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UNIVERSE FABRICATION AS A QUANTUM TUNNELLING PHENOMENON

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Whatever a man prays for, he prays for a miracle. Every prayer reduces itself to this: Great God, grant that two and two be not four.

Ivan Turgenev, from *Prayer*

UNIVERSITY OF ALBERTA

FACULTY OF GRADUATE STUDIES AND RESEARCH

The undersigned certify that they have read, and recommended to the Faculty of Graduate Studies and Research for acceptance, a thesis entitled "Universe Fabrication As A Quantum Tunnelling Phenomenon" submitted by Alick Lachlan Macpherson in partial fulfilment of the requirements for the degree of Master of Science in Theoretical Physics.

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To Catherine; a woman of patience and great warmth.

La fantaisie est un eternel printemps.

ABSTRACT

The thesis considers the possibility that an advanced civilisation might be able to synthesise, in a laboratory, a globule of false vacuum that inflates to become a new universe causally disconnected from our own. On the basis of classical gravitational theory it is shown that this possibility can be ruled out, because it requires acausal behaviour of matter. However, in a quantum context, the phenomenon of tunnelling through gravitational potential barriers implies that universe fabrication is possible in principle, though the probability is extremely small. The bulk of this thesis is devoted to attempts to estimate this probability: the functional integral method and the Hamiltonian approach. I attempt to link and amalgamate these formally rather different approaches.

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The description given to me in New Zealand of my supervisor, Prof. Werner Israel is not the same as the image I now have. In fact it was nowhere near complete. Fortunately, my M.Sc. has allowed me a more complete picture of Werner. He is a brilliant physicist, with a powerful physical insight, but he is also deliberate and has the patience and ability to listen. He is to me a man strong within his mana[1] . Werner has taught me many things and shown me the way to many more. To him I owe a lot, and am very grateful. Thank you Werner.

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[1]Mana is a Maori word that refers to the concept of life force assocciated with status and power.

friend, but also the closest thing I know to a Joycian Irish intellectual. Cheers Des.

The second is Sacha Davidson and she is listed after Des only because S follows D. Sacha, like Des, is a fully competent theoretical physicist, and has been willing to listen to those late night ideas that from a thesis creep. Yet that alone is not all I am grateful to Sacha for. If I left out my thanks for the times, the troubles, and the trust that Sacha and I have shared, it would be a great injustice. She is a friend, confidant, and colleague, for which I am extremely thankful.

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CHAPTER ONE

INTRODUCTION

For as far back in human history as one would care to look, evidence can be found that supports the notion that people have been curious about the composition and origin of the world around them. Fortunately, modern day physics has dispelled most of the myths associated with the composition of the world around us (the observed universe) and the way in which it functions. Yet as to the nature of its origin, one can not be so confident. Although many have entertained various conjectures pertaining to the universe's origin, no one conjecture rises above all the others. As is well known, the scope of these conjectures encompasses everything from theology, through the heuristic, to scientifically respectable models. Each possible explanation attains its own level of plausibility, with perhaps the Big Bang model generally being accepted as the most plausable picture of comological evolution from the moment the universe has cooled below a billion degrees [1]. However, one still lacks a complete answer to the question of the universe's origin.

Whatever the explanation, the author is quite willing to accept the most physically plausible description/scenario, as it is not this issue that this thesis is intended to address. Instead, this thesis is to focus on a subsidiary question. This dilemma is that given that a universe exists (i.e. the observed universe), can one, without knowing the details of its origin, construct a second universe from within the first. More specifically, is it theoretically possible to construct a "new universe" in the laboratory? This is the question that is to be addressed in this thesis.

Yet before one can attempt to answer this question, it must be made more precise, as it is not clear what one means by "construction of a new universe". In fact, at present, the term "new universe" is not even clear. So, in order to clarify things, start by considering the observable universe. It would seem that a "new universe" would have to be defined as something similar to the presently observed universe. That implies that it would have to be a spacetime region that was of a size that was comparable or greater to that of the present size of the universe. Also, this new universe would have to be effectively disconnected from our present universe, otherwise there would be nothing to distinguish the new universe from the old one. Here effectively disconnected means that the spacetime region that constitutes the new universe sooner or later becomes causally disconnected from its parent spacetime (i.e. the laboratory, the universe observed). Note, causally disconnected implies that information from one region cannot reach another region by standard timelike or lightlike propagation. Thus one possible way to achieve a loss of causality is to allow some sort of spacelike propagation. (This could be produced by the presence of a wormhole for example.) [4, 5, 6]

So, in order to construct a "new universe" in the laboratory, the construction should commence within the realm of the observed universe, and finish with the seed spacetime which has expanded to a comparable size, and in the process, disconnected itself from its parent. Thus, one now has a refined form of the question that this thesis is to address: can such a construction be achieved.

This question resolves into three subproblems, namely

1. What is the mechanism by which the seed spacetime attains a size comparable to that of the observed universe?

2. How does one construct a spacetime region that is suitable for the production of a "new universe"?

3. What is the nature of the process whereby effective causal disconnection is achieved?

The answer to the first of these three problems can be found in the theory of inflation, which was first proposed by Alan Guth in 1981 [2]. This theory called inflation was devised in order to solve the what is known as the horizon problem [3], [26, pages 740 and 815] which hindered the Big Bang model's explanation of the state of the observed universe. Without inflation, the Big Bang model, in order to provide predictions of the observed structure of the universe (such as the uniformity of the microwave background), required an exceedingly strong dependence on the initial conditions of the universe. For example, the temperatures associated with the microwave background radiation of two regions of the sky that are separated by 180° are the same to within present experimental accuracy [7, refer to Abbott and Pi, chapter 6 article 1]. Yet these two regions, according to the Big Bang model, have never been in causal contact (i.e their past horizons have not overlapped), which leaves one at a loss to explain why the temperatures are the same. This is one form of the horizon problem. According to Big Bang theory, the only way out of this dilemma is to impose exacting initial conditions of homogeneity and isotropy.

However, one may then be tempted to ask if perhaps these two initial conditions are reasonable. The answer is that they are not, and this can be seen by considering the universe at $t = 10^{-35}$ seconds after the big bang. At this time, according to the Big Bang model, the observed universe would have to be composed of approximately 10^{80} regions that had no prior causal contact with each other. Yet all these regions would have to be bound by the initial conditions of homogeneity and isotropy, so that they have a consistent temperature. Clearly, this is a case of extreme fine tuning, and so the standard Big Bang model is seen as physically unreasonable when extrapolated back to this early time.

For this reason (amongst others), Guth proposed inflation, which was to act as a modifier of the standard Big Bang theory (a more complete review of the theory of inflation is given in [7]. Inflation itself is to be seen as an era in which a spacetime region undergoes very rapid expansion (exponential expansion). In particular, the rate of expansion has to be sufficiently rapid to exceed the rate at which the region's past Hubble radius is expanding. If this is the case, then the region initially inside one horizon could expand to such a size that it encompasses many if not all of the horizons of the individual regions present at the onset of the inflationary era. With such an expansion, one can easily understand the uniformity of the microwave background, as regions that at present seem to be causally disconnected according to Big Bang theory, are actually causally linked by inflation. Thus one has an answer to the horizon problem.

So, inflation is to be seen as an era of very rapid expansion. Therefore, inflation would certainly be a desirable feature to have in any construction process. However, one must ask how such expansion occurs, as one needs some sort of mechanism to propel the expansion. To answer this question, one has to consider the form of the spacetime region that constitutes the seed spacetime (i.e. the region that is to undergo inflation). In particular, the seed spacetime has to be composed of a false vacuum, in order for inflation to occur. A false vacuum state is defined as one in which the vacuum energy density of a region has a local minimum that is different from the global minimum (hence the name "false"). Further, the energy density of a false vacuum is fixed, and takes the value of local minimum, which is usually denoted by ρ.

The stress energy tensor $T_{\mu\nu}$ associated with this state is easily determined (as

$T_{\mu\nu}$ is a covariantly conserved quantity, and Einstein's field equation is assumed), and it has the form $T_{\mu\nu} = -\rho g_{\mu\nu}$. This in turn implies that the pressure of the false vacuum is $P = -\rho$, which is constant and negative. One's first reaction to this is that a constant pressure implies no gradient, and so no mechanical forces are produced. The absence of mechanical forces would seem to indicate that the false vacuum lacks an expansion mechanism. However, this is not the case. The false vacuum region does indeed have an expansion mechanism, and it results from the gravitational effects of the false vacuum pressure. As the pressure is negative, it contributes a negative term to the equation of deceleration (obtained from Einstein's field equation) that exceeds the positive term associated with the energy density. This has the effect of essentially reversing the role of gravity so that it increases the expansion rate with time (instead of the usual deceleration). Thus, given a false vacuum with an appropriate energy density, one can produce a spacetime region that will expand to a size comparable to that of the observed universe in a given amount of time. At this point, it should be noted that the boundary between false and true vacua would be pulled toward the false vacuum region (due to the pressure difference at the boundary), but this does not prevent the expansion, as the volume of the false vacuum that is inflating rapidly.

Also, two further properties of the inflationary model should be noted. First, the expansion due to the false vacuum is not unbounded. The inflationary era will last until the false vacuum decays [8, 9, 10] to the global vacuum energy state - the true vacuum. Secondly, the amount of expansion is generally regarded as considerable, as the energy density for a false vacuum is typically of the order of $\rho \approx 10^{68}$kg m^{-3}($\approx$ (10^{14}GeV)^4). This is large. In order to comprehend the magnitude of such an energy density, one just has to realise that it is roughly equivalent to the energy density one would associate with a large star that had been squashed down to the size of a hydrogen nucleus. Such an energy density then implies that the pressure is not only constant and negative, but is also very large in magnitude, thereby implying an extremely rapid expansion rate.

So it would seem that one does have a mechanism by which a small spacetime region can evolve in a finite time into a spacetime region comparable to the observed universe. Therefore, part one of the three part construction problem is solved.

In expounding the theory of inflation and its explanation of the expansion mechanism, the solution to part two of the construction problem has been disclosed. That is, inflation suggests a form for the required seed spacetime that is to be constructed. Obviously, the seed spacetime is a false vacuum. Requiring one to construct a false vacuum has the advantage that the structure of the seed spacetime in the post-inflationary era will not depend on the initial conditions associated with the pre-inflationary era. That is, one would not have to fine tune the construction process to a particular false vacuum state, as inflation is a generic property of the false vacuum state. Also, the characteristic properties of the false vacuum are independent of any particle theory that one would ascribe to the spacetime. Unfortunately, there is one slight drawback to the use of the false vacuum; a false vacuum state has never been experimentally observed. The reason for this is that one is required to produce very high energy densities, which are out of the range of the present technology. So it would seem that the construction process is hindered not by any theoretical impasse, but rather by technological difficulties. As this is the case, it will be assumed for the remainder of this thesis, that these difficulties will eventually be overcome, so that one can indeed produce the false vacuum region required.

With the expansion and the construction of the seed spacetime settled, attention is now turned to the one remaining question. This last problem concerns the manner in which a potential "new universe" spacetime region would effectively disconnect itself from the parent spacetime (i.e. the laboratory). Unfortunately, no immediate answer is apparent. Further, on reading chapter two of this thesis, one will discover that if one follows the classical laws of physics, there are no seed spacetime scenarios that are both constructible in the laboratory, and have an inflationary era. If one starts by constructing a false vacuum in the laboratory, then it is found that the associated expansion is not sufficient to produce the desired amount of inflation. However, if at some point in its evolution, this solution were interrupted, with the interruption being in the form of a quantum mechanical transition or tunnelling to another classical solution, then one could produce the desired amount of inflation, as long as the new classical solution affords the required amount of inflation. For such a step or interruption of the classical picture, an observer in the laboratory region of the spacetime would see a false vacuum seed expanding, and then suddenly disappearing. All that this observer would see after the seed tunnelled, is a black hole with a characteristic mass equal to that of the seed's mass at the onset of the tunnelling.

With this quantum tunnelling step as an intermediate stage, one now has a process that has a constructible seed spacetime, the required amount of inflation, and a means by which there is an effective causal disconnection from the parent spacetime.

With the addition of the quantum tunnelling stage to the inflationary false vacuum, all the meanings behind the question "Can one construct a new universe in the laboratory?" have been ascertained. Thus, as the problem has been clearly defined, and a possible solution suggested, one can proceed towards a detailed solution of the problem and its physical plausibility. This forms the main substance of this thesis.

However, before launching headlong into this problem, one should stop to consider its development, its history, and the various attempts to solve it. From what has been discussed so far, it seems that the key to the construction process is the presence of a false vacuum state, and hence inflation. Although Guth's 1981 paper [2] introduced inflation, the essential features of the false vacuum had already been realised. For example, Novikov in 1967 [11] discussed the connection between very high densities and gravitational expansion, whilst Gliner [12, 13], and Gliner and Dymnikova [14] were among the first to work with matter that possessed the stress energy tensor of the false vacuum state. Further, the connection between the vacuum energy density and an effective cosmological constant was made in 1968 by Zeldovich [15]. Yet it was the work of Einhorn and Sato (1981) [16] on supercooled phase transitions (which appeared necessary for a false vacuum to occur), that led Guth to his theory of inflation.

With inflation established the way was clear for the evaluation of the classical motion of a spherical false vacuum seed spacetime immersed in a true vacuum, and as a result there have been a number of papers describing the classical physics [17, 18, 19, 20, 29]. The net result of this classical analysis was that construction of a "new universe" seemed impossible, and so, as mentioned, the investigation turns to the use of quantum mechanics, and in particular, quantum mechanical tunnelling. Such a step has produced several possible methods of solution to the problem, and they all centre around the evaluation of the action associated with the spacetime (both seed and laboratory). These different approaches to the quantum tunnelling range from a general evaluation of the action via the Lagrangian of the system [39] through to canonical quantisation [21, 44], and to minisuperspace models [22].

Finally, it should be noted that the false vacuum seed spacetime is not the only way to obtain a new universe. If one considers Reissner-Nordström geometry [23], [26 page 921], or the work of Frolov et. al. [24], then one can discuss the existence of a new universe. Yet such considerations are not really related to the problem at hand as one has lost the notion of constructing the "new universe".

Thus the brief review of the problem's history has been completed, and so the analysis can begin in earnest. The reader will find that the next chapter is devoted to the evaluation of the classical physics of a false vacuum seed spacetime, and its no-go conclusion for the problem of creating a "new universe" in the laboratory. Chapters three and four look at several methods of evaluating the transmission coefficient (transition probability) of the intermediary quantum tunnelling stage, in an effort to decide whether such a process is likely to occur, and also to deduce what exactly happens during the tunnelling stage. With the results of chapter four as guidelines, chapter five then proceeds to attempt to uncover more information about the interpolating geometry of the classically forbidden region (i.e. the geometry over which the tunnelling occurs) by using a somewhat more generalised Hamiltonian formalism. Finally, the results of this thesis are brought together in chapter six, and an overview to the problem of constructing a "new universe" in the laboratory is given.

So, without further ado, let the snark hunt[1] begin.

[1] A snark hunt is defined as an impossible voyage by an improbable crew to find an inconceivable creature. The phrase comes from the poem *The Hunting Of The Snark; An Agony In Eight Fits.* by Charles Lewis Dodgson (Lewis Carroll).

CHAPTER TWO

THE CLASSICAL PHYSICS

2.1 Introduction

In order to look at the creation of a new universe, one must carefully consider the type of spacetime from which it is spawned. The mother spacetime requires a seed or inhomogeneity from which the new universe can be established. As this seed is taken to be small (size wise), then its main feature must be a tendency toward expansion. For this reason, the region of seed spacetime is to be taken as a false vacuum, as it is well known that a false vacuum region that is large enough will undergo inflation [2]. It is this inflationary era that transforms the seed spacetime into a "new universe" (refer back to chapter one). However, the question that has to be addressed is whether a seed spacetime can (given the technology) be constructed by humankind, such that it evolves to a large enough size so that it passes into an inflationary era.

Given that there is a false vacuum seed, there must be something exterior to it, and by Occam's razor[1], the simplest scenario is when the exterior is in the true vacuum state. By Birkhoff's theorem, it can then be assigned a Schwarzschild metric, with parameter m. As usual, m is the mass of the system as seen at infinity. For the seed spacetime, all that is known is that it is a false vacuum, and so has a constant energy density, ρ, and negative pressure. Now, one can, without loss of generality, arrange that the seed spacetime is centred on the coordinate origin. This

[1]Occam's razor is a maxim atttributed to William of Occam (b. ?, d. 1349), stating that in explaining something, assumptions must not be needlessly multiplied.

then requires that at $r = 0$ everything is regular, and in particular, the solution to Einstein's field equations is non-singular, as the energy momentum tensor is defined at the origin. If the restriction to spherical symmetry is also imposed (as otherwise, a description of the seed is difficult) then this leads to the conclusion that de Sitter spacetime is the only suitable coordinate system for the false-vacuum seed.[25]

Using the Schwarzschild and de Sitter coordinate systems, a full description of this inhomogeneous composite spacetime can be given. This assumes that the boundary between the two regions is arbitrarily thin, which will later be seen as an acceptable assumption. However, this system is dynamical, and so will evolve according to the classical equations of motion, which require Einstein's field equations to be solved. In order to do this, one must further assume that the boundary or wall (labelled Σ) between the two spacetime regions also obeys Einstein's field equation. If this is done then the equation of motion for Σ can be obtained, with the one degree of freedom that remains after the application of the symmetry constraints being the radial parameter. By tracing out the trajectories implied by this equation of motion, one then has all the possible classical trajectories of Σ. From these, one can then deduce whether any values of m and/or ρ give a seed spacetime that expands into a new universe.

2.2 The Classical Trajectories Of The Seed Wall

To obtain the classical trajectories, one must of course start with the energy momentum tensor $T_{\mu\nu}$, which due to the inhomogeneity of the spacetime, can be segregated into three parts: the false vacuum region, the wall Σ, and the true vacuum region.

$$T_{\mu\nu} = \begin{cases} -\rho g_{\mu\nu} & \text{false vacuum} \\ S_{\mu\nu}\delta(\Sigma) & \Sigma \\ 0 & \text{Schwarzschild} \end{cases} \tag{2.2.1}$$

Here $S_{\mu\nu}$ is the energy momentum tensor of the wall, and at this stage it is left unspecified. As can be seen, the false vacuum contribution is metric dependent.

Given that Birkhoff's theorem gives the true vacuum to be Schwarzschildian, the coordinates are $x^\mu = (t_S, r, \theta, \phi)$ and the metric for the region is

$$g_{\mu\nu} = \begin{pmatrix} -f_S & 0 & 0 & 0 \\ 0 & \frac{1}{f_S} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix} \tag{2.2.2}$$

with $f_S = (1 - \frac{2m}{r})$. For the false vacuum region a de Sitter metric is to be used, and though there are several choices of coordinates for this spacetime, the one to be used is the static coordinate description, as it mirrors the true vacuum coordinate system. Thus the coordinates are $x^\mu = (t_D, r, \theta, \phi)$ and the metric is

$$g_{\mu\nu} = \begin{pmatrix} -f_D & 0 & 0 & 0 \\ 0 & \frac{1}{f_D} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix} \tag{2.2.3}$$

with $f_D = (1 - \lambda^2 r^2)$. Due to the fact that both coordinate systems have the same angular coordinates, and as a result of the spherical symmetry, the two sphere radius, r for the two systems, must be equal at the surface on which they are matched. Thus, r can be taken as a coordinate common to both spacetimes. However, the time coordinate t will not be the same in the two coordinate patches, and so one must distinguish between t_S and t_D. Also, both of these metrics have a single parameter (m and λ respectively) and each is directly related to the energy density ρ.

As the energy momentum tensor and the geometry of the interior and exterior spacetime regions has been specified, the next step is to write down Einstein's field equation and link the two quantities together. Using the LLSC sign convention [26, the convention is listed inside the front cover] the general form of the field equation is

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi T_{\mu\nu} \qquad (2.2.4)$$

Note; the units are chosen such that the gravitational constant G is set equal to one. In this equation, R is the Ricci scalar and is given by $R = g^{\mu\nu}R_{\mu\nu}$ with $R_{\mu\nu} = R^{\alpha}_{\mu\alpha\nu}$ being the Ricci tensor. Further, the Riemann tensor $R^{\alpha}_{\mu\beta\nu}$ is calculated via

$$R^{\alpha}_{\beta\mu\nu} = \Gamma^{\alpha}_{\beta\nu,\mu} - \Gamma^{\alpha}_{\beta\mu,\nu} + \Gamma^{\alpha}_{\gamma\mu}\Gamma^{\gamma}_{\beta\nu} - \Gamma^{\alpha}_{\gamma\nu}\Gamma^{\gamma}_{\beta\mu} \qquad (2.2.5)$$

where $\Gamma^{\alpha}_{\beta\gamma}$ are the Christoffel symbols. These are generated from the metric via

$$\Gamma^{\alpha}_{\beta\gamma} = \frac{1}{2}g^{\alpha\delta}(g_{\delta\beta,\gamma} + g_{\delta\gamma,\beta} - g_{\beta\gamma,\delta}) \qquad (2.2.6)$$

For the two metrics given, the Christoffel symbols can be written

$$\begin{aligned}
&\Gamma^0_{01} = \Gamma^1_{11} = -\tfrac{f_{,1}}{2f} && \Gamma^1_{00} = \tfrac{ff_{,1}}{2} \\
&\Gamma^1_{22} = -fr && \Gamma^1_{33} = -fr\sin 2\theta \\
&\Gamma^2_{12} = \Gamma^3_{13} = \tfrac{1}{r} && \Gamma^2_{33} = -\sin\theta\cos\theta \\
&\Gamma^3_{23} = \cot\theta &&
\end{aligned} \tag{2.2.7}$$

and after a little algebra the Ricci scalar can be shown to be

$$R = \begin{cases} 12\chi^2 & \text{false vacuum} \\ 0 & \text{Schwarzschild} \end{cases} \tag{2.2.8}$$

Note; as an aside, if one considers the trace of equation 2.2.4 for the false vacuum region, then

$$\begin{aligned}
-R = 8\pi T^\mu_\mu &= -8\pi\rho\delta^\mu_\mu \\
\Rightarrow \qquad \chi^2 &= \frac{8}{3}\pi\rho
\end{aligned} \tag{2.2.9}$$

which gives the relation between the energy density ρ and the de Sitter parameter χ. Unfortunately, the relationship between the Schwarzschild mass parameter m and ρ is not as easily obtained, as the true and false vacua are as yet unlinked.

The fact that Einstein's field equation holds for the two component geometries is well known, and if each is considered separately, no dynamical information is revealed. However, if the field equation is evaluated across Σ (and it is assumed that the field equation holds on Σ) then it will contain information pertaining to the matching of the two spacetimes. This can then translate into dynamical information of Σ. As Σ is, in this simple analysis, treated as a boundary wall, the so called thin wall approximation [27], [29, refer to the paragraph that precedes equation 3.17] is assumed. This approximation requires that Σ be treated as a three surface embedded in the four-space, and so any variation intrinsic to Σ is on a much larger length scale than the negligible thickness of Σ. The existence of Σ implies that there must be a delta function contribution to the energy momentum tensor (otherwise Σ would not be discernible), and hence, by equation (2.2.4), a jump or discontinuity in the geometry at Σ. As mentioned, Σ's energy momentum tensor contribution is the surface energy momentum tensor $S_{\mu\nu}$, and so one can write (2.2.1) as

$$T_{\mu\nu} = -\rho\Theta(\mathcal{R} - r)g_{\mu\nu} + S_{\mu\nu}\delta(\mathcal{R} - r) \qquad (2.2.10)$$

with Θ being a Heaviside step function [28], and $\mathcal{R}$ representing the radius of Σ. With reference to the discontinuity in Σ, one can make a coordinate transformation to a new set of coordinates in which the new metric is continuous across Σ. In doing this, all the geometrical discontinuity is then represented in the normal derivative of the metric at Σ. Such a transformation seems sensible, as it will be shown that the discontinuity is expressible in terms of the extrinsic curvature of Σ. The extrinsic curvature of a surface is just the measure of curvature of an n-dimensional surface relative to the $(n+1)$-dimensional geometry that it is embedded in.

The next step is to develop the coordinate transformation to this new coordinate system, which is known as a Gaussian normal coordinate system (G.N.C.) [25, page 42]. The G.N.C. is such that there are (for this case of a four dimensional embedding space) three coordinates intrinsic to Σ and they are taken to be in a $(2+1)$ spacetime split. As spherical symmetry is maintained, the two angular coordinates θ and ϕ are unaltered by the transformation - only the timelike coordinate is affected. Given that this third intrinsic coordinate is timelike, the obvious choice of coordinate is the proper time parameter τ of an observer at rest with respect to Σ. The fourth and final coordinate in the G.N.C. is, by definition, a coordinate normal to Σ, and in this case it is spacelike. It will be given the label η, and by convention, the outward normal is to be taken as positive. Thus η is a measure of distance along a geodesic orthogonal to Σ. With this set of coordinates ($x^\mu = (\tau, \theta, \phi, \eta)$) the transformed metric has the following restrictions.

$$g^{\eta\eta} = g_{\eta\eta} \quad = 1 \tag{2.2.11}$$

$$g^{\eta\tau} = g_{\eta\tau} \quad = 0 \tag{2.2.12}$$

$$g^{\tau\tau} = g_{\tau\tau} \quad = -1 \qquad \text{only on } \Sigma \tag{2.2.13}$$

So, the transformation equations that give the new metric in terms of the old coordinates are

$$\begin{aligned} g_{\eta\eta} &= 1 = -f(\tfrac{\partial t}{\partial\eta})^2 + \tfrac{1}{f}(\tfrac{\partial r}{\partial\eta})^2 \\ g_{\tau\tau} &= -f(\tfrac{\partial t}{\partial\tau})^2 + \tfrac{1}{f}(\tfrac{\partial r}{\partial\tau})^2 \\ g_{\tau\eta} &= 0 = -f\tfrac{\partial t}{\partial\tau}\tfrac{\partial t}{\partial\eta} + \tfrac{1}{f}\tfrac{\partial r}{\partial\tau}\tfrac{\partial r}{\partial\eta} \end{aligned} \tag{2.2.14}$$

One also has the requirement that the G.N.C. move with Σ, which can be written as

$$r(\tau, \eta = 0) = \mathcal{R}(\tau) \tag{2.2.15}$$

Equations (2.2.14) and (2.2.15) give four equations with four unknown coefficients ($\frac{\partial t}{\partial\eta}$ etc.) which if one designates

$$\frac{\partial r}{\partial\tau} = \dot{\mathcal{R}} \tag{2.2.16}$$

gives a solvable set of equations. In terms f $\dot{\mathcal{R}}$ the other coefficients are

$$\begin{aligned} \tfrac{\partial r}{\partial\eta} &= \beta \qquad \tfrac{\partial t}{\partial\tau} = \pm\tfrac{\beta}{f} \\ \tfrac{\partial t}{\partial\eta} &= \pm\tfrac{\dot{\mathcal{R}}}{f} \quad \tfrac{\partial r}{\partial\tau} = \dot{\mathcal{R}} \end{aligned} \tag{2.2.17}$$

with $\beta = \pm\sqrt{\dot{\mathcal{R}}^2 + f}$. Here the Schwarzschild and de Sitter subscripts have been left off. However, (2.2.17) only gives the coefficients on Σ, which is not sufficient for a G.N.C. - these coefficients must be specified for off-Σ locations as well. This is done by examining the geodesic equations for the neighbourhood of Σ with τ, θ, and ϕ fixed. From this, one gets that

$$\begin{aligned} &\frac{\partial}{\partial\eta}\left(f\frac{\partial t}{\partial\eta}\right) = 0 \\ &\frac{\partial^2 r}{\partial\eta^2} - \frac{1}{2}\frac{df}{dr} = 0 \end{aligned} \tag{2.2.18}$$

When these two equations are integrated, with the on Σ coefficients as the constants of integration, it is found that the coefficients have the same form as (2.2.17) for off-Σ locations. Hence, if $\dot{\mathcal{R}}$ is replaced with $\dot{r}$ then (2.2.17) gives the coefficients for the local coordinate patch around Σ, and a G.N.C. has been defined in this region.

With the G.N.C., the surface Σ is easily specified: it is the $\eta = 0$ hypersurface. The curvature of Σ in the four-space is just the extrinsic curvature of Σ and it is given by

$$K_{\mu\nu} = \xi_{\mu|\nu} \tag{2.2.19}$$

Here $_|$ represents the four-dimensional covariant derivative, and ξ_μ is the normal to Σ, and in the G.N.C. has the form $\xi_\mu = (0, 0, 0, 1)$. Thus, the use of the G.N.C. reduces the extrinsic curvature to

$$K_{\mu\nu} = -\Gamma^{\eta}_{\mu\nu} = \frac{1}{2}\partial_\eta g_{\mu\nu} \tag{2.2.20}$$

and conversely,

$$K^i_j = \Gamma^i_{\eta j} \quad K_{ij} = -\Gamma^{\eta}_{ij} \tag{2.2.21}$$

From 2.2.20, it becomes obvious that the extrinsic curvature is apt to describe the discontinuity in the geometry at Σ; it is just the normal derivative of the metric, and as $g_{\mu\nu}$ is continuous at Σ (in the G.N.C.) this is precisely the geometrical discontinuity implied by the existence of $S_{\mu\nu}$. Evaluating the non-zero components of $K_{\mu\nu}$ in the G.N.C., one gets

$$
\begin{aligned}
K_{\tau\tau} &= \frac{-1}{\dot{r}}\frac{\partial\beta^2}{\partial\tau} \\
K_{\theta\theta} &= r\beta \qquad\qquad (2.2.22) \\
K_{\phi\phi} &= r\beta\sin^2\theta
\end{aligned}
$$

Now in order to utilize this notion of extrinsic curvature, it has to be linked to Einstein's field equation. Such a linking is achieved in the formalism of Gauss and Codazzi [26, section §21.5], [27] which uses 2.2.21 to re-express an n-dimensional field equation in terms of an $(n-1)$-dimensional field equation and the curvature of the hypersurface slicing. For a general four dimensional spacetime the Gauss-Codazzi equations (in G.N.C.) are

$$G^\eta_\eta = -\frac{1}{2}{}^3R + \{(K^m_m)^2 - K^{ij}K_{ij}\} = 8\pi T^\eta_\eta \qquad (2.2.23)$$

$$G^\eta_i = K^m_{i|m} - (K^m_m)_{|i} = 8\pi T^\eta_i \qquad (2.2.24)$$

$$
\begin{aligned}
G^i_j &= {}^3G^i_j - (K^i_j - \delta^i_j K^m_m)_{,\eta} - (K^m_m)K^i_j + \frac{1}{2}\delta^i_j\{K^{mn}K_{mn} + (K^m_m)^2\} \qquad (2.2.25) \\
&= 8\pi T^i_j
\end{aligned}
$$

Given this component form of the field equation, one can substitute for the energy momentum tensor (equation 2.2.10), and check for consistency. For off-Σ evaluation of the field equation (i.e. $\eta \neq 0$), no new dynamical information is obtained. Further, if one considers an evaluation on Σ itself (i.e. $\eta = 0$), then equations 2.2.23 and 2.2.24 are satisfied if g_{ij} is continuous across Σ (which is the case). In order to evaluate equation 2.2.25 across Σ, one must consider the contribution from the $(\delta^1_j K^m_m - K^i_j)_{,\eta}$ which has a delta function contribution at Σ , and so is the only term of G^i_j to survive an integration over the thickness of the wall. As one is using the thin wall approximation, this integration of this term reduces to the jump in $\delta^i_j K^m_m - K^i_j$. The jump in across Σ of a quantity is symbolised by [ ] and is defined by

$$[B] = \lim_{\epsilon \to 0} (B(\eta = +\epsilon) - B(\eta = -\epsilon)) \tag{2.2.26}$$

Thus equations 2.2.10 and 2.2.25 (in the thin wall approximation) give that

$$\begin{aligned} \int_\Sigma G^i_j \sqrt{-{}^4g}\, d^4x &= 8\pi \int_\Sigma T^i_j \sqrt{-{}^4g}\, d^4x \\ \Rightarrow \qquad [K^i_j] - \delta^i_j [K^m_m] &= -8\pi S^i_j \end{aligned} \tag{2.2.27}$$

Considering the trace of equation 2.2.27, one gets

$$[K^m_m] = 4\pi S^m_m = 4\pi S \tag{2.2.28}$$

and equation 2.2.27 can be rewritten as

$$[K_{ij}] = -8\pi(S_{ij} - \frac{1}{2} h_{ij} S) \tag{2.2.29}$$

with h_{ij} as Σ's intrinsic three metric.

