ON THE GALOIS STRUCTURE OF THE S-UNITS FOR CYCLOTOMIC EXTENSIONS OVER $\mathbb{Q}$.

by

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Abstract

Let us consider K/k to be a finite Galois extension of number fields with $G = Gal(K/k)$ and assume that S is finite G-stable set of primes of K which is "large", this means that it contains all archimedean primes, all the ramified primes of K/k and such that the S-class group of K is trivial. K. W. Gruenberg and A. Weiss in [10] proved that the $\mathbb{Z}G$-module of the S-units of K is completely determined up to stable isomorphism by its torsion submodule $\boldsymbol{\mu}$, the set S, a special character ϵ and the Chinburg class $\Omega_m(K/k)$.

The main point of this thesis is to investigate the $\mathbb{Z}G$-module structure of E when $k = \mathbb{Q}$ and K is a cyclotomic extension of $\mathbb{Q}$ by studying in detail the character ϵ.

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Contents

Chapter 1

Introduction.

Let us consider K/k to be a finite Galois extension of number fields with $G = Gal(K/k)$ and assume that S is a finite G-stable set of primes of K which is "large": this means that it contains all Archimedean primes, all the ramified primes of K/k and such that the S-class group of K is trivial.

K.W. Gruenberg and A. Weiss addressed in their joint paper [10] the question of determining the $\mathbb{Z}G$-module structure of the group of the S-units of K.

The belief that one can find explicit models of $\mathbb{Z}G$-modules, which are stably isomorphic to the S-units, by emulating the ideas presented in [9] and [10], is the motivation for the work presented in this thesis.

The structure of the group of units $\mathcal{O}_K^\times$ of the ring of integers $\mathcal{O}_K$, has been of great interest for number theory. In the first half of the 19^{th} century E. Kummer and independently Dirichlet studied the structure of $\mathcal{O}_K^\times$. The work of E. Kummer is based on understanding the arithmetic of cyclotomic fields $K = \mathbb{Q}(\boldsymbol{\mu})$ generated over the field of rational numbers by the set $\boldsymbol{\mu}$ of all roots of the polynomial $X^{2m} - 1$. Dirichlet's work on L-functions and their values at 1 led to the proof that the multiplicative subgroup A of $K^\times$ generated by $\{1 - \mu : \mu \in \boldsymbol{\mu}^\times\}$ has a subgroup of finite index in $\mathcal{O}_K^\times$, where $\boldsymbol{\mu}^\times = \boldsymbol{\mu} \setminus \{1\}$.

The $\mathbb{Z}G$-module E of the S-units of K consists of all nonzero elements u of

$K^\times$ such that

$$v_{\mathfrak{P}}(u) = 0 \text{ for all } \mathfrak{P} \notin S.$$

J. Tate started the study of the module E, which contains $\mathcal{O}_K^\times$ as a submodule. He realized that E is a $\mathbb{Z}G$-module to which the cohomological methods of class field theory can be applied effectively when S is large.

In the second half of the 20^{th} century Tate obtained two major results regarding the cohomology and arithmetic of E. It should be emphasized that throughout the thesis cohomology will always mean Tate cohomology unless it is stated otherwise.

By considering the augmentation map $\acute{i}_0 : \mathbb{Z}S \to \mathbb{Z}$ which sends each $\mathfrak{P}$ in S to 1 and denoting by ΔS the kernel of $\acute{i}_0$, one obtains the following short exact sequence of $\mathbb{Z}G$-modules

$$0 \longrightarrow \Delta S \xrightarrow{i_0} \mathbb{Z}S \xrightarrow{\acute{i}_0} \mathbb{Z} \longrightarrow 0 \tag{1.0.0.1}$$

In [14] published in 1966, Tate proved that the cohomology induced by (1.0.0.1) is isomorphic to the cohomology induced by the short exact sequence

$$0 \longrightarrow E \xrightarrow{j} J \xrightarrow{\acute{j}} C_K \longrightarrow 0 \tag{1.0.0.2}$$

after dimension shifting by 2, where J denotes the group of S-idèles and C_K the group of idèle classes.
He then deduced the existence of an exact sequence of finitely generated $\mathbb{Z}G$-modules

$$0 \longrightarrow E \longrightarrow A \longrightarrow B \longrightarrow \Delta S \longrightarrow 0 \tag{1.0.0.3}$$

where A and B are cohomologically trivial. The proof of this result can be found in [15]. An exact sequence of this type will be called a Tate sequence.

Chinburg in [4], published in 1983, used (1.0.0.3) to define the Chinburg class

$$\Omega_m(K/k) = [A] - [B] \text{ in } K_0(\mathbb{Z}G) \tag{1.0.0.4}$$

and proved that this is an invariant of K. The author conjectured in [5] that

$$\Omega_m(K/k) = W_{K/k},$$

where $W_{K/k}$ is the Cassou-Noguès Fröhlich class of K/k .

For cyclotomic extensions of $\mathbb{Q}$, Chinburg's conjecture was proved in the joint paper of 2003 by D. Burns and C. Greither [3], with $\mathbb{Z}$ replaced by $\mathbb{Z}' = \mathbb{Z}[1/2]$. In 2013 M. Flach completed the proof, in [7], of Chinburg's conjecture for cyclotomic fields, hence, when $K = \mathbb{Q}(\boldsymbol{\mu})$ one can state that $\Omega_m(K/\mathbb{Q}) = 0$.

In [9] the Galois structure of E is studied through its envelopes, which means short exact sequences of finitely generated $\mathbb{Z}G$-modules of the form

$$0 \longrightarrow E \longrightarrow C \longrightarrow L \longrightarrow 0 \tag{1.0.0.5}$$

where C is cohomologically trivial and L is a $\mathbb{Z}G$-lattice, meaning that L is $\mathbb{Z}$-free module.

In the later work [10], published in 1997, the authors proved

1.0.1 Theorem (Theorem B)**.** *The stable isomorphism class of E is determined by the $\mathbb{Z}G$-module $\boldsymbol{\mu}$, the G-set S, a special character $\epsilon : H^2(G, Hom(\Delta S, \boldsymbol{\mu})) \to \mathbb{Q}/\mathbb{Z}$ and the Chinburg invariant $\Omega_m(K/k)$.*

Since the computation of the character ϵ is a crucial factor in this work, we will briefly describe ϵ here.

Given $\mathbb{Z}G$-modules M and N, $Hom(M, N)$ (respectively $M \otimes N$) means $Hom_{\mathbb{Z}}(M, N)$ (resp $M \otimes_{\mathbb{Z}} N$), where G acts by the diagonal action and $M^* = Hom_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$. We also define $[M, N]_G = Hom_G(M, N)/\langle f : f \sim 0\rangle$, where $f \sim 0$ means that there exists a $\mathbb{Z}G$-projective module P and $\mathbb{Z}G$-homomorphisms $f' : M \to P$

and $f'' : P \to N$, such that f is the following

$$f : M \xrightarrow{f'} P \xrightarrow{f''} N.$$

In case there is no confusion we will denote $[M, N]_G$ simply by $[M, N]$.

If L is a $\mathbb{Z}G$-lattice, $H^0(G, Hom(L, M)) \cong [L, M]$. The proof of this fact can be found on pg 270 of [9].

Let us consider a Tate sequence as in (1.0.0.3) and denote the kernel of the map $B \to \Delta S$ by L. Then the Tate sequence can be divided into two short exact sequences

$$0 \longrightarrow E \longrightarrow A \longrightarrow L \longrightarrow 0 \tag{1.0.1.1}$$

$$0 \longrightarrow L \longrightarrow B \longrightarrow \Delta S \longrightarrow 0, \tag{1.0.1.2}$$

where the first exact sequence is a particular "Tate" envelope of E. After applying the functor $Hom(-, \boldsymbol{\mu})$ to the $\mathbb{Z}$-split exact sequence given in (1.0.1.2) and applying cohomology, one obtains a connecting isomorphism

$$\partial'' : H^1(G, Hom(L, \boldsymbol{\mu})) \to H^2(G, Hom(\Delta S, \boldsymbol{\mu})). \tag{1.0.1.3}$$

If we denote by $\alpha : \boldsymbol{\mu} \to E$ the natural inclusion of the roots of unity of K into the S-units, the envelope of E given in (1.0.1.1) induces an envelope of $\boldsymbol{\mu}$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \boldsymbol{\mu} & \longrightarrow & A & \longrightarrow & \bar{A} & \longrightarrow & 0 \\ & & \alpha \downarrow & & = \downarrow & & \bar{f} \downarrow & & \\ 0 & \longrightarrow & E & \longrightarrow & A & \longrightarrow & L & \longrightarrow & 0 \end{array} \tag{1.0.1.4}$$

By applying the exact functor $Hom(L, -)$ to (1.0.1.4) and then applying cohomology, one obtains the following commutative diagram

$$
\begin{array}{ccc}
[L, \bar{A}] & \xrightarrow{\partial'_L} & H^1(G, Hom(L, \boldsymbol{\mu})) \\
\bar{f}_* \downarrow & & \alpha_* \downarrow \\
[L, L] & \xrightarrow{\partial_L} & H^1(G, Hom(L, E)) \\
\tau_L \downarrow & & \\
\mathbb{Q}/\mathbb{Z} & &
\end{array}
\tag{1.0.1.5}
$$

where $\tau_L[l] := Trace_L(l)/|G| + \mathbb{Z}$ for all $[l]$ in $[L, L]$ and the horizontal connecting homomorphisms are in fact isomorphisms.

We now restrict our attention to envelopes of E of the form

$$0 \longrightarrow E \longrightarrow A \longrightarrow \Delta G \otimes \Delta S \longrightarrow 0. \tag{1.0.1.6}$$

From this point on $L = \Delta G \otimes \Delta S$, we can replace (1.0.1.2) by the exact sequence

$$0 \longrightarrow L \xrightarrow{i_1} \mathbb{Z}G \otimes \Delta S \xrightarrow{\acute{\imath}_1} \Delta S \longrightarrow 0, \tag{1.0.1.7}$$

obtained by applying the exact functor $- \otimes \Delta S$ to the augmentation sequence

$$0 \longrightarrow \Delta G \xrightarrow{i_1} \mathbb{Z}G \xrightarrow{\acute{\imath}_1} \mathbb{Z} \longrightarrow 0. \tag{1.0.1.8}$$

The character ϵ can now be described as $\epsilon = \tau_L \partial_L^{-1} \alpha_* (\partial'')^{-1}$.

The original objective of this thesis is to use the *Reconstruction Process*, appearing in §5 of [10], to obtain an explicit description of a $\mathbb{Z}G$-module M stably isomorphic to the S-units E from the following data: the torsion submodule $\boldsymbol{\mu}$, the $\mathbb{Z}G$-lattice L and the character ϵ, for the case when $K = \mathbb{Q}(\boldsymbol{\mu})$ is a cyclotomic extension over the field of rational numbers.

Considering the family of cyclotomic extensions over $\mathbb{Q}$ as our candidate to test the viability of explicitly finding models stably isomorphic to E is motivated by the arithmetic and cohomological knowledge that the literature provides for this family of fields. The fact that $\Omega(\mathbb{Q}(\boldsymbol{\mu})/\mathbb{Q}) = 0$ is an extra incentive.

We expected that the methods applied to the cyclotomic fields case would give us an insight of the $\mathbb{Z}G$-module structure that could be generalized to arbitrary number fields.

The steps to follow are: to construct a particular envelope of $\boldsymbol{\mu}$

$$0 \longrightarrow \boldsymbol{\mu} \xrightarrow{\psi} C \xrightarrow{\pi} \bar{C} \longrightarrow 0,$$

with $[C] - (|S| - 1)[\mathbb{Z}G] = 0$ in $K_0(\mathbb{Z}G)$ and a surjective homomorphism $\bar{f} : \bar{C} \to L$, satisfying

$$\tau_L \ \bar{f}_* = \epsilon \ \partial'_L \ \partial''. \tag{1.0.1.9}$$

The importance of $\bar{f} : \bar{C} \to L$ being a surjective $\mathbb{Z}G$-homomorphism, is that, it induces the following commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \boldsymbol{\mu} & \xrightarrow{\psi} & C & \xrightarrow{\pi} & \bar{C} & \longrightarrow & 0 \\
 & & \downarrow & & \| & & \downarrow \bar{f} & & \\
0 & \longrightarrow & M & \longrightarrow & C & \xrightarrow[f]{} & L & \longrightarrow & 0
\end{array} \tag{1.0.1.10}$$

where $f = \bar{f} \circ \pi$ and $M = ker(f)$. The lower row of diagram (1.0.1.10) and the exact sequence given in (1.0.1.6) are envelopes satisfying a particular case of the *Reconstruction Process*, namely Theorem 4.7 of [10], from which we conclude that, M is stably isomorphic to E.

The setup of this thesis is the following: Chapter 2 contains background theory needed mainly in Chapters 6 and 7.

In chapter 3 we construct a reasonably "small" envelope $0 \to \boldsymbol{\mu} \to C \to \bar{C} \to 0$ of $\boldsymbol{\mu}$, for an arbitrary finite Galois extension K/k of number fields. The envelope presented here satisfies

$$[C] - c[\mathbb{Z}G] = 0 \ \text{ in } \ K_0(\mathbb{Z}G),$$

with $|G|c = dim_{\mathbb{Q}}(\mathbb{Q} \otimes C) = |G|d(G)$, where $d(G)$ is the minimal number of generators of G.

In chapter 4 we prove that for an arbitrary Galois extension of number fields K/k, the character $\epsilon : H^2(G, Hom(\Delta S, \boldsymbol{\mu})) \to \mathbb{Q}/\mathbb{Z}$ can be expressed in terms of global and local invariants maps from class field theory. This result brings advantages in the computability of ϵ, in contrast to the description of ϵ given in [10].

In chapter 5 we start the exposition by considering the general case when K/k is a Galois extension of number fields and studying the long exact sequence in cohomology

$$\cdots \longrightarrow H^2(G, Hom(\mathbb{Z}S, \boldsymbol{\mu})) \xrightarrow{i_1^*} H^2(G, Hom(\Delta S, \boldsymbol{\mu})) \xrightarrow{\partial} H^3(G, \boldsymbol{\mu}) \xrightarrow{\acute{i}_1^*} \cdots \tag{1.0.1.11}$$

induced by the short exact sequence obtained by applying the functor $Hom(-, \boldsymbol{\mu})$ to the $\mathbb{Z}$-split exact sequence (1.0.0.1). We show that ϵ restricted to the image of i_1^* can be expressed only in terms of the local invariants.

After this point, we restrict to the case $K = \mathbb{Q}(\boldsymbol{\mu})$ and show that that 2 annihilates $H^3(G, \boldsymbol{\mu})$.
This last observation leads us to consider working in the category of $\mathbb{Z}'G$-modules, with $\mathbb{Z}' = \mathbb{Z}[1/2]$, the advantage being that ϵ is then completely determined by its restriction to the image of i_1^*, so the global invariant is never needed.

We conclude chapter 5 by proving that, in the category of $\mathbb{Z}'G$-modules, one can reduce the problem of computing ϵ for $K = \mathbb{Q}(\boldsymbol{\mu})$ to computing it for $\tilde{K} = \mathbb{Q}(\tilde{\boldsymbol{\mu}})$, with $\tilde{\boldsymbol{\mu}}$ the set of roots of $x^{2\tilde{m}} - 1$ and $\tilde{m}$ the greatest squarefree divisor of m.

The importance of this observation is that $\tilde{K}$ is tamely ramified for all non archimedean primes different from 2 and that, in this setting, the local invariant maps are less difficult to compute.

In chapter 6 we continue working with $\tilde{K}$. First we compute a set of generators of the group $H^2(G, Hom(\mathbb{Z}S, \boldsymbol{\mu}))$, using the fact that $\mathbb{Z}S \cong \bigoplus_{\mathfrak{P}\in S_*} ind_{G_{\mathfrak{P}}}^{G}\mathbb{Z}$, where S_* is a transversal to the G-orbits in S. Since Shapiro's lemma gives an isomorphism

$$H^2(G, Hom(\mathbb{Z}S, \boldsymbol{\mu})) \cong \bigoplus_{\mathfrak{P}\in S_*} H^2(G_{\mathfrak{P}}, \boldsymbol{\mu}),$$

we are reduced to computing a set of generators for each factor $H^2(G_{\mathfrak{P}}, \boldsymbol{\mu})$, which is done by an application of the Hochschild-Serre spectral sequence. This chapter concludes with the evaluation of ϵ on each generator.

In Chapter 7 we continue to work in the category of $\mathbb{Z}'G$-modules, here we deal with the computation of a suitable $\bar{f} : \bar{C} \to L$ satisfying (1.0.1.9). We approach this by showing that there is a commutative diagram

$$\begin{array}{ccc} H^2(G_{\mathfrak{P}}, \boldsymbol{\mu}) & \longrightarrow & H^2(G, Hom(\Delta S, \boldsymbol{\mu})) \\ \cong\downarrow & & \cong\downarrow \\ H^0(G_{\mathfrak{P}}, Hom(\Delta G, \bar{C})) & \xrightarrow{\alpha_1} & [L, \bar{C}] \end{array}$$

even more, we compute explicitly the vertical isomorphisms. The diagram above suggests that $\bar{f}$ can be computed "locally". Following this idea, we construct a homomorphism $\beta_1 : [\bar{C}, L]_G \to [\bar{C}, \Delta G]_{G_{\mathfrak{P}}}$ and a non-degenerate pairing

$$\tau_{\Delta G}^{G_{\mathfrak{P}}} : [\bar{C}, \Delta G]_{G_{\mathfrak{P}}} \times [\Delta G, \bar{C}]_{G_{\mathfrak{P}}} \to \mathbb{Q}/\mathbb{Z},$$

such that

$$\tau_{\Delta G}^{G_{\mathfrak{P}}}(\beta_1[\bar{f}], z) = \tau_L([\bar{f}], \alpha_1 z), \qquad (1.0.1.12)$$

for all $[\bar{f}]$ in $[\bar{C}, L]$ and all z in $H^0(G_{\mathfrak{P}}, Hom(\Delta G, \bar{C})) \cong [\Delta G, \bar{C}]_{G_{\mathfrak{P}}}$.

Chapter 8 is dedicated to study $\beta_1[f]$. We list the conditions that a $[f_{\mathfrak{P}}] = \beta_1[f]$ in $[\bar{C}, \Delta G]_{G_{\mathfrak{P}}}$ must satisfy so that (1.0.1.12) holds.

Throughout this project we encountered many obstacles, some of which we were able to face by modifying the original setup. As an example of this, we can mention the obstruction that the prime number 2 had on the computation of the character ϵ. In this case, shifting to the category of $\mathbb{Z}'G$-modules allowed us to obtain results without trivializing the project.

This was not the case when dealing with the surjective homomorphism $f : C \to L$. Even in the simplest cases we studied, assuming m to be an odd prime number, where we actually compute $ker(f)$, the answers obtained did not bring any real understanding of E.

The difficulty of computing $ker(f)$, increases drastically with the complexity of m and achieving a general method in this way seemed unrealistic.

We then approached the original question from a different point of view, which led us to the proof of Theorem 1.0.2 presented in the Appendix. It must be emphasized that, Theorem 1.0.2 has been proved, for arbitrary finite Galois extensions K/k of number fields.

We now give a list of results that are necessary to state Theorem 1.0.2.
The isomorphism $\partial'' : H^1(G, Hom(L, \boldsymbol{\mu})) \to H^2(G, Hom(\Delta S, \boldsymbol{\mu}))$, defines an isomorphism

$$(\partial'')^* : H^2(G, Hom(\Delta S, \boldsymbol{\mu}))^* \to H^1(G, Hom(L, \boldsymbol{\mu}))^*. \qquad (1.0.1.13)$$

Similarly $\partial'_L : [L, \bar{C}] \to H^1(G, Hom(L, \boldsymbol{\mu}))$, defines an isomorphism

$$(\partial'_L)^* : H^1(G, Hom(L, \boldsymbol{\mu}))^* \to [L, \bar{C}]^*. \qquad (1.0.1.14)$$

The fact that L is a $\mathbb{Z}G$-lattice implies that the homomorphism

$$[\bar{C}, L] \to [L, \bar{C}]^*, \tag{1.0.1.15}$$

given by $[\bar{f}] \mapsto [\bar{f}]^*$, where $[\bar{f}]^*([g]) = Trace_L(f \circ g)/|G| + \mathbb{Z}$ for $[g]$ in $[L, \bar{C}]$, is an isomorphism. The above discussion implies the existence of $\bar{h} : \bar{C} \to L$ satisfying

$$[\bar{h}]^* = -(\partial'_L)^*(\partial'')^*\epsilon = -\epsilon\partial''\partial'_L.$$

The $\mathbb{Z}G$-map $\pi : C \to \bar{C}$ induces equal functors $Hom(C, -)$ and $Hom(\bar{C}, -)$, on the category of $\mathbb{Z}G$-lattices, hence $[\bar{C}, L] \cong [C, L]$. Letting $h : C \to L$ be the unique homomorphism such that $\bar{h}\pi = h$, we can say that

$$[h]^* = -(\partial'_L)^*(\partial'')^*\epsilon' = -\epsilon'\partial''\partial'_L, \tag{1.0.1.16}$$

under the last identification.
Let us consider the exact sequence given in (1.0.1.7), after applying the exact functor $\Delta G \otimes -$. We obtain the following exact sequence

$$0 \longrightarrow L_2 \longrightarrow N \to L \longrightarrow 0,$$

where $L_2 = \Delta G \otimes L$ and $N \cong \mathbb{Z}G \otimes \Delta G \otimes \Delta S$. The fact that N is a $\mathbb{Z}G$-free module implies that the connecting homomorphism $\partial : [C, L] \to H^1(G, Hom(C, L_2,))$ is an isomorphism.
Let $0 \to \boldsymbol{\mu} \to C \to \bar{C} \to 0$ be an envelope of $\boldsymbol{\mu}$ satisfying

$$[C] - c[\mathbb{Z}G] = \Omega_m(K) \text{ in } Cl(\mathbb{Z}G),$$

where $|G|c = dim_{\mathbb{Q}}(\mathbb{Q} \otimes C)$.
1.0.2 Theorem. *Let $M = M(\epsilon)$ denote the $\mathbb{Z}G$-module in a $\mathbb{Z}$-split 1-extension*

$$0 \longrightarrow L_2 \longrightarrow M \longrightarrow C \longrightarrow 0, \tag{1.0.2.1}$$

with extension class equal to the image of $\epsilon^1 = \partial[h]$ in $H^1(G, Hom(C, L_2))$ ($[h]$ as in (1.0.1.16)). Then $E \oplus (\mathbb{Z}G)^n$ is stably isomorphic to $M(\epsilon)$, with $n := (|G| - 2)(|S| - 1)$, when G is not trivial.

The proof of the theorem above, which forms part of a joint paper in progress by D. Riveros and A. Weiss, is included here as an appendix. It describes a method for finding an explicit model M for the stable isomorphism class of E by having understanding of the factors ΔS, Ω_m, ϵ, n and C. This shows how the character ϵ is highly involved in the $\mathbb{Z}G$-module structure of M. The computations of the local invariant maps done in Chapter 5, for cyclotomic extensions of $\mathbb{Q}$, show that the character ϵ in this case is rarely trivial, hence the same can be said about the sequence in (1.0.2.1) being $\mathbb{Z}G$-split.
The complexity that ϵ reflects on M suggests why the difficulties when computing $ker(f)$ could not be solved by the methods of our original project.

Chapter 2

Group Cohomology

This chapter contains results about group cohomology that will be used in later chapters such as: the Shapiro's isomorphism, restriction, corestriction and inflation maps and a brief presentation of spectral sequences.

We will follow for the first part [13], while the spectral sequence exposition will follow ([1] vol. 2).

2.1 Restriction, corestriction and inflation maps.

Let H and G be groups, $f : H \to G$ a group homomorphism and A a $\mathbb{Z}G$-module. A has a $\mathbb{Z}H$-module structure induced by f in the following way

$$h \cdot a = f(h) \cdot a \quad \text{for all} \quad h \in H. \tag{2.1.0.1}$$

A seen as a $\mathbb{Z}H$-module will be denoted by $f^*(A)$.

If a belongs to A^G and h is an element of H, it follows that $h \cdot a = f(h) \cdot a = a$ hence $A^G \subseteq (f^*A)^H$. Then there is a group homomorphism $H^0(G, A) \to H^0(H, f^*A)$. The fact that $A \mapsto H^q(H, f^*A)$, defines a cohomological δ-functor of the $\mathbb{Z}G$-module A, implies by the universal property of derived functors that $H^0(G, A) \to H^0(H, f^*A)$ extends to a morphism of cohomological δ-functors $\{H^q(G, A), \delta\} \to \{H^q(H, f^*A), \delta\}$. The homomorphism

$H^q(G, A) \to H^q(H, f^*A)$ induced by $f : H \to G$ will be denoted by f^q.

2.1.1 Example. If H is a subgroup of G and $i : H \to G$ is the natural inclusion, $i^q : H^q(G, A) \to H^q(H, i^*A)$ is called restriction map and will be denote $i^q = res^G_H$.

If B is a $\mathbb{Z}H$-module and $j : f^*A \to B$ is a $\mathbb{Z}H$-homomorphism: meaning that

$$h \cdot j(a) = j\left(f(gh) \cdot a\right), \tag{2.1.1.1}$$

for all h in H and a in A, then j defines a group homomorphism $j^q : H^q(H, f^*A) \to H^q(H, B)$ for all non-negative integers q. If $f : H \to G$ and $j : f^*A \to A$ satisfy (2.1.1.1) we say that (f, j) is a compatible pair and the composition

$$H^q(G, A) \xrightarrow{f^q} H^q(H, f^*A) \xrightarrow{j^q} H^q(H, B)$$

is denoted by $(f, j)^*_q$.

2.1.2 Example. Let N be a normal subgroup of G and let $H = G/N$. If A is a $\mathbb{Z}G$-module, then A^N is a $\mathbb{Z}H$-module. If we take $f : G \to H$ to be the natural surjection and $j : A^N \to A$ the inclusion it follows clearly that (f, j) is a compatible pair. We define the inflation map

$$inf = (f, j)^*_q : H^q(G/N, A^N) \to H^q(G, A). \tag{2.1.2.1}$$

2.1.3 Proposition. *Let N be a normal subgroup of G and A a $\mathbb{Z}G$-module, if $H^i(N, A) = 0$ for all $1 \leq i \leq q - 1$ it follows that the following is an exact sequence*

$$0 \to H^q(G/N, A^N) \xrightarrow{inf} H^q(G, A) \xrightarrow{res^G_N} H^q(N, A). \tag{2.1.3.1}$$

Let us now consider H to be a subgroup of G of finite index and A a $\mathbb{Z}G$-module. We can define the the norm map $N_{G/H} : A^H \to A^G$ in the following way: we fix X to be a left transversal of H in G then

$$N_{G/H}(a) = \sum_{x \in X} xa \tag{2.1.3.2}$$

for all a in A^H. It follows clearly that the map $N_{G/H}$ is independent of the choice of X.

2.1.4 Example. For a in A^H, $N_{G/H}(a)$ belongs to A^G implying that $N_{G/H}$ induces a homomorphism $H^0(H, A) \to H^0(G, A)$, which extends uniquely to a homomorphism $\{H^q(H, A)\} \to \{H^q(G, A)\}$ of cohomological δ-functors of $\mathbb{Z}G$-modules. This homomorphism is called corestriction and will be denoted by cor_H^G.

2.2 Shapiro's isomorphism.

We now study a particular case of Shapiro's isomorphism which will be used in later chapters. Let us recall that $M \otimes N$ (respectively $Hom(M, N)$) denotes $M \otimes_{\mathbb{Z}} N$ (resp $Hom_{\mathbb{Z}}(M, N)$) where G acts diagonally. It should be mentioned that in this section H^* will denote regular cohomology while $\hat{H}^*$ denotes Tate cohomology.

Let H be a subgroup of G, $i : H \to G$ the natural inclusion and B a $\mathbb{Z}H$-module. Let us denote by $B^\sharp$ the $\mathbb{Z}G$-module $Hom_{\mathbb{Z}H}(\mathbb{Z}G, B)$, where the action of G is given by

$$(g \cdot \phi)(x) = \phi(x \cdot g)$$

for every ϕ in $B^\sharp$ and g, x in G.
We define $\Theta_B : B^\sharp \to B$ by $\Theta_B(\phi) = \phi(1_G)$. Notice that for all h in H

$$\Theta_B(h\phi) = (h\phi)(1_G) = \phi(h) = h \cdot \phi(1_G) = h\Theta_B(\phi), \tag{2.2.0.1}$$

then Θ_B is a $\mathbb{Z}H$-homomorphism and (i, Θ_B) is a compatible pair which induces the homomorphism

$$(i, \Theta_B)^*_q : H^q(G, B^\sharp) \to H^q(H, B).$$

It is a well know fact that $\mathbb{Z}G$ is a $\mathbb{Z}H$-free module then it is isomorphic to $\mathbb{Z}H \otimes M$ for some $\mathbb{Z}$-free module M. By duality

$$Hom(\mathbb{Z}G, X) \cong Hom(\mathbb{Z}H \otimes M, X) \cong Hom(\mathbb{Z}H, Hom(M, X))$$

for any $\mathbb{Z}$-module X.

2.2.1 Proposition. *1. The homomorphism $(i, \Theta_B)^*_q : \hat{H}^q(G, B^\sharp) \to \hat{H}^q(H, B)$ is an isomorphism called inverse Shapiro's isomorphism, which we will denote by $(sh_B^{-1})^q$.*

2. Define $i_B : B \to B^\sharp$ by $i_B(b) = \phi_b$, where

$$\phi_b(x) = \begin{cases} xb & \text{if } x \in H \\ 0 & \text{if } x \notin H, \end{cases}$$

then i_B is a $\mathbb{Z}H$-homomorphism such that the composite

$$\hat{H}^q(H, B) \xrightarrow{i_B^q} \hat{H}^q(H, B^\sharp) \xrightarrow{cor_H^G} \hat{H}^q(G, B^\sharp)$$

is the inverse of $(sh_B^{-1})^q$, which we will denote by sh_B^q.

Proof. 1. When $q = 0$: let ϕ be an element of $B^\sharp$ then, ϕ belongs to $(B^\sharp)^G$ is equivalent to say that $\phi(gx) = \phi(x)$ for all g, x in G, this implies that

$$\phi(g) = \phi(1_G) = \Theta_B(\phi). \qquad (2.2.1.1)$$

The last equality shows that if h belongs to H

$$h\Theta_B(\phi) = h\phi(1_G) = \phi(h) = \phi(1_G) = \Theta_B(\phi),$$

hence $\Theta_B(\phi)$ is an element of B^H.The fact that $\phi(x) = \phi(1_G)$ for all x in G implies that $(i, \Theta_B)^*_0$ is injective.
Surjectivity can be proved by taking b in B^H and letting ϕ_b be the element of $B^\sharp$ defined by: $\phi_b(x) = b$ for any x in G. Then (2.2.1.1)

implies that $\phi_b \in (B^\sharp)^G$ and $\theta_B(\phi_b) = b$.

The above shows that $(sh_B^{-1})^0$ is an isomorphism from $H^0(G, B^\sharp)$ to $H^0(H, B)$.

The same statement is true for Tate cohomology since $\Theta_B(N_G(B^\sharp)) = N_H(B)$. In order to prove this last equality let X be a right transversal of H in G, then for ϕ in $B^\sharp$ it follows that

$$\begin{aligned}\Theta_B(N_G(\phi)) &= \Theta_B\left(\sum_{g\in G} g\phi\right) = \sum_{g\in G} g\phi(1_g)\\ &= \sum_{g\in G}\phi(g) = \sum_{x\in X}\sum_{h\in H}\phi(hx)\\ &= \sum_{x\in X}\sum_{h\in H} h\phi(x) = \sum_{x\in X} N_H(\phi(x)).\end{aligned}$$

It is clear that for any b in B, $\Theta_B(N_G(i_B(b))) = N_H(b)$.

In order to extend to the case $q \geq 1$ one can easily show that: if B is induced for H, meaning that there is a $\mathbb{Z}$-module M with $B \cong \mathbb{Z}H \otimes M$ then $B^\sharp \cong \mathbb{Z}G \otimes M$. We embed B into an induced module C, if $f : B \to C$ is such an imbedding and we let $D = coker(f)$ one obtains the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B^\sharp & \xrightarrow{f^*} & C^\sharp & \longrightarrow & D^\sharp & \longrightarrow & 0\\ & & \Theta_B\big\downarrow & & \Theta_C\big\downarrow & & \Theta_D\big\downarrow & & \\ 0 & \longrightarrow & B & \underset{f}{\longrightarrow} & C & \longrightarrow & D & \longrightarrow & 0 \end{array} \tag{2.2.1.2}$$

where the top and bottom rows are exact. Commutativity of the left square follows since

$$\Theta_C f^*(\phi) = \Theta_C(f\phi) = f\phi(1_G) = f(\Theta_B(\phi)),$$

for any ϕ in $B^\sharp$. Commutativity of the right square follows by a similar argument. We now apply H-cohomology to diagram (2.2.1.2) to obtain the commutative diagram

$$\begin{array}{ccc} \hat{H}^{q-1}(H,D^\sharp) & \xrightarrow{(\Theta_D)_*} & \hat{H}^{q-1}(H,D) \\ \partial\downarrow\cong & & \partial\downarrow\cong \\ \hat{H}^{q}(H,B^\sharp) & \xrightarrow{(\Theta_B)_*} & \hat{H}^{q}(H,B), \end{array} \tag{2.2.1.3}$$

we complete the last diagram by applying res^G_H and obtain

$$\begin{array}{ccccc} \hat{H}^{q-1}(G,D^\sharp) & \xrightarrow{res^G_H} & \hat{H}^{q-1}(H,D^\sharp) & \xrightarrow{(\Theta_D)_*} & \hat{H}^{q-1}(H,D) \\ \partial\downarrow\cong & & \partial\downarrow\cong & & \partial\downarrow\cong \\ \hat{H}^{q}(G,B^\sharp) & \xrightarrow{res^G_H} & \hat{H}^{q}(H,B^\sharp) & \xrightarrow{(\Theta_B)_*} & \hat{H}^{q}(H,B), \end{array} \tag{2.2.1.4}$$

the composition $(\Theta_D)_* \circ res^G_H$ on the top row of diagram (2.2.1.4) is an isomorphism by induction on q, hence the composition $(\Theta_B)_* \circ res^G_H = (sh_B^{-1})^q$ is also an isomorphism.

2. A straightforward computation shows that $h\phi_b(x) = \phi_{hb}(x)$ for all x in G, h in H and b in B, which proves that i_B is a $\mathbb{Z}H$-homomorphism. In order to prove that $cor^G_H \circ (i_B)_*$ is the inverse of the "inverse Shapiro's isomorphism" we start by showing that the statement is true for $q = 0$. In this case is enough to show that the composition

$$B^H \xrightarrow{i_B} (B^\sharp)^H \xrightarrow{N_{G/H}} (B^\sharp)^G \rightarrowtail (B^\sharp)^H \xrightarrow{\Theta_B} B^H$$

is the identity in B^H. Let X be a fixed left transversal of H in G and b an element of B^H, without losing generality we can assume that $1_G \in X$ then

$$\begin{aligned} \Theta_B(N_{G/H}(i_B(b))) = \Theta_B(N_{G/H}(\phi_b)) &= \Theta_B\left(\sum_{x\in X} x\phi_B\right) \\ &= \sum_{x\in X} x\phi_b(1_G) = \sum_{x\in X}\phi_b(x) = b. \end{aligned}$$

The statement also follows clearly for Tate cohomology. The extension to $q \geq 1$ is obtained by a similar argument as in (1).

□

2.2.2 Claim. *Let H be a subgroup of G and B a $\mathbb{Z}G$-module. Let us define $j_B : B^\sharp \to Hom(ind_H^G\mathbb{Z}, B)$ by $j_B(\phi) := \tilde{\phi}$, where $\tilde{\phi}(x \otimes t) = tx\phi(x^{-1})$: it should be mention that we are assuming G acting on $Hom(ind_H^G\mathbb{Z}, B)$ by diagonal action, then j_B is a $\mathbb{Z}G$-isomorphism.*

Proof. It follows that j_B is well defined since

$$\tilde{\phi}(h \otimes t) = th\phi(h^{-1}) = t\phi(1_G) = \tilde{\phi}(1_G \otimes t)$$

for all h in H and t in $\mathbb{Z}$.
Let g, x be in G, then

$$\begin{aligned}(\tilde{g\phi})(x \otimes t) &= tx(g\phi)(x^{-1}) = tx\phi(x^{-1}g)\\ &= tx\phi((g^{-1}x)^{-1}) = tg(g^{-1}x)\phi((g^{-1}x)^{-1})\\ &= g\tilde{\phi}(g^{-1}x \otimes t) = g\tilde{\phi}(g \cdot (x \otimes t))\\ &= (g\tilde{\phi})(x \otimes t).\end{aligned}$$

This proofs that j_B is a $\mathbb{Z}G$-homomorphism.
We define $j'_B : Hom(ind_H^G, B) \to B^\sharp$ by $j_B(\psi) = \bar{\psi}$, where $\bar{\psi}(x) = x\psi(x^{-1} \otimes 1)$. Proving that j'_B is well defined and that is a $\mathbb{Z}G$-homomorphism can be done in a similar way as done for j_B. If ϕ belongs to $B^\sharp$ it follows that

$$(j'_B j_B)(\phi)(x) = x(j_B\phi)(x^{-1} \otimes 1) = \phi(x),$$

for all x in G.
Now let ψ be an element of $Hom(ind_H^G\mathbb{Z}, B)$, it follows that

$$(j_B j'_B)(\psi)(x \otimes 1) = x(j'_B\psi)(x^{-1}) = \psi(x \otimes 1)$$

which proofs that j'_B is the inverse of j_B. □

Let us denote $i'_B := j_B i_B : B \to Hom(ind_H^G\mathbb{Z}, B)$, hence for b in B $i'_B(b) = \tilde{\phi}_b$ and

$$\tilde{\phi}_b(x \otimes 1) = x\phi_b(x^{-1}) = \begin{cases} b & \text{if } \ x \in H \\ 0 & \text{if } \ x \notin H \end{cases}$$

We will denote by $Sh_B^q = (j_B)_*(sh_B^q)$.
It is not difficult to prove that the following is a commutative diagram

$$\begin{array}{ccccc} & & \hat{H}^q(H,B^*) & \xrightarrow{cor_H^G} & \hat{H}^q(G,B^*) \\ & \nearrow^{(i_B)^*} & \Big\downarrow{\scriptstyle (j_B)_*} & & \Big\downarrow{\scriptstyle (j_B)_*} \\ \hat{H}^q(H,B) & & & & \\ & \searrow_{(i'_B)_*} & & & \\ & & \hat{H}^q(H,Hom(ind_H^G\mathbb{Z},B)) & \xrightarrow[cor_H^G]{} & \hat{H}^q(G,Hom(ind_H^G\mathbb{Z},B)) \end{array}$$

2.2.3 Claim. *The homomorphism* $Sh_B^0 : \hat{H}^0(H,B) \to \hat{H}^0(G,Hom(ind_H^G\mathbb{Z},B))$ *takes* b *in* B^H *to the map* $g \otimes 1 \mapsto gb$ *for all* g *in* G.

Proof. Let b be an element in B^H and X a left transversal of H in G then we obtain

$$\begin{aligned} Sh_B^0(b)(g\otimes 1) &= N_{G/H}(i'_B(b))(g\otimes 1) = \left(\sum_{x\in X} x\tilde{\phi}_b\right)(g\otimes 1) \\ &= \sum_{x\in X} x\tilde{\phi}_b(x^{-1}g) = x_0 b, \end{aligned}$$

where x_0 is the only element in X such that $x_0^{-1}g \in H$ hence $g = x_0h$ for some h in H, which implies $gb = x_0hb = x_0b$. □

2.2.4 Example. Let M and N be $\mathbb{Z}G$-module and G act diagonally on $Hom(M,N)$ then

$$Sh^0_{Hom(M,N)} : \hat{H}^0(H,Hom(M,N)) \to \hat{H}^0(G,Hom(ind_H^G\mathbb{Z},Hom(M,N)))$$

sends ϕ to $\tilde{\phi}$ where $\tilde{\phi}(g\otimes 1) = g\phi$. If x is an element of M it follows that

$$Sh^0_{Hom(M,N)}(\phi)(g\otimes 1)(x) = g\phi(x) = g\phi(g^{-1}x).$$

Let L be a $\mathbb{Z}G$-module and $\psi : Hom(L,Hom(M,N)) \to Hom(L\otimes M,N)$ be the dual isomorphism, which means that for any $f : L \to Hom(M,N)$ $\psi(f)(l\otimes m) = f(l)(m)$. It is not difficult to prove that ψ is a $\mathbb{Z}G$-isomorphism where we are considering G acting diagonally on both $Hom(L, Hom(M, N))$ and $Hom(L \otimes M, N)$. Let us denote

$$Sh^q_{(L,M,N)} = \psi_* Sh^q_{Hom(M,N)} \tag{2.2.4.1}$$

2.2.5 Example. We now consider the augmentation map $\acute{\imath}_0 : \mathbb{Z}G \to \mathbb{Z}$, where every g in G is sent to 1 and denote by $\Delta G = ker(\acute{\imath}_0)$. Let $L = ind_H^G \mathbb{Z}$ and $M = \Delta G$ then

$$Sh^q_{(ind_H^G \mathbb{Z}, \Delta G, N)} : \hat{H}^q(H, Hom(\Delta G, N)) \to \hat{H}^q(G, Hom(ind_H^G \mathbb{Z} \otimes \Delta G, N))$$

at $q = 0$ satisfies

$$\begin{aligned} Sh^0_{(ind_H^G \mathbb{Z}, \Delta G, N)}(\phi)(g \otimes 1 \otimes x) &= \psi_* Sh^0_{Hom(\Delta G, N)}(\phi)(g \otimes 1 \otimes x) \\ &= Sh^0_{Hom(\Delta G, N)}(\phi)(g \otimes 1)(x) = g\phi(g^{-1}x) \end{aligned} \tag{2.2.5.1}$$

for all g in G and x in ΔG.
A straightforward computation shows that $(Sh^0_{(ind_H^G \mathbb{Z}, \Delta G, N)})^{-1}$ applied to ϕ in $Hom(ind_H^G \mathbb{Z} \otimes \Delta G, N)$ is given by

$$(Sh^0_{(ind_H^G \mathbb{Z}, \Delta G, N)})^{-1}(\phi)(x) = \phi(1_G \otimes 1)(x). \tag{2.2.5.2}$$

2.2.6 Remark. It follows by construction that: by fixing L and M

$$Sh^q_{(L,M,-)} : \{\mathbb{Z}G - \text{modules}\} \to \{\text{Groups}\}$$

is a covariant functor sending N to $\hat{H}^q(G, Hom(L \otimes M, N))$.

