The affine vertex superalgebra of $D(2,1;-\frac{v}{w})$ at level 1

by

Devon Stockall

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Department of Mathematical and Statistical Sciences
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Abstract

The affine vertex superalgebra $A = \mathbb{L}^1(D(2,1;-\frac{v}{w}))$ plays a key role as the geometric Langlands kernel VOA for SVOAs associated to $\mathfrak{so}(3)$, $\mathfrak{osp}(1|2)$ and other rank one Lie superalgbras. Since $D(2,1;\alpha)$ is an extension of the direct sum of 3 copies of $\mathfrak{sl}(2)$, A can be naturally realized as an extension of $L_1 = \mathbb{L}^k(\mathfrak{sl}(2)) \otimes \mathbb{L}^l(\mathfrak{sl}(2)) \otimes \mathbb{L}^1(\mathfrak{sl}(2))$ for admissible levels $k = \frac{u}{v} - 2$ and $l = \frac{u}{w} - 2$. Here, I use constructions of gluing VOAs to realize A as an L_1 extension, and the theory of VOA extensions to classify irreducible modules in A-$wtmod_{\geq 0}$. Using the 'Adamovic procedure', an alternate realization of A is given as a subalgebra of $\mathbb{L}^{k-1}(\mathfrak{sl}(2)) \otimes B^l$, where the SVOA B^l is constructed from a 'half-lattice' and $\mathbb{L}^1(\mathfrak{sl}(2))$. This allows calculation of modular S-matrices for A modules induced from relaxed highest weight L_1 modules.

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Chapter 1

Introduction

Vertex operator algebras or VOAs are a structure of great interest, both for physicists and mathematicians. Originally motivated as a formalization of symmetry algebras in two-dimensional conformal field theories, VOA representations provide a tractable way to study certain infinite dimensional Lie algebras and their representation theory, while the VOAs themselves also serve as a sort of 'almost commutative' algebra. The associated categorical behaviour is exactly that relevant to topological invariants of 3 manifolds, realizing the connection between three-dimensional topological quantum field theories and the conformal field theories that appear on their boundaries. As well as their topological appeal, VOAs have geometric relevance, as they appear naturally within the Langlands program for loop groups [32].

VOA representation theory serves as a common point for exciting mathematics, with many results from each of these analogous areas, both in physics and mathematics, translating to the setting of VOAs, perhaps with additional interesting subtleties.

The results of gauge theory extend to relevant constructions for VOAs, providing an example of such a translation. For an affine vertex operator superalgebra $\mathbb{L}^k(\mathfrak{g})$, the 'Hamiltonian reduction' [57] process of gauging unphysical symmetries allows for the construction of certain new SVOAs, called $\mathcal{W}$-algebras [52] and denoted by

$$\mathcal{W}^k(\mathfrak{g}, f)$$

where f is a nilpotent element in $\mathfrak{g}$. When f is the principal nilpotent, this is called the principal $\mathcal{W}$-algebra associated to $\mathfrak{g}$ at level k, and denoted by $\mathcal{W}^k(\mathfrak{g})$. The principal $\mathcal{W}$-algebras exhibit isomorphisms

$$\mathcal{W}^k(\mathfrak{g}) \simeq \mathcal{W}^l({}^L\mathfrak{g})$$

called Feigin-Frenkel duality [30], where ${}^L\mathfrak{g}$ is the Langlands dual to $\mathfrak{g}$. For simply laced $\mathfrak{g}$, the associated $\mathcal{W}$ algebras may alternatively be realized through a coset construction [4].

Further motivated by the study of gauge theory, Gaiotto and Rapcak constructed families of vertex algebras at a 2-dimensional 'corner' intersection of three 4-dimensional gauge theories [37]. Due to symmetries of these gauge theories, these VOAs were conjectured to exhibit a triality, simultaneously generalizing both the Feigin-Frenkel duality and the coset realization of certain $\mathcal{W}$ algebras, called 'hook type', later proven in [21][22].

These 4-dimensional gauge theories also enjoy an 'S-duality', giving a physical analog of the geometric Langlands program [54]. VOAs appear again in this context [8][34]. The effect of these S duality transformations on the level of VOAs can be realized as a convolution, given by a cohomology of the relevant $\mathcal{W}$-algebra with another 'geometric Langlands kernel' VOA [23].

These convolutions act not only to give the desired isomorphisms, but also on the corresponding modules. Then modules for the kernel VOA give functors between categories of modules for related vertex algebras (in both directions), hopefully allowing certain equivalences of blocks of modules and intertwining operators to be proven.

One would hope that these correspondences also hold on the level of conformal blocks. This has already been demonstrated in many cases from the physics perspective by using the path integral formalism to justify certain equivalences of correlators in the relevant conformal field theories [10] [12] [11]. This includes my own work with collaborators [13], demonstrating an equivalence between a coset of a $\mathfrak{sl}(n)$ subregular $\mathcal{W}$ algebra theory and an $\mathfrak{sl}(n|1)$ theory.

The affine vertex superalgebra $\mathbb{L}^1(D(2,1;-\frac{v}{w}))$ is interesting not only in its own right, but especially for its relevance in this context. The role of geometric Langlands kernel VOAs for $\mathfrak{g} = \mathfrak{gl}(2)$, $\mathfrak{g} = \mathfrak{so}(3)$ and $\mathfrak{g} = \mathfrak{osp}(1|2)$ are played by $\mathbb{L}^1(D(2,1;-\frac{1}{k+1}))\otimes\pi$, $\mathbb{L}^1(D(2,1;-2k))$ and $\mathbb{L}^1(D(2,1;-\frac{1}{k+1}))$, respectively, and the cases of other rank 1 $\mathfrak{g}$ are similarly realized using $\mathbb{L}^1(D(2,1;\alpha))$ or some mild extension of it. Then the representation theory of $\mathbb{L}^1(D(2,1;\alpha))$ is exactly the relevant information that would allow us to explore these convolution equivalences of categories of modules, intertwining operators, and conformal blocks.

In this work, I provide a classification for irreducible modules in $\mathbb{L}^k(D(2,1;-\frac{v}{w}))$-*wtmod*$_{\geq 0}$, as well as lower bounded modules in the Ramond sector.

Noting that the Lie superalgebra $D(2,1;\alpha)$ contains $\mathfrak{sl}(2)\oplus\mathfrak{sl}(2)\oplus\mathfrak{sl}(2)$ as a subalgebra, we suspect that $\mathbb{L}^1(D(2,1;-\frac{v}{w}))$ should be realized as an extension for the tensor of three copies of $\mathbb{L}^k(\mathfrak{sl}(2))$ at appropriate levels, obtained using gluing results for vertex algebras. This allows for completion of the classification as modules induced from $\mathbb{L}^k(\mathfrak{sl}(2))\otimes\mathbb{L}^l(\mathfrak{sl}(2))\otimes\mathbb{L}^1(\mathfrak{sl}(2))$ using the theory of VOA extensions. This realization of $\mathbb{L}^1(D(2,1;-\frac{v}{w}))$ is the first major result of this thesis, and the classification is the second.

Theorem 5.5.12. *Suppose that k, l are non-integral admissible levels for $\mathfrak{sl}(2)$*

$$k+2=\frac{u}{v} \qquad\qquad l+2=\frac{u}{w} \tag{1.1}$$

with $u = v + w$. Then we have the following isomorphism as $\mathbb{L}^k(\mathfrak{sl}(2))\otimes\mathbb{L}^l(\mathfrak{sl}(2))\otimes\mathbb{L}^1(\mathfrak{sl}(2))$ modules:

$$\mathbb{L}^1(D(2,1;-\frac{v}{w}))\simeq\bigoplus_{r=1}^{u-1}\mathcal{L}^k_{r,0}\otimes\mathcal{L}^l_{r,0}\otimes\mathcal{L}^1_{\bar{r},0} \tag{1.2}$$

Theorem 5.6.6. *For $v, w\in\mathbb{Z}_{\geq 2}$, a complete list of representatives for isomorphism classes of irreducible modules in $\mathbb{L}^1(D(2,1;-\frac{v}{w})$-wtmod$_{\geq 0}$ (resp. Ramond twisted lower bounded modules for $\mathbb{L}^1(D(2,1;-\frac{v}{w}))$) is given by*

$$(L,N)^{0,0}_{s_1,\lambda_1,s_2,\lambda_2,b}$$

for labels $L, N, s_1, \lambda_1, s_2, \lambda_2$ as in the table 4.2.15 and $b = 0$ (resp. $b = 1$ for

Ramond twisted). These are non-isomorphic for distinct labels.

The 'Adamovic procedure' gives a realization of $\mathbb{L}^k(\mathfrak{sl}(2))$ as a subalgebra in the product of a Virasoro VOA $\mathbb{L}^k(Vir)$ with a 'half-lattice' VOA $\Pi^k(0)$. From this perspective, the relaxed highest weight modules for $\mathbb{L}^k(\mathfrak{sl}(2))$ take a particularly simple form. Along with our previous realization of $\mathbb{L}^1(D(2,1;-\frac{v}{w}))$ modules as induced modules, this gives means to calculate modular data for characters and super characters of $\mathbb{L}^1(D(2,1;-\frac{v}{w}))$ modules using the relevant transformations for $\mathbb{L}^k(\mathfrak{sl}(2))$ and the half-lattice $\Pi^k(0)$. We notice that these modular S-matrices for induced $\mathbb{L}^1(D(2,1;-\frac{v}{w}))$ modules take a particularly nice form: a constant multiple of the S-matrices of the L_1 modules they were induced from, providing additional support for a Verlinde formula for $\mathbb{L}^1(D(2,1;-\frac{v}{w}))$.

Theorem 6.2.6. *Modular transformations of $\mathbb{L}^1(D(2,1;-\frac{v}{w}))$-module characters and supercharacters induced from relaxed highest weight modules are given by*

(supercharacter, local)

$$S\{ch^-[(\mathcal{E},\mathcal{E})^{t_1,t_2}_{s_1,\lambda_1,s_2,\lambda_2,0}]\} = \sum_{t_1',t_2'\in\mathbb{Z}} \int_{-1}^{1}\int_{-1}^{1} \sum_{s_1',s_2'} S^{t_1,t_2}_{s_1,\lambda_1,s_2,\lambda_2,0}{}^{t_1',t_2'}_{s_1',\lambda_1',s_2',\lambda_2',0}\, ch^-[(\mathcal{E},\mathcal{E})^{t_1',t_2'}_{s_1',\lambda_1',s_2',\lambda_2',0}]d\lambda_1'd\lambda_2' \quad (1.3)$$

(supercharacter, Ramond)

$$S\{ch^-[(\mathcal{E},\mathcal{E})^{t_1,t_2}_{s_1,\lambda_1,s_2,\lambda_2,1}]\} = \sum_{t_1',t_2'\in\mathbb{Z}} \int_{-1}^{1}\int_{-1}^{1} \sum_{s_1',s_2'} S^{t_1,t_2}_{s_1,\lambda_1,s_2,\lambda_2,1}{}^{t_1',t_2'}_{s_1',\lambda_1',s_2',\lambda_2',0}\, ch^+[(\mathcal{E},\mathcal{E})^{t_1',t_2'}_{s_1',\lambda_1',s_2',\lambda_2',0}]d\lambda_1'd\lambda_2' \quad (1.4)$$

(character, local)

$$S\{ch^+[(\mathcal{E},\mathcal{E})^{t_1,t_2}_{s_1,\lambda_1,s_2,\lambda_2,0}]\} = \sum_{t_1',t_2'\in\mathbb{Z}} \int_{-1}^{1}\int_{-1}^{1} \sum_{s_1',s_2'} S^{t_1,t_2}_{s_1,\lambda_1,s_2,\lambda_2,0}{}^{t_1',t_2'}_{s_1',\lambda_1',s_2',\lambda_2',1}\, ch^-[(\mathcal{E},\mathcal{E})^{t_1',t_2'}_{s_1',\lambda_1',s_2',\lambda_2',1}]d\lambda_1'd\lambda_2' \quad (1.5)$$

(character, Ramond)

$$S\{ch^+[(\mathcal{E},\mathcal{E})^{t_1,t_2}_{s_1,\lambda_1,s_2,\lambda_2,1}]\}$$
$$= \sum_{t_1',t_2'\in\mathbb{Z}} \int_{-1}^{1}\int_{-1}^{1}\sum_{s_1',s_2'} S^{t_1,t_2}_{s_1,\lambda_1,s_2,\lambda_2,1}{}^{t_1',t_2'}_{s_1',\lambda_1',s_2',\lambda_2',1}\, ch^+[(\mathcal{E},\mathcal{E})^{t_1',t_2'}_{s_1',\lambda_1',s_2',\lambda_2',1}]d\lambda_1' d\lambda_2' \quad (1.6)$$

Where the S matrices are

$$S^{t_1,t_2}_{s_1,\lambda_1,s_2,\lambda_2,b}{}^{t_1',t_2'}_{s_1',\lambda_1',s_2',\lambda_2',b'}$$
$$= \frac{ue^{\frac{2\pi i}{\tau}}}{2\sqrt{2}\sin\left(\frac{w\pi}{u}\right)\sin\left(\frac{v\pi}{u}\right)} S^k_{(t_1,\lambda_1;\Delta_{1,s_1})(t_1',\lambda_1';\Delta_{1,s_1'})} S^l_{(t_2,\lambda_2;\Delta_{1,s_2})(t_2',\lambda_2';\Delta_{1,s_2'})} S_{(p+b)(p'+b')} \quad (1.7)$$

Chapter 2

Categories

Categories of representations for Lie algebras have linear and monoidal structures, which are already quite interesting. In many ways, this setting is too restrictive for interesting categories to appear.

Categories of modules for vertex algebras greatly resemble those for Lie algebras, from which they are often built, but allow for much richer structure, since the associativity and commutativity restrictions on the tensor product of vertex algebra modules is weaker than those on vector spaces, and the tensor product for VOA modules allows more interesting objects to be contained in smaller (and hence more easily studied) monoidal categories.

We begin by introducing the monoidal and enriched category structure that appears in both settings, before proceeding to the richer braided structures that appear in the study of vertex algebra representation theory.

Note that for a category $\mathcal{C}$, I will use the notation $A \in \mathcal{C}$ to indicate that A is an object in the category $\mathcal{C}$.

2.1 Monoidal and enriched categories

Definition 2.1.1. *A monoidal category $\mathcal{V}$ consists of*

1. *A category $\mathcal{V}$*

2. *A functor $\otimes : \mathcal{V} \times \mathcal{V} \to \mathcal{V}$ (product)*

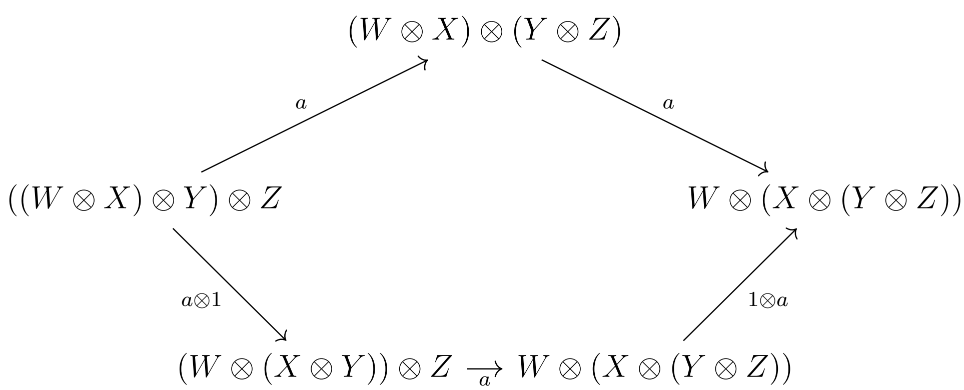
3. An object $\mathbb{1} \in \mathcal{V}$ (unit)

4. Natural isomorphisms

- $a_{XYZ} : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$ *(associators)*
- $l_X : \mathbb{1} \otimes X \to X$ *(left unitor)*
- $r_X : X \otimes \mathbb{1} \to X$ *(right unitor)*

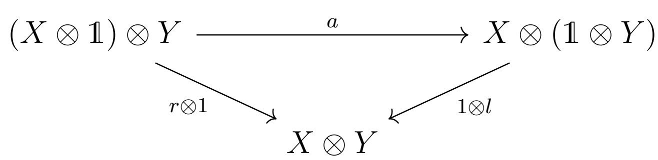
Subject to coherence axioms, expressed by commutativity of the following diagrams

- *(pentagon axiom)*



(2.1)

- *(triangle axiom)*



(2.2)

Definition 2.1.2. *A vector superspace is a $\mathbb{Z}_2$ graded vector space $V = V_{\bar{0}} \oplus V_{\bar{1}}$. Elements $v \in V_{\bar{i}}$ are called homogenous, and we denote their parity by $|v| = i$. Elements in $V_{\bar{0}}$ and $V_{\bar{1}}$ are called even and odd, respectively.*

Example 2.1.1. *Let $\mathbb{F}$ be a field with char$(\mathbb{F}) \neq 2$. Define the category $\underline{\mathcal{SV}ec_{\mathbb{F}}}$ whose objects are vector superspaces $V = V_{\bar{0}} \oplus V_{\bar{1}}$, and morphisms are*

$$Hom_{\underline{\mathcal{SV}ec_{\mathbb{F}}}}(V, W) = Hom_{\mathbb{F}}(V_{\bar{0}}, W_{\bar{0}}) \oplus Hom_{\mathbb{F}}(V_{\bar{1}}, W_{\bar{1}}) \tag{2.3}$$

which we call the <u>*even morphisms*</u>*. We suppress the label* $\mathbb{F}$ *when context allows. Define functor* $\otimes : \underline{\mathcal{SV}ec} \times \underline{\mathcal{SV}ec} \to \underline{\mathcal{SV}ec}$ *by*

$$(V \otimes W)_{\bar{0}} = V_{\bar{0}} \otimes W_{\bar{0}} \oplus V_{\bar{1}} \otimes W_{\bar{1}} \tag{2.4}$$

$$(V \otimes W)_{\bar{1}} = V_{\bar{0}} \otimes W_{\bar{1}} \oplus V_{\bar{1}} \otimes W_{\bar{0}} \tag{2.5}$$

$$(f \otimes g)(v \otimes w) = f(v) \otimes g(w) \tag{2.6}$$

Then $\underline{\mathcal{SV}ec}$ *is a monoidal category.*

Definition 2.1.3. *For monoidal category* $\mathcal{V}$*, a category* $\mathcal{C}$ <u>*enriched over*</u> $\mathcal{V}$ *or* $\mathcal{V}$-<u>*category*</u> *consists of*

1. *a set* $ob(\mathcal{C})$ *of* <u>*objects*</u>
2. <u>*hom-objects*</u> $Hom_{\mathcal{C}}(A, B) \in \mathcal{V}$ *for every* $A, B \in \mathcal{C}$
3. *a* <u>*composition law*</u>
   $\circ : Hom_{\mathcal{C}}(B, C) \otimes Hom_{\mathcal{C}}(A, B) \to Hom_{\mathcal{C}}(A, C)$ *for every* $A, B, C \in \mathcal{C}$
4. *an* <u>*identity element*</u> $j_A : \mathbb{1} \to Hom(A, A)$

These data are subject to associativity and unit consistency axioms. See [55].

Definition 2.1.4. *For* $\mathcal{V}$*-categories* $\mathcal{C}, \mathcal{C}'$*,* $\mathcal{V}$-<u>*functor*</u> $\mathcal{F} : \mathcal{C} \to \mathcal{C}'$ *is a map*

$$\mathcal{F} : ob(\mathcal{C}) \to ob(\mathcal{C}') \tag{2.7}$$

and maps

$$\mathcal{F} : Hom_{\mathcal{C}}(A, B) \to Hom_{\mathcal{C}}(\mathcal{F}(A), \mathcal{F}(B)) \tag{2.8}$$

subject to composition and unit consistency conditions. See [55].

Definition 2.1.5. *For* $\mathcal{V}$*-functors* $\mathcal{F}, \mathcal{F}' : \mathcal{C} \to \mathcal{D}$*, a* $\mathcal{V}$-<u>*natural transformation*</u> $\eta : \mathcal{F} \to \mathcal{F}'$ *is a family of* <u>*components*</u>

$$\eta_A : \mathbb{1} \to Hom_{\mathcal{C}'}(\mathcal{F}(A), \mathcal{F}'(A)) \tag{2.9}$$

subject to unit consistency conditions. See [55].

2.2 Braided tensor categories

Definition 2.2.1. *A tensor category is a monoidal category $\mathcal{C}$ enriched over the category of vector spaces, satisfying the following conditions*

1. *Finite biproducts exist (including the empty biproduct $0 \in ob(\mathcal{C})$)*
2. *Every morphism has a kernel and cokernel*
3. *Every monomorphism is a kernel and every epimorphism is a cokernel*

Definition 2.2.2. *Let $\mathcal{C}$ be a monoidal category with tensor product $\otimes$, unit $\mathbb{1}$ and associators a.*

- *We say that X^* is a left dual of X if there exists morphisms*

$$ev_X : X^* \otimes X \to \mathbb{1} \qquad coev_X : \mathbb{1} \to X \otimes X^* \tag{2.1}$$

such that the following compositions are the identity:

$$X \xrightarrow{coev_X \otimes id_X} (X \otimes X^*) \otimes X \xrightarrow{a} X \otimes (X^* \otimes X) \xrightarrow{id_X \otimes ev_X} X \tag{2.2}$$

$$X^* \xrightarrow{id_{X^*} \otimes coev_X} X^* \otimes (X \otimes X^*) \xrightarrow{a} (X^* \otimes X) \otimes X^* \xrightarrow{ev_X \otimes id_{X^*}} X^* \tag{2.3}$$

- *We say that ${}^*X$ is a right dual to X if there exists morphisms*

$$ev'_X : X \otimes {}^*X \to \mathbb{1} \qquad coev'_X \mathbb{1} \to {}^*X \otimes X \tag{2.4}$$

such that the following compositions are the identity:

$$X \xrightarrow{id_X \otimes coev'_X} X \otimes ({}^*X \otimes X) \xrightarrow{a} (X \otimes^* X) \otimes X \xrightarrow{ev'_X \otimes id_X} X \tag{2.5}$$

$${}^*X \xrightarrow{coev'_X \otimes id_{{}^*X}} ({}^*X \otimes X) \otimes {}^*X \xrightarrow{a} {}^*X \otimes (X \otimes {}^*X) \xrightarrow{id_{{}^*X} \otimes ev'_X} {}^*X \tag{2.6}$$

- *We say that X is rigid if it has left and right duals.*

- *We say that a monoidal category $\mathcal{C}$ is* rigid *if all of its objects are rigid.*

- *For $X, Y \in \mathcal{C}$ with left duals and morphism $f : X \to Y$, we define the* left dual *f^* of f by:*

$$f^* = Y^* \xrightarrow{a^{-1}\circ(id_{Y^*}\otimes coev_X)} (Y^* \otimes X) \otimes X^* \xrightarrow{(id_{Y^*}\otimes f)\otimes id_{X^*}} (Y^* \otimes Y) \otimes X \xrightarrow{ev_Y \otimes id_{X^*}} X^* \tag{2.7}$$

- *For $X, Y \in \mathcal{C}$ with right duals and morphism $f : X \to Y$, we define the* right dual *${}^*f$ of f by:*

$${}^*f = {}^*Y \xrightarrow{a\circ(coev'_X \otimes id_{^*Y})} {}^*X \otimes (X \otimes {}^*Y) \xrightarrow{id_{^*X}(f\otimes id_{^*Y})} {}^*X \otimes (Y \otimes {}^*Y) \xrightarrow{id_{^*X}\otimes ev'_Y} {}^*X \tag{2.8}$$

The complex 'not quite commutative' structure of vertex algebras also imposes some exceptions to commutativity on the categorical level. In particular, the tensor product is commutative or associative only up to isomorphism. This notion gives braiding structure to categories of VOA modules.

Definition 2.2.3. *A* braided monoidal category *is a monoidal category $\mathcal{V}$ with natural isomorphism*

$$R_{X,Y} : X \otimes Y \to Y \otimes X \tag{2.9}$$

subject to coherence axioms expressed by commutativity of the following diagrams:

$$\begin{array}{ccccc} (X \otimes Y) \otimes Z & \xrightarrow{a} & X \otimes (Y \otimes Z) & \xrightarrow{R} & (Y \otimes Z) \otimes X \\ \downarrow{\scriptstyle R\otimes 1} & & & & \downarrow{\scriptstyle a} \\ (Y \otimes X) \otimes Z & \xrightarrow[a]{} & Y \otimes (X \otimes Z) & \xrightarrow[1\otimes R]{} & Y \otimes (Z \otimes X) \end{array} \tag{2.10}$$

$$\begin{array}{ccccc} X\otimes(Y\otimes Z) & \xrightarrow{a^{-1}} & (X\otimes Y)\otimes Z & \xrightarrow{R} & Z\otimes(X\otimes Y) \\ {\scriptstyle 1\otimes R}\downarrow & & & & \downarrow{\scriptstyle a^{-1}} \\ X\otimes(Z\otimes Y) & \xrightarrow[a^{-1}]{} & (X\otimes Z)\otimes Y & \xrightarrow[R\otimes 1]{} & (Z\otimes X)\otimes Y \end{array} \tag{2.11}$$

The braiding structure exactly describes the data needed for 'tying knots' in 3 dimensions [64], hence the name. This structure for VOAs is unsurprising, considering the relationships between vertex algebras and topological quantum field theories. From this perspective, we also expect that categories of VOA modules should inherit more topological structure, such as a twist, which is consistent with the braiding.

Definition 2.2.4.

- *A* <u>*twist*</u> *on a braided rigid monoidal category is* $\theta \in Aut(id_{\mathcal{C}})$ *such that for all objects* $X, Y \in \mathcal{C}$

$$\theta_{X\otimes Y} = (\theta_X \otimes \theta_Y) \circ R_{Y,X} \circ R_{X,Y} \tag{2.12}$$

- *We call a twist a* <u>*ribbon structure*</u> *if* $(\theta_X)^* = \theta_{X^*}$.

- *A* <u>*ribbon category*</u> *is a braided rigid monoidal category with ribbon structure.*

- *On a ribbon category, we can define the* <u>*trace*</u> *of a morphism* $f : X \to X$ *by:*

$$tr(f) = ev_X R_{XX^*}((\theta_X f) \otimes id_{X^*})coev_X : \mathbb{1} \to \mathbb{1} \tag{2.13}$$

By nature as a description of conformal field theories, we also expect that categories of VOA modules should enjoy some form of conformal invariance. In appropriate cases, this appears as an action of the modular group.

Definition 2.2.5.

- *A* <u>*fusion category*</u> *is a semisimple rigid braided tensor category with finitely many isomorphism classes of simple objects.*

- *Suppose $\mathcal{C}$ is a ribbon fusion category. The associated* <u>*S-matrix*</u> *has entries given by $S_{XY} = tr(R_{Y,X} \circ R_{X,Y})$ for X, Y representatives of isomorphism classes of simples.*

- *A* <u>*modular tensor category*</u> *is a ribbon fusion category whose S matrix is nondegenerate.*

2.3 Supercategories

In the following, we study not only vertex algebras, but vertex **super**algebras. This is most natural with both even and odd morphisms, for which we require the notion of supercategory.

Definition 2.3.1.

- *A* <u>*supercategory*</u> *is a category enriched over* <u>$\mathcal{SV}ec$</u>*.*

- *A* <u>*superfunctor*</u> *is an* <u>$\mathcal{SV}ec$</u>*-functor.*

- *For supercategory $\mathcal{S}$, we denote by* <u>$\mathcal{S}$</u> *the* <u>*underlying category*</u>*: the wide subcategory with only even morphisms.*

- *A* <u>*monoidal supercategory*</u> *$\mathcal{SC}$ consists of a supercategory $\mathcal{SC}$ with superfunctor $\otimes : \mathcal{SC} \times \mathcal{SC} \to \mathcal{SC}$ and* <u>$\mathcal{SV}ec$</u>*-natural isomorphisms a, l, r subject to conditions expressed by commutativity of the diagrams 2.1 and 2.2. That is, it is a monoidal category with all data replaced with their 'super' equivalent.*

Example 2.3.1. *Suppose that $\mathcal{C}$ is a braided tensor category.*

- *Let $\mathcal{SC}$ be the category whose objects are ordered pairs $W = (W_{\bar{0}}, W_{\bar{1}})$ with $W_{\bar{0}}, W_{\bar{1}} \in \mathcal{C}$ and morphisms are:*

$$Hom_{\mathcal{SC}}(W^1, W^2) = Hom_{\mathcal{C}}(W^1_{\bar{0}} \oplus W^1_{\bar{1}}, W^2_{\bar{0}} \oplus W^2_{\bar{1}}) \tag{2.1}$$

- *For $W \in \mathcal{SC}$, the* <u>*parity involution*</u> *of W is given by:*

$$P_W = 1_{W_{\bar{0}}} \oplus (-1_{W_{\bar{1}}}) \tag{2.2}$$

- *We say that morphism* $f \in Hom_{\mathcal{SC}}(W^1, W^2)$ *is has parity* $|f| \in \mathbb{Z}/2\mathbb{Z}$ *if* $f \circ P_{W^1} = (-1)^{|f|} P_{W^2} \circ f$.

Note that each $f \in Hom_{\mathcal{SC}}(W^1, W^2)$ *can be written uniquely as* $f = f_{\bar{0}} + f_{\bar{1}}$ *where* $|f_{\bar{i}}| = i$. *Then* $\mathcal{SC}$ *is a supercategory with the parity grading on* $Hom_{\mathcal{SC}}(W^1, W^2)$.

Example 2.3.2. *Following 2.3.1, we construct the category* $\mathcal{SV}ec$. *Our notation is consistent, since the underlying category* $\underline{\mathcal{SV}ec}$ *coincides with* $\underline{\mathcal{SV}ec}$ *introduced in 2.1.1.* $\mathcal{SV}ec \times \mathcal{SV}ec$ *is a supercategory with composition*

$$(f_1, f_2) \circ (g_1, g_2) = (-1)^{|f_2||g_1|}(f_1 \circ g_1, f_2 \circ g_2) \tag{2.3}$$

where f_2, g_1 *are parity homogenous. Define superfunctor* $\boxtimes : \mathcal{SV}ec \times \mathcal{SV}ec \to \mathcal{SV}ec$ *which on objects is as in 2.4, 2.5 and on morphisms is*

$$(f \boxtimes g)(v \boxtimes w) = (-1)^{|g||v|} f(v) \boxtimes g(w) \tag{2.4}$$

Then $\mathcal{SV}ec$ *is a monoidal supercategory.*

Chapter 3

Lie Theory

Much of the representation theory of vertex algebras resembles that of Lie algebras, particularly in the cases of affine vertex algebras that we study here. In the following I will establish notation and provide examples that will be needed for later constructions.

