ESSAYS ON ARBITRAGE THEORY FOR A CLASS OF INFORMATIONAL MARKETS

by

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Abstract

This thesis develops three major essays on Arbitrage Theory, Market's Viability and Informational Markets. The first essay (Chapter 3) elaborates the exact connections among the No-Unbounded-Profit-with-Bounded-Risk (called NUPBR hereafter) condition, the existence of the numéraire portfolio, and market's weak/local viability. These tight relationships together with the financial crisis become our principal financial/economic leitmotif for the development of the next essay.

In the second essay (Chapter 4 – Chapter 6), we focus on quantifying with extreme precision the effect of some additional information/uncertainty on the non-arbitrage concepts. As a result, we describe the interplay of this extra information and the market's parameters for these non-arbitrage concepts to be preserved. Herein, we focus on the classical no-arbitrage and the NUPBR condition. This study contains two main parts. In the first part of this essay (Chapter 4), we analyze practical examples of market models and extra information/uncertainty, for which we construct explicit "classical" arbitrage opportunities generated by the extra information/uncertainty. These examples are built in Brownian filtration and in Poisson filtration as well. The second part (Chapters 5 and 6) addresses the NUPBR condition in two different directions. On the one hand, we describe the pairs of market model and random time for which the resulting informational market model fulfills the NUPBR condition. On the other hand, we characterize the random time models that preserve the NUPBR condition. These results are elaborated for general market models with special attention to practical models such as discrete-time and Lévy market models.

The last essay (Chapter 7) investigates the effect of additional information on the Structure Conditions. These conditions are the alternatives to the non-arbitrage and viability assumption in the Markowitz settings.

Preface

Some of the research conducted for this thesis forms part of an international research collaboration, led by Professor Tahir Choulli at the University of Alberta, with Professor Monique Jeanblanc and Dr. Anna Aksamit at Université d'Evry Val d'Essonne, France and Dr. Junfeng Ma at Bank of Montreal, Canada.

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Chapter 1

Introduction

Since their birth in the PhD thesis of Louis Bachelier (see [16]), mathematical finance and modern finance has known many successful topics that became pillars of both areas. Among these topics, we cite Portfolio Analysis that was pioneered by Merton [109, 110] and Markowitz [108], Arbitrage theory (Arrow and Debreu [13] and Duffie [56], Delbaen and Schachermayer [47, 50]), and Asset Pricing Theory and hedging rules (Black-Scholes [23] and Merton [110]).

1.1 Market's Viability

The market's viability was defined —up to our knowledge— by Harrison and Kreps in [69] (see also Kreps [101] and Jouini et al. [85]) as the setting for which there exists a risk-averse agent who prefers more to less, has continuous preference, and can find an optimal net trade subject to her/his budget constraint. In terms of the popular von Neumann-Morgenstern utility, the market's viability is "essentially" equivalent to the existence of the solution to utility maximization problems and is also related to the economic equilibrium and absence of arbitrage opportunities.

1.2 Arbitrage Theory

Arbitrage theory is a relative modern topic of Finance which becomes fundamental since it was directly linked to Asset Pricing Theory of Black-Scholes-Merton (see [23] and [110]), which was honored by the 1997 Nobel Price in Economics. Probably, the idea of non-arbitrage is best explained by telling a little joke from the book of Delbaen and Schachermayer [51]:

> *"A professor working in Mathematical Finance and a normal person go on a walk and the normal person sees a $100 bill lying on the street. When the normal person wants to pick it up, the professor says: do not try to do that. It is absolutely impossible that there is a $100 bill lying on the street. Indeed, if it were lying on the street, somebody else would have picked it up before you."*

Turning to financial markets, loosely speaking, in a rationale and reasonable market, there is no possibility that one could make profit without taking any risk. In other words, there is no financial strategy that makes profit out of nothing and without any risk (those financial strategies are called arbitrage opportunities). As the joke says, if there were some free snacks and cheap thrills in the financial market, somebody else would have taken the opportunities before you. In the history of the development of non-arbitrage theory, there are several competing definitions (vary in different market models) such as No-arbitrage (NA), No-Unbounded-Profit-with-Bounded-Risk (NUPBR), No-Free-Lunch (NFL), No-Free-Lunch-with-Bounded-Risk (NFLBR), No-Free-Lunch-with-Vanishing-Risk (NFLVR), Asymptotic-Arbitrage, Immediate-Arbitrage, etc..

To keep the story short, the casting from the economic meta-meaning of non-arbitrage into a rigorous mathematical framework goes back to the work of Harrison and Kreps [69] for finite discrete time markets and Harrison–Pliska [70] for complete continuous martingale settings. For their settings, they proved the equivalence between NA and the existence of equivalent martingale measures. Going beyond finite discrete time market model becomes more tricky and mathematically complicated. The key idea brought by Dalang–Morton–Willinger in [44] pushed the study of non-arbitrage theory to the modern mathematical finance by using the powerful tools of measurable selection theorem and Kreps-Yan's separation theorem. The next ground-breaking seminar papers of Delbaen and Schachermayer [47] and [50] established the golden principle of The Fundamental Theorem of Asset Pricing (FTAP, name given by Dybvig and Ross [58]). It was proved that there is *No-Free-Lunch-with-Vanish-Risk* (called NFLVR hereafter) if and only if there exists an *Equivalent Sigma Martingale Measure* (called ESMM hereafter).

The economic interpretation of FTAP is that under an equivalent change of our belief, the financial market is a fair game and evolves as a dynamic of a martingale. The pricing and valuation of contingent claim is brought back to its intrinsic value under this equivalent belief. Since its elaboration, the FTAP has known numerous applications in Finance and Economics including, but not limited to, optimal portfolio problem, market viability, market efficiency and risk management.

Optimal portfolio problem is a rather antique problem in Finance and Economics, where economic agents tend to maximize their benefits, gains and wealth. The 1990 and 1997 Nobel Prize winners in Economics, Harry M. Markowitz and Robert Merton, established the foundation of modern optimal portfolio in their seminar papers [108], [109] and [110]. Turning to mathematical finance field, Markowitz and Merton's framework had been extended, studied and enriched to more general settings using semimartingale theory, stochastic analysis and convex analysis. For those rich topics such as utility maximization, forward utility, and mean-variance hedging, we refer to Kramkov and Schachermayer [98, 99], Schachermayer [128], Delbaen et al.

[46], Frei and Schweizer [64, 65], Karatzas et al. [92, 91], Musiela and Zariphopoulou [112, 114, 113], Choulli and Ma [35], Ma [107], Choulli et al. [36] and the references therein.

There has been an upsurge interest in understanding the relationship among non-arbitrage, market's viability, and utility maximization problem since the time of Arrow and Debreu. In fact, for discrete markets (i.e. when there are only a finite number of trading times and scenarios), Arrow and Debreu proved in [13] (see also [56]), that the market's viability (i.e. utility maximization admits optimal solution for a "nice" von Neumann-Morgenstern utility), absence of arbitrage opportunities and the existence of an equivalent martingale measure are equivalent. These equivalences were extended to the discrete-time case (with infinity many scenarios). However, the question of how the existence of optimal portfolio is connected to the absence of arbitrage (weak or strong form) has been forgotten for the continuous-time context. Recently, Frittelli proposed in [67] an interesting approach for this issue, while his obtained results are not applicable in the context of [105]. Also Karatzas and Kardaras proved in [90] that the NUPBR condition is equivalent to the existence of the numéraire portfolio, and the NUPBR is the least condition for the utility maximization problem having a possible solution.

Thus, the **FIRST** theme of this thesis is to answer the following:

Prob(1.I): How is the market's viability (or utility maximization problem) related to non-arbitrage concepts in general continuous time semi-martingale models and without any assumtpion?

1.3 Information and Its Rôle in Finance and Economics

In Finance and Economics, information means the acquirement of knowledges about prices, costs, inventory, supply and demand of products, which can be exploited to by economic agents to reduce uncertainties in their environment (see Rose [122]). The key to the Economics of uncertainty appears for the first time —up to our knowledge– as Bernoulli's utility principle in 1738 ([19]). However this principle was ignored until 1947, when it was first made respectable and rigorously proved based on a few assumptions of how rational people make their decisions under uncertainty by von Neumann and Morgenstern in [138]. Thus, from the economic standpoint of view, information is a commodity that bears values; and economic agents desire information because it helps them to make decision and maximize their state-dependent utilities, especially when they are facing uncertainties. A good explanation was elaborated by Arrow as follows:

> *"There is a basic assumption about the nature of information contained in the economics of asymmetric information which I certainly wish to retain: that information is scarce to the individual, as well as to society as a whole. Asymmetric information arises because one party cannot obtain freely (or at all) information available to another."*

In 2001, the scientific community and the Nobel Prize Committee recognized the deep importance of the information in Economics and Finance by awarding George Akerlof, Michael Spence, and Joseph E. Stiglitz the 2001 Nobel Price in Economics for their contribution to the field of markets with asymmetric information (see [2], [132], [133], [135] and [68]). Asymmetric information means that one party has more or better information than the other when making decisions, investments and transactions that would lead to adverse selection, moral hazards, and market failure, especially in insurance markets. Adverse selection used in economics, insurance, risk management, and statistics refers to a market process in which undesired results occur when buyers and sellers have access to different information. Moral hazards is a situation where a party will have a tendency to take risks because the costs that could result will not be felt by the party taking the risk. All of these economic weaknesses have the potential to lead to market failure and affect the market's equilibrium (see Riley [119] and Rothschild and Stiglitz [124]). For the testing of asymmetric information in insurance markets, we refer to Abbring et al. [1], Chiappori and Salanie [28], Chiappori et al. [27], Cawley and Philipson [25] and the references therein. For more details about information and uncertainty and their rôles in Economics, we consult Allen [5] and Arrow [9, 10, 11, 12], Antonelli [8], Borch [24], Ross [122], Shackle [131], Stigler [134], Wolpert, S. A. and Wolpert, J. F. [139] and the references therein. For the rôle of information in other fields such as accounting, we refer the reader to Rothenberg [123], Hofmann and Rothenberg [73] and the references therein. In these latter works, the authors studied the relationship between the quality of information available for production decisions and performance evaluation to the choice of whether or not production should occur in teams.

The most common assumption, considered in Finance and Economics, consists of assuming that all the agents share the same information flow on which their portfolio decisions are based. However, this assumption is too restrictive and unrealistic in the real world. Asymmetric information arises quite naturally in Economics since individual's knowledge and availability of information varies from one agent to another. Many mathematical models have been developed to capture the behaviors of two agents with different information levels and are both price-taker and unable to influence the price dynamics of the risky assets. The uninformed (public) trader acts on the basis of the evolution of the market, while the insider trader possesses some additional knowledge and could probably make arbitrage.

Mathematically speaking, the uninformed agents are assumed to have the public information modelled by the filtration $(\mathcal{F}_t)_{t\geq 0}$, while the the extra information is possessed by the financial managers or insiders only. Examples of the extra information can be the IPO price, the possibility of default, the change in the CEO board. Throughout the thesis, the extra information will be modelled mathematically by a ***random time*** τ or an ***honest time***. This random time represents a retirement time, a death time, a default time of firms and/or agents, and any time where an event will occur and affect the markets and/or the agents' decisions. Usually, the random time τ is not adapted to the flow of public information $(\mathcal{F}_t)_{t\geq 0}$ since the availability of the extra information is not possible before its happening, like the Subprime mortgage crisis of 2008.

To incorporate the extra information to the public information $\mathbb{F} := (\mathcal{F}_t)_{t\geq 0}$, there are two main streams in the literature: ***initial enlargement*** and ***progressive enlargement*** that generate an enlarged filtration $\mathbb{G}$ (see Jeulin [83], Jeulin and Yor [84], Pardoux [116], Yoeurp [140], Jacod [77] and Meyer [111]). It turns out that the techniques of enlargement are fitting very well in studying credit risk and defaultable market (see [22], [82, 81], [60], [43], [20], [102] and the references therein).

The **SECOND** main theme of this thesis consists of finding how the non-arbitrage concepts (i.e. NA, NUPBR, NFLVR) would be affected for **informational markets**. Here, informational market means that the agents in the market possess different information levels that they could take advantage of to maximize their utilities, hedge financial outlays, and even make arbitrages. Precisely, we start with an arbitrage-free market (called initial market) and we add some extra information to this initial market. The resulting new market model is called informational market. We will investigate the following three problems for this informational market:

Prob(1.II): Are there any arbitrage opportunities in informational markets?

Prob(1.III): How are those non-arbitrage concepts affected when some extra information is partially/fully incorporated into the market?

Prob(1.IV): What are the necessary and sufficient conditions on the extra information such that the non-arbitrage conditions are preserved?

In this thesis, we focus on two weaker non-arbitrage concepts: ***No-Unbounded-Profit-with-Bounded-Risk*** (**NUPBR**) and ***Structure Conditions*** (**SC**) for three main reasons.

Firstly, for many market models, the NFLVR condition is too much to ask and does not hold in general, while the utility maximization problem might have a solution (see Pikovsky [117], Ruf [125] and the references therein). It turns out that the NUPBR condition is the right non-arbitrage concept that is intimately related to the weakest forms of markets' viability (see Choulli et al. [34] and Kardaras [96] for details about this issue) and is the necessary and sufficient condition for the local existence of the solution to the utility maximization problem.

Secondly, due to Takaoka [137] and Choulli et al. [34], the NUPBR condition possesses the 'dynamic/localization' feature that the NFLVR and others arbitrage concepts lack to possess. By localization feature, we mean that if the property holds locally (i.e. the property holds for the stopped models with a sequence of stopping times increasing to infinity), then it holds globally.

At last, since the seminal work of Markowitz on the optimal portfolio, the quadratic criterion for hedging contingent claims becomes a very popular and important topic in mathematical finance, modern finance, and insurance. In this context, two main competing quadratic approaches were suggested: The local risk minimization and the mean-variance hedging. For more details about these two methods and their relationship, we refer the reader to [72], [26], [57], [49], [21], [80], [129, 130], [41],[103] and the references therein. One important common feature for these methods lies in the assumptions that the market model should fulfill in order that the two methods admit solutions at least locally. These conditions are known as the **Structure Conditions** and sound to be the alternative to non-arbitrage condition in this quadratic context. Indeed, for the case of continuous price processes, it was proved that these conditions are equivalent to NUPBR, or the existence of a local martingale deflator for the market model. For details about these equivalence, we refer the reader to Choulli and Stricker [39]. However, in the general case, the two concepts (i.e. SC and NUPBR) differ tremendously.

1.4 Thesis' Summary

This thesis is based on several research papers co-authored by the candidate during his PhD studies under the supervision of Prof. Tahir Choulli. They include Aksamit/Choulli/Deng/Jeanblanc [3, 4], Choulli/Aksamit/Deng/Jeanblanc [30], Choulli/Deng [31, 32, 33] and Choulli/Deng/Ma [34]. Beside the current chapter, the thesis contains five highly innovative chapters and one chapter for the preliminaries. We keep each chapter of this thesis as independent and self-contained as possible (some chapters do have intersections in notations for the convenience of reader). The organization of these six chapters is detailed in the following.

In Chapter 2, we review the main stochastic tools that will be used throughout the thesis. These tools are essentially from the general theory of stochastic calculus and the theory of enlargement of filtrations. In addition to these mathematical and statistical preliminaries, this chapter introduces the reader to the mathematical formulation of some economic/financial concepts that we will deal with throughout the thesis.

Chapter 3 answers completely **(Prob(1.I))** and closes the existing gaps in this research direction. Our result already draw the attention of many researchers and had been investigated and extended to the context of markets with transaction costs. Our innovative contribution can be described as follows. Firstly, we establish the equivalence among the NUPBR condition, the existence of the numéraire portfolio, and the existence of the solution to the utility maximization under an equivalent probability measure for any nice utility. Furthermore, the equivalent probability measure under which the optimal portfolio exists can be chosen as close to the real-world probability measure as we want (but might not be equal). Secondly, under some mild assumptions on the model and the utility, we prove the equivalence between NUPBR and the local existence of optimal portfolio. These results lead naturally to two new types of viability that we call **weak viability** and **local viability**.

In Chapter 4, we give a positive answer to **(Prob(1.II))**. Precisely, we prove that there exist classic arbitrage opportunities (i.e. NA) for a class of honest times, when the market is complete, in progressive enlargement of filtrations. Furthermore, in the case of Brownian filtration and Poisson filtration, we calculate explicitly the arbitrage strategies. The failure of the classic arbitrage condition leads us to investigate the effect of extra information on the NUPBR condition in full generality.

In Chapter 5, we give a complete and precise answer to **(Prob(1.III))** and **(Prob(1.IV))** for the NUPBR condition when the semi-martingale model S being stopped at a random time. For this non-arbitrage notion, we obtain two principal results. The first result lies in describing the pairs of initial market model and random time for which the resulting stopped model fulfills the NUPBR condition. The second main result characterizes the random time models that preserve the NUPBR property after stopping. Moreover, we construct explicit martingale/pricing densities (deflators) for some classes of local martingales when stopped at random time.

The first originality of Chapter 6 lies in introducing a class of honest times that have the potential to preserve the non-arbitrage concepts for the part after-default of a process. For our family of specific honest times, we give a partial answer to **(Prob(1.III))** and **(Prob(1.IV))**, when dealing with the NUPBR and the part "after-default" of the model. In this setting, we obtain two principal results in the same spirit of Chapter 5. The first result lies in describing the pairs of market model and random time for which the market model fulfills NUPBR condition after an honest time. The second main result characterizes the random time models that preserve the NUPBR condition for the part after an honest time.

In Chapter 7, we give a partial answer to **(Prob(1.III))** and **(Prob(1.IV))** for the so-called Structure Conditions when the extra information is added to the initial model progressively in time. Precisely, for a fixed market model, we prove that Structure Conditions is preserved under a mild condition and we give the necessary and sufficient condition on the random time for which Structure Conditions is preserved for any semi-martingale model.

Chapter 2

Notations and Preliminaries

In this chapter, we introduce notations, definitions of different concepts (mathematical, statistical or financial/economic), and their preliminary analysis. For all unexplained mathematical and statistical/stochastic/probabilistic terms, terminologies and techniques, we refer to Delbaen and Schachermayer [51], Jacod [76], Jacod and Shiryaev [78], Dellacherie and Meyer [54] and He et al [71].

This chapter contains five sections. In section 2.1, we define σ-martingale density. Section 2.2 recalls optional stochastic integral and some properties. In Section 2.4, we give three important non-arbitrage concepts. Section 2.5 introduces the framework of utility maximization problem. In the last section, we present the mathematical models for additional information.

Throughout the remaining chapters of the thesis, our mathematical and economic models start with a stochastic basis $(\Omega, \mathbb{A}, \mathbb{H} := (\mathcal{H}_t)_{t\geq 0}, \mathbb{P})$. Here, Ω is the sample space, $\mathbb{P}$ is the physical measure (economically means the subjective belief), and $\mathbb{H}$ is a filtration that satisfies the usual conditions of right continuity and completeness, and represents the flow of information through time.

Definition 2.1: A process is a family $X = (X_t)_{t\geq 0}$ of measurable mapping from Ω

into some set E. Usually, E is taken as $\mathbb{R}^d$.

If X is a process and T is a random time (i.e. a mapping $\Omega \to \mathbb{R}_+$), we define the stopped process at T, denoted by X^T, by

$$X_t^T := X_{T\wedge t}. \tag{2.1}$$

Definition 2.2: An $\mathbb{H}$-stopping time is a random time such that for all $t \in \mathbb{R}_+$, the set $\{T \leq t\}$ is $\mathcal{H}_t$-measurable.

On the set $\Omega \times \mathbb{R}^+$, we define two σ-fields $\mathcal{O}(\mathbb{H})$ and $\mathcal{P}(\mathbb{H})$ that are generated by all càdlàg $\mathbb{H}$-adapted processes, and all continuous $\mathbb{H}$-adapted processes respectively. If S and T are two stopping times, we define four kinds of stochastic intervals:

$$\begin{aligned}
[\![S,T]\!] &:= \{(\omega,t):\ t\in\mathbb{R}_+,\ S(\omega)\le t\le T(\omega)\},\\
[\![S,T[\![&:= \{(\omega,t):\ t\in\mathbb{R}_+,\ S(\omega)\le t< T(\omega)\},\\
]\!]S,T]\!] &:= \{(\omega,t):\ t\in\mathbb{R}_+,\ S(\omega)< t\le T(\omega)\},\\
]\!]S,T[\![&:= \{(\omega,t):\ t\in\mathbb{R}_+,\ S(\omega)< t< T(\omega)\}.
\end{aligned}$$

Definition 2.3: An $\mathbb{H}$-predictable stopping time is an $\mathbb{H}$-stopping time such that the stochastic interval $[\![0,\tau[\![$ is $\mathbb{H}$-predictable.

Proposition 2.1: *Let T be an $\mathbb{H}$-stopping time, which is the debut $T(\omega) = \inf\{t : (\omega,t) \in A\}$ of an $\mathbb{H}$-predictable set A. If $[\![T]\!] \subset A$, then T is an $\mathbb{H}$-predictable time.*

The next concept is a sort of counterpart of predictable stopping time.

Definition 2.4: An $\mathbb{H}$-stopping time T is called $\mathbb{H}$-totally inaccessible if $P(T = S < +\infty) = 0$ for all $\mathbb{H}$-predictable times S.

The following theorem shows that, loosely speaking, any stopping time can be decomposed into two parts: totally inaccessible part and accessible part.

Theorem 2.1: *Let T be an $\mathbb{H}$-stopping time. There exist a sequence of $\mathbb{H}$-predictable stopping times $(S_n)_{n\geq 1}$ and a unique (up to $\mathbb{P}$-null set) $\mathcal{H}_T$-measurable subset A on $\{T < +\infty\}$, such that the stopping time T_A is totally inaccessible, and that the stopping time T_{A^c} satisfies $[\![T_{A^c}]\!] \subset \bigcup [\![S_n]\!]$. T_A is called the $\mathbb{H}$-totally inaccessible part of T, and T_{A^c} its $\mathbb{H}$-accessible part. They are unique, up to a $\mathbb{P}$-null set.*

Proof. We refer to Dellacherie [52] (Page 58, Théorèm 41) or Jacod and Shiryaev [78] (Page 20, Theorem 2.22) for the proof. □

Definition 2.5: A càdlàg $\mathbb{H}$-adapted process X is called quasi-left-continuous if $\Delta X_T = 0$ a.s. on the set $\{T < +\infty\}$ for every $\mathbb{H}$-predictable time T.

Definition 2.6: A random set A is called $\mathbb{H}$-thin if it is of the form $A = \bigcup_{n\geq 1} [\![T_n]\!]$, where $(T_n)_{n\geq 1}$ is a sequence of $\mathbb{H}$-stopping times; if the sequence $(T_n)_{n\geq 1}$ satisfies $[\![T_n]\!] \cap [\![T_m]\!] = \emptyset$ for all $n \neq m$, it is called the exhausting sequence of A.

Now, we give the definitions of martingale, sub-martingale and super-martingale that are essentially due to Doob [55].

Definition 2.7: An $\mathbb{H}$-adapted càdlàg process X on the stochastic basis $(\Omega, \mathbb{A}, \mathbb{H}, P)$ is called an $\mathbb{H}$-*martingale* (resp. *sub-martingale*, resp. *super-martingale*) if $E|X_t| < +\infty$ and for all $s \leq t$,

$$E[X_t|\mathcal{H}_s] = X_s, \text{ (resp. } E[X_t|\mathcal{H}_s] \geq X_s, \text{ resp. } E[X_t|\mathcal{H}_s] \leq X_s). \qquad (2.2)$$

The set of martingales for a filtration $\mathbb{H}$ will be denoted by $\mathcal{M}(\mathbb{H})$. As usual, $\mathcal{A}^+(\mathbb{H})$ denotes the set of increasing, right-continuous, $\mathbb{H}$-adapted and integrable processes. If $\mathcal{C}(\mathbb{H})$ is a class of processes for the filtration $\mathbb{H}$, we denote by $\mathcal{C}_0(\mathbb{H})$ the set of processes $X \in \mathcal{C}(\mathbb{H})$ with $X_0 = 0$, and by $\mathcal{C}_{loc}$ the set of processes X such that there exists a sequence of $\mathbb{H}$-stopping times, $(T_n)_{n\geq 1}$, that increases to $+\infty$ and the stopped process X^{T_n} belongs to $\mathcal{C}(\mathbb{H})$. We put $\mathcal{C}_{0,loc} = \mathcal{C}_0 \cap \mathcal{C}_{loc}$.

Definition 2.8: An $\mathbb{H}$-*semimartingale* is a càdlàg $\mathbb{H}$-adapted process X of the form $X = X_0 + M + A$, where X_0 is a finite-valued and $\mathcal{H}_0$-measurable random variable, M is an $\mathbb{H}$-local martingale and A is a finite variation process.

If A is predictable, we call X a *special semimartingale* and the decomposition $X = X_0 + M + A$ is called the *canonical decomposition* of X.

For a generic filtration $\mathbb{H}$ and an $\mathbb{H}$-semimartingale X, the set of $\mathbb{H}$-predictable processes H integrable with respect to X in the sense of semimartingale would be denoted by $L(X, \mathbb{H})$ and the stochastic integral as $H \bullet X_t = \int_0^t H_u dX_u$.

Below, we define the optional and predictable projection of a measurable process endowed with suitable integrability properties. For their proof, we refer to Dellacherie and Meyer [54].

Theorem 2.2: *Let X be a positive or bounded $\mathbb{A}$-measurable process. There exist an $\mathbb{H}$-optional process ${}^{o,\mathbb{H}}(X)$ and an $\mathbb{H}$-predictable process ${}^{p,\mathbb{H}}(X)$ such that*

$$\begin{aligned} E\left[X_T I_{\{T<+\infty\}} | \mathcal{H}_T\right] &= {}^{o,\mathbb{H}}(X)_T I_{\{T<+\infty\}} \text{ a.s. for any } \mathbb{H}\text{-stopping time } T, \\ E\left[X_T I_{\{T<+\infty\}} | \mathcal{H}_{T-}\right] &= {}^{p,\mathbb{H}}(X)_T I_{\{T<+\infty\}} \text{ a.s. for any } \mathbb{H}\text{-predictable time } T. \end{aligned}$$

The two processes ${}^{o,\mathbb{H}}(X)$ and ${}^{p,\mathbb{H}}(X)$ are unique up to evanescent set; and they are called the $\mathbb{H}$-optional projection and $\mathbb{H}$-predictable projection of X respectively.

Now, we state another very important concept – dual predictable projection or compensator – that would be used frequently in the thesis.

Theorem 2.3: *Let A be a process in $\mathcal{A}^+_{loc}(\mathbb{H})$. There exists a process $A^{p,\mathbb{H}}$, which is unique up to an evanescent set, and is an $\mathbb{H}$-predictable process in $\mathcal{A}^+_{loc}(\mathbb{H})$ satisfying one of the following three equivalent properties:*

(a) $A - A^{p,\mathbb{H}}$ *is an $\mathbb{H}$-local martingale.*

(b) $E(A^{p,\mathbb{H}}_T) = E(A_T)$ *for all $\mathbb{H}$-stopping times T.*

(c) $E[H \centerdot A^{p,\mathbb{H}}_\infty] = E[H \centerdot A_\infty]$ *for all nonnegative* $\mathbb{H}$*-predictable process* H.

The process $A^{p,\mathbb{H}}$ *is called the dual* $\mathbb{H}$*-predictable projection or compensator of* A.

The concept of dual predictable projection (or compensator) would be frequently used throughout the thesis. To distinguish the effect of filtration and probability measure, we will denote $\langle ., . \rangle^{\mathbb{Q},\mathbb{H}}$ to specify the sharp bracket calculated under the pairs $(\mathbb{Q}, \mathbb{H})$, if confusion may rise. When $\mathbb{P} = \mathbb{Q}$, then we simply write $\langle ., . \rangle^{\mathbb{H}}$. Similarly for the dual projections, we use $(., .)^{p,\mathbb{Q},\mathbb{H}}$ to specify the compensator calculated under $(\mathbb{Q}, \mathbb{H})$, and when $\mathbb{P} = \mathbb{Q}$, we simply write $(., .)^{p,\mathbb{H}}$.