Equation 2.2.29 is the classical equation of motion, and as such holds all the information on the classical trajectories of Σ. Yet this equation is not yet in a workable form, as $S_{\mu\nu}$ is unspecified. In order to elucidate the characteristics of $S_{\mu\nu}$ one must resort to the generic classical property of energy momentum conservation which is formulated as

$$T^{\mu\nu}{}_{|\nu} = 0 \tag{2.2.30}$$

When written in the Gauss-Codazzi formalism, equation 2.2.30 splits into the two following equations.

$$\begin{aligned} T^{i\nu}{}_{|\nu} &= T^{ij}{}_{;j} + T^{i\eta}{}_{,\eta} + 2K^i_j T^{j\eta} + K^m_m T^{i\eta} = 0 \\ T^{\eta\nu}{}_{|\nu} &= T^{\eta j}{}_{;j} + T^{\eta\eta}{}_{,\eta} - K_{ij} T^{ij} + K^m_m T^{\eta\eta} = 0 \end{aligned} \tag{2.2.31}$$

with ; representing the three dimensional covariant derivative. If the jump of equation 2.2.31 is then considered, it can be reduced to

$$0 = [S^{ij}{}_{;j} + 2K^{i}_{j}S^{j\eta}]\delta(\eta) + S^{i\eta}\delta'(\eta) \tag{2.2.32}$$

Here $'$ denotes differentiation with respect to η. In dealing with this expression, one must be careful, as $\delta'(\eta)$ is discontinuous at $\eta = 0$, which is exactly where the jump in K_{ij} is evaluated. This evaluation ambiguity is remedied by requiring $S^{i\eta} = S_{i\eta} = 0$. Thus,

$$S^{ij}{}_{;j} = 0 \tag{2.2.33}$$

Similarly equation 2.2.31 gives

$$0 = [\rho - \tilde{K}_{ij}S^{ij} + K^{m}_{m}S^{\eta\eta}]\delta(\eta) + S^{\eta\eta}\delta'(\eta) \tag{2.2.34}$$

with

$$\tilde{K}_{ij} = \lim_{\epsilon \to 0} \frac{1}{2}(K_{ij}(\eta = +\epsilon) + K_{ij}(\eta = -\epsilon)) \tag{2.2.35}$$

Again, the apparent ambiguity is resolved by taking the δ' coefficient to be zero. That is, $S^{\eta\eta} = 0$, and $\tilde{K}_{ij}S^{ij} = \rho$. Thus

$$S^{\eta\mu} = 0 \tag{2.2.36}$$

Still, one can go further in constraining the form of $S_{\mu\nu}$ by means of the thin wall approximation and spherical symmetry. The spherical symmetry permits the imposition of the constraint that $S_{\mu\nu}$ be rotationally invariant, and so $S_{\mu\nu}$ must be of the form

$$S^{\mu\nu} = \sigma U^{\mu}U^{\nu} - \zeta(h^{\mu\nu} + U^{\mu}U^{\nu}) \tag{2.2.37}$$

with $h^{\mu\nu} = g^{\mu\nu} - \xi^{\mu}\xi^{\nu}$ being the intrinsic metric projected onto Σ. U^{μ} is Σ's four velocity, and in the G.N.C. it has the normalized form $U^{\mu} = (1,0,0,0)$. The two parameters, σ and ζ can be thought of as the surface energy density and the surface tension respectively. Yet by the thin wall approximation $S^{\mu\nu}$ can only depend on quantities intrinsic to the surface, and so from equation 2.2.37, $\sigma = \zeta$ and

$$S^{\mu\nu} = -\sigma h^{\mu\nu}(\eta = 0) \tag{2.2.38}$$

This then gives the dynamical equation of Σ as

$$[K_{ij}] = -4\pi\sigma h_{ij} \tag{2.2.39}$$

From this there are three non-zero component equations, as K_{ij} is diagonal, but $K^{\theta}_{\theta} = K^{\phi}_{\phi}$ by spherical symmetry, so only two of the three equations are independent. On further investigation, a little algebra shows that the $K_{\tau\tau}$ equation is just the proper time derivative of the $K_{\theta\theta}$ equation [29, the paragraph that follows equation 4.28] (due to the energy momentum conservation law (equation 2.2.30) that results from Einstein's field equation), so there is actually just one independent equation of motion (which seems reasonable, as there is only one degree of freedom due to the spherical symmetry). Hence, the equation of motion that is to be analysed is

$$[K_{\theta\theta}] = -4\pi\sigma r^2 \tag{2.2.40}$$

From 2.2.22 $K_{\theta\theta} = r\beta$, so the explicit form of 2.2.40 is

$$\begin{gathered} \beta_D - \beta_S = 4\pi\sigma r \\ \beta = \pm\sqrt{\dot{r}^2 + f} \end{gathered} \tag{2.2.41}$$

In order to use this equation to classify the possible trajectories of Σ, a little rearrangement is required. If one considers the square of $\beta_D = 4\pi\sigma r + \beta_S$ and β^2 is replaced by $\dot{r}^2 + f$ then one obtains an expression that is linear in β_S. By again rearranging and squaring this new expression the β_S dependence is removed and equation 2.2.41 takes the form

$$\dot{r}^2 + V(r, m, \chi) = -1 \tag{2.2.42}$$

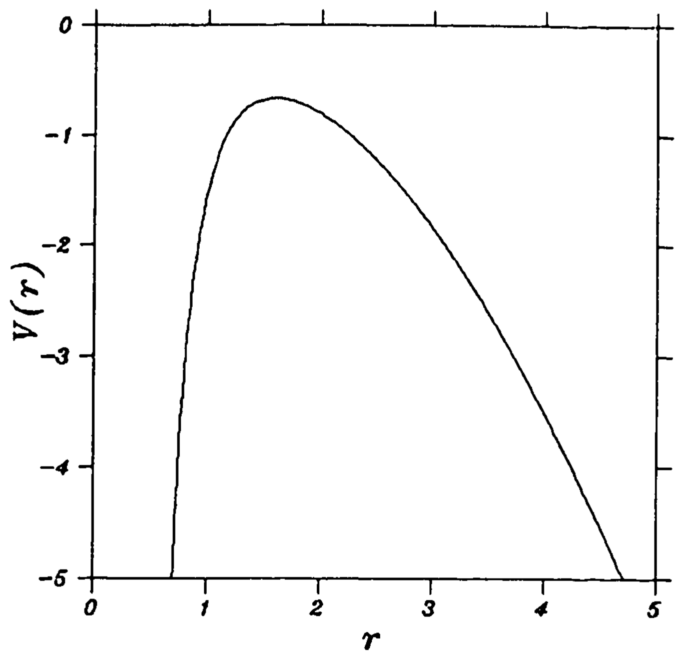

Figure 2.1: The tunnelling potential $V(r, m, \chi)$ with $m = 0.001kg$ and $\chi = 0.001m^{-1}$. (These values place the Schwarzschild and de Sitter horizons at a small and a large radius respectively.)

with

$$V(r, m, \chi) = -\frac{2m}{r} - \frac{1}{4\kappa^2 r^2}(\frac{2m}{r} - (\chi^2 + \kappa^2)r^2)^2 \qquad (2.2.43)$$

with $\kappa = 4\pi\sigma$.

Equation 2.2.42 makes the dynamics of Σ obvious; equation 2.2.42 has the mathematical form that is equivalent to a particle, with rest energy $E = -1$, moving in a one dimensional potential field. Hence, using the analogy to model the situation, one can make a plot of the potential as a function of r (figure 2.1), and as can be seen, $V(r, m, \chi)$ has the form of a potential barrier. This potential barrier creates three general classes of possible Σ trajectories, which have been labelled S1, S2, and S3 (refer to figures 2.2 and 2.3). The S1 class is such that in the particle

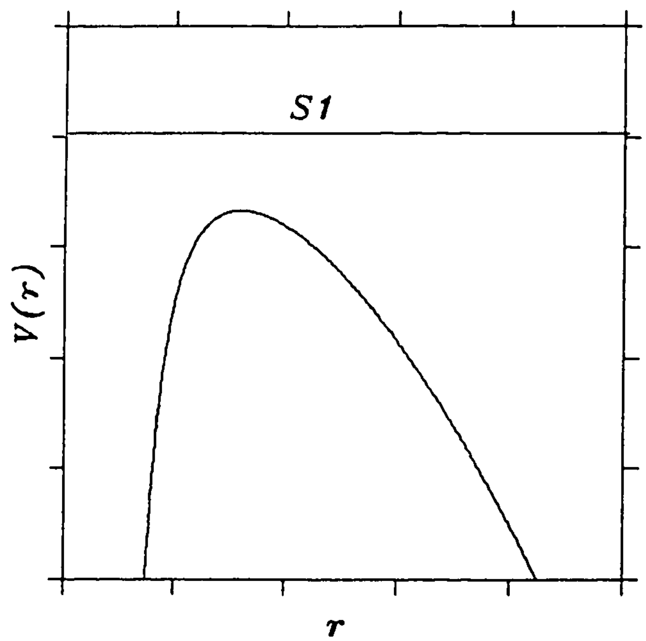

Figure 2.2: The monotonic solution. The trajectory $V = -1$, labelled by S1, is not obstructed by the potential barrier. This allows the seed to expand from the origin, and continue expanding without hinderence, until the false vacuum decays.

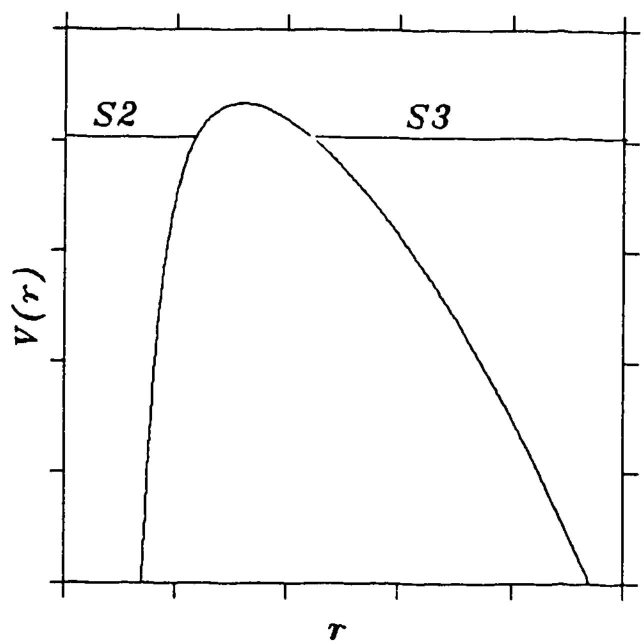

Figure 2.3: The Bounded and Bounce trajectories. For the $V = -1$ trajectory labelled S2, the seed expands to a maximum radius, and then collapses (i.e. a bounded solution). For the S3 solution, the trajectory starts at very large r, collapses to a minimum radius, then expands back out to infinity (i.e. a bounce solution).

analogy, a particle starting from the origin would move outwards, and because it would not encounter the potential barrier, it would continue to move to larger r. This type of solution is referred to as the monotonic solution, and it does possess the expansion property that is required for the evolution of a new universe seed (i.e. it expands to a size large enough to permit an inflationary era). The reason for the continual expansion is that Σ is expanding too rapidly for the pull of the negative pressure of the false vacuum. Thus Σ cannot be brought to a halt, and thence collapse, before the seed attains an inflationary state. Unfortunately, this solution class is not suitable, due to it having a singular initial condition (more on this later).

With S1 being ruled out, S2 and S3 are left as possible Σ trajectory classes. Both S2 and S3 are not defined over all spacetime, but rather, are defined in a certain classically allowed region which is defined by the presence of the potential barrier. The S2 class, if viewed in the analogy, has the particle moving outwards, but with the outward motion being curbed, halted, and then reversed. Thus, the false vacuum seed expands out to a maximum, and then collapses due to the pull of the false vacuum pressure. This type of solution is called a bounded solution.

On the other hand, a trajectory from the S3 class starts with Σ being at a very large radius and collapsing to a minimum and then expanding back out to a large radius. Such a class of solution could physically be seen as one where the spacetime exterior to the seed is the false vacuum, and the interior is the true vacuum (i.e. a reversal of the proposed scenario). This type of solution is referred to as a bounce solution.

In considering these two solution classes as possible solutions to the creation and evolution of a seed spacetime, one must discern whether either class is (given the technology) constructible in the laboratory. That is, it must be checked that the trajectories have non-singular origins or starting points. Unfortunately, the present coordinates (either (t, r, θ, ϕ) or the G.N.C.) are unsuitable for the addressing of such a question. The reason for this is that despite the fact that they are immediately understandable in terms of physical meaning, these coordinates imply the existence of a singularity at the horizon. However, this is a non-physical singularity that is due entirely to the choice of coordinates, as the Kretschmann scalar [26, page 822] is regular at the various horizons. Thus a more suitable coordinate system is required.

As is well known, Kruskal-Szekeres coordinates [30] can be used to describe the true vacuum Schwarzschild region, and in an attempt to maintain the similarity between the interior and the exterior coordinates, Gibbon-Hawking coordinates shall be used to describe the de Sitter region.

The Kruskal-Szekeres coordinates are given by

$$\begin{aligned} U &= \sqrt{\tfrac{r}{2m} - 1}\,\exp(\tfrac{r}{4m})\cosh(\tfrac{t}{4m}) \\ V &= \sqrt{\tfrac{r}{2m} - 1}\,\exp(\tfrac{r}{4m})\sinh(\tfrac{t}{4m}) \end{aligned} \tag{2.2.44}$$

for $r > 2m$ (region I), and

$$\begin{aligned} U &= \sqrt{1 - \tfrac{r}{2m}}\,\exp(\tfrac{r}{4m})\sinh(\tfrac{t}{4m}) \\ V &= \sqrt{1 - \tfrac{r}{2m}}\,\exp(\tfrac{r}{4m})\cosh(\tfrac{t}{4m}) \end{aligned} \tag{2.2.45}$$

for $r < 2m$ (region II). On substitution into the Schwarzschild line element, one gets

$$ds^2 = \frac{32m^3}{r}\exp(\frac{-r}{2m})(dU^2 - dV^2) + r^2 d\Omega_2^2 \tag{2.2.46}$$

which has no singularity at $r = r_0 = 2m$. Such a coordinate transformation maps the Schwarzschild space into the $U+V \geq 0$ region, and imposes a boundary at $U+V = 0$.

This boundary is seen as physical, as one would encounter this boundary in a finite time if travelling along a past directed timelike trajectory. Yet the existence of such a boundary is somewhat artificial, so it is generally accepted that the spacetime described by (U, V) is not restricted by the boundary, but is instead described by all values of U and V and the relation $U^2 - V^2 > -1$. Hence there are two more regions to the spacetime, namely

$$U = -\sqrt{\tfrac{r}{2m} - 1}\exp(\tfrac{r}{4m})\cosh(\tfrac{t}{4m})$$
$$V = -\sqrt{\tfrac{r}{2m} - 1}\exp(\tfrac{r}{4m})\sinh(\tfrac{t}{4m}) \tag{2.2.47}$$

for $r > 2m$ (region III), and

$$U = -\sqrt{1 - \tfrac{r}{2m}}\exp(\tfrac{r}{4m})\sinh(\tfrac{t}{4m})$$
$$V = -\sqrt{1 - \tfrac{r}{2m}}\exp(\tfrac{r}{4m})\cosh(\tfrac{t}{4m}) \tag{2.2.48}$$

for $r < 2m$ (region IV).

Similarly, the de Sitter spacetime can be extended by Gibbons-Hawking coordinates [31], and the apparent singularity at the horizon removed. Again, four different regions are established, and the coordinate transformations for the quadrants are:

$$u = \sqrt{\tfrac{1-\chi r}{1+\chi r}}\cosh(\chi t)$$
$$v = \sqrt{\tfrac{1-\chi r}{1+\chi r}}\sinh(\chi t) \tag{2.2.49}$$

for (region I),

$$u = \sqrt{\tfrac{\chi r-1}{1+\chi r}}\sinh(\chi t)$$
$$v = \sqrt{\tfrac{\chi r-1}{1+\chi r}}\cosh(\chi t) \tag{2.2.50}$$

for (region II),

$$u = -\sqrt{\tfrac{1-\chi r}{1+\chi r}}\cosh(\chi t)$$
$$v = -\sqrt{\tfrac{1-\chi r}{1+\chi r}}\sinh(\chi t) \qquad (2.2.51)$$

for (region III), and

$$u = \sqrt{\tfrac{\chi r-1}{1+\chi r}}\sinh(\chi t)$$
$$v = \sqrt{\tfrac{\chi r-1}{1+\chi r}}\cosh(\chi t) \qquad (2.2.52)$$

for (region IV), with a corresponding line element of

$$ds^2 = \frac{(1+\chi r)^2}{\chi^2}(du^2 - dv^2) + r^2 d\Omega_2^2 \qquad (2.2.53)$$

As in the Schwarzschild case, the singularity at the boundary between regions I and II and regions III and IV has been removed. Both coordinate systems have their salient features given by the associated spacetime diagrams (figures 2.4, 2.5). These diagrams have the angular coordinates θ and ϕ are suppressed, and the point $r = 0$ has been deformed to an $r = 0$ curve. Also, it should be noted that it is not possible to have a trajectory that evolves to a radial size that is different for a de Sitter observer and a Schwarzschild observer. This is because the spherical symmetry implies that the two sphere line elements have the same radius at Σ, making r a coordinate common to both spacetimes.

With these new coordinates established, the next step is to ascertain which regions of the spacetimes the Σ trajectories actually pass through. To do this, a tracing prescription is required. One such prescription is obtained by considering the time rate of change with respect to the proper time τ. From equation 2.2.17

$$\dot{t} = \tfrac{\pm\beta}{f}$$
$$\Rightarrow \qquad \beta = \pm f\dot{t} \qquad (2.2.54)$$

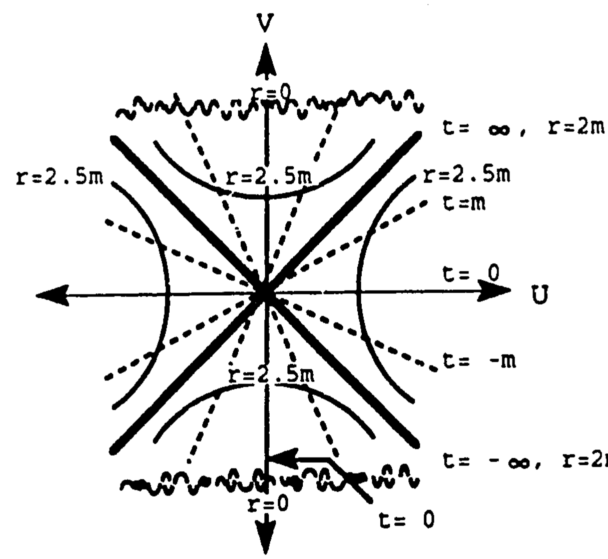

Figure 2.4: The Kruskal-Szekeres coordinate system.

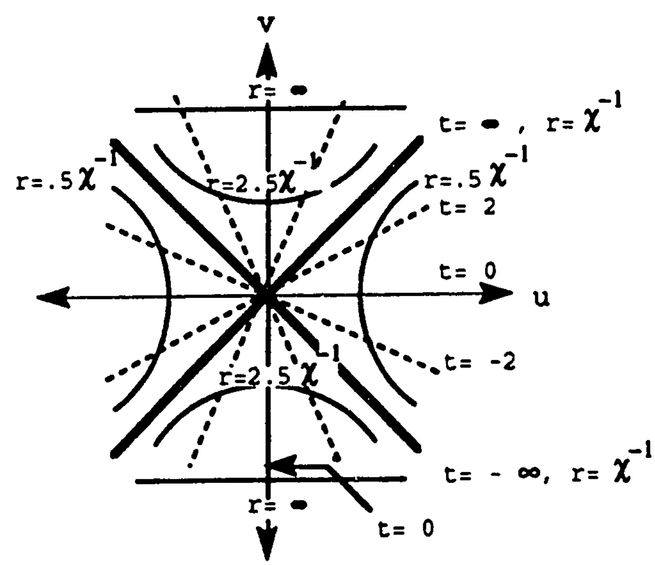

Figure 2.5: The Gibbons-Hawking coordinate system.

which gives that the sign of β depends on the sign of $\dot{t}$. Further, when Σ is traversed, the sign of $\dot{t}$ is related to the change in the polar angle $\arctan \frac{V}{U}$ for the Schwarzschild case and $\arctan \frac{v}{u}$ for the de Sitter space. An increasing polar angle implies that β is strictly positive or strictly negative, depending on the $\pm$ that is generated by the square root. For the Schwarzschild case, with the polar angle increasing, $\dot{t}_S$ is positive, and so β_S is positive. However, in the Gibbons-Hawking diagram $\dot{t}_D$ is negative as Σ is traversed, and so β_D is negative for an increasing polar angle. Note: for either spacetime, if $\beta = 0$ then the polar angle remains unaltered, which means that the trajectory is radial. Thus, if for example, $\beta_S = 0$ for Σ at a radius $r > 2m$ then as the trajectory is tangential to a radial line, it is in fact lightlike at this point. Such a transition (within region I) is seen as not physically acceptable, as β_S does not have zero as a stationary point, and so the trajectory would be changing from timelike to spacelike evolution. This sort of transition lacks a physical explanation. Similarly, one gets another physically unacceptable trajectory if $\beta_D = 0$ for $r < \frac{1}{\chi}$. Hence, $\beta = 0$ provides a way of eliminating some of the possible Σ trajectories.

By squaring equation 2.2.41 and then rearranging, it is easily shown that $\beta_S = 0$ when

$$r^3 = \frac{2m}{\chi^2 + \kappa^2} \tag{2.2.55}$$

and by substituting the expression for β_S used in the derivation of equation 2.2.55 into equation 2.2.41, one finds that $\beta_D = 0$ when

$$r^3 = \frac{2m}{\chi^2 - \kappa^2} \tag{2.2.56}$$

and $\kappa^2 < \chi^2$. For $\kappa^2 \geq \chi^2$, β_D is never zero (i.e. it is always negative. From this, the S2 and S3 classes of Σ trajectories can be split into a more detailed classification, which is obtained by comparing the actual path of the "particle" of rest energy $E = -1$ with its position relative to the $V(r_S)$ and $V(r_D)$ lines. Here $V(r_S)$ and

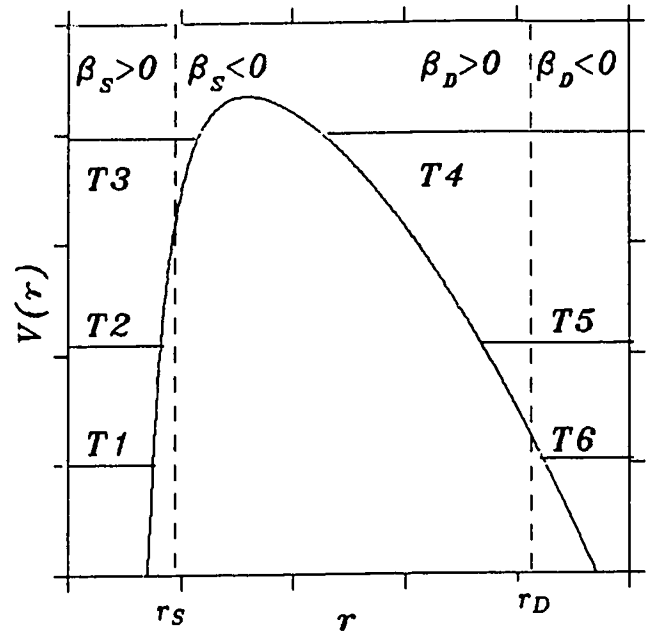

Figure 2.6: A diagrammatic classification of the sub-classes of S2 and S3 trajectories. This diagram is really three diagrams in one, as each horizontal line is to be considered as $V = -1$. Thus the T1, T2, and T3 should be considered separately. Likewise for T4, T5, and T6.

$V(r_D)$ are defined as the values of the potential $V(r)$ (m and χ fixed) for which $\beta_S = 0$ with $r = r_S$, and $\beta_D = 0$ with $r = r_D$ respectively. Depending on the value of ρ (and hence m and χ), the potential will be raised or lowered (and slightly deformed) in such a way that $V = -1$ is above or below either $V = V(r_S)$ and $V = V(r_D)$. Hence, as shown on figures 2.6 and 2.7, the S2 and S3 classes split up into three different cases. For each sub-class there are six S2 and S3 trajectories. However, the T2, and T7 trajectories are of the same structure as the T1 trajectory. Similarly, the other trajectories can be grouped into sets of trajectories of similar structure, and the sets are (T1, T2, T7), (T3, T8, T9), (T4, T5, T10), and (T6, T11, T12). Now these four different Σ trajectories have quite different spacetime

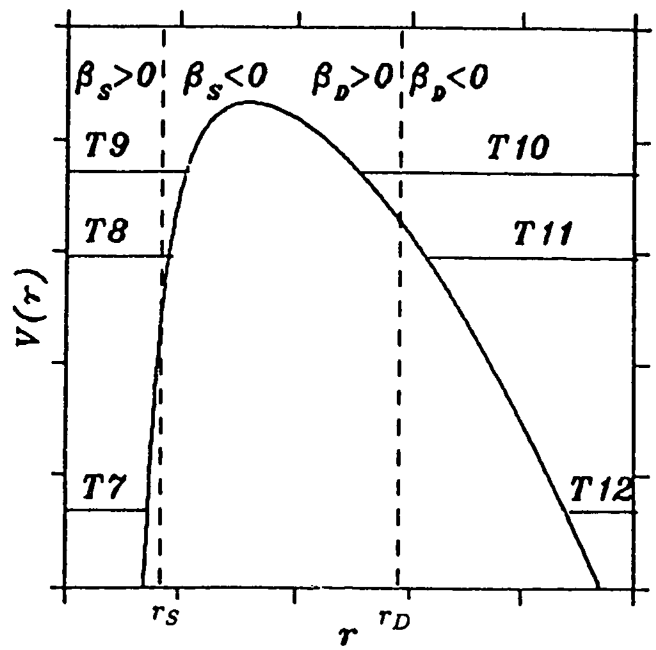

Figure 2.7: More sub-class classifications of the S2 and S3 trajectories. Note, the interpretation of this graph is the same as for figure 2.6.

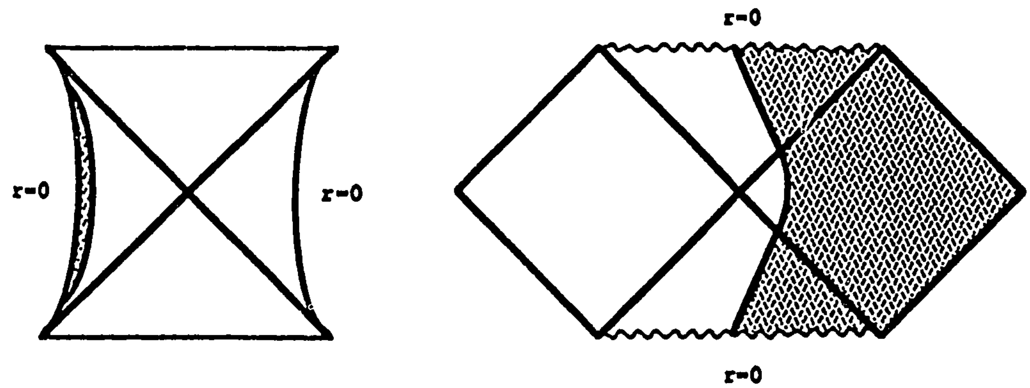

Figure 2.8: The spacetime diagram for a T1 trajectory displayed in composite form. The spacetime is given by the shaded regions.

diagrams, as seen in figures 2.8, 2.9, 2.10, 2.11, and 2.12.

Of these four trajectory groups, some or all may not be suitable for describing a seed spacetime that is constructible in the laboratory, for which the main criterion is a non singular origin or starting point. Yet, from figures 2.8 and 2.9, it appears that all the bounded trajectories start from $r = 0$, which in the Kruskal-Szekeres diagram, is singular, and so plagued by the $r = 0$ Schwarzschild singularity. This singularity must be avoided if the seed spacetime is to be constructible, yet the bounce solutions, with their large r starting condition are also not physically feasible. Fortunately, it may be that not all the bounded trajectories necessarily require an initial singularity. The reason for this lies in a theorem by Penrose [32]. The theorem states that if a spacetime is such that it has

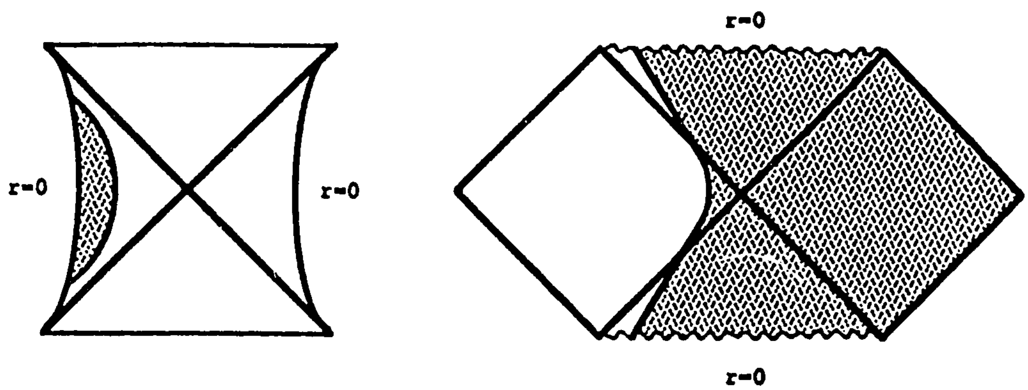

Figure 2.9: The spacetime diagram for a T3 trajectory. For these solutions, Σ never passes through quadrant one of the Kruskal-Szekeres diagram.

