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Minimal Anisotropic Groups of Higher Real Rank

by

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Abstract

The purpose of this thesis is to give a classification of anisotropic algebraic groups over number fields of higher real rank. This will complete the classification of algebraic groups over number fields of higher real rank, which was begun by V. Chernousov, L. Lifschitz and D.W. Morris in their paper "Almost-Minimal Non-Uniform Lattices of Higher Rank". The classification of anisotropic groups of higher real rank is also used to provide a classification of uniform lattices of higher rank contained in semisimple Lie groups with no compact factors. In particular, it is shown that all such lattices sit inside Lie groups of type A_n.

This thesis proceeds as follows: The first chapter provides motivation for the classification and introduces all the main results of the thesis. The second chapter provides relevant definitions and background material for the proof. The next chapters provide a proof of the classification theorem, with chapters 3-5 examining the absolutely simple groups and the final chapter examining the simple groups which are not absolutely simple.

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Contents

Chapter 1

Introduction

1.1 Motivation

Lattices are an important class of discrete subgroups in Lie group theory, with many physical applications. In particular, in the field of crystallography the *space group* is a lattice which describes the symmetry of a given crystal. One of the challenges in studying lattices in Lie groups is that it is not always possible to realize Lie groups as matrix groups, hence we cannot always construct explicit realizations of lattices sitting in the corresponding Lie groups. This forces us to rely on properties which are intrinsic to the lattice. When considering the intrinsic properties of lattices, it is often useful to know which are minimal under inclusion, as these commonly form a base case for induction arguments.

One interesting property that can be examined using induction arguments is: When can a lattice act in a non-trivial, orientation-preserving way on simple smooth manifolds such as the real line? The action of groups on simple manifolds such as the real line and the 1-circle has long been studied and have applications in many diverse areas in mathematics. In particular, Thompson's groups can be realized as homeomorphisms of the circle. These groups have been used in the study of infinite simple groups, homotopy and shape theory, group cohomology, dynamical systems and analysis ([4]).

An intrinsic property of lattices that is also interesting to several areas of mathematics is that of a left ordering. Left orderings of groups are of interest to several branches of mathematics, including algebraic topology and abstract group theory. The canonical example of a group with a left ordering is the additive group of the real numbers with the natural order. In 1968, LaGrange and Rhemtulla were able to prove that if the $\mathbb{Z}$-group rings of two abstract groups are isomorphic and one of the groups is left ordered, then the groups themselves must be isomorphic ([13]). It is well-known that a group can act in a non-trivial, orientation-preserving way on the real line if and only if it can be given a left ordering ([16], Remark 1.5(3), p. 2). In the case of the additive group of the real numbers, one can consider their action on the real line by translation.

The existence of a left ordering (or a non-trivial action on the real line) is a restrictive condition in general. The multiplicative group of the real numbers does not have a left ordering, for example. In 1999, E. Ghys conjectured that no irreducible lattice sitting inside a semisimple Lie group of rank at least 2 has a left ordering. The conjecture remains open in general, but some cases have been proven. In [16], Lifschitz and Morris were able to prove Ghys conjecture for lattices which are non-uniform.

In order to prove Ghys' conjecture for the non-uniform lattices, Lifschitz and Morris employed a two-step strategy. First, together with Chernousov, they classified the (almost) minimal non-uniform lattices under inclusion. If a non-uniform lattice Γ were to have a left ordering, then one could restrict the ordering to arbitrary sublattices of Γ. This would imply that at least one of the minimal lattices would have a left ordering. Next, Lifschitz and Morris proved that none of the lattices obtained in the first step act non-trivially on the real line in an order preserving way. Combining the two steps proves Ghys' conjecture in the non-uniform case.

The first step, examining the minimal non-uniform lattices, was done by translating the problem from a Lie-theoretic one to a question about algebraic groups. Using the Margulis Arithemeticity and Superrigidity theorems, Chernousov, Lifschitz and Morris were able to show that classifying the minimal lattices in Lie groups of higher rank is equivalent to classifying semisimple groups over the rational numbers with higher real rank. Moreover, using Margulis' theorems, one can see that classifying the minimal non-uniform lattices is equivalent to classifying the isotropic groups over the rational numbers with higher real rank while classifying the minimal uniform lattices is equivalent to classifying the anisotropic groups over the rational numbers with higher real rank. Because Lifschitz and Morris were interested in the non-uniform lattices, Chernousov, Lifschitz and Morris restricted themselves to the isotropic algebraic groups.

To prove Ghys' conjecture for the remaining case, the uniform lattices, it is natural to follow the same steps as the proof in the non-uniform case. As stated before, the first step, classifying the minimal non-uniform lattices, is equivalent to classifying minimal anisotropic groups over $\mathbb{Q}$ with higher real rank. That is the focus of this thesis. The second step, proving that none of the corresponding uniform lattices has a left ordering, remains open.

Classifying minimal anisotropic groups over arbitrary fields has been an extremely difficult problem of long standing interest in its own right. The proofs of several important theorems (including the Hasse principle, mentioned below) depend on the idea of finding subgroups of anisotropic groups. For some group types it is possible to find proper semisimple subgroups regardless of the base field. In groups of type G_2 and F_4, for instance, it is possible to construct such subgroups by examining their root systems (specifically, the sub-root system generated by the long roots). In [25], Tits was able to construct an example of a group of type E_8 which does not contain any non-trivial connected subgroups but the maximal tori. More recently, Garibaldi and Gille have constructed groups of trialitarian type D_4 which also have this property ([9]).

Once we begin placing restrictions on the base field, the problem of construction subgroups of semisimple groups becomes easier. Over number fields, for instance, it is possible to show that every group of type E_8 contains a non-trivial semisimple subgroup, which is contrary to the case over arbitrary fields. The case this thesis is focused on, however, is made more difficult by the restriction that subgroups of the groups in question must have appropriate real rank. In this case, virtually nothing was known.

This thesis gives a complete classification of minimal anisotropic groups over number fields that have higher rank over certain completions. The definitions (given below) ensure that when we restrict to the case of the rational numbers we return to the case we began with, namely the anisotropic groups over the rational numbers which have higher real rank. In particular, it is shown that all minimal anisotropic groups have type ${}^{1,2}A_{p-1}$ for some prime p. This is a more complicated list than was obtained by Chernousov, Lifschitz and Morris in the isotropic case, but this still leaves a (relatively) small list of minimal uniform lattices to be considered in order prove Ghys conjecture in the uniform case.

1.2 The Classification

We postpone all relevant definitions to the background section. The classification of minimal anisotropic groups over number fields is broken into two pieces, first the absolutely simple minimal groups:

Theorem 1.1 *If G is an absolutely simple, minimal, anisotropic group over an algebraic number field F, then G is isomorphic to one of the following groups (up to isogeny):*

1. *$\mathbb{SU}_3(L,f)$ for L/F quadratic, f anisotropic hermitian on L^3 with at least one $v \in V^F_{\infty,\mathbb{R}}$ such that $L \otimes F_v \simeq F_v \times F_v$, or*

2. *$\mathbb{SU}(D,\tau)$ a central division algebra of prime degree $p \geq 3$ over L quadratic over F with involution of the second kind τ, or*

3. *$\mathbb{SL}(D)$ for a central division algebra D over F of prime degree $p > 2$.*

Next, we classify the non-absolutely simple groups:

Theorem 1.2 *If G is a minimal anisotropic group over an algebraic number field F that is not absolutely simple, then G is isomorphic to one of the following groups, up to isogeny (let $\epsilon = \pm 1$):*

1. *$R_{K/F}(\mathbb{SL}(D))$ for a central division algebra D of odd prime degree over an extension K such that D does not descend to any P with $F \subset P \subsetneq K$ or,*

2. *$R_{K/F}(\mathbb{SU}(D,\tau))$, where D is a central division algebra of prime degree $p \geq 3$ over a quadratic extension K'/K with involution of the second kind τ such that if (D,τ) descends to P' with $F \subset P \subsetneq K$ and P'/P quadratic, then $P_{w_i} \simeq \mathbb{R}$ and $P_{w_i} \otimes P' \simeq \mathbb{C}$ for all $w_i \in V^P_{\infty,\mathbb{R}}$ lying over at least one $v_0 \in S_G$ and*

   (a) *if $v_0 \in S'_G$ then $(D' \otimes_P P_{w_i}, \tau' \otimes 1) \simeq (\mathrm{M}_n(\mathbb{C}), \epsilon\langle 1, \ldots, 1\rangle)$ for all $w_i \in V^P_{\infty,\mathbb{R}}$ lying over v_0, or*

(b) *if* $v_0 \in S''_G$ *then* $(D' \otimes_P P_{w_i}, \tau' \otimes 1) \simeq (\mathrm{M}_n(\mathbb{C}), \epsilon\langle 1, -1, 1 \ldots, 1\rangle)$ *for at most one* i *and* $(D' \otimes_P P_{w_i}, \tau' \otimes 1) \simeq (\mathrm{M}_n(\mathbb{C}), \epsilon\langle 1, \ldots, 1\rangle)$ *for all others, or*

3. $R_{K/F}(\mathbb{SL}(D))$ *for* D *a quaternion division algebra over* K *such that for every* $F \subset P \subsetneq K$ *such that* D *descends to* P *there exist* $v_0 \in S_G$ *satisfying*

   (a) *If* $v_0 \in S'_G$ *then* $P_{w_i} \simeq \mathbb{R}$ *and* $D' \otimes_P P_{w_i} \simeq \mathbb{H}$ *for all* $w_i \in V^P_{\infty,\mathbb{R}}$ *lying over* v_0 *and*

   (b) *If* $v_0 \in S''_G$ *then there is at most one* $w_i \in V^P_{\infty,\mathbb{R}}$ *lying over* v_0 *such that either* $P_{w_i} \simeq \mathbb{C}$ *or* $D' \otimes_P P_{w_i} \simeq \mathrm{M}_2(\mathbb{R})$, *but not both, or*

4. $R_{K/F}(\mathbb{SU}_3(K', f))$ *for* K'/K *quadratic,* f *hermitian over* K'^3 *such that*

   (a) *For any* $F \subset P \subsetneq K$ *such that* $\mathbb{SU}_3(K', f)$ *descends to* P *we have that there exists a* $v_0 \in S_G$ *such that* $P_{w_i} \simeq \mathbb{R}$ *for all* $w_i \in V^P_{\infty,\mathbb{R}}$ *lying over* v_0 *and*

      i. *If* $\mathbb{SU}_3(K', f)$ *descends to* $\mathbb{SU}_3(P', f')$, *where* $f' = \langle 1, a_2, a_3\rangle$ *then* $P_{w_i} \otimes P' \simeq \mathbb{C}$ *for every* i *and*

         A. *if* $v_0 \in S'_G$ *then the image of* a_j *in* P_{w_i} *is positive for all* i

         B. *if* $v_0 \in S''_G$ *then the image of* a_j *in* P_{w_i} *is negative for at most one* i

      ii. *if* $\mathbb{SU}_3(K', f)$ *descends to* $\mathbb{SU}(D, \tau)$ *where* D *is a central division algebra of degree* 3 *over* P'/P *quadratic with involution* τ *of the second kind then* $P' \otimes P_{w_i} \simeq \mathbb{C}$ *for every* $w_i \in V^P_{\infty,\mathbb{R}}$ *lying over* v_0 *and*

         A. *If* $v_0 \in S'_G$ *then* $(D \otimes P_{w_i}, \tau \otimes 1) \simeq (\mathrm{M}_3(\mathbb{C}), \sigma)$, *where* $\sigma(X) = \overline{X}^T$, *for every* $w_i \in V^P_{\infty,\mathbb{R}}$

B. *If* $v_0 \in S''_G$ *then* $(D \otimes P_{w_i}, \tau \otimes 1) \simeq (\mathrm{M}_3(\mathbb{C}), \sigma)$ *for all but at most one* $w_i \in V^P_{\infty,\mathbb{R}}$ *and for at most one* w_i, $(D \otimes P_{w_i}, \tau \otimes 1) \simeq (\mathrm{M}_3(\mathbb{C}), \sigma \circ \mathrm{Int}(\epsilon\, \mathrm{diag}(1, -1, 1)))$

(b) *For any* $F \subset P \subseteq K$ *such that some subgroup* $\mathbb{SL}(D') \leq \mathbb{SU}_3(K', f)$ *descends to* $\mathbb{SL}(D)$ *over* P *there exists some* $v_0 \in S_G$ *such that*

i. *If* $v_0 \in S'_G$ *then* $P_{w_i} \simeq \mathbb{R}$ *and* $D \otimes P_{w_i} \simeq \mathbb{H}$ *for all* $w_i \in V^P_{\infty,\mathbb{R}}$ *over* v_0 *and*

ii. *if* $v_0 \in S''_G$ *then* $P_{w_i} \simeq \mathbb{C}$ *or* $D \otimes P_{w_i} \simeq \mathrm{M}_2(\mathbb{R})$ *for at most one* $w_i \in V^P_{\infty,\mathbb{R}}$ *over* v_0.

Applying the Margulis Arithmeticity and Superrigidity theorems, this gives the classification of the minimal semisimple real Lie groups with no compact factors containing uniform irreducible lattices of higher rank:

Theorem 1.3 *Every uniform lattice of higher rank contained in a semisimple Lie group with no compact factors contains a subgroup that is isomorphic to a finite index subgroup of a lattice contained in either* $\mathrm{SL}_p(\mathbb{R})^\ell \times \mathrm{SL}_p(\mathbb{C})^m \times \mathrm{SU}_p(\mathbb{C}, f_1) \times \cdots \times \mathrm{SU}_p(\mathbb{C}, f_n)$ *where* f_i *are Hermitian forms of index at least* 1 *or* $\mathrm{SL}_2(\mathbb{R})^n \times \mathrm{SL}_2(\mathbb{C})^m$ *with* $n + m \geq 2$.

Chapter 2

Background

2.1 Lattices in Lie Groups

If G is a real, connected semisimple Lie group, a subgroup Γ of G is called a *lattice* if the induced topology is discrete and the quotient G/Γ has finite Haar measure (see [19], page 221 for details). Given that Lie groups originally arose from considering homeomorphisms of smooth manifolds, it is natural to ask when there are morphisms from subgroups of G to the homeomorphism groups of connected manifolds. The simplest possible case of a connected real manifold is $\mathbb{R}$ itself and morphisms form Γ to the homeomorphisms of $\mathbb{R}$ as a differentiable manifold are equivalent to orientation-preserving actions of Γ on the real line. An interesting class of lattices are the irreducible ones:

Definition 2.1 *A lattice* Γ *is* irreducible *if it contains no subgroup* Γ' *of finite index such that* $\Gamma' = \Gamma_1 \times \Gamma_2$ *with* Γ_i *both infinite.*

Notice that we cannot avoid the consideration of finite index subgroups, since every finite index subgroup of a lattice Γ in G is also a lattice in G. Define the rank of Γ to be the rank of G and define Γ to be of higher rank if the rank of Γ is at least two. In 1999, E. Ghys conjectured the following:

Conjecture 2.2 ([8]) *Suppose that* Γ *is an irreducible lattice of higher rank. Then* Γ *has no non-trivial, orientation preserving action on* $\mathbb{R}$*.*

If Γ is a lattice as in the conjecture, then any action of Γ on the real line can be restricted to any subgroup of Γ. Thus to prove Ghys conjecture it suffices to examine those lattices are minimal under inclusion. Because any finite index subgroup of Γ is also a lattice, however, there are no minimal irreducible lattices of higher rank. If we allow finite-index subgroups, however, we obtain:

Definition 2.3 *A lattice* Γ *of higher rank is* almost minimal *if no proper subgroups of infinite index are also lattices of higher rank.*

It is reasonable, therefore, to consider the following two-step approach to proving Ghys' conjecture:

1. Classify all irreducible almost minimal lattices of higher rank, and
2. prove that no irreducible almost minimal lattices of higher rank have non-trivial, orientation preserving actions on $\mathbb{R}$.

Using two celebrated theorems due to Margulis, the first step can be translated from a question about Lie groups to a question about algebraic groups. Notice that if F is an algebraic group over $\mathbb{Q}$, the real points $F(\mathbb{R})$ of F can be given the structure of a real Lie group. Next, we construct a lattice in $F(\mathbb{R})$. Choose a faithful representation $F \hookrightarrow \mathbb{GL}_n$ and consider $F(\mathbb{Z}) := F(\mathbb{Q}) \cap \mathbb{GL}_n(\mathbb{Z})$. We then have that $F(\mathbb{Z}) \subset F(\mathbb{R})$ is a lattice (see [19], Theorem 4.13, p. 213), and any finite-index subgroup of such a group is called an *arithmetic lattice*. If we choose another representation of F, then we may get another group but it is possible to show that the pre-images of the two subgroups will be *commensurable* in $F(\mathbb{R})$ (i.e. their intersection will have finite index in either group). Because we are only concerned with infinite-index subgroups, this is acceptable.

Theorem 2.4 (Margulis Arithmeticity, [17], IX, Theorem 1.10, p. 298)

Given an irreducible lattice Γ *in a semisimple Lie group of higher rank with no compact factors, one can find a* $\mathbb{Q}$*-algebraic group* F *and a Lie group surjection* $\tau : F(\mathbb{R}) \to G$ *such that:*

1. $\ker(\tau)$ *is compact, and*

2. $\tau(F(\mathbb{Z}))$ *is commensurable with* Γ*.*

This theorem allows us to 'approximate' irreducible lattices in semisimple Lie groups of higher rank by arithmetic lattices. The Margulis superrigidity theorem states that this approximation almost respects inclusion.

Theorem 2.5 (Margulis Superrigidity, [17], IX, Theorem 5.12, p. 327)

Given an embedding of lattices $\gamma : \Gamma' \hookrightarrow \Gamma$ *and algebraic* $\mathbb{Q}$*-groups* F' *and* F *corresponding to* Γ' *and* Γ*, respectively, there exists an morphism of algebraic groups* $\delta : F' \to F$ *that almost extends* γ*.*

By definition, δ *almost* extends γ if the two agree on finite-index subgroups of Γ and Γ'. Again, because we are unconcerned with taking subgroups of finite index, combining the above two theorems gives that finding almost minimal irreducible lattices of higher rank is equivalent to finding minimal simple $\mathbb{Q}$-groups of higher real rank. When considering algebraic groups over $\mathbb{Q}$, there is a dichotomy between the anisotropic groups and the isotropic groups (discussed below). Applying the correspondence given by the arithmeticity and superrigidity theorems, this corresponds exactly to the dichotomy between those cases where G/Γ is compact (in which case Γ is called *uniform*) and those cases where G/Γ is not compact (then Γ is called *non-uniform*).

The minimal, isotropic $\mathbb{Q}$-groups have been classified by Chernousov, Lifschitz and Morris. In fact they classified minimal algebraic groups over number fields, considering isotropic groups that have higher rank over the completion F_v with respect to some archimedean places v on F.

Definition 2.6 ([6], Definition 3.3, p. 7) *Let G be a simple, isotropic group over a number field F and let $V^F_{\infty,\mathbb{R}}$ be the set of real places on F. Let $S_G \subset V^F_{\infty,\mathbb{R}}$ be the set of places such that $\mathrm{Rank}_{F_v}(G) \geq 2$. We say that G is* minimal *if $S_G \neq \emptyset$ and there does not exist a proper, isotropic, simple F-subgroup H of G such that $\mathrm{Rank}_{F_v}(H) \geq 2$ for all $v \in S_G$.*

Under this definition, Chernousov, Lifschitz and Morris proved:

Theorem 2.7 ([6], Theorem 3.4, p. 7) *Suppose G is an isotropic, simple algebraic group over an algebraic number field F such, such that $S_G \neq \emptyset$. If G is minimal, then G is isogenous to either:*

1. $\mathbb{SL}_3$*, or*
2. $\mathbb{SU}_3(L, f, \tau)$*, where*
   - *L is a quadratic extension of F, such that $L \subset F_v$ for some archimedean place v of F,*
   - *τ is the Galois automorphism L over F, and*
   - $f(x_1, x_2, x_3) = \tau(x_1)x_1 - \tau(x_2)x_2 - \tau(x_3)x_3)$*,*

or

3. $R_{K/F}(\mathbb{SU}_3(L, f, \tau))$, *where*

    - K *is a quadratic extension of* F*, such that* $K \not\subset F_v$*, for some archimedean place* v *of* F*,*
    - L *is a quadratic extension of* K*,*
    - τ *is the Galois automorphism* L *over* K*, and*
    - $f(x_1, x_2, x_3) = \tau(x_1)x_1 - \tau(x_2)x_2 - \tau(x_3)x_3$*,*

or

4. $R_{K/F}(\mathbb{SL}_2)$ *for some nontrivial finite extension* K *of* F*, such that either* $[K : F] > 2$*, or* $K \subset F_v$ *for every archimedean place* v *of* F*.*

Using the Marguils arithmeticity and superrigidity theorems, the above statement implies the following:

Theorem 2.8 ([6], Theorem 1.13, p. 3) *Every nonuniform lattice of higher rank contains a subgroup that is isomorphic to a finite index subgroup of a lattice contained in either* $\mathrm{SL}_3(\mathbb{R})$, $\mathrm{SL}_3(\mathbb{C})$ *or a direct product* $\mathrm{SL}_2(\mathbb{R})^m \times \mathrm{SL}_2(\mathbb{C})^n$*, with* $m + n \geq 2$*.*

This provides an answer for the first step in proving Ghys' conjecture for nonuniform lattices. The second step was completed by Lifschitz and Morris in [16], using arguments relying on virtually bounded generation by unipotent elements. The remaining case for Ghys' conjecture is the uniform lattices. The first step is to classify the almost minimal uniform lattices of higher rank, and by the arithemticity and superrigidity theorem this is equivalent to classifying the minimal anisotropic $\mathbb{Q}$-groups of higher real rank. That is the focus of this thesis.

2.2 Algebraic Groups

Assume that all fields are of characteristic 0 unless stated otherwise.

2.2.1 Tori and Rank

A *torus* over $\mathbb{k}$ is a group which becomes isomorphic to a number of copies of the multiplicative group, $\mathbb{G}_{\mathrm{m}}$, over the algebraic closure $\overline{\mathbb{k}}$. If a torus T is isomorphic to a number of copies of $\mathbb{G}_{\mathrm{m}}$ over $\mathbb{k}$, then we call T $\mathbb{k}$-split. On the other hand, if T does not contain a subgroup isomorphic to $\mathbb{G}_{\mathrm{m}}$ we say that T is anisotropic. Given a semisimple group G over $\mathbb{k}$, we define the $\mathbb{k}$-rank of G to be the maximum dimension of a $\mathbb{k}$-split torus that embeds in G. The *absolute rank* of G is the rank of G over $\overline{\mathbb{k}}$. Notice that the absolute rank is the highest possible rank.

If the $\mathbb{k}$-rank of G is equal to the absolute rank of G, then we say that G is $\mathbb{k}$-split. On the other extreme, we say that G is anisotropic over $\mathbb{k}$ if the $\mathbb{k}$-rank of G is 0. Up to quotients by finite central subgroups, simple split groups are categorized by their Dynkin diagrams, of which there are four infinite families, A_n, B_n, C_n and D_n and five exceptional types E_6, E_7, E_8, F_4 and G_2. For example, $\mathbb{SL}_{n+1}$ has type A_n and $\mathbb{SO}_{2\ell+1}(f)$ has type $B_{2\ell+1}$, where $f = \ell \cdot \langle 1, -1\rangle \oplus \langle 1\rangle$.

2.2.2 Forms and Tits' Classification

Assume for this section that G is a simple $\mathbb{k}$-defined algebraic group. Recall that over $\overline{\mathbb{k}}$ G becomes split and split groups are characterized by their Dynkin diagram (or *type*). Given two groups G and G' over $\mathbb{k}$ that become isomorphic to G_0 over $\overline{\mathbb{k}}$, we say that G and G' are $\mathbb{k}$-*forms* of G_0. The forms of G_0 over $\mathbb{k}$ (up to $\mathbb{k}$-isomorphism) are in bijective correspondence with the the first Galois cohomology set $H^1(\mathbb{k}, \mathrm{Aut}(G))$

(see [22], Chapter 3 for more details). Letting Γ be the automorphism group of the Dynkin diagram of G_0, we have the following exact sequence of pointed sets:

$$H^1(\Bbbk, G_0/Z(G_0)) \to H^1(\Bbbk, \mathrm{Aut}(G_0)) \to H^1(\Bbbk, \Gamma)$$

and we say that a form G of G_0 is *inner* if the corresponding cocycle in $H^1(\Bbbk, \mathrm{Aut}(G_0))$ has trivial image in $H^1(\Bbbk, \Gamma)$ and we say that G is *outer* otherwise. If we assume further that G_0 is simply connected (i.e. there are non-trivial surjections $G' \to G_0$ with finite kernel), then we have another exact sequence:

$$H^1(\Bbbk, G_0) \to H^1(\Bbbk, G_0/Z(G_0)) \to H^2(\Bbbk, Z(G_0)).$$

If G is an inner form of G_0 and G corresponds to $[\xi] \in H^1(\Bbbk, \mathrm{Aut}(G))$, then we can consider a pre-image $[\chi]$ of $[\xi]$ in $H^1(\Bbbk, G_0/Z(G_0))$. If the image of $[\chi]$ in $H^2(\Bbbk, Z(G_0))$ is trivial, then we say that G is a *strongly inner* form of G_0.

Given a form G of G_0 corresponding to $[\xi] \in H^1(\Bbbk, \mathrm{Aut}(G))$, if G_0 has type X and the image of $[\xi]$ in $H^1(\Bbbk, \Gamma)$ has order m, we say that G has type ${}^mX$ (in particular, G is an inner form if and only if it is of type ${}^1X$). Given a group G of type ${}^mX$, we call G *quasi-split* if it contains a $\Bbbk$-defined, connected, solvable group B such that G/B is a complete variety. It can be shown that for each type ${}^mX$, there is a unique quasi-split group (up to isogeny) and the rank of a quasi-split group is maximal among groups of type ${}^mX$. In particular, an inner, quasi-split group is split.

Example 2.9 *Assume that $-1 \notin \Bbbk^{\times^2}$ and consider the following quadratic forms over $\Bbbk$:*

$$f_0 := \langle 1,-1,1,-1,1,-1,1,-1,1,-1,1,-1 \rangle$$

$$f_1 := \langle 1,1,1,-1,1,-1,1,-1,1,-1,1,-1 \rangle$$

$$f_2 := \langle 1,1,1,1,1,-1,1,-1,1,-1,1,-1 \rangle$$

also let D be a non-split quaternion algebra with canonical involution τ over $\mathbb{k}$ and f_3 be a 6-dimensional τ-skew-hermitian form with trivial discriminant. Let $G_i = \mathbb{SO}_{12}(f_i)$ for $i = 0, 1, 2$ and let $G_3 = \mathbb{SU}_6(D, f_3, \tau)$. Then G_0 is split of type D_6 and after extension to $\overline{\mathbb{k}}$, G_i all become isomorphic to G_0. One can show that the image of the cocycles correpsonding to G_i in $H^1(\mathbb{k}, \Gamma) = \mathbb{k}^\times/\mathbb{k}^{\times^2}$ are given by the discriminant of the corresponding quadratic or skew-hermitian form. Thus G_0, G_2 and G_3 are all inner, while G_1 is outer by the assumption that $-1 \notin \mathbb{k}^{\times^2}$. It can also be shown that strongly inner forms of G_0 are all of the form $\mathbb{SO}_{12}(f)$ for some quadratic form f, thus G_3 is an inner form but not a strongly inner form.

Because -1 has order two in $\mathbb{k}^\times/\mathbb{k}^{\times^2}$, we have that G_1 is of type ${}^2D_6$. In fact G_1 is quasi-split of type ${}^2D_6$ and the $\mathbb{k}$-rank of G_0 is 5, which is maximal among all groups of type ${}^2D_6$ (since groups of type D_6 with $\mathbb{k}$-rank 6 are split, and hence inner).

In [24], Tits gives a construction of the 'Tits index' of simple groups. Although I will not go into detail of what this involves, these Tits indices classify simple groups up to their anisotropic parts. Tits also gives a list of possible indices that can occur for groups over finite fields, the field of real numbers, p-adic fields and number fields. For each possible Tits index, the general form of groups with such an index is also given. For example, groups of type ${}^1A_n$ over an algebraic number field F are all of the form $\mathbb{SL}(A)$ for a central simple algebra A over F of degree $n + 1$.

2.2.3 Galois Cohomology of Algebraic Groups

Because of the connection between first Galois cohomology sets and forms of algebraic groups, the following results will be necessary. The first is due to Kneser:

Theorem 2.10 ([19], Theorem 6.4, p. 284) *If $\Bbbk$ is a non-archimedean completion of a number field, then $H^1(\Bbbk, G_0) = \{1\}$.*

In particular, this implies that over a non-archimedean completion of a number field there are no non-trivial strongly inner forms of semisimple groups.

If $H \leq G$ is a subgroup, then a cocycle $[\xi] \in H^1(\Bbbk, G)$ is said to have coefficients in H if there is a cocycle $[\gamma] \in H^1(\Bbbk, H)$ such that $[\gamma]$ maps to $[\xi]$ under the natural map $H^1(\Bbbk, H) \to H^1(\Bbbk, G)$. The following theorem, due to Steinberg, gives many important examples when a cocycle has coefficients in a maximal torus:

Theorem 2.11 ([5], Theorem 3.1, p. 301) *Let G_0 be a simple (not necessarily simply connected) linear algebraic group, split or quasi-split over $\Bbbk$ and let $\xi \in Z^1(\Bbbk, G_0)$ be a cocycle with corresponding twisted group $G =_\xi G_0$. For any maximal torus $S \subset G$ over $\Bbbk$ there is an $\Bbbk$-embedding $S \hookrightarrow G_0$ such that the class $[\xi]$ lies in the image of $H^1(\Bbbk, S) \to H^1(\Bbbk, G_0)$.*

In the case that $\Bbbk = \mathbb{R}$, Borovoi proved the following:

Theorem 2.12 ([2], Lemma 1, p. 135) *If G is a connected reductive group over $\mathbb{R}$ and T_a is a maximal anisotropic $\mathbb{R}$-torus of G, then $H^1(\mathbb{R}, T_a) \to H^1(\mathbb{R}, G)$ is a surjection.*

Given an algebraic group G over $\Bbbk$ and an extension $\Bbbk \subset \Bbbk'$, one can define the restriction maps $\mathrm{Res} : H^i(\Bbbk, G) \to H^i(\Bbbk', G_{\Bbbk'})$. In the case that G is the automorphism group of an algebraic object, the restriction maps correspond to the extension of scalars. In the case that G is abelian, we can also define corestriction maps $\mathrm{Cor} : H^i(\Bbbk', G_{\Bbbk'}) \to H^i(\Bbbk, G)$ ([10], p. 62-63).

