial \bar{c}_{n+j}} = \alpha c_{n+j} + \beta c_j - \lambda_q b_j - \sum\limits_{\nu=1}^{q-1} \lambda_\nu b_{\nu n+j}\,, \end{cases} \qquad (0 \le j \le m)$$

and for $m+1 \le j \le n-1$, we see that

$$(3.7)\qquad \frac{\partial G}{\partial \bar{c}_j} = \alpha c_j - \sum_{\nu=0}^{q-1} \lambda_{\nu+1} b_{\nu n+j}\,.$$

Setting the partial derivatives $\dfrac{\partial G}{\partial \bar{c}_\nu}$ equal to zero and then solving the system of equations so obtained simultaneously for c_ν and $c_{\nu+n}$, $\nu = 0,1,\ldots,m$, we obtain

$$(3.8)\qquad c_\nu = (\alpha^2-\beta^2)^{-1}\Big\{\alpha \sum_{j=0}^{q-1} \lambda_{j+1} b_{jn+\nu} - \beta \sum_{j=1}^{q-1} \lambda_j b_{jn+\nu} - \beta\lambda_q b_\nu\Big\}\,,$$

and

$$(3.9)\qquad c_{n+\nu} = (\alpha^2-\beta^2)^{-1}\Big\{\alpha \sum_{j=1}^{q-1} \lambda_j b_{jn+\nu} + \alpha\lambda_q b_\nu - \beta \sum_{j=0}^{q-1} \lambda_{j+1} b_{jn+\nu}\Big\}\,.$$

Also from (3.7) we have

$$(3.10)\qquad c_\nu = \alpha^{-1} \sum_{j=1}^{q} \lambda_j b_{(j-1)n+\nu}\,, \qquad m+1 \le \nu \le n-1.$$

We recall that c_ν, $0 \le \nu \le n+m$, as determined in (3.8)-(3.10), are the coefficients of the polynomial $P^*_{n+m,n}(z,f)$ (cf.(2.2)) which minimizes the expression given by (2.1).

Finally, replacing α, β and λ_j's in c_ν, $(\nu = 0,1,\ldots,n+m)$, by their respective values from (3.5) and (2.5) we obtain the relation (3.1) after some simple arithmetic. □

Remark 3.1. When $q = 1$, the problem (P1) is not uniquely solvable for all $m > -1$. However, $R^*_{n-1,n}(z,f)$ is well-defined for $q = 1$. It turns out that $R^*_{n-1,n}(z,f)$, in this case, interpolates the function $f(z)$ in the n^{th} roots of unity.

§4 PROOF OF THEOREM 2.1. It is known [18] that an integral representation of the rational function $r_{n+m,n}(z,f)$ is given by

$$(4.1) \qquad r_{n+m,n}(z,f) = \frac{1}{2\pi i}\int_\Gamma \frac{(t^n-\sigma^n)f(t)}{(z^n-\sigma^n)(t-z)} \sum_{j=1}^{3} A_j(t,z)dt$$

where Γ is the circle $|t| = R$, $1 < R < \rho$ and $A_j(t,z)$, $j = 1,2,3$, are defined by

$$(4.2) \qquad \begin{cases} A_1(t,z) := \dfrac{t^{m+1}-z^{m+1}}{t^{m+1}}, \quad A_2(t,z) := \dfrac{z^{m+1}(t^{n-m-1}-z^{n-m-1})}{t^n - \sigma^{-n}}, \\ A_3(t,z) := z^n(t^{m+1}-z^{m+1})/t^{m+1}(t^n-\sigma^{-n}) . \end{cases}$$

Also, from (2.2), (2.8) and (3.1), we obtain after some algebraic operations

$$(4.2a) \quad R^*_{n+m,n}(z,f) = \frac{1}{2\pi i}\int_\Gamma \frac{f(t)}{(z^n-\sigma^n)(t-z)(t^{qn}-1)} \sum_{j=1}^{3} A_j(t,z)B_j(t,\sigma)dt$$

with

$$(4.3)\qquad \begin{cases} B_1(t,\sigma) := \sigma^{-(q-2)n}\, B(t,\sigma) - t^{qn}\sigma^{n} \,, \\ B_2(t,\sigma) := (t^{qn}-\sigma^{-qn})(\sigma^{-n}-\sigma^{n})/(1+\sigma^{-qn}) \,, \\ B_3(t,\sigma) := (t^{n} - \sigma^{-n})\{t^{qn} - \sigma^{n}\, B(t,\sigma)\} \,, \end{cases}$$

where

$$B(t,\sigma) := \frac{t^{n}(t^{(q-1)n} - \sigma^{-(q-1)n})(1-\sigma^{-2n})}{(t^{n}-\sigma^{-n})(1-\sigma^{-2(q-1)n})} \,.$$

Therefore,

$$(4.4)\qquad \begin{aligned} &R^{*}_{n+m,n}(z,f) - r_{n+m,n}(z,f) \\ &= \frac{1}{2\pi i}\int_{\Gamma} \sum_{j=1}^{3} \frac{A_j(t,z)K_j(t,\sigma)}{(z^{n}-\sigma^{n})(t-z)(t^{qn}-1)}\, f(t)dt \,, \end{aligned}$$

where $K_j(t,\sigma) := B_j(t,\sigma) - (t^{qn}-1)(t^{n}-\sigma^{n})$, $j = 1,2,3$, can be explicitly rewritten after some simplification as

$$(4.5)\qquad \begin{cases} K_1(t,\sigma) = \sigma^{-(q-2)n}\, B(t,\sigma) - t^{(q+1)n} + t^{n} - \sigma^{n} \,, \\ K_2(t,\sigma) = \dfrac{t^{qn}\sigma^{-n}(1-\sigma^{-qn}) - \sigma^{-qn}(\sigma^{-n}-\sigma^{n})}{1 + \sigma^{-qn}} - t^{(q+1)n} + t^{n} - \sigma^{n}, \\ K_3(t,\sigma) = \dfrac{t^{n}\sigma^{-qn}(\sigma^{2n}-1)(t^{(q-1)n}\sigma^{-(q-1)n}-1)}{1 + \sigma^{-2(q-1)n}} + t^{n} - \sigma^{n} \,. \end{cases}$$

An analysis of the kernels $A_j(t,z)$ and $K_j(t,\sigma)$, $j = 1,2,3$, from (4.2) and (4.5) yields (2.3).

To prove that the result is sharp, we consider the point $\hat{z} = \rho^{1+q}$ and the corresponding function $\hat{f}(z) := (z-\rho)^{-1}$. A direct computation from (3.1), (2.2) and (4.1) for $f = \hat{f}$ shows that

$$(4.6)\qquad R^{*}_{n+m,n}(z,\hat{f}) - r_{n+m,n}(z,\hat{f}) = \sum_{j=1}^{3} \frac{A_j(\rho,z)K_j(\rho,\sigma)}{(z^{n}-\sigma^{n})(z-\rho)(\rho^{qn}-1)} \,.$$

If $\sigma > \rho^{1+q}$, we get after some simple calculus

$$\lim_{n\to\infty} \{R^{*}_{n+m,n}(\rho^{1+q},\hat{f}) - r_{n+m,n}(\rho^{1+q},\hat{f})\} = \frac{\rho^{q}}{\rho^{q}-1} .$$

If $\sigma = \rho^{1+q}$ then the sequence (4.6) is undefined for infinitely many n's for $\hat{z} = \rho^{1+q}\omega_0$ where ω_0 is any primitive root of unity. □

Remark 4.1. If we fix $m = -1$ and let $\sigma \to \infty$ in Theorem 2.1, we get a result of Rivlin ([17], Theorem 1) in the special case when $c = 0$ (cf. I, §4)

Remark 4.2. According to Remark 3.1, Saff-Sharma's result ([18], Theorem 2.3) for the special case $m = -1$ can be retrieved from Theorem 2.1.

§5. SOME EXTENSIONS AND REMARKS. Our next object is to extend Theorem 2.1 in the spirit of (I, (4.2)). For this, we define two different sequences of "help rational functions" which help us to obtain a larger region of equiconvergence. These extensions are obtained from two different expansions of $(t^{qn}-1)^{-1}$.

a) Extension I. Our first extension is based on the following identity:

$$(t^{qn}-1)^{-1} = [t^{qn}-\sigma^{-qn} - (1-\sigma^{-qn})]^{-1} = \sum_{\nu=1}^{\infty} \tilde{F}_{\nu}(t,\sigma)$$

where

(5.1) $$\tilde{F}_\nu(t,\sigma) = \frac{(1-\sigma^{-qn})^{\nu-1}}{(t^{qn}-\sigma^{-qn})^\nu}, \quad \nu = 1,2,\ldots .$$

Using $K_j(t,\sigma)$, $j = 1,2,3$, given by (4.5), we define the rational functions

(5.2) $$\tilde{r}_{n+m,n}(z,f,\nu) := \sum_{j=0}^{n+m} \tilde{c}_j(\nu) z^j/(z^n-\sigma^n), \quad \nu = 1,2,3,\ldots ,$$

where

(5.3) $$\tilde{c}_j(\nu) := \begin{cases} \frac{1}{2\pi i}\int_\Gamma \frac{K_1(t,\sigma)}{t^{j+1}} \tilde{F}_\nu(t,\sigma) f(t)dt, & 0 \le j \le m, \\ \frac{1}{2\pi i}\int_\Gamma \frac{K_2(t,\sigma)\tilde{F}_\nu(t,\sigma)}{t^{m-n+j+2}(t^n-\sigma^{-n})} f(t)dt, & m+1 \le j \le n-1, \\ \frac{1}{2\pi i}\int_\Gamma \frac{K_3(t,\sigma)\tilde{F}_\nu(t,\sigma)}{t^{j+1}(t^n-\sigma^{-n})} f(t)dt, & n \le j \le n+m . \end{cases}$$

For $\nu = 0$, we let

(5.4) $$\tilde{r}_{n+m,n}(z,f,0) := r_{n+m,n}(z,f).$$

Remark 5.1. From (5.3) we can rewrite

$$\tilde{r}_{n+m,n}(z,f,\nu) = \frac{1}{z^n-\sigma^n}\left\{\sum_{j=0}^{m} \tilde{c}_j(\nu)z^j + \sum_{j=m+1}^{n-1} \tilde{c}_j(\nu)z^j + \sum_{j=n}^{n+m} \tilde{c}_j(\nu)z^j\right\},$$

($\nu = 1,2,3,\ldots,$), so that using (4.2) we have

(5.5) $$\tilde{r}_{n+m,n}(z,f,\nu) = \frac{1}{2\pi i}\int_\Gamma \frac{f(t)\tilde{F}_\nu(t,\sigma)}{(z^n-\sigma^n)(t-z)} \sum_{j=1}^{3} A_j(t,z)K_j(t,\sigma)dt.$$

If we define

(5.6) $$\tilde{W}_{n+m,n}(z,f,\ell) := \sum_{\nu=0}^{\ell-1} \tilde{r}_{n+m,n}(z,f,\nu) ,$$

we have the first extension of Theorem 2.1 given by:

Theorem 5.1. Let $m \geq -1$, $q \geq 2$ and $\ell \geq 1$ be three fixed integers and let $\sigma > 1$. If $f \in A_\rho$, $1 < \rho < \infty$, then

$$(5.7)\quad \lim_{n\to\infty} \{R^*_{n+m,n}(z,f) - \tilde{W}_{n+m,n}(z,f,\ell)\} = 0 \begin{cases} |z| < \rho^{\ell q+1} & \text{if } \sigma \geq \rho^{\ell q+1}, \\ |z| \neq \sigma & \text{if } \sigma < \rho^{\ell q+1}, \end{cases}$$

the convergence being uniform and geometric on any compact subset of the regions defined above. Moreover, the result is sharp in the sense of Theorem 2.1.

Proof. The difference in (5.7) can be written as

$$R^*_{n+m,n}(z,f) - \tilde{W}_{n+m,n}(z,f,\ell)$$
$$= R^*_{n+m,n}(z,f) - \tilde{r}_{n+m,n}(z,f,0) - \sum_{\nu=1}^{\ell-1} \tilde{r}_{n+m,n}(z,f,\nu) .$$

Applying (5.4), (4.4), (5.5) and (5.1) to the above relation, we obtain

$$(5.8)\quad R^*_{n+m,n}(z,f) - \tilde{W}_{n+m,n}(z,f,\ell) = \frac{1}{2\pi i}\int_\Gamma \frac{\sum_{j=1}^{3} A_j(t,z)K_j(t,\sigma)}{(z^n-\sigma^n)(t-z)} \sum_{\nu=\ell}^{\infty} \tilde{F}_\nu(t,\sigma)f(t)dt .$$

Since $\sum_{\nu=\ell}^{\infty} \tilde{F}_\nu(t,\sigma) = \frac{(1-\sigma^{-qn})^\ell}{(t^{qn}-\sigma^{-qn})^{\ell-1}(t^{qn}-1)}$, we conclude (5.7) from (5.8) after some computation. As usual, the function $\tilde{f}(z) = (z-\rho e^{i\theta_0})^{-1}$, $0 \leq \theta_0 \leq 2\pi$, does the job to show that the result is sharp.

b) <u>Extension II</u>. In this part we shall rearrange a double sêries in order to construct new "help" rational functions. First, note that for an absolutely convergent series $\sum_{s=1}^{\infty} \sum_{\lambda=1}^{\infty} f(s,\lambda)$ and a fixed integer $q \geq 1$, we have

$$\sum_{s=1}^{\infty} \sum_{\lambda=1}^{\infty} f(s,\lambda) = \sum_{s=1}^{\infty} \sum_{j=1}^{\infty} \sum_{\lambda=1}^{q} f(s,(j-1)q+\lambda)$$

(5.9)

$$= \sum_{\lambda=1}^{q} \sum_{s=1}^{\infty} \sum_{j=1}^{s} f(j,(s-j)q+\lambda),$$

the last expression follows on writing the series $\sum_{s=1}^{\infty} \sum_{j=1}^{\infty} f(s,(j-1)q+\lambda)$, for each fixed λ, as shown below, and then on adding the terms along transverse diagonals

$$\begin{array}{l} f(1,\lambda) + f(1,q+\lambda) + f(1,2q+\lambda) + f(1,3q+\lambda) + \ldots \\ + f(2,\lambda) + f(2,q+\lambda) + f(2,2q+\lambda) + f(2,3q+\lambda) + \ldots \\ + f(3,\lambda) + f(3,q+\lambda) + f(3,2q+\lambda) + f(3,3q+\lambda) + \ldots \\ + f(4,\lambda) + f(4,q+\lambda) + f(4,2q+\lambda) + f(4,3q+\lambda) + \ldots \\ + \ldots. \end{array}$$