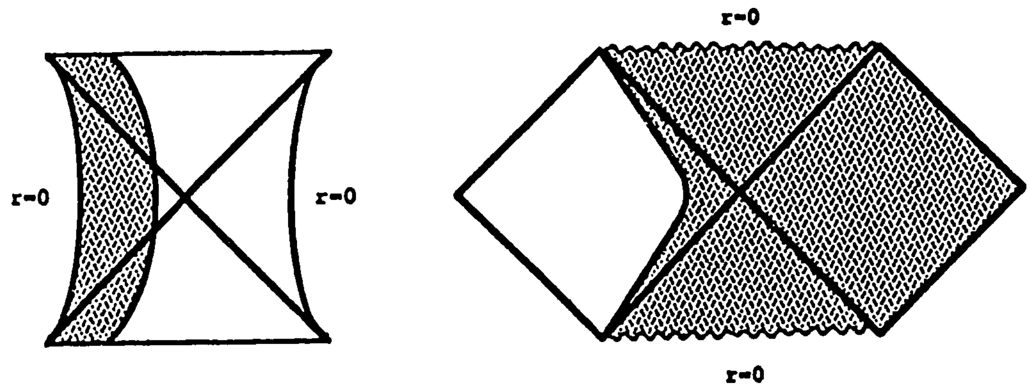

Figure 2.10: The spacetime diagram for a T4 trajectory. This is a bounce solution, and so has both a deflationary and inflationary era (at early and late times respectivly).

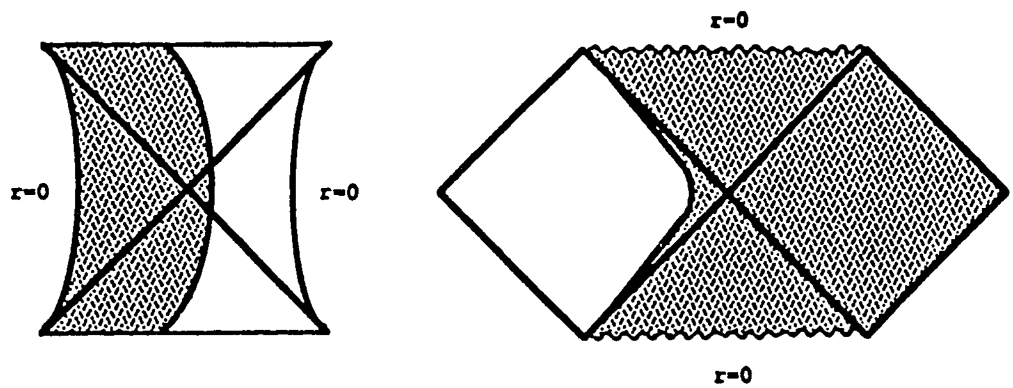

Figure 2.11: The spacetime diagram for a T6 trajectory. Again, one has the deflationary and inflationary eras.

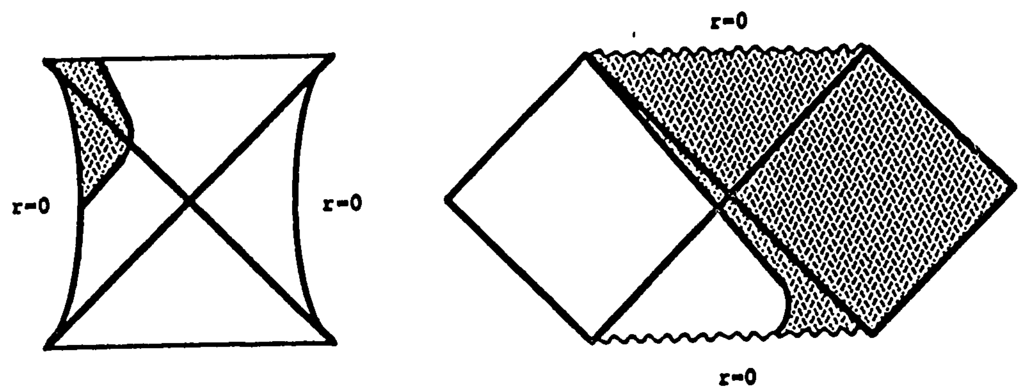

Figure 2.12: The spacetime diagram for the S1 or monotonic solution. Note the late time inflationary era. This is the type of trajectory used to describe the inflationary stage associated with the observed universe.

i a non-compact Cauchy hypersurface,

ii it satisfies the (very) weak energy condition,

iii there exists an anti-trapped surface

then an initial singularity must exist. The (very) weak energy condition is defined as $T_{\mu\nu}U^{\mu}U^{\nu} \geq 0$ for all null vectors U^{μ}, and it is valid as long as the magnitude of the pressure does not exceed that of the energy density ρ. Note, the qualifier (very) is used as U^{μ} is lightlike instead of being timelike as in the standard definition of the weak energy condition.

As the mother spacetime is asymptotically flat, and a well defined Cauchy evolution is assumed, the first requirement is satisfied. Similarly there are no problems with the (very) weak energy condition, as one is dealing with the classically allowed false and true vacuum regions. So this implies that the existence of an initial singularity is dependent on the presence of a closed anti-trapped surface [33]. If the trajectory, at any stage, forms a closed anti-trapped surface, then an initial singularity is required somewhere in the trajectory's past.

For a surface to be anti-trapped, it must have that at any point on that surface, both the in and out going normal incident rays arriving at that point are diverging. On the spacetime diagrams (where a hypersurface is represented by a point), the light rays are shown as 45° lines. In order to tell if the in and out going rays are diverging, consider the light cone formed by the past directed rays of the surface, as shown in figure 2.13. The surface S is then anti-trapped if the radial coordinate increases as one moves along the lightlike lines towards S.

As can be seen, this is the case for the trajectory sub-class T3 (as well as the monotonic solution, S1), and so this trajectory class necessarily has an initial

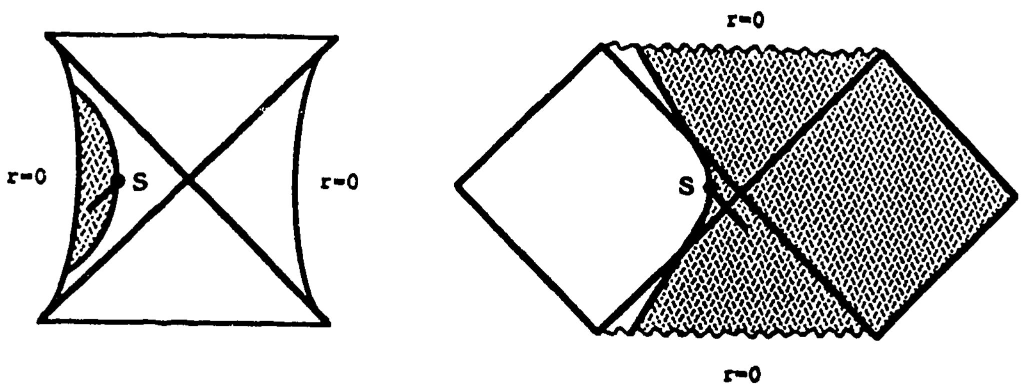

Figure 2.13: An example of a closed anti-trapped surface labelled S.

singularity as a starting condition. Only the T1 bounded solution class lacks any anti-trapped surfaces, and so does not necessarily start from a singularity (i.e. it is theoretically constructible in the laboratory). Further, this trajectory class passes into region I of the Kruskal-Szekeres diagram, which is taken to represent the observed or laboratory spacetime. This means that the seed spacetime T1 could be constructed without an initial singularity, and the construction could be done in a region that is accessible (i.e. the laboratory). In restricting the possibilities to the T1 class, one constrains the parameters ρ (and hence χ), m, and κ. This constraining of the parameters is due to the fact that the T1 class has $r = r_1$ such that $V(r_1) = -1$ and $r_1 < r_S$ (with r_S defined by equation 2.2.55). For a typical T1 trajectory solution, possible values of the parameters (which are those used in figure

2.1) are:

$$\chi = 0.001m^{-1} \quad \Rightarrow \quad \rho = 0.045kg.s^{-2}.m^{-1}$$
$$m = 0.001kg \qquad\qquad \kappa = 1kg.s^{-2} \tag{2.2.57}$$

Note, when considering these parameter values for the T1 class, it should be remembered that the smaller the value of ρ the further out (radially) the de Sitter horizon, and the closer in the Schwarzschild horizon. Also, these values are given under the prescription $G = 1$.

Unfortunately, whilst the T1 class has the right initial conditions, it does not constitute a new universe trajectory, as it is a bounded solution and so eventually collapses back to small r In other words it fails in the second criterion - that of evolving to a stage that tends toward an inflationary era (or at least substantial expansion). Several of the bounce solutions exhibit this trait; namely trajectories T4 and T6, yet neither has a suitable starting condition.

So after studying the classical solutions to the equation of motion for Σ, no trajectories have been found that are suitable for a constructible seed spacetime that is to evolve into a new universe. What is required is a combination of the first part of the T1 solution and the latter part of a bounce solution (either the T4 or T6 solution). Such a combination would resemble that of the monotonic solution, except that it would have an intermediary stage, and no singular starting point. For definiteness, the two classical trajectory regions are to be taken from the subclasses T1 and T6 respective, which then gives a universe fabrication trajectory that diagrammatically is of the form shown in figure 2.14. With this type of trajectory, the intermediary stage would be equivalent to a one dimensional particle tunneling through the potential barrier. However, due to the lack of a theory of quantum gravity, and topological changes, the analogy to tunneling can only be seen as a first

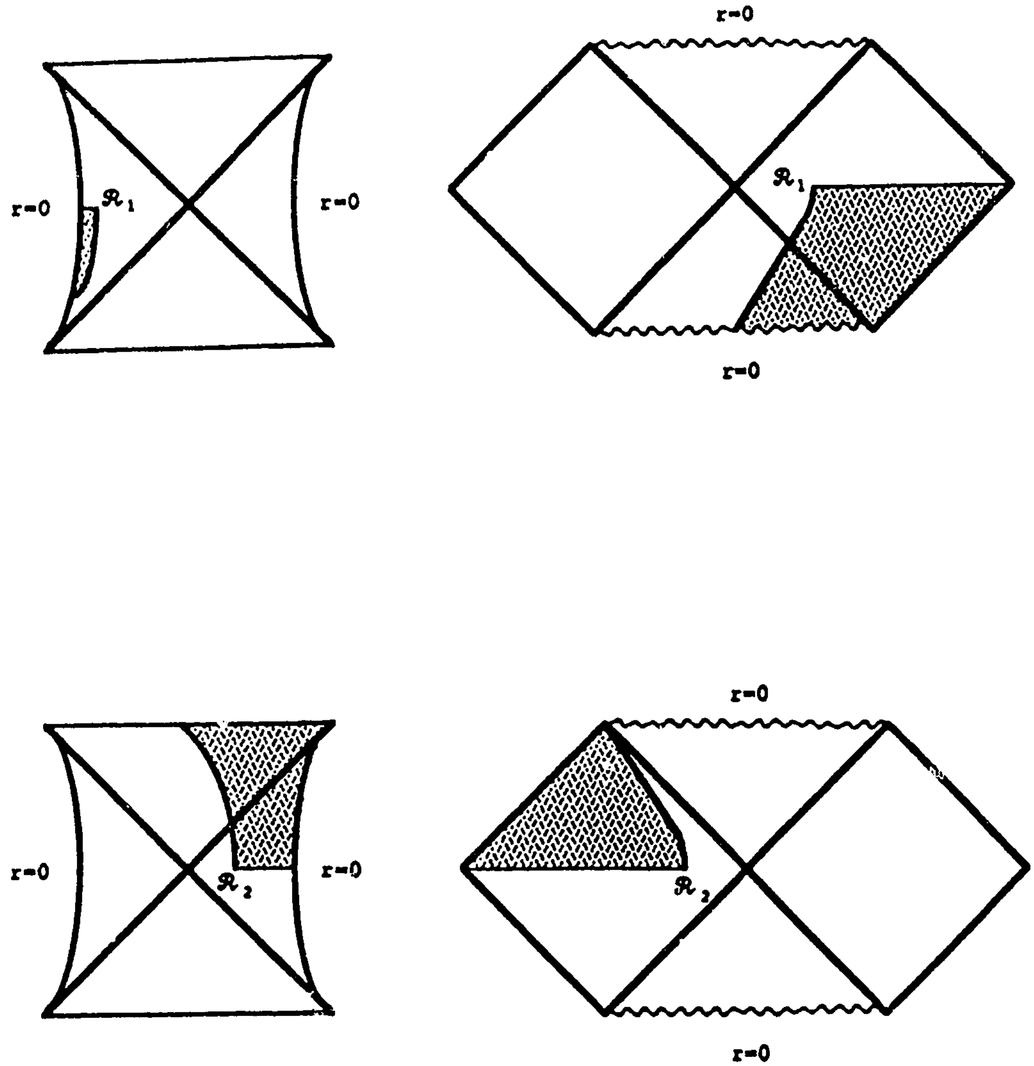

Figure 2.14: The trajectory for a universe fabrication via quantum tunnelling. The upper composite spacetime diagram shows the T1 solution, whilst the lower one is the T6 solution. $\mathcal{R}_1$ and $\mathcal{R}_2$ represent the radius of Σ at the onset and the completion of the intermediary stage. With such a trajectory, it can be seen that both the false vacuum and the true vacuum regions causally disconnect themselves when tunnelling occurs.

order approximation.

2.3 The Transmission Coefficient

Although the classical trajectories of Σ do not give the desired seed spacetime, they do suggest a possible spacetime trajectory that would correspond to a constructible seed spacetime that would evolve into an inflationary era. As mentioned above, this spacetime trajectory is actually a combination of two of the classical trajectories, with the linking between the two being provided by a quantum mechanical tunneling through the classically forbidden region. As the bounded and the bounce segments of this desired trajectory can be adequately described by the classical theory, it is only the transition from one classical turning point (that of the bounded solution) to the other turning point (bounce solution) that requires attention. It is this transition that has to be described quantum mechanically.

In order to describe this crossing of the classically forbidden region, one can use the fact that the classical theory reduces to a particle moving in one dimension in a potential 2.2.42. (This is where Fischler et. al. [44] claims Farhi et. al. [39] goes wrong as they only retain the one degree of freedom). With this analogy in place, one can quantise by treating the position r and the momentum p as operators instead of variables. The system is then aptly given by the Schrodinger equation. Thus, writing Ψ as a shorthand notation for $|\Psi >$, the state vector of the particle (the particle, due to the analogy, is synonymous with Σ), then

$$i\hbar\frac{\partial}{\partial t}\Psi = H\Psi \tag{2.3.58}$$

with H being the Hamiltonian. Classically, H has the form

$$H = \frac{p^2}{2m} + V \tag{2.3.59}$$

and is a constant of the motion, whereby $H = E =$ constant. This latter constraint, when applied to the quantised system, simply gives $H\Psi = E\Psi$ with H now being an operator and E an eigenvalue of H.

Now using standard quantum mechanics, separation of variables and equation 2.3.58 give

$$\Psi(r,t) = \exp(-\frac{iEt}{\hbar})\psi(r) \tag{2.3.60}$$

Due to the conversion of p to an operator ($p \rightarrow -i\hbar\frac{\partial}{\partial r}$), the constraint (equation 2.3.59) has the form $H\psi(r) = E\psi(r)$, and so

$$[\frac{\partial^2}{\partial r^2} + \frac{2m}{\hbar}(E - V(r))]\psi = 0 \tag{2.3.61}$$

If one makes the further substitution $\psi = \exp(\frac{i\phi}{\hbar})$ then

$$-(\frac{\phi'}{\hbar})^2 + \frac{i\phi''}{\hbar} + \frac{2m}{\hbar^2}(E - V(r)) = 0 \tag{2.3.62}$$

with $'$ denoting differentiation with respect to r. The reason for this substitution can now be made clear; it is used to obtain a semi-classical approximation to the quantised theory. The approximation is done by writing ϕ as an expansion in powers of $\hbar$ and then treating $\hbar$ as small. The reason for taking the $\hbar \rightarrow 0$ limit is that in this limit the de Broglie wavelength tends to zero, which is indicative of the classical theory.

As mentioned, $\hbar$ is considered small but non-zero, and if the ϕ expansion is taken only to first order in $\hbar$ then one has

$$\phi = \phi_0 + \hbar\phi_1 \tag{2.3.63}$$

which constitutes what is commonly referred to as the WKB approximation. Substituting this into equation 2.3.62, and keeping terms up $O(\hbar^0)$, one obtains two conditions, namely

$$0 = p^2 - (\phi_0')^2 \tag{2.3.64}$$

and

$$0 = i\phi_0'' - 2\phi_1'\phi_0' \tag{2.3.65}$$

Integrating 2.3.64 gives

$$\phi_0 = \pm \int p dr \tag{2.3.66}$$

which provides a zeroth order form for ψ. Extending to the next order in $\hbar$, 2.3.65 implies

$$\frac{\phi_0''}{\phi_0'} = -2i\phi_1' \tag{2.3.67}$$

and so ϕ_1 has the form

$$\phi_1 = i \ln \sqrt{p} + \text{constant} \tag{2.3.68}$$

Utilizing 2.3.66 and 2.3.68 ψ becomes

$$\psi(r) = \exp \pm \frac{i\phi}{\hbar} = \psi(r_0)\sqrt{\frac{p(r_0)}{p(r)}} \exp \pm \frac{i}{\hbar} \int p dr \approx \psi(r_0) \exp \pm \frac{i}{\hbar} \int p dr \tag{2.3.69}$$

This is the first order WKB form of the wavefunction associated with the one dimensional particle.

With the form of Φ determined, the transmission coefficient or tunneling amplitude can be determined. The transmission coefficient is a measure of the probability of Σ passing from a bounded solution, through the barrier to a bounce solution simply given as the ratio of the wavefunction evaluated at the respective spacetime endpoints of the tunneling. If the transmission coefficient for a wavefunction to propagate from position 1 to position 2 is given by $T(2,1)$ then

$$T(2,1) = \frac{\Psi(2)}{\Psi(1)} \tag{2.3.70}$$

From the WKB approximation, T(2,1) takes the form

$$T(2,1) = \exp \frac{iE}{\hbar}(t_2 - t_1) \exp(\pm \frac{i}{\hbar} \int p dr) \tag{2.3.71}$$

However, this can be rearranged by noting the following.

$$\begin{aligned} \int p dr &= \int p \tfrac{dr}{dt} dt \\ &= \int (H(p,r) + L(p,r)) dt \qquad \text{by equation 2.3.59} \\ &= \int L(r,p) dt + \int H(r,p) dt \end{aligned} \tag{2.3.72}$$

Yet the transmission coefficient is to be calculated for a tunneling from one classical turning point to the other, and for both turning points $\frac{dr}{dt} = 0$ (i.e. according to the classical theory a solution cannot enter the forbidden region as it has no velocity at the tuning points). This in turn gives that $p = 0$ at the turning points, and so by equation 2.3.59, $H(r,p) = E = -L(r, p = 0)$ =constant. When this applied to the integral above,

$$\int p dr = \int L(r,p) dt - \int L(r, p = 0) dt = W_{cl} - W_{cl}|_{static} \tag{2.3.73}$$

Here W_{cl} is the classical action associated with the particle as it traverses the forbidden region (i.e. there is a solution to a Euclidean classical equation of motion which has been obtained by Euclidising the time variable in the classically forbidden region), and $W_{cl}|_{static}$ is the action assigned to a particle that does not tunnel, but rati. r stays at the first turning point for the duration of the tunneling. This in turn gives

$$T(2,1) = \exp \frac{iE}{\hbar}(t_2 - t_1) \exp(\pm \frac{i}{\hbar}(W_{cl} - W_{cl}|_{static})) \tag{2.3.74}$$

As can be seen from equation 2.3.74, the WKB approximation, when applied to the transmission coefficient calculation, has the effect of picking out only the extremal action (i.e. the classical action) contribution, while all the other paths have negligible contributions.

So, in order to evaluate $T(2,1)$, with points 1 and 2 being the classical turning points, the classical action must first be determined.

CHAPTER THREE

THE LAGRANGIAN APPROACH

3.1 The Classical Action

In order to obtain a probability for the tunneling from one classical trajectory to another, it has been shown that the classical action has to be evaluated. However, the form of the action has yet to be derived. Hence, the purpose of this section is to first derive then evaluate the classical action associated with Σ's trajectory.

Due to the composite form of the spacetime, the classical action is to be derived in a piecewise manner, with the gravitational and matter contributions from the three spacetime regions being considered separately. Once the action has been determined, the variational principle can be applied in order to check that the action gives the correct classical equations of motion. With the form of the action established, the numerical evaluation of the transmission coefficient then follows.

In commencing the derivation of the classical action, W, a generic form is assumed, namely the Einstein-Hilbert action integral. This is the standard pure gravity contribution, and it is of the form

$$W = \frac{1}{16\pi} \int {}^4R\sqrt{-{}^4g}\,d^4x \tag{3.1.1}$$

with ${}^4R$ being the four dimensional Ricci scalar, and ${}^4g$ the determinant of the metric. Note, units have been chosen so that the gravitational constant, G, is set equal to one. Yet this form of W is incomplete, as it has only the gravity-matter interaction and it does not account for pure matter contributions. Thus the form of

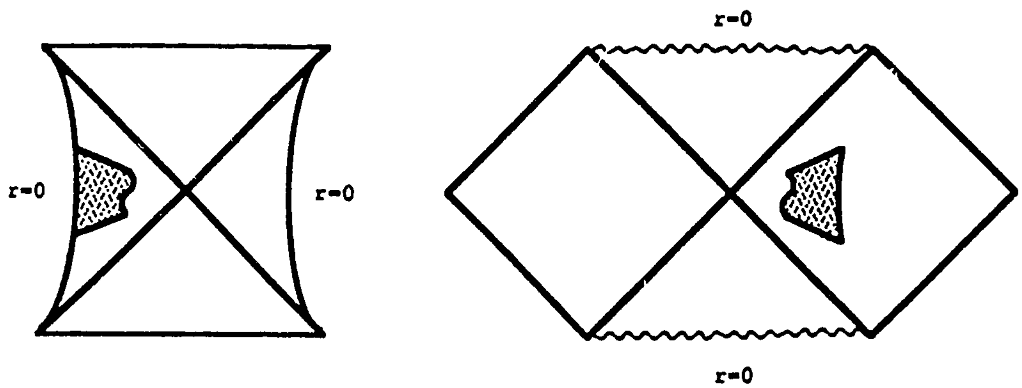

Figure 3.1: A diagrammatic representation of the spacetime region over which the action is to be evaluated.

W should be

$$W = \frac{1}{16\pi}\int \sqrt{-^4g}\,^4R d^4x + \int \mathcal{L}_m d^4x \tag{3.1.2}$$

where $\mathcal{L}_m \equiv L_m\sqrt{-^4g}$ = the lagrangian density of the matter in the system.

With this general form for W, there appear to be no restrictions on the four volume on which W is defined, and in general, the integral is taken over all space. This however is not feasible, as W is eventually evaluated numerically, and so, the four volume of the composite spacetime must be specified. This is done pictorially in figure 3.1 for an arbitrary segment of a Σ trajectory. Note that although the spacetime is most conveniently represented by two spacetime diagrams as in figure 2.8, its actual diagrammatic form can be given as one, e.g. figure 3.2. Now, as the transmission coefficient measures the probability of an initial spacetime configuration evolving to a final configuration, one can choose the initial and final

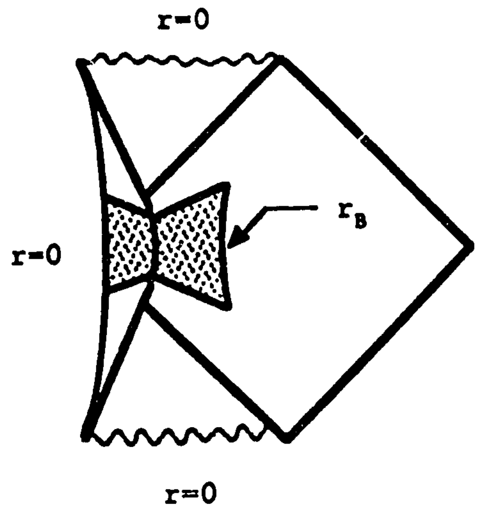

Figure 3.2: A one diagram representation of a composite spacetime trajectory T1.

boundaries to be t = *constant* hypersurfaces. Also, the radial boundary is taken as an $r = r_B$ = *constant* hypersurface, with r_B being asymptotically large. Here the requirement is that r_B be large enough so that the seed spacetime induces only negligible effects on this boundary at r_B. So, if the region over which W is defined is as outlined, then one can proceed with the derivation of W. Note however that in specifying the four volume boundary, one must then consider any possible surface boundary terms that may arise in W. Further, although the choice of boundary surfaces is somewhat arbitrary, the simplicity of these particular choices will become apparent when the surface contributions are evaluated.

As to the actual four volume, its form is to be constrained by the constructibilty requirements. That is, the t^i and t^f hypersurfaces are required to be in either regions I or III of the Kruskal-Szekeres and Gibbons-Hawking diagrams, as this implies that the tunneling of the classical solution starts and ends in the spacetime region over which the constructors have influence (i.e. the laboratory, or its causally disconnected mirror image). This also removes the problem of working in a coordinate set $((t, r, \theta, \phi))$ that is singular on the horizons. Hence, the four volume does not intersect any horizons. However, it should be remembered that the trajectory under consideration is the one associated with universe fabrication, and so, is of the form given by figure 2.14.

Now returning to equation 3.1.2, one has a general form for W over a well defined spacetime interval, but it is still a generic expression, with no allowance made for simplifications due to any inherent symmetries. In particular, the spherical symmetry of the composite spacetime has not been utilized. If this is done then one reduces W from a four dimensional integral to a two dimensional one by integrating over the angular coordinates. Of course this dimensional reduction does not preserve the form of 3.1.2 as ${}^4R$ contains terms that are not a result of manipulations of the reduced two dimensional metric. Fortunately, for spherically symmetric spacetimes one can easily perform the reduction from a $(1+3)$ to a $(1+1)$ dimensional theory, and the pertinent results are

$$\sqrt{-{}^4g} = r^2 \sin\theta \sqrt{-{}^2g} \tag{3.1.3}$$

$$ {}^4R = {}^2R + \frac{2}{r^2}(1 - g^{ab} r_{,a} r_{,b} - 2 r r^{;a}{}_{;a}) \tag{3.1.4}$$

For a more complete description of this reduction via spherical symmetry, consult the paper by Poisson and Israel [35]. When substituted into W, these rearrangements give

$$\begin{aligned} W = \; & \tfrac{1}{4} \int {}^2R r^2 \sqrt{-{}^2g}\, d^2x + \tfrac{1}{2} \int (1 - g^{ab} r_{,a} r_{,b} - 2 r r^{;a}{}_{;a}) \sqrt{-{}^2g}\, d^2x \\ & + 4\pi \int \mathcal{L}_m r^2\, d^2x \end{aligned} \tag{3.1.5}$$

Note, here the latin indices a and b range over the $(1+1)$ spacetime, and ${}^2R$ is the

Ricci scalar for the $(1+1)$ dimensional spacetime.

As can be seen from equation 3.1.5, W has an explicit dependence on second order derivatives (i.e. $r^{;a}{}_{;a} \equiv \Box r$), and in second order time derivatives in particular. This however is not an asset, as actions containing such terms cannot be quantised easily. For example, if one attempts a canonical quantisation, then the generalised coordinate and conjugate momentum associated with $\Box r$ are not well defined. Moreover, a path integral quantisation for such an action founders as the property of intermediate state insertion is lost [36]. That is,

$$W(t_3,t_1) = W(t_3,t_2) + W(t_2,t_1) \tag{3.1.6}$$

with $t_3 > t_2 > t_1$ is not valid. So, if the analysis is to proceed, the terms containing second order time derivatives need to be removed.

If the culprit term in equation 3.1.5 is integrated by parts, then

$$\begin{aligned} -\int r\Box r\,\sqrt{-{}^2g}\,d^2x &= -\int (rr^{;a})_{;a}\,\sqrt{-{}^2g}\,d^2x + \int r^{,a}r_{,a}\,\sqrt{-{}^2g}\,d^2x \\ &= -\int \omega r r^{;a} n_a\,dS + \int r^{,a}r_{,a}\,\sqrt{-{}^2g}\,d^2x \end{aligned} \tag{3.1.7}$$

Here the total divergence has been re-expressed as a boundary surface integral by means of Gauss's divergence theorem [37], which is as follows.

$$\int A^a_{;a}\sqrt{-{}^{n+1}g}\,d^{n+1}x = \int \omega A^a n_a\,dS \tag{3.1.8}$$

with $\omega \equiv n^a n_a = \pm 1$ and n_a is defined as the outward normal.

Using equation 3.1.7, one can write W as

$$\begin{aligned} W = {} & \tfrac{1}{4}\textstyle\int {}^2R\,r^2\sqrt{-{}^2g}\,d^2x + \tfrac{1}{2}\textstyle\int(1+g^{ab}r_{,a}r_{,b})\sqrt{-{}^2g}\,d^2x \\ & +4\pi\textstyle\int \mathcal{L}_m r^2\,d^2x - \textstyle\int \omega r r^{;a} n_a\,dS \end{aligned} \tag{3.1.9}$$

which has no explicit dependence on second order time derivatives. Still, the form of ^2R is as yet undetermined.

Now at this point, one has a choice in the direction the analysis takes, as one has a choice in how ^2R is dealt with. One can use either a direct brute force method which involves using the values of ^4R given by equation 2.2.8, or one can utilize the Gauss-Bonnet theorem and express ^2R as a pure divergence. Whilst the latter approach is more elegant, t[illegible] more direct, with only Σ's contribution to $\int {}^4R\sqrt{-{}^4g}\,d^4x$ requiring add[illegible]n. Therefore, the brute force approach shall be used to continue the ev[illegible]. Nevertheless, the benefits of writing ^2R as a pure divergence will be investigated in section 3.4

3.2 The Brute Force Approach

With this brute force approach to the analysis of the ^2R term in W, contributions from the three spacetime regions are to be considered separately. Further, for all three regions, it is easier to use equations 2.2.8 and 3.1.2 and do the integrations explicitly, than resort to more developed form of the gravitational contribution of W given by equation 3.1.9. Hence,

$$\begin{aligned}
\frac{1}{16\pi}\int {}^4R\sqrt{-{}^4g}\,d^4x &= \frac{1}{4}\int {}^4R\,r^2\sqrt{-{}^2g}\,d^2x \\
&= \frac{1}{4}\left\{\int_{FV}+\int_{\Sigma}+\int_{TV}\right\}({}^4R\,r^2\sqrt{-{}^2g}\,d^2x) \\
&= \int_{FV} 3\chi^2 r^2\sqrt{-{}^2g}\,d^2x + \frac{1}{4}\int_{\Sigma} {}^4R\,r^2\sqrt{-{}^2g}\,d^2x \\
&= \int_{t_D^i}^{t_D^f} dt_D \int_0^{\mathcal{R}} dr\,3\chi^2 r^2 + \frac{1}{4}\int_{\Sigma} {}^4Rr^2\sqrt{-{}^2g}\,d^2x \\
&= \int_{t_D^i}^{t_D^f} dt_D\,\chi^2\mathcal{R}^3 + \frac{1}{4}\int_{\Sigma} {}^4R\,r^2\sqrt{-{}^2g}\,d^2x \qquad (3.2.10)
\end{aligned}$$

Note, $\mathcal{R}$ represents the radial position of the wall. Here the false vacuum contribution has been expressed in terms of the de Sitter parameter and coordinates (equations 2.2.9 and 2.2.3), but one cannot do likewise for the Σ contribution. Instead, one must deal with this ${}^4R$ contribution by using the Gauss-Codazzi formalism. Thus, by appealing to the extrinsic curvature of Σ, one can rewrite ${}^4R$, and if a G.N.C. is used then

$$ {}^4R = {}^3R - (K_{ij}K^{ij} + (K^i{}_i)^2) - 2\frac{\partial K^i{}_i}{\partial \eta} \tag{3.2.11} $$

Note, here the indices i and j range over $\{0, 1, 2\}$, and K_{ij} is the extrinsic curvature of the $\eta = 0$ hypersurface that is Σ. Also, it is not essential that a G.N.C. be used, but it does make for a less cluttered set of equations.