Lemma 2.13 ([10], Proposition 3.3.7, p. 63) *If $\Bbbk'/\Bbbk$ is an algebraic extension of degree n and G is a commutative group over $\Bbbk$, then* $\mathrm{Cor} \circ \mathrm{Res}$ *is given by multiplication by n.*

If F is a number field and V^F is the set of all places on F, then we have the following theorem, known as the Hasse Principle:

Theorem 2.14 ([22], Remark 1, p. 152) *Suppose that G is a simple algebraic group over F, then the product map:*

$$H^1(F, G) \overset{\prod \mathrm{Res}_{F_v}}{\longrightarrow} \prod_{v \in V^F} H^1(F_v, G)$$

is injective if G is adjoint and bijective if G is simply connected.

The following tells us that the image of an element under $\prod \mathrm{Res}_{F_v}$ is trivial for all but finitely many components of the product:

Proposition 2.15 *Let G be a connected group over a number field F. Then G is F_v-quasi-split for almost all places v on F.*

Proof. [19], Theorem 6.7, p. 291. ■

While the Hasse Principle does not hold for tori, we do have the following local-global principle:

Lemma 2.16 ([19], Corollary 2, p. 418) *If T is a torus over F, then the product map*

$$H^1(F, T) \overset{\prod \mathrm{Res}_{F_v}}{\longrightarrow} \prod_{v \in V^F} H^1(F_v, T)$$

is a surjection.

2.2.4 The Weak Approximation Property

I shall refer to the following many times throughout the thesis and is known as the weak approximation property for number fields.

Theorem 2.17 ([19], Theorem 1.4, p. 13) *Given an algebraic number field F and a finite collection of places S on F, the canonical map*

$$F \to \prod_{v \in S} F_v$$

has dense image, where $\prod_{v \in S} F_v$ is given the product topology.

2.2.5 Standard Subgroups

Given a semisimple algebraic group G over $\mathbb{k}$, a reductive $\mathbb{k}$-subgroup $H \leq G$ is called *standard* if there is a maximal torus T of G normalizing H ([5], p. 299). This is equivalent to the statement that H is generated by the root subgroups G_α for the roots in some sub-root system $\Sigma' \subset \Sigma(G, T)$. We will sometimes denote $H = G_{\Sigma'}$.

2.2.6 Weil Restriction

If G is an algebraic group over F and $F \overset{\sigma}{\hookrightarrow} L$ is a morphism of fields, denote by $G_{L,\sigma}$ the extension of scalars of G to L by σ (we omit the σ if it is clear from context).

Given a base field F, a separable extension L of F and an algebraic group H over L, one can define the *Weil restriction* of H to F (denoted by $R_{L/F}(H)$) in the following way: If H is defined by polynomials $f_1, \ldots, f_\ell$ over L and L has an F-basis ω_i, then f_j can be expressed as $f_i = \sum \omega_j g_{ij}$. Let $R_{L/F}(H)$ be the group obtained by considering the F solutions of the g_{ij} (see [19], p. 49-51 for more details). Recall that a group is *absolutely simple* if it is simple over the algebraic closure $\overline{F}$.

Lemma 2.18 *Given an absolutely simple group H over L, $R_{L/F}(H)$ is F-simple.*

As noted previously, the converse also holds:

Lemma 2.19 ([12], Theorem 26.8, p. 365) *Any F-simple group that is not absolutely simple is isomorphic to $R_{L/F}(H)$ for some absolutely simple group H and some finite extension L/F.*

When we compose restriction with extension, we obtain the following:

Lemma 2.20 *Given F, L, H as above, if $L \otimes_F K \simeq L_1 \times \cdots \times L_m$, where L_i are fields, denote σ_i to be the map $L \hookrightarrow L \otimes_F K \xrightarrow{\pi} L_i$ then we have*

$$R_{L/F}(H)_K \simeq R_{L_1/K}(H_{L_1,\sigma_1}) \times \cdots \times R_{L_m/K}(H_{L_m,\sigma_m})$$

Proof. See [19], p. 50. ■

The Galois cohomology of $R_{L/F}(H)$ over F is related to the Galois cohomology of H over L by Shapiro's lemma:

Lemma 2.21 ([10], Corollary 3.3.2, p. 61) *Given an algebraic extension L of F and an algebraic group H over L,*

a) If H is abelian, then $H^i(F, R_{L/F}(H)) \simeq H^i(L, H)$ for all i, and

b) if H is non-abelian, then $H^1(F, R_{L/F}(H)) \simeq H^1(L, H)$.

2.2.7 Central Simple Algebras

For later reference, we begin by recalling some fundamental results in the theory of central simple algebras. The following theorem is referred to as the Skolem-Noether theorem:

Theorem 2.22 ([12], Theorem 1.4, p. 5) *Let A be a central simple algebra over $\mathbb{k}$ and let $B \subset A$ be a simple subalgebra. Every $\mathbb{k}$-algebra homomorphism $\rho : B \to A$ extends to an inner automorphism of A. In particular, every $\mathbb{k}$-algebra automorphism of A is inner.*

The following will be useful in the construction of subgroups of groups of type A_n, it is known as the double centralizer theorem:

Theorem 2.23 ([12], Theorem 1.5, p. 5) *Let A be a central simple algebra over $\mathbb{k}$ and let $B \subset A$ be a simple subalgebra with centre $\mathbb{k}' \supset \mathbb{k}$. The centralizer $C_A(B)$ is a simple subalgebra of A with centre $\mathbb{k}'$ which satisfies*

$$\dim_{\mathbb{k}}(A) = \dim_{\mathbb{k}}(B) \cdot \dim_{\mathbb{k}} C_A(B) \text{ and } C_A C_A(B) = B$$

Moreover, if $\mathbb{k} = \mathbb{k}'$, then multiplication in A defines a canonical isomorphism $A \simeq B \otimes_{\mathbb{k}} C_A(B)$.

For details on the definition of the Brauer group $\mathrm{Br}(\mathbb{k})$ over $\mathbb{k}$ see [10], Chapters 2 and 4. Given a quadratic extension $\mathbb{k}' = \mathbb{k}(\sqrt{a})$, we can consider the regular embedding $R^{(1)}_{\mathbb{k}'/\mathbb{k}}(\mathbb{G}_\mathrm{m}) \hookrightarrow \mathbb{SL}_2(\mathbb{k})$. Taking quotients, this gives rise to a map $R^{(1)}_{\mathbb{k}'/\mathbb{k}}(\mathbb{G}_\mathrm{m})/\mu_2 \to \mathbb{PGL}_2$. Combining this with the fact that $R^{(1)}_{\mathbb{k}'/\mathbb{k}}(\mathbb{G}_\mathrm{m})/\mu_2 \simeq R^{(1)}_{\mathbb{k}'/\mathbb{k}}(\mathbb{G}_\mathrm{m})$ gives an induced map $\phi : H^1(F, R^{(1)}_{\mathbb{k}'/\mathbb{k}}(\mathbb{G}_\mathrm{m})) \to H^1(\mathbb{k}, \mathbb{PGL}_2)$.

Lemma 2.24 *The image of ϕ is the set of quaternion algebras that are split over $\mathbb{k}'$.*

Proof. See [10], Corollary 2.5.5, page 36. ■

It is well-known that the Brauer group is a torsion group ([10], Corollary 4.4.8, p. 99). By breaking the Brauer group into its p-primary components we obtain the following:

Lemma 2.25 ([10], Proposition 4.5.16, p. 105) *Let D be a central division algebra over $\mathbb{k}$. Consider the primary decomposition*

$$\mathrm{Ind}(D) = p_1^{n_1} \cdots p_\ell^{n_\ell}$$

We can find central division algebras D_i over $\mathbb{k}$ such that

$$D \simeq D_1 \otimes_{\mathbb{k}} \cdots \otimes_{\mathbb{k}} D_\ell$$

and $\mathrm{Ind}(D_i) = p_i^{n_i}$ for each i. The D_i are then determined uniquely up to isomorphism.

If $\ell > 1$, then for each i $\mathbb{SL}(D_i)$ embeds in $\mathbb{SL}(D)$ as a proper subgroup. If $\ell = 1 = n_1$, we have:

Lemma 2.26 ([9], Proposition 4.1, page 409) *If D is a central division algebra of prime degree p over F, then $\mathbb{SL}(D)$ has no proper semisimple subgroups.*

If $\mathbb{k}'/\mathbb{k}$ is a field extension, then the diagram:

$$\begin{array}{ccc} H^2(\mathbb{k}, \mathbb{G}_{\mathrm{m}}) & \xrightarrow{\mathrm{Res}_{\mathbb{k}'}} & H^2(\mathbb{k}', \mathbb{G}_{\mathrm{m}}) \\ \simeq \downarrow & & \downarrow \simeq \\ \mathrm{Br}(\mathbb{k}) & \xrightarrow[-\otimes_{\mathbb{k}}\mathbb{k}']{} & \mathrm{Br}(\mathbb{k}') \end{array}$$

commutes. The corestriction map (sometimes also called the norm map) is not as easily described, but it can be calculated in some instances. Assume that $\mathbb{k}'/\mathbb{k}$ is algebraic and $\mathbb{k}$ contains a primitive n-th root of unity ω_n. If $(a, b)_{\omega_n}$ is a cyclic central simple algebra (see [10], Section 2.5) of degree n over $\mathbb{k}'$ with $a \in \mathbb{k}$ then it can be shown that

$$\mathrm{Cor}_{\mathbb{k}}([(a, b)_{\omega_n}) = [(a, N_{\mathbb{k}'/\mathbb{k}}(b))_{\omega_n}].$$

Central Simple Algebras over Number Fields

The general structure of central simple algebras over number fields is known, as described in the following celebrated theorem due to Albert, Brauer, Hasse and Noether:

Theorem 2.27 ([19], Theorem 1.12, p. 38) *Every central simple algebra over a number field is cyclic.*

Given a central simple algebra A over a number field F, it is known that the order of $[A]$ in $H^2(F, \mathbb{G}_{\mathrm{m}})$ is equal to the index of A (see [19], p. 38). It is also well known that there is only one non-trivial element of $\mathrm{Br}(\mathbb{R})$, namely the Hamiltonian Quaternions. This, along with the above results, implies that if A is a central simple algebra over a number field such that the order of $[A]$ in $\mathrm{Br}(F)$ is odd, then A splits over any real completion of F.

For p-adic completions of number fields, we will use the following:

Lemma 2.28 ([14], Remark 2.7, p. 154) *If D is a quaternion algebra over a finite extension K of $\mathbb{Q}_p$, then any quadratic extension K' of K splits D.*

There is also a local-global principle for central simple algebras, known as the Brauer-Hasse-Noether theorem:

Theorem 2.29 ([18], §18.4) *Given a number field F, if V^F is the set of all places on F, then the map*

$$\theta : H^2(F, \mathbb{G}_{\mathrm{m}}) \xrightarrow{\bigoplus \mathrm{Res}_{F_v}} \bigoplus_v H^2(F_v, \mathbb{G}_{\mathrm{m}}).$$

is well-defined and injective.

Involutions of the Second Kind

Given a quadratic etale extension $\mathbb{k}'/\mathbb{k}$, recall that a central simple algebra A with involution of the second kind τ over $\mathbb{k}$ is either:

- a central simple algebra A over $\mathbb{k}'$ with involution τ such that $\mathbb{k}'^{\tau} = \mathbb{k}$ if $\mathbb{k}'$ is a field, or
- an algebra of the form $A = B \times B^{op}$, where B is a central simple algebra over $\mathbb{k}$ and τ is given by the exchange involution $\tau(x, y) = (y^{op}, x^{op})$.

In the case that $\mathbb{k}'$ is a field, we have the following

Lemma 2.30 ([12], Theorem 2, p. 31) *Given a central simple algebra B over $\mathbb{k}'$, there exists an involution of the second kind on B which leaves $\mathbb{k}$ invariant if and only if* $\mathrm{Cor}_{\mathbb{k}}([B])$ *is trivial in* $H^2(\mathbb{k}, \mathbb{G}_{\mathrm{m}})$.

Given a central simple algebra with involution of the second kind over $\mathbb{k}$, the involution τ such that $\mathbb{k}'^{\tau} = \mathbb{k}$ is not unique:

Proposition 2.31 ([12], Proposition 4.18, p. 53) *Let (B, τ) be a central simple $\mathbb{k}$-algebra with involution of the second kind of degree n and let $\mathbb{k}'$ be the centre of B. For every $b \in B$ whose minimal polynomial over $\mathbb{k}'$ has degree n and coefficients in $\mathbb{k}$, there exists an involution of the second kind which leaves b invariant.*

2.3 Minimal Anisotropic Groups of Higher Real Rank

As in the case of isotropic algebraic groups, we do not restrict ourselves to the case of $\mathbb{Q}$-groups specifically, but instead consider the more general case of anisotropic groups over number fields. This requires us to first define minimal algebraic groups over number fields of appropriate rank. As before, let $V^F_{\infty,\mathbb{R}}$ be the set of all real places on F and let F_v be the completion of F with respect to the place v.

Definition 2.32 (Appropriate Real Rank) *Given a group G over a number field F and let*

$$S'_G = \{v \in V^F_{\infty,\mathbb{R}} \mid \operatorname{Rank}_{F_v}(G) = 1\} \textit{ and } S''_G = \{v \in V^F_{\infty,\mathbb{R}} \mid \operatorname{Rank}_{F_v}(G) \geq 2\}.$$

We say that a subgroup $H \leq G$ has appropriate real rank *if* $\operatorname{Rank}_{F_v}(H) = 1$ *for all* $v \in S'_G$ *and* $\operatorname{Rank}_{F_v}(H) \geq 2$ *for all* $v \in S''_G$. *Define* $S_G = S'_G \cup S''_G$.

This allows us to introduce the definition of minimality for the groups we are interested in:

Definition 2.33 (Minimal) *A F-simple group G as above is said to be* minimal *if* $S''_G \neq \emptyset$ *and G contains no proper F-simple subgroups of appropriate real rank.*

For a central simple algebra A over F, let $\mathbb{SL}(A)$ be the elements of A with norm 1. If D is a division algebra over F and $A = \mathrm{M}_n(D)$ we write $\mathbb{SL}_n(D) = \mathbb{SL}(A)$. Similarly, if (A, τ) is a central simple algebra over a quadratic extension L/F with unitary involution τ, write $\mathbb{SU}(A, \tau)$ for the elements x of A such that $\tau(x)x = 1$ and $\mathrm{Nrd}(x) = 1$. In the specific case that $A = \mathrm{M}_n(L)$ and τ corresponds to the hermitian form f, write $\mathbb{SU}_n(L, f)$ for $\mathbb{SU}(A, \tau)$. For the groups of rational points associated to each of these algebraic groups, we write $\mathrm{SL}(A), \mathrm{SL}_n(D), \mathrm{SU}(A, \tau)$ and $\mathrm{SU}_n(L, f)$, respectively.

Notice that all of the groups described above remain simple after extension to the algebraic closure of F. Such groups are called *absolutely simple*. It is well known that every simple group that is not absolutely simple is isogenous to the restriction of scalars (see 2.2.6) of an absolutely simple group (Lemma 2.19), and subgroups of such groups are closely related to the concept of descent:

Definition 2.34 (Descent) *Given an object A (an algebraic group, a central simple algebra, a quadratic form, a hermitian form etc.) over a field K we say that A* descends *to $P \subset K$ if there exists an object A' of the same kind defined over P such that when we extend scalars we have $A'_K \simeq A$.*

Notice that each of the non-absolutely simple groups in Theorem 1.2 is the restriction of scalars of one of the groups listed in Theorem 1.1, except for type 3. Although the technical conditions on the groups in each case appear cumbersome, they are exactly as required to ensure that no subgroup has appropriate real rank.

Chapter 3

Groups of Classical Type

3.1 Orthogonal Groups

In this section we consider groups of the form $\mathbb{SO}_n(f)$ for a bilinear form f of dimension n over F. Up to isogeny, this describes all groups of type B_n and some of type D_n. We will show that no groups of this form are minimal if n is at least 5:

Proposition 3.1 *If $G = \mathbb{SO}_n(f)$ where $n \geq 5$ and $S''_G \neq \emptyset$, then G contains an F-simple subgroup H of (absolute) type $A_1 \times A_1$ with appropriate real rank.*

Before we proceed, we introduce some notation from [12]. Given a quadratic form f with diagonalization $\langle a_1, \ldots, a_n \rangle$, we define the *determinant* of f to be $\det(f) = \prod_{i=1}^{n} a_i$ and the *discriminant* of f to be $\mathrm{disc}(f) = (-1)^{\frac{n(n-1)}{2}} \det(f)$. For $n = 2m$ even, we have that $\mathbb{SO}(f)$ is of inner type D_m if $\mathrm{disc}(f) \equiv 1$ and $\mathbb{SO}(f)$ is of outer type otherwise.

Note that the rank of G over F_v is equal to the Witt index of f over F_v. The following two lemmas are consequences of the Weak Approximation Property :

Lemma 3.2 *Let $h = \langle 1 \rangle \oplus g$ be a n-dimensional quadratic form and $\Omega \subset V^F_{\infty,\mathbb{R}}$ be a subset such that h is isotropic over F_v for each $v \in \Omega$. Then there exists a vector $w \in F^{n-1}$ such that $g(w)$ is negative for all $v \in \Omega$.*

Proof. Choose a diagonalization $g = \langle b_2, \ldots, b_n \rangle$. The fact that h is isotropic over F_v for all $v \in \Omega$ is equivalent to the statement that some b_i is negative in F_v for each v. Let i_v be the first index such that b_{i_v} is negative in F_v. Using the weak approximation property, choose $x_j \in F$, $2 \leq j \leq m$ such that for each $v \in \Omega$

1. For $j \neq i_v$, $0 < |b_j x_j^2|_v < \frac{1}{m-1}$ and

2. $|b_{i_v} x_{i_v}^2|_v > 1$

and let $w = (x_2, \ldots, x_n)$. Then for $v \in \Omega$,

$$g(w) = \sum_i b_j x_j^2 < \sum_{i \neq i_v} |b_i x_i^2|_v - |b_{i_v} x_{i_v}^2|_v < 0 \in F_v$$

■

Lemma 3.3 *Let g be a 3-dimensional form over F, $\alpha \in F^\times$ be arbitrary and let $\Omega \subset V_{\infty,\mathbb{R}}^F$ be a set of places over which g is isotropic. Then there is a two-dimensional sub-form g' of g such that*

1. $\mathrm{disc}(g') \neq \alpha$, *and*

2. g' is hyperbolic over F_v for all $v \in \Omega$.

Proof. Let $g = \langle c_1, c_2, c_3 \rangle$. Using Lemma 3.2, we can assume that $c_1 > 0$ and $c_2 < 0$ in F_v for all $v \in \Omega$. If $c_1 \cdot c_2 \not\equiv \alpha$ let $g' = \langle c_1, c_2 \rangle$. This allows us to assume without loss of generality that $g = \langle c_1, \alpha c_1, c_3 \rangle$. This implies that $\alpha < 0$ in F_v for all $v \in \Omega$.

It then suffices to find $x_1, x_2 \in F$ such that

$$c_1(\alpha c_1 x_1^2 + c_3 x_2^2) \not\equiv \alpha \mod F^{\times^2}$$
$$\alpha c_1 x_1^2 + c_3 x_2^2 < 0 \in F_v \ \forall\ v \in \Omega.$$

Multiplying each of these by $c_1\alpha$ and replacing c_3 by $\tilde{c_3} = \alpha c_1 c_3$, these conditions are equivalent to finding x_1, x_2 such that:

$$x_1^2 + \tilde{c_3}x_2^2 \not\equiv 1 \mod F^{\times^2}$$
$$x_1^2 + \tilde{c_3}x_2^2 > 0 \in F_v \ \forall\ v \in \Omega$$

For $p \neq 2$, in any p-adic completion v_p of an algebraic number field, there are 4 elements of $F_{v_p}^{\times}/F_{v_p}^{\times^2}$ and each of the cosets is an open subset in F_{v_p}. By continuity, this means that if we choose x_{v_p}, y_{v_p} such that $x_{v_p}^2 + \tilde{c_3}y_{v_p}^2 \not\equiv 1 \mod F_{v_p}^{\times^2}$, then we can choose an ϵ such that for any $|x_1 - x_{v_p}|_{v_p} < \epsilon$ and $|x_2 - y_{v_p}|_{v_p} < \epsilon$, x_1, x_2 fulfill the first condition. By the weak approximation property, we can choose $x_1, x_2 \in F^{\times}$ such that $|x_1 - x_{v_p}|_{v_p} < \epsilon$ and $|x_2 - y_{v_p}|_{v_p} < \epsilon$ and $|x_1|_v > 1$, $|x_2|_v < \min\{1/2, 1/2|\tilde{c_3}|_v\}$. Because $x_1^2 + \tilde{c_3}x_2^2 \not\equiv 1 \mod F_{v_p}^{\times^2}$, we must have $x_1^2 + \tilde{c_3}x_2^2 \not\equiv 1 \mod F^{\times^2}$. By the restrictions on $|x_1|_v$ and $|x_2|_v$, the second condition is satisfied by the triangle inequality. ■

Proof of Proposition 3.1. Using the fact that the rank of G over F_v is equal to the Witt index of f over F_v and the Witt cancellation theorem, Lemma 3.2 allows us to assume that if $f = \langle 1, a_2, \ldots, a_m \rangle$, then

1. For all $v \in S_G$, $a_2 < 0 \in F_v$, and
2. for all $v \in S_G''$, $a_3 > 0$ and $a_4 < 0$ in F_v.

Lemma 3.3 allows us to find a sub-form g' of $\langle a_3, a_4, a_5 \rangle$ such that g' is hyperbolic over F_v for all $v \in S_G''$ and $\mathrm{disc}(g') \not\equiv a_2$. This is equivalent to saying that $\mathrm{disc}(\langle 1, a_2 \rangle \oplus g') \not\equiv 1$. Let $H \leq G$ be $\mathbb{SO}_4(\langle 1, a_2 \rangle \oplus g')$, then H is simple of appropriate rank and H has type $D_2 = A_1 \times A_1$, as required. ■

Minimal groups of type $A_1 \times A_1$ will be covered along with other non-absolutely simple groups in a later section. Note that this section covers all groups of type B_n and some groups of type D_n (see [24]).

3.2 Type C_n

3.2.1 Classification over F and $\mathbb{R}$

Groups of this type over F or $\mathbb{R}$ are of one of two forms (up to isogeny):

1. The special unitary group relative to a hermitian form h over a non-split quaternion algebra D of dimension n (denoted $\mathbb{SU}_n(D, h)$), or

2. The symplectic group $\mathbb{S}\mathrm{p}_{2n}$, which is automatically split.

For type 1 groups, the rank of G is equal to the index of h. Because we start with an anisotropic group G over F, G is isomorphic (up to isogeny) to a group of the form $\mathbb{SU}_n(D, h)$ where h is a hermitian form of index 0 and D is non-split. Note that if K is any extension of F such that D splits over K, then G also splits over K, since G_K is of type 2.

3.2.2 Minimality

Proposition 3.4 *No group of type C_n is minimal.*

Proof. As stated above, up to isogeny G is given by $\mathbb{SU}_n(D, f)$. Let τ be the canonical involution on D and $f = \sum_{i=1}^{n} x_i^\tau a_i y_i$, where $a_i \in D^\tau = F$. If $n = 2$, then G has type $C_2 = B_2$ which was covered in the last section, so assume that $n \geq 3$. After normalizing, we can choose $a_1 = 1$. For each $v \in S_G$ such that $D \otimes_F F_v = D_v$ is non-split we have that at least one of $a_i < 0 \in F_v$. Using the same arguments as in Lemma 3.2, we see that after changing bases of D^n we can choose a_2 such that $a_2 < 0 \in F_v$ for all $v \in S_G$.

If $n = 3$, then I claim that $H = \mathbb{SU}_2(D, \langle 1, a_2 \rangle, \tau) \leq G$ has appropriate real rank. First, note that there are three possibilities for the rank of G over F_v:

- $v \in V_{\infty,\mathbb{R}}^F \backslash S_G$ if and only if D_v is non-split and $a_2, a_3 > 0 \in F_v$
- $v \in S'_G$ if and only if D_v is non-split and a_2 or $a_3 < 0 \in F_v$
- $v \in S''_G$ if and only if D_v is split

If $v \in S'_G$, then $\text{Rank}_{F_v}(H) = 1$ by our choice of a_2 and if $v \in S''_G$ then D is split over F_v so H is split over F_v, so H has appropriate real rank. By construction, H is absolutely simple, thus G is not minimal. Therefore we can assume that $n \geq 4$.

Arguing as in Lemma 3.2 above, we can find $b = \sum_{i=3}^n a_i b_i^2$, $b_i \in F$ such that $b > 0 \in F_v$ for all $v \in S''_G$ such that D_v is non-split, and so we can assume that a_3 has this property after changing bases for D^n if necessary. Similarly, we can assume $a_4 < 0$ in F_v for all $v \in S''_G$ such that D_v is non-split. Let $H = \mathbb{SU}_4(D, f', \tau)$, where $f' = \sum_{i=1}^4 x_i^\tau a_i y_i$, so $H \leq G$ is absolutely simple of type C_4. If D_v is split then H_v is automatically of rank at least 2 and if D_v is non-spilt then by the choice of a_i, H_{F_v} has rank at least 2, thus G is not minimal. ■

3.3 Type D_n

3.3.1 Classification over F and $\mathbb{R}$

Over F or $\mathbb{R}$, groups of this type are of two types (up to isogeny):

1. The special unitary group relative to a skew-hermitian form h over a non-split quaternion algebra D of dimension n (denoted $\mathbb{SU}_n(D, h)$), or
2. $\mathbb{SO}_{2n}(f)$ for a quadratic form f.

Because we have already dealt with the orthogonal groups, we will assume that our group G is of the form $\mathbb{SU}_n(D, h)$. As before, let τ be the canonical involution on D.

3.3.2 Background

Let $h(x) = \langle d_1, \ldots, d_n \rangle(x) = \sum_{i=1}^{n} x_i^{\tau} d_i x_i$, where $d_i \in D^0 = \{d \in D \mid \tau(d) = -d\}$ and $x \in D^n$. The *discriminant of* h is $\operatorname{disc}(h) = \prod_{i=1}^{n} d_i^2 \in F^{\times}/F^{\times^2}$ and the *determinant* of h is given by $(-1)^n \operatorname{disc}(h) \in F^{\times}/F^{\times^2}$.

An Exceptional Isomorphism

If $n = 2$, then $\mathbb{SU}_n(D, h)$ is of type $D_2 = A_1 \times A_1$. Groups of type $A_1 \times A_1$ are all isogenous to $R_{L/F}(\mathbb{SL}(A))$, where L is a quadratic etale extension of F and A is a central simple algebra of degree 2 over L. In [12] it is shown that $L = F(\sqrt{\operatorname{disc}(h)})$, thus $\mathbb{SU}_n(D, h)$ is F-simple if and only if $\operatorname{disc}(h)$ is non-trivial (IV, Section 15.B and VI, Section 26.B). Similarly, $\mathbb{SO}_4(q)$ is simple if and only if $\operatorname{disc}(q)$ is non-trivial.

Morita Equvialence

In this section we wish to understand the behaviour of skew-hermitian forms over extensions to fields that split D. Two rings are said to be *Morita Equivalent* if their right-module categories are equivalent. The following is well-known, and is included for notation to be used later on:

Proposition 3.5 *Suppose A is an associative ring with 1, and P is a finitely generated, free A-module. Let $B \simeq \operatorname{Hom}_A(P, P)$, then the category of finitely generated A modules $A - \mathrm{Mod}$ is equivalent to the category of finitely generated B modules $B - \mathrm{Mod}$.*

Proof. Assume for simplicity that $B = \operatorname{End}_A(P)$ (i.e. don't bother writing $\lambda : B \to \operatorname{End}_A(P)$ each time). Define $\mathcal{F} : A - \mathrm{Mod} \to B - \mathrm{Mod}$ by $\mathcal{F}(X) =$

$\mathrm{Hom}_{A-\mathrm{Mod}}(P,X)$, where $\mathcal{F}(X)$ has a B-module structure given by, for $b \in B$, $\phi \in \mathcal{F}(X)$, $\phi \cdot b(p) = \phi(b \cdot p)$, where B has the canonical left action on P.

Define on $P^* := \mathrm{Hom}_{A-\mathrm{Mod}}(P,A)$ the structure of a right B-module via $q \cdot b(p) = q(b \cdot p)$, where $q \in P^*$ and B has the canonical left action on P. Then, for $Y \in B-\mathrm{Mod}$, define $\mathcal{G}(Y) := \mathrm{Hom}_{B-\mathrm{Mod}}(P^*, Y)$, where we give this a right $A-\mathrm{Mod}$-structure via $\phi \cdot a(q) = \phi(a \cdot q)$, where $a \cdot q \in P^*$ is given by $(a \cdot q)(p) = a \cdot (q(p))$.

Then $\mathcal{F}, \mathcal{G}$ give a natural equivalence of categories. ■

Lemma 3.6 *In the notation above, $P \otimes_A P^* \simeq B$ in $B - \mathrm{Mod}$ and $P^* \otimes_B P \simeq A$ in $A - \mathrm{Mod}$.*

Proof. Define $\phi : P \otimes_A P^* \to B$ via $\phi(\sum p_i \otimes q_i)(f) = \sum p_i \cdot q_i(f)$ and $\psi : P^* \otimes_B P \to A$ via $\psi(\sum q_i \otimes p_i) = \sum q_i(p_i)$. That the first is an isomorphism is left as an exercise and for the second, choose $e_1, \ldots, e_n$ to be a basis of P, and let e_i^* be the dual basis. Define $\pi_j \in B$ via

$$\pi_j(e_k) = \begin{cases} 0 & \text{if } k \neq j \\ e_1 & \text{if } k = j \end{cases}$$

then $e_j* = e_1^* \circ \pi_j$, so $e_j^* \otimes e_i = e_1^* \circ \pi_j \otimes_B e_i = e_1^* \otimes_B \pi_j(e_i) = (e_1^* \otimes e_1) \cdot \delta_{ij}$. This means that an inverse to ψ can be given by $\psi^{-1}(a) = (e_1^* \otimes e_1) \cdot a$. ■

Next, consider an involution * on B. A sesquilinear form on a finitely generated projective module M over B is a bi-additive map $h : M \times M \to B$ such that $h(x \cdot \alpha, y) = \alpha^* h(x,y)$, $h(x, y \cdot \alpha) = h(x,y)\alpha$ for every $x, y \in M$, $\alpha \in B$. Give $M^* = \mathrm{Hom}_{B-\mathrm{Mod}}(M, B)$ a $B - \mathrm{Mod}$ structure by $\phi \cdot b(m) = b^* \cdot \phi(m)$. Giving a sesquilinear form is then equivalent to giving a $B - \mathrm{Mod}$ morphism $M \to M^*$.

Assume now that we are in the case that $A = \Bbbk$ is a field. Write V for P, and then $B = \mathrm{End}_{\Bbbk}(V)$ and $\Bbbk \hookrightarrow \mathrm{End}_{\Bbbk}(V)$ via scalar multiplication. Assume that * restricts to the identity map on $\Bbbk$. Given a skew-hermitian form on B, this corresponds to a $B-$Mod morphism $B \xrightarrow{h} B^*$. Applying the Morita equivalence of categories above, we see that this is equivalent to a map $\mathcal{G}(h) : \mathcal{G}(B) \to \mathcal{G}(B^*)$.