With this observation, we have

<u>Lemma 5.1.</u> <u>For</u> $|t| > 1$ <u>and</u> $\sigma > 1$, <u>the following identity holds</u>

(5.10) $$(t^{qn}-1)^{-1} = \sum_{s=1}^{\infty} \sum_{\lambda=1}^{q} F^{*}_{(s-1)q+\lambda}(t,\sigma)$$

where

$$(5.11) \qquad F^{*}_{(s-1)q+\lambda}(t,\sigma) = \sum_{j=1}^{s} \binom{(s-j)q^2+\lambda q+j-2}{j-1} \frac{(-\sigma^{-n})^{j-1}}{(t^n-\sigma^{-n})^{(s-j)q^2+\lambda q+j-1}} .$$

Proof. It is easy to see the validity of the following expansion:

$$(5.12) \qquad (t^{qn}-1)^{-1} = \sum_{s=1}^{\infty} \sum_{\lambda=1}^{\infty} \binom{\lambda q+s-2}{s-1} \frac{(-\sigma^{-n})^{s-1}}{(t^n-\sigma^{-n})^{\lambda q+s-1}} .$$

If we let $f(s,\lambda) = \binom{\lambda q+s-2}{s-1} \frac{(-\sigma^{-n})^{s-1}}{(t^n-\sigma^{-n})^{\lambda q+s-1}}$ in equation (5.9), then (5.10) follows immediately from (5.12) on observing that

$$f(j,(s-j)q+\lambda) = \binom{[(s-j)q+\lambda]q+j-2}{j-1} \frac{(-\sigma^{-n})^{j-1}}{(t^n-\sigma^{-n})^{[(s-j)q+\lambda]q+j-1}} . \qquad \square$$

Now we can define another sequence of help functions. Let

$$(5.13) \qquad r^{*}_{n+m,n}(z,f,\nu) := \sum_{j=0}^{m+n} c^{*}_j(\nu) z^j/(z^n-\sigma^n) \quad , \quad \nu = 1,2,3,\ldots$$

with

$$(5.14) \qquad c^{*}_j(\nu) := \begin{cases} \frac{1}{2\pi i} \int_\Gamma \frac{K_1(t,\sigma)}{t^{j+1}} F^{*}_\nu(t,\sigma) f(t)dt , & 0 \le j \le m , \\ \frac{1}{2\pi i} \int_\Gamma \frac{K_2(t,\sigma)F^{*}_\nu(t,\sigma)}{t^{m-n+j+2}(t^n-\sigma^{-n})} f(t)dt , & m+1 \le j \le n-1, \\ \frac{1}{2\pi i} \int_\Gamma \frac{K_3(t,\sigma)F^{*}_\nu(t,\sigma)}{t^{j+1}(t^n-\sigma^{-n})} f(t)dt , & n \le j \le n+m . \end{cases}$$

For $\nu = 0$, we set

$$r^{*}_{n+m,n}(z,f,0) = r_{n+m,n}(z,f) .$$

On using (4.2), an integral representation of $r^{*}_{n+m,n}(z,f,\nu)$, $\nu \geq 1$, is found to be

$$(5.15)\qquad r^{*}_{n+m,n}(z,f,\nu) = \frac{1}{2\pi i}\int_{\Gamma} \frac{f(t)F^{*}_{\nu}(t,\sigma)}{(z^{n}-\sigma^{n})(t-z)} \sum_{j=1}^{3} A_j(t,z)K_j(t,\sigma)dt \,.$$

For a fixed integer $\ell \geq 1$, we set

$$(5.16)\qquad W^{*}_{n+m,n}(z,f,\ell) := \sum_{\nu=0}^{\ell-1} r^{*}_{n+m,n}(z,f,\nu) \,.$$

With the above notations, we can prove now

<u>Theorem 5.2. Let</u> $m \geq -1$, $q \geq 2$, <u>and</u> $\ell \geq 1$ <u>be three fixed integers and</u> $\sigma > 1$. <u>If</u> $f \in A_{\rho}$, $1 < \rho < \infty$, <u>then</u>

$$(5.17)\qquad \lim_{n\to\infty}\{R^{*}_{n+m,n}(z,f) - W^{*}_{n+m,n}(z,f,\ell)\} = 0 \quad \begin{cases} |z| < \rho^{\ell q+1} & \text{if } \sigma \geq \rho^{\ell q+1}, \\ |z| \neq \sigma & \text{if } \sigma < \rho^{\ell q+1}, \end{cases}$$

<u>the convergence being uniform and geometric on any compact subset of the regions defined above. Moreover, the result is sharp.</u>

<u>Proof</u>. As in (5.8), we use (4.2a), (5.10), (5.15) and (5.16) to obtain

$$(5.18)\qquad \begin{aligned} &R^{*}_{n+m,n}(z,f) - W^{*}_{n+m,n}(z,f,\ell) \\ &= \frac{1}{2\pi i}\int_{\Gamma} \frac{\gamma_{\ell}(t,\sigma)f(t)}{(z^{n}-\sigma^{n})(t-z)} \sum_{j=1}^{3} A_j(t,z)K_j(t,\sigma)dt \end{aligned}$$

where $\gamma_{\ell}(t,\sigma) := \sum_{s=0}^{\infty}\sum_{\lambda=1}^{q} F^{*}_{sq+\lambda}(t,\sigma) - \sum_{\lambda=1}^{\ell-1} F^{*}_{\lambda}(t,\sigma)$ and Γ is the circle $|t| = \rho_1$ with $1 < \rho_1 < \rho$. If we write $\ell-1 := aq+b$ with

$a \geq 0,\ 0 \leq b \leq q-1$, then we have

$$\gamma_\ell(t,\sigma) = \sum_{s=0}^{\infty} \sum_{\lambda=1}^{q} F^*_{sq+\lambda}(t,\sigma) - \sum_{s=0}^{a-1} \sum_{\lambda=1}^{q} F^*_{sq+\lambda}(t,\sigma) - \sum_{\lambda=1}^{b} F^*_{aq+\lambda}(t,\sigma)$$

$$= \sum_{s=a+1}^{\infty} \sum_{\lambda=1}^{q} F^*_{sq+\lambda}(t,\sigma) + \sum_{\lambda=b+1}^{q} F^*_{aq+\lambda}(t,\sigma) .$$

i.e.,

$$\gamma_\ell(t,\sigma) = \sum_{s=1}^{\infty} \sum_{\lambda=1}^{q} F^*_{(s+a)q+\lambda}(t,\sigma) + \sum_{\lambda=b+1}^{q} F^*_{aq+\lambda}(t,\sigma) \tag{5.19}$$

Substituting the value of $F^*_{aq+\lambda}(t,\sigma)$ from (5.11), we can write

$$\sum_{\lambda=b+1}^{q} F^*_{aq+\lambda}(t,\sigma)$$

$$= (t^n-\sigma^{-n})^{-aq^2} \sum_{\lambda=b+1}^{q} (t^n-\sigma^{-n})^{-\lambda q}\Big\{1+ \sum_{j=1}^{a} \binom{(a-j)q^2+\lambda q+j-1}{j} \frac{(-\sigma^{-n})^j}{(t^n-\sigma^{-n})^{(1-q^2)j}}\Big\} .$$

If $\sigma \geq \rho^{\ell q+1}$ and $|t| = \rho_1$, it is easy to see that

$$\Big|\sum_{\lambda=b+1}^{q} F^*_{aq+\lambda}(t,\sigma)\Big|$$

$$\leq (q-b)(\rho_1^n-\sigma^{-n})^{-(aq+b+1)q} \Big\{1 + \frac{(\rho_1^n-\sigma^{-n})^{q^2-1}}{\rho^{\ell qn}} \sum_{j=1}^{a} \binom{(a-j)q^2+\lambda q+j-1}{j} \rho^{-jn}\Big\} ,$$

for all n sufficiently large. Since $aq+b+1 =: \ell$, we obtain

$$\sum_{\lambda=b+1}^{q} F^*_{aq+\lambda}(t,\sigma) = O\big(\rho_1^{-\ell qn}\big) \quad \text{as} \quad n \to \infty . \tag{5.20}$$

It remains to estimate the double summation on the right side of (5.19). For this purpose, we set

$$(5.21)\qquad g(\nu,\mu) = \binom{\mu q+(a+1)q^2+\nu-2}{\nu-1} \frac{(-\sigma^{-n})^{\nu-1}}{(t^n-\sigma^{-n})^{\mu q+(a+1)q^2+\nu-1}} .$$

Then using (5.11), we can rewrite

$$(5.22)\qquad \sum_{s=1}^{\infty}\sum_{\lambda=1}^{q} F^*_{(s+a)q+\lambda}(t,\sigma) = I_1 + I_2 ,$$

where

$$\begin{cases} I_1 := \sum_{s=1}^{\infty}\sum_{\lambda=1}^{q}\sum_{j=1}^{s} g(j,(s-j)q+\lambda) , \\ I_2 := \sum_{s=1}^{\infty}\sum_{\lambda=1}^{\infty}\sum_{j=s+1}^{s+q+1} g(j,(s-j)q+\lambda) . \end{cases}$$

Recalling the identity (5.9), we obtain

$$\sum_{s=1}^{\infty}\sum_{\lambda=1}^{\infty} g(s,\lambda)$$

$$= \frac{1}{(t^n-\sigma^{-n})^{(a+1)q^2}} \sum_{\lambda=1}^{\infty} \frac{1}{(t^n-\sigma^{-n})^{\lambda q}} \sum_{s=1}^{\infty} \binom{\lambda q+(a+1)q^2+s-2}{s-1}\left(\frac{-\sigma^{-n}}{t^n-\sigma^{-n}}\right)^{s-1}$$

$$= (t^n-\sigma^{-n})^{-(a+1)q^2} \sum_{\lambda=1}^{\infty} (t^n-\sigma^{-n})^{-\lambda q} \left(1+\frac{\sigma^{-n}}{t^n-\sigma^{-n}}\right)^{-(\lambda q+(a+1)q^2)}$$

so that

$$(5.23)\qquad I_1 = t^{-(a+1)nq^2}(t^{qn}-1)^{-1} = O\left(\rho_1^{-(a+1)nq^2-qn}\right) .$$

Further, we notice that

$$(5.24)\qquad I_2 := \sum_{s=1}^{\infty}\sum_{\lambda=1}^{q}\sum_{j=1}^{q+1} g(j+s,-jq+\lambda)$$

where in view of (5.21)

$$g(j+s,-jq+\lambda)$$

$$= \binom{(-jq+\lambda)q+(a+1)q^2+j+s-2}{(-jq+\lambda)q+(a+1)q^2-1} \frac{(-\sigma^{-n})^{j-1+s}}{(t^n-\sigma^{-n})^{(a+1-j)q^2+\lambda q+j+s-1}} .$$

Since $h(s) := \sum_{\lambda=1}^{q} \sum_{j=1}^{a+1} \binom{(\lambda-jq)q+(a+1)q^2+j+s-2}{(\lambda-jq)q+(a+1)q^2-1}$ is a polynomial in s of degree at most $(a+1)q^2-1$, it follows that for all n sufficiently large, the function $\sum_{s=1}^{\infty} h(s)(t^n-\sigma^{-n})^{-s}$ is analytic for $|t| > 1$ ([5], Lemma 2). Thus, there is a constant c_0 independent of n such that

$$|\sum_{s=1}^{\infty} h(s)(t^n-\sigma^{-n})^{-s}| \leq c_0 .$$

Since $\sigma \geq \rho^{\ell q+1}$ and $|t| = \rho_1$, it follows from (5.24) and (5.25) after some elementary algebra that for sufficiently large n

$$(5.26) \qquad |I_2| \leq c_0 \rho^{-n(\ell q+1)} (\rho_1^n-\sigma^{-n})^{-aq^2-q} .$$

Recall that $\ell q := (aq+b+1)q \leq (a+1)q^2$. Therefore, combining (5.19), (5.20), (5.22) and (5.26), we observe that

$$(5.27) \qquad |\gamma_\ell(t,\sigma)| \leq \frac{c^*}{\rho_1^{n\ell q}}, \text{ for all sufficiently large } n,$$

where c^* is a constant independent of n. Using (5.18)and(5.27), an analysis of the kernels $A_j(t,z)\, K_j(t,\sigma)$, $j = 1,2,3$, shows that

$$\overline{\lim_{n\to\infty}} \left\{ \max_{\substack{|z|=\tau \\ \tau>1}} |R^*_{n+m,n}(z,f) - W^*_{n+m,n}(z,f,\ell)| \right\}^{1/n} \leq \frac{\tau}{\rho_1^{\ell q+1}} .$$

When $\sigma < \rho^{\ell q+1}$, a similar analysis of $\gamma_\ell(t,\sigma)$ and $A_j(t,z)\cdot K_j(t,\sigma)$ gives us

$$\lim_{n\to\infty} \{R^*_{n+m,n}(z,f) - W^*_{n+m,n}(z,f,\ell)\} = 0 ,$$

for all z with $|z| \neq \sigma$.

The sharpness of the result can be seen by considering

$$f(z) = (z-\rho e^{i\theta_0})^{-1} .$$ □

Remark 5.2. Theorems 5.1 and 5.2 are also valid when $q = 1$ and $m = -1$ (see Remark 3.1). Therefore, another result of Saff-Sharma ([18], Theorem 3.1), under the condition $m = -1$, is a special case of Theorem 5.1.

Remark 5.3. If we fix $m = -1$ and let $\sigma \to \infty$ in either of the Theorems 5.1 and 5.2, we get an extension of Rivlin's result (I, (4.2)) in the special case when $m(n) = qn$. This follows from the fact that (cf. (5.2), (5.13)) for all integers $n \geq 1$, $\nu \geq 0$,

$$\lim_{\sigma\to\infty} \tilde{r}_{n-1,n}(z,f,\nu) = \lim_{\sigma\to\infty} r^*_{n-1,n}(z,f,\nu) = p_{n-1,\nu}(z,f),$$

where $p_{n-1,\nu}(z,f)$ is given by (I, (1.1)).

Finally, we point out that the minimization problem (P1) considered in §2 can also be solved for $m < -1$ and $q \geq 1$. We omit the details.

CHAPTER IV

EQUICONVERGENCE WITH PERTURBED NODES AND POLES

§1. INTRODUCTION. Our object in this chapter is to obtain an analogue of the Saff-Sharma extension ([18], Theorem 2.3) of Walsh's equiconvergence theorem when the nodes of interpolation and the poles of the rational interpolants are slightly perturbed. Our methods are slight variants on those of Szabados and Varga ([23], Theorem 2) who obtained similar results for complex interpolating polynomial sequences.