Now as the thin wall approximation has been assumed,

$$ \int_\Sigma {}^4R\, r^2 \sqrt{-{}^2g}\, d^2x = \lim_{\epsilon \to 0} \int_{-\epsilon}^{+\epsilon} d\eta \int_{\tau^i}^{\tau^f} {}^4R\, r^2\, d\tau \tag{3.2.12} $$

and as Σ is such that for the G.N.C. the metric is continuous and the normal derivative is discontinuous on Σ (pages 16 and 17), then only the last term of 3.2.11 will survive the η integration. Further, as the extrinsic curvature contains all the discontinuity, the η integration will give a delta function contribution, which reduces the volume integral to a surface integral. So,

$$ \int_\Sigma {}^4R\, r^2 \sqrt{-{}^2g}\, d^2x = -2\int \mathcal{R}^2\, [{}^3K] dS \tag{3.2.13} $$

with $[{}^3K]$ being the jump of the extrinsic curvature of the three surface Σ, and dS the one dimensional surface element associated with Σ. Thus the gravitational contribution is

$$ W_{grav} = \int_{t_D^i}^{t_D^f} dt_D\, \chi^2 \mathcal{R}^3 - \frac{1}{2}\int \mathcal{R}^2\, [{}^3K]\, dS \tag{3.2.14} $$

Next, one has to consider the matter contribution to W, which also can be split into a false vacuum, a true vacuum, and a Σ contribution. Obviously, the true vacuum region gives a zero contribution (as there is no matter present). However, the false vacuum has a constant energy density ρ, and the standard contribution to W is

$$\begin{aligned} W^{FV}_{matter} &= \rho \cdot (\text{proper four volume}) \\ &= \rho \int_{FV} \sqrt{-^4g}\, d^4x \\ &= 4\pi\rho \int_{t^i_D}^{t^f_D} \int_0^{\mathcal{R}} r^2\, dr \\ &= \frac{4}{3}\pi\rho \int_{t^i_D}^{t^f_D} \mathcal{R}^3\, dt_D \end{aligned} \tag{3.2.15}$$

This just leaves Σ's matter to be accounted for. Like the false vacuum, Σ is assumed to have a constant energy density, as in the thin wall approximation the wall takes the form of a false vacuum (refer to equation 2.2.38). This suggests that its contribution to the action should mirror that of the false vacuum interior. So, a trial form for W^{Σ}_{matter} is

$$\begin{aligned} W^{\Sigma}_{matter} &= \sigma \cdot (\text{proper three volume}) \\ &= \sigma \int \sqrt{-^3h}\, d^3\xi \end{aligned} \tag{3.2.16}$$

In order to check whether this is the correct form, one must take the variation of W^{Σ}_{matter} and see if it gives the correct energy momentum tensor for Σ. Note, the energy momentum tensor $T^{\mu\nu}$ is generally defined by [38, page 154]

$$\delta W_{matter} = \frac{1}{2} \int d^4x\, T^{\mu\nu} \sqrt{-^4g}\, \delta g_{\mu\nu} \tag{3.2.17}$$

On performing such a variation, [39, refer to Appendix B] one finds that

$$\begin{aligned} \delta W^{\Sigma}_{matter} &= \sigma \int d^3\xi\, \delta\sqrt{-^3h} \\ &= \frac{\sigma}{2} \int d^3\xi\, \sqrt{-^3h}\, \delta h \\ &= \frac{\sigma}{2} \int d^3\xi\, \sqrt{-^3h}\, h^{ij} \frac{\partial X^\mu}{\partial \xi^i} \frac{\partial X^\nu}{\partial \xi^j} \delta^4(x - X(\xi))\, \delta g_{\mu\nu} \end{aligned} \tag{3.2.18}$$

where the embedding of Σ in the four space is given by $x^\mu = X^\mu(\xi)$, with ξ^i being the coordinates intrinsic to Σ. The intrinsic metric is then given by

$$h_{ij} = \frac{\partial X^\mu}{\partial \xi^i}\frac{\partial X^\nu}{\partial \xi^j} g_{\mu\nu} \tag{3.2.19}$$

So, on comparing equation 3.2.18 with the definition of equation 3.2.17, and using the G.N.C. to put things in a concise form, one finds that

$$T^{\mu\nu}_\Sigma = \sigma h^{\mu\nu}\delta(\eta) \tag{3.2.20}$$

Yet this is the negative of the actual wall energy momentum tensor (equation 2.2.38), and so instead of equation 3.2.16 one should use

$$W^\Sigma_{matter} = -\sigma \int_\Sigma \sqrt{-^3h}\, d^3\xi = -4\pi\sigma \int_\Sigma \mathcal{R}^2\, dS \tag{3.2.21}$$

Now if equation 2.2.9, 3.2.14, 3.2.15, and 3.2.21 are combined, then W has the form

$$\begin{aligned} W &= \int_{t^i_D}^{t^f_D} dt_D\, \chi^2\mathcal{R}^3 - \frac{1}{2}\int \mathcal{R}^2\,[^3K]\, dS + \frac{4}{3}\pi\rho\int_{t^i_D}^{t^f_D} \mathcal{R}^3\, dt_D - 4\pi\sigma\int \mathcal{R}^2\, dS \\ &= \frac{3}{2}\int_{t^i_D}^{t^f_D} dt_D\, \chi^2\mathcal{R}^3 - \frac{1}{2}\int \{\mathcal{R}^2\,[^3K] + 8\pi\sigma\mathcal{R}^2\} dS \end{aligned} \tag{3.2.22}$$

Clearly one needs to substitute in for $[^3K]$, and from equation 2.2.22, the trace of the extrinsic curvature of Σ is given as

$$\begin{aligned} ^3K &= -\frac{1}{\dot{\mathcal{R}}}\frac{\partial \beta}{\partial \tau} + \frac{2\beta}{r} \\ &= -\frac{1}{\beta}(\ddot{\mathcal{R}} + \frac{1}{2}f_{,r}) + \frac{2\beta}{\mathcal{R}} \end{aligned} \tag{3.2.23}$$

and hence

$$W = \frac{3}{2}\int_{t^i_D}^{t^f_D} dt_D\, \chi^2\mathcal{R}^3 - \frac{1}{2}\int \{8\pi\sigma\mathcal{R}^2 + \left[(-\frac{1}{\beta}(\ddot{\mathcal{R}} + \frac{1}{2}f_{,r} + \frac{2\beta}{\mathcal{R}})\right]\Big|^S_D\} dS \tag{3.2.24}$$

Thus W has a further second order time derivative dependence ($\ddot{\mathcal{R}}$) that, as outlined on page 50, should be removed Yet this $\ddot{\mathcal{R}}$ term cannot be removed by an integration by parts, and all the terms of the generic Einstein-Hilbert action have been considered. Therefore, the only way that this term can be removed is by addition of a boundary term to W. Such a counter term was originally proposed by Gibbons and Hawking [36], and its form is of no surprise - it is proportional to the extrinsic curvature of the boundary surface integrated over the boundary surface. Formally, the Gibbons-Hawking counter term is

$$\frac{1}{8\pi}\int_{^3B} d^3\xi\, {}^3K\sqrt{-{}^3h} = \frac{1}{8\pi}\int_{^3B} {}^3K\, d^3B \tag{3.2.25}$$

with ${}^3B$ the three dimensional boundary of the four volume over which W is defined. A full description of ${}^3B$ was given on page 46 and it is given diagrammatically in figure 3.1.

For the $t = constant$ sections of the boundary, there is no contribution to W, as ${}^3K_{ij}$ is zero. The reason for this is that for both the Schwarzschild and the de Sitter spacetimes,

$$ {}^3K = n^\mu{}_{|\mu} = \frac{1}{\sqrt{-{}^4g}}\partial_\mu(\sqrt{-{}^4g}n^\mu) = \frac{1}{\sqrt{-{}^4g}}\frac{\partial}{\partial t}(\sqrt{-{}^4g}n^t) = 0 \tag{3.2.26}$$

Here, n^μ is the normal, and $n^\mu = (-\frac{1}{\sqrt{f}}, 0, 0, 0)$. Similarly, one can look at the $r = constant$ hypersurfaces, for which the normal is $n^\mu = (0, \sqrt{f}, 0, 0)$, and find that

$$\begin{aligned} {}^3K &= n^\mu{}_{|\mu} = \frac{1}{\sqrt{-{}^4g}}\frac{\partial}{\partial r}(\sqrt{-{}^4g}n^r) \\ &= \frac{1}{r^2\sin\theta}\frac{\partial}{\partial r}(r^2\sin\theta\sqrt{f}) = \frac{2\sqrt{f}}{r} + \frac{f_{,r}}{2\sqrt{f}} \end{aligned} \tag{3.2.27}$$

This in turn implies that

$$\frac{1}{8\pi}\int_{r=const.} {}^3K dB = \frac{1}{2}\int(\frac{2\sqrt{f}}{r} + \frac{f_{,r}}{2\sqrt{f}})\sqrt{f}r^2\,|_{r=const.}\, dt$$

$$= \int (rf + \frac{f_{,r} r^2}{4})\,|_{r=const.}\, dt \qquad (3.2.28)$$

So, for the $r = 0$ boundary there is no contribution, whilst at the $r = r_B$ boundary

$$\frac{1}{8\pi}\int_{r=r_B} {}^3K\, dB = \int (r_B - \frac{3m}{2}) dt_S = (r_B - \frac{3m}{2})(t_S^f - t_S^i) \qquad (3.2.29)$$

Unfortunately, as one takes the $r_B \to \infty$ limit, this contribution to W blows up. Yet this infinite term has been discussed [36], and a renormalising prescription put forward. It is known as the Gibbons-Hawking prescription, and it requires one to subtract off the contribution from the $r = r_B$ hypersurface embedded in flat space. If this is done then

$$\frac{1}{8\pi}\int_{r=r_B} {}^3K\, dB = -\frac{m}{2}(t_S^f - t_S^i) + O(\frac{1}{r_B}) \qquad (3.2.30)$$

which is regular as $r_B \to \infty$. On consulting figure 3.1, one may think that all the contributions to equation 3.2.25 have been considered, but this is not the case. As can be seen, the normal to the boundary surface change discontinuously at the intersection of $r = constant$ and $t = constant$, and $t_S = constant$ and $t_D = constant$ hypersurfaces. Further, these points of intersection produce additional terms that supplement the Gibbons-Hawking boundary term.

In order to evaluate these terms, the reduced two dimensional approach outlined by Farhi et al [39, page 436] will be used. This approach evaluates the discontinuous normal in terms of a smoothing out of the region of intersection. If these intersections are taken as being represented by an intersection of lines in a two dimensional flat Euclidean space with cartesian coordinates (x, y), then the smoothing out can be attributed to an arbitrary smoothing function $y = f(x)$ of limited range, as shown in figure 3.3. The resulting curve then has a normal n^μ of

$$n^\mu = \frac{1}{\sqrt{1 + (\frac{df}{dx})^2}}(-\frac{df}{dx}, 1) \qquad (3.2.31)$$

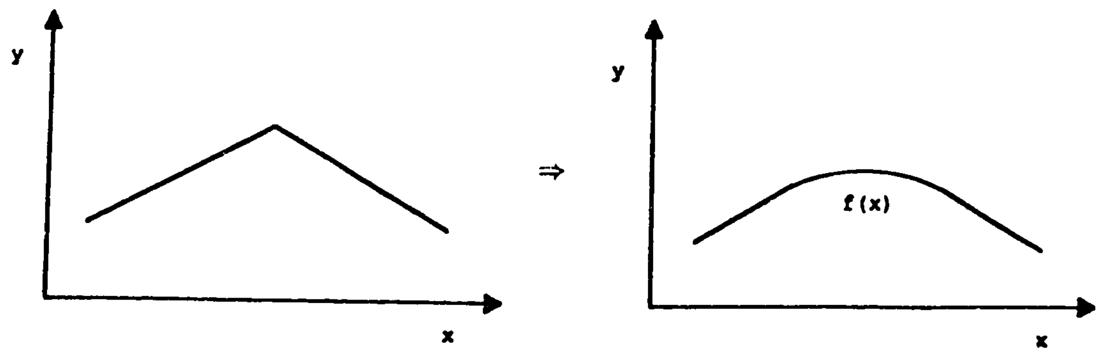

Figure 3.3: The smoothing out of an intersection of two lines.

and so the associated boundary term integral is given as

$$\int K\sqrt{h}\,dx = \int n^{a}{}_{|a}\sqrt{1+(\frac{df}{dx})^2}\,dx \tag{3.2.32}$$

By making the substitution $\cot\theta = \frac{df}{dx}$, the integral can be performed, the result is

$$\int_{intersection} K\,dB = \Delta\theta = \theta_2 - \theta_1 \tag{3.2.33}$$

which is entirely independent of the smoothing function $f(x)$. Now expressing $\Delta\theta$ in terms of workable quantities, one has

$$\Delta\theta = \arccos(n^{(1)}\cdot n^{(2)}) \tag{3.2.34}$$

where $n^{(1)}$ and $n^{(2)}$ are the normal vectors of the two lines (taken at points outside the range of $f(x)$). Now it should be noted that 3.2.34 does not depend on $f(x)$, and so one can retain this result in the limit of $f(x)$ tending to a pointlike function centred on the intersection.

The above result is for two dimensional Euclidean space, which is not quite the same as the four dimensional Lorentzian spacetime of the problem at hand. However, in converting from the simple case to this physical geometry, the analysis is analogous, and the only changes are that $\arccos(n^{(1)} \cdot n^{(2)})$ of 3.2.34 becomes $\cosh^{-1}(-n^{(1)} \cdot n^{(2)})$, and that one must allow for the angular coordinates of the two-sphere. Thus, the contribution to W from the intersection of the two boundary hypersurfaces is

$$W_{intersection} = \frac{1}{8\pi} 4\pi r^2 \cosh^{-1}(-n^{(1)} \cdot n^{(2)}) = \frac{r^2}{2} \cosh^{-1}(-n^{(1)} \cdot n^{(2)}) \qquad (3.2.35)$$

So, using equation 3.2.35 the two types of hypersurface intersections can be evaluated.

The first to be considered is the $t_D = constant$, $t_S = constant$ hypersurface intersection. Obviously, the factor $\cosh^{-1}(-n^{(1)} \cdot n^{(2)}) = \Delta\theta$ needs to be evaluated, but to do this a reference direction is required (so as to measure $\Delta\theta$). The only thing that is common to both hypersurfaces at the point of intersection is Σ itself. Therefore, the four velocity (U^μ) of Σ at the point of intersection is to be taken as the reference direction (refer to figure 3.4). This gives that

$$\Delta\theta = \theta_2 - \theta_1 = \theta_S - \theta_D = \cosh^{-1}(-U^\mu_\Sigma(n^{(S)}_\mu - n^{(D)}_\mu)) \qquad (3.2.36)$$

Now for $t = constant$ surfaces $n^\mu = (\frac{-1}{\sqrt{f}}, 0, 0, 0)$ and $U^\mu_\Sigma = (\dot{t}, \dot{r}, 0, 0)$ (here $\dot{} \equiv$ differentiation with respect to the proper time parameter of Σ), so

$$\begin{aligned} \cosh^{-1}(-U^\mu_\Sigma n_\mu) &= \cosh^{-1}(\dot{t}\sqrt{f}) \\ &= \cosh^{-1}(\frac{\beta}{\sqrt{f}}) \qquad \text{using equation 2.2.17} \\ &= \tanh^{-1}(\frac{\dot{r}}{\beta}) \qquad (3.2.37) \end{aligned}$$

Combining this with equation 3.2.35, one gets

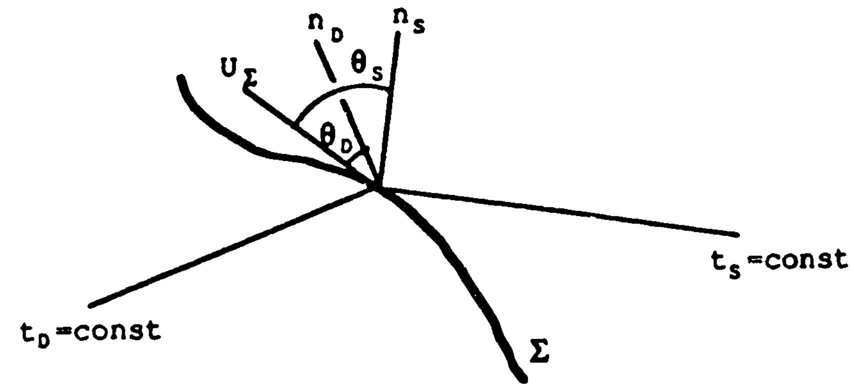

Figure 3.4: The diagrammatic representation of the t_D =constant, t_S =constant hypersurface intersection, and its relation to the four velocity of Σ and the associated angles.

$$W^{\cdot t_S, t_D = constant}_{intersection} = \left\{ \frac{\mathcal{R}^2}{2} \tanh^{-1}(\frac{\dot{\mathcal{R}}}{\beta}) \Big|_D^S \right\} |_{\tau^i}^{\tau^f} \qquad (3.2.38)$$

Here the $|_{\tau^i}^{\tau^f}$ indicates that the contribution from the initial $t = constant$ hypersurface intersection is subtracted from the final intersection contribution.

For the second type of intersection (i.e. the $t = constant$, $r = constant$ hypersurfaces), $n^{(1)} \cdot n^{(2)} = 0$, and so the associated intersection action term is

$$W^{\cdot r, t = constant}_{intersection} = \frac{r^2}{2} \cosh^{-1}(0) = \frac{r^2}{2} \qquad (3.2.39)$$

Yet, there are two such intersection for the $r = 0$ surface and for the $r = r_B$ surface, so the full contribution for each $r = constant$ surface is

$$W^{\cdot r, t = constant}_{intersection} |_{\tau^i}^{\tau^f} = (\frac{r^2}{2} - \frac{r^2}{2})|_{r = constant} = 0 \qquad (3.2.40)$$

With the determination of equation 3.2.40, the last of the Gibbons-Hawking boundary term contributions has been computed. Thus, on combining equations 3.2.24, 3.2.25, 3.2.29, and 3.2.38, one gets the full form of the action, which is

$$\begin{aligned} W &= \frac{3}{2} \int_{t_D^i}^{t_D^f} dt_D \, \chi^2 \mathcal{R}^3 - 4\pi\sigma \int \mathcal{R}^2 \, dS - \frac{1}{2} \int (\frac{\mathcal{R}^2}{\beta}(\dot{\mathcal{R}} + \frac{1}{2} f_{,r}) + 2\beta\mathcal{R})) \Big|_D^S dS \\ &- \int_{t_S^i}^{t_S^f} \frac{m}{2} dt_S + \int_{\tau^i}^{\tau^f} \frac{d}{d\tau} \left(\frac{\mathcal{R}^2}{2} \tanh^{-1}(\frac{\dot{\mathcal{R}}}{\beta}) \Big|_D^S \right) d\tau \end{aligned} \qquad (3.2.41)$$

If one expands the last term in this relation, it is found that

$$\frac{d}{d\tau} \left(\mathcal{R}^2 \tanh^{-1}(\frac{\dot{\mathcal{R}}}{\beta}) \Big|_D^S \right) d\tau = (2\dot{\mathcal{R}} \tanh^{-1}(\frac{\dot{\mathcal{R}}}{\beta}) + \frac{\mathcal{R}^2 \ddot{\mathcal{R}}}{\beta}) \, |_D^S \qquad (3.2.42)$$

and with a little algebra, and equation 2.2.17, W can be written as

$$\begin{aligned} W = & \int_{\tau^i}^{\tau^f} \left\{ \frac{\beta_D}{f_D} \frac{3}{2} \chi^2 \mathcal{R}^3 - 4\pi\sigma\mathcal{R}^2 - (\frac{\mathcal{R}^2 f_{,r}}{4\beta} + \beta\mathcal{R}) \Big|_D^S - \frac{m\beta_S}{2 f_S} \right. \\ & \left. + \mathcal{R}\dot{\mathcal{R}} \tanh^{-1}(\frac{\dot{\mathcal{R}}}{\beta}) \Big|_D^S \right\} d\tau \end{aligned} \qquad (3.2.43)$$

This is now the final form of the action for the composite spacetime, and as expected, it no longer contains second order time derivatives (so no immediately obvious problems arise with its quantisation). Also, W is expressible in terms of one variable (i.e. $\mathcal{R} = \mathcal{R}(\tau)$, Σ's radius) associated with the one degree of freedom parameter.

3.3 Evaluation Of The Transmission Coefficient

Having determined W, an expression for the transmission coefficient is now obtainable (via equation 2.3.74). Yet to evaluate the transmission coefficient, the approach used here is one that differs from the literature (in particular, Farhi et. al. [39]). Instead of integrating over all possible four geometries that interpolate the forbidden region, the classical action is evaluated over a geometry obtained from the euclidised field equation. Whilst such a step may be considered an oversimplification, it does provide a semi-classical evaluation of the transmission coefficient, without getting lost in the interpretation of the interpolating geometry [39, §5]. But first, in order to evaluate T(2,1) it must be clear what the endpoints of the tunneling are. That is, one needs to specify the initial and final configurations (in this case, three dimension $t = constant$ hypersurfaces).

To discern these, it must be remembered what sort of trajectory is being dealt with. From the end of section 2.2, one has that the desired seed wall trajectory is one that has the initial features of a low mass bounded solution, a section where quantum mechanical tunneling permits Σ's passage across the classically forbidden region, and a final era that has the traits of a bounce solution. More specifically, the trajectory of interest is one which has a classically forbidden region sandwiched to the past and the future by a T1, and a T6 trajectory segment respectively. Given that this is Σ's trajectory, then the endpoint configurations associated with T(2,1) are just a matter of choice depending on the interval one wants to evaluate T(2,1) over. If both endpoint configurations are chosen to be the bounded solution era, then T(2,1) is just the transmission coefficient for Σ to evolve as a classical solution, which is of no use in evaluating the seed spacetime for the possibility of it forming a new universe. The same also applies if both endpoints are in the bounce trajectory era. Further, if 1 and 2 are adjacent configurations in one of the two eras, then one should have T(2,1) $\rightarrow$ 1. This is indeed the case, as $W_{cl} - W_{cl|static} = C_{\tau f} - C_{\tau i} = C - C = 0$ for a some value C (here $\mathcal{R}$ and $\dot{\mathcal{R}}$ are the same for the initial and final configurations in this limit), and so $T(2,1) = \exp(0) = 1$ as expected (as $\tau_f \rightarrow \tau_i$).

Thus, the only type of transmission coefficient left to consider is the one for a section of the trajectory that links the classically forbidden segment of Σ's trajectory with either or both the bounded and the bounce eras. Such a transmission coefficient is of the most interest, as the T(2,1) for the case where the initial configuration is in the bounded era and the final configuration is in the bounce era will provide an indication as to whether the tunneling of Σ is physically likely. In order to make this assessment as simple as possible, the two configurations are to be chosen to be at the "edge" of their respective eras - namely, at the classical turning points. This in turn implies that T(2,1) measures the transmission coefficient for Σ to pass through the classically forbidden region. Note, in choosing the classical turning points, one then has $\dot{\mathcal{R}} = 0$ at the endpoint configurations.

Yet to obtain this transmission coefficient, one cannot substitute equation 3.2.43 directly into equation 2.3.74, as W was derived from arguments that were specific to the classically allowed rather than the classically forbidden region. This is emphasised by the fact that Σ has no real time solution to the classical equation of motion in the forbidden region (hence the name). Nevertheless, if the analogy to the one dimensional particle is adhered to, then T(2,1) can be approximated for the tunneling segment if the time coordinate is made imaginary. That is, the transmission coefficient can be approximated if the spacetime is euclidised. To euclidise the composite spacetime, each time coordinate is transformed via

$$t_E = it \tag{3.3.44}$$

and correspondingly, such quantities as velocities are rewritten as

$$\dot{r} = \frac{dr}{dt} = i\frac{dr}{dt_E} = i\dot{r}_E \tag{3.3.45}$$

(Note, the subscript E implies a Euclidean quantity.) These changes allow the action to be written in euclidean form, but before this is done, a few notes on the implications of the euclidising is required.

One euclidises in order to examine the motion of the wall trajectory through the classically forbidden region. Such a step is required as it gives real t_E solutions to Σ's equation of motion for what was the classically forbidden region. Now for the composite spacetime, the euclidised geometries are taken to be described by the euclidean Einstein field equation. Thus the result is that the metric form of the f. e vacuum is euclidean de Sitter whilst the true vacuum region has a euclidean Schwarzschild metric. The actual interpolating geometries that take Σ from one turning point to the other may not be of this form, or there may not even be a fixed metric form [43]. Yet as equation 2.3.74 only requires an evaluation of the classical action, the use of the euclidised classical geometries provides a suitable approximation.

The effect of euclidising on the equation of motion is only minor, as its general form is unchanged; the form of β is all that requires alteration. So, in euclidising equation 2.2.41, one gets

$$\beta_D - \beta_S = 4\pi\sigma\mathcal{R} \qquad \text{with } \beta = \pm\sqrt{f - \dot{\mathcal{R}}_E^2} \tag{3.3.46}$$

or equivalently,

$$\dot{\mathcal{R}}_E^2 = 1 + V(\mathcal{R}, m, \chi) \tag{3.3.47}$$

with $V(\mathcal{R}, m, \chi)$ given by equation 2.2.43. Note, one still has the turning points given by $\dot{\mathcal{R}}_E^2 = 0$.

So now, the euclidised action for the classically forbidden region is

$$\begin{aligned} W = & \int_{\tau^i}^{\tau^f} \left\{ \frac{\beta_D}{f_D} \frac{3}{2} \chi^2 \mathcal{R}^3 - 4\pi\sigma\mathcal{R}^2 - \left(\frac{\mathcal{R}^2 f_{,r}}{4\beta} + \beta\mathcal{R}\right)\Big|_D^S - \frac{m\beta_S}{2f_S} \right. \\ & \left. + \mathcal{R}\dot{\mathcal{R}}_E \arctan\left(\frac{\dot{\mathcal{R}}_E}{\beta}\right)\Big|_D^S \right\} d\tau \end{aligned} \tag{3.3.48}$$

Yet this can be simplified as one requires only the classical action. This means that equation 3.3.46 can be invoked, both to cancel terms, and to re-express $\dot{\mathcal{R}}_E$ in terms of $\mathcal{R}$, m, and χ. Thus one has

$$\begin{aligned} W_{cl} \approx & \int_{\tau^i}^{\tau^f} \left\{ \frac{\beta_D}{f_D} \frac{3}{2} \chi^2 \mathcal{R}^3 - \frac{\mathcal{R}^2 f_{,r}}{4\beta}\Big|_D^S - \frac{m\beta_S}{2f_S} \right. \\ & \left. + \mathcal{R}\sqrt{1+V} \arctan\left(\frac{\sqrt{1+V}}{\beta}\right)\Big|_D^S \right\} d\tau \end{aligned} \tag{3.3.49}$$

with $\beta = \pm\sqrt{f - \dot{\mathcal{R}}_E^2} = \pm\sqrt{f - 1 - V}$. Unfortunately, this expression for the action is not yet suitable for numerical evaluation, as $\mathcal{R} = \mathcal{R}(\tau)$ has not been specified.

From Σ's equation of motion, one has a differential equation for $\mathcal{R}$ in terms of the wall parameter τ, but it is not an easily solved equation. Therefore, an approximation is in order. If it is assumed that the radial width of the forbidden region is very small then Σ's radius changes little over the course of the tunneling. This feature is the one to be exploited, as $\mathcal{R}$ is to be considered constant. Whilst it is perhaps an oversimplification, this approximation reduces the action integral to a trivial integration, for which the result is

$$
\begin{aligned}
W_{cl} \approx \; & \left\{ \frac{\beta_D}{f_D}\frac{3}{2}\lambda^2\mathcal{R}^3 - \frac{\mathcal{R}^2 f_{,r}}{4\beta}\Big|_D^S - \frac{m\beta_S}{2f_S} \right. \\
& \left. +\mathcal{R}\sqrt{1+V}\arctan\left(\frac{\sqrt{1+V}}{\beta}\right)\Big|_D^S \right\} \{\tau^f - \tau^i\}
\end{aligned}
\qquad (3.3.50)
$$

which is of the form $W_{cl} = (constant)\{\tau^f - \tau^i\}$. Further, if the value of the constant is taken as the radial coordinate of a turning point, then $\dot{\mathcal{R}}_E = \sqrt{1+V} = 0$ and the velocity dependent term of 3.3.50 vanishes. However, if the constant is chosen to be $\mathcal{R}_1$, the turning point belonging to the bounded trajectory, then $W_{cl} = W_{cl|static}$ and this implies that the transmission coefficient is just measuring the probability of Σ staying at $\mathcal{R}_1$ for a time corresponding to $\tau^f - \tau^i$. This is not what is wanted, and indeed it makes little sense to choose the constant as $\mathcal{R}_1$. Instead, it should be taken as $\mathcal{R}_2$, as then at least this approximation has some indication of the forbidden region being traversed. If this is done, then one has that for the T1 subclass,

$$
W_{cl} - W_{cl|static} \approx \left\{ \frac{\lambda^2\mathcal{R}^3}{\sqrt{f_D}} - \frac{m}{\sqrt{f_S}} \right\}\Big|_{\mathcal{R}_1}^{\mathcal{R}_2} \{\tau^f - \tau^i\} \qquad (3.3.51)
$$

with the $|_{\mathcal{R}_1}^{\mathcal{R}_2}$ indicating the difference of W_{cl} and $W_{cl|static}$.

Now in order to obtain a numerical value for 3.3.51, the values of $\mathcal{R}_1$ and $\mathcal{R}_2$ need to be specified. As mentioned, one takes $\mathcal{R}_1$ and $\mathcal{R}_2$ to be the classical turning points (so as to simplify the evaluation of the transmission coefficient), and these turning points can be obtained from equation 2.2.42 by solving $V(r) = -1$ for r. Unfortunately the form of $V(r)$ is such that this equation does not afford a general analytic solution. However, one can resort back to numerical and graphical methods, and from the potential (figure 2.1) generated by the parameters given on page 40, one can obtain estimates for $\mathcal{R}_1$ and $\mathcal{R}_2$. Reading from the graph

$$
\mathcal{R}_1 \approx 1.2\text{m} \quad \mathcal{R}_2 \approx 2.0\text{m} \qquad (3.3.52)
$$

These values, along with the parameters, allow $W_{cl} - W_{cl|static}$ to be determined. Before this is done, one should note that the distance $\mathcal{R}_1 - \mathcal{R}_2 \approx 0.8$m may not be small enough to justify the approximation given on page 65. Yet, one must remember that figure 2.1 was geneiated with an exceedingly small value for the false vacuum energy density ρ (so to disdplay the features of the potential). For any realistic value of ρ, $\mathcal{R}_1 - \mathcal{R}_2$ would be very much smaller.