We now apply this to the case where $A = \Bbbk$ is the splitting field of a quaternion algebra D over K and $B = D \otimes_K \Bbbk \simeq \mathrm{End}_{\Bbbk}(V)$ for a two dimensional $\Bbbk$-vector space V. Begin with a skew-hermitian form h on D, i.e. $h(x, y) = \tau(x)dy$ for some λ-symmetric $d \in D$. After extending scalars to $\Bbbk$, we can apply Morita equivalence to this form to obtain a bilinear form, and our aim is to compute this form. We can then extend to the case of a m-dimensional skew-hermitian form on D^m because each of the functors in Morita equivalence are additive.

Choose a basis $\{e_1, e_2\}$ of V and let $\{e_1^*, e_2^*\}$ be the corresponding dual basis of V^* such that with this choice of basis, the natural involution * on B becomes

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}^* = \begin{pmatrix} \delta & -\beta \\ -\gamma & \alpha \end{pmatrix}.$$

Then $\mathcal{G}(B) = \mathrm{Hom}_B(V^*, B) \simeq \{b \in B \mid b \circ n = 0 \; \forall \; n \in \ker(e_1^* \circ)\} \subset B$ via $\phi \mapsto \phi(e_1^*)$. The $\Bbbk$-module structure on $\mathcal{G}(B)$ is given by $\phi \cdot a(e_1^*) = \phi(a \cdot e_1^*) = a\phi(e_1^*)$, i.e. identifying $\Bbbk$ with its image in B, we have that the $\Bbbk$-space structure on $\ker(e_1^* \circ)$ is given by left (or right) multiplication. Similarly, for $\mathcal{G}(B^*)$ we have that elements of $\mathrm{Hom}_B(V^*, B^*)$ are determined by the image of e_1^*, except now we must have that for all $n \in \ker(e_1^* \circ)$, $\phi(e_1^* \circ n) = n^* \circ \phi(e_1^*) = 0 = \phi(e_1^*)^* \circ n$, so $\phi(e_1^*)$ lies in the submodule $\{b \in B^* \mid b^* \circ n = 0 \mid \forall \; n \in \ker(e_1^* \circ)\}$. In our choice of basis, we thus have that

$$\mathcal{G}(B) = \{ \begin{pmatrix} x & 0 \\ y & 0 \end{pmatrix} \mid x, y \in \Bbbk\}, \; \mathcal{G}(B^*) = \{ \begin{pmatrix} 0 & 0 \\ x & y \end{pmatrix} \mid x, y \in \Bbbk\}$$

both with the canonical $\Bbbk$-space structures. We can identify $\mathcal{G}(B^*)$ with $\mathcal{G}(B)^*$ by the following: if $b_2 \in \mathcal{G}(B^*)$ and $b_1 \in \mathcal{G}(B)$, then $b_2(b_1) = x_{21}(b_2 \cdot b_1)$ where x_{21} is the coordinate function. Give $\mathcal{G}(B)$ the basis $\{v_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, v_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\}$. Recall that h is skew-hermitian, so that $d^* = -d$, and then suppose that $d = \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix}$. Under our correspondences, h then corresponds to the bilinear form f on $\mathcal{G}(B)$ given by $f(x, y) = x_{21}(x^*dy)$. In the basis that we have chosen, this is represented by the matrix $\begin{pmatrix} \gamma & -\alpha \\ -\alpha & -\beta \end{pmatrix}$.

It can be shown that this equivalence preserves discriminants and that h is isotropic over K if and only if the corresponding bilinear form has Witt index at least two (see [21], Lemma 3.5, p. 362). These results are summarized below for future reference:

Lemma 3.7 ([21], p. 361-362) *Given a skew-hermitian h on D^n as above, if $F \subset K$ is a field extension splitting D then $h \otimes 1 : (D \otimes_F K)^n \to (D \otimes_F K)$ corresponds to a unique bilinear form b_h on K^{2n}, up to isometry, and $\operatorname{disc}(b_h) = \operatorname{disc}(h)$. Also, h is isotropic over K if and only if b_h has Witt index ≥ 2. This correspondence respects direct sums, i.e. $b_{h \oplus h'} = b_h \oplus b_{h'}$ and on one dimensional forms $\langle d \rangle$, if we choose an isomorphism $D \otimes_F K \simeq \mathrm{M}_2(K)$ and under this isomorphism d corresponds to*

$$\begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix}$$

then there exists a basis of K^2 such that $b_{\langle d \rangle}$ has matrix

$$\begin{pmatrix} \gamma & -\alpha \\ -\alpha & -\beta \end{pmatrix}.$$

Local-Global Principles and Skew-Hermitian Forms

Next, I include some results from [21] about skew-hermitian forms over local fields.

Theorem 3.8 ([21], Theorem 3.6, p. 363) *Let K be a p-adic field and D the unique non-split quaternion algebra over K. For skew hermitian forms over D the following statements hold:*

1. *Two regular forms are isometric if and only if they have the same dimension and determinant.*
2. *Every form of dimension > 3 is isotropic.*
3. *In dimension 1 all regular forms are anisotropic; there are forms of any determinant $\neq 1$.*
4. *For any dimension > 1 there are forms of any determinant. In dimension 2 exactly the forms of determinant 1 are isotropic. In dimension 3, exactly the forms of determinant 1 are anisotropic.*

Theorem 3.9 ([21], Theorem 3.7, p. 364) *Let K be a real closed field and D the unique nonsplit quaternion algebra over K. Every skew hermitian form of dimension > 1 is isotropic and forms of equal dimension are isometric.*

Given $d \in D^0$ it is possible to describe the isotropy type of $b_{\langle d \rangle}$ without appealing to a specific isomorphism $D \otimes_F K \simeq \mathrm{M}_2(K)$. By direct calculation, the form $b_{\langle d \rangle}$ is given (up to scalars) by $\langle 1, -d^2 \rangle$ and so $b_{\langle d \rangle}$ is a hyperplane if and only if $d^2 > 0$.

Now we address local-global properties. First, we have the following, due to Kneser:

Theorem 3.10 ([21], Theorem 4.1, p. 366) *Let K be a global field of characteristic not 2, let D be a quaternion algebra over K and let h be a skew hermitian form over D.*

1. *If* $\dim(h) \geq 3$ *and* h *is locally isotropic, then* h *is isotropic*

2. *If* $\dim(h) \geq 2$ *and if* $d \in D^0$ *which is represented locally by* h*, then* d *is represented globally by* h*.*

The following is a corollary of the Weak Approximation Property:

Lemma 3.11 *Given places* $\{v_1, \ldots, v_\ell\}$ *on* F *and* $d_{v_i} \in D_{v_i} = D \otimes_F F_{v_i}$ *such that* $h_{v_i} = h \otimes 1 : (D_{v_i})^n \to D_{v_i}$ *represents* d_{v_i}*, we can choose* $d \in D$ *such that* h *represents* d *and the one-dimensional skew-hermitian forms corresponding to* d *and* d_{v_i} *on* D_{v_i} *are isometric.*

Proof. Identify D^n with F^{4n} and let $x_{v_i} = (x_{v_i,j})_{j=1}^{4n}$ be the elements of $(D_{v_i})^n$ such that $h_{v_i}(x_{v_i}) = d_{v_i}$. Choose $\delta_i > 0$ such that if $|x_j - x_{v_i,j}|_{v_i} < \delta_i$ and we let $x = (x_j)_{j=1}^4 \in D_{v_i}^n$, then $h(x)^2 \equiv d_{v_i}^2 \mod F_{v_i}^{\times 2}$ and the coefficients of $h_{v_i}(x)$ and d_{v_i} are close enough so that if D_{v_i} is split, the forms represented by the matrices at the end of Lemma 3.7 are isometric. By weak approximation, we can choose $x \in F^{4n}$ so that $|x_j - x_{v_i,j}|_{v_i} < \delta_i$.

Let f be the form corresponding to $d = h(x)$ and f_i be the forms corresponding to d_{v_i}. If F_{v_i} is p-adic and D_{v_i} is non-split, then Theorem 3.8 gives that $f \otimes 1$ and f_i are isometric. If F_{v_i} is real and $D \otimes F_{v_i}$ is non-split, then Theorem 3.9 gives that $f \otimes 1$ and f_i are isometric. If D_{v_i} is split, then f and f_i are isometric by construction. ■

Corollary 3.12 *In the situation above we can choose a diagonalization* $\langle d_1, \ldots, d_n\rangle$ *of* h *such that* $\langle d\rangle \otimes 1$ *and* $\langle d_{v_i}\rangle$ *are isometric as* 1*-dimensional forms on* D_{v_i}*.*

We prove now that no groups of type D_n are minimal for $n \geq 3$. We separate this into three cases, $n = 3$, $n \geq 5$ and $n = 4$:

3.3.3 Case I: $n = 3$

In this case we can find a F-simple subgroup of type $A_1 \times A_1$ of appropriate real rank.

Proposition 3.13 *Keeping the notation at the beginning of this section, if* $n = 3$, *we can choose a diagonalization of* $f = \langle c_1, c_2, c_3 \rangle$ *so that* $\mathbb{SU}_2(D, \langle c_1, c_2 \rangle, \tau) \leq G$ *has appropriate real rank and* $\operatorname{disc}(\langle c_1, c_2 \rangle) \not\equiv 1 \mod F^{\times^2}$.

Proof. For every $v \in V^F_{\infty,\mathbb{R}}$ such that D_v is non-split, Theorem 3.9 gives that any two 2-dimensional skew hermitian forms over D_v are isometric, so we ignore those valuations. Let $\{v_1, \ldots, v_m\}$ be the elements of S'_G for which D_{v_i} is split and notice that D_v is split for every $v \in S''_G$ (by the same theorem). Let $S''_G = \{v_{m+1}, \ldots, v_\ell\}$.

Let $f_{v_i} = f \otimes 1 : D^n_{v_i} \to D_{v_i}$. Then f_{v_i} corresponds to an isotropic quadratic form under Morita equivalence, say f'_{v_i}. This implies $f'_{v_i} \oplus \langle 1, -1 \rangle$ has Witt index at least two, thus by Lemma 3.7 we have that f_{v_i} represents some c_{v_i} such that the one dimensional skew-hermitian form $\langle c_{v_i} \rangle$ corresponds a hyperbolic plane under Morita equivalence. By Corollary 3.12 there then exists $c_1 \in D$ such that f represents c_1 and $\langle c_1 \rangle_{v_i}$ corresponds to $\langle 1, -1 \rangle$ under Morita equivalence for all v_i. Choose d_2, d_3 so that $f = \langle c_1, d_2, d_3 \rangle$. Repeating the same arguments for $\langle d_2, d_3 \rangle$ yields c_2 such that $\langle d_2, d_3 \rangle$ represents c_2 and $\langle c_2 \rangle_{v_i}$ corresponds to an isotropic form over $F^2_{v_i}$ for all $v_i \in S''_G$. Choose c_3 so that $f = \langle c_1, c_2, c_3 \rangle$.

Assume G is of type ${}^1D_3$ and that $\operatorname{disc}(\langle c_1, c_2 \rangle) = 1$. Then we have that $c_1^2 c_2^2 c_3^2 \equiv c_3^2 \equiv 1 \mod F^{\times^2}$. This contradicts the assumption that D is a division algebra over F.

Let G be of type ${}^2D_3$ and assume that $c_1^2 c_2^2 \equiv 1 \mod F^{\times^2}$. I claim that $\langle c_2, c_3 \rangle$ then represents some $d \in D$ such that $\langle d \rangle_{v_i} \simeq \langle c_2 \rangle_{v_i}$ for all v_i and there exists some place v_0 such that $d^2 \not\equiv c_1^2 \mod F^{\times^2}$. If this is true, then replacing c_2 by d completes the proof.

Using Corollary 3.12, it suffices to show that there exists some p-adic place v_0 on F such that $\langle c_2, c_3\rangle_{v_0}$ represents $d_{v_0} \in D_{v_0}$ with $d_{v_0}^2 \not\equiv c_1^2 \mod F_{v_0}^{\times 2}$. Choose any p-adic ($p \neq 2$) place v_0 such that D_{v_0} is split. Suppose that $b_{\langle c_2, c_3\rangle_{v_0}} = \langle \beta_1, \beta_2, \beta_3, \beta_4\rangle$. We then have that $\langle \beta_1, \beta_2, \beta_3, \beta_4, -1\rangle \simeq \langle 1, -1\rangle \oplus \langle r, s, t\rangle$ by the fact that any 5-dimensional quadratic form over a p-adic field is isotropic. From [14], Corollary 2.5, p. 153 we have that $\langle r, s, t\rangle$ represents at least 3 square classes in $F_{v_0}^{\times}/F_{v_0}^{\times 2}$, thus we can choose $y \in F_{v_0}^{\times}$ such that $\langle r, s, t\rangle$ represents $-y$ and $y \not\equiv c_1^2 \mod F_{v_0}^{\times 2}$. Then $\langle \beta_1, \beta_2, \beta_3, \beta_4\rangle \oplus \langle -1, y\rangle$ has Witt index at least 2, and thus by Lemma 3.7, h_{v_0} represents some d_{v_0} such that $\langle d_{v_0}\rangle$ corresponds to $\langle 1, -y\rangle$ under Morita equivalence. Then $d_{v_0}^2 \equiv y \not\equiv c_1^2 \mod F_{v_0}^{\times 2}$, as required. ■

Note that $H = \mathbb{SU}_2(D, \langle c_1, c_2\rangle)$ is F-simple by the restriction that $\mathrm{disc}(\langle c_1, c_2\rangle)$ is non-trivial, thus G is not minimal.

3.3.4 Case II $n \geq 5$

This case is handled by constructing a diagonalization as in the previous case, except now we construct an absolutely simple subgroup of type ${}^{1,2}D_4$.

Proposition 3.14 *In the notation of the previous section, we can find a diagonalization $f = \langle c_1, \ldots, c_n\rangle$ such that $\mathbb{SU}_4(D, \langle c_1, c_2, c_3, c_4\rangle, \tau)$ has appropriate real rank.*

Proof. Arguing as in Proposition 3.13 we can find a diagonalization $f = \langle c_1, \ldots, c_n\rangle$ such that $b_{\langle c_1, c_2\rangle_v}$ has Witt index at least 2 for every $v \in S''_G$ such that D_v is split and Witt index 1 for every $v \in S'_G$. Then $\mathbb{SU}_4(D, \langle c_1, c_2, c_3, c_4\rangle, \tau)$ has appropriate rank over F_v for every v such that D_v is split. For any $v \in V^F_{\infty,\mathbb{R}}$ such that D_v is non-split Theorem 3.9 gives that both G and $\mathbb{SU}_4(D, \langle c_1, c_2, c_3, c_4\rangle, \tau)$ are of higher rank. ■

Letting $H = \mathbb{SU}(D, \langle c_1, c_2, c_3, c_4\rangle, \tau)$ we have that H is absolutely simple of appropriate real rank, hence G is not minimal.

3.3.5 Case III: $n = 4$

Before we begin this case we recall some facts about groups of type ${}^{1,2}D_4$. Suppose that G is of type ${}^2D_4$, then there is a unique quadratic extension of F such that G becomes of inner type ${}^1D_4$, say K. The simply connected quasi-split group of type ${}^2D_4$ (say G_0) is then $\mathbb{S}\mathbf{pin}(f_{qs})$, where f_{qs} is the quasi-split quadratic form $\langle 1, -c, 1, 1, 1, -1, -1, -1\rangle$. The centre of G_0 is $R_{K/F}(\mu_2)$, and so $H^2(F, Z(G_0)) \simeq {}_2\mathrm{Br}(K)$ by Lemma 2.21.

The simply connected split group of type D_4 is $\mathbb{S}\mathbf{pin}(f_s)$, where f_s is the split quadratic form $\langle 1, -1, 1, -1, 1, -1, 1, -1\rangle$. The centre of $\mathbb{S}\mathbf{pin}(f_s)$ is then $\mu_2 \times \mu_2$, and so $H^2(F, Z(G_0)) \simeq {}_2\mathrm{Br}(F) \times {}_2\mathrm{Br}(F)$, again by Lemma 2.21.

Lemma 3.15 *There exists $\alpha \in F$ such that:*

1. *$F(\sqrt{\alpha})$ is a purely imaginary extension of F,*
2. *$-\alpha \notin F^{\times 2}$*
3. *G is split (or quasi-split) over $F(\sqrt{\alpha})$.*

Proof.

Let G correspond to $[\xi] \in H^1(F, \mathbb{PSO}(f_{qs}))$ if G is of type ${}^2D_4$ and $H^1(F, \mathbb{PSO}(f_s))$ if G is of type ${}^1D_4$. Then the image of $[\xi]$ in $H^2(F, Z(G_0))$ is represented by a quaternion algebra $[T]$ over K if G is of type ${}^2D_4$ or a pair of quaternion algebras $([T_1, T_2])$ over F. By Theorem 2.29, there are finitely many places v on K such that $[T]$ is non-trivial over K_v (or finitely many places v on F such that $[T_1]$ or $[T_2]$ is non-trivial over F_v). Let $r_1, \ldots, r_m$ be the non-archimedean places on K such that $[T]$ is non-trivial, s_i be the restriction of r_i to F if G is of type ${}^2D_4$. If G is of type ${}^1D_4$, let $t_1, \ldots, t_\ell$ be the non-archimedean places on F such that $[T_1]$ or $[T_2]$ is non-trivial (if no such r_i or t_i exist, choose any α such that $F(\sqrt{\alpha})$ is purely imaginary and $-\alpha \notin F^{\times^2}$). Choose one non-archimedean place r such that $[T]$ is split over K_r, or one place r such that $[T_1], [T_2]$ are split over F_r.

By Lemma 2.28, we have that $[T]$ (respectively $[T_1], [T_2]$) is split over any quadratic extension of K_{s_i} (respectively F_{t_i}). Assume that $\alpha \in F$ is chosen such that $F(\sqrt{\alpha})$ is purely imaginary, $\alpha \in F$ is non-square in K_{r_i} (respectively F_{t_i}), and $-\alpha$ is non-square in F_r. Note that in the case that G is of type ${}^2D_4$, α is not square in K because α is not square in K_{r_i}.

Let $L = F(\sqrt{\alpha}) \otimes_F K$ if G is of type ${}^2D_4$ and $F(\sqrt{\alpha})$ if G is of type ${}^1D_4$. In the case that G is of type ${}^1D_4$, we have that $L \otimes F_v \simeq \mathbb{C}$ for any $v \in V^F_{\infty,\mathbb{R}}$, and by Lemma 2.28 we have that T_1, T_2 are split over $L \otimes F_{t_i}$ for all i. By Theorem 2.29 we then have that $[T_1], [T_2]$ are split over L. This gives that $\mathrm{Res}_{L/F}([\xi])$ is the image of some $[\gamma] \in H^1(L, \mathbb{S}\mathbf{pin}(f_s))$. Because L has no real completions, the Hasse principle gives that $H^1(L, \mathbb{S}\mathbf{pin}(f_s)) = \{0\}$, thus $\mathrm{Res}_{L/F}([\xi])$ is split, i.e. G splits over L. In the case that G is of type ${}^2D_4$, note that G remains of type ${}^2D_4$ over L, since $K \otimes L$ is a field. Analogous arguments to the case ${}^1D_4$ yield that G is quasi-split over L in this case.

It remains to see that we can choose α such that $F(\sqrt{\alpha})$ has no real completions, α is non-square in K_{r_i} (or F_{t_i}) and $-\alpha$ is non-square in F. It is well-known ([14], Theorem 2.2, p. 152) that for a non-archimedean completion F_v of a number field F, $|F_v^\times / F_v^{\times^2}| = 8$ and if K_w is a quadratic extension of F_v, then the image of $F_v^\times / F_v^{\times^2}$ in $K_w^\times / K_2^{\times^2}$ is non-trivial. It is also well-known that square classes in non-archimedean completions of number fields are open. This means that in the case that G is of type ${}^2D_4$, we can choose $\alpha_{r_i} \in F_{s_i}^\times$ that are not square in K_{r_i} and α_r so that $-\alpha_r$ is non-square in F_r. By the weak approximation property, we can choose $\alpha \in F$ such that α_{r_i} is the same square class as α_{r_i} in F_{s_i} for all i, α is in the same square class as α_r in F_r, and α is negative in F_v for every $v \in V^F_{\infty,\mathbb{R}}$. Then α is as required. The ${}^1D_4$ case is handled analogously. ■

Proposition 3.16 *Up to isogeny, G contains an F-simple subgroup that is of appropriate real rank of the form $R_{F(\sqrt{a})/F}(\mathbb{SO}_4(f'))$ for some $a \not\equiv 1 \mod F^{\times^2}$.*

Note that because G_L is quasi-split, $\mathrm{Res}_{L/F}([D])$ is trivial, so L is a maximal subfield of D. Choose an embedding $L \hookrightarrow D$ and let i be the image of $\sqrt{\alpha}$ under this embedding. The following lemma is due to V. Chernousov and A. Merkurjev ([7]):

Lemma 3.17 *If K is a maximal subfield of D, and f is a skew-hermitian form such that b_f is isotropic over K, then there exists $v \in D^n$ such that $F(f(v,v)) \simeq K$.*

Applying Lemma 3.17 we see that h has a diagonalization $\langle \beta_1 i_1, \beta_2 i_2, \beta_3 i_3, d\rangle$ for some $\beta_j \in F^\times$, $d \in D^0$ and $i_j \in D^0$ such that $F(i_j) \simeq F(i) \subset D$ for each j. By the Skolem-Noether Theorem (2.22) we have that each of the i_j are conjugate to i, say $d_j^{-1} i_j d_j = i$. If $h(v_j) = i_j$ then $h(v_j \cdot d_j) = \mathrm{Nrd}(d_j)\cdot i$ and so replacing v_j by $v_j \cdot d_j$ gives that h has diagonalization $\langle \beta_1 i, \beta_2 i, \beta_3 i, d\rangle$, where $d \in D^0$. Note that the subspaces

$$V_1 = \{d' \in D^0 \mid id' = -d'i\} \text{ and } V_2 = \{d' \in D^0 \mid dd' = -d'd\}$$

both have dimension at least two and D^0 has dimension 3, so $\{0\} \neq V_1 \cap V_2 \subset D^0$. Choose $0 \neq d' \in D^0$ such that $id' = -d'i$ and $dd' = -d'd$, so that $i^{-1}d$ commutes with d' and thus $i^{-1}d \in F(d')$. Note that if $\bar{\cdot} \in \mathrm{Gal}(F(d')/F)$ is the non-trivial automorphism, then the restriction that i anticommutes with d' implies that $i^{-1}xi = \overline{x} = \tau(x)$ for every $x \in F(d')$.

Consider the bilinear form $b = \langle \beta_1, \beta_2, \beta_3, i^{-1}d \rangle$ over $F(d')$, and let $X_b \in \mathrm{M}_4(F(d'))$ correspond to b. Then a matrix g with entries in $F(d')$ is in $\mathbb{SO}(b)$ if and only if $g^T X_b g = X_b$, which is equivalent to $X_b^{-1} g^T X_b = g^{-1}$. If $X_h \in \mathrm{M}_4(D)$ is the matrix corresponding to h, then $X_h = iX_b$, so for $g \in \mathbb{SO}(b)$:

$$X_h^{-1} \tau(g)^T X_h = X_b^{-1} i^{-1} \tau(g)^T i X_b = X_b^{-1} g^T X_b = g^{-1}.$$

Thus g respects the skew-hermitian form h. Considering the F-coefficients of the $F(d')$ entries of g, this gives an embedding $R_{F(d')/F}(\mathbb{SO}(b)) \hookrightarrow G$. Because $\mathrm{Nrd}(i) = -\alpha$, we can replace β_1 above by $-\alpha\beta_1$ and repeat the same procedure. I claim that one of these groups is F-simple.

Lemma 3.18 *At least one of the groups*

$$R_{F(d')/F}(\mathbb{SO}(\langle \beta_1, \beta_2, \beta_3, i^{-1}d \rangle)) \text{ or } R_{F(d')/F}(\mathbb{SO}(\langle -\alpha\beta_1, \beta_2, \beta_3, i^{-1}d \rangle))$$

is F-simple.

Proof. It suffices to prove that $\mathbb{SO}(\langle \beta_1, \beta_2, \beta_3, i^{-1}d \rangle)$ or $\mathbb{SO}(\langle -\alpha\beta_1, \beta_2, \beta_3, i^{-1}d \rangle)$ is $F(d')$-simple (see Lemma 2.18). This is true if and only if $\mathrm{disc}(\langle \beta_1, \beta_2, \beta_3, i^{-1}d \rangle)$ or $\mathrm{disc}(\langle -\alpha\beta_1, \beta_2, \beta_3, i^{-1}d \rangle)$ is non-trivial. Assume that both discriminants are trivial. Then $1 \equiv \beta_1 \cdot \beta_2 \cdot \beta_3 \cdot i^{-1}d \equiv -\alpha \mod F(d')^{\times 2}$. By property 2 of Lemma 3.15, we have that $-\alpha \notin F^{\times 2}$, hence this yields $-\alpha \equiv (d')^2 \mod F^{\times 2}$. By the assumption that d' is purely imaginary and $d'i = -id'$, we have that i, d' is a quaternion basis for D. Thus the norm form of D is given by $\langle 1, -\alpha, \alpha, -\alpha^2 \rangle$, but then D is split over F, a contradiction. ∎

Let $H \leq G$ be $R_{F(d')/F}(\mathbb{SO}(\langle \beta_1, \beta_2, \beta_3, i^{-1}d \rangle))$ if $\beta_1 \cdot \beta_2 \cdot \beta_3 i^{-1}d \not\equiv 1 \mod F(d')^{\times 2}$ and $R_{F(d')/F}(\mathbb{SO}(\langle -\alpha\beta_1, \beta_2, \beta_3, i^{-1}d \rangle))$ if $\beta_1 \cdot \beta_2 \cdot \beta_3 i^{-1}d \equiv 1 \mod F(d')^{\times 2}$.

Lemma 3.19 *H has appropriate real rank over every real valuation on F.*

First, I need the following lemma:

Lemma 3.20 *Suppose we are given* $H = \mathbb{SO}_4(f_1) \times \mathbb{SO}_4(f_2) \leq \mathbb{SO}_8(f)$, *then* $f \simeq \langle c_1 \rangle \cdot f_1 \oplus \langle c_2 \rangle \cdot f_2$.

Proof. Because H is standard of type A_1^4 in G of type D_4, we have that over $\overline{F}$, H is conjugate to $\mathbb{SO}(f|_{V_1}) \times \mathbb{SO}(f|_{V_2})$ for $V_1 \perp V_2$ such that $V_1 \oplus V_2 = V$ (say $gHg^{-1} = \mathbb{SO}(f|_{V_1}) \times \mathbb{SO}(f|_{V_2})$). This means that if we let $W_1 = \{v \in V \mid g_2 v = v \mid \forall g_2 \in \mathbb{SO}(f_2)\}$ and $W_2 = \{v \in V \mid g_1 v = v \ \forall g_1 \in \mathbb{SO}(f_1)\}$, then over $\overline{F}$, $g(W_i \otimes \overline{F}) = V_i \otimes \overline{F}$, hence $W_1 \cap W_2 = \{0\}$ and $W_1 \perp W_2$. Now $\mathbb{SO}(f_i) \leq \mathbb{SO}(f|_{V_i})$, each connected of equal dimension gives that $\mathbb{SO}(f_i) = \mathbb{SO}(f|_{V_i})$, thus there exist c_i such that $\langle c_i \rangle \cdot f_i \simeq f|_{V_i}$. ∎

Consider a $v \in V_{\infty,\mathbb{R}}^F$ such that $D \otimes F_v = D_v$ is split. By Lemma 3.7 we then have that

$$G_{F_v} \simeq \mathbb{SO}(\langle \beta_1, \beta_1, \beta_2, \beta_2, \beta_3, \beta_3 \rangle \oplus \beta_4 \langle 1, -d^2 \rangle).$$

Because i, d' form a quaternion basis for D and we chose i such that i^2 is negative in every F_v for $v \in V_{\infty,\mathbb{R}}^F$ we have that $F(d')$ splits over F_v and

$$H_{F_v} \simeq \mathbb{SO}_4(\langle \beta_1, \beta_2, \beta_3, i^{-1}d \rangle) \times \mathbb{SO}_4(\langle \beta_1, \beta_2, \beta_3, \overline{i^{-1}d} \rangle)$$

where $\overline{\cdot}$ represents conjugation in $F(d')$.

Proof of Lemma 3.19. Let $D = (\alpha, \gamma)$, and note first that $(d')^2 = \gamma \cdot N_{F(\sqrt{\alpha})/F}(x)$ for some x so $(d')^2 < 0 \in F_v$ if and only if D_v is non-split.

We break the valuations $v \in S_G$ into three cases:

1. D_v is non-split: Then $F(d') \otimes_F F_v$ is a subfield of $\mathbb{H} = (-1,-1)_{F_v}$, thus $F(d') \otimes_F F_v \simeq \mathbb{C}$ and $H_{F_v} \simeq R_{\mathbb{C}/\mathbb{R}}(\mathbb{SL}_2 \times \mathbb{SL}_2)$ has F_v-rank 2

2. $v \in S'_G$: In this case D_v is split, β_i all have the same sign and $d^2 > 0 \in F_v$. Applying Lemma 3.20 and Witt cancellation then gives that $\langle 1, -d^2 \rangle \simeq \langle 1, -1 \rangle \simeq \langle i^{-1}d, \overline{i^{-1}d} \rangle$. Thus one of $i^{-1}d$, $\overline{i^{-1}d}$ is positive in F_v and the other negative, so $\mathrm{Rank}_{F_v}(H) = 1$.

3. $\mathrm{Rank}_{F_v}(G) \geq 3$ and D_v is split: In this case, two of β_1, β_2 and β_3 have different signs in F_v and so $\mathrm{Rank}_{F_v}(H) \geq 2$.

4. $\mathrm{Rank}_{F_v}(G) = 2$ and D_v is split: Because $\mathrm{disc}(\langle \beta_1, \beta_1, \beta_2, \beta_2, \beta_3, \beta_3 \rangle \oplus \beta_4 \langle 1, -d^2 \rangle) = -d^2$ and $\mathrm{disc}(\langle 1, 1, 1, 1, 1, 1, -1, -1 \rangle) = 1 \in F_v^\times / F_v^{\times^2}$ we have that $d^2 \equiv -1 \mod F_v^{\times^2}$ in this case. If two of $\beta_1, \beta_2, \beta_3$ have different signs, then $\mathrm{Rank}_{F_v}(H) \geq 2$ so assume that $\beta_1, \beta_2, \beta_3$ are all positive in F_v (the case where $\beta_1, \beta_2, \beta_3$ are all negative is handled analogously). In this case, Lemma 3.20 gives:

$$\langle 1, 1, 1, 1, 1, 1, \beta_4, \beta_4 \rangle \simeq c_1 \langle 1, 1, 1, i^{-1}d \rangle \oplus c_2 \langle 1, 1, 1, \overline{i^{-1}d} \rangle$$

by inspection, the only possibility is that $c_1 = c_2 = 1$ and $\langle -1, -1 \rangle \simeq \langle \beta_4, \beta_4 \rangle \simeq \langle i^{-1}d, \overline{i^{-1}d} \rangle$ by Witt cancellation. Then $H_{F_v} \simeq \mathbb{SO}(\langle 1, 1, 1, -1 \rangle) \times \mathbb{SO}(\langle 1, 1, 1, -1 \rangle)$ has F_v-rank 2.