3.1 Lie superalgebras

Definition 3.1.1. *A superalgebra is a $\mathbb{Z}_2$ graded vector space $A = A_{\bar{0}} \oplus A_{\bar{1}}$, with bilinear product satisfying*

$$A_i A_j \subset A_{i+j}$$

Example 3.1.1. *Suppose that V is a vector superspace. Then $End(V)$ is an associative superalgebra, with product given by composition.*

Example 3.1.2. *Suppose that V is a vector space with bilinear form $\langle \cdot, \cdot \rangle$. Let $T(V)$ be the tensor algebra over V, and 1 be its unit. The associated Clifford algebra is*

$$C(V) = T(V)/\langle u \otimes v + v \otimes u - 2(u, v)1 \rangle \tag{3.1}$$

with the natural inclusion $i : V \to C(V)$. We suppress the tensor product symbol and simply write $u \otimes v = uv$. $T(V)$ is a $\mathbb{Z}$ graded algebra, and the ideal

$\langle u \otimes v + v \otimes u - 2(u,v)1\rangle$ is generated only by even elements. This induces a $\mathbb{Z}_2$ grading on $C(V)$. Then $C(V)$ is an associative superalgebra, satisfying the relations

$$[i(u), i(v)] = 2(u,v)1$$

for $u, v \in V$.

Definition 3.1.2. *A Lie superalgebra is a superalgebra $\mathfrak{g}$ with bilinear product $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ called the Lie bracket, satisfying*

- *super skewsymmetry*

$$[u, v] = -(-1)^{|u||v|}[v, u]$$

- *super Jacobi identity*

$$(-1)^{|x||z|}[x,[y,z]] + (-1)^{|y||x|}[y,[z,x]] + (-1)^{|z||y|}[z,[x,y]]$$

for homogenous $u, v, w \in V$.

Definition 3.1.3. *A homomorphism of vertex superalgebras $\mathfrak{a}, \mathfrak{b}$ is a vector superspace morphism $\phi : \mathfrak{a} \to \mathfrak{b}$ satisfying*

$$\phi([u,v]) = [\phi(u), \phi(v)]$$

Example 3.1.3. *$\mathbb{C}x$ has Lie algebra structure given by Lie bracket $[x,x] = 0$*

Example 3.1.4. *The Lie algebra $\mathfrak{sl}(2)$ is given by 3 dimensional (even) vector space. For basis e, h, f, the Lie bracket is given by*

$$[h,e] = 2e \qquad [h,f] = -2f \qquad [e,f] = h \tag{3.2}$$

Example 3.1.5. *The Lie superalgebra $\mathfrak{osp}(1|2)$ has even part $\mathfrak{sl}(2)$ with basis e, h, f as before, and the odd part has basis x, y with relations*

$$[h,x] = x \qquad [h,y] = -y \qquad [f,x] = -y \qquad [e,y] = -x \tag{3.3}$$

$$[x, x] = 2e \qquad [x, y] = h \qquad [y, y] = -2f \tag{3.4}$$

Example 3.1.6. *Suppose that A is an associative superalgebra. Then A is naturally a Lie superalgebra, with bracket given by*

$$[u, v] = uv - (-1)^{|u||v|} vu \tag{3.5}$$

For homogenous $u, v \in A$. In particular, for vector superspace V, End(V) is naturally a Lie superalgebra.

Example 3.1.7. *The infinite dimensional Lie algebra Der$\mathbb{C}((z))$ of derivations of $\mathbb{C}((z))$ has basis and Lie bracket given by*

$$L_n = -z^{n+1}\partial_z \qquad [L_n, L_m] = (m-n)L_{m+n} \tag{3.6}$$

For $n \in \mathbb{Z}$. This has unique nontrivial one dimensional central extension

$$0 \to \mathbb{C}C \to Vir \to Der\mathbb{C}((z)) \to 0 \tag{3.7}$$

called the Virasoro algebra, where C is central and relations are given by

$$[L_n, L_m] = (m-n)L_{n+m} + \frac{C}{12}(m^3 - m)\delta_{m+n,0} \tag{3.8}$$

We denote by $Vir_{\geq 0}$ the subalgebra generated by C and L_n with $n \geq 0$.

The Virasoro algebra is associated to infinitesimal conformal transformations of a 1 dimensional complex surface, making it essential to the study of conformal field theories and the vertex algebras that describe them.

3.2 Representations

We introduce Lie superalgebra representations and define the typical algebraic constructions associated with them.

Definition 3.2.1. *A representation (V, ϕ) of Lie superalgebra $\mathfrak{g}$ is a vector* *superspace* V *with homomorphism*

$$\phi : \mathfrak{g} \to End(V) \tag{3.1}$$

We will often denote a representation (V, ϕ) *simply by* V*, leaving the homomorphism* ϕ *implicit.*

Example 3.2.1. *The standard representation* $\mathfrak{st}$ *of* $\mathfrak{sl}(2)$ *on two dimensional vector space is given by*

$$e \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad h \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad f \mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{3.2}$$

Example 3.2.2. *The adjoint representation of Lie superalgebra* $\mathfrak{g}$ *is the Lie algebra homomorphism*

$$ad : \mathfrak{g} \to End(\mathfrak{g}) \qquad\qquad ad_v = [v, \cdot] \tag{3.3}$$

The adjoint representation restricted to Lie supalgebra $\mathfrak{g}_{\bar{0}}$ *gives a representation of* $\mathfrak{g}_{\bar{0}}$ *on* $\mathfrak{g}_{\bar{1}}$*.*

Definition 3.2.2.

- *A subrepresentation* W *of* V *is a vector subspace* $W \subset V$ *that is closed under the action of* $\mathfrak{g}$*.*
$$\phi(\mathfrak{g})(W) \subset W$$
- *An irreducible representation* V *is a representation whose only subrepresentations are* 0 *and* V*.*

Definition 3.2.3. *A Lie superalgebra* $\mathfrak{g}$ *is called simple if its adjoint representation is irreducible.*

Definition 3.2.4.

- *Suppose that* $\mathfrak{g}$ *is a Lie superalgebra and* V, W *are* $\mathfrak{g}$ *representations. The direct sum* $V \oplus W$ *is given by the direct sum of vector superspaces, with action given by*
$$g(v, w) = (gv, gw)$$

for $g \in \mathfrak{g}$, $v \in V$ and $w \in W$.

- *A $\mathfrak{g}$ representation V is called completely reducible if it is the direct sum of irreducible representations.*

3.2.1 The Killing form

Definition 3.2.5. *For bilinear form $\langle \cdot, \cdot \rangle : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$, we say that $\langle \cdot, \cdot \rangle$ is*

- *invariant if, for homogenous $u, v, w \in \mathfrak{g}$*

$$\langle u, [v, w] \rangle = \langle [u, v], w \rangle$$

- *supersymmetric if for homogenous $u, v \in \mathfrak{g}$*

$$\langle u, v \rangle = (-1)^{|u||v|} \langle v, u \rangle$$

- *consistent if for $u \in \mathfrak{g}_{\bar{0}}$, $v \in \mathfrak{g}_{\bar{1}}$*

$$\langle u, v \rangle = 0$$

Proposition 3.2.3. *[49] An invariant bilinear form on simple a Lie superalgebra $\mathfrak{g}$ is either non-degenerate or identically 0. Any two invariant bilinear forms on simple $\mathfrak{g}$ are proportional.*

Definition 3.2.6. *[49] Lie superalgebras have an invariant, consistent, supersymmetric bilinear form given by the Killing form.*

$$\langle u, v \rangle = str(ad_u ad_v)$$

Proposition 3.2.4. *[49]*

A Lie superalgebra with nondegenerate Killing form splits into an orthogonal direct sum of Lie superalgebras with nondegenerate Killing forms.

3.2.2 The Casimir element

Definition 3.2.7. *Let $T(\mathfrak{g})$ be the tensor superalgebra over $\mathfrak{g}$ with the induced $\mathbb{Z}_2$ grading. Define the* *universal enveloping superalgebra.*

$$U(\mathfrak{g}) = T(\mathfrak{g})/\langle u \otimes v - (-1)^{|u||v|} v \otimes u - [u, v] \rangle \tag{3.4}$$

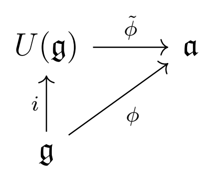
We will suppress the tensor product symbol and write $u \otimes v = uv$.

Theorem 3.2.5 (Poincaré-Birkhoff-Witt)**.** *Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a Lie superalgebra, and $J^1, ..., J^n$, a basis for $\mathfrak{g}_{\bar{0}}$, $J^{n+1}, ..., J^d$ a basis for $\mathfrak{g}_{\bar{1}}$. Then monomials of the form*

$$J^{i_1} \dots J^{i_m} \tag{3.5}$$

with $m \in \mathbb{Z}$, $i_j \leq i_{j+1}$, and for $n < i_j$, $i_j < i_{j+1}$ form a basis for $U(\mathfrak{g})$.

Remark 3.2.6. *$T(\mathfrak{g})$ is an associative superalgebra, and hence a Lie superalgebra as in 3.1.6. The inclusion $\mathfrak{g} \to T(\mathfrak{g})$ induces a homomorphism $i \to U(\mathfrak{g})$. For Lie algebra homomorphism $\phi : \mathfrak{g} \to \mathfrak{a}$, there exists unique $\tilde{\phi}$ satisfying the following commutative diagram*



In particular, a $\mathfrak{g}$ representation is equivalent to a $U(\mathfrak{g})$ module. We will often refer to a $\mathfrak{g}$ representation instead as a $\mathfrak{g}$ module.

Definition 3.2.8. *[63] Let $\langle \cdot, \cdot \rangle$ be a nondegenerate invariant bilinear form on a finite dimensional Lie superalgebra $\mathfrak{g}$. Let $\{J^i\}_{i=1,...,d}$ be a homogenous basis for $\mathfrak{g}$, and $\{J_i\}_{i=1,...,d}$ be a dual basis such that $\langle J^i, J_k \rangle = \delta_{jk}$. The* *Casimir element* *is given by*

$$C = \sum_{i=1}^{d} J_i J^i \in U(\mathfrak{g}) \tag{3.6}$$

This is dependent on the chosen bilinear form. C is central in $U(\mathfrak{g})$.

Proposition 3.2.7. *[63] The Casimir is independent of choice of basis. In particular, choosing basis J_i and noting that $\langle J_i, J^i\rangle = (-1)^{|J^i||J_i|}$, so the dual to J_i is $(-1)^{|J^i||J_i|}J^i$, we have*

$$C = \sum_{i=1}^{d} (-1)^{|J^i||J_i|} J^i J_i$$

Corollary 3.2.8.

$$\sum_{i=1}^{d} [J_i, J^i] = \sum_{i=1}^{d} J_i J^i - (-1)^{|J^i||J_i|} J^i J_i = C - C = 0 \tag{3.7}$$

Example 3.2.9. *The Killing form on $\mathfrak{sl}(2)$ has nontrivial relations given by*

$$\langle h, h\rangle = 2 \qquad\qquad \langle e, f\rangle = 1 \tag{3.8}$$

and the Casimir for $\mathfrak{sl}(2)$ is

$$C = \frac{1}{2}h^2 + ef + fe \tag{3.9}$$

3.3 Classification of classical Lie superalgebras

Simple classical Lie superalgebras have been classified, and this result will be necessary in the following work. See [49] for a complete classification and construction of these Lie superalgebras. I introduce only $D(2,1;\alpha)$ here, as we shall not need the explicit form of the others.

Example 3.3.1. *[49] There is one parameter family $D(2,1;\alpha)$, $\alpha \in \mathbb{C}\backslash\{0,-1\}$ of 17 dimensional simple Lie superalgebras with even part given by 3 copies of $\mathfrak{sl}(2)$ and odd part given by the tensor product of 3 copies of the standard representation*

$$D(2,1;\alpha)_{\overline{0}} = \mathfrak{sl}(2) \oplus \mathfrak{sl}(2) \oplus \mathfrak{sl}(2) \qquad D(2,1;\alpha)_{\overline{1}} = \mathfrak{st} \otimes \mathfrak{st} \otimes \mathfrak{st} \tag{3.1}$$

We follow the conventions and notation of [8][31] aside from a rescaling of h^i

by 2 and rescaling of the e^i, f^i by -1 in the below. For parameter α, we specify

$$\alpha_1^{-1} = -1 - \alpha^{-1} \qquad \alpha_2^{-1} = -1 - \alpha \qquad \alpha_3 = 1 \tag{3.2}$$

For the even part, we give basis e_i, h_i, f_i for $i = 1, 2, 3$, with nonzero brackets

$$[h^i, e^i] = 2e^i \qquad [h^i, f^i] = -2f^i \qquad [e^i, f^i] = h^i \tag{3.3}$$

For odd part, we give basis $\psi(\beta, \gamma, \delta)$ where $\beta, \gamma, \delta = \pm$. Nonzero brackets with the even part are given by

$$[h^1, \psi(\pm, \gamma, \delta)] = \pm\psi(\pm, \gamma, \delta) \qquad [h^2, \psi(\beta, \pm, \delta)] = \pm\psi(\beta, \pm, \delta) \tag{3.4}$$

$$[e^1, \psi(-, \gamma, \delta)] = \psi(+, \gamma, \delta) \qquad [e^2, \psi(\beta, -, \delta)] = \psi(\beta, +, \delta) \tag{3.5}$$

$$[f^1, \psi(+, \gamma, \delta)] = \psi(-, \gamma, \delta) \qquad [f^2, \psi(\beta, +, \delta)] = \psi(\beta, -, \delta) \tag{3.6}$$

$$[h^3, \psi(\beta, \gamma, \pm)] = \pm\psi(\beta, \gamma, \pm) \tag{3.7}$$

$$[e^3, \psi(\beta, \gamma, -)] = \psi(\beta, \gamma, +) \tag{3.8}$$

$$[f^3, \psi(\beta, \gamma, +)] = \psi(\beta, \gamma, -) \tag{3.9}$$

We introduce the notation

$$O^i_{++} = -2e^i \qquad O^i_{--} = -2f^i \qquad O^i_{+-} = O^i_{-+} = -h^i \tag{3.10}$$

$$\epsilon_{+-} = -\epsilon_{-+} = 1 \tag{3.11}$$

Then the bracket on the odd basis is:

$$\begin{aligned}&[\psi(\beta_1, \gamma_1, \delta_1), \psi(\beta_2, \gamma_2, \delta_2)] \\ &\qquad = \alpha_1 O^1_{\beta_1\beta_2}\epsilon_{\gamma_1\gamma_2}\epsilon_{\delta_1\delta_2} + \alpha_2 O^2_{\gamma_1\gamma_2}\epsilon_{\beta_1\beta_2}\epsilon_{\delta_1\delta_2} + \alpha_3 O^3_{\delta_1\delta_2}\epsilon_{\beta_1\beta_2}\epsilon_{\gamma_1\gamma_2}\end{aligned} \tag{3.12}$$

The Killing form on $D(2,1;\alpha)$ is completely degenerate, but there is still invariant form $\langle\cdot,\cdot\rangle$ with nontrivial relations given by:

$$\langle e^i, f^j\rangle = \frac{\delta_{ij}}{\alpha_i} \qquad \langle h^i, h^j\rangle = \frac{2\delta_{ij}}{\alpha_i} \tag{3.13}$$

$$\langle \psi(\beta_1, \gamma_1, \delta_1), \psi(\beta_2, \delta_2, \gamma_2) \rangle = -2\epsilon_{\beta_1\beta_2}\epsilon_{\gamma_1\gamma_2}\epsilon_{\delta_1\delta_2} \tag{3.14}$$

Rescaling all 3 of the parameters $\alpha_1, \alpha_2, \alpha_3$ by a common factor amounts to a rescaling of the odd generators $\psi(\beta, \gamma, \delta)$.

Definition 3.3.1. *A Lie superalgebra $\mathfrak{g}$ is called classical if it is simple and the representation of $\mathfrak{g}_{\bar{0}}$ on $\mathfrak{g}_{\bar{1}}$ is completely reducible.*

Theorem 3.3.2. *[49] A classical Lie superalgebra is isomorphic to either one of the simple Lie algebras $A_n, B_n, C_n, D_n, F_4, G_2, E_6, E_7, E_8$ or to one of $A(m|n), B(m|n), C(n), D(m|n), D(2,1;\alpha), F(4), G(3), P(n)$ or $Q(n)$.*

3.4 Representation categories

We now demonstrate much of the categorical structure of section 2 in the case of Lie superalgebras. The representation theory of vertex algebras will extensively resemble much of the following.

Definition 3.4.1. *For $\mathfrak{g}$ representations V, W, we say that vector space morphism $\phi : V \to W$ is a homomorphism of $\mathfrak{g}$ representations if*

$$g\phi(v) = \phi(gv)$$

for $v \in V$, $g \in \mathfrak{g}$.

Definition 3.4.2. *Suppose that $\mathfrak{g}$ is a Lie superalgebra and V, W are $\mathfrak{g}$ representations. The tensor product $V \otimes W$ is given by the tensor product of vector superspaces, with action given by*

$$g(v \otimes w) = gv \otimes w + v \otimes gw \tag{3.1}$$

for $g \in \mathfrak{g}$, $v \in V$ and $w \in W$.

Definition 3.4.3. *[50] For classical Lie superalgebra $\mathfrak{g}$,*

- *A Cartan subalgebra $\mathfrak{h}$ is a maximal commutative subalgebra of $\mathfrak{g}$.*

Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. For $\alpha \in \mathfrak{h}^*$, denote

$$\mathfrak{g}_\alpha = \{e \in \mathfrak{g} | \forall h \in \mathfrak{h}\ [h, e] = \alpha(h)e\}$$

- *We call nontrivial $\alpha \in \mathfrak{h}^*$ a root of $\mathfrak{g}$ if $g_\alpha \neq 0$. The roots of $\mathfrak{g}$ generate the root lattice, which we denote by $Q \subset \mathfrak{h}^*$.*

Remark 3.4.1. *$\langle \cdot, \cdot \rangle$ induces an invariant bilinear form on $\mathfrak{h}^*$, which we will denote again by $\langle \cdot, \cdot \rangle$. $\mathfrak{g}$ always has simple (even) Lie algebra direct summand $\tilde{g}$. If there are multiple, we choose $\tilde{\mathfrak{g}}$ to have maximal rank, and if all ranks are equal, we choose arbitrary such $\tilde{\mathfrak{g}}$.*

Here and in the following, we normalize the invariant form $\langle \cdot, \cdot \rangle$ such that long roots of $\tilde{\mathfrak{g}}$ have norm 2 with respect to this form, unless otherwise specified.

- *For root $\alpha \in Q$, there is $\alpha^\vee \in \mathfrak{h}$ satisfying $\langle \alpha^\vee, \cdot \rangle = \alpha(\cdot)$. We call $\alpha^\vee$ the coroot associated to α. The coroots generate a coroot lattice, denoted by $Q^\vee$.*

- *The weight lattice P is the lattice dual to $Q^\vee$ with respect to $\langle \cdot, \cdot \rangle$, and the coweight lattice $P^\vee$ is dual to Q.*

- *A Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ is a maximal solvable subalgebra.*

Let $\mathfrak{b}_{\bar{0}}$ be a Borel subalgebra of $\mathfrak{g}_{\bar{0}}$ containing $\mathfrak{h}$ and extend to a Borel subalgebra $\mathfrak{b} = \mathfrak{b}_{\bar{0}} \oplus \mathfrak{b}_{\bar{1}}$ of $\mathfrak{g}$.

- *Diagonalizability of the adjoint representation gives the Cartan decomposition*

$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$$

- *A root α is called positive (resp. negative) if $\mathfrak{g}_\alpha \cap \mathfrak{n}_+ \neq 0$ (resp. $\mathfrak{g}_\alpha \cap \mathfrak{n}_- \neq 0$). Positive roots span the positive root lattice Q_+.*

- *We define the dominant weights*

$$P_+ = \{\lambda \in P | \langle \lambda, \mu \rangle \geq 0 \text{ for all } \mu \in Q_+\}$$

- *For $\mathfrak{g}$ representation V, the vector $v \in V$ is called highest weight if $\mathfrak{n}_+ v = 0$.*

Definition 3.4.4. *A $\mathfrak{g}$ module M is called weight if $\mathfrak{h}$ acts semisimply on $\mathfrak{g}$.*

$$M = \bigoplus_{\lambda \in \mathfrak{h}^*} M_\lambda$$

where $M_\lambda = \{m \in M | hm = \lambda(h)m\}$ is the weight space associated to λ.

Definition 3.4.5. *Suppose that $\mathfrak{g}$ is a Lie superalgebra, $\mathfrak{h}$ is a subalgebra, and V is an $\mathfrak{h}$ representation or equivalently, a $U(\mathfrak{h})$ module. Then*

$$U(\mathfrak{g}) \otimes_{U(\mathfrak{h})} V$$

is naturally a $U(\mathfrak{g})$ module or $\mathfrak{g}$ representation, called the induced module or induced representation, where we define action

$$g(p \otimes v) = gp \otimes v$$

for $g \in \mathfrak{g}$, $p \in U(\mathfrak{g})$ and $v \in V$.

For $\lambda \in \mathfrak{h}^*$, we can define weight representations of highest weight λ. These are constructed as follows:

Definition 3.4.6. *Pick Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, and Borel subalgebra $\mathfrak{b} \supset \mathfrak{h}$. Then $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}_+$. Pick $\lambda \in \mathfrak{h}^*$. Define one dimensional $\mathfrak{b}$ representation $\mathbb{C}|\lambda\rangle$ by*

$$h|\lambda\rangle = \lambda(h)|\lambda\rangle \quad h \in \mathfrak{h} \qquad\qquad \mathfrak{n}_+|\lambda\rangle = 0 \tag{3.2}$$

With $||\lambda\rangle| = \overline{0}$. Define

$$V(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}|\lambda\rangle \tag{3.3}$$

Note that by universal property of tensor product, for any $\mathfrak{g}$ representation V with highest weight vector of weight λ, we have morphism $V(\lambda) \to V$.

$V(\lambda)$ has unique maximal subrepresentation $I(\lambda)$. Then we define

$$L(\lambda) = V(\lambda)/I(\lambda) \tag{3.4}$$

which is an irreducible representation of highest weight λ.

Theorem 3.4.2. *[50] Suppose that $\mathfrak{g}$ is a classical Lie superalgebra excluding $A(n,n)$. Then*

- *Any quotient of $V(\lambda)$ is weight, with finite dimensional weight spaces.*
- *$|\lambda\rangle$ is the unique highest weight vector in $L(\lambda)$.*
- *$Hom(L(\lambda), L(\mu)) \simeq \mathbb{C}\delta_{\lambda,\mu}$.*
- *Any finite dimensional irreducible $\mathfrak{g}$ module is isomorphic to some $L(\lambda)$.*

Then the representations $L(\lambda)$ are a complete list of simple objects in an appropriate category, and they are uniquely determined by their highest weight λ. We proceed by introducing interesting categories of modules for Lie superalgebras.

Theorem 3.4.3. *[50] For (even) Lie algebra $\mathfrak{g}$, $L(\lambda)$ is finite dimensional iff $\lambda \in P_+$.*

Definition 3.4.7. *A $\mathfrak{g}$ representation (ϕ, V) is called integrable if it integrates to a representation of the associated Lie group.*

We denote by $Int_{\mathfrak{g}}$ the category whose objects are integrable representations for $\mathfrak{g}$ and morphisms are homomorphisms of $\mathfrak{g}$ representations.

Proposition 3.4.4. *Simple objects of $Int_{\mathfrak{g}}$ are isomorphic to $L(\lambda)$, $\lambda \in P_+$, and tensor products of simple objects in $Int_{\mathfrak{g}}$ are finite direct sums.*

We sketch explicitly for $\mathfrak{sl}(2)$. By 3.4.2, the simple objects of $Int_{\mathfrak{g}}$ are given by $L(\lambda)$ with $\lambda \in P_+ \simeq \mathbb{Z}_{\geq 0}$. By induction, we confirm for any $\lambda \in \mathfrak{h}^*$

$$ef^i|\lambda\rangle = i(r-i+1)f^{i-1}|\lambda\rangle$$

Then for for $0 \leq n \leq \min\{r, p\}$

$$\sum_{\substack{i,j \in \mathbb{Z}_{\geq 0} \\ i+j=n}} \frac{(r-i)!}{i!} \frac{(r-j)!}{j!} f^i |r\rangle \otimes f^j |p\rangle$$

is annihilated by e, and hence is a highest weight vector in $L(r) \otimes L(p)$ with weight $r + p - 2n$. This gives family of highest weight vectors in $L(r) \otimes L(p)$, generating subrepresentations of $L(r) \otimes L(p)$ of highest weight $r + p - 2n$ that intersect only at 0. Counting dimensions, we have

$$L(r) \otimes L(p) \simeq \bigoplus_{n=0}^{\min\{r,p\}} L(r + p - 2n)$$

Definition 3.4.8. *We define the category $\mathcal{O}$ of $\mathfrak{g}$ representations, whose objects are $\mathfrak{g}$ representations M such that*

1. *M is weight*
2. *M is finitely generated as a $U(\mathfrak{g})$ module*
3. *Each $v \in M$ generates a finite dimensional $\mathfrak{n}_+$ module*

We again sketch monoidal structure for $\mathfrak{g} = \mathfrak{sl}(2)$. Simple objects of $\mathcal{O}$ are isomorphic to L_λ for $\lambda \in \mathfrak{h}^* \simeq \mathbb{C}$. In $L_\lambda \otimes L_\mu$

$$\sum_{\substack{i,j \in \mathbb{Z}_{\geq 0} \\ i+j=n}} i!j! \binom{r-i}{j} \binom{\lambda - j}{i} f^i |\lambda\rangle \otimes f^j |\mu\rangle$$

is annihilated by e and hence is a highest weight vector of weight $\lambda + \mu - 2n$, where

$$\binom{\lambda}{j} = \frac{\lambda(\lambda-1)\dots(\lambda-j+1)}{j(j-1)\dots 2 \cdot 1} \qquad \binom{\lambda}{0} = 1 \tag{3.5}$$

is the binomial coefficient. We note that if $\lambda \in P_+ \simeq \mathbb{Z}_+$ and $\mu \notin P_+$, we get finite direct sum

$$L(r) \otimes L(\mu) \simeq \bigoplus_{n=0}^{r} L(r+\mu-2n)$$

If both $\lambda, \mu \notin P_+$, we obtain **countable** direct sum

$$L(\lambda) \otimes L(\mu) \simeq \bigoplus_{n=0}^{\infty} L(\lambda+\mu-2n)$$

Despite the representations $L(\lambda)$ being quite nice, we notice that their tensor products already give much larger representations which are not in the category $\mathcal{O}$. Then we have to work with much less friendly categories in order to have monoidal structure. This problem is only exacerbated with more interesting simple objects.

Definition 3.4.9. *For weight module M of $\mathfrak{g}$, the* <u>*weight support*</u> *of M is*

$$\{\lambda \in \mathfrak{h}^* | M_\lambda \neq 0\}$$

Definition 3.4.10. *A weight module M with support inside one Q-coset is called* <u>*admissible*</u> *if $\dim M_\lambda$ is uniformly bounded.*

Definition 3.4.11. *[59] Let $\mathfrak{g}$ be a finite dimensional simple Lie superalgebra. A* <u>*coherent family*</u> *is a weight module $\mathcal{C}$ for $\mathfrak{g}$ such that*

- *There exists $d \in \mathbb{Z}_{\geq 0}$, called the* <u>*degree*</u> *of $\mathcal{C}$, such that $\dim \mathcal{C}_\lambda = d$ for all $\lambda \in \mathfrak{h}^*$.*

- *For any $c \in U(\mathfrak{g})^{\mathfrak{h}}$, the function taking $\lambda \in \mathfrak{h}^*$ to $tr_{\mathcal{C}_\lambda} c$ is polynomial in λ.*

Coherent families decompose as

$$\mathcal{C} \simeq \bigoplus_{\lambda \in \mathfrak{h}^*/Q} \mathcal{C}(\lambda) \tag{3.6}$$

where each of the $\mathcal{C}(\lambda)$ is admissible. We demonstrate explicitly for $\mathfrak{sl}(2)$.

Consider one dimensional $\mathbb{C}C \oplus \mathbb{C}h$ module $\mathbb{C}|\lambda, \Delta\rangle$ on which

$$h|\lambda, \Delta\rangle = \lambda|\lambda, \Delta\rangle \qquad\qquad C|\lambda, \Delta\rangle = \Delta|\lambda, \Delta\rangle \tag{3.7}$$

Define $\mathfrak{g}$ module

$$E(\lambda, \Delta) = U(\mathfrak{g}) \otimes_{\mathbb{C}C\oplus\mathbb{C}h} \mathbb{C}|\lambda, \Delta\rangle$$

Then $E(\lambda, \Delta)$ is generated by the action of e, f on $|\lambda, \Delta\rangle$ and finally, define

$$\mathcal{C}(\Delta) = \int_{\lambda\in\mathfrak{h}^*/Q} E(\lambda, \Delta)$$

Then $C(\Delta)$ is a coherent family. We see that tensor products of the $E(\lambda, \Delta)$ are already extremely large, let alone those of coherent families.

3.5 Affine Lie algebras and Virasoro

We may now begin working towards the construction of vertex algebras. To this end, we introduce families of infinite dimensional Lie superalgebras whose representation theory resembles their finite dimensional counterparts.

Definition 3.5.1. *Let $\mathfrak{g}$ be a finite dimensional simple Lie superalgebra with nondegenerate, supersymmetric, invariant bilinear form $\langle\cdot,\cdot\rangle$. Consider $\mathfrak{g}((t)) = \mathfrak{g}\otimes\mathbb{C}((t))$ and for $g \in \mathfrak{g}$ denote $g_n := g \otimes t^n$. We have one dimensional central extension $\hat{\mathfrak{g}}$ of $\mathfrak{g}$, called the affinization of $\mathfrak{g}$*

$$0 \to \mathbb{C}K \to \hat{\mathfrak{g}} \to \mathfrak{g}((t)) \to 0$$

with nontrivial commmutation relation given by:

$$[a_n, b_m] = [a, b]_{n+m} + Kn\langle a, b\rangle\delta_{n+m,0}$$

Remark 3.5.1. *We note that $\mathfrak{g}[t] \oplus \mathbb{C}K$ is subalgebra of $\hat{\mathfrak{g}}$, and $\hat{\mathfrak{g}}$ includes $\mathfrak{g}$ as a Lie subalgebra, given by the 0 modes $g_0 \in \mathfrak{g} \otimes \mathbb{C}$.*

Affine Lie superalgebras also have a family of automorphisms which can be used to twist their action on modules, which we will see often reduces the study of larger categories to that of smaller ones.

Definition 3.5.2. *Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. $\mathfrak{h}$ extends to a Cartan subalgebra $\hat{\mathfrak{h}} = \mathfrak{h} \oplus K\mathbb{C}$ of $\hat{\mathfrak{g}}$. For $\lambda \in P^\vee$, we define the* <u>*spectral flow automorphism*</u> *[7] σ_λ of $\hat{\mathfrak{g}}$ by*

$$\sigma_\lambda(e_n) = e_{n-\alpha(\lambda)} \qquad e \in \mathfrak{g}_\alpha \tag{3.1}$$

$$\sigma_\lambda(h_n) = h_n - K\delta_{n,0}\langle\lambda, h\rangle \qquad h \in \mathfrak{h} \tag{3.2}$$

$$\sigma_\lambda(K) = K \tag{3.3}$$

We can begin to introduce the notation for affine vertex superalgeras and their representation categories.