2.1 σ-martingale Densities

The definition of σ-martingale goes back to Chou [29] (see also Émery [61]) under the name of semi-martingales de la classe $(\sum_m)$. It results naturally when we integrate –in the semimartingale sense– an unbounded and predictable process with respect to a local martingale. We start by defining local martingale density for an $\mathbb{H}$-semimartingale X.

Definition 2.9: Consider an $\mathbb{H}$-semimartingale X and an $\mathbb{H}$-positive local martingale $L > 0$. We call L is the local martingale density for X if the product LX is an $\mathbb{H}$-local martingale.

Definition 2.10: An $\mathbb{H}$-semimartingale is called a σ-martingale if there exists a real-valued $\mathbb{H}$-predictable process ϕ such that $0 < \phi \leq 1$ and $\phi \centerdot X$ is an $\mathbb{H}$-local martingale.

Below, we define σ-martingale density for an $\mathbb{H}$-semimartingale X.

Definition 2.11: Consider an $\mathbb{H}$-semimartingale X and a positive $\mathbb{H}$-local martingale K (i.e. $K > 0$). The process K is called a σ-martingale density for X if the product KX is an $\mathbb{H}$ σ-martingale.

The difference and relationship between σ-martingale and local martingale was discussed by Ansel and Stricker [7].

Proposition 2.2 (Ansel-Stricker): *Let X be an $\mathbb{R}^d$-valued $\mathbb{H}$-local martingale and H be an $\mathbb{H}$-predictable, X integrable $\mathbb{R}^d$-valued integrand in the sense of semimartingale. Then $H \bullet X$ is an $\mathbb{H}$-local martingale if and only if there is an increasing sequence of $\mathbb{H}$-stopping times $(T_n)_{n\geq 1} \uparrow +\infty$ and a sequence of non-positive integrable random variables (θ_n), such that*

$$\Delta(H \bullet X)^{T_n} = H\Delta X^{T_n} \geq \theta_n. \tag{2.3}$$

Theorem 2.4 (Ansel-Stricker): *Let X be an $\mathbb{H}$-local martingale and $H \bullet X$ exists in the sense of semimartingale. If $H \bullet X$ is bounded from below with some constant a, then $H \bullet X$ is an $\mathbb{H}$-local martingale.*

2.2 Optional Stochastic Integral

Here, we recall *optional stochastic integral* (or *compensated stochastic integral*) that is one of our crucial techniques. We refer to Jacod [76] (Chapter III.4.b p. 106-109), also studied in Dellacherie and Meyer [54] (Chapter VIII.2 sections 32-35 p. 356-361) and He et al. [71] for the details. Below, we give the definition and some elementary properties of optional stochastic integral which are important for our benefits.

Definition 2.12: [[76], Definition (3.80)] Let N be an $\mathbb{H}$-local martingale with continuous martingale part N^c, H an $\mathbb{H}$-optional process, and $p \in [1, +\infty)$.

(a) The process H is said to be p-integrable with respect to N if

(a.1) The process ${}^{p,\mathbb{H}}H$ is N^c integrable and (a.2) The process

$$\left(\sum\left(H\Delta N - {}^{p,\mathbb{H}}(H\Delta N)\right)^2\right)^{p/2} \in \mathcal{A}^+_{loc}(\mathbb{H}). \tag{2.4}$$

The set of p-integrable processes with respect to N is denoted by ${}^oL^p_{loc}(N,\mathbb{H})$.

(b) For $H \in {}^oL^p_{loc}(N,\mathbb{H})$, the *optional stochastic integral* of H with respect to N is the unique local martingale, M, that is p-locally integrable and satisfies

$$M^c = {}^{p,\mathbb{H}}H \centerdot N^c \quad \text{and} \quad \Delta M = H\Delta N - {}^{p,\mathbb{H}}(H\Delta N). \tag{2.5}$$

It is denoted by $M = H \odot N$.

It is obvious that the optional stochastic integral is a generalization of the Itô's integral (i.e. stochastic integral with predictable integrands). The conditions, given in the definition, for the existence of optional stochastic integrals are the most general ones, but they are hard to be verified. The following theorem (see [71] and [54]) remedies these two defects to some extent.

Theorem 2.5: *Let M be an $\mathbb{H}$-local martingale and H be an $\mathbb{H}$-optional process. If $\sqrt{H^2 \centerdot [M]} \in \mathcal{A}^+_{loc}(\mathbb{H})$, then $H \odot M$ exists and it is the unique $\mathbb{H}$-local martingale L such that for every bounded $\mathbb{H}$-martingale N, $[L,M] - H \centerdot [M,N] \in \mathcal{M}_{loc}(\mathbb{H})$.*

Lemma 2.1: *Let H be an $\mathbb{H}$-optional process and N be an $\mathbb{H}$-local martingale. Then $H \in {}^oL^2_{loc}(N,\mathbb{H})$ and $H^2 \centerdot [N,N]$ has finite values if and only if ${}^{p,\mathbb{H}}(|H\Delta N|) < +\infty$, and $H^2 \centerdot [N,N] \in \mathcal{A}^+_{loc}(\mathbb{H})$.*

Proof. The proof of the lemma is implied in He et al. [71] (see Theorem 9.10 and the Remarks on pages 232-233). For completeness, we provide the details.

If $H^2 \centerdot [N,N] \in \mathcal{A}^+_{loc}(\mathbb{H})$, it is clear that $H^2 \centerdot [N^c,N^c]$ and $\sum H^2(\Delta N)^2$ are both $\mathbb{H}$-locally integrable. By Yor and Lepingle's inequality (see [142] and [104]) and the elementary inequality

$$\sum \left(H\Delta N - {}^{p,\mathbb{H}}(H\Delta N)\right)^2 \leq 2\sum (H\Delta N)^2 + 2\sum \left({}^{p,\mathbb{H}}(H\Delta N)\right)^2,$$

we deduce $\sum \left(H\Delta N - {}^{p,\mathbb{H}}(H\Delta N)\right)^2 \in \mathcal{A}^+_{loc}(\mathbb{H})$. Therefore H belongs to ${}^{o}L^2_{loc}(N, \mathbb{H})$.

Conversely, if $H \in {}^{o}L^2_{loc}(N, \mathbb{H})$ and $H^2 \centerdot [N, N]$ is finite, we have $H^2 \centerdot [N^c, N^c]$ and $\sum \left(H\Delta N - {}^{p,\mathbb{H}}(H\Delta N)\right)^2$ are both $\mathbb{H}$-locally integrable and $\sum H^2(\Delta N)^2$ is finite. Then, for any interval $\Gamma \in \mathbb{R}_+$, we derive

$$\begin{aligned}\sum I_\Gamma \left({}^{p,\mathbb{H}}(H\Delta N)\right)^2 &\leq 2\sum I_\Gamma \left(H\Delta N - {}^{p,\mathbb{H}}(H\Delta N)\right)^2 + 2\sum I_\Gamma (H\Delta N)^2 \\ &< +\infty.\end{aligned}$$

Therefore, $\sum \left({}^{p,\mathbb{H}}(H\Delta N)\right)^2$ is locally bounded. Hence, it is $\mathbb{H}$-locally integrable. Since $H^2 \centerdot [N, N] \leq H^2 \centerdot [N^c, N^c] + 2\sum \left(H\Delta N - {}^{p,\mathbb{H}}(H\Delta N)\right)^2 + 2\sum \left({}^{p,\mathbb{H}}(H\Delta N)\right)^2$, we conclude that $H^2 \centerdot [N, N]$ is $\mathbb{H}$-locally integrable. This ends the proof of the lemma. □

The optional stochastic integral arises naturally when jumps are incorporated. The following lemma gives two (maybe the most) commonly known examples.

Lemma 2.2: *Let X be an $\mathbb{H}$-semimartingale and M an $\mathbb{H}$-local martingale. Then the following properties hold:*

(a) *The optional integral $I_{\{\Delta M \neq 0\}} \odot M$ is well defined and $I_{\{\Delta M \neq 0\}} \odot M = M^d$.*

(b) *If $[X, M] \in \mathcal{A}_{loc}(\mathbb{H})$, then $(\Delta X) \odot M = [X, M] - \langle X, M \rangle$.*

Proof. (1) Since $I_{\{\Delta M \neq 0\}}\Delta M = \Delta M$ and ${}^{p,\mathbb{H}}(I_{\{\Delta M \neq 0\}}\Delta M) = 0$, the assertion (a) follows immediately from the definition of optional stochastic integral.
(2) The proof of assertion (b) becomes trivial if one notices that $\Delta X \Delta M - {}^{p,\mathbb{H}}(\Delta X \Delta M) = \Delta[X, M] - \Delta\langle X, M \rangle$ and $\{\Delta X \neq 0\}$ is a thin set. This ends the proof of the lemma. □

Below, we recall some important properties of the optional stochastic integral that would be used later.

Proposition 2.3: (a) *The optional stochastic integral* $M = H \odot N$ *is the unique* $\mathbb{H}$*-local martingale such that, for any* $\mathbb{H}$*-local martingale* Y*,*

$$E\left([M,Y]_\infty\right) = E\left(\int_0^\infty H_s d[N,Y]_s\right).$$

(b) *The process* $[M,Y] - H \centerdot [N,Y]$ *is an* $\mathbb{H}$*-local martingale.*

(c) *If* K *is locally bounded and predictable, then the following three stochastic integrals are well defined and are equal* $K \centerdot (H \odot X) = (KH) \odot X = H \odot (K \centerdot X)$.

(d) *If* X *is purely discontinuous, then* $H \odot X$ *is also purely discontinuous, for every process* $H \in {}^oL^p_{loc}(X, \mathbb{H})$.

We end this section by a lemma that would be used in last Chapter 7.

Lemma 2.3: *Let* M *and* N *be two* $\mathbb{H}$*-locally square integrable local martingale,* $H \in {}^oL^2_{loc}(N, \mathbb{H})$*, and* $K \in {}^oL^2_{loc}(M, \mathbb{H})$*. Then, we have*

$$\langle H \odot N, K \odot M\rangle^{\mathbb{H}} = \Big(HK \centerdot [M,N]\Big)^{p,\mathbb{H}} - \sum {}^{p,\mathbb{H}}\Big(H\Delta N\Big)\, {}^{p,\mathbb{H}}\Big(K\Delta M\Big).$$

Furthermore, if N *or* M *is quasi-left continuous, then*

$$\langle H \odot N, K \odot M\rangle^{\mathbb{H}} = \Big(HK \centerdot [M,N]\Big)^{p,\mathbb{H}}.$$

Proof. To this end, we calculate that

$$
\begin{aligned}
[H \odot N, K \odot M] &= \langle H \odot N^c, K \odot M^c \rangle + \sum \Delta (H \odot N) \Delta (K \odot M) \qquad (2.6)\\
&= HK \centerdot [N^c, M^c] + HK \centerdot [M, N]^d \\
&\quad - \sum \Big(H\Delta N - {}^{p,\mathbb{H}}(H\Delta N) \Big)\, {}^{p,\mathbb{H}}(K\Delta M) - \sum {}^{p,\mathbb{H}}(H\Delta N)\, {}^{p,\mathbb{H}}(K\Delta M) \\
&= HK \centerdot [N, M] - \sum \Big(H\Delta N - {}^{p,\mathbb{H}}(H\Delta N) \Big)\, {}^{p,\mathbb{H}}(K\Delta M) \\
&\quad - \sum {}^{p,\mathbb{H}}(H\Delta N)\, {}^{p,\mathbb{H}}(K\Delta M). \qquad (2.7)
\end{aligned}
$$

It is easy to see that ${}^{p,\mathbb{H}}(K\Delta M)$ is locally bounded and due to (2.5), we have $\sum \Big(H\Delta N - {}^{p,\mathbb{H}}(H\Delta N) \Big)\, {}^{p,\mathbb{H}}(K\Delta M) = {}^{p,\mathbb{H}}(K\Delta M)\, H \odot N$ is a local martingale. Therefore, we get

$$
\langle H \odot N, K \odot M \rangle^{\mathbb{H}} = \Big(HK \centerdot [M, N] \Big)^{p,\mathbb{H}} - \sum {}^{p,\mathbb{H}}\Big(H\Delta N \Big)\, {}^{p,\mathbb{H}}\Big(K\Delta M \Big).
$$

If N or M is quasi-left continuous, then ${}^{p,\mathbb{H}}\Big(H\Delta N \Big) = 0$ or ${}^{p,\mathbb{H}}\Big(K\Delta M \Big) = 0$. □

2.3 Jacod's Representation for Local Martingales

In this section, we will recall the modern theory of semimartingale. Although it is much less widely known than the classical semimartingale theory, it would be essential for our purpose. Most of the results presented in this section can be founded in Jacod [76], Jacod and Shiryaev [78] and He et al [71]. However the proofs are not always given in details. We refer to Choulli and Schweizer [37] for a detailed English proof.

Let us consider an auxiliary measurable space $(E, \mathcal{E})$ which we assume to be a *Blackwell space* (i.e., a *separable* space and for any $(E, \mathcal{E})$ measurable random variable ξ admits a *regular condition distribution* with respect to the sub-σ-filed $\mathcal{H}'$ of $\mathcal{H}$). Thoughout this thesis, $(E, \mathcal{E})$ would be $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d)$.

Definition 2.13: A random measure on $\mathbb{R}_+ \times E$ is a family $\mu = \mu(\omega, dt, dx), \omega \in \Omega$ of nonnegative measure on $(\mathbb{R}_+ \times E,\ \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{E})$ satisfying $\mu(\omega, \{0\} \times E) = 0$ identically.

Throughout the thesis, on the space $\left(\widetilde{\Omega} := \Omega \times \mathbb{R}_+ \times E, \widetilde{\mathbb{A}} := \mathbb{A} \otimes \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(E)\right)$, we will consider two σ-fields

$$\widetilde{\mathcal{O}}(\mathbb{H}) = \mathcal{O}(\mathbb{H}) \otimes \mathcal{E} \qquad \text{and} \qquad \widetilde{\mathcal{P}}(\mathbb{H}) = \mathcal{P}(\mathbb{H}) \otimes \mathcal{E}. \tag{2.8}$$

For an $\mathbb{H}$-adapted càdlàg process X, we denote the jump random measure associate with X by μ ($\mu_X^{\mathbb{H}}$ if confusion may arise), which is given by

$$\mu(\omega, dt, dx) := \sum_{s>0} I_{\{\Delta X_s(\omega) \neq 0\}} \delta_{(s, \Delta X_s(\omega))}(dt, dx), \tag{2.9}$$

where δ_a is the Dirac measure at the point a.

For an optional functional W on $\widetilde{\Omega}$, we define the integral process $W \star \mu$ by

$$W \star \mu_t(\omega) := \begin{cases} \int_{[0,t]\times E} W(\omega, s, x)\mu(\omega, ds, dx), & \text{if } \int_{[0,t]\times E} |W(\omega, s, x)|\mu(\omega, ds, dx) < +\infty \\ +\infty, & \text{otherwise.} \end{cases}$$

Another important and useful measure on $(\widetilde{\Omega}, \widetilde{\mathbb{A}})$ is given by

$$M_\mu^P(\widetilde{B}) := E^P\left[\int_{R_+\times E} I_{\widetilde{B}}(\omega, t, x)\ \mu(\omega, dt, dx)\right], \text{ for all } \widetilde{B} \in \widetilde{\mathbb{A}}. \tag{2.10}$$

Thus, by $M_\mu^P[g|\widetilde{\mathcal{P}}(\mathbb{H})]$, we denote the unique $\widetilde{\mathcal{P}}(\mathbb{H})$-measurable function, providing it exists, such that for any bounded $\widetilde{\mathcal{P}}(\mathbb{H})$-measurable function W,

$$M_\mu^P(Wg) := E\left(\int_{R_+}\int_E W(s,x)g(s,x)\ \mu(ds, dx)\right) = M_\mu^P\left(W M_\mu^P\left[g\middle|\widetilde{\mathcal{P}}(\mathbb{H})\right]\right).$$

Remark 2.1: In this thesis, we shall reserve the notation "$\star$" for integrals with respect to random measures.

Definition 2.14: A random measure μ is called $\widetilde{\mathcal{O}}(\mathbb{H})$-*optional* (resp. *predictable*) if the process $W \star \mu$ is $\mathbb{H}$-optional (resp. $\mathbb{H}$-*predictable*) for every $\widetilde{\mathcal{O}}(\mathbb{H})$-measurable (resp. $\widetilde{\mathcal{P}}(\mathbb{H})$-measurable) function W.

For any μ, a jump random measure of a process X, we associate a $\widetilde{\mathcal{P}}(\mathbb{H})$-measurable random measure ν satisfying $W \star \mu - W \star \nu$ is an $\mathbb{H}$-local martingale, for any $\widetilde{\mathcal{P}}(\mathbb{H})$-measurable function W on $\widetilde{\Omega}$ with $|W| \star \mu \in \mathcal{A}^+_{loc}(\mathbb{H})$.

Moreover, there exists a predictable process $A \in \mathcal{A}^+(\mathbb{H})$ and a kernel $F(\omega, t, dx)$ from $(\Omega \times \mathbb{R}_+, \mathcal{P}(\mathbb{H}))$ into $(E, \mathcal{E})$ such that

$$\nu(\omega, dt, dx) = dA_t(\omega)\ F(\omega, t, dx). \tag{2.11}$$

For any $\widetilde{\mathbb{A}}$-measurable function W, we associate the following processes

$$\widehat{W}_t(\omega) := \int_E W(\omega, t, x)\nu(\omega, \{t\} \times dx), \tag{2.12}$$

$$\widetilde{W}_t(\omega) := W(\omega, t, \Delta X_t(\omega)) I_{\{\Delta X \neq 0\}}(\omega, t) - \widehat{W}_t(\omega). \tag{2.13}$$

The resulting random measure $\mu - \nu$ is called a "martingale random measure". For the pair of random measures (μ, ν), we consider two types of integrals that correspond to the sets of integrands denoted by $\mathcal{G}^1_{loc}(\mu, \mathbb{H})$ and $\mathcal{H}^1_{loc}(\mu, \mathbb{H})$ respectively. These two integrals result in two classes of pure jump local martingales. The two sets of integrands are defined by

$$\begin{aligned} &\mathcal{G}^1_{loc}(\mu, \mathbb{H}) := \left\{ W \in \widetilde{\mathcal{P}}(\mathbb{H}) : \quad \sqrt{\textstyle\sum_{s \leq \cdot} \widetilde{W}^2_s} \in \mathcal{A}^+_{loc}(\mathbb{H}) \right\} \quad \text{and} \\ &\mathcal{H}^1_{loc}(\mathbb{H}, \mu) := \left\{ g : g \in \widetilde{\mathcal{O}}(\mathbb{H}), M^P_\mu[g|\mathcal{P}(\mathbb{H})] = 0, \sqrt{g^2 \star \mu} \in \mathcal{A}^+_{loc}(\mathbb{H}) \right\}. \end{aligned} \tag{2.14}$$

The resulting integrals are denoted by $W \star (\mu - \nu)$ and $W \star \mu$ when $W \in \mathcal{G}^1_{loc}(\mu, \mathbb{H})$ and $W \in \mathcal{H}^1_{loc}(\mu, \mathbb{H})$ respectively.

Now, we present the *canonical predictable representation* for a semimartingale, which is based on the stochastic integral with respect to a random measure. For its proof, we refer to Jacod and Shiryaev [78] (see Theorem 2.34, Page 84).

Theorem 2.6: *Let X be an $\mathbb{H}$-semimartingale. Then X has the canonical representation:*

$$X = X_0 + X^c + h \star (\mu - \nu) + (x - h(x)) \star \mu + B, \tag{2.15}$$

where X^c is the continuous martingale part of X, B is a predictable finite variation process and h is a truncation function with the form of $h(x) = xI_{\{|x| \leq 1\}}$.

For the matrix C with entries $C^{ij} := [X^{c,i}, X^{c,j}]$, the triple (B, C, ν) is called *predictable characteristics* of X. Furthermore, we can find a version of the *characteristics triplet* satisfying

$$B = b \centerdot A, \;\; C = c \centerdot A, \;\; \text{and} \;\; \nu(\omega, dt, dx) = dA_t(\omega)F_t(\omega, dx). \tag{2.16}$$

Here, A is an increasing and predictable process, b and c are predictable process and $F_t(\omega, dx)$ is a predictable kernel such that

- $F_t(\omega, \{0\}) = 0, \qquad \int \left(|x|^2 \wedge 1\right) F_t(\omega, dx) \leq 1,$
- $\Delta B_t = \int h(x)\nu(\{t\}, dx), \quad c = 0$ on $\{\Delta A \neq 0\}$,
- $a_t := \nu(\{t\}, \mathbb{R}^d) = \Delta A_t F_t(\mathbb{R}^d) \leq 1.$

If X is a special semimartingale, we have the following:

Corollary 2.6.1: Let X be an $\mathbb{H}$-special semimartingale. Then X has the following decomposition:

$$X = X_0 + X^c + x \star (\mu - \nu) + B, \tag{2.17}$$

where B is a predictable process with finite variation.

For the following representation theorem, we refer to Jacod [76] (Theorem 3.75, page 103) and Jacod and Shiryaev [78] (Lemma 4.24, page 185) and recent results in Choulli and Schweizer [37]. Then the representation of a local martingale with respect to a semimartingale X is given by the following

Theorem 2.7: *[Jacod's representation] Let X be an $\mathbb{H}$-semimartingale, μ is its random jump measure, and ν is the random measure compensator of μ. Consider $N \in \mathcal{M}_{0,loc}(\mathbb{H})$. Then, there exist an $\mathbb{H}$-predictable X^c-integrable process β, $N' \in \mathcal{M}_{0,loc}(\mathbb{H})$ with $[N', X] = 0$, $f \in \mathcal{G}^1_{loc}(\mu, \mathbb{H})$ and $g \in \mathcal{H}^1_{loc}(\mu, \mathbb{H})$ such that*

$$N = \beta \cdot X^c + W \star (\mu - \nu) + g \star \mu + N', \ W := f + \frac{\widehat{f}}{1-a}, \ M^P_\mu[g|\widetilde{P}(\mathbb{H})] = 0,$$

where $f := M^P_\mu[\Delta N|\widetilde{P}(\mathbb{H})]$, $g := \Delta N - f$, $\widehat{f}$ is defined via (2.12) and f has a version such that $\{a = 1\} \subset \{\widehat{f} = 0\}$. Moreover,

$$\Delta N = (f(\Delta X) + g(\Delta X))\, I_{\{\Delta X \neq 0\}} - \frac{\widehat{f}}{1-a} I_{\{\Delta X = 0\}} + \Delta N'. \tag{2.18}$$

Remark 2.2: The Jacod Decomposition Theorem would be frequently used in Chapter 5 and Chapter 6. In the sequel, we shall call (β, W, g, N') or (β, f, g, N') the *Jacod Parameters* of N with respect to X.

The following is a simple but useful result on the conditional expectation with respect to M^P_μ.

Lemma 2.4: *Let f and g be two nonnegative $\widetilde{\mathcal{O}}(\mathbb{H})$-measurable functionals. Then,*

$$M^P_\mu\left(fg\big|\ \widetilde{\mathcal{P}}(\mathbb{H})\right)^2 \leq M^P_\mu\left(f^2\big|\ \widetilde{\mathcal{P}}(\mathbb{H})\right) M^P_\mu\left(g^2\big|\ \widetilde{\mathcal{P}}(\mathbb{H})\right), \qquad M^P_\mu\text{–}a.e. \tag{2.19}$$

Proof. The proof is the same as the one of the regular Cauchy-Schwarz formula, by putting $\bar{f} := f/M^P_\mu\left(f^2\big|\ \widetilde{\mathcal{P}}(\mathbb{H})\right)$ and $\bar{g} := g/M^P_\mu\left(g^2\big|\ \widetilde{\mathcal{P}}(\mathbb{H})\right)$ and using the simple inequality $xy \leq (x^2+y^2)/2$. This ends the proof of the lemma. □

The following lemma is borrowed from Jacod (see Theorem 3.75 in [76] or Proposition 2.2 in [37]).

Lemma 2.5: *Let $\mathcal{E}(N)$ be a positive local martingale and (β, f, g, N') be the Jacod's parameters of N. Then, $\mathcal{E}(N) > 0$ (or equivalently $1 + \Delta N > 0$) implies that*

$$f > 0, \quad M^P_\mu - a.e.$$

Lemma 2.6: **(see Choulli and Schweizer [37]):** *Let f be a $\widetilde{\mathcal{P}}(\mathbb{H})$-measurable functional such that $f > 0$ and*

$$\left((f-1)^2 \star \mu\right)^{1/2} \in \mathcal{A}^+_{loc}(\mathbb{H}). \tag{2.20}$$

Then, the $\mathbb{H}$-predictable process $\left(1 - a + \widehat{f}\right)^{-1}$ is locally bounded, and hence

$$W_t(x) := \frac{f_t(x) - 1}{1 - a_t + \widehat{f}_t} \in \mathcal{G}^1_{loc}(\mu, \mathbb{H}). \tag{2.21}$$

Here, $a_t := \nu(\{t\}, \mathbb{R}^d)$ and $\widehat{f}_t := \int f_t(x)\nu(\{t\}, dx)$.

We end this section by recalling a lemma that is useful when one computes compensator of integral with random measure. It was proved in [71], page 350.

Lemma 2.7: *Let $W \in \mathcal{G}^1_{loc}(\mu, \mathbb{H})$ and $M := W \star (\mu - \nu)$. For any $N \in \mathcal{M}_{loc}(\mathbb{H})$*

such that $[M,N] \in \mathcal{A}_{loc}(\mathbb{H})$*, we have*

$$\langle W \star (\mu - \nu), N \rangle = (VW) \star \nu, \tag{2.22}$$

where $V := M^P_\mu[\Delta N | \widetilde{\mathcal{P}}(\mathbb{H})]$.

2.4 Mathematical Formulation for Non-Arbitrage

In this section, we recall three important non-arbitrage concepts, precisely, No-arbitrage, No-Free-Lunch-with-Vanish-Risk and No-Unbounded-Profit-with-Bounded-Risk. Loosely speaking, in a rationale market, there is no possibility that one could gain profit without taking any risk. We refer to the book of Delbaen and Schachermayer [51] for this rich topic.

Definition 2.15: Let a be a positive real number and X be an $\mathbb{H}$-semimartingale. An X-integrable $\mathbb{H}$-predictable process H is called a-admissible if $H_0 = 0$ and $H \centerdot X_t \geq -a$ for all $t \geq 0$. H is called admissible if it is admissible for some $a \in \mathbb{R}_+$.

Given the semi-martingale X, we denote by $\mathbb{K}_0$ the convex cone of L^0, given by

$$\mathbb{K}_0 := \left\{ H \centerdot X_\infty \;\middle|\; H \text{ is } X-\text{admissible and } \lim_{t\to+\infty} H \centerdot X_t \text{ exists} \right\}.$$

By $\mathbb{C}_0$, we denote the cone of functions dominated by elements of $\mathbb{K}_0$, i.e. $\mathbb{C}_0 := \mathbb{K}_0 - L^0_+$. With $\mathbb{C}$ and $\mathbb{K}$, we denote the corresponding intersections with the space L^∞ of bounded functions $\mathbb{K} := \mathbb{K}_0 \cap L^\infty$ and $\mathbb{C} := \mathbb{C}_0 \cap L^\infty$. By $\overline{\mathbb{C}}$, we denote the closure of $\mathbb{C}$ with respect to the norm topology of L^∞.

Definition 2.16: We say that the $\mathbb{H}$-semi-martingale X satisfies the condition of

(a) *No Arbitrage* (NA) if $\mathbb{C} \cap L^\infty_+ = \{0\}$.

(b) *No Free Lunch with Vanishing Risk* (NFLVR) if $\overline{\mathbb{C}} \cap L^\infty_+ = \{0\}$.

Remark 2.3: (1) It is clear that (b) implies (a). The no-arbitrage property (NA) is equivalent to $\mathbb{K}_0 \cap L^0_+ = \{0\}$ and has an obvious interpretation: there should be no possibility of obtaining a positive profit by trading alone according to an admissible strategy.