■

Proof of Proposition 3.16. Lemma 3.18 gives the construction of a simple subgroup as required and Lemma 3.19 ensures that the subgroup has appropriate real rank. ■

3.4 Type A_{n-1}

3.4.1 Type ${}^1A_{n-1}$

All groups of type ${}^1A_{n-1}$ are isogenous to $\mathbb{SL}_m(D)$ for some central division algebra D. Note that if $G \simeq \mathbb{SL}_m(D)$ and D is a division algebra of degree d, then $m-1$ is the rank of G while the absolute rank of G is given by $n-1 = d \cdot m - 1$. Recall that any division algebra over a number field is cyclic, while the only division algebra over $\mathbb{R}$ is the Hamiltonian quaternions.

Because we begin with an anisotropic group, G will be of the form $\mathbb{SL}(D)$ for a central division algebra D. Over F_v for $v \in V^F_{\infty,\mathbb{R}}$, we have that G either splits and is isomorphic to $\mathbb{SL}_n(F_v)$ or has rank $\frac{n}{2}$ and is isomorphic to $\mathbb{SL}_{n/2}(\mathbb{H})$. If G becomes $\mathbb{SL}_{n/2}(\mathbb{H})$ over F_v then we must have that $\deg(D) = n$ is even. This means that if $\deg(D)$ is odd, then $\mathbb{SL}(D)$ is split over F_v for all $v \in V^F_{\infty,\mathbb{R}}$, in particular $\mathbb{SL}(D)$ attains higher rank over every real completion of F.

Proposition 3.21 *G is minimal if and only if $\deg(D) = p$ for p prime, $p \geq 3$.*

The following two lemmas address the non-minimal cases:

Lemma 3.22 *If $\deg(D)$ is not a power of a prime, then G is not minimal.*

Proof. Let $\deg(D) = p_1^{n_1} \cdots p_k^{n_k}$, then D is of the form $D_1 \otimes \cdots \otimes D_k$, where D_i has degree $p_i^{n_i}$ (Lemma 2.25). Suppose p_1 is odd (or else renumber the p_i), then because $\mathrm{Nrd}_D(d_1 \otimes 1 \otimes \cdots \otimes 1) = \mathrm{Nrd}_{D_1}(d_1)^{\deg(D)/p_i^{n_i}}$, we have that $\mathbb{SL}(D_1) \hookrightarrow \mathbb{SL}(D)$, and $\mathbb{SL}(D_1)$ is of higher rank over all real completions of F. ■

Lemma 3.23 *If $\deg(D) = p^m$ where p is prime and $m > 1$, then $\mathbb{SL}(D)$ is not minimal.*

Proof. Let $K \subset D$ be a maximal subfield which is a cyclic extension of F. Let $\mathrm{Gal}(K/F) = \mathbb{Z}/p^m\mathbb{Z} = \langle \overline{1} \rangle$, and let $\Gamma = \langle \overline{p} \rangle \leq \mathrm{Gal}(K/F)$, $K_0 = K^\Gamma$. Let $T = C_D(K_0)$ and consider $H = R_{K_0/F}(\mathbb{SL}(T)) \leq G$ (see Theorem 2.23). Because T is a central division algebra, $\mathbb{SL}(T)$ is simple over K_0 and so H is F-simple.

If p is odd, then K_0 and T both split over F_v for every $v \in V^F_{\infty,\mathbb{R}}$, so $S_G = S''_G$ and H has appropriate real rank. If $p = 2$ and $m = 2$ then G has type ${}^1A_3 = {}^1D_3$, which was handled previously.

If $p = 2$ and $m \geq 3$ then there are two possibilities for $K_0 \otimes_F F_v$. If $K_0 \otimes_F F_v \simeq \mathbb{C}$, then $H_{F_v} \simeq R_{\mathbb{C}/\mathbb{R}}(\mathbb{SL}_{2^{m-1}})$ has F_v-rank $2^{m-1} - 1 \geq 3$. If $K_0 \otimes_F F_v \simeq F_v \times F_v$ let w_1, w_2 be the two completions of K_0 that restrict to v on F, so

$$H_{F_v} \simeq \mathbb{SL}(T \otimes_{K_0} K_{0,w_1}) \times \mathbb{SL}(T \otimes_{K_0} K_{0,w_2})$$

if T splits over K_{0,w_1} or K_{0,w_2}, then $\mathrm{Rank}_{F_v}(H) \geq 2^{m-1} - 1 \geq 3$. If T becomes $\mathrm{M}_{2^{m-2}}(\mathbb{H})$ over both K_{0,w_1} and K_{0,w_2} then $H_{F_v} \simeq \mathbb{SL}_{2^{m-2}}(\mathbb{H}) \times \mathbb{SL}_{2^{m-2}}(\mathbb{H})$ so H has F_v-rank at least $2^{m-1} \geq 3$. ■

Proof of Proposition 3.21. Recall that if D is a central division algebra over F with $\deg(D) = p \geq 3$ prime, then $\mathbb{SL}(D)$ contains no proper semisimple subgroups (Lemma 2.26) and $\mathbb{SL}(D)$ is split over F_v for all $v \in V^F_{\infty,\mathbb{R}}$. This means that $\mathbb{SL}(D)$ is minimal for any central division algebra D of prime degree $p \geq 3$. Lemmas 3.22 and 3.23 address the converse. ■

3.4.2 Type ${}^2A_{n-1}$

Given a group G of type ${}^2A_{n-1}$ over $\mathbb{k}$, G is isogenous to $\mathbb{SU}(A, \tau)$ where A is a central simple algebra of degree n over the unique quadratic extension $\mathbb{k}'$ of $\mathbb{k}$ such that G becomes inner over $\mathbb{k}'$ and τ is an involution of the second kind on τ such that $\mathbb{k}'^\tau = \mathbb{k}$.

Maximal tori of G are then equivalent to certain commutative subalgebras of A:

Proposition 3.24 ([20], Proposition 2.3) *Any maximal torus in* G *corresponds to a* n*-dimensional,* τ*-invariant commutative etale* $\Bbbk'$*-sub algebra* E *with* $\dim_{\Bbbk}(E^{\tau}) = n$

If $\Bbbk = \mathbb{R}$ then $\Bbbk' = \mathbb{C}$ and $A = \mathrm{M}_n(\mathbb{C})$, with τ corresponding to a hermitian form f on $\mathbb{C}^n$. If $\Bbbk = F$ is a number field and $L = \Bbbk'$ is the unique quadratic extension over which G becomes inner, then there are two possible types for G_{F_v}, given $v \in V^F_{\infty,\mathbb{R}}$. If $L \otimes_F F_v$ is a field we have that G_{F_v} is outer, hence is of the form $\mathbb{SU}_n(\mathbb{C}, f)$ for some hermitian form f. If $L \otimes_F F_v \simeq F_v \times F_v$, however, then G_{F_v} is inner, hence is either of the form $\mathbb{SL}_{(n)/2}(\mathbb{H})$ or $\mathbb{SL}_n(F_v)$.

Proposition 3.25 *For any odd prime* p*,* $\mathbb{SU}(A, \tau)$ *is minimal for any central division algebra* A *of degree* p*.*

Proof. Let L be the centre of A. If G were not minimal, then G would contain a proper F-simple subgroup H. Then $H_L \leq G_L$ would be a proper closed semisimple subgroup of $\mathbb{SL}(A)$ where A is a central division algebra of degree p, but $\mathbb{SL}(A)$ contains no non-trivial semisimple subgroups by Lemma 2.26. ■

I claim that these are all of the possible minimal groups of type ${}^2A_{n-1}$ for $n \neq 3$.

Lemma 3.26 *If* $G \simeq \mathbb{SU}_m(L, f)$ *for a hermitian form* f *over* L*, then* G *is minimal if and only if* $m = 3$ *and* $L \otimes F_v \simeq F_v \times F_v$ *for some* $v \in V^F_{\infty,\mathbb{R}}$*.*

Remark 3.27 *By the assumption* $S''_G \neq \emptyset$ *we have,* $m \geq 3$*.*

Proof. After normalizing, we may assume that $f = \langle 1, a_2, \ldots, a_m \rangle$. If $m \geq 5$, I claim that we can choose a diagonalization of f such that $\langle 1, a_2, a_3, a_4 \rangle$ corresponds to a subgroup of G that has appropriate real rank. To see this, we use the same arguments as in the skew-hermitian case, namely that for any completions F_v such that $L \otimes F_v \simeq \mathbb{C}$, the form f_{F_v} is isotropic, hence represents any $a \in F_v$. Thus, we may use the weak approximation property to replace a_2, a_3, a_4 if necessary so that:

- $a_2 < 0$ in F_v for all $v \in S'_G$
- $a_3 > 0$ and $a_4 < 0$ in F_v for all $v \in S''_G$ such that $L \otimes F_v \simeq \mathbb{C}$.

After this replacement, we have that $\mathbb{SU}_4(L, \langle 1, a_2, a_3, a_4 \rangle)$ is a simple, proper subgroup of G that has appropriate real rank over every F_v, hence G is not minimal.

If $G \simeq \mathbb{SU}_4(L, f)$, then G has type ${}^2A_3 = {}^2D_3$, so G is isomorphic to a group handled in the skew-symmetric section.

Finally, assume $m = 3$. Then G has type A_2 and any subgroup of appropriate real rank must have absolute rank at least 2 (since $S''_G \neq \emptyset$). Assume that G contains a proper simple subgroup H of appropriate real rank. We would then have that H is standard, because the absolute rank of G is equal to that of H, and so the root system of H corresponds to a sub-root system of A_2. Because all the roots of G have equal length, the only possibility is that H is of type $A_1 \times A_1$, but A_2 does not contain two orthogonal roots, a contradiction. ■

The above proposition and lemma deal with the case $\deg(A) = 3$, and if $\deg(A) = 4$ then G is of type $A_3 = D_3$, which was already considered. We have already established that G is minimal in the case where $\deg(A) = 5$, thus it suffices to consider the following:

Proposition 3.28 *If (A, τ) is a central simple algebra with involution τ of the second type over a quadratic extension L/F such that $L^\tau = F$, and $\deg(A) = n > 5$ is not prime, then $\mathbb{SU}(A, \tau)$ is not minimal.*

Our strategy for groups of this type will revolve around the central result from Prasad and Rapinchuk [20]:

Theorem 3.29 ([20], Theorem A, p. 2) *Let L be a global field. Let A be a central simple L algebra of dimension n^2 with an involution τ and let E/L be a field extension of degree n with an involutive automorphism σ such that $\sigma|_L = \tau|_L$. Then the local-global principle for the existence of an embedding $\iota : (E, \sigma) \hookrightarrow (A, \tau)$ holds in each of the following situations:*

1. *τ is an involution of the second kind;*
2. *$A = \mathrm{M}_n(L)$ and τ is an orthogonal involution;*
3. *$A = \mathrm{M}_m(D)$ where D is a quaternion algebra, m is odd, and τ is an orthogonal involution.*

To apply the local-global principle, I claim that it suffices to consider only finitely many places on F, more precisely:

Lemma 3.30 *Let A be a central simple algebra of degree n over L with involution τ of the second type such that $F = L^\tau$ and $\tau|_L = \gamma$. Let K/F be a field extension of degree n, and define $\sigma = 1 \otimes_F \gamma$ on $E = K \otimes_F L$. Then, for any place v on F such that $\mathbb{SU}(A, \tau)_{F_v}$ is split or quasi-split, there exists an embedding:*

$$(E \otimes_F F_v, \sigma \otimes 1) \hookrightarrow (A \otimes_F F_v, \tau \otimes 1)$$

of F_v-algebras that respects involutions.

Proof. First, consider the case where $G = \mathbb{SU}(A, \tau)$ becomes quasi-split but not split over F_v. This means that $L_v = L \otimes F_v$ is a field and $A \otimes_F F_v \simeq \mathrm{M}_n(L_v)$. Let

$K_v = K \otimes F_v$, so that K_v/F_v is etale (but not necessarily a field) and $E_v = K_v \otimes_{F_v} L_v$. Note that in this case, such an embedding is equivalent to finding an embedding of $T_v = \mathbb{SU}(E_v, \sigma \otimes 1)$ in $G_v = \mathbb{SU}(\mathrm{M}_n(L_v), \tau \otimes 1)$ by Proposition 3.24. Because G_v is quasi-split, Theorem 2.11 gives that this is equivalent to finding an inner form H_v of G_v such that $T_v \hookrightarrow H_v$. If we can find an involution τ' on $\mathrm{M}_n(L_v)$ such that $(E_v, \sigma \otimes 1) \hookrightarrow (\mathrm{M}_n(L_v), \tau')$, then setting $H_v = \mathbb{SU}(\mathrm{M}_n(L_v), \tau')$ yields such an inner form.

Choose a generator θ of K/F, and consider the regular embedding $\psi : K_v \hookrightarrow \mathrm{M}_n(F_v) \hookrightarrow \mathrm{M}_n(L_v)$. Then the minimal polynomial of $\psi(\theta)$ has degree n (because K_v/F_v is an etale extension) and coefficients in F_v. By Proposition 2.31, this means that there exists $u \in \mathrm{M}_n(L_v)$ invertible such that $\psi(\theta)$ is fixed under $\tau' = \mathrm{Int}(u) \circ \tau$. Extend ψ to $\tilde{\psi}$ on E_v by defining it to be L_v-linear, then $\tilde{\psi} : E_v \hookrightarrow \mathrm{M}_n(L_v)$ respects involutions, as required.

Second, consider the case where G is split by F_v. In this case, $L_v \simeq F_v \times F_v$ and $A_v \simeq \mathrm{M}_n(F_v) \times \mathrm{M}_n(F_v)^{op}$ with exchange involution $\epsilon(x, y) = (y, x^{op})$. In this case, $E_v \simeq K_v \times K_v$ and $\sigma \otimes 1$ acts on $K_v \times K_v$ via $\sigma \otimes 1(a, b) = (b, a)$. We can then embed $K_v \times K_v \hookrightarrow \mathrm{M}_n(F_v) \times \mathrm{M}_n(F_v)$ via the regular embedding on each component, and this embedding respects involution by inspection. ■

Because G is split or quasi-split over F_v for almost every valuation v by Proposition 2.15, this means that we only have to control finitely many places on F to apply the local-global principle for algebras of the type in Lemma 3.30.

In order to control the finitely many remaining valuations on F, we need the following lemma:

Lemma 3.31 *Given $v_0, \ldots, v_n$ places of F, $K_0, \ldots, K_n$ etale extensions over F_{v_i} of degree m such that K_0/F_{v_0} is a field, there exists a field K/F such that $K \otimes_F F_{v_i} \simeq K_i$.*

Proof. Suppose $K_i \simeq F_{v_i}[x]/(f_i)$ where $f_i \in F_{v_i}[x]$ are monic of degree m with no repeated roots. Choose $0 < \epsilon_i < \min\{|\alpha_{ij} - \alpha_{ik}|_{v_i}/2\}$ where $\{\alpha_{ij}\}$ are the roots of f_i in $\overline{F_{v_i}}$ and choose $g \in F[x]$ such that if $\{\beta_1, \ldots, \beta_m\}$ are the roots of g, then $|\beta_j - \alpha_{ij}|_{v_i} < \epsilon_i$ (see [15], p. 44). In the terminology of [15] we say that β_i is the unique root of g *belonging to* α_{ij} for all i. Assume that $f_i = f_i^{(1)} \cdot f_i^{(s_i)}$ where $f_i^{(j)}$ are irreducible over F_{v_i} and that g decomposes as $g_{v_i}^{(1)} \cdot g_{v_i}^{(t_i)}$ where $g_{v_i}^{(j)}$ are irreducible over F_{v_i}. Let $\alpha_{i,1}$ be a root of $f_i^{(1)}$ and after renumbering assume that β_1 is a root of $g_{v_i}^{(1)}$. By definition of ϵ_i we have that $F_{v_i}(\alpha_{i,1}) \subset F_{v_i}(\beta_1)$ by Krasner's lemma ([15], Proposition 2.3, page 43). Thus $\deg(f_i^{(1)}) \leq \deg(g_{v_i}^{(1)})$. Consider another root β_j of $g_{v_i}^{(1)}$, then $\beta_j = \sigma(\beta_1)$ for some $1 \neq \sigma \in \mathrm{Gal}(\overline{F_{v_i}}/F_{v_i})$ and $|\sigma(\alpha_{i,1}) - \sigma(\beta_1)|_{v_i} = |\alpha_{i,1} - \beta_1|_{v_i} < \epsilon_i$. Then $\sigma(\alpha_{i1})$ is another root of $f_i^{(1)}$, and if $\sigma(\alpha_{i,1}) = \alpha_{ik} \neq \alpha_{ij}$ we have that

$$\begin{aligned} |\alpha_{ij} - \alpha_{ik}|_{v_i} \leq & \quad |\alpha_{ij} - \beta_j|_{v_i} + |\beta_j - \alpha_{ik}|_{v_i} \\ < & \quad 2\epsilon_i \\ < & \quad |\alpha_{ij} - \alpha_{ik}|_{v_i} \end{aligned}$$

which is a contradiction. This means that for every root β_j of $g_{v_i}^{(1)}$, there exists a unique root of $f_i^{(1)}$, so $\deg(g_{v_i}^{(1)}) = \deg(f_i^{(1)})$ and $F_{v_i}(\alpha_{i1}) = F_{v_i}(\beta_1)$. Repeating this inductively shows that if $K = F[x]/(g)$, then $K \otimes_F F_{v_i} \simeq F_i$. Finally, note that if K_0 is a field, then K must be a field. ∎

Corollary 3.32 *Given $v_0, \ldots, v_n$ places of F and towers of algebras:*

$$F_{v_i} \subset J_{v_i}^{(1)} \times \cdots \times J_{v_i}^{(s_i)} \subset K_{v_i}^{(1)} \times \cdots \times K_{v_i}^{(s_i)}$$

such that

1. *$J_{v_i}^{(j)}$ are field extensions of F_{v_i} with $\sum_{j=1}^{s_i} [J_{v_i}^{(j)} : F_{v_i}] = m$ for all i*

2. $K_{v_i}^{(j)}/J_{v_i}^{(j)}$ *are etale extensions with* $\dim_{J_{v_i}^{(j)}}(K_{v_i}^{(j)}) = h$ *for all* i, j

3. $F_{v_0} \subset J_{v_0} \subset K_{v_0}$ *is a tower of field extensions*

Then there exists a tower of field extensions $F \subset J \subset K$ *such that* $J \otimes_F F_{v_i} \simeq \prod J_{v_i}^{(j)}$ *and* $K \otimes_F F_{v_i} \simeq \prod K_{v_i}^{(j)}$.

Proof. First apply Lemma 3.31 to $F_{v_i} \subset \prod J_{v_i}^{(j)}$ to obtain a field extension J as required, and then substitute $J \otimes_F F_{v_i}$ for $\prod J_{v_i}^{(j)}$. This means that we can substitute $J_{w_{i,j}}$ for $J_{v_i}^{(j)}$, where $w_{i,j}$ are all of the valuations on J that restrict to v_i on F.

Next, apply Lemma 3.31 to $J_{w_{ij}} \subset K_{v_i}^{(j)}$ to obtain a field extension $J \subset K$ such that $K \otimes_J J_{w_{ij}} \simeq K_{v_i}^{(j)}$. It remains to show that $K \otimes_F F_{v_i} \simeq \prod K_{v_i}^{(j)}$, but this follows from the fact that:

$$K \otimes_F F_{v_i} \simeq K \otimes_J J \otimes_F F_{v_i} \simeq K \otimes_J \prod J_{w_{ij}} \simeq \prod K \otimes_J J_{w_{ij}} \simeq \prod K_{v_i}^{(j)}$$

■

The next step is to construct algebras $J_v \subset K_v$ over F_v that embed in $(A \otimes_F F_v)^{\tau \otimes 1}$. Let $\{v_1, \ldots, v_\ell\} = V_{\infty,\mathbb{R}}^F$ and let $w_1, \ldots, w_t$ be the non-archimedean valuations on F such that G is not split or quasi split over F_{w_i}. Consider two cases:

Case I $n = 2m$ is even. For the archimedean valuations, $A \otimes_F F_{v_i}$ is isomorphic to either $\mathrm{M}_n(\mathbb{C})$, $\mathrm{M}_n(\mathbb{R}) \times \mathrm{M}_n(\mathbb{R})^{op}$ or $\mathrm{M}_m(\mathbb{H}) \times \mathrm{M}_m(\mathbb{H})^{op}$.

- If $A \otimes_F F_{v_i} \simeq \mathrm{M}_{2m}(\mathbb{R}) \times \mathrm{M}_{2m}(\mathbb{R})$ with exchange involution, let $J_{v_i} = \mathbb{R}^2 \hookrightarrow \mathbb{R}^{2m} = K_{v_i}$ by the map

  $$(j_1, j_2) \mapsto (j_1, \ldots, j_1, j_2, \ldots, j_2).$$

  Let K_{v_i} embed as diagonal matrices in $\mathrm{M}_{2m}(\mathbb{R})$ and compose this embedding with the diagonal embedding of $\mathrm{M}_{2m}(\mathbb{R})$ in $A \otimes_F F_v$. If e_1 is the matrix consisting of 1's along each first m diagonal entries in each component and 0's elsewhere, and if $e_2 = (I_{2m\times 2m}, I_{2m\times 2m}) - e_1$ then J_{v_i} embeds in K_{v_i} via $\mathbb{R}\cdot e_1 + \mathbb{R}\cdot e_2$.

- If $A\otimes_F F_{v_i} \simeq \mathrm{M}_{2m}(\mathbb{C})$ with involution $\tau(X) = f^{-1}\overline{X}^T f$, which corresponds to the hermitian form $f = r\cdot\langle 1,-1\rangle \oplus (2m-2r)\langle 1\rangle$, then let $K_{v_i} = \mathbb{R}^{2m}$ embed in $(A\otimes_F F_{v_i})^{\tau\otimes 1}$ via diagonal matrices. Let e_1 be the diagonal matrix with with the first m entries equal to 1 and the last m entries equal to 0, and $e_2 = I_{2m\times 2m} - e_1$. Then $J_{v_i} = \mathbb{R}^2$ embeds in K_{v_i} via $\mathbb{R}e_1 + \mathbb{R}e_2$.

- If $A\otimes_F F_{v_i} \simeq \mathrm{M}_m(\mathbb{H}) \times \mathrm{M}_m(\mathbb{H})^{op}$, let $K_{v_i} = \mathbb{C}^m$ embed in $A\otimes_F F_{v_i}$ as diagonal matrices in each component, and let $J_{v_i} = \mathbb{C}$ embed in K_{v_i} as scalar matrices in each component.

- If $L\otimes_F F_{w_i} = L_{w_i}$ is a field, by [24] we have that

$$G_{F_{w_i}} \simeq \mathbb{SU}_{2m}(L_{w_i}, f)$$

  where f is the sum of $m-1$ hyperbolic hermitian forms and one anisotropic form $\langle\alpha,\beta\rangle$. By rank considerations, $\mathbb{SU}_2(L_{w_i}, \langle -1, 1\rangle) \simeq \mathbb{SL}_2$ and $\mathbb{SU}_2(\langle\alpha,\beta\rangle) \simeq \mathbb{SL}(Q)$ for some non-split quaternion algebra Q over F_{w_i}. Choose any quadratic extension J_{w_i} of F_{w_i} disjoint from L_{w_i}. By [14], Remark 2.7, p. 154 we have that Q is split over J_{w_i}. By Steinberg's theorem we can therefore embed $R^{(1)}_{J_{w_i}/F_{w_i}}(\mathbb{G}_\mathrm{m})$ in $\mathbb{SL}(Q)$ and $\mathbb{SU}_2(L_{w_i}, \langle -1, 1\rangle)$. This is equivalent to finding embeddings of $J_{w_i}\cdot L_{w_i}$ in $M_2(L_{w_i})$ such that the involutions corresponding to $\langle 1,-1\rangle$ and $\langle\alpha,\beta\rangle$ fix J_{w_i}. Take the diagonal product of these embeddings to construct an embedding $L_{w_i}\cdot J_{w_i} \hookrightarrow M_{2m}(L_{w_i})$ such that $(L_{w_i}\cdot J_{w_i})^{\tau\otimes 1} = J_{w_i}$.

  The double centralizer theorem (Theorem 2.23) gives that $C := C_{A\otimes F_{w_i}}(L_{w_i}\cdot J_{w_i})$ is a central simple algebra over $L_{w_i}\cdot J_{w_i}$ of degree m. The fact that $\tau\otimes 1$ fixes

J_{w_i} means that $\tau \otimes 1|_C$ is an involution of the second kind on C fixing J_{w_i}. Consider an arbitrary subfield E_{w_i} of C such that $[L_{w_i} \cdot J_{w_i} : E_{w_i}] = m$, then [20], Proposition 2.1, p. 5 gives that $K_{w_i} = E_{w_i}^{\tau\otimes 1|_C}$ is a degree m extension of J_{w_i} disjoint from L_{w_i}.

- If $L \otimes_F F_{w_i} \simeq F_{w_i} \times F_{w_i}$, then $A \otimes_F F_{w_i} \simeq A'_{w_i} \times A'^{op}_{w_i}$ with the exchange involution, so we can choose a maximal subfield K_{w_i} of A'_{w_i} and let $J_{w_i} \subset K_{w_i}$ be such that $[K_{w_i} : J_{w_i}] = m$. Then $E_{w_i} = K_{w_i} \times K_{w_i}^{op} \simeq K_{w_i}^2 \subset A_{w_i}$ and $E_{w_i}^{\tau_{w_i}} = K_{w_i}$.

- Finally, if $L \otimes_F F_{w_i} \simeq F_{w_i} \times F_{w_i}$ for all i and $L \otimes_F F_{v_j} \simeq F_{v_j} \times F_{v_j}$ for all j, choose a (non-archimedean) valuation s on F such that $L \otimes_F F_s = L_s$ is a field and choose an arbitrary subfield $E_s \subset A \otimes_F F_s$ such that $\dim_{F_s}(E_s^{\tau_s}) = 2m$ and $E_s \simeq E_s^{\tau_s} \otimes_{F_s} L_s$ and let $K_s = E_s^{\tau_s}$ with $J_s \subset K_s$ an arbitrary subfield with $[K_s : J_s] = m$.

Case II n is odd.

In this case, let p be the smallest prime dividing n and let $n = mp$. For the archimedean valuations, either $A \otimes_F F_{v_i} \simeq \mathrm{M}_n(\mathbb{C})$ or $\mathrm{M}_n(\mathbb{R}) \times \mathrm{M}_n(\mathbb{R})^{op}$.

- If $A \otimes_F F_{v_i} \simeq \mathrm{M}_{pm}(\mathbb{R}) \times \mathrm{M}_{pm}(\mathbb{R})$ with exchange involution, let $J_{v_i} \simeq \mathbb{R}^p \subset \mathbb{R}^n = K_{v_i}$ where K_{v_i} embeds as in the even case, but now let e_i be the matrix with 1's in the $(i-1)m+1$ to im diagonal entries and 0's elsewhere and let J_{v_i} embed in K_{v_i} via $\sum \mathbb{R} e_i$.

- If $A \otimes_F F_{v_i} \simeq \mathrm{M}_{pm}(\mathbb{C})$ with involution $\tau(X) = f^{-1}\overline{X}^T f$, which corresponds to the hermitian form $f = r \cdot \langle 1, -1 \rangle \oplus (pm - 2r)\langle 1 \rangle$, then let $K_{v_i} = \mathbb{R}^n$ embed in $A \otimes_F F_{v_i}^{\tau\otimes 1}$ via diagonal matrices. Let e_i be the matrix with 1's in the

$(i-1)m+1$ to im diagonal entries and 0's elsewhere. Then $J_{v_i} = \mathbb{R}^p$ embeds in K_{v_i} via $\sum \mathbb{R}e_i$.

- For the non-archimedean valuations, choose K_{w_i}, J_{w_i} (and K_s and J_s, if necessary) as in the case that n is even.

Let $F \subset J \subset K$ be a tower of field extensions of F such that $J \otimes_F F_{x_i} \simeq J_{x_i}$ and $K \otimes_F F_{x_i} \simeq K_{x_i}$, where x_i is any archimedean valuation or any non-archimedean valuation listed in the section above. By construction, $E = K \otimes_F L$ is a field and in the notation of Lemma 3.30 there exists an embedding of F_v-algebras with involution

$$(E \otimes_F F_{x_i}, \sigma \otimes 1) \hookrightarrow (A \otimes_F F_{x_i}, \tau \otimes 1).$$

By Lemma 3.30 such an embedding exists for any valuation not among the x_i (since all valuations such that G_{F_v} are not split nor quasi-split are contained in the x_i), thus by [20], there exists an embedding of algebras with involution:

$$(K \otimes_F L, 1 \otimes \gamma) \stackrel{\iota}{\hookrightarrow} (A, \tau).$$

Next I claim that $\iota(K \otimes L) \otimes_F F_v$ and E_v are conjugate by an element of G_{F_v} for every archimedean place v on F. Indeed, $\iota(K \otimes L) \otimes F_v$ and E_v both correspond to unique maximal tori in G_{F_v} it suffices to show that the corresponding tori are conjugate (see [20], Proposition 2.3, p. 6).

If $A \otimes_F F_v \simeq \mathrm{M}_n(\mathbb{C})$, then $\iota(K \otimes L) \otimes F_v$ and E_v both correspond to anisotropic maximal tori in G_{F_v}, hence are conjugate in G_{F_v}. If $A \otimes_F F_v \simeq \mathrm{M}_n(\mathbb{R}) \times \mathrm{M}_n(\mathbb{R})^{op}$, then both $\iota(K \otimes L) \otimes F_v$ and E_v correspond to tori of maximal F_v-rank, hence they are also conjugate by an element of G_{F_v}. Finally, if $A \otimes_F F_v \simeq \mathrm{M}_{n/2}(\mathbb{H}) \times \mathrm{M}_{n/2}(\mathbb{H})$, then $\iota(K \otimes L) \otimes F_v$ and E_v both correspond to maximal tori of maximal rank over F_v

in G_{F_v}, hence they are conjugate as well. By considering eigenvalues with multiplicity we must have that this conjugation takes $\iota(J \otimes L) \otimes F_v$ to J_v.

