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THE UNIVERSITY OF ALBERTA

A NEW APPROACH TO THE ONSET OF INSTABILITY FOR MISCIBLE DISPLACEMENTS

by

Gökhan Coşkuner

A THESIS

SUBMITTED TO THE FACULTY OF GRADUATE STUDIES AND RESEARCH

IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE DEGREE

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(SIGNED) Gökhan Coşkuner

PERMANENT ADDRESS:

Dept. of Mining, Metallurgical
and Petroleum Engineering
University of Alberta
Edmonton, Alberta

DATED December 19 1986

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research, for acceptance, a thesis entitled A NEW APPROACH TO THE ONSET OF INSTABILITY FOR MISCIBLE DISPLACEMENTS submitted by Gökhan Coşkuner in partial fulfilment of the requirements for the degree of Doctor of Philosophy in PETROLEUM ENGINEERING.

Supervisor

External Examiner

Date. December 19, 1986

To the memory of my father and to my mother

ABSTRACT

Many secondary recovery schemes involve the displacement of oil by a miscible fluid. The success of such schemes depends, to a large extent, on whether the displacement is dominated by viscous fingering. That is, if such fingers are present, the displacement efficiency and, as a consequence, the economic return of the recovery scheme is seriously impaired because of macroscopic bypassing of the resident fluid. The presence of viscous fingers depends upon the balance of viscous, gravitational and dispersive forces which exist in the system. If the viscous forces exceed gravitational and dispersive forces, the system is said to be unstable, and viscous fingers develop. Thus, there is a need to develop a dimensionless scaling group (instability number) which can be used to predict the boundary at which a displacement becomes unstable.

This study presents a dimensionless scaling group and its critical value for predicting the onset of instability during a miscible displacement in a porous medium. In order to derive this scaling group, two different approaches were taken. In the first approach, a previously existing theory which is valid for an infinitely long porous medium is extended by employing additional boundary conditions at the walls of the porous medium, and by including in the analysis both the longitudinal and transverse dispersion coefficients.

The second approach is a new one which is based on a variational technique to enable the introduction of appropriate boundary conditions at the inlet and the outlet of the porous medium. As a consequence, it is possible, for the first time, to infer explicitly the effect that the length of a porous medium has on the instability of a miscible displacement. It is demonstrated that, in the limit of an infinitely long porous medium, both the extended theory and the variational theory lead to the same condition for instability. Moreover, in short laboratory systems, the length of the porous medium has an important effect and should be taken into account. However, under field conditions, the analyses which consider the porous medium as being infinitely long are adequate.

The theory developed in this study is used to predict the onset of instability in miscible, Hele-Shaw cell displacements. It is shown that the dimensionless scaling group successfully predicts the onset of instability. Furthermore, experimental observations indicate that although a displacement can appear to be stable initially, it may become unstable as the interface travels through the system. This, as well, agrees with the theory.

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NOMENCLATURE

A_m	Variational coefficients
C	Solvent concentration
$\bar{C}$	Unperturbed solvent concentration
C^*	Solvent concentration perturbation
$\langle\bar{C}\rangle$	Average unperturbed solvent concentration
$\tilde{C}(z)$	Amplitude of solvent concentration perturbation
C_g	Concentration of glycerine
D	Differentiation with respect to z
D'	Molecular diffusion coefficient, m^2/s
D_L	Longitudinal dispersion coefficient, m^2/s
D_T	Transverse dispersion coefficient, m^2/s
g	Gravitational acceleration, m/s^2
h	Distance between the plates, m

k	Absolute permeability, m^2
K	Ratio of transverse to longitudinal dispersion
L	Length of the porous medium, m
L_x	Width of the porous medium, m
L_y	Height of the porous medium, m
M	Gradient of the natural logarithm of unperturbed viscosity, m^{-1}
M_r	Viscosity ratio
P	Pressure, Pa
$\bar{P}$	Unperturbed pressure, Pa
P^*	Peturbation pressure, Pa
$\tilde{P}(z)$	Amplitude of pressure perturbation, Pa
Pe	Peclet number
t	Real time, s

t_o	A short time, s
u	Velocity in the x-direction, m/s
$\bar{u}$	Unperturbed velocity in the x-direction, m/s
u^*	Perturbation velocity in the x-direction, m/s
v	Velocity in the y-direction, m/s
$\bar{v}$	Unperturbed velocity in the y-direction, m/s
v^*	Perturbation velocity in the y-direction, m/s
V	Displacement velocity, m/s
[illegible]	Perturbation velocity vector, m/s
w	Velocity in the z-direction, m/s
$\bar{w}$	Unperturbed velocity in the z-direction, m/s
w^*	Perturbation velocity in the z-direction, m/s
$\tilde{w}(z)$	Amplitude of perturbation velocity in the z-direction, m/s

z	Moving coordinate
z'	Stationary coordinate
Δz	Length of graded viscosity zone, m
α	Eigenvalue in the x-direction, m^{-1}
β	Eigenvalue in the y-direction, m^{-1}
γ	Wavenumber, m^{-1}
δ	Length of transition region, m
δ_{mn}	Dirac delta function
ξ	Any perturbed dependent variable
$\bar{\xi}$	Any unperturbed dependent variable
ξ^*	Perturbation to any of the dependent variables
$\tilde{\xi}(z)$	Amplitude of perturbation to any of the dependent variables
θ	Angle between flow direction and upward vertical, degrees

μ	Viscosity, Pa.s
$\bar{\mu}$	Unperturbed viscosity, Pa.s
μ^*	Viscosity perturbation, Pa.s
μ_o	Oil viscosity, Pa.s
μ_s	Solvent viscosity, Pa.s
ρ	Density, kg/m^3
$\bar{\rho}$	Unperturbed density, kg/m^3
ρ^*	Density perturbation, kg/m^3
ρ_o	Oil density, kg/m^3
ρ_s	Solvent density, kg/m^3
σ	Time constant, s^{-1}
ϕ	Porosity
ϕ^*	Perturbation potential, m^2/s
Ω	Dimensionless length

1. INTRODUCTION

As world oil production fluctuates in response to economic and political imperatives, crude oil shortages may arise again in the future as they have in the past. One way to alleviate such shortages would be to improve recovery efficiencies in known fields. Each one percent increase in recovery efficiency adds about four billion barrels to worldwide reserves. Hence, research effort in the area of enhanced oil recovery could pay large dividends.

Immiscible secondary recovery schemes, even when successful, may be inefficient because capillary forces act to entrap oil within the reservoir. One way to avoid entrapping oil in this way is to use a displacement fluid which is miscible with the oil in the reservoir. However, such an approach, while improving the recovery efficiency in the parts of the reservoir contacted by the displacing fluid, may still be inefficient on an overall basis. This is because of early breakthrough of solvent and poor volumetric sweep efficiency. These two problems arise as a result of the phenomenon of viscous fingering which comes about because of hydrodynamic instability of the displacement front[1,3,4]. Thus, there is a need, when designing a miscible displacement process, to be able to predict whether the displacement is stable or not. In this respect, it would be useful to have at hand a dimensionless scaling group (analogous to the Reynolds number), together with its critical value, which could be used to predict the onset of instability.

There are basically three different approaches to determining the conditions for instability in miscible displacements[5,6]. The first approach involves the application of hydrodynamic stability theory in which small perturbations are superimposed on a stable displacement[7-18]. The conditions for instability are obtained by investigating the conditions under which the perturbations will grow. The small perturbations approach is limited, in that only incipient fingers may be treated. An important reason for using this type of analysis is that it defines the maximum permissible concentration gradient at the displacement front[7,19,20].

The second approach is the so-called "macrostatistical" approach, which is based on a material balance on the displacing fluid coupled with a formula for the fractional flow of solvent[19,21-25]. Invariably, one ends up with non-linear Buckley-Leverett type equations which have to be solved numerically[22,25]; or, alternatively, an empirical approach may be utilized[21,23,24].

Finally, a numerical approach has been used to evaluate the instability of miscible displacements[26-31]. These numerical approaches either try to solve numerically the fundamental flow equations[26-29,31] or they incorporate a macrostatistical approach in a miscible displacement simulator[30]. Because the basic flow equations are non-linear, general statements about the process can not be made. That is, each particular case under investigation has to be treated separately[5,6]. There are also the problems of grid orientation, numerical instability and numerical dispersion associated with the numerical solution techniques used, particularly when the displacement is unstable[32,33]. Finally, the use of conventional mass transport equations, which are valid for equal density, equal viscosity fluids, may be questionable in unstable displacements in which there may be large viscosity and density differences between the oil and the solvent[25,34].

As noted earlier, it would be useful to have a scaling group together with its critical value, which could be used to predict the onset of instability. Such a dimensionless group should include the properties of the fluids and the porous medium as well as the dimensions of the porous medium. The methods which use numerical or empirical approaches to the problem can not be used for this purpose; that is, an analytical solution is needed. Therefore, a hydrodynamic stability theory, i.e. a small perturbations approach, must be utilized to obtain the desired dimensionless scaling group. Consequently, the major purpose of this work is to derive this dimensionless scaling group and its critical value at the onset of instability using small perturbations theory.

2. LITERATURE REVIEW

It has long been recognized that there is a need, when designing a miscible displacement process, to be able to predict whether the displacement is stable or unstable. The need to predict the onset of hydrodynamic instability in miscible displacements has led a number of researchers to undertake a stability analysis. Because of the large number of papers in the literature, only the major works are reviewed in this section.

In 1959, Wooding[7] studied the stability of a viscous fluid in an infinitely long vertical tube filled with porous material which, at steady state, exhibited a small linear variation in density from one boundary to another. By assuming no mechanical dispersion, as well as constant viscosity and diffusivity, he obtained the condition for neutral (marginal) instability using the technique of small perturbations. In a later paper[8], he considered a similar instability problem in a Hele-Shaw cell. In this regard, it has been demonstrated that the criterion obtained by Wooding can be interpreted as the relative amount of time needed to suppress a finger by transverse dispersion as compared to the time available for the finger to grow[4].

In 1961, Perrine[9,10] studied the stability of unsteady-state, two-dimensional unperturbed flow. That is, he allowed the displacement concentration profile to change with time. In the analysis, it was assumed that the density difference was small and that the viscosity was an exponential function of concentration. He took physical dispersion effects into account by introducing a transition region between the two fluids, and he obtained the condition for neutral instability using the small perturbations method. Perrine's approach was criticized by Outmans[23] who suggested that Perrine investigated the stability of a homogeneous fluid rather than that of a miscible displacement. Experiments carried out later by Kyle and Perrine[24] revealed that the length of the stable concentration gradient criterion predicted by Perrine's theory was an order of magnitude larger than that observed in the experiments.

In 1962, Wooding[11] assumed a Hermite polynomial solution to linearized perturbation equations to study the instability of vertical displacement, with an unsteady-state concentration profile, in a porous medium. The condition for stability was obtained by assuming that the fluids were intially separated by a sharp interface, and that the viscosity was constant everywhere except where it appeared in the differentiation with respect to concentration. This differentiation term was also assumed to be constant. Wooding was able to obtain an approximate solution to the problem by using only the first term of the Hermite polynomial approximation at time zero.

In 1965, Schowalter[12] studied a problem similar to that of Perrine, except with a significant density difference between the two fluids. His linearized perturbation equations led to a fourth-order differential equation for perturbed concentration, which he was able to solve numerically to determine the condition for marginal instability.

In 1966, Heller[13] carried out a first-order perturbation analysis of isoconcentration surfaces in a porous medium saturated with a fluid of variable density and viscosity. This analysis enabled him to derive a relationship between the time constant of Fourier components and their wave number. In this relationship, the time constant was a function of the distances from the center of the concentration profile to the inlet and outlet boundaries where the perturbations had been constrained to be zero. However, these distances change with time as the interface travels through the porous medium. That is, the time constant itself is a function of time. Because of this problem, Heller could consider only displacements in which the effect of the distance to the inlet and outlet could either be neglected (infinite or semi-infinite porous medium) or considered constant (concentration profile extends over entire length of porous medium). Therefore, although the boundary conditions at the inlet and outlet were taken into account, the effect of the length of the porous medium on instability could not be deduced explicitly. In undertaking his analysis, Heller assumed that the diffuse front had a linear concentration profile. Moreover, he also supposed that the stability of the entire front could be assessed by examining the amplitude of the velocity perturbation at the centre of the graded front. However, it has since been shown that this point is not necessarily the most unstable point in the transition zone[16].

In 1978, Claridge[17], after comparing Kyle and Perrine's experiments[3,4] with Perrine's theory[9,10] as well as unpublished work by Chuoke, concluded that the criterion for the length of a stable transition zone developed by Chuoke and Perrine was pessimistic (by an order of magnitude) for a system of finite length.

In 1982, Gardner and Ypma[14] presented Chuoke's unpublished work in which small perturbations were introduced at the interface which was initially sharp. In his analysis, Chuoke was able to obtain the critical wavelength for finger growth in a miscible displacement by imposing certain matching conditions at the interface and by using a single dispersion coefficient in the governing equations.

~~In 1984~~, Lee, et al.[15] carried out a linear perturbation analysis of miscible displacement with an unsteady-state solvent concentration profile. They also considered the dispersion coefficients to be functions of displacement velocity. In order to obtain their analytical solution it was necessary to assume that the system was vertical; that the space dependence of the coefficients in the perturbation equations could be ignored; and that the

Fourier modes in all three directions were of exponential form. Their analytical results were similar to those obtained by Perrine[9,10], except for the addition of a longitudinal dispersion term.

In 1984, Peters, et al.[11] treated the transition region between the two fluids in the same way as did Heller[12] by considering the maximum instability to be located at the centre of the transition region. However, they used a complex Laplace-Fourier transform to solve the linearized perturbation equations to obtain a dimensionless number and its value at the stability boundary. The use of a complex Laplace-Fourier transform dictated that they use a linear, unperturbed concentration profile in their analysis. As a consequence, the dimensionless number obtained for prediction of the stability boundary contains an unknown dispersion parameter which has to be evaluated experimentally from the spatial distribution of the fingering pattern.

In 1985, Paterson[13] used a small perturbations approach to study radial displacement in a Hele-Shaw cell by considering viscous dissipation energy. In his analysis, the dispersion, the pressure gradient in the displacing fluid, and the transverse component of the velocity were neglected. As a consequence, it was found that there was no critical length below which fingers did not grow, and that the most unstable finger was a function of plate spacing (permeability) only.

In 1986, Chang and Slattery[18] performed a linear stability analysis for a miscible displacement in a semi-infinite porous medium of finite thickness and unbounded width. This analysis, which utilizes local-volume averaged equations, is similar to those carried out by Perrine[10] and Lee, et al.[13]. It was shown that, when there is a step change in concentration and the viscosity ratio is unfavourable, the displacement can be unstable at the injection boundary.

3. STATEMENT OF THE PROBLEM

From the survey of the literature in the previous section, it becomes clear that previous linear stability studies[7-18], which treated the porous medium as being infinitely long, have identified certain factors as being most influential in the stability of miscible displacements. These are the displacement rate, mobility ratio, density difference, dispersion and the transverse dimensions of the sysytem. However, it is still not possible to say with certainty whether a given miscible displacement is stable or unstable. Thus, there is a need to develop a scaling group which can be used to define the boundary separating stable displacements from those which are unstable. Such a scaling group should include not only the effect of the smaller transverse dimension, which was shown to affect the instability in two dimensions[10,11,15,18], but also the effect of the larger transverse dimension. While it may be acceptable to neglect the larger transverse dimension in field-scale systems, such neglect is probably not warranted in many laboratory studies. The scaling group should also be defined in terms of a variable, unperturbed concentration profile, rather than in terms of a fixed concentration profile[13,16]. This is so that the scaling group can be employed for the design of graded viscosity banks which are used in field projects.

In addition to the influential factors mentioned above, the length of the porous medium has been experimentally identified[2,37-44] as having an important effect on the stability of miscible displacements. In particular, it was concluded that long core systems are needed in order to be able to simulate reservoir conditions properly[2,38,39,43,44]. The effect of length was not introduced into earlier stability analyses because of the need to introduce into these analyses the simplifying assumption that the system was infinitely long. Therefore, the dimensionless scaling group mentioned above should also include the length of the system as one of the parameters to be used in predicting the onset of instability in miscible displacements.

Thus, the objectives of this study were two-fold. The first objective was to derive a scaling group and its critical value at the onset of instability in miscible displacements. The second was to verify the theory experimentally.

In order to derive the scaling group two different approaches were taken. In the first approach, the theory of Chuoke[14], which is valid for an infinitely long porous medium, was extended by employing additional boundary conditions at the walls of the porous medium, and by including in the analysis both the longitudinal and tranverse dispersion coefficients.

The second approach was based on a variational technique which enables one to introduce appropriate boundary conditions at the inlet and the outlet of the porous medium. As a consequence, it was possible, for the first time, to deduce explicitly the effect of length on the instability of a miscible displacement. Moreover, the second approach could be checked against the the first one because, in the limit of an infinitely long porous medium, both approaches should give the same result.

Finally, the stability theory was experimentally verified. In order to test the fluid mechanics aspect of the theory, the experiments were carried out in a simple system, a Hele-Shaw cell. The experiments were conducted at different velocities using fluids which have different viscosity ratios.

4. MODIFIED STABILITY THEORY

In this section the linear perturbation theory of Chuoke[14] is extended in order to obtain a dimensionless scaling group to predict the onset of instability. The modification is accomplished by employing additional boundary conditions in a way similar to that used to determine the analogous scaling group for the displacement of immiscible fluids[45]. The modification also takes into account the fact that the longitudinal dispersion coefficient and the transverse dispersion coefficient are different.

4.1 FLUID DISPLACEMENT

Consider the downward displacement of oil by a solvent in the z' direction in a homogeneous and isotropic porous medium as shown in Figure 1. The fluids are assumed to be miscible in all proportions. The mathematical model describing this situation consists of the continuity equation, three equations of motion[44], and a mass transport equation. For incompressible fluids, these equations are given, respectively, by

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z'} = 0 \tag{1}$$

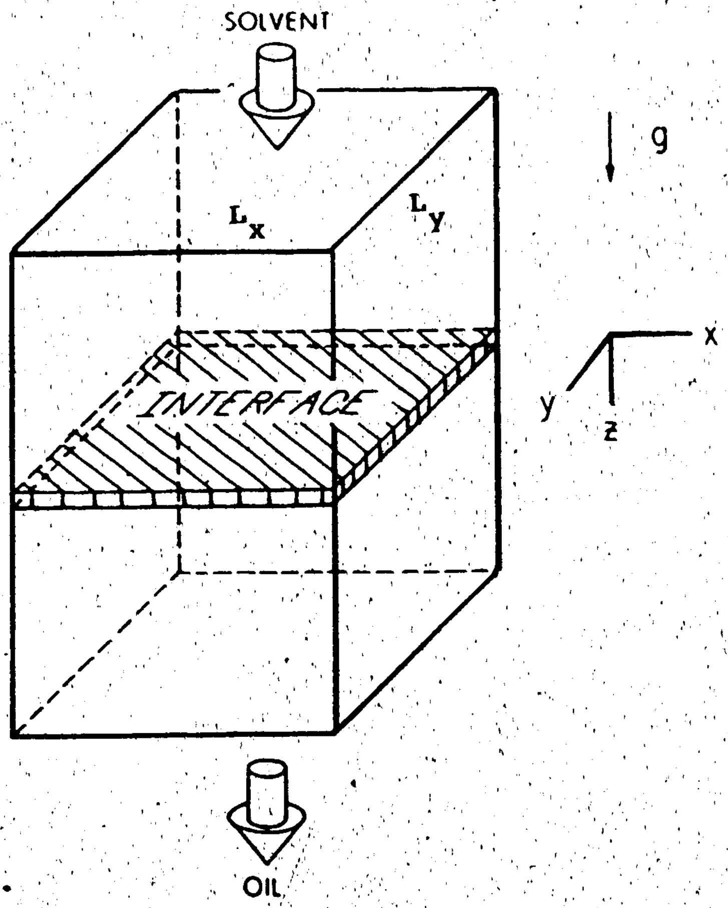

FIG. 1 A POROUS MEDIUM

$$\frac{\rho}{\phi}\frac{\partial u}{\partial t} + \frac{\partial P}{\partial x} + \frac{\mu}{k} u = 0 \tag{2}$$

$$\frac{\rho}{\phi}\frac{\partial v}{\partial t} + \frac{\partial P}{\partial y} + \frac{\mu}{k} v = 0 \tag{3}$$

$$\frac{\rho}{\phi}\frac{\partial w}{\partial t} + \frac{\partial P}{\partial z'} + \frac{\mu}{k} w - \rho g = 0 \tag{4}$$

and

$$\frac{\partial C}{\partial t} + \frac{u}{\phi}\frac{\partial C}{\partial x} + \frac{v}{\phi}\frac{\partial C}{\partial y} + \frac{w}{\phi}\frac{\partial C}{\partial z'} - D_T\left(\frac{\partial^2 C}{\partial x^2} + \frac{\partial^2 C}{\partial y^2}\right) - D_L\frac{\partial^2 C}{\partial z'^2} = 0 \tag{5}$$

It sould be noted that in the mass transport equation

(Equation (5)), a simple form for the diffusivity tensor was assumed[47]. The applied flow rate is directed normal to the interfacial region. Thus, it was assumed that there was a coefficient of longitudinal dispersion, D_L, in the direction of flow, and a coefficient of transverse dispersion, D_T, in any direction normal to the direction of flow. Both of these coefficients are taken at least approximately constant at any given flow rate. To complete the mathematical description of the problem, it is necessary to specify how viscosity and density vary with solvent concentration. Moreover, appropriate boundary and initial conditions must be stated.