(2) The condition of NFLVR has the following economic interpretation: there should be no sequence of final payoffs of admissible integrands, $f_n := H^n \centerdot X_T$ such that the negative parts f_n^- tends to 0 uniformly and such that f_n tends almost surely to a $[0, \infty]$-valued function f_0 satisfying $P[f_0 > 0] > 0$. If (NFLVR) is not satisfied then there is a f_0 in L^∞_+, not identically 0, as well as a sequence $(f_n)_{n\geq 1}$ of elements in $\mathbb{C}$, tending almost surely to f_0, such that for all n, we have that $f_n \geq f_0 - \frac{1}{n}$.

Definition 2.17: The $\mathbb{H}$-semi-martingale X is said to satisfy the *No-Unbounded-Profit-with-Bounded-Risk* (called NUPBR($\mathbb{P}, \mathbb{H}$)) condition if the set

$$\mathbb{K}_1 := \left\{ H \centerdot X_\infty \;\middle|\; H \centerdot X \geq -1 \;\text{ and }\; \lim_{t\to+\infty} H \centerdot X_t \;\text{ exists} \right\}, \qquad (2.23)$$

is bounded in $L^0(\mathbb{P})$ (i.e. bounded in probability under $\mathbb{P}$). When there is no confusion, we simply call it NUPBR.

Remark 2.4: The terminology of NUPBR is also articulated as *The First Kind of No Arbitrage* in Kardaras [94] or *(BK)* in Kabanov [86].

The following connection among (NA), (NUPBR) and (NFLVR) was proved in [86].

Lemma 2.8: *The semimartingale X satisfies (NFLVR) if and only if (NA) and (NUPBR) are satisfied, i.e., NFLVR = NA + NUPBR.*

The fundamental theorem of asset pricing, due to Deblean and Schachermayer's seminal papers [47] and [50], could be read as:

Theorem 2.8: *Let X be an $(\mathbb{H}, \mathbb{P})$-semimartingale. Then X satisfies NFLVR if and* *only if there exists a probability measure $\mathbb{Q} \sim \mathbb{P}$ such that X is a σ-martingale with respect to $\mathbb{Q}$.*

The precise relationship between NUPBR and σ-martingale density was proved by Takaoka [137]; and for the case of continuous semi-martingale, we refer to Choulli and Stricker [39].

Theorem 2.9 (Takaoka): *Let X^T (i.e. X stopped at T) be an $(\mathbb{H}, \mathbb{P})$-semimartingale for a fixed $T \in (0, +\infty)$. Then X^T satisfies the condition of NUPBR if and only if there exists a σ-martingale density Z for X^T, i.e. ZX^T is a σ-martingale.*

2.5 Utility Maximization

In this subsection, we provide the mathematical definitions of the utility and the corresponding admissible set of strategies afterwards.

Definition 2.18: A utility function is a function U satisfying the following:

(a) U is continuously differentiable, strictly increasing, and strictly concave on its effective domain $\mathrm{dom}(U)$.

(b) There exists $u_0 \in [-\infty, 0]$ such that $\mathrm{dom}(U) \subset (u_0, +\infty)$.

The effective domain $\mathrm{dom}(U)$ is the set of $r \in \mathbb{R}$ satisfying $U(r) > -\infty$.

Given a utility function U, an $\mathbb{H}$-semimartingale X, and a probability Q, we define the set of admissible portfolios as follows

$$\begin{aligned} &\mathcal{A}_{adm}(\alpha, U, X, Q) := \\ &\Big\{ H \mid \ H \in L(X),\ H \centerdot X \geq -\alpha \ \ \& \ \ E^Q\Big[U^-(\alpha + (H \centerdot X)_T)\Big] < +\infty \Big\}. \end{aligned} \tag{2.24}$$

When $Q = P$, $X = S$, and U is fixed, we simply denote $\mathcal{A}_{adm}(\alpha, S)$.

The main goal for utility maximization problem is to find the optimal strategy $\widehat{H} \in \mathcal{A}_{adm}(\alpha, U, X, Q)$ that maximizes the following function

$$\mathrm{Max}_{H \in \mathcal{A}_{adm}(x, U, X, Q)}\ E^Q\left[U\left(x + H \centerdot X_T\right)\right] = E^Q\left[U\left(x + \widehat{H} \centerdot X_T\right)\right]. \tag{2.25}$$

2.6 Mathematical Models for Additional Information

In the literature, the additional information is usually modelled by a random time τ (a positive random variable) that would represent the time of default, bankruptcy, retirement, $\cdots$, etc.. To incorporate the information from τ, probabilitist developed two main-streams: initial enlargement of filtration and progressive enlargement of filtration. In the thesis, we will restrict our attention to the informational market in progressive enlargement of filtration.

2.6.1 Progressive Enlargement of Filtration

The framework of the progressive enlargement is suitable for formulating and characterizing the problems associated with an additional information conveyed by observations of occurrence of a random time τ. Below, the elementary properties of progressive enlargement of filtration will be introduced. For their proof, we refer to Jeulin [83]. For a random time $\tau : \Omega \to \overline{\mathbb{R}}_+$, we denote by D the right-continuous process $D_t = I_{\{\tau \le t\}}$, and by $\mathbb{G} = (\mathcal{G}_t, t \ge 0)$ where

$$\mathcal{G}_t = \bigcap_{s>t} \mathcal{H}_s \vee \sigma(D_u, u \le s) \tag{2.26}$$

is the smallest right-continuous filtration which contains $\mathbb{H}$ and makes τ a stopping time. We introduce the $\mathbb{H}$-supermartingale Z and the $\mathbb{H}$-strong supermartingale $\widetilde{Z}$ (without right continuity)

$$Z_t := P(\tau > t|\mathcal{H}_t), \text{ and } \widetilde{Z}_t := P(\tau \ge t|\mathcal{H}_t) \tag{2.27}$$

and we denote by m the $\mathbb{H}$-martingale

$$m = Z + D^{o,\mathbb{H}}, \tag{2.28}$$

where $D^{o,\mathbb{H}}$ is the $\mathbb{H}$-dual optional projection of $I_{[\![\tau,\infty[\![}$. Therefore,

$$\widetilde{Z}_+ = Z, \quad \Delta D^{o,\mathbb{H}} = (\widetilde{Z} - Z)I_{[\![0,+\infty[\![}, \quad \widetilde{Z}_- = Z_- = {}^{p,\mathbb{H}}(\widetilde{Z}), \text{ on } [\![0,+\infty[\![. \tag{2.29}$$

Remark 2.5: The decomposition $Z = m - D^{o,\mathbb{H}}$ is, in general, different from the Doob-Meyer decomposition $Z = M^Z - D^{p,\mathbb{H}}$, where M^Z is a martingale and $D^{p,\mathbb{H}}$ is a predictable increasing process. If τ avoids the $\mathbb{H}$-stopping times, or if all $\mathbb{H}$-martingales are continuous, then $D^{p,\mathbb{H}} = D^{o,\mathbb{H}}$ and $Z = m - D^{o,\mathbb{H}}$ is the Doob-Meyer decomposition of Z.

The following theorem characterizes the precise relationship between $\mathbb{H}$-local martingales and $\mathbb{G}$-local martingales on $[\![0,\tau]\!]$.

Proposition 2.4: *[83] Let τ be a random time. Then the following hold:*

(a) *If X is an $\mathbb{H}$-semimartingale, X^τ (i.e. X stopped at τ) is a $\mathbb{G}$-semimartingale.*

(b) *If X is an $\mathbb{H}$-local martingale, then*

$$\widehat{X}_t = X_{t\wedge\tau} - \int_0^{t\wedge\tau} \frac{1}{Z_{s-}} d\langle X, m\rangle_s^{\mathbb{H}} \tag{2.30}$$

is a $\mathbb{G}$-local martingale.

Next, we shall study the compensator of the jump measure μ under $\mathbb{G}$ and the canonical representation of X^τ in the progressive enlargement of filtration.

Proposition 2.5: *Let μ be the jump measure of X and ν be its $\mathbb{H}$-compensator. Then, on $[\![0,\tau]\!]$, the $\mathbb{G}$-compensator of μ is given by*

$$\nu^{\mathbb{G}} := \left(I_{[\![0,\tau]\!]} \cdot \mu\right)^{p,\mathbb{G}} = \frac{M_\mu^P[\widetilde{Z}|\widetilde{\mathcal{P}}(\mathbb{H})]}{Z_-} I_{[\![0,\tau]\!]} \cdot \nu. \tag{2.31}$$

Proof. Due to Proposition 2.4, for any $W \in \widetilde{\mathcal{P}}(\mathbb{H})$ such that $|W| \star \mu \in \mathcal{A}_{loc}^+$, we know

$$W \star (\mu - \nu)^\tau - \frac{1}{Z_-} I_{[\![0,\tau]\!]} \cdot \langle W \star (\mu - \nu), m\rangle^{\mathbb{H}}$$

is a $\mathbb{G}$-local martingale. Therefore, we calculate from Lemma 2.7 that

$$\begin{aligned} & WI_{[\![0,\tau]\!]} \star \mu - WI_{[\![0,\tau]\!]} \star \nu - \frac{WM_\mu^P[\Delta m|\widetilde{\mathcal{P}}(\mathbb{H})]}{Z_-} I_{[\![0,\tau]\!]} \star \nu \\ & \qquad = WI_{[\![0,\tau]\!]} \star \mu - \frac{WM_\mu^P[\widetilde{Z}|\widetilde{\mathcal{P}}(\mathbb{H})]}{Z_-} I_{[\![0,\tau]\!]} \star \nu \in \mathcal{M}_{loc}(\mathbb{G}, \mathbb{P}), \end{aligned}$$

where we used the fact that $Z_- + \Delta m = \widetilde{Z}$. Therefore, (2.31) follows. □

Theorem 2.10: *Let X be an $\mathbb{H}$-semimartingale with the canonical representation*

$$X = X_0 + X^c + (x-h) \star \mu + h \star (\mu - \nu) + B.$$

Then the canonical representation of X^τ is given by

$$X^\tau = X_0 + \widehat{X^c} + h \star (\mu^{\mathbb{G}} - \nu^{\mathbb{G}}) + (x-h) \star \mu^{\mathbb{G}} + \widetilde{B}, \tag{2.32}$$

where $\widehat{X^c}$ is defined via (2.30) and $\widetilde{B} := B^\tau + \frac{I_{[\![0,\tau]\!]}}{Z_-} \centerdot \langle X^c, m\rangle^{\mathbb{H}} + h \frac{M_\mu^P[\Delta m|\widetilde{\mathcal{P}}(\mathbb{H})]}{Z_-} I_{[\![0,\tau]\!]} \star \nu$.

Proof. By Proposition 2.5, we have

$$\mu^{\mathbb{G}} = I_{[\![0,\tau]\!]} \centerdot \mu \text{ and } \nu^{\mathbb{G}} = \left(1 + \frac{M_\mu^P[\Delta m|\widetilde{\mathcal{P}}(\mathbb{H})]}{Z_-}\right) I_{[\![0,\tau]\!]} \centerdot \nu.$$

Therefore,

$$\begin{aligned} X^\tau &= X_0 + I_{[\![0,\tau]\!]} \centerdot X^c + hI_{[\![0,\tau]\!]} \star (\mu - \nu) + (x-h)\, I_{[\![0,\tau]\!]} \star \mu + I_{[\![0,\tau]\!]} \centerdot B \\ &= X_0 + \widehat{X^c} + h \star (\mu^{\mathbb{G}} - \nu^{\mathbb{G}}) + (x-h) \star \mu^{\mathbb{G}} + \widetilde{B}, \end{aligned}$$

where $\widehat{X^c}$ is defined via (2.30) and $\widetilde{B} := B^\tau + \frac{I_{[\![0,\tau]\!]}}{Z_-} \centerdot \langle X^c, m\rangle^{\mathbb{H}} + h \frac{M_\mu^P[\Delta m|\widetilde{\mathcal{P}}(\mathbb{H})]}{Z_-} I_{[\![0,\tau]\!]} \star \nu$. It remains to show the process $h \frac{M_\mu^P[\Delta m|\widetilde{\mathcal{P}}(\mathbb{H})]}{Z_-} I_{[\![0,\tau]\!]} \star \mu$ is locally integrable. Let $(\sigma_n)_{n\geq 1}$ be the localizing sequence for the $\mathbb{G}$-locally bounded process $Z_-^{-1} I_{[\![0,\tau]\!]}$ (i.e. $Z_-^{-1} I_{[\![0,\tau]\!] \cap [\![0,\sigma_n]\!]} \leq n$); and $(T_n)_{n\geq 1}$ be the localizing sequence of $Var([X,m])$. Then, we derive

$$\begin{aligned}
E\left[|h|\frac{\left|M^P_\mu[\Delta m|\widetilde{\mathcal{P}}(\mathbb{H})]\right|}{Z_-}I_{[\![0,\tau]\!]}\star\mu_{\sigma_n\wedge T_n}\right] &= E\left[|x|I_{\{|x|\leq 1\}}\frac{\left|M^P_\mu[\Delta m|\widetilde{\mathcal{P}}(\mathbb{H})]\right|}{Z_-}I_{[\![0,\tau]\!]}\star\mu_{\sigma_n\wedge T_n}\right]\\
&\leq nE\left[|x|I_{\{|x|\leq 1\}}\left|M^P_\mu[\Delta m|\widetilde{\mathcal{P}}(\mathbb{H})]\right|I_{[\![0,\tau]\!]}\star\mu_{\sigma_n\wedge T_n}\right]\\
&\leq nE\left[\sum_{0\leq u\leq\sigma_n\wedge T_n}|\Delta X_u|I_{\{|\Delta X_u|\leq 1\}}|\Delta m_u|\right]\\
&\leq nE\left[Var([X,m])_{\sigma_n\wedge T_n}\right]<+\infty.
\end{aligned}$$

This ends the proof of the theorem. □

2.6.2 Honest Times

In the theory of enlargement of filtration, one class of random time playing important role is honest time. One crucial feature of honest time is that the (H')-hypothesis (i.e. every $\mathbb{F}$-martingale is a $\mathbb{G}$-semimartingale) is valid on the whole time interval $]0,+\infty[$. For a general random time τ and an $\mathbb{H}$-semimartingale X, the process after τ, $X - X^\tau$ may not be a $\mathbb{G}$-semimartingale.
Below, we recall the definition of honest time (see [17], [83] and the references therein).

Definition 2.19: A random time τ is $\mathbb{H}$-honest, if for any t, there exists an $\mathcal{H}_t$ measurable random variable T_t such that $\tau I_{\{\tau<t\}} = T_t I_{\{\tau<t\}}$.

Now, we are collecting some fundamental results of honest times which will be used later. For their proof, we refer to Jeulin [83].

Proposition 2.6: *Let τ be a random time valued in $\overline{\mathbb{R}}_+ = [0,+\infty]$. The following assertions are equivalent:*

(a) *τ is an $\mathbb{H}$-honest time.*

(b) *τ coincides with an end of an $\mathbb{H}$-optional set on $\{\tau<+\infty\}$.*

(c) *$\widetilde{Z}_\tau = 1, P-a.s.$ on $\{\tau<+\infty\}$.*

(d) $\mathcal{P}(\mathbb{G})$ *is generated by* $\mathcal{P}(\mathbb{H})$ *and* $[\![0,\tau]\!]$.

Remark 2.6: (a) The end of an optional set is honest.

(b) A finite honest time is the end of an optional set.

(c) If τ is honest and A is measurable, the variable τ_A defined by $\tau_A = \tau$ on A, $\tau_A = +\infty$ on A^c, is honest.

(d) A random variable τ is the end of an optional set if and only if there exists a set of finite honest times $(\tau_n)_{n\geq 1}$ such that $\sup_n \tau_n = \tau$.

Proposition 2.7: *Let τ be an $\mathbb{H}$-honest time. Then the following hold:*

(a) *If H is an $\mathcal{P}(\mathbb{G})$-measurable process, then there exist two $\mathcal{P}(\mathbb{H})$-measurable processes J and K such that*

$$HI_{[\![0,+\infty[\![} = JI_{[\![0,\tau]\!]} + KI_{]\!]\tau,+\infty[\![}.$$

(b) *If H is an $\mathcal{O}(\mathbb{G})$-measurable process, then there exist two $\mathcal{O}(\mathbb{H})$-measurable processes U and W, and a progressive measurable process V such that*

$$H = UI_{[\![0,\tau[\![} + VI_{[\![\tau]\!]} + WI_{]\!]\tau,+\infty[\![}.$$

Accordingly, for a measurable process N, we have

(c) *The $\mathbb{G}$-predictable projection of N, denoted by ${}^{p,\mathbb{G}}(N)$, satisfies:*

$${}^{p,\mathbb{G}}(N)\, I_{[\![0,+\infty[\![} = {}^{p,\mathbb{H}}\left(NI_{[\![0,\tau]\!]}\right) \frac{1}{Z_-} I_{]\!]0,\tau]\!]} + {}^{p,\mathbb{H}}\left(NI_{]\!]\tau,+\infty[\![}\right) \frac{1}{1-Z_-} I_{]\!]\tau,+\infty[\![}.$$

(d) *The $\mathbb{G}$-optional projection of N, denoted by ${}^{o,\mathbb{G}}N$, satisfies:*

$${}^{o,\mathbb{G}}(N)\, I_{[\![0,+\infty[\![} = {}^{o,\mathbb{H}}\left(NI_{[\![0,\tau]\!]}\right) \frac{1}{\widetilde{Z}} I_{[\![0,\tau]\!]} + {}^{o,\mathbb{H}}\left(NI_{]\!]\tau,+\infty[\![}\right) \frac{1}{1-\widetilde{Z}} I_{]\!]\tau,+\infty[\![}.$$

The following theorem states the precise relationship between $\mathbb{H}$-local martingale and $\mathbb{G}$-local martingale on the time interval $]0,+\infty[$.

Theorem 2.11: *Let τ be an $\mathbb{H}$-honest time, then:*

(a) $\mathbb{G}$ *satisfies* (H')*-hypothesis, (i.e. every* $\mathbb{F}$*-martingale is a* $\mathbb{G}$*-semimartingale).*

(b) *For all* $\mathbb{H}$*-local martingale* X*,*

$$Y_t := \int_0^{t\wedge\tau} \frac{1}{Z_{s-}} d\langle X, m\rangle_s^{\mathbb{H}} - \int_\tau^{t\wedge\tau} \frac{1}{1-Z_{s-}} d\langle X, m\rangle_s^{\mathbb{H}}$$

is a $\mathbb{G}$*-predictable process with finite variation and* $\overline{X} := X - Y$ *is a* $\mathbb{G}$*-local martingale.*

Below, we investigate the compensator of a random measure on $]\!]\tau, +\infty[\![$.

Proposition 2.8: *Let μ be the jump measure of X and ν its $\mathbb{H}$-compensator. Then, on $]\!]\tau, +\infty[\![$, the $\mathbb{G}$-compensator of μ is given by*

$$\nu_a^{\mathbb{G}} := \left(I_{]\!]\tau,\infty[\![} \cdot \mu\right)^{p,\mathbb{G}} = \frac{1 - M_\mu^P[\widetilde{Z}|\widetilde{\mathcal{P}}(\mathbb{H})]}{1 - Z_-} I_{]\!]\tau,\infty[\![} \cdot \nu \tag{2.33}$$

Proof. Due to Theorem 2.11, for any $|W| \star \mu \in \mathcal{A}_{loc}^+$, we know that $I_{]\!]\tau,\infty[\![} W \star (\mu - \nu) + (1 - Z_-)^{-1} I_{]\!]\tau,\infty[\![} \cdot \langle W \star (\mu - \nu), m\rangle^{\mathbb{H}}$ is a $\mathbb{G}$-local martingale. Therefore, we calculate from Lemma 2.7 that

$$\begin{aligned} & WI_{]\!]\tau,\infty[\![} \star \mu - WI_{]\!]\tau,\infty[\![} \star \nu + \frac{WM_\mu^P[\Delta m|\widetilde{\mathcal{P}}(\mathbb{H})]}{1 - Z_-} I_{]\!]\tau,\infty[\![} \star \nu \\ & \qquad = WI_{]\!]\tau,\infty[\![} \star \mu - \frac{WM_\mu^P[1 - \widetilde{Z}|\widetilde{\mathcal{P}}(\mathbb{H})]}{1 - Z_-} I_{]\!]\tau,\infty[\![} \star \nu \in \mathcal{M}_{loc}(\mathbb{G}, \mathbb{P}), \end{aligned}$$

where we used the fact that $Z_- + \Delta m = \widetilde{Z}$. Therefore, (2.33) follows. □

Theorem 2.12: *The canonical representation of $I_{]\!]\tau,\infty[\![} \cdot X$ is given by*

$$I_{]\!]\tau,\infty[\![} \cdot X = X_0 + \overline{X^c} + h \star (\mu_a^{\mathbb{G}} - \nu_a^{\mathbb{G}}) + (x - h) \star \mu_a^{\mathbb{G}} + \overline{B}, \tag{2.34}$$

where

$$\overline{X^c} := I_{]\!]\tau,\infty[\![} \centerdot X^c + \frac{1}{1-Z_-} I_{]\!]\tau,\infty[\![} \centerdot \langle X^c, m\rangle^{\mathbb{H}},$$

$$\overline{B} := I_{]\!]\tau,\infty[\![} \centerdot B - \frac{1}{1-Z_-} I_{]\!]\tau,\infty[\![} \centerdot \langle X^c, m\rangle^{\mathbb{H}} - h\frac{M^P_\mu[\Delta m|\widetilde{\mathcal{P}}(\mathbb{H})]}{1-Z_-} I_{]\!]\tau,\infty[\![} \star \nu.$$

Proof. We start by showing that $h\frac{M^P_\mu[\Delta m|\widetilde{\mathcal{P}}(\mathbb{H})]}{1-Z_-} I_{]\!]\tau,\infty[\![} \star \mu$ is locally integrable. Let (T_n) be the localizing sequence of $Var([X,m])$. We get

$$\begin{aligned}
E\left[|h|\frac{\left|M^P_\mu[\Delta m|\widetilde{\mathcal{P}}(\mathbb{H})]\right|}{1-Z_-} I_{]\!]\tau,+\infty[\![} \star \mu_{T_n}\right] &= E\left[|x|I_{\{|x|\leq 1\}}\frac{\left|M^P_\mu[\Delta m|\widetilde{\mathcal{P}}(\mathbb{H})]\right|}{1-Z_-} I_{]\!]\tau,+\infty[\![} \star \mu_{T_n}\right] \\
&\leq E\left[|x|I_{\{|x|\leq 1\}}\left|M^P_\mu[\Delta m|\widetilde{\mathcal{P}}(\mathbb{H})]\right| \star \mu_{T_n}\right] \\
&\leq E\left[\sum_{0\leq u\leq T_n} |\Delta X_u| I_{\{|\Delta X_u|\leq 1\}} |\Delta m_u|\right] \\
&\leq E\left[Var([X,m])_{T_n}\right] < +\infty.
\end{aligned}$$

Then, the proof of the theorem mimics the proof of Theorem 2.10. This ends the proof of the theorem. $\square$

Chapter 3

Viability versus Non-Arbitrage

In this chapter, we will discuss the exact relationship among non-arbitrage, viability and numéraire portfolio without any assumption and for the general continuous-time market model. Firstly, we establish the equivalence among the No-Unbounded-Profit-with-Bounded-Risk condition (called NUPBR hereafter), the existence of the numéraire portfolio, and the existence of the solution to the utility maximization under an equivalent probability measure for any nice utility. Secondly, under some mild assumptions on the model and the utility, we prove the equivalence between NUPBR and the "local" existence of optimal portfolio. These results lead naturally to two new types of viability that we call weak viability and local viability.

3.1 Problem Formulation

Here, we will discuss the three financial concepts of non-arbitrage, viability and numéraire portfolio. Among these concepts, the numéraire portfolio is the most recent concept that was introduced by Long in [106]. It is the portfolio with positive value process such that zero is always the best conditional forecast of the numéraire-dominated rate of return of every portfolio.

What does the literature say about this relationship? For discrete-time markets, the economic and financial literature provides the most precise results in this context that we summarize in the following.

Theorem 3.1: *For discrete-time market models, the following are equivalent:*

(a) *The market is viable/Utility Maximization admits solution for a "nice" von Neumann-Morgenstern utility,*

(b) *There are no arbitrage opportunities,*

(c) *There exists an equivalent martingale measure (EMM hereafter),*

(d) *The numéraire portfolio exists.*

The equivalence among (a), (b) and (c) was termed in the financial literature as the Fundamental Theorem of Asset Pricing (FTAP hereafter) by Dybvig and Ross (see Theorems 1 and 2 of [59]). This result goes back to Arrow and Debreu for discrete markets (see [13] and [56]). In mathematical finance the FTAP stands for the equivalence between (b) and (c), and for the rest of the thesis, this meaning will be adopted. The equivalence between (a) and (b) in discrete-time for smooth utilities was proved by [88] and [118]. The utility maximization problem has been intensively investigated, under the assumption that (c) holds. This condition allows authors to use the two rich machineries of martingale theory and convex duality. These works can be traced back to [92], [66], [46] and [98], and the references therein to cite few. The main results in this literature focus on finding assumptions on the utility function for which duality can hold, and/or the solutions to the primal problem and its dual problem will exist.

The question of how the existence of optimal portfolio is connected to the absence of arbitrage (weak or strong form) has been forgotten for the continuous-time context. Recently, Frittelli proposed in [67] an interesting approach for this issue, while his obtained results are not applicable in the context of [105].

The equivalence between (b) and (c) goes back to Kreps in [101], Harrison–Pliska in [70], and Dalang–Morton–Willinger in [44]. To obtain an analogous equivalence in the most general framework, Delbaen and Schachermayer had to strengthen the non-arbitrage condition (by considering NFLVR) while weakening the EMM (by considering σ-martingale measures). Their approach established the very general version of the FTAP in their seminal works [47] and [50].

Our Goal: The main aim of this chapter is to elaborate the equivalence among all four assertions of Theorem 3.1 for the most general continuous-time framework under no assumption by choosing adequate notions and formulations. This main result is detailed in Section 3.2, and is based on a key lemma that is important in itself. In fact this lemma closes the existing gap in the tight connection between (a) and (b) without changing the underlying probability measure (the original belief). The proof and the extensions of this lemma are given in Section 3.3.

3.2 NUPBR, Weak Viability and Numéraire Portfolio

In order to elaborate our main results, we start with describing the mathematical framework and formalizing mathematically the economic concepts used throughout this chapter. Our mathematical model, herein, is the same as the one defined in Chapter 2, Section 2.5. For the reader's convenience, we recall the most important ingredients of this model. Our model is based on a filtered probability space $(\Omega, \mathbb{A}, \mathbb{F}, P)$, where the filtration, $\mathbb{F} := (\mathcal{F}_t)_{0\leq t\leq T}$, satisfies the usual conditions of right continuity and completeness. On this stochastic basis, we consider a d-dimensional semi-martingale $(S_t)_{0\leq t\leq T}$, that represents the discounted price of d risky assets. The set of $\mathbb{F}$-predictable processes H that are S-integrable (i.e. $H \bullet S$ exists) in the sense of semimartingale will be denoted by $L(S)$.

Throughout this section, we will focus on utility functions U satisfying the Inada's conditions and the Kramkov and Schachermayer's asymptotic elasticity assumption defined in [100]:

$$\mathrm{dom}(U) = (0, +\infty), \quad U'(0) = +\infty, \quad U'(\infty) = 0, \text{ and } \limsup_{x\to\infty} \frac{xU'(x)}{U(x)} < 1. \quad (3.1)$$

These utilities were termed by "nice" von Neumann-Morgenstern utilities. For any $x > 0$, we define the set of wealth processes obtained from admissible strategies with initial capital x by

$$\mathcal{X}(x) := \left\{ X \geq 0 \;\middle|\; \exists\; H \in L(S),\; X = x + H \cdot S \;\right\}. \quad (3.2)$$

Given a utility function U, a semimartingale X, and a probability Q, we define the set of admissible portfolios as follows

$$\begin{aligned} &\mathcal{A}_{adm}(\alpha, U, X, Q) := \\ &\Big\{ H \;|\;\; H \in L(X),\; H \cdot X \geq -\alpha \;\;\&\;\; E^Q\Big[U^-(\alpha + (H \cdot X)_T)\Big] < +\infty \Big\}. \end{aligned} \quad (3.3)$$

When $Q = P$, $X = S$, and U is fixed, we simply denote $\mathcal{A}_{adm}(\alpha, S)$.
Below, for the reader's convenience, we recall the definitions of NUPBR, σ-martingale density and numéraire portfolio.