In order to avoid confusion, the numerical evaluation of equation 3.3.51 is to be carried out in Planck units, and so in this unit system

$$\mathcal{R}_1^P \approx 7.5 \times 10^{30} \quad \mathcal{R}_2^P \approx 1.3 \times 10^{31} \tag{3.3.53}$$

(Here the ^{P} indicates a quantity given in Planck units.) Substituting in to equation 3.3.51, one gets

$$W_{cl} - W_{cl|static} \approx 2.0 \times 10^{72}\{\tau_f^P - \tau_i^P\} \tag{3.3.54}$$

and so the transmission coefficient $T(\mathcal{R}_2, \mathcal{R}_1)$ can be determined via equation 2.3.74.

On substituting the numerical results obtained from the above approximation, one finds that $T(\mathcal{R}_2, \mathcal{R}_1)$ is

$$T(\mathcal{R}_2, \mathcal{R}_1) \approx \exp -(2.0 \times 10^{72})\{\tau_f^P - \tau_i^P\} \tag{3.3.55}$$

As can be seen from equation 3.3.55, the value of the transmission coefficient is very small. This however, was to be expected. Further, discussion on this small tunnelling probability is left until the conclusion, as there a more complete overview is attainable, and the implications of the magnitude of the transmission coefficient can be realised.

3.4 The Covariant Approach

Having evaluated the action in the step by step brute force manner of the previous section, one is left with the desire to redo the calculations in a more elegant way. (By elegance, it is meant that the form of W is more concise, general, and possibly covariant.) This move toward elegance was hinted at in the final paragraph of section 3.1. There it was suggested that the two dimensional Ricci scalar ${}^2R$ be written as a pure divergence, courtesy of the Gauss-Bonnett theorem [26, page 309], [34]. In particular, the work of Horndeski [40] on dimensionally dependent divergences is to be used, but with the dimension set to $n = 2$.

If one has an arbitrary non-null vector field ξ^a from which a second vector field V^a is defined, and has the form

$$\begin{aligned} V^a &\equiv \frac{-2}{\xi^m\xi_m}\delta^{ab}_{cd}\xi^c\xi^d{}_{;b} \\ &= \frac{2}{\xi^m\xi_m}(\xi^b\xi^a{}_{;b} - \xi^a\xi^b{}_{;b}) \end{aligned} \tag{3.4.56}$$

Then from Theorem 1 of [40]

$${}^2R = V^a{}_{;a} \tag{3.4.57}$$

This is the form of ${}^2R$ that was sought (i.e. a total divergence). Utilizing this, one can re-arrange the first term of equation 3.1.9 by Gauss' divergence theorem (equation 3.1.8). Thus

$$\begin{aligned} \int {}^2R\, r^2 \sqrt{-{}^2g}\, d^2x &= \int r^2 V^a{}_{;a}\sqrt{-{}^2g}\, d^2x \\ &= \int_S \omega r^2 V^a n_a\, dS - 2\int r r_{,a} V^a \sqrt{-{}^2g}\, d^2x \end{aligned} \tag{3.4.58}$$

Here S is the usual boundary surface of the composite spacetime. Substituting this into the action, one gets

$$\begin{aligned} W = & \ \tfrac{1}{2}\int(1 + g^{ab}r_a r_b - r r_{,a}V^a)\sqrt{-^2g}\,d^2x + 4\pi\int \mathcal{L}_m r^2\,d^2x \\ & -\tfrac{1}{2}\int \omega(2rr^{,a}n_a + \tfrac{1}{2}r^2V^a n_a)\,dS \end{aligned} \qquad (3.4.59)$$

Now taking advantage of the fact that ξ^a is an arbitrary vector field, one can choose ξ^a such that on the boundary surface S, it is proportional to the normal of the boundary surface, n^a. Note, this restricts the form of ξ^a slightly, but it still maintains it arbitrariness within the volume. So, if $\xi^a = \lambda(x^b)n^a$ then the associated boundary term of equation 3.4.59 has an integrand that can be written as

$$\begin{aligned} V^a n_a &= \frac{-2}{\xi^m\xi_m}(\delta^{ab}_{cd}\xi^c\xi^d{}_{;b})\frac{\xi_a}{\lambda} \\ &= \frac{-2}{\lambda\xi^m\xi_m}(\xi^a\xi_a\xi^b{}_{;b} - \xi_a\xi^b\xi^a{}_{;b}) \\ &= \frac{-2}{\lambda}\xi_{a;b}(g^{ab} - \frac{\xi^a\xi^b}{\xi^m\xi_m}) \end{aligned} \qquad (3.4.60)$$

The term in the bracket has the appearance of a projection, and indeed if one takes

$$\Delta_{ab} = g_{ab} - \frac{\xi_a\xi_b}{\xi^c\xi_c} \qquad (3.4.61)$$

then on S,

$$\Delta_{ab}n^b = n^a - \frac{\lambda^2 n_a n_b n^b}{\xi^c\xi_c} = n_a - n_a = 0 \qquad (3.4.62)$$

So Δ_{ab} is indeed a projection onto S (it projects perpendicularly to n^a). Therefore,

$$V^a n_a = -\frac{2}{\lambda}\xi_{a;b}\Delta^{ab} = -2\Delta^{ab}n_{a;b} = -2K^a_a \qquad (3.4.63)$$

The last step in 3.4.63 involves an extrinsic curvature similar to the one defined by equation 2.2.19; the difference is that this present extrinsic curvature is related to the one dimensional boundary surface.

Combining equation 3.4.63 with the other surface term in W, one gets

$$-\frac{1}{2}\int \omega(2rr^a - \frac{1}{2}r^2V^a)n_a\,dS = -\frac{1}{2}\int \omega(\frac{2}{r}\frac{\partial r}{\partial x^a}n^a + K^a_a)r^2\,dS$$
$$= -\frac{1}{2}\int \omega\,{}^3K^i_i r^2\,dS \qquad (3.4.64)$$

Here the two surface terms are incorporated into the one three surface term (still integrated over the one surface). Using equation 3.4.64, the action becomes

$$W = \tfrac{1}{2}\int(1 + g^{ab}r_{,a}r_{,b} - rr_{,a}V^a)\sqrt{-^2g}\,d^2x + 4\pi\int \mathcal{L}_m r^2\,d^2x$$
$$-\tfrac{1}{2}\int \omega\,{}^3K^i_i r^2\,dS \qquad (3.4.65)$$

The benefit of writing the action in this form is now immediate, due to the fact that W has to be extended to its quantisable form. This means that the complete form of W requires the addition of the Gibbons-Hawking counter-term (page 75) From the form of the counter-term (equation 3.2.25) it is clear that it exactly cancels the surface term in the action. Thus the complete form of W is,

$$W = \frac{1}{2}\int(1 + g^{ab}r_{,a}r_{,b} - rr_{,a}V^a)\sqrt{-^2g}\,d^2x + 4\pi\int \mathcal{L}_m r^2\,d^2x \qquad (3.4.66)$$

with

$$V^a = \frac{-2}{\xi^m\xi_m}\delta^{ab}_{cd}\xi^c\xi^d{}_{;b} \quad \text{and} \quad \xi^a = \lambda(x^a)n^a \text{ on the boundary.} \qquad (3.4.67)$$

Once again, the action has been written in a form that is independent of second order time derivatives (given that $\mathcal{L}_m$ contains none), and indeed of second order derivatives generally. Further, this form of W is much more concise, and it is also covariant. This means that it is applicable to any spacetime. (Compare with the brute force approach, which relied on the form of the spacetime.) For this reason, W as given by equation 3.4.66 is to be used in the check on the form of W. That is, by extremising the variation of W one should get the classical equations of motion of the variables defined in W (in this case, Einstein's field equation), and to calculate δW equation 3.4.66 is used. Also, equation 3.4.66 has two dynamical variables, namely $r(x^a)$ and g^{ab}, and one gauge variable ξ^a. So, there should be two classical equations of motion and one trivial identity that follow from the variation of W.

If Einstein's equation is assumed, then by Poisson and Israel's spherical symmetry split of four space [35, refer to Appendix A.], one gets the following decomposition.

$$2rr_{;ab} + (1 - g^{cd}r_{,c}r_{,d} - 2r\Box r)g_{ab} = -8\pi r^2 T_{ab} \quad (3.4.68)$$

and

$$r\Box r - \frac{1}{2}{}^2Rr^2 = 8\pi r^2 T^\theta_\theta = 8\pi r^2 T^\phi_\phi = 8\pi r^2 P \quad (3.4.69)$$

(Note, $P = g^{\theta\theta}T_{\theta\theta}$ = tangential pressure.) These are the two classical equations of motion that one should obtain through a variation of the action.

Taking the first of the dynamical variables, namely $r(x^a)$, one has that W's variation be extremised, which is written as

$$\frac{\delta W}{\delta r} = 0 \quad (3.4.70)$$

By the variational principle [41, see chapter 12], [42], this extremum can be re-expressed in the Euler-Lagrange equation form [48, chapter 7]

$$0 = \frac{\partial \mathcal{L}}{\partial r} - \partial_a \left(\frac{\partial \mathcal{L}}{\partial r_{,a}} \right) \quad (3.4.71)$$

Here $\mathcal{L}$ is the total lagrangian density. Substituting in for $\mathcal{L}$ (i.e. for W) from equation 3.4.66, one gets

$$0 = -\frac{1}{2}r_{,a}V^a\sqrt{-{}^2g} + 8\pi r\mathcal{L}_m - (g^{ab}r_{,b} - \frac{1}{2}rV^a)_{;a}\sqrt{-{}^2g} \quad (3.4.72)$$

which can be rearranged into the form

$$r\Box r - \frac{1}{2}{}^2Rr^2 = 8\pi r^2 \frac{\mathcal{L}_m}{\sqrt{-{}^2g}} \tag{3.4.73}$$

This has the form of one of the classical equations of motion (equation 3.4.69) except for the $\frac{\mathcal{L}_m}{\sqrt{-{}^2g}}$ term. Yet $P = g^{00}T_{00}$, and by resorting to the energy momentum tensor definition (equation 3.2.17) one has

$$\begin{aligned}
P &= g^{00}\left(\frac{-2}{\sqrt{-{}^4g}}\frac{\delta W_m}{\delta g^{00}}\right) \\
&= g_{00}\left(\frac{2}{\sqrt{-{}^4g}}\frac{\delta W_m}{\delta g_{00}}\right) \\
&= \frac{2g_{00}\mathcal{L}_m}{r^2\sin\theta\sqrt{-{}^2g}}\frac{\delta}{\delta g_{00}}\left(\sqrt{g_{00}g_{\phi\phi}}\right) \\
&= \frac{g_{00}\mathcal{L}_m}{r^2\sin\theta\sqrt{-{}^2g}}\frac{g_{\phi\phi}}{\sqrt{g_{00}g_{\phi\phi}}} \\
&= \frac{\mathcal{L}_m}{\sin\theta\sqrt{-{}^2g}}\frac{r^2\sin^2\theta}{r^2\sin\theta} \\
&= \frac{\mathcal{L}_m}{\sqrt{-{}^2g}}
\end{aligned} \tag{3.4.74}$$

So equation 3.4.73 is indeed one of the classical equations of motion.

In order to obtain the other classical equation of motion, it should just be a matter of re-applying the variational procedure for the dynamical variable g^{ab}. Again the starting point is the extremum

$$\frac{\delta W}{\delta g^{ab}} = \frac{\delta W_{grav}}{\delta g^{ab}} + \frac{\delta W_m}{\delta g^{ab}} = 0 \tag{3.4.75}$$

The definition of the energy momentum tensor deals with the variation of the matter contribution, as one has

$$\frac{\delta W_m}{\delta g^{ab}} = -\frac{4\pi}{2}r^2\sqrt{-{}^2g}T_{ab} \tag{3.4.76}$$

As to the gravitational term, its variation is not as simple. Whilst equation 3.4.66 is more elegant, and has W expressed as the two dimensional integral of a

Lagrangian density, the second order derivatives have been removed, so it is hard to match the result of the variation to equation 3.4.68. Instead, it is easier to use the general form given in equation 3.1.5. Thus one requires

$$\frac{\delta}{\delta g^{ab}}\left(\frac{1}{4}\int {}^2R\, r^2\sqrt{-{}^2g}\, d^2x + \frac{1}{2}\int(1 - g^{ab}r_{,a}r_{,b} - 2r\Box r)\sqrt{-{}^2g}\, d^2x\right) \qquad (3.4.77)$$

In order to take the variation, write ${}^2R = g^{ab}R_{ab}$, and then the first term in the variation becomes

$$\begin{aligned}\delta\left(\frac{1}{4}\int {}^2R\, r^2\sqrt{-{}^2g}\, d^2x\right) &= \delta\left(\frac{1}{4}\int g^{ab}R_{ab}\, r^2\sqrt{-{}^2g}\, d^2x\right)\\ &= \frac{1}{4}\int r^2(R_{ab} - \frac{1}{2}g_{ab}{}^2R)\sqrt{-{}^2g}\,\delta g^{ab}\, d^2x\\ &= +\frac{1}{4}\int g^{ab}\delta^2 R_{ab} r^2\sqrt{-{}^2g}\, d^2x \end{aligned} \qquad (3.4.78)$$

using the standard variations of the metric and determinant [41, page 364] [38, page 88]. Now, a two dimensional spacetime is conformally flat, so $R_{ab} - \frac{1}{2}g_{ab}{}^2R$ is identically zero. Therefore,

$$\delta\left(\frac{1}{4}\int {}^2R\, r^2\sqrt{-{}^2g}\, d^2x\right) = \frac{1}{4}\int g^{ab}\delta^2 R_{ab}r^2\sqrt{-{}^2g}\, d^2x \qquad (3.4.79)$$

Now, the variation of the Ricci tensor is obviously non-zero and it has the form [41, consult §12.4]

$$\delta R_{ab} = \delta\Gamma^c_{ab;c} - \delta\Gamma^c_{ac;b} \qquad (3.4.80)$$

which immediately suggests an integration by parts (so to re-express the variation in terms of δg^{ab}). Hence,

$$r^2\delta R_{ab} = (r^2)_{;b}\,\delta\Gamma^c_{ac} - (r^2)_{;c}\,\delta\Gamma^c_{ab} + \text{divergence} \qquad (3.4.81)$$

(Note, one can drop the divergences as they take the form of dynamically irrelevant surface terms.) Still, the variation is not in terms of δg^{ab}. To correct this, write

$h_{ab} \equiv \delta g_{ab}$ and integrate the variation of the Christoffel symbols by parts. A little algebra gives

$$\begin{aligned} r^2\delta R_{ab} &= (r^2)_{;c(b}\, h^c_{a)} - \frac{1}{2}(r^2)^{;c}{}_{;c}\, h_{ab} - \frac{1}{2}(r^2)_{;ba}\, h \\ \Rightarrow \qquad g^{ab}r^2\delta R_{ab} &= (r^2)_{;ab}\, h^{ab} - \frac{1}{2}(r^2)^{;c}{}_{;c}\, h - \frac{1}{2}(r^2)^{;c}{}_{;c}\, h + \text{divergence} \\ &= -(r^2)_{;ab}\, \delta g^{ab} + (r^2)^{;c}{}_{;c}\, g_{ab}\, \delta g^{ab} \end{aligned} \qquad (3.4.82)$$

(Note, $h^{ab} = -\delta g^{ab}$ and $h = -g_{ab}\delta g^{ab}$). Using this result, the complete variation of $W_{grav.}$ is

$$\begin{aligned} \frac{\delta W_{grav.}}{\delta g^{ab}} &= \int(\frac{1}{4}((-r^2)_{;ab} + (r^2)^{;c}_{;c} g_{ab} - (1 - g^{cd}r_{,c}r_{,d} - 2rr_{;cd}g^{cd})g_{ab})) \\ &\quad -\frac{1}{2}r_{,a}r_{,b} - rr_{;ab} + (rr_{;ab} + r_{;a}r_{;b}) \\ &\quad -\frac{1}{2}(rr_{;cd}g^{cd} + r_{;c}r_{;d}g^{cd})g_{ab})\sqrt{-^2g}\, d^2x \end{aligned} \qquad (3.4.83)$$

(Note, in order to obtain the correct variation, one must not forget to include such terms as $\frac{\delta}{\delta g^{ab}}(rr_{ab})$.) Thus, under arbitrary variations in g^{ab} one gets that

$$\begin{aligned} 8\pi r^2 T_{ab} &= (-r^2)_{;ab} + (r^2)^{;c}_{;c} g_{ab} - (1 - g^{cd}r_{,c}r_{,d} - 2rr_{;cd}g^{cd})g_{ab} - 2r_{,a}r_{,b} \\ &\quad -4rr_{;ab} + 4(rr_{;ab} + r_{;a}r_{;b}) - 2(rr_{;cd}g^{cd} + r_{;c}r_{;d}g^{cd})g_{ab} \end{aligned} \qquad (3.4.84)$$

which can be easily rearranged to give the second classical equation of motion, equation 3.4.68. So, it appears that the action (equation 3.4.66) is of the correct form, in as much that it implies the Einstein field equations (which are taken as the starting assumption).

Yet there is still the variation of W with respect to the gauge variable ξ^a. However, as ξ^a is a gauge variable, and the two classical equations of motion have already been obtained from the variational principle, only a trivial identity is expected from the ξ^a variation. From the variational principle

$$\frac{\delta W}{\delta \xi^a} = 0 \qquad \Rightarrow \qquad 0 = \frac{\partial \mathcal{L}}{\partial \xi^a} - \partial_b\left(\frac{\partial \mathcal{L}}{\partial \xi^a{}_{;b}}\right) \qquad (3.4.85)$$

So, by direct calculation (Note, here ξ^a is chosen to be normalised in order to simplify calculations, yet it does not compromise the arbitrariness of ξ^a) one has

$$\frac{\partial \mathcal{L}}{\partial \xi^a} = r r_{,b} \xi^b{}_{;a} - r r_{,a} \xi^b{}_{;b} \tag{3.4.86}$$

$$\frac{\partial \mathcal{L}}{\partial \xi^a{}_{;b}} = r r_{,c} \delta^b_a \xi^c - r r_{,a} \xi^b \tag{3.4.87}$$

$$\partial_b \left(\frac{\partial \mathcal{L}}{\partial \xi^a{}_{;b}} \right) = r r_{,b} \xi^b{}_{;a} - r r_{,a} \xi^b{}_{,b} \tag{3.4.88}$$

On substituting these expressions into 3.4.85, it is clear that the variation gives $0 = 0$, which is in line with what was expected.

Thus, it has been shown that for the action given (either equation 3.1.5, 3.1.9 or 3.4.66) one can obtain the classical equations of motion which are the components of Einstein's field equation. As the field equation has been the starting point of the analysis, W has been shown to be of the correct form for the composite spacetime.

3.5 Evaluation Of The Covariant Action

Given that section 3.4 was spent developing a more elegant form for the action, with the results culminating in equation 3.4.66, the obvious question is begged. Can the use of equation 3.4.66, with its more elegant (and covariant) form, simplify the evaluation of W and of the transmission coefficient? The immediate answer would be that the covariant form, and absence of surface terms would provide for easy evaluation of W. Unfortunately, this is not the case. The reason for this is the boundary restriction on $\xi^a = \lambda(x^b) n^a$. This requirement means that ξ^a is not completely arbitrary, but is restricted. Nevertheless choosing a ξ^a that satisfies the boundary condition is not difficult, and for example, one immediate candidate is

$$\xi^a = g^{ab} F_{,b} \qquad \text{with} \quad F = (t - t^i)(t - t^f) r (r - r_B) \tag{3.5.89}$$

The form of F is due to the boundary surface (which was discussed on page 47) and its rectangular or box like shape. Such a shape cannot be described by a single parameter (as could a spherical boundary), but instead, the r = *constant* and t = *constant* hypersurfaces have to be incorporated into the form of ξ^a. Hence such choices as 3.5.89

Yet it is the actual evaluation of W that causes problems, as one is required to integrate

$$rr_{,a}V^a \tag{3.5.90}$$

over the proper two volume, with V^a given by 3.4.67. In order to do this integral, one selects a convenient coordinate system (e.g. Schwarzschild or de Sitter), and [illegible] to do the integration over the two coordinates. However, all the obvious [illegible] choices lead to rather difficult integrals; most of which are elliptic in [illegible]acter (due to the dependence of V^a on $\xi^a\xi^b_{;c}$ and $\xi^a\xi_a$). Again, the cause of the problem is the form of ξ^a which results from the box shaped boundary.

Thus the action, when expressed in this elegant form, is quite difficult to evaluate. Therefore, whilst the use of the Gauss-Bonnet theorem may provide a more elegant form of W for which the application of the variation principle is simple (due to covariance and the three generalised coordinates), it loses to the brute force approach in terms of physical simplicity, and evaluatibility of W.

CHAPTER FOUR

THE STANDARD HAMILTONIAN APPROACH

4.1 Introduction

As mentioned in the introduction of chapter one, the explicit evaluation of the action via the Lagrangian formalism is not the only approach that can be used in order to determine the transmission coefficient for Σ passing through the classically forbidden region. One has the choice of using an alternative treatment of the tunneling process. This alternative approach uses the Hamiltonian formalism, which has its foundation in the generalised coordinates and conjugate momenta of the system. One of the main reasons for turning to this formalism is that it links up with the attempt at canonical quantisation of the system. If such a quantisation can be performed, then through the solution of the Hamilton-Jacobi equation of the system, one can obtain a first order approximation to the transmission coefficient. This last step is achieved by using Hamilton's principal function (the solution to the Hamilton-Jacobi equation) to give a WKB approximation to the wavefunction, and from this the transmission coefficient is obtained by taking the ratio of wavefunctions.

Unlike the evaluation of T(2,1) in section 2.3, the emphasis here is on the quantisation of the system. This however, is not a cut and dried matter, as the method of quantisation is not obvious. Several different quantisation schemes are possible [46, 47, 49, 50], but in the discussion that follows, attention will be centred on the Dirac and ADM schemes.

4.2 The Slicing Of Spacetime

In order to start this Hamiltonian analysis one assumes the same starting point as the previous method; the basic action of the spacetime is given by equation 3.1.2. This generic action makes no assumptions about t. , form of the spacetime over which it is defined. In particular, it does not have to adhere to a specific choice of metric. Therefore, all one has is that the spacetime has an energy-momentum tensor, which describes the form of the spacetime. For the case at hand, the spacetime is a seed of false vacuum in a sea of true vacuum.

This property of generality has two beneficial features. First, some of the rigidity in the spacetime is lost due to the fact that one can retain additional degrees of freedom within the formulation of the problem. This means that during the quantum tunneling era, the metric of the seed spacetime can adopt any form so long as the energy-momentum tensor is maintained, or there may in fact be no fixed metric form. Such a variation in the metric form would seem to imply that the occurrence of tunneling is linked with topological changes during the course of the transition. (Note that this link to topological changes is only applicable to a classically forbidden region, as in such a region the classical equations of motion have no solution, yet there exists an interpolating geometry.). The second feature is subservient to the former, and is that in order to describe the spacetime as given by equation 3.1.2, one has a large range of choice in the form of the metric.

Utilising these features, one can easily write down a very general metric, which is only restricted by the assumption of spherical symmetry (which is used to simplify the analysis). So, splitting off the spherical symmetry contribution, the line element takes the form

$$ds^2 = g_{ab}dx^a dx^b + u^2(x^a)d\Omega_2^2 \tag{4.2.1}$$

with $d\Omega_2^2$ representing the two sphere line element, and $u(x^a)$ the two sphere radius. However, it is the two metric g_{ab} that has to be specified, and generally it contains three independent components (given that it is symmetric in its indices) which must be identified. Identification is necessary, as one of the crucial requirements of canonical quantisation schemes is that time is treated as a special coordinate. That is, the time variable is singled out in order to make the description of the system dynamical. Such an identification of the time variable corresponds to foliating spacetime with surfaces and having the connection between hypersurfaces given by the evolution of the time variable. With canonical quantisation and the Hamiltonian formalism, the time coordinate assumes a higher status than the coordinates, as the theory is given in terms of spatial configurations (or snapshots) that evolve via the laws of physics, to different spatial configurations. This is a direct result of trying to visualise the four dimensional spacetime as a foliated structure.

Now in order to identify the time coordinate, we need only specify the form of g_{ab} explicitly. That is, the three independent components of g_{ab} must be identified. Note, as g_{ab} has three independent components, it implies that the system g_{ab} described has three degrees of freedom. The standard form for breaking up g_{ab} is

$$g_{ab} = \begin{pmatrix} l^2v^2 - n^2 & l^2v \\ l^2v & l^2 \end{pmatrix} \tag{4.2.2}$$

which gives the line element as

$$ds^2 = (l^2v^2 - n^2)dt^2 + 2l^2vdtdr + l^2dr^2 + u^2(t,r)d\Omega_2^2 \tag{4.2.3}$$

with n, l, v, and u being functions of the coordinates $x^a = (t, r)$. (Note, here t and r represent generic temporal and radial coordinates respectively.) In this notation $n(x^a)$ is commonly referred to as the lapse function (as ndt is a measure of proper time between hypersurfaces), and v as the shift vector (which is one dimensional in this case, hence no index). A full description of the geometrical properties of this metric construction can be found in §21.4 of reference [26]

On taking equation 4.2.2 the Ricci scalar can be calculated by the standard procedure, and with a little algebra it can be shown that

$$\begin{aligned}\sqrt{-{}^4g} &= u^2 \sin\theta \sqrt{-{}^2g} = nlu^2 \sin\theta \qquad (4.2.4)\\ \sqrt{-{}^4g}\,{}^4R &= 2\sin\theta\{\frac{2}{n}(vlu)'(\dot{u} - vu') - \frac{2}{n}(lr)^{\cdot}(\dot{u} - nu') + \frac{2}{l}(nu)'u' \\ &\quad +ln + \frac{l}{n}(\dot{u} - vu')^2 - \frac{nu'}{l}\} \qquad (4.2.5)\\ &= 2\sin\theta\,{}^2\mathcal{L}\end{aligned}$$

with $\dot{}$ representing differentiation with respect to the time coordinate and $'$ representing differentiation with respect to r. Equation 4.2.5 gives the gravitational contribution to the action in terms of the four degrees of freedom (n, l, v, u). For the expanded version of equation 4.2.5, consult appendix A, which contains the result of the computer generated calculation of the Ricci scalar. Thus the generic action has the form

$$\begin{aligned}W &= \frac{1}{16\pi}\int {}^4R\sqrt{-{}^4g}\, d^4x + W_m\\ &= \frac{1}{2}\int {}^2\mathcal{L}\, dt\, dr + W_m \qquad (4.2.6)\end{aligned}$$

Yet this form of W is not immediately beneficial, as no obvious features spring forth. In fact the only thing worthy of note is that the gravitational contribution is independent of time derivatives of the lapse and shift (i.e. it is independent of $\dot{n}$ and $\dot{v}$).

However, the reason for re-expressing W in this form is that it is to be used in the canonical quantisation of the system. Yet before discussing this specific case, a general outline of the quantisation procedure seems appropriate.

4.3 Canonical Quantisation

The reason for re-expressing W (equations 4.2.5 and 4.2.6) has not been made clear (other than to say that it aids the canonical quantisation). Therefore, an explanation is in order.

The general definition of the action of the system in terms of the system's Lagrangian density is

$$W = \int \mathcal{L}\, dt \tag{4.3.7}$$

with t being some time coordinate. From the variational principle one gets the standard Euler-Lagrange equations, and if $\mathcal{L}$ is expressed in terms of generalised coordinates q_i, and the generalised velocities $\dot{q}_i$, then the form of the Euler-Lagrange equations is

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} = 0 \tag{4.3.8}$$

Here $i = 0, 1, ..., N$ with N being the number of generalised coordinates. Now the Euler-Lagrange equations give N second order coupled differential equations, which, generally, are difficult to solve.

Fortunately, on switching to the Hamiltonian formulation, this set of second order differential equations can be replaced by an equivalent first order set. To see this, start by defining the momenta conjugate to the generalised coordinates. Following the standard definition [48, chapter 8] the conjugate momenta are

$$\pi_{q_i} = \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \tag{4.3.9}$$

Such a definition allows the system to be described either by the $(q_i, \dot{q}_i)$ or by the phase space coordinates (q_i, π_{q_i}), with the former used in the Lagrangian formulation. If one chooses to switch to the phase space coordinates, a Legendre transformation is required [48, §8.1], [49]. As a result of this coordinate transformation, one obtains (by analogy to thermodynamic arguments) the Hamiltonian $\mathcal{H}$, which is by definition a function of the phase space coordinates. $\mathcal{H}$ has the form

$$\mathcal{H}(q_i, \pi_{q_i}) = \dot{q}_i \pi_{q_i} - \mathcal{L}(q_i, \dot{q}_i) \tag{4.3.10}$$

Clearly, in order to express $\mathcal{H}$ purely in terms of q_i and π_{q_i} (i.e. eliminate $\dot{q}_i$), an expression for $\dot{q}_i$ must be obtainable. That is, equation 4.3.9 must be invertible. If this is the case, then $\mathcal{H}(q_i, \pi_{q_i})$ can be defined, and so with the help of equation 4.3.10, the action becomes

$$W = \int (\dot{q}_i \pi_{q_i} - \mathcal{H}(q_i, \pi_{q_i}))\, dt \tag{4.3.11}$$

which can be expressed as a function of q_i and π_{q_i}. This is desirable as then there is no trouble in the quantisation, which involves treating q_i and π_{q_i} as operators. Further, one is now working in phase space where there are $2N$ independent variables (i.e q_i and π_{q_i}) instead of N. When the variational principle is applied to this form of W, gets $2N$ coupled differential equations. However, the benefit is that this set of rential equations contains only first order equations, and is commonly referred to as the canonical equations of Hamilton [48, refer to §8.1], and are equivalent to the Euler-Lagrange equations (equation 4.3.8).

Yet one must be careful, as this system gives $2N$ equations of motion, but it may not have $2N$ degrees of freedom associated with the $2N$ canonical variables. Instead, the system may have some constraining equations, which would restrict the number of independent variables. To deal with these constraints and their implications it is advisable to first classify them. To this end, the classification scheme to be used is the one given by Dirac in his papers on generalised Hamiltonian dynamics [49]. It is as follows:

1. Primary Constraints. These are defined as relations where the π_{q_i}'s are not all independent functions of the $\dot{q}_i$'s. Thus, if the π_{q_i}'s only involve $N - M$ independent functions of the $\dot{q}_i$'s then there will be M independent relations

$$\phi_m(q_i, \pi_{q_i}) = 0 \qquad m = 1, 2 \ldots M \quad M \leq N \tag{4.3.12}$$

2. Secondary Constraints. These are obtained from the primary constraints by differentiation with respect to the time coordinate. If differentiation produces something other than a trivial result $(0 = 0)$ then one has a secondary constraint. Such constraints are usually expressed as

$$\chi_k(q_i, \pi_{q_i}) = 0 \qquad k = 1, 2, \ldots \tag{4.3.13}$$

   Note, one of the main distinctions between these two classes of constraints is that the primary constraints occur in the equations of motion, whilst the secondary ones don't.

3. First Class Constraints. If the Poisson brackets of a function with the Hamiltonian and all the primary and secondary constraints all vanish, then the function is said to be first class.

4. Second Class Constraint. If a function does not satisfy the requirements of a first class function then it is second class.

It should be noted that for a Poisson bracket to vanish, it is necessary that it vanish weakly. In this context, a weak equation is one where the equation is satisfied such that the Poisson bracket remains a well defined object. The reason for this definition is that given that primary constraints exist, then their use in the determination of the Poisson bracket causes it to be ill-defined (as the Poisson bracket assumes all the q_i's and π_{q_i}'s are independent). Therefore a weak equation or equality is one where the Poisson brackets are evaluated without any assistance from the primary or secondary constraints. Naturally, equations 4.3.12 and 4.3.13 are weak equations.