2.3 Spectral sequences.

For this concise exposition of spectral sequences we will follow ([1] vol.2).
We begin by constructing the spectral sequence of a filtered chain complex, then present the spectral sequence of a double complex and apply this to obtain the Hochschild-Serre spectral sequence associated to a group extension.

2.3.1 Spectral sequence induced by a filtered chain complex.

Let $\{X, \delta\}$ be a cochain complex, which means that $X = \bigoplus_{n\in\mathbb{Z}} X_n$ and δ is a family of maps

$$\cdots \xrightarrow{\delta_{n-1}} X_n \xrightarrow{\delta_n} X_{n+1} \xrightarrow{\delta_{n+1}} X_{n+2} \xrightarrow{\delta_{n+2}} \cdots$$

satisfying $\delta_{n+1} \circ \delta_n = 0$. We will denote by $H^n(X) = ker(\delta_i)/im(\delta_{i-1})$.
Lets us assume that there is a filtration of $\{X, \delta\}$ given by

$$X = F^0X \supseteq F^1X \supseteq \cdots \supseteq \bigcap_i F^iX = \{0\}, \qquad (2.3.1.1)$$

where each $\{F^iX, \delta^i\}$ (δ^i being the restriction of δ to F^iX) is a subcomplex of $\{X, \delta\}$. Let us denote by $\bar{\delta}^i : F^iX/F^{i+1}X \to F^iX/F^{i+1}X$ the natural map induced by δ^i. It follows clearly that $\left\{F^iX/F^{i+1}X, \bar{\delta}^i\right\}$ is a cochain complex. For each $i \geq 0$ the short exact sequence

$$0 \longrightarrow F^{i+1}X \xrightarrow{i^1} F^1X \xrightarrow{j^1} F^iX/F^{i+1}X \longrightarrow 0$$

induces long exact sequences of the form

$$\cdots \to H^n(F^{i+1}X) \xrightarrow{i^1_*} H^n(F^iX) \xrightarrow{j^1_*} H^n(F^iX/F^{p+1}X) \xrightarrow{\partial^1} H^{n+1}(F^{i+1}X) \to \cdots$$

All this sequences can be arranged in the following diagram

$$
\begin{array}{ccccccc}
& \downarrow & & \downarrow & & & \\
\xrightarrow{\partial^1} & H^{n-1}(F^{p+1}X) & \to H^{n-1}(F^{p+1}X/F^{p+2}X) \to & H^n(F^{p+2}X) & \to & H^n(F^{p+2}X/F^{p+3}X) & \to \\
& {\scriptstyle i^1_*}\downarrow & & \downarrow & & & \\
\longrightarrow & H^{n-1}(F^pX) & \xrightarrow{j^1_*} H^{n-1}(F^pX/F^{p+1}X) \xrightarrow{\partial^1} & H^n(F^{p+1}X) & \to & H^n(F^{p+1}X/F^{p+2}X) & \to \\
& \downarrow & & {\scriptstyle i^1_*}\downarrow & & & \\
\longrightarrow & H^{n-1}(F^{p-1}X) & \to H^{n-1}(F^{p-1}X/F^pX) \longrightarrow & H^n(F^pX) & \xrightarrow{j^1_*} & H^n(F^pX/F^{p+1}X) & \longrightarrow \\
& \downarrow & & \downarrow & & &
\end{array}
\tag{2.3.1.2}
$$

Letting $E_1^{p.q} := H^{p+q}(F^pX/F^{p+1}X)$ and $D_1^{p,q} := H^{p+q}(F^pX)$ one obtains the following exact triangle

$$
\begin{array}{ccc}
D_1^{*,*} & \xrightarrow{i^1_*} & D_1^{*,*} \\
{\scriptstyle \partial^1}\nwarrow & & \swarrow{\scriptstyle j^1_*} \\
& E_1^{*,*} &
\end{array}
\tag{2.3.1.3}
$$

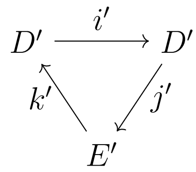
The above diagram is an example of an exact couple.

2.3.2 Definition. *An exact couple $\{D, E, i, j, k\}$ is an exact triangle of the form*

$$
\begin{array}{ccc}
D & \xrightarrow{i} & D \\
{\scriptstyle k}\nwarrow & & \swarrow{\scriptstyle j} \\
& E &
\end{array}
\tag{2.3.2.1}
$$

Given an exact couple $\{D, E, i, j, k\}$ define $d = jk : E \to E$, the fact that $kj = 0$ implies that $d^2 = 0$ hence we can form $H^*(E, d)$.

2.3.3 Definition. *If* $\{D, E, i, j, k\}$ *is an exact couple, its derived couple* $\{D', E', i', j', k'\}$



(2.3.3.1)

is given by $D' = im(i)$ *and* $E' = H^*(E, d)$ *where:* i' *is the restriction of* i *to* D', $j'(i(x)) = [j(x)]$ *and* $k'[x] = k(x)$.

It can be proved that the derived couple defined above is actually an exact couple. By iterating this process one obtains the spectral sequence induced by the exact couple $\{D, E, i, j, k\}$ as follows.

2.3.4 Definition. *Given an exact couple* $\{D, E, i, j, k\}$ *we define the exact couple* $\{D_n, E_n, i^n, j^n, k^n\}$ *as the* $(n-1)$*th-derived couple of the original exact couple. The sequence* $\{(E_n, d^n)\}_{n\geq 1}$ *is called the spectral sequence associated to* $\{D, E, i, j, k\}$.

If we consider $\{D, E, i, j, k\}$ where D and E are objects in the category of doubly graded modules and assume that

$$\begin{aligned} deg(i) &= (-1, 1) \\ deg(j) &= (0, 0) \\ deg(k) &= (1, 0). \end{aligned}$$

It follows that $deg(d) = deg(jk) = deg(k) + deg(j) = (1, 0)$, it can also be proved that for all $n \geq 1$.

$$\begin{aligned} deg(i^n) &= deg(i^{n-1}) = (-1, 1) \\ deg(j^n) &= deg(j^{n-1}) - deg(i^{n-1}) = (n-1, 1-n) \\ deg(k^n) &= deg(k^{n-1}) = (1, 0) \\ deg(d^n) &= deg(k^n) + deg(j^n) = (n, 1-n). \end{aligned}$$

Considering the filtration of the chain complex X given in (2.3.1.1), we would like to mention how does the spectral sequence induced by the exact couple given in (2.3.1.3) actually gives information about $H^*(X)$.

Note that each $D^{p,q}_{n+1} \subseteq D^{p,q}_n$ for all $n \geq 1$. Let $D^{p,q}_\infty = \cap_n D^{p,q}_n$, on the other hand $E^{p,q}_2$ is a subquotient of $E^{p,q}_1$ this implies that there are subgroups $Z^{p,q}_2$ and $B^{p,q}_2$ of $E^{p,q}_1$ with $Z^{p,q}_2 = ker(d^1)$ and $B^{p,q}_2 = im(d^1)$ such that $E^{p,q}_2 = Z^{p,q}_2/B^{p,q}_2$. We can continue this process and find subgroups $Z^{p,q}_i$ and $B^{p,q}_i$ of $E^{p,q}_1$ with $Z^{p,q}_i/B^{p,q}_{i-1} = ker(d^{i-1})$ and $B^{p,q}_i/B^{p,q}_{i-1} = im(d^{i-1})$ satisfying $E^{p,q}_i \cong Z^{p,q}_i/B^{p,q}_i$. If we set $Z^{p,q}_1 = E^{p,q}_1$ and $B^{p,q}_1 = \{0\}$ one obtains

$$E^{p,q}_1 = Z^{pq,}_1 \supseteq Z^{p,q}_2 \supseteq Z^{p,q}_3 \supseteq \cdots \supseteq B^{p,q}_3 \supseteq B^{p,q}_2 \supseteq B^{p,q}_1 = \{0\}\,. \qquad (2.3.4.1)$$

Define $Z^{p,q}_\infty = \cap_n Z^{p,q}_n$, $B^{p,q}_\infty = \cup_n B^{p,q}_n$ and $E^{p,q}_\infty = Z^{p,q}_\infty/B^{p,q}_\infty$.

The group $H^*(X)$ itself has a "canonical" filtration induced by (2.3.1.1) namely

$$H^*(X) = F^0H^*(X) \supseteq F^1H^*(X) \supseteq \cdots\,,$$

where $F^pH^{p+q}(X) = im\left(H^{p+q}(F^pX) \to H^{p+q}(X)\right)$.
2.3.5 Theorem. *With the notation given above it follows that*

$$F^pH^{p+q}(X)/F^{p+1}H^{p+q}(X) \cong E^{p,q}_\infty. \qquad (2.3.5.1)$$

The proof of Theorem 2.3.5 can be found in [1].
If one assumes that $H^n(F^{n+1}X) = \{0\}$ then $H^n(X)$ has a finite filtration

$$H^n(X) = F^0H^n(X) \subseteq F^1H^n(X) \subseteq \cdots \subseteq F^nH^n(X) \subseteq F^{n+1}H^n(X) = \{0\}\,.$$

2.3.6 Remark. From the filtration given above one can conclude that finding a set of generators for $H^n(X)$ can be reduced, by Theorem 2.3.5, to obtain a set of generators of $E^{p,q}_\infty$ for all p satisfying $0 \leq p \leq n$ and $p + q = n$. Since $E^{p,q}_2$ surjects onto $E^{p,q}_\infty$ one can restrict to find generators for the groups $E^{p,q}_2$ with $p + q = n$.

2.3.7 Spectral sequence induced by a double complex.

The difficulty of computing spectral sequences depends highly on how much control one has over $E_2^{p,q}$. In this section we study the spectral sequence induced by a double complex, this is a particular case in which the terms $E_2^{p,q}$ have concrete expression. The Hochschild-Serre spectral sequence is a particular example of this kind of spectral sequences.

2.3.8 Definition. *A double complex is a collection of modules and module homomorphisms arranged as follows*

$$\begin{array}{ccccccccc}
\vdots & & \vdots & & \vdots & & \vdots & & \\
\partial_0\uparrow & & \partial_0\uparrow & & \partial_0\uparrow & & \partial_0\uparrow & & \\
E_0^{0,2} & \underset{\partial_1}{\longrightarrow} & E_0^{1,2} & \underset{\partial_1}{\longrightarrow} & E_0^{2,2} & \underset{\partial_1}{\longrightarrow} & E_0^{3,2} & \underset{\partial_1}{\longrightarrow} & \cdots \\
\partial_0\uparrow & & \partial_0\uparrow & & \partial_0\uparrow & & \partial_0\uparrow & & \\
E_0^{0,1} & \underset{\partial_1}{\longrightarrow} & E_0^{1,1} & \underset{\partial_1}{\longrightarrow} & E_0^{2,1} & \underset{\partial_1}{\longrightarrow} & E_0^{3,1} & \underset{\partial_1}{\longrightarrow} & \cdots \\
\partial_0\uparrow & & \partial_0\uparrow & & \partial_0\uparrow & & \partial_0\uparrow & & \\
E_0^{0,0} & \underset{\partial_1}{\longrightarrow} & E_0^{1,0} & \underset{\partial_1}{\longrightarrow} & E_0^{2,0} & \underset{\partial_1}{\longrightarrow} & E_0^{3,0} & \underset{\partial_1}{\longrightarrow} & \cdots
\end{array} \tag{2.3.8.1}$$

such that the maps satisfy $\partial_0^2 = \partial_1^2 = \partial_0\partial_1 + \partial_1\partial_0 = 0$. *We will denote the double complex by* $\{E_0, \partial_0, \partial_1\}$.

Given a double complex $\{E_0, \partial_0, \partial_1\}$ one can construct a graded $\mathbb{Z}$-module which is call the total complex of E_0 denoted by $X = Tot(E_0)$, where $X^n = \bigoplus_{p+q=n} E_0^{p,q}$ with differential $\delta = \partial_0 + \partial_1 : X^n \to X^{n+1}$. By definition it follows that $\delta^2 = 0$, then $\{Tot(E_0), \delta\}$ is a cochain complex.

We define a filtration of the cochain complex $X = Tot(E_0)$ by

$$F^pX = \bigoplus_{i\geq p} E_0^{p,q}$$

and denote by

$$D_0^{p,q} := F^pX^{p+q} = \bigoplus_{\substack{i+j=p+q,\\ i\geq p}} E_0^{i,j}.$$

Notice that $F^pX^{p+q}/F^{p+1}X^{p+q} \cong E_0^{p,q}$, in particular one can think of $F^pX/F^{p+1}X$ as the pth-column of diagram (2.3.8.1). It follows that the differential in this quotient is induced by ∂_0 hence

$$E_1^{p,q} = H^{p+q}(F^pX/F^{p+1}X) \cong H(E_0^{p,q}, \partial_0) \tag{2.3.8.2}$$
$$D_1^{p,q} = H^{p+q}(F^pX) = H(D_0^{p,q}, \partial_0 + \partial_1). \tag{2.3.8.3}$$

If we identify $F^pX^{p+q}/F^{p+1}X^{p+q}$ with $E_0^{p,q}$, the following short exact sequence

$$0 \longrightarrow F^{p+1}X^{p+q} \longrightarrow F^pX^{p+q} \longrightarrow E_0^{p,q} \longrightarrow 0$$

induces the following exact couple

$$\begin{array}{ccc} D_1^{**} & \xrightarrow{\ i_1\ } & D_1^{**} \\ {\scriptstyle k_1}\nwarrow & & \swarrow{\scriptstyle j_1} \\ & E_1^{**} & \end{array}$$

In order to describe E_2 we must first compute $d_1 = j_1k_1$.
Taking into consideration the following commutative diagram with exact rows

$$\begin{array}{ccccc} D_0^{p+1,q-1} & \xrightarrow{\ \alpha_{p,q}\ } & D_0^{p,q} & \xrightarrow{\ \beta_{p,q}\ } & E_0^{p,q} \\ \downarrow{\scriptstyle \partial_0+\partial_1} & & \downarrow{\scriptstyle \partial_0+\partial_1} & & \downarrow{\scriptstyle \partial_0+\partial_1} \\ D_0^{p+1,q} & \xrightarrow[\ \alpha_{p,q+1}\ ]{} & D_0^{p,q+1} & \xrightarrow[\ \beta_{p,q+1}\ ]{} & E_0^{p,q+1} \end{array} \tag{2.3.8.4}$$

with $\alpha_{p,q} : D_0^{p+1,q-1} \to D_0^{p,q}$ and $\beta_{p,q} : D_0^{p,q} \to E_0^{p,q}$ the natural inclusion and projection respectively.
An element $[x]$ in $E_1^{p,q}$ is the class of $x \in E_0^{p,q}$ with $\partial_0 x = 0$. Let $(x, 0, \ldots, 0) \in D_0^{p,q}$ then $\beta_{p,q}(x, 0, \ldots, 0) = x$. The fact that $\partial_0 x = 0$ implies that $(\partial_0 + \partial_1)(x, 0, \ldots, 0) = (0, \partial_1 x, \ldots, 0) \in D_0^{p,q+1}$. Let $(\partial_1 x, 0, \ldots, 0) \in D_0^{p+1,q}$ clearly $\alpha_{p,q+1}(\partial_1 x, 0, \ldots, 0) = (0, \partial_1 x, \ldots, 0)$ hence

$$k_1[x] = [(\partial_1 x, 0, \ldots, 0)]$$
$$d_1[x] = j_1k_1[x] = j_1[(\partial_1 x, 0, \ldots, 0)] = [\partial_1 x].$$

From this we conclude that for the double complex in (2.3.8.1), the second term of the spectral sequence induced by the exact couple given in (2.3.8.2) has the following expression.

$$E_2^{p,q} = H^p(H^q(E_0, d_0), d_1). \tag{2.3.8.5}$$

2.3.9 The spectral sequence of a group extension.

We are now in position to present the Hochschild-Serre spectral sequence which will be use in chapter 6 to find a set of generators for $H^2(G_{\mathfrak{P}}, \boldsymbol{\mu})$.

Let G be a group, H a normal subgroup of G and M a $\mathbb{Z}G$-module. Let us denote by $\bar{G} = G/H$. Fixing a $\mathbb{Z}\bar{G}$-projective resolution of $\mathbb{Z}$

$$\cdots \xrightarrow{\delta_3} P_2 \xrightarrow{\delta_2} P_1 \xrightarrow{\delta_1} P_0 \xrightarrow{\delta_0} \mathbb{Z}$$

and a $\mathbb{Z}G$-projective resolution of $\mathbb{Z}$

$$\cdots \xrightarrow{\delta_3'} Q_2 \xrightarrow{\delta_2'} Q_1 \xrightarrow{\delta_1'} Q_0 \xrightarrow{\delta_0'} \mathbb{Z}.$$

We can construct a double complex E_0 by letting

$$E_0^{i,j}; = Hom_{\bar{G}}(P_i, Hom_H(Q_j, M)),$$

where the differentials $\partial_0 : E_0^{i,j} \to E_0^{i,j+1}$ and $\partial_1 : E_0^{i,j} \to E_0^{i+1,j}$ are defined by $\partial_0 = (-1)^p(\delta_{j+1}')^*$ and $\partial_1 = \delta_i^*$. A simple computation shows that $\partial_0^2 = \partial_1^2 = \partial_0\partial_1 + \partial_1\partial_0 = 0$.
By (2.3.8.5) one obtains that the second term of the spectral sequence induced by the double complex E_0 defined above is

$$E_2^{p,q} = H^p(\bar{G}, H^q(H, M)).$$

We conclude this exposition with the computation of $H^*(Tot(E_0))$.
2.3.10 Claim. *Let G, H, $\bar{G}$ and M as above. Let A be a $\mathbb{Z}\bar{G}$-module and B*

a $\mathbb{Z}G$-module then

$$Hom_{\mathbb{Z}\bar{G}}(A, Hom_{\mathbb{Z}H}(B, M)) \cong Hom_{\mathbb{Z}G}(A \otimes B, M)$$

Proof. Let us define the maps

$$\begin{aligned}\gamma &: Hom_{\mathbb{Z}\bar{G}}(A, Hom_{\mathbb{Z}H}(B, M)) \to Hom_{\mathbb{Z}G}(A \otimes B, M)\\ \beta &: Hom_{\mathbb{Z}G}(A \otimes B, M) \to Hom_{\mathbb{Z}\bar{G}}(A, Hom_{\mathbb{Z}H}(B, M))\end{aligned}$$

by: $(\gamma f)(a \otimes b) = f(a)(b)$ and $(\beta f)(a)(b) = f(a \otimes b)$.
It follows immediately that γ is well defined. In order to show that β is also well defined we choose a $\mathbb{Z}G$-homomorphism $f : A \otimes B \to M$ and $g \in G$, let $\bar{g}$ denote the image of g under the natural projection $G \to \bar{G}$, then for any $a \in A$ and $b \in B$

$$\begin{aligned}(\beta f)(\bar{g} \cdot a)(b) &= f(\bar{g} \cdot a \otimes b)\\ &= gf(a \otimes g^{-1}b)\\ &= g(\beta f)(a)(g^{-1}b)\\ &= \bar{g} \cdot (\beta f)(a)(b).\end{aligned}$$

The last equality shows that $(\beta f)(\bar{g} \cdot a) = \bar{g} \cdot (\beta f)(a)$ hence βf is a $\mathbb{Z}\bar{G}$-homomorphism. It remains to show that $\beta f(a)$ is a $\mathbb{Z}H$-homomorphism, for this let $h \in H$ then

$$\begin{aligned}(\beta f)(a)(hb) &= f(a \otimes hb)\\ &= f(h \cdot a \otimes b)\\ &= hf(a \otimes b)\\ &= h(\beta f)(a)(b).\end{aligned}$$

It follows clearly that γ and β are inverse maps from each other, which concludes the proof. □

Let $\tilde{E} = \bigoplus_{i,j} P_i \otimes Q_j$ and $\tilde{E}^n = \bigoplus_{i+j=n} P_i \otimes Q_j$. If we define $\tilde{\delta}_n : \tilde{E}_n \to \tilde{E}_{n+1}$ such that for any $x \in P_i$ and $y \in Q_j$

$$\tilde{\delta}_n(x \otimes y) = \delta_i x \otimes y + (-1)^i x \otimes \delta'_j y,$$

then by corollary 2.7.3 of ([1] vol 1) it follows that

$$\cdots \xrightarrow{\tilde{\delta}_3} \tilde{E}^2 \xrightarrow{\tilde{\delta}_2} \tilde{E}^1 \xrightarrow{\tilde{\delta}_1} \tilde{E}^0 \xrightarrow{\tilde{\delta}_0} \mathbb{Z}$$

is a $\mathbb{Z}G$-projective resolution of $\mathbb{Z}$.

2.3.11 Claim. *Let X^n be defined as in the previous subsection. The following diagram commutes*

$$\begin{array}{ccc} Hom_{\mathbb{Z}G}(\tilde{E}^n, M) & \xrightarrow{\tilde{\delta}^*_{n+1}} & Hom_{\mathbb{Z}G}(\tilde{E}^{n+1}, M) \\ \beta \downarrow & & \downarrow \beta \\ X^n & \xrightarrow[d=\partial_0+\partial_1]{} & X^{n+1} \end{array}$$

where β is the isomorphism of claim (2.3.10), hence β induces an isomorphism

$$\beta_* : H^*(G, M) \to H^*(Tot(E_0)).$$

The proof of Claim 2.3.11 is omitted here. Let $\gamma^{i,j}$ be the isomorphism given in Claim 2.3.10 with $A = P_i$ and $B = Q_j$ then $\gamma^{i,j}$ induces a natural homomorphism

$$\gamma^{i,j}_* : E^{i,j}_2 \to H^{i+j}(G, M). \qquad (2.3.11.1)$$

It follows: by Theorem 2.3.5 and the observation after it that

$$\Gamma = \bigoplus_{i+j=n} \gamma^{i,j}_* : \bigoplus_{i+j=n} E^{i,j}_2 \to H^n(G, M), \qquad (2.3.11.2)$$

is a surjection. This last observation show that the spectral sequence associated to the group extension

$$H \rightarrowtail G \twoheadrightarrow \bar{G}$$

induces a filtration of $H^n(G, M)$ given by the groups $H^{n-i}(\bar{G}, H^i(H, M))$ for $0 \leq i \leq n$.

Chapter 3

Construction of a ”small” envelope of $\boldsymbol{\mu}$.

Let us consider K/k to be a finite Galois extension of number fields with $G = Gal(K/k)$. We denote by $\boldsymbol{\mu} := \boldsymbol{\mu}_K$ the group of roots of unity of K and by $\bar{K} := k(\boldsymbol{\mu})$. It follows that $\boldsymbol{\mu}$ is cyclic of even order. We will assume that $|\boldsymbol{\mu}| = 2m$ for some positive integer m.
In this chapter we construct a ”small” envelope of $\boldsymbol{\mu}$, by ”small” we mean that $dim_{\mathbb{Q}}(\mathbb{Q} \otimes C) = |G|d(G)$ where $d(G)$ is the minimal number of generators of G.
The construction presented here follows the idea of [9] where the existence of envelopes of finitely generated $\mathbb{Z}G$-modules is proved.

There is an isomorphism of groups $\bar{a} : G \to (\mathbb{Z}/2m\mathbb{Z})^{\times}$ defined by

$$g \cdot u = u^{\bar{a}(g)} \quad \text{for all} \quad g \in G \quad \text{and all} \quad u \in \boldsymbol{\mu}. \tag{3.0.0.1}$$

Let us denote by $a : G \to \mathbb{Z}$ the lift of $\bar{a}$ which satisfies $-m < a(g) < m$ for all g in G.
This lift allow us to construct the distinguished element Θ in $\mathbb{Q}G$ given by

$$\Theta = 1/2m \sum_{g \in G} a(g^{-1})g$$

and the $\mathbb{Z}$-submodule $Y = \mathbb{Z}G + \mathbb{Z}\Theta$ of $\mathbb{Q}G$.

By fixing a generator μ of $\boldsymbol{\mu}$ one can show that Y fits in the following exact sequence

$$0 \longrightarrow \mathbb{Z}G \xrightarrow{\alpha} Y \xrightarrow{\beta} \boldsymbol{\mu} \longrightarrow 0, \tag{3.0.0.2}$$

where α is the natural inclusion and

$$\beta(y + t\Theta) = \mu^t \text{ for all } y \in \mathbb{Z}G \text{ and all } t \in \mathbb{Z}.$$

It follows that β is well defined. In order to prove this let y_1 in $\mathbb{Z}G$ and t_1 an integer such that $y + t\Theta = y_1 + t_1\Theta$, which implies that $(t_1 - t)\Theta = y - y_1$ belongs to $\mathbb{Z}G$. Then $t_1 \equiv t \pmod{2m}$ or $t_1 = t + 2mk$ for some integer k this implies

$$\beta(y_1 + t_1\Theta) = \mu^{t_1} = \mu^{t+2mk} = \mu^t = \beta(y + t\Theta).$$

To show that β is a homomorphism of $\mathbb{Z}G$-modules we need to prove that $\beta(g\Theta) = g \cdot \mu = \mu^{\bar{a}(g)}$ for any g in G.

The fact that a is a lift of $\bar{a}$ implies that for all g, h in G there exists $l_{g,h} \in \mathbb{Z}$ such that $a(g)a(h) - a(gh) = 2ml_{g,h}$, hence

$$\begin{aligned} g \cdot \Theta &= \frac{1}{2m} \sum_{h \in G} a(h^{-1})gh = \frac{1}{2m} \sum_{h \in H} a(h^{-1}g)h \\ &= \frac{1}{2m} \sum_{h \in G} \left[a(h^{-1})a(g) - 2ml_{h^{-1},g}\right] h \\ &= a(g)\Theta - \sum_{h \in G} l_{h^{-1},g}h. \end{aligned}$$

If we let $L_g = \sum_{h \in G} l_{h^{-1},g}h$ by the definition of β we obtain that

$$\beta(g\Theta) = \beta(a(g)\Theta - L_g) = \mu^{a(g)} = g \cdot \mu = g \cdot \beta(\Theta).$$

For any $\mathbb{Z}G$-module M, let us dente $Hom(M, \mathbb{Z})$ by M^o, while M^* will denote

$Hom(M, \mathbb{Q}/\mathbb{Z})$. The following is a well known result. We sketch a proof here for the reader's convenience.

3.0.1 Claim. *Given an exact sequence of* $\mathbb{Z}G$*-modules*

$$0 \longrightarrow L \xrightarrow{\alpha} M \xrightarrow{\beta} T \longrightarrow 0, \tag{3.0.1.1}$$

where L *and* M *are lattices and* T *is a torsion module, there exists an exact sequence of* $\mathbb{Z}G$ *modules*

$$0 \longrightarrow M^o \xrightarrow{\alpha^o} L^o \xrightarrow{\gamma} T^* \longrightarrow 0. \tag{3.0.1.2}$$

Proof. If f is an element of M^o we can define $\alpha^o(f) = f\alpha$. In order to define the map γ, we will start by showing that for any given $f : L \to \mathbb{Z}$ there exist a unique $\bar{f} : M \to \mathbb{Q}$ making the following diagram commute.

$$\begin{array}{ccc} L & \xrightarrow{\alpha} & M \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \bar{f}} \\ \mathbb{Z} & \longrightarrow & \mathbb{Q} \end{array} \tag{3.0.1.3}$$

For every m in M there is a nonzero integer z such that $\beta(zm) = z\beta(m) = 0$ in T, hence exactness of (3.0.1.1) gives the existence of an element l in L satisfying $\alpha(l) = zm$. We define

$$\bar{f}(m) = z^{-1} f(l), \tag{3.0.1.4}$$

notice that if z_1 is a different nonzero integer with $\beta(z_1 m) = 0$ and l_1 in L satisfies $\alpha(l_1) = z_1 m$ then

$$z_1\alpha(l) = z_1 zm = zz_1 m = z\alpha(l_1).$$

By injectivity of α one obtains $z_1 l = zl_1$, which gives $z_1 f(l) = zf(l_1)$ or equivalently $z^{-1}f(l) = z_1^{-1}f(l_1)$. This last equality proves that $\bar{f}$ is well defined.

Assume that $h : M \to \mathbb{Q}$ makes diagram (3.0.1.3) commute. For m in M let z be a nonzero integer annihilating $\beta(m)$ and l in L with $\alpha(l) = zm$, then

$zh(m) = h(\alpha(l)) = f(l)$ which gives $h(m) = z^{-1}f(l) = \bar{f}(m)$. This proves the uniqueness of $\bar{f} : M \to \mathbb{Q}$ making diagram (3.0.1.3) commute.

We now show that diagram (3.0.1.3) can be completed in a unique way to the following commutative diagram

$$\begin{array}{ccccc} L & \xrightarrow{\alpha} & M & \xrightarrow{\beta} & T \\ \downarrow f & & \downarrow \bar{f} & & \downarrow \tilde{f} \\ \mathbb{Z} & \longrightarrow & \mathbb{Q} & \longrightarrow & \mathbb{Q}/\mathbb{Z}, \end{array} \tag{3.0.1.5}$$

where the map $\tilde{f} : T \to \mathbb{Q}/\mathbb{Z}$ is defined by choosing for any t in T an element m in M with $\beta(m) = t$, such element m exists by surjectivity of β hence

$$\tilde{f}(t) = \bar{f}(m) + \mathbb{Z}.$$

Let m_1 be any other element in M with $\beta(m_1) = t$ then $m_1 - m$ belongs to $ker(\beta) = im(\alpha)$, which gives the existence of l in L satisfying $m_1 = m + \alpha(l)$. Since $\bar{f}(\alpha(l)) = f(l)$ which belongs to $\mathbb{Z}$, it follows that

$$\begin{aligned} \bar{f}(m_1) + \mathbb{Z} &= \bar{f}(m + \alpha(l)) + \mathbb{Z} \\ &= \bar{f}(m) + f(l) + \mathbb{Z} \\ &= \bar{f}(m) + \mathbb{Z}, \end{aligned}$$

this proves that $\tilde{f}$ is well defined, uniqueness can be proved in a similar way as done for diagram (3.0.1.3). We define $\gamma(f) = \tilde{f}$.

We now focus on showing that (3.0.1.2) is exact. By applying the functor $(\ )^o$ to (3.0.1.1) one obtains the following exact sequence

$$0 \longrightarrow T^o \xrightarrow{\beta^o} M^o \xrightarrow{\alpha^o} L^o,$$

since $T^o = 0$ it follows that α^o is injective. Surjectivity of γ follows from the fact that M is $\mathbb{Z}$-projective hence, if $\tilde{f} \in T^*$ there exists $\bar{f} : M \to \mathbb{Q}$ making the right square of diagram (3.0.1.5) commute, this implies that $\bar{f}(\alpha(l)) + \mathbb{Z} =$

$\tilde{f}(\beta\alpha(l)) = 0$ in $\mathbb{Q}/\mathbb{Z}$ for all $l \in L$. It follows that $\bar{f}\alpha$ takes values only in $\mathbb{Z}$, one can define $f(l) = \bar{f}(\alpha(l))$ for all l in L and clearly f makes the left square of diagram (3.0.1.5)commute.
Uniqueness of this diagram implies that $\gamma f = \tilde{f}$ hence γ is surjective, notice that this argument also shows that $ker(\gamma) \subseteq im(\alpha^o)$. It follows clearly that $im(\alpha^o) \subseteq ker(\gamma)$, which completes the proof. □

If we apply the previous claim to the exact sequence (3.0.0.2) we obtain a short exact sequence

$$0 \longrightarrow Y^o \xrightarrow{\alpha^o} (\mathbb{Z}G)^o \xrightarrow{\gamma} \boldsymbol{\mu}^* \longrightarrow 0, \tag{3.0.1.6}$$

where $\gamma(f)(\mu) = f(2m\Theta)/2m + \mathbb{Z}$.

Let us denote by k_G the element in $(\mathbb{Z}G)^o$ given by

$$k_G(g) = \begin{cases} 1 & \text{if} \quad g = 1_G \\ 0 & \text{if} \quad g \neq 1_G \end{cases}$$

and extend it $\mathbb{Z}$-linearly. The previous map induces the following isomorphism of $\mathbb{Z}G$-modules.

3.0.2 Claim. *The map* $k : \mathbb{Z}G \to (\mathbb{Z}G)^o$ *given by*

$$k(\sum_{g \in G} x_g g) = \sum_{g \in G} x_g (g k_G)$$

is an isomorphism of $\mathbb{Z}G$*-modules.*

Proof. Given $f \in (\mathbb{Z}G)^o$ let $x = \sum_{g \in G} f(g) g \in \mathbb{Z}G$ and $h \in G$, then

$$k(x)(h) = \sum_{g \in G} f(g)(g k_G)(h) = \sum_{g \in G} f(g) k_G(g^{-1}h) = f(h)$$

which proves surjectivity of k. On the other hand if $x = \sum_{g \in G} x_g g$ satisfies that $k(x)$ is the zero element in $(\mathbb{Z}G)^o$. It follows that

$$0 = k(x)(h) = \sum_{g\in G} x_g(gk_G)(h) = \sum_{g\in G} x_g k_G(g^{-1}h) = x_h$$

for all $h \in G$, which proves that k is injective. □

The previous Claim and diagram (3.0.1.6) gives us the following commutative diagram

$$\begin{array}{ccccc} Z & \xrightarrow{\alpha_o} & \mathbb{Z}G & & \\ \cong \Big\downarrow \rho & & \cong \Big\downarrow k & & \\ Y^o & \xrightarrow[\alpha^o]{} & (\mathbb{Z}G)^o & \xrightarrow[\gamma]{} & \boldsymbol{\mu}^*, \end{array} \tag{3.0.2.1}$$

where Z is the pull-back of $k : \mathbb{Z}G \to (\mathbb{Z}G)^o$ and $\alpha^o : Y^o \to (\mathbb{Z}G)^o$.

3.0.3 Claim. *Given $z = \sum_{g\in G} z_g g$ in $\mathbb{Z}G$, z in Z if, and only, if*

$$\sum_{g\in G} z_g a(g^{-1}) \equiv 0 \pmod{2m\mathbb{Z}}.$$

Proof. Given $z = \sum_{g\in G} z_g g \in \mathbb{Z}G$ $z \in Z$ if, and only if, $k(z) \in Ker(\gamma)$, which is equivalent to say by the definition of γ that $k(z)(2m\Theta)/2m \in \mathbb{Z}$. Since

$$\begin{aligned} k(z)(2m\Theta) &= \sum_{g\in G} z_g(gk_G)(2m\Theta) = \sum_{g\in G} z_g k_G(2mg^{-1}\Theta) \\ &= \sum_{g\in G} z_g k_G\left(\sum_{h\in G} a(h^{-1})g^{-1}h\right) = \sum_{g\in G} z_g a(g^{-1}), \end{aligned}$$

It follows that $k(z)(2m\Theta)/2m \in \mathbb{Z}$ if, and only if, $\sum_{g\in G} z_g a(g^{-1}) \equiv 0 \pmod{2m\mathbb{Z}}$. □

The description of Z given above allow us to find a set of generators of Z as $\mathbb{Z}$-module as well as a $\mathbb{Z}G$-module.

3.0.4 Claim. *Let us denote by $G^\times = G \setminus \{1_G\}$. Then*

a) $\{2m\} \cup \{g - a(g^{-1}) : g \in G^\times\}$ *is a $\mathbb{Z}$-basis for Z.*

b) *If $\{g_1, g_2, \ldots, g_n\} \subset G^\times$ generates G, $\{2m\} \cup \{g_i - a(g_i^{-1}) : i = 1, 2, \ldots, n\}$ generate Z as a $\mathbb{Z}G$-module.*

Proof. Given $z = \sum_{g \in G} z_g g \in Z$, we have that

$$\begin{aligned} z &= \sum_{g \in G} z_g \left(a(g^{-1}) + g - a(g^{-1}) \right) \\ &= \sum_{g \in G} z_g a(g^{-1}) + \sum_{g \in G} z_g \left(g - a(g^{-1}) \right). \end{aligned}$$

By Claim 3.0.4, $\sum_{g \in G} z_g a(g^{-1})$ belongs to $2m\mathbb{Z}$, which proves (a).

In order to prove statement (b) it is enough to observe that for all $g, h \in G$

$$\begin{aligned} gh - a(h^{-1}g^{-1}) &= gh - ga(h^{-1}) + ga(h^{-1}) - a(h^{-1}g^{-1}) \\ &\equiv gh - ga(h^{-1}) + ga(h^{-1}) - a(h^{-1})a(g^{-1}) \mod 2m\mathbb{Z}G \\ &\equiv g\left(h - a(h^{-1})\right) + a(h^{-1})\left(g - a(g^{-1})\right) \mod 2m\mathbb{Z}G. \end{aligned}$$

□

Let $F = \bigoplus_{0 \le i \le n} \mathbb{Z}G$ with the standard $\mathbb{Z}G$-basis $\{e_i : 0 \le i \le n\}$ and let $\pi : F \to Z$ be defined by

$$\pi(e_i) = \begin{cases} 2m & \text{if } \; i = 0 \\ g_i - a(g_i^{-1}) & \text{if } \; 1 \le i \le n \end{cases}$$

We obtain a short exact sequence

$$0 \longrightarrow X \xrightarrow{i_1} F \xrightarrow{\pi} Z \longrightarrow 0, \tag{3.0.4.1}$$

where X is the kernel of π and i_1 is the natural inclusion. If we apply the functor $(\ )^o$ to the exact sequence given in (3.0.4.1) we obtain the following commutative diagram with exact top row

$$\begin{array}{ccccc} Z^o & \overset{\pi^o}{\rightarrowtail} & F^o & \overset{i_1^o}{\twoheadrightarrow} & X^o \\ \cong \big\uparrow \rho^o & & & & \\ (Y^o)^o & & \cong \big\uparrow k_F & & \\ \cong \big\uparrow ev & & & & \\ Y & \xrightarrow{\psi} & F & & \end{array} \tag{3.0.4.2}$$

The vertical isomorphism $Y \to (Y^o)^o$ sends $y \mapsto ev_y$, where $ev_y(f) = f(y)$ for any $f \in Y^o$, and $k_F : F \to F^o$ is the map obtained generalizing k.

The diagram above gives us an injective map $\psi : Y \to F$. Putting together diagrams (3.0.4.1) and (3.0.0.2) gives us a commutative diagram

$$\begin{array}{ccccc} \mathbb{Z}G & \overset{\alpha}{\rightarrowtail} & Y & \overset{\beta}{\twoheadrightarrow} & \boldsymbol{\mu} \\ {\scriptstyle =}\downarrow & & \downarrow{\scriptstyle \psi} & & \\ \mathbb{Z}G & \underset{\psi\alpha}{\rightarrowtail} & F & & \\ & & \downarrow & & \\ & & X^o & & \end{array} \tag{3.0.4.3}$$

where X^o is the cokernel of $\psi : Y \to F$. We can complete the last diagram by taking the push-out of $\beta : Y \to \boldsymbol{\mu}$ and $\psi : Y \to F$ and obtain the following commutative diagram

$$\begin{array}{ccccc} \mathbb{Z}G & \overset{\alpha}{\rightarrowtail} & Y & \overset{\beta}{\twoheadrightarrow} & \boldsymbol{\mu} \\ {\scriptstyle =}\downarrow & & \downarrow{\scriptstyle \psi} & & \downarrow \\ \mathbb{Z}G & \underset{\psi\alpha}{\rightarrowtail} & F & \twoheadrightarrow & C \\ & & \downarrow & & \downarrow \\ & & X^o & \underset{\cong}{\longrightarrow} & \bar{C} \end{array} \tag{3.0.4.4}$$

We should mention that the right column from this last diagram is the desired envelope of $\boldsymbol{\mu}$, where the cohomologically trivial module C can be identified with the cokernel of $\psi\alpha$.

It is clear that the map $\boldsymbol{\mu} \to C$ is completely determined by the class of $\psi(\Theta)$, the snake lemma proves that ψ is injective and that $\bar{C}$ is isomorphic to X^o.

Our final task is to give an explicit description of C and $\bar{C}$. For this we should study in more detail the homomorphism ψ.