The stable solution, for the situation described above, is defined by

$$\bar{u},\ \bar{v} = 0 \tag{6}$$

$$\bar{w} = V = \text{constant} \tag{7}$$

$$\frac{\partial \bar{P}}{\partial x} = \frac{\partial \bar{P}}{\partial y} = 0 \tag{8}$$

$$\frac{\partial \bar{P}}{\partial z'} + \frac{\bar{\mu}}{k} V - \bar{\rho} g = 0 \tag{9}$$

and

$$\frac{\partial \bar{C}}{\partial t} + \frac{V}{\phi} \frac{\partial \bar{C}}{\partial z'} - D_L \frac{\partial^2 \bar{C}}{\partial z'^2} = 0 \tag{10}$$

where the variables superimposed with a bar represent unperturbed quantities.

4.2 SMALL PERTURBATIONS THEORY

In accordance with the small perturbations method, the stability of the flow system is examined by assuming that the dependent variables of the system are slightly perturbed about their mean or unperturbed values‘’,‘’. Hence, the resultant values of the dependent variables are given by the following set of equations :

$$\xi = \bar{\xi} + \xi^{*} \tag{11a}$$

where

$$\xi = C,\ u,\ v,\ w,\ P \tag{11b}$$

and where the bar represents, as mentioned before, the stable solution, and the asterisk represents the arbitrary small perturbation. Note that ξ is defined, in turn, by each of the different variables listed in Equation (11b). Next, the set of Equations (11) is substituted into Equations (1) to (5) and the stable solution, Equations (9) and (10), is subtracted from them. If Equations (6) to (8) are substituted and the non-linear terms are neglected in the resulting equations, one obtains the following equations :

$$\frac{\partial u^*}{\partial x} + \frac{\partial v^*}{\partial y} + \frac{\partial w^*}{\partial z} = 0 \tag{12}$$

$$\frac{\bar{\rho}}{\phi}\frac{\partial u^*}{\partial t} = -\frac{\partial P^*}{\partial x} - \frac{\bar{\mu}}{k}u^* \tag{13}$$

$$\frac{\bar{\rho}}{\phi}\frac{\partial v^*}{\partial t} = -\frac{\partial P^*}{\partial y} - \frac{\bar{\mu}}{k}v^* \tag{14}$$

$$\frac{\bar{\rho}}{\phi}\frac{\partial w^*}{\partial t} + \frac{\partial P^*}{\partial z'} + \frac{\bar{\mu}}{k}w^* + \frac{V}{k}\mu^* - g\rho^* = 0 \tag{15}$$

and

$$\frac{\partial C^*}{\partial t} + \frac{V}{\phi}\frac{\partial C^*}{\partial z'} + \frac{w^*}{\phi}\frac{\partial \bar{C}}{\partial z'} - D_T\left(\frac{\partial^2 C^*}{\partial x^2} + \frac{\partial^2 C^*}{\partial y^2}\right) - D_L\frac{\partial^2 C^*}{\partial z'^2} = 0 \tag{16}$$

Now, if the viscosity and the density are expanded in a Taylor series about the mean solvent concentration, $\bar{C}$, it follows that

$$\mu = \bar{\mu} + \mu^* = \mu\ (\bar{C}) + \frac{d\mu}{dC}\ C^* + \ldots \qquad (17a)$$

and that

$$\rho = \bar{\rho} + \rho^* = \rho\ (\bar{C}) + \frac{d\rho}{dC}\ C^* + \ldots \qquad (17b)$$

If the second and higher order terms in the Taylor series expansion are neglected, Equations (17a) and (17b) lead to

$$\mu^* = \frac{d\mu}{dC}\ C^* \qquad (18a)$$

and

$$\rho^* = \frac{d\rho}{dC}\ C^* \qquad (18b)$$

where the relationships $\bar{\mu}=\mu(\bar{C})$ and $\bar{\rho}=\rho(\bar{C})$ were used. If Equations (18a) and (18b) are inserted into Equation (15), it can be shown that

$$\frac{\rho}{\phi}\frac{\partial w^*}{\partial t} = -\frac{\partial P^*}{\partial z'} - \frac{\mu}{k} w^* - C^* \left(\frac{V}{k}\frac{d\mu}{dC} - \frac{d\rho}{dC} g\right) \tag{19}$$

Moreover, one can define a moving coordinate system given by

$$z = z' - \frac{Vt}{\phi} \tag{20}$$

If Equations (12) to (14), (16) and (19) are now transformed into the new coordinate system, they become

$$\frac{\partial u^*}{\partial x} + \frac{\partial v^*}{\partial y} + \frac{\partial w^*}{\partial z} = 0 \tag{21}$$

$$\frac{\bar{\rho}}{\phi}\frac{\partial u^*}{\partial t} = -\frac{\partial P^*}{\partial x} - \frac{\bar{\mu}}{k}u^* \tag{22}$$

$$\frac{\bar{\rho}}{\phi}\frac{\partial v^*}{\partial t} = -\frac{\partial P^*}{\partial y} - \frac{\bar{\mu}}{k}v^* \tag{23}$$

$$\frac{\bar{\rho}}{\phi}\frac{\partial w^*}{\partial t} = -\frac{\partial P^*}{\partial z} - \frac{\bar{\mu}}{k}w^* - C^*\left(\frac{V}{k}\frac{d\mu}{dC} - \frac{d\rho}{dC}g\right) \tag{24}$$

and

$$\frac{\partial C^*}{\partial t} + \frac{w^*}{\phi}\frac{\partial \bar{C}}{\partial z} - D_T\left(\frac{\partial^2 C^*}{\partial x^2} + \frac{\partial^2 C^*}{\partial y^2}\right) - D_L\frac{\partial^2 C^*}{\partial z^2} = 0 \tag{25}$$

It should be noted that, although the unperturbed velocities are not functions of time, the perturbations imposed upon them are functions of time. Therefore, it was necessary to use the time derivative term in the equations of motion, Equations (2), (3) and (4).

The coefficients of the linearized perturbation equations (Equations (21) to (25)) are, in general, functions of space and time, i.e. $\bar{\mu}=\bar{\mu}(\bar{C})$, $\bar{\rho}=\bar{\rho}(\bar{C})$ and $\bar{C}=\bar{C}(z,t)$; consequently, an analytical solution cannot be found. However, in order to make the problem tractable, it may not be unreasonable to assume that these coefficients are constants[34,44]. Such an approach is justified provided that the differences in viscosity and density of the oil and the solvent are not large[7,11]. Alternatively, a "local analysis" that is intended to apply only in the neighbourhood of each cross-section can be used[7,14,38]. This allows one to treat the coefficients $\bar{\mu}$, $\bar{\rho}$ and $\partial\bar{C}/\partial z$ as being constants in the neighbourhood of any given point. It has been shown that a local analysis can be used safely without introducing any significant errors into hydrodynamic stability problems[14].

For a linear system of perturbation equations, a general disturbance can be represented by a Fourier series as the sum of any number of basic modes. Therefore, a Fourier component of the disturbance is given by

$$\xi^* = \tilde{\xi}(z)\, e^{i(\alpha x + \beta y) + \sigma t} \tag{26}$$

This equation describes a general disturbance which is periodic in the x and y directions with an amplitude given by $\tilde{\xi}(z)$. If the appropriate form of Equation (26) is substituted into Equations (21) to (25), the perturbation variables $\tilde{u}(z)$, $\tilde{v}(z)$, and $\tilde{P}(z)$ may be eliminated from the resulting equations. If this is done, then the following two ordinary differential equations relating the concentration perturbation, $\tilde{C}(z)$, and the velocity perturbation, $\tilde{w}(z)$, are obtained :

$$\left[-\left(\frac{\bar{\rho}\sigma}{\phi\gamma^2} + \frac{\bar{\mu}}{k\gamma^2}\right)D^2 - \left(\frac{\sigma}{\phi\gamma^2}\frac{d\bar{\rho}}{dz} + \frac{1}{k\gamma^2}\frac{d\bar{\mu}}{dz}\right)D + \left(\frac{\bar{\mu}}{k} + \frac{\bar{\rho}\sigma}{\phi}\right)\right]\tilde{w} = -N_C\tilde{C} \tag{27}$$

and

$$\left(D^2 - \frac{D_T\gamma^2}{D_L} - \frac{\sigma}{D_L}\right)\tilde{C} = \frac{1}{\phi D_L}\frac{\partial \bar{C}}{\partial z}\tilde{w} \tag{28}$$

where

$$N_c = \frac{V}{k}\frac{d\mu}{dC} - \frac{d\rho}{dC}\,g \qquad (29)$$

and

$$r^2 = \alpha^2 + \beta^2 \qquad (30)$$

and where D represents differentiation with respect to the z direction.

A displacement process is considered to be stable if small perturbations to the stable solution are damped with time, while an unstable displacement is obtained when the perturbations grow with time. Whether a perturbation decays or grows with time depends upon the time constant, σ, which in general may be complex. The real part of σ is called the "stability" coefficient and the imaginary part is called the "phase shift" coefficient[18]. As the imaginary part of the time function is limited in magnitude to one, stability is determined by the stability coefficient. That is, the displacement is stable whenever the stability coefficient is negative, and unstable whenever the stability coefficient is positive. The condition of interest is that of marginal instability which occurs when the stability coefficient equals zero. If the phase shift coefficient is also zero, a stationary (rather than an oscillatory) pattern of motion prevails at the onset of instability. When this is true, the principle of exchange of stabilities is valid[19] ($\sigma=0$). That is, under these conditions, the condition of marginal instability divides perturbations which are damped aperiodically from those which grow aperiodically. On the other hand, if the phase shift coefficient is not zero, at the onset of instability an oscillatory motion prevails which is termed overstability.

Although the principle of exchange of stabilities has not been proven for the system under consideration, it was still thought to apply because the instabilities observed in the laboratory have been of the non-oscillatory type[12,47]. Therefore, it has been assumed usually that the principle of exchange of stabilities applies to the instability between two miscible fluids[7,12,44]. Consequently, in what follows, it is assumed that the principle of exchange of stabilities is valid: that is, it is assumed that the time constant is real and the marginal state is characterized by $\sigma=0$. If this assumption is made, and if one solves for $\tilde{C}(z)$ from Equation (27) and inserts it into Equation (28), then it can be shown that

$$(D^2 - K\gamma^2)(D^2 + MD - \gamma^2)\tilde{w} = -R^2\gamma^2\,\tilde{w} \qquad (31)$$

where

$$M = \frac{d\ell n\,\bar{\mu}}{dz} \qquad (32)$$

$$R^2 = -\frac{k\ N_c}{\bar{\mu}\phi D_L}\left(\frac{\partial\bar{C}}{\partial z}\right) \qquad (33)$$

and

$$K = \frac{D_T}{D_L} \qquad (34)$$

Equation (31), an ordinary differential equation, has to be solved in order to determine the criterion for marginal instability.

In some displacements the displacing and the displaced fluid viscosities are not very different. Under these conditions, it may not be unreasonable to assume that the change of the natural logarithm of the unperturbed viscosity with distance (M) is negligible, so that M is approximately equal to zero. If this assumption is made, then the general solution of Equation (31) is given by

$$\tilde{w}(z) = A_1 e^{D_1 z} + A_2 e^{D_2 z} + A_3 e^{D_3 z} + A_4 e^{D_4 z} \tag{35}$$

where D_1, D_2, D_3 and D_4 are defined by the following expressions

$$D_1 = \gamma \left\{ \frac{1+K}{2} + \left[\frac{(1+K)^2}{4} - K - \frac{R^2}{\gamma^2} \right]^{\frac{1}{2}} \right\}^{\frac{1}{2}} \tag{36}$$

$$D_3 = \gamma\left\{\frac{1+K}{2} - \left[\frac{(1+K)^2}{4} - K - \frac{R^2}{\gamma^2}\right]^{\frac{1}{2}}\right\}^{\frac{1}{2}} \tag{37}$$

$$D_2 = - D_1 \tag{38a}$$

and

$$D_4 = - D_3 \tag{38b}$$

The unknown constants A_1, A_2, A_3 and A_4 can be determined by application of the appropriate matching conditions at the interfacial region. Thus, if it is assumed that the perturbation velocity should disappear far away from the interface, i.e. that

$$\tilde{w} = 0 \qquad \text{as} \qquad z \to \pm\infty \tag{39}$$

then the perturbation velocity may be shown, in the negative and positive domains, respectively, to be

$$\tilde{w} = A_1 e^{D_1 z} + A_3 e^{D_3 z} \quad \text{for} \quad z<0 \tag{40a}$$

and

$$\tilde{w} = A_2 e^{-D_1 z} + A_4 e^{-D_3 z} \quad \text{for} \quad z>0 \tag{40b}$$

As all the equations describing the miscible displacement of one fluid by another within a porous medium are written at the macroscopic level, it may be assumed that velocity, pressure and concentration are continuous, single-valued functions of position. If such is the case, a number of matching conditions may be imposed on the solution. Thus, if it is demanded that the perturbation velocity be continuous, i.e. that

$$\tilde{w}\Big|_{z=-\varepsilon} = \tilde{w}\Big|_{z=+\varepsilon} \quad \text{as } \varepsilon \to 0 \tag{41}$$

it follows, from Equations (40a) and (40b), that

$$A_1 - A_2 + A_3 - A_4 = 0 \tag{42}$$

If continuity of pressure is also required in the transition region, i.e. if

$$\tilde{P}\Big|_{z=-\epsilon} = \tilde{P}\Big|_{z=+\epsilon} \quad \text{as } \epsilon \to 0 \tag{43}$$

then by combining the appropriate form of Equation (26) with Equations (21), (22) and (23) and by using Equations (40a) and (40b) in the resulting equations, it may be shown that

$$D_1A_1 + D_1A_2 + D_3A_3 + D_3A_4 = 0 \tag{44}$$

Moreover, if continuity of concentration in the interfacial region is required, i.e., if it is required that

$$\tilde{C}\Big|_{z=-\epsilon} = \tilde{C}\Big|_{z=+\epsilon} \quad \text{as } \epsilon \to 0. \tag{45}$$

then from Equation (27) together with (40a) and (40b), it follows that

$$A_1(D_1{}^2-\gamma^2) + A_2(\gamma^2-D_1) + A_3(D_3{}^2-\gamma^2) + A_4(\gamma^2-D_3) = 0 \tag{46}$$

where it was assumed that

$$\left.\frac{d\mu}{dC}\right|_{z=-\varepsilon} = \left.\frac{d\mu}{dC}\right|_{z=+\varepsilon} \quad \text{as } \varepsilon \to 0 \tag{47a}$$

and that

$$\left.\frac{d\rho}{dC}\right|_{z=-\varepsilon} = \left.\frac{d\rho}{dC}\right|_{z=+\varepsilon} \quad \text{as } \varepsilon \to 0 \tag{47b}$$

In miscible liquid-liquid displacement, the concentration is distributed, relative to a plane $z=Vt/\phi$ moving with a mean fluid velocity V/ϕ, by a process resembling molecular diffusion, but with the diffusivity dictated by a dispersion coefficient*. Thus, if D_L is constant and if it is demanded that the dispersive flux as well as the concentration be continuous, i.e., if it is demanded that

$$\frac{\partial \tilde{C}}{\partial z}\bigg|_{z=-\epsilon} = \frac{\partial \tilde{C}}{\partial z}\bigg|_{z=+\epsilon} \quad \text{as } \epsilon \rightarrow 0 \tag{48}$$

then by combining Equations (27), (40a) and (40b), it may be demonstrated that

$$A_1(D_1{}^3-\gamma^2 D_1) + A_2(D_1{}^3-\gamma^2 D_1) + A_3(D_3{}^3-\gamma^2 D_3) + A_4(D_3{}^3-\gamma^2 D_3) = 0 \tag{49}$$

Equations (42), (44), (46) and (49) constitute a set of four linear homogeneous equations in the four unknowns A_1, A_2, A_3 and A_4. These equations have a unique solution, provided that the determinant of the coefficient matrix vanishes. That is, by setting the determinant of the coefficients of this set of equations equal to zero, it may be shown that

$$-\frac{R^2}{K} = \gamma^2 \tag{50}$$

Therefore, at the onset of instability the condition given by Equation (50) should hold true.

4.3 BOUNDARY CONDITIONS

At this point, it is necessary to define the wavenumber, γ, which is given by Equation (30), in terms of the dimensions of the porous medium. This allows the instabilities to be limited to a cut-off wavenumber which is determined by the geometrical boundaries of the system. Because the arbitrary disturbance is separable, and because it has a cellular form in the x-y plane (see Equation (26)), it satisfies the Helmholtz equation[11,34]; that is,

$$\frac{\partial^2 \xi^*}{\partial x^2} + \frac{\partial^2 \xi^*}{\partial y^2} + \gamma^2 \xi = 0 \tag{51}$$

with the boundary conditions that the normal derivative of ξ^* vanishes at the bounding, impermeable walls of the porous medium[11]. In other words, one is consider~~ing a single~~ Fourier component of the disturbance in the x-y plane. Consequently, the following boundary conditions apply:

$$\frac{\partial \xi^*}{\partial x} = 0 \quad \text{at } x = 0 \text{ and } x = L_x \tag{52}$$

$$\frac{\partial \xi^*}{\partial y} = 0 \quad \text{at } y = 0 \text{ and } y = L_y \tag{53}$$

Equation (51) can be solved by separation of variables to determine the actual form of ξ^* in the x-y plane. If the boundary conditions defined by Equations (52) and (53) are applied to this solution, it can be shown that

$$\xi^* = \tilde{\xi}(z) \, \mathrm{Cos}\,\alpha x \; \mathrm{Cos}\,\beta y \; e^{\sigma t} \tag{54}$$

where

$$\alpha = \frac{m \pi}{L_x}; \qquad m = 0, 1, 2 \ldots \tag{55a}$$

and

$$\beta = \frac{n\pi}{L_y} ; \quad n = 0, 1, 2 \ldots \tag{55b}$$

Next, the wavenumber is defined by inserting Equations (55a) and (55b) into Equation (30). If this is done, it can be shown that

$$\gamma^2 = \left(\frac{m\pi}{L_x}\right)^2 + \left(\frac{n\pi}{L_y}\right)^2 \tag{56}$$

It should be noted that Equation (56) can also be obtained by postulating the existence of a velocity potential and combining the equation for velocity which results with the continuity equation. The consequent equation is Laplace's equation which can be solved with the boundary conditions that the normal velocity vanishes at the side walls of the porous medium. However, as the velocity potential exists only under limited conditions, this approach was not utilized.

4.4 CONDITION FOR MARGINAL INSTABILITY

The condition for marginal instability is given by Equation (50); that is, a given displacement will be stable provided that

$$\frac{R^2}{K} < \gamma^2 \qquad (57)$$

Substitution of Equations (29), (38) and (56) into Equation (57) yields, for the stability criterion,

$$\frac{k\left(\frac{V}{k}\frac{d\mu}{dC} - \frac{d\rho}{dC}g\right)}{K\bar{\mu}\phi D_L}\left(\frac{\partial \bar{C}}{\partial z}\right) < \pi^2\left(\frac{m^2}{L_x^2} + \frac{n^2}{L_y^2}\right) \qquad (58)$$

The onset of instability is dictated by the minimum non-zero eigenvalue[illegible]. That is, choosing m=n=1 and bringing right-hand side to left-hand side gives, for the onset of instability,

$$I_{sr} = \pi^2 \qquad (59a)$$

where the instability number, I_{sr}, is defined as follows :

$$I_{sr} = \frac{-k\left(\frac{V}{k}\frac{d\mu}{dC} - \frac{d\rho}{dC}g\right)}{K\bar{\mu}\phi D_L}\left(\frac{\partial\bar{C}}{\partial z}\right)\frac{L_x^2\,L_y^2}{L_x^2 + L_y^2} \qquad (59b)$$

4.5 UNPERTURBED CONCENTRATION PROFILE

Equations (59a) and (59b) are perfectly general as they are written, and can be used to predict the stability of a displacement at any given time and location , provided it is possible to evaluate $\partial\bar{C}/\partial z$ at that time and location. However, it is unlikely that the time and location at which a given displacement becomes unstable will be known with any degree of precision. Thus, if the dimensionless group defined by Equation (59b) is to be useful in predicting the onset of instability, there is a need to select a representative concentration profile at which the stability of the displacement can be evaluated.

In order to select a specific, representative concentration profile, two simplifying assumptions are made. First, it is assumed that the stability of the displacement is dictated by the maximum value of $\partial\bar{C}/\partial z$ ($\bar{C}=0.5$) within the frontal region. In this regard, it should be noted that this point is not necessarily the most unstable point in the transition region''. Second, it is assumed that it is permissible to select a specific, arbitrary point in time at which to evaluate $\partial\bar{C}/\partial z$.

Two approaches, one conservative and one optimistic, can be taken in selecting the time at which to evaluate $\partial\bar{C}/\partial z$. In the first, it is assumed that the stability of the displacement is dictated by the maximum value of $\partial\bar{C}/\partial z$ ($\bar{C}$=0.5) evaluated at some small time, t_o. In this worst case approach[13,14], the maximum concentration can be approximated by

$$\frac{\partial\bar{C}}{\partial z} = -\frac{1}{\delta} = -\frac{1}{2\sqrt{\pi D_L t_o}} \tag{60a}$$

While the use of Equation (60a) leads to a conservative estimate of the point at which a displacement becomes unstable, it does have one major drawback. That is, the value of t_o has to be evaluated experimentally from the spatial distribution of the fingering pattern[14] (see Equation (67)). This limits the usefulness of Equation (59b) for predicting the onset of instability, because experimental work would have to be undertaken for a particular type of system before the instability of the displacement could be predicted with any confidence.