§2 NOTATIONS AND MAIN RESULT. Let $\omega_{j,n}$ $(j = 1,\ldots,n)$ denote the n^{th} roots of unity and let $\mathcal{S}_{\rho,\sigma}$ $(\rho,\sigma > 1)$ denote the class of infinite triangular matrices S whose n^{th} row S^n is given by $S^n = \{\sigma_{k,n}\}_{k=1}^{n}$ where

$$(2.1) \qquad |\sigma_{k,n} - \sigma\omega_{k,n}| < \rho^{-n}, \quad k = 1,\ldots,n \, ; \; n = 1,2,3,\ldots \, .$$

For any fixed integer $m \geq -1$, we introduce another class $\mathcal{F}_{\rho,m}$ of triangular matrices whose n^{th} row $Z^n = \{z_{k,n'}\}_{k=1}^{n'}$, $(n' = n+m+1)$, satisfies the inequality

$$(2.2) \qquad |z_{k,n'} - \omega_{k,n'}| < \rho^{-n}, \quad k = 1,\ldots,n'(=n+m+1) \, ; \; n = 1,2,\ldots$$

When $\sigma_{k,n} - \sigma\omega_{k,n} = 0$, $(k = 1,2,\ldots,n)$, we shall denote the matrix by S^* and when $z_{k,n'} = \omega_{k,n'}$, $(k = 1,2,\ldots,n')$, we shall denote the matrix by Z^*. We shall denote by $\hat{S}$ the matrix whose n^{th} row $\hat{S}^n$ is given by

$$\hat{S}^n = \{\underbrace{0,\ldots,0}_{m+1}, \bar{\sigma}_{1,n}^{-1}, \bar{\sigma}_{2,n}^{-1}, \ldots, \bar{\sigma}_{n,n}^{-1}\}.$$

We shall associate with the rows S^n and Z^n, the monic polynomials

$$(2.3) \qquad \gamma(z,S^n) = \prod_{k=1}^{n} (z-\sigma_{k,n}) \quad \text{and} \quad \gamma(z,Z^n) = \prod_{k=1}^{n'} (z-z_{k,n'}).$$

Similarly, we have the monic polynomials

$$\gamma(z,S^{*n}) = z^n - \sigma^n, \quad \gamma(z,Z^{*n}) = z^{n'}-1, \quad \gamma(z,\hat{S}^{*n}) = z^{m+1}(z^n - \sigma^{-n}).$$

If $f \in A_\rho$, $Z \in \mathcal{F}_{\rho,m}$ and $S \in \mathcal{G}_{\rho,\sigma}$, let $R_{n+m,n}(z,f,Z,S)$ be the rational function of the form

$$(2.4) \qquad R_{n+m,n}(z,f,Z,S) = \frac{B_{n+m,n}(z,f,Z)}{\gamma_n(z,S)}, \quad (B_{n+m,n}(z,f,Z) \in \pi_{n+m}),$$

which interpolates $f(z)$ in the nodes $\{z_{k,n'}\}_{k=1}^{n'}$, the zeros of $\gamma(z,Z^n)$. From this it follows that

$$(2.5) \qquad R_{n+m,n}(z,f,Z,S) = \frac{1}{2\pi i}\int_\Gamma \frac{\gamma(t,S^n)}{\gamma(z,S^n)}\left(\frac{\gamma(t,Z^n)-\gamma(z,Z^n)}{\gamma(t,Z^n)}\right)\frac{f(t)}{t-z}\,dt,$$

where Γ is a circle $|t| = \rho_1$, $1 < \rho_1 < \rho$.

If we replace Z by $\hat{S}$ in (2.4), then from a theorem of Walsh (I, §12, Theorem A) we know that the rational function $R_{n+m,n}(z,f,\hat{S},S)$ is the best L_2-approximant to f on $|z| = 1$ over all rational functions of the form (2.4).

We shall prove

Theorem 2.1. *Let $\sigma > 1$ and let the integer $m \geq -1$ be fixed. If $f \in A_\rho$, $1 < \rho < \infty$, and if S and Z are infinite triangular matrices in $\mathcal{S}_{\rho,\sigma}$ and $\mathcal{F}_{\rho,m}$ respectively, then*

$$(2.6)\qquad \lim_{n\to\infty} \Delta(z,f,Z^n,S^n) = 0 \begin{cases} |z| < \rho^2, & \text{if } \sigma \geq \rho^2, \\ |z| \neq \sigma, & \text{if } \sigma < \rho^2, \end{cases}$$

where

$$(2.7)\qquad \Delta(z,f,Z^n,S^n) := R_{n+m,n}(z,f,Z,S) - R_{n+m,n}(z,f,\hat{S},S).$$

The convergence in (2.6) is uniform and geometric on compact subsets of the region described above.

§3 REPRESENTATION OF $\Delta(z,f,Z^n,S^n)$. The proof of Theorem 2.1 requires some estimates similar to those given by Szabados and Varga in [23]. For this purpose, we write

$$(3.1)\qquad \Delta(z,f,Z^n,S^n) = \sum_{j=1}^{5} \Delta_{n,j}(z,f)$$

where

$$(3.2)\quad \begin{cases} \Delta_{n,1}(z,f) := R_{n+m,n}(z,f,Z,S) - R_{n+m,n}(z,f,Z^*,S), \\ \Delta_{n,2}(z,f) := R_{n+m,n}(z,f,Z^*,S) - R_{n+m,n}(z,f,Z^*,S^*), \\ \Delta_{n,3}(z,f) := R_{n+m,n}(z,f,Z^*,S^*) - R_{n+m,n}(z,f,\hat{S}^*,S^*), \\ \Delta_{n,4}(z,f) := R_{n+m,n}(z,f,\hat{S}^*,S^*) - R_{n+m,n}(z,f,\hat{S}^*,S), \\ \Delta_{n,5}(z,f) := R_{n+m,n}(z,f,\hat{S}^*,S) - R_{n+m,n}(z,f,\hat{S},S). \end{cases}$$

The third difference $\Delta_{n,3}(z,f)$ in (3.2) is the same as that considered by Saff and Sharma ([18], Theorem 2.3).

The lemma given below is based on the formulae (2.4) and (2.5).

<u>Lemma 3.1.</u> <u>If</u> $f \in A_\rho$, <u>then the differences</u> $\Delta_{n,j}$'s <u>defined in (3.2) have the following integral representation:</u>

$$\Delta_{n,1}(z,f) = \frac{1}{2\pi i}\int_\Gamma \frac{\gamma(t,S^n)}{\gamma(z,S^n)}\left(\frac{z^{n+m+1}-1}{t^{n+m+1}-1}\right)\left(\frac{f(t)}{z-t}\right)V_{n,m}(t,z)dt,$$

$$\Delta_{n,2}(z,f) = \frac{1}{2\pi i}\int_\Gamma \frac{t^n-\sigma^n}{z^n-\sigma^n}\left(\frac{t^{n+m+1}-z^{n+m+1}}{t^{n+m+1}-1}\right)\left(\frac{f(t)}{t-z}\right)U_n(t,z)dt ,$$

$$\Delta_{n,4}(z,f) = \frac{1}{2\pi i}\int_\Gamma \frac{t^n-\sigma^n}{\sigma^n-z^n}\left(\frac{\gamma(t,\hat{S}^{*n}) - \gamma(z,\hat{S}^{*n})}{\gamma(t,\hat{S}^{*n})}\right)\left(\frac{f(t)}{t-z}\right)\dot{U}_n(t,z)dt ,$$

$$\Delta_{n,5}(z,f) = \frac{1}{2\pi i}\int_\Gamma \frac{\gamma(t,S^n)}{\gamma(z,S^n)}\left(\frac{\gamma(z,\hat{S}^{*n})}{\gamma(t,\hat{S}^{*n})}\right)\left(\frac{f(t)}{t-z}\right)W_{n,m}(t,z)dt ,$$

<u>where</u>

$$(3.3)\quad \begin{cases} V_{n,m}(t,z) := \left(\dfrac{t^{n+m+1}-1}{z^{n+m+1}-1}\right) \dfrac{\gamma(z,Z^n)}{\gamma(t,Z^n)} - 1, \\ U_n(t,z) := \left(\dfrac{z^n-\sigma^n}{t^n-\sigma^n}\right) \dfrac{\gamma(t,S^n)}{\gamma(z,S^n)} - 1, \\ W_{n,m}(t,z) := \left(\dfrac{t^n-\sigma^{-n}}{z^n-\sigma^{-n}}\right) \dfrac{t^{m+1}\gamma(z,\hat{S}^n)}{z^{m+1}\gamma(t,\hat{S}^n)} - 1. \end{cases}$$

§4. SOME UPPER BOUNDS.

Lemma 4.1. Let $\rho,\sigma > 1$ and let m be an integer ≥ -1. Suppose (i) $1 < |t| < \rho$, if $\sigma \geq \rho$ and (ii) $\sigma < |t| < \rho$, if $\sigma < \rho$. Then we have

$$(4.1)\qquad |V_{n,m}(t,z)| < c_1 n\rho^{-n} \qquad \text{for } |z| > 1,$$

$$(4.2)\qquad \begin{cases} |U_n(t,z)| < c_2 n\rho^{-n}, \\ \qquad\qquad\qquad\qquad \text{for } |z| > 1,\ |z| \neq \sigma \\ |W_{n,m}(t,z)| \leq c_3 n\rho^{-n}, \end{cases}$$

where c_1, c_2, c_3 are positive constants independent of n.

Proof. Using the definition of $\gamma(z,Z^n)$ we can rewrite $V_{n,m}(t,z)$ as

$$V_{n,m}(t,z) = \prod_{k=1}^{n'} \left\{1 + \frac{z_{k,n'} - \omega_{k,n'}}{t - z_{k,n'}}\right\}\left\{1 + \frac{\omega_{k,n'} - z_{k,n'}}{z - \omega_{k,n'}}\right\} - 1$$

where $\omega_{k,n'}$, $(n' = n+m+1)$, are the n'th roots of unity. Set

$$a_{n'} := \max_{1\leq k\leq n'} |z_{k,n'}|, \quad n = 1,2,3 \ldots$$

If we let $|t| = \rho_1$, then for sufficiently large n, $|t| = \rho_1 > a_{n'}$, and $|t-a_{n'}| \geq \rho_1 - a_{n'}$. Also, for $|z| > 1$, we have $|z-\omega_{k,n'}| > |z|-1$. This together with (2.2) gives us

$$|V_{n,m}(t,z)| \leq [(1+ \frac{\rho^{-n}}{\rho_1 - a_{n'}})(1+ \frac{\rho^{-n}}{|z|-1})]^{n'} - 1 .$$

From (2.2) we see that $a_{n'} \to 1$ as $n \to \infty$. If we set $d_0 := \min(\rho_1 - 1, |z|-1)$, then for sufficiently large n, it is easy to see that

$$|V_{n,m}(t,z)| \leq (1+ \frac{\rho^{-n}}{d_0})^{2n'} - 1 \leq \frac{6(n+m+1)}{d_0 \rho^n} .$$

This proves (4.10).

In order to prove (4.2), we observe that

$$U_n(t,z) = \prod_{k=1}^{n} \{1+ \frac{\sigma_{k,n} - \sigma\omega_{k,n}}{z - \sigma_{k,n}}\}\{1+ \frac{\sigma\omega_{k,n} - \sigma_{k,n}}{t - \sigma\omega_{k,n}}\} - 1,$$

and

$$W_{n,m}(t,z) = \prod_{k=1}^{n} \{1+ \frac{\bar{\sigma}_{k,n}^{-1} - \sigma^{-1}\omega_{k,n}}{t - \bar{\sigma}_{k,n}^{-1}}\}\{1+ \frac{\sigma^{-1}\omega_{k,n} - \bar{\sigma}_{k,n}^{-1}}{z - \sigma^{-1}\omega_{k,n}}\} - 1.$$

Following closely the analysis given above for the proof of (4.1) we can deduce the relations (4.2) on using (2.1). □

<u>Lemma 4.2.</u> <u>Under the hypotheses (i) and (ii) of Lemma 4.1, we have the following estimates for all sufficiently large</u> n:

(4.3) $$\left| \frac{\gamma(z,\hat{S}^n)}{\gamma(t,\hat{S}^n)} \right| \leq c_3 \left|\frac{z^n - \sigma^{-n}}{t^n - \sigma^{-n}}\right| \quad \text{for } |z|, |t| > \sigma^{-1},$$

<u>and</u>

$$(4.4)\qquad \left|\frac{\gamma(t,S^n)}{\gamma(z,S^n)}\right| \le c_4 \left|\frac{t^n-\sigma^n}{z^n-\sigma^n}\right| \quad , \text{ for } |z| \;,\; |t| \ne \sigma \;,$$

<u>where the positive constants</u> c_3 <u>and</u> c_4 <u>are independent of</u> n.

<u>Proof</u>. If we write $z - \bar{\sigma}^{-1}_{k,n}$ as

$$z - \bar{\sigma}^{-1}_{k,n} = (z-\sigma^{-1}\omega_{k,n})\left\{1+ \frac{\sigma^{-1}\omega_{k,n} - \bar{\sigma}^{-1}_{k,n}}{z - \sigma^{-1}\omega_{k,n}}\right\}$$

then

$$\frac{\gamma(z,\hat{S}^n)}{z^{m+1}} := \prod_{k=1}^{n} (z-\bar{\sigma}^{-1}_{k,n}) = (z^n-\sigma^{-n}) \prod_{k=1}^{n} \left\{1+ \frac{\sigma^{-1}\omega_{k,n} - \bar{\sigma}^{-1}_{k,n}}{z - \sigma^{-1}\omega_{k,n}}\right\} .$$

Similarly, we have

$$\frac{\gamma(t,\hat{S}^n)}{t^{m+1}} = (t^n-\sigma^{-n}) \prod_{k=1}^{n} \left\{1+ \frac{\sigma^{-1}\omega_{k,n} - \bar{\sigma}^{-1}_{k,n}}{t - \sigma^{-1}\omega_{k,n}}\right\} .$$

From (2.1), it is easy to see that $|\bar{\sigma}^{-1}_{k,n} - \sigma^{-1}\omega_{k,n}| < \sigma^{-1}\rho^{-n}$. Hence

$$\left|\frac{t^{m+1}\gamma(z,\hat{S}^n)}{z^{m+1}\gamma(t,\hat{S}^n)}\right| \le \left|\frac{z^n-\sigma^{-n}}{t^n-\sigma^{-n}}\right| \left\{\frac{1+\sigma^{-1}\rho^{-n}(|z|-\sigma^{-1})^{-1}}{1-\sigma^{-1}\rho^{-n}(|t|+\sigma^{-1})^{-1}}\right\}^n ,$$

where the second factor on the right side approaches 1 as $n \to \infty$. This shows that there exists a positive constant c_3, independent of n, such that

$$\left| \frac{\gamma(z,\hat{S}^n)}{\gamma(t,\hat{S}^n)} \right| \le c_3 \left| \frac{z^n - \sigma^{-n}}{t^n - \sigma^{-n}} \right| , \quad \text{for all sufficiently large } n,$$

which is the desired estimate (4.3).