Definition 3.1: The semi-martingale S is said to satisfy the *No-Unbounded-Profit-with-Bounded-Risk* (called NUPBR(P), hereafter) condition if the following set

$$\mathbb{K}_1 := \{(H \cdot S)_T \mid H \centerdot S \geq -1\}, \quad (3.4)$$

is bounded in $L^0(P)$ (i.e. bounded in probability under P).

Definition 3.2: A σ-martingale density for S is any positive local martingale, Z, such that there exists a real-valued predictable process ϕ satisfying $0 < \phi \leq 1$ and $Z(\phi \cdot S)$ is a local martingale. The set of σ-martingale densities for S will be denoted by

$$\mathcal{Z}_{loc}(S) := \{Z \in \mathcal{M}_{loc}(P) \;\Big|\; Z > 0, \quad ZS \text{ is a } \sigma\text{-martingale } \}. \tag{3.5}$$

Definition 3.3: Let Q be a probability measure. A process $\widetilde{X} \in \mathcal{X}(x_0)$ is called a numéraire portfolio under Q if $\widetilde{X} > 0$ and for every $X \in \mathcal{X}(x_0)$, the relative wealth process $X/\widetilde{X}$ is a Q-supermartingale .

If $Q = P$, then $\widetilde{X}$ is simply called the numéraire portfolio.

In the following, we will state our principal theorem of this chapter, and discuss its novelties by comparing it to the existing literature. Afterwards, we will provide its proof and related technical results.

3.2.1 The Main Theorem and Its Interpretations

Below, we state the principal result of this chapter.

Theorem 3.2: *The following properties are equivalent:*

(i) *S satisfies the NUPBR condition.*

(ii) *The σ-martingale density for S is not empty.*

(iii) *There exists a probability $Q \sim P$, such that for any utility U satisfying (3.1) and any $x \in dom(U)$, there exists $\widehat{\theta} \in \mathcal{A}_{adm}(x, U, S, Q)$ such that*

$$\max_{\theta \in \mathcal{A}_{adm}(x,U,S,Q)} E^Q U\Big(x + (\theta \cdot S)_T\Big) = E^Q U\left(x + (\widehat{\theta} \cdot S)_T\right) < +\infty. \tag{3.6}$$

(iv) *For any $\epsilon > 0$, there exists $\widehat{Q}_\epsilon \sim P$ such that $E|\frac{d\widehat{Q}_\epsilon}{dP} - 1| \le \epsilon$, and for any utility U satisfying (3.1) and any $x \in dom(U)$, there exists $\widehat{\theta}_\epsilon \in \mathcal{A}_{adm}(x, U, S, \widehat{Q}_\epsilon)$ such that*

$$\max_{\theta \in \mathcal{A}_{adm}(x,U,S,\widehat{Q}_\epsilon)} E^{\widehat{Q}_\epsilon} U\Big(x + (\theta \cdot S)_T\Big) = E^{\widehat{Q}_\epsilon} U\left(x + (\widehat{\theta}_\epsilon \cdot S)_T\right) < +\infty. \tag{3.7}$$

(v) *For any* $\epsilon \in (0,1)$*, there exist* $\widetilde{Q}_\epsilon \sim P$ *and* $\widetilde{\theta}_\epsilon \in \mathcal{A}_{\epsilon,1} := \mathcal{A}_{adm}(1, \log, S, \widetilde{Q}_\epsilon)$ *such that* $E|\frac{d\widetilde{Q}_\epsilon}{dP} - 1| \leq \epsilon$ *and*

$$\max_{\theta \in \mathcal{A}_{\epsilon,1}} E^{\widetilde{Q}_\epsilon} \log\Big(1 + (\theta \cdot S)_T\Big) = E^{\widetilde{Q}_\epsilon} \log\Big(1 + (\widetilde{\theta}_\epsilon \cdot S)_T\Big) < +\infty. \tag{3.8}$$

(vi) *The numéraire portfolio exists.*

Interpretations of the Main Theorem:

(a) The innovation of this theorem lies in the equivalence among assertions (i), (iii), (iv), (v) and (vi). The equivalence between (i) and (ii) is exactly Theorem 2.6 of [137] on which our proof relies heavily on the one hand. On the other hand, by adding assertion (ii), we show how Theorem 3.1 becomes in our general context. In our view, Theorem 3.2 is very important from the financial economic side and the mathematical finance side. Below, we will detail these two views.

(b) From the financial/economic side, our theorem is a generalization of Theorem 3.1 to the most complex market model with no assumption. In fact, by substituting the viability under an equivalent belief and the NUPBR condition to assertions (a) and (b) of Theorem 3.1 respectively, we obtained similar important result for continuous-time framework. Furthermore, our statement (*iii*) claims that any agent whose preference fulfills (3.1) can find optimal net trade under the same equivalent belief. This belief can be chosen as close to the real-world belief as we want (but might not be equal). This enhances our economic interpretation of the statement (*iii*) given by the following.

Definition 3.4: A market is ***weakly viable*** when there exist an agent —whose utility fulfills (3.1)— and an initial capital for which the corresponding optimal portfolio exists under an equivalent probability measure.

It is worth mentioning that the equivalence between assertions (b) and (d) of Theorem 3.1 was proved by Long in [106], and was extended afterwards to general and different contexts by many scholars. For details, we refer the reader to the works of Artzner [14], Becherer [18], Christensen and Larsen [42], Karatzas and Kardaras [90], Korn et al. [97] and the references therein.

(c) From the mathematical finance perspective, the equivalence between (i) and (vi) of the theorem was established in [18], [42] and [90]. For this part, our originality lies in the method used to prove this equivalence. In fact our approach is much shorter, and much less technical than the one of [90] (our proof does not use the semimartingale characteristics nor the measurable selection theorem that are very powerful tools but not easy to handle). Furthermore, Becherer and Christensen/Larsen (see [18] and [42]) connected these two assertions to the existence of growth-optimal portfolio and the existence of the solution to the log-utility maximization. A summary of these results is given by Hulley and Schweizer (see Theorem 2.3. of [74]), where the authors stated that the assertions (i), (vi), and

(vii) The growth-optimal portfolio X^{go} exists,

are equivalent. If furthermore

$$\sup\Big\{ E\Big[\log X_T\Big] \;\Big|\; X \in \mathcal{X}(1), \quad X_- > 0, \ \text{ and } E\left[(\log X_T)^-\right] < \infty\Big\} < \infty, \quad (3.9)$$

then the properties (i), (vi), and (vii) are also equivalent to

(viii) The existence of the solution of the log-utility maximization.

(d) Theorem 3.2 proposes a new formulation for which the equivalence among the above four properties holds without any assumption and for any utility satisfying (3.1) –not only the log utility–. This formulation uses the appropriate change of probability. More importantly, the set of equivalent probabilities —under which utilities satisfying (3.1) admit optimal portfolios— is variation-dense. The change of probability measure has been known as a powerful probabilistic technique used in stochastic calculus to overcome integrability difficulties. Thus, mathematically speaking, the change of probability in Theorem 3.2 is a natural and adequate formulation that allowed us to establish the exact connection between the viability and the NUPBR with no assumption —such as (3.9)— on the model. It is worth mentioning that the utility maximization problem might have no solution (even for the log utility) for models satisfying the NUPBR (see Example 4.3 in [42]). This explains the intuitive mathematical idea that motivated this change of probability. Our next question in this discussion is: What is the economic meaning of this probability change? In order to answer this question, we need to go back to financial economics, where scholars call the probability measures by agents' subjective believes. In this literature, the change of probability measures/believes has been well received and adopted since a while, and the robust/uncertainty models and the random utility theory are among the successful areas of economics in which the change of probability is central. In this spirit of random utility theory, our assertion (iii) says that the market's viability is achieved by a random field utility for which (3.1) is fulfilled pathwise. In mathematical terms, assertion (iii) is equivalent to

(iii') There exists a random field utility $\widetilde{U}(\omega, x)$ and $\widetilde{\theta} \in \mathcal{A}_{adm}(x, \widetilde{U})$ such that $\widetilde{U}(\omega, .)$ is a utility fulfilling (3.1) and

$$\max_{\theta \in \mathcal{A}_{adm}(x,\widetilde{U})} E\widetilde{U}\Big(x + (\theta \cdot S)_T\Big) = E\widetilde{U}\left(x + (\widetilde{\theta} \cdot S)_T\right).$$

For other situations —where the change of probability is economicly motivated and strongly supported— and for the random utility theory literature, we refer the reader to [35] and the references therein.

3.2.2 Proof of the Main Theorem

The proof of the main theorem is based essentially on five lemmas that we start with. The first three lemmas are dealing with the Fatou convergence of processes that was defined in Definition 5.2 of Föllmer and Kramkov [62], while the fourth lemma deals with a supermartingale property. The fifth lemma states our second contribution in this chapter.

The following lemma is a variant of Kolmos' argument and is borrowed from [47].

Lemma 3.1: *(see Lemma A1.1 in [47])*

Let (f_n) be a sequence of $[0,+\infty[$ valued measurable functions on a probability space $(\Omega,\mathcal{F},P)$. There is a sequence $g_n \in conv(f_l,\ l \geq n)$ such that g_n converges almost surely to a $[0,+\infty]$ valued function g, and the following properties hold:

(a) *If $conv(f_n, n \geq 1)$ is bounded in L^0, then g is finite almost surely,*

(b) *If there are $c > 0$ and $\delta > 0$ such that for all n*

$$P(f_n > c) > \delta,$$

then $P(g > 0) > 0$.

Definition 3.5: Let $\mathcal{J}$ be a dense subset of $\mathbb{R}_+$. A sequence of processes (X_n) is called Fatou convergent on $\mathcal{J}$ to a process X if (X_n) is uniformly bounded from below, and if for any $t \geq 0$ we have

$$X_t = \limsup_{s\downarrow t, s\in\mathcal{J}} \limsup_{n\to\infty} X^n_s = \liminf_{s\downarrow t, s\in\mathcal{J}} \liminf_{n\to\infty} X^n_s.$$

If $\mathcal{J} = \mathbb{R}_+$, the sequence $(X_n)_{n\geq 1}$ is called simply Fatou convergent.

The dynamic version of Lemma 3.1 can be found in Föllmer and Kramkov [62] and is recalled below.

Lemma 3.2: (a) *Let $(X_n)_{n\geq 1}$ be a sequence of non-negative supermartingales. Let $\mathcal{J}$ be a dense countable subset of $\mathbb{R}_+$. Then, there exists a sequence $Y_n \in conv(X^n, X^{n+1}, \cdots)$ and a supermartingale Y such that $Y_0 \leq 0$ and $(Y_n)_{n\geq 1}$ is Fatou convergent on $\mathcal{J}$ to Y.*

(b) *Let $(A_n)_{n\geq 1}$ be a sequence of increasing processes such that $A^n_0 = 0$. There exists a sequence $B^n \in conv(A^n, A^{n+1}, \cdots)$ and an increasing process B with values in $\overline{R}_+$ such that $(B^n)_{n\geq 1}$ is Fatou convergent to B. If there are $T > 0, a > 0$ and $\delta > 0$ such that $P(A^n_T > \alpha) > \delta$ for all $n \geq 1$, then $P(B_T > 0) > 0$.*

The importance of this lemma lies mainely in the optional decomposition of Kramkov (see [100]), that we will use in our proof. In fact, we will use its weakest form that was elaborated by Stricker and Yan in Theorem 2.1 of [136], where the authors used the set $\mathcal{Z}_{loc}(S)$ instead of the set of σ-martingale measures. As a direct consequence of Lemma 3.2 and Theorem 2.1 of [136], we obtain the following

Lemma 3.3: *Suppose that $\mathcal{Z}_{loc}(S) \neq \emptyset$. Let $(\theta_n)_{n\geq 1}$ be such that $\theta_n \in L(S)$ and $\theta_n \cdot S \geq -1$. Then, there exist $\phi_n \in conv(\theta_k,\ k \geq n)$, $\widehat{\theta} \in L(S)$, and a nondecreasing process C such that $\widehat{\theta} \cdot S \geq -1$, $C_0 = 0$, and*

$$1 + \phi_n \cdot S \quad \textit{is Fatou convergent to} \quad 1 + \widehat{\theta} \cdot S - C. \tag{3.10}$$

The following lemma is dealing with a supermartingale property.

Lemma 3.4: *Let X be any RCLL semimartingale, and $\widetilde{\pi} \in L(X)$ such that $\mathcal{E}(\widetilde{\pi} \cdot X) > 0$. Then, the following are equivalent:*

(a) *For any $\pi \in L(X)$ such that $\mathcal{E}(\pi \cdot X) \geq 0$, and any stopping time, τ, we have*

$$E\Big[\frac{\mathcal{E}(\pi \cdot X)_\tau}{\mathcal{E}(\widetilde{\pi} \cdot X)_\tau}\Big] \leq 1. \tag{3.11}$$

(b) *For any $\pi \in L(X)$ such that $\mathcal{E}(\pi \cdot X) \geq 0$, the ratio $\mathcal{E}(\pi \cdot X)/\mathcal{E}(\widetilde{\pi} \cdot X)$ is a supermartingale.*

Proof. The proof of (b) $\Longrightarrow$ (a) is obvious and will be omitted. Suppose that assertion (a) holds, and consider $\pi \in L(X)$ such that $\mathcal{E}(\pi \cdot X) \geq 0$. Then, for any pair of stopping times, τ and σ, such that $\tau \leq \sigma$ $P-a.s.$ and $A \in \mathcal{F}_\tau$, we put

$$\overline{\pi} := \widetilde{\pi} I_{]\!]0,\tau_A]\!]} + \pi I_{]\!]\tau_A,+\infty[\![}, \quad \tau_A := \begin{cases} \tau & \text{on } A \\ +\infty & \text{on } A^c \end{cases}.$$

Then, we easily calculate

$$\frac{\mathcal{E}(\overline{\pi}\cdot X)_\sigma}{\mathcal{E}(\widetilde{\pi}\cdot X)_\sigma} = \frac{\mathcal{E}(\widetilde{\pi}\cdot X)_\tau}{\mathcal{E}(\widetilde{\pi}\cdot X)_\sigma}\frac{\mathcal{E}(\pi\cdot X)_\sigma}{\mathcal{E}(\pi\cdot X)_\tau} I_A + I_{A^c}.$$

Therefore, a direct application of (3.11) for $\overline{\pi}$ and σ, we obtain

$$E\Big\{\frac{\mathcal{E}(\widetilde{\pi}\cdot X)_\tau}{\mathcal{E}(\widetilde{\pi}\cdot X)_\sigma}\frac{\mathcal{E}(\pi\cdot X)_\sigma}{\mathcal{E}(\pi\cdot X)_\tau} I_A\Big\} \le P(A),$$

for any $A \in \mathcal{F}_\tau$. Hence, the supermartingale property for $\mathcal{E}(\pi\cdot X)\Big(\mathcal{E}(\widetilde{\pi}\cdot X)\Big)^{-1}$ follows immediately, and the proof of the lemma is achieved. □

Now, we state our Key Lemma in this chapter that will be interpreted economically, proved, and extended to other types of utilities in Section 3.3.

The Key Lemma Let U be a utility function satisfying (3.1). Suppose that there exists a sequence of stopping times $(T_n)_{n\ge1}$ that increases stationarily to T and $x_n > 0$ such that

$$\sup_{\theta\in\mathcal{A}_{adm}(x_n,S^{T_n})} EU\Big(x_n + (\theta\cdot S)_{T_n}\Big) < +\infty, \qquad \forall\ n\ge 1. \tag{3.12}$$

Then, the following are equivalent:

(i) There exists a sequence of stopping times $(\tau_n)_{n\ge1}$ that increases stationarily to T such that for any $n\ge1$ and any initial wealth $x_0 > 0$, there exists $\widehat{\theta}^{(n)} \in \mathcal{A}_{adm}(x_0, S^{\tau_n})$ satisfying

$$\max_{\theta\in\mathcal{A}_{adm}(x_0,S^{\tau_n})} EU\Big(x_0 + (\theta\cdot S)_{\tau_n}\Big) = EU\Big(x_0 + (\widehat{\theta}^{(n)}\cdot S)_{\tau_n}\Big) < +\infty. \tag{3.13}$$

(ii) S satisfies the NUPBR condition.

The remaining part of this section is devoted to the proof of Theorem 3.2.

Proof of Theorem 3.2:

The proof of this theorem will be achieved after three steps. The first step will focus on proving $(i) \iff (ii) \iff (iii)$. The second step will prove $(i) \iff (iv) \iff (v)$, while the last step will deal with $(v) \Longrightarrow (vi) \Longrightarrow (i)$.

1) The equivalence $(i) \Longleftrightarrow (ii)$ is exactly Takaoka's result (see Theorem 2.6 in [137]), and its proof will be omitted.

The proof of $(i) \Longleftrightarrow (iii)$ boils down to the proof of $(i) \Longrightarrow (iii)$, since the reverse implication follows directly from the Key Lemma by considering Q instead of P and taking $\tau_n = T$ for all $n \geq 1$. Suppose that assertion (i) holds. Then, due to the equivalence between (i) and (ii), we consider $Z \in \mathcal{Z}_{loc}(S)$ (i.e. a σ-martingale density for S) and put

$$Q := \frac{Z_T}{E\left[Z_T\right]} \cdot P \sim P. \tag{3.14}$$

Let U be a utility function satisfying (3.1) and $x \in \mathrm{dom}(U)$. For any $\theta \in \mathcal{A}_{adm}(x, U, S, Q)$, $Z(x + \theta \cdot S)$ is a nonnegative local martingale, and hence a supermartingale. Then, the concavity of U leads to

$$E^Q U\left(x + (\theta \cdot S)_T\right) \leq U(x/E\left[Z_T\right]) < +\infty, \qquad \forall\ \theta \in \mathcal{A}_{adm}(x, U, S, Q). \tag{3.15}$$

Therefore, a direct application of the Key Lemma under Q implies the existence of a sequence of stopping times $(\tau_n)_{n\geq 1}$ that increases stationarily to T and a sequence $\widehat{\theta}^{(n)} \in \mathcal{A}_{adm}(x, U, S^{\tau_n}, Q)$ such that

$$\sup_{\theta \in \mathcal{A}_{adm}(x,U,S^{\tau_n},Q)} E^Q U\left(x + (\theta \cdot S)_{\tau_n}\right) = E^Q U\left(x + (\widehat{\theta}^{(n)} \cdot S)_{\tau_n}\right). \tag{3.16}$$

Thus, thanks to Lemma 3.3, we deduce the existence of $(a_l)_{l\geq 1}$ ($a_l \in (0,1)$), $\widehat{\theta} \in L(S)$, and a nondecreasing RCLL process C such that $C_0 = 0$,

$$\sum_{l=n}^{m_n} a_l = 1, \text{ and } x + \sum_{l=n}^{m_n} a_l \widehat{\theta}^{(l)} \cdot S^{\tau_l} \quad \text{is Fatou convergent to} \quad x + \widehat{\theta} \cdot S - C. \tag{3.17}$$

Hence, assertion (iii) will follow immediately once we prove that $\widehat{\theta}$ belongs to

$\mathcal{A}_{adm}(x, U, S, Q)$ and it is the optimal solution to (3.6). We start by proving the admissibility of $\widehat{\theta}$. Due to Fatou's lemma and the concavity of U, we get

$$
\begin{aligned}
E^Q U^-(x + \widehat{\theta} \cdot S_T) &\leq \liminf_n E^Q U^- \left(x + \sum_{l=n}^{m_n} a_l \widehat{\theta}^{(l)} \cdot S_{\tau_l} \right) \\
&\leq \liminf_n \sum_{l=n}^{m_n} a_l E^Q U^- \left(x + \widehat{\theta}^{(l)} \cdot S_{\tau_l} \right).
\end{aligned} \tag{3.18}
$$

If $U(\infty) \leq 0$, then we have

$$
\sum_{l=n}^{m_n} a_l E^Q U^- \left(x + \widehat{\theta}^{(l)} \cdot S_{\tau_l} \right) = -\sum_{l=n}^{m_n} a_l E^Q U \left(x + \widehat{\theta}^{(l)} \cdot S_{\tau_l} \right) \leq -U(x) < +\infty,
$$

and the admissibility of $\widehat{\theta}$ follows immediately from this inequality and (3.18). Suppose that $U(+\infty) > 0$. Then, there exists a real number r such that $U(r) > 0$, and the following hold

$$
\begin{aligned}
&\liminf_n \sum_{l=n}^{m_n} a_l E^Q U^- \left(x + \widehat{\theta}^{(l)} \cdot S_{\tau_l} \right) \\
&\leq \liminf_n \sum_{l=n}^{m_n} a_l E^Q U \left(r + x + \widehat{\theta}^{(l)} \cdot S_{\tau_l} \right) - U(x) \\
&\leq U \left(\frac{r + x}{E[Z_T]} \right) - U(x) < +\infty.
\end{aligned} \tag{3.19}
$$

A combination of these inequalities and (3.18) completes the proof of $\widehat{\theta} \in \mathcal{A}_{adm}(x, U, S, Q)$. Furthermore, we get $U(x + \widehat{\theta} \cdot S_T) \in L^1(Q)$. Next, we will prove the optimality of the strategy $\widehat{\theta}$. To this end, we will start by proving

$$
E^Q U \left(x + \widehat{\theta} \cdot S_T \right) \geq \limsup_n E^Q U \left(x + \sum_{l=n}^{m_n} a_l \widehat{\theta}^{(l)} \cdot S_{\tau_l} \right). \tag{3.20}
$$

If $U(+\infty) \leq 0$, then the above inequality follows from Fatou's lemma. Suppose that $U(+\infty) > 0$. In this case, by mimicking the proof of Lemma 3.2 of [98], we easily prove that

$$\left\{U(y_n):\ y_n := x + \sum_{l=n}^{m_n} a_l \widehat{\theta}^{(l)} \cdot S_{\tau_l},\ n \geq 1\right\} \text{ is } Q\text{-uniformly integrable.} \tag{3.21}$$

Denote the inverse of U by $\phi : (U(0+), U(+\infty)) \to (0, +\infty)$. Then we derive $E^Q[\phi(U(y_n))] \leq x/E(Z_T)$ and due to l'Hospital rule and (3.1) we have

$$\lim_{x \to U(+\infty)} \frac{\phi(x)}{x} = \lim_{y \to +\infty} \frac{y}{U(y)} = \lim_{y \to +\infty} \frac{1}{U'(y)} = +\infty.$$

Then, the uniform integrability of the sequence $(U(y_n))_{n \geq 1}$ follows from the La-Vallée-Poussin argument. Then, (3.20) follows immediately from this uniform integrability and (3.17). Therefore, we obtain

$$\begin{aligned}
E^Q U\left(x + \widehat{\theta} \cdot S_T\right) &\geq \limsup_n E^Q U\left(x + \sum_{l=n}^{m_n} a_l \widehat{\theta}^{(l)} \cdot S_{\tau_l}\right) \\
&\geq \limsup_n \sum_{l=n}^{m_n} a_l E^Q U\left(x + \widehat{\theta}^{(l)} \cdot S_{\tau_l}\right) \\
&\geq \limsup_n \sum_{l=n}^{m_n} a_l E^Q U\left(x + \epsilon\theta \cdot S_{\tau_l}\right) \qquad (3.22) \\
&\geq \liminf_n \sum_{l=n}^{m_n} a_l E^Q U\left(x + \epsilon\theta \cdot S_{\tau_l}\right) \\
&\geq E^Q U\left(x + \epsilon\theta \cdot S_T\right) \qquad (3.23) \\
&\geq (1-\varepsilon)U(x) + \varepsilon E^Q U\left(x + \theta \cdot S_T\right),
\end{aligned}$$

for any $\theta \in \mathcal{A}_{adm}(x, U, S, Q)$, and any $\epsilon \in (0,1)$. It is clear that the optimality of $\widehat{\theta}$ follows immediately from the above inequalities by letting ϵ increases to one. It is obvious that (3.22) follows from (3.16), while (3.23) follows from Fatou's lemma and $U(x + \epsilon(\theta \cdot S)_{\tau_n}) \geq U((1-\epsilon)x) > -\infty$. This proves assertion (iii), and the proof of $(i) \iff (iii)$ is achieved.

2) Herein, we will prove $(i) \iff (iv) \iff (v)$. Since the log-utility satisfies (3.1), then it is easy to see that the proof of $(i) \iff (v)$ is similar to the proof of $(i) \iff$

(iv). Thus, we will focus on proving this latter equivalence only.

Suppose that assertion (i) holds. Then, assertion (iv) follows immediately as soon as we find Q_δ equivalent to P whose density converges to one in $L^1(P)$ when δ goes to zero, and the utility maximization problem admits solution under Q_δ for any $\delta \in (0,1)$. To prove this latter claim, we put

$$q := \frac{Z_T}{E[Z_T]}, \quad q_\delta := \frac{q}{\delta + q}, \quad Z_\delta := \frac{q_\delta}{E[q_\delta]} := q_\delta C_\delta, \quad Q_\delta := Z_\delta \cdot P \sim P, \qquad (3.24)$$

for any $\delta \in (0,1)$. By examining closely the proof of $(i) \Longrightarrow (iii)$, we can easily conclude that the utility maximization problem admits solution under Q_δ whenever Q_δ satisfies similar inequality as in (3.15). Thus, for any utility U satisfying (3.1), any $x \in dom(U)$, any $\delta \in (0,1)$, and any $\theta \in \mathcal{A}_{adm}(x, U, S, Q_\delta)$, we derive

$$E^{Q_\delta} U\left(x + (\theta \cdot S)_T\right) \leq U\left(E^{Q_\delta}\left[x + (\theta \cdot S)_T\right]\right) \leq U\left(\frac{E\left[Z_T[x + (\theta \cdot S)_T]\right]}{\delta E(q_\delta) E(Z_T)}\right)$$
$$\leq U\left(\frac{x}{\delta E(q_\delta) E[Z_T]}\right) < +\infty.$$

Hence, this allows us to conclude that for any $\delta \in (0,1)$ and any utility U satisfying (3.1), the utility maximization problem admits solution under Q_δ. To conclude the proof of $(i) \Longrightarrow (iv)$, we will prove that Z_δ converges to one in $L^1(P)$ when δ goes to zero. Thanks to

$$1 > (C_\delta)^{-1} = E\left(\frac{q}{\delta + q}\right) \geq E\left[\frac{q}{1+q}\right] =: \Delta_0,$$

we deduce that Z_δ is positive, bounded by $(\Delta_0)^{-1}$, and converges almost surely to one when δ goes to zero. Then, for any $\epsilon > 0$, the dominated convergence theorem implies the existence of $\delta := \delta(\epsilon) > 0$ such that $E|Z_{\delta(\epsilon)} - 1| < \epsilon$. This ends the proof of $(i) \Longrightarrow (iv)$. The reverse implication follows from $(iv) \Longrightarrow (iii) \Longrightarrow (i)$, and the proof of $(i) \Longleftrightarrow (iv) \Longleftrightarrow (v)$ is completed.