By diagram (3.0.4.2) we obtain that $\pi^o\rho^o = (\rho\circ\pi)^o$ hence

$$\rho\pi(e_i) = \begin{cases} 2mk_G & \text{if } \ i=0 \\ g_ik_G - a(g_i^{-1})k_G & \text{if } \ 1\le i\le s \end{cases}$$

We now compute the following composition

$$Y \xrightarrow{ev} (Y^o)^o \xrightarrow{\rho^o} Z^o \xrightarrow{\pi^o} F^o.$$

Given $y\in Y$ let $F_y\in F^o$ be the image of y under the composition given above, which means that for any $x\in F$, $F_y(x) = \left((\rho\pi)(x)\right)(y)$. Let $\lambda\in(Y^o)^o$ then $(\pi^o\rho^o)(\lambda) = (\rho\pi)^o(\lambda) = \lambda\rho\circ\pi$ and $F_y(x) = ev_y\rho\pi$.
We now prove that if $y=\sum_{g\in G} y_g g$

$$F_y(ge_i) = \begin{cases} 2my_g & \text{if } \ i=0 \\ y_{gg_i} - a(g_i^{-1})y_g & \text{if } \ 1\le i\le n \end{cases} \tag{3.0.4.5}$$

Proof. Let $i=0$, then

$$ev_y\rho\pi(ge_0) = 2m(gk_G)(y) = 2mk_G(g^{-1}y) = 2my_g.$$

Now let $1\le i\le n$, then

$$\begin{aligned} ev_y\rho\pi(ge_i) &= \left(gg_ik_G - a(g_i^{-1})gk_G\right)(y) \\ &= k_G((gg_i)^{-1}y) - a(g_i^{-1})k_G(g^{-1}y) \\ &= y_{gg_i} - a(g_i^{-1})y_g. \end{aligned}$$

□

3.0.5 Claim. *The homomorphism $\psi: Y\to F$ is defined as follows: for any $y\in Y$*

$$\psi(y) = \left(2my, y(g_1^{-1} - a(g_1^{-1})), \ldots, y(g_n^{-1} - a(g_n^{-1}))\right)$$

Proof. It is enough to show that ψ given above satisfies $k_F\psi(y) = F_y =$

$ev_y(\psi\pi)^o$.
We will proof this equality by evaluating at the standard basis of F.
Let $y = \sum_{h\in G} y_h h$ in Y and $g \in G$, then we have

$$\begin{aligned} k_F\psi(y)(ge_0) &= 2m(yk_G)(g) = 2m\sum_{h\in G} y_h(hk_G)(g) \\ &= 2m\sum_{h\in G} y_h k_G(h^{-1}g) = 2my_g \end{aligned}$$

by the first case of (3.0.4.5) we obtain $k_F\psi(y)(ge_0) = F_y(ge_0)$.

We will proceed in a similar way for the other cases, let $1 \leq i \leq n$

$$\begin{aligned} k_F\psi(y)(ge_i) &= (g_i^{-1}yk_G)(g) - a(g_i^{-1})(yk_G)(g) \\ &= \sum_{h\in G} y_h k_G(h^{-1}g_i g) - a(g_i^{-1})\sum_{h\in G} y_h k_G(h^{-1}g) \\ &= y_{g_i g} - a(g_i^{-1})y_g, \end{aligned}$$

one more time (3.0.4.5) shows that $k_F\psi(y)(ge_i) = F_y(ge_i)$ when $1 \leq i \leq n$. This concludes the proof. $\square$

Let A_Y and $A_{\mathbb{Z}G}$ be the following $\mathbb{Z}G$-submodules of F

$$\begin{aligned} A_Y &= \left\langle \left(2my, y(g_1^{-1} - a(g_1^{-1})), \ldots, y(g_n^{-1} - a(g_n^{-1}))\right) : y \in Y \right\rangle \\ A_{\mathbb{Z}G} &= \left\langle \left(2mx, x(g_1^{-1} - a(g_1^{-1})), \ldots, x(g_n^{-1} - a(g_n^{-1}))\right) : x \in \mathbb{Z}G \right\rangle \end{aligned}$$

Then $C = F/A_{\mathbb{Z}G}$ and $\bar{C} = F/A_Y$.

Chapter 4

The special character ϵ.

As stated in Theorem B in [10] the character $\epsilon : H^2(G, Hom(\Delta S, \boldsymbol{\mu})) \rightarrow \mathbb{Q}/\mathbb{Z}$ is one of the factors that determines the stable isomorphism class of the S-units E.

Theorem A, proved on pg 955 of [10], reformulates ϵ using class field theoretic data through what the authors called *the arithmetic trace formula*. The disadvantage that Theorem A brings to our program is: that *the arithmetic trace formula* is given by a character $t_E : H^1(G, Hom(L, E)) \rightarrow \mathbb{Q}/\mathbb{Z}$.

This chapter is dedicated to the study of the character ϵ. The main result obtained here states that: ϵ satisfies an equation that reminiscent the equation given in Theorem A for t_E.
This expression will simplify the computation of ϵ.

Much of the work done in this chapter follows the ideas presented in [14] and in [10] hence we will continue using the notation introduced by the mentioned authors.

We will denote the group of S-idèles of K by J and the group of all idèle classes of K by C_K, on the other hand we will let $L = \Delta G \otimes \Delta S$, $I = \Delta G \otimes \mathbb{Z}S$, $\bar{I} = \mathbb{Z}G \otimes \Delta S$ and $M = \mathbb{Z}G \otimes \mathbb{Z}S$. With the above notation one obtains the following commutative diagram, that will be used continuously in this chapter

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & L & \xrightarrow{i_0} & \bar{I} & \xrightarrow{\acute{i}_0} & \Delta S & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle i_1} & & \downarrow{\scriptstyle i_1} & & \downarrow{\scriptstyle i_1} & & \\
0 & \longrightarrow & I & \xrightarrow{i_0} & M & \xrightarrow{\acute{i}_0} & \mathbb{Z}S & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \acute{i}_1} & & \downarrow{\scriptstyle \acute{i}_1} & & \downarrow{\scriptstyle \acute{i}_1} & & \\
0 & \longrightarrow & \Delta G & \xrightarrow{i_0} & \mathbb{Z}G & \xrightarrow{\acute{i}_0} & \mathbb{Z} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
\qquad (4.0.0.1)
$$

where all rows and columns are exact.

4.1 The character ϵ and *the arithmetic trace formula.*

We begin this section by describing the the character $t_E : H^1(G, Hom(\Delta S, E)) \to \mathbb{Q}/\mathbb{Z}$, we will show later how t_E determines ϵ and conclude by recalling *the arithmetic trace formula.*

4.1.1 The characters t_E and ϵ.

Let N be a $\mathbb{Z}G$-module, by fixing an envelope of N

$$0 \longrightarrow N \longrightarrow C \longrightarrow M \longrightarrow 0 \qquad (4.1.1.1)$$

one obtains that the connecting homomorphism $\partial : H^0(G, Hom(M, M)) \to H^1(G, Hom(M, N))$ is in fact an isomorphism, since $Hom(M, C)$ is a cohomologically trivial $\mathbb{Z}G$-module. It has been proved on §5 of [9] that $H^0(G, Hom(M, Q)) \cong [M, Q]$ whenever M is a $\mathbb{Z}G$-lattice, which implies that there is a connecting

isomorphism $\partial_M : [M, M] \to H^1(G, Hom(M, N))$.
We denote by $\tau_M : [M, M] \to \mathbb{Q}/\mathbb{Z}$ the map given by

$$\tau_M[m] = \frac{Trace_M(m)}{|G|} + \mathbb{Z}, \tag{4.1.1.2}$$

this map allow us to define the trace map $t_N : H^1(G, Hom(M, N)) \to \mathbb{Q}/\mathbb{Z}$ associated to the envelope (4.1.1.1) as the following composition

$$t_N : H^1(G, Hom(M, N)) \xrightarrow{\partial_M^{-1}} [M, M] \xrightarrow{\tau_M} \mathbb{Q}/\mathbb{Z}.$$

Every envelope $0 \to N \to C \to M \to 0$ induces a commutative diagram

$$\begin{array}{ccc} [M, M] & \xrightarrow{\partial_M} & H^1(G, Hom(M, N)) \\ & {\scriptstyle \tau_M} \searrow \quad \swarrow {\scriptstyle t_N} & \\ & \mathbb{Q}/\mathbb{Z} & \end{array} \tag{4.1.1.3}$$

By fixing a short Tate envelope $0 \to E \to A \to L \to 0$ the above gives us a commutative diagram

$$\begin{array}{ccc} [L, L] & \xrightarrow{\partial_L} & H^1(G, Hom(L, E)) \\ & {\scriptstyle \tau_L} \searrow \quad \swarrow {\scriptstyle t_E} & \\ & \mathbb{Q}/\mathbb{Z} & \end{array} \tag{4.1.1.4}$$

from where we define

$$t_E = \tau_L \circ \partial_L^{-1}$$

Before showing the relation between t_E and ϵ, we will recall a known result of group cohomology.

4.1.2 Claim. *Let A be an induced $\mathbb{Z}G$-module then $Hom(A, B)$ and $A \otimes B$ are induced, hence cohomologically trivial.*

The proof of Claim 4.1.2 can be found on pg 141 of [13].

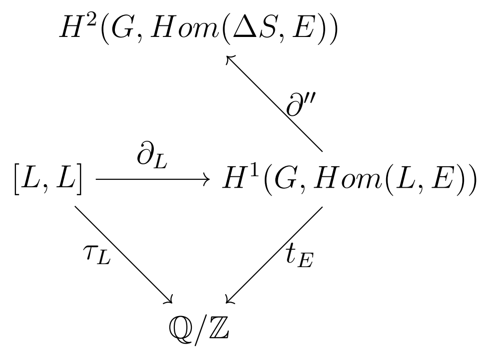
Let us consider the top row of diagram (4.0.0.1)

$$0 \longrightarrow L \xrightarrow{i_0} \bar{I} \xrightarrow{\acute{i}_0} \Delta S \longrightarrow 0, \tag{4.1.2.1}$$

by applying the functor $Hom(-, E)$ to the $\mathbb{Z}$-split exact sequence (4.1.2.1) one obtains the following short exact sequence

$$0 \to Hom(\Delta S, E) \longrightarrow Hom(\bar{I}, E) \longrightarrow Hom(L, E) \longrightarrow 0.$$

It follows from Claim 4.1.2 that $Hom(\bar{I}, E)$ is cohomologically trivial, hence the connecting homomorphism in cohomology induced by the last exact sequence $\partial'' : H^1(G, Hom(L, E)) \to H^2(G, Hom(\Delta S, E))$ is an isomorphism, which gives the following diagram



(4.1.2.2)

Let $\alpha : \boldsymbol{\mu} \to E$ be the natural inclusion and denote by $\alpha_* : H^2(G, Hom(\Delta S, \boldsymbol{\mu})) \to H^2(G, Hom(\Delta S, E))$ the group homomorphism induced by α. The character ϵ is given by

$$\epsilon = t_E(\partial'')^{-1}\alpha_*. \tag{4.1.2.3}$$

4.1.3 The arithmetic trace formula.

We now follow the ideas presented in [14]. Given two short exact sequences of $\mathbb{Z}G$-modules

$$0 \longrightarrow X \xrightarrow{\beta_1} Y \xrightarrow{\beta_1'} Z \longrightarrow 0$$

$$0 \longrightarrow X' \xrightarrow{\beta_2} Y' \xrightarrow{\beta_2'} Z' \longrightarrow 0$$

Let $Hom((Y),(Y'))$ denote the $\mathbb{Z}G$-module of all triples (f_1, f_2, f_3) of $\mathbb{Z}$-homomorphisms making the following diagram commute.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \xrightarrow{\beta_1} & Y & \xrightarrow{\beta_1'} & Z & \longrightarrow & 0 \\ & & \downarrow f_1 & & \downarrow f_2 & & \downarrow f_3 & & \\ 0 & \longrightarrow & X' & \xrightarrow[\beta_2]{} & Y' & \xrightarrow[\beta_2']{} & Z' & \longrightarrow & 0 \end{array} \tag{4.1.3.1}$$

Let us denote by $u^1 : Hom((Y),(Y')) \to Hom(X, X')$, $u^2 : Hom((Y),(Y')) \to Hom(Y, Y')$ and $u^3 : Hom((Y),(Y')) \to Hom(Z, Z')$ the natural projections on each component. It is not difficult to prove that the following sequence

$$0 \longrightarrow Hom((Y),(Y')) \xrightarrow{(u^1, u^2)} Hom(X, X') \oplus Hom(Y, Y') \xrightarrow{\beta} Hom(X, Y') \longrightarrow 0 \tag{4.1.3.2}$$

where $\beta(f, g) = \beta_2 f - g\beta_1$ is exact.

If we apply the last construction to the exact sequences

$$0 \longrightarrow L \xrightarrow{i_1} I \xrightarrow{\acute{i}_1} \Delta G \longrightarrow 0$$

$$0 \longrightarrow E \xrightarrow{j} J \xrightarrow{\acute{j}} C_K \longrightarrow 0$$

we obtain the short exact sequence

$$0 \longrightarrow Hom((I),(J)) \xrightarrow{(u^1,u^2)} Hom(L,E) \oplus Hom(I,J) \xrightarrow{\beta} Hom(L,J) \longrightarrow 0, \tag{4.1.3.3}$$

which induces in cohomology a long exact sequence

$$\cdots \xrightarrow{\partial} H^1(G, Hom((I),(J))) \xrightarrow{u_*} H^1(G, Hom(L,E)) \oplus H^1(G, Hom(I,J)) \xrightarrow{\beta_*} H^1(G, Hom(L,J)) \xrightarrow{\partial} \cdots \tag{4.1.3.4}$$

where u_* denotes $(u^1, u^2)_*$.

We will use this sequence to describe t_E in terms of local and global invariant maps from class field theory.

Before achieving this, we need the following result.

4.1.4 Claim. *For the long exact sequence given in (4.1.3.4) follows that*

i) $u_* : H^1(G, Hom(I),(J))) \to H^1(G, Hom(L,E)) \oplus H^1(G, Hom(I,J))$ *is injective and*

ii) $i_1^* : H^1(G, Hom(I,J)) \to H^1(G, Hom(L,J))$ *is surjective.*

Proof.

i) Injectivity of u_* follows from the exactness of (4.1.3.4) and the fact that $[L,J]=0$, which proof can be found on pg 971 of [10].

ii) Surjectivity of $i_1^* : H^1(G, Hom(L,I)) \to H^1(G, Hom(L,J))$ is shown on pg 970 of [10]. □

4.1.5 Remark. Part ii) of Claim 4.1.4 implies that

$$\beta_* : H^1(G, Hom(L,E)) \oplus H^1(G, Hom(I,J)) \to H^1(G, Hom(L,J))$$

is surjective. This fact together with part i) of Claim 4.1.4, allow us to write the long exact sequence given in (4.1.3.4) as the following short exact sequence

$$0 \longrightarrow H^1(G, Hom((I),(J))) \xrightarrow{u_*} H^1(G, Hom(L,E)) \oplus H^1(G, Hom(I,J)) \xrightarrow{\beta_*} H^1(G, Hom(L,J)) \longrightarrow 0 \tag{4.1.5.1}$$

Let us fix $\mathfrak{P} \in S_*$ and denote by $G_{\mathfrak{P}}$ the decomposition group associated to $\mathfrak{P}$. We consider the following commutative diagram of $\mathbb{Z}G_{\mathfrak{P}}$-modules with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Delta G_{\mathfrak{P}} & \rightarrow & \mathbb{Z}G_{\mathfrak{P}} & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \Delta G & \longrightarrow & \mathbb{Z}G & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 \end{array}$$

where $\mathbb{Z}G_{\mathfrak{P}} \to \mathbb{Z}$ and $\mathbb{Z}G \to \mathbb{Z}$ are the augmentation maps.

By the snake lemma one has that the cokernel of $\Delta G_{\mathfrak{P}} \to \Delta G$ is isomorphic to the cokernel of $\mathbb{Z}G_{\mathfrak{P}} \to \mathbb{Z}G$ which is $\mathbb{Z}G_{\mathfrak{P}}$-free, this implies that $ind_{G_{\mathfrak{P}}}^G \Delta G \cong ind_{G_{\mathfrak{P}}}^G \Delta G_{\mathfrak{P}} \oplus F$ with F a $\mathbb{Z}G$-free module. The fact that $ind_{G_{\mathfrak{P}}}^G \Delta G \cong ind_{G_{\mathfrak{P}}}^G \mathbb{Z} \otimes \Delta G$ gives that $I = \Delta G \otimes \mathbb{Z}S$ is isomorphic as $\mathbb{Z}G$-module to $I' \oplus F'$ with $I' = \bigoplus_{\mathfrak{P} \in S_*} ind_{G_{\mathfrak{P}}}^G \Delta G_{\mathfrak{P}}$ and F' a $\mathbb{Z}G$-free module.
We will denote by $i_{\mathfrak{P}} : \Delta G_{\mathfrak{P}} \to I$ the $\mathbb{Z}G_{\mathfrak{P}}$-homomorphism under the above identification and by $k^{\mathfrak{P}} : J \to K_{\mathfrak{P}}^{\times}$ the natural $\mathbb{Z}G_{\mathfrak{P}}$-projection for every $\mathfrak{P} \in S_*$.

Given y in $H^1(G, Hom(I,J))$ let $y_{\mathfrak{P}}$ in $H^1(G_{\mathfrak{P}}, Hom(\Delta G, K_{\mathfrak{P}}^*))$ be the image of y under the following composition

$$H^1(G, Hom(I,J)) \xrightarrow{res_{G_{\mathfrak{P}}}^G} H^1(G_{\mathfrak{P}}, Hom(I,J)) \xrightarrow{i_{\mathfrak{P}}^*} H^1(G_{\mathfrak{P}}, Hom(\Delta G_{\mathfrak{P}}, J)) \xrightarrow{k_*^{\mathfrak{P}}} H^1(G_{\mathfrak{P}}, Hom(\Delta G_{\mathfrak{P}}, K_{\mathfrak{P}}^{\times})).$$

As mentioned before, we will denote by $u^3 : Hom((I),(J)) \to Hom(\Delta G, C_K)$ the $\mathbb{Z}G$-homomorphism defined by $u^3(f_1, f_2, f_3) = f_3$, then u^3 induces a homomorphism

$$u^3_* : H^1(G, Hom((I),(J))) \to H^1(G, Hom(\Delta G, C_K)).$$

We are now in position to state *the arithmetic trace formula* for the character t_E.
Let x in $H^1(G, Hom(L, E))$, part ii) of Claim 4.1.4 ensures the existence of y in $H^1(G, Hom(I, J))$ such that $j_* x = i_1^* y$ or equivalently $(x, y) \in ker(\beta_*)$, exactness of the sequence (4.1.3.4) implies that there exists a unique element t in $H^1(G, ((I),(J)))$ with $u_* t = (x, y)$, let $z = u^3_* t \in H^1(G, Hom(\Delta G, C_K))$ then

$$t_E\ x = inv\ \partial' z - \sum_{\mathfrak{P} \in S_*} inv_{\mathfrak{P}}\ \partial'_{\mathfrak{P}}\ y_{\mathfrak{P}} \tag{4.1.5.2}$$

where $\partial' : H^1(G, hom(\Delta G, C_K)) \to H^2(G, C_K)$ and $\partial'_{\mathfrak{P}} : H^1(G_{\mathfrak{P}}, Hom(\Delta G, K^\times_{\mathfrak{P}})) \to H^2(G_{\mathfrak{P}}, K^\times_{\mathfrak{P}})$ are the connecting isomorphisms induced by $0 \to \Delta G \to \mathbb{Z}G \to \mathbb{Z} \to 0$ and S_* is a transversal to the G-orbits of S.
The proof of the last equation can be found on §9 [10].

4.2 The character ϵ'.

In this section we will begin by defining a character $\epsilon' : H^2(G, Hom(\Delta G, E)) \to \mathbb{Q}/\mathbb{Z}$ following the ideas presented in the previous section. We will conclude by showing that $\epsilon' = t_E(\partial'')^{-1} = \epsilon$.

From the exact sequences

$$0 \longrightarrow \Delta S \xrightarrow{i_1} \mathbb{Z}S \xrightarrow{\acute{i}_1} \mathbb{Z} \longrightarrow 0$$
$$0 \longrightarrow E \xrightarrow{j} J \xrightarrow{\acute{j}} C_K \longrightarrow 0,$$

we obtain the short exact sequence

$$0 \longrightarrow Hom((\mathbb{Z}S),(J)) \xrightarrow{(\bar{u}^1,\bar{u}^2)} Hom(\Delta S, E) \oplus Hom(\mathbb{Z}S, J) \xrightarrow{\bar{\beta}} Hom(\Delta S, J) \to 0. \tag{4.2.0.1}$$

In this case we are using bars to avoid confusion with the maps from the previous section.
The short exact sequence given in (4.2.0.1) induces in cohomology a long exact sequence

$$\begin{aligned} \cdots \xrightarrow{\partial} H^2(G, Hom((\mathbb{Z}S),(J))) &\xrightarrow{\bar{u}_*} H^2(G, Hom(\Delta S, E)) \oplus H^2(G, Hom(\mathbb{Z}S, J)) \\ &\xrightarrow{\bar{\beta}_*} H^2(G, Hom(\Delta S, J)) \xrightarrow{\partial} \cdots \end{aligned} \tag{4.2.0.2}$$

We now prove a similar result as Claim 4.1.4.
4.2.1 Claim. *For the long exact sequence constructed above follows that*

i) $\bar{u}_* : H^2(G, Hom((\mathbb{Z}S),(J))) \to H^2(G, Hom(\Delta S, E)) \oplus H^2(G, Hom(\mathbb{Z}S, J))$ *is injective and*

ii) $\acute{\imath}_1^* : H^2(G, Hom(\mathbb{Z}S, J)) \to H^2(G, Hom(\Delta S, J))$ *is surjective.*

Proof.

i) Injectivity of $\bar{u}_*$ follow from the exactness of (4.2.0.2) and the fact that $[L, J] = 0$. By applying the functor $Hom(-, J)$ to the $\mathbb{Z}$-split exact sequence given in (4.1.2.1), one obtains a connecting isomorphism $\partial : [L, J] \to H^1(G, Hom(\Delta S, J))$ hence $0 = im(\partial) = ker(\bar{u}_*)$.

ii) Surjectivity of $\acute{\imath}_1^*$ follows from the fact that $i_1^* : H^1(G, Hom(L, I)) \to H^1(G, Hom(L, J))$ is surjective in the following way: by applying $Hom(-, J)$ to the top and middle rows of diagram 4.0.0.1 one obtains the following commutative diagram with exact rows.

$$\begin{array}{ccccccc}
0 \longrightarrow & Hom(\mathbb{Z}S, J) & \overset{\acute{\imath}_0^*}{\to} & Hom(M, J) & \overset{i_0^*}{\longrightarrow} & Hom(I, J) & \to 0 \\
& i_1^* \big\downarrow & & i_1^* \big\downarrow & & i_1^* \big\downarrow & \\
0 \longrightarrow & Hom(\Delta S, J) & \overset{\acute{\imath}_0^*}{\to} & Hom(\bar{I}, J) & \overset{i_0^*}{\longrightarrow} & Hom(L, J) & \to 0
\end{array} \tag{4.2.1.1}$$

By applying G-cohomology to diagram(4.2.1.1) we get a commutative diagram

$$\begin{array}{ccc}
H^2(G, Hom(\mathbb{Z}S, J)) & \overset{\acute{\imath}_1^*}{\longrightarrow} & H^2(G, Hom(\Delta S, J)) \\
\partial_M \big\uparrow & & \big\uparrow \partial_{\bar{I}} \\
H^1(G, Hom(I, J)) & \underset{i_1^*}{\longrightarrow} & H^1(G, Hom(L, J))
\end{array}$$

where the vertical maps are isomorphisms, since $Hom(\bar{I}, J)$ and $Hom(M, J)$ are cohomologically trivial by Claim 4.1.2. The diagram above proves the surjectivity of $\acute{\imath}_1^* : H^2(G, Hom(\mathbb{Z}S, J)) \to H^2(G, Hom(\Delta S, J))$. □

4.2.2 Remark. Part ii) of Claim 4.2.1 implies that

$$\bar{\beta}_* : H^2(G, Hom(\Delta S, E)) \oplus H^2(G, Hom(\mathbb{Z}S, J)) \to H^2(G, Hom(\Delta S, J))$$

is surjective. This together with part i) of Claim 4.2.1 allow us to write the long exact sequence given in (4.2.0.2) as the following short exact sequence

$$\begin{aligned}
0 \longrightarrow H^2(G, Hom((\mathbb{Z}S), (J))) &\overset{\bar{i}_*}{\longrightarrow} H^2(G, Hom(\Delta S, E)) \oplus H^2(G, Hom(\mathbb{Z}S, J)) \\
\overset{\bar{\beta}_*}{\longrightarrow} H^2(G, Hom(\Delta S, J)) &\longrightarrow 0
\end{aligned} \tag{4.2.2.1}$$

Fixing $\mathfrak{P} \in S_*$let us denote by $\hat{\imath}_{\mathfrak{P}}$ the $\mathbb{Z}G_{\mathfrak{P}}$-homomorphism obtained by the composition

$$\hat{\imath}_{\mathfrak{P}} : \mathbb{Z} \longrightarrow ind_{G_{\mathfrak{P}}}^G \mathbb{Z} \longrightarrow \mathbb{Z}S.$$

Given y in $H^2(G, Hom(\mathbb{Z}S, J))$ let $y_{\mathfrak{P}}$ in $H^2(G_{\mathfrak{P}}, K^*_{\mathfrak{P}})$ be the image of y under the following composition

$$\begin{array}{ccc} H^2(G, Hom(\mathbb{Z}S, J)) \xrightarrow{res^G_{G_{\mathfrak{P}}}} H^2(G_{\mathfrak{P}}, Hom(\mathbb{Z}S, J)) & & \\ \quad\quad\quad\quad \downarrow \hat{i}^*_{\mathfrak{P}} & & \\ H^2(G_{\mathfrak{P}}, J) & \xrightarrow{k^{\mathfrak{P}}_*} & H^2(G_{\mathfrak{P}}, K^{\times}_{\mathfrak{P}}). \end{array}$$

Let $\bar{u}^3 : Hom((\mathbb{Z}S), (J)) \to Hom(\mathbb{Z}, C_K)$ be the $\mathbb{Z}G$-homomorphism defined by $\bar{u}^3(f_1, f_2, f_3) = f_3$, then $\bar{u}^3$ induces a homomorphism

$$\bar{u}^3_* : H^2(G, Hom((\mathbb{Z}S), (J))) \to H^2(G, C_K).$$

Let x in $H^2(G, Hom(\Delta S, E))$, part ii) of Claim 4.2.1 ensures the existence of y in $H^2(G, Hom(\mathbb{Z}S, J))$ such that $j_*x = i_1^*y$ or equivalently $(x, y) \in ker(\bar{\beta}_*)$, exactness of the sequence (4.2.2.1) implies that there is a unique element t in $H^2(G, ((\mathbb{Z}S), (J)))$ with $\bar{u}_*t = (x, y)$, let $z = \bar{u}^3_*t \in H^2(G, C_K)$ then we define

$$\epsilon' x = inv\ z - \sum_{\mathfrak{P} \in S_*} inv_{\mathfrak{P}}\ y_{\mathfrak{P}}. \quad (4.2.2.2)$$

4.3 Compatibility of t_E and ϵ'

By considering the following two exact sequences

$$0 \longrightarrow \bar{I} \xrightarrow{i_1} M \xrightarrow{\acute{i}_1} \mathbb{Z}G \longrightarrow 0$$
$$0 \longrightarrow E \xrightarrow{j} J \xrightarrow{\acute{j}} C_K \longrightarrow 0,$$

we can construct the $\mathbb{Z}G$-module $Hom((M), (J))$ which fits into the following exact sequence, as done in the previous sections

$$0 \longrightarrow Hom((M), (J)) \xrightarrow{(\hat{u}^1, \hat{u}^2)} Hom(\bar{I}, E) \oplus Hom(M, J) \xrightarrow{\hat{\beta}} Hom(\bar{I}, J) \longrightarrow 0 \quad (4.3.0.1)$$

The exact sequences given in (4.1.3.3), (4.2.0.1) and (4.3.0.1) can be arranged into the following diagram

$$
\begin{array}{ccccc}
Hom((I),(J)) & \overset{u_*}{\rightarrowtail} & Hom(L,E)\oplus Hom(I,J) & \overset{\beta_*}{\twoheadrightarrow} & Hom(L,J) \\
\kappa^* \uparrow & & i_1^* \uparrow & & i_0^* \uparrow \\
Hom((M),(J)) & \overset{\hat{u}_*}{\rightarrowtail} & Hom(\bar{I},E)\oplus Hom(M,J) & \overset{\hat{\beta}_*}{\twoheadrightarrow} & Hom(\bar{I},J) \\
\lambda^* \uparrow & & \acute{\imath}_0^* \uparrow & & \acute{\imath}_0 \uparrow \\
Hom((\mathbb{Z}S),(J)) & \overset{\bar{u}_*}{\rightarrowtail} & Hom(\Delta S,E)\oplus Hom(\mathbb{Z}S,J) & \overset{\bar{\beta}_*}{\longrightarrow} & Hom(\Delta S,J)
\end{array}
\tag{4.3.0.2}
$$

where $\lambda^* : Hom((\mathbb{Z}S),(J)) \to Hom((M),(J))$ is defined by $(f_1, f_2, f_3) \mapsto (f_1\acute{\imath}_0, f_2\acute{\imath}_0, f_3\acute{\imath}_0)$, similarly $\kappa^* : Hom((M),(J)) \to Hom((I),(J))$ is given by $(g_1, g_2, g_3) \mapsto (g_1 i_0, g_2 i_0, g_3 i_0)$, exactness of diagram (4.3.0.2) follows straightforward.

It is clear that the middle and right columns of diagram (4.3.0.2) are exact and by the snake lemma follows that the left column is also exact.
Exactness of the middle row of diagram (4.3.0.2) implies that $Hom((M),(J))$ is a cohomologically trivial $\mathbb{Z}G$-module, hence diagram (4.3.0.2) induces in cohomology the following commutative diagram

$$
\begin{array}{ccc}
H^2(G, Hom((\mathbb{Z}S),(J)) & \overset{\partial}{\longleftarrow} & H^1(G, Hom((I),(J)) \\
\bar{u}* \downarrow & & u_* \downarrow \\
H^2(G, Hom(\Delta S, E)) \oplus H^2(G, Hom(\mathbb{Z}S, J)) & \overset{\partial}{\longleftarrow} & H^1(G, Hom(L,E)) \oplus H^1(G, Hom(I,J)) \\
\bar{\beta}_* \downarrow & & \beta_* \downarrow \\
H^2(G, Hom(\Delta S, J)) & \overset{\partial}{\longleftarrow} & H^1(G, Hom(L,J))
\end{array}
\tag{4.3.0.3}
$$

where the columns are exact and the diagonal maps are isomorphisms.

Let x in $H^2(G, Hom(\Delta S, E))$, assume that x' is the preimage of x under the connecting isomorphism $\partial : H^1(G, Hom(L, E)) \to H^2(G, Hom(\Delta S, E))$. We choose y' in $H^1(G, Hom(I, J))$ such that $j_* x' = i_1^* y'$, if $y = \partial y'$ with $\partial : H^1(G, Hom(I, J)) \to H^2(G, Hom(\mathbb{Z}S, J))$, commutativity of diagram (4.3.0.3) implies that (x, y) belongs to the kernel of $\bar{\beta}_*$. We denote by t' and t the unique elements in $H^1(G, Hom((I), (J)))$ and $H^2(G, Hom((\mathbb{Z}S), (J)))$ respectively satisfying $u_* t' = (x', y')$ and $\bar{u}_* t = (x, y)$, once more commutativity of diagram (4.3.0.3) gives $\partial t' = t$. We will denote by $z' = u_*^3 t'$ and by $z = \bar{u}_*^3 t$

We are in position to state the main result of this chapter.

4.3.1 Claim. *With the notation given above it follows that*

$$
\begin{aligned}
t_E\ x' &= inv\ \partial'\ z' - \sum_{\mathfrak{P} \in S_*} inv_{\mathfrak{P}}\ \partial'_{\mathfrak{P}}\ y'_{\mathfrak{P}} \\
&= inv\ z - \sum_{\mathfrak{P} \in S_*} inv_{\mathfrak{P}}\ y_{\mathfrak{P}} = \epsilon'\ x
\end{aligned}
$$

We prove Claim 4.3.1 by showing first that $\partial'\ z' = z$ and then proving that $\partial'_{\mathfrak{P}}\ y'_{\mathfrak{P}} = y_{\mathfrak{P}}$ for each $\mathfrak{P} \in S_*$.

Let us consider the following diagram with exact rows

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & Hom((\mathbb{Z}S), (J)) & \underset{\lambda^*}{\longrightarrow} & Hom((M), (J)) & \underset{\kappa^*}{\longrightarrow} & Hom((I), (J)) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle u_*^3} & & \downarrow{\scriptstyle \hat{u}_*^3} & & \downarrow{\scriptstyle \bar{u}_*^3} & & \\
0 & \longrightarrow & Hom(\mathbb{Z}, C_K) & \overset{\acute{\imath}_0^*}{\longrightarrow} & Hom(\mathbb{Z}G, C_K) & \overset{\acute{\imath}_0^*}{\longrightarrow} & Hom(\Delta G, C_K) & \longrightarrow & 0
\end{array}
\tag{4.3.1.1}
$$

Let (f_1, f_2, f_3) be an element of $Hom((\mathbb{Z}S), (J))$, notice that

$$
\begin{aligned}
\hat{u}_*^3 \lambda_* (f_1, f_2, f_3) &= \hat{u}_*^3 (f_1 \acute{\imath}_0, f_2 \acute{\imath}_0 f_3 \acute{\imath}_0) = f_3 \acute{\imath}_0 \\
&= \acute{\imath}_0^* f_3 = \acute{\imath}_0^* u_*^3 (f_1, f_2, f_3).
\end{aligned}
$$

This last equality proves commutativity of the left square of diagram (4.3.1.1),

commutativity of the right square follows in a similar way.
By applying G-cohomology to diagram (4.3.1.1) we obtain the following commutative diagram

$$\begin{array}{ccc} H^1(G, Hom((I),(J))) & \xrightarrow{\partial} & H^2(G, Hom((\mathbb{Z}S),(J))) \\ \Big\downarrow{\scriptstyle u^3_*} & & \Big\downarrow{\scriptstyle \bar{u}^3_*} \\ H^1(G, Hom(\Delta G, C_K)) & \xrightarrow{\partial'} & H^2(G, C_K) \end{array} \tag{4.3.1.2}$$

Commutativity of the diagram (4.3.1.2) implies that

$$\partial'\ z' = \partial'\ u^3_* t' = \bar{u}^3_* \partial t' \tag{4.3.1.3}$$

$$\bar{u}^3_* t = z. \tag{4.3.1.4}$$

In order to prove that $\partial'_{\mathfrak{P}}\ y'_{\mathfrak{P}} = y_{\mathfrak{P}}$ for all primes we need the following claim.

4.3.2 Claim. *If G is a group, N a $\mathbb{Z}G$-module, S a finite set of subgroups of G and for each $H \in S$, M_H is a $\mathbb{Z}H$-module. Let $M = \bigoplus_{H\in S} Ind^G_H M_H$, $i_H : M_H \to M$ be the natural $\mathbb{Z}H$-monomorphism and $l_H : Ind^G_H M_H \to M$ be the natural $\mathbb{Z}G$-monomorphism. Then the following diagram commutes for all $n \geq 1$.*

$$\begin{array}{ccc} H^n(G, Hom(M,N)) & \xrightarrow{res^G_H} & H^n(H, Hom(M,N)) \\ \Big\downarrow{\scriptstyle l^*_H} & & \Big\downarrow{\scriptstyle i^*_H} \\ H^n(G, Hom(Ind^G_H M_H, N)) & \xrightarrow[Sh]{} & H^n(G, Hom(M_H, N)) \end{array}$$

Proof. Let us recall the dual $\mathbb{Z}G$-isomorphism $\psi : Hom(L, Hom(M,N)) \to Hom(L \otimes M, N)$, introduced at the end of page 19. In this case we have that

$$\psi^{-1} : Hom(ind^G_H M_H, N) \to Hom_{\mathbb{Z}H}(\mathbb{Z}G, Hom(M_H, N)),$$

where f in $Hom(ind^G_H M_H, N)$ is sent to $\psi^{-1} f$, the element defined by $\psi^{-1} f(g) = f_g : M_H \to N$ where $m \mapsto f(g \otimes_H m)$ for all m in M_H.

We will also recall the homomorphism $\Theta_B : B^\sharp \to B$ defined at the end of page 14 and consider the case when $B = Hom(M_H, N)$, in this case

$$\Theta_{Hom(M_H,N)} : Hom_{\mathbb{Z}H}(\mathbb{Z}G, Hom(M_H, N)) \to Hom(M_H, N)$$

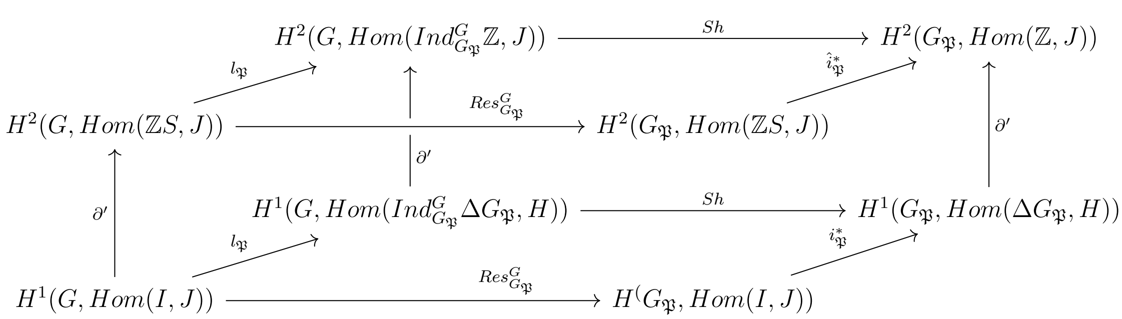
is simply evaluation at 1_G.
Since $Sh = (res^G_H, \Theta_{Hom(M_H,N)})^* \psi_*$, our statement is equivalent to prove that the following diagram commutes

$$\begin{array}{ccc}
H^n(G, Hom(M,N)) & \xrightarrow{res^G_H} & H^n(H, Hom(M,N)) \\
l^*_H \downarrow & & \downarrow i^*_H \\
H^n(G, Hom(ind^G_H M_H, N)) & & H^n(H, Hom(M_H, N)) \\
\psi_* \downarrow & & \uparrow \Theta^*_{Hom(M_H,N)} \\
H^n(G, Hom(M_H, N)^*) & \xrightarrow[res^G_H]{} & H^n(H, Hom(M_H, N)^*)
\end{array}$$

Commutativity of the diagram above can be easily done at the level of cocycles.

□

We now consider the following diagram, which we state that is commutative.



(4.3.2.1)

Commutativity of the back face follows from functoriality of "Shapiro's isomorphism", while commutativity of the the top and lower faces follows by a direct application of Claim 4.3.2.
Commutativity of the left face follows by applying G-cohomology to the fol-

lowing commutative diagram

$$
\begin{array}{ccccc}
I & \overset{i_1}{\rightarrowtail} & M & \overset{\acute{i}_1}{\twoheadrightarrow} & \mathbb{Z}S \\
\uparrow l_{\mathfrak{P}} & & \uparrow l_{\mathfrak{P}} & & \uparrow l_{\mathfrak{P}} \\
Ind_{G_{\mathfrak{P}}}^{G} \Delta G_{\mathfrak{P}} & \rightarrowtail & \mathbb{Z}G & \twoheadrightarrow & Ind_{G_{\mathfrak{P}}}^{G} \mathbb{Z}
\end{array}
$$

Finally commutativity of

$$
\begin{array}{ccc}
H^2(G_{\mathfrak{P}}, J) & \xrightarrow{k_*^{\mathfrak{P}}} & H^2(G_{\mathfrak{P}}, K_{\mathfrak{P}}^x) \\
\uparrow \partial' & & \uparrow \partial' \\
H^1(G_{\mathfrak{P}}, Hom(\Delta G_{\mathfrak{P}}, J)) & \xrightarrow[k_*^{\mathfrak{P}}]{} & H^1(G_{\mathfrak{P}}, Hom(\Delta G_{\mathfrak{P}}, K_{\mathfrak{P}}^x))
\end{array}
\tag{4.3.2.2}
$$

Implies that

$$
\begin{aligned}
\partial'[y'_{\mathfrak{P}}] &= \partial' k_*^{\mathfrak{P}} i_{\mathfrak{P}}^* Res_{G_{\mathfrak{P}}}^G [y'] && \text{by (4.2.0.1)} \\
&= k_*^{\mathfrak{P}} \partial' i_{\mathfrak{P}}^* Res_{G_{\mathfrak{P}}}^G [y'] && \text{by diagram (4.3.2.2)} \\
&= k_*^{\mathfrak{P}} i_{\mathfrak{P}}^* Res_{G_{\mathfrak{P}}}^G \partial' [y'] && \text{by diagram (4.3.2.1)} \\
&= k_*^{\mathfrak{P}} \hat{i}_{\mathfrak{P}}^* Res_{G_{\mathfrak{P}}}^G [y] \\
&= [y_{\mathfrak{P}}] && \text{by 4.1.3.2}
\end{aligned}
\tag{4.3.2.3}
$$

Equations (4.3.2.3) and (4.3.1.3) prove Claim 4.3.1.

Claim 4.3.1 states that if x is a two cocycle in $H^2(G, Hom(\Delta S, \boldsymbol{\mu}))$ and x' is a 1-cocycle in $H^1(G, Hom(L, \boldsymbol{\mu}))$ such that $\alpha_* x = \partial'' x'$, then by 4.1.2.3

$$
\begin{aligned}
\epsilon(x) &= t_E (\partial'')^{-1} \alpha_*(x) \\
&= t_E(x') = \epsilon'(x).
\end{aligned}
\tag{4.3.2.4}
$$

Chapter 5

The character ϵ on the group $H^2(G, Hom(\Delta S, \boldsymbol{\mu}))$.

The possibility of computing ϵ depends highly on our ability of finding generators of $H^2(G, Hom(\Delta S, \boldsymbol{\mu}))$, on which ϵ can be evaluated.