Similarly, we can prove (4.4) by repeating the above argument for $\gamma(t,S^n)$ and $\gamma(z,S^n)$. □

§5. REGION OF CONVERGENCE OF $\{\Delta_{n,j}(z,f)\}_{j=1}^{5}$. As pointed out earlier, the two sequences $\{\Delta_{n,3}(z,f)\}_1^\infty$ and $\{\Delta^\sigma_{1,n,m}(z,f)\}_1^\infty$ (cf. I, (12.4)) are identical. We shall show in the following proposition that the sequences $\{\Delta_{n,j}(z,f)\}_{n=1}^\infty$, $(j = 1,2,\dots,5)$, have the same region of convergence.

<u>Proposition 5.1.</u> <u>Let</u> $\sigma > 1$, $\rho > 1$ <u>and an integer</u> $m \ge -1$ <u>be given.</u> <u>If</u> $Z \in \mathcal{F}_{\rho,m}$, $S \in \mathcal{G}_{\rho,\sigma}$ <u>and</u> $f \in A_\rho$ <u>then for</u> $j = 1,2,3,4,5$

$$(5.1) \qquad \lim_{n\to\infty} \Delta_{n,j}(z,f) = 0 \quad \begin{cases} |z| < \rho^2, & \text{if } \sigma \ge \rho^2, \\ |z| \ne \sigma, & \text{if } \sigma < \rho^2, \end{cases}$$

<u>where</u> $\Delta_{n,j}(z,f)$ <u>is the same as defined in (3.2).</u> <u>The convergence in (5.1) is uniform and geometric on any compact subset of the regions described above.</u>

<u>Proof.</u> For $j = 3$, (5.1) is known [18]. Next we consider the case when $j = 1$. From Lemma (3.1) we know that

$$\Delta_{n,1}(z,f) = \frac{1}{2\pi i} \int_\Gamma \frac{\gamma(t,S^n)}{\gamma(z,S^n)} \left(\frac{z^{n+m+1}-1}{t^{n+m+1}-1}\right)\left(\frac{f(t)}{z-t}\right) V_{n,m}(t,z)dt$$

where Γ is a circle $|t| = \rho_1$, $1 < \rho_1 < \rho$. An application of (4.1) and (4.4) to the above integral shows that

$$(5.2) \qquad |\Delta_{n,1}(z,f)| \leq \frac{nC}{\rho^n} \left(\frac{|z|^{n+m+1}+1}{\rho_1^{n+m+1}-1}\right) \frac{\rho_1^n + \sigma^n}{|z^n-\sigma^n|} ,$$

for all n sufficiently large. Here C is a positive constant independent of n. If $\sigma \geq \rho^2$ and $\tau \geq \rho$, we get

$$\overline{\lim_{n\to\infty}} \left\{ \max_{|z|=\tau} |\Delta_{n,1}(z,f)|\right\}^{1/n} \leq \frac{\tau}{\rho_1\rho} .$$

Letting $\rho_1 \to \rho$, we conclude that

$$\lim_{n\to\infty} \Delta_{n,1}(z,f) = 0 , \text{ for } |z| < \rho^2 .$$

If $\sigma < \rho^2$, again it follows from (5.2) that

$$\lim_{n\to\infty} \Delta_{n,1}(z,f) = 0 , \text{ for all } z \text{ with } |z| \neq \sigma.$$

This completes the proof for the sequence $\{\Delta_{n,j}(z,f)\}_1^n$.

We omit the proof for the remaining three cases when $j = 2,4$ and 5. □

§6 PROOF OF THEOREM 2.1. From (3.1) we know that

$$\Delta(z,f,Z^n,S^n) = \sum_{j=1}^{5} \Delta_{n,j}(z,f) .$$

Therefore,

$$|\Delta(z,f,Z^n,S^n)| \leq \sum_{j=1}^{5} |\Delta_{n,j}(z,f)| \ .$$

A straightforward calculation on using Proposition 5.1 shows that

$$\lim_{n\to\infty} \Delta(z,f,Z^n,S^n) = 0 \quad \begin{cases} |z| < \rho^2 & , \text{ if } \sigma \geq \rho^2 , \\ |z| \neq \sigma & , \text{ if } \sigma < \rho^2 . \end{cases}$$

This completes the proof. □

CHAPTER V

QUANTITATIVE ESTIMATES AND SHARPNESS RESULTS

§1. INTRODUCTION. In the present chapter, we derive some quantitative estimates for the sequence $\{\Delta^{\sigma}_{\ell,n,m}(z,f)\}_{n=1}^{\infty}$ introduced earlier in (I, (12.9)). We are motivated in our efforts by a recent paper of V. Totik [25] on complex interpolating polynomials. Our object is to make precise some of the results in [18] on the behaviour of $\max\limits_{|z|=R} |\Delta^{\sigma}_{\ell,n,m}(z,f)|^{1/n}$ as $n \to \infty$.

The main results are stated in §2. §3 deals with the representation of $\Delta^{\sigma}_{\ell,n,m}(z,f)$ in terms of the Taylor coefficients of f. The last three sections are devoted to the proof of our main results.

§2. STATEMENT OF NEW RESULTS. Let R be a positive real number different from σ. For $\sigma > \rho$, set (cf. I, (12.9))

$$(2.1) \qquad F_{\ell}(R,\sigma) := \overline{\lim_{n\to\infty}}\left\{\max_{|z|=R} |\Delta^{\sigma}_{\ell,n,m}(z,f)|\right\}^{1/n}$$

and

$$(2.2) \qquad K_{\ell}(R,\sigma,\rho) := \rho^{-\ell} \max(\min\{\frac{R}{\rho}, \frac{\sigma}{\rho}\},1) .$$

Then we have the following result:

<u>Theorem 2.1</u>. <u>Let</u> $1 < \rho < \sigma$ <u>and let</u> $R > 0$ <u>be a real number</u>

different from σ. If $f \in A_\rho$, then

(2.3) $$F_\ell(R,\sigma) = K_\ell(R,\sigma,\rho) ,$$

where $K_\ell(R,\sigma,\rho)$ and $F_\ell(R,\sigma)$ are given by (2.2) and (2.1) respectively.

Remark 2.1. For fixed ρ and σ, the value of the function $K_\ell(R,\sigma,\rho)$ can be described as

(2.4) $$K_\ell(R,\sigma,\rho) = \begin{cases} \sigma\rho^{-\ell-1} , & \text{if } R > \sigma , \\ R\rho^{-\ell-1} , & \text{if } \rho \leq R < \sigma , \\ \rho^{-\ell} , & \text{if } 0 < R < \rho . \end{cases}$$

The relation (2.3) does not hold when $\sigma = \rho$. For this, consider the following example:

Example 2.1. Let $f(z) = (z-\rho)^{-1}$. Then (cf. [18], (3.15)),

$$\Delta^\sigma_{\ell,n,m}(z,f) = \frac{(\rho^n-\sigma^n)\{\alpha_{n,m}(z)\beta_{n,m}(\rho) - \alpha_{n,m}(\rho)\beta_{n,m}(z)\}}{(z^n-\sigma^n)(z-\rho)(\rho^{n+m+1}-1)\alpha_{n,m}(\rho)} \left(\frac{\alpha_{n,m}(\rho)}{\beta_{n,m}(\rho)}\right)^\ell$$

which is identically zero on the set $\{z: |z| \neq \sigma\}$ if $\sigma = \rho$.

Also, for $R = 0$, Theorem 2.1 is no longer true. We shall justify this statement in the next section (see Remark 3.2).

If ω is any fixed primitive root of unity, then $z = \sigma\omega$ is a singular point of the function $\Delta^\sigma_{\ell,n,m}(z,f)$ for infinitely many n's (see Example 2.1). Because of this, we have excluded the case $R = \sigma$ in the relation (2.3).

It may be noted that $K_\ell(R,\sigma,\rho) \neq K_\ell(R,\sigma,\rho')$ if $\rho \neq \rho'$.

Thus as in ([25], Cor. 1), Theorem 2.1 gives

Corollary 2.1. Let σ and ρ' be fixed numbers with $\sigma > \rho' > 1$ and let $f(z)$ be an analytic function in $|z| < \rho'$. If, for any fixed integer $\ell \geq 1$ and any real number ρ, $(\rho' \leq \rho < \sigma)$, the relation (2.3) holds for some $R > 0$, then $f \in A_\rho$.

Our next concern is to study the pointwise behaviour of $\Delta^{\sigma}_{\ell,n,m}(z,f)$ in the complex plane when $\sigma \geq \rho^2$. If we set

$$Y_1 := \{z: \rho < |z|,\ |z| \neq \sigma\} \quad \text{and} \quad Y_2 = \{z : |z| < \rho\},$$

then we shall prove

Theorem 2.2. Let $\ell \geq 1$, $m \geq -1$ be integers and $\sigma \geq \rho^2$. For each $f \in A_\rho$, $1 < \rho < \infty$, we have

$$(2.5) \qquad \overline{\lim_{n\to\infty}} |\Delta^{\sigma}_{\ell,n,m}(z,f)|^{1/n} = K_\ell(|z|,\sigma,\rho), \qquad z \in Y_j,$$

except at most at $\ell - [\frac{j}{2}]$ points in Y_j, $(j=1,2)$, where $K_\ell(|z|,\sigma,\rho)$ is given by (2.4).

Remark 2.2. From the above theorem, we note that if $\sigma > \rho^{\ell+1}$, then the sequence $\{\Delta^{\sigma}_{\ell,n,m}(z,f)\}_{n=1}^{\infty}$, $f \in A_\rho$, cannot be bounded at more than ℓ points in the region $Z = \{z : |z| > \rho^{\ell+1}, |z| \neq \sigma\}$. This gives an analogue of a result of Saff and Varga [19] on the sharpness of some equiconvergence results for interpolating polynomials.

The next result shows that in some sense Theorem 2.2 cannot be improved.

<u>Theorem 2.3. Let</u> $\rho > 1$, $\sigma \geq \rho^2$ <u>and let integers</u> $\ell \geq 1$ and $m \geq -1$ <u>be fixed. Given any set</u> $\{z_k\}$ <u>of</u> $\ell - [\frac{j}{2}]$ <u>distinct points in the region</u> Y_j , $(j=1,2)$, <u>there exist rational functions</u> $f_j \in A_\rho$ $(j=1,2)$ <u>for which</u>

$$\overline{\lim_{n\to\infty}} |\Delta^{\sigma}_{\ell,n,m}(z_k,f_j)|^{1/n} < K_\ell(|z_k|,\sigma,\rho), \qquad (j=1,2)$$

<u>for every</u> $k = 1,2,\ldots,\ell-[\frac{j}{2}]$. ○

Finally, we remark that Theorems 2.1-2.3 hold when $m < -1$.

§3. REPRESENTATION OF $\Delta^{\sigma}_{\ell,n,m}(z,f)$. The proofs of the above theorems will be based on a representation of $\Delta^{\sigma}_{\ell,n,m}(z,f)$ in terms of the Taylor coefficients of f which is given by (3.15) in Lemma 3.2.

In order to establish (3.15), we recall that (cf. [18], Cor. 3.2)

$$(3.1) \qquad (z^n-\sigma^n)\Delta^{\sigma}_{\ell,n,m}(z,f) = \sum_{\nu=\ell}^{\infty} P_{n+m,n}(z,f,\nu)$$

where

$$(3.2) \qquad P_{n+m,n}(z,f,\nu) = \frac{1}{2\pi i}\int_{\Gamma} \frac{(t^n-\sigma^n)f(t)K_{n,m}(z,t)}{\alpha_{n,m}(t)\beta_{n,m}(t)}\left(\frac{\alpha_{n,m}(t)}{\beta_{n,m}(t)}\right)^{\nu} dt$$

with

$$(3.3) \qquad K_{n,m}(t,z) := \frac{\alpha_{n,m}(z)\beta_{n,m}(t) - \alpha_{n,m}(t)\beta_{n,m}(z)}{t-z} .$$

Here Γ is a circle $|t| = \rho_1$, $1 < \rho_1 < \rho$. From the definitions of

$\beta_{n,m}(t)$ and $\alpha_{n,m}(t)$ in (I, (12.5)), we easily see that

$$(3.4)\qquad \begin{cases} (\beta_{n,m}(t))^{-\nu-1} = t^{-(\nu+1)(n+m+1)} \sum_{s=0}^{\infty} \binom{\nu+s}{s} \sigma^{-ns} t^{-ns} , \\ (\alpha_{n,m}(t))^{\nu-1} = \sum_{j=0}^{\nu-1} (-1)^j \binom{\nu-1}{j} \sigma^{-nj} t^{(m+1)j} , \\ K_{n,m}(t,z) = t^{n+m} \sum_{k=0}^{n+m} b_{n,k}(z,\sigma) t^{-k} z^k , \end{cases}$$

where

$$(3.5)\qquad b_{n,k}(z,\sigma) := \begin{cases} 1 - \sigma^{-n} z^{m+1} , & 0 \le k \le n-1 , \\ 1 - \sigma^{-n} z^{-n} , & n \le k \le n+m . \end{cases}$$

Using (3.2)-(3.5), we see after some elementary computation that

$$(3.6)\qquad P_{n+m,n}(z,f,\nu) = \sum_{j=0}^{\nu-1} (-1)^j \binom{\nu-1}{j} \sigma^{-nj} \sum_{s=0}^{\infty} \binom{\nu+s}{s} \sigma^{-ns} S_{n+m}(z,j,s) ,$$

where

$$(3.7)\qquad S_{n+m}(z,j,s) = \sum_{k=0}^{n+m} \left(a_{N(k)-n} - \sigma^n a_{N(k)}\right) b_{n,k}(z,\sigma) z^k ,$$

with $N(k) := \nu(n+m+1) - j(m+1) + ns + k$. Setting

$$(3.8)\qquad I_{\nu,k,n} \equiv I_{\nu,k,n}(f) := \sum_{j=0}^{\nu-1} (-1)^j \binom{\nu-1}{j} \sigma^{-nj} \sum_{s=0}^{\infty} \binom{\nu+s}{s} \sigma^{-ns} a_{N(k)},$$

($\nu = 1,2,\ldots$), it follows from (3.5)-(3.7) that

$$(3.9)\qquad \begin{aligned} P_{n+m,n}(z,f,\nu) = {} & \sum_{k=0}^{n+m} (I_{\nu,k-n,n} - \sigma^n I_{\nu,k,n}) z^k \\ & - \sigma^{-n} \sum_{k=0}^{n-1} (I_{\nu,k-n,n} - \sigma^n I_{\nu,k,n}) z^{k+m+1} \\ & - \sigma^{-n} \sum_{k=0}^{m} (I_{\nu,k,n} - \sigma^n I_{\nu,k+n,n}) z^k . \end{aligned}$$

Lemma 3.1. For a given integer $\ell \geq 1$ and for $1 < \rho_1 < \rho$, the following estimates hold

$$(3.10)\quad \begin{cases} I_{\ell,k,n} = a_{\ell(n+m+1)+k} + O(\sigma^{-n}\rho_1^{-\ell n-k}), \\ \sum_{\nu=\ell+1}^{\infty} I_{\nu,k,n} = a_{(\ell+1)(n+m+1)+k} + O(\sigma^{-n}\rho_1^{-(\ell+1)n-k}), \end{cases}$$

$(0 \leq k \leq n+m)$, where the constants depend on σ, ρ, ρ_1, f and m.