Definition 3.5.3. *Let $\hat{\mathfrak{g}}$ be an affine Lie superalgebra, and M a $\hat{\mathfrak{g}}$ module.*

- *M is called* <u>*smooth*</u> *if for every $m \in M$ and $x \in \mathfrak{g}$, there exists $N \in \mathbb{Z}$ such that $x_n m = 0$ for all $n > N$.*
- *M is called* <u>*lower bounded*</u> *if the generalized C eigenspaces satisfy, for all $\Delta \in \mathbb{C}$, $M_{\Delta+m} = 0$ for $m \in \mathbb{Z}$ sufficiently negative.*
- *M is said to have* <u>*level*</u> *$k \in \mathbb{C}$ if K acts as multiplication by k.*
- *Suppose M is weight. $v \in M$ is called* <u>*relaxed highest weight*</u> *if $t\mathfrak{g}[t]v = 0$.*
- *M is called a* <u>*relaxed highest weight module*</u> *if it is generated by a relaxed highest weight vector.*

Definition 3.5.4. *We define the following categories of $\hat{\mathfrak{g}}$ modules*

- *$\mathbb{V}^k(\mathfrak{g})$-wtmod: the category of finitely generated smooth weight $\hat{\mathfrak{g}}$ modules at level k, such that M is bigraded by weight and conformal weight,*

  $$M = \bigoplus_{\lambda,\Delta} M_{\lambda,\Delta}$$

  with each $M_{\lambda,\Delta} < \infty$ and for each λ, there exists h_λ such that $M_{\lambda,\Delta} = 0$ when $Re(\Delta) < Re(h_\lambda)$.

- $\mathbb{V}^k(\mathfrak{g})$-*wtmod*$_{\geq 0}$: *the full subcategory of* $\mathbb{V}^k(\mathfrak{g})$-*wtmod whose objects are lower bounded.*

- $\mathbb{V}^k(\mathfrak{g})$-*wtmod*$_{KL}$: *The full subcategory of* $\mathbb{V}^k(\mathfrak{g})$-*wtmod*$_{\geq 0}$ *of objects that are in the category* $\mathcal{O}$ *as* $\mathfrak{g}$ *modules, and have finite dimensional conformal weight spaces.*

Definition 3.5.5. *Let* M *be a smooth* $\hat{\mathfrak{g}}$ *module. Define the spectral flow twist*

$$\sigma_\lambda^*(M)$$

to be isomorphic to M *as a vector space, with isomorphism* σ_λ^*, *and action given by*

$$x\sigma_\lambda^*(m) = \sigma_\lambda^*(\sigma_{-\lambda}(x)m)$$

for $x \in \hat{\mathfrak{g}}$ *and* $m \in M$. *Note that* $\sigma_\lambda^*(M)$ *is again a smooth* $\hat{\mathfrak{g}}$ *module.*

Theorem 3.5.2. *[36] Suppose that* $\mathfrak{g}$ *is an (even) Lie algebra,* $k \neq 0$ *and* $M \in \mathbb{V}^k(\mathfrak{g})$-*wtmod is irreducible. Then there exists* $\lambda \in P^\vee$ *such that* $\sigma_\lambda(M) \in \mathbb{V}^k(\mathfrak{g})$-*wtmod*$_{\geq 0}$.

We may construct $\hat{\mathfrak{g}}$ modules from $\mathfrak{g}$ modules in a natural way. Here we give this construction and establish our notation.

Definition 3.5.6. *Suppose that* N *is a weight module for* $\mathfrak{g}$. *For* $k \in \mathbb{C}$ *non-critical,* N *is naturally a* $\mathfrak{g}[t] \oplus \mathbb{C}K$ *module of level* k, *with*

$$t\mathfrak{g}[t]v = 0 \tag{3.4}$$

This induces to $\hat{\mathfrak{g}}$ *module, which we denote by:*

$$M^k(N) = U(\hat{\mathfrak{g}}) \otimes_{\mathfrak{g}[t]\oplus\mathbb{C}K} N \tag{3.5}$$

Definition 3.5.7. *Suppose that* M *is a* $\hat{\mathfrak{g}}$ *module with minimal conformal weight.*

- *Denote by* M_{top} *its subspace of minimal conformal weight.*

- *Let I be the sum of all submodules that intersect M_{top} trivially. Define the almost irreducible quotient of M to be M/I.*
- *Denote the almost irreducible quotient of $M^k(N)$ by $L^k(N)$.*
- *For $\lambda \in P$, we denote*

$$\mathbb{V}^k(\lambda) = M^k(L(\lambda)) \qquad \mathbb{L}^k(\lambda) = L^k(L(\lambda)) \tag{3.6}$$

While *Vir* is not an affine Lie algebra, we may construct modules for it in a very similar fashion to what we have done in the affine case.

Definition 3.5.8. *For $c, h \in \mathbb{C}$, consider one dimensional $Vir_{\geq 0}$ module $\mathbb{C}|h\rangle$, on which*

$$L_0|h\rangle = h|h\rangle \qquad C|h\rangle = c|h\rangle \qquad Vir_{>0}|h\rangle = 0 \tag{3.7}$$

Define the Vir module

$$\tilde{\mathbb{V}}^c(h) = U(Vir) \otimes_{Vir_{\geq 0}} \mathbb{C}|h\rangle \tag{3.8}$$

Note this has PBW basis. Denote the simple graded quotient of $\tilde{\mathbb{V}}^c(h)$ by $\tilde{\mathbb{L}}^c(h)$.

Chapter 4

Vertex algebras

4.1 Basic notions

We may now introduce the machinery needed for vertex algebras. This begins with formal distributions $A(z_1, ..., z_n) \in R[[z_1^{\pm}, ..., z_n^{\pm}]]$ for ring R. Notice that the product of formal distributions is not always well defined, but for $A(z_1, ..., z_n) \in R[[z_1^{\pm}, ..., z_n^{\pm}]]$ and $B(w_1, ..., w_m) \in R[[w_1^{\pm}, ..., w_m^{\pm}]]$, the product $A(z_1, ..., z_n)B(w_1, ..., w_m)$ is always a well defined object of $R[[z_1^{\pm}, ..., z_n^{\pm}, w_1^{\pm}, ..., w_m^{\pm}]]$. The delta distribution, which becomes critical in the study of 'local operators' that appear in conformal field theories, is an example of such a distribution in $R[[z^{\pm}, w^{\pm}]]$.

Definition 4.1.1. *The formal delta distribution is*

$$\delta(z-w) = \sum_{n\in\mathbb{Z}} z^n w^{-n-1} \tag{4.1}$$

For any formal distribution $A(w) \in R[[z^{\pm}]]$ we may calculate the product $A(w)\delta(z-w) = A(z)\delta(z-w)$. In particular $(z-w)\delta(z-w) = 0$. Differentiating and proceeding by induction, we see

$$(z-w)^{n+1}\partial_w^n \delta(z-w) = 0 \tag{4.2}$$

Lemma 4.1.1. *[51] Let $F(z, w) \in R[[z^{\pm}, w^{\pm}]]$ formal distributions such that*

$(z-w)^N F(z-w) = 0$. *Then* $F(z,w)$ *can be written uniquely as*

$$F(z,w) = \sum_{n=0}^{N-1} f_n(w) \partial_w^n \delta(z-w) \tag{4.3}$$

A 'local operator' as it appears in physics, should be a vector space morphism, depending on an 'insertion point' parameter z, which does not 'interact' with operators inserted at distinct point $w \neq z$. This notion is formalized in the following way:

Definition 4.1.2. *Let* V *be a vector superspace. Then* $End(V)[[z^{\pm}]]$ *is a vector superspace.*

- *A field on* V *is* $A \in End(V)[[z^{\pm}]]$ *satisfying* $A(z)v \in V((z))$ *for all* $v \in V$.
- *The supercommutator of fields is:*

$$[A(z), B(w)] = A(z)B(w) - (-1)^{|A||B|} B(w)A(z) \tag{4.4}$$

- *We say that fields* $A(z), B(w)$ *are mutually local if there is* $N \in \mathbb{Z}_{\geq 0}$ *such that:*

$$(z-w)^N [A(z), B(w)] = 0 \tag{4.5}$$

  as distributions in $End(V)[[z^{\pm}, w^{\pm}]]$.

- *The normally ordered product of fields is given by:*

$$: A(z)B(w) := A(z)_+ B(w) + (-1)^{|A||B|} B(w) A(z)_- \tag{4.6}$$

  where

$$A(z)_- = \sum_{n<0} A_n z^n \qquad A(z)_+ = \sum_{n\geq 0} A_n z^n \tag{4.7}$$

In this way, operators 'do not interact' if they commute, and they commute unless $z = w$. Lemma 4.1.1 then tells us that a commutator of mutually local fields is given by the sum of products of fields in one parameter with derivatives of the delta distribution. We may are now prepared to introduce the notion of a vertex superalgebra.

Definition 4.1.3. *A* *vertex superalgebra* *is a quadruple* $(V, |0\rangle, T, Y)$ *such that*

1. *(space of states)* $V = V_{\bar{0}} \oplus V_{\bar{1}}$ *is a vector superspace*

2. *(vacuum vector)* $|0\rangle \in V_{\bar{0}}$

3. *(translation operator) Even* $T : V \to V$

4. *(vertex operator) Even* $Y(\cdot, z) : V \to End[[z^{\pm}]]$

$$Y(v, z) = \sum_{n\in\mathbb{Z}} v_{(n)} z^{-n-1}$$

5. *(vacuum axioms)*

   (a) $Y(|0\rangle, z) = Id_V$

   (b) $Y(v, z)|0\rangle \in V[[z]]$

   (c) $Y(v, 0)|0\rangle = v$

6. *(translation axioms)*

   (a) $[T, Y(v, z)] = \partial_z Y(v, z)$

   (b) $T|0\rangle = 0$

7. *(locality) All fields* $Y(v, z)$ *are mutually local.*

From a physics perspective, this gives a formal notion of the symmetry algebra for a 2 dimensional quantum field theory: a vector space of even bosonic and odd fermionic states, associated local fields acting on a vector space, a means to 'translate' the insertion points of local fields, and a 'homogenous' vacuum state.

From a mathematical perspective, this is an 'almost commutative superalgebra' with derivation T, in the sense that we have bilinear product given by the operator Y, which is commutative when $z \neq w$, and identity object $|0\rangle$.

We introduce a natural (and physically necessary) notion of $\frac{1}{2}\mathbb{Z}$ grading on a vertex superalgebra.

Definition 4.1.4. *Suppose that V is a $\frac{1}{2}\mathbb{Z}$ graded vector space $V = \bigoplus_{n\in\frac{1}{2}\mathbb{Z}} V_n$.*

- *We say that linear operator $\phi : V \to V$ is homogenous of degree m if $\phi(V_n) \subset V_{n+m}$.*
- *Suppose that V is a vertex algebra. V is a $\frac{1}{2}\mathbb{Z}$-graded vertex algebra if*
  1. *$|0\rangle \in V_0$*
  2. *T is homogenous of degree 1*
  3. *For $v \in V_m$ and associated field $Y(v,z) = \sum_{n\in\mathbb{Z}} v_n z^{-n-m}$, the modes v_n are homogenous of degree n.*

Keeping with the analogy of a vertex superalgebra as an almost commutative superalgebra, we would like to have some notion of associativity. This role is played by the Jacobi identity. It can be proven that this is equivalent to locality in the presence of the other vertex superalgebra axioms.

Theorem 4.1.2 (Jacobi Identity)**.** *[33] For V a vertex algebra and homogeous $a, b, c \in V$, the three expressions*

$$Y(a,z)Y(b,w)c \in V((z))((w)) \tag{4.8}$$

$$(-1)^{|a||b|}Y(b,w)Y(a,z)c \in V((w))((z)) \tag{4.9}$$

$$Y(Y(a,z-w)b,w)c \in V((w))((z-w)) \tag{4.10}$$

are expansions in their respective domain of the same element of

$$V[[z,w]][z^{-1}, w^{-1}, (z-w)^{-1}]$$

From a physics perspective, this notion of associativity gives us means to expand products of fields inserted near each other as sums of fields inserted only at one point, weighted by the distance between insertions $z - w$.

Proposition 4.1.3. *[51] Let $\phi(z), \varphi(w)$ be homogenous fields on vector superspace V. The following are equivalent:*

1. *In $End(V)[[z^{\pm 1}, w^{\pm 1}]]$ we have:*

$$(z-w)^N[\phi(z), \varphi(w)] = 0$$

2. *For some fields $\gamma_n(w)$ and $(z-w)^{-1}$ expanded in positive powers of w/z, we have:*

$$\phi(z)\varphi(w) = \sum_{n=0}^{N-1} \frac{\gamma_n(w)}{(z-w)^{n+1}} + :\phi(z)\varphi(w): \tag{4.11}$$

The same equality holds for $(-1)^{|\varphi||\phi|}\varphi(w)\phi(z)$, with $(z-w)^{-1}$ expanded in positive powers of z/w.

Proof. *We prove for homogenous fields ϕ, φ. Note that*

$$\begin{aligned} &\phi(z)\varphi(w) = \phi(z)_-\varphi(w) + \phi(z)_+\varphi(w) \\ &= \phi(z)_-\varphi(w) - (-1)^{|\phi||\varphi|}\varphi(w)\phi(z)_- + (-1)^{|\phi||\varphi|}\varphi(w)\phi(z)_- + \phi(z)_+\varphi(w) \\ &= [\phi(z)_-, \varphi(w)] + :\phi(z)\varphi(w): \end{aligned} \tag{4.12}$$

Assume 1. above. By 4.1.1 we have fields $\gamma_n(w)$ such that:

$$[\phi(z), \varphi(w)] = \sum_{n=0}^{N-1} \frac{1}{n!}\gamma_n(w)\partial_w^n \delta(z-w) \tag{4.13}$$

Looking at only negative powers in z we have:

$$[\phi(z)_-, \varphi(w)] = \sum_{n=0}^{N-1} \frac{1}{n!}\gamma_n(w)\partial_w^n \delta(z-w)_-$$

Then the first form of 4.11 follows by noting that $\frac{1}{n!}\partial_w^n\delta(z-w)_-$ is the expansion of $(z-w)^{n+1}$ in $\mathbb{C}((z))((w))$. The formula for $\varphi(w)\phi(z)$ is obtained similarly using $(-1)^{|\varphi||\phi|}\varphi(w)\phi(z) = [\phi_+(z), \varphi(w)] + :\phi(z)\varphi(w):$.

Assuming both forms of 4.11 above and taking the commutator, we see that the normal order portions cancel and we are left with the sum of $\delta(z-w)_-$, and $\delta(z-w)_+$, recovering the expression in 1. above.

We call 4.11 the operator product expansion of fields $\phi(z), \varphi(w)$. By the previous theorem, data of operator product expansions is equivalent to the data of commutators of fields.

Comparing the expressions 4.11 with 4.10 for $Y(a,z), Y(b,w)$, and using

Taylor's theorem to expand $Y(a, z)$, we obtain:

$$\sum_{n\in\mathbb{Z}} \frac{Y(a_{(n)}b, w)}{(z-w)^{n+1}} = \sum_{n=0}^{N-1} \frac{\gamma_n(w)}{(z-w)^{n+1}} + \sum_{m=0}^{\infty} \frac{(z-w)^m}{m!} : \partial_w^m Y(a, w) Y(b, w) :$$

Then the following expressions hold for $n \geq 0$:

$$Y(a_{(n)}b, w) =: \partial_w^{-n-1} Y(a, w) Y(b, w) :$$

$$\gamma_n(w) = Y(a_{(n)}b, w) \tag{4.14}$$

This motivates the following theorems.

Theorem 4.1.4. *[33] For vertex algebra V and $a, b \in V$, we have mode commutation relations:*

$$[a_{(m)}, b_{(k)}] = \sum_{n\geq 0} \binom{m}{n} (a_{(n)}b)_{(m+k-n)} \tag{4.15}$$

Proof. *With 4.13 and 4.14 we have:*

$$\begin{aligned}[\phi(z), \varphi(w)] = \sum_{n=0}^{\infty} \frac{1}{n!} Y(a_{(n)}b, w) \partial_w^n (z-w) \\ = \sum_{n=0}^{\infty} \sum_{m\in\mathbb{Z}} \binom{m}{n} Y(a_{(n)}b, w) z^{-m-1} w^{m-n}\end{aligned}$$

Theorem 4.1.5 (Reconstruction)**.** *[33] Let V be a vector space, $|0\rangle \in V$ a nonzero vector, and T an endomorphism of V. Suppose that $\{J^\alpha\}_{\alpha\in S}$ is a collection of vectors in V, and we are given fields*

$$J^\alpha(z) = \sum_{n\in\mathbb{Z}} J^\alpha_{(n)} z^{-n-1}$$

Satisfying the following conditions

1. *For all α, $J^\alpha(z)|0\rangle = J^\alpha + z(...)$*

2. *$T|0\rangle = 0$ and $[T, J^\alpha(z)] = \partial_z J^\alpha(z)$*

3. All fields $J^\alpha(z)$ *are mutually local*

4. V is spanned by vectors

$$J^{\alpha_1}_{j_1}...J^{\alpha_m}_{j_m}|0\rangle \qquad\qquad j_i < 0$$

Then

$$Y(J^{\alpha_1}_{j_1}....J^{\alpha_m}_{j_m}|0\rangle, z) = \frac{1}{(-j_1-1)!...(-j_m-1)!} :\partial_z^{-j_1-1}J^{\alpha_1}(z)...\partial_z^{-j_m-1}J^{\alpha_m}(z): \tag{4.16}$$

Defines the structure of a vertex algebra on V*, and this is the unique vertex algebra structure on* V *satisfying* $(1),(2),(3),(4)$ *such that* $Y(J^\alpha, z) = J^\alpha(z)$*.*

This theorem gives two powerful tools: first, it gives a uniqueness result that allows us demonstrate that two vertex superalgebras are isomorphic by looking only at a generating set and a limited number of relations. Secondly, it gives us means to practically construct the large amounts of data associated to a vertex superalgebra. In order to recover all relations from the generating ones, we will need some means to calculate the commutator of a field with a normally ordered product. This is given by Wick's theorem.

Theorem 4.1.6. *For homogenous fields* $A(z), B(z)$ *and* $C(w)$*, we have*

$$[:A(z)B(z):, C(w)] = (-1)^{|B||C|} :[A(z), C(w)]B(z): + :A(z)[B(z), C(w)]: \tag{4.17}$$

where the normal ordered product is with respect to powers of z*.*

Proof.

$$\begin{aligned}
&[: A(z)B(z) :, C(w)] \\
&= \sum_{\substack{m,l \\ n<0}} [A_{(n)}B_{(m)}, C_{(l)}] z^{-n-m-2} w^{-l-1} + \sum_{\substack{m,l \\ n\geq 0}} [B_{(m)}A_{(n)}, C_{(l)}] z^{-n-m-2} w^{-l-1} \\
&= \sum_{\substack{m,l \\ n<0}} \left((-1)^{|B||C|} [A_{(n)}, C_{(l)}] B_{(m)} + A_{(n)} [B_{(m)}, C_{(l)}] \right) z^{-n-m-2} w^{-l-1} \\
&+ \sum_{\substack{m,l \\ n\geq 0}} \left((-1)^{|A|(|B|+|C|)} [B_{(m)}, C_{(l)}] A_{(n)} + (-1)^{|B||A|} B_{(m)} [A_{(n)}, C_{(l)}] \right) z^{-n-m-2} w^{-l-1} \\
&=: A(z)[B(z), C(w)] : + (-1)^{|B||C|} : [A(z), C(w)] B(z) : \qquad (4.18)
\end{aligned}$$

Where the red portion becomes the second term in the last line and the blue portion becomes the first.

Example 4.1.7. *Suppose that $\mathfrak{g}$ is a Lie superalgebra with nondegenerate, invariant bilinear form $\langle \cdot, \cdot \rangle$, and $k \in \mathbb{C}$.*

- *Let $\{J^i\}_{i=1,\ldots,d}$ be a basis of $\mathfrak{g}$ with $J^i \in \mathfrak{g}_{\bar{0}}$ for $i \leq n$ and $J^i \in \mathfrak{g}_{\bar{1}}$ for $i > n$.*

- *Consider affinization $\hat{\mathfrak{g}}$ and module $\mathbb{V}^k(0)$, note it has PBW basis.*

- *Define operator T on $\mathbb{V}^k(0)$ by:*

$$T|0\rangle = 0 \qquad [T, J^\alpha_n] = nJ_{n-1} \qquad (4.19)$$

- *Define fields:*

$$J^i(z) = \sum_{n \in \mathbb{Z}} J^i_n z^{-n-1} \qquad (4.20)$$

  using bracket of modes, these fields have commutator:

$$[J^\alpha(z), J^\beta(w)] = [J^\alpha, J^\beta](w)\delta(z-w) + K\langle J^\alpha, J^\beta \rangle \partial_w \delta(z-w) \qquad (4.21)$$

By reconstruction, this defines vertex superalgebra structure $(\mathbb{V}^k(0), |0\rangle, T, Y)$, *which we call the universal affine vertex superalgebra associated to $\mathfrak{g}$, and denote by $\mathbb{V}^k(\mathfrak{g})$.*

This also defines vertex algebra structure on $\mathbb{L}^k(0)$, which we call the affine vertex superalgebra associated to $\mathfrak{g}$, and denote by $\mathbb{L}^k(\mathfrak{g})$.

As it turns out, affine vertex superalgebras are the prototypical example, as they demonstrate many of the properties present in all vertex superalgebras, and all known vertex superalgebras are given somehow by beginning with an affine vertex superalgebra and performing various constructions to obtain one vertex superalgebra from another.

We may perform a similar construction using the Virasoro algebra in place of an affine Lie algebra.

Example 4.1.8.

- *Consider the Vir module $\tilde{\mathbb{V}}^c(0)$.*
- *Define operator $T = L_{-1}$.*
- *Define field:*

$$T(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2} \tag{4.22}$$

By bracket of the modes, this has commutator:

$$[T(z), T(w)] = \frac{C}{12}\partial_w^3\delta(z-w) + 2T(w)\partial_w\delta(z-w) + \partial_w T(w)\cdot\delta(z-w) \tag{4.23}$$

by reconstruction, this defines vertex algebra $(\tilde{\mathbb{V}}^c(0), |0\rangle, T, Y)$, which we will call the universal Virasoro vertex algebra and denote by $\tilde{\mathbb{V}}^c(Vir)$.

This also defines vertex algebra structure on $\tilde{\mathbb{L}}^c(0)$, which we call the Virasoro vertex algebra and denote by $\tilde{\mathbb{L}}^c(Vir)$.

Because the Virasoro algebra plays the role of conformal symmetry in two dimensions, the Virasoro vertex algebra should be present in any vertex superalgebra consisting of symmetries for some 2 dimensional conformal field theory.

Definition 4.1.5. *Consider $\frac{1}{2}\mathbb{Z}$ graded vertex algebra $V = \bigoplus_{n\in\frac{1}{2}\mathbb{Z}} V_n$. We call V a conformal vertex algebra or vertex operator algebra if there exists conformal vector $\omega \in V_2$ with*

$$Y(\omega, z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2} \tag{4.24}$$

Such that:

1. *The modes L_n satisfy the defining relations of the Virasoro algebra.*
2. *$L_{-1} = T$.*
3. *L_0 is the 'grading operator' on V, $L_0|_{V_n} = nId$. We call the L_0 eigenvalues conformal weights.*

As before, affine vertex superalgebras are the prototypical example, as (nearly all) affine vertex superalgebras have a natural conformal structure.

Theorem 4.1.9 (Segal-Sugawara). *Let $k \neq -h^\vee$ where $h^\vee$ is the dual Coexeter number of $\mathfrak{g}$. Pick parity homogenous basis $\{J^i\}_{i=1,\ldots,d}$ of $\mathfrak{g}$ and dual basis $\{J_i\}_{i=1,\ldots,d}$ with respect to the invariant bilinear form $\langle\cdot,\cdot\rangle$. Define:*

$$J^i(z) = \sum_{n\in\mathbb{Z}} J^i_{(n)} z^{-n-1} \qquad\qquad J_i(z) = \sum_{n\in\mathbb{Z}} J_{i,(n)} z^{-n-1} \tag{4.25}$$

We define the following weight two vector:

$$\omega = \frac{1}{2(k+h^\vee)} \sum_{i=1}^{d} J_{i,(-1)} J^i_{(-1)} |0\rangle \tag{4.26}$$

Then ω is a conformal vector in $V^k(\mathfrak{g})$ with central charge:

$$c(k) = \frac{k \dim \mathfrak{g}}{k+h^\vee} \tag{4.27}$$

The Virasoro field has form:

$$T(z) = Y(\omega, z) = \frac{1}{2(k+h^\vee)} \sum_{i=1}^{d} :J_i(z)J^i(z): \tag{4.28}$$

Then every affine vertex superalgebra associated to simple Lie superalgbra is conformal away from critical level $k = -h^\vee$. Note that the mode L_0 is the Casimir of $\mathfrak{g}$, with modifications in higher degrees.

Proof. *Put $S = (k + h^\vee)\omega$ so $S(z) = \sum_{n\in\mathbb{Z}} S_n z^{-n-2} = (k + h^\vee)L(z)$. For $J \in \mathfrak{g}$, we claim that*

$$\begin{aligned} &S_n J_{(-1)}|0\rangle = 0 && n \geq 1 \\ &S_0 J_{(-1)}|0\rangle = (k + h^\vee)J_{(-1)}|0\rangle && \\ &S_{-1} J_{(-1)}|0\rangle = (k + h^\vee)J_{-2}|0\rangle && \end{aligned}$$

We note the following relations

$$S_n = \frac{1}{2}\sum_{i=1}^{d}\left(\sum_{m<0} J_{i\ (m)}J^i_{(n-m)} + (-1)^{|J^i||J_i|}\sum_{m\geq 0} J^i_{(n-m)}J_{i\ (m)}\right)$$

$$J^i_{(l)}J_{(-1)}|0\rangle = (-1)^{|J^i||J|}J_{(-1)}J^i_{(l)}|0\rangle + [J^i, J]_{(l-1)}|0\rangle + kl\delta_{l,1}\langle J^i, J\rangle|0\rangle$$

and similarly for lower indices J_i. In particular, we have the following for $l \geq 2$:

$$J^i_{(l)}J_{(-1)}|0\rangle = 0 \qquad J^i_{(1)}J_{(-1)}|0\rangle = k\langle J^i, J\rangle|0\rangle \qquad J^i_{(0)}J_{(-1)}|0\rangle = [J^i, J]_{(-1)}|0\rangle \tag{4.29}$$

and similarly for the lower indices J_i. Assume that J is homogenous. We consider the case $n \geq 2$. In this case, all terms in $S_n J_{(-1)}|0\rangle$ except those with $m = 0, 1$ vanish from the rightmost mode in S_n. The terms $m = 0, 1$ vanish from the action of $J^i_{(n-m)}$.

We consider the case $n = 1$. All terms vanish from the rightmost mode in S_n except those with $m = 1, 0$. The term with $m = 1$ again vanishes from the *action of of $J^i_{(n-m)}$. Recalling 3.2.8*

$$S_1 J_{(-1)}|0\rangle = \frac{1}{2}\sum_{i=1}^{d}(-1)^{|J^i||J_i|}J^i_{(1)}J_{i\ (0)}J_{(-1)}|0\rangle$$
$$= \frac{1}{2}\sum_{i=1}^{d}(-1)^{|J^i||J^i|}k\langle J^i,[J_i,J]\rangle|0\rangle = \frac{k}{2}\left\langle\sum_{i=1}^{d}(-1)^{|J^i||J_i|}[J^i,J_i],J\right\rangle|0\rangle$$
$$= \frac{k}{2}\left\langle\sum_{i=1}^{d}[J_i,J^i],J\right\rangle|0\rangle = 0 \quad (4.30)$$

We now consider the case $n = 0$. Again, all terms except those with $m = 0, 1, -1$ immediately vanish due to the above relations. We are left with

$$S_0 J_{(-1)}|0\rangle = \frac{1}{2}\sum_{i=1}^{d}\left(J_{i\ (-1)}J^i_{(1)} + (-1)^{|J_i||J^i|}\left(J^i_{(-1)}J_{i\ (1)} + J^i_{(0)}J_{i\ (0)}\right)\right)J_{(-1)}|0\rangle$$
$$= \frac{k}{2}\sum_{i=1}^{d}\left(J_{i\ (-1)}\langle J^i_{(-1)},J\rangle + J^i_{(-1)}\langle J,J_i\rangle\right)|0\rangle + h^\vee J_{(-1)}|0\rangle = (k+h^\vee)J_{(-1)}|0\rangle \quad (4.31)$$

where we note that the $m = 0$ term is exactly the Casimir constructed out of 0 *modes, so it acts as the Casimir does on the adjoint representation. Finally, we consider the case $n = -1$. All terms except those with $m = 1, 0, -1, -2$ vanish from the action of the rightmost mode. We are left with*

$$
\begin{aligned}
&S_{-1}J_{(-1)}|0\rangle \\
&= \frac{1}{2}\sum_{i=1}^{d}\left(J_{i\ (-2)}J^i_{(1)} + J_{i\ (-1)}J^i_{(0)} + (-1)^{|J^i||J_i|}\left(J^i_{(-1)}J_{i\ (0)} + J^i_{(-2)}J_{i\ (1)}\right)\right)J_{(-1)}|0\rangle \\
&\quad = kJ_{(-2)}|0\rangle + \frac{1}{2}\sum_{i=1}^{d}\left(J_{i\ (-1)}[J^i, J]_{(-1)} + (-1)^{|J^i||J_i|}J^i_{(-1)}[J_i, J]_{(-1)}\right)|0\rangle \\
&= kJ_{(-2)}|0\rangle + \frac{1}{2}\sum_{i,k=1}^{d}\left(J_{i\ (-1)}\langle[J^i, J], J_k\rangle J^k_{(-1)} + (-1)^{|J^i||J_i|+|J||J_i|}J^i_{(-1)}\langle[J^k, J], J_i\rangle J_{k\ (-1)}\right)|0\rangle \\
&\quad = kJ_{(-2)}|0\rangle + \frac{1}{2}\sum_{i,k=1}^{d}\left((-1)^{|J^i||J_i|}\langle J^k, [J_i, J]\rangle[J^i, J_k]_{(-2)}\right)|0\rangle \\
&\quad = kJ_{(-2)}|0\rangle + \frac{1}{2}\sum_{i=1}^{d}\left((-1)^{|J^i||J_i|}[J^i_{(0)}, [J_{i\ (0)}, J_{(-2)}]]\right)|0\rangle \\
&\quad = kJ_{(-2)}|0\rangle + \frac{1}{2}\sum_{i=1}^{d}(-1)^{|J^i||J_i|}J^i_{(0)}J_{i\ (0)}, J_{(-2)}|0\rangle = (k+h^\vee)J_{(-2)}|0\rangle \qquad (4.32)
\end{aligned}
$$

Using the OPE formula 4.1.3, these results gives us:

$$
\begin{aligned}
L(z)J(w) &= \sum_{n=-1}^{\infty}\frac{Y(L_nJ_{(-1)}|0\rangle, w)}{(z-w)^{n+2}} + :L(z)J(w): \\
&= \frac{J(w)}{(z-w)^2} + \frac{\partial_w J(w)}{z-w} + :L(z)J(w): \qquad (4.33)
\end{aligned}
$$

Using Wick theorem, we obtain the OPE:

$$
L(z)L(w) \sim \frac{c}{2(z-w)^4} + \frac{2L(w)}{(z-w)^2} + \frac{\partial_w L(w)}{z-w} \qquad (4.34)
$$

With $c = \frac{ksdim\mathfrak{g}}{k+h^\vee}$*. The mode commutation relations 4.1.4 then give us the* L_{-1} *derivative and* L_0 *grading properties from* $L(z)J(w)$*, and the Virasoro relations of modes from* $L(z)L(w)$*.*

We can now present a number of examples of vertex superalgebras that will be necessary for later constructions here. The simplest of these is the free bosons associated to a lattice, the typical Lie theoretic example of $\mathfrak{sl}(2)$, and finally the free fermionic counterpart of the free boson, which we may associate to a Clifford algebra.