By considering the exact sequence

$$0 \longrightarrow \boldsymbol{\mu} \overset{\acute{i}_1^*}{\longrightarrow} Hom(\mathbb{Z}S, \boldsymbol{\mu}) \overset{i_1^*}{\longrightarrow} Hom(\Delta S, \boldsymbol{\mu}) \to 0,$$

obtained after applying the functor $Hom(-, \boldsymbol{\mu})$ to the $\mathbb{Z}$-split augmentation sequence

$$0 \longrightarrow \Delta S \overset{i_1}{\longrightarrow} \mathbb{Z}S \overset{\acute{i}_1}{\longrightarrow} \mathbb{Z} \longrightarrow 0,$$

one obtains after applying Tate cohomology, the following long exact sequence

$$\cdots \overset{\acute{i}_1^*}{\longrightarrow} H^2(G, Hom(\mathbb{Z}S, \boldsymbol{\mu})) \overset{i_1^*}{\longrightarrow} H^2(G, Hom(\Delta S, \boldsymbol{\mu})) \overset{\partial}{\longrightarrow} H^3(G, \boldsymbol{\mu})$$
$$\overset{\acute{i}_1^*}{\longrightarrow} H^3(G, Hom(\mathbb{Z}S, \boldsymbol{\mu})) \overset{i_1^*}{\longrightarrow} \cdots \tag{5.0.0.1}$$

In this chapter we will study the character ϵ by understanding its behaviour on the image of $i_1^* : H^2(G, Hom(\mathbb{Z}S, \boldsymbol{\mu})) \to H^2(G, Hom(\Delta S, \boldsymbol{\mu}))$, then we will concentrate on

$$coker(i_1^*) \cong ker\left(\acute{i}_1^* : H^3(G, \boldsymbol{\mu}) \to H^3(G, Hom(\mathbb{Z}S, \boldsymbol{\mu}))\right).$$

Through the study of $coker(i_1^*)$ we will show that 2 annihilates the Tate cohomology of G with values in $\boldsymbol{\mu}$, hence we will reformulate our program by passing to the ring $\mathbb{Z}' = \mathbb{Z}[1/2]$.
We conclude this chapter by showing a partial reduction formula to the square-free case.

5.1 ϵ restricted to the image of i_1^*.

In this section we will prove that for $\tilde{x}$ in $H^2(G, Hom(\mathbb{Z}S, \boldsymbol{\mu}))$, $\epsilon(i_1^*\tilde{x})$ can be computed only in terms of the local invariants, more precise

$$\epsilon(i_1^*\tilde{x}) = -\sum_{\mathfrak{P}\in S_*} inv_{\mathfrak{P}}\ (j_*\tilde{x})_{\mathfrak{P}}. \tag{5.1.0.1}$$

In order to simplify the notation we will dimension shift the argument and work with the character t_E defined in chapter 4.

Let $\tilde{x}$ be a fix element in $H^1(G, Hom(I, E))$ and fix a 1-cocycle $g \mapsto \tilde{x}_g$ whose class in cohomology is $\tilde{x}$, this induces the following two 1-cocycles

$$g \mapsto x_g = \tilde{x}_g i_1$$
$$g \mapsto y_g = j\tilde{x}_g$$

If we denote by x (respectively y) to be the class of the cocycle $g \mapsto x_g$ (resp $g \mapsto y_g$) it follows immediately that (x, y) belongs to the $ker(\beta_*)$, since $j_*x = j_*i_1^*\tilde{x} = i_1^*j_*\tilde{x} = i_1^*y$.

We can consider the following diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & L & \xrightarrow{i_1} & I & \xrightarrow{\acute{\imath}_1} & \Delta G & \longrightarrow & 0 \\
 & & {\scriptstyle x_g}\big\downarrow & & {\scriptstyle y_g}\big\downarrow & & {\scriptstyle z_g}\big\downarrow & & \\
0 & \longrightarrow & E & \xrightarrow{j} & J & \xrightarrow{\acute{\jmath}} & C_K & \longrightarrow & 0
\end{array} \tag{5.1.0.2}$$

where the left square commutes. We now prove the existence of $z_g : \Delta G \to C_K$ making the above diagram commutes. Let a in ΔG and b in I such that $\acute{\imath}_1(b) = a$, define

$$z_g(a) = \acute{\jmath} y_g(b).$$

The map z_g is well defined. To prove this assume that b_1 in I also satisfies $\acute{\imath}_1(b_1) = a$, then exactness of the top row ensures the existence of c in L such that $b = b_1 + i_1(c)$, which implies

$$\begin{aligned}
\acute{\jmath} y_g(b) &= \acute{\jmath} y_g(b_1) + \acute{\jmath} y_g(i_1(c)) \\
&= \acute{\jmath} y_g(b_1) + \acute{\jmath} j x_g(c) \\
&= \acute{\jmath} y_g(b_1).
\end{aligned}$$

By definition of z_g follows the commutativity of the right square of diagram (5.1.0.2).
It follows clearly that $g \mapsto z_g$ is a 1-cocycle with values in $Hom(\Delta G, C_K)$ since $\delta z = \acute{\jmath}_* \delta y = 0$. We now take the 1-cocycle with values in $Hom((I),(J))$ defined by $g \mapsto (x_g, y_g, z_g)$, then its class t in $H^1(G, Hom((I),(J)))$ satisfies $u_*(t) = (x, y)$. Commutativity of diagram (5.1.0.2) gives that $z_g \acute{\imath}_1 = \acute{\jmath} y_g = \acute{\jmath} j \tilde{x}_g = 0$, which implies $z_g = 0$ since $\acute{\imath}_1 : I \to \Delta G$ is a surjection. The fact that $z_g = 0$ for all g in G implies that $u_*^3(t) = 0$ in $H^1(G, Hom(\Delta G, C_K))$ which proves (5.1.0.1).

The dimension shifting described in chapter 4, which relates the characters ϵ and t_E, gives that for any $\tilde{x}$ in $H^2(G, Hom(\mathbb{Z}S, \boldsymbol{\mu}))$

$$\epsilon(\tilde{x}) = -\sum_{\mathfrak{P} \in S_*} inv_{\mathfrak{P}}\ (j_* \tilde{x})_{\mathfrak{P}}. \tag{5.1.0.3}$$

5.2 About the cokernel of i_1^*.

We will now concentrate on studying

$$\begin{aligned} &coker\left(i_1^*: H^2(G, Hom(\mathbb{Z}S, \boldsymbol{\mu})) \to H^2(G, Hom(\Delta S, \boldsymbol{\mu}))\right) \\ \cong &ker\left(\acute{i}_1^*: H^3(G, \boldsymbol{\mu}) \to H^3(G, Hom(\mathbb{Z}S, \boldsymbol{\mu}))\right). \end{aligned}$$

It is necessary for our purpose to to describe the group $H^3(G, \boldsymbol{\mu})$.
We begin with the following two technical claims.

5.2.1 Claim. *i) Let q be an odd prime number, $G = Aut(\mathbb{Z}/q^r\mathbb{Z})$ with $r \geq 1$, then $(\mathbb{Z}/q^r\mathbb{Z})$ is a cohomologically trivial $\mathbb{Z}G$-module.*

ii) Let $G = Aut(\mathbb{Z}/2^{r+1}\mathbb{Z})$ for $r \geq 1$ and G_1 be the kernel of the natural projection $G \to Aut(\mathbb{Z}/4\mathbb{Z})$, then $\mathbb{Z}/2^{r+1}\mathbb{Z}$ is a cohomologically trivial $\mathbb{Z}G_1$-module.

Proof. We will only proof here (i) since the proof of (ii) follows a similar argument. One can prove by induction on s that

$$(1+q)^{q^{s-1}} \equiv 1 + q^s \pmod{q^{s+1}},$$

for all $s \geq 1$. Notice that for the case $s = 1$ the equality follows immediately. By induction hypothesis there exist an integer A such that

$$\begin{aligned} (1+q)^{q^s} &= \left[(1+q)^{q^{s-1}}\right]^q = \left[1 + q^s + Aq^{s+1}\right]^q \\ &= (1+q^s)^q + \sum_{j=1}^{q} \binom{q}{j} (1+q^s)^{q-j} (Aq^{s+1})^j, \end{aligned}$$

hence it would be enough to show that q^{s+2} divides $(1+q^s)^q - (1+q^{s+1})$, which follows easily from the equality

$$(1+q^s)^q - (1+q^{s+1}) = \sum_{j=2}^{q} \binom{q}{j} q^{sj}.$$

We identify G with $(\mathbb{Z}/q^r\mathbb{Z})^\times$ and denote the kernel of the map $G \to (\mathbb{Z}/q^s\mathbb{Z})^\times$ by G_s for $1 \leq s \leq r$, then G_s is cyclic with generator the class of $(1+q)^{q^{s-1}}$ or equivalently the class of $(1+q^s)$ and $|G_s| = q^{r-s}$. The following two equalities hold

$$\begin{aligned}(\mathbb{Z}/q^r\mathbb{Z})^{G_s} &= \{x \in \mathbb{Z}/q^r\mathbb{Z} : (1+q^s)x \equiv x \pmod{q^r}\} \\ &= q^{r-s}(\mathbb{Z}/q^r\mathbb{Z}) \quad \text{and}\end{aligned}$$

$$\begin{aligned}\hat{G}_s = \sum_{j=0}^{q^{r-s}-1}(1+q)^{q^{s-1}j} &= \frac{(1+q)^{q^{s-1}q^{r-s}}-1}{(1+q)^{q^{s-1}}-1} \\ = \frac{(1+q)^{q^{r-1}}-1}{(1+q)^{q^{s-1}}-1} &= \frac{q^r x_r}{q^s x_s} = q^{r-s}\frac{x_r}{x_s},\end{aligned}$$

for some integers x_r and x_s coprime to q. This implies that $H^0(G, \mathbb{Z}/q^r\mathbb{Z}) = 0$. The Herbrand quotient and the fact the G_s is cyclic imply that $H^i(G, \mathbb{Z}/q^r\mathbb{Z}) = 0$ for all integers i. Clearly $\mathbb{Z}/q^r\mathbb{Z}$ is a cohomologically trivial $\mathbb{Z}G_1$-module and since every subgroup H of G satisfies $H \cap G_1 = G_s$ for some $s \geq 1$, one obtains that $H^i(H \cap G_1, \mathbb{Z}/q^r\mathbb{Z}) = 0$ for all i. The inflation restriction exact sequence given in (2.1.3.1) implies that

$$inf : H^i\left(H/H \cap G_1, (\mathbb{Z}/q^r\mathbb{Z})^{H \cap G_1}\right) \to H^i(H, \mathbb{Z}/q^r\mathbb{Z})$$

is an isomorphism. The second isomorphism theorem gives $H/H \cap G_1 \cong HG_1/G_1$ which is a subgroup of $G/G_1 \cong (\mathbb{Z}/q\mathbb{Z})^\times$ that has order coprime to q, implying that $H^i\left(H/H \cap G_1, (\mathbb{Z}/q^r\mathbb{Z})^{H \cap G_1}\right) = 0$, from where one concludes that $\mathbb{Z}/q^r\mathbb{Z}$ is a cohomologically trivial $\mathbb{Z}G$-module. □

We can identify $\boldsymbol{\mu}$ with $\mathbb{Z}/2m\mathbb{Z}$ where G acts according to the group homomorphism $\bar{a} : G \to Aut\,(\mathbb{Z}/2m\mathbb{Z})$ defined in (3.0.0.1). We denote by $N^{(o)} := \prod G^o_{\mathfrak{P}}$, where $G^o_{\mathfrak{P}}$ denotes the ramification subgroup of the decomposition group $G_{\mathfrak{P}}$, and the product is taken over all primes $\mathfrak{P}$ in S_* such that $\mathfrak{P}|p$ and p an odd prime dividing m.
It follows that:

$$\boldsymbol{\mu}^{N^{(o)}} = \boldsymbol{\mu}(2),$$

where $\boldsymbol{\mu}(q)$ denotes the q-primary component of $\boldsymbol{\mu}$ for any prime q, and that $K^{N^{(o)}} = \mathbb{Q}(\boldsymbol{\mu}(2))$, which is $\mathbb{Q}$ when m is odd.

Let $\boldsymbol{\upsilon} = \prod \boldsymbol{\mu}(p)$, where the product is taken over all odd primes $p|m$ hence

$$\boldsymbol{\mu} = \boldsymbol{\upsilon} \times \boldsymbol{\mu}(2) \text{ and}$$
$$\boldsymbol{\upsilon}^{N^{(o)}} = 1.$$

5.2.2 Claim. *$H^i(N^{(o)}, \boldsymbol{\upsilon}) = 0$ for all $i \geq 1$.*

Proof. By part (i) of Claim 5.2.1 follows that for all odd primes $p|m$, $\boldsymbol{\mu}(p)$ is a cohomologically trivial $\mathbb{Z}G^o_{\mathfrak{P}}$-module. We will use this observation and proceed by induction on the number of primes dividing m.
If $m = 2^{r_2}$, $\boldsymbol{\upsilon} = 1$ and the claim follow immediately.

Now let p be the smallest odd prime dividing m, we can write $m = p^{r_p}m'$ with p not dividing m' and $r_p \geq 1$. If $\mathfrak{P}$ in S_* is the prime of K above p, every prime dividing $|G^o_{\mathfrak{P}}| = p^{r_p-1}(p-1)$ is less than or equal to p. Notice that $\boldsymbol{\upsilon} = \boldsymbol{\upsilon}(p) \times \boldsymbol{\upsilon}_{m'}$ where $\boldsymbol{\upsilon}_{m'} = \prod \boldsymbol{\upsilon}(q)$ and the product is taken over all odd primes q dividing m with $q > p$. It follows that $\boldsymbol{\upsilon}(p)$ and $\boldsymbol{\upsilon}_{m'}$ are cohomologically trivial $\mathbb{Z}G_{\mathfrak{P}}$-modules, hence so is $\boldsymbol{\upsilon}$. Let us define $N^{(o)}_{m'} = N^{(o)}/G^o_{\mathfrak{P}}$. Since $\boldsymbol{\upsilon}^{G^o_{\mathfrak{P}}} = \boldsymbol{\upsilon}_{m'}$, the inflation-restriction exact sequence gives isomorphisms

$$inf : H^i(N^{(0)}_{m'}, \boldsymbol{\upsilon}_{m'}) \overset{\cong}{\rightarrow} H^i(N^{(o)}, \boldsymbol{\upsilon}).$$

By hypothesis of induction $H^i(N^{(o)}, \boldsymbol{\upsilon}) \cong H^i(N^{(0)}_{m'}, \boldsymbol{\upsilon}_{m'}) = 0$, which completes the proof. □

We now denote by $\tilde{m}$ the largest squarefree divisor of m, $\tilde{\boldsymbol{\mu}}$ the set of all roots of the polynomial $x^{2\tilde{m}} - 1$, $\tilde{K} = \mathbb{Q}(\tilde{\boldsymbol{\mu}})$ and $\tilde{G} = Gal(\tilde{K}/\mathbb{Q})$. We let N be the kernel of the natural surjection $G \to \tilde{G}$. For s and t integers we will denote by $(s,t) = gcd(s,t)$.
5.2.3 Claim. *The group $\boldsymbol{\mu}$ is a cohomologically trivial $\mathbb{Z}N$-module.*

Proof. Let us write $m = \prod_{p|m} p^{r_p}$ such that all $r_p \geq 1$, then

$$|N| = \frac{(2,m)\prod_{p|m}\phi(p^{r_p})}{(2,m)\prod_{p|m}\phi(p)} = \prod_{p|m} p^{r_p-1}.$$

It follows that N is the product of its p-primary subgroups $N(p)$, all of which are cyclic hence so is N. Every subgroup M of N can be seen as $M = \prod_{p|m} M(p)$ with $M(p) = M \cap N(p)$. The fact that N is cyclic reduces the statement to show that $H^1(M, \boldsymbol{\mu}) = 0$ for any subgroup M of N. Let us fix p an odd prime dividing m and denote by $\boldsymbol{\mu}_p$ the set of all p^{r_p}th-roots of unity, then $\boldsymbol{\mu} = \boldsymbol{\mu}_p \times \boldsymbol{\mu}'_p$ with $m' = m/p^{r_p}$ and $\boldsymbol{\mu}'_p \cong \mathbb{Z}/2m'\mathbb{Z}$. One obtains that

$$\begin{aligned} H^1(M(p), \boldsymbol{\mu}) &\cong H^1(M(p), \boldsymbol{\mu}_p) \oplus H^1(M(p), \boldsymbol{\mu}'_p) \\ &\cong H^1(M(p), \boldsymbol{\mu}_p) = 0. \end{aligned}$$

The last equality follows from part (i) of Claim 5.2.1. The inflation-restriction sequence gives

$$H^1(M, \boldsymbol{\mu}) \cong H^1(M/M(p), \boldsymbol{\mu}^{M(p)}),$$

with $\boldsymbol{\mu}^{M(p)} \cong \boldsymbol{\mu}_p^{M(p)} \times \boldsymbol{\mu}'_p$. It follows that $H^1(M/M(p), \boldsymbol{\mu}^{M(p)}) \cong H^1(M/M(p), \boldsymbol{\mu}'_p)$. We can now use induction on the number of odd primes dividing m to obtain that $H^1(M, \boldsymbol{\mu}) \cong H^1(M(2), \boldsymbol{\mu}(2))$ which is cohomologically trivial by (ii) of Claim 5.2.1. □

The next is a series of results relating the cohomology of G and the cohomology of $\tilde{G}$.

5.2.4 Claim. *$H^i(G, \boldsymbol{\mu}) \cong H^i(\tilde{G}, \tilde{\boldsymbol{\mu}})$ for all $i \geq 1$.*

Proof. By Claim 5.2.3 it follows that $H^i(N, \boldsymbol{\mu}) = 0$ for all $i \geq 1$, then the inflation-restriction exact sequence gives isomorphisms

$$inf : H^i(G/N, \boldsymbol{\mu}^N) \xrightarrow{\cong} H^i(G, \boldsymbol{\mu}).$$

The fact that $\tilde{G} \cong G/N$ and that $\boldsymbol{\mu}^N = \tilde{\boldsymbol{\mu}}$ concludes the proof. □

5.2.5 Claim. *$H^i(G, \boldsymbol{\mu}) \cong H^i(\tilde{G}, \boldsymbol{\mu}_{2(2,m)})$, where $\boldsymbol{\mu}_{2(2,m)}$ denotes the set of all roots of the polynomial $X^{2(2,m)} - 1$.*

Proof. In claim 5.2.2 we proved that $H^i(N^{(o)}, \boldsymbol{\upsilon}) = 0$ for all $i \geq 1$, then the inflation restriction exact sequence gives that the homomorphism

$$inf : H^i(G/N^{(o)}, \boldsymbol{\upsilon}^{N^{(o)}}) \xrightarrow{\cong} H^i(G, \boldsymbol{\upsilon})$$

is actually an isomorphism. Since $\boldsymbol{\upsilon}^{N^{(o)}} = 1$, the above isomorphism implies that $H^i(G, \boldsymbol{\upsilon}) = 0$ for all $i \geq 1$. We can decompose $\boldsymbol{\mu}$ by $\boldsymbol{\mu} = \boldsymbol{\mu}(2) \times \boldsymbol{\upsilon}$. The last observation gives that $H^i(G, \boldsymbol{\mu}) \cong H^i(G, \boldsymbol{\mu}(2))$. By Claim 5.2.3 $\boldsymbol{\mu}(2)$ is a cohomologically trivial $\mathbb{Z}N$-module, once more the inflation-restriction exact sequence gives isomorphisms

$$inf : H^1(\tilde{G}, \boldsymbol{\mu}(2)^N) \xrightarrow{\cong} H^1(G, \boldsymbol{\mu}(2))$$

for all $i \geq 1$. The statement then follows after noticing that $\boldsymbol{\mu}(2)^N = \boldsymbol{\mu}_{2(2,m)}$.
□

The last claim says that $H^3(G, \boldsymbol{\mu}) \cong H^3(G, \mathbb{F}_2)$ in the case when m is odd and that $H^3(G, \boldsymbol{\mu}) \cong H^3(G, \boldsymbol{\mu}_4)$ in the case when m is even, where $\boldsymbol{\mu}_4$ is the set of roots of the polynomial $X^4 - 1$.

Claim 5.2.4 gives us the hope that one can reduce the exact sequence given in (5.0.0.1) to a similar exact sequence in terms of $\tilde{G}$, we show here a first step towards this.

Let $\tilde{S} = \left\{ \tilde{\mathfrak{P}} = \mathfrak{P} \cap \tilde{K} : \mathfrak{P} \in S \right\}$ and define the natural surjective G-map $S \to \tilde{S}$ where $\mathfrak{P} \mapsto \tilde{\mathfrak{P}}$, this maps extends to a surjective $\mathbb{Z}G$-homomorphism $\mathbb{Z}S \to \mathbb{Z}\tilde{S}$. By taking coinvariants one obtains a $\mathbb{Z}G$-isomorphism

$$\mathbb{Z}S_N \xrightarrow{\cong} \mathbb{Z}\tilde{S}.$$

We now notice that $gcd\left((N : N_{\mathfrak{P}}) : \mathfrak{P} \in S_*\right) = 1$ since $N_{\mathfrak{P}}$ contains $N(p)$, then there exist integers $\alpha_{\mathfrak{P}}$ such that

$$\sum_{\mathfrak{P} \in S_*} \alpha_{\mathfrak{P}} (N : N_{\mathfrak{P}}) = 1.$$

By fixing a left transversal $X_{\mathfrak{P}}$ of $N_{\mathfrak{P}}$ in N, let $x_{\mathfrak{P}} = \sum_{n \in X_{\mathfrak{P}}} n\mathfrak{P}$ and define the homomorphism $\mathbb{Z} \to \mathbb{Z}S$ where $1 \mapsto \sum_{\mathfrak{P} \in S_*} \alpha_{\mathfrak{P}} x_{\mathfrak{P}}$, we conclude that the augmentation sequence

$$0 \longrightarrow \Delta S \longrightarrow \mathbb{Z}S \longrightarrow \mathbb{Z} \longrightarrow 0$$

$\mathbb{Z}N$-splits, which implies that the following diagram with exact rows commutes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Delta S_N & \rightarrow & \mathbb{Z}S_N & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 \\ & & \downarrow & & =\downarrow & & =\downarrow & & \\ 0 & \longrightarrow & \Delta\tilde{S} & \longrightarrow & \mathbb{Z}\tilde{S} & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 \end{array}$$

We can identify ΔS_N with $\Delta\tilde{S}$ and consider the following commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Hom(\mathbb{Z}, \tilde{\boldsymbol{\mu}}) & \longrightarrow & Hom(\mathbb{Z}\tilde{S}, \tilde{\boldsymbol{\mu}}) & \longrightarrow & Hom(\Delta\tilde{S}, \tilde{\boldsymbol{\mu}}) & \rightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \rightarrow & Hom(\mathbb{Z}, \boldsymbol{\mu})^N & \rightarrow & Hom(\mathbb{Z}S, \boldsymbol{\mu})^N & \longrightarrow & Hom(\Delta S, \boldsymbol{\mu})^N & \rightarrow & 0 \end{array}$$

By taking Tate cohomology we obtain the following commutative diagram

$$\begin{array}{ccccccc} H^2(G, Hom(\mathbb{Z}S, \boldsymbol{\mu})) & \longrightarrow & H^2(G, Hom(\Delta S, \boldsymbol{\mu})) & \longrightarrow & H^3(G, \boldsymbol{\mu}) & \longrightarrow & H^3(G, Hom(\mathbb{Z}S, \boldsymbol{\mu})) \\ inf\uparrow\cong & & inf\uparrow\cong & & inf\uparrow\cong & & inf\uparrow\cong \\ H^2\left(\tilde{G}, Hom(\mathbb{Z}S\boldsymbol{\mu})^N\right) & \longrightarrow & H^2\left(\tilde{G}, Hom(\Delta S, \boldsymbol{\mu})^N\right) & \rightarrow & H^3(\tilde{G}, \tilde{\boldsymbol{\mu}}) & \rightarrow & H^3\left(\tilde{G}, Hom(\mathbb{Z}S\boldsymbol{\mu})^N\right) \end{array} \tag{5.2.5.1}$$

From Claim 5.2.3 follows that $\boldsymbol{\mu}$ is a cohomologically trivial $\mathbb{Z}N$-module, then for any lattice L one has, by pg 141 of [13], that $Hom(L, \boldsymbol{\mu})$ is also $\mathbb{Z}N$-cohomologically trivial. This implies that the inflation maps, which are the vertical maps of diagram (5.2.5.1), are actually isomorphisms.

To make sense of the bottom row of the diagram (5.2.5.1) one must understand $Hom(\mathbb{Z}S, \boldsymbol{\mu})^N$ as a $\mathbb{Z}\tilde{G}$-module.

5.2.6 Claim. *Let* $S_{\mathbb{Q}} = \left\{p : p \text{ is a prime of } \mathbb{Q} \text{ such that } \tilde{\mathfrak{P}}|p \text{ for some } \tilde{\mathfrak{P}} \in \tilde{S}\right\}$.

It follows that

$$Hom(\mathbb{Z}S, \boldsymbol{\mu})^N \cong \bigoplus_{p \in S_{\mathbb{Q}}} ind_{\tilde{G}_{\tilde{\mathfrak{P}}}}^{\tilde{G}} \boldsymbol{\mu}^{N_{\mathfrak{P}}}.$$

In order to prove Claim 5.2.6 we need the following claim.

5.2.7 Claim. *Let H be a subgroup of G and denote the image of H under the surjection $G \to \tilde{G}$ by $\tilde{H}$, it follows that*

$$Hom(ind_H^G \mathbb{Z}, M)^N \cong ind_{\tilde{H}}^{\tilde{G}}(M^{N \cap H})$$

for any $\mathbb{Z}G$-module M.

Proof. Let us choose subsets X and Y of G such that 1_G belongs to both X and Y and satisfying

$$G = \bigsqcup_{x \in X} xNH$$
$$N = \bigsqcup_{y \in Y} y(N \cap H),$$

then $\tilde{G} = \bigsqcup_{x \in X} \tilde{x}\tilde{H}$ and $G = \bigsqcup_{x \in X} \bigsqcup_{y \in Y} xyH$, which implies that $ind_H^G \mathbb{Z}$ has $\mathbb{Z}$-basis $\{xy \otimes_{\mathbb{Z}H} 1 : x \in X, y \in Y\}$.

Let W be the $\mathbb{Z}$-span of $\{y \otimes_{\mathbb{Z}H} 1 : y \in Y\}$, hence W is a $\mathbb{Z}(NH)$-submodule of $ind_H^G \mathbb{Z}$ since $h \cdot (y \otimes_{\mathbb{Z}H} 1) = hyh^{-1} \otimes_{\mathbb{Z}H} 1$. Being hyh^{-1} an element of N there are unique $y_1 \in Y$ and $n_1 \in N \cap H$ such that $hyh^{-1} = y_1 n_1$, hence $h \cdot (y \otimes_{\mathbb{Z}H} 1) = y_1 \otimes_{\mathbb{Z}H} 1$.

It also follows that $ind_H^G \mathbb{Z} = \bigoplus_{x \in X} xW$.

If we fix a x in X, xW is a $\mathbb{Z}(x(NH)x^{-1})$-submodule, notice that $x(NH)x^{-1} = NxHx^{-1}$ hence

$$Hom(ind_H^G \mathbb{Z}, M) = \bigoplus_{x \in X} Hom(xW, M),$$

where $Hom(xW, M)$ is the submodule of $Hom(ind_H^G \mathbb{Z}, M)$

$$\left\{ f \in Hom(ind_H^G \mathbb{Z}, M) : f|_{x'W} = 0, \text{ for all } x' \neq x \right\}.$$

Seeing as $\mathbb{Z}\tilde{G}$-modules one has that

$$Hom(ind_H^G\mathbb{Z}, M)^N \cong \bigoplus_{x\in X} Hom(xW, M)^N.$$

It is not difficult to prove that $Hom(xW, M) = xHom(W, M)$, hence $Hom(xW, M)^N = \tilde{x}Hom(W, M)^N$ which implies

$$\begin{aligned} Hom(ind_H^G\mathbb{Z}, M)^N &\cong \bigoplus_{x\in X} \tilde{x}Hom(W, M)^N \\ & \bigoplus_{x\in X} ind_{\tilde{H}}^{\tilde{G}}\left(Hom(W, M)^N\right). \end{aligned}$$

The claim follows from the fact $Hom(W, M)^N \cong M^{N\cap H}$ as $\mathbb{Z}\tilde{H}$-modules. In order to prove this last statement notice that the homomorphism $ind_{N\cap H}^N\mathbb{Z} \to W$, where $y \otimes_{\mathbb{Z}N\cap H} 1_G \mapsto y \otimes_{\mathbb{Z}H} 1_G$, is an isomorphism hence by Frobenius reciprocity one has that as $\mathbb{Z}$-modules

$$Hom(W, M)^N \cong Hom(ind_{N\cap H}^N\mathbb{Z}, M)^N \cong Hom(\mathbb{Z}, M)^{N\cap H} \cong M^{N\cap H}$$

as wanted. □

Proof of claim 5.2.6. Let S_p (respectively $S_{\tilde{\mathfrak{P}}}$) be the set of all primes of K above p (resp, the set of all primes of K above $\tilde{\mathfrak{P}}$) then $\mathbb{Z}S = \bigoplus_{p\in S_\mathbb{Q}} \mathbb{Z}S_p$. There is for each $p \in S_\mathbb{Q}$ a unique $\mathfrak{P} \in S_*$ above p and an isomorphism $ind_{G_\mathfrak{P}}^G\mathbb{Z} \to \mathbb{Z}S_p$ where $g\otimes_{\mathbb{Z}G_\mathfrak{P}}1 \mapsto g\mathfrak{P}$. We now apply Claim 5.2.7 with $H = G_\mathfrak{P}$ and $M = \boldsymbol{\mu}$, since $G_\mathfrak{P}\cap N = N_\mathfrak{P}$ we obtain $Hom(\mathbb{Z}S_p, \boldsymbol{\mu})^N \cong ind_{\tilde{G}_{\tilde{\mathfrak{P}}}}^{\tilde{G}}(\boldsymbol{\mu}^{N_\mathfrak{P}})$. □

The last two claims suggest that the following diagram commutes

$$\begin{array}{ccccc} \bigoplus_{\mathfrak{P}\in S_*} H^2(G_\mathfrak{P}, \boldsymbol{\mu}) & \xrightarrow{Sh} & H^2(G, Hom(\mathbb{Z}S, \boldsymbol{\mu})) & \longrightarrow & H^2(G, Hom(\Delta S, \boldsymbol{\mu})) \\ inf\uparrow\cong & & inf\uparrow\cong & & inf\uparrow\cong \\ \bigoplus_{p\in S_\mathbb{Q}} H^2(\tilde{G}_{\tilde{\mathfrak{P}}}, \boldsymbol{\mu}^{N_\mathfrak{p}}) & \xrightarrow{Sh} & H^2(\tilde{G}, Hom(\mathbb{Z}S, \boldsymbol{\mu})^N) & \to & H^2(\tilde{G}, Hom(\Delta S, \boldsymbol{\mu})^N) \end{array} \tag{5.2.7.1}$$

where the inflation maps are all isomorphisms, in fact we only need to prove commutativity of the left square of diagram 5.2.7.1.

Since $\mathbb{Z} \cong \bigoplus_{\mathfrak{P}\in S_*} ind_{G_\mathfrak{P}}^G\mathbb{Z}$ as a $\mathbb{Z}G$-module and from the fact that as $\mathbb{Z}\tilde{G}$-

module $\mathbb{Z}S \cong \bigoplus_{p\in S_{\mathbb{Q}}} \mathbb{Z}S_p$ and $\mathbb{Z}S_p \cong ind_{G_{\mathfrak{P}}}^{G}\mathbb{Z}$ one obtains the following commutative diagram

$$\begin{array}{ccc} \bigoplus_{\mathfrak{P}\in S_*} H^2(G, Hom(ind_{G_{\mathfrak{P}}}^{G}\mathbb{Z}, \boldsymbol{\mu})) & \xrightarrow{\cong} & H^2(G, Hom(\mathbb{Z}S, \boldsymbol{\mu})) \\ inf \uparrow \cong & & inf \uparrow \cong \\ \bigoplus_{p\in S_{\mathbb{Q}}} H^2(\tilde{G}, Hom(ind_{G_{\mathfrak{P}}}^{G}\mathbb{Z}, \boldsymbol{\mu})^N) & \xrightarrow{\cong} & H^2(\tilde{G}, Hom(\mathbb{Z}S, \boldsymbol{\mu})^N) \end{array} \tag{5.2.7.2}$$

In order to show that the left square of diagram 5.2.7.1 commutes, it would be enough to show that for each p in $S_{\mathbb{Q}}$, if $\mathfrak{P}$ is the only prime in S_* above p, the next diagram commutes.

$$\begin{array}{ccc} H^2(G_{\mathfrak{P}}, \boldsymbol{\mu}) & \xleftarrow[\cong]{Sh_{\mathfrak{P}}^{-1}} & H^2(G, Hom(ind_{G_{\mathfrak{P}}}^{G}\mathbb{Z}, \boldsymbol{\mu})) \\ inf \uparrow \cong & & inf \uparrow \cong \\ H^2(\tilde{G}_{\tilde{\mathfrak{P}}}, (\boldsymbol{\mu})^{N_{\mathfrak{P}}}) & \xleftarrow[\cong]{Sh_p^{-1}} & H^2(\tilde{G}, Hom(ind_{G_{\mathfrak{P}}}^{G}\mathbb{Z}, \boldsymbol{\mu})^N) \end{array} \tag{5.2.7.3}$$

We will prove in more generality the following

5.2.8 Claim. *Let G be a group, H a subgroup of finite index and N a normal subgroup of G. Let us denote by $\tilde{G} = G/N$ and the image of H under the projection $G \to \tilde{G}$ by $\tilde{H}$. Then the following diagram commutes*

$$\begin{array}{ccc} H^2(H, M) & \xleftarrow[\cong]{Sh^{-1}} & H^2(G, Hom(ind_H^G\mathbb{Z}, M)) \\ inf \uparrow \cong & & inf \uparrow \cong \\ H^2(\tilde{H}, (M)^{N\cap H}) & \xleftarrow[\cong]{Sh^{-1}} & H^2(\tilde{G}, Hom(ind_H^G\mathbb{Z}, M)^N) \end{array} \tag{5.2.8.1}$$

for any $\mathbb{Z}H$-module M.

We begin by describing the isomorphism

$$Sh^{-1} : H^2(G, Hom(ind_H^G\mathbb{Z}, M)) \to H^2(H, M).$$

Let us denote by $\{B^G_\bullet, \delta_\bullet\}$ the bar resolution of G (the reader can find the definition in subsection 6.1.3) hence we can define

$$\psi_n : Hom_G(B^G_n, Hom(ind^G_H \mathbb{Z}, M)) \to Hom_H(B^G_n, M)$$

in the following way: given x in $Hom_G(B^G_n, Hom(ind^G_H \mathbb{Z}, M))$ let

$$\psi_n x_{g_1,\ldots,g_n} = x_{g_1,\ldots,g_n}(1_G \otimes_H 1). \tag{5.2.8.2}$$

It follows clearly that $\delta^*_{n+1}\psi_n x = \psi_{n+1}\delta^*_n x$, then $\{\psi_n\}$ is a chain map which induces group homomorphisms

$$\psi_n : H^n(G, Hom(ind^G_H \mathbb{Z}, M)) \to H^n(H, M).$$

It is not difficult to show that $\psi_n = Sh^{-1}$.

The following lemma is a well known result, we present the proof here for computational reasons.

5.2.9 Lemma. *Let G be a group, H a subgroup of finite index and M a $\mathbb{Z}H$-module. Let us denote by $coind^G_H M = Hom_{\mathbb{Z}H}(\mathbb{Z}G, M)$ where G acts by*

$$(gf)(s) = f(sg)$$

for all g, s in G. Then $ind^G_H M \cong conind^G_H M$ as $\mathbb{Z}G$-modules.

Proof. We begin by fixing a finite set X of G such that $G = \bigsqcup_{x \in X} Hx$. We define $\psi : coind^G_H M \to ind^G_H M$ by

$$\psi(f) = \sum_{x \in X} x^{-1} \otimes_H f(x). \tag{5.2.9.1}$$

Notice that $(hx)^{-1} \otimes_H f(hx) = x^{-1} \otimes_H f(x)$, hence ψ is well defined and independent of the choice of X. We now prove that ψ is a $\mathbb{Z}G$-homomorphism. Let g be an arbitrary element of G hence

$$\begin{aligned}\psi(gf) &= \sum_{x \in X} x^{-1} \otimes_H (gf)(x) = \sum_{x \in X} x^{-1} \otimes_H f(xg) \\ &= \sum_{x \in X} gg^{-1}x^{-1} \otimes_H f(xg) = g \sum_{x \in X} (xg)^{-1} \otimes_H f(xg) = g\psi(f).\end{aligned}$$

In order to show that ψ is an isomorphism we describe its inverse. Let $z = \sum_{x\in X} x^{-1} \otimes_H m_x$ in $ind_H^G M$. We define $\phi : ind_H^G M \to coind_H^G M$ where $\phi(z) = \phi_z : \mathbb{Z}G \to M$ is the map defined by

$$\phi_z(g) = hm_x \quad \text{where} \quad hx = g. \tag{5.2.9.2}$$

It follows that ϕ well defined since $\phi_z(h_0 g) = h_0\phi_z(g)$ for all h_0 in H.
We now prove that ϕ is a $\mathbb{Z}G$-homomorphism. For this we fix g in G and show that

$$\phi_{gz}(s) = g\phi_z(s) = \phi_z(sg) \tag{5.2.9.3}$$

for all s in G.
Let us assume that $s = h^0 x^0$ for some h^0 in H and x^0 in X. Then $sg = h^0 h' x'$ for some h' in H and x' in X, hence $\phi_z(sg) = h^0 h' m_{x'}$.
Notice that for each x in X there exist h_x in H and y_x in X such that $xg^{-1} = h_x y_x$ it follows that

$$\begin{aligned} gz &= g\sum_{x\in X} x^{-1} \otimes_H m_x = \sum_{x\in X} (xg^{-1})^{-1} \otimes_H m_x \\ &= \sum_{x\in X} y_x^{-1} \otimes_H h_x^{-1} m_x. \end{aligned}$$

Since $s = h^0 x^0$ it follows that $x^0 = y_x = h_x^{-1} x g^{-1}$ is equivalent to $h_0 x^0 = h_0 h_x^{-1} x g^{-1}$, hence

$$h^0 h' x' = sg = h^0 x^0 g = h^0 h_x^{-1} x.$$

We can conclude that $h_x^{-1} = h'$ and $x = x'$ hence $\phi_{gz}(s) = h^0 h' m_{x'}$, which proofs (5.2.9.3).
It follows clearly that $\psi\phi(z) = z$ while

$$\begin{aligned} \phi\psi(f)(s) &= \phi_{\sum_{x\in X}}(s) = h^0 f(x^0) \\ &= f(h^0 x^0) = f(s), \end{aligned}$$

which concludes the proof. □

It follows from the proof of claim 5.2.7 that the $\mathbb{Z}\tilde{G}$-isomorphism

$$Hom(ind_H^G\mathbb{Z}, \boldsymbol{\mu})^N \to ind_{\tilde{H}}^{\tilde{G}}(\boldsymbol{\mu}^{N\cap H})$$

can be expressed by

$$f \mapsto 1_{\tilde{G}} \otimes_{\tilde{H}} f(1_G \otimes_H 1).$$

The isomorphism above induces an isomorphism between $H^2(\tilde{G}, Hom(ind_H^G\mathbb{Z}, M)^N)$ and $H^2(\tilde{G}, ind_{\tilde{H}}^{\tilde{G}}(M^{N\cap H}))$.

The lower horizontal isomorphism of diagram 5.2.8.1 is given by the following composition

$$\begin{array}{ccc} H^2(\tilde{G}, Hom(ind_H^G\mathbb{Z}, M)^N) & \longrightarrow & H^2(\tilde{G}, ind_{\tilde{H}}^{\tilde{G}}(M^{N\cap H})) \\ & \swarrow & \\ H^2(\tilde{G}, coind_{\tilde{H}}^{\tilde{G}}(M^{N\cap H})) & \xrightarrow{Sh^{-1}} & H^2(\tilde{H}, M^{N\cap H}) \end{array} \tag{5.2.9.4}$$

At the level of cocycles the composition above sends the 2-cocycle $\tilde{g}_1, \tilde{g}_2 \mapsto x_{\tilde{g}_1,\tilde{g}_2}$ to the 2-cocycle $\tilde{g}_1, \tilde{g}_2 \mapsto x''_{\tilde{g}_1,\tilde{g}_2}$ where

$$\begin{aligned} x''_{\tilde{g}_1,\tilde{g}_2} &= \phi_{1_{\tilde{G}}\otimes_{\tilde{H}} x_{\tilde{g}_1,\tilde{g}_2}(1_G\otimes_H 1)}(1_{\tilde{G}}) \\ &= x_{\tilde{g}_1,\tilde{g}_2}(1_G \otimes_H 1). \end{aligned} \tag{5.2.9.5}$$

We are in position to prove claim 5.2.8. Let $\tilde{g}_1, \tilde{g}_2 \mapsto x_{\tilde{g}_1,\tilde{g}_2}$ be a 2-cocycle whose class x is in $H^2(\tilde{G}, Hom(ind_{\tilde{H}}^{\tilde{G}}\mathbb{Z}, M)^N)$. By (5.2.8.2) and the definition of the inflation map one obtains that the class x' of the 2-cocycle $g_1, g_2 \mapsto x'_{g_1,g_2}$ given by

$$x'_{g_1,g_2} = x_{\tilde{g}_1,\tilde{g}_2}(1_G \otimes_H 1),$$

satisfies $Sh^{-1} \circ inf(x) = x'$. It also follows clearly by (5.2.9.5) that $inf \circ Sh^{-1}(x) = inf(x'') = x'$, which proves the commutativity of diagram 5.2.8.1.