Proof. (i) From the Cauchy integral formula and the definition of $N(k)$ we can rewrite (3.8) for $\nu = \ell$, and after some simplification we obtain

$$(3.11)\qquad I_{\ell,k,n} = \frac{1}{2\pi i}\int_\Gamma \frac{f(t)}{t^{\ell(n+m+1)+k+1}}\left(\frac{(1-\sigma^{-n}t^{m+1})^{\ell-1}}{(1-\sigma^{-n}t^{-n})^{\ell+1}}\right)dt .$$

Since $\sigma > 1$ and $|t| = \rho_1 > 1$, we have

$$\frac{(1-\sigma^{-n}t^{m+1})^{\ell-1}}{(1-\sigma^{-n}t^{-n})^{\ell+1}} = 1 + \frac{(1-\sigma^{-n}t^{m+1})^{\ell-1} - (1-\sigma^{-n}t^{-n})^{\ell+1}}{(1-\sigma^{-n}t^{-n})^{\ell+1}} = 1 + O(\sigma^{-n}),$$

so that from (3.11), we obtain the first relation in (3.10).

(ii) Summing up the relation (3.8) over ν from $\ell+1$ to ∞ and using (3.11), we obtain

$$\sum_{\nu=\ell+1}^{\infty} I_{\nu,k,n} = \frac{1}{2\pi i}\int_\Gamma \frac{f(t)}{t^{(\ell+1)(n+m+1)+k+1}}\left(\frac{(1-\sigma^{-n}t^{m+1})^{\ell}}{(1-\sigma^{-n}t^{-n})^{\ell+2}}\right)\frac{t^{m+1}(t^n-\sigma^n)}{t^{n+m+1}-1}\,dt .$$

With an argument similar to the above one, we can easily derive the second relation in (3.10), which completes the proof. □

Lemma 3.2. Let $\rho > \rho_1 > 1$ and let ℓ be a positive integer. If $\sigma > \rho$, then (cf. (3.9))

$$P_{n+m,n}(z,f,\ell) = \sum_{k=0}^{n+m} (a_{\ell(n+m+1)-n+k} - \sigma^n a_{\ell(n+m+1)+k}) z^k + O\left(\rho_1^{-\ell n} \sum_{k=0}^{n+m} \left(\frac{|z|}{\rho_1}\right)^k\right) . \tag{3.12}$$

Proof. From Lemma 3.1, we get

$$I_{\ell,k-n,n} - \sigma^n I_{\ell,k,n} = a_{\ell(n+m+1)-n+k} - \sigma^n a_{\ell(n+m+1)+k} + O(\rho_1^{-(\ell-1)n-k}\sigma^{-n}) + O(\rho_1^{-\ell n-k}) . \tag{3.13}$$

Since $\overline{\lim_{r\to\infty}} |a_r|^{1/r} = \rho^{-1}$, we have $|a_r| = O(\rho_1^{-r})$, $1 < \rho_1 < \rho$. If $\sigma > \rho$, it can be deduced easily from (3.11) that

$$I_{\ell,k-n,n} - \sigma^n I_{\ell,k,n} = O(\sigma^n \rho_1^{-\ell n-k}) . \tag{3.14}$$

Thus (3.12) follows from (3.9) and (3.13). □

Now we can represent $\Delta^{\sigma}_{\ell,n,m}(z,f)$ explicitly in terms of the Taylor coefficients of $f(z)$ as below:

Lemma 3.3. Let $m \geq -1$ and $\ell \geq 1$ be fixed integers and let $f \in A_\rho$, $1 < \rho < \infty$. If $\sigma > \rho$, then for every $\varepsilon > 0$, the following relation holds:

$$(z^n - \sigma^n)\Delta^{\sigma}_{\ell,n,m}(z,f) = \sum_{k=0}^{n+m} \{a_{\ell(n+m+1)-n+k} - \sigma^n a_{\ell(n+m+1)+k}\} z^k + G_n(\rho_1,\sigma,z) , \tag{3.15}$$

where $\rho_1 = \rho - \varepsilon$ and

(3.16) $$G_n(\rho_1,\sigma,z) = O\{\sigma^n \rho_1^{-(\ell+1)n} \sum_{k=0}^{n+m} (\frac{|z|}{\rho_1})^k\} .$$

Proof. We can rewrite (3.1) as

(3.17) $$(z^n - \sigma^n)\Delta^\sigma_{\ell,n,m}(z,f) = P_{n+m,n}(z,f,\ell) + \sum_{\nu=\ell+1}^{\infty} P_{n+m,n}(z,f,\nu) .$$

If we sum up the relation (3.9) over ν from $\ell+1$ to ∞, and then use the second formula in (3.10), we obtain after some computation

(3.18) $$\sum_{\nu=\ell+1}^{\infty} P_{n+m,n}(z,f,\nu) = O(\sigma^n \rho_1^{-(\ell+1)n} \sum_{k=0}^{n+m} (\frac{|z|}{\rho_1})^k) ,$$

which dominates $O(\rho_1^{-\ell n} \sum_{k=0}^{n+m} (\frac{|z|}{\rho_1})^k)$ when $\sigma > \rho$. Hence (3.15) follows from (3.17), (3.12) and (3.18). □

Remark 3.2. We pointed out in §2 that, in general, Theorem 2.1 is not true for $R = |z| = 0$. For this, notice that (3.15) and (3.16) give

(3.19) $$-\sigma^n \Delta^\sigma_{\ell,n+m}(0,f) = a_{\ell(n+m+1)-n} - \sigma^n a_{\ell(n+m+1)} + O\{\sigma^n \rho_1^{-(\ell+1)n}\} .$$

It may happen for some $f \in A_\rho$ that $a_{\ell(n+m+1)-n} = a_{\ell(n+m+1)} = 0$ for all n, in which case $F_\ell(0,\sigma) \leq \rho^{-(\ell+1)}$.

The proof of Theorem 2.2 essentially depends on the representation (3.21) for the function $H_n(z)$ given by

(3.20) $$H_n(z) := \sigma\Delta^\sigma_{\ell,n,m}(z,f) - \left(\frac{z^{n+1} - \sigma^{n+1}}{z^n - \sigma^n}\right) z^\ell \Delta^\sigma_{\ell,n+1,m}(z,f).$$

Lemma 3.4. Let $\ell \geq 1$, $1 < \rho_1 < \rho$ and $\sigma \geq \rho^2$. If $f \in A_\rho$, then

$$(3.21) \qquad (z^n - \sigma^n) H_n(z) = -\sigma^{n+1} \sum_{k=0}^{\ell-1} a_{\ell(n+m+1)+k} z^k$$

$$+ \sigma^{n+1} \sum_{k=0}^{\ell} a_{(\ell+1)(n+m+1)+k} z^k + G_{n+1}(\rho_1, \sigma, z),$$

where $G_n(\rho_1, \sigma, z)$ is given by (3.16).

Proof. From (3.15), after using the fact $a_{\ell(n+m+1)-n+k} = O(\rho_1^{-(\ell-1)n-k})$, $0 \leq k \leq n+m$, we can write

$$(z^n - \sigma^n)\Delta^\sigma_{\ell,n,m}(z,f) = -\sigma^n \sum_{k=0}^{n+m} a_{\ell(n+m+1)+k} z^k + O\{\rho_1^{-(\ell-1)n} \sum_{k=0}^{n+m} (\frac{|z|}{\rho_1})^k\}$$

$$+ O\{\sigma^n \rho_1^{-(\ell+1)n} \sum_{k=0}^{n+m} (\frac{|z|}{\rho_1})^k\}.$$

Since $\sigma \geq \rho^2$, it follows that

$$(3.22) \qquad (z^n - \sigma^n)\Delta^\sigma_{\ell,n,m}(z,f) = -\sigma^n \sum_{k=0}^{n+m} a_{\ell(n+m+1)+k} z^k + O\{\sigma^n \rho_1^{-(\ell+1)n} \sum_{k=0}^{n+m} (\frac{|z|}{\rho_1})^k\}.$$

Similarly,

$$(z^{n+1}-\sigma^{n+1})\Delta^{\sigma}_{\ell,n+1,m}(z,f) = -\sigma^{n+1}\sum_{k=0}^{n+m+1} a_{\ell(n+m+2)+k}z^k$$

(3.23)

$$+ O\{\sigma^n\rho_1^{-(\ell+1)n}\sum_{k=0}^{n+m+1}(\frac{|z|}{\rho_1})^k\} .$$

If we multiply (3.22) and (3.23) by σ and z^{ℓ} respectively, and subtract the resulting equations, then from (3.20), after simple calculations we obtain

$$(z^n-\sigma^n)H_n(z) = -\sigma^{n+1}\sum_{k=0}^{n+m} a_{\ell(n+m+1)+k}\, z^k + \sigma^{n+1}\sum_{k=\ell}^{n+m+\ell+1} a_{\ell(n+m+1)+k}\, z^k$$

(3.24)

$$+ G_{n+1}(\rho_1,\sigma,z),$$

where $G_n(\rho_1,\sigma,z)$ is given by (3.16). From (3.24), we easily derive (3.21) .

§4. PROOF OF THEOREM 2.1. Since $\sigma > \rho$, $\Delta^{\sigma}_{\ell,n,m}(z,f)$ can be estimated from (3.15) to yield

$$|\Delta^{\sigma}_{\ell,n,m}(z,f)| \leq \frac{C\,\sigma^n\rho_1^{-\ell n}}{|z^n-\sigma^n|}\sum_{k=0}^{n+m}(\frac{|z|}{\rho_1})^k + \frac{G_n(\rho_1,\sigma,|z|)}{|z^n-\sigma^n|} ,$$

where C is a positive constant independent of n. If we let $|z| = R$, then (3.16) yields

$$|\Delta^{\sigma}_{\ell,n,m}(z,f)| \leq \frac{C\sigma^n\rho_1^{-\ell n}}{|R^n-\sigma^n|}\sum_{k=0}^{n+m}(\frac{R}{\rho_1})^k + \begin{cases} O\left(\frac{\sigma^n}{|R^n-\sigma^n|}\,(\frac{R}{\rho_1^{\ell+2}})^n\right), & \text{if } R \geq \rho , \\ O\left(\frac{\sigma^n}{|R^n-\sigma^n|}(\frac{1}{\rho_1})^{(\ell+1)n}\right), & \text{if } 0 < R < \rho . \end{cases}$$

A straightforward analysis now gives us

$$\max_{|z|=R} |\Delta^{\sigma}_{\ell,n,m}(z,f)| \le C \begin{cases} \sigma^n \rho_1^{-(\ell+1)n} & , \text{ if } R > \sigma , \\ R^n \rho_1^{-(\ell+1)n} & , \text{ if } \rho \le R < \sigma , \\ \rho_1^{-\ell n} & , \text{ if } 0 < R < \rho . \end{cases}$$

Since $\varepsilon > 0$ is arbitrary and $\rho_1 = \rho - \varepsilon$, we obtain (cf. (2.1))

$$F_\ell(R,\sigma) \le \begin{cases} \sigma\rho^{-(\ell+1)} & , R > \sigma , \\ R\rho^{-(\ell+1)} & , \rho \le R < \sigma , \\ \rho^{-\ell} & , 0 < R < \rho , \end{cases}$$

that is,

(4.1) $$F_\ell(R,\sigma) \le K_\ell(R,\sigma,\rho) \quad , \quad R > 0 \; , \; R \ne \sigma .$$

In order to prove the reverse inequality, we shall consider two cases 1) $R \ge \rho$ and 2) $0 < R < \rho$. Let $\varepsilon > 0$ be so small that

(4.2) $$\rho^{\ell+1} < (\rho-\varepsilon)^{\ell+2} =: \rho_1^{\ell+2} .$$

<u>Case 1.</u> ($R \ge \rho$). Given integers $\ell \ge 1$ and $m \ge -1$, we set for any integer q, $n = [\frac{q}{\ell+1}] - m$. Then q can be expressed as $q = \ell(n+m+1) + k_1$, $n + m - \ell \le k_1 \le n + m$. This shows that $a_q \ne 0$ for some k_1 and for infinitely many n. If we divide both sides of (3.15) by z^{k_1+1} and then integrate over $|z| = R$, we see on using (3.16) and Cauchy's theorem, that

$$\frac{1}{2\pi i} \int_{|z|=R} \frac{(z^n-\sigma^n)\Delta^{\sigma}_{\ell,n,m}(z,f)}{z^{k_1+1}} dz$$

(4.3)

$$= a_{\ell(n+m+1)-n+k_1} - \sigma^n a_{\ell(n+m+1)+k_1} + O\left(\frac{\sigma^n}{R^{k_1}} \left(\frac{R}{\rho_1^{\ell+2}}\right)^n\right) .$$

Since $|z^n - \sigma^n| \le R^n + \sigma^n$ for $|z| = R$, it follows from the definition (2.1) that

$$\left|\int_{|z|=R} \frac{(z^n-\sigma^n)\Delta^{\sigma}_{\ell,n,m}(z,f)}{z^{k_1+1}(R^n+\sigma^n)}\,dz\right| \le \frac{1}{R^{k_1+1}}\int_{|z|=R}\left\{\max_{|z|=R}|\Delta^{\sigma}_{\ell,n,m}(z,f)|\right\}|dz|$$
$$\le cR^{-k_1}(F_\ell(R,\sigma)+\varepsilon)^n .$$