Example 4.1.10. *Let L be a finite rank lattice with basis $\{h^i\}_{i=1,..,d}$ and symmetric bilinear form $\langle \cdot, \cdot \rangle : L \times L \to \mathbb{Z}$.*

Consider commutative Lie algebra $\mathfrak{h} = L \otimes_{\mathbb{Z}} \mathbb{C}$. The form $\langle \cdot, \cdot \rangle$ induces to a bilinear form on $\mathfrak{h}$, which we again denote by $\langle \cdot, \cdot \rangle$.

Then $\pi_L = \mathbb{V}^1(\mathfrak{h})$ is the Heisenberg or bosonic vertex algebra associated to the lattice L. It has generating fields $h^i(z)$ with relations:

$$h^i(z)h^j(w) \sim \frac{\langle h^i, h^j \rangle}{(z-w)^2} \tag{4.35}$$

In particular, if L is the rank one lattice generated by h with form $\langle h, h \rangle = 1$, then we call $\mathbb{V}^1(\mathfrak{h})$ simply the free boson.

Example 4.1.11. *The affine vertex algebras $\mathbb{V}^k(\mathfrak{sl}(2))$, $\mathbb{L}^k(\mathfrak{sl}(2))$ have generating fields $e(z), h(z), f(z)$ with nontrivial relations:*

$$e(z)f(w) \sim \frac{h(w)}{(z-w)} + \frac{k}{(z-w)^2} \qquad h(z)e(w) \sim \frac{2e(w)}{z-w} \tag{4.36}$$

$$h(z)f(w) \sim -\frac{2f(w)}{z-w} \qquad h(z)h(w) \sim \frac{2k}{(z-w)^2} \tag{4.37}$$

The Segal-Sugawara Virasoro field is given by:

$$T(z) = \frac{1}{2(k+2)} \left(\frac{1}{2} : h(z)h(z) : + : e(z)f(z) : + : f(z)e(z) : \right)$$

Example 4.1.12. *We introduce odd modes $\psi_{(n)}$ for $n \in \mathbb{Z}$ with nontrivial (anti)commutation relations*

$$[\psi_{(n)}, \psi_{(m)}] = \delta_{n+m+1} \tag{4.38}$$

The free fermionic Fock module F has basis:

$$\psi_{(n_1)} \dots \psi_{(n_m)} |0\rangle$$

with $n_1 < \cdots < n_m$. We define the field:

$$\psi(z) = \sum_{n\in\mathbb{Z}} \psi_{(n)} z^{-n-1}$$

with nontrivial OPEs given by:

$$\psi(z)\psi(w) \sim \frac{1}{z-w} \tag{4.39}$$

This has Virasoro field:

$$T(z) = -\frac{1}{2} : \psi\partial\psi : (z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2} \tag{4.40}$$

of central charge $c = \frac{1}{2}$ and we define the translation operator $T = L_{-1}$ and grading operator L_0, with respect to which ψ has weight $\frac{1}{2}$. By reconstruction, this defines conformal vertex superalgebra structure $(F, |0\rangle, T, Y)$, which we will denote again by F and call the Majorana free fermion.

Example 4.1.13. *We consider the tensor product of two Majorana free fermions $F_1 \otimes F_2$, denoting the generators by ψ_1 and ψ_2. We define:*

$$\psi(z) = \frac{1}{\sqrt{2}}(\psi_1(z) - i\psi_2(z)) \qquad \psi^\dagger(z) = \frac{1}{\sqrt{2}}(\psi_1(z) + i\psi_2(z)) \tag{4.41}$$

with nontrivial OPEs given by:

$$\psi(z)\psi^\dagger(w) \sim \frac{1}{z-w} \tag{4.42}$$

which we call the charged free fermions. This has family of Virasoro fields:

$$T(z) = -a : \psi(z)\partial\psi^\dagger(z) : -(1-a) : \psi^\dagger(z)\partial\psi(z) :$$

for $a \in [0,1]$, with central charge $1 - 12(a - \frac{1}{2})^2$. With respect to this conformal structure, ψ has conformal weight a and $\psi^\dagger$ has conformal weight $1-a$.

In particular, taking $a = 1$ and denoting $b = \psi$ and $c = \psi^\dagger$, we have

Virasoro field:

$$T(z) = - : b\partial c : (z)$$

with which b *has weight 1 and* c *has weight 0. We call this sVOA the* <u>*fermionic ghosts*</u> *and denote it by* bc.

Similarly, we denote by bc^{tw} *the charged free fermions with conformal structure corresponding to* $a = 0$, *with respect to which* b *has weight 0 and* c *has weight one.*

4.2 Modules

The conformal vertex superalgebra only plays the role of the *symmetries* of a 2 dimensional conformal field theory. Then we also need some notion for how such an algebra should act on other 'states'. This is again natural from the almost commutative algebra perspective.

Definition 4.2.1. *Let* $(V, |0\rangle, T, Y)$ *be a conformal vertex superalgebra. Then*

- *A* <u>*weak* V*-module*</u> *is* (M, Y_M) *such that*

  1. M *is a vector superspace.*

  $$M = M_{\overline{0}} \oplus M_{\overline{1}} \tag{4.1}$$

  2. $Y_M : V \otimes M \to M[[z^{\pm 1}]]$ *is even.*

  $$v \otimes m \mapsto Y_M(v, z)m = \sum_{n \in \mathbb{Z}} v_n m z^{-n-1} \tag{4.2}$$

  3. $Y_M(|0\rangle, z) = Id_M$.
  4. *For any* $v \in V$, $m \in M$,

  $$Y_M(v, z)m \in M((z))$$

  5.

  $$Y_M(Tv, z) = \partial_z Y_M(v, z)$$

6. *For parity homogenous $a, b \in V$:*

$$\begin{aligned}\delta((z_1 - z_2) - z_0)Y_M(a, z_1)Y_M(b, z_2)& \\ - (-1)^{|a||b|}\delta((-z_2 - z_1) - z_0)Y_M(b, z_2)Y_M(a, z_1)& \\ = \delta((z_1 - z_0) - z_2)Y_M(Y(a, z_0)b, z_2)&\end{aligned} \quad (4.3)$$

7. *If we denote:*

$$Y_M(\omega, z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$$

Then the L_n satisfy the Virasoro relations, with c is the central charge of V.

$$[L_n, L_m] = (m - n)L_{m+n} + \frac{m^3 - m}{12}\delta_{m+n,0}c \quad (4.4)$$

- *A generalized V-module is a weak V-module such that:*

1. *M is a $\mathbb{C}$-graded vector space.*

$$M = \coprod_{h\in\mathbb{C}} M_{[h]} \quad (4.5)$$

2. *For $i = \bar{0}, \bar{1}$,*

$$M_i = \coprod_{h\in\mathbb{C}} M_{i,[h]} \qquad M_{i,[h]} := M_i \cap M_{i,[h]} \quad (4.6)$$

3. *$M_{[h]}$ is the generalized eigenspace of L_0 with generalized eigenvalue h.*

- *A grading-restricted generalized V module is a lower bounded generalized V module M with $M_{[h]} < \infty$ for any $h \in \mathbb{C}$.*

In the following, we shall refer to a generalized module simply as a module. We can now introduce the standard algebraic notions for vertex superalgebra modules.

Remark 4.2.1. *A $\mathbb{V}^k(\mathfrak{g})$ module is equivalent to a smooth $\hat{\mathfrak{g}}$ module, with Y_M given by the fields 4.20 and 4.16.*

Definition 4.2.2. *Let V be a vertex superalgebra. A V module M is indecomposable if it is nonzero and can not be written as a direct sum of two nonzero submodules.*

Definition 4.2.3. *For (M_1, Y_{M_1}), (M_2, Y_{M_2}) generalized V modules, a parity-homogenous homomorphism $f : M_1 \to M_2$ of vertex superalgebra modules is parity homogenous linear map satisfying:*

$$f(Y_{M_1}(v,z)m) = (-1)^{|f||v|}Y_{M_2}(v,z)f(m) \tag{4.7}$$

for all parity homogenous $v \in V$ and $m \in M$. A homomorphism of generalized modules is a sum of even and odd homomorphisms.

For the physically relevant quantities: the 'correlation functions', it is also necessary to have a notion of product for two states in different modules. This is given by an intertwining operator.

Definition 4.2.4. *For M_1, M_2, M_3 modules for conformal vertex superalgebra V, a parity homogenous intertwining operator of type $\binom{M_3}{M_1\ M_2}$ is parity homogenous linear map:*

$$\mathcal{Y} : M_1 \otimes M_2 \to M_3[\log z]\{z\} \tag{4.8}$$

$$m_1 \otimes m_2 \mapsto \mathcal{Y}(m_1,z)m_2 = \sum_{n\in\mathbb{C}}\sum_{k\in\mathbb{N}}(w_1)^{\mathcal{Y}}_{n:k}w_2 z^{-n-1}(\log z)^k \tag{4.9}$$

satisfying the following conditions:

1. *For $m_1 \in M_1$, $m_2 \in M_2$, $n \in \mathbb{C}$:*

$$(m_1)^{\mathcal{Y}}_{n+m,k}m_2 = 0$$

for sufficiently large $m \in \mathbb{N}$, independent of k.

2. For $m_1 \in M_1$ and $v \in V$:

$$(-1)^{|\mathcal{Y}||v|}\delta((z_1 - z_2) + z_0)Y_{M_3}(v, z_1)\mathcal{Y}(m_1, z_2) \\ - (-1)^{|m_1||v|}\delta((z_2 - z_1) - z_0)\mathcal{Y}(m_1, z_2)Y_{M_2}(v, z_1) \\ = \delta((z_1 - z_0) - z_2)\mathcal{Y}(Y_{M_1}(v, z_0)m_1, z_2) \quad (4.10)$$

for parity homogenous $v \in V$, $m_1 \in M_1$.

3. For any $m_1 \in W_1$:

$$\mathcal{Y}(Tm_1, z) = \partial_z \mathcal{Y}(m_1, z)$$

An intertwining operator of type $\binom{M_3}{M_1\ M_2}$ is a sum of an even and odd intertwining operator of this type.

There is a natural template which we would hope to use to construct monoidal and braiding structures on categories of conformal vertex algebra modules. Verifying that such structures do exist becomes one of the most challenging problems in the study of VOAs.

These notions were introduced by Huang, Lepowsky and Zhang in[41]-[48], and we call this the HLZ tensor category structure.

We would like to think of products of intertwining operators

$$\mathcal{Y}_1(v_1, z_1)...\mathcal{Y}_m(v_m, z_m)|0\rangle$$

as giving us some physical state depending on insertion points $z_1, ..., z_m \in \mathbb{C}$. However, $\mathcal{Y}(v, z)m$ need only live in some completion of the 'target' space of the intertwining operator $\mathcal{Y}$.

Definition 4.2.5. *For $M = \bigoplus_{h \in \mathbb{C}} M_{[h]}$, we define the algebraic closure to be:*

$$\overline{M} = \Pi_{h \in \mathbb{C}} M_{[h]} \quad (4.11)$$

This is naturally a vector superspace, with $\overline{M}_i = \overline{M_i}$ for $i = \overline{0}, \overline{1}$. Denote the projection by:

$$\pi_h : \overline{M} \to M_{[h]}$$

Definition 4.2.6. *For M_1, M_2, M_3 modules for conformal vertex superalgebra V, a* <u>*parity homogenous $P(z)$ intertwining map*</u> *of type $\binom{M_3}{M_1\ M_2}$ is parity homogenous linear map:*

$$I : M_1 \otimes M_2 \to \overline{M_3} \tag{4.12}$$

satisfying the following conditions:

1. *For $m_1 \in M_1$, $m_2 \in M_2$, $n \in \mathbb{C}$:*

   $$\pi_{n-m}(I(m_1 \otimes m_2)) = 0$$

   For sufficiently large $m \in \mathbb{N}$

2. $$\begin{aligned}(-1)^{|I||v|}\delta((x_1 - z) - x_0)Y_{M_3}(v, x_1)I(m_1 \otimes m_2)\\ - (-1)^{|v||m_1|}\delta((z - x_1) + x_0)I(m_1 \otimes Y_{W_2}(v, x_1)m_2)\\ = -\delta((x_1 - x_0) - z)I(Y_{M_1}(v, x_0)m_1 \otimes m_2)\end{aligned} \tag{4.13}$$

   for $m_2 \in M_2$ and parity homogenous $v \in V$, $m_1 \in M_1$.

A <u>*$P(z)$-intertwining map*</u> *of type $\binom{M_3}{M_1\ M_2}$ is a sum of an even and odd $P(z)$ intertwining map of this type.*

The physically relevant objects are not the 'states' themselves: the vectors of in a module. Rather it is their corresponding fields: the intertwining operators. Then the tensor product of vector spaces underlying modules (as we would expect for Lie algebras) is too large. Here we arrive at the appropriate notion of tensor product.

Definition 4.2.7. *Suppose that M_1, M_2 are V modules in the category $\mathcal{C}$ and $z \in \mathbb{C}^\times$. A* <u>*$P(z)$ tensor product*</u> *of W_1 and W_2 in $\mathcal{C}$ is*

1. *A module $M_1 \boxtimes_{P(z)} M_2 \in \mathcal{C}$.*

2. *An even $P(z)$ intertwining map $\boxtimes_{P(z)}$ of type $\binom{M_1 \boxtimes_{P(z)} M_2}{M_1\ M_2}$.*

3. *For any $M_3 \in \mathcal{C}$ and $P(z)$ intertwining map I of type $\binom{M_3}{M_1\ M_2}$, there exists unique homomorphism $\eta_I : M_1 \boxtimes_{P(z)} M_2 \to M_3$ so that the following commutes:*

$$\begin{array}{ccc} \overline{M_1 \boxtimes_{P(z)} M_2} & \xrightarrow{\overline{\eta_I}} & \overline{M_3} \\ {\scriptstyle \boxtimes_{P(z)}}\big\uparrow & \nearrow_{I} & \\ M_1 \otimes M_3 & & \end{array}$$

Where $\overline{\eta_I}$ is the natural extension of η_I to $\overline{M_1 \boxtimes_{P(z)} M_2}$.

If we have morphisms $f_1 : M_1 \to M_1'$ and $f_2 : M_2 \to M_2'$, then $\boxtimes_{P(z)} \circ (f_1 \otimes f_2)$ is an intertwining operator of type $\binom{W_1' \boxtimes_{P(z)} W_2'}{W_1\ W_2}$. Define

$$f_1 \boxtimes_{P(z)} f_2\, W_1 \boxtimes_{P(z)} W_2 \to W_1' \boxtimes_{P(z)} W_2'$$

to be the unique morphism induced by the universal property of the $P(z)$ tensor product.

We may now introduce the means by which we might arrive at a natural braided tensor category structure on categories of vertex superalgebra modules, and sufficient conditions for such a structure to arise.

Remark 4.2.2. *When it is possible to obtain the structure of a monoidal category on $\mathcal{C}$, we pick in particular the $P(1)$ tensor product, which we simply denote by $\boxtimes$.*

Definition 4.2.8. *For V a conformal vertex algebra and M a V-module, a* <u>*twist*</u> *on M is defined by:*

$$\theta_M = e^{2\pi i L_0}$$

Definition 4.2.9. *Let V be a conformal vertex superalgebra and M be a V module. We define:*

$$C_2(M) = \{v_{(-2)}m | v \in V,\ m \in M\} \tag{4.14}$$

We say that M is <u>*C_2 cofinite*</u> *if $C_2(M)$ has finite codiminsion in M.*

Definition 4.2.10. *We say that a conformal vertex algebra V is* <u>*rational*</u> *if*

1. *There are only finitely many irreducible V modules.*

2. *Every V module is a finite direct sum of irreducible V modules.*

3. *All fusion rules for V are finite.*

Theorem 4.2.3. *[39] Let V be a simple conformal vertex algebra satisfying:*

1. $V_n = 0$ *for* $n < 0$, $V_0 = \mathbb{C}|0\rangle$, *and the contragradient* $V^\vee \simeq V$.

2. *V is rational.*

3. *V is C_2 cofinite.*

Then the category of V modules is a modular tensor category with respect to the HLZ structure.

4.2.1 Fock modules

We begin with the simplest examples of braided tensor category structure for vertex algebras: those associated to the Heisenberg vertex algebras π_L.

Example 4.2.4. *Suppose that L is a finite rank lattice with symmetric bilinear form $\langle \cdot, \cdot \rangle : L \times L \to \mathbb{Z}$, $\mathfrak{h} = L \otimes_{\mathbb{Z}} \mathbb{C}$ and π_L the associated Heisenberg vertex algebra. Let $\lambda \in \mathfrak{h}$, and identify λ with $\langle \lambda, \cdot \rangle \in \mathfrak{h}^*$. Then we define the* <u>*Fock modules*</u>*:*

$$\pi_L(\lambda) = \mathbb{V}^1(\lambda)$$

The $\pi_L(\lambda)$ are form a complete list of isomorphism classes of simple π_L modules, and the gradation is given by $\deg|\lambda\rangle = \frac{\langle \lambda, \lambda \rangle}{2}$ *and for* $h \in \mathfrak{h}$, $\deg h_{(n)} = -n$.

Proposition 4.2.5. *[27] Let $\lambda, \mu \in \mathfrak{h}^*$. Then the tensor product of $\pi_L(\lambda)$ and $\pi_L(\mu)$ exists, and we have the isomorphism:*

$$\pi_L(\lambda) \boxtimes \pi_L(\mu) \simeq \pi_L(\lambda + \mu) \tag{4.15}$$

Theorem 4.2.6. *[17][25] Let $\mathcal{C}$ be the semisimple category generated by the Fock modules $\pi_L(\lambda)$. Then $\mathcal{C}$ is a vertex tensor category with respect to the HLZ structure.*

4.2.2 Admissible level $\mathbb{L}^k(\mathfrak{sl}(2))$

Definition 4.2.11. *We define the following categories of affine vertex algebra modules*

- $\mathbb{L}^k(\mathfrak{g})$*-wtmod: the full subcategory of* $\mathbb{V}^k(\mathfrak{g})$*-wtmod whose objects are* $\mathbb{L}^k(\mathfrak{g})$ *modules.*
- $\mathbb{L}^k(\mathfrak{g})$*-wtmod*$_{\geq 0}$*:* $\mathbb{L}^k(\mathfrak{g})$*-wtmod* $\cap$ $\mathbb{V}^k(\mathfrak{g})$*-wtmod*$_{\geq 0}$.
- $\mathbb{L}^k(\mathfrak{g})$*-wtmod*$_{KL}$*:* $\mathbb{L}^k(\mathfrak{g})$*-wtmod* $\cap$ $\mathbb{V}^k(\mathfrak{g})$*-wtmod*$_{KL}$.

This brings us to the details most relevant for the study of representation theory of $\mathbb{L}^1(D(2,1;-\frac{v}{w}))$.

Definition 4.2.12. *We say that the level $k \in \mathbb{C}$ is admissible if $k = \frac{u}{v} - 2$ with $(u,v) = 1$ and $u \in \mathbb{Z}_{\geq 2}$ and $v \in \mathbb{Z}_{\geq 1}$. In the following, we will assume that k is an admissible level.*

Remark 4.2.7. *Let $\omega \in P^\vee$ be the fundamental weight of $\mathfrak{sl}(2)$. For $\lambda \in \mathbb{C}$, we identify λ with $\langle \lambda\omega, \cdot \rangle \in \mathfrak{h}^*$. Define:*

$$\lambda_{r,s} = r - 1 - \frac{u}{v} s \tag{4.16}$$

Remark 4.2.8. *We note that for modules of highest weight λ, the form of the Segal-Sugawara conformal vector gives minimal L_0 eigenvalue:*

$$\Delta = \frac{(\lambda+1)^2 - 1}{4(k+2)} \tag{4.17}$$

If $|\lambda, \Delta\rangle$ is a state with weight λ and conformal weight Δ, then weights and conformal weights of the spectral flow are given by:

$$h_0(\sigma^t)^*|\lambda, \Delta\rangle = (\lambda + tk)(\sigma^t)^*|\lambda, \Delta\rangle \tag{4.18}$$

$$L_0(\sigma^t)^*|\lambda, \Delta\rangle = \left(\Delta + \frac{1}{2}t\lambda + \frac{1}{4}t^2 k\right)(\sigma^t)^*|\lambda, \Delta\rangle \tag{4.19}$$

We establish notation and give an overview of each of the categories defined above in the case of admissible level $\mathfrak{sl}(2)$, including their simple objects, tensor category structure, and fusion products.

Theorem 4.2.9. *[14] For admissible level k, the category $\mathbb{L}^k(\mathfrak{sl}(2))$-wtmod$_{KL}$ is a semisimple rigid braided tensor category, and any irreducible object is isomorphic to one of*

$$\mathcal{L}_{r,0} = \mathbb{L}^k(\lambda_{r,0}) \tag{4.20}$$

for $r \in \{1, ..., u-1\}$.

Definition 4.2.13. *We introduce the notation:*

$$\mathcal{D}^+_{r,s} = L^k(L(\lambda_{r,s})) \tag{4.21}$$

Define the modules $\mathcal{D}^-_{r,s}$ to be the conjugates of $\mathcal{D}^+_{r,s}$, obtained by twisting the action of $\mathbb{L}^k(\mathfrak{sl}(2))$ by the Weyl reflection of $\mathfrak{sl}(2)$.

Theorem 4.2.10. *[2] For admissible level k, any irreducibles in $\mathbb{L}^k(\mathfrak{sl}(2))$-wtmod$_{\geq 0}$ is isomorphic to one of*

$$\mathcal{L}_{r,0} \qquad \mathcal{D}^\pm_{r,s} \qquad \mathcal{E}_{\lambda;\Delta_{r,s}} = L^k(E(\lambda, \Delta_{r,s})) \tag{4.22}$$

for $r \in \{1, ..., u-1\}$ and $s \in \{1, ..., v-1\}$ and $\lambda \neq \lambda_{r,s} \mod 2\mathbb{Z}$.

Theorem 4.2.11. *[2] For admissible level k, any irreducible object in $\mathbb{L}^k(\mathfrak{sl}(2))$-wtmod is isomorphic to one of the spectral flows*

$$\sigma^t(\mathcal{L}_{r,0}) \qquad \sigma^t(\mathcal{D}^\pm_{r,s}) \qquad \sigma^t(\mathcal{E}_{\lambda,\Delta_{r,s}}) \tag{4.23}$$

for $t \in \mathbb{Z}$,(with some redundancy), where we define $\sigma^t = \sigma^*_{t\omega}$.

Proposition 4.2.12. *[1] There exist indecomposable modules $\sigma^t(\mathcal{E}^\pm_{r,s})$ satisfying the nonsplit exact sequences:*

$$0 \longrightarrow \sigma^t(\mathcal{D}^+_{r,s}) \longrightarrow \sigma^t(\mathcal{E}^+_{r,s}) \longrightarrow \sigma^t(\mathcal{D}^-_{u-r,v-s}) \longrightarrow 0 \tag{4.24}$$

$$0 \longrightarrow \sigma^t(\mathcal{D}^-_{r,s}) \longrightarrow \sigma^t(\mathcal{E}^-_{r,s}) \longrightarrow \sigma^t(\mathcal{D}^+_{u-r,v-s}) \longrightarrow 0 \tag{4.25}$$

for $r \in \{1, ..., u-1\}$, $s \in \{1, ..., v-1\}$. There exist indecomposable modules

$\sigma^t(\mathcal{P}^{\pm}_{r,s})$ satisfying the nonsplit exact sequences:

$$0 \longrightarrow \sigma^t(\mathcal{E}^{+}_{r,s}) \longrightarrow \sigma^t(\mathcal{P}^{+}_{r,s}) \longrightarrow \sigma^{t+1}(\mathcal{E}^{+}_{r,s+1}) \longrightarrow 0 \tag{4.26}$$

$$0 \longrightarrow \sigma^t(\mathcal{E}^{-}_{r,s}) \longrightarrow \sigma^t(\mathcal{P}^{-}_{r,s}) \longrightarrow \sigma^{t+1}(\mathcal{E}^{-}_{r,s+1}) \longrightarrow 0 \tag{4.27}$$

for $r \in \{1, ..., u-1\}$ and $s \in \{1, ..., v-2\}$.

Theorem 4.2.13. *[7] For admissible k, $\mathbb{L}^k(\mathfrak{sl}(2))$-wtmod is finite length and any indecomposable projective module is isomorphic to one of*

$$\begin{array}{lllll} \sigma^t(\mathcal{E}_{\lambda;\Delta_{r,s}}) & \lambda \neq \lambda_{r,s} \mod 2\mathbb{Z} & r \in \{1, ..., u-1\} & s \in \{1, ..., v-1\} \\ \sigma^t(\mathcal{P}^{\pm}_{r,s}) & & r \in \{1, ..., u-1\} & s \in \{1, ..., v-2\} \end{array}$$

Theorem 4.2.14. *[6] $\mathbb{L}^k(\mathfrak{sl}(2))$-wtmod is a vertex tensor category, with the following fusion products when $v > 1$.*

$$\mathcal{L}_{r,0} \boxtimes \mathcal{L}_{r',0} \simeq \bigoplus_{r''=1}^{u-1} N^{(u)\ \ r''}_{r,r'} \mathcal{L}_{r'',0} \tag{4.28}$$

$$\mathcal{L}_{r,0} \boxtimes \mathcal{D}^{\pm}_{r',s'} \simeq \bigoplus_{r''=1}^{u-1} N^{(u)\ \ r''}_{r,r'} \mathcal{D}^{\pm}_{r'',s'} \tag{4.29}$$

$$\mathcal{L}_{r,0} \boxtimes \mathcal{E}_{\lambda;\Delta_{r',s'}} \simeq \bigoplus_{r''=1}^{u-1} N^{(u)\ \ r''}_{r,r'} \mathcal{E}_{\lambda+\lambda_{r,0};\Delta_{r'',s'}} \tag{4.30}$$

Where we define:

$$N^{Vir\ \ (r'',s'')}_{(r,s)(r',s')} = N^{(u)\ \ r''}_{r,r'} N^{(v)\ \ s''}_{s,s'} \tag{4.31}$$

$$N^{(u)\ \ r''}_{r,r'} = \begin{cases} 1 & \text{if } |r-r'|+1 \leq r'' \leq \min\{r+r'-1, 2u-r-r'-1\} \\ & \text{and } r+r'+r'' \text{ is odd} \\ 0 & \text{otherwise} \end{cases} \tag{4.32}$$

We make note of the following identities:

$$N^{(u)\ r''}_{1,r'} = \delta_{r',r''} \qquad\qquad N^{(u)\ u-r''}_{r,u-r'} = N^{(u)\ r''}_{r,r'} \tag{4.33}$$

and for $r' \neq 1, u-1$:

$$N^{(u)\ r''}_{2,r'} = \delta_{r'+1,r''} + \delta_{r'-1,r''} \tag{4.34}$$

Remark 4.2.15. *Notice that many properties of irreducible modules in $\mathbb{L}^k(\mathfrak{sl}(2))$-wtmod, including generating conformal weights and the forms of some fusion products, depend only on their coherent family and not on the particular form of the module.*

It is then practical for us to introduce some notation so that these forms can be treated uniformly, and specify these properties. We denote:

$$\sigma^t(M_{\lambda,\Delta_{r,s}}) \tag{4.35}$$

where M stands in for the labels $\mathcal{L}, \mathcal{D}^\pm, \mathcal{E}$ and the labels $\lambda, \Delta_{r,s}$ are as in the following table.

	$\mathcal{L}$	$\mathcal{D}^\pm$	$\mathcal{E}$
λ	$\lambda_{r,0}$	$\lambda_{r,s}$ $s \neq 0$	$\lambda \in \mathbb{R}/2\mathbb{Z}$ $\quad \lambda \neq \lambda_{r,s} \mod 2$
Δ	$\Delta_{r,0}$	$\Delta_{r,s}$ $s \neq 0$	$\Delta_{r,s}$

We denote:

$$\Delta(\sigma^t(M_{\lambda,\Delta_{r,s}})) = \Delta_{r,s} + \frac{1}{2}t\lambda + \frac{1}{4}t^2 k \in \mathbb{C}/\mathbb{Z} \tag{4.36}$$

to be the conformal weight of any vector in $\sigma^t(M_{\lambda,\Delta_{r,s}})$ modulo integer, in this case calculated using the conformal weight of a generating vector. The following fusion products also depend only on the coherent family.

$$\mathcal{L}_{r,0} \boxtimes \sigma^t(M_{\lambda,\Delta_{1,s}}) \simeq \sigma^t(M_{\lambda+r-1,\Delta_{r,s}}) \tag{4.37}$$

$$\mathcal{L}_{r,0} \boxtimes \sigma^t(M_{\lambda,\Delta_{u-1,s}}) \simeq \sigma^t(M_{\lambda+r-1,\Delta_{u-r,s}}) \tag{4.38}$$

$$\mathcal{L}_{2,0} \boxtimes \sigma^t(M_{\lambda,\Delta_{1,s}}) \simeq \sigma^t(M_{\lambda+1,\Delta_{2,s}}) \tag{4.39}$$

$$\mathcal{L}_{2,0} \boxtimes \sigma^t(M_{\lambda,\Delta_{u-1,s}}) \simeq \sigma^t(M_{\lambda-1,\Delta_{u-2,s}}) \tag{4.40}$$

and for $r \neq 1, u-1$

$$\mathcal{L}_{2,0} \boxtimes \sigma^t(M_{\lambda,\Delta_{r,s}}) \simeq \sigma^t(M_{\lambda-1,\Delta_{r-1,s}}) \oplus \sigma^t(M_{\lambda+1,\Delta_{r+1,s}}) \tag{4.41}$$

Proposition 4.2.16. *We have the following isomorphism as $\mathbb{L}^1(\mathfrak{sl}(2)) \otimes \mathbb{L}^1(\mathfrak{sl}(2))$ modules*

$$bc^{\otimes 2} \simeq \mathcal{L}^1_{1,0} \otimes \mathcal{L}^1_{1,0} \oplus \mathcal{L}^1_{2,0} \otimes \mathcal{L}^1_{2,0}$$

Proof. *Denote generators of $bc^{\otimes 2}$ by b_1, c_1 and b_2, c_2. We give generators for $\mathcal{L}^1_{1,0} \otimes \mathcal{L}^1_{1,0}$ as:*

$$h_1 =: b_1c_1 : + : b_2c_2 : \qquad e_1 =: b_1b_2 : \qquad f_1 = - : c_1c_2 : \tag{4.42}$$

$$h_2 =: b_1c_1 : - : b_2c_2 : \qquad e_2 =: b_1c_2 : \qquad f_2 = - : c_1b_2 : \tag{4.43}$$

It can be confirmed using Wick's theorem that these satisfy generating relations for $\mathfrak{sl}(2)$. By uniqueness from reconstruction, this gives a $\mathbb{L}^1(\mathfrak{sl}(2)) \otimes \mathbb{L}^1(\mathfrak{sl}(2))$ subalgebra of $bc^{\otimes 2}$.