3) In this last part, we will prove $(v) \Longrightarrow (vi) \Longrightarrow (i)$. Suppose that assertion (v)

holds (and hence we have $\mathcal{Z}_{loc}(S) \neq \emptyset$). Then, it is easy to see that assertion (v) implies the existence of the numéraire portfolio under each Q_ϵ. Therefore, for any $n \geq 1$, there exist $0 < Z_n = C_n q_n$ (here $q_n = \frac{n}{n+q^{-1}}$ where q is given by (3.24)) that converges to one in $L^1(P)$, and W_n the numéraire portfolio for S under $Q_n := Z_n \cdot P$. Hence, a direct application of Lemma 3.3 leads to the existence of $(\beta_n)_{n\geq 1}$ ($\beta_n \in (0,1)$), $\widetilde{\theta} \in L(S)$, and a nondecreasing and RCLL process C such that $C_0 = 0$,

$$\sum_{k=n}^{m_n} \beta_l = 1, \text{ and } \sum_{k=n}^{m_n} \beta_k W_k \text{ is Fatou convegent to } \widetilde{W} = x + \widetilde{\theta} \cdot S - C =: \widehat{W} - C.$$

Let $W \in \mathcal{X}(x)$ be a wealth process, $b \in (0,1)$, $\alpha > 1$, and τ be a stopping time. Then, there exists a sequence of stopping times $(\tau_k)_{k\geq 1}$ that decreases to τ and takes values in $\left(\mathbb{Q}^+ \cap [0,T[\right) \cup \{T\}$ such that

$$\text{on} \quad \{\tau < T\} \qquad T \geq \tau_k > \tau, \qquad \text{and on } \{\tau = T\} \quad \tau_k = T.$$

Due to Fatou's Lemma, we obtain

$$\begin{aligned} E\left(\tfrac{W_\tau}{\widehat{W}_\tau} \wedge \alpha\right) \leq E\left(\tfrac{W_\tau}{\widetilde{W}_\tau} \wedge \alpha\right) &\leq \liminf_n \liminf_k E\left(\frac{W_{\tau_k}}{\sum_{l=n}^{m_n} \beta_l W_l(\tau_k)} \wedge \alpha\right) \\ &\leq \liminf_n \liminf_k E\Big(\left[\sum_{l=n}^{m_n} \beta_l \frac{W_{\tau_k}}{W_l(\tau_k)}\right] \wedge \alpha\Big). \end{aligned}$$

Since $q_n := \frac{n}{n+q^{-1}}$ is increasing in n, then for any $l \geq n$ and any k we have

$$\{E(q_n|\mathcal{F}_{\tau_k}) > b\} \subset \{E(q_l|\mathcal{F}_{\tau_k}) > b\} = \{1 < b^{-1}\frac{Z_l(\tau_k)}{C_l} = E(q_l\big|\mathcal{F}_{\tau_k})b^{-1}\}.$$

Hence, we derive

$$
\begin{aligned}
E\Big(\Big[\sum_{l=n}^{m_n}\beta_l\frac{W_{\tau_k}}{W_l(\tau_k)}\Big]\wedge\alpha\Big) &= E\Big(\Big[\sum_{l=n}^{m_n}\beta_l\frac{W_{\tau_k}}{W_l(\tau_k)}\Big]\wedge\alpha I_{\{E(q_n|\mathcal{F}_{\tau_k})\le b\}}\Big)+\\
&\quad +E\Big(\Big[\sum_{l=n}^{m_n}\beta_l\frac{W_{\tau_k}}{W_l(\tau_k)}\Big]\wedge\alpha I_{\{E(q_n|\mathcal{F}_{\tau_k})>b\}}\Big)\\
&\le \alpha P\Big(E(q_n|\mathcal{F}_{\tau_k})\le b\Big)+b^{-1}E\Big(\sum_{l=n}^{m_n}\beta_l\frac{Z_l(\tau_k)}{C_l}\frac{W_{\tau_k}}{W_l(\tau_k)}\Big)\\
&\le \alpha P\Big(E(q_n|\mathcal{F}_{\tau_k})\le b\Big)+b^{-1}\sum_{l=n}^{m_n}\frac{\beta_l}{C_l}.
\end{aligned}
$$

Since both C_n and q_n converge to one when n goes to infinity, and $E(q_n|\mathcal{F}_{\tau_k})$ converges to $E(q_n|\mathcal{F}_\tau)$ when k goes to infinity, then it is obvious that

$$
\alpha P\Big(E(q_n|\mathcal{F}_{\tau_k})\le b\Big)+b^{-1}\sum_{l=n}^{m_n}\frac{\beta_l}{C_l}\quad \text{converges to}\quad b^{-1},
$$

when k and afterwards n goes to infinity. Hence, we deduce that

$$
E\left(\frac{W_\tau}{\widehat{W}_\tau}\wedge\alpha\right)\le b^{-1},
$$

for any $b\in(0,1)$, any $\alpha>1$, and any stopping time τ. Thus, by taking b to one and α to $+\infty$ and using Fatou's lemma, we deduce that

$$
E\left(\frac{W_\tau}{\widehat{W}_\tau}\right)\le 1,\qquad \text{for any stopping time } \tau.
$$

A straightforward application of Lemma 3.4 leads to the conclusion that $\widehat{W}$ is the numéraire portfolio under P. This completes the proof of assertion (vi).

The proof of the remaining implication (i.e. $(vi)\Longrightarrow(i)$) is easy, and will be detailed below for the sake of completeness. Suppose that there exists a numéraire portfolio W^*. Then, for any $\theta\in L(S)$ such that $1+\theta\cdot S\ge 0$,

$$
\frac{1+\theta\cdot S}{W^*}\quad \text{is a nonnegative supermartingale.}
$$

As a result, for all $c > 0$, we obtain

$$P\Big(\frac{1+(\theta\cdot S)_T}{W^*_T} > c\Big) \le c^{-1}E\Big\{\frac{1+(\theta\cdot S)_T}{W^*_T}\Big\} \le c^{-1}.$$

This clearly implies the boundedness of $\mathcal{K}(1)$ in probability and hence S satisfies the NUPBR. This ends the proof of the theorem. □

3.3 The Proof of The Key Lemma and Its Extensions

This section contains three subsections, where we prove the Key Lemma, and develop two of its extensions. The condition (3.12), in the Key Lemma, is vital for the analysis of the utility maximization problem (see [92], [93], and [98] and the references therein). Furthermore, (3.12) is irrelevant for the most innovative part of our lemma which is $(i) \Longrightarrow (ii)$. The reverse implication follows from the seminal work of Kramkov and Schachermayer (see [98]), and for the sake of completeness, details will be provided in the proof below. Below, in three parts a), b) and c) we will discuss the meaning of the Key Lemma, and the importance of its extensions.

(a) What is the meaning of the Key Lemma? In virtue of Theorem 3.2, the Key Lemma proposes —under assumption (3.12)— an alternative to the equivalence between the NUPBR and the weak viability when working with the real-world probability measure is not an option. This lemma claims that, under mild assumptions, one can use the original belief P and look for the optimal portfolio "*locally*" instead of globally. The result of the lemma supports our definition of market's local viability as the market's viability up to a sequence of stopping times that increases stationarily to T (respectively increases to infinity for the infinite horizon context). Furthermore, as mentioned in the introduction, this lemma closes the existing gap in quantifying the tightest relationship between the absence of arbitrage and the utility maximization *à la* Delbaen and Schachermayer (i.e. without changing measure, but by weakening and/or strengthening the concepts under consideration).

(b) Can NFLVR be substituted into NUPBR in the Key Lemma? The stability of the NUPBR under the localization is a direct consequence of Takaoka's Theorem (see Theorem 2.6 in [137]). In contrast to the NUPBR, Non-Arbitrage or NFLVR can hold locally and fail globally. Thus, the existence of the optimal portfolio might not illuminate arbitrage opportunuities in the model and hence NFLVR might be violated. For the sake of completeness, below we provide an example.

Example 3.1 Consider the market model where there is one stock on the finite time horizon $[0,1]$, with $S_0 = 1$ and S satisfying the stochastic differential equation $dS_t = (1/S_t)dt + d\beta_t$. Here β is a standard one-dimensional Brownian motion, and hence S is the three-dimensional Bessel process. This example was considered in many papers starting with [48], [6], and [90]. Precisely, in [90] (see Example 4.6), the authors proved that this market model admits arbitrage opportunities and the numéraire portfolio given by $1/S = \mathcal{E}(-\frac{1}{S} \cdot \beta)$ (which is a local martingale). Here, with simple calculation, we will prove that the log-utility maximization problem admits solution for this model. It is worth mentioning that —in general— the existence of numéraire portfolio does not guarantee the existence of the optimal portfolio (for more details about this fact, we refer the reader to Example 4.3 of [42]). If we put

$$dX_t := dS_t/S_t = (1/S_t^2)dt + (1/S_t)d\beta_t,$$

then it is easy to calculate

$$\log(\mathcal{E}(X)_T) = \int_0^T \frac{1}{S_u} d\beta_u + \frac{1}{2}\int_0^T \frac{1}{S_u^2} du, \quad \text{and} \quad E\int_0^T \frac{1}{S_u^2} du \leq 8 + T < +\infty.$$

This proves that $\frac{1}{S} \cdot \beta$ is a square integrable martingale and $\log(\mathcal{E}(X)_T)$ is an integrable random variable. Hence, by combining this with the supermartingale property of $\frac{1+\theta\cdot S}{\mathcal{E}(X)}$ —for any $\theta \in L(S)$ such that $1+\theta\cdot S>0$— we derive

$$E\left[\log(1+\theta\cdot S_1)\right]-E\left[\log(\mathcal{E}(X)_1)\right]=E\left[\log\left(\frac{1+\theta\cdot S_1}{\mathcal{E}(X)_1}\right)\right]\leq \log\left(E\left[\frac{1+\theta\cdot S_1}{\mathcal{E}(X)_1}\right]\right)\leq 0.$$

Thus, for this model, the optimal portfolio for the log-utility exists, while there is no equivalent martingale measure.

(c) Why are the extensions of The Key Lemma important? In our view, the Key Lemma is important for two reasons. The first reason is its role in the proof of Theorem 3.2 which is vital. Then, the extension of this theorem to the framework of lemmas 3.5-3.6 will be an obvious motivation for the extension of the Key Lemma. However, extending Theorem 3.2 certainly will add technical complexity in the formulation that will make our result difficult to interpret since the main ideas will be buried with technical conditions. The second reason — which is our main leitmotif for extending the Key Lemma — resides in studying the dependence structures of the optimal portfolio on the model's factors such as initial wealth, horizon, ..., etcetera. The minimal assumption for the development of these structures is the existence of the optimal portfolio (at least locally), and the use of convex duality —that requires the NUPBR condition— is crucial for the analysis. Our Lemmas 3.5-3.6 claim that the NUPBR holds automatically in this context. Furthermore, for the exponential case, the martingale density —dual process— possesses a nice property of local integrability.

3.3.1 Proof of The Key Lemma:

We will start proving the easier part of the lemma, which is $(ii) \Longrightarrow (i)$. Suppose that S satisfies the NUPBR condition. Thanks to Takaoka's Theorem (see Theorem 2.6 in [137]), we conclude the existence of a local martingale $Z>0$ and a real-valued predictable process φ such that $0<\varphi\leq 1$ and $Z(\varphi\cdot S)\in\mathcal{M}_{loc}(P)$. Then, for any $\theta\in L(S)$ we have $\theta\cdot S=\theta^{\varphi}\cdot S^{\varphi}$ where $\theta^{\varphi}:=\theta/\varphi$ and $S^{\varphi}:=\varphi\cdot S$. Thus, without loss of generality, we assume that ZS is a local martingale. Consider a sequence of stopping times, $(\sigma_n)_{n\geq 1}$, that increases stationarily to T such that both Z^{σ_n} and $Z^{\sigma_n}S^{\sigma_n}$ are martingales. Put

$$Q_n := Z_{\sigma_n} \cdot P \quad \text{ and } \quad \tau_n := T_n \wedge \sigma_n \uparrow T.$$

Then, Q_n is an equivalent martingale measure for S^{σ_n}. Since $\theta I_{[\![0,\tau_n]\!]} \in \mathcal{A}_{adm}(x_n, S^{T_n})$ whenever $\theta \in \mathcal{A}_{adm}(x_n, S^{\tau_n})$, we derive

$$\sup_{\theta\in\mathcal{A}_{adm}(x_n,S^{\tau_n})} EU\Big(x_n+(\theta\cdot S)_{\tau_n}\Big) \leq \sup_{\psi\in\mathcal{A}_{adm}(x_n,S^{T_n})} EU\Big(x_n+(\psi\cdot S)_{T_n}\Big) < +\infty, \qquad \forall n.$$

Therefore, a direct application of Theorems 2.1 and 2.2 of [98] implies that for any $n \geq 0$ and any initial wealth $x_0 > 0$, there exists an x_0-admissible optimal strategy $\widehat{\theta}^{(n)}$ for S^{τ_n}, such that

$$\max\nolimits_{\theta\in\mathcal{A}_{adm}(x_0,S^{\tau_n})} EU\Big(x_0 + (\theta \cdot S)_{\tau_n}\Big) = EU\Big(x_0 + (\widehat{\theta}^{(n)} \cdot S)_{\tau_n}\Big) < +\infty.$$

This proves assertion (i). In the remaining part of the proof, we will focus on proving $(i) \Longrightarrow (ii)$. Suppose that assertion (i) holds, and consider $x_0 = 1 + r$ such that $r \in \mathrm{dom}(U)$. Then, there exists $\widehat{\theta}^{(n)} \in \mathcal{A}_{adm}(1 + r, S^{\tau_n})$ such that

$$\max\nolimits_{\theta\in\mathcal{A}_{adm}(1+r,S^{\tau_n})} EU\Big(1 + r + (\theta \cdot S)_{\tau_n}\Big) = EU\Big(1 + r + (\widehat{\theta}^{(n)} \cdot S)_{\tau_n}\Big) < +\infty.$$

For the sake of simplicity, we put $\tau := \tau_n$ and $\widehat{\theta} := \widehat{\theta}^{(n)}$ in what follows. In order to prove the NUPBR for S^τ, we proceed by assuming that

$$\mathcal{K} := \{(H \cdot S)_\tau | H \text{ is a 1-admissible strategy for } S^\tau\}$$

is not bounded in $L^0(P)$. Therefore, there exist a sequence of 1-admissible strategy $(\theta^m)_{m\geq 1}$, a sequence of positive real numbers, $(c_m)_{m\geq 1}$, that increases to $+\infty$, and

$\alpha > 0$ such that

$$P\Big((\theta^m \cdot S)_\tau \geq c_m\Big) > \alpha > 0.$$

Consider a sequence of positive numbers, $(\delta_m)_{m\geq 1}$, such that

$$0 \leq \delta_m \to 0, \quad \text{and} \quad \delta_m c_m \to +\infty.$$

Then, put

$$X_m := \delta_m(\theta^m \cdot S)_\tau \geq -\delta_m, \qquad \text{for all} \;\; m \geq 1.$$

Hence, an application of Kolmos's argument to $(X_m + \delta_m)_{m\geq 1}$ (see Lemma 3.1) leads to the existence of a sequence of random variables, $(g_k)_{k\geq 1}$, such that

$$0 \leq g_k := \sum_{m=k}^{N_k} \alpha_m X_m + \sum_{m=k}^{N_k} \alpha_m \delta_m \in \text{conv}\Big(X_m + \delta_m,\; m \geq k\Big),$$

and g_k converges almost surely to $\widetilde{X} \geq 0$, with $P(\widetilde{X} > 0) > 0$.
Since $y_k := \sum_{m=k}^{N_k} \alpha_m \delta_m$ converges to zero, we conclude that

$$-y_k \leq \widetilde{X}_k := \sum_{m=k}^{N_k} \alpha_m \delta_m (\theta^m \cdot S)_\tau \;\; \text{converges to} \;\; \widetilde{X} \quad P-a.s., \;\; \text{and}$$

$$-(1+r)(1-y_k) \leq \widehat{X}_k := (1-y_k)(\widehat{\theta} \cdot S)_\tau \;\; \text{converges to} \qquad (\widehat{\theta} \cdot S)_\tau \;\; P-a.s.$$

Consider the new trading strategies

$$\widetilde{\theta}^{(k)} := \sum_{m=k}^{N_k} \alpha_m \delta_m \theta_m + \Big(1 - \sum_{m=k}^{N_k} \alpha_m \delta_m\Big)\widehat{\theta} = \sum_{m=k}^{N_k} \alpha_m \delta_m \theta_m + (1-y_k)\widehat{\theta}.$$

Then, it is easy to check that $1 + r + \widetilde{\theta}^{(k)} \cdot S_\tau = 1 + r + \widetilde{X}_k + \widehat{X}_k \geq y_k r > 0$ (due mainly to $-y_k \leq \widetilde{X}_k$ and $-(1+r)(1-y_k) \leq \widehat{X}_k$). Furthermore, due to the concavity of U, we have

$$
\begin{aligned}
U\Big(1+r+(\widetilde{\theta}^{(k)}\cdot S)_\tau\Big) &= U\Big(1+r+\widetilde{X}_k+\widehat{X}_k\Big)\\
&= U\Big(1+r+\widetilde{X}_k+(1-y_k)(\widehat{\theta}\cdot S)_\tau\Big)\\
&\geq U\Big(1+r-y_k+(1-y_k)(\widehat{\theta}\cdot S)_\tau\Big)\\
&= U\Big(y_k r+(1-y_k)\Big[1+r+(\widehat{\theta}\cdot S)_\tau\Big]\Big)\\
&\geq y_k U(r)+(1-y_k)U\Big(1+r+(\widehat{\theta}\cdot S)_\tau\Big).
\end{aligned}
$$

This implies that $\widetilde{\theta}^{(k)} \in \mathcal{A}_{adm}(1+r, S^\tau)$. On the one hand, a combination of the previous inequality and Fatou's lemma implies that

$$
\begin{aligned}
& E\left\{U\left(1+r+\widetilde{X}+(\widehat{\theta}\cdot S)_\tau\right)-U\left(1+r+(\widehat{\theta}\cdot S)_\tau\right)\right\}\\
&= E\left\{\lim_k \left[U\Big(1+r+\widetilde{X}_k+\widehat{X}_k\Big)-(1-y_k)U\Big(1+r+(\widehat{\theta}\cdot S)_\tau\Big)-y_kU(r)\right]\right\}\\
&= E\left\{\lim_k \left[U\Big(1+r+(\widetilde{\theta}^{(k)}\cdot S)_\tau\Big)-(1-y_k)U\Big(1+r+(\widehat{\theta}\cdot S)_\tau\Big)-y_kU(r)\right]\right\}\\
&\leq \liminf_k E\left\{U\Big(1+r+(\widetilde{\theta}^{(k)}\cdot S)_\tau\Big)-(1-y_k)U\Big(1+r+(\widehat{\theta}\cdot S)_\tau\Big)-y_kU(r)\right\}\\
&\leq \liminf_k E\left\{U\Big(1+r+(\widehat{\theta}\cdot S)_\tau\Big)-(1-y_k)U\Big(1+r+(\widehat{\theta}\cdot S)_\tau\Big)-y_kU(r)\right\}=0.
\end{aligned}
\tag{3.25}
$$

On the other hand, since $P(\widetilde{X}>0)>0$ and U is strictly increasing, we get

$$
E\left\{U\Big(1+r+\widetilde{X}+(\widehat{\theta}\cdot S)_\tau\Big)\right\} > E\left\{U\Big(1+r+(\widehat{\theta}\cdot S)_\tau\Big)\right\}.
$$

This is a contradiction with (3.25), and the NUPBR for S^τ is fulfilled. Then, the global NUPBR for S is a direct consequence of Takaoka's Theorem (Theorem 2.6 of [137]), and the proof of the lemma is completed. □

3.3.2 Extension to the Case of Real-valued Wealth Processes

In this subsection, we will consider a locally bounded semimartingale S and a utility function U satisfying

$$\mathrm{dom}(U) = \mathbb{R}, \quad U'(\infty) = 0, \quad U'(-\infty) = \infty, \tag{3.26}$$

and

$$AE_{+\infty}(U) := \limsup_{x \to +\infty} \frac{xU'(x)}{U(x)} < 1, \quad AE_{-\infty}(U) := \liminf_{x \to -\infty} \frac{xU'(x)}{U(x)} > 1. \tag{3.27}$$

These conditions are used by Schachermayer in [128], which are essential in his proofs. In the current setting, the set of admissible strategies will be

$$\Theta(x, S) := \Big\{ H \text{ admissible} \quad \Big| \quad E\Big[\Big|U(x + H \cdot S_T)\Big|\Big] < +\infty \Big\}. \tag{3.28}$$

Let us point out that when $U(\infty) < +\infty$, the required integrability in the definition of $\Theta(x, S)$ is superfluous and in that case, we simply put $\Theta(S)$.

Lemma 3.5: *Let U be a utility function satisfying (3.26)–(3.27). Suppose that there exist a sequence of stopping times $(T_n)_n$ that increases stationarily to T and $x_n \in \mathbb{R}$ such that*

$$\sup_{\theta \in \Theta(x_n, S^{T_n})} EU\Big(x_n + (\theta \cdot S)_{T_n}\Big) < U(+\infty), \qquad \forall n.$$

Then, the following properties are equivalent:

(i) There exists a sequence of stopping times $(\tau_n)_{n \geq 1}$ that increases stationarily to T such that for any $n \geq 0$ and any initial wealth x_0, there exist $W_n^* \in L^0$ and $(\widehat{\theta}^m)_{m \geq 1} \in \Theta(x_0, S^{\tau_n})$ satisfying

$$\max_{\theta \in \Theta(x_0, S^{\tau_n})} EU\Big(x_0 + (\theta \cdot S)_{\tau_n}\Big) = EU(W_n^*) < U(+\infty),$$

and

$$\lim_m E\left[\Big|U(W_n^*) - U(x_0 + \widehat{\theta}^m \cdot S_{\tau_n})\Big|\right] = 0.$$

(ii) S satisfies the NUPBR.

Proof. We start by proving the easier part of the lemma which is $(ii) \Longrightarrow (i)$. Similarly as in the proof of $(ii) \Longrightarrow (i)$ of the Key Lemma, we use Takaoka's Theorem (see Theorem 2.6 in [137]) and assume without loss of generality that there exists a positive local martingale Z, such that $ZS \in \mathcal{M}_{loc}(P)$. Consider a localizing sequence for Z and ZS that we denote by $(\sigma_n)_{n\geq 1}$. Then, Z^{σ_n} and $Z^{\sigma_n}S^{\sigma_n}$ are both martingales. Put

$$Q_n := Z^{\sigma_n} \cdot P \qquad \text{and} \qquad \tau_n := T_n \wedge \sigma_n \uparrow T.$$

Thus, Q_n is an equivalent martingale measure for S^{σ_n} and for S^{τ_n} as well. Since $\theta I_{[\![0,\tau_n]\!]} \in \Theta(x_n, S^{T_n})$ whenever $\theta \in \Theta(x_n, S^{\tau_n})$, we get

$$\sup_{\theta\in\Theta(x_n,S^{\tau_n})} EU\Big(x_n + (\theta\cdot S)_{\tau_n}\Big) \leq \sup_{\varphi\in\Theta(x_n,S^{T_n})} EU\Big(x_n + (\varphi\cdot S)_{T_n}\Big) < U(+\infty), \qquad \forall n.$$

Thus, a direct application of Theorem 2.2 in [128] leads to conclude that for any $n \geq 0$ and any initial wealth x_0, there exist $W_n^* \in L^0$ and $\widehat{\theta}^m \in \Theta(x_0, S^{\tau_n})$ such that

$$\lim_m E\Big[\Big|U(W_n^*) - U(x_0 + \widehat{\theta}^m \cdot S_{\tau_n})\Big|\Big] = 0,$$

and

$$\max_{\theta\in\Theta(x_0,S^{\tau_n})} EU\Big(x_0 + (\theta\cdot S)_{\tau_n}\Big) = EU(W_n^*) < U(+\infty).$$

This proves assertion (i). Now we focus on proving the reverse implication and start with assuming that assertion (i) holds. Without loss of generality, we take $x_0 > 1$, and due to

$$\lim_m E\Big[\Big|U(W_n^*) - U\Big(x_0 + (\widehat{\theta}^m \cdot S)_{\tau_n}\Big)\Big|\Big] \longrightarrow 0,$$

we obtain the convergence in probability of $U\Big(x_0 + (\widehat{\theta}^m \cdot S)_{\tau_n}\Big)$ to $U(W_n^*)$. Thus, $\Big(x_0 + (\widehat{\theta}^m \cdot S)_{\tau_n}\Big)$ converges in probability to W_n^* when m goes to infinity. By taking a subsequence, w.l.o.g. we assume that the sequence converges almost surely to W_n^*. Then, for any $\lambda \in (0,1)$ and any $\theta \in \Theta(x_0, S_{\tau_n})$, we derive

$$\begin{aligned} \lambda f(\lambda, m) := \Big\{ & U\Big(x_0 + (\widehat{\theta}^m \cdot S)_{\tau_n} + \lambda((\theta - \widehat{\theta}^m)\cdot S)_{\tau_n}\Big) \\ & -\lambda U(W_n^*) - (1-\lambda)U\Big(x_0 + (\widehat{\theta}^m \cdot S)_{\tau_n}\Big)\Big\} \\ & \geq \lambda\Big[U(x_0 + \theta \cdot S_{\tau_n}) - U(W_n^*)\Big]. \end{aligned} \tag{3.29}$$

Since the RHS term in the above inequality is integrable, then due to Fatou's lemma, this implies that

$$E[(x_0 + \theta \cdot S_{\tau_n} - W_n^*)U^{'}(W_n^*)] = E[\underline{\lim}_{\lambda\downarrow 0} \lim_m f(\lambda, m)]$$

$$\leq \underline{\lim}_{\lambda\downarrow 0} \lim_m \tfrac{1-\lambda}{\lambda}\Big\{EU(W_n^*) - EU(x_0 + (\widehat{\theta}^m \cdot S)_{\tau_n})\Big\} = 0.$$

By combining the inequality $0 \leq \xi U^{'}(\xi) \leq U(\xi) - U(0)$ (for any $\xi \in L_+^0(P)$) and (3.30), we obtain

$$\begin{aligned} & E[(x_0 + (\theta \cdot S)_{\tau_n})U^{'}(W_n^*)] \leq E[W_n^* U^{'}(W_n^*)] < +\infty, \text{ and} \\ & x_0 E[U^{'}(W_n^*)] \leq E[W_n^* U^{'}(W_n^*)] < +\infty. \end{aligned} \tag{3.30}$$

Consider the probability measure

$$R := \frac{U^{'}(W_n^*)}{E[U^{'}(W_n^*)]} \cdot P \sim P.$$

Then, (3.30) becomes

$$E^R\left[x_0 + \theta\cdot S_{\tau_n}\right] \leq E^R[W_n^*] < +\infty. \tag{3.31}$$

Thus, $\{1 + (\theta\cdot S)_{\tau_n} \mid \theta\in L(S) \text{ and } (\theta\cdot S)\geq -1\}$ is bounded in $L^1(R)$, and the NUPBR for S^{τ_n} follows. Then, again Theorem 2.6 of [137] implies the global NUPBR for S, and the proof of the lemma is achieved. □

3.3.3 Extension to the Case of Exponential Utility

Even though the exponential utility is a particular case of Subsection 3.3.2, it deserves special attention for two reasons. The first reason lies in the popularity of the exponential utility, while the second reason lies in our belief that for this case, when S is locally bounded, we may obtain more precise results with less assumptions. Throughout this section the set of admissible strategies for the model (X, Q) will be denoted by $\Theta(X, Q)$, and is given by

$$\Theta(X,Q) := \left\{\theta\in L(X)\;\middle|\;\theta\cdot X \text{ are uniformly bounded in } (\omega,t)\right\}.$$

When $Q\sim P$, we simply write $\Theta(X) := \Theta(X, P)$ for short. Then, the set of local martingale densities that are locally in $L\log L$ will be denoted by

$$\mathcal{Z}_{f,loc}(X,Q) := \left\{Z>0\;\middle|\; Z,\; ZX\in\mathcal{M}_{loc}(Q),\; Z\log(Z) \text{ is } Q\text{-locally integrable}\right\}. \tag{3.32}$$

when $Q = P$, we simply write $\mathcal{Z}_{f,loc}(X)$.

Definition 3.6: Let $Z = \mathcal{E}(N)\geq 0$, where $N\in\mathcal{M}_{0,loc}(P)$. If

$$V^{(E)}(N) := \frac{1}{2}\langle N^c\rangle + \sum\Big[(1+\Delta N)\log(1+\Delta N) - \Delta N\Big], \tag{3.33}$$

is locally integrable, then its compensator is called the entropy-Hellinger process of Z and is denoted by $h^E(Z,P)$ (see [40] for details).