Notice that $k_1/n \to 1$ so that $q/n \to \ell+1$, as $n \to \infty$. On dividing both sides of (4.3) by $(R^n+\sigma^n)$ and taking n^{th} roots, we get

$$\frac{1}{R}(F_\ell(R,\sigma)+\varepsilon) \ge \overline{\lim_{n\to\infty}}\left|\frac{a_{q-n}-\sigma^n a_q}{R^n+\sigma^n} + O\left\{\frac{\sigma^n}{(R^n+\sigma^n)R^{k_1}}\left(\frac{R}{\rho_1^{\ell+2}}\right)^n\right\}\right|^{1/n}$$
$$= \overline{\lim_{n\to\infty}}\left|\frac{\sigma^n}{R^n+\sigma^n}a_q\right|^{1/n} .$$

Since $\rho_1^{-(\ell+2)} < \rho^{-(\ell+1)} = \overline{\lim_{n\to\infty}}|a_q|^{1/n}$ and

$$\lim_{n\to\infty}\left\{\frac{\sigma^n}{R^n+\sigma^n}\right\}^{1/n} = \begin{cases} \frac{\sigma}{R} & \text{if } R > \sigma, \\ 1 & \text{if } R < \sigma, \end{cases}$$

we deduce that

$$\frac{1}{R}(F_\ell(R,\sigma)+\varepsilon) \ge \begin{cases} \frac{\sigma}{R}\rho^{-(\ell+1)}, & \text{if } R > \sigma, \\ \rho^{-(\ell+1)}, & \text{if } R < \sigma . \end{cases}$$

In other words,

(4.4) $$F_\ell(R,\sigma) \ge \rho^{-\ell}\min\left\{\frac{\sigma}{\rho}, \frac{R}{\rho}\right\} = K_\ell(R,\sigma,\rho), \quad R \ge \rho.$$

<u>Case 2</u>. $(0 < R < \rho)$. When $0 < R < \rho$, we set for any integer q $n = [\frac{q}{\ell}] - m - 1$, i.e., $q = \ell(n+m+1) + k_2$, $0 \le k_2 \le \ell-1$. Here we

observe that $k_2/n \to 0$ and $q/n \to \ell$, as $n \to \infty$. On following the method used above after (4.3) and taking into account the estimate of $G_n(\rho,\sigma,z)$ when $|z| = R < \rho$, we conclude that

$F_\ell(R,\sigma) + \epsilon \geq \overline{\lim_{n\to\infty}} \left|\frac{\sigma^n}{R^n+\sigma^n} a_q\right|^{1/n} = \rho^{-\ell}$. Thus we have

$$F_\ell(R,\sigma) \geq K_\ell(R,\sigma,\rho) \quad (4.5)$$

From (4.1), (4.4) and (4.5), we obtain the relation (2.3).

§5. PROOF OF THEOREM 2.2. First we remark that if for some z_0 $\overline{\lim_{n\to\infty}} |\Delta^\sigma_{\ell,n,m}(z_0,f)|^{1/n} < K_\ell(|z_0|,\sigma,\rho)$, then it follows from (3.20) that

$$\overline{\lim_{n\to\infty}} |H_n(z_0)|^{1/n} < K_\ell(|z_0|,\sigma,\rho). \quad (5.1)$$

We shall show that there can not be more than ℓ (or $\ell-1$) points in Y_1 (or Y_2) for which (2.5) fails. The proof by contradiction follows the line of proof of V. Totik [25] (cf. Saff and Varga [19]) and is by contradiction.

Case 1 ($|z| > \rho$, $|z| \neq \sigma$). Using (3.16), we can rewrite (3.21) as

$$(z^n-\sigma^n)H_n(z) = \sigma^{n+1} \sum_{k=0}^{\ell} a_{(\ell+1)(n+m+1)+k}\, z^{k+n+m+1} + O((\sigma\rho_1^{-\ell})^n)$$

$$+ O((q|z|\rho_1^{-(\ell+2)})^n).$$

Next choose $\epsilon > 0$ satisfying (4.2). Then there exists $\eta > 0$ such that

$$H_n(z) = \frac{\sigma^{n+1}}{z^n - \sigma^n} \sum_{k=0}^{\ell} a_{(\ell+1)(n+m+1)+k}\, z^{k+n+m+1}$$

(5.2)

$$+ \begin{cases} O\left((\sigma\rho^{-(\ell+1)} - \eta)^n\right) , & \text{if } |z| < \sigma , \\ O\left((|z|\rho^{-(\ell+1)} - \eta)^n\right) , & \text{if } \rho < |z| < \vartheta . \end{cases}$$

Assume that $\{z_j\}_{j=1}^{\ell+1}$ are $\ell+1$ distinct points in the region Y_1 where (2.5) fails, i.e.,

$$\overline{\lim_{n\to\infty}} |\Delta^{\sigma}_{\ell,n,m}(z_j,f)|^{1/n} < K_{\ell}(|z_j|,\sigma,\rho) \quad , \; 1 \le j \le \ell+1 .$$

Without loss of generality, we may assume that $|z_j| > \sigma$ for $j = 1,\ldots,\lambda$ and $|z_j| < \sigma$ for $j = \lambda+1,\ldots,\ell+1$. Then, from (5.2), we have

$$\overline{\lim_{n\to\infty}} |H_n(z_j)|^{1/n} < \begin{cases} \sigma\rho^{-(\ell+1)} , & \text{if } 1 \le j \le \lambda , \\ |z_j|\rho^{-(\ell+1)} , & \text{if } \lambda+1 \le j \le \ell+1 . \end{cases}$$

This together with (5.2) shows that there are numbers $\eta_1 > 0$ and $C \ge 1$ such that for all $n \ge 1$,

$$(5.3) \qquad |\beta_{j,n}| < \begin{cases} C(\sigma\rho^{-(\ell+1)} - \eta_1)^n , & \text{if } 1 \le j \le \lambda , \\ C(|z_j|\rho^{-(\ell+1)} - \eta_1)^n , & \text{if } \lambda+1 \le j \le \ell+1 , \end{cases}$$

where

$$(5.4) \qquad \sum_{k=0}^{\ell} a_{(\ell+1)(n+m+1)+k}\, z_j^k = \frac{z_j^n - \sigma^n}{\sigma^n z_j^{n+m+1}} \beta_{j,n} \quad , \quad j = 1,2,\ldots,\ell+1 .$$

The coefficient matrix in the above system of $\ell+1$ equations is Vandermondian which is nonsingular since all the z_j's are distinct. Solving the system (5.4) for a_{Λ_n+k}, where $\Lambda_n := (\ell+1)(n+m+1)$, we obtain

$$a_{\Lambda_n+k} = \sum_{j=1}^{\ell+1} c_{j,k} \frac{z_j^n - \sigma^n}{\sigma^n z_j^{n+m+1}} \beta_{j,n} , \qquad k = 0,1,\ldots,\ell ,$$

where $c_{j,k}$ are constants independent of n. Thus from (5.3), we have

$$\overline{\lim_{n\to\infty}} |a_{\Lambda_n+k}|^{1/(\Lambda_n+k)} \le \max(\xi_1,\xi_2) , \quad 0 \le k \le \ell ,$$

where

$$\begin{cases} \xi_1 = \overline{\lim_{n\to\infty}} \{ \max\limits_{1\le j\le\lambda} |\frac{z_j^n - \sigma^n}{\sigma^n z_j^n} (\sigma\rho^{-(\ell+1)} - \eta_1)^n| \}^{1/[\Lambda_n+k]} < 1/\rho , \\ \xi_2 = \overline{\lim_{n\to\infty}} \{ \max\limits_{\lambda+1\le j\le\ell+1} |\frac{z_j^n - \sigma^n}{\sigma^n z_j^n} (|z_j|\rho^{-(\ell+1)} - \eta_1)^n| \}^{1/[\Lambda_n+k]} < 1/\rho , \end{cases}$$

that is,

$$\overline{\lim_{n\to\infty}} |a_{(\ell+1)(n+m+1)+k}|^{1/[(\ell+1)(n+m+1)+k]} < 1/\rho$$

independently of $0 \le k \le \ell$. This contradicts the fact that $\overline{\lim_{n\to\infty}} |a_n|^{1/n} = \rho^{-1}$. Hence the relation (2.6) holds in the region Y_1 for all but at most ℓ points.

Case 2 ($|z| < \rho$). On using (3.16), we can rewrite (3.21) for $|z| < \rho$ as follows:

$$(z^n - \sigma^n)H_n(z) = -\sigma^{n+1} \sum_{k=0}^{\ell-1} a_{\ell(n+m+1)+k} z^k + O((\sigma\rho_1^{-(\ell+1)})^n)$$
$$+ O((|z|\rho_1^{-(\ell+1)})^n) .$$

Choosing $\rho_1 < \rho$ so close to ρ that $\rho_1^{\ell+1} > \rho^\ell$, we get

$$H_n(z) = O((\rho^{-\ell}-\eta)^n) - \frac{\sigma^{n+1}}{z^n - \sigma^n} \sum_{k=0}^{\ell-1} a_{\ell(n+m+1)+k} z^k ,$$

where η is a sufficiently small positive number. If we assume that

$$\overline{\lim_{n\to\infty}} |\Delta^{\sigma}_{\ell,n,m}(z_j,f)|^{1/n} < \rho^{-\ell} , \quad 1 \le j \le \ell,$$

where $\{z_j\}_1^\ell$ are ℓ distinct points in the region Y_2 we arrive at a contradiction on following the procedure used for Case 1.

This completes the proof of Theorem 2.2. □

§6. PROOF OF THEOREM 2.3. (i) We shall first prove the theorem when the set $\{z_j\}$ lies in Y_1. Let $z_1,\ldots,z_\ell$ be ℓ distinct points with $\sigma \ne |z_j| > \rho$, $1 \le j \le \ell$. Then the system of $\ell+1$ equations

$$(6.1) \qquad \sum_{k=0}^{\ell} \mu_k z_j^k = 0 , \quad 1 \le j \le \ell$$

with $\mu_\ell := 1$ has a unique solution $\mu_0,\ldots,\mu_{\ell-1}$. Set

$$(6.2) \qquad f_1(z) := \Big(\sum_{k=0}^{\ell} \mu_k z^k\Big)\Big(1-\big(\tfrac{z}{\rho}\big)^{\ell+1}\Big) = \sum_0^\infty a_j z^j \quad \text{(say)}.$$

Then f_1 is a rational function with $\ell+1$ poles on $|z| = \rho$ which implies that $f_1 \in A_\rho$. Since $\big(1-(\frac{z}{\rho})^{\ell+1}\big)^{-1} = \sum_{q=0}^{\infty} (\frac{z}{\rho})^{(\ell+1)q}$, we find from (6.2) that

$$a_{(\ell+1)q+k} = \mu_k \rho^{-(\ell+1)q}, \quad 0 \le k \le \ell ; \; q = 0,1,2,\ldots,$$

Thus, from (6.1), we have

$$(6.3)\qquad \sum_{k=0}^{\ell} a_{(\ell+1)q+k}\, z_j^k = \sum_{k=0}^{\ell} \mu_k \rho^{-(\ell+1)q} z_j^k = 0\ ,$$
$$1 \le j \le \ell\ ,\ q = 0,1,2,\ldots,\ .$$

For any integer n, we can determine integers r and s so that $\ell n+s = (\ell+1)(r-m-1)$, where $0 \le s \le \ell$. More precisely, $r = -\left[\frac{-\ell n}{\ell+1}\right]+m+1$ (or $r = \left[\frac{\ell n}{\ell+1}\right]+m+2$, unless $s \equiv 0 \pmod{\ell+1}$), where $[x]$ denotes the integer part of x. That is, $\ell(n+m+1) + s + m + 1 = (\ell+1)r$, $0 \le s \le \ell$. Consider the following decomposition:

$$\sum_{k=0}^{n+m} a_{\ell(n+m+1)+k}\, z_j^k = \sum_{k=0}^{s+m} a_{\ell(n+m+1)+k}\, z_j^k$$
$$+ \sum_{q=r}^{n+m} z_j^{(\ell+1)q-\ell(n+m+1)} \sum_{k=0}^{\ell} a_{(\ell+1)q+k}\, z_j^k\ ,$$

in which the second term on the right side vanishes by (6.3). Therefore,

$$(6.4)\qquad \sum_{k=0}^{n+m} a_{\ell(n+m+1)+k}\, z_j^k = \sum_{k=0}^{s+m} a_{\ell(n+m+1)+k}\, z_j^k = O(\rho^{-\ell n}),\quad 1 \le j \le \ell\ .$$

Since $\sigma \ge \rho^2$, from (3.22) and (6.4) we have

$$(z_j^n-\sigma^n)\Delta^{\sigma}_{\ell,n,m}(z_j,f_1) = O(\sigma^n\rho^{-\ell n}) + O((\sigma|z_j|\rho_1^{-(\ell+2)})^n)\ ,$$

where $\rho_1 := \rho-\varepsilon$ satisfies (4.2). Thus,

$$(6.5)\qquad \Delta^{\sigma}_{\ell,n,m}(z_j,f_1) = O\left(\frac{\sigma^n\rho^{-\ell n}}{|z_j^n-\sigma^n|} + \frac{(\sigma|z_j|\rho_1^{-(\ell+2)})^n}{|z_j^n-\sigma^n|}\right)\ .$$

Notice that $\rho^{-\ell} = \rho\rho^{-(\ell+1)} < |z_j|\rho^{-(\ell+1)}$, and $\rho_1^{-(\ell+2)} < \rho^{-(\ell+1)}$ by (4.2). With this observation, (6.5) yields

$$\Delta^{\sigma}_{\ell,n,m}(z_j,f_1) = O\left(\frac{\sigma^n}{|\sigma^n - z_j^n|}\cdot\frac{\delta^n|z_j|^n}{\rho^{(\ell+1)n}}\right) ,$$

where $\delta = \max\left\{\frac{\rho}{|z_j|} , \frac{\rho^{\ell+1}}{\rho_1^{\ell+2}}\right\} < 1$. Hence, for every $j = 1,2,\ldots,\ell$, we get

$$\overline{\lim_{n\to\infty}} |\Delta^{\sigma}_{\ell,n,m}(z_j,f_1)|^{1/n} < \begin{cases} \sigma/\rho^{\ell+1} & , \text{ if } |z_j| > \sigma , \\ |z_j|/\rho^{\ell+1} & , \text{ if } \rho < |z_j| < \sigma . \end{cases}$$

This proves Theorem 2.3 for the region Y_1.