We also check with Wick's theorem that b_1, c_1, b_2, c_2 transform in the tensor product of two copies of the standard representation of $\mathfrak{sl}(2)$. Then $bc^{\otimes 2}$ has $\mathbb{L}^1(\mathfrak{sl}(2)) \otimes \mathbb{L}^1(\mathfrak{sl}(2))$ submodule $\mathcal{L}^1_{1,0} \otimes \mathcal{L}^1_{1,0} \oplus \mathcal{L}^1_{2,0} \otimes \mathcal{L}^1_{2,0}$.

Since $bc^{\otimes 2}$ is generated by this submodule under the action of $\mathbb{L}^1(\mathfrak{sl}(2)) \otimes \mathbb{L}^1(\mathfrak{sl}(2))$, we conclude the isomorphism.

4.2.3 Virasoro minimal models

We now give an overview of results for the Virasoro vertex algebra at central charges relevant for our work at admissible level. For $\mathfrak{sl}(2)$ admissible levels $k = \frac{u}{v} - 2$ with $v \neq 1$, consider the central charge values:

$$c_{u,v} = 13 - 6\left(\frac{u}{v} + \frac{v}{u}\right) = 13 - 6\left(k + 2 + \frac{1}{k+2}\right) \tag{4.44}$$

For such admissible levels k, we introduce the notation:

$$\mathbb{V}^k(\mathit{Vir}) := \tilde{\mathbb{V}}^{c_{u,v}}(\mathit{Vir}) \qquad\qquad \mathbb{L}^k(\mathit{Vir}) := \tilde{\mathbb{L}}^{c_{u,v}}(\mathit{Vir}) \tag{4.45}$$

Definition 4.2.14. *Denote by* $\mathcal{O}_k^{fin}$ *the category of finitely generated* $\mathbb{L}^k(\mathit{Vir})$ *modules.*

Theorem 4.2.17. *Let* $k = \frac{u}{v} - 2$ *be a nonintegral* $\mathfrak{sl}(2)$ *admissible level. Then* $\mathbb{L}^k(\mathit{Vir})$ *is rational [65] and* C_2 *cofinite [28]. In particular, by 4.2.3,* $\mathcal{O}_k^{fin}$ *is a modular tensor category. The modules*

$$\mathcal{M}^k_{r,s} = \tilde{\mathbb{L}}^{c_{u,v}}(h_{r,s}) \tag{4.46}$$

are representatives for a complete list of isomorphism classes of irreducibles in $\mathcal{O}_k^{fin}$*, where we define:*

$$h_{r,s} = \frac{(vr-us)^2-(v-u)^2}{4vu} \qquad 1 \le r \le u-1, \quad 1 \le s \le v-1 \tag{4.47}$$

Fusion products are given by:

$$\mathcal{M}_{r,s} \boxtimes \mathcal{M}_{r',s'} = \sum_{r''=1}^{u-1} \sum_{s''=1}^{v-1} N^{\mathit{Vir}\ (r'',s'')}_{(r,s)(r',s')} \mathcal{M}_{r'',s''} \tag{4.48}$$

Definition 4.2.15. *Denote by* $\mathcal{O}_{k,L}^{fin}$ *the full subcategory of* $\mathcal{O}_k^{fin}$ *whose objects are direct sums of the modules* $\mathcal{M}^k_{r,1}$.

Theorem 4.2.18. *Let* $k = \frac{u}{v} - 2$ *be a nonintegral* $\mathfrak{sl}(2)$ *admissible level. Then* $\mathcal{O}_{k,L}^{fin}$ *is a rigid braided tensor subcategory of* $\mathcal{O}_k^{fin}$ *and the* $\mathcal{M}^k_{r,1}$ *are representatives for a complete list of isomorphism classes of irreducibles.*

Proof. $\mathcal{O}_{k,L}^{fin}$ *is a full subcategory of a modular tensor category, and closed under direct sum, duals, and fusion products.*

Chapter 5

Constructing new vertex algebras

It is generally quite difficult to find explicit examples of vertex superalgebras. In order to obtain more exotic algebras than those already constructed, it is desirable to have some means to construct new examples of vertex superalgebras from known ones. Here, I give an overview of some of these constructions, and details relevant to study the representation theory of $\mathbb{L}^1(D(2,1;-\frac{v}{w}))$.

5.1 The coset construction

Definition 5.1.1. *Suppose that we have vertex algebra V with vertex subalgebra $W \subset V$. We define the coset or commutant as:*

$$Com(W,V) = \{v \in V | \forall m,n \in \mathbb{Z},\ w \in W\ [w_{(m)}, v_{(n)}] = 0\}$$

Note that $|0\rangle \in Com(W,V)$, T preserves $Com(W,V)$, and if $v \in Com(W,V)$, then $Y(v,z)$ preserves the subspace $Com(W,V)$. Then $Com(W,V)$ is a vertex subalgebra of V.

Theorem 5.1.1. *[38][4] Suppose that $k = \frac{u}{v} - 2$ is an admissible level for $\mathfrak{sl}(2)$, and consider $\mathbb{L}^{k-1}(\mathfrak{sl}(2)) \otimes \mathbb{L}^1(\mathfrak{sl}(2))$ and the subalgebra $\mathbb{L}^k(\mathfrak{sl}(2))$ generated by the diagonal action of $\hat{\mathfrak{g}}$ on the vacuum vector $|0\rangle_{k-1} \oplus |0\rangle_1$. Then*

- *For l satisfying*
$$\frac{1}{k+2}+\frac{1}{l+2}=1 \tag{5.1}$$
we have the following isomorphism of conformal vertex algebras:
$$\mathbb{L}^l(Vir) \simeq Com(L^k(\mathfrak{sl}(2)), \mathbb{L}^{k-1}(\mathfrak{sl}(2)) \otimes \mathbb{L}^1(\mathfrak{sl}(2))) \tag{5.2}$$

- $\mathbb{L}^k(\mathfrak{sl}(2)) \otimes \mathbb{L}^l(Vir)$ *is a vertex subalgebra of* $\mathbb{L}^{k-1}(\mathfrak{sl}(2)) \otimes \mathbb{L}^1(\mathfrak{sl}(2))$

- *We have the following isomorphism of* $\mathbb{L}^k(\mathfrak{sl}(2)) \otimes \mathbb{L}^l(Vir)$ *modules:*
$$\mathcal{L}^{k-1}_{r,0} \otimes \mathcal{L}^1_{r',0} \simeq \bigoplus_{\substack{r''=1 \\ r''-r-r'-1 \ even}}^{u-1} \mathcal{L}^k_{r'',0} \otimes \mathcal{M}^l_{r'',r} \tag{5.3}$$

5.2 Hamiltonian reduction

In physics, we often run into theories with non-physical 'gauge' symmetries. These redundant symmetries can be removed via Hamiltonian reduction, which, can be realized for vertex superalgebras as a cohomology. We will only need this construction in the simplest case of $\mathbb{L}^k(\mathfrak{sl}(2))$.

Definition 5.2.1. *For affine vertex algebra $\mathbb{L}^k(\mathfrak{sl}(2))$ and $\mathbb{L}^k(\mathfrak{sl}(2))$ module M, consider the vertex algebra $C^\bullet_k(\mathfrak{sl}(2)) = \mathbb{L}^k(\mathfrak{sl}(2)) \otimes bc$, and $C^\bullet_k(\mathfrak{sl}(2))$ module $C^\bullet_k(M) = M \otimes bc$, each with additional charge gradation given by char $b = -$char $c = 1$. Define the odd field*
$$Q_s(z) = e(z)b(z) + \frac{\partial_z^s b(z)}{s!} \tag{5.1}$$
The OPE $Q_s(z)Q_s(w)$ is regular at $z=0$, so $Q_{s\ (0)}^2 = 0$. Then $C^\bullet_k(M)$ is a complex with differential $Q_{s\ (0)}$. We denote the cohomology of $(C^\bullet_k(M), Q_{s\ (0)})$ as follows, and introduce in particular notation in the case $M = \mathbb{L}^k(\mathfrak{sl}(2))$:
$$H^\bullet_{k,s}(M) \qquad\qquad \mathcal{W}_k(\mathfrak{sl}(2)) = H^0_{k,0}(\mathfrak{sl}(2)) \tag{5.2}$$
$\mathcal{W}_k(\mathfrak{sl}(2))$ obtains the structure of a $\mathbb{Z}$ graded vertex algebra, called the <u>*$\mathcal{W}$-algebra*</u>

associated to $\mathfrak{sl}(2)$ at level k, and $H^0_{k,s}(M)$ is naturally a $\mathcal{W}_k(\mathfrak{sl}(2))$ module.

Theorem 5.2.1. *[35]Suppose that $k = \frac{u}{v} - 2$ is a non-integral admissible level for $\mathfrak{sl}(2)$. We have the following isomorphism of conformal vertex algebras:*

$$\mathcal{W}_k(\mathfrak{sl}(2)) \simeq \mathbb{L}^k(\mathit{Vir}) \tag{5.3}$$

and $\mathbb{L}^k(\mathit{Vir})$ module isomorphisms:

$$H^0_{k,s}(\mathcal{L}_{r,0}) \simeq \mathcal{M}^k_{r,s+1} \tag{5.4}$$

There is an appropriate sense in which tensoring by an affine Lie algebra at level one commutes with Hamiltonian reduction. Applying this to the coset result 5.1.1, we obtain the following theorem.

Theorem 5.2.2. *[3] Suppose that k and l non-integral admissible levels for $\mathbb{L}^k(\mathfrak{sl}(2))$ satisfying:*

$$\frac{1}{k+2} + \frac{1}{l+2} = 1 \tag{5.5}$$

Then we have the following isomorphism of conformal vertex algebras:

$$\mathbb{L}^{k-1}(\mathit{Vir}) \otimes \mathbb{L}^1(\mathfrak{sl}(2))^{tw} \simeq \bigoplus_{\substack{r=1 \\ r\ odd}}^{u-1} \mathcal{M}^k_{r,1} \otimes \mathcal{M}^l_{r,1} \tag{5.6}$$

and the following isomorphism as $\mathbb{L}^k(\mathit{Vir}) \otimes \mathbb{L}^l(\mathit{Vir})$ modules:

$$\mathcal{M}^{k-1}_{s,s'} \otimes {\mathcal{L}^1_{p,0}}^{tw} \simeq \bigoplus_{\substack{r=1 \\ r-s-s'-p\ even}}^{u-1} \mathcal{M}^k_{r,s'} \otimes \mathcal{M}^l_{r,s} \tag{5.7}$$

for $p = 1, 2$, where by $\mathbb{L}^1(\mathfrak{sl}(2))^{tw}$ we mean the affine vertex algebra of $\mathfrak{sl}(2)$ with Urod conformal structure, and by ${\mathcal{L}^1_{p,0}}^{tw}$ we mean the module $\mathcal{L}^1_{p,0}$, viewed as an $\mathbb{L}^1(\mathfrak{sl}(2))^{tw}$ module. See [3] for details.

Remark 5.2.3. *The relations of 4.2.16 also give an isomorphism of conformal vertex superalgebras*

$$bc \otimes bc^{tw} \simeq \mathbb{L}^1(\mathfrak{sl}(2)) \otimes \mathbb{L}^1(\mathfrak{sl}(2))^{tw} \oplus \mathcal{L}^1_{2,0} \otimes {\mathcal{L}^1_{2,0}}^{tw} \tag{5.8}$$

We introduce the notation:

$$bc^{\otimes 2,tw} = bc \otimes bc^{tw} \tag{5.9}$$

5.3 Orbifolds

Definition 5.3.1. *Suppose that V is an SVOA, and G is a subgroup of the automorphism group of G. Then $V^G = \{v \in V | gv = v\}$ has the structure of an SVOA, called the orbifold.*

Definition 5.3.2. *Suppose that V is an SVOA. A simple current of V is a V module J which is invertible with respect to the tensor product. That is, there exists V modules J^{-1} with $J \boxtimes J^{-1} \simeq V$.*

Theorem 5.3.1. *[17][60] Suppose that V is a VOA with HLZ structure. Suppose G is a finitely generated abelian subgroup of the automorphism group of V, and V decomposes as:*

$$V = \bigoplus_{\chi \in I} V_\chi \tag{5.1}$$

Where $\chi \in I$ are one dimensional representations of G, and

$$V_\chi = \{v \in V | gv = \chi(g)v\}$$

Then the orbifold $V_1 = V^G$ is a simple vertex subalgebra of V, and the V_χ are simple currents for V_1.

5.4 Vertex algebra extensions

In many of the earlier sections, I have given the analogy of vertex superalgebras as 'almost commutative' superalgebras. This perspective is most apparent on the from their categories of modules, where many results of commutative algebra have direct analogs for vertex superalgebras.

This machinery is key for the exploration of $\mathbb{L}^1(D(2,1;\alpha))$ representation theory. Since the Lie superalgebra $D(2,1;\alpha)$ is itself an extension of 3 copies of $\mathfrak{sl}(2)$, our goal is to realize $\mathbb{L}^1(D(2,1;-\frac{v}{w}))$ as an extension of 3 copies of $\mathbb{L}^k(\mathfrak{sl}(2))$ at appropriate levels.

We now define the notion of a superalgebra in a braided tensor category, and explore how such an algebra is related to extensions of vertex algebras.

Definition 5.4.1. *A superalgebra in the braided tensor category $\mathcal{C}$ is an object $A = (A_{\bar{0}}, A_{\bar{1}})$ in $\mathcal{SC}$ with even morphisms $\mu : A \boxtimes A \to A$ and $\iota_A : \mathbb{1} \to A$ in $\mathcal{SC}$ satisfying:*

1. *Associativity: $\mu \circ (\mu \boxtimes 1_A) \circ a_{A,A,A} = \mu \circ (1_A \boxtimes \mu)$ as morphisms $A \boxtimes (A \boxtimes A) \to A$.*
2. *Supercommutativity $\mu \circ R_{A,A} = \mu$ as morphisms $A \boxtimes A \to A$.*
3. *Unit: $\mu \circ (\iota_A \boxtimes 1_A) \circ l_A^{-1} = 1_A$ as morphisms $A \to A$.*

Definition 5.4.2. *Given a superalgebra A in $\mathcal{C}$, define RepA to be the category with objects (W, μ_W) where $W \in \mathcal{SC}$ and $\mu_W : A \boxtimes W \to W$ is an even morphism satisfying:*

1. *Associativity: $\mu_W \circ (\mu \boxtimes 1_W) \circ a_{A,A,W} = \mu_W \circ (1_A \boxtimes \mu_W)$ as morphisms $A \boxtimes (A \boxtimes W) \to W$.*
2. *Unit: $\mu_W \circ (\iota_A \boxtimes 1_W) \circ l_W^{-1} = 1_W$ as morphisms $W \to W$.*

RepA is a monoidal supercategory with tensor product $\boxtimes_A$ based on the construction of [56], coming from the monoidal structure on $\mathcal{C}$. See [19] section 2.3 for details.

Definition 5.4.3. *Define $Rep^0 A$ to be the full subcategory of RepA consisting of objects (W, μ_W) satisfying:*

$$\mu_W \circ R_{W,A} \circ R_{A,W} = \mu_W : A \boxtimes W \to W$$

$Rep^0 A$ is a braided monoidal supercategory with tensor product $\boxtimes_A$, unit (A, μ), left unit isomorphism l^A, right unit isomorphism r^A, associativity mophisms a^A, and even braiding isomorphisms R^A. See [19] section 2.6 for details. We call this the KO braided tensor category structure on $Rep^0 A$.

Definition 5.4.4.

- *We call* $W \in Rep^0 A$ *a local or Neveu-Schwarz* A *module.*
- $W \in RepA$ *is said to Ramond if the double braiding acts as the parity involution* $R_{W,A} \circ R_{A,W} = P_W$.

Definition 5.4.5. *The induction functor* $\mathcal{F} : \mathcal{SC} \to RepA$ *is given on objects* $W \in \mathcal{SC}$ *by:*

$$\mathcal{F}(W) = A \boxtimes W \qquad \mu_{\mathcal{F}(W)} = (\mu \boxtimes 1_W) \circ a_{A,A,W} \tag{5.1}$$

and on morphisms $f \in Hom_{\mathcal{SC}}(W_1, W_2)$ *by*

$$\mathcal{F}(f) = 1_A \boxtimes f \tag{5.2}$$

$\mathcal{F}$ *is a tensor functor [56, 19].*

Definition 5.4.6. *We let* $\mathcal{SC}^0$ *denote the full subcategory of objects in* $\mathcal{SC}$ *that induce to* $Rep^0 A$.

Proposition 5.4.1 ([19] proposition 2.65)**.** *For* $M \in \mathcal{SC}$, $M \in \mathcal{SC}^0$ *if and only if* $R_{M,A} \circ R_{A,M} = 1_{A \otimes M}$.

Theorem 5.4.2 ([19] theorem 2.67)**.** $\mathcal{SC}^0$ *is an* $\mathbb{F}$*-linear braided monoidal supercategory with structures induced from those on* $\mathcal{SC}$*, and induction defines a braided tensor functor* $\mathcal{F} : \mathcal{SC}^0 \to Rep^0 A$.

Definition 5.4.7. *The restriction functor* $\mathcal{G} : RepA \to \mathcal{SC}$ *is given for objects* (W, μ_W) *and morphism* f *by:*

$$(W, \mu_W) \mapsto W \qquad f \mapsto f \tag{5.3}$$

When $\mathcal{SC}$ *is a category of modules for some vertex algebra* V *and it is not otherwise clear from context, we will denote* $\mathcal{G}$ *by* $\mathcal{G}_V$.

Lemma 5.4.3 ([29] lemma 7.8.12)**.** *Induction is left adjoint to restriction. That is for* $M \in \mathcal{SC}$ *and* $W \in RepA$ *there is natural isomorphism*

$$Hom_{RepA}(\mathcal{F}(M), W) \simeq Hom_{\mathcal{SC}}(M, \mathcal{G}(W)) \tag{5.4}$$

called Frobenius reciprocity.

This gives the necessary categorical notions for extension. The following theorems give the relationship between a superalgebra A in a category $\mathcal{C}$ of V modules, a vertex superalgebra extension of V, and the two natural braided tensor category stuctures on $\mathrm{Rep}^0 A$, coming from the 'categorical side' (the KO structure) and the 'vertex algebra side' (the HLZ structure).

Remark 5.4.4. *For the following theorems, suppose that V is a $\mathbb{Z}$ graded vertex operator algebra, and $\mathcal{C}$ is an abelian category of V modules with vertex tensor category (and hence braided tensor category) structure of [41]-[48], which we call the HLZ braided tensor category structure.*

Theorem 5.4.5. *[40, 15] Vertex operator superalgebra extensions A of V in $\mathcal{C}$ such that $V \subset A_{\bar{0}}$ are precisely superalgebras (A, μ, ι_A) in the braided tensor category $\mathcal{C}$ which satisfy:*

1. *ι_A is injective.*
2. *A is $\frac{1}{2}\mathbb{Z}$-graded by conformal weights: $\theta_A^2 = 1_A$ where $\theta_A = e^{2\pi i L_{A,(0)}}$.*
3. *μ has no monodromy: $\mu \circ (\theta_A \boxtimes \theta_A) = \theta_A \circ \mu$.*

Theorem 5.4.6. *Suppose that $\mathcal{C}$ is a category of grading-restricted generalized V modules, and suppose that W is a vertex superalgebra extension of V such that $V \subset W_{\bar{0}}$, with corresponding superalgebra object $A \in \mathcal{C}$. Then:*

1. *$Rep^0 A$ is the category of grading-restricted generalized W-modules in $\mathcal{C}$. [40, 15]*
2. *$Rep^0 A$ has HLZ braided tensor category structure, and this is isomorphic to the KO monoidal tensor category structure introduced in 5.4.3. [19]*

Finally, for the study of representation theory, we will need sufficient conditions for an induced module to be simple.

Proposition 5.4.7. *[19] Let A be a simple vertex operator superalgebra extension of simple vertex algebra V with $V \subset A_{\bar{0}}$ and*

$$A = \bigoplus_{i \in I} A_i$$

Suppose that M is a simple V module. If

1. *Each A_i is a simple V module.*

2. *Each nonzero $A_i \boxtimes_V M$ is simple.*

3. *$A_i \boxtimes_V M \not\simeq A_j \boxtimes_V M$ for $i \neq j$.*

Then $\mathcal{F}(M)$ is a simple object in RepA.

This leads us to the first example of vertex algebra extension: the lattice vertex algebras.

Example 5.4.8. *Let L be a finite rank integral lattice with symmetric bilinear form $\langle \cdot, \cdot \rangle$, and π_L the Heisenberg vertex algebra associated to the lattice L.*

1. *Define:*
$$V_L = \bigoplus_{\lambda \in L} \pi_L(\lambda) \tag{5.5}$$

2. *Define $\mu : V_L \boxtimes V_L \to V_L$ such that:*
$$\mu|_{\pi_L(\lambda) \boxtimes \pi_L(\nu)} : \pi_L(\lambda) \boxtimes \pi_L(\nu) \to V_L$$
is given by the isomorphism $\pi_L(\lambda) \boxtimes \pi_L(\nu) \to \pi_L(\lambda + \nu)$.

3. *Define $\iota_{V_L} : \pi_L \to V_L$ to be the natural inclusion, noting that $\pi_L \simeq \pi_L(0)$.*

It is clear that ι_{V_L} is injective, and each $\pi_L(\lambda)$ is $\frac{1}{2}\mathbb{Z}$ graded since $\langle \lambda, \lambda \rangle / 2 \in \frac{1}{2}\mathbb{Z}$. Finally,

$$\frac{\langle \lambda + \nu, \lambda + \nu \rangle}{2} \equiv \frac{\langle \lambda, \lambda \rangle}{2} + \frac{\langle \nu, \nu \rangle}{2} \mod \mathbb{Z}$$

So $\mu \circ (\theta_{V_L} \boxtimes \theta_{V_L}) = \theta_{V_L} \circ \mu$. Then V_L is a conformal vertex superalgebra extension of π_L, called the lattice vertex algebra associated to L.

For our purposes, we will make use of a 'half lattice' vertex algebra, constructed again as an extension of a Heisenberg vertex algebra associated to a lattice, but in this case, extended only along one direction in the lattice.

Example 5.4.9. *Consider the lattice L generated by c, d with symmetric bilinear form*

$$\langle c, d\rangle = 2 \qquad\qquad \langle c, c\rangle = \langle d, d\rangle = 0 \tag{5.6}$$

with $\mathfrak{h} = \mathbb{C}\otimes_{\mathbb{Z}} L$ and π_L the Heisenberg vertex algebra associated to this lattice. We define new conformal vector:

$$\omega = \left(\frac{1}{2}c_{(-1)}d_{(-1)} - \frac{1}{2}d_{(-2)} + \frac{k}{4}c_{(-2)}\right)|0\rangle \tag{5.7}$$

with central charge $\overline{c} = 6k + 4$, and in the following, the conformal structure on π_L is given by this ω. Identify $\lambda \in \mathfrak{h}$ with $\langle \lambda, \cdot\rangle \in \mathfrak{h}^*$.

1. *Define:*
$$\Pi^k(0) = \bigoplus_{n\in\mathbb{Z}} \pi_L(nc) \tag{5.8}$$

2. *Define $\mu : \Pi^k(0) \boxtimes \Pi^k(0) \to \Pi^k(0)$ such that:*
$$\mu|_{\pi_L(nc)\boxtimes\pi_L(mc)} : \pi_L(nc) \boxtimes \pi_L(mc) \to \Pi^k(0)$$
is given by the isormorphism $\pi_L(nc) \boxtimes \pi_L(mc) \to \pi_L((n+m)c)$.

3. *Define $\iota_{\Pi^k(0)} : \pi_L \to \Pi^k(0)$ to be the natural inclusion, noting that $\pi_L \simeq \pi_L(0)$.*

It is clear that the inclusion is injective. Note that $L_0|nc\rangle = n|nc\rangle$, so the twist on $\Pi^k(0)$ is trivial, and μ has no monodromy. Then $\Pi^k(0)$ is a $\mathbb{Z}$ graded conformal vertex algebra extensions of π_L.

Proposition 5.4.10. *Define the Heisenberg field $h = \frac{k}{2}c + d$. Then $\langle c, h\rangle = 2$, $\langle d, h\rangle = k$ and $\langle h, h\rangle = 2k$. For every $r \in \mathbb{Z}$ and $\lambda \in \mathbb{C}$, the induced $\Pi^k(0)$ module:*

$$\Pi^k_{(t)}(\lambda) = \mathcal{F}\left(\pi_L\left(\frac{\lambda}{2}c + \frac{t}{2}h\right)\right) \tag{5.9}$$

Whose restriction as to a π_L module decomposes as:

$$\mathcal{G}(\Pi^k_{(t)}(\lambda)) = \bigoplus_{n\in\mathbb{Z}} \pi_L\left(nc + \frac{\lambda}{2}c + \frac{t}{2}h\right) \tag{5.10}$$

is an irreducible $\Pi(0)$ module on which $c_{(0)}$ acts as tId [1], and conformal weights are:

$$(t+1)\frac{\lambda}{2} + \frac{t^2k}{2} + \mathbb{Z}$$

Proof. *Note that*

$$L_0 = \frac{1}{2}d_{(0)} - \frac{k}{4}c_{(0)} + \frac{1}{2}\sum_{n\in\mathbb{Z}} :c_{(n)}d_{(-n)}:$$

where the normal order product of modes is defined as the composition with positive modes on the right. In particular

$$L_0\left|nc + \frac{\lambda}{2}c + \frac{t}{2}h\right\rangle = \left((t+1)n + (t+1)\frac{\lambda}{2} + \frac{t^2k}{4}\right)\left|nc + \frac{\lambda}{2}c + \frac{t}{2}h\right\rangle$$

and $\deg c_{(n)} = \deg d_{(n)} = -n$. *The conformal weight claim follows.*

Example 5.4.11. *Consider the vertex algebra $B_0 = \mathbb{L}^1(\mathfrak{sl}(2))\otimes\Pi(0)\otimes\mathbb{L}^1(\mathfrak{sl}(2))$. Define odd module $B_1 = \mathcal{L}^1_{2,0}\otimes\Pi_{(0)}(1)\otimes\mathcal{L}^1_{2,0}$, and notice that $B_a\boxtimes B_b \simeq B_{\overline{a+b}}$*

1. *Define:*

$$B = B_0 \oplus B_1$$

2. *Define $\mu : B\boxtimes B \to B$ such that:*

$$\mu|_{B_a\boxtimes B_b} : B_a\boxtimes B_b \to B$$

   is given by the isomorphism $B_a\boxtimes B_b \simeq B_{\overline{a+b}}$.

3. *Define $\iota_B : B_0 \to B$ to be the inclusion.*

It is clear that the inclusion is injective. The twist θ_{B_a} is $(-1)^a$. Then $\theta_{B_a}\boxtimes\theta_{B_b} = (-1)^{a+b} = \theta_{B_{a+b}}$, so μ is monodromy free on the restriction to $B_a\boxtimes B_b$, and hence monodromy free. Then B is a $\frac{1}{2}\mathbb{Z}$ graded conformal vertex *superalgebra extension of B_0. We introduce the following notation for induced modules:*

$$B^k_{(t)}(\lambda; a, b) = \mathcal{F}(\mathcal{L}^1_{a,0} \otimes \Pi^k_{(t-1)}(\lambda + k) \otimes \mathcal{L}^1_{b,0}) \tag{5.11}$$

Whose restriction to a B_0 module decomposes as:

$$\mathcal{L}^1_{a,0} \otimes \Pi^k_{(t-1)}(\lambda + k) \otimes \mathcal{L}^1_{b,0} \oplus \mathcal{L}^1_{a+1,0} \otimes \Pi^k_{(t-1)}(\lambda + k + 1) \otimes \mathcal{L}^1_{b+1,0} \tag{5.12}$$

Theorem 5.4.12. *[9] Let $k = \frac{u}{v} - 2$ and $\tilde{l} = \frac{2u-v}{u} - 2$ be admissible levels for $\mathfrak{sl}(2)$. Then $\mathbb{L}^k(\mathfrak{osp}(1|2))$ is a $\mathbb{L}^k(\mathfrak{sl}(2)) \otimes \mathbb{L}^{\tilde{l}}(Vir)$ extension with branching rule;*

$$\mathbb{L}^k(\mathfrak{osp}(1|2)) \simeq \bigoplus_{r=1}^{u-1} \mathcal{L}^k_{r,0} \otimes \mathcal{M}^{\tilde{l}}_{1,r} \tag{5.13}$$

Let ω be the fundamental weight of $\mathfrak{osp}(1|2)$. Then:

$$\mathcal{G}(\mathbb{L}^k(s\omega)) \simeq \bigoplus_{r=1}^{u-1} \mathcal{L}^k_{r,0} \otimes \mathcal{M}^{\tilde{l}}_{s,r} \tag{5.14}$$

Theorem 5.4.13. *[18] Let $k = \frac{u}{v} - 2$ be an admissible level for (2), and suppose that we have irreducible, lower bounded relaxed highest weight module M for $\mathbb{L}^k(\mathfrak{osp}(1|2))$. Then:*

$$\mathcal{G}_{\mathbb{L}^k(\mathfrak{sl}(2)) \otimes \mathbb{L}^{\tilde{l}}(Vir)}(M) \simeq \bigoplus_{i=1}^{u-1} N^k_{\lambda;\Delta_{r,s_1}} \otimes \mathcal{M}^{\tilde{l}}_{r,s_2} \tag{5.15}$$

For some labels N, λ as in the table of 4.2.15 with $s_1 \in \{1, ..., v-1\}$, $s_2 \in \{1, ..., 2u - v\}$.

Theorem 5.4.14. *[16] Suppose that V is a VOA with HLZ braided tensor category structure. Suppose that J is an order 2 simple current. If $R_{JJ} = 1$, then $V \oplus J$ has the structure of a vertex algebra, and if $R_{JJ} = -1$, then $V \oplus J$ has the structure of a vertex superalgebra, with:*

$$(V \oplus J)_{\bar{0}} = V \qquad (V \oplus J)_{\bar{1}} = J \tag{5.16}$$

Proposition 5.4.15. *The Majorana free fermion is a simple current extension of* $\mathbb{L}^{-\frac{5}{4}}(Vir)$*, by order 2 simple current, with*

$$F = \mathbb{L}^{-\frac{5}{4}}(Vir) \oplus \mathcal{M}_{2,1}^{-\frac{5}{4}}$$

$$F_{\overline{0}} = \mathbb{L}^{-\frac{5}{4}}(Vir) \qquad\qquad F_{\overline{1}} = \mathcal{M}_{2,1}^{-\frac{5}{4}} \tag{5.17}$$

Then the category

$$Rep^0_{\mathcal{O}^{fin}}(F)$$

has HLZ vertex tensor category structure. By 5.4.14, we have $R_{F_{\overline{1}},F_{\overline{1}}} = -1$.