These are the four main classes of constraints, but as the definitions show, they are not mutually exclusive; for example, a primary constraint can also be first class. Nevertheless, one does have a complete categorization of the constraints, and so the task is now one of considering the effects they produce. This is done by solving the non-trivial equations and the Poisson brackets that the constraints produce. Then under quantisation (i.e. treating the canonical variables as operators) one gets a set of constraining operator equations which act on the wavefunction of the system. Apart from complications associated with operator ordering, the quantisation proceeds as with normal quantum mechanics. Further if no gauge fixing is done, so that the system contains more dynamical degrees of freedom than required, then this canonical quantisation is referred to as Dirac quantisation [49]. On the other hand, if one gauge fixes (usually by hand) so that one has the minimal dynamical degrees of freedom, then the quantisation scheme follows that of Arnowitt, Deser, and Misner (ADM) [50].

If the ADM version is applied to the composite spacetime, then due to the symmetries (eg. spherical), the system should reduce to just one degree of freedom; the radial position of Σ. Such a gauge fixing would simplify the analysis of the action, but unfortunately, there does not seem to be a clear choice for fixing the gauge [44]. Therefore, this scheme will not be adopted in favour of the procedure of Dirac.

4.4 Dirac Quantisation of W

In applying the method explained in the previous section to the action integral, one needs to re-express W in the form

$$W = \frac{1}{2}\int(\pi_{q_i}\dot{q}_i - \mathcal{H})\,dt\,dr \;+\; W_m \tag{4.4.14}$$

After a little algebra, one gets from the explicit form of $\mathcal{L}$ given in ion 4.2.5 that the conjugate momenta are

$$\begin{aligned} \pi_n &= \tfrac{\partial\mathcal{L}}{\partial\dot{n}} = 0 \qquad & \pi_u &= \tfrac{\partial\mathcal{L}}{\partial\dot{u}} = \tfrac{1}{n}((vlu)' - (lu)^{\cdot}) \\ \pi_v &= \tfrac{\partial\mathcal{L}}{\partial\dot{v}} = 0 \qquad & \pi_l &= \tfrac{\partial\mathcal{L}}{\partial\dot{l}} = \tfrac{-u}{n}(\dot{u} - vu') \end{aligned} \tag{4.4.15}$$

which allows equation 4.4.14 to be written as

$$W = \frac{1}{2}\int(\pi_l\dot{l} + \pi_u\dot{u} - n\mathcal{H}_n - v\mathcal{H}_v)\,dt\,dr \;+\; W_m(p_i, q_i^m) \tag{4.4.16}$$

with

$$\mathcal{H}_n = \frac{l\pi_l^2}{2u} - \frac{\pi_l\pi_u}{u} + \frac{1}{2}[(\frac{2uu'}{l})' - \frac{u'^2}{l} - l] + \mathcal{H}_n^m \tag{4.4.17}$$

and

$$\mathcal{H}_v = u'\pi_u - l\pi_l' + \mathcal{H}_v^m \tag{4.4.18}$$

Here q_i^m refers to the matter degrees of freedom, and p_i the associated conjugate momenta. If the matter action W_m contains no momentum dependence other than p_i (as is the case for true and false vacua), then this form of W has two immediate features; the vanishing of both π_n and π_v. Therefore,

$$\pi_n = 0 = \pi_v \tag{4.4.19}$$

correspond to two primary constraints. Further, from these one obtains two secondary constraints, as the time derivatives of these momenta result in non-vanishing

Poisson brackets of the momenta and $\mathcal{H}$. Following the standard definition [51], the Poisson bracket of operators A and B is

$$[A, B] = \frac{\partial A}{\partial q_i}\frac{\partial B}{\partial \pi_{q_i}} - \frac{\partial A}{\partial \pi_{q_i}}\frac{\partial B}{\partial q_i} \tag{4.4.20}$$

and so

$$[\pi_n, \mathcal{H}] = -\frac{\partial \mathcal{H}}{\partial n} \tag{4.4.21}$$

However, $-\frac{\partial \mathcal{H}}{\partial n} = \dot{\pi}_n$ (by the canonical equations of Hamilton [48, §8.1]), and π_n can be shown to be zero. This can be done by writing $\frac{\partial}{\partial t}(\pi_n)$, and then considering the Euler-Lagrange equation (equation 4.3.8) which gives $\frac{\partial \mathcal{L}}{\partial \dot{n}} = \pi_n = 0$, thereby implying that $\dot{\pi}_n = 0$. Thus $[\pi_n, \mathcal{H}] = 0$, equation 4.4.21 implies the secondary constraint

$$\mathcal{H}_n = 0 \tag{4.4.22}$$

Similarly, the Poisson bracket $[\pi_v, \mathcal{H}]$ produces a second secondary constraint, namely

$$\mathcal{H}_v = 0 \tag{4.4.23}$$

In terms of the constraint classification of the previous section, it should be noted that both equation 4.4.22 and 4.4.23 are first class constraints.

With the constraints as given above, their physical implications need to be extracted.

To do this, one must realise that one is dealing with a quantum system, with the canonical variables as operators, and so the system is described by a wavefunction Ψ. If no constraints were applicable, Ψ would be a function of all the generalised coordinates. However, due to the primary constraints, Ψ must be independent of n and v (this can be seen if one considers the Euler-Lagrange equations for n and v).

Thus $\Psi = \Psi(l, u, q_m)$, which implies that Ψ depends only on the spatial geometry of the foliated spacetime, and the matter content.

Next, the secondary constraints have to be considered. Yet

$$\mathcal{H}_n \Psi = 0 = \mathcal{H}_v \Psi \tag{4.4.24}$$

does not appear to be of much help. Only if one confronts the Hamilton-Jacobi equation and uses a WKB approximation does the usefulness of equation 4.4.24 become apparent.

As mentioned in the introduction to this chapter, the easiest way to solve a system of coupled equation characterised by the Hamiltonian $\mathcal{H}$, is to transform the generalised coordinates to a new set where the canonical coordinates are cyclic [48, §10]. This corresponds to a transformation to a system that has a Hamiltonian that is identically zero. That is, if $\bar{\mathcal{H}}$ is the transformed Hamiltonian, then $\bar{\mathcal{H}} = 0$, and

$$0 = \mathcal{H}(q, \pi_q, t) + \frac{\partial F}{\partial t} \tag{4.4.25}$$

with F being the generating function of the transformation [48, chapters 9 and 10]. Clearly, this equation is a function of the phase space variables (q_i, π_{q_i}). However, using the transformation, one can write equation 4.4.25 as a partial differential equation of the generalised coordinates q_i and t. In particular, the transformation of the Hamiltonian gives several coordinate transformation equations ([48, §9.1, equation 9-17, page 383]), of which, the most pertinent one is

$$\pi_{q_i} = \frac{\partial F}{\partial q_i} \tag{4.4.26}$$

Substituting this into equation 4.4.25 gives a partial differential equation of q_i and t, which is of the form

$$0 = \mathcal{H}(q_i, \frac{\partial F}{\partial q_i}, t) + \frac{\partial F}{\partial t} \tag{4.4.27}$$

and it is this equation that is commonly referred to as the Hamilton-Jacobi equation. Further, its solution is labelled as Hamilton's principal function.

If the principal function is designated by $\alpha(q_i, t)$ then it can be shown ([48, §10.1, page 439]) that

$$\alpha(q_i, t) = F \tag{4.4.28}$$

and so

$$\pi_{q_i} = \frac{\partial \alpha}{\partial q_i} \tag{4.4.29}$$

This in turn implies that

$$\alpha(q_i, t) = \int dr \int \pi_{q_i} dq_i \tag{4.4.30}$$

with the integration with respect to r being due to the fact that $\mathcal{H}$ is a density defined over the two-space (i.e. $W = \int \mathcal{L}\, d^2x$). Taking equation 4.4.29, and using it to replace the momenta in a WKB expansion of Ψ, one has (cf. equation 2.3.69), to first order in $\hbar$

$$\Psi(q, \pi_q) = \exp(\frac{i}{\hbar}\alpha(l, u, q_m^i, t)) \tag{4.4.31}$$

With this WKB form for Ψ, the transmission coefficient is then just given by equation 2.3.70, with the result being

$$T(r_2, r_1) \approx \frac{\Psi(r_2)}{\Psi(r_1)} = \exp(\frac{i}{\hbar}(\alpha(r_2) - \alpha(r_1)) \tag{4.4.32}$$

Under this WKB approximation, the secondary constraints can be written as

$$\mathcal{H}_n(q, \frac{\partial \alpha}{\partial l}, \frac{\partial \alpha}{\partial u}) = 0 \tag{4.4.33}$$

and

$$\mathcal{H}_v(q, \frac{\partial \alpha}{\partial l}, \frac{\partial \alpha}{\partial u}) = 0 \tag{4.4.34}$$

and from these, all the derivatives of σ can be specified. Then, in order to obtain α itself, one just has to integrate equation 4.4.30. So, the principal function is

$$\begin{aligned}\alpha &= \int_0^\infty dr \int \pi_{q_i}\, dq_i \\ &= \int_0^\infty dr \int (\pi_l\, dl + \pi_u\, du) \qquad (4.4.35)\end{aligned}$$

Once α has been determined, the form of the wavefunction can be given by equation 4.4.31. The transmission coefficient then just becomes the ratio of the wavefunction at the initial and final spacetime configurations (as given by equation 4.4.32).

4.5 Determining The Principal Function For The Composite Spacetime

It is now clear what is required of this Hamiltonian approach, as the mathematical skeleton has been laid bare. To flesh out this Hamiltonian formalism as applied to the composite spacetime, the matter contribution of the action must be specified. With recourse to the discussion of the matter contributions given in section 3.2, one can write W as

$$\begin{aligned}W &= \frac{1}{16\pi}\int {}^4R\sqrt{-{}^4g}\, d^4x - \rho\int \sqrt{-{}^4g}\, d^4x + \sigma\int_\Sigma \sqrt{-{}^3g}\, d^3\xi \\ &= \frac{1}{4}\int {}^2Ru^2\sqrt{-{}^2g}\, d^2x - 4\pi\rho\int u^2\sqrt{-{}^2g}\, d^2x + 4\pi\sigma\int_\Sigma u^2\, d\tau \qquad (4.5.36)\end{aligned}$$

Here τ is the proper time parameter on Σ. From equation 4.2.3, if one restricts the line element to that of the wall Σ, then in the thin wall approximation

$$ds^2 = (-n^2 + l^2(\dot{u} + v)^2)\, dt^2 = -d\tau^2 \qquad (4.5.37)$$

This gives $d\tau = \sqrt{n^2 - l^2(\dot{u} + v)^2}\, dt$, and so the action be written entirely in terms of the canonical coordinates. Thus, by comparing equations 4.4.14, 4.4.16, and

4.5.37 one obtains the following form for W

$$W = \frac{1}{2}\int(\pi_l \dot{l} + \pi_u \dot{u} - n\mathcal{H}_n - \nu\mathcal{H}_v)\,dt\,dr \qquad (4.5.38)$$

with

$$\begin{aligned}\mathcal{H}_n = \; & \frac{l\pi_l^2}{2u^2} - \frac{\pi_l\pi_u}{u} + \frac{1}{2}[(\frac{2uu'}{l})' - \frac{u'^2}{l} - l] + 4\pi\rho l u^2\Theta(\hat{u} - u) \\ & + \delta(\hat{u} - u)\sqrt{\frac{\hat{p}^2}{l^2} + 16\pi^2\sigma^2\hat{u}^4}\end{aligned} \qquad (4.5.39)$$

and

$$\mathcal{H}_v = u'\pi_u - l\pi_l' - \delta(\hat{u} - u)\hat{p} \qquad (4.5.40)$$

Note, $\hat{u}$ is the two sphere radius of Σ, and $\hat{p}$ is its momentum conjugate. Like equation 4.4.16 the action has the canonical form , with $\mathcal{H}_n$ and $\mathcal{H}_v$ acting as Lagrange multipliers, and the constraint classification remaining unchanged. However, the difference between equation 4.4.16 and equation 4.5.38 is that the latter includes the matter explicitly, and as such is specific to the composite spacetime scenario.

In order to solve for the principal function (equation 4.4.35) one requires the form of π_l and π_u, which are obtainable from their definitions (equation 4.3.9). However the use of equation 4.3.9 is not necessary (and is in fact quite algebraically messy), as one is free to appeal to the two secondary constraint (equations 4.5.39 and 4.5.40). Both depend on π_u, and so π_u can be eliminated via a suitable combination of these constraints. The appropriate combination is

$$0 = \frac{u'}{l}\mathcal{H}_n + \frac{\pi_l}{ul}\mathcal{H}_v \qquad (4.5.41)$$

which in expanded form is

$$\begin{aligned}0 \;=\; & \frac{u'\pi_l^2}{2u^2} + \frac{u'}{2l}[(\frac{2uu'}{l})' - \frac{u'^2}{l} - l] + 4\pi\rho u^2 u'\Theta(\hat{u} - u) \\ & + \delta(\hat{u} - u)\frac{u'}{l}\sqrt{\frac{\hat{p}^2}{l^2} + 16\pi^2\sigma^2\hat{u}^4} - \frac{\pi_l\pi_l'}{u} - \frac{\pi_l}{lu}\delta(\hat{u} - u)\hat{p}\end{aligned} \qquad (4.5.42)$$

$$
\begin{aligned}
&= -(\frac{u}{2}(1-f))' + (\frac{4\pi\rho}{3}u^3\Theta(\hat{u}-u))' \\
&\quad +\delta(\hat{u}-u)(\frac{u'}{l}\sqrt{\frac{\hat{p}^2}{l^2}+16\pi^2\sigma^2\hat{u}^4}+\frac{\pi_l}{lu}\hat{p})
\end{aligned}
\tag{4.5.43}
$$

with $f = g^{ab}u_{,a}u_{,b}$ as the geometrical mass function of Poisson and Israel [35]. In order to interpret this constraint combination rewrite f as

$$
f = 1 - \frac{2m(x^a)}{u} \tag{4.5.44}
$$

with $m(x^a)$ taken as a mass function. Equations 4.5.43 and 4.5.44 then give that

$$
m(x^a) = \frac{\pi_l^2}{2u} + \frac{u}{2}[1-(\frac{u'}{l})^2] \tag{4.5.45}
$$

and the constraint (equation 4.5.43) takes the form

$$
m(x^a) = \frac{2\pi\rho}{3}u^2\Theta(\hat{u}-u) + \frac{2}{u}\int\delta(\hat{r}-r)(\frac{u'}{l}\sqrt{\frac{\hat{p}^2}{l^2}+16\pi^2\sigma^2\hat{u}^4}+\frac{\pi_l}{lu}\hat{p})\,dr \tag{4.5.46}
$$

However, the matter contributions of the system can be incorporated into a general mass function by defining $M(x^a)$ as

$$
\begin{aligned}
M(x^a) = {} & m(x^a) - \tfrac{2\pi\rho}{3}u^2\Theta(\hat{u}-u) \\
& -\tfrac{2}{u}\int\delta(\hat{r}-r)(\tfrac{u'}{l}\sqrt{\tfrac{\hat{p}^2}{l^2}+16\pi^2\sigma^2\hat{u}^4}+\tfrac{\pi_l}{lu}\hat{p})\,dr
\end{aligned}
\tag{4.5.47}
$$

Note, this definition simplifies the form of the constraint to $M(x^a)' = 0$.

At this point, the physical interpretation of $M(x^a)$ is unclear. The $M(x^a)' = 0$ constraint implies that $M(x^a)$ is constant in the radial coordinate r. Yet from equations 4.5.44 and 4.5.45, it can be seen that $m(x^a)$ reduces to the Schwarzschild mass parameter m (this is most easily ascertained if one considers the case of a static slicing of the spacetime ($\pi_l = 0 = \pi_u$). The other contributions to $M(x^a)$ simply subtract off the matter content of the composite spacetime. Therefore $M(x^a)$ is an adjusted mass parameter of the system. With this interpretation of $M(x^a)$, one then gets two boundary conditions on $M(x^a)$. The first is a result of requiring the origin to be non-singular, which implies (due to equations 4.5.44 and 4.5.47) that $M(r = 0) = 0$. The second comes from the fact Birkhoff's theorem implies the exterior true vacuum region is Schwarzschildian with mass parameter m. Thence, $M(r \to \infty) = m$. Also, using the fact that as the composite spacetime is composed of true and false vacuum regions, $M(x^a)$ is constant within a given region, one gets the following conditions on $M(x^a)$.

$$M(x^a) = \begin{cases} 0 & u < \hat{u} \\ m & u > \hat{u} \end{cases} \tag{4.5.48}$$

Note, the matter terms in equation 4.5.47 counteract any contribution from $m(x^a)$ so that equation 4.5.48 holds. Substituting this into equations 4.5.45 and 4.5.47 one gets that

$$\pi_l^2 = u^2\left(\frac{u'^2}{l^2} - 1 + \frac{8\pi\rho u^2}{3}\right) = u^2\left(\frac{u'^2}{l^2} - f_D\right) \qquad u < \hat{u} \tag{4.5.49}$$

and

$$\pi_l^2 = u^2\left(\frac{u'^2}{l^2} - 1 + \frac{2m}{u}\right) = u^2\left(\frac{u'^2}{l^2} - f_S\right) \qquad u > \hat{u} \tag{4.5.50}$$

which gives π_l in the two regions (here f_D and f_S are as defined by equations 2.2.2 and 2.2.3). From the second secondary constraint (equation 4.5.40) one can determine the form of π_u as a function of l, u, u', ρ, and m.

Specifying π_l and π_u in both the false and true vacuum regions is not quite enough to permit the determination of Hamilton's principal function. The reason for this is that equations 4.5.49 and 4.5.50 don't include contributions from Σ.

To include Σ's contribution to the differential form of the principal function (equation 4.4.29), the jump in the momenta must be evaluated via the Σ junction conditions. To do this, it is assumed that the foliating of the spacetime is such that the spatial geometry is continuous across Σ. Note, this seems physically reasonable, and due to the form of equation 4.2.3 it is just a matter of choosing wisely. With this assumption, one can then take the jump of equations 4.5.49 and 4.5.50 at Σ (note, the jump is as defined by equation 2.2.26), and obtain

$$\frac{\pi_l[\pi_l]}{u} - \frac{uu'[u']}{l^2} = \frac{u'E}{l^2} - \frac{\pi_l\hat{p}}{ul} \tag{4.5.51}$$

with

$$E = \sqrt{\hat{p}^2 + 16\pi^2\sigma^2 l^2 \hat{u}^4} \tag{4.5.52}$$

Now by matching terms of π_l and u' in equation 4.5.51, one gets

$$[\pi_l] = \frac{-\hat{p}}{l} \tag{4.5.53}$$

$$[u'] = \frac{-E}{u} \tag{4.5.54}$$

which are the junction conditions for Σ.

For the set of equations given by the constraints (equations 4.5.49, 4.5.50, 4.5.52, 4.5.53, and 4.5.54) there are six different parameters to be determined (namely $u'(\hat{r}+\epsilon)$, $u'(\hat{r}-\epsilon)$, $\pi_l(\hat{r}+\epsilon)$, $\pi_l(\hat{r}-\epsilon)$, $\hat{p}$, and $\hat{E}$), but only five constraining equations . So, in order to solve this set of equations, one of the parameters has to be specified. One such choice is one that corresponds to choosing the frame of reference that the set of equations is to be solved in. The easiest and most obvious choice is the rest frame of Σ, and this has $\hat{p} = 0$. With this choice, the junction conditions then give

$$\begin{aligned} \pi_l^2(\hat{r}+\epsilon) &= \pi_l^2(\hat{r}-\epsilon) \\ &= 4\pi^2\sigma^2\hat{u}^4\{(2\mu - 1 - \lambda) + 8\mu - 4(\frac{\mu}{4\pi\sigma m})^{\frac{2}{3}}\} \end{aligned} \tag{4.5.55}$$

$$u'(\hat{r}+\epsilon) = \frac{-m\hat{l}}{4\pi\sigma\hat{u}^2} + 2\pi\sigma\hat{l}\hat{u}(\lambda+1) \tag{4.5.56}$$

$$u'(\hat{r} - \epsilon) = \frac{-m\dot{l}}{4\pi\sigma\hat{u}^2} + 2\pi\sigma\dot{l}\hat{u}(\lambda - 1) \quad (4.5.57)$$

with

$$\lambda = \frac{\rho}{6\pi\sigma^2} \qquad \mu = \frac{m}{16\pi^2\sigma^2\hat{u}^3} \quad (4.5.58)$$

This result, when combined with equations 4.5.49, and 4.5.50 gives π_l and u' over the entire range of the composite spacetime. Also, equations 4.5.55, 4.5.56, and 4.5.57 give the motion of Σ, which has some of the features described in section 2.2. The characteristics of the motion are as follows.

- π_l^2 can be positive or negative, corresponding to the momentum being in classically allowed or classically forbidden regions.
- π_l^2 is positive as $\hat{u} \to 0$ and ∞, and so the initial segments of trajectories like the bounded and the bounce solutions of section 2.2 are permissible.
- If m is sufficiently large , then π_l^2 is never less than zero, and so the associated Σ trajectory would have no classical turning points. This corresponds to the monotonic solution of section 2.2.

Now the second and third characteristics listed imply that for m less than the critical value, a classically forbidden region exists, which is bounded by two classical turning points.

So, given that two classical turning points for the motion of Σ exist, one can legitimately ask what the transmission coefficient is between these two turning point configurations. To start such a calculation, label the turning points by r_1 and r_2 with $r_1 < r_2$. The transmission coefficient that is to be determined is $T(r_2, r_1)$ and is given by equation 4.4.32, with r_1 and r_2 being the two end points. Note, one feature of this Hamiltonian method is that one can calculate $T(r_2, r_1)$ without worrying about the types of trajectories involved (as was the case with the direct method of section 3.3). Also, as the turning points are the end points for the tunneling calculation, one has by definition of a turning point, that the endpoints have the condition that all the momenta are zero. That is,

$$\dot{p} = \pi_l = \pi_u = 0 \tag{4.5.59}$$

4.6 Evaluation of $T(r_2, r_1)$.

In order to evaluate $T(r_2, r_1)$ one appeals to equations 4.4.31 and 4.4.32 which give

$$T(r_2, r_1) \approx \frac{\Psi(r_2)}{\Psi(r_1)} = \exp(\frac{i}{\hbar}(\alpha(r_2) - \alpha(r_1)) \tag{4.6.60}$$

with

$$\Psi(r) = \exp(\frac{i}{\hbar}\alpha(r)) \tag{4.6.61}$$

As can be seen from these equations, the determination of the principal function $\alpha(r)$ is of central importance. Further, $T(r_2, r_1)$ involves evaluating $\alpha(r)$ over the range (r_1, r_2). Turning attention to the principal function, equation 4.4.35 gives

$$\alpha(r) = \int dr \int (\pi_l dl + \pi_u du) \tag{4.6.62}$$

Note, unlike equation 4.4.35, the range of the r integration has not been given, but as it is $T(r_2, r_1)$ that is being determined, the range is (r_1, r_2). In applying equation 4.6.62 it should first be noted that the region over which $T(r_2, r_1)$ is evaluated is the classically forbidden region, and for this region $\pi_l^2 < 0$ (equations 4.5.49 and 4.5.50) which implies that π_l is imaginary. Similarly for π_u. This gives equation 4.6.62 an imaginary integrand. To remove the apparent imaginary factor from the integrand (which is due to being in the classically forbidden region), multiply through by i (this factor of i comes from equation 4.6.61) to get

$$\pi_{q_i} = i\tilde{\pi}_{q_i} \tag{4.6.63}$$

with $\tilde{\pi}_u$ being real. Under this change in notation, equation 4.6.61 becomes

$$\Psi(r) = \exp\frac{\tilde{\alpha}}{\hbar} \tag{4.6.64}$$

From this equation it is clear that if $\tilde{\alpha}(r)$ is real, then the wavefunction has the form of exponential decay (or growth) which is consistent with standard quantum mechanical tunneling through a barrier.

Now as equation 4.6.62 has an integration over the whole range of r (r_1, r_2) and Σ lies within this range, $\tilde{\alpha}(r)$ is broken into three components; an interior, an exterior, and a wall component. So, define

$$\begin{aligned}\tilde{\alpha}(r) &= \tilde{\alpha}_1(r) + \tilde{\alpha}_2(r) + \tilde{\alpha}_3(r) \\ &= (\int_{r_1}^{\hat{r}} + \int_{\Sigma} + \int_{\hat{r}}^{r_2})\, dr \int (\tilde{\pi}_l\, dl + \tilde{\pi}_u\, du) \end{aligned}\tag{4.6.65}$$

and consider each component separately. As discussed by Fischler et al. [44] the evaluation of $\tilde{\alpha}(r)$ can be done in two steps. The first step is the integration over the variation in $l(r)$ and $u(r)$, whilst the second is the integration over the radial coordinate r.

In attempting this first step, consider $\tilde{\alpha}_1(r)$. This contribution is from the false vacuum region, and so one can consider Σ and the geometry in its immediate neighbourhood to be held fixed, whilst l and u are varied over the false vacuum region. Thus with $\hat{r}$ fixed

$$\begin{aligned}\tilde{\alpha}_1(r) &= \int_{r_1}^{\hat{r}} dr \int (\tilde{\pi}_l\, dl + \tilde{\pi}_u\, du) \\ &= \int_{r_1}^{\hat{r}} dr \int \left(\frac{u}{l}\sqrt{(1-\frac{4\pi\rho}{3})l^2u^2 - u'^2}\, dl \right. \\ &\quad \left. + \frac{(1-\frac{4\pi\rho}{3})l^2u^2 - u'^2 + \frac{u'l'}{l} - uu''}{\sqrt{(1-\frac{4\pi\rho}{3})l^2u^2 - u'^2}}\, du\right) \\ &= \int_{r_1}^{\hat{r}} dr \int (\frac{u}{l}\sqrt{f_D l^2 - u'^2}\, dl + \frac{f_D l^2 - u'^2 + \frac{u'l'}{l} - uu''}{\sqrt{f_D l^2 - u'^2}}\, du) \end{aligned}\tag{4.6.66}$$

Here the momenta have been converted to real quantities via equation 4.6.63, and $\tilde{\pi}_u$ has been obtained from the second secondary constraint (i.e. $\tilde{\pi}_u = \frac{l\tilde{\pi}_l}{u'}$). Also $f_D = 1 - \frac{4\pi\rho}{3}u^2 = 1 - \chi^2 u^2$ (equations 2.2.3 and 2.2.9). In order to obtain $\tilde{\alpha}_1(r)$ from this variation over l and u, integrate first with respect to l, then with respect to u. Thus, integrating with respect to l along a path of fixed u up to an l associated with the r_1 turning point, which is characterised by $l = \frac{u'}{\sqrt{f_D}}$ one has

$$\int \frac{u}{l}\sqrt{f_D l^2 - u'^2}\, dl \tag{4.6.67}$$

Note, as u is fixed, the last term in equation 4.6.66 drops out. With reference to Gradshteyn and Ryzhik [45], one finds that the general form of such an integral is given by

$$\int \frac{\sqrt{cx^2 + a}}{x}\, dx = \sqrt{cx^2 + a} + aI \tag{4.6.68}$$

with

$$I = \begin{cases} \frac{1}{2\sqrt{a}} \ln \frac{u - \sqrt{a}}{u + \sqrt{a}} & a > 0, c > 0 \\ \frac{1}{2\sqrt{a}} \ln \frac{\sqrt{a} - u}{\sqrt{a} + u} & a > 0, c < 0 \\ \frac{1}{\sqrt{-a}} \cos^{-1}\left(\frac{\sqrt{\frac{-a}{c}}}{x}\right) & a < 0, c > 0 \end{cases} \tag{4.6.69}$$

and the case that is applicable to equation 4.6.67 is equation 4.6.69. Therefore

$$\int \frac{u}{l}\sqrt{f_D l^2 - u'^2}\, dl = \sqrt{f_D l^2 - u'^2} - uu' \cos^{-1}\left(\frac{u'}{l\sqrt{f_D}}\right) \tag{4.6.70}$$

To complete the integration over the variation in l and u, one has to keep l fixed at $l = \frac{u'}{\sqrt{f_D}}$ and integrate over u to a standard configuration like $l = 1$ and $u = r$. With such a path there is no contribution from this second section, as due to l being fixed at $l = \frac{u'}{\sqrt{f_D}}$, all the momenta are zero. Hence,

$$\tilde{\alpha}_1(r) = \int_{r_1}^{\hat{r}} \left(\sqrt{f_D l^2 - u'^2} - uu' \cos^{-1}\left(\frac{u'}{l\sqrt{f_D}}\right)\right) dr \tag{4.6.71}$$

Similarly, one can evaluate $\tilde{\alpha}_3(r)$ with the same sort of argument; the only differences being that as one is dealing with the exterior f_S replaces f_D, and the range of the r integration is different. Thus $\tilde{\alpha}_3(r)$ is

$$\tilde{\alpha}_3(r) = \int_{\hat{r}}^{r_2} \left(\sqrt{f_S l^2 - u'^2} - uu' \cos^{-1}\left(\frac{u'}{l\sqrt{f_S}} \right) \right) dr \qquad (4.6.72)$$

With $\tilde{\alpha}_1(r)$ and $\tilde{\alpha}_3(r)$ determined, attention must be focused on $\tilde{\alpha}_2(r)$, which by definition must depend entirely on the geometry of Σ. This implies that the geometry is not fixed at Σ and so the variation of l and u extends to Σ in this case. Using the same type of path of integration as before, one gets

$$\begin{aligned} \tilde{\alpha}(r)_2 &= \int_{r_1}^{r_2} d\hat{r} \int (\tilde{\pi}_l \, d\hat{l} + \tilde{\pi}_u \, d\hat{u}) \\ &= \int_{r_1}^{r_2} \left(\hat{u}\{\cos^{-1}\left(\frac{u'(\hat{r}-\epsilon)}{\hat{l}\sqrt{\hat{f}_D}} \right) - \cos^{-1}\left(\frac{u'(\hat{r}+\epsilon)}{\hat{l}\sqrt{\hat{f}_S}})\}\right) d\hat{r} \end{aligned} \qquad (4.6.73)$$

Here it should be noted that for equations 4.6.71, 4.6.72, and 4.6.73 the inverse cosine is taken to have values in the interval $[0, \pi]$.

With $\tilde{\alpha}_1(r)$, $\tilde{\alpha}_2(r)$, and $\tilde{\alpha}_3(r)$ determined, the transmission coefficient (equation 4.6.60) can now be written explicitly. As $T(r_2, r_1)$ depends on the quantity $\tilde{\alpha}(r_2) - \tilde{\alpha}(r_1)$, this can now be determined using the three component pieces evaluated over the turning points. Fortunately, as the turning points are the endpoints of the tunneling, the turning point condition (equation 4.5.59) simplifies the form of $\tilde{\alpha}_1(r)$ and $\tilde{\alpha}_3(r)$. Principally, this condition sets the first term on the right hand side of equations 4.6.71 and 4.6.72 to zero, whilst the integrand of the second term in these equations reduces to $-uu' \cos^{-1}(\pm 1)$. The $\pm$ in the inverse cosine is dependent on the sign of u' relative to the sign of l and can be ascertained using equations 4.5.56 and 4.5.57. Also, as the inverse cosine is defined over the principal branch $\cos^{-1}(+1) = 0$ and $\cos^{-1}(-1) = \pi$. However, care should be taken, as the relative sign of u' may change over the range of the integration (r_1 to r_2). Hence, the evaluation of $\tilde{\alpha}_1(r)$ and $\tilde{\alpha}_3(r)$ requires a little more attention.