Proof of Proposition 3.28. Let $P = J \otimes_F L$ and consider

$$H = R_{J/F}(\mathbb{SU}(C_A(P), \tau|_{C_A(P)})) \leq G.$$

Then H is a proper simple subgroup, and I claim that H has appropriate real rank. To see this, note that if $v \in V^F_{\infty,\mathbb{R}}$ is such that $J \otimes_F F_v \simeq \prod J_v^{(i)}$ where $J_v^{(i)}$ are field extensions of F_v, then $H_{F_v} \simeq \prod R_{J_v^{(i)}/F_v}(\mathbb{SU}(C_A(P), \tau|))_{J_v^{(i)}} \simeq \prod R_{J_v^{(i)}/F_v}(\mathbb{SU}(C_A(P) \otimes_J J_v^{(i)}, \tau| \otimes 1))$.

First, consider the case that $J \otimes_F F_v \simeq \mathbb{C}$. This implies that $A \otimes_F F_v \simeq \mathrm{M}_{n/2}(\mathbb{H}) \times \mathrm{M}_{n/2}(\mathbb{H})$, and because $J \otimes_F F_v$ is conjugate to J_v, we have that $C_{A \otimes_F F_v}(J_v \otimes L)$ consists of scalar matrices in each component. Thus $\mathbb{SU}(C_{A \otimes_F F_v}(J \otimes_F L \otimes F_v), \tau| \otimes 1) \simeq \mathbb{SL}_{n/2}(\mathbb{C})$, and so $H_{F_v} \simeq R_{\mathbb{C}/\mathbb{R}}(\mathbb{SL}_{n/2}(\mathbb{C}))$ has rank $\frac{n}{2} - 1 \geq 2$, as required.

Next, assume that $J \otimes_F F_v$ is not a field. Then $J \otimes_F F_v \simeq \mathbb{R}^p$ if $n = pm$, where p is the smallest prime dividing n and up to conjugation (and possibly renumbering) $J_v^{(i)} = \mathbb{R}e_i$. To calculate $\mathbb{SU}(C_A(P) \otimes_J J_v^{(i)}, \tau| \otimes 1)$, consider the following chain of isomorphisms:

$$\begin{aligned} \bigoplus C_A(P) \otimes_J J_v^{(i)} \simeq \quad C_A(P) \otimes_J (\prod J_v^{(i)}) &\simeq \quad C_A(P) \otimes_J J \otimes_F F_v \\ \simeq \quad C_A(P) \otimes_F F_v &\simeq \quad C_{A \otimes_F F_v}(P \otimes_F F_v) \\ \simeq \quad C_{A \otimes_F F_v}(P_v) &\simeq \quad \prod_i C_{e_i \cdot A \otimes_F F_v e_i}(\mathbb{R}e_i \cdot L_v) \end{aligned}$$

all of the isomorphisms respect components and involutions (because we conjugate by an element of G_{F_v}), thus $H_{F_v} \simeq \prod \mathbb{SU}(C_{e_i A \otimes_F F_v e_i}(\mathbb{R}e_i \cdot L_v))$.

If $A \otimes_F F_v \simeq \mathrm{M}_n(\mathbb{R}) \times \mathrm{M}_n(\mathbb{R})$, then this means that $H_{F_v} \simeq \prod_{i=1}^p \mathbb{SL}_m(\mathbb{R})$, which has higher rank. If $A \otimes_F F_v \simeq \mathrm{M}_n(\mathbb{C})$ and $\tau \otimes 1$ corresponds to the hermitian form with diagonalization $r \cdot \langle 1, -1 \rangle \oplus (pm - 2r)\langle 1 \rangle$, then $H_{F_v} \simeq \mathbb{SU}_m(\mathbb{C}, f_1) \times \cdots \times \mathbb{SU}_m(\mathbb{C}, f_p)$, where $f = f_1 \oplus \cdots \oplus f_p$ and f_1 is taken from the first m coefficients of the diagonalization of f, f_2 from the second, and so on. If $r = 1$, then both G_{F_v} and $\mathbb{SU}_m(\mathbb{C}, f_1)$ have rank 1, thus H_{F_v} has rank 1. If $r \geq 2$ and $m > 3$, then $\mathbb{SU}_m(\mathbb{C}, f_1)$ has rank ≥ 2, thus H_{F_v} is of higher rank. If $r \geq 2$ and $m = 3$, then $\mathbb{SU}_m(\mathbb{C}, f_1)$ has rank 1, as does $\mathbb{SU}_m(\mathbb{C}, f_2)$, thus H_{F_v} is of higher rank as well.

Combining these cases shows that H has appropriate real rank and thus G is not minimal. ■

Chapter 4

Exceptional Groups Splitting over Quadratic Extensions

The purpose of this section is to prove that absolutely simple groups of type E_7, E_8, F_4 and G_2 are not minimal. Unless otherwise stated, G will be simply connected throughout this section. The approach for these four cases will rely on the following observation:

Lemma 4.1 *Any group of type E_7, E_8, F_4 or G_2 over F becomes split over a purely imaginary quadratic extension K.*

Proof. Notice that the automorphism groups of the Dynkin diagrams for each of these groups are trivial, and that any group of type E_8, F_4 or G_2 has trivial centre, while a simply connected group of type E_7 has centre μ_2. If we choose any purely imaginary quadratic extension K of V, then Theorem 2.14 immediately gives that G_K is split for any G of type E_8, F_4 or G_2.

If G is of type E_7, assume that $G = {}^{\xi}G_0$, where G_0 is split and simply connected of type E_7 and $[\xi] \in H^1(F, \overline{G_0})$. Then the image of $[\xi]$ in $H^2(F, Z(G_0))$ corresponds to a quaternion algebra over F, say D. By the weak approximation property, there exists an $a \in F$ such that the image of a in F_v is negative for all $v \in V^F_{\infty,\mathbb{R}}$ and a is non-square in F_v for all non-archimedean places v on F such that $D \otimes_F F_v$ is non-split. Let

$K = F(\sqrt{a})$, then D becomes split over K by the Hasse principle for central simple algebras. This means that $\mathrm{Res}_{K/F}([\xi])$ is in the image of $H^1(K, G_0) \to H^1(K, \overline{G_0})$, but $H^1(K, G_0)$ is trivial by Theorem 2.10 so $\mathrm{Res}_{K/F}([\xi])$ is trivial, i.e. G is split over K. ■

Remark 4.2 *Note that in the case that G has type G_2, we can choose $K = F(\sqrt{a})$ with a positive in F_v for all $v \in S_G$ such that G splits over K. Recall from Tits' classification [24] that in this case $S_G = S''_G$, i.e. G is split over F_v for all $v \in S_G$. By the weak approximation property, we may choose $a \in F$ such that the image of a in F_v is positive for all $v \in S_G = S''_G$ and the image of a in F_v is negative for all $v \in V^F_{\infty,\mathbb{R}} \backslash S_G$. Let $K = F(\sqrt{a})$, then if $w \in V^K_{\infty,\mathbb{R}}$ lies over $v \in S_G$ we have*

$$\mathrm{Res}_{K_w/K} \circ \mathrm{Res}_{K/F}([\xi]) = \mathrm{Res}_{K_w/F_v} \circ \mathrm{Res}_{F_v/F}([\xi]) = \mathrm{Res}_{K_w/F_v}(1) = 1$$

and if $w \in V^K_{\infty,\mathbb{R}}$ lies over $v \in V^F_{\infty,\mathbb{R}} \backslash S_G$ then K_w is algebraically closed so

$$\mathrm{Res}_{K_w/K} \circ \mathrm{Res}_{K/F}([\xi]) = 1$$

automatically. Applying the above corollary gives that G splits over K, and $K \hookrightarrow F_v$ for all $v \in S_G$.

I introduce some notions developed by Chernousov in [5] relating to groups splitting over quadratic extensions. Let G be an F-defined group splitting over a quadratic extension K/F, τ be the non-trivial element of $\mathrm{Gal}(K/F)$ and T be some maximal F-torus in G splitting over K (see [5], §4, p. 302 for remarks on the existence of T).

Lemma 4.3 ([5], Lemma 4.1, p. 303) *Let T be as above, then for any $\alpha \in \Sigma(G, T)$, we have $\tau(\alpha) = -\alpha$.*

Definition 4.4 ([5], p. 304) *If we choose a Chevalley basis* $\{H_{\alpha_1}, \dots, H_{\alpha_n}, X_\alpha, \alpha \in \Sigma(G,T)\}$ *then the above lemma implies that* $\tau(X_\alpha) = c_\alpha X_{-\alpha}$ *for some* $c_\alpha \in F$. *Call* $\{c_\alpha \mid \alpha \in \Sigma(G,T)\}$ *the* structure constants *of* G *with respect to* T.

Note that these structure constants depend of the choice of T:

Lemma 4.5 ([5], Lemmas 4.6 and 4.7, p. 305) *Any two maximal* F*-tori (say* T *and* T'*) of* G *splitting over* K *are isomorphic over* F *and* $T' = gTg^{-1}$ *for some* $g \in G(K)$ *such that* $t = g^{-1+\tau} \in T(K)$. *Moreover, if* $t = h_{\alpha_1}(t_1) \cdots h_{\alpha_n}(t_n)$ *then* $t_i \in F^\times$ *and the structure constants* $\{c'_\alpha\}$ *with respect to* T' *are related to the structure constants* $\{c_\alpha\}$ *with respect to* T *by:*

$$c'_\alpha = t_1^{-\langle \alpha, \alpha_1 \rangle} \cdots t_n^{-\langle \alpha, \alpha_n \rangle} \cdot c_\alpha$$

Because $\tau(\alpha) = -\alpha$, we have that G_α is F-defined, and furthermore it can be shown that:

Lemma 4.6 ([5], Lemma 4.11, p. 306) $G_\alpha \simeq \mathbb{SL}_1(D_\alpha)$, *where* D_α *is the quaternion algebra* (d, c_α).

Using this, we can already eliminate one type of algebraic group from the list of possible minimal algebraic groups:

Proposition 4.7 *Every anisotropic group* G *of type* G_2 *over* F *contains an absolutely simple subgroup* H *of type* A_2 *of appropriate real rank.*

Proof. Choose a as in remark 4.2 and T splitting over $K = F(\sqrt{a})$. Recall the notion of a standard subgroup and the notation $G_{\Sigma'}$ from 2.2.5. Let Σ' be the root sub-system of long roots in $\Sigma(G,T)$ and let $H = G_{\Sigma'}$. For any $v \in S_G$ we have that T is split over F_v, thus H is split over F_v. ■

Assume again that G is any simple group splitting over quadratic extension and that τ, T are as above. The structure constants defined above are very useful in determining the isotropy of G over F_v for $v \in V^F_{\infty,\mathbb{R}}$. From Lemma 4.6, we obtain:

Lemma 4.8 *Given $v \in V^F_{\infty,\mathbb{R}}$ such that $K \otimes_F F_v \simeq \mathbb{C}$:*

1. *G is anisotropic over F_v if and only if c_α are negative in F_v for all $\alpha \in \Sigma(G,T)$*
2. *If $\langle \alpha, \beta \rangle = 0$ and $c_\alpha, c_\beta > 0$ in F_v, then G has higher rank over F_v.*

Proof. Only the "if" direction of the first statement requires proof, the others are immediate. Note that over $F_v \simeq \mathbb{R}$, a semisimple group is anisotropic if and only if it is compact. If all structure constants are negative in F_v, then every root subgroup is anisotropic. By Tychonoff's theorem, the arbitrary product of these root subgroups is compact. Because G_{F_v} is generated by the root subgroups, this gives that G_{F_v} is the image of a compact set and therefore compact. ∎

From [24], there are three possibilities for the rank of a group G of type F_4 over any field. Over a completion F_v for $v \in V^F_{\infty,\mathbb{R}}$, I claim that the sign of the structure constants completely determines the rank of G over F_v.

Lemma 4.9 *If G is anisotropic over F of type F_4, $T \leq G$ is a maximal F-defined torus splitting over K as in Lemma 4.1 and $\{c_\alpha\}$ are the structure constants of G with respect to T, then for $v \in V^F_{\infty,\mathbb{R}}$:*

1. *$c_\alpha < 0$ in F_v for all α if and only if G_{F_v} is anisotropic.*
2. *Over F_v, $c_\alpha < 0$ for all long roots α and $c_\beta > 0$ for some short root β if and only if G has F_v-rank 1.*
3. *At least one long root α has $c_\alpha > 0$ in F_v if and only if G is F_v-split.*

Proof. The first statement is Lemma 4.8(1). Assume that for some $\alpha \in \Sigma(G,T)$ with length 2 we have $c_\alpha > 0$ in F_v. I claim that G_{F_v} is then split.

Let $\Sigma' \leq \Sigma(G,T)$ be the sub-root system generated by the long roots, so that Σ' has type D_4, and let $H = G_{\Sigma'}$. Then because $\mathrm{Gal}(K/F)$ stabilizes $\{\pm\alpha\}$ for each $\alpha \in \Sigma(G,T)$, we have that H is of type ${}^1D_4$. By the assumption that $c_\alpha > 0$ for some long root α, we also have that H is F_v-isotropic. From [24], we therefore have that $\mathrm{Rank}_{F_v}(H) \geq 2$, thus $\mathrm{Rank}_{F_v}(G) \geq 2$ and so G is split over F_v.

To complete the proof of the lemma, it suffices to prove that if G is split over F_v then $c_\alpha > 0$ for some long root $\alpha \in \Sigma(G,T)$. Assume that G is split over F_v and let T' be a maximal torus in G split over F_v. If $c_\alpha < 0$ in F_v for all $\alpha \in \Sigma'$, then H is anisotropic over F_v. Let B be a Borel subgroup of G containing T'. Note that $(B \cap H)^0$ is reductive. Because reductive groups are unirational (see [23], Corollary 13.3.9, p. 231), this means that $(B \cap H)^0(F_v)$ is non-empty. Choose a F_v-rational point $x \in (B \cap H)^0(F_v)$ of infinite order, then $\overline{\langle x \rangle}$ is a connected diagonalizable subgroup of H defined over F_v, a contradiction. ■

4.1 Modification of Structure Constants

Recall that the structure constants are dependent on the choice of maximal F-torus $T \leq G$ splitting over K, and that Lemma 4.5 gives a formula for how the structure constants change if we choose another $T' \leq G$. We can use this to 'modify' structure constants by replacing T with gTg^{-1} for specifically chosen $g \in G(K)$. In particular, Lemma 4.6 gives that $G_\alpha(K) \simeq \mathbb{SL}_{2,K}$ for all $\alpha \in \Sigma(G,T)$, so given $y \in K^\times$ we can define

$$g_\alpha = \begin{pmatrix} 1 & \frac{\tau(y)}{c_\alpha - y\tau(y)} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix}$$

and show that $g_\alpha^{1-\tau} = h_\alpha(\frac{c_\alpha}{c_\alpha - y\tau(y)})$. If $\{c_\beta\}$ are the structure constants of G with respect to T and we replace T by $T' = g_\alpha T g_\alpha^{-1}$ and let $\{c'_\beta\}$ be the structure constants of G with respect to T', then Lemma 4.5 implies that $c'_\beta = \left(\frac{c_\alpha}{c_\alpha - y\tau(y)}\right)^{\langle\beta,\alpha\rangle} c_\beta$.

Given that Lemma 4.8 is interested only in the sign of c_β in F_v for $v \in V^F_{\infty,\mathbb{R}}$ (which I denote by $\mathrm{Sign}_v(c_\beta)$), this is all we are interested in modifying when we modifying structure constants. We can do this for each $v \in V^F_{\infty,\mathbb{R}}$ independently:

Lemma 4.10 *Given $\alpha \in \Sigma(G,T)$, $v \in V^F_{\infty,\mathbb{R}}$ such that $\mathrm{Sign}_v(c_\alpha) = 1$, we can choose $g_\alpha \in G_\alpha(K)$ such that, if $\{c'_\beta\}$ are the structure constants of G with respect to $g_\alpha T g_\alpha^{-1}$, then*

1. $\mathrm{Sign}_w(c'_\beta) = \mathrm{Sign}_w(c_\beta)$ *for all* $w \neq v \in V^F_{\infty,\mathbb{R}}$ *and*
2. $\mathrm{Sign}_v(c'_\beta) = (-1)^{\langle\beta,\alpha\rangle}\,\mathrm{Sign}_v(c_\beta)$ *for all* β.

Proof. By the weak approximation property, we can choose $y \in F$ such that $|y^2|_w < |c_\alpha|_w$ for all $w \neq v \in V^F_{\infty,\mathbb{R}}$ and $|c_\alpha|_v < |y^2|_v$. Define g_α as above. Replacing T by $T' = g_\alpha T g_\alpha^{-1}$, we get that $c'_\beta = \left(\frac{c_\alpha}{c_\alpha - y^2}\right)^{\langle\beta,\alpha\rangle} c_\beta$. Our choice of y gives that c'_β has the desired sign in F_v for all $v \in V^F_{\infty,\mathbb{R}}$. ■

We call a modification of the form above a *modification of T by α with respect to v*.

Proposition 4.11 *Every anisotropic group G of type F_4 over F contains an absolutely simple subgroup H of type B_3 of appropriate real rank.*

Proof. Let Σ' be the root subsystem of $\Sigma(G,T)$ generated by $\{\alpha_1, \alpha_2, \alpha_3\}$ and let $H = G_{\Sigma'}$ (throughout the proof I use Bourbaki's explicit realization of root systems ([3], Plate I-IX, with identical notation). Then H is a proper, absolutely simple subgroup of G, so it suffices to show that H has appropriate real rank.

Claim 4.12 *We can choose T in such a way that* $\mathrm{Sign}_v(c_{\alpha_3}) = 1$ *for all* $v \in S_G$ *and* $\mathrm{Sign}_v(c_{\alpha_1}) = 1$ *for all* $v \in S''_G$.

First I claim that we can modify T so that $\mathrm{Sign}_v(c_{\alpha_1}) = 1$ for all $v \in S''_G$. If $v \in S''_G$, then by Lemma 4.8 we have that $\mathrm{Sign}_v(c_\alpha) = 1$ for some long root $\alpha \in \Sigma(G,T)$. The possibilities for $\langle \alpha_1, \alpha \rangle$ are $0, \pm 1$ and ± 2. If there exists a long root α such that $\mathrm{Sign}_v(c_\alpha) = 1$ and $\langle \alpha_1, \alpha \rangle = \pm 2$, then $\alpha = \pm \alpha_1$, so assume no such α exists. If there exists such an α such that $\langle \alpha_1, \alpha \rangle = \pm 1$, then modifying T by α with respect to v yields T as desired.

If there does not exist an α with $\mathrm{Sign}_v(c_\alpha) = 1$ and $\langle \alpha_1, \alpha \rangle = \pm 1$, but there does exist α with $\mathrm{Sign}_v(c_\alpha) = 1$ and $\langle \alpha_1, \alpha \rangle = 0$, then α must be of the form $\pm(\epsilon_1 + \epsilon_2)$ or $\pm \epsilon_3 \pm \epsilon_4$. If $\alpha = \pm \epsilon_3 \pm \epsilon_4$ let $\alpha' = \epsilon_2 + \epsilon_4$ and if $\alpha = \pm(\epsilon_1 + \epsilon_2)$ let $\alpha' = \epsilon_2 + \epsilon_3$. In either case, we have that $\langle \alpha', \alpha \rangle = \pm 1$ and $\langle \alpha_1, \alpha' \rangle = \pm 1$, so modifying T by α' with respect to v returns us to the case that there exists a long root α with $\mathrm{Sign}_v(c_\alpha) = 1$ and $\langle \alpha_1, \alpha \rangle = \pm 1$.

Assume that $v \in S''_G$ and we have done the modifications above so that $\mathrm{Sign}_v(c_{\alpha_1}) = 1$. If $\mathrm{Sign}_v(c_{\alpha_3}) = 1$, then T is as required. If $\mathrm{Sign}_v(c_{\alpha_3}) = -1$ and there exists a short root β such that $\mathrm{Sign}_v(c_\beta) = 1$ and $\langle \alpha_3, \beta \rangle = \pm 1$, then modifying T by β with respect to v gives T as required. If no such β exists, let $\beta' = \frac{1}{2}(\epsilon_1 + \epsilon_2 - \epsilon_3 + \epsilon_4)$, then $\langle \beta', \alpha_1 \rangle = 1 = \langle \alpha_3, \beta' \rangle$ and $\langle \alpha_1, \beta' \rangle = 2$. Modifying T by α_1 with respect to v gives a new T such that $\mathrm{Sign}_v(c_{\beta'}) = 1$. Next, modifying T by β' with respect to v gives another T such that $\mathrm{Sign}_v(c_{\alpha_3}) = 1$ and $\mathrm{Sign}_v(c_{\alpha_1})$ is unchanged (because $\langle \alpha_1, \beta' \rangle = 2$). This new T is such that $\mathrm{Sign}_v(c_{\alpha_1}) = 1 = \mathrm{Sign}_v(c_{\alpha_3})$ for all $v \in S''_G$.

Assume now that $v \in S'_G$. If $\mathrm{Sign}_v(c_\beta) = 1$ for $\beta = \pm\frac{1}{2}(\epsilon_1 \pm \epsilon_2 \pm \epsilon_3 \pm \epsilon_4)$, then $\langle \beta, \alpha_3 \rangle = \pm 1$, so we can modify T by β with respect to v to obtain $\mathrm{Sign}_v(c'_{\alpha_3}) = 1$. If $\mathrm{Sign}_v(c_\beta) = -1$ for all β of the form above, then we must have that $\mathrm{Sign}_v(\epsilon_i) = 1$ for some $i \neq 4$ by the assumption that some short root has positive associated structure constant. Fix $\beta = \frac{1}{2}(\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4)$. Then $\mathrm{Sign}_v(c_\beta) = -1$ by assumption, and we have $\langle \epsilon_i, \alpha_3 \rangle = 0$, $\langle \beta, \epsilon_i \rangle = 1$ for all i. This means that if we modify T first by ϵ_i and then by β with respect to v, we will have $\mathrm{Sign}_v(c''_{\alpha_3}) = 1$. This proves the claim.

Combining Lemma 4.8 with the above claim yields that H has appropriate real rank, thus H is not minimal. ■

Proposition 4.13 *Any anisotropic group G of type E_7 over F contains an absolutely simple subgroup H of type A_3 of appropriate real rank.*

Remark 4.14 *By [24], $S_G = S''_G$ for G of type E_7.*

Proof. For a maximal F-defined torus T of G, define $\Sigma' \subset \Sigma(G,T)$ to be the sub-root system generated by $\{\alpha_5, \alpha_6, \alpha_7\}$, and let $H = G_{\Sigma'}$. Clearly, H is an absolutely simple proper subgroup of type A_3 and it remains to show that H has appropriate real rank. By Lemma 4.8, it suffices to prove the following:

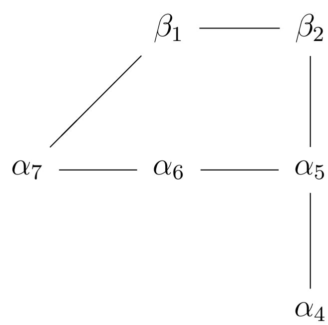
Claim 4.15 *We can choose T so that $c_{\alpha_5}, c_{\alpha_7} > 0$ in F_v for all $v \in S_G$.*

By Lemma 4.8, we may always choose some $\alpha \in \Sigma(G,T)$ such that $\mathrm{Sign}_v(c_\alpha) = 1$. After modification, we can say that $\mathrm{Sign}_v(c_{\alpha_7}) = 1$. Indeed, assume that $\mathrm{Sign}_v(c_{\alpha_7}) = -1$. If there exists α with $\langle \alpha_7, \alpha \rangle = \pm 1$, then modification of T by α with respect to v reverses the sign of c_{α_7}. If $\langle \alpha_7, \alpha \rangle \in \{0, \pm 2\}$ for all $\alpha \in \Sigma(G,T)$ with $\mathrm{Sign}_v(c_\alpha) = 1$, then choose an α with $\mathrm{Sign}_v(c_\alpha) = 1$, define $\kappa = \epsilon_7 - \epsilon_8 + \epsilon_6 - \epsilon_5$ and let α' be

$$\begin{cases} \epsilon_j + \epsilon_6 \text{ if} & \alpha = \pm\epsilon_j \pm \epsilon_k,\ j < k \in \{1,2,3,4\} \\ \epsilon_4 + \epsilon_6 \text{ if} & \alpha = \pm(\alpha_5 + \alpha_6) \\ \frac{1}{2}(\kappa + \sum_{i=1}^{4} \epsilon_i) \text{ if} & \alpha = \pm(\epsilon_7 - \epsilon_8) \\ \frac{1}{2}(\kappa + (-1)^{\nu(4)} + \sum_{i=1}^{3} (-1)^{1-\nu(i)} \epsilon_i) \text{if} & \alpha = \frac{1}{2}(\epsilon_7 - \epsilon_8 \pm (\epsilon_5 + \epsilon_6) + \sum_{i=1}^{4} (-1)^{\nu(i)} \epsilon_i) \end{cases}$$

Then modifying T by α with respect to v returns us to the case where there exists α' with $\mathrm{Sign}_v(c_{\alpha'}) = 1$ and $\langle \alpha_7, \alpha' \rangle = \pm 1$, and so we can modify T again so that $\mathrm{Sign}_v(c_{\alpha_7}) = 1$.

Now, assuming that we have modified T so that $\mathrm{Sign}_v(c_{\alpha_7}) = 1$, I claim that we can modify T further so that $\mathrm{Sign}_v(c_{\alpha_5}) = 1$ as well. To see this, let $\beta_1 = \epsilon_1 - \epsilon_6$ and $\beta_2 = \frac{1}{2}(\epsilon_8 - \epsilon_7 + \epsilon_6 + \epsilon_5 + \epsilon_4 - \epsilon_3 - \epsilon_2 - \epsilon_1)$. Recall that if $\mathrm{Sign}_v(c_\alpha) = 1$, then modifying T by α with respect to v only affects $\mathrm{Sign}_v(\beta)$ for those β with $\langle \beta, \alpha \rangle$ odd. In the following graph, the nodes correspond to roots, and edges connect roots such that $\langle \alpha, \beta \rangle$ is odd:



If $\mathrm{Sign}_v(c_{\alpha_5}) = 1$, then no modification is necessary. If $\mathrm{Sign}_v(c_{\alpha_5}) = -1$, but $\mathrm{Sign}_v(c_{\beta_2})$ or $\mathrm{Sign}_v(c_{\alpha_4}) = 1$, then modify T by β_2 or α_4 with respect to v to change the sign of c_{α_5} in F_v. Assume then that $\mathrm{Sign}_v(c_{\alpha_5}) = \mathrm{Sign}_v(c_{\alpha_4}) = \mathrm{Sign}_v(c_{\beta_2}) = -1$. If $\mathrm{Sign}_v(c_{\alpha_6}) = \mathrm{Sign}_v(c_{\beta_1}) = 1$, then modifying T first by α_6, then by β_1 with respect to v reverses $\mathrm{Sign}_v(c_{\alpha_7})$ twice and $\mathrm{Sign}_v(c_{\alpha_5})$ once, so after modification $\mathrm{Sign}_v(c_{\alpha_7}) = 1 = \mathrm{Sign}_v(c_{\alpha_5})$. If $\mathrm{Sign}_v(c_{\alpha_6}) = \mathrm{Sign}_v(c_{\beta_1}) = -1$, then modifying by α_7 with respect to v returns us to the case where $\mathrm{Sign}_v(c_{\alpha_6}) = \mathrm{Sign}_v(c_{\beta_1}) = 1$.

If $\mathrm{Sign}_v(c_{\beta_1}) = 1$ and $\mathrm{Sign}_v(c_{\alpha_6}) = -1$, then modifying T by α_7 with respect to v gives $\mathrm{Sign}_v(c_{\beta_1}) = -1$ and $\mathrm{Sign}_v(c_{\alpha_6}) = 1$. Therefore the only case left to consider is the case where:

$$\mathrm{Sign}_v(c_{\alpha_7}) = \mathrm{Sign}_v(c_{\alpha_6}) = 1$$

$$\mathrm{Sign}_v(c_{\beta_1}) = \mathrm{Sign}_v(c_{\beta_2}) = \mathrm{Sign}_v(c_{\alpha_5}) = \mathrm{Sign}_v(c_{\alpha_4}) = -1$$

In this case, if we modify T with respect to v by roots in the following order $\alpha_6, \alpha_5, \beta_2, \beta_1, \alpha_4$, then $(\mathrm{Sign}_v(c_{\alpha_7}), \mathrm{Sign}_v(c_{\alpha_5}))$ changes in the following way:

$$(1,-1) \overset{\alpha_6}{\to} (-1,1) \overset{\alpha_5}{\to} (-1,1) \overset{\beta_2}{\to} (-1,-1) \overset{\beta_1}{\to} (1,-1) \overset{\alpha_4}{\to} (1,1)$$

so after modification, $\mathrm{Sign}_v(c_{\alpha_7}) = 1 = \mathrm{Sign}_v(c_{\alpha_5})$, as required. ■

Proposition 4.16 *Any anisotropic group G of type E_8 over F contains an absolutely simple subgroup H of type A_3 of appropriate real rank.*

Proof. As in the previous case, define Σ' to be the subsystem of $\Sigma(G,T)$ generated by $\{\alpha_5, \alpha_6, \alpha_7\}$. Also as in the previous case, from [24] we have $S_G = S''_G$ for groups of type E_8, so it suffices to prove that we can choose some maximal F-torus T of G so that $\mathrm{Sign}_v(c_{\alpha_5}) = \mathrm{Sign}_v(c_{\alpha_7}) = 1$ for all $v \in S_G$.

Let $\Sigma'' \subset \Sigma(G,T)$ be the subsystem of $\Sigma(G,T)$ of type E_7 generated by $\{\alpha_1, \dots, \alpha_7\}$. To reduce the proof to the previous case, it suffices to show that it is possible to choose a maximal F-torus T of G so that $\mathrm{Sign}_v(\alpha) = 1$ for some root $\alpha \in \Sigma''$. Indeed, if we can show that some $\alpha \in \Sigma''$ has $\mathrm{Sign}_v(c_\alpha) = 1$, then we can modify T with respect to each v by roots in Σ'' as described in the previous proof to obtain $\mathrm{Sign}_v(c_{\alpha_5}) = \mathrm{Sign}_v(c_{\alpha_7}) = 1$ for all $v \in S''_G$.

Let $\beta_1 = \epsilon_6 + \epsilon_8$ and $\beta_2 = \epsilon_6 - \epsilon_8$, then $\langle \sum_{i=1}^{8} (-1)^{\nu(i)} \epsilon_i, \beta_j \rangle \neq 0 \mod 2$ for $j = 1$ or 2. Next, if $\alpha = \pm\epsilon_i \pm \epsilon_j$ and $\langle \beta_j, \alpha \rangle \equiv 0 \mod 2$, then $\langle \alpha, \alpha_i \rangle \neq 0 \mod 2$ for some $1 \leq i \leq 7$. This means that no matter what, for every $\alpha \in \Sigma(G,T)$, there exists a $\gamma \in \Sigma(G,T)$ and a $\delta \in \Sigma''$ such that $\langle \alpha, \gamma \rangle \equiv \langle \gamma, \delta \rangle \equiv 1 \mod 2$.