Proof. *Recall that the conformal structure in* F *has central charge* $c = \frac{1}{2}$*, so* F *is a module for* $\tilde{\mathbb{L}}^{\frac{1}{2}}(Vir) = \mathbb{L}^{-\frac{5}{4}}(Vir)$ *module. Since* $\mathcal{O}^{fin}$ *is semisimple, we need only confirm which representations* $\mathcal{M}_{r,s}$ *appear.*

ψ *has weight* $\frac{1}{2}$ *in* F*, and* $\psi_{(-1)}|0\rangle$ *has minimal half-integer degree. Since each mode* L_n *is homogenous of integer degree, we conclude that* $\psi_{(-1)}|0\rangle$ *is a highest weight vector for* $\mathbb{L}^{-\frac{5}{4}}(Vir)$ *of weight* $\frac{1}{2} = h_{2,1}$*. Then* $\mathcal{M}_{2,1}$ *is a summand of* F*.*

We note that $-\frac{5}{4} + 2 = \frac{3}{4}$*, so* $u = 3,\ v = 4$*, and* $N_{2,2}^{(3),r''} = \delta_{r'',1},\ N_{1,r'}^{(4),\ r''} = \delta r, r''$*, so that* $\mathcal{M}_{2,1}^{-\frac{5}{4}}$ *is indeed an order 2 simple current.*

F is generated by ψ*, and we can confirm using the commutation relations if modes* ψ_n *and* L_m *that* $\mathbb{L}^{-\frac{5}{4}}(Vir) \oplus \mathcal{M}_{2,1}^{-\frac{5}{4}}$ *is closed under the action of* ψ*. The result follows.*

5.5 Gluing

In our case, we will make use of a particular kind of extension, obtained using by 'gluing together' modules of two other vertex algebras, each in equivalent categories with reversed braiding. A converse also holds: if such a gluing is an extension, then there is also a braid reversed equivalence of categories.

We shall see that manipulating the two extensions of this form given in 5.3 and 5.7 using this machinery will give us the desired realization of

$\mathbb{L}^1(D(2,1;-\frac{v}{w}))$.

Theorem 5.5.1. *[20] Suppose that*

- *U and V are conformal vertex superalgebras.*
- *$\mathcal{U}$ and $\mathcal{V}$ are locally finite abelian categories of modules for U and V respectively, with HLZ braided tensor category structure.*
- *$\mathcal{U}$ is semisimple.*
- *$\mathcal{V}$ is closed under submodules and quotients.*

Define $\langle \mathcal{U} \otimes \mathcal{V} \rangle$ to be the full subcategory of $U \otimes V$ modules whose objects are isomorphic to direct sums of $M \otimes N$ with $M \in \mathcal{U}$ and $N \in \mathcal{V}$. Then $\langle \mathcal{U} \otimes \mathcal{V} \rangle$ admits HLZ vertex tensor category structure and is braided tensor equivalent to the Deligne product $\mathcal{U} \otimes \mathcal{V}$.

This theorem confirms that under certain finiteness and closure conditions, we can work with the HLZ structure over a product of vertex superalgebras just as we would expect. In particular, the categories $\mathcal{O}_k^{fin}$ and $\mathbb{L}^k(\mathfrak{sl}(2))$-$wtmod_{KL}$ satisfy these conditions.

Remark 5.5.2. *For theorems 5.5.3 and 5.5.9, suppose that $\mathcal{U}, \mathcal{V}$ are rigid, locally finite module categories for self-dual vertex operator algebras, U, V respectively, that admit HLZ vertex tensor category structure as in [41]-[48].*

Suppose that $\mathcal{U}$ is semisimple with $\{U_i\}_{i \in I}$ representatives of equivalence classes of simple modules in $\mathcal{U}$, with $U_0 = U$.

Suppose $\mathcal{V}$ is closed under submodules and quotients.

Theorem 5.5.3. *[20] Suppose that $\mathcal{U}, \mathcal{V}$ are ribbon categories satisfying 5.5.2 and*

$$A = \bigoplus_{i \in I} U_i \otimes V_i$$

is a $\frac{1}{2}\mathbb{Z}$ graded conformal vertex algebra extension of $U \otimes V$, such that $V_i \in \mathcal{V}$ satisfy $\dim Hom_{\mathcal{V}}(V, V_i) = \delta_{i,0}$, and there is a partition $I = I_0 \sqcup I_1$ such that:

$$\bigoplus_{i \in I_j} U_i \otimes V_i = \bigoplus_{n \in \frac{j}{2} + \mathbb{Z}} A_{(n)}$$

Define $\overline{\mathcal{V}}$ to be the full subcategory of $\mathcal{V}$ whose objects are isomorphic to direct sums of the $\mathcal{V}_i$. Then:

1. *$\overline{\mathcal{V}}$ is a ribbon subcategory of $\mathcal{V}$ with distinct irreducible objects $\{V_i\}_{i\in I}$.*
2. *There is braid-reversed tensor equivalence $\tau : \mathcal{U} \to \overline{\mathcal{V}}$ such that $\tau(U_i) \simeq V_i^*$.*

We are now prepared to make use of the relations of 5.3 and 5.7 to construct braid-reversed and braided equivalences necessary for later constructions.

Remark 5.5.4. *In the following, let k, l be non-integral admissible levels for $\mathfrak{sl}(2)$*

$$k+2 = \frac{u}{v} \qquad\qquad l+2 = \frac{u}{w} \tag{5.1}$$

With $u = v + w$. In particular, we have:

$$\frac{1}{k+2} + \frac{1}{l+2} = \frac{v+w}{u} = 1 \tag{5.2}$$

so that the relations 5.3 and 5.7 are satisfied. We establish the notation:

$$KL_k = \mathbb{L}^k(\mathfrak{sl}(2))\text{-}wtmod_{KL} \tag{5.3}$$

For modules M_r indexed by $r \in \{1, ..., p\}$ and $s \in \mathbb{Z}$, we use the notation:

$$M_{\overline{s}} = M_r \qquad r \in \{1, ..., p\} \qquad r = s \mod p \tag{5.4}$$

Lemma 5.5.5. *Suppose that k, l are admissible levels for $\mathfrak{sl}(2)$ satisfying the conditions in 5.5.4. Then there exists braid reversed equivalence:*

$$\tau_1 : KL_k \to \overline{\mathcal{O}_l^{fin} \otimes KL_1 \otimes \mathcal{O}_{-\frac{5}{4}}^{fin}} \tag{5.5}$$

such that:

$$\tau_1(\mathcal{L}_{r,0}^k) \simeq (\mathcal{M}_{r,1}^l \otimes \mathcal{L}_{\overline{r},0}^1 \otimes F_{\overline{r+1}})^* \tag{5.6}$$

where $\overline{\mathcal{O}_l^{fin} \otimes KL_1 \otimes \mathcal{O}_{-\frac{5}{4}}^{fin}}$ *is the full rigid braided tensor subcategory of* $\mathcal{O}_l^{fin} \otimes KL_1 \otimes \mathcal{O}_{-\frac{5}{4}}^{fin}$ *whose objects are isomorphic to direct sums of the* $\mathcal{M}_{r,1}^l \otimes \mathcal{L}_{\overline{r},0}^1 \otimes F_{\overline{r+1}}$.

Lemma 5.5.6. *Suppose that* k, l *are admissible levels for* $\mathfrak{sl}(2)$ *satisfying the conditions in 5.5.4. Then there exists braid reversed equivalence:*

$$\tau_2 : \mathcal{O}_{k,L}^{fin} \to \overline{KL_l \otimes KL_1 \otimes \mathcal{O}_{-\frac{5}{4}}^{fin}} \tag{5.7}$$

Such that:

$$\tau_2(\mathcal{M}_{r,1}^k) \simeq (\mathcal{L}_{r,0}^l \otimes \mathcal{L}_{r,0}^1 \otimes F_{\overline{r+1}})^* \tag{5.8}$$

where $\overline{KL_l \otimes KL_1 \otimes \mathcal{O}_{-\frac{5}{4}}^{fin}}$ *is the full rigid braided tensor subcategory of* $KL_l \otimes KL_1 \otimes \mathcal{O}_{-\frac{5}{4}}^{fin}$ *whose objects are isomorphic to direct sums of the* $\mathcal{L}_{r,1}^l \otimes \mathcal{L}_{\overline{r},0}^1 \otimes F_{\overline{r+1}}$.

Proof (Lemmas 5.5.5 and 5.5.6). *Recall the relations from 4.2.16. We obtain*

$$\mathbb{L}^{k-1}(\mathfrak{sl}(2)) \otimes bc^{\otimes 2} \otimes F \simeq \mathbb{L}^{k-1}(\mathfrak{sl}(2)) \otimes (\mathcal{L}_{1,0}^1 \otimes \mathcal{L}_{1,0}^1 \oplus \mathcal{L}_{2,0}^1 \otimes \mathcal{L}_{2,0}^1) \otimes F \tag{5.9}$$

We look at the even vertex subalgebra of this. By 5.3

$$\begin{aligned}(\mathbb{L}^{k-1}(\mathfrak{sl}(2)) \otimes bc^{\otimes 2} \otimes F)_{\overline{0}} &\simeq \mathbb{L}^{k-1}(\mathfrak{sl}(2)) \otimes (\mathcal{L}_{1,0}^1 \otimes \mathcal{L}_{1,0}^1 \otimes F_{\overline{0}} \oplus \mathcal{L}_{2,0}^1 \otimes \mathcal{L}_{2,0}^1 \otimes F_{\overline{1}}) \\ &\simeq \bigoplus_{\substack{r=1 \\ r \ odd}}^{u-1} \mathcal{L}_{r,0}^k \otimes \mathcal{M}_{r,1}^l \otimes \mathcal{L}_{1,0}^1 \otimes F_{\overline{0}} \oplus \bigoplus_{\substack{r=1 \\ r \ even}}^{u-1} \mathcal{L}_{r,0}^k \otimes \mathcal{M}_{r,1}^l \otimes \mathcal{L}_{2,0}^1 \otimes F_{\overline{1}} \end{aligned} \tag{5.10}$$

This gives $\frac{1}{2}\mathbb{Z}$*-graded vertex algebra extension for* $\mathbb{L}^k(\mathfrak{sl}(2)) \otimes \mathbb{L}^l(Vir) \otimes \mathbb{L}^1(\mathfrak{sl}(2)) \otimes F_{\overline{0}}$

By 4.2.9, KL_k *is a semisimple ribbon category, with simple objects* $\mathcal{L}_{r,0}^k$ *for* $r \in \{1, ..., u-1\}$

By 4.2.9, KL_1 *is a semisimple ribbon category with simple objects* $\mathcal{L}_{r,0}^1$ *for* $r \in \{1, 2\}$*, respectively.*

By 4.2.3, the $\mathcal{O}_l^{fin}$ *is a modular tensor category with simple objects* $\mathcal{M}_{r,1}^l$ *for* $r \in \{1, ..., u-1\}$.

By 4.2.3, $\mathcal{O}_{-\frac{5}{4}}^{fin}$ *is a modular tensor category.*

Then the category $\mathcal{O}_l^{fin} \otimes KL_1 \otimes \mathcal{O}_{-\frac{5}{4}}^{fin}$ *of modules for* $\mathbb{L}^l(Vir) \otimes \mathbb{L}^1(\mathfrak{sl}(2)) \otimes F_{\overline{0}}$ *is a ribbon category with* $\mathcal{M}_{r,1}^l \otimes \mathcal{L}_{\overline{r},0}^1 \otimes F_{\overline{r+1}}$ *satisfying homspace conditions by simplicity.*

The category $Kl_k \otimes KL_1 \otimes \mathcal{O}_{-\frac{5}{4}}^{fin}$ *of modules for* $\mathbb{L}^k(\mathfrak{sl}(2)) \otimes \mathbb{L}^1(\mathfrak{sl}(2)) \otimes F_{\overline{0}}$ *is a ribbon category with* $\mathcal{L}_{r,1}^k \otimes \mathcal{L}_{\overline{r},1}^1 \otimes F_{\overline{r+1}}$ *satisfying homspace conditions by simplicity.*

Since conformal weights of any two vectors in $\mathcal{L}_{r,0}^k \otimes \mathcal{M}_{r,1}^l \otimes \mathcal{L}_{\overline{r},0}^1 \otimes F_{\overline{r+1}}$ *differ by integer, it is clear that we can pick a partition of the labels* r *into* $\mathbb{Z}$ *graded and* $\mathbb{Z} + \frac{1}{2}$ *graded portions. Then both results follows from 5.5.3, although we note that the roles of* k *and* l *are exchanged in the second equivalence.*

Lemma 5.5.7. *Suppose that* k, l *are admissible levels for* $\mathfrak{sl}(2)$ *satisfying the conditions in 5.5.4. Then there exists braid reversed equivalence:*

$$\tau_3 : \mathcal{O}_{k,L}^{fin} \to \overline{\mathcal{O}_l^{fin} \otimes KL_1 \otimes \mathcal{O}_{-\frac{5}{4}}^{fin}} \tag{5.11}$$

such that:

$$\tau_3(\mathcal{M}_{r,1}^k) \simeq (\mathcal{M}_{r,1}^l \otimes \mathcal{L}_{\overline{r},0}^1 \otimes F_{\overline{r+1}})^* \tag{5.12}$$

Proof. *Recall the relations* 4.2.16

$$\mathbb{L}^{k-1}(Vir) \otimes bc^{\otimes 2,tw} \otimes F \simeq \mathbb{L}^{k-1}(Vir) \otimes (\mathcal{L}_{1,0}^1 \otimes \mathcal{L}_{1,0}^1{}^{tw} \oplus \mathcal{L}_{2,0}^1 \otimes \mathcal{L}_{2,0}^1{}^{tw}) \otimes F \tag{5.13}$$

We look at the even vertex subalgebra of this. By 5.7, setting $s = s' = 1$

$$\begin{aligned}(\mathbb{L}^{k-1}(Vir) \otimes bc^{\otimes 2,tw} \otimes F)_{\overline{0}} &\simeq \mathbb{L}^{k-1}(Vir) \otimes (\mathcal{L}_{1,0}^1 \otimes \mathcal{L}_{1,0}^1{}^{tw} \otimes F_{\overline{0}} \oplus \mathcal{L}_{2,0}^1 \otimes \mathcal{L}_{2,0}^1{}^{tw} \otimes F_{\overline{1}}) \\ &\simeq \bigoplus_{\substack{r=1 \\ r \text{ odd}}}^{u-1} \mathcal{M}_{r,1}^k \otimes \mathcal{M}_{r,1}^l \otimes \mathcal{L}_{1,0}^1 \otimes F_{\overline{0}} \oplus \bigoplus_{\substack{r=1 \\ r \text{ even}}}^{u-1} \mathcal{M}_{r,1}^k \otimes \mathcal{M}_{r,1}^l \otimes \mathcal{L}_{2,0}^1 \otimes F_{\overline{1}}\end{aligned} \tag{5.14}$$

This gives $\frac{1}{2}\mathbb{Z}$*-graded vertex algebra extension for* $\mathbb{L}^k(Vir) \otimes \mathbb{L}^l(Vir) \otimes \mathbb{L}^1(\mathfrak{sl}(2)) \otimes F_{\overline{0}}$.

By 4.2.3, the $\mathcal{O}_k^{fin}$ *is a modular tensor category with irreducible objects* $\mathcal{M}_{r,1}^k$ *for* $r \in \{1, ..., u-1\}$*, and all other relevant categories are as in the previous lemmas.*

Since conformal weights of any two vectors in $\mathcal{M}^k_{r,1} \otimes \mathcal{M}^l_{r,1} \otimes \mathcal{L}^1_{\bar{r},0} \otimes F_{\bar{0}}$ differ by integer, it is clear that we can pick a partition of the labels r into $\mathbb{Z}$ graded and $\mathbb{Z} + \frac{1}{2}$ graded portions. The result then follows from theorem 5.5.3.

Proposition 5.5.8. *Suppose that k is a non-integral admissible level for $\mathfrak{sl}(2)$. Then there is braided equivalence:*

$$\mu_k : KL_k \to \mathcal{O}^{fin}_{k,L} \tag{5.15}$$

such that:

$$\mu_k(\mathcal{L}^k_{r,0}) \simeq \mathcal{M}^k_{r,1} \tag{5.16}$$

Proof. *Suppose that k is an admissible level. Then l satisfying $\frac{1}{k+2} + \frac{1}{l+2} = 1$ is also a non-integral admissible level for $\mathfrak{sl}(2)$. Let τ_1 and τ_3 be as in 5.5.5 and 5.5.7, respectively. Then*

$$\mu_k = \tau_3^{-1} \circ \tau_1 : KL_k \to \mathcal{O}^{fin}_{k,L} \tag{5.17}$$

is a braided equivalence satisfying the conditions of the proposition.

Now that we have obtained the necessary braid-reversed and braided equivalences, we present the result needed to use them in construction of an extension of $\mathbb{L}^k(\mathfrak{sl}(2)) \otimes \mathbb{L}^l(\mathfrak{sl}(2) \otimes \mathbb{L}^1(\mathfrak{sl}(2))$.

Theorem 5.5.9. *[20] Suppose that $\mathcal{U}, \mathcal{V}$ satisfy 5.5.2, and $\tau : \mathcal{U} \to \mathcal{V}$ is a braid-reversed tensor equivalence with twists satisfying $\theta_{\tau(U_i)} = \pm\tau(\theta^{-1}_{U_i})$. Then*

1. $$A = \bigoplus_{i \in I} U_i^* \otimes \tau(U_i)$$

   is a simple $\frac{1}{2}\mathbb{Z}$ graded conformal vertex algebra extension of $U \otimes V$.

2. *The multiplication rules of A satisfy $M^{U_k \otimes \tau(U_k)^*}_{U_i \otimes \tau(U_i)^*, U_j \otimes \tau(U_j)^*} = 1 \iff U_k$ occurs as a submodule of $U_i \otimes U_j$.*

Remark 5.5.10. *We introduce the notation*

$$L_r := \mathcal{L}^k_{r,0} \otimes \mathcal{L}^l_{r,0} \otimes \mathcal{L}^1_{\bar{r},0} \tag{5.18}$$

In the following, we denote by $\Delta_{min}(N)$ the minimal conformal weight of vectors in sVOA module N, when it exists.

Each L_r has highest weight vector $|\lambda_{r,0}\rangle := |\lambda^k_{r,0}\rangle \otimes |\lambda^l_{r,0}\rangle \otimes |\lambda^1_{r,0}\rangle$, which is of minimal conformal weight in L_r. Then:

$$\Delta_{min}(L_r) = \frac{r^2-1}{4}\left(\frac{1}{k+2}+\frac{1}{l+2}\right)+\frac{\overline{r}^2-1}{12} = \frac{r^2-1}{4}+\frac{\overline{r}^2-1}{12} \tag{5.19}$$

Where we define:

$$\overline{r} = \begin{cases} 1 & r \text{ is odd} \\ 0 & r \text{ is even} \end{cases}$$

Note that $\Delta_{min}(L_r) \in \mathbb{Z}$, so each L_r is $\mathbb{Z}$ graded.

Lemma 5.5.11. *Suppose that k, l are admissible levels for $\mathfrak{sl}(2)$ satisfying the conditions in 5.5.4. Then*

$$A = \bigoplus_{r=1}^{u-1} \mathcal{L}^k_{r,0} \otimes \mathcal{L}^l_{r,0} \otimes \mathcal{L}^1_{\overline{r},0} = \bigoplus_{r=1}^{u-1} L_r \tag{5.20}$$

is a simple vertex superalgebra extension of $L_1 = L^k(\mathfrak{sl}(2)) \otimes L^l(\mathfrak{sl}(2)) \otimes L^1(\mathfrak{sl}(2))$ with:

$$A_{\overline{0}} = \bigoplus_{\substack{r=1 \\ r \text{ odd}}}^{u-1} L_r \qquad\qquad A_{\overline{1}} = \bigoplus_{\substack{r=1 \\ r \text{ even}}}^{u-1} L_r \tag{5.21}$$

Proof. *Let μ_k be as in 5.5.8 and τ_2 be as in 5.5.6. Then*

$$\tau_2 \circ \mu_k : KL_k \to \overline{KL_l \otimes KL_1 \otimes \mathcal{O}^{fin}_{-\frac{5}{4}}} \tag{5.22}$$

is a braid reversed equivalence such that

$$\tau_2 \circ \mu_k(\mathcal{L}^k_{r,0}) \simeq (\mathcal{L}^l_{r,0} \otimes \mathcal{L}^1_{r,0} \otimes F_{\overline{r+1}})^* \tag{5.23}$$

The twist condition of 5.5.9 is satisfied since the conformal weights 5.19 are integer, and conformal weights of $F_{\overline{r+1}}$ are half integer. Then we get vertex *algebra extension*

$$A_{\bar{0}} \otimes F_{\bar{0}} \oplus A_{\bar{1}} \otimes F_{\bar{1}} \tag{5.24}$$

of $L_1 \otimes F_{\bar{0}}$. *This has* $\mathbb{Z}_2$ *action given by the* $\mathbb{Z}_2$ *action on* F, *where the nontrivial automorphism acts on* $A_{\bar{0}} \otimes F_{\bar{0}}$ *as id and on* $A_{\bar{1}} \otimes F_{\bar{1}}$ *as* $-id$. *The orbifold* $A_{\bar{0}} \otimes F_{\bar{0}}$ *is a vertex algebra extension of* $L_1 \otimes F_{\bar{0}}$, *and by 5.3.1* $A_{\bar{1}} \otimes F_{\bar{1}}$ *is an order 2 simple current for* $A_{\bar{0}} \otimes F_{\bar{0}}$. *Since* $A_{\bar{0}} \otimes F_{\bar{0}} \oplus A_{\bar{1}} \otimes F_{\bar{1}}$ *is an (even) vertex algebra, we conclude that* $id = R_{A_{\bar{1}} \otimes F_{\bar{1}}, A_{\bar{1}} \otimes F_{\bar{1}}}$ *by 5.4.14.*

We take the $F_{\bar{0}}$ *coset. Noting that* $Com(F_{\bar{0}}, F_{\bar{0}}) \simeq \mathbb{C}|0\rangle$, *we have*

$$L_1 \simeq Com(F_{\bar{0}}, L_1 \otimes F_{\bar{0}}) \subset Com(F_{\bar{0}}, A_{\bar{0}} \otimes F_{\bar{0}}) \simeq A_{\bar{0}}$$

Finally, $A_{\bar{1}} \simeq A_{\bar{1}} \otimes |0\rangle$ *is an* $A_{\bar{0}}$ *submodule, and an order 2 simple current. By 5.5.1 we have:*

$$R_{A_{\bar{1}}, A_{\bar{1}}} \otimes (-id_{F_{\bar{1}}}) = R_{A_{\bar{1}}, A_{\bar{1}}} \otimes R_{F_{\bar{1}}, F_{\bar{1}}} = R_{A_{\bar{1}} \otimes F_{\bar{1}}, A_{\bar{1}} \otimes F_{\bar{1}}} = id$$

Then $R_{A_{\bar{1}}, A_{\bar{1}}} = -id$, *and by 5.4.14,* $A = A_{\bar{0}} \oplus A_{\bar{1}}$ *is a vertex superalgebra extension of* $A_{\bar{0}}$ *with odd part* $A_{\bar{1}}$.

Theorem 5.5.12. *Suppose that* k, l *are admissible levels for* $\mathfrak{sl}(2)$ *satisfying the conditions in 5.5.4. Then we have the following isomorphism as* $\mathbb{L}^k(\mathfrak{sl}(2)) \otimes \mathbb{L}^l(\mathfrak{sl}(2)) \otimes \mathbb{L}^1(\mathfrak{sl}(2))$ *modules:*

$$\mathbb{L}^1(D(2,1;-\frac{v}{w})) \simeq \bigoplus_{r=1}^{u-1} L_r = A \tag{5.25}$$

Proof. *We begin by identifying weight one vectors for* A. *We note that* $\Delta_{min}(L_r)$ *is always an integer, so* A *is integer graded. We obtain* $\Delta_{min}(L_1) = 0$, $\Delta_{min}(L_2) = 1$, *and for* $r > 2$, *we have* $\Delta_{min}(L_r) > 1$. *Note the fusion product is as follows for* $r \neq 1, u-1$

$$L_2 \boxtimes L_r \simeq \bigoplus_{i_1, i_2 = \pm 1} \mathcal{L}^k_{r+i_1,0} \otimes \mathcal{L}^l_{r+i_2,0} \otimes \mathcal{L}^1_{\overline{r+1},0} \tag{5.26}$$

Then every L_r *appears as a summand in the fusion product of* L_2 *with itself* *sufficiently many times.*

Pick $m, n \in \mathbb{N}$ such that L_r (resp. L_j) appears as a summand in $L_2^{\boxtimes n}$ (resp. $L_2^{\boxtimes m}$). Then $L_r \boxtimes L_j$ appears as a summand in $L_2^{\boxtimes m+n}$. Let $i : L_r \boxtimes L_j \to L_2^{\boxtimes m+n}$ be the inclusion, and $\pi : L_2^{\boxtimes m+n} \to L_r \boxtimes L_j$ be the projection.

Let $Y : A \otimes A \to A[[z]]$ be the vertex operator on A. Then $Y|_{L_r \otimes L_j}$ is an intertwiner of type $\binom{A}{L_r \ L_j}$. Then by universal property of $P(1)$ tensor product, there exists morphism $\phi : L_2^{\boxtimes m+n} \to A$ such that the following diagram commutes:

$$\begin{array}{ccc} \overline{L_2^{\boxtimes m+n}} & \xrightarrow{\phi \circ \pi} & \overline{A} \\ {\scriptstyle i \circ \boxtimes} \uparrow & \nearrow {\scriptstyle Y|_{L_r \otimes L_j}} & \\ L_r \otimes L_j & & \end{array}$$

Then A is weakly generated by L_2, and certainly it is weakly generated by $L_1 \oplus L_2$. But L_1 is strongly generated by the vacuum and its weight one vectors, and L_2 is strongly generated by its top level, which is also weight 1. Then A is strongly generated by its vacuum and weight one subspace.

Then A must be an affine vertex superalgebra $A \simeq \mathbb{L}^\nu(\mathfrak{g})$ for some Lie superalgebra $\mathfrak{g}$, and $\mathfrak{g}$ must be simple by simplicity of A.

We recall that the summand L_r is even for odd r and odd for even r. By our earlier computation, all even weight one vectors fall in L_1 and all odd weight one vectors fall in L_2. But 0 modes of weight one vectors in L_1 transform as $\mathfrak{sl}(2) \otimes \mathfrak{sl}(2) \otimes \mathfrak{sl}(2)$, and 0 modes of the top level of L_2 transform in the tensor of 3 copies of the standard representation $\mathfrak{st} \otimes \mathfrak{st} \otimes \mathfrak{st}$.

By the classification of simple Lie superalgberas [49], we conclude that $\mathfrak{g} = D(2,1;\alpha)$ for some α, and $A \simeq \mathbb{L}^\nu(D(2,1;\alpha))$. We need only confirm the parameters ν and α. Comparing relations from $\mathbb{L}^\nu(D(2,1;\alpha))$ with those from the subalgebra $\mathbb{L}^1(\mathfrak{sl}(2))$:

$$\partial_w \delta(z-w) = [h^3(z), h^3(w)] = \frac{\nu}{\alpha_3} \partial_w \delta(z-w) \tag{5.27}$$

So we conclude that $\nu = \alpha_3 = 1$. Finally, comparing relations from $\mathbb{L}^\nu(D(2,1;\alpha)$

with those from the subalgebra $\mathbb{L}^k(\mathfrak{sl}(2))$*:*

$$k\partial_w(z-w) = [h^1(z), h^2(w)] = \frac{1}{\alpha_1}\partial_w(z-w) \tag{5.28}$$

We conclude that $-\alpha^{-1} - 1 = \alpha_1^{-1} = k$*, so that* $\alpha = -\frac{1}{k+1}$*. Recalling that* $k = \frac{v+w}{v} - 2 = \frac{w}{v} - 1$*, we prefer to write this as* $\alpha = -\frac{v}{w}$*.*

5.6 Irreducible modules for $\mathbb{L}^1(D(2,1;-\frac{v}{w}))$

We are finally prepared to use the machinery introduced in section 5.4 in our construction and classification of irreducible modules for $\mathbb{L}^1(D(2,1;-\frac{v}{2}))$. We begin by putting restrictions on labels that may appear for inclusions of L_1 modules in $\mathbb{L}^1(D(2,1;-\frac{v}{w}))$ modules.