In the second approach, it is assumed that a representative concentration profile can be determined by averaging the unperturbed concentration profile over the life of the displacement. If this optimistic approach is used, partial account is taken of the dampening effect of the dispersion on the already developed fingers. Thus, this approach seems to be more realistic than the first one. Given the validity of this approach, the average value of the maximum, unperturbed concentration gradient ($\bar{C}=0.5$) can be shown to be

$$\frac{\partial \bar{C}}{\partial z} = -\frac{1}{\delta} = -\sqrt{\frac{V}{\pi D_L L \Phi}} \qquad (60b)$$

where L is the length of the flow system. It should be noted that, while Equation (60a) contains an unknown parameter, t_o, all the parameters in Equation (60b) are known, or can be estimated readily, for any given system. Moreover, as noted earlier, the use of Equations (60a) and (60b) is not necessary to the application of Equation (59a). Rather, they have been introduced to simplify, by employing a representative concentration profile, the use of Equation (59a) to predict the onset of viscous fingering.

4.6 COMPARISON WITH EXISTING THEORIES

In this section, the criterion for the onset of instability (Equation (59)) is compared, firstly with that given by Peters et al.[14], and secondly with that given by Perrine[9,10].

In order to evaluate the scaling group on the left hand side of Equation (59a), one must be able to specify how viscosity and density vary with concentration in the transition region, A number of models of varying complexity are available in the literature for this purpose[24-30]. Because of its simplicity, Peters et al. used a model which assumes an initially sharp interface. As a consequence, differentiation of density and viscosity with respect to concentration leads, respectively, to the jump conditions

$$\frac{d\rho}{dC} = \frac{\rho_s - \rho_o}{C_s - C_o} \tag{61}$$

and

$$\frac{d\mu}{dC} = \frac{\mu_s - \mu_o}{C_s - C_o} \tag{62}$$

Moreover, it was also assumed that

$$\bar{\mu} = \frac{\mu_o + \mu_s}{2} \tag{63}$$

If Equations (61) to (63) are inserted into Equation (59), the condition for the onset of instability becomes

$$\frac{2(M_r-1)\ (V-V_c)\ L_x^2\ L_y^2}{(M_r+1)\ D^*\ (L_x^2 + L_y^2)} = \frac{\pi^2}{2} \tag{64}$$

where

$$V_c = \frac{k\ (\rho_o - \rho_s)\ g}{\mu_o - \mu_s} \tag{65}$$

$$M_r = \frac{\mu_o}{\mu_s} \tag{66}$$

and

$$D^* = \phi D_T \delta \tag{67}$$

The stability criterion given by Equation (64) is identical to that obtained by Peters et al. except for the right-hand side. These authors give the marginal stability boundary as $4\pi^2$ which makes a difference of a factor of approximately one order of magnitude in the critical value of the stability boundary. This is probably due to the specific form of the complex Laplace-Fourier transform used by Peters, et al. in the solution of the linearized perturbation equations.

It is also possible to compare Equation (59a) with Perrine's theory[9,10]. In particular, if the thickness (L_x) of the porous medium is much smaller than its width (L_y), the stability criterion given by Equation (59a) degenerates to that given by Perrine[9]. While the condition that L_x is much smaller than L_y may be a good approximation in field-scale systems, it probably does not apply to most laboratory sand packs. Therefore, Equation (59a) should be used when evaluating laboratory experiments.

4.7 DESIGN OF GRADED VISCOSITY BANKS

If a miscible enhanced recovery operation is to be economically feasible, it is important to avoid the continuous injection of an expensive solvent. This can be accomplished by injecting a small slug of solvent and then chasing it with a less expensive fluid which is miscible with the solvent slug. The success of such a scheme depends, to a large extent, upon whether the displacement is stable or unstable. As frontal instability results from unfavourable mobility ratios that exist in the reservoir, it can be reduced by controlling the mobility of the displacing fluid. It has been shown experimentally that one way to achieve this goal is to gradually change the viscosity of injected fluid from that of the displaced fluid to that of the displacing fluid[34,41]. Because the viscosity of the mixure depends upon composition, the desired change in viscosity can be achieved by gradually changing the composition of the fluids in the system from that of high viscosity to that of low viscosity. A basic economic requirement for such a system is to use as little solvent as possible. Therefore, it is important to know how steep the concentration gradient, $\partial\bar{C}/\partial z$, has to be in order to control the effects of viscous fingering.

The modified theory can be used to predict the length of the transition region which should be introduced initially into the reservoir between the solvent and the oil to prevent viscous fingering. If it is assumed that the concentration changes linearly, then the length of the stable transition region can be solved for from Equation (59a). However, in general, there may be uncertainities in the determination of the variables used in the instability number, I_{sr}. That is, the thickness, the width, the permeability and the porosity of the reservoir may not be known very accurately. Moreover, the value of the dispersion coefficients which pertain in the reservoir may be different from those measured in the laboratory. Therefore, it is helpful to specify an upper and a lower bound for the marginal stability boundary. This leads to the determination of upper and lower bounds for the slug size.

In order to be able to determine an upper bound (optimistic estimate) for the marginal instability boundary, it is assumed that the transverse dispersion is equal to the longitudinal dispersion; i.e., $K=1$. If this value of K is substituted into Equation (31), again with the assumption of $M \cong 0$, it follows that

$$(D^2 - \gamma^2)^2 \tilde{w} = - R^2 \gamma^2 \tilde{w} \tag{68}$$

Equation (68) can be solved by taking the same approach used to solve Equation (31); that is, by demanding continuity across the interfacial region. Thus, at the upper marginal instability boundary

$$I'_{sr} = \pi^2 \tag{69a}$$

where the new instability number, I'_{sr}, is defined as follows

$$I'_{sr} = \frac{-k\left(\frac{V}{k}\frac{d\mu}{dC} - \frac{d\rho}{dC}g\right)}{\bar{\mu}\ \phi\ D_L}\left(\frac{\partial \bar{C}}{\partial z}\right)\frac{L_x^2 L_y^2}{L_x^2 + L_y^2} \tag{69b}$$

A lower bound (conservative estimate) for the marginal instability boundary is obtained by assuming that the transverse dispersion is very small compared to the longitudinal dispersion, i.e. $K \cong 0$. Substitution of this value of K and the assumption of $M \cong 0$ into Equation (31) leads to

$$D^2 (D^2 - \gamma^2)\, \tilde{w} = - R^2\, \gamma^2\, \tilde{w} \tag{70}$$

Solving this equation with the same boundary conditions used to solve Equation (31), one obtains for the conservative estimate of the marginal instability boundary

$$I'_{sr} = \frac{\pi^2}{4} \tag{71}$$

Equations (69) and (71) can be used to obtain an estimate for the upper and lower bounds of the length of the graded solvent slug. Thus, if it is assumed that the concentration changes linearly from zero to one, then the term $\partial \bar{C}/\partial z$ can be replaced by $1/\Delta z$. Consequently, the length of the solvent transition region, Δz, for a stable displacement is given by, in view of Equation (69),

$$\Delta z = \frac{-k \left(\frac{V}{k}\frac{d\mu}{dC} - \frac{d\rho}{dC} g\right)}{\bar{\mu}\ \phi\ D_L\ \pi^2} \frac{L_x^2 L_y^2}{L_x^2 + L_y^2} \tag{72}$$

as an estimate for the lower bound, and in view of Equation (71),

$$\Delta z = \frac{-4k \left(\frac{V}{k}\frac{d\mu}{dC} - \frac{d\rho}{dC} g\right)}{\bar{\mu}\ \phi\ D_L\ \pi^2} \frac{L_x^2 L_y^2}{L_x^2 + L_y^2} \tag{73}$$

as an estimate for the upper bound.

Kyle and Perrine[34] carried out some miscible displacement experiments with a graded viscosity zone in a rectangular sand pack of dimensions 0.25in x 9.50in x 10ft. The displacements were carried out vertically in the downward direction and the viscosity ratio was 4.85. Although the theory of Perrine[3,10] gave the length of the stable transition region, Δz, as 295 ft, Kyle and Perrine's experiments demonstrated that such was not the case. That is, at Δz=20 ft the displacement was stable until late in the test when some small viscous fingers developed, and at Δz=50 ft the displacement was stable. Claridge[11] showed that for the parameters used in the Kyle and Perrine experiments, the theory of Chuoke[14] also gave Δz=295 ft for a stable displacement. Consequently, he concluded that the grading criterion developed by Perrine and Chuoke from theoretical considerations was pessimistic for systems of finite length. Here, Equations (72) and (73) are used to compare the theory with Kyle and Perrine's experiments. Given that Δz=295 ft from Perrine's theory[10], it follows that

$$\frac{-k\,(\frac{V}{k}\frac{d\mu}{dC} - \frac{d\rho}{dC}\,g)}{\bar{\mu}\;\phi\;\pi^2\;D_T}\,L_x^{\,2} = 295\ \text{ft} \tag{74}$$

Thus,

$$\frac{-k\left(\frac{V}{k}\frac{d\mu}{dC}-\frac{d\rho}{dC}g\right)}{\bar{\mu}\ \phi\ D_T} = 6729690\ \text{ft}^{-1} \qquad (75)$$

where L_x was taken as 0.25 in. In order to use Equations (72) and (73), an estimate of D_L is also needed. For the sand pack used, the particle diameter was 0.01 cm. By assuming typical laboratory values for the displacement rate (10 ft/D) and the molecular diffusion coefficient (1×10^{-5} cm^2/s), it can be shown that[51]

$$D_L \cong 15\ D_T \qquad (76)$$

Therefore, by substituting Equation (76) into Equations (72) and (73), and by using the value obtained from Equation (75), as well as the values of L_x=0.25 in. and L_y=9.50 in. in the resulting equations, an estimate of the upper and lower bounds for Δz can be obtained. If this is done, it can be shown that the length of the transition region for a stable displacement for Kyle and Perrine's experiments lies in the interval 19.7 ft $\le \Delta z \le$ 78.6 ft. That is, there is very good agreement between the modified theory and the experimental work of Kyle and Perrine. As a consequence, it seems that the modifications introduced into Chuoke's theory have significantly improved the theory to the point where it can be used with some confidence, when designing a miscible, enhanced oil recovery process.

Next, a hypothetical field example given by Claridge[11] is considered. He assumes a horizontal displacement where L_y=660 ft, V/ϕ=3.528x10^{-4} cm/s (1 ft/D), D_T=1x10^{-4} cm²/s and $M_r=\mu_o/\mu_w=10$. Given these conditions, Claridge determined, from both Perrine's and Chouke's theory, that Δz=11,000,000 ft for a stable displacement which is unattainable in practice. However, if in addition to the above variables, it is assumed that the height of the reservoir, L_x, is 30 ft and that for a typical field displacement[11] $D_L \cong 20D_T$; then it can be shown from Equations (72) and (73) that 1125 ft ≤ Δz ≤ 4500 ft. These values are more reasonable but still difficult to attain in the field, and given the good agreement with the Kyle and Perrine experiments, the actual value of Δz should lie in this interval. It should also be noted that these values are sensitive to the ratio of D_L to D_T. Nevertheless, if it is assumed that the reservoir is thinner (L_x=10 ft; M_r=10; 125 ft ≤ Δz ≤ 500 ft) or that the mobility ratio is smaller (L_x=30 ft; M_r=1.5; 198 ft ≤ Δz ≤ 794 ft) then the resulting value of Δz may be attainable under field conditions.

Therefore, given the underlying assumptions, the modified theory suggests that the use of a graded viscosity zone is practical in situations where the mobility ratio is slightly unfavourable and/or where the reservoir is thin.

5. VARIATIONAL APPROACH

In this section, a variational technique (direct method) is employed which removes the need to assume that the porous medium is infinitely long. This approach uses a linear stability analysis, as does the modified instability theory (Section 4), to obtain the perturbation equations. The solution of these equations is obtained through the variational technique which enables one to introduce the appropriate boundary conditions at the inlet and the outlet of the porous medium. As a consequence, a dimensionless number for predicting the onset of instability, in a porous medium which has a finite length, can be derived.

5.1 PERTURBATION EQUATIONS

The same flow system that was used in Section 4 is considered again (see Figure 1). Thus, the equations describing the flow are given by Equations (1) to (5). Assuming that the base state is given by Equations (6) to (10), small perturbations are introduced into the dependent variables of the system. If the resulting equations are linearized, then a Fourier component of any arbitrary disturbance can be represented by Equation (26). When this equation is utilized, after some manipulations, the two perturbation equations given by Equations (27) and (28), which are repeated here, are obtained :

$$\left[-\left(\frac{\bar{\rho}\sigma}{\phi\gamma^2}+\frac{\bar{\mu}}{k\gamma^2}\right)D^2-\left(\frac{\sigma}{\phi\gamma^2}\frac{d\bar{\rho}}{dz}+\frac{1}{k\gamma^2}\frac{d\bar{\mu}}{dz}\right)D+\left(\frac{\bar{\mu}}{k}+\frac{\bar{\rho}\sigma}{\phi}\right)\right]\tilde{w}=-N_C\tilde{C} \qquad (27)$$

and

$$(D^2-\frac{D_T}{D_L}\gamma^2-\frac{\sigma}{D_L})\tilde{C}=\frac{1}{\phi D_L}\frac{\partial\bar{C}}{\partial z}\tilde{w} \qquad (28)$$

If it is assumed that the principle of exchange of stabilities applies in this case, i.e. that the condition of marginal instability occurs when the time constant is equal to zero, then Equations (27) and (28) may be shown to become

$$(D^2+MD-\gamma^2)\tilde{w}=\frac{k\gamma^2N_c}{\bar{\mu}}\tilde{C} \qquad (77)$$

and

$$(D^2 - K\gamma^2)\tilde{C} = \frac{1}{\phi D_L} \frac{\partial \bar{C}}{\partial z} \tilde{w} \qquad (78)$$

where the parameters N_C, γ^2, M and K are defined, respectively, by Equations (29), (30), (32) and (34). Equations (77) and (78), which are coupled, linear ordinary differential equations, have to be solved to determine the criterion for marginal instability. At the inlet and the outlet boundaries of the porous medium, the perturbations are presumed to be zero''. It should be noted that, under actual field conditions, fluids are mixed to a certain extent in the injection well as they travel down the wellbore, and in the production well at the sandface. Therefore, the velocity and concentration perturbation at the injection and the production wells may be assumed to be zero. These conditions, which are expressed in the Eulerian coordinates at z'=0 (inlet) and z'=L (outlet), become

$$\tilde{C} = 0 \quad \text{at} \quad z = -\frac{Vt}{\phi} \quad \text{and} \quad z = L - \frac{Vt}{\phi} \qquad (79a)$$

and

$$\tilde{w} = 0 \quad \text{at} \quad z = -\frac{Vt}{\phi} \quad \text{and} \quad z = L - \frac{Vt}{\phi} \tag{79b}$$

in Lagrangian coordinates (see Equation (20)). Unfortunately, it is very difficult to find an analytical solution to Equations (77) and (78) with boundary conditions specified at locations which are functions of time in Lagrangian coordinates. However, the problem can be simplified if a solution to the problem is sought for times only slightly larger than zero. Such an analysis can be utilized because it was assumed that the principle of exchange of stabilities is valid for the system under consideration here. That is, the perturbations either grow or decay in a continuous, non-oscillatory fashion. Therefore, any arbitrary small perturbation which grows or decays within the infitesimally small time period after time zero will continue to do so for the whole duration of the displacement. Consequently, it is assumed that the rate of change of concentration perturbation, which is the difference between the resultant concentration and the unperturbed concentration, with respect to distance ($\partial C^*/\partial z'$) is negligible for times only slightly larger than zero. Under these conditions, the linearized perturbation equations can be written in an identical form to those written in Lagrangian coordinates. Alternatively, for infitesimally small times ($t \to 0$), at the inlet $z \cong z' = 0$ and at the outlet $z \cong z' = L$ (see Equation (20)). Thus Equations (79a) and (79b) now become

$$\tilde{C} = 0 \quad \text{at } z = 0 \text{ and } z = L \tag{80a}$$

and

$$\tilde{w} = 0 \quad \text{at } z = 0 \text{ and } z = L \tag{80b}$$

5.2 VARIATIONAL SOLUTION

The approximate method of solving Equations (77) and (78), together with boundary conditions (80a) and (80b), is similar to the variational scheme used to analyse the stability of natural convection[49]. A similar scheme is also used in the theory of elasticity[52,53] (direct method or Ritz method). Thus, the solution of Equations (77) and (78), which satisfy the boundary conditions (80a) and (80b), can be expanded in a sine series in the following form

$$\tilde{w}(z) = \sum_{m=1}^{\infty} A_m \operatorname{Sin} \frac{m\pi z}{L} \tag{81}$$

If $\tilde{w}(z)$ is given by Equation (81), then from Equation (78) it may be shown that

$$(D^2 - K\gamma^2)\tilde{C} = \frac{1}{\phi D_L} \frac{\partial \bar{C}}{\partial z} \sum_{m=1}^{\infty} A_m \operatorname{Sin} \frac{m\pi z}{L} \tag{82}$$

As Equation (82) is a linear, ordinary differential equation, the solution for $\tilde{C}(z)$ may be assumed to take the form

$$\tilde{C}(z) = \frac{1}{\phi D_L} \frac{\partial \bar{C}}{\partial z} \sum_{m=1}^{\infty} A_m C_m(z) \tag{83}$$

where $C_m(z)$ is the solution of the equation

$$(D^2 - K\gamma^2)C_m = \mathrm{Sin}\,\frac{m\pi z}{L} \tag{84}$$

which satisfies the boundary conditions

$$C_m = 0 \qquad \text{at } z = 0 \text{ and } z = L \tag{85}$$

Substitution of Equations (81) and (83) into Equation (77) leads to

$$(D^2 + MD - \gamma^2) \sum_{m=1}^{\infty} A_m \,\mathrm{Sin}\,\frac{m\pi z}{L} = \frac{k\gamma^2 N_c}{\phi\bar{\mu}D(L)} \frac{\partial\bar{C}}{\partial z} \sum_{m=1}^{\infty} A_m C_m(z) \tag{86}$$

Undertaking this differentiation yields

$$\sum_{m=1}^{\infty} A_m \gamma_m \,\mathrm{Sin}\,\frac{m\pi z}{L} - \sum_{m=1}^{\infty} A_m \beta_m \,\mathrm{Cos}\,\frac{m\pi z}{L} = R^2\gamma^2 \sum_{m=1}^{\infty} A_m C_m(z) \tag{87}$$

where

$$R^2 = -\frac{kN_c}{\phi\bar{\mu}D_L}\frac{\partial\bar{C}}{\partial z} \qquad (33)$$

$$\gamma_m = \frac{m^2\pi^2}{L^2} + \gamma^2 \qquad (88a)$$

and

$$\beta_m = \frac{Mm\pi}{L} \qquad (88b)$$

Next, Equation (87) is multiplied by Sin($n\pi z/L$) where $n=1,2,3,\ldots$ and integrated over the range of z. Thus, it follows that

$$\sum_{m=1}^{\infty}\left[\frac{\delta_{mn}\gamma_m L}{2R^2\gamma^2} - \frac{2n\beta_m L}{\pi R^2\gamma^2(n^2-m^2)} - \theta_{mn}\right]A_m = 0 \qquad (89a)$$

where δ_{mn} is the Dirac delta function and where

$$\theta_{mn} = \int_0^L C_m(z) \operatorname{Sin} \frac{n\pi z}{L} \, dz \qquad (89b)$$

The second term in Equation (89a) should be set equal to zero when m and n are both odd or even. This comes about because of the integration carried out to obtain Equation (89a).

Equation (32) represents a system of linear, homogeneous equations for the variational coefficients A_m. These equations have non-trivial solutions provided that the determinant of the coefficient matrix vanishes. This determinant is called the "secular" determinant and is given by

$$\det \left| \frac{\delta_{mn} \gamma_m L}{2R^2 \gamma^2} - \frac{2n\beta_m L}{\pi R^2 \gamma^2 (n^2 - m^2)} - \theta_{mn} \right| = 0 \qquad (90)$$

Again in Equation (90), the second term should be set equal to zero when m and n are both even or odd. Restricting the analysis to the first n coefficients, $A_1, A_2, \ldots, A_n$ in Equation (81), and setting the others equal to zero, means that a trial function with n parameters is being considered that a trial function with n parameters is being considered for $\tilde{w}(z)$. Therefore, the resulting secular determinant will be of order n.

Before proceeding any further, the parameter θ_{mn} must be defined. To this end, Equation (84) is solved subject to boundary conditions (86). If this is done, it can be shown that

$$C_m(z) = -\frac{1}{\gamma_m^*} \text{Sin}\,\frac{m\pi z}{L} \tag{91}$$

where

$$\gamma_m^* = \frac{m^2\pi^2}{L^2} + K\gamma^2 \tag{92}$$

Then, Equation (91) is inserted into (89) and the integration is carried out to obtain the parameter θ_{mn} as follows

$$\theta_{mn} = -\frac{L\,\delta_{mn}}{2\gamma_m^*} \tag{93}$$

Therefore, the final form of the secular determinant is given by

$$\det \left| \frac{\delta_{mn}\gamma_m L}{2R^2\gamma^2} - \frac{2n\beta_m L}{\pi R^2\gamma^2(n^2-m^2)} + \frac{\delta_{mn} L}{2\gamma_m^*} \right| = 0 \qquad (94)$$

It should be noted that the present method of solution is applicable even if the variational principle does not underlie the problem[illegible]. But the variational principle ensures that, by keeping more and more terms in the Fourier expansion for $\tilde{w}(z)$ (see Equation (81)), we approach the true condition for the onset of instability monotonically from above.