If $\text{Sign}_v(c_\delta) = 1$, we are done. If $\text{Sign}_v(c_\gamma) = 1$, modify T by γ with respect to v to obtain that $\text{Sign}_v(c'_\delta) = 1$. If $\text{Sign}_v(c_\gamma) = -1$, modify T by α with respect to v. This ether reverses the sign of c_δ with respect to v, or it returns us to the previous case. In any event, $\text{Sign}_v(c_\delta) = 1$, with $\delta \in \Sigma''$. ■

Chapter 5

Modification of Cocycles

5.1 Modification of Cocycles Lemma

There are two types of absolutely simple groups left to examine, ${}^{3,6}D_4$ and ${}^{1,2}E_6$. Let G_0 be a simply connected quasi-split group of the above type, let $\overline{G_0} = G_0/Z(G_0)$ be the corresponding quasi-split adjoint group and for a subgroup $H \leq G_0$, let $\overline{H}$ be the image of H in $\overline{G_0}$. We use the following technique for both types: Let G correspond to $[\xi] \in H^1(F, \overline{G_0})$ and assume that we can construct a maximal torus $T \leq G_0$ such that:

1. $[\xi]$ is in the image of $H^1(F, \overline{T}) \to H^1(F, \overline{G_0})$,

2. T normalizes a proper, simple subgroup H of G_0, and

3. $(T \cap H)^0$ has appropriate real rank.

If these conditions hold, then a twisted copy of H lies in G by conditions (1) and (2), and H has appropriate real rank by condition (3), thus G is not minimal.

Assume that we can construct a torus T satisfying properties (2) and (3) and we can find $[\mu] \in H^1(F, \overline{T})$ such that $[\xi]$ and $[\mu]$ have the same image in $H^2(F, Z(G_0))$

under the commuting diagram with exact rows:

$$\begin{array}{ccccc} H^1(F,G_0) & \xrightarrow{\pi_1} & H^1(F,\overline{G_0}) & \xrightarrow{\delta_1} & H^2(F,Z(G_0)) \\ \uparrow \iota_1 & & \uparrow \iota_2 & & \uparrow = \\ H^1(F,T) & \xrightarrow[\pi_2]{} & H^1(F,\overline{T}) & \xrightarrow[\delta_2]{} & H^2(F,Z(G_0)) \end{array} \qquad (1)$$

We wish to 'modify' $[\mu] \in H^1(F,\overline{T})$ by an element $[\alpha] \in H^1(F,T)$ to get $[\mu]\cdot\pi_2([\alpha]) \in H^1(F,\overline{T})$ so that $\iota_2([\mu]\cdot\pi_2([\alpha])) = [\xi]$. More precisely:

Lemma 5.1 (Modification of Cocycles) *Given $G_0, \overline{G_0}, T, \overline{T}, [\xi]$ as above, if there exist:*

1. *$[\mu] \in H^1(F,\overline{T})$ with $\delta_2([\mu]) = \delta_1([\xi])$, and*
2. *$[\nu_v] \in H^1(F_v,\overline{T})$ with $\iota_2([\nu_v]) = [\xi_v]$ for each archimedean place v*

then there exists a $[\gamma] \in H^1(F,\overline{T})$ such that $\iota_2([\gamma]) = [\xi]$.

Proof. Keep the notation of diagram (1). By the Hasse principle for $H^1(F,\overline{G_0})$ (see Theorem 2.14), it suffices to show that we can choose $[\gamma] \in H^1(F,\overline{T})$ such that $\iota_2([\gamma_v]) = [\xi_v]$ for any valuation v on F.

Fist, I claim that $\iota_2([\mu_v]) = [\xi_v]$ for any non-archimedean place v. From the condition that $\delta_2([\mu]) = \delta_1([\xi])$, we see that $\iota_2([\mu_v]) \in \delta_1^{-1}(\delta_1([\xi_v]))$. From [22], Chapter 1, Section 5, $\delta_1^{-1}(\delta_1([\xi_v]))$ is in bijective correspondence with $H^1(F_v, {}_{\xi}G_0)/\sim$ for some equivalence relation $\sim$. Because we assume that ${}_{\xi}G_0$ is simply connected and v is non-archimedean, Kneser's theorem gives that $H^1(F_v, {}_{\xi}G_0) = \{1\}$ and so $\delta_1^{-1}(\delta_1([\xi_v])) = \{[\xi_v]\}$, i.e. $\iota_2([\mu_v]) = [\xi_v]$.

Next, given $v \in V^F_{\infty,\mathbb{R}}$, condition (2) gives that $\delta_2([\nu_v]) = \delta_1([\xi_v])$ and condition (1) gives that $\delta_2([\mu_v]) = \delta_1([\xi_v])$, so $\delta_2([\nu_v]) = \delta_2([\mu_v])$. By the exactness of the bottom row in diagram (1), we get that $[\mu_v] = [\nu_v] \cdot \pi_2([\lambda_v])$ for some $[\lambda_v] \in H^1(F_v, T)$. From Lemma 2.16, the map $H^1(F,T) \overset{\prod \mathrm{Res}_{F_v}}{\longrightarrow} \prod_{v \in V^F_{\infty,\mathbb{R}}} H^1(F_v, T)$ is surjective. This means that we can choose $[\alpha] \in H^1(F,T)$ such that $[\alpha_v] = [\lambda_v]$ for all $v \in V^F_{\infty,\mathbb{R}}$.

I claim that $[\gamma] := [\mu] \cdot \pi_2([\alpha])$ has $\iota_2([\gamma_v]) = [\xi_v]$ for every v. For v non-archimedean, note that

$$\delta_1(\iota_2([\gamma_v])) = \delta_2([\gamma_v]) = \delta_2([\mu_v]) \cdot \delta_2(\pi_2([\alpha_v])) = \delta_2([\mu_v]) = \delta_1([\xi_v])$$

but we have shown that the fibre of $[\xi_v]$ under δ_1 is just $\{[\xi_v]\}$, so $\iota_2([\gamma_v]) = [\xi_v]$ for every non-archimedean v. Finally, for $v \in V^F_{\infty,\mathbb{R}}$ we have

$$\iota_2([\gamma_v]) = \iota_2([\mu_v] \cdot \pi_2([\alpha_v])) = \iota_2([\mu_v] \cdot \pi_2([\lambda_v])) = \iota_2([\nu_v]) = [\xi_v]$$

by construction. ■

5.2 Type ${}^{3,6}D_4$

5.2.1 Groups of type D_4 over $\mathbb{R}$

First, we recall some facts about groups of type D_4 over $\mathbb{R}$: Because there exist no cubic field extensions of $\mathbb{R}$, any group G of type D_4 over $\mathbb{R}$ is of type ${}^{1,2}D_4$. By Tits' classification, any simply connected group of type ${}^1D_4$ over $\mathbb{R}$ is isomorphic to a group of the form $\mathbb{S}\mathbf{pin}(f_i)$, where f_i is one of:

$$\begin{aligned} f_0 &= \sum_{i=1}^{8} x_i^2, \\ f_2 &= \sum_{i=1}^{6} x_i^2 - y_1^2 - y_2^2, \text{ or} \\ f_4 &= \sum_{i=1}^{4} x_i^2 - \sum_{i=1}^{4} y_i^2 \end{aligned}$$

up to multiplication by ± 1. Let G_0 be the split simply connected group of type ${}^1D_4$, so $G_0 \simeq \mathbb{S}\mathbf{pin}(f_4)$. Note that f_4 is a Pfister form over $\mathbb{R}$ and recall that a Pfister form over $\mathbb{R}$ is either split or anisotropic. This gives that $\mathbb{S}\mathbf{pin}(f_0)$ and G_0 are the two distinct groups corresponding to cocycles from the set $H^1(\mathbb{R}, G_0)$, and $\mathbb{S}\mathbf{pin}(f_2)$ corresponds to a cocycle in $H^1(\mathbb{R}, \overline{G_0})$ that is not contained in the image of $H^1(\mathbb{R}, G_0)$.

If G has type ${}^2D_4$, then G is also isomorphic to a group of the form $\mathbb{S}\mathbf{pin}(f_i)$, except now f_i has discriminant -1, thus f_i is either:

$$\begin{aligned} f_1 &= \sum_{i=1}^{7} x_i^2 - y_1^2, \text{ or} \\ f_3 &= \sum_{i=1}^{5} x_i^2 - \sum_{i=1}^{3} y_i^2 \end{aligned}$$

up to multiplication by ± 1.

5.2.2 Construction of T

Let G now be a simply connected group of type ${}^{3,6}D_4$ corresponding to $[\xi] \in H^1(F, \overline{G_0})$, where G_0 is now the simply connected quasi-split group of type ${}^{3,6}D_4$. Let E be a cubic extension of F over which G has type ${}^{1,2}D_4$. Then $Z(G_0) \simeq R^{(1)}_{E/F}(\mu_2)$ and so $H^2(F, Z(G_0)) \simeq \ker({}_2\mathrm{Br}(E) \xrightarrow{\mathrm{Cor}} {}_2\mathrm{Br}(F))$ where N is the norm map. Recall that the order of an element of $\mathrm{Br}(E)$ is equal to its index (see Section 2.2.7), so the image of $[\xi]$ in $H^2(F, Z(G_0))$ is represented by the isomorphism class of a quaternion algebra $[(a,b)_E]$. Because the corestriction of $[(a,b)_E]$ is trivial we can choose a, b such that $a \in F$ and $N_{E/F}(b) = 1$ (see [10], Section 7.3, p. 195). Applying the weak approximation property to the norm form of $[(a,b)_E]$, it is not difficult to see that we can also choose a so that $F(\sqrt{a})$ has no real completions.

The following result is proven the proof of [6], Theorem 6.1:

Theorem 5.2 *There exists a subgroup $H < G_0$ of type $A_1 \times A_1 \times A_1 \times A_1$ that is isogenous to $R_{P/F}(SL_2)$ for some quartic field extension P/F that is contained in $E(\sqrt{b}, \sqrt{\sigma(b)}, \sqrt{\sigma^2(b)})$ where $\sqrt{\sigma^i(b)}$ are the Galois conjugates of $\sqrt{b}$ in the normal closure of E over F.*

Let $\tilde{H} = R_{P/F}(\mathbb{SL}_2)$, H the image of $\tilde{H}$ in G_0, $\overline{H}$ the image of $\tilde{H}$ in $\overline{G_0}$ and $\overline{H}' = \tilde{H}/Z(\tilde{H})$. If we consider the sequence of projections

$$\tilde{H} \xrightarrow{\phi_1} H \xrightarrow{\phi_2} \overline{H} \xrightarrow{\phi_3} \overline{H}'$$

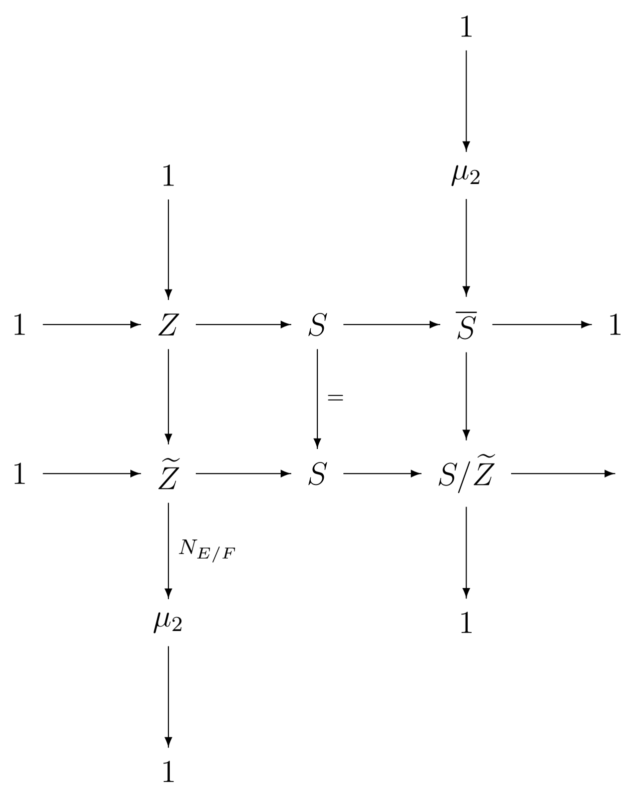
then $\ker(\phi_1)$ is the diagonal embedding of μ_2 into $Z(\tilde{H})$ over the algebraic closure, $\ker(\phi_2) = Z(G_0)$ and $\ker(\phi_3) = Z(\overline{H}) \simeq Z(H)/Z(G_0) \simeq \mu_2$.

In [6], Chernousov, Lifschitz and Morris construct a maximal torus $T_0 \leq G_0$ that is the almost direct product of $\mathbb{G}_\mathrm{m}$ and $R_{E/F}(R^{(1)}_{E(\sqrt{b})/E}(\mathbb{G}_\mathrm{m}))$. Let $\tilde{T_0} = \mathbb{G}_\mathrm{m} \times R_{E/F}(R^{(1)}_{E(\sqrt{b})/E}(\mathbb{G}_\mathrm{m}))$, then $\tilde{T_0} \to T_0 \leq G_0$ via the product map. Let $\overline{T_0}$ be the image of T_0 in $\overline{G_0}$. If $\alpha_1, \ldots, \alpha_4$ are a basis of $\Sigma(G_0, T_0)$, then $R_{E/F}(R^{(1)}_{E(\sqrt{b})/E}(\mathbb{G}_\mathrm{m})) = T_0 \cap G_{\alpha_1,\alpha_3,\alpha_4}$ and $H = G_\Phi$, where $\Phi = \{\alpha_2, \alpha_2 + \alpha_3 + \alpha_4, \alpha_2 + \alpha_1 + \alpha_4, \alpha_2 + \alpha_1 + \alpha_3\}$. In this notation, we have:

Lemma 5.3 *There exists a cocycle $[\mu_{\overline{T_0}}] \in H^1(F, \overline{T_0})$ such that $[\mu_{\overline{T_0}}] \mapsto [(a,b)_E]$ under $H^1(F, \overline{T_0}) \to H^1(F, \overline{G_0}) \to H^2(F, Z(G_0))$*

Proof. Consider the sub-torus $S \leq T_0$ given by $S = R_{E/F} R^{(1)}_{E(\sqrt{b})/E} \mathbb{G}_{\mathrm{m}}$ and let $\overline{S}$ be the image of S in $\overline{G_0}$. I claim that there exists a $[\mu_{\overline{S}}] \in H^1(F, \overline{S})$ that maps to $[(a,b)_E] \in H^2(F, Z(G_0))$. The image of $[\mu_{\overline{S}}]$ under the map $H^1(F, \overline{S}) \to H^1(F, \overline{T_0})$ is then the cocycle we are looking for.

To see that $[\mu_{\overline{S}}]$ exists, consider the F-defined subgroups $\widetilde{Z}$, $Z \leq S$ where Z is the centre of G_0 and $\widetilde{Z}$ is the 2-torsion part of S, also given by $R_{E/F}(\mu_2)$. Note that over $\overline{F}$, $\widetilde{Z}$ has the form $\mu_2 \times \mu_2 \times \mu_2$, the norm map is given by the product of the entries and Z is the kernel of this map. Using this, we have an interlocking diagram of exact sequences:



which induces the following exact sequences of Galois cohomology sets with corresponding morphisms:

$$\begin{array}{ccccc}
H^1(F,S) & \longrightarrow & H^1(F,\overline{S}) & \longrightarrow & H^2(F,Z) \\
\downarrow{\scriptstyle =} & & \downarrow & & \downarrow \\
H^1(F,S) & \longrightarrow & H^1(F,S/\widetilde{Z}) & \longrightarrow & H^2(F,\widetilde{Z}) \\
 & & \downarrow & & \downarrow{\scriptstyle N_{E/F}} \\
 & & H^2(F,\mu_2) & \xrightarrow{=} & H^2(F,\mu_2)
\end{array}$$

Assume that there is $[\mu_{S/\widetilde{Z}}] \in H^1(F, S/\widetilde{Z})$ that maps to $[(a,b)_E]$ under $H^1(F, S/\widetilde{Z}) \to H^2(F, \widetilde{Z})$ in the diagram above. The norm of $(a,b)_E$ is trivial by assumption, so $[\mu_{S/\widetilde{Z}}]$ is the image of some $[\mu_{\overline{S}}] \in H^1(F, \overline{S})$. We have a section $\lambda : \mu_2 \to \widetilde{Z}$ given by the diagonal embedding, and so $H^2(F,Z) \to H^2(F, \widetilde{Z})$ is injective. This, combined with the commutativity of the upper-right hand square, shows that $[\mu_{\overline{S}}] \mapsto [(a,b)_E] \in H^2(F,Z)$.

It remains to prove that there exists a $[\mu_{S/\widetilde{Z}}] \in H^1(F, S/\widetilde{Z})$ such that $[\mu_{S/\widetilde{Z}}] \mapsto [(a,b)_E] \in H^2(F, \widetilde{Z})$. Note that, by Shapiro's Lemma,

$$H^1(F, S/\widetilde{Z}) \to H^2(F, \widetilde{Z})$$

is equivalent to

$$H^1(E, R^{(1)}_{E(\sqrt{b})/E}(\mathbb{G}_{\mathrm{m}})/\mu_2) \to H^2(E, \mu_2).$$

Thus Lemma 2.24 gives the existence of $[\mu_{S/\widetilde{Z}}] \in H^1(F, S/\widetilde{Z})$. ■

Let $[\mu_{\overline{H}}]$ be the image of $[\mu_{\overline{T_0}}]$ in $H^1(F, \overline{H})$, $[\mu_{\overline{H}'}]$ its image in $H^1(F, \overline{H}')$ and $[(r,s)_P]$ the image of $[\mu_{\overline{H}'}]$ under the isomorphism $H^1(F, \overline{H}') \to H^2(F, R_{P/F}(\mu_2)) \simeq {}_2\mathrm{Br}(P)$. Choose $p \in P$ such that $[(r,s)_P]$ splits over $P(\sqrt{p})$, and define $\tilde{T} =$

$R_{P/F}(R^{(1)}_{P(\sqrt{p})/P}(\mathbb{G}_{\mathrm{m}}))$ embedded in $\tilde{H}$ via the regular representation. Let T be the image of $\tilde{T}$ in H, $\overline{T}$ the image of $\tilde{T}$ in $\overline{H}$ and $\overline{T}'$ the image of $\tilde{T}$ in $\overline{H}'$. Then:

Lemma 5.4 *There exists* $[\mu] \in H^1(F, \overline{T})$ *such that* $[\mu] \mapsto [\mu_{\overline{H}}]$ *under* $H^1(F, \overline{T}) \to H^1(F, \overline{H})$

Proof. Consider the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & Z(\overline{H}) & \longrightarrow & \overline{T} & \longrightarrow & \overline{T}' & \longrightarrow & 1 \\
 & & \Big\downarrow{=} & & \Big\downarrow & & \Big\downarrow & & \\
1 & \longrightarrow & Z(\overline{H}) & \longrightarrow & \overline{H} & \longrightarrow & \overline{H}' & \longrightarrow & 1
\end{array}$$

This induces the following commutative diagram with exact rows:

$$\begin{array}{ccccccc}
H^1(F, Z(\overline{H})) & \xrightarrow{\iota_1} & H^1(F, \overline{T}) & \xrightarrow{\pi_1} & H^1(F, \overline{T}') & \xrightarrow{\delta_1} & H^2(F, Z(\overline{H})) \\
\Big\downarrow{=} & & \Big\downarrow{\iota_2} & & \Big\downarrow{\iota_4} & & \Big\downarrow{=} \\
H^1(F, Z(\overline{H}) & \xrightarrow[\iota_3]{} & H^1(F, \overline{H}) & \xrightarrow[\pi_2]{} & H^1(F, \overline{H}') & \xrightarrow[\delta_2]{} & H^2(F, Z(\overline{H})
\end{array} \qquad (2)$$

By Shapiro's Lemma, $H^1(F, \overline{T}') \xrightarrow{\iota_4} H^1(F, \overline{H}')$ is isomorphic to

$$H^1(P, R^{(1)}_{P(\sqrt{p})/P}(\mathbb{G}_{\mathrm{m}})) \to H^1(P, \mathbb{PSL}_2)$$

and so by Lemma 2.24, there exists a $[\mu'] \in H^1(F, \overline{T}')$ such that $\iota_4([\mu']) = [\mu_{\overline{H}'}]$. The assumption that $\iota_4([\mu']) = [\mu_{\overline{H}'}]$ gives that $\delta_1([\mu']) = \delta_2([\mu_{\overline{H}'}]) = 1$, and so there exists a $[\mu''] \in H^1(F, \overline{T})$ such that $\pi_1([\mu'']) = [\mu']$. By the commutativity of diagram (2), $\pi_2(\iota_2([\mu''])) = \pi_2([\mu_{\overline{H}}])$ and so from [22], Chapter 1, Section 5 we find that there exists a $[\theta] \in H^1(F, Z(\overline{H}))$ such that $\iota_3([\theta]) \cdot \iota_2([\mu'']) = [\mu_{\overline{H}}]$. If we define $[\mu] = \iota_1([\theta]) \cdot [\mu'']$, then $\iota_2([\mu]) = \iota_2\iota_1([\theta]) \cdot \iota_2([\mu'']) = \iota_3([\theta]) \cdot \iota_2([\mu'']) = [\mu_{\overline{H}}]$. ∎

5.2.3 Modification of $[\mu]$

By Lemma 5.4 and the commutativity of the following diagram:

$$\begin{array}{ccc} H^1(F,\overline{T}) & \longrightarrow & H^2(F,Z(G_0)) \\ \downarrow & \nearrow & \\ H^1(F,\overline{H}) & & \end{array}$$

we have that $[\mu] \mapsto [(a,b)_E]$ under $H^1(F,\overline{T}) \to H^2(F,Z(G_0))$. In this section we modify $[\mu]$ as in Section 5.2.1 to obtain a cocycle $[\gamma] \in H^1(F,\overline{T})$ such that $[\gamma] \mapsto [\xi]$ under $H^1(F,\overline{T}) \to H^1(F,\overline{G_0})$. In order to do this, we need cocycles $[\nu_v] \in H^1(F_v,\overline{T})$ for each $v \in V^F_{\infty,\mathbb{R}}$ such that $[\nu_v] \mapsto [\xi_v]$ under $H^1(F_v,\overline{T}) \to H^1(F_v,\overline{G_0})$. We break this into two cases:

$E \otimes_F F_v \simeq F_v \times F_v \times F_v$

In order to understand how $\overline{T}$ behaves over F_v, it is necessary to understand the structure of $P \otimes_F F_v$. Recall that H is isogenous to $R_{P/F}(\mathbb{SL}_2)$, and so in order to understand $P \otimes_F F_v$, it is instructive to examine H over F_v. In order to examine H, we need to remember that $H = G_\Phi$ where $\Phi = \{\alpha_2, \alpha_2+\alpha_3+\alpha_4, \alpha_2+\alpha_1+\alpha_3, \alpha_2+\alpha_1+\alpha_4\} \subset \Sigma(G,T_0)$ has Galois action described in [6]. I claim that the sign of b under each of the maps $E \hookrightarrow E\otimes_F F_v \xrightarrow{\pi_i} F_v$ determines the Galois action of $\mathrm{Gal}(\mathbb{C}/F_v)$ on Φ, hence the structure of H and thus the structure of $P \otimes_F F_v$.

Lemma 5.5 *In the notation above, let b_1, b_2, b_3 be the images of b under the maps $E \hookrightarrow E\otimes_F F_v \xrightarrow{\pi_i} F_v$. If at least one of b_1, b_2, b_3 are negative then $P \otimes_F F_v \simeq \mathbb{C}\times\mathbb{C}$, while if all of b_1, b_2, b_3 are positive then $P\otimes_F F_v \simeq F_v\times F_v\times F_v\times F_v$.*

Proof. Suppose that b_1, b_2, b_3 are all positive in F_v. In this case,

$$\begin{aligned} R_{E/F}(R^{(1)}_{E(\sqrt{b})/E}(\mathbb{G}_{\mathrm{m}}))_{F_v} &\simeq R^{(1)}_{F_v(\sqrt{b_1})/F_v}(\mathbb{G}_{\mathrm{m}}) \times R^{(1)}_{F_v(\sqrt{b_2})/F_v}(\mathbb{G}_{\mathrm{m}}) \times R^{(1)}_{F_v(\sqrt{b_3})/F_v}(\mathbb{G}_{\mathrm{m}}) \\ &\simeq \mathbb{G}_{\mathrm{m}} \times \mathbb{G}_{\mathrm{m}} \times \mathbb{G}_{\mathrm{m}} \end{aligned}$$

thus T_0 is split over F_v. This gives that all $\alpha \in \Sigma(G_0, T_0)$ are fixed under $\mathrm{Gal}(\mathbb{C}/F_v)$. This means that Φ is fixed under $\mathrm{Gal}(\mathbb{C}/F_v)$, hence $\tilde{H}_{F_v} \simeq \mathbb{SL}_2 \times \mathbb{SL}_2 \times \mathbb{SL}_2 \times \mathbb{SL}_2$, and so $P \otimes_F F_v \simeq F_v \times F_v \times F_v \times F_v$.

Suppose now that one of b_1, b_2, b_3 is negative. Up to renumbering, we may assume then that b_1, b_2 are negative while b_3 is positive (because $N_{E/F}(b) = 1$). In this case

$$\begin{aligned} R_{E/F}(R^{(1)}_{E(\sqrt{b})/E}(\mathbb{G}_{\mathrm{m}}))_{F_v} &\simeq R^{(1)}_{F_v(\sqrt{b_1})/F_v}(\mathbb{G}_{\mathrm{m}}) \times R^{(1)}_{F_v(\sqrt{b_2})/F_v}(\mathbb{G}_{\mathrm{m}}) \times R^{(1)}_{F_v(\sqrt{b_3})/F_v}(\mathbb{G}_{\mathrm{m}}) \\ &\simeq R^{(1)}_{\mathbb{C}/F_v}(\mathbb{G}_{\mathrm{m}}) \times R^{(1)}_{\mathbb{C}/F_v}(\mathbb{G}_{\mathrm{m}}) \times \mathbb{G}_{\mathrm{m}} \end{aligned}$$

and thus (again, up to renumbering) $1 \neq \tau \in \mathrm{Gal}(\mathbb{C}/F_v)$ acts by

$$\begin{aligned} \alpha_1 &\mapsto \alpha_1 \\ \alpha_3 &\mapsto -\alpha_3 \\ \alpha_4 &\mapsto -\alpha_4 \end{aligned}$$

and if $\tilde{\alpha}$ is a root of maximal height, $\tilde{\alpha} \mapsto \tilde{\alpha}$ (since this was true over F). This means that $\alpha_2 \mapsto \alpha_2 + \alpha_1 + \alpha_3$ and $\alpha_1 + \alpha_2 + \alpha_4 \mapsto \alpha_2 + \alpha_3 + \alpha_4$ and so Φ has type $(A_1 \times A_1) \times (A_1 \times A_1)$ with $\mathrm{Gal}(\mathbb{C}/F_v)$ permuting the factors within the brackets. This gives that $\tilde{H}_{F_v} \simeq R_{\mathbb{C}/F_v}(\mathbb{SL}_2) \times R_{\mathbb{C}/F_v}(\mathbb{C}/F_v)$, thus $P \otimes_F F_v \simeq \mathbb{C} \times \mathbb{C}$. ■

By our restriction that $E \otimes_F F_v \simeq F_v \times F_v \times F_v$, we have that G_{F_v} is of type ${}^1D_4$. By the classification given in Section 5.2.1, we have that G_{F_v} is either of rank 0, 2 or 4. Recall that $[\xi]_v$ is in the image of $H^1(F_v, G_0) \to H^1(F_v, \overline{G_0})$ if and only if G has rank 0 or 4. This is true if and only if $(a, b_1)_{F_v}$, $(a, b_2)_{F_v}$ and $(a, b_3)_{F_v}$ are all split, which is equivalent to the condition that b_1, b_2, b_3 are all positive (since $F(\sqrt{a})$ is purely imaginary by assumption). This, combined with Lemma 5.5 gives that:

Lemma 5.6 *If G_{F_v} has rank 2, then $\tilde{T}$ has the form*

$$R_{\mathbb{C}/F_v}(\mathbb{G}_{\mathrm{m}}) \times R_{\mathbb{C}/F_v}(\mathbb{G}_{\mathrm{m}})$$

and at least one of b_1, b_2, b_3 are negative in F_v.

If G_{F_v} is anisotropic or split then b_1, b_2, b_3 are all positive in F_v. Moreover, if we let $\psi_{i,v}$ be the composition

$$P \hookrightarrow P \otimes_F F_v \simeq F_v \times F_v \times F_v \times F_v \xrightarrow{\pi_i} F_v$$

then $\tilde{T}_{F_v}$ has the form

$$R^{(1)}_{F_v(\sqrt{\psi_{1,v}(p)})/F_v}(\mathbb{G}_{\mathrm{m}}) \times R^{(1)}_{F_v(\sqrt{\psi_{2,v}(p)})/F_v}(\mathbb{G}_{\mathrm{m}}) \times R^{(1)}_{F_v(\sqrt{\psi_{3,v}(p)})/F_v}(\mathbb{G}_{\mathrm{m}}) \times R^{(1)}_{F_v(\sqrt{\psi_{4,v}(p)})/F_v}(\mathbb{G}_{\mathrm{m}})$$

Notice that if b_1, b_2, b_3 are all positive in F_v, then the structure of $\overline{T}_{F_v}$ depends on the sign of $\psi_{i,v}(p)$. The following lemma allows us to control these signs.

Lemma 5.7 *There exists $p \in P$ such that $P(\sqrt{p})$ splits $[(r,s)_P]$ and $\psi_{i,v}(p) > 0$ in F_v if and only if $[\xi]$ is trivial over F_v.*

Proof. Recall the definition of $[\mu_{\overline{H}}]$ and $[\mu_{\overline{H}'}]$, defined immediately before Lemma 5.4.