(ii) If $z_1,\ldots,z_{\ell-1}$ are $\ell-1$ distinct points with $z_{\ell-1} \neq 0$ in the region Y_2, we solve the system of ℓ equations

$$\sum_{k=0}^{\ell-1} \mu_k z_j^k = 0 \quad , \quad 1 \leq j \leq \ell-1 ,$$

with $\mu_{\ell-1} = 1$. Set $f_2(z) := \left(\sum_{k=0}^{\ell-1} \mu_k z^k\right)\Big/\left(1 - \left(\frac{z}{\rho}\right)^{\ell}\right)$. Then, on repeating the above argument with suitable changes, we conclude that

$$\overline{\lim_{n\to\infty}} |\Delta^{\sigma}_{\ell,n,m}(z_j,f_2)|^{1/n} < \rho^{-\ell} \quad , \quad 1 \leq j \leq \ell-1 .$$

This completes the proof. □

CHAPTER VI

A CONVERSE RESULT IN THE THEORY OF EQUICONVERGENCE OF INTERPOLATING RATIONAL FUCNTIONS

§1. INTRODUCTION. Since the first extension [5] of Walsh's theorem in 1981, there have been in the last few years a number of direct theorems on the theory of equiconvergence of certain schemes of interpolatory polynomial sequences. A recent paper of Saff and Sharma [18] also gives some direct theorems, but it deals with the equiconvergence of two schemes of rational interpolants. Our object in this chapter is to obtain a sort of converse of this theorem on the lines of a corresponding theorem due to Szabados [22], which is related to the Lagrange interpolant and the Taylor sections of an analytic function.

We observe in Chapter I that (12.3) reduces to a classic extension [5] of a theorem of Walsh when $\sigma \to \infty$. In this case, it is known that

$$\Delta^{\infty}_{\ell,n,m}(z,f) = L_{n+m}(z,f) - \sum_{\nu=0}^{\ell-1} P^{*}_{n+m}(z,f,\nu) \tag{1.1}$$

where $L_{n+m}(z,f) \in \pi_{n+m}$ is Lagrange interpolant to $f(z)$ in the $(n+m+1)^{th}$ roots of unity, and

$$P^{*}_{n+m}(z,f,\nu) := \sum_{j=0}^{n+m} a_{\nu(n+m+1)+j} z^{j}, \quad (\nu = 0,1,2,\ldots), \tag{1.2}$$

with $f(z) := \sum_{k=0}^{\infty} a_k z^k$ and $\overline{\lim_{n\to\infty}} |a_n|^{1/n} = \rho^{-1}$.

Let A^{*}_{ρ} (or $A^{*}_{\rho}C$), $\rho \geq 1$, denote the set of all functions which are analytic in $|z| < \rho$ (or analytic in $|z| < \rho$ and continuous in $|z| \leq \rho$). We shall say that <u>a sequence</u> $\{S_n(z)\}_{n=1}^{\infty}$ <u>is U.B. in</u>

$|z| < \gamma^r$ if $\{S_n(z)\}_{n=1}^{\infty}$ is uniformly bounded in every closed subset of $|z| < \gamma^r$.

The following theorem is due to Szabados [22]:

Theorem A. Let $\ell \geq 1$ and $m \geq -1$ be fixed integers. If $f \in A_1^* C$ and if $\{\Delta_{\ell,n,m}^{\infty}(z,f)\}_{n=1}^{\infty}$ is U.B. in $|z| < \rho^{\ell+1}$ for some $\rho > 1$, then $f \in A_\rho^*$.

We shall prove an analogue of the above theorem when $\sigma > 1$ is finite.

§2. PRELIMINARIES AND STATEMENT OF MAIN RESULT.

Let $f \in A_1^* C$ and let,

(2.1) $r_{n+m,n}(z,f,0) = P_{n+m,n}(z,f,0)/(z^n-\sigma^n)$, $(P_{n+m,n}(z,f,0) \in \pi_{n+m})$

be the best L_2-approximant to $f(z)$ on $|z| = \delta$, $\sigma^{-1} < \delta < 1$, among rational functions of the form $p_{n+m}(z)/(z^n-\sigma^n)$, $p_{n+m}(z) \in \pi_{n+m}$.

Recall that (I, (12.5))

(2.2) $\alpha_{n,m}(z) := 1-z^{m+1}\sigma^{-n}$, $\beta_{n,m}(z) := z^{m+1}(z^n - \sigma^{-n})$.

We define rational functions $r_{n+m,n}(z,f,\nu)$, $(\nu = 1,2,\ldots)$, of the form

(2.3) $r_{n+m,n}(z,f,\nu) = P_{n+m,n}(z,f,\nu)/(z^n-\sigma^n)$, $(P_{n+m,n}(z,f,\nu) \in \pi_{n+m})$,

which are determined recursively from the Jacobi type expansion (cf.

[18], (3.7))

$$f(z) = \sum_{\nu=0}^{\infty} \left(\frac{\beta_{n,m}(z)}{\alpha_{n,m}(z)}\right)^{\nu} r_{n+m,n}(z,f,\nu) \ , \quad |z| < 1 \ .$$

If for a positive integer ℓ, we set

$$\Delta^{\sigma}_{\ell,m,n}(z,f) = R_{n+m,n}(z,f) - \sum_{\nu=0}^{\ell-1} r_{n+m,n}(z,f,\nu) \ , \tag{2.4}$$

where $R_{n+m,n}(z,f)$ is defined in (I, (12.2)), then we state our main result as follows:

<u>Theorem 2.1.</u> <u>Let</u> $m \geq -1$ <u>and</u> $\ell \geq 1$ <u>be fixed integers, and let</u> $f \in A_1^* C$. <u>If, for some</u> $\rho > 1$ <u>and for some</u> $\sigma \geq \rho^{\ell+1}$, <u>the sequence</u> $\{\Delta^{\sigma}_{\ell,m,n}(z,f)\}_{n=1}^{\infty}$ <u>given by</u> (2.4) <u>is U.B. in</u> $|z| < \rho^{\ell+1}$, <u>then</u> $f \in A_{\rho}^*$.

<u>Remark 2.1.</u> Theorem 2.1 may be looked upon as a partial converse of the statement (I, (12.4)). A natural question which arises at this point is the following: If $1 < \sigma < \rho^{\ell+1}$ and if $\{\Delta^{\sigma}_{\ell,n,m}(z,f)\}_{n=1}^{\infty}$ is uniformly bounded on every compact subset of the domain $\{z: |z| \neq \sigma\}$, does $f \in A_{\rho}^*$? We assert that, in general, the answer is in the negative. This is easily seen on taking $\hat{f}(z) = (z-\eta)^{-1}$ where we choose $\alpha \in (0,1)$ such that $\sigma < \rho^{(\ell+1)\alpha} =: \eta^{\ell+1}$. Then $\hat{f} \in A_{\eta}$ and from the Saff-Sharma Theorem (cf. I, (12.4)) we have $\Delta^{\sigma}_{\ell,n,m}(z,\hat{f}) \to 0$ on every compact subset of $\{z: |z| \neq \sigma\}$. But $\hat{f} \in A_{\rho^{\alpha}}$ and so $\hat{f} \notin A_{\rho}$ since $\rho^{\alpha} < \rho$.

Remark 2.2. Theorem 2.1 is also valid if we consider $m < -1$. (See [18] or (I, Remark 12.3) for the construction of the rational functions $r_{n+m,n}(z,f,\nu)$, $\nu = 0,1,2,\ldots$, when $m < -1$.)

§3. SOME LEMMAS. In this section, we shall first compare some polynomial interpolatory processes with some rational ones, and then show that the sequences $\{\Delta^{\infty}_{\ell,n+m}(z,f)\}_{n=1}^{\infty}$ and $\{\Delta^{\sigma}_{\ell,n,m}(z,f)\}$, $(\sigma \geq \rho^{\ell+1})$, given by (1.1) and (2.4) respectively are either both bounded or both unbounded in the region $|z| < \sqrt{\sigma}$. It will enable us to show that f is analytic in $|z| < \min(\rho, \sigma^{\frac{1}{2(\ell+1)}})$, which is the main idea that underlies the proof of Theorem 2.1.

Lemma 3.1. Let the integer $m \geq -1$ and let $\sigma > 1$. If $f \in A^{*}_{1}C$, then

$$\lim_{n\to\infty} \{L_{n+m}(z,f) - R_{n+m,n}(z,f)\} = 0 \quad ; \quad \text{for} \quad |z| < \sqrt{\sigma} \tag{3.1}$$

where $L_{n+m}(z,f)$ and $R_{n+m,n}(z,f) = B_{n+m,n}(z,f)/(z^n - \sigma^n)$ are defined by (1.1) and (2.4) respectively. Moreover, the convergence in (3.1) is uniform and geometric on every closed subset of the region $|z| < \sqrt{\sigma}$.

Proof. Let ω be a primitive $(n+m+1)^{th}$ root of unity. From the definition of Lagrange interpolating polynomial, we have

$$L_{n+m}(z,f) = \sum_{k=0}^{n+m} \left(\frac{z^{n+m+1}-1}{z - \omega^k}\right)\left(\frac{\omega^k}{n+m+1}\right) f(\omega^k)$$

and

$$B_{n+m,n}(z,f) = \sum_{k=0}^{n+m} \left(\frac{z^{n+m+1}-1}{z - \omega^k}\right)\left(\frac{\omega^k}{n+m+1}\right)(\omega^{kn}-\sigma^n)f(\omega^k) .$$

This gives us

$$R_{n+m,n}(z,f) - L_{n+m}(z,f) = \sum_{k=0}^{n+m} \left(\frac{z^{n+m+1}-1}{z - \omega^k}\right)\left(\frac{\omega^k f(\omega^k)}{n+m+1}\right) \frac{\omega^{kn}-z^n}{z^n - \sigma^n}$$

$$= \sum_{k=0}^{n+m} \sum_{j=0}^{n+m} z^{n+m-j} \omega^{k(j+1)}\left(\frac{f(\omega^k)}{n+m+1}\right)\left(\frac{\omega^{kn}-z^n}{z^n-\sigma^n}\right) .$$

Since $f \in A_1^*C$, there is an $M > 0$ so that $|f(t)| \leq M$ for every $|t| \leq 1$. Let $|z| = \tau$, $\tau \geq 1$. Then, from the above relation, we have

$$|L_{n+m}(z,f) - R_{n+m,n}(z,f)| \leq M(n+m+1)\tau^{n+m}\left(\frac{\tau^n+1}{|\tau^n-\sigma^n|}\right).$$

If $\sigma > \tau$, we obtain

$$\overline{\lim_{n\to\infty}}\{ \max_{|z|=\tau} |R_{n+m,n}(z,f) - L_{n+m}(z,f)|\}^{1/n} \leq \frac{\tau^2}{\sigma}$$

which completes the proof. □

<u>Lemma 3.2. Let</u> $m \geq -1$ <u>be a fixed integer and</u> $\sigma > 1$. <u>If</u> $f \in A_1^*C$, <u>then the conclusion of Lemma 3.1 remains valid if</u> $L_{n+m}(z,f)$ <u>and</u> $R_{n+m,n}(z,f)$ <u>are replaced by</u> $P^*_{n+m}(z,f,0)$ <u>and</u> $r_{n+m,n}(z,f,0)$

(cf. (1.2) and (2.1)) respectively.

Proof: It is known that $r_{n+m,n}(z,f,0)$ interpolates $f(z)$ in the zeros of $z^{m+1}(z^n-\sigma^{-n})$ (I, Remark 12.1). Therefore,

$$r_{n+m,n}(z,f,0) = \frac{1}{2\pi i}\int_{|t|=\delta} \frac{t^n-\sigma^n}{z^n-\sigma^n}\left(\frac{t^{m+1}(t^n-\sigma^{-n})-z^{m+1}(z^n-\sigma^{-n})}{t^{m+1}(t^n-\sigma^{-n})}\right)\frac{f(t)}{t-z}\,dt,$$

where $\sigma^{-1} < \delta < 1$. Also, we can write

$$P^*_{n+m}(z,f,0) = \frac{1}{2\pi i}\int_{|t|=\delta} \frac{f(t)}{t-z}\left(\frac{t^{n+m+1}-z^{n+n+1}}{t^{n+m+1}}\right)dt.$$

An elementary calculation now shows that

$$r_{n+m,n}(z,f,0) - P^*_{n+m}(z,f,0)$$

(3.2) $$= \frac{1}{2\pi i}\int_{|t|=\delta} \frac{f(t)K_n(t,z)dt}{(t-z)(t^n-\sigma^{-n})(z^n-\sigma^n)t^{n+m+1}}$$

where

$$K_n(t,z) := (t^{n+m+1}-z^{n+m+1})(t^{2n}-t^nz^n-1)-t^n(t^{m+1}-z^{m+1})$$

(3.3) $$- \sigma^{-n}(t^n-z^n)(t^{n+m+1}-t^nz^{m+1}-z^{n+m+1}).$$

Since $\sup_{|t|\le 1}|f(t)| \le M$ for some $M > 0$, from (3.2) we obtain

$$|r_{n+m,n}(z,f,0) - P^*_{n+m}(z,f,0)|$$

(3.4) $$\le \frac{M}{2\pi(\delta^n-\sigma^{-n})|z^n-\sigma|}\int_{|t|=\delta}\left|\frac{K_n(t,z)}{t-z}\right||dt|,$$

whereas

$$\frac{K_n(t,z)}{t-z} = (t^{2n}-t^n z^n-1)\sum_{j=0}^{n+m} t^j z^{n+m-j} - t^n \sum_{j=0}^{m} t^j z^{m-j}$$

(3.5)

$$- \sigma^{-n}(t^{n+m+1} - t^n z^{m+1} - z^{n+m+1}) \sum_{j=0}^{n-1} t^j z^{n-j-1} .$$

If $|z| = \tau \geq 1$, and $|t| = \delta < 1$, then

$$\left|\frac{K_n(t,z)}{t-z}\right| \leq (\delta^{2n}+\delta^n\tau^n-1)(n+m+1)z^{n+m} - \delta^n(m+1)\tau^m$$

$$-\sigma^{-n}(\delta^{n+m+1} + \delta^n\tau^{m+1} + \tau^{n+m+1})n\tau^{n-1} .$$

Notice that the relation (3.4) holds for all $\delta \in (\sigma^{-1},1)$. Using (3.5) and then letting $\delta \to 1$ yields

$$|r_{n+m,n}(z,f,0) - P^*_{n+m}(z,f,0)| \leq \frac{CM(n+m+1)|z|^{2n}}{(1-\sigma^{-n})|z^n-\sigma^n|} .$$

Here C is constant independent of n. If $\sigma > |z| = \tau$, then it is easy to see that

$$\lim_{n\to\infty} \{ \sup_{|z|=\tau} |r_{n+m,n}(z,f,0) - P^*_{n+m}(z,f,0)|\}^{1/n} \leq \frac{\tau^2}{\sigma}$$

which proves the lemma. □

Lemma 3.3. Let $m \geq -1$ be a fixed integer and $\sigma > 1$. If $f \in A_1^*C$, then the conclusion of Lemma 3.1 remains valid if $L_{n+m}(z,f)$ and $R_{n+m,n}(z,f)$ are replaced by $P^*_{n+m}(z,f,\nu)$ and $r_{n+m,n}(z,f;\nu)$, $(\nu = 1,2,3,\ldots)$, (cf. (1.5)&(2.3)) respectively.