5.3 FIRST APPROXIMATION

The actual condition for marginal instability can be obtained by evaluating the secular determinant which has an order of n. That is, by taking more and more terms in the Fourier expansion (81) for $\tilde{W}(z)$ and evaluating the corresponding secular determinant, the exact condition for marginal instability can be obtained. However, Chandrasekhar[illegible] has shown that even the vanishing of the upper left element (first element) of the determinant gives a very good approximation to the exact condition of marginal instability in natural convection. Thus, following

Chandrasekhar's analysis, a first approximation to the exact condition of marginal instability can be obtained by setting the element (1,1) of the secular determinant equal to zero and ignoring all the others. This corresponds to $\mathrm{Sin}(\pi z/L)$ as a trial function for $\tilde{w}(z)$. If this approach is taken, the condition for marginal instability becomes

$$\frac{L\gamma_{m=1}}{2R^2\gamma^2} - \frac{L}{2\gamma^*_{m=1}} = 0 \qquad (95)$$

If Equations (88a) and (92) are inserted into Equation (95), it can be shown that

$$-R^2 = \left(\frac{\pi^2}{L^2\gamma^2} + K\right)\left(\frac{\pi^2}{L^2\gamma^2} + 1\right)\gamma^2 \qquad (96)$$

If the porous medium is infinitely long, this expression degenerates to

$$-R^2 = K\gamma^2 \qquad (97)$$

The condition of marginal instability given by Equation (97) is identical to that obtained previously (see Equation (50)) by assuming an infinitely long porous medium and $M \simeq 0$. Hence, it is concluded that the first approximation does not take into account the change of viscosity with distance in the transition region. However, it can be used for a porous medium with a finite length.

The explicit expression for the first approximation to the condition of marginal instability is obtained by substituting Equations (29), (33) and (56) into Equation (96). The onset of instability is again dictated by the smallest non-zero eigenvalue. Therefore, as a first approximation to the condition for the onset of instability, the following expression is obtained

$$\frac{k\left(\frac{V}{k}\,\frac{d\mu}{dC} - \frac{d\rho}{dC}\,g\right)}{\bar{\mu}\,\phi\,D_L}\left(\frac{\partial \bar{C}}{\partial z}\right)\frac{L_x^{\,2}L_y^{\,2}}{L_x^{\,2}+L_y^{\,2}} = \pi^2\left(\frac{1}{\Omega} + K\right)\left(\frac{1}{\Omega} + 1\right) \tag{98}$$

where the dimensionless length, Ω, is defined as

$$\Omega = \frac{L^2(L_x^{\,2}+L_y^{\,2})}{L_x^{\,2}L_y^{\,2}} \tag{99}$$

5.4 SECOND APPROXIMATION

An improved (second) approximation can be obtained if the first two terms in Equation (81) are chosen. For this case, the top left 2x2 portion of the nxn secular determinant is considered. When the determinant of this 2x2 matrix is set equal to zero, a quadratic equation in R^2 results, and consequently the two conditions for the onset of instability are given by

$$\frac{k\left(\frac{V}{k}\frac{d\mu}{dC} - \frac{d\rho}{dc}g\right)}{\bar{\mu}\phi D_L}\left(\frac{\partial\bar{C}}{\partial z}\right) = \frac{B \mp \sqrt{B^2 - 4AG}}{2A\gamma^2} \tag{100}$$

where

$$A = \frac{1}{(\pi^2/L^2 + K\gamma^2)(4\pi^2/L^2 + K\gamma^2)} \tag{101a}$$

$$B = \frac{(\pi^2/L^2 + \gamma^2)}{(4\pi^2/L^2 + K\gamma^2)} + \frac{(4\pi^2/L^2 + \gamma^2)}{(\pi^2/L^2 + K\gamma^2)} \tag{101b}$$

$$G = \left(\frac{\pi^2}{L^2} + \gamma^2\right)\left(\frac{4\pi^2}{L^2} + \gamma^2\right) - \frac{128M^2}{9\ L^2} \qquad (101c)$$

and where γ is again defined (see Equation (56)) as the minimum non-zero eigenvalue (m=n=1). It should be noted that Equation (100) degenerates to (97), if the porous medium is infinitely long. Moreover, the first solution, the one with a minus sign, is equivalent to the condition given by the first approximation, Equation (98), when M is negligible. The second solution, the one with a plus sign, defines a different criterion. The two solutions may provide bounds on the actual solution.

While third-order and/or higher approximations would be of interest, it does not appear to be possible to solve explicitly for the roots of the very complicated analytical expressions which result. Consequently, no higher approximations are presented in this study.

6. EXPERIMENTS

In order to test the fluid mechanics aspects of the theory, the experiments were carried out in a simple system, a Hele-Shaw cell. The use of a Hele-Shaw cell not only simplifies the problem, as it is an idealized, two-dimensional porous medium, but it also enables attention to be focused on the most important aspect of the problem, namely, the interaction of forces within the interfacial region[35,36]. Thus, a Hele-Shaw cell provides an experimentally realizable physical system[37] for studying viscous fingering as it contains a great deal of physics without the complication of irregular geometry[38,39]. For these reasons, Hele-Shaw cells are also widely used in gaining insight into various fluid mechanics problems[40-43]. The modification of the theory for flow in a Hele-Shaw cell is given in Appendix 1.

6.1 DESCRIPTION OF EXPERIMENTAL APPARATUS

The experimental equipment consisted of two Hele-Shaw cells of lengths 100 cm and 20 cm joined by a dam which separated the two cells. An "O" ring, which was cut to obtain a long, straight barrier in the dam assembly, prevented the fluids from coming into contact with each other (see Figure 2). Each Hele-Shaw cell was made up of flat sheets of glass of 10 mm in thickness. The glass sheets were clamped together by an aluminum housing and separated by strips of teflon to provide a plate separation of 1 mm.

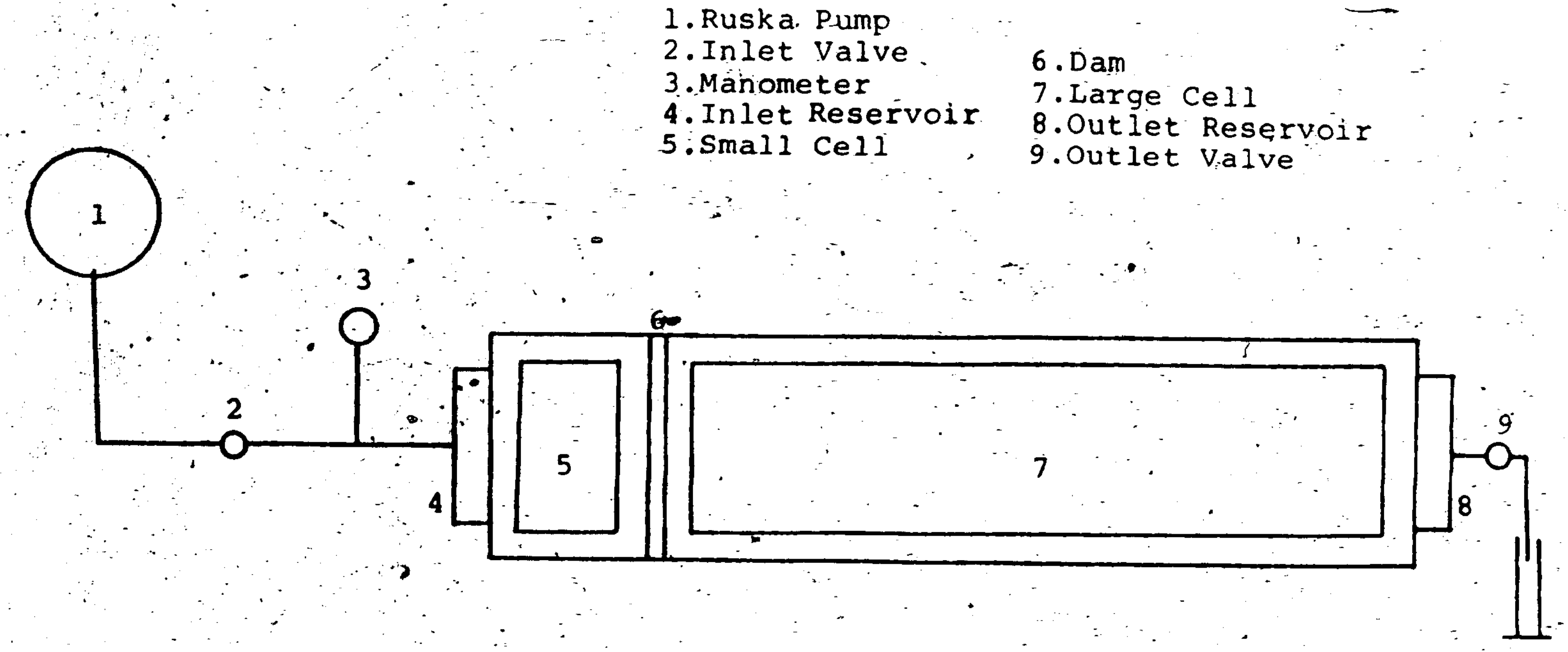

FIG.2 SCHEMATIC DRAWING OF EXPERIMENTAL APPARATUS

Both cells were 20 cm wide. Inlet and outlet ends were connected to small reservoirs of solvent and oil, respectively. These reservoirs had screens in them to disperse the injected and produced fluids uniformly along the width of the cells, thus providing a uniform entrance and discharge of fluids. A Ruska Pump was used to obtain a constant injection rate of the desired magnitude over the range of 8.5 to 560 cc/hr.

6.2 EXPERIMENTAL PROCEDURE

Forty-four miscible displacement experiments were carried out using water as displacing fluid and glycerine as the displaced fluid. These fluids were completely miscible in all proportions. The displaced fluid, glycerine, was premixed with water to achieve the desired viscosity contrast, and dyed with a water soluble dye to distinguish it visually from the water.

The Hele-Shaw cell was tested for bulging under pressure by using a caliper which could measure changes within 0.025 mm. The glass plates which made up the cell did not change their separation under the pressure developed by the maximum rate which was attained in the experiments. Moreover, the Reynolds number $(Vh\rho/\mu)$ for each experiment was well below one, indicating that the creeping flow assumption was met. Therefore, the analogy between flow in a Hele-Shaw cell and that in a two-dimensional porous medium was valid.

Before each run, both cells were cleaned using acetone. After assembling the cells with the dam in place, the small cell was filled with water and the large cell with a glycerine mixture of the desired viscosity. The next step was to remove carefully the "O" ring separating the two fluids so that no deformation of the interface occurred. At this point, an initially straight interface was observed. The "O" ring groove in the dam initiated the perturbations after the experiment started. As the interfacial region moved away from the dam into the large cell, the zero percent concentration line (i.e. 100% glycerine premix) was traced onto the glass plate at various intervals. After the experiments, these isoconcentration lines were transferred onto tracing paper for later analysis and digitization.

7. RESULTS AND DISCUSSION

The results of the forty-four Hele-Shaw cell experiments carried out are summarized in Table 1. The values of the dimensionless instability number, I_{sr}, were calculated using the left-hand side of Equation (1.39) in Appendix 1. This equation is the Hele-Shaw cell equivalent of Equation (98). The right-hand side of Equation (1.39) gives the value of the instability number at which a displacement becomes unstable.

The parameters in Equation (1.39) were calculated in the following manner. The longitudinal dispersion coefficient in a Hele-Shaw cell is given by[64]

$$D_L = D' + 7.9512 \frac{h^2 V^2}{210 D} \qquad (102)$$

The average unperturbed viscosity, $\bar{\mu}$, was calculated by taking the arithmetic average of the viscosity of the displaced and displacing fluids (see Equation (63)). The magnitude of the slope of the unperturbed concentration profile was calculated using Equation (60b) which is the same as Equation (1.43) in Appendix 1.

In order to calculate the values of $d\rho/dC$ and $d\mu/dC$, a curve was first fitted to the density and viscosity versus concentration data of aqueous solutions of glycerine which were obtained from the literature[65] (see Figures 3 and 4).

TABLE 1

HELE-SHAW CELL EXPERIMENTS

No.	μ_o/μ_s	VELOCITY (cm/min)	θ	I_{sr}	EXPERIMENTAL OBSERVATION
1	4.73	0.291	0	437.4	Unstable
2	4.73	0.404	0	267.5	Unstable
3	4.73	0.807	0	88.17	Unstable
4	4.73	1.179	0	53.43	Unstable
5	4.73	1.570	0	34.98	Unstable
6	4.73	0.807	0	88.17	Unstable
7	4.73	0.404	0	267.5	Unstable
8	4.73	0.291	0	437.4	Unstable
9	4.73	0.212	0	699.7	Unstable
10	4.73	0.105	0	1959.	Unstable
11	4.73	0.088	0	2519.	Unstable
12	4.73	0.088	0	2519.	Unstable
13	4.73	0.062	0	4030.	Unstable
14	4.73	0.073	0	3293.	Unstable
15	38.77	2.567	89.10	51.53	Unstable
16	38.77	2.561	88.16	54.61	Unstable
17	38.77	0.933	85.36	34.61	Unstable
18	38.77	0.170	85.40	1442.	Unstable
19	38.77	0.166	92.84	-465.0	Stable

TABLE 1 (Continued)

HELE-SHAW CELL EXPERIMENTS

No.	μ_o/μ_s	VELOCITY (cm/min)	θ	I_{sr}	EXPERIMENTAL OBSERVATION
20	38.77	0.252	92.86	-110.4	Stable
21	38.77	0.245	91.87	-22.55	Stable
22	38.77	0.283	91.87	2.108	Stable
23	38.77	0.502	91.87	25.23	Unstable
24	38.77	0.669	91.87	21.52	Unstable
25	38.77	0.402	91.87	24.27	Unstable
26	38.77	0.333	91.87	16.43	Unstable
27	38.77	0.282	91.87	1.703	Stable
28	10.00	0.845	91.87	-50.08	Stable
29	10.00	1.002	91.87	-24.96	Stable
30	10.00	1.168	91.87	-11.38	Stable
31	10.00	1.285	91.87	-5.846	Stable
32	10.00	1.546	91.87	0.703	Stable
33	10.00	1.933	91.87	4.119	Unstable
34	10.00	2.573	91.87	5.026	Unstable
35	10.00	3.209	91.87	4.617	Unstable
36	10.00	3.855	91.87	4.024	Unstable
37	4.73	1.002	91.00	-72.12	Stable

TABLE 1 (Continued)

HELE-SHAW CELL EXPERIMENTS

No.	μ_o/μ_s	VELOCITY (cm/min)	θ	I_{sr}	EXPERIMENTAL OBSERVATION
38	4.73	1.169	91.00	-36.80	Stable
39	4.73	1.282	91.00	-25.05	Stable
40	4.73	1.544	91.00	-9.620	Stable
41	4.73	1.937	91.00	2.270	Unstable
42	4.73	2.577	91.00	3.992	Unstable
43	4.73	2.577	91.00	3.992	Unstable
44	4.73	3.221	91.00	4.665	Unstable

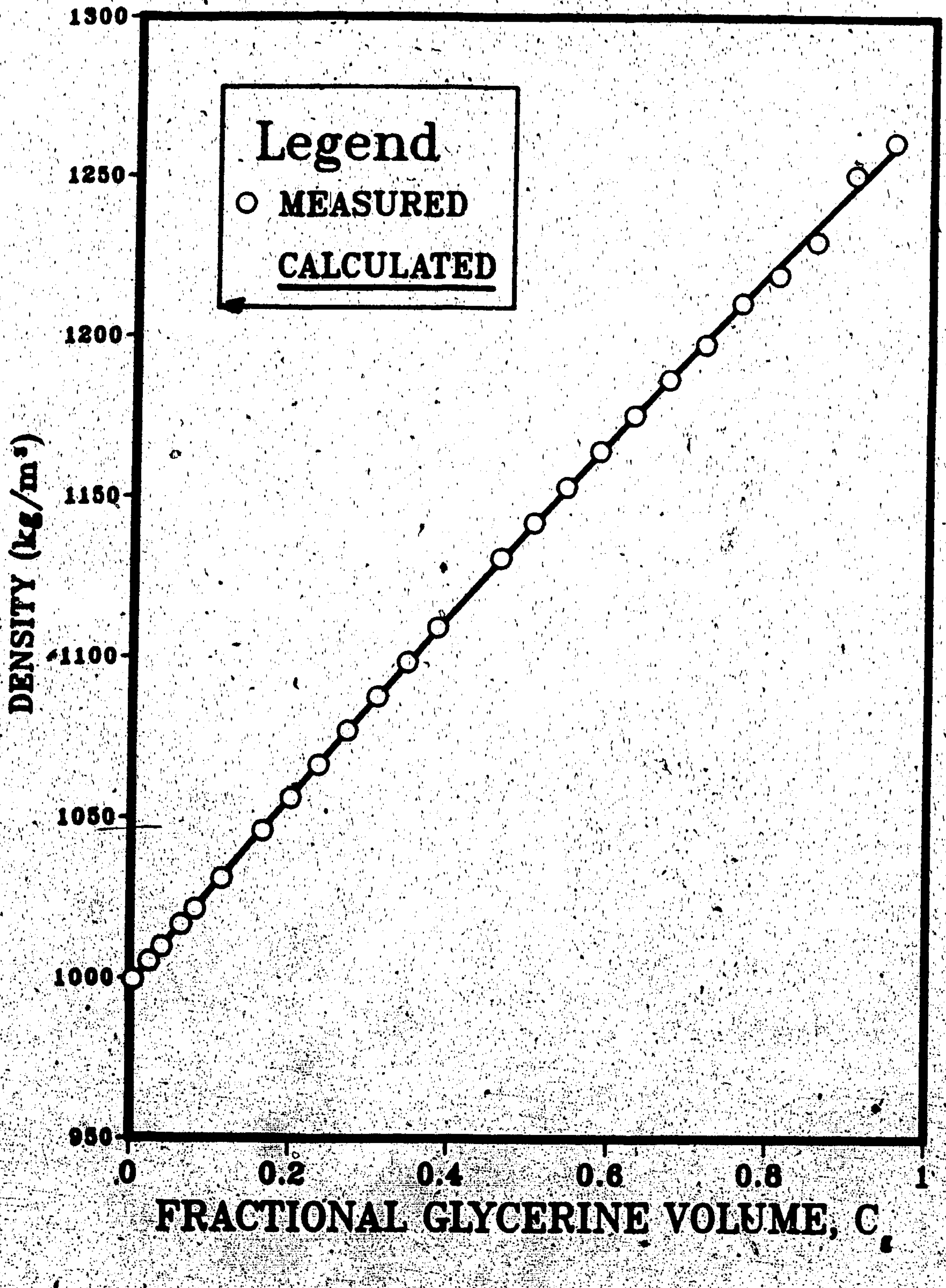

FIG.3 DENSITY OF AQUEOUS SOLUTIONS OF GLYCERINE

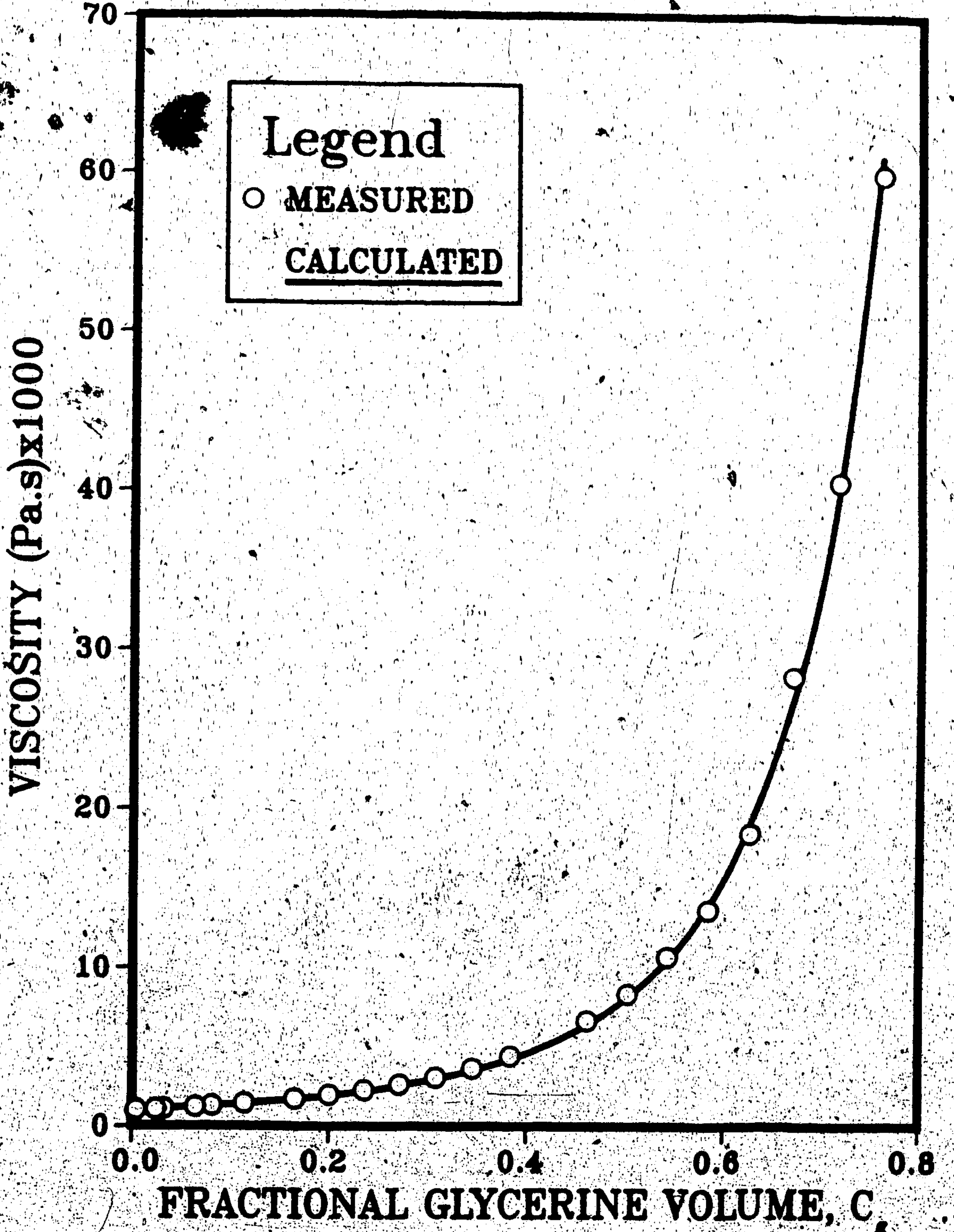

FIG.4 VISCOSITY OF AQUEOUS SOLUTIONS OF GLYCERINE

The functions fitted are, for the density,

$$\rho = 997.93 + 297.71\ C_g - 23.86\ C_g^2 \quad (\mp 1.43\ \mathrm{kg/m^3}) \tag{103}$$

and, for the viscosity,

$$\mu = 1763^q \ (\mp 0.47\ \mathrm{cp}),\ \text{for } C_g < 0.80 \tag{104}$$

where

$$q = \frac{C_g}{C_g + 2.5871\ (1-C_g)} \tag{105}$$

The function fitted for the viscosity data is similar to those suggested previously in the literature[26,66].