Let $\Psi_1 \subset V^F_{\infty,\mathbb{R}}$ be the set of all places of F such that b_1, b_2, b_3 are all positive in F_v but $[\xi]_v$ is non-trivial. Let $([(r_1,s_1)_{F_v}], [(r_2,s_2)_{F_v}], [(r_3,s_3)_{F_v}], [(r_4,s_4)_{F_v}])$ be the image of $[(r,s)_P]$ under the isomorphism $H^1(F_v, \overline{H}') \simeq H^2(F_v, \mu_2) \times \cdots \times H^2(F_v, \mu_2)$. Given a quaternion algebra over the real numbers, it is always possible to find a pure quaternion q such that $q^2 = -1$. For $v \in \Psi_1$, choose $x_{i,v}, y_{i,v}, z_{i,v} \in F_v$ such that

$$r_i x_{i,v}^2 + s_i y_{i,v}^2 - r_i s_i z_{i,v}^2 = -1.$$

Let Ψ_2 be the set of all places of F such that $[\xi]_v$ is split. For every such v, I claim that $[(r_1, s_1)_{F_v}], [(r_2, s_2)_{F_v}], [(r_3, s_3)_{F_v}]$ and $[(r_4, s_4)_{F_v}]$ are split. To see this, recall the definition of S from the proof of 5.3, and consider the short exact sequence

$$1 \to Z(G_0) \to S \to \overline{S} \to 1.$$

Recall also that $[\mu_{\overline{H}'}]$ was the image of a cocycle $[\mu_{\overline{S}}] \in H^1(F, \overline{S})$ that mapped to $[(a, b)_E]$ under $H^1(F, \overline{S}) \to H^2(F, Z(G_0))$. Because $[(a, b)_E]$ is split over F_v, this means that $[\mu_{\overline{S}}]$ is the image of some $[\mu_S] \in H^1(F_v, S)$, but by the definition of S, $S_{F_v} \simeq \mathbb{G}_{\mathrm{m}} \times \mathbb{G}_{\mathrm{m}} \times \mathbb{G}_{\mathrm{m}}$. This means that $[\mu_{\overline{S}}]$ is split over F_v by Hilbert 90, hence $[\mu_{\overline{H}'}]$ is also split over F_v, and thus $[(r_1, s_1)_{F_v}], [(r_2, s_2)_{F_v}], [(r_3, s_3)_{F_v}]$ and $[(r_4, s_4)_{F_v}]$ are split as claimed.

Because $[(r_i, s_i)_{F_v}]$ are split, there exist pure quaternions $q_i \in (r_i, s_i)$ such that $q_i^2 = 1$. For $v \in \Psi_2$, choose $x_{i,v}, y_{i,v}, z_{i,v} \in \mathbb{R}$ such that

$$r_i x_{i,v}^2 + s_i y_{i,v}^2 - r_i s_i z_{i,v}^2 = 1.$$

Next, choose $\epsilon > 0$ such that if $|x'_{i,v} - x_{i,v}| + |y'_{i,v} - y_{i,v}| + |z'_{i,v} - z_{i,v}| < \epsilon$, then

$$|r_i x'^2_{i,v} + s_i y'^2_{i,v} - r_i s_i z'^2_{i,v} - r_i x_{i,v}^2 - s_i y_{i,v}^2 + r_i s_i z_{i,v}^2| < \frac{1}{2}$$

applying the weak approximation property then provides $x, y, z \in P$ such that

$$|\psi_{i,v}(x) - x_{i,v}| + |\psi_{i,v}(y) - y_{i,v}| + |\psi_{i,v}(z) - z_{i,v}| < \epsilon$$

and so if we let $p = rx^2 + sy^2 - rsz^2$, p satisfies the conditions of the lemma. ■

Recall that there are three possibilities for G_{F_v}. Either G_{F_v} is split, anisotropic, or of rank 2. If G_{F_v} is split, then $[\xi]_v$ is trivial, so we can let $[\nu_v] = 1$ and then $[\nu_v] \mapsto [\xi]_v$. If G_{F_v} is anisotropic, then by our choice of p, T_{F_v} is anisotropic and thus

T_{F_v} is isomorphic to a maximal torus of G_{F_v}. By Steinberg's theorem, we therefore have an embedding $\phi : \overline{T}_{F_v} \hookrightarrow \overline{G_0}_{F_v}$ and $[\nu'_v] \in H^1(F_v, \phi(\overline{T}_{F_v}))$ such that $[\nu'_v] \mapsto [\xi]_v$.

Any two anisotropic maximal tori in $\overline{G_0}_{F_v}$ are conjugate ([11], Theorem 32.1), hence the image of $H^1(F_v, \overline{T}_{F_v})$ and $H^1(F_v, \phi(\overline{T}_{F_v}))$ in $H^1(F_v, \overline{G_0}_{F_v})$ are the same and there exists a $[\nu_v] \in H^1(F_v, \overline{T}_{F_v})$ such that $[\nu_v] \mapsto [\xi]_v$.

Finally, we must consider the case that G_{F_v} has rank 2. In this case Lemma 5.6 gives that $P \otimes_F F_v \simeq \mathbb{C} \times \mathbb{C}$ and $\tilde{T}_{F_v} \simeq R_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_\mathrm{m}) \times R_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_\mathrm{m})$. Recall the definition of T_0, then the action of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ on $\Sigma(G_0, T_0)$ is described in Lemma 5.5 and up to renumeration the subsets $\Phi_1 = \{\alpha_2, \alpha_2 + \alpha_1 + \alpha_3\}$ and $\Phi_2 = \{\alpha_2 + \alpha_3 + \alpha_4, \alpha_2 + \alpha_1 + \alpha_4\}$ are $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$-stable. Let G_i be the subgroup of G_{0,F_v} generated by G_α where $\alpha \in \Phi_i$. Finally, recall that G_0 is split over F_v in this case and hence $G_{0,F_v} \simeq \mathbb{S}\mathbf{pin}(f_4)$ (where f_4 is defined in Section 5.2.1). The following is a slight re-phrasing of Lemma 3.20 to our situation.

Lemma 5.8 *Given (V, f_4) as above, there exist $V_1, V_2 \subset V$ such that $V = V_1 \oplus V_2$, $V_2 = V_1^\perp$ under $(\ ,\ )_f$ and if $g_1 = f_4|_{V_1}$, $g_2 = f_4|_{V_2}$ then $f = g_1 \oplus g_2$ and up to isogeny $G_i \leq G_{0,F_v}$ is given by $\mathbb{SO}(g_i) \leq \mathbb{SO}(f_4)$.*

For a given 4-dimensional quadratic form g over a field F, recall that $\mathbb{S}\mathbf{pin}(g) \simeq R_{F(\sqrt{\mathrm{disc}(g)})/F}(\mathbb{SL}(T))$, where T is a quaternion algebra over $F(\sqrt{\mathrm{disc}(g)})$ (see Section 3.3.2). Recalling that $G_i \simeq R_{\mathbb{C}/\mathbb{R}}(\mathbb{SL}_2)$ from Lemma 5.5, this gives that g_i have nontrivial discriminant, so up to multiplication by ± 1, $g_1 = \langle 1, 1, 1, -1 \rangle = g_2$. The above lemma gives that $g_1 \oplus g_2 = \langle 1, -1, 1, -1, 1, -1, 1, -1 \rangle$, so up to renumbering, $g_1 = \langle 1, 1, 1, -1 \rangle$ and $g_2 = \langle 1, -1, -1, -1 \rangle$.

Let T' be the image of T in $\mathbb{SO}(f_4)$. Consider $z = (1, -1) \in \mathbb{SO}(g_1) \times \mathbb{SO}(g_2) \leq \mathbb{SO}(f_4)$. Let $[\nu'_v] \in H^1(F_v, \mathbb{PSO}(f_4)) = H^1(F_v, \overline{G_0}_{F_v})$ be given by $(\nu'_v)_\tau = \overline{z} \in \mathbb{PSO}(f_4)$

if $\tau \in \mathrm{Gal}(\mathbb{C}/\mathbb{R})$ is non-trivial. By definition of T, $T' \cap \mathbb{SO}(g_2)$ is a maximal torus in $\mathbb{SO}(g_2)$, thus $Z(\mathbb{SO}(g_2)) \leq T' \cap \mathbb{SO}(g_2)$, and so $z \in T'$. Therefore there exists $[\nu_v] \in H^1(F_v, \overline{T}_{F_v})$ such that $[\nu_v] \mapsto [\nu_v']$.

Lemma 5.9 *Under $H^1(F_v, \overline{T}_{F_v}) \to H^1(F_v, \overline{G_0}_{F_v})$, $[\nu_v] \mapsto [\xi]_v$.*

Proof. It suffices to show that ${}_{\nu_v'}G_{0,F_v} \simeq G$. This property is invariant under taking quotients by a central subgroup, so it suffices to show that ${}_{\nu_v'}\mathbb{SO}(f_4) \simeq \mathbb{SO}(\sum_{i=1}^{6} x_i^2 - x_7^2 - x_8^2)$. Identifying g_i on V_i with their corresponding 4×4 matrices, we have that

$$\mathbb{SO}(f_4)(\mathbb{C}) = \left[x = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \;\middle|\; x_{ij} \in \mathrm{M}_{4\times 4}(\mathbb{C}),\ \det(x) = 1,\ \begin{matrix} x_{11}^T g_1 x_{11} + x_{21}^T g_2 x_{21} = g_1 \\ x_{11}^T g_1 x_{12} + x_{21}^T g_2 x_{22} = 0 \\ x_{12}^T g_1 x_{12} + x_{22}^T g_2 x_{22} = g_2 \end{matrix} \right]$$

and the new action of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ on ${}_{\nu_v'}\mathbb{SO}(f_4)$ is given by

$${}^{\tau'}x = z\,{}^{\tau}x z^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \overline{x_{11}} & \overline{x_{12}} \\ \overline{x_{21}} & \overline{x_{22}} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} \overline{x_{11}} & -\overline{x_{12}} \\ -\overline{x_{21}} & \overline{x_{22}} \end{pmatrix}$$

so x is fixed under the new $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$-action if and only if $x_{11}, x_{22} \in \mathrm{M}_{4\times 4}(\mathbb{R})$ and $x_{12}, x_{21} \in \mathrm{M}_{4\times 4}(i\mathbb{R})$. If we let $g = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$, then direct calculation shows that x is stabilized by the twisted action of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ if and only $\mathrm{Int}(g)(x)$ preserves the form represented by the matrix $\begin{pmatrix} g_1 & 0 \\ 0 & -g_2 \end{pmatrix}$. Thus ${}_{\nu_v'}\mathbb{SO}(f_4) \simeq \mathbb{SO}(g_1 \oplus (-g_2)) = \mathbb{SO}(\sum_{i=1}^{6} x_i^2 - x_7^2 - x_8^2)$. ■

Remark 5.10 *By our choice of p, $\overline{T}_{F_v}$ has higher rank for all $v \in V_{\infty,\mathbb{R}}^F$ such that $F_v \otimes_F E \simeq F_v \times F_v \times F_v$ and $v \in S_G''$.*

This completes the examination of the case that $E \otimes_F F_v \simeq F_v \times F_v \times F_v$ and we are left with the case:

$E \otimes_F F_v \simeq F_v \times \mathbb{C}$

In this case, we have $[(a, b)_E]$ has norm $[(a, b_1)_{F_v}] \cdot \mathrm{Res}_{\mathbb{C}/\mathbb{R}}([\mathrm{M}_2(\mathbb{C})]) = [(a, b_1)_{F_v}]$ where b_1 is the image of b under the map

$$E \hookrightarrow E \otimes_F F_v \xrightarrow{\pi_1} \mathbb{R} \times \mathbb{C}.$$

By the restriction that $N_{E/F}([(a, b)_E]) = 1$, we therefore get that $(a, b_1)_E$ becomes split over F_v. Because we chose a such that $F(\sqrt{a})$ is purely imaginary, $\mathrm{Sign}_v(a) = -1$, thus $\mathrm{Sign}_v(b_1) = 1$. This tells us the structure of $P \otimes_F F_v$:

Lemma 5.11 *If $E \otimes_F F_v \simeq F_v \times \mathbb{C}$, then $P \otimes_F F_v \simeq \mathbb{R} \times \mathbb{R} \times \mathbb{C}$.*

Proof. Recall that if G_0 is as in [6], then $G_{0,\alpha_1,\alpha_3,\alpha_4}$ has the maximal torus $R_{E/F}(R^{(1)}_{E(\sqrt{b})/E}(\mathbb{G}_\mathrm{m}))$ which becomes $\mathbb{G}_\mathrm{m} \times R_{\mathbb{C}/F_v}(\mathbb{G}_\mathrm{m})$ over F_v. Thus up to relabeling $\mathrm{Gal}(\mathbb{C}/F_v)$ acts by fixing α_1 and sending $\alpha_3 \mapsto \pm\alpha_4$. From [6] we see that $\tilde{\alpha}$ is fixed so $\mathrm{Gal}(\mathbb{C}/F_v)$ acts on $\Phi = \{\alpha_2, \alpha_2 + \alpha_3 + \alpha_4, \alpha_2 + \alpha_1 + \alpha_3, \alpha_2 + \alpha_1 + \alpha_4\}$ by fixing two elements and permuting the other two (which are fixed and which are permuted depends on the sign of $\alpha_3 \mapsto \pm\alpha_4$). This gives that $\tilde{H}_{F_v} \simeq \mathbb{SL}_2 \times \mathbb{SL}_2 \times R_{\mathbb{C}/F_v}(\mathbb{SL}_2)$, thus $P \otimes_F F_v \simeq \mathbb{R} \times \mathbb{R} \times \mathbb{C}$. ■

As in the case $E \otimes_F F_v \simeq F_v \times F_v \times F_v$, it is necessary to understand the sign of p under the maps $\psi_{i,v} : P \hookrightarrow P \otimes_F F_v \xrightarrow{\pi_i} F_v$ where $i = 1, 2$. How the sign of $\psi_{i,v}(p)$ is controlled will depend on the form that $\overline{G}$ takes over F_v. From the restriction that $E \otimes_F F_v \simeq F_v \times \mathbb{C}$, we have that $\overline{G}$ is of type ${}^2D_4$ over F_v, and so Tits' classification gives two possibilities; $\overline{G}_{F_v}$ is quasi-split of rank 3 or $\overline{G}_{F_v}$ has rank 1.

Let $\Psi_3 \subset V^F_{\infty,\mathbb{R}}$ be the set of all places of F such that $E \otimes_F F_v \simeq F_v \times \mathbb{C}$ and G becomes quasi-split over F_v and $\Psi_4 \subset V^F_{\infty,\mathbb{R}}$ be the set of all places of F such that $E \otimes_F F_v \simeq F_v \times \mathbb{C}$ and G has rank 1 over F_v.

Lemma 5.12 *There exists $p \in P$ satisfying the conditions of Lemma 5.7 such that $\psi_{i,v}(p)$ is positive in F_v if $v \in \Psi_3$ and negative in F_v if $v \in \Psi_4$.*

Proof. The proof is identical to the proof of Lemma 5.7 with one exception. Recall the definitions of S and $\overline{S}$ from Lemma 5.3. We do not have that S is split in this case, however, we still have that $H^1(F_v, S) = H^1(F_v, \mathbb{G}_{\mathrm{m}} \times R_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{\mathrm{m}})) = 1$, and the same arguments as in Lemma 5.7 then give that $[(r_1, s_1)_{F_v}]$ and $[(r_2, s_2)_{F_v}]$ defined as in Lemma 5.7 are split (here there are no $[(r_3, s_3)_{F_v}]$ or $[(r_4, s_4)_{F_v}]$, as $P \otimes_F F_v \simeq F_v \times F_v \times \mathbb{C}$). ■

Now, choosing p as in Lemma 5.12, I claim that there exist $[\nu_v] \in H^1(F_v, \overline{T}_{F_v})$ that map to $[\xi]_v$ for all $v \in \Psi_3 \cup \Psi_4$. This is proven in an analogous manner to the case where $E \otimes_F F_v \simeq F_v \times F_v \times F_v$, with a few exceptions. Namely, in this case $G_{0,F_v} \simeq \mathbb{S}\mathbf{pin}(f_3)$. Recall the definition of $T_0 \leq G_0$ and the $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$-action described in Lemma 5.11. Up to re-numerating, if we let G_1 be the subgroup of G_0 generated by the root subgroups corresponding to $\{\alpha_2, \alpha_2 + \alpha_3 + \alpha_4\}$ then $G_1 \simeq \mathbb{SL}_2 \times \mathbb{SL}_2$, and if we let G_2 be the subgroup generated by the root subgroups corresponding to$\{\alpha_2 + \alpha_1 + \alpha_4, \alpha_2 + \alpha_1 + \alpha_3\}$ then $G_2 \simeq R_{\mathbb{C}/\mathbb{R}}(\mathbb{SL}_2)$.

Lemma 5.13 *Given (V, f_3) with f_3 defined as in Section 5.2.1, there exist $V_1, V_2 \subset V$ such that $V = V_1 \oplus V_2$, $V_2 = V_1^{\perp}$ under $(\ ,\ )_{f_3}$ and if $g_1 = f_3|_{V_1}$, $g_2 = f_3|_{V_2}$ then $f = g_1 \oplus g_2$ and up to isogeny $G_i \leq G_{0,F_v}$ is given by $\mathbb{SO}(g_i) \leq \mathbb{SO}(f_3)$.*

Proof. As in Lemma 5.8. ■

Recall that we have $\mathbb{S}\mathbf{pin}(g_i) \simeq R_{F_v(\sqrt{\mathrm{disc}(g_i)})/F_v}(\mathbb{SL}(T))$ where T is a quaternion algebra over $F_v(\sqrt{\mathrm{disc}(g_i)})$. Because G_1 is split, g_1 is as well, while G_2 has no F_v-defined subgroups of type A_1, thus g_2 has non-trivial discriminant. This means that up to multiplication by ± 1, we have that

$$g_1 = x_1^2 - x_2^2 + x_3^2 - x_4^2$$
$$g_2 = y_1^2 + y_2^2 + y_3^2 - y_4^2$$

and the criterion that $g_1 \oplus g_2 = f_3$ means that we can choose g_i as above.

If G_{F_v} has rank 3, then $G_{F_v} \simeq G_{0,F_v}$, so $[\xi]_v$ is trivial and $1 \in H^1(F_v, \overline{T})$ maps to $[\xi]_v$. If G_{F_v} has rank 1, then recall that by our choice of p, we have that $T_1 = T \cap G_1 \simeq R^{(1)}_{\mathbb{C}/F_v}(\mathbb{G}_{\mathrm{m}}) \times R^{(1)}_{\mathbb{C}/F_v}(\mathbb{G}_{\mathrm{m}})$. Let $S_1 = \mathbb{S}\mathbf{pin}(x_1^2 + x_3^2) \times \mathbb{S}\mathbf{pin}(-x_2^2 - x_4^2) \leq G_1$. Because any two anisotropic tori over $\mathbb{R}$ are conjugate, if $\overline{T_1}$ and $\overline{S_1}$ are the images of T_1 and S_1 in $\mathbb{PSO}(g_1)$, we have that the image of $H^1(F_v, \overline{T_1})$ and $H^1(F_v, \overline{S_1})$ in $H^1(F_v, \mathbb{PSO}(g_1))$ is the same. Let T_1' and S_1' be the images of T_1 and S_1 in $\mathbb{SO}(g_1)$, and let $z_1 = (1, -1) \in S_1'$. If we let $[\gamma_v] \in H^1(F_v, \overline{S_1})$ be given by $(\gamma_v)_\tau = \overline{z_1} \in \overline{S_1}$, let $[\gamma_v'] \in H^1(F_v, \overline{T_1})$ be chosen such that $\mathrm{im}([\gamma_v']) = \mathrm{im}([\gamma_v]) \in H^1(F_v, \mathbb{PSO}(g_1))$.

Let $[\nu_v] \in H^1(F_v, \overline{T})$ be the image of $[\gamma_v']$ under the map $H^1(F_v, \overline{T_1}) \to H^1(F_v, \overline{T}_{F_v})$. Let $g_{11} = x_1^2 + x_2^2$, $g_{12} = -x_2^2 - x_4^2$, so that $g_1 = g_{11} \oplus g_{12}$. As in 5.9, direct calculation shows that ${}_{\nu_v}\mathbb{SO}(f_3) \simeq \mathbb{SO}(f_1)$, thus:

Lemma 5.14 *In the situation above, $[\nu_v] \mapsto [\xi]_v$ under $H^1(F_v, \overline{T}_{F_v}) \to H^1(F_v, \overline{G_0}_{F_v})$.*

Remark 5.15 *For every $v \in S_G$ such that $E \otimes_F F_v \simeq F_v \times \mathbb{C}$, we have that T_{F_v} has rank 1 whenever $v \in S_G'$ and T_{F_v} is of higher rank whenever $v \in S_G''$.*

5.2.4 Concluding Argument

Thus far we have constructed a torus $\overline{T} \leq \overline{G_0}$ such that:

1. there exists $[\gamma] \in H^1(F, \overline{T})$ that maps to $[\xi] \in H^1(F, \overline{G_0})$,

2. $T \leq H$, where $H \leq G_0$ is a simple group of type $A_1 \times A_1 \times A_1 \times A_1$ and,

3. T has appropriate real rank.

Arguing as in Section 5.1, G is not minimal.

Remark 5.16 *In [1] B. Allison showed how to construct all central simple Lie algebras of type D_4 over an algebraic number field. These results can also be used to obtain subgroups of G of type $A_1 \times A_1 \times A_1 \times A_1$, at least one of which has appropriate real rank. We keep the original proof here because the same technique (modification of cocycles) is used to prove that groups of type ${}^{1,2}E_6$ are not minimal.*

5.3 Type ${}^{1,2}E_6$

Let G_0 be a split (or quasi-split) simply connected group of type ${}^{1,2}E_6$ over F. If G_0 is of outer type, let L/F be the unique quadratic extension over which G_0 becomes inner. It is well-known ([19], p. 332) that $Z(G_0) = \mu_3$ if G_0 is of inner type and $Z(G_0) = R^{(1)}_{L/K}(\mu_3)$ if G_0 is of outer type. This gives that $H^2(F, Z(G_0))$ is ${}_3\mathrm{Br}(F)$ if G_0 is inner and $\ker({}_3\mathrm{Br}(L) \overset{\mathrm{Res}}{\to}_3 Br(F))$ if G_0 is outer. Combining Lemma 2.30 with the fact that the index of a central simple algebra over a number field is equal to its exponent in the Brauer group (Section 2.2.7), we see that $\ker({}_3\mathrm{Br}(L) \overset{\mathrm{Res}}{\to}_3 Br(F))$ is in bijective correspondence with degree 3 division algebras D with involutions of the second kind over F such that $Z(D) = L$.

5.3.1 Construction of a Special Torus

Let T_0 be a F-defined split or quasi-split torus of G_0 and let $\Sigma' \subset \Sigma(G_0, T_0)$ be the root system generated by roots $\{\alpha_1, \alpha_3, \alpha_4, \alpha_5, \alpha_6\}$. Let H_0 be the subgroup of G_0 generated by the root subgroups corresponding to $\alpha \in \Sigma'$. If G_0 is split, then H_0 is split of type ${}^1A_5$, i.e. is isogenous to $\mathbb{SL}_6$. If G_0 is quasi-split, then H_0 is quasi-split of type ${}^2A_5$, i.e. is isogenous to $\mathbb{SU}_6(L, f_q)$ for a quasi-split hermitian form of dimension 6. Let $\tilde{\alpha}$ be the root of maximal height in either case, and let $G_{\tilde{\alpha}}$ be the root group corresponding to $\tilde{\alpha}$ (so $G_{\tilde{\alpha}} \simeq \mathbb{SL}_2$ and $G_{\tilde{\alpha}}$ commutes with H_0).

Let G be a simply connected, anisotropic group of type ${}^{1,2}E_6$ over F and assume that G corresponds to $[\xi] \in H^1(F, \overline{G_0})$. Let $[D]$ be the image of $[\xi]$ in $H^2(F, Z(G_0))$.

Type ${}^1E_6$

In this case, let M be any degree 3 Galois subfield of D (this exists because all central simple algebras over number fields are cyclic). Let $P = F(\sqrt{-1})$ and consider $M \cdot P$. Define T_1 to be the the image of $R^{(1)}_{M \cdot P/F}(\mathbb{G}_{\mathrm{m}})$ in H_0 via the regular embedding and let T_2 be the image of $R^{(1)}_{P/F}(\mathbb{G}_{\mathrm{m}})$ in $G_{\tilde{\alpha}}$. Define T to be the almost-direct product $T_1 \cdot T_2$.

Lemma 5.17 *T defined as above satisfies condition 1 of Lemma 5.1, i.e. there exists a $[\mu] \in H^1(F, \overline{T})$ with $\delta_2([\mu]) = [D]$ (in the notation of Lemma 5.1).*

Proof. Consider the following diagram:

$$\begin{array}{ccccc}
\mu_2 & \longrightarrow & \mu_2 & & \\
\downarrow & & \downarrow & & \\
\mu_6 & \longrightarrow & T_1 \times T_2 & \longrightarrow & \overline{T} \\
\downarrow & & \downarrow & & \downarrow \\
Z(G_0) & \longrightarrow & T & \longrightarrow & \overline{T}
\end{array}$$

with exact columns and rows. This gives a diagram of interconnected long exact sequences with segment:

$$\begin{array}{ccccccc}
H^1(F, T_1 \times T_2) & \longrightarrow & H^1(F, \overline{T}) & \xrightarrow{\phi_1} & H^2(F, \mu_6) & \xrightarrow{\phi_2} & H^2(F, T_1 \times T_2) \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle \phi_3} & & \\
H^1(F, T) & \longrightarrow & H^1(F, \overline{T}) & \xrightarrow{\phi_4} & H^2(F, \mu_3) & &
\end{array}$$

By commutativity, $\mathrm{im}(\phi_4) = \mathrm{im}(\phi_3 \circ \phi_1) = \phi_3(\ker(\phi_2))$. Using Shapiro's lemma, we have that $H^2(F, T_1 \times T_2) = \ker(Br(M \cdot P) \xrightarrow{N} Br(F)) \times \ker(Br(P) \xrightarrow{Norm} Br(F))$. Recall that elements of ${}_6Br(F)$ can be written in the form $[D_1 \otimes D_2]$ with D_1 cubic cyclic and D_2 a quaternion algebra because F is a number field. The map $\mu_6 \to T_1 \times T_2$ takes $\xi_6 \mapsto (\xi_6, \xi_6^3)$, so $\phi_2([D_1 \otimes D_2]) = ([D_1 \otimes_F D_2 \otimes_F M \cdot P], [D_1 \otimes_F D_2 \otimes F]^3) = ([D_1 \otimes_F D_2 \otimes_F M \cdot P], [D_2 \otimes_F P])$. If $[D_1 \otimes_F D_2]$ is in the kernel of this map, then D_2 is split by P and $D_1 \otimes_F D_2$ is split by $M \cdot P$. The first condition gives that D_1 is split by $M \cdot P$, and so the fact that the degree of D_1 is relatively prime to the degree of P over F, we have that D_1 is split over M. This means that the kernel of ϕ_2 is given by $\{[D_1 \otimes D_2] \in {}_6Br(F) \mid [D_1 \otimes M] = 1 = [D_2 \otimes P]\}$. The map $\mu_6 \to \mu_3$ is given by squaring, so $\phi_3([D_1 \otimes_F D_2]) = [D_1 \otimes_F D_2]^2 = [D_1]^{-1}$. Combining these results gives that $[D]$ is in the image of ϕ_4 if and only if $[D]^{-1}$ contains M as a maximal subfield, which is true because $[D]$ is assumed to contain M and $[D]^{-1} = [D^{op}]$. Thus the existence of $[\mu]$ is proven. ■

Type $^2E_6$

Let $[(D, \tau)]$ correspond to the image of $[\xi]$ in $H^2(F, Z(G_0))$, and let σ be the involution on $M_2(D)$ corresponding to the τ-hermitian form $\langle 1, -1 \rangle$ on D^2. Recall from the classification of minimal groups of type ^2A_n that given local constructions $E_v \subset \mathrm{M}_2(D) \otimes_F F_v$ such that $E_v^{\sigma_v}$ has dimension n for every $v \in V^F_{\infty,\mathbb{R}}$, there exists a subfield $E \subset \mathrm{M}_2(D)$ such that $(E \otimes_F F_v, \sigma \otimes 1) \simeq (E_v, \sigma_v)$ (Lemma 3.31, Corollary 3.32 and Theorem 3.29). We break the local construction into the following cases:

If $\mathrm{Rank}_{F_v}(G) = 0$, then by Tits' classification, G remains outer over F_v in this case, thus $(\mathrm{M}_2(D) \otimes F_v, \sigma \otimes 1) \simeq (\mathrm{M}_6(\mathbb{C}), \langle 1, -1, 1, -1, 1, -1 \rangle)$. Let $E_v = \mathbb{C}^6$ embed via diagonal matrices, so $E_v^{\sigma_v} = \mathbb{R}^6$ and the maximal torus of $\mathbb{SU}_6(\mathbb{C}, \langle 1, -1, 1, -1, 1, -1 \rangle)$ corresponding to E_v is anisotropic.

If G_{F_v} is isotropic of outer type, we have that

$$(\mathrm{M}_2(D) \otimes F_v, \sigma \otimes 1) \simeq (\mathrm{M}_6(\mathbb{C}), \langle -1, -1, -1, 1, 1, 1 \rangle).$$

Note that $\mathrm{M}_3(\mathbb{R}) \times \mathrm{M}_3(\mathbb{R}) \subset \mathrm{M}_6(\mathbb{C})^{\sigma_v}$ in this case, so we can embed $F_v = (\mathbb{R} \times \mathbb{C}) \times (\mathbb{R} \times \mathbb{C}) \subset \mathrm{M}_6(\mathbb{C})^{\tau_v}$ by first embedding $\mathbb{R} \times \mathbb{C} \subset \mathrm{M}_3(\mathbb{R})$ via the regular representation along the diagonal and then taking the product of this embedding with itself. We then let $E_v = F_v \otimes_{\mathbb{R}} \mathbb{C} \hookrightarrow \mathrm{M}_6(\mathbb{C})$ via $(\mathrm{M}_3(\mathbb{R}) \times \mathrm{M}_3(\mathbb{R}) \otimes \mathbb{C} \hookrightarrow \mathrm{M}_6(\mathbb{C})$. Then

$$\{x \in E_v \mid x\sigma_v(x) = 1 = \mathrm{Nrd}(x)\} = \{(z_1, z_2, z_2^{-1}, z_1^{-1}, z_4, z_4^{-1}) \mid N_{\mathbb{C}/\mathbb{R}}(z_1) = 1\}$$

so the maximal torus of $\mathbb{SU}_6(\mathbb{C}, \langle 1,-1,1,-1,1,-1\rangle)$ corresponding to E_v in this case has F_v-rank 2.

If G_{F_v} is isotropic of inner type, let $E_v = \mathbb{C}^3 \times \mathbb{C}^3 \hookrightarrow \mathrm{M}_6(\mathbb{R}) \times \mathrm{M}_6(\mathbb{R})^{op}$ with exchange involution (embedded via the regular embedding). Then the maximal torus of $\mathbb{SL}_6(\mathbb{R})$ corresponding to E_v is

$$\{(z_1, z_2, z_3) \mid N_{\mathbb{C}/\mathbb{R}}(z_1 z_2 z_3) = 1\}$$

which has rank 2 over $\mathbb{R}$.