Proof. An integral representation of $r_{n+m,n}(z,f,\nu)$, $\nu \geq 1$, is given by (cf. I, (12.7))

$$(3.6)\qquad r_{n+m,n}(z,f,\nu) = \frac{1}{2\pi i}\int_{|t|=\delta} \frac{t^n-\sigma^n}{z^n-\sigma^n}\left(\frac{(\alpha_{n,m}(z))^{\nu-1}}{(\beta_{n,m}(z))^{\nu+1}}\right)\frac{H_n(t,z,\nu)}{t-z}\,dt,$$

where $\sigma^{-1} < \delta < 1$ and

$$H_n(t,z,\nu) := \alpha_{n,m}(z)\beta_{n,m}(t) - \alpha_{n,m}(t)\beta_{n,m}(z)$$

(3.7)

$$= t^{n+m+1} - z^{n+m+1} - \sigma^{-n}(tz)^{m+1}(t^n - z^n) - \sigma^{-n}(t^{m+1} - z^{m+1}) .$$

Also, from (1.2) we have

$$(3.8)\qquad P^*_{n+m}(z,f,\nu) = \frac{1}{2\pi i}\int_{|t|=\delta} \frac{f(t)}{t^{(\nu+1)(n+m+1)}}\left(\frac{t^{n+m+1}-z^{n+m+1}}{t-z}\right)dt .$$

Since $\{\alpha_{n,m}(t)\}^{\nu-1} = 1 + \sum_{j=1}^{\nu-1}(-1)^j\binom{\nu-1}{j}(t^{m+1}\sigma^{-n})^j$, using (3.6) and (3.7) we can rewrite $r_{n+m,n}(z,f,\nu)$ as

$$(3.9)\qquad r_{n+m,n}(z,f,\nu) = Q_{n+m,n}(z,f,\nu) + T_{n+m,n}(z,f,\nu)$$

with

$$Q_{n+m,n}(z,f,\nu) = \frac{1}{2\pi i}\int_{|t|=\delta} \frac{t^n-\sigma^n}{z^n-\sigma^n}\left(\frac{f(t)}{(t^{m+1}(t^n-\sigma^{-n}))^{\nu+1}}\right)\frac{t^{n+m+1} - z^{n+m+1}}{t - z}\,dt ,$$

(3.10)

$$T_{n+m,n}(z,f,\nu) = \frac{1}{2\pi i}\int_{|t|=\delta} \frac{t^n-\sigma^n}{z^n-\sigma^n}\left(\frac{f(t)J_n(t,z)}{(t^{m+1}(t^n-\sigma^{-n}))^{\nu+1}}\right)dt,$$

where

$$J_n(t,z) := H_n(t,z,\nu)\sum_{j=1}^{\nu-1}\binom{\nu-1}{j}(-t^{m+1}\sigma^{-n})^j$$

$$- \sigma^{-n}((tz)^{m+1}(t^n-z^n) + t^{m+1}-z^{m+1}) .$$

Now one can easily see after some computation that

$$T_{n+m,n}(z,f,\nu) = O\left(\frac{1+|z|^n}{|z^n-\sigma^n|}\right), \tag{3.11}$$

and

$$\overline{\lim_{n\to\infty}}\left\{\sup_{\substack{|z|=\tau\\ \tau<\sqrt{\sigma}}} |Q_{n+m,n}(z,f,\nu) - P^*_{n+m,n}(z,f,\nu)|\right\}^{1/n} \le \frac{\tau^2}{\sigma}. \tag{3.12}$$

(3.12) follows from (3.10) and (3.8) on mimicking the procedure starting at (3.1) in Lemma 3.2. Therefore, from (3.9)-(3.12), we conclude that

$$\overline{\lim_{n\to\infty}}\left\{\sup_{\substack{|z|=\tau\\ \tau<\sqrt{\sigma}}} |r_{n+m,n}(z,f,\nu) - P^*_{n+m}(z,f,\nu)|\right\}^{1/n} \le \frac{\tau^2}{\sigma}.$$

□

Remark 3.1. If ℓ is a fixed positive integer, then it follows directly from Lemma 3.3 that

$$\overline{\lim_{n\to\infty}}\left\{\sup_{\substack{|z|=\tau\\ \tau<\sqrt{\sigma}}} \left|\sum_{\nu=0}^{\ell-1} r_{n+m,n}(z,f,\nu) - \sum_{\nu=0}^{\ell-1} P^*_{n+m}(z,f,\nu)\right|\right\}^{1/n} \le \frac{\tau^2}{\sigma}.$$

Next, we prove

Lemma 3.4. Let $\ell \ge 1$ and $m \ge -1$ be fixed integers and $\sigma > 1$. Suppose $f \in A_1^*C$. Then $\{\Delta^{\infty}_{\ell,n,m}(z,f)\}_{n=1}^{\infty}$ is U.B. in $|z| < \sqrt{\sigma}$ if and only if the sequence $\{\Delta^{\sigma}_{\ell,n,m}(z,f)\}_{n=1}^{\infty}$ is also U.B. in the same region, where $\Delta^{\infty}_{\ell,n,m}(z,f)$ and $\Delta^{\sigma}_{\ell,n,m}(z,f)$ are given by (1.1) and (2.4).

Proof. From the triangle inequality and the definition of $\Delta^{\infty}_{\ell,n,m}(z,f)$ and $\Delta^{\sigma}_{\ell,n,m}(z,f)$, we note that

$$\left| |\Delta^{\sigma}_{\ell,n,m}(z,f)| - |\Delta^{\infty}_{\ell,n,m}(z,f)| \right| \le |\Delta^{\sigma}_{\ell,n,m}(z,f) - \Delta^{\infty}_{\ell,n,m}(z,f)|$$

$$\le |R_{n+m,n}(z,f) - L_{n+m}(z,f)|$$

$$+ \left| \sum_{\nu=0}^{\ell-1} r_{n+m,n}(z,f,\nu) - \sum_{\nu=0}^{\ell-1} P^{*}_{n+m}(z,f,\nu) \right| .$$

An application of Lemma 3.1 and Remark 3.1 now gives the desired result. □

Remark 3.2. If $\sigma \ge \rho^{2(\ell+1)}$, then Lemma 3.4 also holds if $|z| < \sqrt{\sigma}$ is replaced by $|z| < \rho^{\ell+1}$. For this, it is enough to note that the Lemmas 3.1-3.3 are valid for the region $|z| < \rho^{\ell+1} < \sqrt{\sigma}$.

§4. PROOF OF THEOREM 2.1. First assume that $\sigma \ge \rho^{2(\ell+1)}$. By the hypothesis of Theorem 2.1, $\{\Delta^{\sigma}_{\ell,n,m}(z,f)\}_{n=1}^{\infty}$ is U.B. in $|z| < \rho^{\ell+1}$. From Remark 3.2, it follows that $\{\Delta^{\infty}_{\ell,n,m}(z,f)\}_{n=1}^{\infty}$ is U.B. in $|z| < \rho^{\ell+1}$ too. Thus, $f \in A^{*}_{\rho}$ by Theorem A.

Next consider $\rho^{\ell+1} \le \sigma < \rho^{2(\ell+1)}$. Then $\{\Delta^{\sigma}_{\ell,n,m}(z,f)\}_{n=1}^{\infty}$ being a U.B. sequence in $|z| < \rho^{\ell+1}$ is also U.B. in $|z| < \sqrt{\sigma}$. Now from Lemma (3.4), it implies that the sequence $\{\Delta^{\infty}_{\ell,n,m}(z,f)\}_{n=1}^{\infty}$ is U.B. in $|z| < \sqrt{\sigma}$. If we let $\xi^{\ell+1} := \sqrt{\sigma}$, then $f \in A^{*}_{\xi}$ (cf. Theorem A).

Notice that $\xi > 1$. Let $\rho_1 := \sup\{\eta: f \in A^*_\eta\}$. Then $\rho_1 > 1$, $f \in A^*_{\rho_1}$ and f has a singularity on $|z| = \rho_1$.

The proof will be completed by showing that $\rho_1 \geq \rho$. Assume that $\rho_1 < \rho$. Then the set $D^* = \{z: \rho_1^{\ell+1} < |z| < \rho^{\ell+1}\}$ contains infinitely many points, and $\{\Delta^\sigma_{\ell,n,m}(z,f)\}_{n=1}^\infty$ being U.B. in $|z| < \rho^{\ell+1}$, is bounded at each point of D^*. On the other hand, $\sigma \geq \rho^{\ell+1} > \rho_1^{\ell+1}$. Thus, $\{\Delta^\sigma_{\ell,n,m}(z,f)\}_{n=1}^\infty$ can not be bounded at more than ℓ points in the region $|z| > \rho_1^{\ell+1}$ (cf. V, Remark 2.2). This contradicts the boundedness of $\{\Delta_{\ell,n,m}(z,f)\}_{n=1}^\infty$ at each point of D^*. Therefore $\rho_1 \geq \rho$. □

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APPENDIX

The following observations and reflections are inspired by the work of Edrei, Saff and Varga [8] and by Edrei's study of Lindelöf functions. I gratefully acknowledge the benefit of some discussions with Professor Edrei and of his kind suggestions during his brief visit here in March '85.

The problem of the distribution of the zeros of sections of power-series of analytic functions "has a long and respectable history". Since Walsh's theorem on equiconvergence ([28] p. 153) shows a close relationship between the Lagrange interpolant of an analytic function in the roots of unity and the partial sections of its Taylor series, it seems worthwhile to ask whether something can be said about the asymptotic distribution of the zeros of Lagrange interpolants in the roots of unity. Thus if $f(z) = (\rho-z)^{-1}$ then, in the notation of Chapter I, we have $L_n(z,f) = \frac{\rho^{n+1}}{\rho^{n+1}-1} p_n(z,f)$, but if $f(z) = e^z$, we have $L_n(z,f) = \sum_{k=0}^{n} a_{nk} z^k$ and $p_n(z,f) = \sum_{k=0}^{n} z^k/k!$ where $a_{nk} = \sum_{\lambda=0}^{\infty} \frac{1}{(\lambda n+k)!}$, $k = 0,1,\ldots,n$. Now it is not clear in this case what relation exists between the zeros of $p_n(z,f)$ and those $L_n(z,f)$ as $n \to \infty$.

Since $f(z) = e^z$ is entire, it seems suitable to consider the Lagrange interpolant $L_n^{\sigma}(z,f)$ which interpolates $f(z)$ in the zeros of $z^{n+1} - \sigma^{n+1} = 0$, where $\sigma > 1$. We shall show that when suitably normalized, the difference $[L_n^{\sigma}(z,f) - p_n(z,f)]/a_n z^n$ tends to zero as $n \to \infty$. This shows that asymptotically the distribution of the zeros of $L_n^\sigma(z,f)$ and $p_n(z,f)$ is similar when $f(z) = e^z$.

More precisely, we shall show that if $f(z) = e^z$, $z = nw$, with $0 < s_0 < |w| \leq 1$, and $\sigma = ns_0/2$, we have

$$\text{(A.1)} \qquad \lim_{n\to\infty} \frac{L_n^\sigma(nw,f) - p_n(nw,f)}{a_n(nw)^n} = 0 \text{ , where } a_n = 1/n! \text{ .}$$

<u>Proof.</u> It is easy to see that

$$\text{(A.2)} \qquad L_n^\sigma(z,f) - p_n(z,f) = \frac{1}{2\pi i}\int_\Gamma \frac{f(t)\sigma^{n+1}(t^{n+1}-z^{n+1})}{(t-z)t^{n+1}(t^{n+1}-\sigma^{n+1})}\,dt \text{ .}$$

Writing $\tau = n(1+\varepsilon)$, $\varepsilon > 0$ and replacing z in (A.2) by nw with $0 < s_0 < |w| \leq 1$, we have

$$I = |[L_n^\sigma(nw,f) - p_n(nw,f)]/a_n(nw)^n|$$

$$\leq \frac{e^\tau\sigma^{n+1}(\tau^{n+1}-n^{n+1}|w|^{n+1})\tau}{(\tau-n|w|)\tau^{n+1}(\tau^{n+1}-\sigma^{n+1})|a_n|n^n|w|^n} \text{ .}$$

If $w = se^{i\varphi}$, $0 < s_0 \leq s < 1$, then $\left(\text{since } a_n = \frac{1}{n!} \sim \frac{e^n}{n^n\sqrt{2\pi n}}\right)$ we get

$$I \leq \frac{e^\tau\sigma^{n+1}(\tau^{n+1}-n^{n+1}s^{n+1})\tau\sqrt{2\pi n}\,e^{-n}(1+\eta_n)}{(\tau-ns)\tau^{n+1}(\tau^{n+1}-\sigma^{n+1})s^n} \text{ ,} \qquad (\text{where } \eta_n \to 0)$$

$$= \frac{e^{n(1+\varepsilon)}\sigma^{n+1}\{1-(\frac{s}{1+\varepsilon})^{n+1}\}\sqrt{2\pi n}\, e^{-n}(1+\eta_n)}{(1-\frac{s}{1+\varepsilon})n^{n+1}(1+\varepsilon)^{n+1}(1-(\frac{\sigma}{n(1+\varepsilon)})^{n+1})s^n} .$$

Since $\{1-(\frac{s}{1+\varepsilon})^{n+1}\}/(1-\frac{s}{1+\varepsilon}) < n+1$, on substituting $\sigma = s_0 n/2$ and simplifying, we get

$$I \leq \frac{\sqrt{2\pi n}\,(n+1)e^{n\varepsilon}(1+\eta_n)}{2^{n+1}(1-\frac{s_0}{2(1+\varepsilon)})^{n+1}}$$

which tends to zero as $n \to \infty$, since $e^{n\varepsilon}\cdot e^{-n\log 2} = e^{-n(\log 2-\varepsilon)}$. □

It may be interesting to examine more systematically the distribution of the zeros of Lagrange interpolants of other entire transcendental functions of finite order and exponential type on equidistributed nodes (in the sense of H. Weyl).