Once the value of $\bar{\mu}$ was calculated from Equation (63), the average concentration, $<\bar{C}>$, corresponding to $\bar{\mu}$ was calculated from Equation (104). Having found $<\bar{C}>$, the values of $d\rho/dC$ and $d\mu/dC$ were calculated by differentiating

Equations (103) and (104) at this average concentration. Moreover, as the diffusivity of glycerine in water is a function of concentration, the molecular diffusion coefficient, D', was also taken to be the value at the average concentration, $<\bar{C}>$, from the data given in the literature[illegible]. The variables used in the calculation of I_{sr} are given in Table 2. It should be noted that the parameters used in the calculation of the instability number are sensitive to the way in which the data is averaged. There may be other equally plausible ways of averaging such data[illegible]. These ways could result in different values of I_{sr} being estimated for the same experimental conditions. Two different ways of averaging the data were tried but the method used here was thought to be more plausible than a sharp interface approximation. Moreover, it gave the best agreement with the experimental results.

Finally, the ratio of transverse to longitudinal dispersion, K, in a coarse sand pack within the velocity range used in this study is known to be 0.2[27,48]. Therefore, it was assumed that, in a Hele-Shaw cell, a similar value would pertain. That is, K was assumed to be approximately 0.2. While it is thought that K may be a weak function of velocity in general, it was assumed to be essentially constant in this study because no data was available to estimate its actual value. Given that this value of K is a good approximation, the marginal instability boundary is reached when $I_{sr}=2.463$; i.e., the right-hand side of

TABLE 2
PROPERTIES OF THE FLUIDS USED

μ_o/μ_s	$\bar{\mu}\times10^3$ (Pa.s)	$\langle\bar{C}\rangle$	$D'\times10^{10}$ (m^2/s)	$d\rho/dC$ (kg/m^3)	$d\mu/dC\times10^3$ (Pa.s)
4.73	2.86	0.702	5.0	283.50	12.391
10.00	5.50	0.567	3.5	277.04	29.475
38.77	19.88	0.367	1.5	267.50	153.52

Equation (1.39) equals 2.463. As a consequence, theoretically, displacements whose instability numbers are smaller than 2.463 should be stable, or, alternatively, a displacement which has an instability number larger than 2.463 should be unstable. The experiments carried out in this study covered a viscosity ratio range of approximately 5 to 40 and a velocity range of approximately 0.05 to 4 cm/min. The velocity range was dictated by the limitations of the pump. Because of the step changes in the volumetric output of the pump, it was not possible to achieve values of the instabilility number which were closer to the stability boundary than that for Run 41. In Table 1, it can be seen that the agreement between the theoretically predicted onset of instability and the experimental observations is very good with the exception of Run 41. The disagreement of Run 41 may be due to the fact that the value of K may be somewhat smaller than 0.2 for the experiments reported in Table 1. As the instability number is very close to the stability boundary in this run, even a small change in K would change the boundary enough so that Run 41 falls in the unstable region. Some of the other experiments whose instability numbers are close to the instability boundary are Runs 22, 27, 32, 33, 34, 35, 36, 42, 43 and 44. All of these experiments as well as the rest of them, agree favourably with the theory. Therefore, the instability boundary was predicted correctly by the first approximation for the set of experiments conducted in this study.

The marginal instability boundary was also estimated from the second approximation using Equation (1.41) in Appendix 1. The term containing M (see Equation (1.42b)) was one to two orders of magnitude smaller than the rest of the terms; thus, this term was neglected. Under these conditions, as mentioned earlier, the boundary obtained using the negative root in Equation (1.41) was equivalent to that given by the first approximation (I_{sr}=2.463). Hence, the negative root of the second approximation agrees very well with the experimental observations. The other boundary, the one obtained using the positive root, was given when I_{sr}=4.122. Although the second approximation should be more accurate, a marginal instability boundary of 4.122 (for the positive root) seems to agree less favourably with the experiments than does the one obtained from the first approximation. That is, five runs (Runs 33, 36, 41, 42 and 43) disagree with the positive root of the second approximation as opposed to one run (Run 41) with the first approximation (which is also equivalent to the negative root of the second approximation). A better agreement could have been obtained by dec~~reasing the~~ value of K which was assumed to be the same as that in a coarse sand pack. That is, if the value of K had been reduced slightly (to K=0.18), this would have lowered the instability boundary sufficiently (to I_{sr}=2.258) so that all the runs including Run 41 would have agreed with the first approximation as well as with the negative root of the second approximation. However, such adjustments were not undertaken as there was no experimental data available to estimate K in a Hele-Shaw cell.

One of the two roots for the second approximation may be extraneous. Because no theoretical guidance is available as to which of the two roots is extraneous, it is assumed that the positive root should be excluded. This is due to the fact that the negative root gives the most conservative estimate of the stability boundary. Moreover, this choice gives the best agreement between the theoretically predicted stability boundary and that observed in the laboratory.

To investigate whether M is significant under field conditions, it is necessary to assume typical values for the various parameters. Hence, it is assumed that the viscosity function given by Peaceman and Rachford[24,30] is representative of field conditions, and that the following values are typical of those encountered in the field[19,20] : V= 1 ft/D; D_L= 2x10^{-4} cm²/s; L= 1000 ft; L_x= 30 ft; L_y= 660 ft. Then, by making use of the viscosity function together with the above values, it can be shown that the second term (the one containing M) in Equation (101c) is two orders of magnitude smaller than the first term (see Appendix 2).

Similar results are obtained under laboratory conditions except that the second term is three orders of magnitude smaller than the first. As a consequence, for a typical miscible displacement, the effect of M on the prediction of the onset of instability can be neglected. In addition, this result also justifies the assumption of $M \cong 0$ in the modified theory.

The unperturbed concentration profile, $\partial\bar{C}/\partial z$, is the one on which small perturbations are superimposed. As it is analytically difficult to handle a concentration profile which changes with time, it was replaced by a linear concentration profile which was averaged over the life of an experiment (see Equation (60b)). It was found that the use of this average concentration profile resulted in good agreement between the experimentally observed instability boundary and that predicted theoretically. One possible explanation for this is as follows: as the interfacial region advances, small fingers may be created on it; however, widening of the transition region will smear out some of these small fingers before they have a chance to grow. This process starts with the injection of solvent and continues throughout the experiment. Thus, it seems that using an unperturbed concentration profile averaged over the life of the experiment takes this phenomenon, albeit approximately, into account.

The condition for the onset of instability obtained by extending Chuoke's analysis is the same as that obtained from the variational method (first and second approximation (both roots)), provided the system is infinitely long. This suggests that the physically reasonable matching conditions used in the modified theory (Equations (41), (43), (45) and (48)) are appropriate for two reasons. First, they result in the same stability boundary as obtained from the more rigorous variational method. Second, the predicted stability boundary is in good agreement with that obtained experimentally.

The coefficients of the perturbation equations (Equations (21) to (25)) are, in general, functions of space and time. However, these coefficients were treated as being constants in both the modified theory and the variational approach. When using the small perturbations method to predict the stability of miscible liquid-liquid displacements in porous media, it is generally assumed that the perturbation equations have constant coefficients and, consequently, that solutions having the form of Equation (26) can be utilized[8,10,11,12,14,16]. This seems to be a reasonable approximation for the set of experiments carried out in this study as the agreement between the experiments and the theory was good.

In Equation (1.39)(first approximation), as well as in Equation (1.41) (second approximation), the right-hand side asymptotically approaches a constant value as the dimensionless length, Ω, becomes large. As a consequence, the effect of length on instability can be deduced by plotting the criterion for instability as a function of dimensionless length. This was done for the Hele-Shaw cell used in this study. Thus, R^2, which is given by the right-hand side of the first and second approximations (positive root) from Equations (1.37) and (1.41), respectively, and that given for an infinitely long system from Equation (1.38) (equivalent to Equations (57) and (97) in a porous medium), were plotted as a function of dimensionless length defined by Equation (1.40). This plot is shown in Figure 5. It should be noted that, for the experiments reported here, M was assumed to be zero as it was very small. Consequently, the curve for the positive root of the second approximation can be used for all the experiments.

The region that lies below each curve represents the combination of R^2 and Ω for which a stable displacement can be conducted. It is also seen that, after a certain length is reached, the marginal instability boundary becomes independent of the length of the system. As a result, under field conditions for which the dimensionless length is large, the onset of instability can be determined from an analysis which assumes the porous medium as being infinitely long (see Chapter 4). However, in the laboratory, the dimensionless length may fall in the region which lies before the point where all the curves converge to $K\alpha^2$ (see Equation (1.38)). Moreover, in this region the marginal instability boundary is a very strong function of length. In

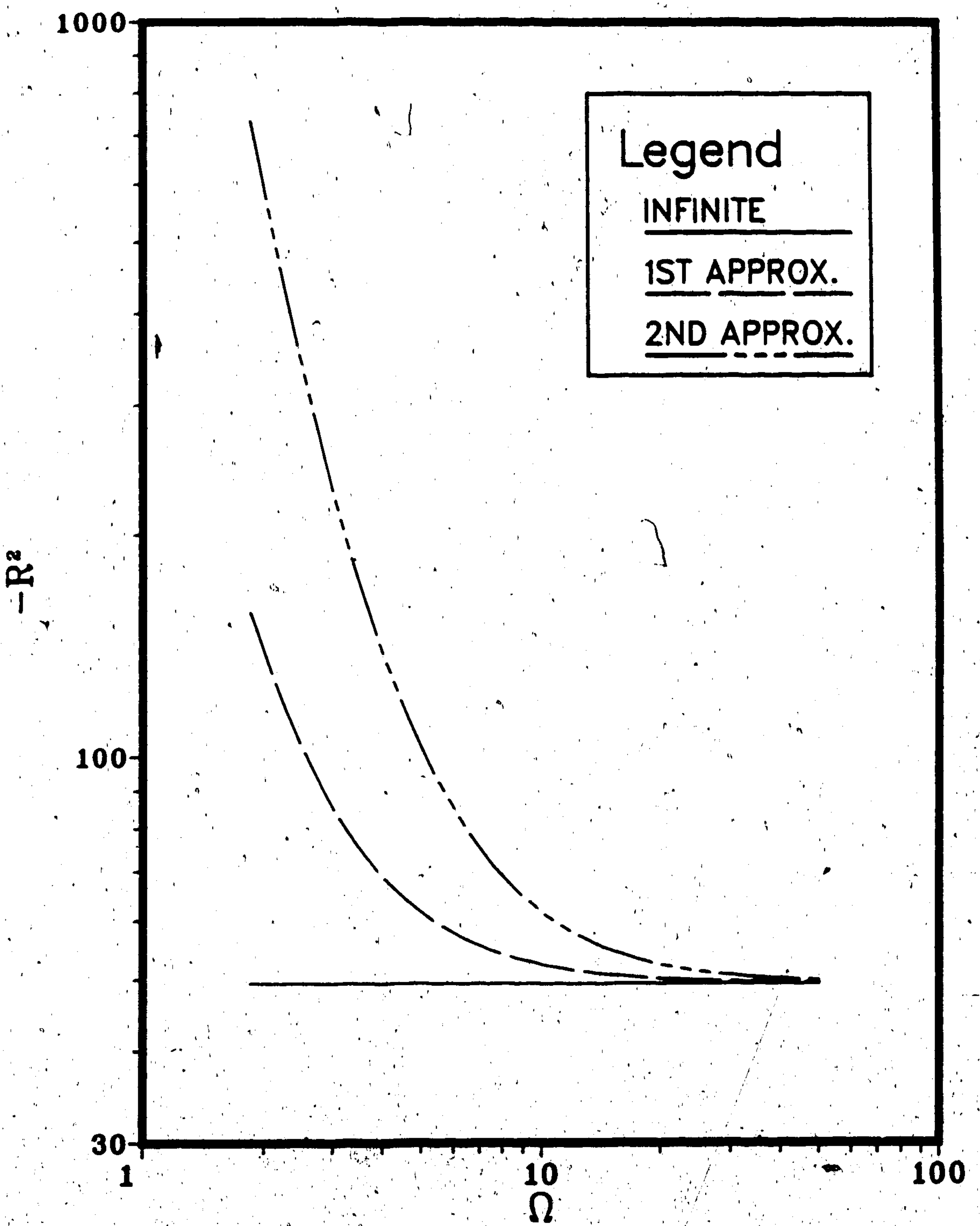

FIG. 5 STABILITY PLOT FOR A HELE-SHAW CELL

particular, the instability boundary in the laboratory may be a few orders of magnitude larger than that in the field. That is, there is a possibility that experiments conducted in a laboratory model might be carried out under stable flow conditions whereas, in the prototype, an unstable flow regime would prevail under similar flow conditions. As a consequence, the laboratory results would not be applicable to the prototype. Therefore, in scaling field conditions in the laboratory, care must be taken to ensure that both flow systems fall as close as possible to each other on a stability plot such as that given in Figure 5 so that the laboratory model represents the intended field conditions.

The variational analysis presented in this study reveals that fingers are more readily formed in a longer system than in a shorter system under similar conditions of flow, provided that the transverse dimensions are the same in both systems. This is due to the fact that perturbations in the flow direction having longer wavelengths than the length of the porous medium can not be observed in that porous medium. As a consequence, the system behaves as if it were stable. However, in a longer porous medium, these perturbations will affect the displacement, causing it to become unstable.

The prediction that shorter systems should be more stable agreed with experimental observations. This can be confirmed by observing the configuration of the zero percent isoconcentration lines (original glycerine premix) at different times. Some of these plots, which are typical for the experiments carried out in this study, are shown in Figures 6, 7, 8, 9 and 10. Additional plots are also included in Appendix 3 (Figures 12 to 16). These plots include two stable experiments, Runs 22 (Fig.9) and 27 (Fig.15) and eight unstable experiments, Runs 4 (Fig.6), 5 (Fig.7), 9 (Fig.8), 13 (Fig.12), 23 (Fig.13), 26 (Fig.14), 33 (Fig.10) and 34 (Fig.16). It is seen that for unstable displacements, initially, the 0% isoconcentration lines are almost straight. However, they become distorted after a certain distance has been travelled†. This means that if the Hele-Shaw cell were shorter than the distance travelled before the perturbations were observed, then the displacement would have appeared to be stable. Alternatively, if the cell were longer, Runs 22 and 27 would have been unstable because the length of the system is inversely proportional to the critical value of the instability number at which the displacement becomes unstable (see Equation (1.39)). Therefore, it is concluded that the experiments seem to agree with theory on the effect of length.

Because the perturbation equations were linearized, the analysis presented here does not apply, once the perturbations reach a finite size. Thus neither the modified

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† It should be noted that in these figures, the ratio of width (x axis) to length (z axis) is not true to life. That is, the width is exaggerated somewhat. Thus, the curvature of the interface in unstable displacements in the figures is not as pronounced as it actually is.

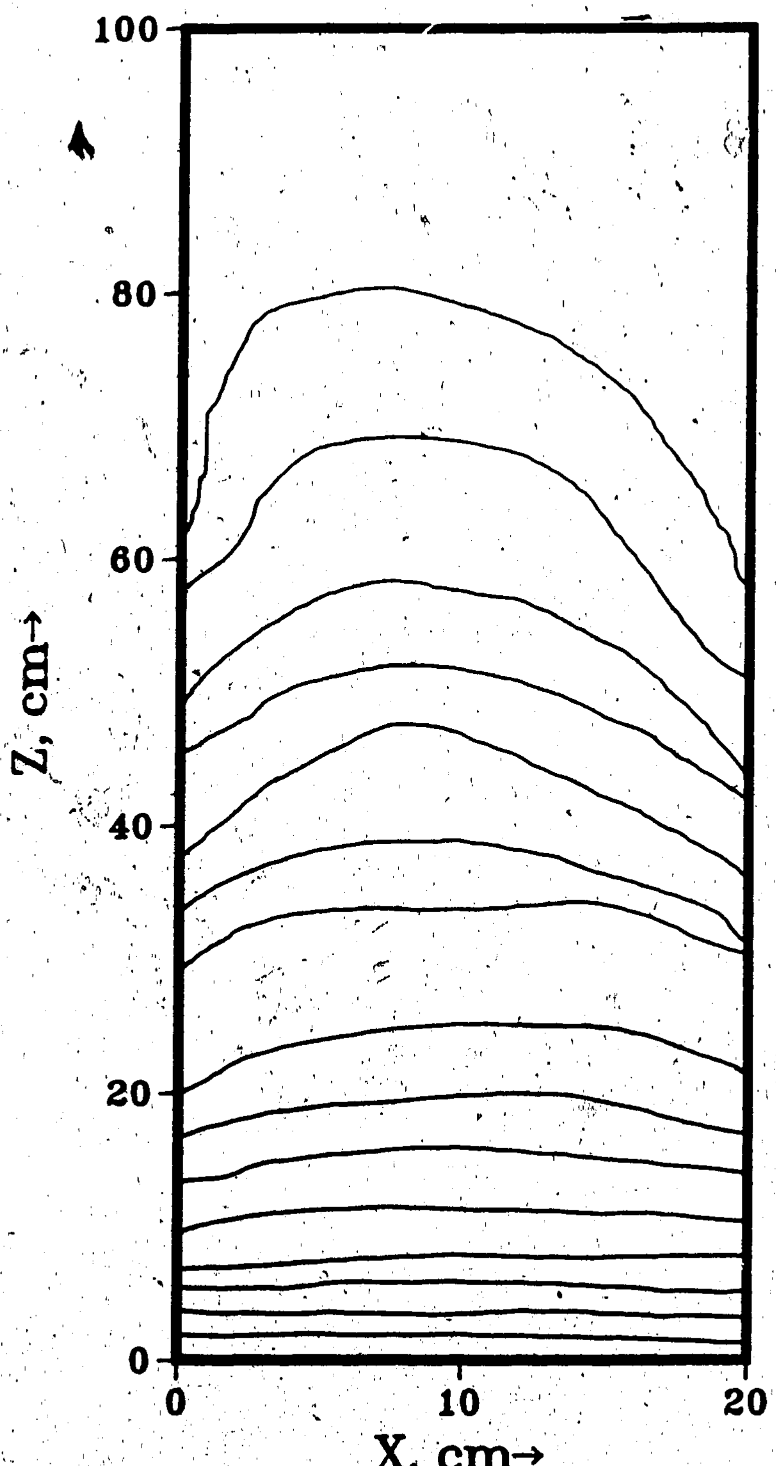

FIG. 6 0% SOLVENT ISOCONCENTRATION LINES AT DIFFERENT TIMES FOR RUN 4 (Isr = 53.43)