Lemma 3.6: *Suppose S is locally bounded. Then the following are equivalent:*

(i) There exist a sequence of stopping times $(\tau_n)_{n\geq 1}$ increasing stationarily to T and $\widehat{\theta}^n \in L(S^{\tau_n})$ such that $E\left(\sup_{0\leq t\leq \tau_n} \exp\left[-(\widehat{\theta}^n \cdot S)_t\right]\right) < +\infty$ and

$$\inf_{\theta\in\Theta(S^{\tau_n})} E\left(e^{-(\theta\cdot S)_{\tau_n}}\right) = E\left(e^{-(\widehat{\theta}^n\cdot S)_{\tau_n}}\right). \tag{3.34}$$

(ii) $\mathcal{Z}_{f,loc}(S) \neq \emptyset$.

Proof. We start by proving $(ii) \implies (i)$. Suppose that assertion (ii) holds, and consider $Z \in \mathcal{Z}_{f,loc}(S)$. Then, there exists a sequence of stopping times, $(\tau_n)_{n\geq 1}$, that increases stationarily to T such that Z^{τ_n} is a martingale and $h^E_{t\wedge\tau_n}(Z,P)$ is bounded. Therefore, due to Theorem 3.7 or Proposition 3.6 in [40], we deduce that $Q^n := Z_{\tau_n}\cdot P$ is an equivalent martingale measure for S^{τ_n} satisfying the reverse Hölder condition $R_{L\log L}(P)$ (for the definition of reverse Hölder condition, we refer to [46] or [40]). Thus, Theorem 2.1 of [87] implies the existence of the optimal solution $\widehat{\theta}^n \in L(S^{\tau_n})$ for (3.34) such that $\exp\left[-(\widehat{\theta}^n\cdot S)_{\tau_n}\right] = E\exp\left[-(\widehat{\theta}^n\cdot S)_{\tau_n}\right] Z^{(E,n)}_{\tau_n}$ on the one hand. Here, $Z^{(E,n)}$ is the minimal entropy martingale density for S^{τ_n} which is an LlogL-integrable martingale and hence $E\left(\sup_{0\leq t\leq\tau_n} Z^{(E,n)}_t\right) < +\infty$. On the other hand, by Lemma 3.2 of [46], we conclude the existence of a positive constant C_n such that $\exp\left[-(\widehat{\theta}^n\cdot S)_{t\wedge\tau_n}\right] \leq C_n Z^{E,n}_{t\wedge\tau_n}$. This ends the proof of assertion (i).

In the remaining part of this proof, we will prove $(i) \implies (ii)$. Suppose that assertion (i) holds and put

$$U^{(n)}_t := \exp\left(-\widehat{\theta}^n\cdot S_{t\wedge\tau_n}\right). \tag{3.35}$$

Then by mimicking the proof of Lemma 4.1 in [46], we deduce that there exists a sequence of bounded strategies $(\theta^{(N)})_{N\geq 1} \subset \Theta\left(S^{\tau_n}\right)$ such that

$$\lim_{N\longrightarrow+\infty} e^{-(\theta^{(N)}\cdot S)_{\tau_n}} = U^{(n)}_{\tau_n} \quad P-a.s. \ \ \& \sup_{0\leq t\leq T} e^{-\theta^{(N)}\cdot S_{\tau_n}} \leq 6 \sup_{0\leq t\leq T} U^{(n)}_t \in L^1(P). \tag{3.36}$$

Therefore, $\exp\left[-(\theta^{(N)}\cdot S)_{\tau_n}\right]$ converges to $U^{(n)}_{\tau_n}$ in L^1 when N goes to $+\infty$. For an arbitrary but fixed $\theta\in\Theta(S^{\tau_n})$ and any $\lambda\in(0,1)$, we denote

$$\phi_{\lambda,N} := -\lambda\theta + \theta^{(N)} \in \Theta(S^{\tau_n}),$$

and derive

$$\begin{aligned}\frac{Ee^{-\theta^{(N)}\cdot S_{\tau_n}} - Ee^{-\phi_\lambda\cdot S_{\tau_n}}}{\lambda} &= \frac{-Ee^{-\widehat{\theta}^n\cdot S_{\tau_n}} + Ee^{-\theta^{(N)}\cdot S_{\tau_n}}}{\lambda} + \frac{-Ee^{-\phi_\lambda\cdot S_{\tau_n}} + Ee^{-\widehat{\theta}^n\cdot S_{\tau_n}}}{\lambda}\\ &\leq \frac{-Ee^{-\widehat{\theta}^n\cdot S_{\tau_n}} + Ee^{-\theta^{(N)}\cdot S_{\tau_n}}}{\lambda} \to 0, \ \text{as } N \text{ goes to} +\infty.\end{aligned}$$

Due to (3.36) and the boundedness of $(\theta\cdot S)$, it is easy to check that $\left(e^{-(\theta^{(N)}\cdot S)_{\tau_n}} - e^{-(\phi_{\lambda,N}\cdot S)_{\tau_n}}\right)/\lambda$ converges to $-e^{-\widehat{\theta}^n\cdot S_{\tau_n}}\left(\theta\cdot S_{\tau_n}\right)$ in $L^1(P)$ when λ and N go to zero and infinity respectively. By combining all the above remarks, we obtain

$$E^{Q_n}\left[-(\theta\cdot S)_{\tau_n}\right] \leq 0, \quad \text{where} \quad Q_n := \frac{\exp\left[-(\widehat{\theta}^n\cdot S)_{\tau_n}\right]}{E\left(\exp\left[-(\widehat{\theta}^n\cdot S)_{\tau_n}\right]\right)}\cdot P. \tag{3.37}$$

Since θ is arbitrary in $\Theta(S^{\tau_n})$, we conclude that Q_n is an equivalent martingale measure for S^{τ_n}. The density process of this martingale measure will be denoted by

$$\widehat{Z}^n_t := \frac{E\left(\exp\left[-(\widehat{\theta}^n\cdot S)_{\tau_n}\right]\Big|\mathcal{F}_t\right)}{E\left(\exp\left[-(\widehat{\theta}^n\cdot S)_{\tau_n}\right]\right)} =: \mathcal{E}_t\left(\widehat{N}^{(n)}\right).$$

For any $\theta\in\Theta(S^{\tau_n})$, and any $\lambda\in(0,1)$, on the one hand, the convexity of e^x leads to conclude that $((\theta\cdot S)_{\tau_n} - (\widehat{\theta}^n\cdot S)_{\tau_n})\exp(-(\widehat{\theta}^n\cdot S)_{\tau_n})$ is bounded from below by $-\exp(-(\theta\cdot S)_{\tau_n})\in L^1(P)$. On the other hand, again the convexity of e^x combined with Fatou's lemma and the minimality of $\widehat{\theta}^n$ imply that

$$E\left(e^{-(\widehat{\theta}\cdot S)_{\tau_n}}((\theta-\widehat{\theta}^n)\cdot S)_{\tau_n}\right) \leq \lim_{\lambda\to 0} E\left(e^{-\widehat{\theta}\cdot S_{\tau_n}}\frac{1-\exp\left[-\lambda((\theta-\widehat{\theta}^n)\cdot S)_{\tau_n}\right]}{\lambda}\right) \leq 0,$$

This proves that $(\widehat{\theta}^n\cdot S)_{\tau_n}\exp\left[-(\widehat{\theta}^n\cdot S)_{\tau_n}\right]\in L^1(P)$. By combining this with

$$\widehat{Z}^n_{\tau_n}\log(\widehat{Z}^n_{\tau_n}) = \frac{-(\widehat{\theta}^n\cdot S)_{\tau_n}\exp(-(\widehat{\theta}^n\cdot S)_{\tau_n}) - \exp(-(\widehat{\theta}^n\cdot S)_{\tau_n})\log\left(E\left[\exp(-(\widehat{\theta}^n\cdot S)_{\tau_n})\right]\right)}{E\left(\exp\left[-(\widehat{\theta}^n\cdot S)_{\tau_n}\right]\right)},$$

we deduce that $\widehat{Z}^n_{\tau_n}\log(\widehat{Z}^n_{\tau_n})$ is integrable, and hence $\widehat{Z}^n$ is a martingale density for S^{τ_n} that is $L\log L$-integrable. Then, by putting

$$\widehat{N} := \sum_{n=1}^{+\infty} I_{]\!]\tau_{n-1},\tau_n]\!]}\cdot\widehat{N}^{(n)},$$

and applying Lemma 3.7 below, assertion *(ii)* follows immediately. This ends the proof of the lemma. □

Lemma 3.7: *Let $(\tau_n)_{n\geq 1}$ be a sequence of stopping times that increases stationarily to T, and $(N^{(n)})_n$ be a sequence of local martingales. Then, the process*

$$N := \sum_{n=1}^{+\infty} I_{]\!]\tau_{n-1},\tau_n]\!]}\cdot N^{(n)}, \quad (\tau_0 = 0),$$

is a local martingale satisfying the following.

(i) If $\mathcal{E}(N^{(n)}) > 0$ for any $n\geq 1$, then $\mathcal{E}(N) > 0$.

(ii) If $V^{(E)}(N^{(n)})\in\mathcal{A}^+_{loc}(P)$ for any $n\geq 1$, then $V^{(E)}(N)\in\mathcal{A}^+_{loc}(P)$.

(iii) If $\mathcal{E}(N^{(n)})$ is a σ-martingale density for S^{τ_n} for any $n\geq 1$, then $\mathcal{E}(N)$ is a σ-martingale density for S.

Proof. It is obvious that

$$N^{\tau_n} = \sum_{k=1}^{n} I_{]\!]\tau_{k-1},\tau_k]\!]} \cdot N^{(k)} \in \mathcal{M}_{0,loc}(P).$$

This proves that $N \in \left(\mathcal{M}_{0,loc}(P)\right)_{loc} = \mathcal{M}_{0,loc}(P)$, and $\mathcal{E}(N) > 0$ since

$$1 + \Delta N = 1 + \Delta N^{(n)} > 0 \quad \text{on} \quad]\!]\tau_{n-1}, \tau_n]\!], \qquad n \geq 1.$$

Then, due to the definition of the operator $V^{(E)}$ given by (3.33), it is also easy to remark that $V^{(E)}(I_{]\!]\sigma,\tau]\!]} \cdot M) = I_{]\!]\sigma,\tau]\!]} \cdot V^{(E)}(M)$ for any local martingale M (with $1 + \Delta M \geq 0$) and any pair of stopping times τ and σ such that $\tau \geq \sigma$. Thus, we get

$$\left(V^{(E)}(N)\right)^{\tau_n} = \sum_{k=1}^{n} I_{]\!]\tau_{k-1},\tau_k]\!]} \cdot V^{(E)}(N^{(k)}) \in \mathcal{A}^+_{loc}(P).$$

Hence, we deduce (thanks to Lemma 1.35 of [78]) that $V^{(E)}(N) \in \left(\mathcal{A}^+_{loc}(P)\right)_{loc} = \mathcal{A}^+_{loc}(P)$. This ends the proof of assertion (i) and (ii) of the lemma. To prove the last assertion, we first remark that $\mathcal{E}(M)$ is a σ-martingale density for S if and only if there exists a predictable process φ such that $0 < \varphi \leq 1$ and

$$\varphi \cdot S + \varphi \cdot [S, M] \in \mathcal{M}_{0,loc}(P).$$

Therefore, since $\mathcal{E}(N^{(n)})$ is a σ-martingale density for S^{τ_n} for each $n \geq 1$, then there exists ϕ_n such that $0 < \phi_n \leq 1$ and

$$Y_n := \phi_n \cdot S + \phi_n \cdot [S^{\tau_n}, N^{(n)}] \in \mathcal{M}_{0,loc}(P), \qquad \forall \;\; n \geq 1. \tag{3.38}$$

Put $\phi := \sum_{k=1}^{+\infty} I_{]\!]\tau_{k-1},\tau_k]\!]} \phi_k$. Thus, it is easy to prove that $0 < \phi \leq 1$, and

$$\left(\phi \cdot S + \phi \cdot [S, N]\right)^{\tau_n} = \sum_{k=1}^{n} I_{]\!]\tau_{k-1},\tau_k]\!]} \cdot Y_k \in \mathcal{M}_{0,loc}(P).$$

Hence, $\phi \cdot S + \phi \cdot [S, N] \in (\mathcal{M}_{0,loc}(P))_{loc} = \mathcal{M}_{0,loc}(P)$, and hence $\mathcal{E}(N)$ is a σ-martingale density for S. This ends the proof of the lemma. □

Conclusions

In this chapter, we established the equivalence among the NUPBR condition, the existence of the numéraire portfolio, market's weak viability and local viability. These results together with the next chapter (Chapter 4) constitute an important motivation for Chapters 5, 6 and 7. In fact, the results of these chapters explain that one needs to check the validity of the NUPBR condition for any model before thinking about finding an optimal portfolio in the weakest form possible. Recently, in mathematical finance, there has been an upsurge interest in developing optimization problems, hedging and pricing rules for models with additional information. In virtue of the results of this chapter, these investigations may end to nonsense if the NUPBR is violated under this extra information.

Chapter 4

Examples of Informational Arbitrages: Explicit Descriptions

This chapter presents some practical examples that admit classic arbitrage opportunities in informational markets, i.e. some extra information is incorporated into the markets. The extra information could be the occurrence time of a default event, the knowledge that only insiders could get, the last passage time of a process, $\cdots$, etc. For these markets, we calculate explicitly the arbitrage opportunities.

The financial market in which some assets, with prices adapted with respect to a reference filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ (public information), are traded. One then assumes that an agent who has some extra information, and may use those strategies that are predictable with respect to a larger filtration $\mathbb{G}$, i.e. $\mathbb{F} \subset \mathbb{G}$. This extra information is modeled by the knowledge of some random time τ. We restrict our study to progressive enlargement of filtration setting, and we pay a particular attention to honest times.

Our goal is to detect if the knowledge coming from τ allows for some arbitrage, i.e., if by using $\mathbb{G}$-predictable strategies, the agent can make profit without taking any risk.

This chapter is organized as follows: Section 4.1 presents the problem and the general theorem. In Section 4.2, we study the case of Brownian filtration; while Section 4.3 deals with the case of Poisson filtration. Those fact are illustrated by many examples, where we exhibit these arbitrages in a closed form. In Section 4.4, we deal with arbitrage in a two period discrete time model. In the last Section 4.5, we study some examples of non-honest times.

4.1 General Theorem

We consider a filtered probability space $(\Omega, \mathcal{A}, \mathbb{F} := (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ where the public information filtration $\mathbb{F}$ satisfies the usual hypotheses with $\mathcal{F}_\infty := \bigvee_{t\geq 0} \mathcal{F}_t \subset \mathcal{A}$, and a random time τ (i.e., a positive $\mathcal{A}$-measurable random variable and τ would vary in different examples below). We assume that the financial market where a risky asset with price S (an $\mathbb{F}$-adapted positive process) and a non-risky asset S^0 (assumed, for simplicity, to have a constant price so that the risk-free interest rate is null) are traded is arbitrage free. More precisely, without loss of generality we assume that S is a $(\mathbb{P}, \mathbb{F})$-(local) martingale.

For a generic filtration $\mathbb{H}$ and an $\mathbb{H}$-semimartingale X, we denote $L(X)$ the set of $\mathbb{H}$-predictable processes φ integrable with respect to X in the sense of semimartingale, i.e, $L(X) := \{\varphi \in \mathcal{P}(\mathbb{H}) : \quad \varphi \centerdot X \text{ is well defined}\}$.

We denote by $\mathbb{G} := (\mathcal{G}_t)_{t\geq 0}$ the progressively enlarged filtration of $\mathbb{F}$ by τ, i.e., the smallest right-continuous filtration that contains $\mathbb{F}$ and makes τ a stopping time defined by

$$\mathcal{G}_t = \bigcap_{\epsilon>0} \mathcal{F}_{t+\epsilon} \vee \sigma(\tau \wedge (t+\epsilon)), \quad t \geq 0.$$

Let us recall the $\mathbb{F}$-supermartingale Z and the strong supermartingale $\widetilde{Z}$ (without right-continuity)

$$Z_t := P(\tau > t|\mathcal{F}_t), \text{ and } \widetilde{Z}_t := P(\tau \geq t|\mathcal{F}_t), \tag{4.1}$$

and the $\mathbb{F}$-martingale m given by

$$m = Z + D^{o,\mathbb{F}}. \tag{4.2}$$

Here, $D^{o,\mathbb{F}}$ is the $\mathbb{F}$-dual optional projection of $I_{]\!]\tau,\infty]\!]}$. Therefore,

$$\widetilde{Z}_+ = Z, \quad \Delta D^{o,\mathbb{F}} = (\widetilde{Z} - Z)I_{]\!]0,+\infty]\!]}, \quad \widetilde{Z}_- = Z_- = {}^{p,\mathbb{F}}(\widetilde{Z}), \text{ on }]\!]0,+\infty]\!].$$

Remark 4.1: The definitions of $Z, \widetilde{Z}$ and m depend on the random time τ and the filtration $\mathbb{F}$. In the following examples, they vary from one to another. With an abuse of notations, we will use $Z, \widetilde{Z}$ and m without specifying random time and filtration, when there is no confusion.

For the reader's convenience, we recall below the definition of honest time (see [17], [83] and the references therein).

Definition 4.1: A random time τ is honest, if for any t, there exists an $\mathcal{F}_t$ measurable random variable T_t such that $\tau I_{\{\tau<t\}} = T_t I_{\{\tau<t\}}$.

A trivial remark is that, in the particular case where τ is an $\mathbb{F}$-stopping time, the enlarged filtration $\mathbb{G}$ and the reference filtration $\mathbb{F}$ are identical. Therefore, no-arbitrage conditions hold.

We denote by $\mathcal{T}_s$ the set of all $\mathbb{F}$-stopping times, $\mathcal{T}_h$ the set of all $\mathbb{F}$-honest times, and $\mathcal{R}$ the set of random times given by

$$\mathcal{R} := \Big\{\sigma \ \text{ r.t.} \ \big| \ \exists\, \Gamma \in \mathcal{A} \text{ and } T \in \mathcal{T}_s \text{ such that } \sigma = T 1\!\!1_{\Gamma} + \infty 1\!\!1_{\Gamma^c}\Big\}. \tag{4.3}$$

Proposition 4.1: *The following inclusions hold*

$$\mathcal{T}_s \subset \mathcal{R} \subset \mathcal{T}_h. \tag{4.4}$$

Proof. The first inclusion is clear. For the inclusion $\mathcal{R} \subset \mathcal{T}_h$, we give, for ease of the reader two different proofs. Let us take $\sigma \in \mathcal{R}$.

(1) On $\{\sigma < t\} = \{T < t\} \cap \Gamma$, we have $\sigma = T \wedge t$ and $T \wedge t$ is $\mathcal{F}_t$-measurable. Thus, σ is an honest time.

(2) Since $\widetilde{Z}_t = 1\!\!1_{\{T \geq t\}} \mathbb{P}\{\Gamma | \mathcal{F}_t\} + \mathbb{P}\{\Gamma^c | \mathcal{F}_t\}$, we derive

$$\begin{aligned} 1\!\!1_{\{\sigma<\infty\}} \widetilde{Z}_\sigma &= 1\!\!1_\Gamma 1\!\!1_{\{T<\infty\}} \widetilde{Z}_T = 1\!\!1_\Gamma 1\!\!1_{\{T<\infty\}} \{1\!\!1_{\{T \geq T\}} \mathbb{P}\{\Gamma | \mathcal{F}_T\} + \mathbb{P}\{\Gamma^c | \mathcal{F}_T\}\} \\ &= 1\!\!1_\Gamma 1\!\!1_{\{T<\infty\}} = 1\!\!1_{\{\sigma<\infty\}}. \end{aligned}$$

Therefore, $\widetilde{Z}_\sigma = 1$, on the set $\{\sigma < \infty\}$. This ends the proof of the proposition. □

The following theorem is our principal result for honest times in this chapter.

Theorem 4.1: *Assume $(S, \mathbb{F})$ is a complete market, τ is an honest time and $m = 1 + \varphi \centerdot S$, where m is defined in (4.2) and $\varphi \in L(S)$. Then the following hold.*

(a) *If $\tau \notin \mathcal{R}$, then the $\mathbb{G}$-predictable process $\varphi^b = \varphi 1\!\!1_{[\![0,\tau]\!]}$ is a classical arbitrage strategy in the market "before τ", i.e., in $(S^\tau, \mathbb{G})$.*

(b) *If τ is not an $\mathbb{F}$-stopping time, and if $\{\tau = \infty\} \in \mathcal{F}_\infty$, then the $\mathbb{G}$-predictable process $\varphi^a = -\varphi 1\!\!1_{]\!]\tau,\nu]\!]}$, with $\mathbb{G}$-stopping time defined as*

$$\nu := \inf \left\{ t > \tau : \widetilde{Z}_t \leq \frac{1 - \Delta D^{o,\mathbb{F}}_\tau}{2} \right\}, \tag{4.5}$$

is a classical arbitrage strategy in the market "after τ", i.e., in $(S - S^\tau, \mathbb{G})$.

Proof. (a) From $m = \widetilde{Z} + D^{o,\mathbb{F}}_-$ and $\widetilde{Z}_\tau = 1$, we deduce that $m_\tau \geq 1$. Since $\tau \notin \mathcal{R}$, we have $\mathbb{P}(m_\tau > 1) = \mathbb{P}(D^{o,\mathbb{F}}_{\tau-} > 0) > 0$. Therefore, the process $\varphi^b = \varphi 1\!\!1_{[\![0,\tau]\!]}$ is an arbitrage strategy in $(S^\tau, \mathbb{G})$.

(b) From $m = Z + D^{o,\mathbb{F}}$, we have for $t > \tau$, $m_t - m_\tau = Z_t - Z_\tau \geq -1$. On the other hand, using $m = \widetilde{Z} + D^{o,\mathbb{F}}_-$, we get, for $t > \tau$, $m_t - m_\tau = \widetilde{Z}_t - 1 + \Delta D^{o,\mathbb{F}}_\tau$. Assumption $\{\tau = \infty\} \in \mathcal{F}_\infty$ ensures that $\widetilde{Z}_\infty = 1\!\!1_{\{\tau=\infty\}}$ and in particular $\{\tau < \infty\} \subset \{\widetilde{Z}_\infty = 0\}$. So, $\mathbb{G}$-stopping time ν defined in (4.5) satisfies $\{\nu < \infty\} = \{\tau < \infty\}$. Then,

$$m_\nu - m_\tau = \widetilde{Z}_\nu - 1 + \Delta D^{o,\mathbb{F}}_\tau \leq \frac{\Delta D^{o,\mathbb{F}}_\tau - 1}{2} \leq 0,$$

and, as τ is not an $\mathbb{F}$-stopping time,

$$\mathbb{P}(m_\nu - m_\tau < 0) = \mathbb{P}(\Delta D^{o,\mathbb{F}}_\tau < 1) > 0.$$

Hence $-\int_\tau^{t\wedge\nu} \varphi_s dS_s = m_{\tau\wedge t} - m_{t\wedge\nu}$ is the value of an admissible self-financing strategy $\varphi^a = -\varphi 1\!\!1_{]\!]\tau,\nu]\!]}$ with initial value 0 and terminal value $m_\tau - m_\nu \geq 0$ satisfying $\mathbb{P}(m_\tau - m_\nu > 0) > 0$. This ends the proof of the theorem. □

We present here two basic examples, in order to show in a first step how arbitrages can occur in a Brownian filtration, and in a second step that discontinuous models present some difficulties.

Example 4.1 (**Brownian Case**) Let $dS_t = S_t \sigma dW_t$, be the price of the risky asset, where W is a Brownian motion and σ a constant. This martingale S goes to 0 a.s. when t goes to infinity. Hence the random time $\tau = \sup\{t : S_t = S^*\}$ where $S^* = \sup_{s\geq 0} S_s$ is a finite honest time, and obviously leads to an arbitrage before τ: at time 0, buy one share of S (at price S_0), borrow S_0, then, at time τ, reimburse the loan S_0 and sell the share of the asset at price S_τ. The gain is $S_\tau - S_0 > 0$ with an initial wealth null. There are also arbitrages after τ: at time τ, take a short position on S, i.e., hold a self financing portfolio with value V such that $dV_t = -dS_t, V_\tau = 0$. Usually shortselling positions are not admissible, since $V_t = -S_t + S_\tau$ is not bounded below. Here $-S_t + S_\tau$ is positive, hence shortselling is an arbitrage opportunity.

Example 4.2 (**Poisson case**) Let N be a Poisson process with intensity λ and $M_t := N_t - \lambda t$. We define the price process S as $dS_t = S_{t-}\psi dM_t, S_0 = 1$ with ψ a constant satisfying $\psi > -1$ and $\psi \neq 0$, so that

$$S_t = \exp(-\lambda\psi t + \ln(1+\psi)N_t)\,.$$

Since $\frac{N_t}{t}$ goes to λ a.s. when t goes to infinity and $\ln(1+\psi) - \psi < 0$, S_t goes to 0 a.s. when t goes to infinity. The random time

$$\tau = \sup\{t \,:\, S_t = S^*\}$$

with $S^* = \sup_{s \geq 0} S_s$ is a finite honest time. The arbitrage strategies are:

(a) If $\psi > 0$, then $S_\tau \geq S_0$ and an arbitrage opportunity is realized at time τ, with a long position in the stock. If $\psi < 0$, then the arbitrage is not so obvious. We shall discuss that with more details in Section 4.3.

(b) There are arbitrages after τ, selling at time τ a contingent claim with payoff 1, paid at the first time ϑ after τ when $S_t > \sup_{s \leq \tau} S_s$. For $\psi > 0$, it reduces to $S_\tau = \sup_{s \leq \tau} S_s$, and, for $\psi < 0$, one has $S_{\tau-} = \sup_{s \leq \tau} S_s$. At time $t_0 = \tau$, the non informed buyer will agree to pay a positive price, the informed seller knows that the exercise will be never done.

4.2 Classical Arbitrages in a Brownian Filtration

In this section, we develop practical market models S and honest times τ within the Brownian filtration for which we compute explicitly the arbitrage opportunities for both before and after τ. For other examples of honest times, and associated classical arbitrages we refer the reader to Fontana et al. [63] (note that the arbitrages constructed in [63] are different from our arbitrages). Throughout this subsection, we assume given a one-dimensional Brownian motion W and $\mathbb{F}$ is its augmented natural filtration. The market model is represented by the bank account whose process is the constant one and one stock whose price process is given by

$$S_t = \exp(\sigma W_t - \frac{1}{2}\sigma^2 t), \qquad \sigma > 0 \text{ given.} \tag{4.6}$$

Remark 4.2: In this subsection, only the market S is fixed. The random time τ changes from one to another. With an abuse of notations, τ appears in different examples, when there is no confusion.

It is worth mentioning that in this context of Brownian filtration, for any process V with locally integrable variation, its $\mathbb{F}$-dual optional projection is equal to its $\mathbb{F}$-dual predictable projection, i.e., $V^{o,\mathbb{F}} = V^{p,\mathbb{F}}$.

4.2.1 Last Passage Time at a Given Level

Proposition 4.2: *Consider the market S in (4.6) and the following random times*

$$\tau := \sup\{t \,:\, S_t = a\} \qquad \text{and} \qquad \nu := \inf\{t > \tau \;\big|\; S_t \leq \frac{a}{2}\},$$

where $0 < a < 1$. Then, the following assertions hold.