To specify when $u'(\hat{r}-\epsilon)$ and $u'(\hat{r}+\epsilon)$ change sign, consider equations 4.5.56 and 4.5.57. If one adopts the classification scheme of Farhi et al., there are two values of the mass parameter m for which a sign change could occur (here it is assumed that m is below the critical value, so that a classically forbidden region exists). It should also be noted that this classification is equivalent to one given by Blau et al. [29]. So, under this classification, identify two characteristic mass parameter values, namely m_D and m_S, with

$$m_D < m_S < m_{\text{critic}} \tag{4.6.74}$$

On considering the inner turning point, one has that $u'(\hat{r}-\epsilon) > 0$ whilst $u'(\hat{r}+\epsilon) > 0$ for $m < m_S$, and $u'(\hat{r}+\epsilon) < 0$ for $m > m_S$. Similarly, the outer turning point configuration has $u'(\hat{r}+\epsilon)$ always negative with $u'(\hat{r}-\epsilon) > 0$ for $m > m_D$, and $u'(\hat{r}-\epsilon) < 0$ for $m < m_D$. Note, this use of the two values of the mass parameter, gives the four trajectory types discussed in both section 2.2 and reference [29].

With this classification in place, one can proceed with the evaluation of $\tilde{\alpha}_1(r)$ and $\tilde{\alpha}_3(r)$. From above, these contributions are of the form

$$\int_{r_1}^{r_2} a u u' \, dr \tag{4.6.75}$$

with $a = \cos^{-1}(\pm 1)$: the value of a depends on the value of m. Combining the value of $\cos^{-1}(\frac{u'}{l\sqrt{f}})$ with 4.6.75, one gets

$$\tilde{\alpha}_1(r_2) + \tilde{\alpha}_3(r_2) - \tilde{\alpha}_1(r_1) - \tilde{\alpha}_3(r_1) = \begin{cases} \frac{\pi}{2}(u_2^2 - u_1^2) & m > m_S \\ \frac{\pi}{2}(u_2^2 - u_S^2) & m_S > m > m_D \\ \frac{\pi}{2}(u_D^2 - u_S^2) & m_D > m \end{cases} \quad (4.6.76)$$

Note, here $u_D \equiv u(r_D)$ and $u_S \equiv u(r_S)$, with r_D and r_S being defined as the radial coordinates associated with m_D and m_S.

The other contribution to $\tilde{\alpha}$ is $\tilde{\alpha}_2(r)$, but for this contribution Σ cannot be treated as fixed, and so

$$\begin{aligned} \tilde{\alpha}_2(r) &= \int_{r_1}^{r_2} (\cos^{-1}\left(\frac{u'(\hat{r}-\epsilon)}{l\sqrt{f_D}}\right) - \cos^{-1}\left(\frac{u'(\hat{r}+\epsilon)}{l\sqrt{f_S}}\right)) \, d\hat{u} \\ &= \int_{r_1}^{r_2} (\cos^{-1}\left(\frac{\frac{-m\hat{l}}{4\pi\sigma\hat{u}^2} + 2\pi\sigma\hat{l}\hat{u}(\lambda-1)}{l\sqrt{f_D}}\right) \\ &\quad - \cos^{-1}\left(\frac{\frac{-m\hat{l}}{4\pi\sigma\hat{u}^2} + 2\pi\sigma\hat{l}\hat{u}(\lambda+1)}{l\sqrt{f_S}}\right)) \, d\hat{u} \end{aligned} \quad (4.6.77)$$

with λ given by equation 4.5.58.

Thus, from equation 4.6.60

$$\begin{aligned} T(r_2, r_1) &= \exp(\frac{-1}{\hbar}(\tilde{\alpha}_1(r_2) + \tilde{\alpha}_2(r_2) + \tilde{\alpha}_3(r_2) - \tilde{\alpha}_1(r_1) - \tilde{\alpha}_2(r_1) - \tilde{\alpha}_3(r_1)) \\ &= \frac{\pi}{2}(u_D^2 - u_S^2) + \int_{r_1}^{r_2} (\cos^{-1}\left(\frac{\frac{-m\hat{l}}{4\pi\sigma\hat{u}^2} + 2\pi\sigma\hat{l}\hat{u}(\lambda-1)}{l\sqrt{f_D}}\right) \\ &\quad - \cos^{-1}\left(\frac{\frac{-m\hat{l}}{4\pi\sigma\hat{u}^2} + 2\pi\sigma\hat{l}\hat{u}(\lambda+1)}{l\sqrt{f_S}}\right)) \, d\hat{u} \end{aligned} \quad (4.6.78)$$

In this last line the constructibility requirements of section 2.2 have been invoked, as one requires a composite spacetime of very low m in order for the trajectory to exist in region I of the Kruskal-Szekeres diagram (i.e. the laboratory region of the spacetime).

CHAPTER FIVE

THE HAMILTONIAN METHOD WITH TIME LEFT IMPLICIT

5.1 Introduction

Given that the Lagrangian and Hamiltonian methods for dealing with the tunneling of a seed spacetime have been thoroughly discussed in the previous chapters, a method that highlights the benefits of both formalisms is now to be attempted. The main focus of this new method is to not single out time explicitly, yet still retain the generalised coordinates and the associated phase space (so to allow for an applicable quantisation procedure). This notion of having no explicit time dependence in a Hamiltonian formalism, is not a new idea (e.g. see the work by Weiss in the area of continuum mechanics [52]), however it is one that still requires some form of structure to the spacetime. The most natural choice of structure is that in which the composite spacetime is to be seen as a foliated structure, one can not do away with the idea of hypersurfaces evolving from one configuration (snapshot) to another.

So, in order to develop this approach, a canonical Hamiltonian formalism similar to that of chapter four is used as a starting point. Thus, this method will use a $(2+2)$ split of the four metric, with one of the two metrics being a two sphere (as usual, spherical symmetry has been assumed). The other two metric, which is orthogonal to the two sphere, is then just a Lorentzian $(1+1)$ metric. However, unlike the previous method, the Lorentzian two space will not be sliced up in the standard ADM way, which identifies the time variable and then specifies the three independent components of the two metric g_{ab}. This is the standard practice of choosing the lapse and the shift functions (refer to page 78), and the surface-one metric associated with the family of "time" = constant surfaces. In the previous method, these three independent functions are used as generalised coordinates in the subsequent Hamiltonian analysis. Hence it is this complete and explicit specification of all the degrees of freedom of g_{ab} that has to be avoided if this implicit time coordinate method is to proceed. Further, it seems that the simplicity of the Hamiltonian analysis is hidden by the choice of foliation.

5.2 The Implicit Time Version Of The Hamiltonian Formalism

As mentioned above, in order to produce a simpler and or conceptually clearer resolution to this problem of the tunneling of the trajectory of Σ, a canonical Hamiltonian-styled formalism is sought. Such a formalism would, by design, not explicitly specify the form of the $(1+1)$ metric. Although this lack of explicitness may cause some problems, it has a degree of sensibility to it, as the tunneling of the seed spacetime trajectory across the classically forbidden region is discussed in the semi-classical approximation. This approximation only requires that the geometry be of some Lorentzian form (i.e. $(1+1)$). Note, the spherical symmetry is assumed to be maintained in the classically forbidden region. Yet with this somewhat arbitrary geometry, the notion of the two space being built up by stacking a family of spacelike surfaces is retained (a choice between timelike and spacelike hypersurfaces had to be made, as one needs a consistently defined normal). However, one should note that if the foliating hypersurfaces are specified then one has regressed back to the Hamiltonian approach of chapter four. Hence, although the two space is foliated, the foliation is to be treated as being (somewhat) arbitrary, as the hypersurfaces are given as spacelike, with a normal n^a.

This inexplicit foliation is a direct result of the motivation of this approach. That is, the hypersurfaces are left implicit as one wishes to make the physics of the problem more apparent. In addressing this concern, only the two quantities inherent to any spacetime slice are to be used; the outward normal and the intrinsic surface metric. Hence in shifting from the ADM description of the foliation to one based on these properties, a reduction in the number of generalised coordinates occurs (as the normal replaces the lapse and the shift function). As mentioned in the Misner, Thorne, and Wheeler book *Gravitation* [26, §21.4, page 507], such a reduction results in a spacetime description that is "a structure deprived of rigidity". Nevertheless, such flexibility may be what is needed for the description of the tunneling through the classically forbidden region. This loss of rigidity fits in with the notion expressed in the previous chapter (page 78) that the tunneling is really to be seen as a result of topological change of the geometry of the spacetime. That is, the proposition is that the tunneling is caused by a deforming of the geometry of spacetime.

Before considering the problem of the composite spacetime, it seems wise to attempt a test case first so that this new method may be checked. Such a procedure has been adopted by Fischler et. al. [44], as for a simple test case, calculated quantities can be checked against known results. Further, for this problem of a non-explicit time coordinate foliation of the spacetime, the easiest and most well known scenario is a true vacuum or empty space case.

5.3 The Empty Space Case

The action associated with pure gravity is generally taken as the Einstein-Hilbert action, which for the true vacuum spherically symmetric spacetime is of the form

$$W = \frac{1}{16\pi} \int {}^4R \sqrt{-{}^4g}\, d^4x \qquad (5.3.1)$$

with ${}^4R \equiv$ the four-dimensional Ricci scalar. Using the spherical symmetry one has a four-metric as given by equation 4.2.1, and so the theory is dimensionally reduced to two dimensions (refer to page 49). As with the Lagrangian method, one has to write ${}^4R$ in terms of a two geometry, and so again by Poisson and Israel [35] one has

$$ {}^4R = {}^2R + \frac{2}{u^2}(1 - 2u\Box u - u^{,a}u_{,a}) \tag{5.3.2} $$

Remember, $u = u(x^a) =$ the two sphere radius, and $\Box u \equiv u^{;a}{}_{;a}$. Again, the terms that are extra to the purely gravitational two dimensional theory are to be treated like source terms. As shown earlier, the action, after the dimensional reduction, has the form

$$ W = \frac{1}{4}\int \{{}^2Ru^2 + 2(1 - 2u\Box u - u^{,a}u_{,a})\}\sqrt{-{}^2g}\, d^2x \tag{5.3.3} $$

However, as with the method of chapter three, this is not to be taken as the complete form of the action, as the contributions from the boundary surfaces have not been considered. Also, this is one of the features that distinguishes this approach from that of chapter four, which doesn't consider surface terms. The reason for the inclusion of these boundary terms is threefold, and at risk of repetition, the reasons are as follows:

- These boundary surface contributions to W are genuine contributions, even though they themselves are not dynamically relevant.
- Inclusion of such terms allows the extremisation of the action ($\delta W = 0$) to be satisfied for arbitrary boundary surface metrics as well as arbitrary spacetime metrics [36].
- Without the addition of these terms the Lagrangian contains second order time derivatives of standard coordinates (this is not true for the phase space

coordinates of chapter four), which in turn causes problems with quantisation and intermediate state insertion (equation 3.1.6. For more details, see page 50).

So following convention, the standard boundary surface term contribution is the Gibbons-Hawking surface term (equation 3.2.25), and when this is added to the action, one has

$$W = \frac{1}{4}\int \{{}^2Ru^2 + 2(1 - 2u\Box u - u^{,a}u_{,a})\}\sqrt{-{}^2g}\,d^2x + \frac{1}{8\pi}\int {}^3K\sqrt{-{}^3h}\,d^3\xi \quad (5.3.4)$$

with ${}^3K$ being the trace of the extrinsic curvature, and ${}^3h$ the determinant of the intrinsic three surface metric.

Now to deal with this boundary term, one has from spherical symmetry that

$$\begin{aligned} {}^3K &= {}^3K_i^i \qquad\qquad i \in \{1,2,3\} \\ &= {}^3K_1^1 + 2\,{}^3K_2^2 \end{aligned} \quad (5.3.5)$$

but this is not in a workable form for this analysis. Like the ${}^2R$, equation 5.3.5 has to be expressed in terms of the foliating surface metric and its normal (i.e. the replacements for the ADM lapse, shift and intrinsic metric). Starting with the surface term, one has that the extrinsic curvature of a hypersurface is given by

$${}^3K_{ij} = e_i^\mu e_j^\nu n_{\mu|\nu} \quad (5.3.6)$$

(Note, $|$ ≡ four dimensional covariant derivative.) As the two-sphere is orthogonal to g_{ab}, the normal to the foliating surfaces is of the form $n_\mu = (n_1, n_2, 0, 0)$. With the normal as such, the K_1^1 term in equation 5.3.5 can be written as

$$\begin{aligned} {}^3K_1^1 &= e^{\mu\,1}e_1^\nu(n_{\mu,\nu} - {}^4\Gamma^c_{\mu\nu}n_c) \\ &= e^{a\,1}e_1^b(n_{a,b} - {}^2\Gamma^c_{ab}n_c) + e^{x\,1}e_1^{y\,4}\Gamma^c_{xy}n_c \end{aligned} \quad (5.3.7)$$

Note that here the indices x and y belong to the two-sphere part of the metric, and as g_{ab} is orthogonal to the two-sphere metric $e_1^y = 0$, giving

$$^3K_1^1 = e^{a\,1}e_1^b n_{a;b} = {}^1K_1^1 = \mathcal{K} \qquad (5.3.8)$$

($_;$ $\equiv$ the two dimensional covariant derivative). $\mathcal{K}$ is the trace of the extrinsic curvature of the one-surface, and as it is a one-surface, there is only one extrinsic curvature component.

Having established equation 5.3.8, one can move on to the two-Ricci scalar ^2R, which can be written in terms of the extrinsic curvature of the foliating hypersurfaces. From the Gauss-Codazzi formalism discussed on pages 18 and 52, one has

$$^{(m)}R = {}^{(m-1)}R + (K_{\mu\nu}K^{\mu\nu} - (K^{\mu}{}_{\mu})^2) - 2K_{;n} \qquad (5.3.9)$$

with $_{;n}$ $\equiv$ m-dimensional covariant derivative taken with respect to the normal (i.e. $\frac{\partial}{\partial n} = n^{\mu}\frac{\partial}{\partial x^{\mu}}$). For $m = 2$

$$\begin{aligned} {}^2R &= {}^1R + (K_{ab}h^{ab} - K) - 2K_{;n} \\ &= -2K_{;n} \\ &= -2\mathcal{K}_{;n} \end{aligned} \qquad (5.3.10)$$

Note, the Ricci scalar for a line ^1R, is identically zero. Substituting equation 5.3.10 into equation 5.3.4, one has, with the aid of Gauss's Theorem equation 3.1.8,

$$W = \frac{1}{2}\int \sqrt{-{}^2g}(1 - 2u\Box u - u^{,a}u_{,a} - \mathcal{K}_{;n}u^2)\,d^2x + \int_B \sqrt{-{}^1h}\,{}^3Ku^2 d\xi \qquad (5.3.11)$$

Yet this form of W will not suffice as it is an action that is the two-space integral of a Lagrangian density that is sought. Also, equation 5.3.11 contains second order derivatives, and it is not clear whether these lie entirely in the foliating surface or not. In order to resolve this, one needs to consider the terms containing $\Box u$ and

${}^3K$ more closely This can be done by applying Gauss's theorem to the remaining surface term, with the result that it can be written as

$$\begin{aligned}
\int_B {}^3K u^2 dB &= \int \sqrt{-{}^1h}(\omega\, {}^3K u^2 n^a n_a)\, d\xi \\
&= \int \sqrt{-{}^2g}({}^3K u^2 n^a)_{;a}\, d^2x && (5.3.12) \\
&= \int \sqrt{-{}^2g}(({}^3K)_{;a} u^2 n^a + {}^3K 2uu_{;a} n^a + {}^3K u^2 n^a_{;a})\, d^2x && (5.3.13)
\end{aligned}$$

Hence, on combining equations 5.3.5, 5.3.8, 5.3.10, and 5.3.13, the action is of the form

$$\begin{aligned}
W &= \frac{1}{2}\int \sqrt{-{}^2g}(1 - 2u\Box u - u^{,a}u_{,a} + 2({}^3K^2_2)_{;a} n^a u^2 + 2uu_{,a}n^a\mathcal{K} \\
&\quad +4uu_{,a}n^a\, {}^3K^2_2 + u^2 n^a{}_{;a}\mathcal{K} + {}^3K^2_2 2u^2 n^a_{;a})\, d^2x && (5.3.14)
\end{aligned}$$

Using equation 5.3.6, and given that $n_\mu e^\mu_i = 0$ by definition, K_{ij} can be shown to have the equivalent form (equation 2.2.20) of

$$K_{ij} = \frac{1}{2} g_{ij,\mu} n^\mu \qquad (5.3.15)$$

as this surface plus normal description is in a sense a G.N.C. system (see the definition on page 16), and so,

$$\begin{aligned}
{}^3K^2_2 &= \frac{1}{2} g^{22} g_{22,a} n^a && (5.3.16) \\
&= \frac{u_{,a}n^a}{u} && (5.3.17)
\end{aligned}$$

This implies that

$$\begin{aligned}
W &= \frac{1}{2}\int \sqrt{-{}^2g}(1 - 2u\Box u - u^{,a}u_{,a} + 2(\frac{u_{,a}n^a}{u})_{;b} n^b u^2 + 2uu_{,a}n^a\mathcal{K} \\
&\quad +4uu_{,b}n^b \frac{u_{,a}n^a}{u} + u^2 n^a{}_{;a}\mathcal{K} + 2\frac{u_{,b}n^b}{u} u^2 n^a_{;a})\, d^2x \\
&= \frac{1}{2}\int \sqrt{-{}^2g}(1 - 2u\Box u - u^{,a}u_{,a} + 2uu_{;ab}n^a n^b + 2uu_{,a}n^a{}_{;b}n^b - 2u_{,a}u_{,b}n^a n^b
\end{aligned}$$

$$+2uu_{,a}n^a\mathcal{K} + 4u_{,b}n^b u_{,a}n^a + u^2 n^a{}_{;a}\mathcal{K} + 2u_{,a}n^a u n^b_{;b})\, d^2x$$

$$= \frac{1}{2}\int\sqrt{-{}^2g}(1 - 2uu_{;ab}(g^{ab} - n^a n^b) - u_{,a}u_{,b}g^{ab} + 2uu_{,a}n^a{}_{;b}n^b$$
$$+2uu_{,a}n^a\mathcal{K} + 2u_{,b}n^b u_{,a}n^a + u^2 n^a{}_{;a}\mathcal{K} + 2u_{,a}n^a u n^b_{;b})\, d^2x \qquad (5.3.18)$$

However, equation 5.3.18 is not the final form of W. To see this, consider $uu_{;a}n^a{}_{;b}n^b$. Due to $K_{ab} = n_{a;b}$, this term can be written as

$$uu_{;a}n^a{}_{;b}n^b = u^{;a}K_{ab}n^b \qquad (5.3.19)$$

Now switching to the equivalent definition of K_{ab} which is in terms of the Lie derivative with respect to the normal, one has

$$\begin{aligned} K_{ab} &= \frac{1}{2}\pounds_n h_{ab} \\ &= \frac{1}{2}(h_{ab;c}n^c + h_{cb}n^c{}_{;a} + h_{ca}n^c{}_{;b}) \\ \Rightarrow \quad K_{ab}n^b &= \frac{1}{2}(h_{ab;c}n^b n^c + h_{cb}n^b n^c{}_{;a} + h_{ca}n^b n^c{}_{;b}) \qquad (5.3.20) \\ K_{ab}n^b &= \frac{1}{2}(h_{ab;c}n^b n^c - h_{ac;b}n^b n^c) = 0 \qquad (5.3.21) \end{aligned}$$

which is as expected. Thus the term in W given by equation 5.3.19 drops out. Note, the simplification occurring in passing from equation 5.3.20 to equation 5.3.21 is due to the fact that $h_{ab}n^b = 0$ by definition (as h_{ab} is the intrinsic surface metric), and hence, $h_{ab;c}n^b = -h_{ab}n^b{}_{;c}$. Further, the factor $g_{ab} - n^a n^b$ is just the projection of the two metric onto a spacelike surface (the foliating surfaces) with normal n^a. This projection is then just the intrinsic surface metric, so $h_{ab} = g_{ab} - n^a n^b$. This in turn gives W as

$$W = \frac{1}{2}\int\sqrt{-{}^2g}(1 - 2uu_{;ab}h^{ab} - u_{,a}u_{,b}h^{ab} + 2uu_{,a}n^a{}_{;b}n^b + 2uu_{,a}n^a\mathcal{K}$$
$$+u_{,b}n^b u_{,a}n^a + u^2 n^a{}_{;a}\mathcal{K} + 2u_{,a}n^a u\mathcal{K})\, d^2x \qquad (5.3.22)$$

From equation 5.3.22 it is clear that W contains no second order derivatives that are taken with respect to the normal (given that $\mathcal{K}$ contains none). This indicates that the Gibbons-Hawking surface term has succeeded in removing all the second order "time" derivatives. This also indicates that in the language of the Hamiltonian formalism, there exists a primary constraint that is equivalent to equation 4.4.19.

Yet equation 5.3.22 is not the final form of W as one can simplify the terms involving $\mathcal{K}$. From equation 5.3.16 applied to the one-extrinsic curvature $\mathcal{K}$, one gets that there is no contribution from these terms as the splitting has a G.N.C. form. Hence, the action can be written as

$$W = \frac{1}{2}\int \sqrt{-{}^2g}(1 - u^{,a}u_{,a} + 2u_{,a}u_{,b}n^a n^b - 2uu_{;ab}h^{ab})\, d^2x \qquad (5.3.23)$$

which is a purely gravitational action. This form of W can be written as the two-integral of a Lagrangian density. That is, $W = \int \mathcal{L}\, d^2x$ with

$$\mathcal{L} = \frac{1}{2}\sqrt{-{}^2g}(1 - u_{,a}u_{,b}g^{ab} - 2uu_{;ab}h^{ab} + 2u_{,a}u_{,b}n^a n^b) \qquad (5.3.24)$$

This puts the action in a form that is applicable to a Hamiltonian formalism, and so one can retain some of the guidelines of the method of the previous chapter. However, before proceeding any further, it seems advisable to tighten up the notation a little. To do this, consider the following definitions:

$$\dot{u} \equiv u_{,a}n^a \qquad (5.3.25)$$

$$h_{ab} = e^{\xi}_{a}e_{b\xi} \qquad (5.3.26)$$

Note, ξ is the index for the one-surface intrinsic coordinate, and e^{ξ}_{a} are the basis vectors that form the surface metric. These definitions allow $\mathcal{L}$ to be written in an abbreviated form, namely

$$\mathcal{L} = \frac{1}{2}\sqrt{-{}^2g}(1 - u_{,\xi}u^{,\xi} + \dot{u}^2 - 2uu^{;\xi}{}_{;\xi}) \qquad (5.3.27)$$

Having the Lagrangian density specified, the next step in the procedure is to determine the associated Hamiltonian density. As stated in chapter four, the Hamiltonian density is given by

$$\mathcal{H} = \pi_q \dot{q} - \mathcal{L}(q, \pi_q) \qquad (5.3.28)$$

with q and π_q being the set of phase space coordinates (refer to page 80). So, in order to determine $\mathcal{H}$, one must first identify the generalised coordinates to be used. Just from looking at the form of $\mathcal{L}$ (equation 5.3.27) an obvious choice for a generalised coordinate is $u(x^a)$. As a second generalised coordinate, choose $\sqrt{-^2g}$; the reason for such a choice is that there should be no loss of metric information from the theory. All the information associated with the structure of the two-dimensional spacetime in the form of the components of the metric (which in the ADM formalism forms the set of generalised coordinates) should be contained in the determinant $\sqrt{-^2g}$. Another reason for this choice is that it does not appear to be possible to re-express $\sqrt{-^2g}$ in terms of the one-metric of the family of foliating hypersurfaces unless the $(1+1)$ Lorentzian metric, along with the family of foliating hypersurfaces, is specified. This would go against the motivation of this particular approach, as in specifying the form of the metric, one has return the analysis to a version of the ADM formalism (i.e. the time coordinate would be explicitly identified). Hence $\sqrt{-^2g} \equiv g_0$ is taken as a second generalised coordinate. No further generalised coordinates are apparent or necessary.

The conjugate momenta for these coordinates are then given by

$$\pi_q = \frac{\partial \mathcal{L}}{\partial q_{,a} n^a} = \frac{\partial \mathcal{L}}{\partial \dot{q}} \qquad (5.3.29)$$

and so the two conjugate momenta are

$$\pi_u = \frac{\partial \mathcal{L}}{\partial u_{,a} n^a} = \frac{\partial \mathcal{L}}{\partial \dot{u}} = g_0 \dot{u} \qquad (5.3.30)$$

and

$$\pi_{g_0} = \frac{\partial \mathcal{L}}{\partial g_{0\,,a} n^a} = \frac{\partial \mathcal{L}}{\partial \dot{g}_0} = 0 \tag{5.3.31}$$

Substituting equations 5.3.30 and 5.3.31 into equation 5.3.28, one gets the Hamiltonian density as

$$\begin{aligned} \mathcal{H} &= \pi_u \dot{u} - \frac{1}{2} g_0 (1 - 2u u^{;\xi}{}_{;\xi} - u^{;\xi} u_{,\xi} - \dot{u}^2) \\ &= \frac{\pi_u^2}{2g_0} - \frac{g_0}{2}(1 - 2u u^{;\xi}{}_{;\xi} - u^{;\xi} u_{,\xi}) \end{aligned} \tag{5.3.32}$$

One immediately obvious point is that to write $\mathcal{H}$ in the form given by equation 5.3.32, the phase space coordinates are such that $\dot{q} = \dot{q}(q, \pi_q)$. As mentioned in the previous chapter, this is necessary if a Hamiltonian formalism is to be applied to the problem, but it is one of the difficulties that lies in the Lagrangian approach of chapter three, as well as the work of Farhi et al. [39, §6]. So as $\mathcal{H}$ is given by equation 5.3.32, there is no problem treating the q's and the π_q's as independent variables. Further, from equation 5.3.31 and equation 5.3.32 it is clear that $\mathcal{H}$ is independent of π_{g_0}, and so equation 5.3.31 forms a primary constraint (c.f. primary constraint definition given on page 82)

Now using the Dirac quantisation procedure (as opposed to the ADM quantisation scheme which requires a fixing of the foliating surfaces and therefore is not suitable here) and the primary constraint, one can determine an associated secondary constraint. This is done by considering the Poisson bracket of π_{g_0} and $\mathcal{H}$. In a similar argument to that of section 4.4 of chapter four, one has that

$$\dot{\pi}_{g_0} = [\pi_{g_0}, \mathcal{H}] = 0 \tag{5.3.33}$$

due to the Euler-Lagrange equation (equation 4.3.8). From this result, and the expansion of the Poisson bracket, one obtains the secondary constraint

$$0 = \frac{\partial \mathcal{H}}{\partial g_0} \tag{5.3.34}$$

On substituting in for $\mathcal{H}$ from equation 5.3.32 and rearranging, this constraint takes the form

$$\pi_u^2 = -g_0^2(1 - 2uu^{;\xi}{}_{;\xi} - u^{;\xi}u_{,\xi}) \qquad (5.3.35)$$

As can be seen, this secondary constraint is a momentum constraint that is similar in form to equations 4.5.49 and 4.5.50 of chapter four (which is to be expected, as equation 5.3.35 should contain the same amount of information as the combination of the lapse and shift secondary constraints given by equation 4.5.43). However, it must be checked to see if they are indeed equivalent. If the two momentum constraints are equivalent, then the formalism outlined in this chapter does give a suitable description of the composite spacetime (at least for this test case).

5.4 The Comparison With The Standard ADM Approach

In order to check this constraint with those of the previous chapter, consider the metric function f defined by equation 4.5.44. For f as given, one has

$$\begin{aligned} f &\equiv 1 - \frac{2m(x^a)}{u} \equiv g^{ab}u_{,a}u_{,b} && (5.4.36)\\ &= u^{;\xi}u_{,\xi} - \dot{u}^2 \\ &= u^{;\xi}u_{,\xi} - \frac{\pi_u^2}{g_0} \\ &= u^{;\xi}u_{,\xi} + (1 - u^{;\xi}u_{,\xi} - 2uu^{;\xi}{}_{;\xi})\dot{u}^2 \\ &= 1 - 2uu^{;\xi}{}_{;\xi} && (5.4.37) \end{aligned}$$

Using the definition given in equation 5.4.36, one gets that equation 5.4.37 implies

$$m(x^a) = u^2u^{;\xi}{}_{;\xi} \qquad (5.4.38)$$

which is a measure of the mass parameter of the system. But, the test case under consideration is that of a completely vacuous spacetime, thereby implying that

$m(x^a) = 0$ Thus, the use of the momenta constraint to get equation 5.4.38 has resulted in the condition

$$u^{;\xi}{}_{;\xi} = 0 \tag{5.4.39}$$

for the test case spacetime. Before continuing, one should consider whether this result is sensible. In particular, if $m(x^a)$ is taken to be zero for a vacuous spacetime, does the corresponding geometrical relation (equation 5.4.38) appear correct. Even though the form of the metric is not specified (for the general discussion), this test case is a spherically symmetric spacetime that is completely void of matter and so $u(x^a)$ can be taken as ξ, the intrinsic surface coordinate. It is then that clear that equation 5.4.39 is satisfied. Hence, the analysis seems to give results that are acceptable.

Equation 5.4.39 can also be applied to the momentum constraint (equation 5.3.35), thereby providing a simplified form for π_u^2, namely

$$\pi_u^2 = -g_0^2(1 - u^{;\xi}_{;\xi}) \tag{5.4.40}$$

for this true vacuum test case. In this equation it is clear that the value of the $u^{;\xi}u_{;\xi}$ will determine whether π_u is real or imaginary (i.e. whether π_u is classically allowed or classically forbidden for a given value of u). Obviously the transition between these to states is where $\pi_u = 0$. If equation 5.4.40 admits both real and imaginary values of π_u, then the spacetime geometry is composed of both classically allowed and classically forbidden regions.

In the ADM formalism, one can consider the empty space case, and obtain an expression for the conjugate momenta involved. Such a calculation was done by Fischler et al. [44]. Unfortunately, a direct comparison of the momentum constraints obtained from these two Hamiltonian approaches is not possible as equation 5.4.40 contains the factor $g_0 = \sqrt{-{}^2g}$, which can not be simplified. However, one can compare the positions of the turning points and the relative sign of the various regions as predicted by the two approaches. From the work of Fischler et al. or by taking the result straight from equations 4.5.49 and 4.5.50 (with all the mass contributions removed) one has that the position of the classical turning points are given by

$$\left(\frac{u'}{l}\right)^2 - 1 = 0 \tag{5.4.41}$$

with $'$ denoting differentiation with respect to the spatial coordinate of the two space (i.e. differentiation with respect to r).

Now the two space has, in the ADM formalism, a line element given by

$$ds^2 = -n^2dt^2 + l^2(dr + vdt)^2 + u^2d\Omega_2^2 \tag{5.4.42}$$

and with this general form of the metric, the arbitrariness in the functions n, l, v, and u allows the foliating surfaces to be described by $t = constant$. If this is the case, then $'$ refers to an in-surface derivative, and l^2 is the surface one-metric. This in turn implies that $(u')^2 = u_{,\xi}u_{,\xi}$, and so $u^{,\xi}u_{,\xi} = (u')^2g^{\xi\xi} = (\frac{u'}{l})^2$. Hence, equations 4.5.49 and 4.5.50, and 5.4.40 predict the same turning point positions. Also, as $g_0^2 > 0$ the regions that are classically allowed and classically forbidden are the same for both approaches, which implies that for this test case this second non-explicit metric Hamiltonian approach gives the same physical results as the ADM approach.