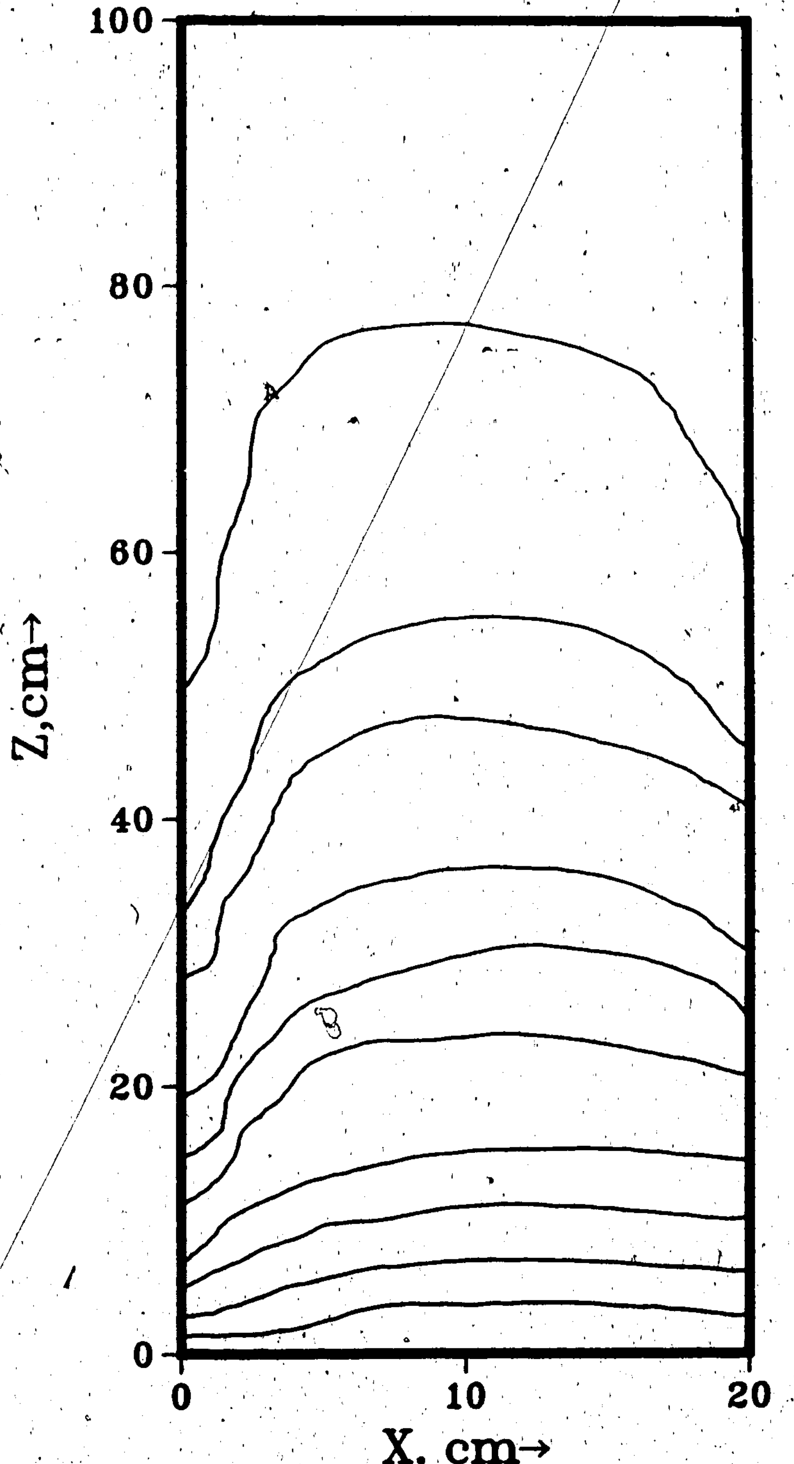

FIG.7 0% SOLVENT ISOCONCENTRATION LINES AT DIFFERENT TIMES FOR RUN 5 (Isr = 34.98)

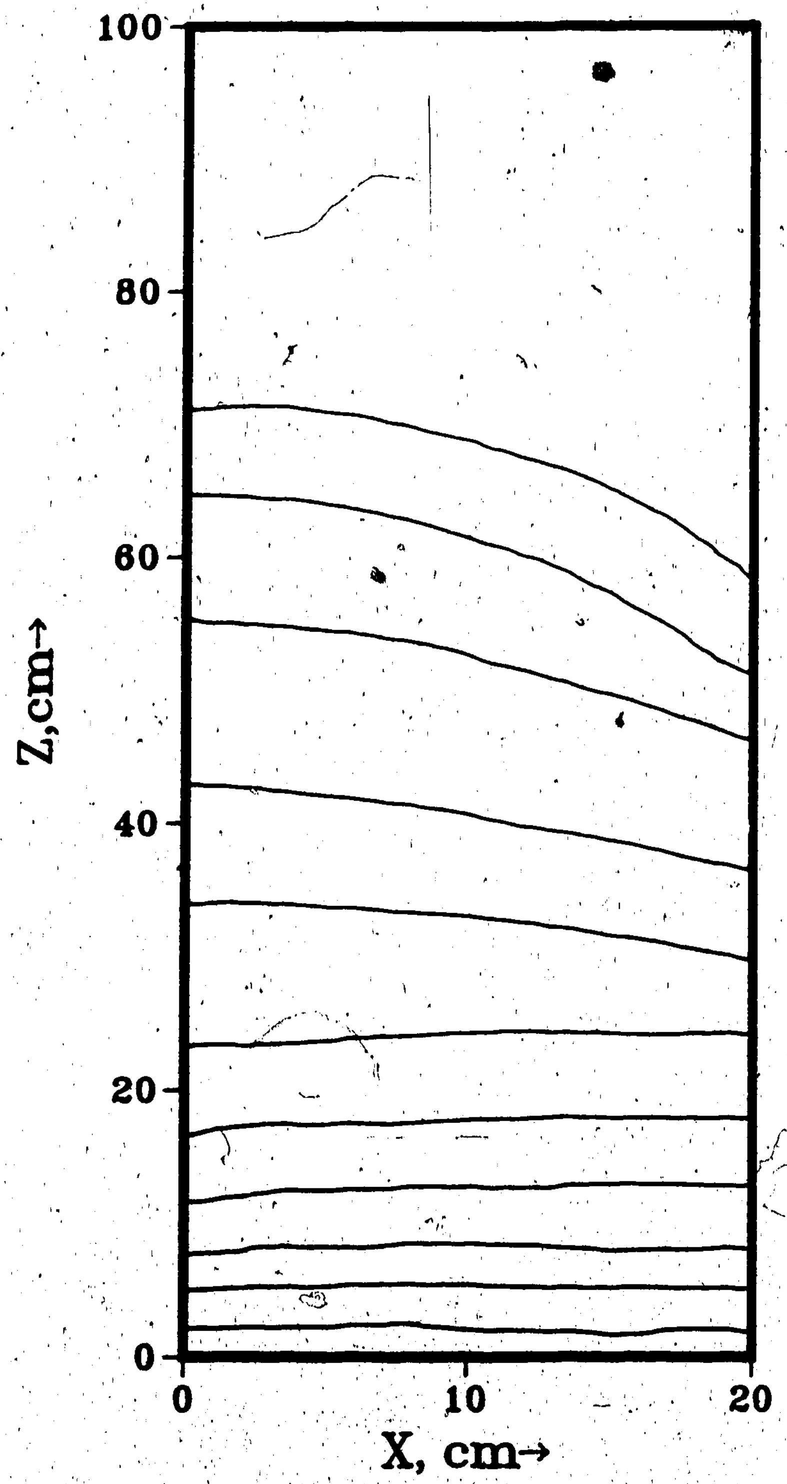

FIG. 8 0% SOLVENT ISOCONCENTRATION LINES AT DIFFERENT TIMES FOR RUN 9 (Isr = 699.7)

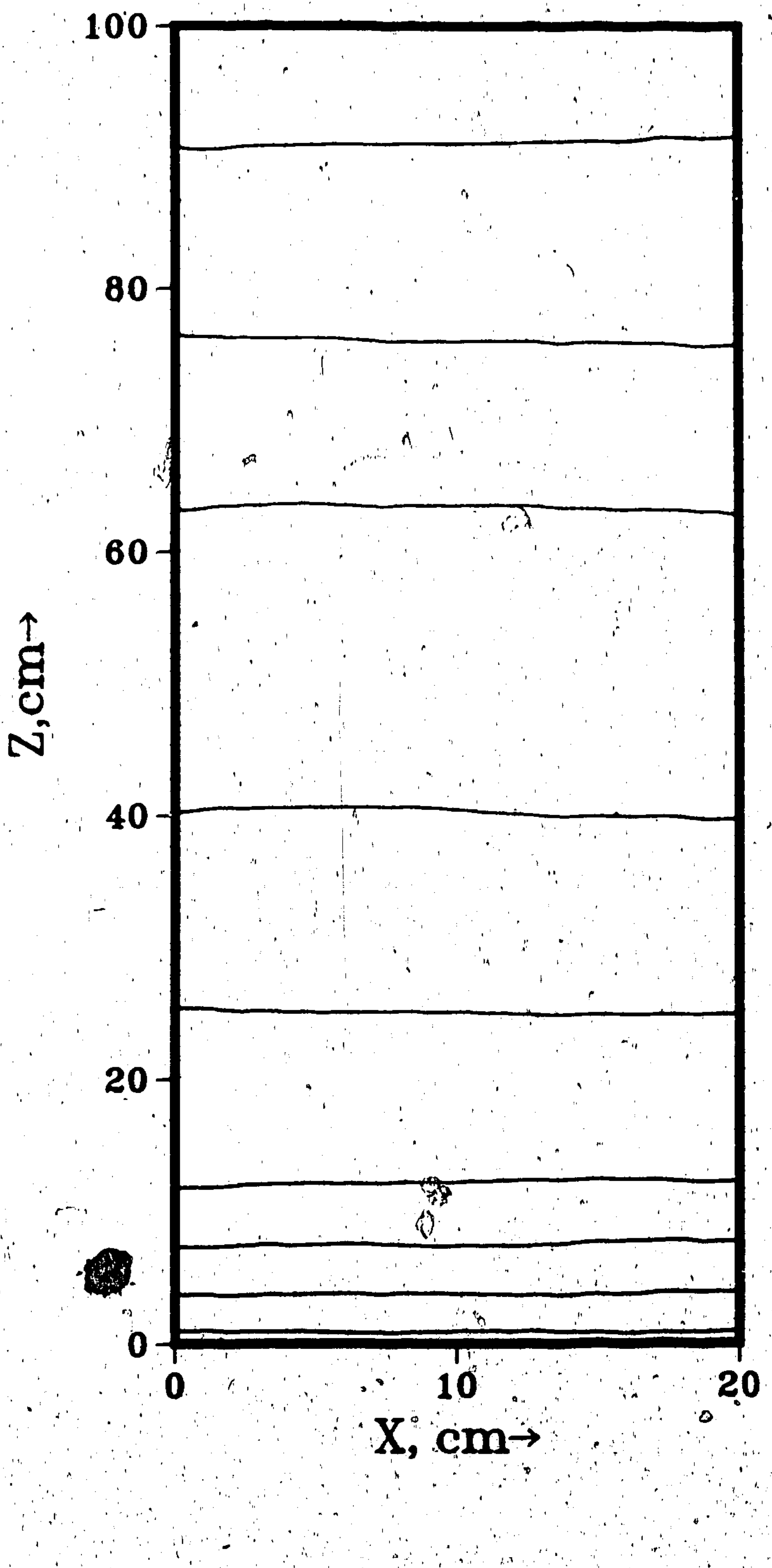

FIG.9 0% SOLVENT ISOCONCENTRATION LINES AT DIFFERENT TIMES FOR RUN 22 (lsr = 2.108)

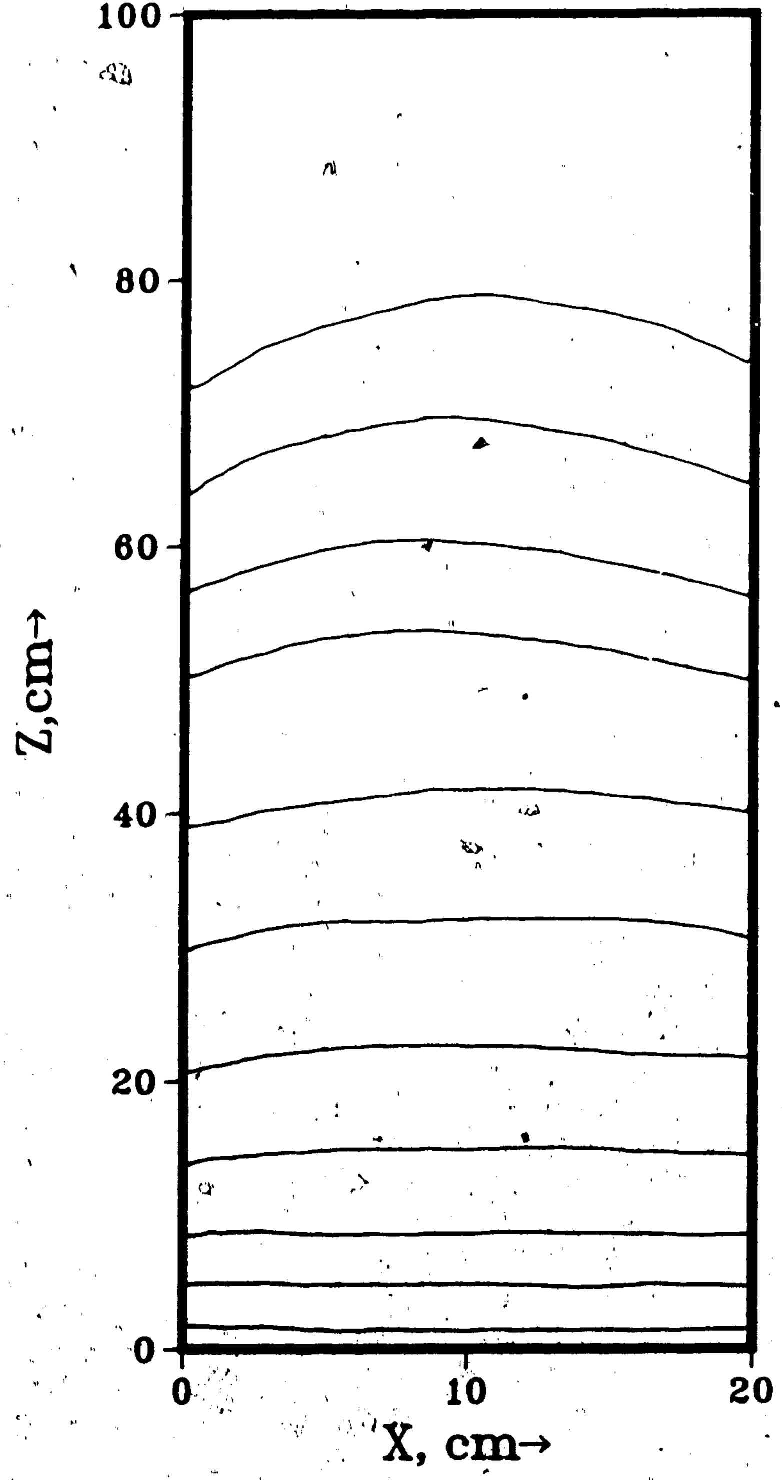

FIG. 10 0% SOLVENT ISOCONCENTRATION LINES AT DIFFERENT TIMES FOR RUN 33 (Isr = 4.199)

theory nor the variational theory can be used to interpret the behaviour of unstable displacements. Consequently, an attempt was made to interpret the behaviour of the unstable isoconcentration lines in the light of different dimensionless numbers which were thought to be possibly pertinent to miscible displacements. The values of these dimensionless groups for the experiments shown in Figures 6 to 10 and 12 to 16 are given in Table 3.

The first dimensionless number used was the instability number, I_{sr}. As the displacements with instability numbers above the critical instability number are unstable, it was thought that an increase in the instability number would have caused an increase in the amount of instability. However, when Runs 5 (Fig.7, I_{sr}=34.98), 9 (Fig.8, I_{sr}=699.7), 13 (Fig.12, I_{sr}=4030.) and 34 (Fig.16, I_{sr}=5.026) are compared, this does not seem to be the case. That is, even an increase of three orders of magnitude in the instability number does not cause the isoconcentration lines to become significantly more unstable. A possible explanation for this lack of correlation between the values of the instability number and the amount of instability is as follows: the configuration of the isoconcentration lines and their development in unstable displacements depends on the relationship between the frequency of the perturbations and the corresponding growth rate. As there is no relationship available between the frequency of the arbitrary perturbations introduced into each experiment and

TABLE 3
VALUES OF VARIOUS DIMENSIONLESS NUMBERS

Run No.	I_{sr}	Pe	N_f	Experimental Observation
4	53.43	392.8	13.34	Unstable
5	34.98	523.4	10.09	Unstable
9	699.7	70.66	74.36	Unstable
13	4030.	20.48	238.9	Unstable
22	2.108	315.2	16.79	Stable
23	25.23	557.8	5.329	Unstable
26	16.43	370.0	14.27	Unstable
27	1.703	313.9	16.83	Stable
33	4.119	920.3	5.742	Unstable
34	5.026	1225.	4.310	Unstable

the instability number for that experiment, the amount of instability can not be correlated with the instability number.

The second dimensionless group used was the Peclet number, Pe, defined as follows:

$$Pe = \frac{Vh}{D'} \tag{106}$$

The Peclet number is the ratio of convective to diffusive forces in the system'!'. Thus, it may be expected that the amount of instability in the system can be interpreted in terms of the Peclet number. It can be seen from Runs 4 (Fig.6, Pe=392.8) and 34 (Fig.16, Pe=1225.) that an increase in the Peclet number (hence, in convective forces) does not cause the the isoconcentration lines to become more distorted. It can be seen also from Runs 23 (Fig.13, Pe=557.8) and 33 (Fig.10, Pe=920.3) that, although the Peclet number increases by two fold, the amount of instability is practically the same. Therefore, the Peclet number, too, can not be used to interpret the unstable experiments. However, that such is the case is not surprising because the Peclet number is a measure of the amount of molecular diffusion with respect to the amount of convection. As a consequence, one might expect that it would dictate the smoothness of the finger surfaces (local perturbations). This can not be demonstrated using the isoconcentration plots due to smoothing and/or introduction of noise which took place during digitization of the traced profiles.

Finally, a dimensionless scaling group proposed by O[illegible]nga and van der Poel[37] was used for the purpose of interpreting the unstable displacements. This dimensionless group is the ratio of the time required to eliminate the fingers by transverse dispersion to the time required to produce a finger and is defined as follows:

$$N_f = \frac{L_x^2 \; V}{D_T \; L} \qquad (107)$$

It should be noted that the dimensionless number defined in Equation (107) is proportional to the Bodenstein number'', $Bo=Vh/D_T$. If this number is small, transverse dispersion is relatively important. As a consequence, the formation of a uniform transition zone is promoted. If it is large, dispersion will not play a significant role; and, consequently, an unstable displacement results''. Therefore, it might be concluded that larger values of N_f would cause the amount of instability to increase. However, this was not the case for the experiments conducted in this study. For instance, when Runs 4 (Fig.6, $N_f=13.34$) and 9 (Fig.8, $N_f=74.09$) are compared, it can be seen that an increase in N_f does not result in an increase in the amount of fingering. In addition, when Runs 13 (Fig.12, $N_f=238.9$) and 34 (Fig.16, $N_f=4.310$) are compared, it is seen that the amount of instability does not seem to be very different while the values of N_f differ by two orders of magnitude. Consequently, there is no correlation between the amount of instability and the dimensionless group N_f.

Hence, none of the three dimensionless numbers, I_{sr}, Pe and N_f, which often arise in the scaling of miscible displacements can be used to interpret the behaviour of the unstable displacements. Thus, it would appear that the structure of unstable isoconcentration lines is strongly dependent on the frequency of the initial perturbations. This may not be unexpected as the partial differential equations decribing the miscible displacement of one fluid by another in porous media are non-linear. In this regard, it is important to note that one of the characteristics of highly non-linear problems is that the dynamic history is very sensitive to small changes in the initial conditions[70]. Therefore, depending on the frequency and the amplitude of the arbitrary initial perturbations, every unstable displacement will probably exhibit a different configuration of the isoconcentration lines.

Because the process of dispersion and the amount of disorder differ considerably, it is to be expected that displacements conducted in a porous medium will be different from those conducted in a Hele-Shaw cell. Thus, how far one may extend the analogy between flow in a Hele-Shaw cell and that in a porous medium is uncertain[71]. However, the investigation of the unstable displacement problem in a Hele-Shaw cell is expected to be of help in understanding many of the general properties of unstable flows in finite systems. Moreover, a detailed analysis of experimentally realizable physical systems is necessary in order to understand interfacial instability in general[36].

The analysis of unstable displacement presented in this study does not account for the effects of reservoir heterogeneity. Therefore, the instability boundary derived from this theory corresponds to an optimistic value in an homogeneous reservoir. In natural porous media, it is quite possible to encounter large scale heterogeneities. As might be expected, heterogeneities make prevention of viscous fingering more difficult. Consequently, the effects of permeability stratification, gravity override, the presence of free gas in the reservoir, and the arrangement and spacing of wells must also be taken into account when such a theory is applied to field projects. Moreover, it should be noted that the parameter limits within which instabilites can be triggered by macroscopic perturbations to the system may be considerably broader. Within a certain range of parameters, a flow that may be stable to infitesimally small disturbances may be unstable to macroscopic perturbations. This distinction is important, as heterogeneities may introduce macroscopic perturbations into an otherwise homogeneous system''.

The theory developed herein is probably not valid if it is possible for a gravity tongue to form. This was not a problem in this stuy because of the small separation between the glass plates which formed the Hele-Shaw cell. However, when a miscible displacement is carried out in a dipping layer, a tilted plane interface between the solvent and the oil will occur. Under these conditions, a growing unstable gravity tongue is super-imposed upon the tilted plane interface''. When such is the case, the theory developed here may not be valid.

The theoretical analysis presented here studies the behaviour of small perturbations superimposed on a stable concentration profile. In general, in any experiment, the interfacial region has to experience a disturbance before the instability (or stability) of the displacement can be evaluated. As a consequence, in a perfectly homogeneous porous medium, a stable displacement may be obtained even though theoretically the displacement should be unstable, because the interfacial region was not perturbed. However, in the experiments conducted in this study, it is thought that perturbations were always introduced at the beginning of each experiment because of the existence of the "O" ring groove in the dam assembly. That is, the groove introduced small perturbations to the interface at the beginning of every experiment. Therefore, the true behaviour of the displacements could be evaluated properly for the experiments reported in Table 1.