5.3 The new model.

Recall the augmentation sequence

$$0 \longrightarrow \Delta G \xrightarrow{i_0} \mathbb{Z}G \xrightarrow{\acute{\imath}_0} \mathbb{Z} \longrightarrow 0.$$

5.3.1 Claim. *Let c in centre($\mathbb{Z}G$), with $\acute{\imath}_0(c) = 0$ and M a finitely generated $\mathbb{Z}G$-module, then the action of c on M annihilates all Tate cohomology.*

Proof. If one proves the statement for $\mathbb{Z}G$-lattices then it follows for any $\mathbb{Z}G$-module. This follows since there exists a $\mathbb{Z}G$-projective module P and a surjective $\mathbb{Z}G$-homomorphism $P \to M$. Letting $L = ker(P \to L)$ one obtains the following commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \longrightarrow & P & \longrightarrow & M & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle c_L} & & \downarrow{\scriptstyle c_P} & & \downarrow{\scriptstyle c_M} & & \\ 0 & \longrightarrow & L & \longrightarrow & P & \longrightarrow & M & \longrightarrow & 0 \end{array}$$

By applying Tate cohomology one obtains the following commutative square

$$\begin{array}{ccc} H^n(G,M) & \xrightarrow[\cong]{\partial} & H^{n+1}(G,L) \\ \downarrow{\scriptstyle (c_M)_*} & & \downarrow{\scriptstyle (c_L)_*} \\ H^n(G,M) & \xrightarrow[\cong]{\partial} & H^{n+1}(G,L) \end{array}$$

where the connecting homomorphisms are actually isomorphisms. If $(c_L)_*$ is the zero map, commutativity of the last diagram implies that $(c_M)_*$ is the zero map.

In order to prove the statement for for lattices let us assume that M has no torsion, recalling that $M^o = Hom(M, \mathbb{Z})$ there exists a projective module P and a surjection $P \to M^o$, letting $L = ker(P \to M^o)$ one obtains the following exact sequence

$$0 \longrightarrow L \longrightarrow P \longrightarrow M^o \longrightarrow 0,$$

we now apply the functor $(\ )^o$ to the last exact sequence and using the fact that $(\ )^o$ is an exact contravariant functor in the category of $\mathbb{Z}G$-lattices, that sends $\mathbb{Z}G$-projectives to $\mathbb{Z}G$-projectives, one obtains that

$$0 \longrightarrow (M^o)^o \rightarrow P^o \longrightarrow L^o \longrightarrow 0$$

is an exact sequence. Since $(M^o)^o \cong M$ the following diagram with exact row commutes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & P^o & \longrightarrow & L^o & \longrightarrow & 0 \\ & & c_M \downarrow & & c_{P^o} \downarrow & & c_{L^o} \downarrow & & \\ 0 & \longrightarrow & M & \longrightarrow & P^o & \longrightarrow & L^o & \longrightarrow & 0 \end{array}$$

Once more, we take Tate cohomology and obtain a commutative square

$$\begin{array}{ccc} H^n(G, L^o) & \xrightarrow[\cong]{\partial} & H^{n+1}(G, M) \\ (c_{L^o})_* \downarrow & & (c_M)_* \downarrow \\ H^n(G, L^o) & \xrightarrow[\cong]{\partial} & H^{n+1}(G, M) \end{array}$$

where the horizontal maps are connecting isomorphism. The last diagram shows that it would be enough to prove that $(c_M)_* : H^0(G, M) \rightarrow H^0(G, M)$ is the zero map for an arbitrary lattice M.
There is a natural surjection $M^G \rightarrow H^0(G, M)$, If $b = \sum_{g \in G} b_g g$ is an element in $\mathbb{Z}G$ and m is an element of M^G one has that

$$\begin{aligned} b \cdot m &= \sum_{g \in G} b_g (g \cdot m) = \left(\sum_{g \in G} b_g \right) m \\ &= \acute{\imath}_0(b) m. \end{aligned}$$

Clearly if c belongs to $ker(\acute{\imath}_0)$, the last equation shows that $c_{M^G} : M^G \rightarrow M^G$ is given by $c_{M^G}(m) = \acute{\imath}_0(c) m = 0$. Finally commutativity of the following diagram

$$\begin{array}{ccc} M^G & \twoheadrightarrow & H^0(G,M) \\ {\scriptstyle (c_{M^G})_*}\downarrow & & {\scriptstyle (c_M)_*}\downarrow \\ M^G & \twoheadrightarrow & H^0(G,M) \end{array}$$

shows that $(c_M)_* : H^0(G,M) \to H^0(G,M)$ is the zero map as wanted. □

Notice that Claim 5.3.1 is still true if c is in $centre(\mathbb{Z}G)$ and $\acute{\imath}_0(c) \equiv 0 \pmod{|G|}$ since $|G|M^G$ is a subset of $\hat{G}M^G$.

5.3.2 Claim. *$H^*(G, \boldsymbol{\mu})$ is annihilated by 2.*

Proof. Let $\bar{c}$ in G be the complex conjugation then $\bar{c}(\mu) = \mu^{-1}$ for all μ in $\boldsymbol{\mu}$. Let $c = 1_G - \bar{c}$ and notice that this element satisfies the conditions of Claim 5.3.1 hence $c_* : H^n(G, \boldsymbol{\mu}) \to H^n(G, \boldsymbol{\mu})$ is the zero map.
It follows that $c(\mu) = \mu(\mu^{-1})^{-1} = \mu^2$ for all $\mu \in \boldsymbol{\mu}$ hence c_* is given by multiplication by 2 on $H^n(G, \boldsymbol{\mu})$, which proves the statement. □

The fact that 2 annihilates the Tate cohomology of G with coefficients in $\boldsymbol{\mu}$ led us to consider working in the category of $\mathbb{Z}'G$-modules, where $\mathbb{Z}' = \mathbb{Z}[1/2]$. We will show next the advantages of this change of categories.

Given a $\mathbb{Z}G$-module M, let $M' = \mathbb{Z}' \otimes M$ considered as a $\mathbb{Z}'G$-module in the natural way. Given a $\mathbb{Z}'G$-module N we will denote by $H^n_{\mathbb{Z}'}(G,N) = Ext^n_{\mathbb{Z}'G}(\mathbb{Z}', N)$.
The next claim is a well know fact, hence we omit the proof here.

5.3.3 Claim. *Let M be a finitely generated $\mathbb{Z}G$-module then*

$$\mathbb{Z}' \otimes H^n(G,M) \cong H^n_{\mathbb{Z}'}(G, M').$$

From Claims 5.3.3 and 5.3.2 we obtain that $H^n_{\mathbb{Z}'}(G, \boldsymbol{\mu}') = 0$ for all n in $\mathbb{Z}$, then the exact sequence given in (5.1.0.1) under this new model gives an isomorphism

$$i_1^* : H^2_{\mathbb{Z}'}(G, Hom(\mathbb{Z}'S, \boldsymbol{\mu}')) \xrightarrow{\cong} H^2_{\mathbb{Z}'}(G, Hom_{\mathbb{Z}'}((\Delta S)', \boldsymbol{\mu}')).$$

By Claims 5.3.3 and 5.2.3 one obtains that $\boldsymbol{\mu}'$ is a cohomologically trivial $\mathbb{Z}'N$-module, the same statement holds for $Hom_{\mathbb{Z}'}(\mathbb{Z}'S, \boldsymbol{\mu}')$ and $Hom_{\mathbb{Z}'}((\Delta S)', \boldsymbol{\mu}')$, hence the commutative diagram given in (5.2.7.1) can be rewritten in the following way

$$\begin{array}{ccccc}
\bigoplus_{\mathfrak{P}\in S_*} H^2_{\mathbb{Z}'}(G_{\mathfrak{P}}, \boldsymbol{\mu}') & \xrightarrow{Sh} & H^2_{\mathbb{Z}'}(G, Hom_{\mathbb{Z}'}(\mathbb{Z}'S, \boldsymbol{\mu}')) & \xrightarrow{i_1^*} & H^2_{\mathbb{Z}'}(G, Hom_{\mathbb{Z}'}((\Delta S)', \boldsymbol{\mu}')) \\
inf \uparrow \cong & & inf \uparrow \cong & & inf \uparrow \cong \\
\bigoplus_{p\in S_{\mathbb{Q}}} H^2_{\mathbb{Z}'}(\tilde{G}_{\tilde{\mathfrak{P}}}, (\boldsymbol{\mu}')^{N_{\mathfrak{P}}}) & \xrightarrow{Sh} & H^2_{\mathbb{Z}'}(\tilde{G}, Hom_{\mathbb{Z}'}(\mathbb{Z}'S, \boldsymbol{\mu}')^N) & \xrightarrow{i_1^*} & H^2_{\mathbb{Z}'}(\tilde{G}, Hom_{\mathbb{Z}'}((\Delta S)', \boldsymbol{\mu}')^n)
\end{array}$$

where all maps are isomorphism.

The last diagram shows that under this new set up, one can always reduce the computation of ϵ to compute local invariant in a tamely ramified sub extensions, we will see in the next chapter that the computation of local invariants in tamely ramified extension of $\mathbb{Q}$ can be done in an algorithmic way.

Chapter 6

Generator of $H^2(G_{\mathfrak{P}}, \boldsymbol{\mu})$.

One way to have sufficient control over the character ϵ is to find generators for $H^2(G, Hom(\Delta S, \boldsymbol{\mu}))$ where ϵ can be evaluated. A first step towards achieving this is to find generators for $H^2(G, Hom(\mathbb{Z}S, \boldsymbol{\mu}))$ which is isomorphic (by Shapiro's Lemma) to $\bigoplus_{\mathfrak{P} \in S_*} H^2(G_{\mathfrak{P}}, \boldsymbol{\mu})$.

The main objective of this chapter will be to study the groups $H^2(G_{\mathfrak{P}}, \boldsymbol{\mu})$ for a fix $\mathfrak{P}$ in S_*.
In the case when $\mathfrak{P}$ is non archimedean, we will use the filtration given by the Hochschild-Serre spectral sequence associated to the group extension

$$G^o_{\mathfrak{P}} \rightarrowtail G_{\mathfrak{P}} \twoheadrightarrow \bar{G}_{\mathfrak{P}},$$

where $G^o_{\mathfrak{P}}$ denotes the ramification subgroup of $G_{\mathfrak{P}}$ and $\bar{G}_{\mathfrak{P}} = G_{\mathfrak{P}}/G^o_{\mathfrak{P}}$, to show that finding generators for the groups $H^{2-i}(\bar{G}_{\mathfrak{P}}, H^i(G^o_{\mathfrak{P}}, \boldsymbol{\mu}))$ for $i = 0, 1, 2$ will induce a set of generators for $H^2(G_{\mathfrak{P}}, \boldsymbol{\mu})$.

This chapter is organized as follows. In the first section we will use the Hochschild-Serre spectral sequence for the particular case when $\boldsymbol{\mu} \cong \mathbb{Z}/2m\mathbb{Z}$ with m an odd squarefree positive integer, to find explicit generators of $H^{2-i}(\bar{G}_{\mathfrak{P}}, H^i(G^o_{\mathfrak{P}}, \boldsymbol{\mu}))$.

In the second section we keep the assumption that m is an odd squarefree positive integer and evaluate $inv_{\mathfrak{P}}$ in the set of generators found in the previous

section.
It should be mentioned that in this chapter H^* will denote regular cohomology while $\hat{H}^*$ denotes Tate cohomology.

6.1 Set of generators for $H^2(G_{\mathfrak{P}}, \boldsymbol{\mu})$.

From this point on in this chapter, we will assume that $\boldsymbol{\mu} \cong \mathbb{Z}/2m\mathbb{Z}$ for m an odd squarefree positive integer.

We mention at the beginning of the chapter that for a fix $\mathfrak{P} \in S_*$, $G^o_{\mathfrak{P}}$ will denote the ramification subgroup of $G_{\mathfrak{P}}$ and that $\bar{G}_{\mathfrak{P}} = G_{\mathfrak{P}}/G^o_{\mathfrak{P}}$, then we have proved in Section 3 of Chapter 2 that the second term of the Hochschild-Serre spectral sequence associated to the group extension

$$G^o_{\mathfrak{P}} \rightarrowtail G_{\mathfrak{P}} \twoheadrightarrow \bar{G}_{\mathfrak{P}},$$

has the following form $E_2^{p,q} = H^p(\bar{G}_{\mathfrak{P}}, H^q(G^o_{\mathfrak{P}}, \boldsymbol{\mu}))$.

We want to begin this section by showing that for any prime number p which divides m, $\boldsymbol{\mu}$ can be factor by $\boldsymbol{\mu} = \boldsymbol{\mu}_p \oplus \boldsymbol{\mu}'_p$ where $gcd\left(|\boldsymbol{\mu}_p|, |G^o_{\mathfrak{P}}|\right) = 1$ and $\mathfrak{P} \in S_*$ is the prime above p. Also that $G_{\mathfrak{P}} = G^o_{\mathfrak{P}} \times \langle\sigma_p\rangle$ where σ_p is an element in $G_{\mathfrak{P}}$ such that its image $\bar{\sigma}_p$ under the map $G_{\mathfrak{P}} \to \bar{G}_{\mathfrak{P}}$ is a Frobenius automorphism. This will imply that

$$E_2^{p,q} \cong H^p(\langle\sigma_p\rangle, H^q(G^o_{\mathfrak{P}}, \boldsymbol{\mu}'_p)).$$

Since we are interested in $E_2^{p,q}$ when $p+q = 2$ our next task will be to compute $H^i(G^o_{\mathfrak{P}}, \boldsymbol{\mu}'_p)$ and $H^{2-i}(\langle\sigma_p\rangle, H^i(G^o_{\mathfrak{P}}, \boldsymbol{\mu}'_p))$ for $i = 0, 1, 2$.
We will conclude this section by presenting explicit generators for $H^2(G_{\mathfrak{P}}, \boldsymbol{\mu})$.

6.1.1 Two observations.

Let p be a prime number dividing m and $\mathfrak{P}$ be the prime in S_* above p, there is a $\mathbb{Z}G$-homomorphism

$$\rho : \boldsymbol{\mu} \to K_{\mathfrak{P}}^{\times} \tag{6.1.1.1}$$

given by $u \mapsto u + \mathfrak{P}$ for all $u \in \boldsymbol{\mu}$.
Let us choose a generator μ of $\boldsymbol{\mu}$ and denote $m_p = m/p$. Since $ker(\rho)$ consist of all the pth-roots of unity of E, $ker(\rho) = \langle \mu_p \rangle$ where $\mu_p = \mu^{2m_p}$. We will denote by $\boldsymbol{\mu}_p = ker(\rho)$.

The following exact sequence

$$\boldsymbol{\mu}_p \overset{i}{\rightarrowtail} \boldsymbol{\mu} \twoheadrightarrow im(\rho) \tag{6.1.1.2}$$

$\mathbb{Z}G^o_{\mathfrak{P}}$-splits. In order to prove this, let α be an integer such that $1 + 2m_p\alpha \equiv 0 \pmod p$ and define $\psi : \boldsymbol{\mu} \to \boldsymbol{\mu}_p$ by $\mu \mapsto \mu_p^{-\alpha}$ which is clearly a $\mathbb{Z}G^o_{\mathfrak{P}}$-homomorphism, it follows that $\psi i = Id_{\boldsymbol{\mu}_p}$ since

$$\psi i(\mu_p) = \mu_p^{-2m_p\alpha} = \mu_p,$$

by the choice of α.

Let $\mu'_p = \mu\mu_p^{\alpha}$ and $\boldsymbol{\mu}'_p = \langle \mu'_p \rangle \cong im(\rho)$ then $\boldsymbol{\mu} = \boldsymbol{\mu}_p \oplus \boldsymbol{\mu}'_p$.

Since $|\boldsymbol{\mu}_p| = p$ and $|G^o_{\mathfrak{P}}| = p - 1$ it follows that $gcd(|\boldsymbol{\mu}_p|, |G^o_{\mathfrak{P}}|) = 1$ from where we conclude that

$$H^n(G^o_{\mathfrak{P}}, \boldsymbol{\mu}) \cong H^n(G^o_{\mathfrak{P}}, \boldsymbol{\mu}'_p) \quad \text{for all } n \geq 0. \tag{6.1.1.3}$$

6.1.2 Claim. *If $\sigma_{\mathfrak{P}} \in G_{\mathfrak{P}}$ is such that $\bar{\sigma}_p$ is a Frobenius automorphism*

$$G_{\mathfrak{P}} \cong G^o_{\mathfrak{P}} \oplus \langle \sigma_{\mathfrak{P}} \rangle.$$

6.1.3 The groups $H^i(G^o_{\mathfrak{P}}, \boldsymbol{\mu}'_p)$ for $i = 0, 1, 2$.

This subsection will focus on finding generators x^i for the groups $H^i(G^o_{\mathfrak{P}}, \boldsymbol{\mu}'_p)$ for $i = 0, 1, 2$.
We should mention at this point that throughout this chapter we will change the notation that has been used in previous chapters, regarding element in cohomology groups and cocycles in order to make the equations easier to read.

We will use two type of projective resolutions namely the bar resolution and the cyclic resolution.
Given a finite group G we will denote by $\{B^G_\bullet, \delta_\bullet\}$ to be the bar resolution

$$\cdots \xrightarrow{\delta_3} B^G_2 \xrightarrow{\delta_2} B^G_1 \xrightarrow{\delta_1} B^G_0 \xrightarrow{\epsilon^G} \mathbb{Z},$$

where $B^G_0 = \mathbb{Z}G$, B^G_n is the $\mathbb{Z}G$-free module with $\mathbb{Z}G$-basis $\{[g_1|\cdots|g_n] : g_1, \ldots, g_n \in G\}$ for $n \geq 1$, ϵ^G is the augmentation map and

$$\delta_n[g_1|\cdots|g_n] = g_1[g_2|\cdots|g_n] + \sum_{i=1}^{n-1}(-1)^i[g_1|\cdots|g_ig_{i+1}|\cdots|g_n] + (-1)^n[g_1|\cdots|g_{n-1}].$$

In the case when G is cyclic let us fix a generator s of G, $\{C^G_\bullet, \delta_\bullet\}$ will denote the cyclic resolution

$$\cdots \xrightarrow{\delta_3} C^G_2 \xrightarrow{\delta_2} C^G_1 \xrightarrow{\delta_1} C^G_0 \xrightarrow{\epsilon^G} \mathbb{Z},$$

where $C^G_n = \mathbb{Z}G$, ϵ^G is the augmentation map, and δ_n is multiplication by $s-1$ for n odd while $\delta_n = N_G$ for n even.

The following is a series of known results in cohomology.
6.1.4 Claim. *Let G be a finite group and M an abelian group. If G acts trivially on M it follows*

i) $H^0(G, M) = M$

ii) $H^1(G, M) \cong Hom_{\mathbb{Z}}(G, M)$

iii) $\hat{H}^0(G, M) = M/|G|M$.

The proof of this statement follows by using the bar resolution of G. Also notice that if G is cyclic and M is a $\mathbb{Z}G$-module, the cyclic resolution of G implies that $\hat{H}^n(G,M)$ depends only on the parity of n.

6.1.5 Claim. *Let G be a cyclic group and M a $\mathbb{Z}G$-module then $\hat{H}^n(G,M) \cong \hat{H}^{n+2}(G,M)$ for all n in $\mathbb{Z}$.*

6.1.6 Remark. Since $G^o_{\mathfrak{P}}$ is cyclic of order $p-1$, which acts trivially on $\boldsymbol{\mu}'_p$ whenever p divides m, it follows that

i) $H^0(G^o_{\mathfrak{P}}, \boldsymbol{\mu}'_p) = \boldsymbol{\mu}'_p$

ii) $H^1(G^o_{\mathfrak{P}}, \boldsymbol{\mu}'_p) \cong Hom_{\mathbb{Z}}(G^o_{\mathfrak{P}}, \boldsymbol{\mu}'_p)$

iii) $H^2(G^o_{\mathfrak{P}}, \boldsymbol{\mu}'_p) \cong \hat{H}^o(G^o_{\mathfrak{P}}, \boldsymbol{\mu}'_p) = \boldsymbol{\mu}'_p/|G^o_{\mathfrak{P}}|\boldsymbol{\mu}'_p$.

From remark 6.1.6 one obtains that $|H^0(G^o_{\mathfrak{P}}, \boldsymbol{\mu}'_p)| = 2m_p$ and if we denote $v_p = gcd(2m_p, p-1)$ then $|H^1(G^o_{\mathfrak{P}}, \boldsymbol{\mu}'_p)| = |H^2(G^o_{\mathfrak{P}}, \boldsymbol{\mu}'_p)| = v_p$.

We will now concentrate on finding generators for each of this cohomology groups.

Let G be a finite cyclic group and consider the following exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Q} \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0, \tag{6.1.6.1}$$

where G acts trivially on each module. The fact that $|G|\mathbb{Q} = \mathbb{Q}$ and that $Hom_{\mathbb{Z}}(G,\mathbb{Q}) = 0$ implies that $\mathbb{Q}$ is $\mathbb{Z}G$-cohomologically trivial. After applying Tate cohomology to the sequence (6.1.6.1) one obtains a connecting isomorphisms

$$\partial_n : \hat{H}^n(G,\mathbb{Q}/\mathbb{Z}) \to \hat{H}^{n+1}(G,\mathbb{Z}), \tag{6.1.6.2}$$

for all n. In the particular case when $n = 1$ we obtain

$$H^2(G,\mathbb{Z}) \cong H^1(G,\mathbb{Q}/\mathbb{Z}) \cong Hom_{\mathbb{Z}}(G,\mathbb{Q}/\mathbb{Z}).$$

Let s be a fix generator of G and define $\bar{\theta}_s : G \to \mathbb{Q}/\mathbb{Z}$ by

$$\bar{\theta}_s(s^i) = \frac{i}{|G|} + \mathbb{Z}. \tag{6.1.6.3}$$

It follows that $\bar{\theta}_s$ is a generator of $Hom_{\mathbb{Z}}(G, \mathbb{Q}/\mathbb{Z})$.

We now define $\tilde{\theta}_s : B_1^G \to \mathbb{Q}/\mathbb{Z}$ by

$$\tilde{\theta}_s[s^i] = \frac{i}{|G|} + \mathbb{Z}. \tag{6.1.6.4}$$

It is not difficult to show that $\delta\tilde{\theta}_s = 0$ and that $[\tilde{\theta}_s]$ is a generator of $H^1(G, \mathbb{Q}/\mathbb{Z})$. It also follows clearly that ∂_2 given in (6.1.6.2) sends $[\tilde{\theta}_s]$ to $[\theta_s]$ where θ_s in $Hom_{\mathbb{Z}G}(B_2^G, \mathbb{Z})$ is given by

$$\theta_s[s^i|s^j] = \begin{cases} 0 & \text{if } \ i + j < |G| \\ 1 & \text{if } \ i + j \geq |G| \end{cases}$$

We must mention here that we choose $0 \leq i, j < |G|$.

6.1.7 Claim. *Let G be acyclic group and M a $\mathbb{Z}G$-module. The generator $[\theta_s]$ of $H^2(G, \mathbb{Z})$ induces an isomorphism, which by abuse of notation will be denoted by $[\theta_s] : H^n(G, M) \to H^{n+2}(G, M)$ for every integer n, given by the following composition*

$$H^n(G, M) \xrightarrow{[\theta_s]\sqcup} H^{n+2}(G, \mathbb{Z} \times M) \xrightarrow{\phi_*} H^{n+2}(G, M) \tag{6.1.7.1}$$

where $\phi : \mathbb{Z} \times M \to M$ maps (t, m) to tm and $\sqcup$ is the cup product.

From this point on s_p denotes a generator of the cyclic group $G^o_{\mathfrak{P}}$.
Let $\tilde{x}^0 = \mu'_p$ in $H^0(G^o_{\mathfrak{P}}, \boldsymbol{\mu}'_p)$, $\tilde{x}^1$ in $Hom_{\mathbb{Z}}(B_1^{G^o_{\mathfrak{P}}}, \boldsymbol{\mu}'_p)$ be defined by

$$\tilde{x}^1[s_p^i] = (\mu'_p)^{it_p} \quad \text{where} \quad t_p = 2m_p/v_p,$$

and $\hat{x}^0 = (\mu'_p)^{t_p}$ in $\hat{H}^0(G^o_{\mathfrak{P}}, \boldsymbol{\mu}'_p)$.

By Claim 6.1.4 it follows that $[\tilde{x}^0]$ and $[\tilde{x}^1]$ are generators of $H^0(G^o_{\mathfrak{P}}, \boldsymbol{\mu}'_p)$ and $H^1(G^o_{\mathfrak{P}}, \boldsymbol{\mu}'_p)$ respectively, it also follows that $[\hat{x}^0]$ generates $\hat{H}^0(G^o_{\mathfrak{P}}, \boldsymbol{\mu}'_p)$.

We can now apply Claim 6.1.7 for $n=0$ to conclude that $\tilde{x}^2 = \phi_*(\theta_{s_{\mathfrak{P}}} \sqcup \hat{x}^0)$, which has the following expression

$$\tilde{x}^2[s_p^i|s_p^j] = \begin{cases} 1 & \text{if } \ i+j < |G_{\mathfrak{P}}^o| \\ (\mu_p')^{t_p} & \text{if } \ i+j \geq |G_{\mathfrak{P}}^o| \end{cases}$$

is a 2-cocycle whose class $[\tilde{x}^2]$ generates $H^2(G_{\mathfrak{P}}^o, \boldsymbol{\mu}_p')$.

It will be shown in the next section that we actually require cocycles x^i in $Hom_{\mathbb{Z}G_{\mathfrak{P}}^o}(B_i^{G_{\mathfrak{P}}}, \boldsymbol{\mu}_p')$ such that their classes $[x^i]$ generate $H^i(G_{\mathfrak{P}}^o, \boldsymbol{\mu}_p')$ for $i = 0, 1, 2$.

Let G be a finite group and H a subgroup of G. Fixing X a right transversal of H in G one can define a map $\phi^X : G \to H$ by $\phi^X(hx) = h$ for all h in H and x in X, then ϕ^X extends to a $\mathbb{Z}H$-homomorphism $\phi_n^X : B_n^G \to B_n^H$ as follows

$$\phi_n^X[h_1x_1|\cdots|h_nx_n] = [h_1|\cdots|h_n].$$

it can easily be proved that $\delta_n^H \phi_n^X = \phi_{n-1}^X \delta_n^G$. Since $\left\{B_\bullet^G, \delta_\bullet\right\}$ is a $\mathbb{Z}H$-projective resolution for $\mathbb{Z}$ the following Claim holds.

6.1.8 Claim. *Let G,H and X as above. For any $\mathbb{Z}G$-module M the homomorphism*

$$(\phi_n^X)^* : H^n(H, M) \to H^n(H, M)$$

is an isomorphism for all integers n.

The proof of this Claim can be found on pg 113 [12].

By Claim 6.1.2 we can take $X = \left\{\sigma_p^j : 0 \leq j < f_{\mathfrak{P}}\right\}$, then $\phi^X : G_{\mathfrak{P}} \to G_{\mathfrak{P}}^o$ is given by $\phi^X(s_p^i\sigma_p^j) = s_p^i$. By Claim 6.1.8 if $x^i = \tilde{x}^i \circ \phi_i^x$ it follows that $[x^i]$ is a generator of $H^i(G_{\mathfrak{P}}^o, \boldsymbol{\mu}_p')$ for $i = 0, 1, 2$. We can explicitly compute

$$x^0(s_p^i\sigma_p^j) = \tilde{x}^0(s_p^i) = \mu_p' \tag{6.1.8.1}$$

$$x^1[s_p^i\sigma_p^j] = \tilde{x}^1[s_p^i] = (\mu_p')^{it_p} \tag{6.1.8.2}$$

$$\begin{aligned} x^2[s_p^{i_1}\sigma_p^{j_1}|s_p^{i_2}\sigma_p^{j_2}] &= \tilde{x}^2[s_p^{i_1}|s_p^{i_2}] \\ &= \begin{cases} 1 & \text{if } \ i+j < |G_{\mathfrak{P}}^o| \\ (\mu_p')^{t_p} & \text{if } \ i+j \geq |G_{\mathfrak{P}}^o| \end{cases} \end{aligned} \tag{6.1.8.3}$$

6.1.9 The groups $H^{2-i}(\bar{G}_{\mathfrak{P}}, H^i(G_{\mathfrak{P}}^o, \mu))$.

Our next task is to compute the groups $H^{2-i}(\bar{G}_{\mathfrak{P}}, H^i(G_{\mathfrak{P}}^o, \mu))$ for $i = 0, 1, 2$.

We start by noticing that

$$\sigma_{\mathfrak{P}}(\mu_p')^{t_p} = (\mu_p')^{pt_p} = (\mu_p')^{t_p},$$

the equality above follows since $2m_p$ divides $(p-1)t_p$.

This last observation implies by (6.1.8.2) (respectively (6.1.8.3)) that the action of $\bar{G}_{\mathfrak{P}}$ on $H^1(G_{\mathfrak{P}}^o, \boldsymbol{\mu}_p')$ (resp $H^2(G_{\mathfrak{P}}^o, \boldsymbol{\mu}_p')$) is trivial.

By Claim 6.1.4 (i) one can state that $H^0(\bar{G}_{\mathfrak{P}}, H^2(G_{\mathfrak{P}}^o, \boldsymbol{\mu}_p')) \cong H^2(G_{\mathfrak{P}}^o, \boldsymbol{\mu}_p')$ meaning that we can define the 0-cocycle y^0 in $Hom_{\bar{G}_{\mathfrak{P}}}(B_0^{\bar{G}_{\mathfrak{P}}}, H^2(G_{\mathfrak{P}}^o, \boldsymbol{\mu}_p'))$ in the following way

$$y^0(\bar{\sigma}_p) = [x^2], \tag{6.1.9.1}$$

where $\bar{\sigma}_p$ denotes the image of σ_p under the natural projection of $G_{\mathfrak{P}}$ onto $\bar{G}_{\mathfrak{P}}$.

Being $H^2(G_{\mathfrak{P}}^o, \boldsymbol{\mu}_p')$ cyclic with generator $[x^2]$ it follows that $[y^0]$ is a generator of $H^0(\bar{G}_{\mathfrak{P}}, H^2(G^o_{\mathfrak{P}}, \boldsymbol{\mu}'_p))$.

Claim 6.1.4 (ii) gives $H^1(\bar{G}_{\mathfrak{P}}, H^1(G^0_{\mathfrak{P}}, \boldsymbol{\mu}'_p)) \cong Hom_{\mathbb{Z}}(\bar{G}_{\mathfrak{P}}, H^1(G^o_{\mathfrak{P}}, \boldsymbol{\mu}'_p))$. Since $\bar{G}_{\mathfrak{P}}$ is cyclic of order $f_{\mathfrak{P}}$ and $H^1(G^o_{\mathfrak{P}}, \boldsymbol{\mu}'_p)$ is cyclic of order v_p, if we let $v'_p = gcd(f_{\mathfrak{P}}, v_p)$ and $t'_p = v_p/v'_p$, we can define the 1-cocycle y^1 as follows

$$y^1[\bar{\sigma}_p] = t_p[x^1]. \tag{6.1.9.2}$$

Then $[y^1]$ generates $H^1(\bar{G}_{\mathfrak{P}}, H^1(G^o_{\mathfrak{P}}, \boldsymbol{\mu}'_p))$.

In order to compute $H^2(\bar{G}_{\mathfrak{P}}, H^0(G^o_{\mathfrak{P}}, \boldsymbol{\mu}'_p))$ we must first compute the group $\hat{H}^0(\bar{G}_{\mathfrak{P}}, H^0(G^o_{\mathfrak{P}}, \boldsymbol{\mu}'_p))$. Notice that $(\boldsymbol{\mu}'_p)^{\bar{G}_{\mathfrak{P}}} = \langle (\mu'_p)^{t_p} \rangle$ and that

$$N_{\bar{G}_{\mathfrak{P}}}(\mu'_p) = (\mu'_p)^{\sum_{i=0}^{f_{\mathfrak{P}}-1} p^i}.$$

The $\mathbb{Z}G$-homomorphism $\rho : \boldsymbol{\mu} \to K^\times_{\mathfrak{P}}$ given in (6.1.1.1) and the fact that $\boldsymbol{\mu}'_p \cong im(\rho)$ give that $|\boldsymbol{\mu}'_p| = 2m_p$ divides $|K^\times_{\mathfrak{P}}| = p^{f_{\mathfrak{P}}} - 1$. We recall that $t_p v_p = 2m_p$, which implies that t_p divides $p^{f_{\mathfrak{P}}} - 1$. Since $gcd(t_p, p-1) = 1$ it follows that t_p must divide $\sum_{i=0}^{f_{\mathfrak{P}}-1} p^i$. Let $n_p = \left(\sum_{i=0}^{f_{\mathfrak{P}}-1} p^i\right)/t_p$, $v''_p = gcd(v_p, n_p)$ and $t''_p = v_p/v''_p$, then the class of the 0-cocycle $\hat{y}^0$ given by

$$\hat{y}^0(\bar{\sigma}_p) = t''_p[x^0]$$

is a generator of $\hat{H}^0(\bar{G}_{\mathfrak{P}}, H^0(G^o_{\mathfrak{P}}, \boldsymbol{\mu}'_p))$. Applying Claim 6.1.7 one can show that the class of the 2-cocycle y^2 given by

$$y^2[\bar{\sigma}_p^{i_1} | \bar{\sigma}_p^{i_2}] = \begin{cases} 0 & \text{if } i_1 + i_2 < |\bar{G}_{\mathfrak{P}}| \\ t''_p[x^0] & \text{if } i_1 + i_2 \geq |\bar{G}_{\mathfrak{P}}| \end{cases} \tag{6.1.9.3}$$

generates $H^2(\bar{G}_{\mathfrak{P}}, H^0(G^o_{\mathfrak{P}}, \boldsymbol{\mu}'_p))$.

6.1.10 Set of generators of $H^2(G_{\mathfrak{P}}, \boldsymbol{\mu})$ for $\mathfrak{P}$ a ramified prime.

Let $\mathfrak{P}$ in S_* be the prime above the prime number p which divides m. Following the idea presented at the end of section 3 of chapter 2, we can construct the

$\mathbb{Z}G$-projective resolution of $\mathbb{Z}$ $\tilde{E}^n = \bigoplus_{i=0}^n B_i^{\bar{G}_\mathfrak{P}} \otimes B_{n-i}^{G_\mathfrak{P}}$. Let $\phi_n : B_n^{G_\mathfrak{P}} \to \tilde{E}_n$ be defined by

$$\phi_n[g_1|\cdots|g_n] = 1 \otimes [g_1|\cdots|g_n] + \sum_{i=1}^{n-1} [\bar{g}_1|\cdots|\bar{g}_{n-i}] \otimes g_1 \cdots g_{n-i}[g_{n-i+1}|\cdots|g_n]$$
$$+ [\bar{g}_1|\cdots|\bar{g}_n] \otimes g_1 \cdots g_n$$

for $n \geq 0$, where $\bar{g}$ denotes the image of g in $G_\mathfrak{P}$ under the natural projection $G_\mathfrak{P} \to \bar{G}_\mathfrak{P}$. Keeping the notation given in section 2.3.9, it is not difficult to show that $\tilde{\delta}_n \phi_n = \phi_{n-1}\delta_n$.
Let us define $\psi_n : \tilde{E}^n \to B_n^{G_\mathfrak{P}}$ as follows

$$\psi|_{B_i^{\bar{G}_\mathfrak{P}} \otimes B_{n-i}^{G_\mathfrak{P}}} = \begin{cases} Id_{B_0^{G_\mathfrak{P}}} & \text{if } \ i = n \\ 0 & \text{if } \ i \neq n \end{cases}$$

It is not difficult to show that $\{\psi_n\}$ is a chain map and that ϕ_n^* and ψ_n^* are inverse chain maps to each other.

The composition $\phi_2^* \circ \Gamma$ (Γ given in 2.3.11.2) induces a surjective homomorphism.

$$\bigoplus_{i=0}^{2} H^{2-i}(\bar{G}_\mathfrak{P}, H^i(G_\mathfrak{P}^o, \boldsymbol{\mu})) \to H^2(G_\mathfrak{P}, \boldsymbol{\mu}).$$

Let $z^i \in Hom_{G_\mathfrak{P}}(B_2^{G_\mathfrak{P}}, \boldsymbol{\mu})$ be the 2-cocycle given by

$$z^i = \phi_2^* \circ \Gamma y^i = \phi_2^* \circ \gamma_*^{i,2-i} y^i,$$

then $\{[z^i] : 0 \leq i \leq 2\}$ is a set of generators of $H^2(G_\mathfrak{P}, \boldsymbol{\mu})$.

We conclude this section by given an explicit description of the 2-cocycles z^i

for $i = 0, 1, 2$.

$$z_p^0[s_p^{i_1}\sigma_p^{j_1}|s_p^{i_2}\sigma_p^{j_2}] = \gamma_*^{0,2} y_0(1 \otimes [s_p^{i_1}\sigma_p^{j_1}|s_p^{i_2}\sigma_p^{j_2}])$$
$$= x^2[s_p^{i_1}\sigma_p^{j_1}|s_p^{i_2}\sigma_p^{j_2}]$$
$$= \begin{cases} 1 \ \text{ i } \ i_1 + i_2 < |G_{\mathfrak{P}}^o| \\ (\mu_p')^{t_p} \ \text{ if } \ i_1 + i_2 \geq |G_{\mathfrak{P}}^o| \end{cases} \tag{6.1.10.1}$$

$$z_p^1[s_p^{i_1}\sigma_p^{j_1}|s_p^{i_2}\sigma_p^{j_2}] = \gamma_*^{1,1} y_1([\sigma_p^{j_1}] \otimes s_p^{i_1}\sigma_p^{j_1}[s_p^{i_2}\sigma_p^{j_2}])$$
$$= j_1 t_p' x^1(s_p^{i_1}\sigma_p^{j_1}[s_p^{i_2}\sigma_p^{j_2}])$$
$$= (\mu_p')^{j_1 i_2 t_p t_p'} \tag{6.1.10.2}$$

$$z_p^2[s_p^{i_1}\sigma_p^{j_1}|s_p^{i_2}\sigma_p^{j_2}] = \gamma_*^{2,0} y_2([\sigma_p^{i_1}|\sigma_p^{j_2}] \otimes s_p^{i_1+i+2}\sigma_p^{j_1+j_2})$$
$$= \begin{cases} 1 \ \text{ if } \ j_1 + j_2 < |\bar{G}_{\mathfrak{P}}| \\ (\mu_p')^{t_p''} \ \text{ if } \ j_1 + j_2 \geq |\bar{G}_{\mathfrak{P}}| \end{cases} \tag{6.1.10.3}$$

6.1.11 The unramified case.

Another case to consider is when $\mathfrak{Q}$ in S_* is unramified. Let q be the prime number below $\mathfrak{Q}$, in this case, $G_{\mathfrak{Q}}^o$ is trivial and the cyclic group $\bar{G}_{\mathfrak{Q}} = G_{\mathfrak{Q}}$. The only non trivial group $E_2^{i,2-i}$ associated to the Hochschild-Serre spectral sequence is $E_2^{2,0} = H^2(G_{\mathfrak{Q}}, H^0(G_{\mathfrak{Q}}^o, \boldsymbol{\mu})) = H^2(G_{\mathfrak{Q}}, \boldsymbol{\mu}) \cong \hat{H}^0(G_{\mathfrak{Q}}, \boldsymbol{\mu})$.
We can proceed exactly as in the previous subsection with the only exception of assuming $m_q = m$.
From the computations done before one has $\boldsymbol{\mu}^{G_{\mathfrak{Q}}} = \langle \mu^{t_q} \rangle$ and that $\hat{H}^0(G_{\mathfrak{Q}}, \boldsymbol{\mu}) = \langle \mu^{t_q t_q''} \rangle$. Finally by applying Claim 6.1.7 one gets that the class of the 2-cocycle z_q^2 given by

$$z_q^2[\sigma_{\mathfrak{Q}}^{i_1}|\sigma_{\mathfrak{Q}}^{i_2}] = \begin{cases} 1 \ \text{ if } \ i_1 + i_2 < |G_{\mathfrak{P}}| \\ \mu^{t_q t_q''} \ \text{ if } \ i_1 + i_2 \geq |G_{\mathfrak{P}}| \end{cases} \tag{6.1.11.1}$$

generates the group $H^2(G_{\mathfrak{Q}}, \boldsymbol{\mu})$.

6.1.12 The archimedean case.

Let $\bar{c}$ be complex conjugation in G, we will denote by G_∞ the decomposition subgroup associated to $\bar{c}$. Since G_∞ is cyclic of order 2 it follows by Claim 6.1.7 that the class of the 2-cocycle

$$z_\infty^2[c^{i_1}|c^{i_2}] = \begin{cases} 1 & \text{if } \; i_1 + i_2 < |G_\infty| \\ -1 & \text{if } \; i_1 + i_2 \geq |G_\infty| \end{cases} \tag{6.1.12.1}$$

generates the group $H^2(G_\infty, \boldsymbol{\mu})$.

6.2 Computation of $inv_{\mathfrak{P}}$.

We remain under the assumption that m is an odd squarefree positive integer. We will describe a method to evaluate local invariant maps, this method can be found in a paper by A. Weiss (in progress).