Lemma 5.6.1. *Suppose that* $\tilde{M}$ *is an irreducible* $\mathbb{L}^1(D(2,1,-\frac{v}{w}))$ *module and that* M *is a nontrivial irreducible* L_1 *module with* L_1 *morphism* $M \to \mathcal{G}(\tilde{M})$*. Then* M *has the form*

$$M = \sigma^{t_1}(L^k_{\lambda_1,\Delta_{r_1,s_1}}) \otimes \sigma^{t_2}(N^l_{\lambda_2,\Delta_{r_2,s_2}}) \otimes \mathcal{L}^1_{p,0}$$

with $p \in \{1,2\}$*,* $N, L \in \{\mathcal{L}, \mathcal{D}^\pm, \mathcal{E}\}$ *and* $r_2 \in \{r_1, u - r_1\}$*.*

Proof. *We first note that all irreducible modules for* L_1 *have the appropriate form, and we need only confirm the restriction on the parameters* r_1, r_2*. Consider first the case that all* $\psi(\beta\gamma\delta)_{(0)}$ *acts trivially on* M*.*

$$e^1_{(0)} = -\frac{1}{2}[\psi(+++)_{(0)}, \psi(+--)_{(0)}] \tag{5.1}$$

$$f^1_{(0)} = -\frac{1}{2}[\psi(-++)_{(0)}, \psi(---)_{(0)}] \tag{5.2}$$

$$h^1_{(0)} = [e^1_{(0)}, f^1_{(0)}] \tag{5.3}$$

and similarly for $i = 2, 3$*. Then all of the generating* 0 *modes in* L_1 *act trivially on* M*, so* $M = 0$*, a contradiction.*

Consider the case that not all of the zero modes $\psi(\beta\gamma\delta)_{(0)}$ *act trivially on* M*. Pick* $\psi(\beta\gamma\delta)$ *and* $m \in M$ *such that* $\psi(\beta\gamma\delta)_{(0)}m \neq 0$*. The module map*

$Y_{\tilde{M}}(\cdot,z)$ induces intertwining operator $\mathcal{Y}$ of type $\binom{\mathcal{G}(M)}{L_2\ M}$. By universal property, there exists morphism

$$\phi : L_2 \boxtimes_{L_1} M \to \mathcal{G}(\tilde{M}) \qquad\qquad \mathcal{Y} = \phi \circ Y_{L_2 \boxtimes M} \tag{5.4}$$

By nontriviality:

$$0 \neq \psi(\beta\gamma\delta)^{\mathcal{Y}}_{(0)} m = \phi\psi(\beta\gamma\delta)^{L_2\boxtimes M}_{(0)} m$$

$\psi(\beta\gamma\delta)^{L_2\boxtimes M}_{(0)}$ and is homogenous of degree 0. Note also $L_2 \otimes_{L_1} M \simeq \bigoplus_{i=1}^{4} M_i$ where M_i are indecomposable or trivial. Noting that indecomposable L_1 modules are $\mathbb{Z}+q$ graded for some $q \in \mathbb{C}$, we have

$$\Delta(M) \equiv \Delta(m) = \Delta(\psi(\beta\gamma\delta)^{L_2\boxtimes M}_{(0)} m) \equiv \Delta(M_i) \mod \mathbb{Z} \tag{5.5}$$

for some $i = 1,..,4$ with M_i nontrivial. We recall the fusion products in 4.2.15 and treat the cases $r = 1, u-1$ cases uniformly with the others, since the absence of a second module in fusion in these cases puts stronger conformal weight restrictions on our parameters than the $r = 2,...,u-2$ case, which we will see is already sufficient. We define the relevant quantity for our conformal weight condition:

$$\begin{aligned}\nu^k_{t,r,s}(i) &= \Delta(\sigma^t(M^k_{\lambda,\Delta_{r,s}})) - \Delta(\sigma^t(M^k_{\lambda+i,\Delta_{r+i,s}}))\\ &= \Delta_{r,s} + \frac{1}{2}t\lambda + \frac{1}{4}t^2k - \Delta_{r+i,s} - \frac{1}{2}t(\lambda+i) - \frac{1}{4}t^2k\\ &= -\frac{2ir+i^2}{4(k+2)} + \frac{i(s-t)}{2}\end{aligned} \tag{5.6}$$

The conformal weight condition 5.5 reduces to

$$\begin{aligned}&-\frac{i_1r_1v+i_2r_2w}{2u} + \frac{i_1(s_1-t_1)+i_2(s_2-t_2)}{2} - \frac{1}{4} \pm \frac{1}{4}\\ &\qquad = \nu^k_{t_1,r_1,s_1}(i_1) + \nu^l_{t_2,r_2,s_2}(i_2) + \Delta(\mathcal{L}^1_{r_3,0}) - \Delta(\mathcal{L}^1_{\overline{r_3+1},0}) \in \mathbb{Z}\end{aligned} \tag{5.7}$$

for some $i_1, i_2 = \pm 1$, where we note that $\Delta(\mathcal{L}^1_{r_3,0}) - \Delta(\mathcal{L}^1_{\overline{r_3+1},0}) = \pm\frac{1}{4}$ and $i_1^2 = i_2^2 = 1$. Note that the contribution in blue is always a half integer, so 5.7 *implies the necessary condition:*

$$\frac{v}{u}(i_1r_1 - i_2r_2) + \frac{v+w}{u}i_2r_2 = \frac{i_1r_1v + i_2r_2w}{u} \in \mathbb{Z} \tag{5.8}$$

By coprimeness of u and v, this is equivalent to $i_1r_1 - i_2r_2 \in u\mathbb{Z}$. Since $r_1, r_2 \in \{1, ..., u-1\}$, we conclude that $r_2 \in \{r_1, u - r_1\}$. The result follows.

The simple forms of fusion products for modules labeled by 1 or $u-1$ suggests that we should begin here in our classification. We confirm that modules induced in this way are simple.

Lemma 5.6.2. *The induction $\mathcal{F}(M)$ is simple in RepA when M is an L_1 module of the form:*

$$M = \sigma^{t_1}(L^k_{\lambda_1,\Delta_{r_1,s_1}}) \otimes \sigma^{t_2}(N^l_{\lambda_2,\Delta_{r_2,s_2}}) \otimes \mathcal{L}^1_{p,0} \tag{5.9}$$

with $r_1, r_2 \in \{1, u-1\}$, and labels L, N, $\lambda_1, \lambda_2, \Delta_{r_1,s_1}, \Delta_{r_2,s_2}$ as in the table of 4.2.15.

Proof. *$A = \bigoplus_{r=1}^{u-1} L_r$ is a simple vertex superalgebra extension of L_1, each L_r is simple, and M is a simple L_1 module. Considering first the case with labels $r_1 = r_2 = 1$, we have*

$$L_r \boxtimes_{L_1} M \simeq \sigma^{t_1}(L^k_{\lambda_1+r-1;\Delta_{r,s_1}}) \otimes \sigma^{t_2}(N^l_{\lambda_2+r-1;\Delta_{r,s_2}}) \otimes \mathcal{L}^1_{\overline{p+r-1},0} \tag{5.10}$$

are distinct simple L_1 modules. Then by 5.4.7, we find that $\mathcal{F}(M)$ is a simple A module. The other cases follow similarly, noting that $N^{(u)\ r''}_{r,r'} = N^{(u)\ u-r''}_{r,u-r'}$.

We will need to introduce more machinery to confirm that modules with these labels do indeed appear as L_1-summands in irreducible $\mathbb{L}^1(D(2,1;-\frac{v}{w}))$ modules.

Lemma 5.6.3. *Let $k = \frac{u}{v} - 2$ and $l = \frac{u}{w} - 2$ be admissible levels for $\mathfrak{sl}(2)$. Then we have the following isomorphism as $\mathbb{L}^k(\mathfrak{osp}(1|2)) \otimes \mathbb{L}^{l+1}(\mathfrak{sl}(2))$ modules*

$$\mathbb{L}^1(D(2,1;-\frac{v}{w})) \simeq \bigoplus_{r=1}^{v+w+1} \mathbb{L}^k(r\omega) \otimes \mathcal{L}^{l+1}_{r,0} \tag{5.11}$$

Proof. *We first note*

$$\frac{1}{\tilde{l}+2} = 1 - \frac{1}{l+2+1} = \frac{u}{2u-v}$$

The result follows by applying 5.1.1 and 5.4.12.

$$\mathbb{L}^1(D(2,1;-\frac{v}{w})) \simeq \bigoplus_{r=1}^{u-1} \mathcal{L}^k_{r,0} \otimes \mathcal{L}^l_{r,0} \otimes \mathcal{L}^1_{\overline{r},0}$$
$$\simeq \bigoplus_{r=1}^{u-1} \bigoplus_{\substack{r''=1 \\ r''\ odd}}^{u+w+1} \mathcal{L}^k_{r,0} \otimes \mathcal{L}^{l+1}_{r'',0} \otimes \mathcal{M}^{\tilde{l}}_{r'',r} \simeq \bigoplus_{r''=1}^{u+w+1} \mathbb{L}^k(r''\omega) \otimes \mathcal{L}^{l+1}_{r'',0} \quad (5.12)$$

Proposition 5.6.4. *Consider L_1 module of the form*

$$M = \sigma^{t_1}(L^k_{\lambda,\Delta_{1,s_1}}) \otimes \sigma^{t_2}(N^l_{\lambda,\Delta_{r_2,s_2}}) \otimes \mathcal{L}^1_{p,0} \quad (5.13)$$

for labels $L, N, \lambda_1, \lambda_2, \Delta_{r,s_1}, \Delta_{r,s_2}$ as in the table of 4.2.15, $p = 1,2$ and $r_2 \in \{1, u-1\}$.

Then the induction $\mathcal{F}(M)$ is a local A module if and only if $r_2 = 1$ and

$$p = t_1 + s_1 + t_2 + s_2 + 1 \mod 2\mathbb{Z} \quad (5.14)$$

The induction $\mathcal{F}(M)$ is a Ramond twisted A modules if and only if $r_2 = 1$ and

$$p = t_1 + s_1 + t_2 + s_2 \mod 2\mathbb{Z} \quad (5.15)$$

Proof. *We recall the following special case of the fusion products 4.28:*

$$\mathcal{L}^1_{r,0} \times \mathcal{L}^1_{p,0} \simeq \mathcal{L}^1_{\overline{p+r-1},0} \quad (5.16)$$

M induces to a local module iff $R_{M,A} \circ R_{A,M} = 1$. A is integer graded so it has trivial twist. The twist on M is given by a constant $e^{2\pi i \Delta(M)}$. Balancing then gives

$$\theta_{A\otimes M} = R_{M,A} \circ R_{A,M} \circ (\theta_A \otimes \theta_M) = e^{2\pi i \Delta(M)} R_{M,A} \circ R_{A,M}$$

Each of the summands of $L_r \otimes M$ is also irreducible so

$$\theta_{A\otimes M}|_{L_r\otimes M} = e^{2\pi i\Delta(L_r\otimes M)}$$

We conclude that $A \otimes M$ is local if and only if $\Delta(L_r \otimes M) = \Delta(M)$ mod $\mathbb{Z}$ for all $r = 1, ..., u-1$, or equivalently $\Delta(L_{r+1} \otimes M) - \Delta(L_r \otimes M) \in \mathbb{Z}$ since $L_1 \otimes M \simeq M$. We define the quantity

$$\begin{aligned}\nu^k_{t,r,s} &= \Delta(\sigma^t(M^k_{\lambda+r,\Delta_{r+1,s}})) - \Delta(\sigma^t(M^k_{\lambda+r-1,\Delta_{r,s}})) \\ &= \Delta_{r+1,s} + \frac{1}{2}t(\lambda+r) + \frac{1}{4}t^2k - \Delta_{r,s} - \frac{1}{2}t(\lambda+r-1) - \frac{1}{4}t^2k \\ &= \frac{2r+1}{4(k+2)} + \frac{1}{2}(t-s) \qquad (5.17)\end{aligned}$$

Noting that

$$\Delta^{\frac{1}{r+p},0} - \Delta^{\frac{1}{r+p-1},0} \equiv -\frac{r+p}{2} + \frac{1}{4} \mod \mathbb{Z}$$

The locality condition for the module induced from module of the form 5.13 with $r_2 = 1$ reduces to:

$$\begin{aligned}&\nu^k_{t_1,r,s_1} + \nu^l_{t_2,r,s_2} + \Delta^{\frac{1}{r+p},0} - \Delta^{\frac{1}{r+p-1},0} \\ &\equiv \frac{2r+1}{4} + \frac{t_1-s_1+t_2-s_2}{2} - \frac{r+p}{2} + \frac{1}{4} \mod \mathbb{Z} \\ &= \frac{1}{2}(t_1-s_1+t_2-s_2-p+1) \in \mathbb{Z} \qquad (5.18)\end{aligned}$$

We note that bosonic states correspond to odd r and fermionic states to even r. Then the induction is Ramond twisted when

$$\nu^k_{t_1,r,s_1} + \nu^l_{t_2,r,s_2} + \Delta^{\frac{1}{r+p},0} - \Delta^{\frac{1}{r+p-1},0} = \frac{1}{2}(t_1-s_1+t_2-s_2-p+1) \in \mathbb{Z} + \frac{1}{2} \qquad (5.19)$$

Similarly, the locality condition for modules induced from 5.13 with $r_2 = u-1$

reduces to:

$$\begin{aligned}\nu^k_{t_1,r,s_1} + \nu^l_{t_2,u-r,s_2} + \Delta^1_{\overline{r+p},0} - \Delta^1_{\overline{r+p-1},0} \\ \equiv \frac{2(u-2r)}{4(l+2)} + \frac{1}{2}(t_1 - s_1 + t_2 - s_2 - p + 1) \mod \mathbb{Z} \\ = \frac{1}{2}w - \frac{rw}{u} + \frac{1}{2}(t_1 - s_1 + t_2 - s_2 - p + 1) \in \mathbb{Z} \quad (5.20)\end{aligned}$$

*for **all** $r = 1, ..., u-1$, and similarly, the Ramond condition reduces to*

$$\frac{1}{2}w - \frac{rw}{u} + \frac{1}{2}(t_1 - s_1 + t_2 - s_2 - p + 1) \in \mathbb{Z} + \frac{1}{2} \quad (5.21)$$

In particular, fixing $r = 1$, these conditions can hold only if $\frac{w}{u} \in \frac{1}{2}\mathbb{Z}$. Then by coprimeness of w and u, we conclude $u = 2$, so $r_2 = u - 1 = 1$. This case is exhausted by the first case $r_2 = 1$.

Theorem 5.6.5. *Let $k = \frac{u}{v} - 2$ and $l = \frac{u}{w} - 2$ be admissible levels for $\mathfrak{sl}(2)$. Suppose that M is an $\mathbb{L}^1(D(2,1;-\frac{v}{w}))$ module, with $M \in \mathbb{L}^1(D(2,1;-\frac{v}{w}))$-wtmod$_{\geq 0}$ or lower bounded and Ramond. Then*

$$M \simeq \mathcal{F}\left(L^k_{\lambda_1,\Delta_{1,s_1}} \otimes N^l_{\lambda_2,\Delta_{1,s_2}} \otimes \mathcal{L}^1_{p,0}\right)$$

for some $p \in \{1,2\}$ and for some labels $L, N, \lambda_1, \lambda_2, \Delta_{1,s_1}, \Delta_{1,s_2}$ as in the table of 4.2.15.

Proof. *$M \in \mathbb{L}^1(D(2,1;-\frac{v}{w}))$-wtmod$_{\geq 0}$ (resp. M is lower bounded and Ramond) for $\mathbb{L}^1(D(2,1;-\frac{v}{w}))$. We consider its restriction to a $\mathbb{L}^k(\mathfrak{osp}(1|2)) \otimes \mathbb{L}^{l+1}(\mathfrak{sl}(2))$ module. Then*

$$\mathcal{G}_{\mathbb{L}^k(\mathfrak{osp}(1|2))\otimes\mathbb{L}^{l+1}(\mathfrak{sl}(2))}(M) \simeq \bigoplus_{i\in I} M_i$$

Where M_i are nontrivial indecomposable $\mathbb{L}^k(\mathfrak{osp}(1|2)) \otimes \mathbb{L}^{l+1}(\mathfrak{sl}(2))$ modules. Fix i, and pick an irreducible $\mathbb{L}^k(\mathfrak{osp}(1|2)) \otimes \mathbb{L}^{l+1}(\mathfrak{sl}(2))$ module $M^k_{\mathfrak{osp}} \otimes M^{l+1}_{\mathfrak{sl}}$ with nontrivial morphism $M^k_{\mathfrak{osp}} \otimes M^{l+1}_{\mathfrak{sl}} \to M_i$. Since $\mathbb{L}^{l+1}(\mathfrak{sl}(2))$ is even, we conclude that $M^k_{\mathfrak{osp}}$ is local (resp. Ramond twisted). We consider the further *restriction of* $M^k_{\mathfrak{osp}}$ *as a* $\mathbb{L}^k(\mathfrak{sl}(2)) \otimes \mathbb{L}^{\tilde{l}}(Vir)$ *module. By 5.4.13 we see that*

$$\mathcal{G}_{\mathbb{L}^k(\mathfrak{sl}(2))\otimes\mathbb{L}^{\tilde{l}}(Vir)}(M^k_{\mathfrak{osp}}) \simeq \bigoplus_{r=1}^{u=1} N^k_{\lambda;\Delta_{r,s_1}} \otimes M^{\tilde{l}}_{r,s_2}$$

for some $N \in \{\mathcal{L}, \mathcal{D}^{\pm}, \mathcal{E}\}$*. Then:*

$$\mathcal{G}_{\mathbb{L}^k(\mathfrak{sl}(2))\otimes\mathbb{L}^{l+1}(\mathfrak{sl}(2))\times\mathbb{L}^{\tilde{l}}(Vir)}(M^k_{\mathfrak{osp}} \otimes M^{l+1}_{\mathfrak{sl}}) \simeq \bigoplus_{r=1}^{u-1} N^k_{\lambda;\Delta_{r,s}} \otimes M^{l+1}_{\mathfrak{sl}} \otimes \mathcal{M}^{\tilde{l}}_{s,r}$$

Since the isomorphism giving the decomposition of $\mathbb{L}^k(\mathfrak{sl}(2)\otimes\mathbb{L}^l(\mathfrak{sl}(2))\otimes\mathbb{L}^1(\mathfrak{sl}(2))$ *modules as* $\mathbb{L}^k(\mathfrak{sl}(2)\otimes\mathbb{L}^{l+1}(\mathfrak{sl}(2))\otimes\mathbb{L}^{\tilde{l}}(Vir)$ *modules does not affect the* $\mathbb{L}^k(\mathfrak{sl}(2))$ *factor,* $\mathcal{G}_{L_1}(M)$ *must have summand of the form*

$$\tilde{M} = N^k_{\lambda_1;\Delta_{1,s_1}} \otimes M^l_{\lambda_2;\Delta_{r_2,s_2}} \otimes \mathbb{L}^1_{p,0}$$

By 5.6.1 we conclude that $r_2 \in \{1, u-1\}$ *since* M *is irreducible. By 5.6.4, we conclude that* $r_2 = 1$ *and* $p = s_1 + s_2 + 1$ *(resp.* $p + s_1 + s_2$*) since* M *is local (resp. Ramond twisted).* $\tilde{M}$ *is not necessarily irreducible, since* $N_{\lambda_1;\Delta_{r,s}}$ *may have the form* $\mathcal{E}_{\lambda_{r,s};\Delta_{r,s}}$*, and similarly for* L*. But in this case, we may pick irreducible* M' *with the same* r_1, r_2 *labels and inclusion* $M' \to \tilde{M}$*. Then there is nontrivial* L_1 *morphism*

$$M' \to \mathcal{G}_{L_1}(M)$$

By Frobenius reciprocity, there is $\mathbb{L}^1(D(2,1;-\frac{v}{w}))$ *morphism:*

$$\mathcal{F}(M') \to M$$

By 5.6.2, $\mathcal{F}(M')$ *is irreducible, and we conclude that* $M \simeq \mathcal{F}(M')$ *by Schur's lemma.*

For L_1 module M of the form

$$M = \sigma^{t_1}(L^k_{\lambda,\Delta_{1,s_1}}) \otimes \sigma^{t_2}(N^l_{\lambda,\Delta_{1,s_2}}) \otimes \mathcal{L}^1_{\overline{s_1+s_2+t_1+t_2+1+b},0} \tag{5.22}$$

for labels $b \in \{0,1\}$, $L, N, \lambda_1, \lambda_2, \Delta_{r,s_1}, \Delta_{r,s_2}$ as in the table of 4.2.15, we introduce notation for the induced module:

$$(L, N)^{t_1,t_2}_{s_1,\lambda_1,s_2,\lambda_2,b} = \mathcal{F}(M) \tag{5.23}$$

Whose restriction to an L_1 module takes the form:

$$\bigoplus_{r=1}^{u-1} \sigma^{t_1}(L^k_{\lambda_1+r-1;\Delta_{r,s_1}}) \otimes \sigma^{t_2}(N^l_{\lambda_2+r-1;\Delta_{r,s_2}}) \otimes \mathcal{L}^1_{\overline{p+b+r-1},0} \tag{5.24}$$

For $p = s_1 + s_2 + t_2 + t_2 + 1$.

Theorem 5.6.6. *A complete list of representatives for isomorphism classes of irreducible modules in $\mathbb{L}^1(D(2,1;-\frac{v}{w})$-wtmod$_{\geq 0}$ (resp. Ramond twisted lower bounded modules for $\mathbb{L}^1(D(2,1;-\frac{v}{w}))$ is given by*

$$(L, N)^{0,0}_{s_1,\lambda_1,s_2,\lambda_2,b}$$

for labels $L, N, s_1, \lambda_1, s_2, \lambda_2$ as in the table 4.2.15 and $b = 0$ (resp. $b = 1$ for Ramond twisted). These are non-isomorphic for distinct labels.

Proof. *Suppose that M' is irreducible in $\mathbb{L}^1(D(2,1;-\frac{v}{w}))$-wtmod$_{\geq 0}$ (resp. Ramond twisted and lower bounded), By 5.6.5, $M' \simeq (L, N)^{0,0}_{s_1,\lambda_1,s_2,\lambda_2,b}$ for labels $L, N, \lambda_1, \lambda_2$ as in the table of 4.2.15. By 5.6.4, $b = 0$ (resp. $b = 1$ for Ramond twisted).*

As L_1 modules, by 5.24, $\mathcal{G}\left((L, N)^{0,0}_{s_1,\lambda_1,s_2,\lambda_2,b}\right)$ share no direct summands for distinct labels. Then the $(L, N)^{0,0}_{s_1,\lambda_1,s_2,\lambda_2,b}$ are non-isomorphic for distinct labels.

This completes the classification of irreducibles in $\mathbb{L}^1(D(2,1;-\frac{v}{w}))$-wtmod$_{\geq 0}$, as well as Ramond twisted lower bounded modules.

We expect that some analog of 3.5.2 should hold for Lie superalgebras, which would also complete the classification for irreducibles in $\mathbb{L}^1(D(2,1;-\frac{v}{w}))$-*wtmod* and the Ramond sector. These should take the form $(L, N)^{t_1,t_2}_{s_1,\lambda_1,s_2,\lambda_2,b}$ with $b = 0$ for local and $b = 1$ for Ramond twisted. I have already demonstrated that these are irreducible.

Chapter 6

Characters and modular transformations

Definition 6.0.1. *In the following section, we use the following notation for the delta distribution and the Dedekind eta function (respectively):*

$$\delta(z) = \sum_{n\in\mathbb{Z}} z^n \qquad\qquad \eta(q) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n) \tag{6.1}$$

We can now introduce the notion of characters for vertex algebra modules. For superalgebras, however, the invariant form does not come from a trace, but the supertrace. Then we must introduce a notion of character constructed in this way also, and notation to distinguish our two notions of character.

Definition 6.0.2. *Suppose that V is a vertex superalgebra, and M is a V module graded by $L_0, h^1_{(0)}, ..., h^n_{(0)}$ eigenvalues. Then the supercharacter of M is:*

$$ch^-[M] = str_M q^{L_0 - \frac{c}{24}} z_1^{h^1_{(0}} \dots z_n^{h^n_{(0)}} \tag{6.2}$$

and the character of M is:

$$ch^+[M] = tr_M q^{L_0 - \frac{c}{24}} z_1^{h^1_{(0}} \dots z_n^{h^n_{(0)}} \tag{6.3}$$

When V is an even vertex algebra and M is even, these two notions coincide and we shall denote them just by $ch[M]$.

Proposition 6.0.1. *[62] Let $k = \frac{u}{v} - 2$ be an admissible level for $\mathfrak{sl}(2)$. Then the $\mathbb{L}^k(Vir)$ modules $\mathcal{M}^k_{r,s}$ have characters given by:*

$$ch[\mathcal{M}^k_{r,s}] = \frac{1}{\eta(q)} \sum_{j \in \mathbb{Z}} \left[q^{(2uvn+vr-us)^2/4uv} - q^{(2uvn+vr+us)^2/4uv} \right] \tag{6.4}$$

Proposition 6.0.2. *For $\lambda \in \mathbb{C}$, the $\Pi^k(0)$ modules $\Pi^k_{(t)}(\lambda)$ have characters given by:*

$$ch[\Pi_{(t)}(\lambda)](z,q) = \frac{z^{tk+\lambda} q^{(t+1)\frac{\lambda-k}{2} + \frac{(t+1)^2}{4}k} \delta(z^2 q^{t+1})}{\eta(q)^2} \tag{6.5}$$

Proof. *The Heisenberg field h was introduced with $\langle h, c \rangle = 2$, $\langle h, d \rangle = k$, and $\langle h, h \rangle = 2k$. The mode h_0 commutes with all modes $d_{(n)}$ and $c_{(n)}$. Then on $M = \pi_L \left(nc + \frac{\lambda}{2} c + \frac{t}{2} h \right)$ h_0 acts as $2n + \lambda + tk$. Along with the conformal weight calculations in 5.4.10, we have*

$$ch_M(z,q) = tr z^h q^{L_0 - \frac{k}{4} - \frac{1}{6}} = \frac{z^{2n+\lambda+tk} q^{(t+1)n + (t+1)\frac{\lambda-k}{2} + \frac{(t+1)^2 k}{4}}}{\eta(q)^2}$$

The result follows by summing over n.

Proposition 6.0.3. *[53] $\mathbb{L}^1(\mathfrak{sl}(2))$ modules for $p \in 1, 2$ have characters given by:*

$$ch[\mathcal{L}^1_{p,0}](z,q) = \frac{1}{\eta(q)} \sum_{n \in \mathbb{Z}} z^{2n + \overline{p+1}} q^{\left(n + \frac{\overline{p+1}}{2}\right)^2} \tag{6.6}$$

and $\mathbb{L}^1(\mathfrak{sl}(2))^{tw}$ modules have characters:

$$ch[\mathcal{L}^1_{p,0}{}^{tw}](z,q) = \frac{1}{\eta(q)} \sum_{n \in \mathbb{Z}} z^{2n+\overline{p}} q^{\left(n + \frac{\overline{p}}{2}\right)^2} \tag{6.7}$$

Proposition 6.0.4. *[24] Let $k = \frac{u}{v} - 2$ be an admissible level for $\mathfrak{sl}(2)$ with $v > 1$. Then the $\mathbb{L}^k(\mathfrak{sl}(2))$ modules $\sigma^t(\mathcal{E}_{\lambda,\Delta_{r,s}})$ have characters:*

$$ch[\sigma^t(\mathcal{E}^k_{\lambda;\Delta_{r,s}})](z,q) = \frac{z^{\lambda+tk} q^{\frac{t^2 k}{4} + \lambda \frac{t}{2}} \delta(z^2 q^t)}{\eta(q)^2} ch[\mathcal{M}^k_{r,s}](q) \tag{6.8}$$

Remark 6.0.5. *Noting that the right hand side is not graded by h_0 eigenvalues,* *the result of 5.7 gives the following character identity:*

$$ch[\mathcal{M}^{k-1}_{s,s'}](q)\,ch[\mathcal{L}^{1}_{p,0}{}^{tw}](1,q) \quad = \sum_{\substack{r=1 \\ r-s-s'-p \text{ even}}}^{u-1} ch[\mathcal{M}^{k}_{r,s'}](q)\,ch[\mathcal{M}^{l}_{r,s}](q) \quad (6.9)$$

Proposition 6.0.6. *We have the following identity of $\mathbb{L}^k(\mathfrak{sl}(2)) \otimes \mathbb{L}^l(Vir)$ module characters:*

$$ch[\sigma^t(\mathcal{E}^{k-1}_{\lambda;\Delta_{s_2,s_1}})](z,q)\,ch[\mathcal{L}^{1}_{\overline{p+t},0}](z,q)$$
$$= \sum_{\substack{r=1 \\ r+s_1+s_2+p \text{ odd}}}^{u-1} ch[\sigma^t(\mathcal{E}^{k}_{\lambda+p-1;\Delta_{r,s_1}})](z,q)\,ch[\mathcal{M}^{l}_{r,s_2}](q) \quad (6.10)$$

Proof. *We first note*

$$\left(n+\frac{p+t}{2}\right)^2 = \left(n+\frac{p}{2}\right)^2 + tn + t\frac{p}{2} + \frac{t^2}{4}$$

and absorb $z^{2n}q^{tn}$ in $\delta(z^2q^t)$

$$\delta(z^2q^t)\,ch[\mathcal{L}^{1}_{\overline{p+t},0}{}^{tw}](z,q) = \frac{\delta(z^2q^t)}{\eta(q)}\sum_{n\in\mathbb{Z}} z^{2n+p+t}q^{\left(n+\frac{p+t}{2}\right)^2}$$
$$= \frac{z^{t+p}q^{\frac{t^2}{4}+\frac{t^2}{4}+t\frac{p}{2}}\delta(z^2q^t)}{\eta(q)}\sum_{n\in\mathbb{Z}} q^{\left(n+\frac{p}{2}\right)^2} = z^{t+p}q^{\frac{t^2}{4}+\frac{t^2}{4}+t\frac{p}{2}}\delta z^2q^t\,ch[\mathcal{L}^{1}_{p,0}](1,q) \quad (6.11)$$

$$
\begin{aligned}
& ch[\sigma^t(\mathcal{E}^{k-1}_{\lambda;\Delta_{s_2,s_1}})](z,q)\, ch[\mathcal{L}^1_{\overline{p+t},0}{}^{tw}](z,q) \\
&\quad = \frac{z^{\lambda+t(k-1)} q^{\frac{t^2}{4}(k-1)+\lambda\frac{t}{2}} \delta(z^2q^t)}{\eta(q)^2} ch[\mathcal{M}^k_{r,s}](q)\, ch[\mathcal{L}^1_{\overline{p+t},0}{}^{tw}](z,q) \\
&\quad = \frac{z^{\lambda+p+tk} q^{\frac{t^2}{4}k+(\lambda+p)\frac{t}{2}} \delta(z^2q^t)}{\eta(q)^2} ch[\mathcal{M}^k_{r,s}](q)\, ch[\mathcal{L}^1_{\overline{p},0}{}^{tw}](1,q) \\
&\quad = \frac{z^{\lambda+p+tk} q^{\frac{t^2}{4}k+(\lambda+p)\frac{t}{2}} \delta(z^2q^t)}{\eta(q)^2} \sum_{\substack{r=1 \\ r+s_1+s_2+p \ even}}^{u-1} ch[\mathcal{M}^k_{r,s_1}](q)\, ch[\mathcal{M}^l_{r,s_2}](q) \\
&\quad = \sum_{\substack{r=1 \\ r+s_1+s_2+p \ even}}^{u-1} ch[\sigma^t(\mathcal{E}^k_{\lambda+p;\Delta_{r,s_1}})](z,q)\, ch[\mathcal{M}^l_{r,s_2}](q) \qquad (6.12)
\end{aligned}
$$

The result follows noting that $ch[\mathcal{L}^1_{p+1,0}](z,q) = ch[\mathcal{L}^1_{p,0}{}^{tw}](z,q)$

6.1 The Adamovic construction

Theorem 6.1.1. *[1] Let $k = \frac{u}{v} - 2$ be an admissible level for $\mathfrak{sl}(2)$. Then there exists injective homomorphism of conformal vertex algebras:*

$$\mathbb{L}^k(\mathfrak{sl}(2)) \hookrightarrow \mathbb{L}^k(Vir) \otimes \Pi^k(0) \qquad (6.1)$$

Then $\mathbb{L}^k(Vir) \otimes \Pi^k(0)$ modules are $\mathbb{L}^k(\mathfrak{sl}(2))$ modules by restriction.