Finally, knowledge of whether a displacement is stable or unstable is of critical importance to various practitioners of petroleum engineering. For example, it is of interest to the reservoir engineer because, if the displacement process is unstable, the recovery efficiency of the process will be significantly reduced, the amount depending primarily upon the viscosity ratio. Such information is also of interest to the reservoir modeller, in as much as, if the displacement is unstable, conventional methods of simulating miscible processes may not be appropriate because they are based on the conventional mass transport equations which are valid only for equal density, equal viscosity fluids. Moreover, it is important to realize that non-linear differential (or difference) equations, such as those used to describe unstable, miscible displacement, can have aperiodic solutions that appear to be random, even though no random quantities appear in the equations[10]. Thus, the usefulness of numerically simulating such systems becomes questionable because simple non-linearities may lead to chaotic solutions. While conventional wisdom holds that larger and faster super computers will enable engineers to make more precise predictions of system behaviour, the reality may turn out to be quite different. That is, for non-linear problems having the chaotic dynamics observed experimentally in this study, the time history is very sensitive to minor changes in the initial conditions and, consequently precise knowledge of the future can not be ascertained.

Whether a displacement process is stable or unstable is also important in laboratory work. That is, the interpretation of scaled model studies of proposed miscible enhanced oil recovery schemes may not apply to the prototype because the instability in the model is not properly scaled to that in the reservoir. In this regard, it is imperative that the flow system modelled in the laboratory fall as close as possible, on a stability plot, to that in the field. Thus, it would appear that knowledge of whether a given displacement is stable or unstable is of considerable importance to <u>both</u> the research scientist and the practising engineer. The theory developed herein enables the acquisition of such knowledge.

SUMMARY AND CONCLUSIONS

A dimensionless scaling group and its critical value at the onset of instability for miscible displacements in porous media has been derived. A linearized perturbation analysis was employed to obtain the perturbation equations which were, in turn, solved using a variational technique. The previous stability analyses were not able to include explicitly the effect of the length of the porous medium on the instability of miscible fluid displacements because of the necessity of assuming an infinitely long porous medium in order to obtain a solution. However, the use of the variational method enables one to introduce the appropriate boundary conditions at the inlet and the outlet of the porous medium. As a consequence, it was possible to explicitly deduce the effect of length on instability.

Furthermore, another stability criterion was obtained by modifying an existing theory by employing additional boundary conditions at the walls of the porous medium, and by including in the analysis both the longitudinal and the transverse dispersion coefficients. The modified theory implicitly assumed an infinitely long porous medium. It was demonstrated that, at the limit of an infinitely long porous medium, the variational theory yields the same stability criterion as does the modified theory.

A Hele-Shaw cell was used to carry out forty-four miscible displacement experiments in which the viscosity ratio, flow rate and flow angle were changed to validate the theory. Keeping in mind the limitations of the theory, and for the experiments conducted herein, the following conclusions may be drawn:

1. The marginal instability boundary was predicted correctly by the variational theory.
2. The experiments confirm the theoretical prediction that fingers are more readily formed in longer systems than in shorter ones, provided that the flow conditions and the transverse dimensions are similar.
3. The effect of M (change of natural logarithm of unperturbed viscosity with distance) on the onset of instability may be neglected for typical field displacements and in the laboratory.
4. The marginal instability boundary is a very strong function of length for shorter systems. However, it becomes independent of length once a certain length is reached.
5. If laboratory experiments are to be properly scaled, it is necessary that the instability number for both the prototype and the model fall close to each other on a stability plot.
6. It appears that the use of a graded viscosity bank to prevent viscous fingering in field displacements is feasible only for thin reservoirs and/or slightly unfavourable mobility ratios.

SUGGESTIONS FOR FURTHER STUDY

The following studies are suggested to extend the present study:

1. The theoretical basis for the stability analysis carried out in this study would be strengthened if it could be proven that the principle of exchange of instabilities is valid for miscible displacements in porous media.
2. As flow in the immediate vicinity of the wellbore is radial in nature, consideration should be given to undertaking a similar stability analysis for such systems.
3. Because the flow in a Hele-Shaw cell and a porous medium are not completely analogous, experiments to verify the theory should be carried out in actual porous media. In this regard, the following suggestions are offered:
   a. The character of the interface can be studied by utilizing a porous medium which consists of sand (or glass beads) packed in between two sheets of glass. This type of set up would allow visual observation, especially if the refractive indices of the glass beads and one of the fluids were matched.
   b. In a rectangular sand pack, the breakthrough recovery can be plotted versus the instability number to see when the displacement becomes unstable. Furthermore, the relationship between the breakthrough recovery and instability number can be investigated in the unstable region to see whether a

pseudostable region can be found.

c. Experiments similar to those carried out by Kyle and Perrine[48] can be carried out to check the validity of upper and lower bounds for graded vicosity banks.

d. Fluids with different transport properties can be used to further validate the theory.

4. There is no experimental data on the ratio of transverse to longitudinal dispersion in a Hele-Shaw cell. Some experimental work in this area is needed to predict the instability boundary accurately.

5. An attempt should be made to see whether the theory can be extended to include non-Newtonian fluids. This would be useful to determine the stability of polymer slugs chased by water.

6. An attempt should be made to see whether the force potential defined by Lusczynski[72] can be used to develop a theory capable of dealing with fingers subsequent to their initiation.

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APPENDIX 1

DERIVATION OF INSTABILITY BOUNDARY FOR A HELE-SHAW CELL

Consider the displacement of oil by solvent in a Hele-Shaw cell which makes an angle θ with the upward vertical (see Figure 11). The mathematical decription of the flow system is given by

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \tag{1.1}$$

$$\rho \frac{\partial u}{\partial t} + \frac{\partial P}{\partial x} + \frac{12\mu}{h^2} u = 0 \tag{1.2}$$

$$\rho \frac{\partial w}{\partial t} + \frac{\partial P}{\partial z} + \frac{12\mu}{h^2} w + \rho g \cos\theta = 0 \tag{1.3}$$

and

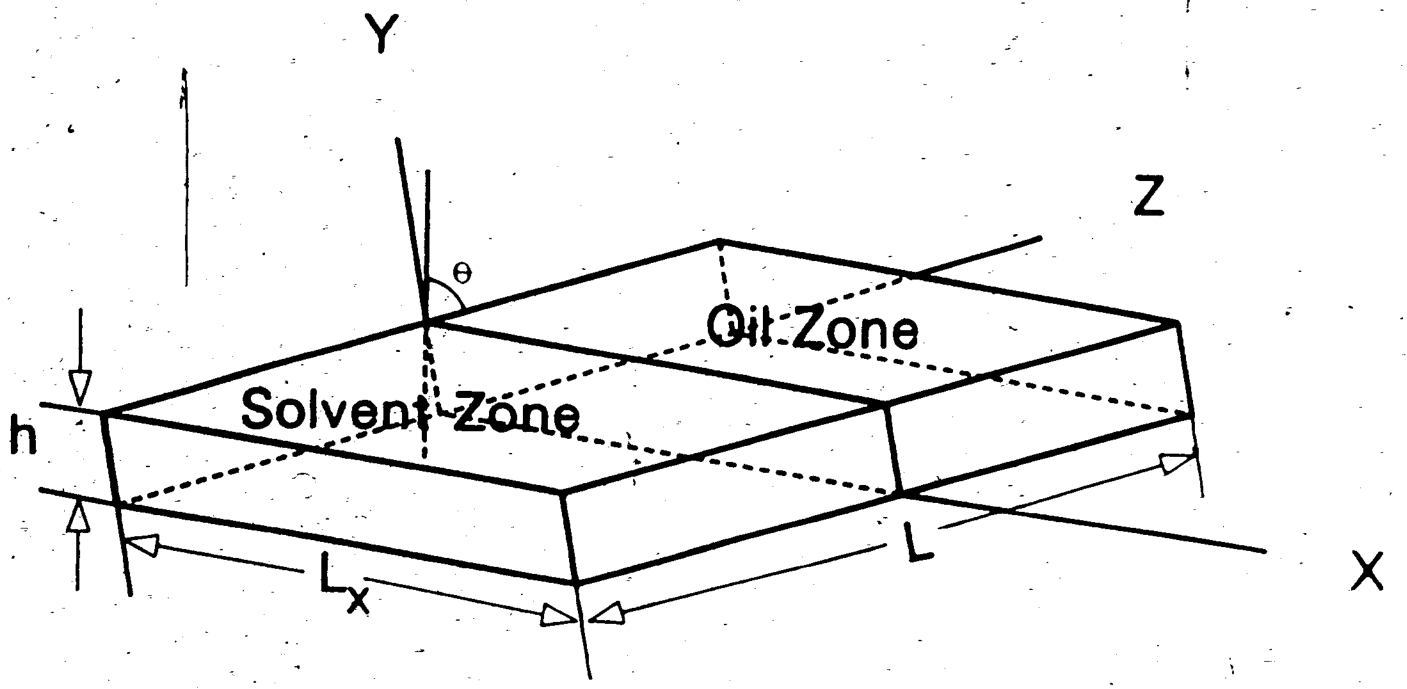

FIG.11 A HELE-SHAW CELL

$$\frac{\partial C}{\partial t} + u\,\frac{\partial C}{\partial x} + w\,\frac{\partial C}{\partial z'} - D_T\,\frac{\partial^2 C}{\partial x^2} - D_L\,\frac{\partial^2 C}{\partial z'^2} = 0 \qquad (1.4)$$

Next, it is assumed that the system is slightly perturbed so that the resulting dependent variables are given by

$$\xi = \bar{\xi} + \xi^* \qquad (1.5a)$$

where

$$\xi = C,\ u,\ w,\ P \qquad (1.5b)$$

The stable solution is defined by the following equations for the system under consideration

$$\bar{u} = 0 \qquad (1.6)$$

$$\bar{w} = V = \text{constant} \tag{1.7}$$

$$\frac{\partial \bar{P}}{\partial x} = 0 \tag{1.8}$$

$$\frac{\partial \bar{P}}{\partial z'} + \bar{\rho} g \operatorname{Cos}\theta + \frac{12\bar{\mu}}{h^2} V = 0 \tag{1.9}$$

and

$$\frac{\partial \bar{C}}{\partial t} + V \frac{\partial \bar{C}}{\partial z'} - D_L \frac{\partial^2 \bar{C}}{\partial z'^2} = 0 \tag{1.10}$$

In order to investigate the stability of this system, appropriate forms of Equations (1.5), together with Equations (1.6) to (1.10), are inserted into Equations (1.1) to (1.4). Then, if the non-linear terms are neglected in the resulting equations, the following set of linear partial differential equations, in the moving coordinates given by $z=z'-V/t$, are obtained

$$\frac{\partial u^*}{\partial x} + \frac{\partial w^*}{\partial z} = 0 \tag{1.11}$$

$$\bar{\rho}\,\frac{\partial u^*}{\partial t} + \frac{\partial P^*}{\partial x} + \frac{12\bar{\mu}}{h^2}\,u^* = 0 \tag{1.12}$$

$$\bar{\rho}\,\frac{\partial w^*}{\partial t} + \frac{\partial P^*}{\partial z} + C^*\left(\frac{12V}{h^2}\cdot\frac{d\mu}{dC} + \frac{d\rho}{dC}\,g\cos\theta\right) + \frac{12\bar{\mu}}{h^2}\,w^* = 0 \tag{1.13}$$

and

$$\frac{\partial C^*}{\partial t} + w^*\,\frac{\partial \bar{C}}{\partial z} - D_T\,\frac{\partial^2 C^*}{\partial x^2} - D_L\,\frac{\partial^2 C^*}{\partial z^2} = 0 \tag{1.14}$$

The general solution of these equations can be expressed by a Fourier series as the sum of any number of basic modes. Thus, a Fourier component of the disturbance is given by

$$\xi^* = \tilde{\xi}(z)\, e^{i\alpha x + \sigma t} \tag{1.15}$$

If the appropriate form of Equation (15) is inserted into the perturbation equations and if the perturbation variables $\tilde{u}(z)$ and $\tilde{P}(z)$ are eliminated from the resulting equations, the following two equations are obtained when the principle of exchange of instabilities is utilized :

$$(D^2 + MD - \alpha^2)\tilde{w} = N_c \alpha^2 \tilde{C} \tag{1.16}$$

and

$$(D^2 - K\alpha^2)\tilde{C} = \beta\, \tilde{w} \tag{1.17}$$

where

$$N_c = \frac{V}{\bar{\mu}} \frac{d\mu}{dC} + \frac{d\rho}{dC} \frac{g\mathrm{Cos}\theta h^2}{12\bar{\mu}} \tag{1.18}$$

$$K = \frac{D_T}{D_L} \tag{1.19}$$

$$M = \frac{d\ln\bar{\mu}}{dz} \tag{1.20}$$

and

$$\beta = \frac{1}{D_L} \frac{\partial \bar{C}}{\partial z} \tag{1.21}$$

The boundary conditions which apply at the inlet and the outlet of the Hele-Shaw cell are

$$\bar{C} = 0 \quad \text{at} \quad z = 0 \quad \text{and} \quad z = L \tag{1.22a}$$

$$\tilde{w} = 0 \quad \text{at} \quad z = 0 \quad \text{and} \quad z = L \qquad (1.22b)$$

The solution of Equations (1.16) and (1.17) which satisfy the boundary conditions (1.22a) and (1.22b) can be expanded in a Sine series in the following form :

$$\tilde{w}(z) = \sum_{m=1}^{\infty} A_m \operatorname{Sin} \frac{m\pi z}{L} \qquad (1.23)$$

By inserting Equations (1.23) into (1.17), the solution for $\tilde{C}(z)$ can be assumed to be

$$\tilde{C}(z) = \beta \sum_{m=1}^{\infty} A_m C_m(z) \qquad (1.24)$$

where $C_m(z)$ satisfies, from (1.17),

$$(D^2 - K\alpha^2)C_m = \text{Sin}\ \frac{m\pi z}{L} \tag{1.25}$$

with the boundary conditions

$$C_m = 0 \quad \text{at} \quad z = 0 \ \text{and} \ z = L \tag{1.26}$$

Substituting Equations (1.23) and (1.24) into Equation (1.16), and carrying out the differentiations yields

$$\sum_{m=1}^{\infty} (A_m\alpha_m \text{Sin}\ \frac{m\pi z}{L} - A_m\beta_m \text{Cos}\ \frac{m\pi z}{L}) = R^2\alpha^2 \sum_{m=1}^{\infty} A_m C_m \tag{1.27}$$

where

$$\alpha_m = \frac{m^2\pi^2}{L^2} + \alpha^2 \tag{1.28}$$

$$\beta_m = \frac{Mm\pi}{L} \tag{1.29}$$

and

$$-R^2 = N_C\beta = \frac{1}{D_L}\frac{\partial\bar{C}}{\partial z}\left(\frac{V}{\bar{\mu}}\frac{d\mu}{dC} + \frac{d\rho}{dC}\frac{gCos\theta h^2}{12\bar{\mu}}\right) \tag{1.30}$$

Next, Equation (1.27) is multiplied by $Sin(n\pi z/L)$ where $n=1,2,...,$ and integrated over the range of z. If this is done, it follows that

$$\sum_{m=1}^{\infty}\left[\frac{\delta_{mn}\alpha_m L}{2R^2\alpha^2} - \frac{2\beta_m nL}{\pi R^2\alpha^2(n^2-m^2)} - I_{mn}\right]A_m = 0 \tag{1.31}$$

where

$$I_{mn} = \int_0^L C_m Sin\,\frac{n\pi z}{L}\,dz \tag{1.32}$$

The second term in Equation (1.31) should be set equal to zero when m and n are both odd or even.

The non-trivial solution to Equation (1.31) is obtained, for the variational coeficients, A_m, if the determinant of the coefficient matrix vanishes. That is, the secular determinant is given by

$$\det. \left| \frac{\delta_{mn}\alpha_m L}{2R^2\alpha^2} - \frac{2\beta_m nL}{\pi R^2\alpha^2(n^2-m^2)} - I_{mn} \right| = 0 \qquad (1.33)$$

Next, Equation (1.25), subject to boundary conditions (1.26), is solved and the solution is inserted into Equation (1.32). If this is done, the final form for the secular determinant becomes

$$\det \left| \frac{\delta_{mn}\alpha_m L}{2R^2\alpha^2} - \frac{2\beta_m nL}{\pi R^2\alpha^2(n^2-m^2)} + \frac{\delta_{mn} L}{2\alpha_m^*} \right| = 0 \qquad (1.34)$$

where

$$\alpha_m^* = \frac{m^2\pi^2}{L^2} + K\alpha^2 \qquad (1.35)$$

A cut-off eigenvalue can be defined in terms of the dimensions of the Hele-Shaw cell. In order to do this, the Helmholtz eqation is utilized in connection with the boundary conditions that the normal derivative of ξ^* vanishes at the side walls. Therefore, the solution leads to

$$\alpha = \frac{m'\pi}{L_x} \; ; \; m' = 0, 1, 2, \ldots \qquad (1.36)$$

A first approximation to the exact condition of instability can be obtained by setting the first element of the secular determinant equal to zero. If this is done, the condition of marginal instability can be shown to be

$$-R^2 = \frac{1}{\alpha^2}\left(\frac{\pi^2}{L^2} + \alpha^2\right)\left(\frac{\pi^2}{L^2} + K\alpha^2\right) \qquad (1.37)$$

For an infinitely long porous medium, this condition degenarates to

$$-R^2 = K\alpha^2 \tag{1.38}$$

The condition of marginal instability is dictated by the minimum non-zero eigenvalue. Thus, combining Equations (1.30) and (1.36) (with m'=1) with Equation (1.37) leads to

$$-\frac{L_x^2}{\bar{\mu} D_L}\left(V\frac{d\mu}{dC} + \frac{h^2 g \cos\theta}{12}\frac{d\rho}{dC}\right)\left(\frac{\partial\bar{C}}{\partial z}\right) = \pi^2\left(\frac{1}{\Omega^2} + 1\right)\left(\frac{1}{\Omega^2} + K\right) \tag{1.39}$$

where the dimensionless length, Ω, is defined as

$$\Omega = \frac{L}{L_x} \tag{1.40}$$

A second approximation can be obtained by considering the top left 2x2 portion of the secular determinant and ignoring all the other terms. The solution of this determinant yields

$$-\frac{1}{\bar{\mu} D_L}\left(V\frac{d\mu}{dC} + \frac{h^2 g \cos\theta}{12}\frac{d\rho}{dC}\right)\left(\frac{\partial \bar{C}}{\partial z}\right) = \frac{B \mp \sqrt{B^2-4A}}{2\alpha^2} \qquad (1.41)$$

where

$$A = \left(\frac{\pi^2}{L^2} + \alpha^2\right)\left(\frac{\pi^2}{L^2} + K\alpha^2\right)\left(\frac{4\pi^2}{L^2} + \alpha^2\right)\left(\frac{4\pi^2}{L^2} + K\alpha^2\right) +$$

$$\left(\frac{64M^2}{9L^2}\right)\left(\frac{\pi^2}{L^2} + K\alpha^2\right)\left(\frac{4\pi^2}{L^2} + K\alpha^2\right) \qquad (1.42a)$$

$$B = \left(\frac{\pi^2}{L^2} + \alpha^2\right)\left(\frac{\pi^2}{L^2} + K\alpha^2\right) + \left(\frac{4\pi^2}{L^2} + \alpha^2\right)\left(\frac{4\pi^2}{L^2} + K\alpha^2\right) \qquad (1.42b)$$

and where α is defined by Equation (1.36) with $m'=1$. It should be noted that when the Hele-Shaw cell is infinitely long, Equation (1.41) degenerates to (1.38). Moreover, the solution with a minus sign, (see Equation (1.41)) (negative root), is equivalent to that obtained from the first approximation (Equation (1.39)), if M is negligible. The solution with a plus sign (positive root) gives a different boundary.

Finally, the unperturbed concentration profile can be averaged over the life span of the experiment to partially take into account the dampening effect of dispersion on the already developed fingers. If this approach is taken, it can be shown that

$$\frac{\partial \bar{C}}{\partial z} = -\sqrt{\frac{V}{\pi D_L L}} \qquad (1.43)$$

APPENDIX 2

CONDITIONS UNDER WHICH M CAN BE NEGLECTED

The change of unperturbed viscosity with distance, M, is defined by

$$M = \frac{d\ln\bar{\mu}}{dz} \tag{32}$$

and by using chain rule,

$$\frac{d\ln\bar{\mu}}{dz} = \frac{d\ln\mu}{dC}\bigg|_{\bar{C}} \frac{\partial\bar{C}}{\partial z} \tag{2.1}$$

Equation (69b) can be inserted into (2.1) to obtain

$$-\frac{d\ln\bar{\mu}}{dz} = \frac{d\ln\mu}{dC}\bigg|_{\bar{C}} \left(\sqrt{\frac{V}{\pi D_L L}}\right) \tag{2.2}$$

For a typical field displacement[19,20]:

$V=3.528\times10^{-6}$ m/s (1 ft/D)

$D_L = 20D_T = (20)(1\times10^{-8}) = 2\times10^{-7}$ m²/s

L=300 m (1000 ft)

L_x=10 m (30 ft)

L_y=200 m (660 ft)

Thus

$$\frac{\partial C}{\partial z} = -\ 0.137\ \mathrm{m}^{-1} \tag{2.3}$$

It is also assumed that the viscosity function given by Peaceman and Rachford[24] for the oil-gasoline system applies, i.e.

$$\ln\mu = x\ln\mu_s + (1-x)\ln\mu_o \tag{2.4}$$

where

μ_o=0.213 g/cm.s

μ_s=0.00539 g/cm.s

and where

$$x = \frac{C}{C + 0.52(1-C)} \tag{2.5}$$

Thus,

$$\frac{d\ln\mu}{dC} = (-3.677)\,\frac{0.52}{(0.48C+0.52)^2} \tag{2.6}$$

Therefore, from Equations (2.2), (2.3) and (2.6)

when $\bar{C}=0.80$, $d\ln\bar{\mu}/dz=0.320\ m^{-1}$,

when $\bar{C}=0.50$, $d\ln\bar{\mu}/dz=0.453\ m^{-1}$,

when $\bar{C}=0.10$, $d\ln\bar{\mu}/dz=0.811\ m^{-1}$,

Thus, in the field, assume that a typical maximum value for $d\ln\bar{\mu}/dz$ is given by

$$\frac{d\ln\bar{\mu}}{dz} = 1.0\ m^{-1} \tag{2.7}$$

The term containing M in the second approximation from Equation (101c) becomes, in SI units,

$$\frac{128M^2}{9L^2} = 1.580 \times 10^{-4}\ m^{-4} \tag{2.8}$$

and the other term becomes

$$\left(\frac{\pi^2}{L^2}+\gamma^2\right)\left(\frac{4\pi^2}{L^2}+\gamma^2\right) = 9.855 \times 10^{-3} \qquad (2.9)$$

Thus, the difference between the two terms is approximately three orders of magnitude.