(a) *The model "before τ" $(S^\tau, \mathbb{G})$ admits a classical arbitrage opportunity given by the $\mathbb{G}$-predictable process*

$$\varphi^b = \frac{1}{a} \mathbb{1}_{\{S \leq a\}} I_{]\!]0,\tau]\!]}.$$

(b) *The model "after τ" $(S - S^\tau, \mathbb{G})$ admits a classical arbitrage opportunity given by $\mathbb{G}$-predictable process*

$$\varphi^a = -\frac{1}{a} \mathbb{1}_{\{S \leq a\}} I_{]\!]\tau,\nu]\!]}.$$

Proof. It is clear that τ is a finite honest time, and does not belong to the set $\mathcal{R}$ defined in (4.3). Thus τ fulfills the assumptions of Theorem 4.1. We now compute the predictable process φ such that $m = 1 + \varphi \centerdot S$. To this end, we calculate Z as follows. Using Jeanblanc et al. [79][exercise 1.2.3.10], we derive

$$\mathbb{P}\left(\tau \le t | \mathcal{F}_t\right) = \mathbb{P}\left(\sup_{t<u} S_u \le a | \mathcal{F}_t\right) = \mathbb{P}\left(\sup_u \widetilde{S}_u \le \frac{a}{S_t} | \mathcal{F}_t\right) = \Psi\left(\frac{a}{S_t}\right)$$

where $\widetilde{S}_u = \exp(\sigma \widetilde{W}_u - \frac{1}{2}\sigma^2 u)$, $\widetilde{W}$ is a Brownian motion independent of $\mathbb{F}$ and $\Psi(x) = \mathbb{P}\left(\sup_u \widetilde{S}_u \le x\right) = \mathbb{P}(\frac{1}{U} \le x) = \mathbb{P}(\frac{1}{x} \le U) = (1 - \frac{1}{x})^+$, where U is a random variable with uniform law. Thus we get $Z_t = 1 - (1 - S_t/a)^+$ (in particular $Z_\tau = \widetilde{Z}_\tau = 1$ and τ is honest), and

$$dZ_t = \mathbb{1}_{\{S_t \le a\}} \frac{1}{a} dS_t - \frac{1}{2a} d\ell_t^a$$

where ℓ^a is the local time of S at the level a (see page 252 of He et al. [71] for the definition of the local time). Therefore, we deduce that

$$m = 1 + \varphi \centerdot S, \quad \text{where } \varphi_t := \mathbb{1}_{\{S_t \le a\}} \frac{1}{a}.$$

Note that $\nu := \inf\{t > \tau \mid S_t \le \frac{a}{2}\} = \inf\{t > \tau \mid 1 - (1 - \frac{S_t}{a})^+ \le \frac{1}{2}\}$, so ν coincides with (4.5). Theorem 4.1 ends the proof of the proposition. □

4.2.2 Last Passage Time at a Level before Maturity

Our second example, in this subsection, takes into account finite horizon. In this example, we introduce the following notation

$$H(z, y, s) := e^{-zy} \mathbb{N}\left(\frac{zs - y}{\sqrt{s}}\right) + e^{zy} \mathbb{N}\left(\frac{-zs - y}{\sqrt{s}}\right), \tag{4.7}$$

where $\mathbb{N}(x)$ is the cumulative distribution of the standard normal distribution.

Proposition 4.3: *Consider the market S in (4.6) and the following random time*

$$\tau_1 := \sup\{t \leq 1 \,:\, S_t = b\}$$

where b is a real number, $0 < b < 1$. Let V and β be given by

$$V_t := \alpha - \gamma t - W_t, \quad \textit{where } \alpha = \frac{\ln b}{\sigma} \textit{ and } \gamma = -\frac{\sigma}{2}$$
$$\beta_t := e^{\gamma V_t}\left(\gamma H(\gamma, |V_t|, 1-t) + sgn(V_t)\, H'_x(\gamma, |V_t|, 1-t)\right),$$

with H defined in (4.7), and let ν be as in (4.5). Then, the following hold.

(a) *The model "before τ_1", i.e. $(S^{\tau_1}, \mathbb{G})$ admits a classical arbitrage opportunity given by the $\mathbb{G}$-predictable process*

$$\theta^b := \frac{1}{\sigma S_t}\beta_t I_{[\![0,\tau_1]\!]}.$$

(b) *The model "after τ_1", i.e. $(S - S^{\tau_1}, \mathbb{G})$ admits a classical arbitrage opportunity given by $\mathbb{G}$-predictable process*

$$\theta^a := -\frac{1}{\sigma S_t}\beta_t I_{]\!]\tau_1,\nu]\!]}.$$

Proof. The proof of this proposition follows from Theorem 4.1 as long as we can write the martingale m as a stochastic integral with respect to S. This is the main focus of the remaining part of this proof. The time τ_1 is a finite honest time. From

$$\tau_1 = \sup\{t \leq 1 \,:\, \gamma t + W_t = \alpha\} = \sup\{t \leq 1 \,:\, V_t = 0\}.$$

and setting $T_0(V) = \inf\{t \,:\, V_t = 0\}$, we obtain, using standard computations (see Jeanblanc et al. [79] p. 145-148)

$$1 - Z_t = \mathbb{P}(\tau_1 \leq t|\mathcal{F}_t) = (1 - e^{\gamma V_t} H(\gamma, |V_t|, 1-t))\mathbb{1}_{\{T_0(V) \leq t \leq 1\}} + \mathbb{1}_{\{t>1\}},$$

where H is given in (4.7). In particular $Z_\tau = \widetilde{Z}_\tau = 1$ and τ is an honest time. Using Itô's lemma, we obtain the decomposition of $1 - e^{\gamma V_t} H(\gamma, |V_t|, 1-t)$ as a semimartingale. The martingale part of Z is given by $dm_t = \beta_t dW_t = \frac{1}{\sigma S_t}\beta_t dS_t$. This ends the proof of the proposition. □

4.3 Classic Arbitrages in a Poisson Filtration

Throughout this subsection, we suppose given a Poisson process N, with intensity rate $\lambda > 0$, and natural filtration $\mathbb{F}$. The stock price process is given by

$$dS_t = S_{t-}\psi dM_t, \quad M_t := N_t - \lambda t, \ \ S_0 = 1, \tag{4.8}$$

or equivalently $S_t = \exp(-\lambda\psi t + \ln(1+\psi)N_t)$, where $\psi > -1$ and $\psi \neq 0$. In what follows, we introduce the notations

$$\alpha := \ln(1+\psi), \quad \mu := \frac{\lambda\psi}{\ln(1+\psi)} \ \text{ and } \ Y_t := \mu t - N_t, \tag{4.9}$$

so that $S_t = \exp(-\ln(1+\psi)Y_t)$. We associate to the process Y its ruin probability $\Psi(x)$ given by, for $x \geq 0$,

$$\Psi(x) = \mathbb{P}(T^x < \infty), \quad \text{with} \quad T^x = \inf\{t : x + Y_t < 0\}\,. \tag{4.10}$$

Remark 4.3: In this subsection, only the market S is fixed. The random time τ changes from one to another. With an abuse of notations, τ appears in different examples, when there is no confusion.

4.3.1 Last Passage Time at a Given Level

Proposition 4.4: *Consider the market S in (4.8) with the notations (4.9) and (4.10). Suppose that $\psi > 0$ and φ given by*

$$\varphi := \frac{\Psi(Y_- - a - 1)1\!\!1_{\{Y_- \geq a+1\}} - \Psi(Y_- - a)1\!\!1_{\{Y_- \geq a\}} + 1\!\!1_{\{Y_- < a+1\}} - 1\!\!1_{\{Y_- < a\}}}{\psi S_-}.$$

For $0 < b < 1$, consider the following random time

$$\tau := \sup\{t : \ S_t \geq b\} = \sup\{t : Y_t \leq a\}, \ \textit{where} \ \ a := -\ln(b)/\alpha. \tag{4.11}$$

Then the following assertions hold.

(a) *The random time τ is an honest time.*

(b) *The model "before τ" $(S^\tau, \mathbb{G})$ admits a classical arbitrage opportunity given by the $\mathbb{G}$-predictable process $\varphi^b := \varphi I_{[\![0,\tau]\!]}$.*

(c) *The model "after τ" $(S - S^\tau, \mathbb{G})$ admits a classical arbitrage opportunity given by the $\mathbb{G}$-predictable process $\varphi^a := -\varphi I_{]\!]\tau,\nu]\!]}$, with ν as in* (4.5).

Proof. Since $\psi > 0$, one has $\mu > \lambda$ so that Y goes to $+\infty$ as t goes to infinity, and τ is finite. The supermartingale Z associated with the time τ is

$$Z_t = \mathbb{P}(\tau > t|\mathcal{F}_t) = \Psi(Y_t - a)1\!\!1_{\{Y_t \geq a\}} + 1\!\!1_{\{Y_t < a\}} = 1 + 1\!\!1_{\{Y_t \geq a\}}\left(\Psi(Y_t - a) - 1\right),$$

where Ψ is defined in (4.10). We set $\theta = \dfrac{\mu}{\lambda} - 1$, and deduce that $\Psi(0) = (1+\theta)^{-1}$ (see Asmussen [15]). Define $\vartheta_1 = \inf\{t > 0 : Y_t = a\}$ and then, for each $n > 1$, $\vartheta_n = \inf\{t > \vartheta_{n-1} : Y_t = a\}$. It can be proved that the times $(\vartheta_n)_n$ are $\mathbb{F}$-predictable stopping times. The $\mathbb{F}$-dual optional projection $D^{o,\mathbb{F}}$ of the process $1\!\!1_{[\![\tau,\infty[\![}$ equals

$$D^{o,\mathbb{F}} = \frac{\theta}{1+\theta}\sum_n 1\!\!1_{[\![\vartheta_n,\infty[\![}.$$

Indeed, for any $\mathbb{F}$-optional process U, we have

$$\mathbb{E}(U_\tau) = \mathbb{E}\left(\sum \mathbb{1}_{\{\tau=\vartheta_n\}} U_{\vartheta_n}\right) = \mathbb{E}(\sum \mathbb{E}\left(\mathbb{1}_{\{\tau=\vartheta_n\}}|\mathcal{F}_{\vartheta_n}\right) U_{\vartheta_n})$$

and $\mathbb{E}(\mathbb{1}_{\{\tau=\vartheta_n\}}|\mathcal{F}_{\vartheta_n}) = \mathbb{P}(T^0 = \infty) = 1 - \Psi(0) = 1 - (1+\theta)^{-1}$.

As a result the process $D^{o,\mathbb{F}}$ is predictable, and hence $Z = m - D^{o,\mathbb{F}}$ is the Doob-Meyer decomposition of Z. Thus we can get

$$\Delta m = Z - {}^{p,\mathbb{F}}Z,$$

where ${}^{p,\mathbb{F}}Z$ is the $\mathbb{F}$-predictable projection of Z. To calculate ${}^{p,\mathbb{F}}Z$, we write the process Z in a more adequate form. To this end, we first remark that

$$\begin{aligned}
\mathbb{1}_{\{Y \geq a\}} &= \mathbb{1}_{\{Y_- \geq a+1\}} \Delta N + (1 - \Delta N) \mathbb{1}_{\{Y_- \geq a\}} \\
\mathbb{1}_{\{Y < a\}} &= \mathbb{1}_{\{Y_- < a+1\}} \Delta N + (1 - \Delta N) \mathbb{1}_{\{Y_- < a\}}.
\end{aligned}$$

Then, we obtain

$$\begin{aligned}
\Delta m &= \left(\Psi(Y_- - a - 1)\mathbb{1}_{\{Y_- \geq a+1\}} - \Psi(Y_- - a)\mathbb{1}_{\{Y_- \geq a\}}\right) \Delta N \\
&\qquad + \left(\mathbb{1}_{\{Y_- < a+1\}} - \mathbb{1}_{\{Y_- < a\}}\right) \Delta N \\
&= \psi S_- \varphi \Delta M = \varphi \Delta S.
\end{aligned}$$

Since the two martingales m and S are purely discontinuous, we deduce that $m = 1 + \varphi \centerdot S$. Therefore, the proposition follows from Theorem 4.1. □

4.3.2 Time of Supremum on Fixed Time Horizon

The second example requires the following notations $S^*_t := \sup_{s \leq t} S_s$ and

$$\Psi^{(1)}(x,t) := \mathbb{P}(S^*_t > x), \quad \widetilde{\Psi}(x,t) := \mathbb{P}\left(\sup_{s<t} S_s < x\right), \quad \widehat{\Psi}(t) := \widetilde{\Psi}(x,1). \tag{4.12}$$

Proposition 4.5: *Consider the market S in (4.8) with the notations (4.9) and (4.12) and the random time τ defined by*

$$\tau = \sup\{t \leq 1 : S_t = S_t^*\}, \tag{4.13}$$

where $S_t^* = \sup_{s\leq t} S_s$. Then, the following assertions hold.

(a) *The random time τ is an honest time.*

(b) *For $\psi > 0$, define the $\mathbb{G}$-predictable process η as*

$$\begin{aligned}\eta_t := & \mathbb{1}_{\{t<1\}}\left[\Psi^{(1)}\left(\max(\frac{S_{t-}^*}{S_{t-}(1+\psi)},1),1-t\right) - \Psi^{(1)}\left(\frac{S_{t-}^*}{S_{t-}},1-t\right)\right] \\ & + \mathbb{1}_{\{S_{t-}^* < S_{t-}(1+\psi)\}}\,\widehat{\Psi}(1-t) \\ & + \left[\mathbb{1}_{\{\max(S_{1-}^*,S_{1-}(1+\psi))=S_0\}} - \mathbb{1}_{\{\max(S_{1-}^*,S_{1-})=S_0\}}\right]\mathbb{1}_{\{t=1\}}.\end{aligned} \tag{4.14}$$

Then, $\eta^b := \eta\mathbb{1}_{[\![0,\tau]\!]}$ is an arbitrage opportunity for the model $(S^\tau, \mathbb{G})$, and $\eta^a := -\eta I_{]\!]\tau,\nu]\!]}$ is an arbitrage opportunity for the model $(S - S^\tau, \mathbb{G})$. Here $\Psi^{(1)}$ and $\widehat{\Psi}$ are defined in (4.12), and ν is defined similarly as in (4.5).

(c) *For $-1 < \psi < 0$, define the $\mathbb{G}$-predictable process*

$$\eta_t^{(1)} := \frac{\psi I_{\{S_t^*=S_{t-}\}}\widehat{\Psi}(\frac{1}{1+\psi},1-t) + \Psi^{(1)}(\frac{S_t^*}{S_{t-}(1+\psi)},1-t) - \Psi^{(1)}(\frac{S_t^*}{S_{t-}},1-t)}{\psi S_{t-}}. \tag{4.15}$$

Then, $\eta^{(1),b} := \eta^{(1)}\mathbb{1}_{[\![0,\tau]\!]}$ is an arbitrage opportunity for the model $(S^\tau, \mathbb{G})$, and $\eta^{(1),a} := -\eta^{(1)} I_{]\!]\tau,\nu]\!]}$ is an arbitrage opportunity for the model $(S - S^\tau, \mathbb{G})$.

Proof. Note that, if $-1 < \psi < 0$ the process S^* is continuous, $S_\tau < S_\tau^* = \sup_{t\in[0,1]} S_t$ on the set $(\tau < 1)$ and $S_{\tau-} = S_{\tau-}^* = \sup_{t\in[0,1]} S_t$. If $\psi > 0$, $S_{\tau-} < S_{\tau-}^* < \sup_{t\in[0,1]} S_t$ on the set $(\tau < 1)$.

Define the sets $(E_n)_{n=0}^\infty$ such that $E_0 = \{\tau = 1\}$ and $E_n = \{\tau = T_n\}$ with $n \geq 1$. The sequence $(E_n)_{n=0}^\infty$ forms a partition of Ω. Then, $\tau = \mathbb{1}_{E_0} + \sum_{n=1}^\infty T_n\mathbb{1}_{E_n}$. Note that τ is not an $\mathbb{F}$ stopping time since $E_n \notin \mathcal{F}_{T_n}$ for any $n \geq 1$.

The supermartingale Z associated with the honest time τ is

$$\begin{aligned}Z_t = \mathbb{P}(\tau > t|\mathcal{F}_t) = \mathbb{P}(\sup_{s\in(t,1]} S_s > \sup_{s\in[0,t]} S_s|\mathcal{F}_t) = \mathbb{P}(\sup_{s\in[0,1-t]} \widehat{S}_s > \frac{S_t^*}{S_t}|\mathcal{F}_t)\\ = 1\!\!1_{(t<1)}\Psi^{(1)}(\frac{S_t^*}{S_t}, 1-t),\end{aligned}$$

with $\widehat{S}$ an independent copy of S and $\Psi^{(1)}(x,t)$ is given by (4.12).

As $\{\tau = T_n\} \subset \{\tau \leq T_n\} \subset \{Z_{T_n} < 1\}$, we have

$$Z_\tau = 1\!\!1_{\{\tau=1\}} Z_1 + \sum_{n=1}^{\infty} 1\!\!1_{\{\tau=T_n\}} Z_{T_n} < 1, \quad \text{and} \quad \{\widetilde{Z} = 0 < Z_-\} = \emptyset.$$

In the following we will prove assertion (b). Thus, we suppose that $\psi > 0$, and we calculate

$$\begin{aligned}D_t^{o,\mathbb{F}} &= \mathbb{P}(\tau = 1|\mathcal{F}_1) 1\!\!1_{\{t\geq 1\}} + \sum_n \mathbb{P}(\tau = T_n|\mathcal{F}_{T_n}) 1\!\!1_{\{t\geq T_n\}}\\ &= 1\!\!1_{\{S_1^*=S_0,\, t\geq 1\}} + \sum_n 1\!\!1_{\{T_n<1,\, S_{T_n-}^*<S_{T_n}\}} \mathbb{P}(\sup_{s\in[T_n,1[} S_s \leq S_{T_n}|\mathcal{F}_{T_n}) 1\!\!1_{\{t\geq T_n\}}\\ &= 1\!\!1_{\{S_1^*=S_0,\, t\geq 1\}} + \sum_n 1\!\!1_{\{T_n<1,\, S_{T_n-}^*<S_{T_n-}(1+\psi)\}}\, \widehat{\Psi}(1-T_n) 1\!\!1_{\{t\geq T_n\}},\end{aligned}$$

with $\widehat{\Psi}$ is given by (4.12). As before, we write

$$\begin{aligned}D_t^{o,\mathbb{F}} &= 1\!\!1_{\{S_1^*=S_0\}} 1\!\!1_{\{t\geq 1\}} + \sum_{s\leq t} 1\!\!1_{\{s<1\}} 1\!\!1_{\{S_{s-}^*<S_{s-}(1+\psi)\}}\, \widehat{\Psi}(1-s)\Delta N_s\\ &= 1\!\!1_{\{S_1^*=S_0\}} 1\!\!1_{\{t\geq 1\}} + \int_0^{t\wedge 1} 1\!\!1_{\{S_{s-}^*<S_{s-}(1+\psi)\}}\, \widehat{\Psi}(1-s)\, dM_s\\ &\quad +\lambda \int_0^{t\wedge 1} 1\!\!1_{\{S_{s-}^*<S_{s-}(1+\psi)\}}\, \hat{\Psi}(1-s) ds.\end{aligned}$$

Remark that we have

$$1\!\!1_{\{S_1^*=S_0\}} = \left[1\!\!1_{\{\max(S_{1-}^*, S_{1-}(1+\psi))=S_0\}} - 1\!\!1_{\{\max(S_{1-}^*, S_{1-})=S_0\}}\right] \Delta M_1 + 1\!\!1_{\{\max(S_{1-}^*, S_{1-})=S_0\}}$$

and

$$\Delta m = \Delta Z + \Delta D^{o,\mathbb{F}} = Z - {}^{p,\mathbb{F}}Z + \Delta D^{o,\mathbb{F}} - {}^{p,\mathbb{F}}(\Delta D^{o,\mathbb{F}}).$$

Then we re-write the process Z as follows

$$Z = 1\!\!1_{[\![0,1[\![}\Psi^{(1)}\left(\max(\frac{S^*_-}{S_-(1+\psi)},1),1-t\right)\Delta M + (1-\Delta M)I_{[\![0,1[\![}\Psi^{(1)}\left(\frac{S^*_-}{S_-},1-t\right).$$

This implies that

$$Z - {}^{p,\mathbb{F}}Z = 1\!\!1_{[\![0,1[\![}\left[\Psi^{(1)}\left(\max(\frac{S^*_-}{S_-(1+\psi)},1),1-t\right) - \Psi^{(1)}\left(\frac{S^*_-}{S_-},1-t\right)\right]\Delta M.$$

Thus by combining all these remarks, we deduce that

$$m = m_0 + \eta \centerdot S, \quad \text{where } \eta \ \text{ is given by } \ (4.14).$$

Then, the assertion (b) follows immediately from Theorem 4.1.

Next, we will prove assertion (c). Suppose that $-1 < \psi < 0$, and we calculate

$$\begin{aligned}
D^{o,\mathbb{F}}_t &= \mathbb{P}(\tau = 1|\mathcal{F}_1)1\!\!1_{\{t\geq 1\}} + \sum_n \mathbb{P}(\tau = T_n|\mathcal{F}_{T_n})1\!\!1_{\{t\geq T_n\}}\\
&= 1\!\!1_{\{S^*_1=S_1,\, t\geq 1\}} + \sum_n 1\!\!1_{\{T_n<1,\, S^*_{T_n}=S_{T_n-}\}}\mathbb{P}(\sup_{s\in[T_n,1[} S_s < S_{T_n-}|\mathcal{F}_{T_n})1\!\!1_{\{t\geq T_n\}}\\
&= 1\!\!1_{\{S^*_1=S_1\}}1\!\!1_{\{t\geq 1\}} + \sum_n 1\!\!1_{\{T_n<1\}}1\!\!1_{\{S^*_{T_n}=S_{T_n-}\}}\widetilde{\Psi}(\frac{S_{T_n-}}{S_{T_n}},1-T_n)1\!\!1_{\{t\geq T_n\}},
\end{aligned}$$

with $\widetilde{\Psi}(x,t)$ is given by (4.12). In order to find the compensator of $D^{o,\mathbb{F}}$, we write

$$\begin{aligned}
D^{o,\mathbb{F}}_t &= 1\!\!1_{\{S^*_1=S_1\}}1\!\!1_{\{t\geq 1\}} + \sum_{s\leq t} 1\!\!1_{\{s<1\}}1\!\!1_{\{S^*_s=S_{s-}\}}\widetilde{\Psi}(\frac{1}{1+\psi},1-s)\,\Delta N_s\\
&= 1\!\!1_{\{S^*_1=S_1\}}1\!\!1_{\{t\geq 1\}} + \int_0^{t\wedge 1} 1\!\!1_{\{S^*_s=S_{s-}\}}\widetilde{\Psi}(\frac{1}{1+\psi},1-s)\,dM_s\\
&\quad +\lambda\int_0^{t\wedge 1} 1\!\!1_{\{S^*_s=S_{s-}\}}\widetilde{\Psi}(\frac{1}{1+\psi},1-s)\,ds.
\end{aligned}$$

As a result, due to the continuity of the process S^*, we get

$$D_t^{o,\mathbb{F}} - {}^{p,\mathbb{F}}(D^{o,\mathbb{F}})_t = I_{\{S_t^*=S_{t-}\}}\widetilde{\Psi}(\frac{1}{1+\psi}, 1-t)\Delta M_t,$$
$$Z_t - {}^{p,\mathbb{F}}Z_t = \left[\Psi^{(1)}(\frac{S_t^*}{S_{t-}(1+\psi)}, 1-t) - \Psi^{(1)}(\frac{S_t^*}{S_{t-}}, 1-t)\right]\Delta N_t.$$

This implies that

$$\Delta m_t = Z_t - {}^{p,\mathbb{F}}Z_t + D_t^{o,\mathbb{F}} - {}^{p,\mathbb{F}}(D^{o,\mathbb{F}})_t = \psi 1\!\!1_{\{S_t^*=S_{t-}\}}\widetilde{\Psi}(\frac{1}{1+\psi}, 1-t)\}\Delta N_t$$
$$+\left\{\Psi^{(1)}\left(\frac{S_t^*}{S_{t-}(1+\psi)}, 1-t\right) - \Psi^{(1)}\left(\frac{S_t^*}{S_{t-}}, 1-t\right)\right\}\Delta N_t.$$

Since m and S are pure discontinuous $\mathbb{F}$-local martingales, we conclude that m can be written in the form of

$$m = m_0 + \eta^{(1)} \centerdot S, \quad \text{where} \quad \eta^{(1)} \quad \text{is given by (4.15)}$$

and the proof of the assertion (c) follows immediately from Theorem 4.1. This ends the proof of the proposition. □

4.3.3 Time of Overall Supremum

Below, we will present our last example of this subsection. The analysis of this example is based on the following three functions, where $S^* = \sup_{u\in[0,+\infty]} S_u$.

$$\Psi^{(2)}(x) = \mathbb{P}(S^* > x), \quad \widehat{\Psi^{(2)}} = \mathbb{P}(S^* \leq 1), \quad \widetilde{\Psi^{(2)}}(x) = \mathbb{P}(S^* < x). \qquad (4.16)$$

Proposition 4.6: *Consider the market S in (4.8) with the notations (4.9) and (4.16) and the random time $\tau^{(2)}$ given by*

$$\tau^{(2)} = \sup\{t : S_t = S_t^*\}. \qquad (4.17)$$

Then, the following assertions hold.

(a) *The random time $\tau^{(2)}$ is an honest time.*

(b) *For $\psi > 0$, define the $\mathbb{G}$-predictable process φ as*

$$\varphi_t := \frac{\mathbb{1}_{\{S^*_{t-} < S_{t-}(1+\psi)\}} \widehat{\Psi^{(2)}} + \Psi^{(2)}\left(\max(\frac{S^*_{t-}}{S_{t-}(1+\psi)}, 1)\right) - \Psi^{(2)}(\frac{S^*_{t-}}{S_{t-}})}{S_{t-}\psi}. \quad (4.18)$$

Then, $\varphi^b := \varphi \mathbb{1}_{[\![0,\tau^{(2)}]\!]}$ is an arbitrage opportunity for the model $(S^{\tau^{(2)}}, \mathbb{G})$, and $\varphi^a := -\varphi I_{]\!]\tau^{(2)},\nu]\!]}$ is an arbitrage opportunity for the model $(S - S^{\tau^{(2)}}, \mathbb{G})$. Here $\Psi^{(2)}$ and $\widehat{\Psi^{(2)}}$ are defined in (4.16), and ν is defined in similar way as in (4.5).

(c) *For $-1 < \psi < 0$, define the $\mathbb{G}$-predictable process φ as*

$$\varphi^{(2)} := \frac{\Psi^{(2)}(\frac{S^*}{S_-(1+\psi)}) - \Psi^{(2)}(\frac{S^*}{S_-}) + \mathbb{1}_{\{S^*=S_-\}} \widetilde{\Psi^{(2)}}(\frac{1}{1+\psi})\psi}{\psi S_-}. \quad (4.19)$$

Then, $\varphi^{(2),b} := \varphi^{(2)} \mathbb{1}_{[\![0,\tau^{(2)}]\!]}$ is an arbitrage opportunity for the model $(S^{\tau^{(2)}}, \mathbb{G})$, and $\varphi^{(2),a} := -\varphi^{(2)} I_{]\!]\tau^{(2)},\nu]\!]}$ is an arbitrage opportunity for the model $(S - S^{\tau^{(2)}}, \mathbb{G})$. Here again ν is defined as in (4.5).

Proof. It is clear that $\tau^{(2)}$ is an $\mathbb{F}$-honest time. Let us note that $\tau^{(2)}$ is finite and, as before, if $-1 < \psi < 0$, $S_{\tau^{(2)}} < S^*_{\tau^{(2)}} = \sup_t S_t$ and S^* is continuous and if $\psi > 0$, $S_{\tau^{(2)}} = S^*_{\tau^{(2)}} = \sup_t S_t$.

The supermartingale Z associated with the honest time $\tau^{(2)}$ is

$$Z_t = \mathbb{P}(\sup_{s\in(t,\infty]} S_s > \sup_{s\in[0,t]} S_s | \mathcal{F}_t) = \mathbb{P}(\sup_{s\in[0,\infty]} \widehat{S}_s > \frac{S^*_t}{S_t} | \mathcal{F}_t) = \Psi^{(2)}\left(\frac{S^*_t}{S_t}\right),$$

with $\widehat{S}$ an independent copy of S and $\Psi^{(2)}$ is given by (4.16). As a result, we deduce that $Z_{\tau^{(2)}} < 1$.