Theorem 6.1.2. *[1] Let $k = \frac{u}{v} - 2$ be an admissible level for $\mathfrak{sl}(2)$. Then there is injective homomorphism of $\mathbb{L}^k(\mathfrak{sl}(2))$ modules:*

$$\mathcal{L}^k_{r,0} \hookrightarrow \mathcal{M}^k_{r,1} \otimes \Pi^k_{(0)}(\overline{r} - 1) \qquad (6.2)$$

Theorem 6.1.3. *[1] Let k be an admissible level for $\mathfrak{sl}(2)$. Then there is $\mathbb{L}^k(\mathfrak{sl}(2))$ module isomorphism:*

$$\sigma^t(\mathcal{E}^k_{\lambda;\Delta_{r,s}}) \simeq \mathcal{M}^k_{r,s} \otimes \Pi^k_{(t-1)}(\lambda + k) \qquad (6.3)$$

Theorem 6.1.4. *Let $k = \frac{u}{v} - 2$ and $l = \frac{u}{w} - 2$ be admissible levels for $\mathfrak{sl}(2)$. Then there are injective homomorphism of conformal vertex superalgebras:*

$$\mathbb{L}^1(D(2,1;-\frac{v}{w})) \to \mathbb{L}^{k-1}(\mathfrak{sl}(2)) \otimes B^l \tag{6.4}$$

$$\mathbb{L}^1(D(2,1;-\frac{v}{w})) \to \mathbb{L}^{l-1}(\mathfrak{sl}(2)) \otimes B^k \tag{6.5}$$

Proof. *Using the injections of 6.1.2, and the relations of 5.1.1 on both the even and odd parts separately, we have:*

$$\mathbb{L}^1(D(2,1;-\frac{v}{w})) \simeq \bigoplus_{r=1}^{u-1} \mathcal{L}^k_{r,0} \otimes \mathcal{L}^l_{r,0} \otimes \mathcal{L}^1_{\bar{r},0} \to \bigoplus_{r=1}^{u-1} \mathcal{L}^k_{r,0} \otimes \mathcal{M}^l_{r,1} \otimes \Pi^l_{(0)}\,(\bar{r}-1) \otimes \mathcal{L}^1_{\bar{r},0}$$
$$\simeq \mathbb{L}^{k-1}(\mathfrak{sl}(2)) \otimes \mathcal{L}^1_{1,0} \otimes \Pi^l(0) \otimes \mathcal{L}^1_{1,0} \oplus \mathbb{L}^{k-1}(\mathfrak{sl}(2)) \otimes \mathcal{L}^1_{2,0} \otimes \Pi^l_{(0)}(1) \otimes \mathcal{L}^1_{2,0}$$
$$\simeq \mathbb{L}^{k-1}(\mathfrak{sl}(2)) \otimes B^l \tag{6.6}$$

We may similarly obtain an injective homomorphism

$$\mathbb{L}^1(D(2,1;-\frac{v}{w})) \to \mathbb{L}^{l-1}(\mathfrak{sl}(2)) \otimes B^k$$

by instead using the injection $\mathcal{L}^k_{r,0} \to \mathcal{M}^k_{r,1} \otimes \Pi^k_{(0)}\,(\bar{r}-1)$.

Lemma 6.1.5. *Let $k = \frac{u}{v} - 2$ and $l = \frac{u}{w} - 2$ be admissible levels for $\mathfrak{sl}(2)$. We have the following identity of $L^1(D(2,1;-\frac{v}{w}))$ module characters:*

$$\begin{aligned} &ch^+[(\mathcal{E},\mathcal{E})^{t_1,t_2}_{s_1,\lambda_1,s_2,\lambda_2,b}](z_1,z_2,z_3,q) \\ &\quad = ch^+[\sigma^{t_1}(\mathcal{E}^{k-1}_{\lambda_1+s_1+s_2+1,\Delta_{s_2,s_1}}) \otimes B^l_{(t_2)}(\lambda_2+a; s_1+s_2+t_1+a, p+a+b)] \end{aligned} \tag{6.7}$$

$$\begin{aligned} &ch^-[(\mathcal{E},\mathcal{E})^{t_1,t_2}_{s_1,\lambda_1,s_2,\lambda_2,b}](z_1,z_2,z_3,q) \\ &= (-1)^a ch^-[\sigma^{t_1}(\mathcal{E}^{k-1}_{\lambda_1+s_1+s_2+1,\Delta_{s_2,s_1}}) \otimes B^l_{(t_2)}(\lambda_2+a; s_1+s_2+t_1+a, p+a+b)] \end{aligned} \tag{6.8}$$

For $a, b \in \{0, 1\}$.

Proof.

$$ch[\sigma^{t_1}(\mathcal{E}^{k-1}_{\lambda_1+s_1+s_2+1,\Delta_{s_2,s_1}})](z_1,q)\,ch^{\pm}[B^l_{(t_2)}(\lambda_2+a;a+s_1+s_2+t_1,b)](z_2,z_3,q)$$
$$= ch[\sigma^{t_1}(\mathcal{E}^{k-1}_{\lambda_1-s_1-s_2+1;\Delta_{s_2,s_1}})](z_1,q)\,ch[\mathcal{L}^1_{a+s_1+s_2+t_1+1,0}\otimes\Pi^l_{(t_2-1)}(\lambda_2+a+l)\otimes\mathcal{L}^1_{p+a+b,0}](z_2,z_3,q)$$
$$\pm ch[\sigma^{t_1}(\mathcal{E}^{k-1}_{\lambda_1-s_1-s_2;\Delta_{s_2,s_1}})](z_1,q)\,ch[\mathcal{L}^1_{a+s_1+s_2+t_1+1,0}\otimes\Pi^l_{(t_2-1)}(\lambda_2+a+l+1)\otimes\mathcal{L}^1_{b+1,0}](z_2,z_3,q)$$

$$= \sum_{\substack{r=1\\ r+a\ odd}}^{u-1} ch[\sigma^{t_1}(\mathcal{E}^k_{\lambda_1+a;\Delta_{r,s_1}})](z_1,q)\,ch[\sigma^{t_2}(\mathcal{E}^l_{\lambda_2+a;\Delta_{r,s_2}})](z_2,q)\,ch[\mathcal{L}^1_{p+a+b,0}](z_3,q)$$
$$\pm \sum_{\substack{r=1\\ r+a\ even}}^{u-1} ch[\sigma^{t_1}(\mathcal{E}^k_{\lambda_1+a+1;\Delta_{r,s_1}})](z_1,q)\,ch[\sigma^{t_2}(\mathcal{E}^l_{\lambda_2+a+1;\Delta_{r,s_2}})](z_2,q)\,ch[\mathcal{L}^1_{p+a+b+1,0}](z_3,q)$$

Noting that in the first summand, we have $a = r-1 \mod 2$, and in the second summand, we have $a+1 = r-1 \mod 2$, we obtain in the super character case

$$= (-1)^a ch^- \sum_{r=1}^{u-1} \sigma^{t_1}(\mathcal{E}^k_{\lambda_1+r-1;\Delta_{r,s_1}}) \otimes \sigma^{t_2}(\mathcal{E}^l_{\lambda_2+r-1;\Delta_{r,s_2}}) \otimes \mathcal{L}^1_{p+b+r-1,0} \tag{6.9}$$

and in the character case we have

$$= ch^+ \sum_{r=1}^{u-1} \sigma^{t_1}(\mathcal{E}^k_{\lambda_1+r-1;\Delta_{r,s_1}}) \otimes \sigma^{t_2}(\mathcal{E}^l_{\lambda_2+r-1;\Delta_{r,s_2}}) \otimes \mathcal{L}^1_{p+b+r-1,0} \tag{6.10}$$

Proposition 6.1.6. *Let $k = \frac{u}{v}-2$ and $l = \frac{u}{w}-2$ be admissible levels for $\mathfrak{sl}(2)$. The characters and supercharacters of $\mathbb{L}^1(D(2,1;-\frac{v}{w}))$ modules M of the form $(\mathcal{E},\mathcal{E})^{t_1,t_2}_{s_1,\lambda_1,s_2,\lambda_2,b}$ are linearly independent for non-isomorphic module. We have the isomorphism of $\mathbb{L}^1(D(2,1;-\frac{v}{w}))$ modules*

$$(\mathcal{E},\mathcal{E})^{t_1,t_2}_{s_1,\lambda_1,s_2,\lambda_2,b}$$
$$\simeq \sigma^{t_1}(\mathcal{E}^{k-1}_{\lambda_1+s_1+s_2+1,\Delta_{s_2,s_1}}) \otimes B^l_{(t_2)}(\lambda_2+a;s_1+s_2+t_1+a,p+a+b) \tag{6.11}$$

Proof. *Consider linear dependence $\sum_{i\in I} c_i ch^{\pm}[M_i] = 0$ for modules $M_i =$*

$(\mathcal{E},\mathcal{E})^{t_{1,i},t_{2,i}}_{s_{1,i},\lambda_{1,i},s_{2,i},\lambda_{2,i},b_i}$ *nonisomorphic for distinct* $i \in I$*. WLOG we assume that* I *is minimal in that there is no linear dependence summing over* $I' \subsetneq I$*. These characters and supercharacters take the form:*

$$z_1^{\lambda_{1,i}+r-1+t_{1,i}k} q^{\frac{t_{1,i}^2 k}{4}+\lambda_{1,i}+r-1+\frac{t_{1,i}}{2}} z_2^{\lambda_{2,i}+t_{2,i}l} q^{\frac{t_{2,i}^2 l}{4}+\lambda_{2,i}\frac{t_{2,i}}{2}}$$
$$\sum_{r=1}^{u-1} ch[\mathcal{M}^k_{r,s_{1,i}}](q)\, ch[\mathcal{M}^l_{r,s_{2,i}}](q) \frac{\delta(z_1^2 q^{t_{1,i}})\delta(z_2^2 q^{t_{2,i}})}{\eta(q)^4} \sum_{n\in\mathbb{Z}} z_3^{2n+p_i+b_i+r} q^{\left(n+\frac{p_i+b_i+r}{2}\right)^2} \tag{6.12}$$

We first look at the support of distributions in these characters. Since these are distinct for distinct t_1, t_2*, any minimal linear dependence must have contributions only from modules with matching labels* $t_{1,i} = t_{1,j}$*,* $t_{2,i} = t_{2,j}$*. We notice that the* z_1, z_2, z_3 *contributions are the same except for the shift* $z_1^{\lambda_1+t_1k} z_2^{\lambda_2+t_2l} z_3^b$*. In particular, since* t_1 *and* t_2 *labels match, we conclude that* λ_1, λ_2 *and* b *labels match. The remaining matching of labels then follows from independence of Virasoro characters.*

The isomorphism of modules follows from 6.1.5 *and the inclusion 6.1.3.*

6.2 Modular transformations

Remark 6.2.1. *We write define the parameters* τ, ξ *such that:*

$$q = e^{2\pi i\tau} \qquad\qquad z = e^{2\pi i\xi} \tag{6.1}$$

Proposition 6.2.2. $\mathbb{L}^1(\mathfrak{sl}(2))$ *modules have* S *matrices given by:*

$$S_{a,a'} = \frac{1}{\sqrt{2}} e^{2i\pi\frac{\xi^2}{\tau}} (-1)^{(a+1)(a'+1)} \tag{6.2}$$

$\mathbb{L}^1(\mathfrak{sl}(2))^{tw}$ *modules have* S *matrices given by:*

$$S^{tw}_{a,a'} = \frac{1}{\sqrt{2}} e^{2\pi i\frac{\xi^2}{\tau}} (-1)^{aa'} \tag{6.3}$$

Proof. *We use the conventions of [61] for theta functions. Notice that* $ch[\mathcal{L}^1_{1,0}](\xi,\tau) = \frac{\theta_3(2\xi,2\tau)}{\eta(q)}$ *and* $ch[\mathcal{L}^1_{2,0}](\xi,\tau) = \frac{\theta_2(2\xi,2\tau)}{\eta(q)}$.

$$\begin{aligned} ch[\mathcal{L}^1_{1,0}]\left(\frac{\xi}{\tau},-\frac{1}{\tau}\right) &= \frac{1}{\eta(-\frac{1}{\tau})}\theta_3\left(2\frac{\xi}{\tau},-\frac{2}{\tau}\right) = \frac{1}{\sqrt{2}}e^{2\pi i\frac{\xi^2}{\tau}}\frac{\theta_3(\xi,\frac{\tau}{2})}{\eta(q)} \\ &= \frac{1}{\sqrt{2}}e^{2\pi i\frac{\xi^2}{\tau}}\left(ch[\mathcal{L}^1_{1,0}]+ch[\mathcal{L}^1_{2,0}]\right) \end{aligned} \tag{6.4}$$

$$\begin{aligned} ch[\mathcal{L}^1_{2,0}]\left(\frac{\xi}{\tau},-\frac{1}{\tau}\right) &= \frac{1}{\eta(-1/\tau)}\theta_2\left(2\frac{\xi}{\tau},-\frac{1}{\tau}\right) = \frac{1}{\sqrt{2}}e^{2\pi i\frac{\xi^2}{\tau}}\frac{\theta_4(\xi,\tau/2)}{\eta(q)} \\ &= \frac{1}{\sqrt{2}}e^{2\pi i\frac{\xi^2}{\tau}}(ch[\mathcal{L}^1_{1,0}]-ch[\mathcal{L}^1_{2,0}]) \end{aligned} \tag{6.5}$$

Theorem 6.2.3. *[26] Let* $k=\frac{u}{v}-2$ *be an admissible level for* $\mathfrak{sl}(2)$ *with* $v>1$. *Then* S *transformations for Virasoro module characters are given by:*

$$S\{ch[\mathcal{M}^k_{r,s}]\} = \sum_{r',s'} S^{Vir}_{(r,s)(r',s')}ch[\mathcal{M}^k_{r',s'}] \tag{6.6}$$

with S *matrices given by:*

$$S^{Vir}_{(r,s)(r',s')} = -2\sqrt{\frac{2}{uv}}(-1)^{rs'+r's}\sin\frac{v\pi rr'}{u}\sin\frac{u\pi ss'}{v} \tag{6.7}$$

Theorem 6.2.4. *[24] Let* $k=\frac{u}{v}-2$ *be an admissible level for* $\mathfrak{sl}(2)$ *with* $v>1$. *Then the* S *transformation for standard* $\mathbb{L}^k(\mathfrak{sl}(2))$ *modules is given by:*

$$S\{ch[\sigma^t(\mathcal{E}^k_{\lambda,\Delta_{r,s}})]\} = \sum_{t'\in\mathbb{Z}}\sum_{r',s'}\int_{-1}^{1} S_{(t,\lambda;\Delta_{r,s})(t',\lambda';\Delta_{r',s'})}ch[\sigma^{t'}(\mathcal{E}^k_{\lambda',\Delta_{r',s'}})]d\lambda' \tag{6.8}$$

with S *matrices given by:*

$$S_{(t,\lambda;\Delta_{r,s})(t',\lambda';\Delta_{r',s'})} = \frac{1}{2}\frac{|\tau|}{-i\tau}e^{-i\pi(ktt'+t\lambda'+t'\lambda)}S^{Vir}_{(r,s)(r',s')} \tag{6.9}$$

Proposition 6.2.5. *S transformations for $\Pi^k(0)$ modules are given by:*

$$S\{ch[\Pi^k_{(t-1)}(\lambda+k)]\} = \int_{-1}^{1} S^{\Pi}_{(t,\lambda)(t',\lambda')} ch[\Pi^k_{(t'-1)}(\lambda'+k)] d\lambda' \quad (6.10)$$

with S matrices:

$$S^{\Pi}_{(t,\lambda)(t',\lambda')} = \frac{1}{2}\frac{|\tau|}{-i\tau} e^{-i\pi(ktt'+t\lambda'+t'\lambda)} \quad (6.11)$$

Proof. *The argument is exactly as in [24] theorem 6.*

Theorem 6.2.6. *Modular transformations of characters and supercharacters of $\mathbb{L}^1(D(2,1;-\frac{v}{w}))$ modules induced from relaxed highest weight modules for $\mathbb{L}(\mathfrak{sl}(2))$ are given by:*

(supercharacter, local)

$$S\{ch^-[(\mathcal{E},\mathcal{E})^{t_1,t_2}_{s_1,\lambda_1,s_2,\lambda_2,0}]\} = \sum_{t'_1,t'_2\in\mathbb{Z}} \int_{-1}^{1}\int_{-1}^{1} \sum_{s'_1,s'_2} S^{t_1,t_2\ \ t'_1,t'_2}_{s_1,\lambda_1,s_2,\lambda_2,0\, s'_1,\lambda'_1,s'_2,\lambda'_2,0} ch^-[(\mathcal{E},\mathcal{E})^{t'_1,t'_2}_{s'_1,\lambda'_1,s'_2,\lambda'_2,0}] d\lambda'_1 d\lambda'_2 \quad (6.12)$$

(supercharacter, Ramond)

$$S\{ch^-[(\mathcal{E},\mathcal{E})^{t_1,t_2}_{s_1,\lambda_1,s_2,\lambda_2,1}]\} = \sum_{t'_1,t'_2\in\mathbb{Z}} \int_{-1}^{1}\int_{-1}^{1} \sum_{s'_1,s'_2} S^{t_1,t_2\ \ t'_1,t'_2}_{s_1,\lambda_1,s_2,\lambda_2,1\, s'_1,\lambda'_1,s'_2,\lambda'_2,0} ch^+[(\mathcal{E},\mathcal{E})^{t'_1,t'_2}_{s'_1,\lambda'_1,s'_2,\lambda'_2,0}] d\lambda'_1 d\lambda'_2 \quad (6.13)$$

(character, local)

$$S\{ch^+[(\mathcal{E},\mathcal{E})^{t_1,t_2}_{s_1,\lambda_1,s_2,\lambda_2,0}]\} = \sum_{t'_1,t'_2\in\mathbb{Z}} \int_{-1}^{1}\int_{-1}^{1} \sum_{s'_1,s'_2} S^{t_1,t_2\ \ t'_1,t'_2}_{s_1,\lambda_1,s_2,\lambda_2,0\, s'_1,\lambda'_1,s'_2,\lambda'_2,1} ch^-[(\mathcal{E},\mathcal{E})^{t'_1,t'_2}_{s'_1,\lambda'_1,s'_2,\lambda'_2,1}] d\lambda'_1 d\lambda'_2 \quad (6.14)$$

(character, Ramond)

$$S\{ch^+[(\mathcal{E},\mathcal{E})^{t_1,t_2}_{s_1,\lambda_1,s_2,\lambda_2,1}]\} = \sum_{t'_1,t'_2\in\mathbb{Z}} \int_{-1}^{1}\int_{-1}^{1} \sum_{s'_1,s'_2} S^{t_1,t_2}_{s_1,\lambda_1,s_2,\lambda_2,1}{}^{t'_1,t'_2}_{s'_1,\lambda'_1,s'_2,\lambda'_2,1} ch^+[(\mathcal{E},\mathcal{E})^{t'_1,t'_2}_{s'_1,\lambda'_1,s'_2,\lambda'_2,1}]d\lambda'_1 d\lambda'_2 \quad (6.15)$$

Where the S matrices are:

$$S^{t_1,t_2}_{s_1,\lambda_1,s_2,\lambda_2,b}{}^{t'_1,t'_2}_{s'_1,\lambda'_1,s'_2,\lambda'_2,b'} = \frac{ue^{\frac{2\pi i}{\tau}}}{2\sqrt{2}\sin\left(\frac{w\pi}{u}\right)\sin\left(\frac{v\pi}{u}\right)} S^k_{(t_1,\lambda_1;\Delta_{1,s_1})(t'_1,\lambda'_1;\Delta_{1,s'_1})} S^l_{(t_2,\lambda_2;\Delta_{1,s_2})(t'_2,\lambda'_2;\Delta_{1,s'_2})} S_{(p+b)(p'+b')} \quad (6.16)$$

Proof. *We will calculate S matrices using the character identity in proposition 6.1.5. We must first calculate modular transformations for the* $B^l_{(t_2)}(\lambda_2+a;s_1+s_2+t_1+a,p+a+b)$*, which is given by the S transformations of characters of each of its constituent parts. We begin by identifying any cancellations that might occur from its even and odd portions. To this end, we identify relations between S matrices for the even and odd portions, corresponding to replacing the parameter a with* $a+1$*. We first notice for* $p=s_1+s_2+t_1+t_2+1$ *and* $p'=s_1+s_2+t_1+t_2+1$ *we have*

$$S_{(s_1+s_2+t_1+a)(s'_1+s'_2+t'_1+a')} S_{(p+a+b)(p'+a'+b')} S^{\Pi}_{(t_2,\lambda_2+a)(t'_2,\lambda'_2+a')} = (-1)^{b'+1} S_{(s_1+s_2+t_1+a+1)(s'_1+s'_2+t'_1+a')} S_{(p+a+1+b)(p'+a'+b')} S^{\Pi}_{(t_2,\lambda_2+a+1)(t'_2,\lambda'_2+a')} \quad (6.17)$$

We need only determine the S transformation for $ch^{\pm}[B^l_{(t_2)}(\lambda_2;s_1+s_2+t_1,p+b)]$*, as the additional parameter corresponds only to parity reversal. In the following, the new parameter a gives the sum over the even and odd portions* *of* $B^l_{(t_2)}(\lambda_2; s_1+s_2+t_1, p+b)$.

$$\begin{aligned} &S\{ch^{\pm}[B^l_{(t_2)}(\lambda_2; s_1+s_2+t_1, p+b)]\} \\ &= \sum_{t_2'\in\mathbb{Z}} \int_{-1}^{1} d\lambda_2' \sum_{a,a',b'=0,1} \Big(S_{(s_1+s_2+t_1+a)(s_1'+s_2'+t_1'+a')} S_{(p+a+b)(p'+a'+b')} S^{\Pi}_{(t_2,\lambda_2+a)(\lambda_2'+a')} \\ &\qquad \cdot (\pm 1)^a ch[\mathcal{L}^1_{s_1'+s_2'+t_1'+a',0}] ch[\Pi^l_{(t_2')}(\lambda_2'+l+a')] ch[\mathcal{L}^1_{p'+a'+b'}] \Big) \end{aligned} \tag{6.18}$$

The sum over a then results in a factor $(1 \pm (-1)^{b'+1})$. *In the supercharacter case, this is* $2\delta_{b',0}$, *and in the character case this is* $2\delta_{b',1}$. *We will note that entirely similarly to before, if we replace the parameter* a' *by* $a'+1$ *in the second line, we have:*

$$\begin{aligned} &S_{(s_1+s_2+t_1+a)(s_1'+s_2'+t_1'+a')} S_{(p+a+b)(p'+a'+b')} S^{\Pi}_{(t_2,\lambda_2+a)(t_2',\lambda_2'+a')} \\ &\quad = (-1)^{b+1} S_{(s_1+s_2+t_1+a)(s_1'+s_2'+t_1'+a'+1)} S_{(p+a+b)(p'+a'+1+b')} S^{\Pi}_{(t_2,\lambda_2+a)(t_2',\lambda_2'+a'+1)} \end{aligned} \tag{6.19}$$

Then in the supercharacter case, the sum over b' *fixes* $b'=0$. *Summing over* a' *then gives:*

$$\begin{aligned} &S\{ch^{-}[B^l_{(t_2)}(\lambda_2; s_1+s_2+t_1, p+b)]\} \\ &\qquad = 2\sum_{t_2'\in\mathbb{Z}} \int_{-1}^{1} d\lambda_2' \Big(S_{(s_1+s_2+t_1)(s_1'+s_2'+t_1')} S_{(p+b)(p')} S^{\Pi^l}_{(t_2,\lambda_2)(t_2',\lambda_2')} \\ &\cdot \Big(ch[\mathcal{L}^1_{s_1'+s_2'+t_1',0}] ch[\Pi^l_{(t_2')}(\lambda_2'+l)] ch[\mathcal{L}^1_{p'}] \\ &\qquad - (-1)^b ch[\mathcal{L}^1_{s_1'+s_2'+t_1'+1,0}] ch[\Pi^l_{(t_2')}(\lambda_2'+l+1)] ch[\mathcal{L}^1_{p'+1}] \Big) \Big) \end{aligned} \tag{6.20}$$

Then in the case $b = 0$ we have:

$$S\{ch^-[B^l_{(t_2)}(\lambda_2; s_1+s_2+t_1, p)]\}$$
$$= 2\sum_{t_2'\in\mathbb{Z}}\int_{-1}^{1} d\lambda_2' S_{(s_1+s_2+t_1)(s_1'+s_2'+t_1')}S_{(p)(p')}S^{\Pi^l}_{(t_2,\lambda_2)(t_2',\lambda_2')}\,ch^-[B^l_{(t_2')}(\lambda_2'; s_1'+s_2'+t_1', p')] \tag{6.21}$$

For $b = 1$ we have:

$$S\{ch^-[B^l_{(t_2)}(\lambda_2; s_1+s_2+t_1, p+1)]\}$$
$$= 2\sum_{t_2'\in\mathbb{Z}}\int_{-1}^{1} d\lambda_2' S_{(s_1+s_2+t_1)(s_1'+s_2'+t_1')}S_{(p+1)(p')}S^{\Pi^l}_{(t_2,\lambda_2)(t_2',\lambda_2')}\,ch^+[B^l_{(t_2')}(\lambda_2'; s_1'+s_2'+t_1', p')] \tag{6.22}$$

Then in the character case, the sum over b' fixes $b' = 1$. Summing over a' then gives:

$$S\{ch^+[B^l_{(t_2)}(\lambda_2; s_1+s_2+t_1, p+b)]\}$$
$$= 2\sum_{t_2'\in\mathbb{Z}}\int_{-1}^{1} d\lambda_2' \Big(S_{(s_1+s_2+t_1)(s_1'+s_2'+t_1')}S_{(p+b)(p'+1)}S^{\Pi^l}_{(t_2,\lambda_2)(t_2',\lambda_2')}$$

$$\cdot\Big(ch[\mathcal{L}^1_{s_1'+s_2'+t_1',0}]\,ch[\Pi^l_{(t_2')}(\lambda_2'+l)]\,ch[\mathcal{L}^1_{p'+1}]$$
$$-(-1)^b ch[\mathcal{L}^1_{s_1'+s_2'+t_1'+1,0}]\,ch[\Pi^l_{(t_2')}(\lambda_2'+l+1)]\,ch[\mathcal{L}^1_{p'}]\Big)\Big) \tag{6.23}$$

In the $b = 0$ case we have:

$$S\{ch^+[B^l_{(t_2)}(\lambda_2; s_1+s_2+t_1, p)]\}$$
$$= 2\sum_{t_2'\in\mathbb{Z}}\int_{-1}^{1} d\lambda_2' S_{(s_1+s_2+t_1)(s_1'+s_2'+t_1')}S_{(p)(p'+1)}S^{\Pi^l}_{(t_2,\lambda_2)(t_2',\lambda_2')}\,ch^-[B^l_{(t_2')}(\lambda_2'; s_1'+s_2'+t_1', p'+1)] \tag{6.24}$$

and in the $b = 1$ case we have:

$$
\begin{aligned}
&S\{ch^+[B^l_{(t_2)}(\lambda_2; s_1 + s_2 + t_1, p + 1)]\} \\
&= 2 \sum_{t'_2 \in \mathbb{Z}} \int_{-1}^{1} d\lambda'_2 S_{(s_1+s_2+t_1)(s'_1+s'_2+t'_1)} S_{(p+1)(p'+1)} S^{\Pi^l}_{(t_2,\lambda_2)(t'_2,\lambda'_2)} ch^+[B^l_{(t'_2)}(\lambda'_2; s'_1+s'_2+t'_1, p'+1)]
\end{aligned}
\tag{6.25}
$$

Noting that $z_1 = 1$ corresponds to $\xi_1 = \pm 1$, this gives the S matrices:

$$
\begin{aligned}
&2S_{(s_1+s_2+t_1)(s'_1+s'_2+t'_1)} S_{(p+b)(p'+b')} S^{\Pi^l}_{(t_2,\lambda_2)(t'_2,\lambda'_2)} \\
&\qquad = \sqrt{2} e^{\frac{2\pi i}{\tau}} (-1)^{(s_1+s_2+t_1+1)(s'_1+s'_2+t'_1+1)} S_{(p+b)(p'+b')} S^{\Pi^l}_{(t_2,\lambda_2)(t'_2,\lambda'_2)}
\end{aligned}
\tag{6.26}
$$

for the B contribution in character, supercharacter, twisted and untwisted cases (with appropriate choices of b and b' for each of these cases). We will now focus on the $\sigma^{t_1}(\mathcal{E}_{\lambda_1+s_1+s_2+1;\Delta_{s_2,s_1}})$ contribution. We note that

$$
S^{\Pi^{k-1}}_{(t_1,\lambda_1)(t'_1,\lambda'_1)} = (-1)^{t_1 t'_1 + t_1(s'_1+s'_2+1) + t'_1(s_1+s_2+1)} S^{\Pi^k}_{(t_1,\lambda_1)(t'_1,\lambda'_1)}
$$

and we have the following identity:

$$
\begin{aligned}
&S^{Vir,k}_{(1,s_1)(1,s'_1)} S^{Vir,l}_{(1,s_2)(1,s'_2)} \\
&= 4\sqrt{\frac{4}{u^2 vw}} (-1)^{s_1+s'_1+s_2+s'_2} \sin\left(\frac{v\pi}{u}\right) \sin\left(\frac{u\pi s_1 s'_1}{v}\right) \sin\left(\frac{w\pi}{u}\right) \sin\left(\frac{u\pi s_2 s'_2}{w}\right) \\
&\frac{8}{u}\sqrt{\frac{2}{vw}} (-1)^{s_1+s'_1+s_2+s'_2+s_1 s'_1+s_2 s'_2} \sin\left(\frac{v\pi}{u}\right) \sin\left(\frac{w\pi s_1 s'_1}{v}\right) \sin\left(\frac{w\pi}{u}\right) \sin\left(\frac{v\pi s_2 s'_2}{w}\right) \\
&\qquad = \frac{4}{u} (-1)^{(s_1+s_1+1)(s'_1+s'_2+1)} \sin\left(\frac{w\pi}{u}\right) \sin\left(\frac{v\pi}{u}\right) S^{Vir,k-1}_{(s_2,s'_2)(s_1,s'_1)}
\end{aligned}
\tag{6.27}
$$

Then:

$$
\begin{aligned}
&S^{k-1}_{(t_1,\lambda_1+s_1+s_2+1;\Delta_{s_2,s_1})(t'_1,\lambda'_1+s'_1+s'_2+1;\Delta_{s'_2,s'_1})} = S^{Vir,\ k-1}_{(s_2,s'_2)(s_1,s'_1)} S^{\pi^{k-1}}_{(t_1,\lambda_1+s_1+s_2+1)(t'_1,\lambda'_1+s'_1+s'_2+1)} \\
&\qquad = \frac{(-1)^{(s_1+s_2+t_1+1)(s'_1+s'_2+t'_1+1)} u}{4\sin\left(\frac{w\pi}{u}\right)\sin\left(\frac{v\pi}{u}\right)} S^{Vir,k}_{(1,s_1)(1,s'_1)} S^{Vir,l}_{(1,s_2)(1,s'_2)} S^{\Pi^k}_{(t_1,\lambda_1)(t'_1,\lambda'_1)}
\end{aligned}
\tag{6.28}
$$

Combining with the contribution from B, we obtain S matrices of the form

$$\frac{ue^{\frac{2\pi i}{\tau}}}{2\sqrt{2}\sin\left(\frac{w\pi}{u}\right)\sin\left(\frac{v\pi}{u}\right)} S^k_{(t_1,\lambda_1;\Delta_{1,s_1})(t'_1,\lambda'_1;\Delta_{1,s'_1})} S^l_{(t_2,\lambda_2;\Delta_{1,s_2})(t'_2,\lambda'_2;\Delta_{1,s'_2})} S_{(p+b)(p'+b')} \tag{6.29}$$

again, with appropriate choice of b and b' for each of the character, supercharacter, twisted and untwisted cases.

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