For a typical laboratory displacement, if the following values apply:

$V=3.528 \times 10^{-6}$ m/s

$D_L=15D_T=1.5 \times 10^{-7}$ m²/s

$L=1$ m

$L_x=0.05$ m

$L_y=0.10$ m

then it may be shown that

$$\frac{128M^2}{9L^2} = 3.556 \times 10^4 \ m^{-4} \qquad (2.10)$$

$$\left(\frac{\pi^2}{L^2}+\gamma^2\right)\left(\frac{4\pi^2}{L^2}+\gamma^2\right) = 2.460 \times 10^7 \qquad (2.11)$$

Thus, in this case the difference is approximately three orders of magnitude.

APPENDIX 3

ADDITIONAL ISOCONCENTRATION PROFILES

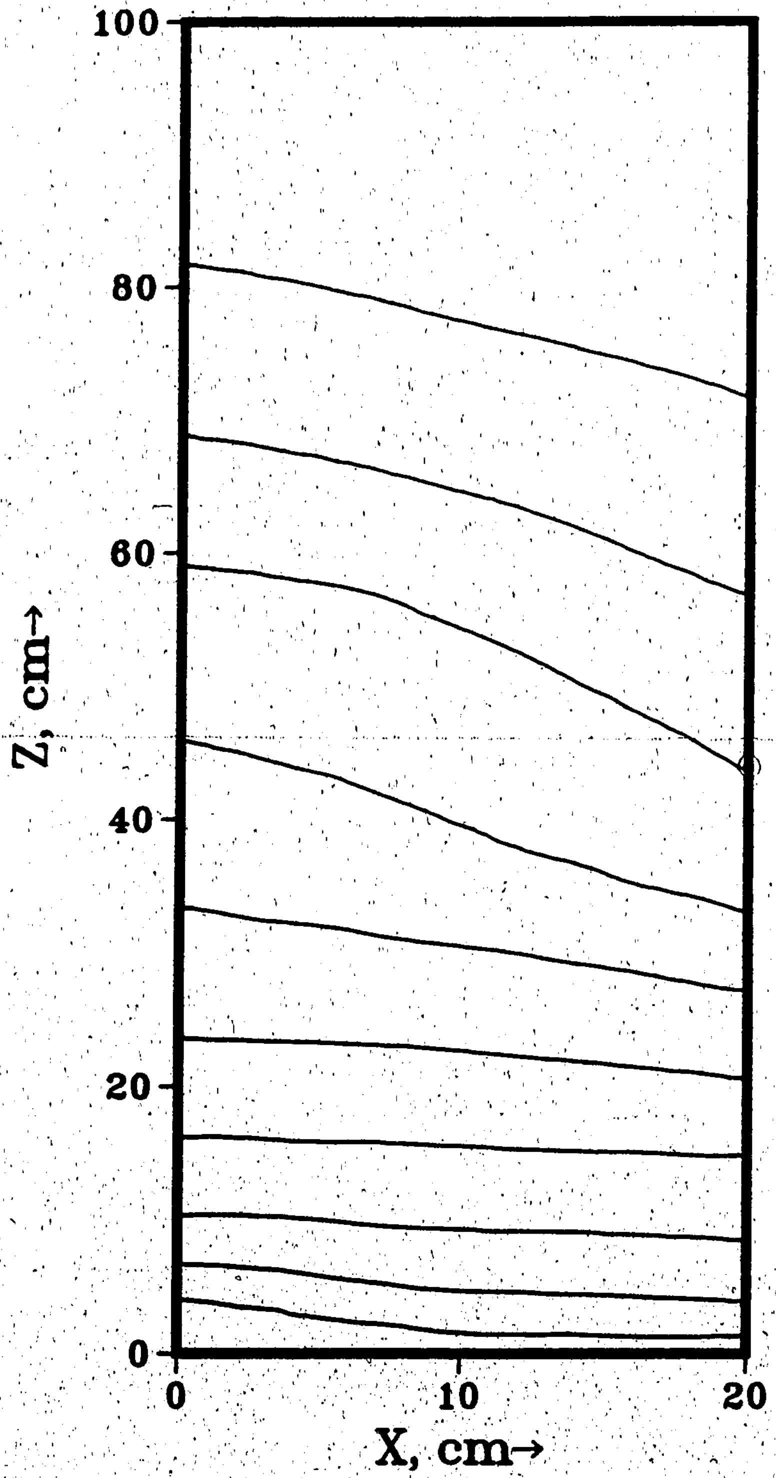

FIG. 12 0% SOLVENT ISOCONCENTRATION LINES AT DIFFERENT TIMES FOR RUN 13 (Isr = 4030.)

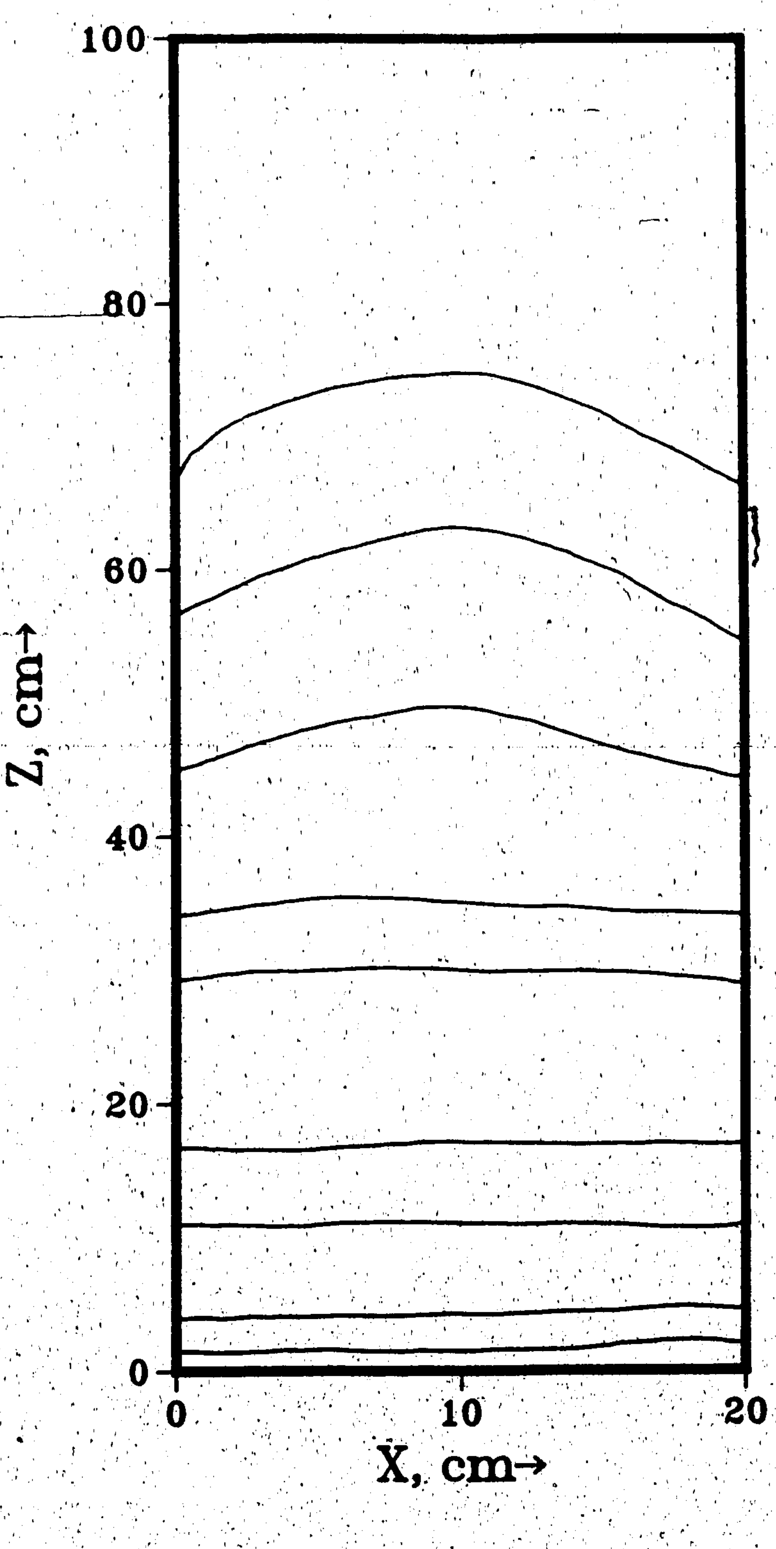

FIG. 13 0% SOLVENT ISOCONCENTRATION LINES AT DIFFERENT TIMES FOR RUN 23 (Isr = 25.23)

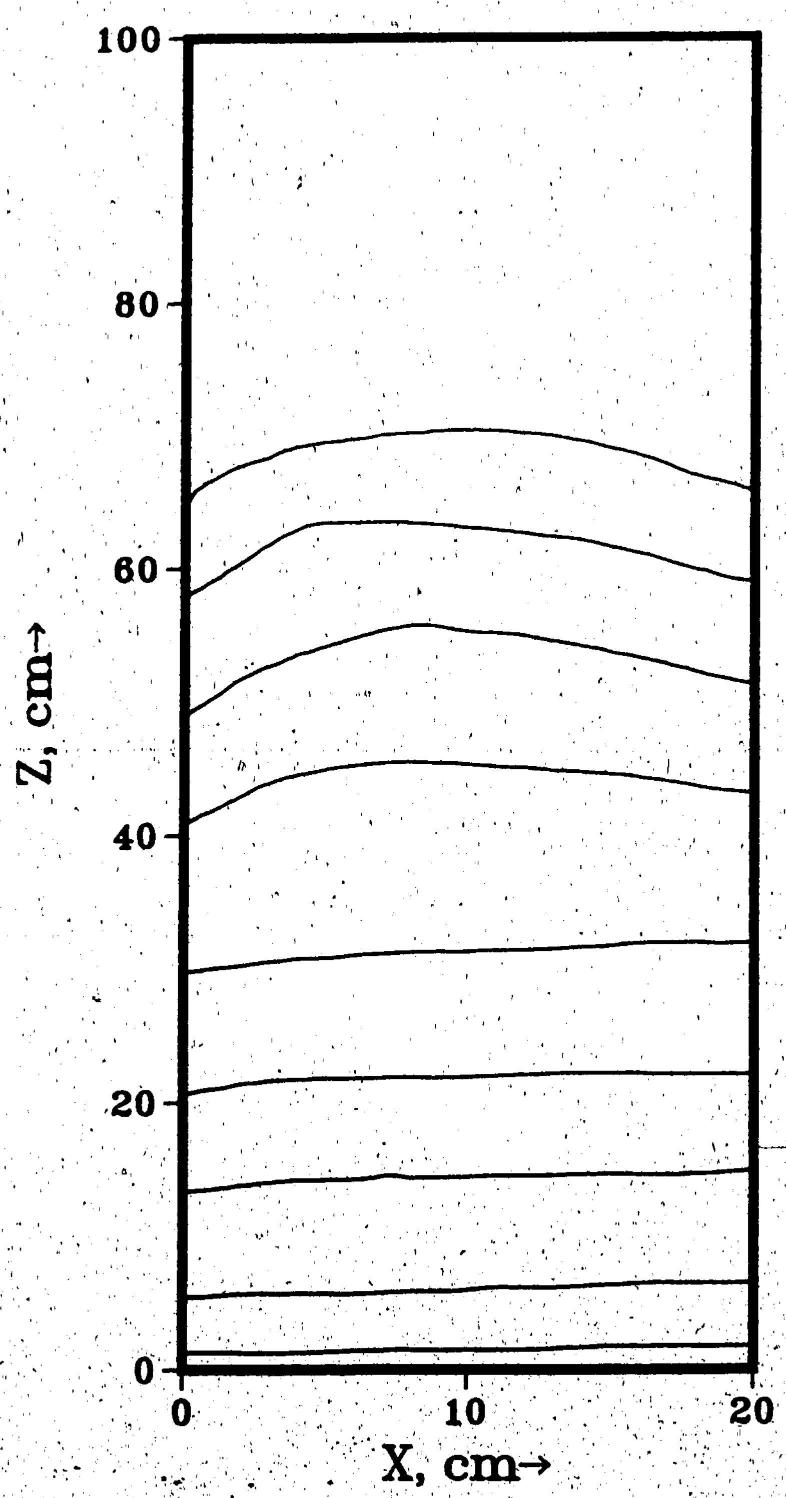

FIG. 14 0% SOLVENT ISOCONCENTRATION LINES AT DIFFERENT TIMES FOR RUN 26 (lsr = 16.43)

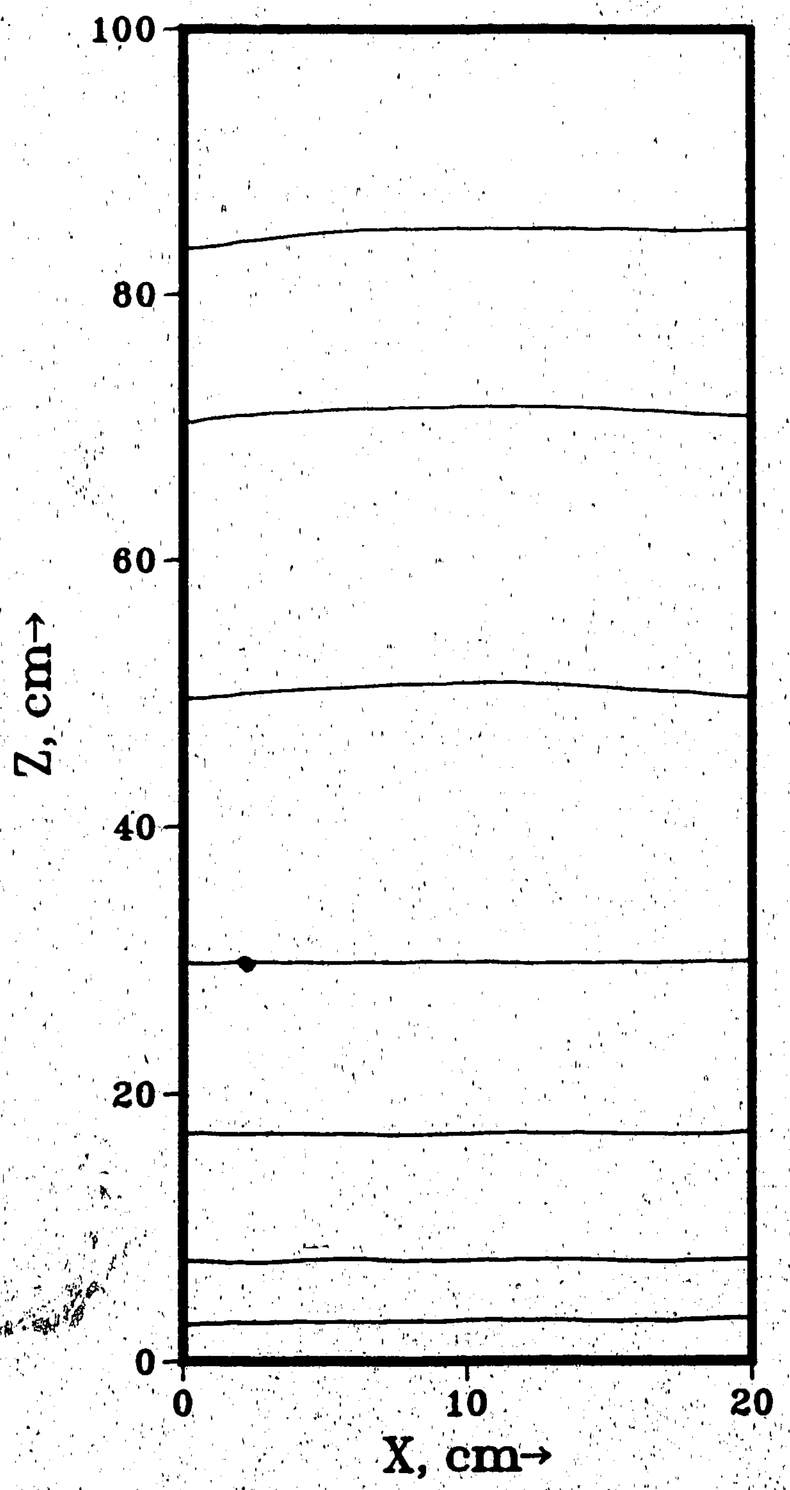

FIG. 15 0% SOLVENT ISOCONCENTRATION LINES
AT DIFFERENT TIMES FOR RUN 27 (ler = 1.703)

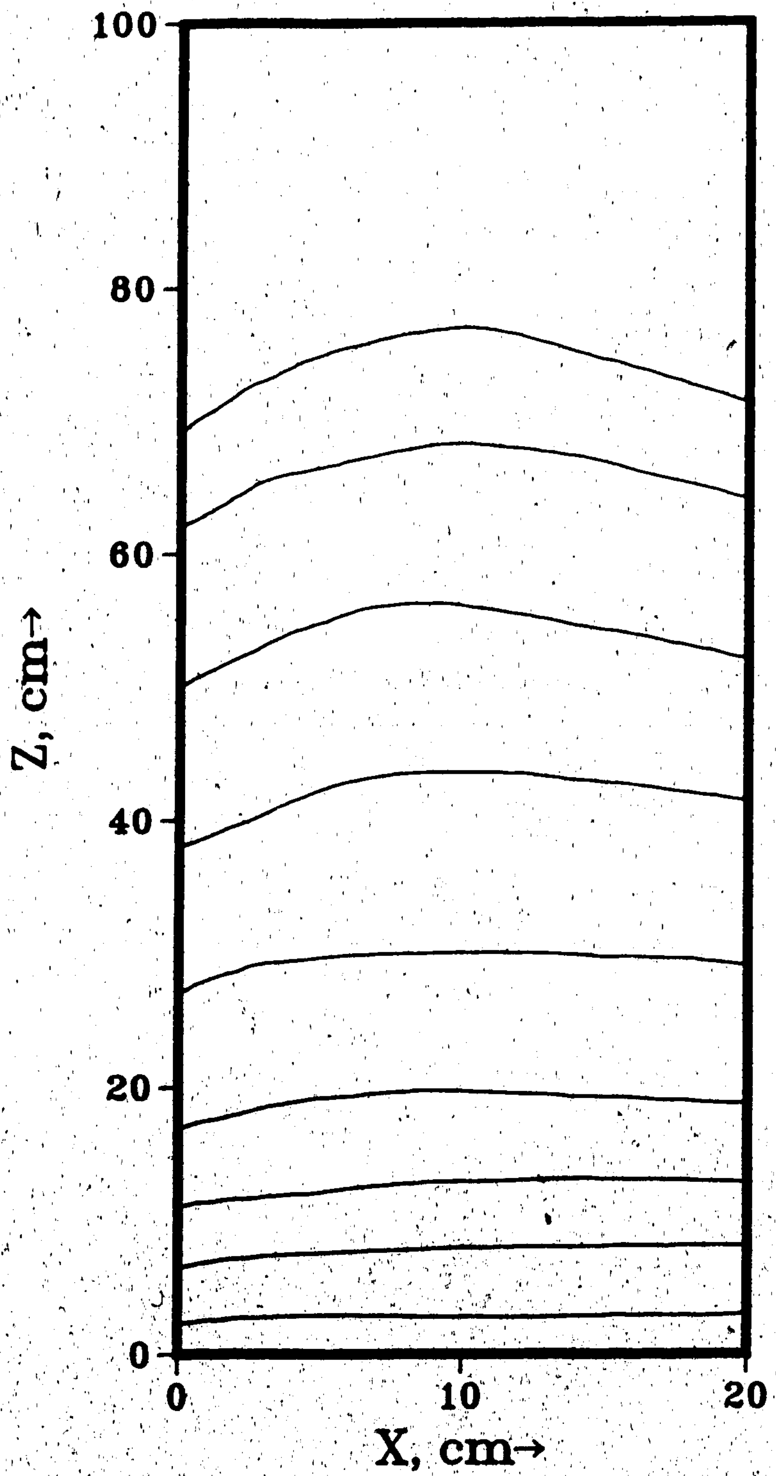

FIG. 16 0% SOLVENT ISOCONCENTRATION LINES AT DIFFERENT TIMES FOR RUN 34 (Isr = 5.026)