Let p be a prime which divides m and $\mathfrak{P}$ the prime in S_* above p. For x a 2-cocycle whose class $[x]$ is in $H^2(G_{\mathfrak{P}}, \boldsymbol{\mu})$, we fix s in $G_{\mathfrak{P}}$ to be a generator of $G^o_{\mathfrak{P}}$. By Claim 6.1.2 there exists σ in $G_{\mathfrak{P}}$ satisfying $G_{\mathfrak{P}} = \langle s \rangle \oplus \langle \sigma \rangle$ and $\langle \bar{\sigma} \rangle = \bar{G}_{\mathfrak{P}}$.
We will denote by

$$\bar{x}_s = \big(x[\sigma|s]^{-1} x[s|\sigma]\big) \prod_{u \in G^o_{\mathfrak{P}}} x[u|s]. \tag{6.2.0.1}$$

It follows that $\bar{x}_s$ belongs to $\mathbb{F}_p^\times$. Let d be the least positive integer such that $\bar{x}_s^d = 1$.

Let $\bar{a} : G \to (\mathbb{Z}/2m\mathbb{Z})^\times$ be the group homomorphism defined in (3.0.0.1) and denote by $\tilde{a}(s)$ the image of $\bar{a}(s)$ under the projection $(\mathbb{Z}/2m\mathbb{Z})^\times \to \mathbb{F}_p^\times$, since $\tilde{a}(s)$ generates $\mathbb{F}_p^\times$

$$\langle \tilde{a}(s)^{-(p-1)/d} \rangle = \langle \bar{x}_s \rangle.$$

There exists an integer h relatively prime to d with $\tilde{a}(s)^{-(p-1)h/d} = \bar{x}_s$.

One can conclude that

$$inv_{\mathfrak{P}}([x]) = -\frac{h}{d} + \mathbb{Z}. \tag{6.2.0.2}$$

We will now evaluate $inv_{\mathfrak{P}}$ on the 2-cocycles z^i given in subsection 6.1.10. We start by considering the 2-cocycle z^2. By (6.1.10.3) one can say that

$$\bar{z}^2_{s_p} = \left(z^2[\sigma_p|s_p]^{-1} z^2[s_p|\sigma_p]\right) \prod_{u \in G^o_{\mathfrak{P}}} z^2[u|s_p] = 1,$$

which immediately implies that

$$inv_{\mathfrak{P}}([z^2_p]) = 0 + \mathbb{Z}. \tag{6.2.0.3}$$

We can now look at z^0, in this case (6.1.10.1) gives

$$\bar{z}^0_{s_p} = \left(z^0[\sigma_p|s_p]^{-1} z^0[s_p|\sigma_p]\right) \prod_{u \in G^o_{\mathfrak{P}}} z^0[u|s_p] = (\mu'_p)^{t_p}.$$

It follows that d in this case is $v_p = gcd(2m_p, p-1)$. Notice that this value does not depend of the choice of s_p or σ_p, hence we can choose s_p to be a generator of $G^o_{\mathfrak{P}}$ with the property that $\tilde{a}(s_p)^{-(p-1)/d} = \bar{z}^1_{s_p}$. This implies that

$$inv_{\mathfrak{P}}([z^0_p]) = -\frac{1}{v_p} + \mathbb{Z}. \tag{6.2.0.4}$$

We consider now the cocycle z^1, (6.1.10.2) shows that

$$\bar{z}^1_{s_p} = \left(z^1[\sigma_p|s_p]^{-1} z^1[s_p|\sigma_p]\right) \prod_{u \in G^o_{\mathfrak{P}}} z^1[u|s_p] = (\mu'_p)^{-t_p t'_p}.$$

Since $t_p t'_p = 2m_p/v'_p$, we can conclude that d is v'_p. As in the previous case we can assume h to be 1 hence

$$inv_{\mathfrak{P}}([z^1_p]) = -\frac{1}{v'_p} + \mathbb{Z}. \tag{6.2.0.5}$$

If q is a prime number that does not divide m and $\mathfrak{Q}$ is the unique prime in

S_* above q and $\bar{c}$ denotes complex conjugation, it follows that

$$inv_{\mathfrak{Q}}([z_q^2]) = 0 + \mathbb{Z} \tag{6.2.0.6}$$

$$inv_\infty[z_\infty^2] = \frac{1}{2} + \mathbb{Z}. \tag{6.2.0.7}$$

Chapter 7

Localizing.

In this chapter we introduce the "local parts" for a $[f]$ in $[L, \bar{C}]$.

We begin by proving the existence of a commutative diagram

$$\begin{array}{ccc} H^2(G_{\mathfrak{P}}, \boldsymbol{\mu}) & \longrightarrow & H^2(G, Hom(\Delta S, \boldsymbol{\mu})) \\ \partial_{\mathfrak{P}}^{-1} \Big\downarrow \cong & & \partial^{-1} \Big\downarrow \cong \\ \hat{H}^0(G_{\mathfrak{P}}, Hom(\Delta G, \bar{C})) & \xrightarrow[\alpha_1]{} & [L, \bar{C}] \end{array}$$

for each $\mathfrak{P}$ in S_*.
We then give an explicit description of the isomorphism

$$\partial_{\mathfrak{P}}^{-1} : H^2(G_{\mathfrak{P}}, \boldsymbol{\mu}) \to \hat{H}^0(G_{\mathfrak{P}}, Hom(\Delta G, C)).$$

We conclude by defining a $\mathbb{Z}G$-homomorphism $\beta_1 : [\bar{C}, L]_G \to [\bar{C}, \Delta G]_{G_{\mathfrak{P}}}$ and a non-degenerate pairing

$$\tau_{\Delta G}^{G_{\mathfrak{P}}} : \hat{[}\bar{C}, \Delta G]_{G_{\mathfrak{P}}} \times [\Delta G, \bar{C}]_{G_{\mathfrak{P}}} \to \mathbb{Q}/\mathbb{Z},$$

such that the following holds

$$\tau_{\Delta G}^{G_{\mathfrak{P}}}(\beta_1[f], z) = \tau_L(f_*(\alpha_1 z))$$

for all z in $\hat{H}^0(G_{\mathfrak{P}}, Hom(\Delta G, \bar{C})) \cong [\Delta G, \bar{C}]_{G_{\mathfrak{P}}}$.

7.1 A commutative diagram.

The main objective of this section is to show that there exist a commutative diagram

$$\begin{array}{ccc} H^2(G_{\mathfrak{P}}, \boldsymbol{\mu}) & \longrightarrow & H^2(G, Hom(\Delta S, \boldsymbol{\mu})) \\ \partial^{-1} \Big\downarrow \cong & & \partial^{-1} \Big\downarrow \cong \\ H^0(G_{\mathfrak{P}}, Hom(\Delta G, \bar{C})) & \longrightarrow & [L, \bar{C}] \end{array} \tag{7.1.0.1}$$

where $\boldsymbol{\mu} \to C \to \bar{C}$ is an envelope of $\boldsymbol{\mu}$ and $\mathfrak{P}$ is an arbitrary prime in S_*. Let $L = \Delta G \otimes \Delta S$ and $\bar{I} = \mathbb{Z}G \otimes \Delta S$. We consider $0 \to L \to \bar{I} \to \Delta S \to 0$ to be the exact sequence obtained by applying the exact functor $- \otimes \Delta S$ to the augmentation sequence $0 \to \Delta G \to \mathbb{Z}G \to \mathbb{Z} \to 0$.

We will list a couple of known result in cohomology for which we will only sketch their proof.

7.1.1 Claim. *Given L_1 and L_2 $\mathbb{Z}G$-lattices, ϕ in $Hom(L_1, L_2)$ and N a $\mathbb{Z}G$-module there is commutative diagram*

$$\begin{array}{ccc} H^{q+1}(G, Hom(L_2, N)) & \xrightarrow{\phi^*} & H^{q+1}(G, Hom(L_1, N)) \\ \cong \Big\uparrow \partial & & \cong \Big\uparrow \partial \\ H^q(G, Hom(\Delta G \otimes L_2, N)) & \xrightarrow{\phi^*} & H^q(G, Hom(\Delta G \otimes L_1, N)) \end{array}$$

for all integers q.

Proof. Let us start by mentioning that the functors $- \otimes L_i$ are exact for $i = 1, 2$. By applying this functors to the exact sequence $0 \to \Delta G \to \mathbb{Z}G \to \mathbb{Z} \to 0$ we obtain the following commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Delta G \otimes L_1 & \longrightarrow & \mathbb{Z}G \otimes L_1 & \longrightarrow & L_1 & \longrightarrow & 0 \\ & & \phi \Big\downarrow & & \phi \Big\downarrow & & \phi \Big\downarrow & & \\ 0 & \longrightarrow & \Delta G \otimes L_2 & \longrightarrow & \mathbb{Z}G \otimes L_2 & \longrightarrow & L_2 & \longrightarrow & 0 \end{array} \tag{7.1.1.1}$$

Using the fact that $Ext^1_{\mathbb{Z}}(L_i, N) = 0$ for $i = 1, 2$, we obtain that diagram (7.1.1.1) induces a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Hom(L_2, N) & \longrightarrow & Hom(\mathbb{Z}G \otimes L_2, N) & \longrightarrow & Hom(\Delta G \otimes L_2, N) & \longrightarrow & 0 \\ & & \phi^* \downarrow & & \phi^* \downarrow & & \phi^* \downarrow & & \\ 0 & \longrightarrow & Hom(L_1, N) & \longrightarrow & Hom(\mathbb{Z}G \otimes L_1, N) & \longrightarrow & Hom(\Delta G \otimes L_1, N) & \longrightarrow & 0 \end{array} \tag{7.1.1.2}$$

with exact rows. We now apply G-cohomology to diagram (7.1.2.1) and since $Hom(\mathbb{Z}G \otimes L_i, N)$ are cohomologically trivial for $i = 1, 2$, we obtain the desired diagram. □

7.1.2 Claim. *Let $\boldsymbol{\mu} \to N \to M$ be an envelope of $\boldsymbol{\mu}$, L_1, L_2 $\mathbb{Z}G$-lattices and ϕ an element of $Hom(L_1, L_2)$, there exist commutative diagrams*

$$\begin{array}{ccc} H^{q+1}(G, Hom(L_2, \boldsymbol{\mu})) & \xrightarrow{\phi^*} & H^{q+1}(G, Hom(L_1, \boldsymbol{\mu})) \\ \cong \uparrow \partial & & \cong \uparrow \partial \\ H^q(G, Hom(L_2, M)) & \xrightarrow{\phi^*} & H^q(G, Hom(L_1, M)) \end{array}$$

for all integers q.

Proof. Since the functors $Hom(L_i, -)$ are exact for $i = 1, 2$ one obtains the following commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Hom(L_2, \boldsymbol{\mu}) & \longrightarrow & Hom(L_2, N) & \to & Hom(L_2, M) & \longrightarrow & 0 \\ & & \phi^* \downarrow & & \phi^* \downarrow & & \phi^* \downarrow & & \\ 0 & \longrightarrow & Hom(L_1, \boldsymbol{\mu}) & \longrightarrow & Hom(L_1, N) & \to & Hom(L_1, M) & \longrightarrow & 0 \end{array} \tag{7.1.2.1}$$

by applying G-cohomology to diagram (7.1.2.1) and using the fact that $Hom(L_i, N)$ are cohomologically trivial, by ([13] pg 152), one obtains the desired diagram. □

As mention before we fix

$$0 \longrightarrow \boldsymbol{\mu} \longrightarrow C \longrightarrow \bar{C} \longrightarrow 0 \tag{7.1.2.2}$$

to be an envelope of $\boldsymbol{\mu}$, hence we obtain the following commutative diagram

$$\begin{array}{ccc} H^2(G, Hom(\mathbb{Z}S, \boldsymbol{\mu})) & \longrightarrow & H^2(G, Hom(\Delta S, \boldsymbol{\mu})) \\ \partial \uparrow \cong & & \partial \uparrow \cong \\ H^1(G, Hom(\Delta G \otimes \mathbb{Z}S, \boldsymbol{\mu})) & \longrightarrow & H^1(G, Hom(L, \boldsymbol{\mu})) \\ \partial \uparrow \cong & & \partial \uparrow \cong \\ [\Delta G \otimes \mathbb{Z}S, \bar{C}] & \longrightarrow & [L, \bar{C}] \end{array} \tag{7.1.2.3}$$

where the upper square is obtained by applying claim 7.1.1 with $L_1 = \Delta S$, $L_2 = \mathbb{Z}S$ and $\phi : \Delta S \hookrightarrow \mathbb{Z}S$ the natural inclusion, while the lower square is obtained by applying claim 7.1.2 to (7.1.2.2) with $L_1 = L$ and $L_2 = \Delta G \otimes \mathbb{Z}S$.

Since $\mathbb{Z}S \cong \bigoplus_{\mathfrak{P} \in S_*} ind_{G_{\mathfrak{P}}}^G \mathbb{Z}$, we will denote by $\phi_{\mathfrak{P}} : \mathbb{Z}S \to ind_{G_{\mathfrak{P}}}^G \mathbb{Z}$ the natural projection for each $\mathfrak{P} \in S_*$. Letting $L_1 = \mathbb{Z}S$ and $L_2 = ind_{G_{\mathfrak{P}}}^G \mathbb{Z}$, claim 7.1.1 and claim 7.1.2 give the following commutative diagram

$$\begin{array}{ccc} H^2(G, Hom(ind_{G_{\mathfrak{P}}}^G \mathbb{Z}, \boldsymbol{\mu})) & \xrightarrow{\phi_{\mathfrak{P}}^*} & H^2(G, Hom(\mathbb{Z}S, \boldsymbol{\mu})) \\ \partial \uparrow \cong & & \partial \uparrow \cong \\ H^1(G, Hom(\Delta G \otimes ind_{G_{\mathfrak{P}}}^G \mathbb{Z}, \boldsymbol{\mu})) & \xrightarrow{\phi_{\mathfrak{P}}^*} & H^1(G, Hom(\Delta G \otimes \mathbb{Z}S, \boldsymbol{\mu})) \\ \partial \uparrow \cong & & \partial \uparrow \cong \\ [\Delta G \otimes ind_{G_{\mathfrak{P}}}^G \mathbb{Z}, \bar{C}] & \xrightarrow{\phi_{\mathfrak{P}}^*} & [\Delta G \otimes \mathbb{Z}S, \bar{C}] \end{array} \tag{7.1.2.4}$$

If we identify $\boldsymbol{\mu}$ naturally with $Hom(\mathbb{Z}, \boldsymbol{\mu})$, functoriality of the Shapiro's isomorphism $Sh_{(L,M,N)}$ defined in 2.2.4.1, gives the following commutative diagram

$$\begin{array}{ccc} H^2(G_{\mathfrak{P}}, \boldsymbol{\mu})) & \longrightarrow & H^2(G, Hom(ind_{G_{\mathfrak{P}}}^G \mathbb{Z}, \boldsymbol{\mu})) \\ \partial \uparrow \cong & & \partial \uparrow \cong \\ H^1(G_{\mathfrak{P}}, Hom(\Delta G, \boldsymbol{\mu})) & \longrightarrow & H^1(G, Hom(\Delta G \otimes ind_{G_{\mathfrak{P}}}^G \mathbb{Z}, \boldsymbol{\mu})) \\ \partial \uparrow \cong & & \partial \uparrow \cong \\ H^0(G_{\mathfrak{P}}, Hom(\Delta G, \bar{C})) & \longrightarrow & [\Delta G \otimes ind_{G_{\mathfrak{P}}}^G \mathbb{Z}, \bar{C}] \end{array} \tag{7.1.2.5}$$

Diagrams (7.1.2.3), (7.1.2.4) and (7.1.2.5) prove the existence of diagram

(7.1.0.1).

7.2 $\partial_{\mathfrak{P}}^{-1} : H^2(G_{\mathfrak{P}}, \boldsymbol{\mu}) \to H^0(G_{\mathfrak{P}}, Hom(\Delta G, \bar{C}))$.

In this this section we will compute the inverse of the isomorphism

$$H^0(G_{\mathfrak{P}}, Hom(\Delta G, \bar{C})) \xrightarrow{\cong} H^2(G_{\mathfrak{P}}, \boldsymbol{\mu}),$$

obtained as the composition of the left column of diagram (7.1.2.5).
This will be done by computing separately the isomorphisms

$$\partial^{-1} : H^2(G_{\mathfrak{P}}, \boldsymbol{\mu}) \to H^1(G_{\mathfrak{P}}, Hom(\Delta G, \boldsymbol{\mu})) \text{ and}$$

$$\partial^{-1} : H^1(G_{\mathfrak{P}}, Hom(\Delta G, \boldsymbol{\mu})) \to H^0(G_{\mathfrak{P}}, Hom(\Delta G, \bar{C})).$$

Let G be a finite group, H a subgroup of G and M a $\mathbb{Z}G$-module. By fixing X a right transversal of H in G one obtains an H-map $\phi_H^X : G \to H$ defined by

$$\phi_H^X(hx) = h \text{ for all } h \in H \text{ and } x \in X.$$

This map induces in a natural way a $\mathbb{Z}H$-homomorphism $\phi_H^X : \mathbb{Z}G \to \mathbb{Z}H$.

From this point on and until the end of this section $\phi_H^X = \phi$ if there is no confusion. We should also mention here that $(B_\bullet^G, \delta)$ denotes the bar resolution for the group G introduced in subsection 6.1.3.

7.2.1 $\partial^{-1} : H^2(G_{\mathfrak{P}}, \boldsymbol{\mu}) \to H^1(G_{\mathfrak{P}}, Hom(\Delta G, \boldsymbol{\mu}))$.

7.2.2 Claim. *Let $h_1, h_2 \mapsto x_{h_1,h_2}$ be a 2-cocycle whose class x belongs to $H^2(H, M)$, define $\hat{x}$ in $Hom_H(B_1^H, Hom(\Delta G, M))$ in the following way: if*

$l=\sum_{g\in G} a_g g$ in $\mathbb{Z}G$ with $\sum_{g\in G} a_g = 0$,

$$\hat{x}_h(l) = \sum_{g\in G} a_g \phi(g) x_{\phi(g)^{-1},h}$$

for all h in H. It follows that $\delta\hat{x}=0$ (which implies that $h\mapsto \hat{x}_h$ is a 1-cocycle) and $\partial\hat{x}=x$.

Proof. Notice that x being a 2-cocycle implies that

$$h_1 x_{h_2,h_3} - x_{h_1h_2,h_3} + x_{h_1,h_2h_3} - x_{h_1,h_2} = 0 \qquad (7.2.2.1)$$

for all h_1, h_2 and h_3 in H.
We recall that M is naturally isomorphic to $Hom(\mathbb{Z}, M)$ as a $\mathbb{Z}G$-module, hence the short exact sequence

$$0 \longrightarrow \Delta G \xrightarrow{i_G} \mathbb{Z}G \xrightarrow{\pi_G} \mathbb{Z} \longrightarrow 0$$

induces the following commutative diagram with exact rows

$$\begin{array}{ccccc}
Hom_H(B_2^H, Hom(\mathbb{Z},M)) & \overset{\pi_G^*}{\rightarrowtail} & Hom_H(B_2^H, Hom(\mathbb{Z}G,M)) & \overset{i_G^*}{\twoheadrightarrow} & Hom_H(B_2^H, Hom(\Delta G,M)) \\
\delta\uparrow & & \delta\uparrow & & \delta\uparrow \\
Hom_H(B_1^H, Hom(\mathbb{Z},M)) & \overset{\pi_G^*}{\rightarrowtail} & Hom_H(B_1^H, Hom(\mathbb{Z}G,M)) & \overset{i_G^*}{\twoheadrightarrow} & Hom_H(B_1^H, Hom(\Delta G,M))
\end{array}$$

Let $\bar{x}$ in $Hom_H(B_1^H, Hom(\mathbb{Z}G, M))$ be given by

$$\bar{x}_h(g) = \phi(g) x_{\phi(g)^{-1},h},$$

it follows clearly that $i_G^*\bar{x} = \hat{x}$: it remains to show that $\delta\bar{x} = \pi_G^* x$. In order to prove this let h_1, h_2 in H and g in G, then

$$\begin{aligned}
(\delta\bar{x})_{h_1,h_2}(g) &= (h_1\bar{x}_{h_2})(h_1^{-1}g) - \bar{x}_{h_1h_2}(g) + \bar{x}_{h_1}(g) \\
&= \phi(g)x_{\phi(g)^{-1}h_1,h_2} - \phi(g)x_{\phi(g)^{-1},h_1h_2} + \phi(g)x_{\phi(g)^{-1},h_1} \quad \text{by (7.2.2.1)} \\
&= x_{h_1,h_2} \\
&= x_{h_1,h_2}(\pi_G(g)) \\
&= \pi_G^* x_{h_1,h_2}(g),
\end{aligned}$$

which conclude the proof by the definition of ∂. $\square$

If we identify $\boldsymbol{\mu}$ with $\mathbb{Z}/2m\mathbb{Z}$ as $\mathbb{Z}G$-modules, where the action of G is given by

$$g\,(t+2m\mathbb{Z}) = a(g)t + 2m\mathbb{Z},$$

being a the lift of the isomorphism $\bar{a} : G \to (\mathbb{Z}/2m\mathbb{Z})^{\times}$ defined in (3.0.0.1). Every 2-cocycle in $Hom_{G_{\mathfrak{P}}}(B_2^{G_{\mathfrak{P}}}, \mathbb{Z}/2m\mathbb{Z})$ is determined by a function (not uniquely) $b : G_{\mathfrak{P}} \times G_{\mathfrak{P}} \to \mathbb{Z}$ satisfying

$$x_{h_1,h_2} = b(h_1,h_2) + 2m\mathbb{Z} \quad \text{and} \tag{7.2.2.2}$$

$$a(h_1)b(h_2,h_3) - b(h_1h_2,h_3) + b(h_1,h_2h_3) - b(h_1,h_2) \equiv 0 \pmod{2m\mathbb{Z}} \tag{7.2.2.3}$$

for all h_1, h_2 and h_3 in $G_{\mathfrak{P}}$, hence by claim 7.2.2 one obtains that the class of the 1-cocycle given by

$$\begin{aligned}\hat{x}_h(l) &= \sum_{g\in G} a_g \phi(g) x_{\phi(g)^{-1},h} \\ &= \sum_{g\in G} a_g a(\phi(g)) b(\phi(g)^{-1},h) + 2m\mathbb{Z},\end{aligned} \tag{7.2.2.4}$$

is the preimage of x under the isomorphism $\partial : H^1(G_{\mathfrak{P}}, Hom(\Delta G, \boldsymbol{\mu})) \to H^2(G_{\mathfrak{P}}, \boldsymbol{\mu})$.

7.2.3 $\partial^{-1} : H^1(G_{\mathfrak{P}}, Hom(\Delta G, \boldsymbol{\mu})) \to H^0(G_{\mathfrak{P}}, Hom(\Delta G, \bar{C}))$.

Let $0 \to \boldsymbol{\mu} \to C \to \bar{C} \to 0$ be the envelope of $\boldsymbol{\mu}$ constructed in chapter 3 and assume that $\{g_i : 1 \le i \le n\}$ is a fix set of generators of G.

Let $h \mapsto \hat{x}_h$ be a 1-cocycle with values in $Hom(\Delta G, \mathbb{Z}/2m\mathbb{Z})$, there exists a function $c : G_{\mathfrak{P}} \times \Delta G \to \mathbb{Z}$ satisfying

$$\hat{x}_{h_1}(l_1) = c(h_1, l_1) + 2m\mathbb{Z} \tag{7.2.3.1}$$

$$c(h_1, l_1 + l_2) \equiv c(h_1, l_1) + c(h_1, l_2) \pmod{2m\mathbb{Z}} \text{ and} \tag{7.2.3.2}$$

$$a(h_1)c(h_2, h_1^{-1}l_1) - c(h_1h_2, l_1) + c(h_1, l_1) \equiv 0 \pmod{2m\mathbb{Z}} \tag{7.2.3.3}$$

for all h_1, h_2 in $G_{\mathfrak{P}}$ an all l_1, l_2 in ΔG.

For a fix l in ΔG let us define the following elements D_l^i in $\mathbb{Q}G$ for $0 \leq i \leq n$, as follows:

$$D_l^0 = \frac{1}{2m} \sum_{g \in G} a(g^{-1}\phi(g))c(\phi(g)^{-1}, \phi(g)^{-1}l)g, \tag{7.2.3.4}$$

and for $1 \leq i \leq n$

$$D_l^i = \frac{-1}{2m} \sum_{g \in G} a(g^{-1}\phi(g))c(\phi(g)^{-1}\phi(gg_i^{-1}), \phi(g)^{-1}l)g. \tag{7.2.3.5}$$

7.2.4 Claim. *With the notation given above, let $\tilde{x}$ in $Hom(\Delta G, \bar{C})$ be given by*

$$\tilde{x}(l) = \left(2mD_l^0, \left(D_l^0\left(g_i^{-1} - a(g_i^{-1})\right)\right)_{1\leq i \leq n}\right) + \left(0, \left(D_l^i g_i^{-1}\right)_{1 \leq i \leq n}\right) + A_Y.$$

It follows that $\tilde{x}$ belongs to $Hom(\Delta G, \bar{C})^{G_{\mathfrak{P}}}$ and $\partial\tilde{x} = \hat{x}$.

Proof. We need to show that $\tilde{x}$ is well defined. For this let

$$\tilde{x}_i(l) = \begin{cases} 2mD_l^0 & \text{if } i = 0 \\ D_l^0\left(g_i^{-1} - a(g_i^{-1})\right) + D_l^i g_i^{-1} & \text{if } 1 \leq i \leq n \end{cases}$$

We will now proof that $\tilde{x}_i(l)$ belongs to $\mathbb{Z}G$. Since

$$\begin{aligned}
\tilde{x}_i(l) &= \frac{g_i^{-1} - a(g_i^{-1})}{2m} \sum_{g\in G} a(g^{-1}\phi(g))c(\phi(g)^{-1}, \phi(g)^{-1}l)g \\
&- \frac{g_i^{-1}}{2m} \sum_{g\in G} a(g^{-1}\phi(g))c(\phi(g)^{-1}\phi(gg_i^{-1}), \phi(g)^{-1}l)g \\
&= \frac{1}{2m} \sum_{g\in G} a(g^{-1}\phi(g)) \left(c(\phi(g)^{-1}, \phi(g)^{-1}l) - c(\phi(g)^{-1}\phi(gg_i^{-1}), \phi(g)^{-1}l)\right) gg_i^{-1} \\
&- \frac{1}{2m} \sum_{g\in G} a(g_i^{-1})a(g^{-1}\phi(g))c(\phi(g)^{-1}.\phi(g)^{-1}l)g \\
&= \frac{1}{2m} \sum_{g\in G} a(g_i^{-1}g^{-1}\phi(gg_i)^{-1} \left(c(\phi(gg_i)^{-1}, \phi(gg_i)^{-1}l) - c(\phi(gg_i)^{-1}\phi(g), \phi(gg_i)^{-1}l)\right) g \\
&- \sum_{g\in G} a(g_i^{-1})a(g^{-1}\phi(g))c(\phi(g)^{-1}, \phi(g)^{-1}l)g,
\end{aligned}$$

it would be enough to show that for all g in G

$$\begin{aligned}
&a(g_i^{-1}g^{-1}\phi(gg_i)^{-1} \left(c(\phi(gg_i)^{-1}, \phi(gg_i)^{-1}l) - c(\phi(gg_i)^{-1}\phi(g), \phi(gg_i)^{-1}l)\right) \\
&- a(g_i^{-1})a(g^{-1}\phi(g))c(\phi(g)^{-1}, \phi(g)^{-1}l) \equiv 0 \pmod{2m\mathbb{Z}}. \qquad (7.2.4.1)
\end{aligned}$$

It follows from (7.2.3.3) that the left hand side of (7.2.4.1) is equivalent to

$$\left(a(g_i^{-1}g^{-1}) - a(g_i^{-1})a(g^{-1})\right)\left(c(1,l) - c(\phi(g),l)\right) \pmod{2m\mathbb{Z}},$$

which is equivalent to 0 (mod $2m\mathbb{Z}$), since $a(st) - a(s)a(t) \equiv 0 \pmod{2m\mathbb{Z}}$ for all s, t in G.

We will now show that $\tilde{x}$ is a 0-cocycle. A simple computation shows that $hD_l^i = D_{hl}^i$ for all $h \in G_{\mathfrak{P}}$ and $1 \le i \le n$. In order to show that $h\tilde{x}(l) = \tilde{x}(hl)$ it would be enough to prove that

$$\left(2m(hD_l^0 - D_{hl}^0), \left(\left(hD_l^0 - D_{hl}^0\right)\left(g_i^{-1} - a(g_i^{-1})\right)\right)_{1\le\imath\le n}\right) \in A_Y$$

or equivalently that $hD_l^0 - D_{hl}^0$ is an element of Y, which follows since

$$\begin{aligned} hD_l^0 - D_{hl}^0 &= \frac{1}{2m}\sum_{g\in G} a(g^{-1}\phi(g))c(\phi(g)^{-1}, \phi(g)^{-1}l)hg \\ &- \frac{1}{2m}\sum_{g\in G} a(g^{-1}\phi(g))c(\phi(g)^{-1}.\phi(g)^{-1}hl)g \\ &= \frac{1}{2m}\sum_{g\in G} a(g^{-1}\phi(g))\left[c(\phi(g)^{-1}h, \phi(g)^{-1}hl) - c(\phi(g)^{-1}, \phi(g)^{-1}hl)\right]g \\ &\equiv \frac{1}{2m}\sum_{g\in G} a(g^{-1}\phi(g))a(\phi(g)^{-1})c(h, hl)g \pmod{Y} \qquad \text{by (7.2.3.3)} \\ &\equiv c(h, hl)\Theta \equiv 0 \pmod{Y}. \end{aligned}$$

In order to show that $\partial\tilde{x} = \hat{x}$ we should recall that the envelope

$$0 \longrightarrow \boldsymbol{\mu} \xrightarrow{\gamma} C \xrightarrow{\pi} \bar{C} \longrightarrow 0$$

of $\boldsymbol{\mu}$ given in Chapter 3 induces a the following commutative diagram

$$\begin{array}{ccccc} Hom_{G_{\mathfrak{P}}}(B_1^{G_{\mathfrak{P}}}, Hom(\Delta G, \boldsymbol{\mu})) & \overset{\gamma_*}{\rightarrowtail} & Hom_{G_{\mathfrak{P}}}(B_1^{G_{\mathfrak{P}}}, Hom(\Delta G, C)) & \overset{\pi_*}{\twoheadrightarrow} & Hom_{G_{\mathfrak{P}}}(B_1^{G_{\mathfrak{P}}}, Hom(\Delta G, \bar{C})) \\ \delta\uparrow & & \delta\uparrow & & \delta\uparrow \\ Hom_{G_{\mathfrak{P}}}(B_0^{G_{\mathfrak{P}}}, Hom(\Delta G, \boldsymbol{\mu})) & \overset{\gamma_*}{\rightarrowtail} & Hom_{G_{\mathfrak{P}}}(B_0^{G_{\mathfrak{P}}}, Hom(\Delta G, C)) & \overset{\pi_*}{\twoheadrightarrow} & Hom_{G_{\mathfrak{P}}}(B_0^{G_{\mathfrak{P}}}, Hom(\Delta G, \bar{C})) \end{array}$$

Define $\bar{x}$ in $Hom_{G_{\mathfrak{P}}}(B_0^{G_{\mathfrak{P}}}, Hom(\Delta G, C))$ by $\bar{x}(l) = \big(\tilde{x}_0, (\tilde{x}_i)_{1\le i\le n}\big) + A_{\mathbb{Z}G}$.

It follows immediately that $\pi_*\bar{x} = \tilde{x}$. It only remains to show that $\delta\bar{x} = \gamma_*\hat{x}$, which is equivalent to prove that

$$\begin{aligned} h\tilde{x}(h^{-1}l) - \tilde{x}(l) &= \gamma_*\hat{x}(l) \\ &= c(h, l)\left(2m\Theta, \left(\left(g_i^{-1} - a(g_i^{-1})\right)\Theta\right)_{1\le i\le n}\right) + A_{\mathbb{Z}G} \end{aligned}$$

for all h in $G_{\mathfrak{P}}$.

Since $hD_l^i = D_{hl}^i$ for $1 \le i \le n$, it would be enough to prove that

$$2m(hD^0_{h^{-1}l} - D^0_l) \equiv c(h,l)\Theta \pmod{2m\mathbb{Z}G}.$$

Notice that

$$\begin{aligned}
2m\left(hD^0_{h^{-1}l} - D^0_l\right) &= \sum_{g\in G} a(g^{-1}\phi(g))\left[c(\phi(g)^{-1}, \phi(g)^{-1}h^{-1}l)hg - c(\phi(g)^{-1}, \phi(g)^{-1}l)g\right] \\
&\equiv \sum_{g\in G} a(g^{-1}\phi(g))a(\phi(g)^{-1})c(h,l)g \pmod{2m\mathbb{Z}G} \quad \text{by (7.2.3.3)} \\
&\equiv \sum_{g\in G} a(g^{-1})c(h,l)g \pmod{2m\mathbb{Z}G} \\
&\equiv c(h,l)\Theta \pmod{2m\mathbb{Z}G}.
\end{aligned}$$

□

For any l in ΔG the following holds

$$\begin{aligned}
D^0_l &= \frac{1}{2m}\sum_{g\in G} a(g^{-1}\phi(g))c(\phi(g)^{-1}, \phi(g)^{-1}l)g \\
&\equiv \frac{1}{2m}\sum_{g\in G} a(g^{-1})\left[c(1,l) - c(\phi(g),l)\right]g \pmod{\mathbb{Z}G} \qquad \text{by (7.2.3.3)} \\
&\equiv \frac{-1}{2m}\sum_{g\in G} a(g^{-1})c(\phi(g),l)g \pmod{\mathbb{Z}G}
\end{aligned}$$

This last series of equivalences show that if we let

$$\bar{D}^0_l = \frac{-1}{2m}\sum_{g\in G} a(g^{-1})c(\phi(g),l)g \tag{7.2.4.2}$$

and (by abuse of notation) if we denote by

$$\tilde{x}_i(l) = \begin{cases} 2m\bar{D}^0_l & \text{if } \ i=0 \\ \bar{D}^0_l(g_i^{-1} - a(g_i^{-1})) + D^i_l g_i^{-1} & \text{if } \ 1 \le i \le n \end{cases} \tag{7.2.4.3}$$

then $\tilde{x}(l) = (\tilde{x}_i(l))_{0\le i\le n} + A_Y$.

7.3 The pairing $\tau_{\Delta G}^{G_{\mathfrak{P}}}$.

The character $\epsilon : H^2(G, Hom(\Delta S, \boldsymbol{\mu})) \to \mathbb{Q}/\mathbb{Z}$ is determined, after dimension shifting, by the non-degenerate pairing $[\bar{C}, L] \times [L, \bar{C}] \to \mathbb{Q}/\mathbb{Z}$ where

$$([f],[z]) \mapsto \tau_L(f_*[z]) = \tau_L([f \circ z]) = \frac{Trace_L(f \circ z)}{|G|} + \mathbb{Z}$$

for all $[f]$ and $[z]$ in $[\bar{C}, L]$ and $[L, \bar{C}]$ respectively.

In section 7.1 we proved the existence of the commutative diagram (7.1.0.1), let us denote by

$$\alpha_1 : H^0(G_{\mathfrak{P}}, Hom(\Delta G, \bar{C})) \to [L, \bar{C}]$$

the homomorphism of the bottom row. We will prove in this section the existence of a homomorphism

$$\beta_1 : [\bar{C}, L]_G \to [\bar{C}, \Delta G]_{G_{\mathfrak{P}}}$$

and a non-degenerate pairing

$$\tau_{\Delta G}^H : [\bar{C}, \Delta G]_{G_{\mathfrak{P}}} \times [\Delta G, \bar{C}]_{G_{\mathfrak{P}}} \to \mathbb{Q}/\mathbb{Z}$$

such that it satisfies

$$\tau_{\Delta G}^H(\beta_1[f], z) = \tau_L(f_*\alpha_1[z]) \tag{7.3.0.1}$$

for any $[f]$ in $[\bar{C}, L]_G$ and $[z]$ in $H^0(G_{\mathfrak{P}}, Hom(\Delta G, \bar{C})) \cong [\Delta G, \bar{C}]_{G_{\mathfrak{P}}}$.

The existence of $\tau_{\Delta G}^H$ and β_1 will allow us to compute the character ϵ in a "local" way.

It should be mentioned at this point that we will change some of the notation established in previous chapters, with the idea that the proofs presented in this section become clearer to the reader.

Let us start by considering the short $\mathbb{Z}$-split exact sequence

$$0 \longrightarrow \Delta S \xrightarrow{i_1} \mathbb{Z}S \xrightarrow{\acute{i}_1} \mathbb{Z} \longrightarrow 0,$$

and applying to it the functor $\Delta G \otimes -$, to obtain the short exact sequence

$$0 \longrightarrow L \xrightarrow{i_1 \otimes id_{\Delta G}} \Delta G \otimes \mathbb{Z}S \xrightarrow{\acute{i}_1 \otimes id_{\Delta G},} \Delta G \longrightarrow 0. \tag{7.3.0.2}$$

We denote $I = \mathbb{Z}S \otimes \Delta G$. Recall that $\mathbb{Z}S \cong \bigoplus_{\mathfrak{P} \in S_*} ind^G_{G_{\mathfrak{P}}}\mathbb{Z}$, then for each $\mathfrak{P}$ in S_*, let $\phi_{\mathfrak{P}}$ be the natural projection from $\mathbb{Z}S$ onto $ind^G_{G_{\mathfrak{P}}}\mathbb{Z}$.

In order to simplify the notation we will fix a prime $\mathfrak{P}_{\mathfrak{o}}$ in S_* and denote by $H = G_{\mathfrak{P}_{\mathfrak{o}}}$, $I_H = ind^G_{G_{\mathfrak{P}_{\mathfrak{o}}}}\mathbb{Z} \otimes \Delta G$, $\phi_H = \phi_{\mathfrak{P}_{\mathfrak{o}}}$, $\hat{i}_1 = i_1 \otimes id_{\Delta G} : L \to I$ and $\hat{\phi}_H = \phi_H \otimes id_{\Delta G} : I \to I_H$.

Let $\tau_I : [\bar{C}, I] \times [I, \bar{C}] \to \mathbb{Q}/\mathbb{Z}$ be defined by

$$\tau_I([f],[z]) = \frac{Trace_I(f \circ z)}{|G|} + \mathbb{Z}$$

7.3.1 Claim. *Given* $[f]$ *in* $[\bar{C}, L]_G$ *and* $[z]$ *in* $[I, \bar{C}]_G$ *it follows that*

$$\tau_I((\hat{i}_1)_*[f],[z]) = \tau_L(f_*(\hat{i}_1)^*[z]).$$

Proof. The statement follows clearly from the equality

$$Trace_I(\hat{i}_1 \circ f \circ z) = Trace_L(f \circ z \circ \hat{i}_1).$$

□

We can now define $\tau_{I_H} : [\bar{C}, I_H] \times [I_H, \bar{C}] \to \mathbb{Q}/\mathbb{Z}$ by

$$\tau_{I_H}([f],[z]) = \frac{Trace_{I_H}(f \circ z)}{|G|} + \mathbb{Z}.$$

7.3.2 Claim. *Given* $[f]$ *in* $[\bar{C}, I_H]_G$ *and* $[z]$ *in* $[I_H, \bar{C}]_G$ *it follows that*

$$\tau_{I_H}((\hat{\phi}_H)_*[f], [z]) = \tau_I([f], (\hat{\phi}_H)^*[z]).$$

Proof. It is enough to show that

$$Trace_{I_H}(\hat{\phi}_H \circ f \circ z) = Trace_I(f \circ z \circ \hat{\phi}_H),$$

which follows by properties of the trace. □

We want to define the following dual maps and prove that they are actually $\mathbb{Z}G$-isomorphisms

$$\psi_1 : Hom(ind_H^G \mathbb{Z}, Hom(\Delta G, \bar{C})) \to Hom(I_H, \bar{C})$$
$$\psi_2 : Hom(\bar{C}, I_H) \to Hom(ind_H^G \mathbb{Z}, Hom(\bar{C}, \Delta G)).$$

In order to give the expression of ψ_2 we will need the following remark.
7.3.3 Remark. $Hom(ind_H^G \mathbb{Z}, \Delta G)$ is isomorphic to I_H as $\mathbb{Z}G$-modules.

Proof. Let $\phi_2 : I_H \to Hom(ind_H^G \mathbb{Z}, \Delta G)$ be defined in the following way: fix X to be a left transversal of H in G and for every element x of X let $\bar{x} := x \otimes 1$. If $Y = \{y_i : i \in I\}$ is a $\mathbb{Z}$-basis for ΔG it follows that $\{\bar{x} \otimes y_i : x \in X, i \in I\}$ is a $\mathbb{Z}$-basis for I_H, then

$$\begin{aligned} &\phi_2(\bar{x} \otimes y_i) = \Phi_{\bar{x} \otimes y_i} : ind_H^G \mathbb{Z} \to \Delta G \quad \text{where,} \\ &\Phi_{\bar{x} \otimes y_i}(\sum_{z \in X} a^z \bar{z}) = a^x y_i. \end{aligned} \tag{7.3.3.1}$$

We start by showing that ϕ_2 is a $\mathbb{Z}G$-homomorphism.