Let $E \subset \mathrm{M}_2(D)$ be a maximal sub-field such that $(E \otimes_F F_v, \sigma \otimes 1) \simeq (E_v, \sigma_v)$ for each $v \in V^F_{\infty,\mathbb{R}}$. Note that $E = K \otimes_F L$ for some degree 3 field extension K of F with τ acting on the second component by [20], Proposition 2.1, p. 5. Let T_1 be the maximal torus in $H' = \mathbb{SU}_2(D, \langle 1,-1\rangle)$ given by

$$\{x \in E \mid x\tau(x) = 1 = \mathrm{Nrd}(x)\}$$

Lemma 5.18 *There exists an embedding $T_1 \hookrightarrow H_0 \le G_0$ and a $[\mu] \in H^1(F, \overline{T_1})$ such that $\delta_2([\mu]) = \delta_1([D])$.*

Proof. Let $\tilde{H_0} = H_0/Z(H_0)$ and $\tilde{T_1} = T_1/Z(H_0)$. Then H' is a form of H_0, hence there exists $[\lambda'] \in H^1(F, \tilde{H_0})$ such that ${}^{\lambda'}H_0 = H'$. By Steinberg's theorem (Theorem 2.11), there exists an embedding $T_1 \hookrightarrow H_0$ such that $[\lambda'] \in \mathrm{im}(H^1(F, \tilde{T_1}) \to H^1(F, \tilde{H_0}))$. Let $[\mu'] \in H^1(F, \tilde{T_1})$ be chosen such that $[\mu'] \mapsto [\lambda']$. Let $[\chi']$ be the image of $[\mu']$ in $H^2(F, Z(H_0))$. Note that H' becomes quasi-split over K, hence $[\lambda']$ (and $[\chi']$) become split over K as well. This means that $|[\chi']|$ divides 3 in $H^2(F, Z(H_0))$.

Note that $Z(H_0) = R^{(1)}_{L/F}(\mu_6)$ and $Z(G_0) = R^{(1)}_{L/F}(\mu_3)$ fit in the exact sequence:

$$1 \to Z(G_0) \to Z(H_0) \to \mu_2 = R^{(1)}_{L/F}(\mu_2) \to 1 \quad (*)$$

and this sequence splits. We can use this to construct the following diagram with exact columns:

$$\begin{array}{ccccc}
H^2(F,\mu_2) & \xrightarrow{=} & H^2(F,\mu_2) & \xrightarrow{=} & H^2(F,\mu_2) \\
\uparrow & & \uparrow & & \uparrow \\
H^1(F,\tilde{T_1}) & \longrightarrow & H^1(F,\tilde{H}) & \longrightarrow & H^2(F,Z(H_0)) \\
\uparrow & & \uparrow & & \uparrow \\
H^1(F,\overline{T_1}) & \longrightarrow & H^1(F,\overline{H_0}) & \longrightarrow & H^2(F,Z(G_0))
\end{array}$$

Because $[\chi']$ has order dividing 3, its image in $H^2(F,\mu_2)$ is trivial, and because the diagram commutes, this means that there are $[\mu] \in H^1(F,\overline{T_1})$ and $[\lambda] \in H^1(F,\overline{H_0})$ such that $[\mu] \mapsto [\mu']$ and $[\lambda] \mapsto [\lambda']$ under the maps in the diagram. Let $[\chi]$ be the image of $[\lambda]$ in $H^2(F,Z(G_0))$ and consider the diagram:

$$\begin{array}{ccc}
H^2(F,Z(G_0)) & \longrightarrow & H^2(F,Z(H_0)) \\
\downarrow{\scriptstyle \mathrm{Res}} & & \downarrow{\scriptstyle \mathrm{Res}} \\
H^2(L,Z(G_0)) & \longrightarrow & H^2(L,Z(H_0))
\end{array}$$

the horizontal arrows are injections because the sequence $(*)$ is exact. The vertical arrow on the left hand side is injective because $\mathrm{Cor} \circ \mathrm{Res}$ is multiplication by $[L : F] = 2$ and $H^2(F,Z(G_0))$ is a 3-torsion group. Thus, to prove that $[\lambda] \in H^1(F,\overline{H_0})$ maps to $[D]$ in $H^2(F,Z(G_0))$ it suffices to show that $[\chi]_L = [D]_L$. Recall that if $[\alpha] \in H^1(F,\mathbb{PGL}_n)$ has ${}^{\alpha}\mathbb{SL}_n = \mathbb{SL}(A)$ for A a central simple algebra of degree n (not necessarily a division algebra), then $[A] = \mathrm{im}([\alpha]) \in H^2(F,\mu_n) = {}_n\mathrm{Br}(F)$.

The proof is then completed by noticing that ${}^{\lambda}(H_0)_L = \mathbb{SL}_2(D)$ and $H^2(L,Z(G_0)) \hookrightarrow H^2(L,Z(H_0))$. ■

To define T_2, choose $a \in F$ such that a is positive in F_v for all $v \in V^F_{\infty,\mathbb{R}}$ such that G_{F_v} is split or quasi-split and negative otherwise and let $T_2 = R^{(1)}_{F(\sqrt{a})/F}(\mathbb{G}_{\mathrm{m}})$, embedded in $G_{\tilde{\alpha}}$ via the regular embedding. As before, let $T = T_1 \cdot T_2$. Then T satisfies the first criteria of Lemma 5.1 by Lemma 5.18.

5.3.2 Modification of $[\mu]$

Notice that in both cases, T normalizes H_0, an absolutely simple subgroup of type A_5. By construction, $(T \cap H_0)$ also has appropriate real rank. To proceed as in Section 5.1, it therefore suffices to show that T satisfies the second criteria of Lemma 5.1, i.e that $[\xi]_v$ is in the image of $H^1(F_v, T) \to H^1(F_v, G_0)$ for every $v \in V^F_{\infty,\mathbb{R}}$. In the case that G_v is split, we may choose the trivial cocycle in $H^1(F_v, T)$. In the case that G_v is anisotropic, T is anisotropic over F_v by construction, and so $H^1(F_v, T) \twoheadrightarrow H^1(F_v, G_0)$ by Theorem 1 of [2]. Thus it remains to show that in the cases where G_v is isotropic but not split.

If G_v is inner, then note that $|H^1(F_v, G_0)| = 2$, so it suffices to prove that the image of $H^1(F_v, T)$ in $H^1(F_v, G_0)$ is non-trivial. Also, if G_v is outer of rank 2, then T_v is also rank 2, and so any twist by a cocycle in T_v will also have rank at least 2. We have that $|H^1(F_v, G_0)| = 3$ by Tits' classification, with one element being trivial and another corresponding to the anisotropic group. If $1 \neq [\chi]$ is in the image of $H^1(F_v, T)$ in $H^1(F_v, G_0)$, then ${}^{\chi}G_0$ is neither split nor anisotropic, hence must be equal to $[\xi]_v$. Thus it suffices to prove that the image of $H^1(F_v, T)$ in $H^1(F_v, G_0)$ is non-trivial as well.

Lemma 5.19 *If T is a non-split maximal torus in a split or quasi-split group G_0 of type E_6 over $\mathbb{R}$, then the image of $H^1(F_v, T) \to H^1(F_v, G_0)$ is non-trivial.*

Proof. In the case that G_0 is inner over $\mathbb{R}$, then T has rank 2 over $\mathbb{R}$, and thus the anisotropic part of T_a over $\mathbb{R}$ has rank 4, hence is maximal anisotropic (see Proposition 5.21 below). Thus $H^1(\mathbb{R}, T_a) \twoheadrightarrow H^1(\mathbb{R}, G_0)$ by [2], in particular the image of $H^1(\mathbb{R}, T) \to H^1(\mathbb{R}, G_0)$ is non-trivial.

In the case that G_0 is outer over $\mathbb{R}$ let $T = T_1 \cdot T_2$ where T_1 is split of rank 2 over $\mathbb{R}$ and T_2 is anisotropic of rank 4. Then $C_{G_0}(T_2)$ is a reductive group, hence $C_{G_0}(T_2) = H \cdot S$, where S is a torus in G_0 containing T_2 and H is semi-simple.

Claim 5.20 $S = T_2$

Suppose not. If H is trivial, then $C_{G_0}(T_2) = T$, but G_0 contains a maximal anisotropic torus containing T_2, and T has rank 2, a contradiction.

If H has rank 1, then $C_{G_0}(T_2) = \mathbb{SL}_2 \cdot S$. Let T_a be a maximal torus of G_0 which is anisotropic over $\mathbb{R}$ and contains T_2, then $T_a \subset \mathbb{SL}_2 \cdot S$ yields that $T_a \cap S$ has dimension 5 and S is anisotropic. In particular $C_{G_0}(T_2)$ has rank 1, but $T \subset C_{G_0}(T_2)$ has rank 2, a contradiction. This proves the claim.

Because H is standard of rank 2 there are two possibilities, H is of type $A_1 \times A_1$ or A_2 (if H has type G_2 or B_2, then H would have roots of different lengths, which is impossible). In either case, H contains a split subgroup of type A_1. If $\tilde{\alpha}$ is the root of maximal height in E_6, then we may assume after conjugation that $G_{\tilde{\alpha}} \leq H$. Then $T_2 \subset C_{G_0}(H) \subset C_{G_0}(G_{\tilde{\alpha}})$, and so we can consider $C_{C_{G_0}(G_{\tilde{\alpha}})}(T_2)$. Then $C_{C_{G_0}(G_{\tilde{\alpha}})}(T_2) = H' \cdot S'$, where H' is semisimple and S' is a torus containing T_2, as before.

Note that $C = C_{G_0}(G_{\tilde{\alpha}})$ is standard in G_0 of type ${}^2A_5$. Thus C contains an anisotropic torus of rank 5. Arguing as in the claim, we see that $S' = T_2$ and $H' \simeq \mathbb{SL}_2$. Let $\tilde{\beta}$ be the root of maximal height in A_5. After conjugation by an element of C we may assume that $H' = G_{\tilde{\beta}}$. Then $C_C(H') = H'' \cdot S''$, where H'' is of type ${}^2A_3$ and S'' is anisotropic of dimension 1. Then $T_2 \cap H''$ is a maximal torus of H'' which is also maximal. By [2], we then have that there exists an element $[\alpha]$ of $H^1(\mathbb{R}, T_2 \cap H'')$ such that ${}^{\alpha}H''$ is compact. It suffices to show that the image of $[\alpha]$ in $H^1(\mathbb{R}, G_0)$ is non-trivial.

To see this, first note that because ${}^{\alpha}H'' \leq {}^{\alpha}C$ is standard, if ${}^{\alpha}H'' = \mathbb{SU}(\mathbb{C}, f_4)$ for a compact hermitian form f_4, then ${}^{\alpha}C = \mathbb{SU}(\mathbb{C}, f_4 \oplus f_2)$ for some hermitian 2-form f_2. Thus the maximum possible rank of ${}^{\alpha}C$ is 2, so the image of $[\alpha]$ in $H^1(\mathbb{R}, C)$ is non-trivial.

To complete the proof, it suffices to show that if $[\alpha] \in H^1(\mathbb{R}, C)$ maps to the trivial cocycle in $H^1(\mathbb{R}, G_0)$, then $[\alpha]$ is trivial. Recall that C commutes with $G_{\tilde{\alpha}}$ by definition of C, and so for any $[\alpha] \in H^1(\mathbb{R}, C)$ we have that ${}^{\alpha}G_{\tilde{\alpha}} = G_{\tilde{\alpha}}$. Let T_0 be a split torus sitting in $G_{\tilde{\alpha}}$, and consider $C_{{}^{\alpha}G_0}(T_0)$. Because ${}^{\alpha}C \leq C_{{}^{\alpha}G_0}(T_0)$ and $C_{{}^{\alpha}G_0}(T_0)$ is reductive, we have that $C_{{}^{\alpha}G_0}(T_0) = T_0 \cdot {}^{\alpha} C$. Thus the maximum possible rank of any torus containing T_0 is $1 + 2 = 3$, but if ${}^{\alpha}G_0$ is split, then T_0 is contained in a maximal split torus in ${}^{\alpha}G_0$ which has rank 4, a contradiction. ■

5.4 Ansiotropic Tori in E_6 over $\mathbb{R}$

The following was used in the proof of Lemma 5.19:

Proposition 5.21 *Over $\mathbb{R}$, any maximal anisotropic torus of a split group G_0 of type E_6 has absolute rank 4.*

Proof. Because all maximal anisotropic tori are conjugate, it suffices to prove that there exists an anisotropic torus of rank 4 in G_0 that is not properly contained in a larger anisotropic torus. Using the numbering found in [3], Plate I-IX, consider the subgroup H_0 of type ${}^1D_4$ generated by the root subgroups $G_{\alpha_2}, G_{\alpha_3}, G_{\alpha_4}, G_{\alpha_5}$. This is isogenous to the group $\mathbb{SO}_8(\sum_{i=1}^4 x_i^2 - \sum_{i=1}^4 y_i^2)$, and thus contains an anisotropic torus of rank 4 (take products of the $\mathbb{SO}(x_i^2 + x_{i+1}^2)$). Call this torus T.

Claim 5.22 *$C_{G_0}(T)$ is a torus.*

Note that this claim holds over F if it holds over $\overline{F}$. For the purposes of the proof of this claim, take now a maximal torus of G_0 that includes T, and consider the root system of G_0 with respect to this torus over the closure. Because T is a torus, $C_{G_0}(T)$ is reductive, hence $C_{G_0}(T)$ is the almost direct product of a central torus and its derived subgroup. The derived subgroup is generated by those root subgroups that commute with T, of which I claim there are none. This is proven by computing

$$h_{\alpha_2}(t_2)h_{\alpha_3}(t_3)h_{\alpha_4}(t_4)h_{\alpha_5}(t_5)X_\alpha(h_{\alpha_2}(t_2)h_{\alpha_3}(t_3)h_{\alpha_4}(t_4)h_{\alpha_5}(t_5))^{-1}$$

and showing that this is not X_α for any α. Indeed, if this is true for some α, then $\langle \alpha_i, \alpha \rangle = 0$, for $i = 2, 3, 4, 5$. If $\alpha = \sum_{i=1}^8 c_i \epsilon_i$ (again, in the notation of [3], Plate I-IX), then these equations give:

$$c_1 = -c_2,\ c_1 = c_2,\ c_2 = c_3,\ c_3 = c_4$$

which imply $c_1 = c_2 = c_3 = c_4 = 0$, which is impossible for any root $\alpha \in E_6$. This proves the claim.

Any torus is contained in a maximal torus, and so there is a maximal torus contained in $C_{G_0}(T)$, call it S. Because $C_{G_0}(T)$ is also a torus, we must have that $C_{G_0}(T) = S$. Assume that S contains a split torus of rank 2. If there is an anisotropic torus properly containing T, say S', then we would have that $S' \subset C_{G_0}(T) = S$, and so S could have rank at most 1, a contradiction. Thus, it suffices to prove that S contains a split torus of rank 2.

Note that if $C_{G_0}(H_0)$ contains a split torus of rank 2, then $C_{G_0}(T)$ does as well. In order for an element $\prod h_{\alpha_i}(t_i)$ (recalling that we take roots with respect to a F-split torus again) to commute with H_0, we have the following restrictions on t_i:

$$t_2^2 t_4 = 1,\ t_1 t_3^2 t_4 = 1,\ t_3 t_4^2 t_2 t_5 = 1,\ t_6 t_4 t_5 = 1$$

and elements of the form $h_{\alpha_1}(s^2t^2)h_{\alpha_2}(s)h_{\alpha_3}(t)h_{\alpha_4}(s^{-2})h_{\alpha_5}(t^{-1})h_{\alpha_6}(s^2t)$ form a two dimensional split torus that commutes with H_0 (and thus with T). ■

Chapter 6

Non-Absolutely Simple Groups

Collecting the results from Chapters 2-4 completes the proof of Theorem 1.1. It remains to prove Theorem 1.2. Thus, we consider G that is not absolutely simple. Recall that simple algebraic groups over number fields that are not absolutely simple are the restriction of scalars of absolutely simple groups over finite extension of F (Lemma 2.19). Moreover, the following lemma shows that we may restrict ourselves to the case where G is the restriction of a minimal absolutely simple group.

Lemma 6.1 *If $G = R_{K/F}(H)$ where H is an absolutely simple group over K of absolute rank at least 2 and H is not minimal, then G is not minimal.*

Proof. Choose a subgroup $H' \leq H$ that has appropriate real rank over K. Consider $G' = R_{K/F}(H') \leq G$. This is proper because H' is. For $v \in V^F_{\infty,\mathbb{R}}$

$$G'_{F_v} = R_{K_{w_1}/F_v}(H'_{K_{w_1}}) \times \cdots \times R_{K_{w_s}/F_v}(H'_{K_{w_s}})$$

where w_i are the valuations on K that restrict to v on F. Assume $v \in S'_G$. If $K_{w_i} \simeq \mathbb{C}$ for some i, then G_{F_v} has a factor of the form $R_{K_{w_i}/F_v}(H_{K_{w_i}})$ which has rank at least 2, which contradicts $v \in S'_G$. If $K_{w_i} \simeq \mathbb{R}$ for each i, then $H_{K_{w_i}}$ has rank 1 for some i, so $H'_{K_{w_i}}$ has rank 1 as well, thus G' has F_v-rank 1.

If $v \in S''_G$ and $w_i \in S''_H$ for some i, then $H'_{K_{w_i}}$ has higher rank, hence so does G'_{F_v}. Also, if $K_{w_i} \simeq \mathbb{C}$ for some i, then G' also has F_v-rank at least two because

$R_{K_{w_i}/F_v}(H')$ does. Thus, we may assume that no w_i is in S''_H and no w_i has $K_{w_i} \simeq \mathbb{C}$. This gives that at least two w_i are in $S'_H = S'_{H'}$, so G' has appropriate F_v-rank. ■

Notice that $\mathbb{SL}(D)$ and $\mathbb{SU}(D,\tau)$ are simply connected and have no F-defined proper semisimple subgroups for $\deg(D) = p$ prime. The following lemma strongly limits the possible simple subgroups $R_{K/F}(G)$ when G has no semisimple K-defined subgroups.

Lemma 6.2 *Suppose that $G = R_{K/F}(H)$, where H is defined over K, simply connected and has no proper semi-simple subgroups defined over K. Then every F-simple proper subgroups of G is isomorphic to $R_{P/F}(H')$ where $F \subset P \subsetneq K$, H' is defined over P and H'_K is isomorphic to H_K. In particular, if G has proper F-simple subgroups, H admits descent to a subfield $P \subset K$.*

Proof. Suppose that $G' \leq G$ is a non-trivial proper semi-simple subgroup of G as above. Let $K \otimes_F K \simeq K \times K'$, where K' is an etale extension of K and $G_K \simeq H_K \times R_{K'/K}(H_1)$ for some H_1 defined over K'. Let π be the projection $G_K \twoheadrightarrow H_K$. Then $\pi(G'_K)$ is a semi-simple subgroup of H_K, thus $\pi(G'_K)$ is either trivial or all of H_K.

Assume that the image of G'_K under π is trivial. Over $\overline{K}$, $G_{\overline{K}}$ becomes

$$H_{\overline{K}} \times \cdots \times H_{\overline{K}},$$

with $\Gamma = \mathrm{Gal}(\overline{K}/K)$ permuting the components of $G_{\overline{K}}$ transitively. Let $1 \neq g = (g_1, \ldots, g_n) \in G'_K(\overline{K})$ and suppose that $g_j \neq 1$. Because Γ permutes the components of $G_{\overline{K}}$ transitively, there exists a $\sigma \in \Gamma$ such that the first component of $\sigma(g)$ is $\sigma(g_j)$. Then $\pi(\sigma(g)) = \sigma(g_j) \neq 1$, but $\sigma(g) \in G'_F(K)$ because G' is F-defined, and so $\pi(\sigma(g)) = 1$, a contradiction.

If G' is absolutely simple then the kernel of π is finite, so setting $H' = G'$ and $P = F$ we have that π is a finite covering of H_K by H'_K. By the assumption that H is simply connected, we obtain that π is an isomorphism.

If G' is not absolutely simple, then $G' = R_{F'/F}(H')$ for some H' absolutely simple over F'. Suppose $F' \otimes_F K \simeq K_1 \times \cdots \times K_\ell$ with K_i/K finite field extensions. Then

$$G'_K \simeq R_{K_1/K}(H'_{K_1}) \times \cdots \times R_{K_\ell/K}(H'_{K_\ell}).$$

Let π_i be the composition $R_{K_i/K}(H'_{K_i}) \hookrightarrow G'_K \overset{\pi}{\twoheadrightarrow} H_K$. If the images of all of the π_i are trivial, then the image of π is trivial, which is impossible. Thus, because H_K contains no proper semi-simple subgroups and $R_{K_i/K}(H'_{K_i})$ are K-simple, some π_i is an K-defined isogeny. By the assumption that H_K is simply connected, we get that π_i is an isomorphism. If K_i/K is a non-trivial field extension, then π_i is an isomorphism between one group which is absolutely simple and one that is not, which is impossible. Thus $K_i = K$ and π_i is an isomorphism $H'_K \twoheadrightarrow H_K$. Identifying P with the image of F' in $K_i = K$, we see that H' is defined over P and $G' = R_{P/K}(H')$, as required. ■

This lemma allows us to handle several cases:

Proposition 6.3 *If $G = R_{K/F}(\mathbb{SL}(D))$ for a central division algebra D/K of prime degree $p \geq 3$, then G is minimal if and only if D does not descend to any sub-field $F \subset P \subsetneq K$.*

Proof. Assume that D does not descend. By Lemma 6.2, G contains no proper F-simple subgroups in this case. If D does descend, then $H = R_{P/F}(\mathbb{SL}(D'))$ is a proper F-simple subgroup of appropriate real rank. Indeed, by the assumption that D' has prime degree $p \geq 3$, we must have that D' is split over P_w for all $w \in V^P_{\infty,\mathbb{R}}$. ■

Proposition 6.4 *If G is of the form $R_{K/F}(\mathbb{SL}(D))$ for D a quaternion algebra over*

K, then G is minimal if and only if for every $F \subset P \subsetneq K$ such that D descends to P there exist $v_0 \in S_G$ such that

- *If $v_0 \in S'_G$ then $P_{w_i} \simeq \mathbb{R}$ and $D' \otimes_P P_{w_i} \simeq \mathbb{H}$ for all w_i lying over v_0 and*
- *If $v_0 \in S''_G$ then there is at most one w_i lying over v_0 such that either $P_{w_i} \simeq \mathbb{C}$ or $D' \otimes_P P_{w_i} \simeq \mathrm{M}_2(\mathbb{R})$.*

Proof. Using Lemma 6.2 we find that all possible F-simple subgroups correspond to $F \subset P \subsetneq K$ such that D descends to P, and the conditions imposed upon such P exactly yield that the corresponding subgroup cannot have appropriate real rank. ∎

Example 6.5 *Let $K = \mathbb{Q}(\sqrt[3]{2}, \sqrt{3})$, $D = (-1,-1)$, $F = \mathbb{Q}$ and $G = R_{K/\mathbb{Q}}(\mathbb{SL}(D))$. Then K has two real completions and two complex, so*

$$G_{\mathbb{R}} \simeq \mathbb{SL}(D) \times \mathbb{SL}(D) \times R_{\mathbb{C}/\mathbb{R}}(\mathbb{SL}_2(\mathbb{C})) \times R_{\mathbb{C}/\mathbb{R}}(\mathbb{SL}_2(\mathbb{C}))$$

has $\mathbb{R}$-rank 2. For any field $\mathbb{Q} \subset P \subsetneq K$, we have that D descends to P, but P has at most one complex completion, thus $R_{P/\mathbb{Q}}(\mathbb{SL}(D))$ has $\mathbb{R}$-rank at most 1 and therefore by Lemma 6.2, G is minimal.

Proposition 6.6 *If $G = R_{K/F}(\mathbb{SU}(D,\tau))$ for D a central division algebra of degree $p \geq 3$ over K'/K quadratic with involution of the second kind τ such that $K'^{\tau} = K$, then G is minimal if and only if for all $F \subset P \subsetneq K$ such that D descends to a central simple algebra (D', τ') over a quadratic extension P'/P with involution of the second kind τ' with $P'^{\tau'} = P$, there exists some $v_0 \in S_G$ such that $P_{w_i} \simeq \mathbb{R}$ and $P_{w_i} \otimes P' \simeq \mathbb{C}$ for all w_i lying v_0, and*

1. *if $v_0 \in S'_G$ then $(D' \otimes_P P_{w_i}, \tau' \otimes 1) \simeq (\mathrm{M}_n(\mathbb{C}), \pm\langle 1, \ldots, 1\rangle)$ for all w_i lying over v_0, or*

2. *if* $v_0 \in S''_G$ *then* $(D' \otimes_P P_{w_i}, \tau' \otimes 1) \simeq (\mathrm{M}_n(\mathbb{C}), \pm\langle 1, -1, 1 \ldots, 1\rangle)$ *for at most one* i *and* $(D' \otimes_P P_{w_i}, \tau' \otimes 1) \simeq (\mathrm{M}_n(\mathbb{C}), \pm\langle 1, \ldots, 1\rangle)$ *for all others.*

Proof. Using Lemma 6.2 we find that all possible simple subgroups correspond to $F \subset P \subsetneq K$ such that D' exists as above, and the conditions imposed upon such P exactly guarantee that the corresponding subgroup cannot have appropriate real rank. ■

It remains to consider the restrictions of absolutely simple groups of the form $\mathbb{SU}_3(K', f)$ for K'/K a quadratic extension and f a 3-dimensional Hermitian form over K'. Notice that there exist proper, non-trivial, K-simple subgroups $H \leq \mathbb{SU}_3(K', f)$, but because A_2 does not contain a root system of type $A_1 \times A_1$, these can only be of absolute rank 1.

Proposition 6.7 *If* G *is of the form* $R_{K/F}(\mathbb{SU}_3(K', f))$ *for* K'/K *quadratic,* f *hermitian over* K'^3*, then* G *is minimal if and only if:*

1. *For any* $F \subset P \subsetneq K$ *such that* $\mathbb{SU}_3(K', f)$ *descends to* P *we have that there exists a* $v_0 \in S_G$ *such that* $P_{w_i} \simeq \mathbb{R}$ *for all* w_i *lying over* v_0 *and*
   (a) *If* $\mathbb{SU}_3(K', f)$ *descends to* $\mathbb{SU}_3(P', f')$*, where* $f' = \langle 1, a_2, a_3\rangle$ *then* $P_{w_i} \otimes P' \simeq \mathbb{C}$ *for every* w_i *and*
      i. *if* $v_0 \in S'_G$ *then the image of* a_j *in* P_{w_i} *is positive for all* i
      ii. *if* $v_0 \in S''_G$ *then the image of* a_j *in* P_{w_i} *is negative for at most one* i
   (b) *if* $\mathbb{SU}_3(K', f)$ *descends to* $\mathbb{SU}(D, \tau)$ *where* D *is a central division algebra of degree* 3 *over* P'/P *quadratic with involution* τ *of the second kind then* $P' \otimes P_{w_i} \simeq \mathbb{C}$ *for every* i *and*

i. *If* $v_0 \in S'_G$ *then* $(D \otimes P_{w_i}, \tau \otimes 1) \simeq (\mathrm{M}_3(\mathbb{C}), \sigma)$, *where* $\sigma(X) = \overline{X}^T$, *for every* w_i

ii. *If* $v_0 \in S''_G$ *then* $(D \otimes P_{w_i}, \tau \otimes 1) \simeq (\mathrm{M}_3(\mathbb{C}), \sigma)$ *for all but at most one* w_i *and for at most one* w_i, $(D \otimes P_{w_i}, \tau \otimes 1) \simeq (\mathrm{M}_3(\mathbb{C}), \sigma \circ \mathrm{Int}(\mathrm{diag}(1, -1, 1)))$ *or* $(\mathrm{M}_3(\mathbb{C}), \sigma \circ \mathrm{Int}(\mathrm{diag}(1, -1, -1)))$

2. *For any* $F \subset P \subseteq K$ *such that some subgroup* $\mathbb{SL}(D') \leq \mathbb{SU}_3(K', f)$ *descends to* $\mathbb{SL}(D)$ *over* P *there exists some* $v_0 \in S_G$ *such that*

   (a) *If* $v_0 \in S'_G$ *then* $P_{w_i} \simeq \mathbb{R}$ *and* $D \otimes P_{w_i} \simeq \mathbb{H}$ *for all* w_i *over* v_0 *and*

   (b) *if* $v_0 \in S''_G$ *then* $P_{w_i} \simeq \mathbb{C}$ *or* $D \otimes P_{w_i} \simeq \mathrm{M}_2(\mathbb{R})$ *for at most one* w_i *over* v_0.

Proof. Arguing as in Lemma 6.2, let $G' \leq G$ be an F-defined, F-simple subgroup, and $G_K = \mathbb{SU}_3(K', f) \times R_{K'/K}(H_1)$. Let $\pi : G_K \to \mathbb{SU}_3(K', f)$ be projection on the first component. If $\pi(G'_K) = 1$, then as before $G' = 1$, a contradiction. This means that $\pi(G'_K)$ is either all of $\mathbb{SU}_3(K', f)$ or isomorphic to $\mathbb{SL}(D)$ for a quaternion algebra D defined over K. If $\pi(G'_K) \leq \mathbb{SL}(D) \leq \mathbb{SU}_3(K', f)$ and $g = (g_1, \ldots, g_n) \in G'_K(\overline{K})$, then for any g_i there exists $\sigma \in \Gamma$ such that $\sigma(g_i)$ is the first component of $\sigma(g)$. Because $\mathbb{SL}(D)$ and G'_K are K-defined, we therefore have that $g_i \in \mathbb{SL}(D)$. This means that $G' \leq R_{K/F}(\mathbb{SL}(D))$, so we can apply Lemma 6.2 to find that G' is isomorphic to $R_{P/F}(\mathbb{SL}(D'))$ for some D' over P. The conditions listed in item (2) are exactly what is necessary to ensure that no subgroup of this form has appropriate real rank.

Assume that $\pi(G'_K) = \mathbb{SU}_3(K', f)$. If G'_K is absolutely simple then π is an isomorphism, and setting $F = P$, the conditions in 1 ensure that any such subgroup does not have appropriate real rank. If G' is not absolutely simple, $G' \simeq R_{F'/F}(H')$

for some absolutely simple H'. Then

$$G'_K = R_{K_1/K}(H'_{K_1}) \times \cdots \times R_{K_m/K}(H'_{K_m})$$

Let π_i be the restriction of π to $R_{K_i/K}(H'_{K_1})$. Because $R_{K_i/K}(H'_{K_i})$ are K-simple, we must have that $\ker(\pi_i)$ is either finite or all of $R_{K_i/K}(H'_{K_i})$. Assume that some π_i is surjective. Then π_i is an isomorphism because $\mathbb{SU}_3(K', f)$ is simply connected. Arguing as in Lemma 6.2, we have that $K_i = K$ and $H'_K \simeq \mathbb{SU}_3(K', f)$ and the conditions listed in (1) are exactly the conditions required to ensure that G' does not have appropriate real rank.

Assume π_i is not surjective for any i. The image of π_i cannot be trivial for all i, or else the image of π would be trivial, thus there exists some i for which the image of π_i is $\mathbb{SL}(D)$ for some quaternion algebra D over K. This means that H'_{K_i} has type A_1, so $\pi_i : R_{K_i/K}(\mathbb{SL}(D_1)) \to \mathbb{SL}(D)$ is a surjection with finite kernel. This means that π_i must be an isomorphism, and G' is again of the form $R_{P/F}(\mathbb{SL}(D))$ for a quaternion algebra D. The conditions listed in 2 are exactly what is required for such a subgroup not to have appropriate real rank. ■

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