Continuing with this test case a little further, on can study the Hamilton-Jacobi equation in an attempt to solve for the principal function. As discussed in chapter four (page 86), the Hamilton-Jacobi equation is generated by a canonical transformation which transforms the Hamiltonian to zero. The usual form of the Hamilton-Jacobi equation is

$$0 = \mathcal{H}(q_i, \frac{\partial \alpha}{\partial q_i}, t) + \frac{\partial \alpha}{\partial t} \tag{5.4.43}$$

with α being the principal function. Rephrasing this in terms of the formalism at hand, one has

$$\begin{aligned} 0 &= \mathcal{H}(u, g_0, \frac{\partial \alpha}{\partial u}) + \alpha_{,a} n^a \\ &= \mathcal{H}(u, g_0, \frac{\partial \alpha}{\partial u}) + \dot{\alpha} \end{aligned} \tag{5.4.44}$$

with the principal function given by the canonical transformation, namely

$$\pi_u = \frac{\partial \alpha}{\partial u} \tag{5.4.45}$$

$$\Rightarrow \quad \alpha = \int dx^{\xi} \int \pi_u \, du \tag{5.4.46}$$

So, for the test case,

$$\alpha = \int dx^{\xi} \int g_0 \sqrt{u^{,\xi} u_{,\xi} - 1} \, du \tag{5.4.47}$$

Note, the integration over the intrinsic metric coordinate is due to the fact that $\mathcal{L}$ is defined as density (i.e. $W = \int \mathcal{L} d^2x$).

Yet equation 5.4.47 is not the complete form of the principal function, as any "time" dependence would only appear as a constant of integration. To get the full form of α, remember that $\mathcal{H}$ is a constant of the motion ($\mathcal{H}_{,a}n^a = \dot{\mathcal{H}} = [\mathcal{H}, \mathcal{H}] = 0$), and so $\mathcal{H} = c_1 =$ constant. Then from equation 5.4.44, $c_1 = -\dot{\alpha}$, which implies that the full form of α is

$$\alpha = c_1 n_a x^a + \int g_0 \sqrt{u^{,\xi} u_{,\xi} - 1} \, du \tag{5.4.48}$$

This is the complete form of the principal function. Unfortunately, the integration in equation 5.4.48 can not be done explicitly, as $u^{,\xi}u_{,\xi}$ needs to be specified as a function of u. Nevertheless one can use equations 2.3.70 and 4.4.31 to write down the form of the wavefunction associated with the system, and the transmission coefficient from one spacetime point to another, as

$$T(2,1) = \exp(\frac{i}{\hbar}(\alpha(2) - \alpha(1)) \tag{5.4.49}$$

However, nothing further is gained by looking at T(2,1) for this empty space case, and so the test case scenario has outlived its usefulness. The next step is to take this altered Hamiltonian formalism and apply it to the problem of the composite spacetime.

5.5 Adding The Matter

With the formalism for the non-explicit time form of the Hamiltonian approach having been successfully applied to the test case of a pure true vacuum space time, the next step is to take this formalism and apply it to the problem at hand. Thus, the composite spacetime (as described on page 47) is to be examined.

In order to apply the formalism to this spacetime, one needs to write down the action associated with it. As usual, the action can be split into two contributions; the gravitational and the matter contribution. That is, if the action for the system is W, then

$$W = W_G + W_m \tag{5.5.50}$$

Fortunately, as a result of the choice of test case, some of the results of the previous section can be used to specify the gravitational contribution to W. In particular, one has $W_G = \int \mathcal{L}_G\, d^2x$ with $\mathcal{L}_G$ given by equation 5.3.27. So, instead of worrying about the geometrical contributions to the action for this spacetime, one only has to determine the matter contributions to W. Once W_m has been specified in terms of the intrinsic metric of the foliating surfaces, and the outward normal, it is just a matter of working through the formalism.

Fortunately, in specifying the matter contribution to W, one can repair to the results of chapter three. There it was shown (equations 3.2.15 and 3.2.21) that the matter contribution for the composite spacetime is of the form

$$W_m = \int \rho\Theta(\hat{u} - u)\sqrt{-^4g}\, d^4x - \int_{\Sigma} \sigma\sqrt{-^3h}\, d^3\xi \qquad (5.5.51)$$

However as can be seen, the last term in W_m is a type of surface term, and its presence in equation 5.5.50 prevents one writing W in the form $W = \int \mathcal{L}\, d^2x$. Thence, this surface term needs to be converted to a volume integral. Normally, one would be tempted to use Gauss's theorem (equation 3.1.8), but this is not appropriate, as σ (the surface energy density of Σ) is only defined on Σ. So, instead, one makes use of the thin wall approximation and the notions of proper volume and proper area, and writes

$$\begin{aligned} \int \sigma\sqrt{-^3h} d^3\xi &= \int \sigma dA \\ &= \int \sigma\delta(\Sigma) dV \\ &= \int \sigma\delta(\Sigma)\sqrt{-^4g} d^2x \end{aligned} \qquad (5.5.52)$$

Here $\delta(\Sigma)$ is a delta function that only has a contribution when the point of evaluation is on the hypersurface Σ. Now this equation gives that

$$\begin{aligned} W_m &= \int (\rho\Theta(\hat{u} - u) - \sigma\delta(\Sigma))\sqrt{-^4g}\, d^4x \\ &= 4\pi \int (\rho\Theta(\hat{u} - u) - \sigma\delta(\Sigma))u^2\sqrt{-^2g}\, d^2x \end{aligned} \qquad (5.5.53)$$

and immediately one has the form of the Lagrangian density for the matter. It is

$$\mathcal{L}_m = 4\pi(\rho\Theta(\hat{u} - u) - \sigma\delta(\Sigma))u^2\sqrt{-^2g} \qquad (5.5.54)$$

With $\mathcal{L}_m$ defined, the Lagrangian density is then just the combination of equation 5.3.27 and equation 5.5.54. Thus, the Lagrangian density for the composite spacetime is

$$\mathcal{L} = \frac{1}{2}\sqrt{-^2g}(1 - u_{,\xi}u^{,\xi} + \hat{u}^2 - 2uu^{;\xi}{}_{;\xi} - 8\pi(\sigma\delta(\Sigma) - \rho\Theta(\hat{u} - u))u^2) \qquad (5.5.55)$$

with the notation of equations 5.3.25 and 5.3.26 being used. Having obtained equation 5.5.55, it is now just a matter of applying the Hamiltonian formalism to $\mathcal{L}$.

From the form of $\mathcal{L}$, one can see that the choice of generalised coordinates is straightforward, and are the same as the generalised coordinates used in the test case example. That is, for the same reasons as given on page 109, the generalised coordinates are $u(x^a)$ and $g_0 = \sqrt{-^2g}$. To extend the generalised coordinate system to the set of phase space coordinates required for the Hamiltonian method, one needs the conjugate momenta. From the definition of conjugate momentum (equation 4.3.9) one has that

$$\pi_u = g_0 \dot{u} \tag{5.5.56}$$

$$\pi_{g_0} = 0 \tag{5.5.57}$$

These conjugate momenta are the same as for the test case, and obviously there is no problem with inverting the $\pi(q,\dot{q})$'s to give $\dot{q} = \dot{q}(q,\pi_q)$. This in turn allows the full set of phase space coordinates (q,π_q) to be treated as independent variables, and so, one can construct the Hamiltonian density. Applying the above results to equation 4.3.10, one has

$$\begin{aligned} \mathcal{H} &= \dot{q}\pi_q - \mathcal{L}(q,\pi_q) \\ &= \frac{\pi_u^2}{2g_0} - \frac{1}{2}\sqrt{-^2g}(1 - u_{,\xi}u^{,\xi} - 2uu^{;\xi}{}_{;\xi} \\ &\quad -8\pi(\sigma\delta(\Sigma) - \rho\Theta(\hat{u} - u))u^2) \end{aligned} \tag{5.5.58}$$

With the Hamiltonian density defined, the next step in the process is to look for constraints. Despite the addition of the matter terms, it is obvious that one has primary and secondary constraints that are similar to those of the test case. Clearly, there is a primary constraint which is given by equation 5.5.57, and by the same reasoning as for the test case, one has a secondary constraint, given by equation 5.3.34 via equation 5.3.33. With some rearrangement, this secondary constraint can be written as

$$\pi_u^2 = -g_0^2(1 - 2uu^{;\xi}{}_{;\xi} - u^{,\xi}u_{,\xi} - 8\pi(\sigma\delta(\Sigma) - \rho\Theta(\hat{u} - u))u^2) \qquad (5.5.59)$$

As with the test case, this constraint retains the features ascribed to the momenta variables of the ADM description; a classically forbidden region, types of solutions that depend on the mass parameter associated with the seed spacetime, and π_u^2 positive for the $\hat{u} \to 0, \infty$ limits.

However, the primary and secondary constraints are not the only constraints on the system. One gets additional constraints which are second class due to the physical state of the composite spacetime. That is, due to the non-uniformity of the matter distribution over the spacetime region for which the action is defined, one has additional restrictions on the phase space coordinates. As the composite is a false vacuum interior and a true vacuum exterior, the extra restrictions will result from the discontinuities that occur at the interface between these two regions. This implies that these second class constraints are a measure of the contribution due to Σ, and they are commonly referred to as junction conditions.

5.6 The Junction Conditions

The wall between the two vacuum regions has been idealised by the use of the thin wall approximation, and is treated as a surface layer. This surface layer has a non-zero surface energy density and a perfect fluid equation of state (refer to equation 2.2.38). In order to phrase these surface layer attributes into a second class constraint, one must focus on the discontinuity of the matter as one crosses Σ. Thus these constraints are truly junction conditions.

So, it is now clear that given that there is a physical surface layer, there exists a junction condition, and hence additional constraints. Yet although these additional constraints may exist, it may not be obvious to the reader as to why they are required. In order to clarify why they are necessary, one need only consider the evaluation of the transmission coefficient. According to equation 4.6.60, T(2,1) depends on the exponential of Hamilton's principal function, which is just the solution to the Hamilton-Jacobi equation. However, it has been shown that in solving the Hamilton-Jacobi equation one uses a canonical transformation of the Hamiltonian, and as a result of this transformation, one obtains a differential expression for the principal function, α. From equation 4.4.30 the general expression for α is

$$\alpha = \int dx^{\xi} \int \pi_{q_i} dq_i \qquad (5.6.60)$$

and it is this expression that displays the need for the junction conditions. The junction conditions are required as this expression for α has an integration over all space of an integrand that is a function of the momenta. Yet from the secondary constraint (or indeed the definition of the conjugate momenta), one has that π_u depends on $u_{;\xi}$. But, it is this $u_{;\xi}$ dependence that is the root of the problem, as it is precisely this quantity that is discontinuous at the surface layer, thus not allowing the integration over all space to be completed until satisfactory junction conditions have been established (compare this with the argument for the junction conditions in the ADM approach, given on page 92). Thus, in order to evaluate T(2,1), one requires Σ junction conditions.

To actually determine the junction conditions, the variation of the secondary constraint across Σ must be examined (as this will give a measure of the discontinuities). From equation 5.5.59 one has that

$$\pi_u^2 = -g_0^2(1 - 2uu^{;\xi}{}_{;\xi} - u^{;\xi}u_{;\xi} - 8\pi(\sigma\delta(\Sigma) - \rho\Theta(\hat{u} - u))u^2) \qquad (5.6.61)$$

However, this can be compared to a second expression of π_u^2 which is obtained from the geometrical mass definition used in the ADM method. That is, an alternative form of equation 5.6.61 can be obtained through the use of equation 5.4.37, with the result being that

$$f = 1 - \frac{2m(x^a)}{u} = u^{;\xi} u_{,\xi} - \frac{\pi_u^2}{g_0} \tag{5.6.62}$$

and

$$\pi_u^2 = -g_0(f - u^{;\xi} u_{,\xi}) \tag{5.6.63}$$

Comparing this result to equation 5.6.61 one gets that the geometrical mass function is

$$m(x^a) = u^2 u^{;\xi}{}_{;\xi} + 4\pi u^3(\sigma\delta(\Sigma) - \rho\Theta(\hat{u} - u)) \tag{5.6.64}$$

Now to measure the discontinuity across Σ, one needs to examine the discontinuity in $m(x^a)$, and from equation 5.6.63 one has

$$\begin{aligned} [\pi_u^2] &= [u^\xi u_\xi] + \frac{2}{u}[m(x^a)] \\ \Rightarrow \qquad [m(x^a)] &= u\pi_u[\pi_u] - u^2 u^{;\xi}[u_{;i}] \end{aligned} \tag{5.6.65}$$

whilst equation 5.6.64 gives

$$[m(x^a)] = u^2[u^{;\xi}{}_{;\xi}] + 4\pi u^3\sigma \tag{5.6.66}$$

So on equating these two expressions for $[m(x^a)]$ one gets that

$$\pi_u[\pi_u] - uu^{;\xi}[u_{;\xi}] = u[u^{;\xi}{}_{;\xi}] + 4\pi u^2\sigma \tag{5.6.67}$$

which gives a relationship between the quantities that are discontinuous at Σ (i.e. π_u and $u_{;\xi}$). This is the junction condition that is required.

5.7 The Problem

The result of determining the junction condition (equation 5.6.67) is that unlike the ADM analysis, one obtains a junction condition that does not separate into a piece that contains just the π_u discontinuity, and a piece with the $u_{;\xi}$ discontinuity. The lack of such a separation means that one can not give an exact description of the π_u as Σ is crossed (i.e. one cannot say in what ratio the right hand side of equation 5.6.67 contributes to $[\pi_u]$ and to $[u_{;\xi}]$). This then implies that the integrand in equation 5.6.60 can not be explicitly specified, and so Hamilton's principal function can not be determined. If α can not be determined, then in this formalism, one does not have a method for evaluating T(2,1).

So, it seems that this attempt to construct a non-explicit time version of the Hamiltonian approach which only requires the spacetime to be foliated cannot provide a satisfactory procedure for the evaluation of T(2,1) for any initial and any final configurations. This inadequacy is due to the fact that the loss of the rigidity of the structure of the spacetime (i.e. just assuming a foliated spacetime, and working with the normal) results in a loss in the number of constraining equations, so that there are insufficient equations for the variables of the problem (i.e. π_u and $u^{;\xi}$ on either side of Σ, and σ.) This then implies that there has been a loss of information in the formulation of this approach (as the ADM method works fine), which leaves one uncertain as to the quantitative analysis of the various terms associated with the junction conditions.

If one could establish from the secondary constraint a relation similar to that of equation 4.5.43, then the explicit evaluation of the junction condition would be possible. However, the price paid for the combining of the primary constraints associated with the lapse and the shift function into one primary constraint (equation

5.5.57) is that one does not have a separation of the quantities that are discontinuous. This then effectively halts the analysis, and prevents one from proceeding to the evaluation of T(2,1). So, in terms of the motivation for this non-explicit time approach, the approach has failed in its effort to produce the result. This in turn suggests that if the quantum mechanical tunneling of a seed spacetime is to be discussed, one has to be careful to explicitly specify the form of the metric for the classically forbidden region (i.e. the way that the spacetime is foliated), even though one is allowing for a deformation of the metric as the seed spacetime tunnels. As such, it appears that the ADM foliation provides a more than adequate specification of the metric form. Further from the above result, it seems that if one is restricted to a Lorentzian form of metric, then one has to explicitly pick out a time coordinate, and follow the standard Hamiltonian procedure. If one tries to veer away from this explicit time, rigid foliation form of the metric, then one has to suffer a description that has a loss of information. Such a description then runs the risk of being incomplete, with the net result that the description is insufficient. This has been shown to be the case with the approach outlined in this chapter.

CHAPTER SIX

CONCLUSION

As each of the avenues outlined in the introduction have been explored, one is now in a position to critically assess the problem of constructing a "new universe" spacetime region. Further, whilst the complete picture has not been given, the work discussed in this thesis is sufficient to allow some modest conclusions. However, the concluding remarks are such that as several different approaches to the problem have been discussed, the essential and the salient features will be in a chapter by chapter review. This will then be followed by a general overview of the problem.

So, in order to begin , consider the results of chapter two, which provides the classical description of the problem. The main result of this chapter is that classically, one cannot construct a universe in the laboratory. To see this one need only look at the various possible trajectories for the boundary between the seed spacetime and the laboratory. Generally one gets three classes of trajectory (the S1, S2, and S3 classes, page 22), but only one of them starts off with a small radial size and then proceeds to evolve into an inflationary era. This is the S1 class. Unfortunately, any trajectory belonging to this class is required by a singularity theorem due to Penrose, to evolve from an initial singularity. Thus, the S1 class of trajectories is unacceptable for the discussion at hand, as the initial singularity prerequisite makes it impossible to initiate this expansion in a laboratory.

In fact the only trajectory class that avoids the implications of Penrose's initial singularity theorem is a set of trajectories in the low mass parameter range (i.e. small m) of the S2 class. This sub-class has been given the classification T1.

Therefore, it is conceivable that a trajectory of classification T1 could be constructed in the laboratory (i.e. region I of the Kruskal-Szekeres diagram), yet if allowed to evolve in a purely classical manner, it would just expand to a maximum size (which would not be very large due to the small value of m) and then collapse. This also is unacceptable as a solution to the problem of constructing a "new universe" in the laboratory.

One other salient feature can be ascertained from the results of the classical trajectory analysis, and that is that for this particular composite spacetime, Σ's equation of motion is analogous to the motion of a particle in a one dimensional potential. The reason why this analogy is important is that it provides an escape to the no-go result of classical physics. This so-called escape route is one in which the problem is re-formulated in terms of a quantum mechanical tunnelling from one classical solution to another (i.e. from a T1 to either a T4 or T6 trajectory class). Such a tunnelling process would then produce a trajectory that has the desired constructibilty and inflationary features of a constructible "new universe". Of course the obvious question was whether such a trajectory is possible. To answer this question one needed to determine the transmission coefficient associated with the tunnelling process, which neatly led one into the work of chapter three.

Thus, the main purpose of chapter three was not to elucidate any elaborate scheme, but rather, to provide a straightforward evaluation of the transmission coefficient for the tunnelling of the trajectory wall from one classical turning point to another. As shown by equation 2.3.74, this involved the evaluation of the action over this interval. By taking the action to be the standard Einstein-Hilbert action plus the Gibbons-Hawking boundary surface term, one obtained a Lagrangian density given by equation 3.2.43.

However, this expression was not quite appropriate for the evaluation of the transmission coefficient, as the Lagrangian density has to be integrated over the classically forbidden range of the spacetime. Further, this classically forbidden region has no real time solutions to the equation of motion, so one was forced to euclidise this spacetime region in order to proceed. In doing this, one obtained euclidean field equations with euclidean solutions, which in turn, permitted one to write done a euclidean action for the spacetime region. Whether this euclidean geometry is an apt description of the interpolating geometry of the classically forbidden region is not clear. Yet in this semi-classical limit, such an approximation appears sufficient.

Despite having a form for the euclidean action, which meant that the transmission coefficient for Σ to tunnel through the classically forbidden region could be evaluated, there was still one computational difficulty. The euclidean action depended on one knowing the velocity of Σ as a function of $\mathcal{R}$ during its trek across the forbidden region. From the equation of motion, one can write down such a relation for $\dot{\mathcal{R}}$, but the substitution of this expression into the action integral only served to complicate the integral to such a degree that it was not readily integrable. Under a turning point velocity approximation which dropped the troublesome terms in W, which were essentially corrections resulting from the boundary terms, a simplified form of the action was obtained. Using this approximate form, and substituting in for the parameters associated with the T1 trajectory classification, an estimate for the transmission coefficient was obtained. This estimate is given by equation 3.3.55.

Although only an approximation to the transmission coefficient, this estimate shows that the probability of a trajectory tunnelling through the forbidden region is particularly small. This seems to imply that the possibility of constructing a spacetime region that would evolve into a "new universe" is virtually zero. That is, such a quantum mechanical tunnelling between two classical trajectory solutions to the classical equation of motion is very very unlikely.

In order to bypass some of the approximations used in the brute force evaluation of the transmission coefficient, and thereby obtain a more precise evaluation, the latter part of chapter three was devoted to developing a covariant form of the action for this composite spacetime. Such a derivation was most easily obtained when one considered the four dimensional Lorentzian spacetime to be a two dimensional Lorentzian spacetime with additional source terms. Further, as was shown in chapter three, this covariant form of the action is equivalent to the Einstein-Hilbert action with Gibbons-Hawking surface term. Unfortunately, due to the form of the spacetime region over which the action is defined, this covariant form of the action resulted in an expression for the transmission coefficient that was rather more difficult to evaluate. That is, the boundary construction caused, in this covariant formulation, the action to be composed of a combination of several integrals that had elliptic integrands. This only served to increase the complexity of the expression of T(2,1). So, whilst the covariant action is more general and in some sense simpler, it did not appear to aid in the evaluation of the transmission coefficient, and so was not really useful.

As the transmission coefficient was determined by the brute force Lagrangian approach, (albeit under several simplifying assumptions) and was found to be very close to zero, implying an almost zero probability of any tunnelling, one would be tempted to take this as the final conclusion. However, such a stance was not taken. The reason for this was that whilst the result of the Lagrangian approach is believable, it does not lend itself to a clear quantisation procedure (i.e. the invertibility of $\dot{r}$ in terms of r and p). Yet it is the quantum mechanical description of the system that provides the basis for the investigation into the viability of constructing a new universe in the laboratory (as classical physics does not allow such a process). Therefore, in order to avoid this problem of quantisability, one was lead to the Hamiltonian approach, which constituted the work in chapter four.

To move from the Lagrangian approach to one that is more receptive to quantisation, the action was re-expressed using equations 4.3.7 and 4.3.10. This has the effect of rephrasing the problem in terms of the Hamiltonian density of the system, and if quantisation is to be permitted then this density should be expressible in terms of the phase space coordinates. It was found that the Hamiltonian density for the composite spacetime was indeed expressible in these coordinates, and thereby suitable for quantisation. That is, there was no ambiguity due to factors such as $\ddot{r}$ which plagued the lagrangian approach. Hence, by switching to the Hamiltonian formalism, a quantisable action was obtained.

Applying the Hamiltonian formalism generally requires that one specify the spacetime structure, as the Hamiltonian approach attaches special significance to the time coordinate. For this chapter, the standard form of describing an arbitrary spacetime was use, namely the ADM description, which treats the spacetime as a foliated structure. Slicing up the spacetime is done by prescribing to it the ADM line element. This line element introduced extra arbitrariness, as the lapse, shift and intrinsic metric were not specified, but this was what was required in order to discuss tunnelling through the classically forbidden region. The reason for this being that it is not clear what form the geometry assumes in this region. The benefit in using this description is that one retains extra degrees of freedom (in this case four), and they are treated as the generalised coordinates for the Dirac quantisation.

With the generalised coordinates specified, the conjugate momenta were obtained, the quantisability verified, and the Hamiltonian density determined. Unlike the Lagrangian approach, only the Einstein-Hilbert action was required in order to obtain an acceptable action - inclusion of the Gibbons-Hawking boundary term was not necessary. Further, it was found that for this particular spacetime there were primary constraints associated with the lapse and the shift. These in turn gave secondary constraints, which when treated as operator equations acting on the wavefunction of the system yielded constraining equations on the phase space coordinates.

Rearrangement of these constraint equations permitted one to show that they implied the five different trajectory classes of the Lagrangian method. Also, an expression for the radial derivative of the mass of the form $M'(x^a) = 0$ (see page 90) was obtained, which is compatible with the physical situation (i.e. the true and false vacuum regions, and Σ, which under the thin wall approximation, had a constant surface energy density.)

The usefulness of the secondary constraint equations was not necessarily obvious, but on examining the Hamilton-Jacobi equation of the system, their usefulness was clear. These constraints served to constrain Hamilton's principal function, which is the solution to the Hamilton-Jacobi equation. It was also shown that under the WKB approximation, the principal function was related to the transmission coefficient (equation 4.4.32). Further, the principal function associated with the tunnelling across the classically forbidden region was found to contain contributions from both the regions of the forbidden spacetime geometry that were interior and exterior to Σ as well as a contribution from Σ. Thus a second form for the transmission coefficient was obtained (equation 4.6.78).

This second form differs from the first in that it depends not on the (euclidean) time that Σ takes to traverse the forbidden region, but rather on the width of the forbidden region that has to be tunnelled through. Also, it contains terms that involve some of the generalised coordinates. This implies that the transmission coefficient cannot be explicitly evaluated until the arbitrariness of the ADM metric has been removed. To do this one would need to specify explicitly the nature of the foliation of spacetime, yet the Hamiltonian approach gives no indication of the form of the ADM metric. One could make educated guesses [44], but then at this stage that is all they would be. The formulation of this quantum tunnelling is, at this stage, not quite complete or sufficient, as the geometry that interpolates between the two classical regions of spacetime is still a mystery.

Given the partial success of both the Lagrangian and Hamiltonian methods, chapter five was devoted to an attempt to combine the beneficial features of each. Thus, what was attempted was an approach that gave a readily quantisable action as well as some suggestion of the geometry of the forbidden region. In order to do this a Hamiltonian formalism was invoked (as this gives the quantisation), but it was formed not from the purely gravitational Einstein-Hilbert action. Instead, the Gibbons-Hawking boundary term was added. Addition of this term not only aided the quantisation,but it also helped to constrain the form of the metric description of spacetime (i.e. compare with the covariant approach to the action discussed in chapter three).

Further, in an attempt to simplify the formalism and allow for a general geometry the forbidden region, the spacetime foliation was not specified by the ADM prescription. The reason for this is that whilst it utilises an arbitrary metric, it imposes a degree of rigidity on the spacetime structure. Instead, the foliating surfaces were taken to be described by the normal and the intrinsic surface metric. Using the normal as a descriptive tool was equivalent to using the lapse and the shift functions in the ADM method. However, such a step resulted in a reduction in the number of degrees of freedom.

In order to check that the method being attempted gave something sensible,the action was determined for the test case of an empty spacetime. The main point that resulted from this test case evaluation was that as one no longer had the four degrees of freedom (as with ADM), the number of generalised coordinates was also reduced. Further, due to the presence of the $\sqrt{-^2g}$ factor in the Lagrangian density, and its irreducibility in terms of the normal and the intrinsic surface metric of the foliation, one was forced to take $\sqrt{-^2g}$ as a generalised coordinate. The advantage in doing this was that one recovered the same type of primary constraint as those of the ADM method. Yet, on the negative side, the choice of $\sqrt{-^2g}$ as a generalised coordinate meant a concealing of information concerning the metric (this is because the determinant ^2g can not be written as a function of the metric components, as the metric components are not explicitly given).

Fortunately, for the test case, the reduced set of generalised coordinates caused no apparent problems, as secondary constraints were obtained that were equivalent to those of an empty space ADM analysis. That is, a secondary constraint was obtained that predicted the same sort of partitioning into classically allowed and classically forbidden regions. However, when matter was added (i.e. the composite spacetime) the loss of metric information became apparent. In particular, one was prohibited from obtaining explicit junction conditions for the metric quantities that varied discontinuously at Σ (equation 5.6.67). Having an implicit junction condition was not itself a problem, but it caused the evaluation of the transmission coefficient to be halted. This was because one could not explicitly evaluate the principal function (as its contribution from Σ was not clear).

Such a result implies that the use of $\sqrt{-^2g}$ as a generalised coordinate (or even ^2g) is not suitable, and so it suggests that the normal-intrinsic metric description of the foliation is not sufficient. Thus one is forced back to the ADM description of the foliation, which leaves no hint as to the nature of the geometry during the course of the tunnelling (i.e. during the traversing of classically forbidden region).

So, if one takes a step back and looks not at each approach individually, but rather, considers them collectively, then the overall picture is one that is not quite complete. Certainly, it is clear that classically, one can not construct a "new universe" in the laboratory. Also, it appears that whilst such a construction may be possible quantum mechanically, its occurrence is highly unlikely.

Aside from the actual technological details associated with producing a false vacuum region, the main concern is the interpretation of the classically forbidden region during the course of the tunnelling. Understanding this geometrical problem is where future research efforts should be directed. In particular, it would be hoped that one could move away from the simple and somewhat ad hoc notion of euclidising the time coordinate, as this step appears to be more a matter of convenience. Instead, the idea of treating the forbidden region as one with a varying geometry should be pursued. For this, an ADM styled foliation could be retained, as it provides the means to quantisation. Nevertheless, one would still be presented with several options.

The first option is that one could attempt to move away from the explicit Lorentzian foliation which singles out the time coordinate and attaches added importance to it. The other option is to look for additional constraints that are neither primary nor secondary, but second class. Such constraints could then serve to further constrain the junction conditions and the principal function, so that the expression for T(2,1) becomes tractable. Note, an example of such a second class constraint was used in the latter part of chapter five.

Of course, one may be tempted to say that the quantum tunnelling argument is either too simple or that it is a result of an over-extended analogy to a particle moving in a potential. Yet if this is so, then one is left with no real path towards a semi-classical treatment of the problem. This would then imply that one has to turn either to the conclusions of the classical physics, or else to the search for a quantum theory of gravity. Both of these options are not overly palatable, and so it is hoped that the investigation into some sort of middle ground would continue.

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APPENDIX

^4R FOR THE ADM METRIC

Using the algebraic computer package MACSYMA ©Symbolics Inc., one can produce the general form for the Ricci Scalar associated with the ADM line element (equation 4.2.3). This expanded form of ^4R is as follows;

$$
\begin{aligned}
{}^4R = \; & \tfrac{2}{l^3n^3u^3}\{2l^3nu\ddot{u} + l^3n\dot{u}^2 - 2l^3nvu'\dot{u} - 2l^3nv'u\dot{u} + 2l^3n'vu\dot{u} - 2l^2l'nvu\dot{u} \\
& -2l^3\dot{n}u\dot{u} + 2l^2\dot{l}nu\dot{u} + 2l^3nv^2u'' - 2ln^3uu'' - 4l^3nvu(\dot{u})' + l^3nv^2(u')^2 \\
& -ln^3(u')^2 - 2l^3n\dot{v}uu' + 4l^3nvv'uu' - 2l^3n'v^2uu' + 2l^2l'nv^2uu' + 2l^3\dot{n}vuu' \\
& -2l^2\dot{l}nvuu' - 2ln^2n'uu' + 2l'n^3uu' - l^2l'n\dot{v}u^2 + l^3nvv''u^2 - l^3n(\dot{v})'u^2 \\
& +l^3n(v')^2u^2 - l^3n'vv'u^2 + 3l^2l'nvv'u^2 + l^3\dot{n}v'u^2 - 2l^2\dot{l}nv'u^2 - l^2l'n'v^2w^2 \\
& +l^2l''nv^2u^2 + l^2l'\dot{n}vu^2 + l^2\dot{l}n'vu^2 - 2l^2(\dot{l})'nvu^2 - l^2\dot{l}\dot{n}u^2 - ln^2n''u^2 \\
& +l'n^2n'u^2 + l^2\ddot{l}nu^2 + l^3n^3\}
\end{aligned}
\tag{A.1}
$$

Note, $\dot{}$ and $'$ respectively represent differentiation with respect to the time coordinate and the radial coordinate.