Let g in G a and z in X, notice that

$$\begin{aligned}\Phi_{g\cdot(\bar{x}\otimes y_i)}(\bar{z}) &= \begin{cases} gy_i & \text{if } \bar{z} = g\bar{x} \\ 0 & \text{if } \bar{z} \neq g\bar{x} \end{cases} \\ &= \begin{cases} gy_i & \text{if } g^{-1}\bar{z} = \bar{x} \\ 0 & \text{if } g^{-1}\bar{z} \neq \bar{x} \end{cases} \\ &= (g\Phi_{x\otimes y_i})(\bar{z})\end{aligned}$$

in order to prove that ϕ_2 is bijective we define $\tilde{\phi}_2 : Hom(ind_H^G\mathbb{Z}, \Delta G) \to I_H$ by

$$\tilde{\phi}_2(f) = \sum_{x\in X} \bar{x} \otimes f(\bar{x}).$$

Notice that for any g in G, one has

$$\begin{aligned}\tilde{\phi}_2(gf) &= \sum_{x\in X} \bar{x} \otimes (gf)(\bar{x}) = \sum_{x\in X} \bar{x} \otimes gf(g^{-1}\bar{x}) \\ &= g\left(\sum_{x\in X} g^{-1}\bar{x} \otimes f(g^{-1}\bar{x})\right) = g\tilde{\phi}_2(f)\end{aligned}$$

which proves that $\tilde{\phi}_2$ is a $\mathbb{Z}G$-homomorphism. Let f in $Hom(ind_H^G\mathbb{Z}, \Delta G)$ then

$$\phi_2\tilde{\phi}_2(f) = \sum_{x\in X} \phi_2(\bar{x} \otimes f(\bar{x})) = \sum_{x\in X} \Phi_{\bar{x}\otimes f(\bar{x})} = f. \tag{7.3.3.2}$$

In order to prove this last equality we can write

$$f(\bar{z}) = \sum_{i\in I} b_{\bar{z}}^i y_i$$

for any z in X. This implies $\Phi_{\bar{x}\otimes f(\bar{x})} = \sum_{i\in I} \Phi_{\bar{x}\otimes b_{\bar{x}}^i y_i}$, then for any element w in X we obtain

$$\begin{aligned}\sum_{x\in X} \Phi_{\bar{x}\otimes f(\bar{x})}(w) &= \sum_{i\in I}\sum_{x\in X} \Phi_{\bar{x}\otimes b_{\bar{x}}^i}(w) \\ &= \sum_{i\in I} b_{\bar{w}}^i y_i = f(\bar{w}).\end{aligned}$$

If we now take arbitrary $x \in X$ and $y_i \in Y$ it follows that

$$\tilde{\phi_2}\phi_2(\bar{x} \otimes y_i) = \sum_{z \in X} \bar{x} \otimes \phi_2(\bar{x} \otimes y_i)(\bar{z}) = \bar{z} \otimes y_i, \tag{7.3.3.3}$$

it is clear that equations (7.3.3.2) and (7.3.3.3) prove that $\tilde{\phi_2}$ is the inverse of ϕ_2, which proves the bijectivity of ϕ_2. □

We are now in position to define ψ_1 an ψ_2. For any f in $Hom(ind_H^G \mathbb{Z}, Hom(\Delta G, \bar{C}))$ let $\psi_1(f)(\bar{x} \otimes y_i) = f(\bar{x})(y_i)$, a straightforward computations show that ψ_1 is a $\mathbb{Z}G$-homomorphism and that is bijective. Now let f be in $Hom(\bar{C}, M_H)$, for any x in X and c in $\bar{C}$, let us define

$$((\psi_2 f)(\bar{x}))(c) = (\phi_2(f(c)))(\bar{x}). \tag{7.3.3.4}$$

We will prove that ψ_2 is a $\mathbb{Z}G$-homomorphism. Let g in G

$$\begin{aligned}((\psi_2(gf))(\bar{x}))(c) &= (\phi_2((gf)(c)))\,(\bar{x}) = \big(\phi_2(gf(g^{-1}c))\big)\,(\bar{x}) \\ &= (g\phi_2)(f(g^{-1}c))(\bar{x}) = g\phi_2(f(g^{-1}c))(g^{-1}\bar{x}) \\ &= g\psi_2(f)(g^{-1}\bar{x})(g^{-1}c) = \big(g\psi_2(f)(g^{-1}\bar{x})\big)\,(c) \\ &= (g\psi_2(f)(\bar{x}))\,(c).\end{aligned}$$

The proof that ψ_2 is bijective is omitted here.

The isomorphisms ψ_1 and ψ_2 induced in cohomology isomorphisms

$$\begin{aligned}(\psi_1)_* &: H^0(G, Hom(ind_H^G \mathbb{Z}, Hom(\Delta G, \bar{C}))) \to H^0(G, Hom(I_H, \bar{C})) \\ (\psi_2)_* &: H^0(G, Hom(\bar{C}, I_H)) \to H^0(G, Hom(ind_H^G \mathbb{Z}, Hom(\bar{C}, \Delta G))).\end{aligned}$$

Let us define the following non-degenerate pairing

$$\tau_\psi : [ind_H^G \mathbb{Z}, Hom(\bar{C}, \Delta G)]_G \times [ind_H^G \mathbb{Z}, Hom(\Delta G, \bar{C})]_G \to \mathbb{Q}/\mathbb{Z}$$

for any $[f]$ and $[z]$ by

$$\tau_\psi([f], [z]) = \frac{\sum_{x \in X} Trace_{\Delta G}(f(\bar{x}) \circ z(\bar{x}))}{|G|} + \mathbb{Z}.$$

7.3.4 Claim. *Given* $[f]$ *in* $[\bar{C}, I_H]_G$ *and* $[z]$ *in* $[ind_H^G\mathbb{Z}, Hom(\Delta G, \bar{C})]_G$*, then the next equality holds*

$$\tau_\psi((\psi_2)_*[f],[z]) = \tau_{I_H}([f],(\psi_1)_*[z]).$$

Proof. We start by understanding $Trace_{I_H}(f \circ \psi_1(z))$. For any x in X and y_i in Y one can write

$$(f \circ \psi_1(z))(\bar{x} \otimes y_i) = f(z(\bar{x})(y_i)) = \sum_{w \in X} \sum_{j \in I} a_{\bar{w}}^j (\bar{w} \otimes y_j) \tag{7.3.4.1}$$

for suitable integers $a_{\bar{w}}^j$, keeping the same notation we now fix x in X and define $\eta_{\bar{x}}^{f,z} : \Delta G \to \Delta G$ by

$$\eta_{\bar{x}}^{f,z}(y_i) = \sum_{j \in I} a_{\bar{x}}^j y_j, \tag{7.3.4.2}$$

we obtain that

$$Trace_{I_H}(f \circ \psi_1(z)) = \sum_{x \in X} Trace_{\Delta G}(\eta_{\bar{x}}^{f,z}). \tag{7.3.4.3}$$

On the other hand if we consider $(\psi_2 f)(\bar{x}) \circ z(\bar{x}) : \Delta G \to \Delta G$, it follows that

$$\begin{aligned}
(\psi_2 f)(\bar{x}) \circ z(\bar{x})(y_i) &= \phi_2(f(z(\bar{x})(y_i)))(\bar{x}) \\
&= \phi_2((f \circ \psi_1 z)(\bar{x} \otimes y_i))(\bar{x}) \\
&= \sum_{w \in X} \sum_{j \in I} a_{\bar{w}}^j \phi_2(\bar{w} \otimes y_j)(\bar{x}) && \text{by (7.3.4.1)} \\
&= \sum_{w \in X} \sum_{j \in I} a_{\bar{w}}^j \Phi_{\bar{w} \otimes y_j}(\bar{x}) \\
&= \sum_{j \in J} a_{\bar{x}}^j y_j \\
&= \eta_{\bar{x}}^{f,z}(y_i) && \text{by (7.3.4.2).}
\end{aligned}$$

The last equation shows that as linear endomorphisms of ΔG, $(\psi_2 f)(\bar{x}) \circ z(\bar{x})$

and $\eta_{\bar{x}}^{f,z}$ are equal then by (7.3.4.3) one obtains

$$\begin{aligned} Trace_{I_H}(f \circ \psi_1 z) &= \sum_{x \in X} Trace_{\Delta G}(\eta_{\bar{x}}^{f,z}) \\ &= \sum_{x \in X} Trace_{\Delta G}((\psi_2 f)(\bar{x}) \circ z(\bar{x})). \end{aligned}$$

The last equality proves the statement. □

We now recall from the Section 2 of Chapter 2 the Shapiro's isomorphisms

$$\begin{aligned} Sh_{(ind_H^G \mathbb{Z}, \Delta G, \bar{C})} &: [\Delta G, \bar{C}]_H \to [ind_H^G \mathbb{Z}, Hom(\Delta G, \bar{C})]_G \\ Sh^{-1}_{(ind_H^G \mathbb{Z}, \bar{C}, \Delta G)} &: [ind_H^G \mathbb{Z}, Hom(\bar{C}, \Delta G)]_G \to [\bar{C}, \Delta G]_H. \end{aligned}$$

In order to simplify the notation we will denote by $Sh = Sh_{(ind_H^G \mathbb{Z}, \Delta G, \bar{C})}$ and by $Sh^{-1} = Sh^{-1}_{(ind_H^G \mathbb{Z}, \bar{C}, \Delta G)}$.

We conclude this section by defining the non-degenerate pairing

$$\tau_{\Delta G}^H : [\bar{C}, \Delta G]_H \times [\Delta G, \bar{C}]_H \to \mathbb{Q}/\mathbb{Z}$$

by setting

$$\tau_{\Delta G}^H([f], [z]) = \frac{Trace_{\Delta G}(f \circ z)}{|H|} + \mathbb{Z},$$

and proving equation (7.3.0.1).

7.3.5 Claim. *Given* $[f]$ *in* $[ind_H^G \mathbb{Z}, Hom(\bar{C}, \Delta G)]_G$ *and* $[z]$ *an element of* $[\Delta G, \bar{C}]_H$, *the next equality holds*

$$\tau_{\Delta G}^H(Sh^{-1}[f], [z]) = \tau_\psi([f], Sh[z]).$$

Proof. We start by computing $\tau_\psi([f], Sh[z])$.

$$\begin{aligned} \tau_\psi([f], Sh[z]) &= \frac{\sum_{x \in X} Trace_{\Delta G}(f(\bar{x}) \circ (Sh(z)(\bar{x})))}{|G|} + \mathbb{Z} \\ &= \frac{\sum_{x \in X} Trace_{\Delta G}(f(\bar{x}) \circ xz)}{|G|} + \mathbb{Z}. \end{aligned}$$

On the other hand one has that

$$\tau^H_{\Delta G}(Sh^{-1}[f],[z]) = \frac{Trace_{\Delta G}(f(\bar{1_G}) \circ z)}{|H|} + \mathbb{Z}$$

In order to prove the claim, it would be enough to show that for all x in X $Trace_{\Delta G}(f(\bar{x}) \circ xz) = Trace_{\Delta G}(f(\bar{1_G}) \circ z)$. This follows simply from the fact that $f(\bar{x}) = xf(\bar{1_G})$, hence for any $y \in \Delta G$, $(xz)(y) = xz(x^{-1}y)$, which implies

$$f(\bar{x})((xz)(y)) = ((xf)(\bar{1_G}))\,(xz(x^{-1}y)) = xf(\bar{1_G})(z(x^{-1}y)).$$

From the above equality one can conclude that $f(\bar{y}) \circ yz$ and $f(\bar{1_G}) \circ z$ are similar endomorphisms of ΔG and then have the same trace as needed. □

We can conclude this section by mentioning that the $\mathbb{Z}G$-homomorphism β_1 is the composition $Sh^{-1}(\psi_2)_*(\hat{\phi}_H)_*(\hat{i}_H)_*$.

Chapter 8

Conditions over $f_{\mathfrak{P}}$.

In this final chapter we will give a set of conditions that the candidate $f : C \to$ must satisfy in order for (1.0.1.9) to hold. We restrict to the case when m is an odd squarefree positive integer, in particular we will assume that

$$m = \prod_{i=1}^{n} p_i$$

with each p_i an odd prime and $p_i \neq p_j$, whenever $i \neq j$.
Let $\mathfrak{P}_i$, be the only prime in S_* above p_i and fix s_i, σ_i in $G_{\mathfrak{P}_i}$ such that s_i generate $G^o_{\mathfrak{P}_i}$ and $\bar{\sigma}_i$, the image of σ_i under the natural projection $G_{\mathfrak{P}} \to G_{\mathfrak{P}}/G^o_{\mathfrak{P}}$, is a Frobenius element. We have that

$$G_{\mathfrak{P}_i} = \langle s_i \rangle \oplus \langle \sigma_i \rangle \tag{8.0.0.1}$$

as in (6.1.2).

Since G is the product of the subgroups $G^o_{\mathfrak{P}_i}$ for $1 \leq i \leq n$, it follows that $\Sigma = \{s_i : 1 \leq i \leq n\}$ is a set of generators for G. By chapter 3 we can construct an envelope of $\boldsymbol{\mu}$

$$0 \longrightarrow \boldsymbol{\mu} \longrightarrow C \longrightarrow \bar{C} \longrightarrow 0,$$

where $C = F/A_{\mathbb{Z}G}$, $\bar{C} = F/A_Y$, $F = \bigoplus_{i=0}^{n} \mathbb{Z}G$ and $A_{\mathbb{Z}G}$ and A_Y the submodules of F given by

$$A_{\mathbb{Z}G} = \left\langle \left(2mx, x(g_1^{-1} - a(g_1^{-1})), \ldots, x(g_n^{-1} - a(g_n^{-1}))\right) : x \in \mathbb{Z}G \right\rangle$$
$$A_Y = \left\langle \left(2my, y(g_1^{-1} - a(g_1^{-1})), \ldots, y(g_n^{-1} - a(g_n^{-1}))\right) : y \in Y \right\rangle .$$

If we assume to work in the category of $\mathbb{Z}'G$-modules, obtaining $[f]$ in $[\bar{C}, L]_G$ satisfying (1.0.1.9), can be achieved by computing its 'local parts" introduced in the previous chapter, which means that for each $\mathfrak{P}$ in S_* one needs to compute $[f_{\mathfrak{P}}]$ in $[\bar{C}, \Delta G]_{G_{\mathfrak{P}}}$ such that

$$\tau_{\Delta G}^{G_{\mathfrak{P}}}([f_{\mathfrak{P}}], [z]) = inv_{\mathfrak{P}}(\partial_{\mathfrak{P}}[z]) \tag{8.0.0.2}$$

for all $[z]$ in $[\Delta G, \bar{C}]_{G_{\mathfrak{P}}}$, where

$$\partial_{\mathfrak{P}} : [\Delta G, \bar{C}]_{G_{\mathfrak{P}}} \to H^2(G_{\mathfrak{P}}, \boldsymbol{\mu}) \tag{8.0.0.3}$$

is the inverse of the isomorphism given in section 7.2.

We should recall that $\partial_{\mathfrak{P}}^{-1}$ depends on the choice of fix a map $\phi_{\mathfrak{P}} : G \to G_{\mathfrak{P}}$ (as mentioned at the beginning of 7.2). if $\mathfrak{P}$ in S_* and p is the prime number such that $p\mathbb{Z} = \mathfrak{P} \cap \mathbb{Q}$ we denote

$$\Sigma_p = \begin{cases} \Sigma \setminus \{s_i\} & \text{if} \quad p = p_i \quad \text{for some i} \\ \Sigma \quad \text{else} \end{cases}$$

If $p = p_i$ for some i and since every g in G can be expressed in the following form: $g = s_i^n \prod s_j^{n_j}$ where the product is taken over all s_j in Σ_p, then $\phi_{\mathfrak{P}}(g) = s_i^n \phi\left(\prod s_j^{n_j}\right)$. If there is no confusion we will denote $\phi_{\mathfrak{P}}$ by simply ϕ.

We will discuss the conditions that $f_{\mathfrak{P}} : \bar{C} \to \Delta G$ must satisfy so that (8.0.0.2) holds in three different cases: non archimedean, unramified and ramified.

Before looking at each particular case we make the following observation.

Let us denote $G^{\times} = G \setminus \{1_G\}$ and $l_g = 1_G - g$, we obtain that $\{l_g : g \in G^{\times}\}$ is a $\mathbb{Z}$-basis for ΔG.

Recalling from chapter 7 that for each $\mathfrak{P}$ in S_*, if $[x]$ is an element in $H^2(G_{\mathfrak{P}}, \boldsymbol{\mu})$ and $b : G_{\mathfrak{P}} \times G_{\mathfrak{P}} \to \mathbb{Z}$ is a function such that

$$x_{h_1,h_2} = b(h_1, h_2) + 2m\mathbb{Z}$$

for all h_1, h_2 in $G_{\mathfrak{P}}$, then $\partial_{\mathfrak{P}}^{-1}[x] = [\tilde{x}]$ where

$$\tilde{x}(l) = \bar{D}_l^0(2m, s_1^{-1} - a(s_1^{-1}), \ldots, s_n^{-1} - a(s_n^{-1})) + (0, D_l^1 s_1^{-1}, \ldots, D_l^n s_n^{-1}) + A_y.$$

A straightforward computation shows that

$$\bar{D}_{l_g}^o = \frac{1}{2m} \sum_{h \in G} a(h^{-1}) a(\phi(g)) b(\phi(g)^{-1}, \phi(h)) h. \tag{8.0.0.4}$$

Since ΔG is a lattice it follows that $[C, \Delta G]_{G_{\mathfrak{P}}} \cong [\bar{C}, \Delta G]_{G_{\mathfrak{P}}}$.
In order to define $f_{\mathfrak{P}} : C \to \Delta G$ we can first consider to define $\bar{f}_{\mathfrak{P}} : F \to \Delta G$ satisfying

$$\bar{f}_{\mathfrak{P}}(2m, s_1^{-1} - a(s_1^{-1}), \ldots, s_n^{-1} - a(s_n^{-1})) = 0. \tag{8.0.0.5}$$

Finally let us denote by $\tilde{X}$ the element in $[\Delta G, F]$ given by

$$\tilde{X}(l) = \bar{D}_l^0(2m, s_1^{-1} - a(s_1^{-1}), \ldots, s_n^{-1} - a(s_n^{-1})) + (0, D_l^1 s_1^{-1}, \ldots, D_l^n s_n^{-1}),$$

it follows clearly that

$$\tau_{\Delta G}^{G_{\mathfrak{P}}}([f_{\mathfrak{P}}], [x]) = \frac{Trace_{\Delta G}(\bar{f}_{\mathfrak{P}} \circ \tilde{X})}{|G_{\mathfrak{P}}|} + \mathbb{Z}'.$$

8.1 Conditions over $f_{\mathfrak{P}}$.

We give a list of equations that $f_{\mathfrak{P}}$ must satisfy so that (8.0.0.2) holds for each of the three different cases.

8.1.1 The archimedean case.

By (6.1.12.1) we have that, $H^2(G_\infty, \boldsymbol{\mu})$ is generated by a single element of order 2; namely z_∞^2. It follows immediately that for any $f_\infty : C \to \Delta G$

$$\frac{Trace_{\Delta G}(f_\infty \circ \tilde{z}_\infty^2)}{|G_\infty|} \equiv inv_\infty(z_\infty^2) \pmod{\mathbb{Z}'}.$$

We will denote by $\{e_i : 0 \le i \le m\}$ to be the standard $\mathbb{Z}G$-basis for F. It follows that $\bar{f}_\infty : F \to \Delta G$ must only satisfy (8.0.0.5).

Let $\bar{f}_\infty(e_i) = \sum_{g \in G^\times} a_g^i l_g$ and for a moment assume that $\bar{f}_\infty$ is actually a $\mathbb{Z}G$-homomorphism, then the left hand side of (8.0.0.5) can be written as

$$2m\bar{f}_\infty(e_0) + \sum_{i=1}^{n}\left(s_i^{-1}\bar{f}_\infty(e_i) - a(s_i^{-1})\bar{f}_\infty(e_i)\right) =$$
$$2m\sum_{g\in G^\times} a_g^0 l_g - \sum_{i=1}^{n}\sum_{g\in G^\times} a(s_i^{-1})a_g^i l_g + \sum_{i=1}^{n}\sum_{g\in G^\times} a_g^i s_i^{-1} l_g =$$
$$\sum_{g\in G^\times}\left(2ma_g^o - \sum_{i=1}^{n} a(s_i^{-1})a_g^i\right) l_g + \sum_{i=1}^{n}\left[\sum_{g\in G^\times\setminus\{s_i\}} a_g^i(l_{s_i^{-1}g} - l_{s_i^{-1}}) - a_{s_i}^i l_{s_i^{-1}}\right].$$

If we let

$$b_g^i = \begin{cases} a^i_{s_i^{-1}g} & \text{if } \ g \ne s_i^{-1} \\ -\sum_{g\in G^\times} a_g^i & \text{if } \ g = s_i^{-1} \end{cases}$$

it follows that

$$2m\bar{f}_\infty(e_0) + \sum_{i=1}^{n}\left(s_i^{-1}\bar{f}_\infty(e_i) - a(s_i^{-1})\bar{f}_\infty(e_i)\right) =$$
$$\sum_{g\in G^\times}\left[2ma_g^0 + \sum_{i=1}^{n} b_g^i - a(s_i^{-1})a_g^i\right] l_g. \tag{8.1.1.1}$$

If g does not belong to $\{s_1^{-1}, \cdots, s_n^{-1}\}$ one can define $a_g^i = a(g)$ for $1 \le i \le n$

then

$$
\begin{aligned}
A_g^0 &= \sum_{i=1}^{n} b_g^i - a(s_i^{-1})a_g^i = \sum_{i=1}^{n} a^i_{s_i^{-1}g} - a(s_i^{-1})a_g^i \\
&= \sum_{i=1}^{n} a(s_i^{-1}g) - a(s_i^{-1})a(g) \equiv 0 \pmod{2m},
\end{aligned}
$$

hence one can chose $a_g^0 = -A_g^0/2m$.

If $g = s_i^{-1}$, then as before let $a_g^i = a(g)$. In this case

$$
\begin{aligned}
A_i^0 &= \sum_{i=1}^{n} b_g^i - a(s_i^{-1})a_g^i \\
&= -\sum_{g\in G^\times} a_g^i - \sum_{j=1}^{n} a(s_j^{-1})a^j_{s_i^{-1}} + \sum_{j=1, j\neq i}^{n} a^j_{s_j^{-1}s_i^{-1}} \\
&= \sum_{j=1, j\neq i}^{n} \big(a(s_j^{-1}s_i^{-1}) - a(s_j^{-1})a(s_i^{-1})\big) - a(s_i^{-1})a(s_i) - \sum_{g\in G^\times} a(g) \\
&\equiv 0 \pmod{2m},
\end{aligned}
$$

let $a^0_{s_i} = -A_i^0/2m$.

With the choices made for the integers a_g^i, for $0 \leq i \leq n$ and $g \in G^\times$, it follows that (8.1.1.1) is 0 as wanted.

8.1.2 The unramified case.

Let us consider $\mathfrak{Q}$ in S_* to be unramified, where q is the prime number satisfying $q\mathbb{Z} = \mathfrak{Q} \cap \mathbb{Q}$, by (6.1.11.1) z_q^2 generates $H^2(G_{\mathfrak{Q}}, \boldsymbol{\mu})$. If $\tilde{z}_q^2 = \partial_{\mathfrak{Q}}^{-1} z_q^2$, then $\bar{f}_{\mathfrak{Q}}$ must satisfy apart from (8.0.0.5) that, $Trace_{\Delta G}(\bar{f}_{\mathfrak{Q}} \circ \tilde{Z}_q^2) \equiv 0 \pmod{|G_{\mathfrak{Q}}|}$. This two conditions can easily be achieved by defining the $\mathbb{Z}G$-homomorphism $\bar{f}_{\mathfrak{Q}} = |G_{\mathfrak{Q}}|\bar{f}_\infty$.

8.1.3 The ramified case.

In this section we return to the notation established in chapter 6. Let us choose one of the primes appearing in the factorization of m and denoted by $p = p_k$, Let $\mathfrak{P}$ be the unique prime in S_* above p_k and let s_k, σ_k elements in $G_{\mathfrak{P}}$ satisfying (8.0.0.1).

In subsection 6.1.10 we found three generator for the group $H^2(G_{\mathfrak{P}}, \boldsymbol{\mu})$ namely $[z^0]$, $[z^1]$ and $[z^2]$. We will recall the expression of the cocycles z^0, z^1 and z^2 here. In order to do this lets us define one more time the following integers.

Let $m_p = m/p$, $v_p = gcd(p-1, 2m_p)$ and $t_p = 2m_p/v_p$. We also choose α to be an integer satisfying

$$1 + 2m_p\alpha \equiv 0 \pmod{p}.$$

If we fix μ to be a generator of $\boldsymbol{\mu}$, let $\mu'_p = \mu^{1+2m_p\alpha}$. With this we can define the first 2-cocycle z^0 by

$$z^0[s_k^{i_1}\sigma_k^{j_1}|s_k^{i_2}\sigma_k^{j_2}] = \begin{cases} 1 \ \text{ if } \ i_1 + i_2 < p-1 \\ (\mu'_p)^{t_p} \ \text{ if } \ i_1 + i_2 \geq p-1 \end{cases} \tag{8.1.3.1}$$

Now let $v_p = gcd(f_{\mathfrak{P}}, v_p)$, $t'_p = v_p/v'_p$. If $n_p = \left(\sum_{i=0}^{f_{\mathfrak{P}}-1} p^i\right)/t_p$ and $v''_p = gcd(v_p, n_p)$, by letting $t''_p = v_p/v''_p$, we can now describe the remaining two cocycles:

$$z^1[s_k^{i_1}\sigma_k^{j_1}|s_k^{i_2}\sigma_k^{j_2}] = (\mu'_p)^{j_1 i_2 t_p t'_p} \tag{8.1.3.2}$$

$$z^2[s_k^{i_1}\sigma_k^{j_1}|s_k^{i_2}\sigma_k^{j_2}] = \begin{cases} 1 \ \text{ if } \ j_1 + j_2 < f_{\mathfrak{P}} \\ (\mu'_p)^{t''_p} \ \text{ if } \ j_1 + j_2 \geq f_{\mathfrak{P}} \end{cases} \tag{8.1.3.3}$$

By (7.2.2.2) there are functions $b_i : G_{\mathfrak{P}} \times G_{\mathfrak{P}} \to \mathbb{Z}$ such that

$$z_p^i[h_1|h_2] = b_i(h_1, h_2) + 2m\mathbb{Z}$$

for $i = 0, 1, 2$. Let us assume that $\bar{f}_{\mathfrak{P}} : F \to \Delta G$ is a $\mathbb{Z}G_{\mathfrak{P}}$-homomorphism satisfying (8.0.0.5).

1. If $[\tilde{z}_p^0] = \partial_{\mathfrak{P}}^{-1}[z_p^0]$ then we can define

$$b_0(s_k^{i_1}\sigma_k^{j_1}, s_k^{i_2}\sigma_k^{j_2}) = \begin{cases} 0 \ \text{ if } \ i_1 + i_2 < p - 1 \\ (1 + 2m_p\alpha)t_p \ \text{ if } \ i_1 + i_2 \geq p - 1 \end{cases}$$

   It follows by (8.0.0.4) that $v_p\bar{D}_{l_g}^0$ belongs to $\mathbb{Z}G$.
   Let $\bar{Z}^0$ in $[\Delta G, F]$ be given by

$$\bar{Z}_{l_g}^0 = v_p(0, D^1 s_1^{-1}, \ldots, D_{l_g}^n s_n^{-1}).$$

   It follows that

$$Trace_{\Delta G}(\bar{f}_{\mathfrak{P}} \circ \tilde{Z}_p^0) = \frac{Trace_{\Delta G}(\bar{f}_{\mathfrak{P}} \circ \bar{Z}^0)}{v_p},$$

   hence we will need that

$$Trace_{\Delta G}(\bar{f}_{\mathfrak{P}} \circ \bar{Z}^0) \equiv -1 \pmod{v_p|G_{\mathfrak{P}}|}. \tag{8.1.3.4}$$

2. Let $[\tilde{z}_p^1] = \partial_{\mathfrak{P}}^{-1}[z_p^1]$ then we can define

$$b_1(s_k^{i_1}\sigma_k^{j_1}, s_k^{i_2}\sigma_k^{j_2}) = (1 + 2m_p\alpha)j_1 i_2 t_p t'_p.$$

   Since $t_p t'_p = 2m_p/v'_p$ it follows by (8.0.0.4) that $v'_p\bar{D}_{l_g}^0$ belongs to $\mathbb{Z}G$.
   Let $\bar{Z}^1$ in $[\Delta G, F]$ be given by

$$\bar{Z}_{l_g}^1 = v'_p(0, D^1 s_1^{-1}, \ldots, D_{l_g}^n s_n^{-1}).$$

   We can conclude that

$$Trace_{\Delta G}(\bar{f}_{\mathfrak{P}} \circ \tilde{Z}_p^1) = \frac{Trace_{\Delta G}(\bar{f}_{\mathfrak{P}} \circ \bar{Z}^1)}{v'_p},$$

   in this case one would need to solve

$$Trace_{\Delta G}(\bar{f}_{\mathfrak{P}} \circ \bar{Z}^1) \equiv -1 \pmod{v'_p|G_{\mathfrak{P}}|}. \tag{8.1.3.5}$$

3. Finally let $[\tilde{z}_p^2] = \partial_{\mathfrak{P}}^{-1}[z_p^2]$ we can define

$$b_2(s_k^{i_1}\sigma_k^{j_1}, s_k^{i_2}\sigma_k^{j_2}) = \begin{cases} 0 \text{ if } j_1 + j_2 < |G_{\mathfrak{P}}^o| \\ (1+2m_p\alpha)t_p'' \text{ if } j_1 + j_2 \geq |G_{\mathfrak{P}}^o| \end{cases}$$

We conclude by (8.0.0.4) that $2m_p\bar{D}_{l_g}^0$ belongs to $\mathbb{Z}G$.
Let $\bar{Z}^2$ in $[\Delta G, F]$ be given by

$$\bar{Z}_{l_g}^2 = 2m_p(0, D^1 s_1^{-1}, \ldots, D_{l_g}^n s_n^{-1}).$$

It follows that

$$Trace_{\Delta G}(\bar{f}_{\mathfrak{P}} \circ \tilde{Z}_p^2) = \frac{Trace_{\Delta G}(\bar{f}_{\mathfrak{P}} \circ \bar{Z}^2)}{2m_p},$$

hence we will need that

$$Trace_{\Delta G}(\bar{f}_{\mathfrak{P}} \circ \bar{Z}^2) \equiv 0 \pmod{2m_p|G_{\mathfrak{P}}|}. \tag{8.1.3.6}$$

The candidate $\bar{f}_{\mathfrak{P}} : F \to \Delta G$ must then satisfy equation (8.0.0.5) and congruences (8.1.3.4),(8.1.3.5) and (8.1.3.6). The complexity of the linear algebra system needed to solve this four equations increases with p.

8.2 The homomorphism $f : C \to L$.

We present here a method to compute a "global" $f : C \to L$, satisfying 1.0.1.9. In order to achieve this we begin by recalling some facts and notation introduced in previous chapters.

From the construction of the envelope of $\boldsymbol{\mu}$ in chapter 3, we consider the middle exact row from diagram 3.0.4.4

$$\mathbb{Z}G \longrightarrow F \xrightarrow{\pi} C.$$

Following the notation of the previous section $\{e_i : 0 \leq i \leq n\}$ will denote the standard $\mathbb{Z}G$-basis of the free $\mathbb{Z}G$-module F. Since $ker(\pi)$ is the $\mathbb{Z}G$-

submodule generated by the element

$$(2m, s_1^{-1} - a(s_1^{-1}), \ldots, s_n^{-1} - a(s_n^{-1})),$$

any $\bar{f} : F \to L$ satisfying $ker(\pi) \subset ker(\bar{f})$ induces a homomorphism $f : C \to L$.

We now consider the fact that $\mathbb{Z}S \cong \bigoplus_{\mathfrak{P} \in S_*} ind_{G_{\mathfrak{P}}}^G \mathbb{Z}$ and denote by $\bar{\phi}_{\mathfrak{P}}$ the projection form $\mathbb{Z}S$ onto $ind_{G_{\mathfrak{P}}}^G \mathbb{Z}$. Let $I_{\mathfrak{P}} := ind_{G_{\mathfrak{P}}}^G \otimes \Delta G$ and

$$\hat{\phi}_{\mathfrak{P}} = \bar{\phi}_{\mathfrak{P}} \otimes id_{\Delta G} : \mathbb{Z}S \otimes \Delta G \to I_{\mathfrak{P}}.$$

For each $\mathfrak{P}$ in S_* we fix $X_{\mathfrak{P}}$ to be a left transversal of $G_{\mathfrak{P}}$ in G, and for x in $X_{\mathfrak{P}}$ we denote by $\bar{x} = x \otimes_{G_{\mathfrak{P}}} 1$, then $\bar{X}_{\mathfrak{P}} = \{\bar{x} : x \in X_{\mathfrak{P}}\}$ forms a $\mathbb{Z}$-basis of $ind_{G_{\mathfrak{P}}}^G \mathbb{Z}$.

There is a $\mathbb{Z}G$-isomorphism defined in 7.3.3.1

$$\phi_2^{\mathfrak{P}} : I_{\mathfrak{P}} \to Hom(ind_{G_{\mathfrak{P}}}^G \mathbb{Z}, \Delta G),$$

where $\phi_2^{\mathfrak{P}}(\bar{x} \otimes l_g) = \Phi_{\bar{x} \otimes l_g} : ind_{G_{\mathfrak{P}}}^G \mathbb{Z} \to \Delta G$ is defined by

$$\Phi_{\bar{x} \otimes l_g}(\sum_{z \in X_{\mathfrak{P}}} a^z \bar{z}) = a^x l_g.$$

The isomorphism $\phi_2^{\mathfrak{P}}$ induces a $\mathbb{Z}G$-isomorphism

$$\psi_2^{\mathfrak{P}} : Hom(M, I_{\mathfrak{P}}) \to Hom(ind_{G_{\mathfrak{P}}}^G \mathbb{Z}, Hom(M, \Delta G)),$$

where M is a $\mathbb{Z}G$-module. The definition of $\psi_2^{\mathfrak{P}}$ is given in 7.3.3.4. We recall that if $f : M \to I_{\mathfrak{P}}$

$$(\psi_2^{\mathfrak{P}} f)(\bar{x})(m) = \phi_2^{\mathfrak{P}}(f(m))(\bar{x})$$

for all m in M and $\bar{x}$ in $\bar{X}_{\mathfrak{P}}$.

We finally consider the Shapiro isomorphism computed in 2.2.5.1

$$Sh^0_{\mathfrak{P}} = Sh^0_{Hom(M,\Delta G)} : [M, \Delta G]_{G_{\mathfrak{P}}} \to [ind^G_{G_{\mathfrak{P}}}\mathbb{Z}, Hom(M, \Delta G)]_G.$$

With this setup, it follows clearly that the following diagram commutes

$$\begin{array}{ccccccccc}
[F,\Delta G]_{G_{\mathfrak{P}}} & \overset{Sh^0_{\mathfrak{P}}}{\longrightarrow} & [ind^G_{G_{\mathfrak{P}}}\mathbb{Z}, Hom(F,\Delta G)]_G & \overset{(\psi_2^{\mathfrak{P}})_*}{\longleftarrow} & [F, I_{\mathfrak{P}}]_G & \overset{(\hat{\phi}_{\mathfrak{P}})_*}{\longleftarrow} & [F, \mathbb{Z}S \otimes \Delta G]_G & \leftarrow & [F, L]_G \\
\pi_* \downarrow & & \pi_* \downarrow & & \pi_* \downarrow & & \pi_* \downarrow & & \pi_* \downarrow \\
[C,\Delta G]_{G_{\mathfrak{P}}} & \overset{Sh^0_{\mathfrak{P}}}{\longrightarrow} & [ind^G_{G_{\mathfrak{P}}}\mathbb{Z}, Hom(C,\Delta G)]_G & \overset{(\psi_2^{\mathfrak{P}})_*}{\longleftarrow} & C, I_{\mathfrak{P}}]_G & \overset{(\hat{\phi}_{\mathfrak{P}})_*}{\longleftarrow} & [C, \mathbb{Z}S \otimes \Delta G]_G & \leftarrow & [C, L]_G
\end{array}$$

where the homomorphisms $[F, L] \to [F, \mathbb{Z}S \otimes \Delta G]$ and $[C, L] \to [C, \mathbb{Z}S \otimes \Delta G]$ are induced by the natural inclusion $L \to \mathbb{Z}S \otimes \Delta G$.

Let us assume that $\bar{f} : F \to L$ is a $\mathbb{Z}G$-homomorphism described by

$$\bar{f}(e_i) = \sum_{g \in G^\times} \sum_{\mathfrak{P} \in S_*} \sum_{x \in X_{\mathfrak{P}}} b^{i,g}_{\mathfrak{P},x}(x\mathfrak{P}) \otimes l_g,$$

where $\sum_{\mathfrak{P} \in S_*} \sum_{x \in X_{\mathfrak{P}}} b^{i,g}_{\mathfrak{P},x} = 0$ for all g in $G^\times$. We can consider $(\hat{\phi}_{\mathfrak{P}})_* \bar{f}$ to be defined by

$$(\hat{\phi}_{\mathfrak{P}})_*(\bar{f})(e_i) = \sum_{g \in G^\times} \sum_{x \in X_{\mathfrak{P}}} b^{i,g}_{\mathfrak{P},x} \bar{x} \otimes l_g,$$

we can evaluate $(\hat{\phi}_{\mathfrak{P}})_* \bar{f}$ at $(\psi_2^{\mathfrak{P}})_*$ to obtain the expression

$$\begin{aligned}
(\psi_2^{\mathfrak{P}})_*(\hat{\phi}_{\mathfrak{P}})_*(\bar{f})(\bar{z})(e_i) &= \phi_2^{\mathfrak{P}}(\bar{f}(e_i))(\bar{z}) \\
&= \Phi_{\sum_{g \in G^\times} \sum_{x \in X_{\mathfrak{P}}} b^{i,g}_{\mathfrak{P},x}}(\bar{z}) \\
&= \sum_{g \in G^\times} b^{i,g}_{\mathfrak{P},x} l_g.
\end{aligned}$$

If we consider the maps $\bar{f}_{\mathfrak{P}} : F \to \Delta G$, defined in subsections 8.1.1, 8.1.2 and 8.1.3, where we are assuming $\mathfrak{P}$ is archimedean, unramified and ramified respectively, and

$$\bar{f}_{\mathfrak{P}}(e_i) = \sum_{g \in G^{\times}} a^i_g l_g,$$

then $\bar{f} : F \to L$ must satisfy that, $ker(\pi_*) \subset ker(\bar{f})$ and that for each $\mathfrak{P}$ in S_*

$$Sh^0_{\mathfrak{P}} \bar{f}_{\mathfrak{P}} = (\psi^{\mathfrak{P}}_2)_* (\hat{\phi}_{\mathfrak{P}})_* (\bar{f}).$$

This last equation can be written as

$$\begin{gathered}(Sh^0_{\mathfrak{P}} \bar{f}_{\mathfrak{P}})(\bar{x})(e_i) = (\psi^{\mathfrak{P}}_2)_* (\hat{\phi}_{\mathfrak{P}})_* (\bar{f})(\bar{x})(e_i) \\ \sum_{g \in G^{\times}} a^i_g l_g = \sum_{g \in G^{\times}} b^{i,g}_{\mathfrak{P},x} l_g,\end{gathered}$$

for each $\mathfrak{P}$ in S_*, x in $X_{\mathfrak{P}}$ and $0 \leq i \leq n$.

The last set of equations gives a method to compute a ”global” $\bar{f} : F \to L$ from the local maps $\bar{f}_{\mathfrak{P}} : F \to \Delta G$.

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Appendix A

Appendix

The material presented in this appendix is a joint work by D. Riveros and A.Weiss.

It presents a new approach to the $\mathbb{Z}G$-structure of the S-units and set a program to compute models of $\mathbb{Z}G$-modules M stably isomorphic to E.

APPENDIX

GALOIS STRUCTURE OF S-UNITS

D. Riveros and A. Weiss

Let K/k be a finite Galois extension of number fields with Galois group G and let S be a finite G-stable set of primes of K containing all archimedean primes. Assume that S is *large* in the sense that it contains all ramified primes of K/k and that the S-class group of K is trivial. Let E denote the G-module of S-units of K and $\boldsymbol{\mu}$ the roots of unity in K. The purpose of this paper is to specify the stable isomorphism class of the G-module E in a much more explicit way than in Theorem B of [GW2].

More precisely, and continuing in the notation of [GW2], we recall that [T1], [T2] defines a canonical 2-extension class of G-modules, represented by Tate sequences

$$0 \to E \to A \to B \to \Delta S \to 0,$$

with A a finitely generated cohomologically trivial $\mathbb{Z}G$-module, B a finitely generated projective $\mathbb{Z}G$-module and ΔS the kernel of the G-map $\mathbb{Z}S \to \mathbb{Z}$ which sends every element of S to 1. From this [C1] obtains the Chinburg $\Omega(3)$-class

$$\Omega_m := [A] - [B]$$

in the locally free class group $\mathrm{Cl}(\mathbb{Z}G) \subseteq K_0(\mathbb{Z}G)$, which is an invariant of K/k that is independent of S, and conjectures that Ω_m equals the root number class in $\mathrm{Cl}(\mathbb{Z}G)$.

The method of [GW2] analyzes the G-module E in terms of a *fixed* envelope of $\boldsymbol{\mu}$. This is an exact sequence

$$0 \to \boldsymbol{\mu} \to \boldsymbol{\omega} \to \overline{\boldsymbol{\omega}} \to 0, \qquad (0.1)$$

with $\boldsymbol{\omega}$ cohomologically trivial and $\overline{\boldsymbol{\omega}}$ the $\mathbb{Z}G$-lattice obtained from $\boldsymbol{\omega}$ by factoring by its $\mathbb{Z}$-torsion. By Theorem B, the G-module E is determined, up to stable isomorphism, by knowledge of the G-set S, the G-module $\boldsymbol{\mu}$, the Chinburg class $\Omega_m(K/k) \in \mathrm{Cl}(\mathbb{Z}[G])$, and an arithmetically defined character

$$\varepsilon \in H^2\big(G, \mathrm{Hom}(\Delta S, \boldsymbol{\mu})\big)^*,$$

where $\_^*$ means $\mathrm{Hom}(\_, \mathbb{Q}/\mathbb{Z})$.

Let $L_1 := \Delta G \otimes \Delta S$ and $L_2 := \Delta G \otimes L_1$ with $\otimes = \otimes_{\mathbb{Z}}$ and diagonal action by G. Choose the envelope $\boldsymbol{\omega}$ to be related to the Chinburg class by the condition

$$[\boldsymbol{\omega}] - w[\mathbb{Z}G] = \Omega_m(K/k) \text{ in } \mathrm{Cl}(\mathbb{Z}G), \tag{0.2}$$

with $|G|w$ equal to the $\mathbb{Q}$-dimension of $\mathbb{Q} \otimes \boldsymbol{\omega}$. We will construct a canonical isomorphism $H^2\big(G, \mathrm{Hom}(\Delta S, \boldsymbol{\mu})\big)^* \to H^1\big(G, \mathrm{Hom}(\boldsymbol{\omega}, L_2)\big)$ so that our main result is the

Theorem. *Let $M = M(\varepsilon)$ denote the G-module in a $\mathbb{Z}$-split 1-extension*

$$0 \to L_2 \to M \to \boldsymbol{\omega} \to 0$$

with extension class equal to the image $\varepsilon^{(1)}$ of $-\varepsilon$ in $H^1\big(G, \mathrm{Hom}(\boldsymbol{\omega}, L_2)\big)$. Then $E \oplus (\mathbb{Z}G)^n$ is stably isomorphic to $M(\varepsilon)$, with $n := (|G| - 2)(|S| - 1) + w$ when $G \neq 1$.

This improves Theorem B by explaining *how* its data determines M, a *model* for the stable isomorphism class of E. The remaining problem becomes not only to understand the ingredients $\Delta S, \boldsymbol{\omega}, \Omega_m, \varepsilon, n$ of the Theorem, but to do so in a way that improves M into a better approximation of E. As a first example of this, we show how to get a smaller n, and an M', in Corollary 4.1. There is also a continuing discussion on the relation of the Theorem with [GW2], including a Proposition 2.2, and especially on the role of the distinguished character ε, in Remark 4.3 and Lemma 4.4.

Our proof of the Theorem, based on [GW2], is presented in three sections: the first recalling relevant results, the second reformulating the Theorem in their terms, and the third containing a proof. The last section discusses some basic aspects of the many new problems that arise.

1 Review of [GW2]

Applying $\_ \otimes \Delta S$ to the ($\mathbb{Z}$-split) augmentation sequence $0 \to \Delta G \to \mathbb{Z}G \to \mathbb{Z} \to 0$ gives the ($\mathbb{Z}$-split) G-module sequence

$$0 \to L_1 \to \mathbb{Z}G \otimes \Delta S \to \Delta S \to 0, \tag{1.1}$$

with $\mathbb{Z}G \otimes \Delta S$ a free $\mathbb{Z}G$-module, and $L_1 := \Delta G \otimes \Delta S$. Applying $\mathrm{Hom}(\_, \boldsymbol{\mu})$ to this gives the exact G-module sequence

$$0 \to \mathrm{Hom}(\Delta S, \boldsymbol{\mu}) \to \mathrm{Hom}(\mathbb{Z}G \otimes \Delta S, \boldsymbol{\mu}) \to \mathrm{Hom}(L_1, \boldsymbol{\mu}) \to 0,$$

inducing the connecting isomorphism in Tate cohomology

$$\partial_1 : H^1\big(G, \mathrm{Hom}(L_1, \boldsymbol{\mu})\big) \to H^2\big(G, \mathrm{Hom}(\Delta S, \boldsymbol{\mu})\big) \tag{1.2}$$

and defining $\varepsilon_1 := \varepsilon \circ \partial_1 \in H^1\big(G, \mathrm{Hom}(L_1, \boldsymbol{\mu})\big)^*$.

Similarly, applying $\mathrm{Hom}(L_1, \_\,)$ to our fixed envelope (0.1) of $\boldsymbol{\mu}$ and then G-cohomology gives the

$$\partial_0' : \widehat{H}^{\,0}\Big(G, \mathrm{Hom}(L_1, \overline{\boldsymbol{\omega}})\big) \to H^1\big(G, \mathrm{Hom}(L_1, \boldsymbol{\mu})\big), \tag{1.3}$$

and defines $\varepsilon_0 := \varepsilon_1 \circ \partial_0' \in \widehat{H}^{\,0}\big(G, \mathrm{Hom}(L_1, \overline{\boldsymbol{\omega}})\big)^*$.

We now use the isomorphism

$$\widehat{H}^{\,0}\big(G, \mathrm{Hom}(\overline{\boldsymbol{\omega}}, L_1)\big) \to H^1\big(G, \mathrm{Hom}(L_1, \overline{\boldsymbol{\omega}})\big)^*, \tag{1.4}$$

from (1.2) of loc.cit, that sends $[f]$ to $[f]^*$ with $[f]^*$ represented by the element $g \mapsto (1/|G|)$ trace $(f \circ g) + \mathbb{Z}$ of $\mathrm{Hom}_G(L_1, \overline{\boldsymbol{\omega}})^*$. It follows that

$$\varepsilon_0 = [f]^* \text{ for some } G\text{-homomorphism } f : \overline{\boldsymbol{\omega}} \to L_1\,. \tag{1.5}$$

Extension classes in Tate cohomology are as in §11 of [GW1] (cf. Remark after 11.1): a $\mathbb{Z}$-split 1-extension $(M) : 0 \to X \to M \to Y \to 0$ of G-modules remains exact on applying $\mathrm{Hom}(Y, \_\,)$, and the connecting homomorphism

$$\partial_{(M)} : \widehat{H}^{\,0}\big(G, \mathrm{Hom}(Y, Y)\big) \to H^1\big(G, \mathrm{Hom}(Y, X)\big) \tag{1.6}$$

on its G-cohomology allows the definition $\xi_{(M)} := \partial_{(M)}(\mathrm{id}_Y) \in H^1\big(G, \mathrm{Hom}(Y, X)\big)$ of the extension class of (M). Note that $(M) \mapsto \xi_{(M)}$ induces a bijection between the set of equivalence classes of $\mathbb{Z}$-split 1-extensions (M) and $H^1\big(G, \mathrm{Hom}(Y, X)\big)$.

The notational deviation L_1, ε_1 from the L, ε of [GW2] in (1.1) is intended to separate the role of ε_1 which is at the centre of the envelope focus of loc.cit. (so *every* ε after the first two pages there is now ε_1), from that of the more fundamental ε. The basic idea, only partially realized by Theorem B, is to use the homotopy class $[f]$ to 'reconstruct' E : the formation in Proposition 5.1 of the 'homotopy' kernel M' of f_0 doesn't provide a description of M'. This defect is here addressed by using extension classes.

We will use, near (3.4), the notation $[L_1, N] = \widehat{H}^{\,0}\big(G, \mathrm{Hom}(L_1, N)\big)$ from (5.1) of [GW1] to evoke the homotopy language. Given an envelope $(C) : 0 \to M \to C \to L_1 \to 0$, with $\mathbb{Z}$-torsion $j : \boldsymbol{\mu} \hookrightarrow M$, applying $\mathrm{Hom}(L_1, \_\,)$ and G-cohomology gives an isomorphism

$$\partial_{(C)} : [L_1, L_1] \to H^1\big(G, \mathrm{Hom}(L_1, M)\big), \tag{1.7}$$

of right $[L_1, L_1]$-modules. Then $\tau_1 \partial_{(C)}^{-1} j_*$ is in $H^1\big(G, \mathrm{Hom}(L_1, \boldsymbol{\mu})\big)^*$ and we say, following (1.6) of [GW2], that (C) is *linked* to its $\mathrm{Aut}_G(\boldsymbol{\mu})$-orbit. This orbit is here insensitive to the choice of j, because $\mathrm{Aut}_G(\boldsymbol{\mu}) = \mathrm{Aut}(\boldsymbol{\mu})$ since $\boldsymbol{\mu}$ cyclic implies that $\mathrm{Aut}(\boldsymbol{\mu})$ is abelian.

2 Reformulation

First, applying _ $\otimes L_1$ to the augmentation sequence, as in (1.1), gives a $\mathbb{Z}$-split G-module sequence

$$0 \to L_2 \to \mathbb{Z}G \otimes L_1 \overset{p_1}{\to} L_1 \to 0, \tag{2.1}$$

with $\mathbb{Z}G \otimes L_1$ $\mathbb{Z}G$-free and $L_2 := \Delta G \otimes L_1$. Thus applying $\mathrm{Hom}(\boldsymbol{\omega}, \_)$, as in §1, and then G-cohomology gives the connecting isomorphism

$$\delta_0 : \widehat{H}^{\,0}\big(G, \mathrm{Hom}(\boldsymbol{\omega}, L_1) \to H^1(G, \mathrm{Hom}(\boldsymbol{\omega}, L_2)\big). \tag{2.2}$$

Our reformulation starts from the trivial observation that the G-map $\boldsymbol{\omega} \to \overline{\boldsymbol{\omega}}$ of (1.1) induces an equality of the functors $\mathrm{Hom}(\overline{\boldsymbol{\omega}}, \_) \to \mathrm{Hom}(\boldsymbol{\omega}, \_)$ on $\mathbb{Z}G$-lattices X. Then

$$\widehat{H}^{\,0}\big(G, \mathrm{Hom}(\overline{\boldsymbol{\omega}}, L_1)\big) = \widehat{H}^{\,0}\big(G, \mathrm{Hom}(\boldsymbol{\omega}, L_1)\big) \tag{2.3}$$

allows us to rewrite (1.4) as an isomorphism

$$\widehat{H}^{\,0}\big(G, \mathrm{Hom}(\boldsymbol{\omega}, L_1)\big) \to \widehat{H}^{\,0}\big(G, \mathrm{Hom}(L_1, \overline{\boldsymbol{\omega}})\big)^* \tag{2.4}$$

that sends $[h]$ to $[h]^*$ with $[h]^*$ represented by the element $g \mapsto (1/|G|)\,\mathrm{trace}\,(\overline{h} \circ g) + \mathbb{Z}$, of $\mathrm{Hom}_G(L_1, \overline{\boldsymbol{\omega}})^*$. It follows that

$$\varepsilon_0 = [h]^* \text{ for some } h \in \mathrm{Hom}(\boldsymbol{\omega}, L_1)^G. \tag{2.5}$$

We now define the isomorphism before the Theorem of the introduction to be the composition of the isomorphisms

$$\begin{aligned} H^2\big(G, \mathrm{Hom}(\Delta S, \boldsymbol{\mu})\big)^* \to H^1\big(G, \mathrm{Hom}(L_1, \boldsymbol{\mu})\big)^* \to \widehat{H}^{\,0}\big(G, \mathrm{Hom}(L_1, \overline{\boldsymbol{\omega}})\big)^* \\ \leftarrow \widehat{H}^{\,0}\big(G, \mathrm{Hom}(\boldsymbol{\omega}, L_1)\big) \to H^1\big(G, \mathrm{Hom}(\boldsymbol{\omega}, L_2)\big) \end{aligned} \tag{2.6}$$

of $(1.2)^*$, $(1.3)^*$, (2.4), (2.2), and observe that it takes $-\varepsilon$ to $-\delta_0([h])$.

It follows that $\varepsilon^{(1)} = -\delta_0([h])$ in the statement of the Theorem of the introduction, which is therefore equivalent to the following reformulation.

Theorem 2.1. *Let* $[h] \in \widehat{H}^{\,0}\big(G, \mathrm{Hom}(\boldsymbol{\omega}, L_1)\big)$ *be the image of* ε *under the composite of the first three maps in* (2.6), *and let* δ_0 *be the last map of that composite, as in* (2.2). *Let* M *be the* G*-module in a* $\mathbb{Z}$*-split* 1*-extension*

$$0 \to L_2 \to M \to \boldsymbol{\omega} \to 0$$

with extension class equal to $-\delta_0([h])$ *in* $H^1\big(G, \mathrm{Hom}(\boldsymbol{\omega}, L_2)\big)$. *Then* $E \oplus (\mathbb{Z}G)^n$ *is stably isomorphic to* M, *with* $n := (|G| - 2)(|S| - 1) + w$ *when* $G \neq 1$.

In particular, the class ε *and the extension class of* $M(\varepsilon)$ *determine each other uniquely.*

The envelope focus of [GW2] overemphasizes ε_1 for our purposes. We eventually need to restate Theorem A in terms of ε : see Remark 4.3. The connection between ε and ε_1 is a consequence of the relationship between Tate sequences and Tate envelopes, or, more precisely, between the Tate canonical class $\alpha_3 \in H^2\big(G, \mathrm{Hom}(\Delta S, E)\big)$ and Tate envelopes. Thus, following the last four paragraphs of Tate's proof of Theorem 5.1 of Chapter 2 in [T2], we select a *special* Tate sequence representing α_3 and *define* the Tate envelope to be the left half of this special Tate sequence.

Proposition 2.2. *A Tate envelope* $0 \to E \to A \to L_1 \to 0$ *has*

$$\Omega_m = A - (|S| - 1)[\mathbb{Z}G] \quad \text{in} \quad \mathrm{Cl}(\mathbb{Z}G).$$

Proof. We specialize Tate's initial exact sequence by selecting the one

$$0 \to L_2 \to B' \to B \to \Delta S \to 0, \tag{2.7}$$

obtained by splicing (1.1) and (2.1); Tate's first paragraph ends with isomorphisms

$$\widehat{H}^{\,r}\big(G, \mathrm{Hom}(L_2, E)\big) \simeq \widehat{H}^{\,r+2}\big(G, \mathrm{Hom}(\Delta S, E)\big),$$

for all $r \in \mathbb{Z}$, in our notation. The second paragraph chooses $\alpha \in \mathrm{Hom}_G(L_2\,, E)$ corresponding to $\alpha_3 \in H^2\big(G, \mathrm{Hom}(\Delta S, E)\big)$ and deduces, from his (5.2), that α induces isomorphisms $\widehat{H}^{\,r}(G, L_2) \to \widehat{H}^{\,r}(G, E)$, for all r; the third paragraph extends α to a surjective $\alpha : L_2 \oplus F \to E$, with F free, and replaces $L_2 \to B'$ in (2.7) by $L_2 \oplus F \to B' \oplus F$ to get a new (2.7) and the exact sequence $0 \to \ker(\alpha) \to L_2 \oplus F \to E \to 0$. The fourth paragraph deduces that $\ker(\alpha)$, and thus $A := (B' \oplus F)/\ker(\alpha)$, is cohomologically trivial. Combining with the new (2.7) gives the Tate sequence $0 \to E \to A \to B \to \Delta S \to 0$, the left half $0 \to E \to A \to L_1 \to 0$ of which is our Tate envelope.

Now $B = \mathbb{Z}G \otimes \Delta S \simeq (\mathbb{Z}G)^{|S|-1}$ implies that $\Omega_m = [A] - [B] = [A] - (|S| - 1)[\mathbb{Z}G]$. $\square$

3 Proof of the reformulated Theorem

The proof is now straightforward. We assume that $G \neq 1$ (since the $G = 1$ case, while true with the obvious interpretation, is trivial), and start by fixing an envelope

$$0 \to \boldsymbol{\mu} \to \boldsymbol{\omega} \to \overline{\boldsymbol{\omega}} \to 0,$$

satisfying (0.1) and (0.2). The existence of such an $\boldsymbol{\omega}$ follows from (2.1) in [GW1] and (39.12), (32.13) in [CR]: start with any envelope $0 \to \boldsymbol{\mu} \to C \to \overline{C} \to 0$, define c by $|G|c = \dim \mathbb{Q}\otimes C$, and observe that $\Omega_m - ([C] - c[\mathbb{Z}G]) = [P] - [\mathbb{Z}G]$ in $\mathrm{Cl}(\mathbb{Z}G)$, for some projective $\mathbb{Z}G$-module P with $\dim \mathbb{Q} \otimes P = |G|$, hence $C' := C \oplus P$ gives an envelope $0 \to \boldsymbol{\mu} \to C' \to \overline{(C')} \to 0$ with $\Omega_m = [C'] - c'[\mathbb{Z}G]$, as required.

Letting $[h]$, with $h \in \mathrm{Hom}(\boldsymbol{\omega}, L_1)^G$, be as in the assertion of Theorem 2.1, define $\eta : (\mathbb{Z}G \otimes L_1) \oplus \boldsymbol{\omega} \to L_1$ by $\eta\big((x,y)\big) = p_1(x) + h(y)$, and form the big diagram

$$
\text{(3.1)} \qquad
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & L_2 & \hookrightarrow & \ker(\eta) & \xrightarrow{p_0} & \boldsymbol{\omega} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \to & \mathbb{Z}G \otimes L_1 & \hookrightarrow & (\mathbb{Z}G \otimes L_1) \oplus \boldsymbol{\omega} & \to & \boldsymbol{\omega} & \to & 0 \\
 & & \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle \eta} & & & & \\
 & & L_1 & = & L_1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
$$

as follows: start from the commutative square containing p_1 and η, use it to form the bottom two rows with the additional map sending (x,y) to y, and then get the top row by taking kernels, and using (2.1) as the first column. We put $M := \ker(\eta)$ and focus first on the column and then on the row containing M.

Now let $0 \to E \to A \to L_1 \to 0$ be a fixed Tate envelope, and form the envelope

$$
\text{(3.2)} \qquad 0 \to (\mathbb{Z}G)^n \oplus E \to (\mathbb{Z}G)^n \oplus A \to L_1 \to 0,
$$

from it by adding $(\mathbb{Z}G)^n = (\mathbb{Z}G)^n$. This is an envelope with $\mathbb{Z}$-torsion $\boldsymbol{\mu}$ and lattice L_1, as is the middle column

$$
\text{(3.3)} \qquad 0 \to M \to (\mathbb{Z}G \otimes L_1) \oplus \boldsymbol{\omega} \to L_1 \to 0,
$$

of (3.1). We now apply Theorem 4.7 of [GW2] to show that the left ends of these envelopes are stably isomorphic. This requires two conditions to be verified.

The quicker condition to check is that $[(\mathbb{Z}G \otimes L_1) \oplus \boldsymbol{\omega}]$ is equal to $[(\mathbb{Z}G)^n \oplus A]$ in $\mathrm{Cl}(\mathbb{Z}G)$. Now $\mathbb{Z}G \otimes L_1 \simeq (\mathbb{Z}G)^{(|G|-1)(|S|-1)}$, because it's $\mathbb{Z}G$-free; and (0.2) applies to $[\boldsymbol{\omega}]$, hence $[(\mathbb{Z}G \otimes L_1) \oplus \boldsymbol{\omega}] = (|G|-1)(|S|-1)[\mathbb{Z}G] + w[\mathbb{Z}G] + \Omega_m$. Similarly, the second expression equals $n[\mathbb{Z}G] + (|S|-1)[\mathbb{Z}G] + \Omega_m$, by Proposition 2.2. These agree by the choice of n.

The other condition is that both of these envelopes are linked to the *same* $\mathrm{Aut}_G(\boldsymbol{\mu})$-orbit on $H^1\big(G, \mathrm{Hom}\,(L_1, \boldsymbol{\mu})\big)^*$, which we will show is $\varepsilon_1 \mathrm{Aut}_G(\boldsymbol{\mu})$.

First, by definition, the Tate envelope is linked to $\tau_1 \partial_{(A)}^{-1} j_*$; and with $j : \boldsymbol{\mu} \hookrightarrow E$ the inclusion, which is $t_E j_*$ by definition of the trace character t_E in §7, i.e the 'restriction' ε_1

of t_E to $H^1\big(G, \mathrm{Hom}\,(L_1, \boldsymbol{\mu})\big)$. To get the same conclusion for the envelope (3.2), consider the commutative diagram defined by inclusion of the Tate envelope into (3.2), and apply $\mathrm{Hom}\,(L_1, \_\,)$ and G-cohomology to get the commutative square, with all maps isomorphisms, inside the commutative diagram

$$\begin{array}{ccccccc}
 & & H^1\big(G, \mathrm{Hom}\,(L_1\,, E)\big) & \overset{\partial_{(A)}}{\longleftarrow} & \widehat{H}^{\,0}\big(G, \mathrm{Hom}\,(L_1, L_1)\big) & & \\
 & {\scriptstyle j_*}\nearrow & & & & \searrow{\scriptstyle \tau_1} & \\
H^1\big(G, \mathrm{Hom}\,(L_1, \boldsymbol{\mu})\big) & & \big\downarrow \simeq & & \Vert & & \mathbb{Q}/\mathbb{Z} \\
 & {\scriptstyle j_*}\searrow & & & & \nearrow{\scriptstyle \tau_1} & \\
 & & H^1\big(G, \mathrm{Hom}\,(L_1, (\mathbb{Z}G)^n \oplus E)\big) & \underset{\partial_{((\mathbb{Z}G)^n \oplus A)}}{\longleftarrow} & \widehat{H}^{\,0}\big(G, \mathrm{Hom}\,(L_1, L_1)\big) & &
\end{array}$$

with left triangle from composing the inclusions $\boldsymbol{\mu} \hookrightarrow E$ and $E \hookrightarrow (\mathbb{Z}G)^n \oplus E$. The top composite from $H^1\big(G, \mathrm{Hom}\,(L_1, \boldsymbol{\mu})\big)$ to $\mathbb{Q}/\mathbb{Z}$ is equal to ε_1, by the first sentence of this paragraph, hence so is the bottom one.

Next, to see that the envelope (3.3) is linked to ε_1, consider the commutative diagram

$$\begin{array}{ccccccccc}
0 \to & \boldsymbol{\mu} & \to & \boldsymbol{\omega} & \to & \overline{\boldsymbol{\omega}} & \to 0 \\
 & \big\downarrow {\scriptstyle j'} & & \big\downarrow {\scriptstyle k} & & \big\downarrow {\scriptstyle \overline{h}} & \\
0 \to & M & \hookrightarrow & C & \overset{\eta}{\to} & L_1 & \to 0
\end{array}$$

with top row the envelope $(\boldsymbol{\omega})$ of (0.1), (0.2), bottom row the vertical envelope (C) of (3.1) with $C = (\mathbb{Z}G \otimes L_1) \oplus \boldsymbol{\omega}$, and $k(y) = (0, y)$ for all $y \in \boldsymbol{\omega}$. Here, forming the right square first defines j'. Applying $\mathrm{Hom}\,(L_1\,, \_\,)$ and G-cohomology gives the commutative square

$$\text{(3.4)} \qquad \begin{array}{ccc}
[L_1, \overline{\boldsymbol{\omega}}] & \overset{\partial_{(\boldsymbol{\omega})}}{\longrightarrow} & H^1\big(G, \mathrm{Hom}\,(L_1, \boldsymbol{\mu})\big) \\
\big\downarrow {\scriptstyle [\mathrm{id}_{L_1}, \overline{h}]} & & \big\downarrow {\scriptstyle (j')_*} \\
[L_1, L_1] & \underset{\partial_{(C)}}{\longrightarrow} & H^1\big(G, \mathrm{Hom}\,(L_1, M)\big)
\end{array}$$

with horizontal isomorphisms and (C) linked to $\tau_1 \partial_{(C)}^{-1} (j')_* \in H^1\big(\mathrm{Hom}\,(L_1, \boldsymbol{\mu})\big)^*$, by the definition (1.7), with $\tau_1 : [L_1, L_1] \to \mathbb{Q}/\mathbb{Z}$. Our hypothesis on $[h]$ implies the $[\overline{h}]^* = \varepsilon_1 \partial_0'$ by (2.5), (1.5) and (1.3), with $\partial_0' = \partial_{(\boldsymbol{\omega})}$, i.e. $[\overline{h}]^* = \varepsilon_1 \partial_{(\boldsymbol{\omega})}$.

Now, quoting [GW2], $\varepsilon_1 \in H^1\big(G, \mathrm{Hom}\,(L_1, \boldsymbol{\mu})\big)^*$ implies that $\varepsilon_1 = \tau_1 \theta$ for some right $[L_1, L_1]$-homomorphism $\theta : H^1\big(\mathrm{Hom}\,(L_1, \boldsymbol{\mu})\big) \to [L_1, L_1]$, by (1.3). Then $\theta \partial_{(\boldsymbol{\omega})}$ is a right $(L_1, L_1]$-homomorphism: $[L_1, \overline{\boldsymbol{\omega}}] \to [L_1, L_1]$ so that $[\overline{h}] \in [\overline{\boldsymbol{\omega}}, L_1]$ having $[\overline{h}]^* = \varepsilon_1 \partial_{(\boldsymbol{\omega})} = \tau_1 \theta \partial_{(\boldsymbol{\omega})}$, by the previous paragraph, implies that $\theta \partial_{(\boldsymbol{\omega})} = [\mathrm{id}_{L_1}\,, \overline{h}]$, by (1.4).

Combining with (3.4) above gives $\tau_1\partial_{(C)}^{-1}(j')_* = \tau_1[\mathrm{id}_{L_1}, \overline{h}]\partial_{(\boldsymbol{\omega})}^{-1} = \tau_1\theta = \varepsilon_1$, as required.

Finally, we must show that the top row

$$(M) : 0 \to L_2 \hookrightarrow M \to \boldsymbol{\omega} \to 0$$

of the big diagram (3.1) has extension class $-\delta_0([h])$, in the notation of (1.6).

To get a 1-cocycle representing $-\delta_0([h])$, one applies $\mathrm{Hom}\,(\boldsymbol{\omega}, \_)$ to (2.1), getting the exact sequence $0 \to \mathrm{Hom}\,(\boldsymbol{\omega}, L_2) \to \mathrm{Hom}\,(\boldsymbol{\omega}, \mathbb{Z}G \otimes L_1) \to \mathrm{Hom}\,(\boldsymbol{\omega}, L_1) \to 0$, chooses a pre-image of h in $\mathrm{Hom}\,(\boldsymbol{\omega}, \mathbb{Z}G \otimes L_1)$, say the map $1 \otimes h$ taking every $y \in \boldsymbol{\omega}$ to $1 \otimes h(y)$, and then forms the 1-cocycle $g \mapsto (1 \otimes h) - g(1 \otimes h)$ (with $g \in G$) taking values in $\mathrm{Hom}\,(\boldsymbol{\omega}, L_2)$, namely $[(1\otimes h) - g\cdot(1\otimes h)](y) = (1\otimes h)(y) - g\cdot(1\otimes h)(g^{-1}y) = 1\otimes h(y) - g\cdot\big(1\otimes h(g^{-1}y)\big) = 1 \otimes h(y) - g \otimes g\cdot h(g^{-1}y) = 1 \otimes h(y) - g \otimes h(y) = (1-g) \otimes h(y) \in \Delta G \otimes L_1 = L_2$.

On the other hand, the extension class $\xi_{(M)}$ of (M) is, by definition, obtained from (M) by applying $\mathrm{Hom}\,(\boldsymbol{\omega}, \_)$ to (M), getting $0 \to \mathrm{Hom}\,(\boldsymbol{\omega}, L_2) \to \mathrm{Hom}\,(\boldsymbol{\omega}, M) \to \mathrm{Hom}\,(\boldsymbol{\omega}, \boldsymbol{\omega}) \to 0$, lifting $\mathrm{id}_{\boldsymbol{\omega}}$ to some $s \in \mathrm{Hom}\,(\boldsymbol{\omega}, M)$, and forming the class of the 1-cocycle $g \mapsto gs - s$ with values in $\mathrm{Hom}\,(\boldsymbol{\omega}, L_2)$. Setting $s(y) = \big(-1\otimes h(y), y\big)$ works, since $\eta\big(s(y)\big) = p_1\big(-1\otimes h(y)\big) + h(y) = 0$ and $p_0\big(s(y)\big) = y$. Now $(gs-s)(y) = g\big(-1 \otimes h(g^{-1}y), g^{-1}y\big) - \big(-1 \otimes h(y), y\big) = \big(-g\otimes h(y), y\big) + \big(1\otimes h(y), -y\big) = \big((1-g)\otimes h(y), 0\big)$, which is the image of $(1-g)\otimes h(y) \in L_2$. This agrees with the 1-cocycle of the previous paragraph. ☐

4 Discussion

We begin with a consequence of the Theorem, for which we prepare with a naturality property of the Gruenberg resolution. We start with a subset, of d elements g_i of $G\backslash\{1\}$, which *generates* G, form the free group F on x_i, $1 \le i \le d$, and define the relation module R_d by the exact sequence

$$0 \to R_d \to \mathbb{Z}G \otimes_{\mathbb{Z}F} \Delta F \to \Delta G \to 0 \tag{4.1}$$

(cf. [HS] p. 199 and 218). Here the middle term is $\simeq (\mathbb{Z}G)^d$ since ΔF is $\mathbb{Z}F$-free on the $(x_i - 1)$'s, and the right map sends the $\mathbb{Z}G$-basis $1 \otimes_F (x_i - 1)$ to $g_i - 1$.

In the special case $d = |G| - 1$, write $\mathcal{R}, \mathcal{F}$ for R_d, F respectively. For general d, the inclusion $F \to \mathcal{F}$ induces a map from the relation sequence for R_d to $\mathcal{R}$, which on middle terms is an inclusion of the respective $\mathbb{Z}G$-bases so has cokernel $\simeq (\mathbb{Z}G)^{|G|-1-d}$, yielding the exact sequence $0 \to R_d \to \mathcal{R} \to (\mathbb{Z}G)^{|G|-1-d} \to 0$ on the left terms.

Similarly, the relation module sequence for $\mathcal{R}$ maps to the exact sequence obtained by applying $\_ \otimes \Delta G$ to the augmentation sequence, with middle map matching $\mathbb{Z}G$-bases by $1 \otimes_{\mathcal{F}} (x_i - 1) \mapsto 1 \otimes (g_i - 1)$, inducing an isomorphism $\mathcal{R} \to \Delta G \otimes \Delta G$. This implies that

$$0 \to R_d \xrightarrow{\beta} \Delta G \otimes \Delta G \to (\mathbb{Z}G)^m \to 0 \tag{4.2}$$

is exact with an explicit map β and $m = |G| - 1 - d$, when $G \neq 1$.

Let $d(G)$ be the minimal number of generators of G, and set $R := R_{d(G)}$, to state the

Corollary 4.1. *There is an explicit G-homomorphism $\beta' : R \otimes \Delta S \to L_2$ so that the induced isomorphism $\beta'_* : H^1\big(G, \mathrm{Hom}(\boldsymbol{\omega}, R \otimes \Delta S)\big) \to H^1\big(G, \mathrm{Hom}(\boldsymbol{\omega}, L_2)\big)$ has the following property: let M' be the G-module in a $\mathbb{Z}$-split 1-extension*

$$0 \to R \otimes \Delta S \to M' \to \boldsymbol{\omega} \to 0$$

with extension class mapping to $\varepsilon^{(1)}$ under β'_*. Then $E \oplus (\mathbb{Z}G)^{n'}$ is stably isomorphic to M' with $n' = \big(d(G) - 1\big)(|S| - 1) + w$ when $G \neq 1$.

Proof. By $L_2 = \Delta G \otimes L_1 = \Delta G \otimes (\Delta G \otimes \Delta S) \simeq (\Delta G \otimes \Delta G) \otimes \Delta S$, applying $\_ \otimes \Delta S$ to (4.2) gives the exact sequence

$$0 \to R \otimes \Delta S \xrightarrow{\beta'} L_2 \to (\mathbb{Z}G)^{n-n'} \to 0, \tag{4.3}$$

defining β'. This follows from $(\mathbb{Z}G)^m \otimes \Delta S \simeq (\mathbb{Z}G)^{m(|S|-1)}$ with $m(|S| - 1) = n - n'$.

Now the extension class of the 1-extension (M') has the property that its pushout along β' has extension class $\varepsilon^{(1)}$ so there is a commutative diagram

$$\begin{array}{ccccccc} 0 \to & R \otimes \Delta S & \to & M' & \to & \boldsymbol{\omega} \to 0 \\ & \downarrow{\scriptstyle \beta'} & & \downarrow & & \| \\ 0 \to & L_2 & \to & M & \to & \boldsymbol{\omega} \to 0. \end{array}$$

Since β' has cokernel $(\mathbb{Z}G)^{n-n'}$ so does the middle arrow, hence there's an exact sequence $0 \to M' \to M \to (\mathbb{Z}G)^{n-n'} \to 0$. Thus, by the Theorem, $E \oplus (\mathbb{Z}G)^n \approx M \approx M' \oplus (\mathbb{Z}G)^{n-n'}$, which implies that $E \oplus (\mathbb{Z}G)^{n'} \approx M'$. □

Remark 4.2. R has no non-zero projective summand if G is solvable or, more generally, when G has generation gap $= 0$ (cf. (24) in [G]), in which case we cannot expect bigger $\mathbb{Z}G$-free summands from the above approach. Note that R_d is determined up to stable isomorphism by d, as follows from (4.1) by Schanuel's lemma. Corollary 4.1 is a first step toward the important goal of excising as many $\mathbb{Z}G$-free summands of M as explicitly as possible. There are many aspects of this problem but still no systematic approach.

There has been considerable work on Chinburg's conjecture as a special case of the Equivariant Tamagawa Number Conjecture; a recent reference is [B] (cf. Corollary 2.8 and Remark 2.9). Since Chinburg's conjecture predicts that $\Omega_m = 0$ whenever G has no irreducible symplectic representation (cf. §3 of [C2]), an envelope $\boldsymbol{\omega}$ of $\boldsymbol{\mu}$ with $[\boldsymbol{\omega}] - w[\mathbb{Z}G] = 0$ and $w = d(G)$ as in Chapter 3, starting at page 31, is a useful ingredient for examples.

On the other hand, the condition (0.2) on $\boldsymbol{\omega}$ could be replaced in the Theorem by

$$[\boldsymbol{\omega}] - w[\mathbb{Z}G] \equiv \Omega_m \mod B[\varepsilon_1],$$

as the appeal to Theorem 4.7 of [GW2] in its proof shows. This shows that the full strength of Chinburg's conjecture may not be needed.

Remark 4.3. The emphasis on ε_1 in [GW2] comes from the envelope focus. In particular, Theorem A for ε_1 is proved by this method, but its statement depends on the local and global invariant maps on H^2, where ε becomes more central. Theorem A can be translated from ε_1 to ε by using the formalism of [T1], in the direction of the last paragraph of the Remark on p. 971 of [GW2].

More precisely, let $\mathrm{Hom}\big((\mathbb{Z}S),(J)\big)$ be the G-module consisting of all triples (f_1, f_2, f_3) of $\mathbb{Z}$-homomorphisms so that the diagram

$$\begin{array}{ccccccccc} 0 & \to & \Delta S & \to & \mathbb{Z}S & \to & \mathbb{Z} & \to & 0 \\ & & {\scriptstyle f_1}\downarrow & & {\scriptstyle f_2}\downarrow & & {\scriptstyle f_3}\downarrow & & \\ 0 & \to & E & \to & J & \to & C_K & \to & 0 \end{array}$$

commutes. This leads to an exact sequence

$$0 \to H^2\big(G, \mathrm{Hom}((\mathbb{Z}S),(J))\big) \to H^2\big(G, \mathrm{Hom}(\Delta S, E)\big) \oplus H^2\big(G, \mathrm{Hom}(\mathbb{Z}S, J)\big) \to H^2\big(G, \mathrm{Hom}(\Delta S, J)\big) \to 0$$

allowing us to study the trace character $T_E : H^2\big(G, \mathrm{Hom}(\Delta S, E)\big) \to \mathbb{Q}/\mathbb{Z}$ defined by dimension shifting t_E using the exact sequence (1.1). This implies that $\varepsilon = T_E \circ j_*$, with $j : \boldsymbol{\mu} \hookrightarrow E$ the inclusion, but now the point is that T_E can be described in terms of the H^2-sequence above without further dimension shifting.

More precisely, given $x \in H^2\big(G, \mathrm{Hom}(\Delta S, E)\big)$, there exists $y \in H^2\big(G, \mathrm{Hom}(\mathbb{Z}S, J)\big)$ so (x, y) maps to 0 in $H^2\big(G, \mathrm{Hom}(\Delta S, E)\big)$, hence there is a unique $T \in H^2\big(G, \mathrm{Hom}\big((\mathbb{Z}S),(J)\big)\big)$ mapping to (x, y). Taking a 2-cocycle of triples representing T and projecting on the third component gives a 2-cocycle defining $z \in H^2\big(G, \mathrm{Hom}(\mathbb{Z}, C_K)\big)$. Then by Chapter 4, starting at page 41,

$$T_E(x) = \mathrm{inv}(z) - \sum_{\mathfrak{p} \in S_*} \mathrm{inv}_{\mathfrak{p}}(y_{\mathfrak{p}}), \tag{4.4}$$

where S_* is a transversal to the G-orbits on $S, y_{\mathfrak{p}} = k_{\mathfrak{p}}(\mathrm{res}\ y) i_{\mathfrak{p}}$ with $k_{\mathfrak{p}} : J \to K_{\mathfrak{p}}^{\times}$ the projection and $i_{\mathfrak{p}} : \mathbb{Z} \to \mathbb{Z}S$ with $i_{\mathfrak{p}}(1) = \mathfrak{p}$.

This description has the weakness that the existence of y apparently depends on the vanishing of $H^3(G, J)$. This situation is improved by the

Lemma 4.4. *The map* $H^2\big(G, \mathrm{Hom}(\mathbb{Z}S, J)\big) \to H^2\big(G, \mathrm{Hom}(\Delta S, J)\big)$ *has a special splitting.*

Proof. The S-idele group J is a finite product, over $\mathfrak{p} \in S_*$, of components $V_\mathfrak{p} := \prod_\mathfrak{q} K_\mathfrak{q}^\times$, with $\mathfrak{q}$ running through the G-orbit of $\mathfrak{p}$, up to a large cohomologically trivial component of unit ideles. So it suffices to show that $H^2\big(G, \mathrm{Hom}(\mathbb{Z}S, V_\mathfrak{p})\big) \to H^2\big(G, \mathrm{Hom}(\Delta S, V_\mathfrak{p})\big)$ is split for each $\mathfrak{p}$.

If H is a subgroup of G, and B any H-module, define the coinduced G-module coind (B), from H to G, to be $\mathrm{Hom}_{\mathbb{Z}H}(\mathbb{Z}G, B)$ with $g \in G$ acting on elements φ by $(g\varphi)(z) = \varphi(zg)$ for all $z \in \mathbb{Z}G$ (cf VII, §5 of [S]). If D is any $\mathbb{Z}G$-lattice, viewing $\mathrm{Hom}\big(D, \mathrm{coind}\,(B)\big)$ as G-module and $\mathrm{Hom}(\mathrm{res}\, D, B)$ as H-module, both by diagonal action, then there are natural Shapiro isomorphisms

$$H^n\big(G, \mathrm{Hom}(D, \mathrm{coind}\, B)\big) \to H^n\big(H, \mathrm{Hom}(\mathrm{res}\, D, B)\big).$$

Take $H = G_\mathfrak{p}$, $B = K_\mathfrak{p}^\times$ and identify $\mathrm{coind}\, K_\mathfrak{p}^\times$ with $V_\mathfrak{p}$, via $\varphi \mapsto \prod_t \big(t \cdot \varphi(t^{-1})\big)$, with t a choice of representatives of $G/G_\mathfrak{p}$. This choice doesn't matter, since $(th) \cdot \varphi\big((th)^{-1}\big) = t \cdot \big(h \cdot \varphi(h^{-1}t^{-1})\big) = t \cdot \varphi(t^{-1})$ for $h \in G_\mathfrak{p}$. The map is bijective, since the components $tK_\mathfrak{p}^\times$ of $V_\mathfrak{p}$ are disjoint, and is a G-homomorphism because $g\big(\prod_t (t \cdot \varphi(t^{-1})\big) = \prod_t \big((gt) \cdot \varphi(gt)^{-1}g)\big) = \prod_t \big((gt) \cdot (g\varphi)(gt)^{-1})\big) = \prod_t \big(t \cdot (g\varphi)(t^{-1})\big)$.

This identifies our map of the first paragraph with the top row of the commutative square

$$\begin{array}{ccc} H^2\big(G, \mathrm{Hom}(\mathbb{Z}S, \mathrm{coind}\, K_\mathfrak{p}^\times)\big) & \stackrel{a^*}{\to} & H^2\big(G, \mathrm{Hom}(\Delta S, \mathrm{coind}\, K_\mathfrak{p}^\times)\big) \\ {\scriptstyle sh}\downarrow & & {\scriptstyle sh}\downarrow \\ H^2\big(G_\mathfrak{p}, \mathrm{Hom}(\mathrm{res}\, \mathbb{Z}S, K_\mathfrak{p}^\times)\big) & \stackrel{a^*}{\to} & H^2\big(G_\mathfrak{p}, \mathrm{Hom}(\mathrm{res}\, \Delta S, K_\mathfrak{p}^\times)\big), \end{array} \tag{4.5}$$

with vertical isomorphisms, and horizontal maps from $0 \to \Delta S \stackrel{a}{\hookrightarrow} \mathbb{Z}S \stackrel{a'}{\to} \mathbb{Z} \to 0$. This exact sequence is $G_\mathfrak{p}$-split, by the $G_\mathfrak{p}$-map $\lambda_\mathfrak{p} : d \mapsto d - a'(d)\mathfrak{p}$ having $\lambda_\mathfrak{p} \circ a = \mathrm{id}_{\Delta S}$. Thus $\lambda_\mathfrak{p}$ induces $H^2\big(\mathrm{Hom}(\mathrm{res}\, \Delta S, K_\mathfrak{p}^\times)\big) \to H^2\big(G_\mathfrak{p}, \mathrm{Hom}(\mathrm{res}\, \mathbb{Z}S, K_\mathfrak{p}^\times)\big)$ splitting the bottom a^* of the commutative square, which then completes the argument. □

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