In the following, we will prove assertion (b). If $\psi > 0$, by putting $(T_n)_n$ the sequence of jumps of the Poisson process N, we derive

$$\begin{aligned} D_t^{o,\mathbb{F}} &= \sum_n \mathbb{P}(\tau^{(2)} = T_n|\mathcal{F}_{T_n})\mathbb{1}_{\{t\geq T_n\}} \\ &= \sum_n \mathbb{1}_{\{S^*_{T_n-}<S_{T_n}\}}\mathbb{P}(\sup_{s\geq T_n} S_s \leq S_{T_n}|\mathcal{F}_{T_n})\mathbb{1}_{\{t\geq T_n\}} \\ &= \sum_n \mathbb{1}_{\{S^*_{T_n-}<S_{T_n-}(1+\psi)\}}\widehat{\Psi^{(2)}}\mathbb{1}_{\{t\geq T_n\}}, \end{aligned}$$

with $\widehat{\Psi^{(2)}} = \mathbb{P}(\sup_s S_s \leq 1)$ given by (4.16).

To get the compensator of $D^{o,\mathbb{F}}$, we rewrite it as

$$\begin{aligned} D_t^{o,\mathbb{F}} &= \sum_{s\leq t} \mathbb{1}_{\{S^*_{s-}<S_{s-}(1+\psi)\}}\widehat{\Psi^{(2)}}\Delta N_s \\ &= \int_0^t \mathbb{1}_{\{S^*_{s-}<S_{s-}(1+\psi)\}}\widehat{\Psi^{(2)}}dM_s + \lambda\int_0^t \mathbb{1}_{\{S^*_{s-}<S_{s-}(1+\psi)\}}\widehat{\Psi^{(2)}}ds. \end{aligned}$$

Now as we did for the previous propositions, we calculate the jumps of m. To this end, we re-write Z as follows

$$Z = \left[\Psi^{(2)}\left(\max(\frac{S^*_-}{S_-(1+\psi)},1)\right) - \Psi^{(2)}\left(\frac{S^*_-}{S_-}\right)\right]\Delta M + \Psi^{(2)}\left(\frac{S^*_-}{S_-}\right).$$

This implies that

$$Z - {}^{p,\mathbb{F}}Z = \left[\Psi^{(2)}\left(\max(\frac{S^*_-}{S_-(1+\psi)},1)\right) - \Psi^{(2)}\left(\frac{S^*_-}{S_-}\right)\right]\Delta M.$$

Hence, we derive

$$\Delta m = \left[\mathbb{1}_{\{S^*_{s-}<S_{s-}(1+\psi)\}}\widehat{\Psi^{(2)}} + \Psi^{(2)}\left(\max(\frac{S^*_-}{S_-(1+\psi)},1)\right) - \Psi^{(2)}\left(\frac{S^*_-}{S_-}\right)\right]\Delta M.$$

Since both martingales m and M are purely discontinuous, we deduce that $m = m_0 + \varphi \centerdot S$, where φ is given by (4.18). Then, the assertion (b) follows immediately from Theorem 4.1.

In the following, we prove assertion (c). If $-1 < \psi < 0$, we calculate

$$
\begin{aligned}
D_t^{o,\mathbb{F}} &= \sum_n \mathbb{P}(\tau^{(2)} = T_n | \mathcal{F}_{T_n}) \mathbb{1}_{\{t \geq T_n\}} \\
&= \sum_n \mathbb{1}_{\{S^*_{T_n} = S_{T_n-}\}} \mathbb{P}\left(\sup_{s \geq T_n} S_s < S_{T_n-} | \mathcal{F}_{T_n} \right) \mathbb{1}_{\{t \geq T_n\}} \\
&= \sum_n \mathbb{1}_{\{S^*_{T_n} = S_{T_n-}\}} \widetilde{\Psi^{(2)}} \left(\frac{S_{T_n-}}{S_{T_n}} \right) \mathbb{1}_{\{t \geq T_n\}},
\end{aligned}
$$

with $\widetilde{\Psi^{(2)}}(x)$ given in (4.16). Therefore,

$$
\begin{aligned}
D_t^{o,\mathbb{F}} &= \sum_{s \leq t} \mathbb{1}_{\{S^*_s = S_{s-}\}} \widetilde{\Psi^{(2)}} \left(\frac{1}{1+\psi} \right) \Delta N_s \\
&= \int_0^t \mathbb{1}_{\{S^*_s = S_{s-}\}} \widetilde{\Psi^{(2)}} \left(\frac{1}{1+\psi} \right) dM_s + \lambda \int_0^t \mathbb{1}_{\{S^*_s = S_{s-}\}} \widetilde{\Psi^{(2)}} \left(\frac{1}{1+\psi} \right) ds.
\end{aligned}
$$

Since in the case of $\psi < 0$, the process S^* is continuous, we obtain

$$
\begin{aligned}
Z - {}^{p,\mathbb{F}}Z &= \left[\Psi^{(2)} \left(\frac{S^*}{S_-(1+\psi)} \right) - \Psi^{(2)} \left(\frac{S^*}{S_-} \right) \right] \Delta N, \\
D^{o,\mathbb{F}} - {}^{p,\mathbb{F}}(D^{o,\mathbb{F}}) &= \mathbb{1}_{\{S^* = S_-\}} \widetilde{\Psi^{(2)}} \left(\frac{1}{1+\psi} \right) \Delta M.
\end{aligned}
$$

Therefore, we conclude that

$$
\begin{aligned}
\Delta m &= Z - {}^{p,\mathbb{F}}Z + D^{o,\mathbb{F}} - {}^{p,\mathbb{F}}(D^{o,\mathbb{F}}) \\
&= \left\{ \Psi^{(2)} \left(\frac{S^*}{S_-(1+\psi)} \right) - \Psi^{(2)} \left(\frac{S^*}{S_-} \right) + \mathbb{1}_{\{S^* = S_-\}} \widetilde{\Psi^{(2)}} \left(\frac{1}{1+\psi} \right) \psi \right\} \Delta N.
\end{aligned}
$$

This implies that the martingale m has the form of $m = 1 + \varphi^{(2)} \centerdot S$, where $\varphi^{(2)}$ is given by (4.19) and assertion (c) follows immediately from Theorem 4.1, and the proof of the proposition is completed. □

4.4 Classic Arbitrages in Discrete Time

In this section, we consider a two period market model, where we calculate explicitly the arbitrage opportunities. On the stochastic basis $(\Omega, \mathbb{A}, \mathbb{F} := (\mathcal{F}_n)_{0\leq n\leq 2}, \mathbb{P})$, we assume given a risky asset $S := (S_n)_{0\leq n\leq 2}$, where $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$ represents the uncertainties and the natural filtration $\mathbb{F} := (\mathcal{F}_n)_{0\leq n\leq 2}$ is given by

$$\mathcal{F}_0 = \{\emptyset, \Omega\},\ \mathcal{F}_1 = \{\emptyset, \Omega, \{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}\}, \text{and } \mathcal{F}_2 = \sigma(\{\emptyset, \Omega, \{\omega_1\}, \{\omega_2\}, \{\omega_3\}, \{\omega_4\}\}).$$

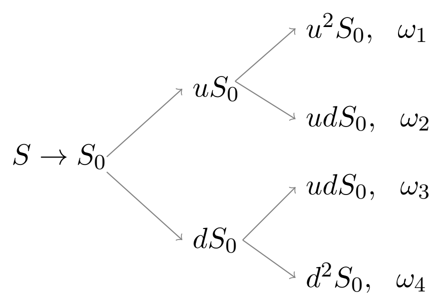
Let u and d be two constants such that $u > 1$ and $0 < d < 1$. Assume that

$$S_1(\{\omega_1, \omega_2\}) = uS_0, \quad S_1(\{\omega_3, \omega_4\}) = dS_0,$$
$$S_2(\{\omega_1\}) = u^2 S_0,\ S_2(\{\omega_2\}) = udS_0,\ S_2(\{\omega_3\}) = udS_0, S_2(\{\omega_4\}) = d^2 S_0.$$

The probability that the stock price will increase (or decrease) is p (or $q = 1 - p$). For the sake of simplicity, we assume $pu + (1 - p)d = 1$, i.e. S is an $\mathbb{F}$-martingale under the physical probability

$$\mathbb{P} = (\mathbb{P}(\omega_1), \mathbb{P}(\omega_2), \mathbb{P}(\omega_3), \mathbb{P}(\omega_4)) = (p^2, pq, pq, q^2).$$

The evolution of the stock price S through time is illustrated as



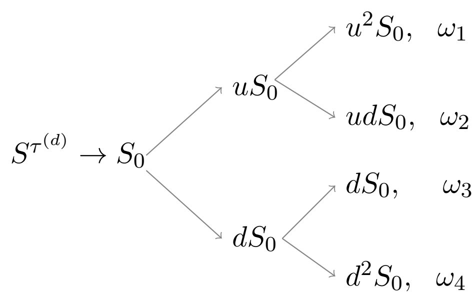
Consider the random time

$$\tau^{(d)} = \begin{cases} 1, & \text{on } \{\omega_3\} \\ 2, & \text{otherwise.} \end{cases} \tag{4.20}$$

Apparently, $\tau^{(d)}$ is not an $\mathbb{F}$-stopping time since $\{\tau^{(d)} = 1\} \notin \mathcal{F}_1$. A straightforward calculation shows $S^{\tau^{(d)}} := (S_{n\wedge\tau^{(d)}})_{0\le n\le 2}$ takes the following values:

$$S_0^{\tau^{(d)}} = S_0, \quad S_1^{\tau^{(d)}}(\{\omega_1,\omega_2\}) = uS_0, \quad S_1^{\tau^{(d)}}(\{\omega_3,\omega_4\}) = dS_0,$$
$$S_2^{\tau^{(d)}}(\{\omega_1\}) = u^2S_0,\; S_2^{\tau^{(d)}}(\{\omega_2\}) = udS_0,\; S_2^{\tau^{(d)}}(\{\omega_3\}) = dS_0, S_2^{\tau^{(d)}}(\{\omega_4\}) = d^2S_0.$$

The evolution of the stock price $S^{\tau^{(d)}}$ through time is illustrated as



The progressive enlarged filtration $\mathbb{G}$ associated to the random time $\tau^{(d)}$ is given by

$$\mathcal{G}_0 = \{\emptyset,\Omega\},\; \mathcal{G}_1 = \sigma(\emptyset,\Omega,\{\omega_1,\omega_2\},\{\omega_3\},\{\omega_4\}), \text{and } \mathcal{G}_2 = \sigma(\emptyset,\Omega,\{\omega_1\},\{\omega_2\},\{\omega_3\},\{\omega_4\}).$$

For the random time $\tau^{(d)}$, we define the stopping times:

$$R_1 := \inf\{n\ge 1 : Z_n = 0\}, \quad R_2 := \inf\{n \ge 1 : Z_{n-1} = 0\} \text{ and } R_3 := \inf\{n\ge 1 : \widetilde{Z}_n = 0\},$$
$$\sigma_1 := \inf\{n\ge 1 : Z_n < 1\}, \quad \sigma_2 := \inf\{n\ge 1 : Z_{n-1} < 1\} \text{ and } \sigma_3 := \inf\{n\ge 1 : \widetilde{Z}_n < 1\}. \tag{4.21}$$

Lemma 4.1: *For the above market, the following properties hold:*

(a) *The processes D, m, Z and $\widetilde{Z}$ take the form of*

$$D_0 = 0,\ D_1 = p\mathbb{1}_{\{\omega_3,\omega_4\}},\ D_2 = p\mathbb{1}_{\{\omega_3,\omega_4\}} + \mathbb{1}_{\{\omega_1,\omega_2,\omega_4\}}.$$
$$m_0 = 1,\ m_1 = 1,\ m_2 = p\mathbb{1}_{\{\omega_3,\omega_4\}} + \mathbb{1}_{\{\omega_1,\omega_2,\omega_4\}}.$$
$$Z_0 = 1,\ Z_1 = 1 - p\mathbb{1}_{\{\omega_3,\omega_4\}},\ Z_2 = 0.$$
$$\widetilde{Z}_0 = 1,\ \widetilde{Z}_1 = 1,\ \widetilde{Z}_2 = \mathbb{1}_{\{\omega_1,\omega_2,\omega_4\}}.$$

(b) $\widetilde{Z}_{\tau^{(d)}} = 1,\ \tau^{(d)} = \sup\{n \geq 0 :\ \widetilde{Z}_n = 1\}$, *and* $\tau^{(d)}$ *is an honest time.*

(c) *The stopping times in (4.21) are given by*

$$R_1 = 2,\ R_2 = +\infty,\ R_3(\omega_3) = 2,\ R_3(\omega_1,\omega_2,\omega_4) = +\infty,$$
$$\sigma_1(\omega_1,\omega_2) = 2, \sigma_1(\omega_3,\omega_4) = 1,$$
$$\sigma_2(\omega_1,\omega_2) = +\infty, \sigma_2(\omega_3,\omega_4) = 2, \sigma_3(\omega_3) = 2,\ \sigma_3(\omega_1,\omega_2,\omega_4) = +\infty.$$

Proof. By the definitions of Z and $\widetilde{Z}$, we calculate that

$$Z_0 = P(\tau^{(d)} > 0) = 1,\ Z_1 = P(\tau^{(d)} > 1|\mathcal{F}_1) = \mathbb{1}_{\{\omega_1,\omega_2\}} + \mathbb{1}_{\{\omega_3,\omega_4\}}(1-p),\ Z_2 = 0$$
$$\widetilde{Z}_0 = P(\tau^{(d)} \geq 0) = 1,\ \widetilde{Z}_1 = P(\tau^{(d)} \geq 1|\mathcal{F}_1) = 1,\ \widetilde{Z}_2 = P(\tau^{(d)} \geq 2|\mathcal{F}_2) = \mathbb{1}_{\{\omega_1,\omega_2,\omega_4\}}.$$

The calculation for D and m is similar. The proofs of the assertions (b) and (c) are straightforward. Hence, we omit them here. □

Theorem 4.2: *Under the current settings, the following properties hold:*

(a) *For the market $S^{\tau^{(d)}}$, there are arbitrage opportunities by using public information $\mathbb{F}$ and extra information $\mathbb{G}$.*

(b) *In the market S, there are arbitrage opportunities only in $\mathbb{G}$.*

(c) $\{\Delta S_2 \neq 0\} \cap \{\widetilde{Z}_2 = 0\} \cap \{Z_1 > 0\} = \{\omega_3\}$ *and* $\{\Delta S_2 \neq 0\} \cap \{\widetilde{Z}_2 = 1\} \cap \{Z_1 < 1\} = \{\omega_4\}$.

Proof. In the market $S^{\tau^{(d)}}$, apparently, by taking $H_1 = 0,\ H_2(\{\omega_1,\omega_2\}) = 0,\ H_2(\{\omega_3,\omega_4\}) =$

-1, one can make an arbitrage. While in the market S, by taking $H_1^{\mathbb{G}} = 0,\ H_2^{\mathbb{G}}(\{\omega_1, \omega_2\}) = 0,\ H_2^{\mathbb{G}}(\{\omega_3\}) = 1, H_2^{\mathbb{G}}(\{\omega_4\}) = -1$, only the one with information $\mathbb{G}$ can make an arbitrage since the strategies $(H_n^{\mathbb{G}})_{1\leq n\leq 2}$ are only $\mathbb{G}$-predictable. Assertion (c) is straightforward to verify. This ends the proof of the theorem. □

4.5 Classic Arbitrages for Non-honest Times

In this section, we develop a number of practical examples of market models and examples of random times that are not honest times and we study the existence of classical arbitrages.

4.5.1 In a Brownian filtration: Emery's Example

We present here an example where τ is a pseudo stopping time (see Nikeghbali and Yor [115] for the definition). Let S be defined through $dS_t = \sigma S_t dW_t$, where W is a Brownian motion and σ a constant. Consider the random time

$$\tau = \sup\{t \leq 1 \,:\, S_1 - 2S_t = 0\}.$$

Proposition 4.7: *For the above settings, there is no arbitrage opportunity before τ, i.e. $(S^\tau, \mathbb{G})$; while arbitrage opportunity exists after τ, i.e. $(S - S^\tau, \mathbb{G})$.*

Proof. Note that

$$\{\tau \leq t\} = \{\inf_{t\leq s\leq 1} 2S_s \geq S_1\} = \left\{\inf_{t\leq s\leq 1} 2\frac{S_s}{S_t} \geq \frac{S_1}{S_t}\right\}$$

Since $S_s/S_t, s \geq t$ and S_1/S_t are independent from $\mathcal{F}_t$, we derive

$$\mathbb{P}\left(\inf_{t\leq s\leq 1} 2\frac{S_s}{S_t} \geq \frac{S_1}{S_t}|\mathcal{F}_t\right) = \mathbb{P}\left(\inf_{t\leq s\leq 1} 2S_{s-t} \geq S_{1-t}\right) = \Psi(1-t)$$

where $\Psi(u) = \mathbb{P}(\inf_{s\leq u} 2S_s \geq S_u)$. It follows that the supermartingale Z is a deterministic decreasing function. Hence τ is a pseudo stopping time (see [115]) and S is a $\mathbb{G}$-martingale up to time τ and there are no arbitrages up to τ.

There are obviously arbitrages after τ, since, at time τ, one knows the value of S_1 and $S_1 > S_\tau$. In fact, for $t > \tau$, one has $S_t > S_\tau$, and the arbitrage occurs at any time before 1.

4.5.2 In a Poisson Filtration

This subsection develops similar examples of random times – as in the Brownian filtration of the previous subsection – and shows that the effects of these random times on the market's economic structure differ tremendously from the one of the previous subsection. In this section, we will work with a Poisson process N having intensity λ and the compensated martingale $M_t = N_t - \lambda t$. Denote

$$T_n := \inf\{t \geq 0 : N_t \geq n\}, \text{ and } H_t^n := \mathbb{1}_{\{T_n \leq t\}}, \quad n = 1, 2. \tag{4.22}$$

The stock price S is described by

$$dS_t = S_{t-}\psi dM_t, \quad \text{where,} \quad \psi > -1, \text{ and } \psi \neq 0, \tag{4.23}$$

or equivalently, $S_t = S_0 \exp(-\lambda\psi t + \ln(1+\psi)N_t)$. Then,

$$\begin{aligned} M_t^1 &:= H_t^1 - \lambda(t \wedge T_1) := H_t^1 - A_t^1, \\ M_t^2 &:= H_t^2 - (\lambda(t \wedge T_2) - \lambda(t \wedge T_1))) := H_t^2 - A_t^2 \end{aligned} \tag{4.24}$$

are two $\mathbb{F}$-martingales. Remark that if $\psi \in (-1, 0)$, between T_1 and T_2, the stock price increases; if $\psi > 0$, between T_1 and T_2, the stock process decreases. This would be the starting point of the existence of arbitrages.

Example 4.3 (**Convex Combination of two jump times**) Below, we present an example of random time that avoids stopping times and the non-arbitrage property fails.

Proposition 4.8: *Consider the market S in (4.23) with notations (4.22) and (4.24) and the random time*

$$\tau = k_1 T_1 + k_2 T_2, \quad \textit{where} \;\; k_1 + k_2 = 1 \;\textit{and}\;\; k_1, k_2 > 0,$$

which avoids $\mathbb{F}$-stopping times. Then the following properties hold:

(a) *The random time τ is not an honest time.*

(b) $\widetilde{Z}_\tau = Z_\tau = e^{-\lambda k_1 (T_2 - T_1)} < 1$, *and* $\{\widetilde{Z} = 0 < Z_-\} = [\![T_2]\!]$.

(c) *There is a classical arbitrage before τ, given by*

$$\varphi_t^{(1)} := -e^{-\lambda \frac{k_2}{k_1}(t - T_1)} \left(1\!\!1_{\{N_{t-} \geq 1\}} - 1\!\!1_{\{N_{t-} \geq 2\}} \right) \frac{1}{\psi S_{t-}} 1\!\!1_{\{t \leq \tau\}}. \tag{4.25}$$

(d) *There exist arbitrages after τ: if $\psi \in (-1, 0)$, buy at τ and sell before T_2; if $\psi > 0$, short sell at τ and buy back before T_2.*

Proof. First, we compute the supermartingale Z:

$$\mathbb{P}(\tau > t | \mathcal{F}_t) = 1\!\!1_{\{T_1 > t\}} + 1\!\!1_{\{T_1 \leq t\}} 1\!\!1_{\{T_2 > t\}} \mathbb{P}(k_1 T_1 + k_2 T_2 > t | \mathcal{F}_t).$$

On the set $E = \{T_1 \leq t\} \cap \{T_2 > t\}$, the quantity $\mathbb{P}(k_1 T_1 + k_2 T_2 > t | \mathcal{F}_t)$ is $\mathcal{F}_{T_1}$-measurable. It follows that, on E,

$$\mathbb{P}(k_1 T_1 + k_2 T_2 > t | \mathcal{F}_t) = \frac{\mathbb{P}(k_1 T_1 + k_2 T_2 > t, T_2 > t | \mathcal{F}_{T_1})}{\mathbb{P}(T_2 > t | \mathcal{F}_{T_1})} = e^{-\lambda \frac{k_1}{k_2}(t - T_1)},$$

where we used the independence property of T_1 and $T_2 - T_1$. Therefore,

$$Z_t := \mathbb{P}(\tau > t | \mathcal{F}_t) = 1\!\!1_{\{T_1 > t\}} + 1\!\!1_{\{T_1 \leq t\}} 1\!\!1_{\{T_2 > t\}} e^{-\lambda \frac{k_1}{k_2}(t - T_1)}.$$

Since $Z_t = (1-H^1_t)+H^1_t(1-H^2_t)e^{-\lambda\frac{k_1}{k_2}(t-T_1)}$ and $e^{-\lambda(t-T_1)}dH^1_t = dH^1_t$, we deduce

$$\begin{aligned} dZ_t &= e^{-\lambda\frac{k_1}{k_2}(t-T_1)}(-H^2 dH^1_t - H^1_t dH^2_t) - \lambda\frac{k_1}{k_2}H^1_t(1-H^2_t)e^{-\lambda\frac{k_1}{k_2}(t-T_1)}dt \\ &= -e^{-\lambda\frac{k_1}{k_2}(t-T_1)}dH^2_t - \lambda\frac{k_1}{k_2}H^1_t(1-H^2_t)e^{-\lambda\frac{k_1}{k_2}(t-T_1)}dt \\ &= dm_t - e^{-\lambda\frac{k_1}{k_2}(t-T_1)}dA^2_t - \lambda\frac{k_1}{k_2}H^1_t(1-H^2_t)e^{-\lambda\frac{k_1}{k_2}(t-T_1)}dt, \end{aligned}$$

where $dm_t = -e^{-\lambda\frac{k_1}{k_2}(t-T_1)}dM^2_t$. Hence,

$$m_\tau = 1 - \int_0^\tau e^{-\lambda\frac{k_1}{k_2}(t-T_1)}dM^2_t = 1 + \int_{T_1}^\tau e^{-\lambda\frac{k_1}{k_2}(t-T_1)}\lambda dt > 1.$$

Now we start proving the proposition.

(i) Since τ avoids stopping times, $Z = \widetilde{Z}$. Note that $\widetilde{Z}_\tau = Z_\tau = e^{-\lambda k_1(T_2-T_1)} < 1$. Hence, τ is not an honest time. Thus, assertions (a) and (b) hold.

(ii) Now, we prove assertion (c). We will describe explicitly the arbitrage strategy. Note that $\{T_2 \le t\} = \{N_t \ge 2\}$. We deduce that

$$M^2_t = \mathbb{1}_{\{T_2\le t\}} - A^2_t = \mathbb{1}_{\{N_t\ge 2\}} - A^2_t = \mathbb{1}_{\{N_{t-}\ge 1\}}\Delta N_t + \mathbb{1}_{\{N_{t-}\ge 2\}}(1-\Delta N_t) - A^2_t.$$

Hence,

$$\Delta M^2_t = \left(\mathbb{1}_{\{N_{t-}\ge 1\}} - \mathbb{1}_{\{N_{t-}\ge 2\}}\right)\Delta N_t = \left(\mathbb{1}_{\{N_{t-}\ge 1\}} - \mathbb{1}_{\{N_{t-}\ge 2\}}\right)\Delta M_t.$$

Since M^2 and M are both purely discontinuous, we have $m_t = 1 + \phi \centerdot M_t = 1 + \varphi^{(1)} \centerdot S_t$, where

$$\phi_t = -e^{-\lambda\frac{k_1}{k_2}(t-T_1)}\left(I_{\{N_{t-}\ge 1\}} - I_{\{N_{t-}\ge 2\}}\right), \text{ and } \varphi^{(1)}_t = \phi_t\frac{1}{\psi S_{t-}}.$$

(iii) Arbitrages after τ: At time τ, the value of T_2 is known for the one who has $\mathbb{G}$ information. If $\psi > 0$, then the price process decreases before time T_2, while waiting up to time T_2 does not lead to an arbitrage. Setting $\Delta = T_2 - \tau$ (which is known at time τ), there is an arbitrage selling short S at time τ for a delivery at time $\tau + \frac{1}{2}\Delta$. The strategy is admissible, since between T_1 and T_2, the quantity S_t is bounded by $S_0(1+\varphi)$. This ends the proof of the proposition. □

Example 4.4 (**Minimum of two scaled jump times**) We give now an example of a non honest random time, which does not avoid $\mathbb{F}$ stopping time and induces classical arbitrage opportunities.

Proposition 4.9: *Consider the market S in (4.23) with notations (4.22) and (4.24) and the random time*

$$\tau_2 := T_1 \wedge aT_2, \quad \textit{where} \;\; 0 < a < 1 \;\; \textit{and} \;\; \beta = \lambda(1/a - 1).$$

Then, the following properties hold:

(a) τ_2 *is not an honest time and does not avoid $\mathbb{F}$-stopping times.*

(b) $Z_{\tau_2} = 1\!\!1_{\{T_1 > aT_2\}} e^{-\beta a T_2}(\beta a T_2 + 1) < 1$ *and* $\widetilde{Z}_{\tau_2} = e^{-\beta a T_2}(\beta a T_2 + 1) < 1$, *and* $\{\widetilde{Z} = 0 < Z_-\} = \emptyset$.

(c) $m \equiv 1$ *and M^τ is a $\mathbb{G}$-local martingale for any $\mathbb{F}$-local martingale M.*

Proof. First, let us compute the supermartingale Z,

$$\begin{aligned} Z_t &= 1\!\!1_{\{T_1>t\}} \mathbb{P}(aT_2 > t | \mathcal{F}_t) = 1\!\!1_{\{T_1>t\}} \frac{\mathbb{P}(aT_2 > t, T_1 > t)}{\mathbb{P}(T_1 > t)} \\ &= 1\!\!1_{\{T_1>t\}} e^{\lambda t}\, \mathbb{E}(1\!\!1_{T_1>t} e^{-\lambda(\frac{t}{a} - T_1)^+}) \\ &= 1\!\!1_{\{T_1>t\}} e^{\lambda t} \int_t^{t/a} e^{-\lambda(\frac{t}{a} - x)} \lambda e^{-\lambda x} dx + 1\!\!1_{\{T_1>t\}} e^{\lambda t} \int_{t/a}^{\infty} \lambda e^{-\lambda y} dy \\ &= 1\!\!1_{\{T_1>t\}} e^{-\beta t}(\beta t + 1)\,. \end{aligned}$$

In particular $Z_{\tau_2} = 1\!\!1_{\{T_1 > aT_2\}} e^{-\beta a T_2}(\beta a T_2 + 1) < 1$. Similar computation as above leads to $\widetilde{Z}_t = Z_{t-} = 1\!\!1_{\{T_1 \geq t\}} e^{-\beta t}(\beta t + 1)$. This proves assertions (a) and (b). Since $\Delta m = \widetilde{Z} - Z_-$, we have $\Delta m \equiv 0$. Hence m is a constant equal to 1 since m is a purely discontinuous $\mathbb{F}$-martingale. This proves assertion (c) and the proof of the proposition is achieved. □

Conclusions

In this chapter, we treated the question whether the no-arbitrage conditions are stable with respect to progressive enlargement of filtration. Precisely, we proved that there exist classic arbitrage opportunities for many models of honest times when the market is complete. Furthermore, in the case of Brownian filtration and Poisson filtration, we calculated explicitly the arbitrage strategies. One may further investigate similar problem without assuming market completeness and consider other examples/classes of non-honest random times.

The failure of the classic arbitrage condition leads us to investigate the stability of the NUPBR condition in Chapters 5 and 6.

Chapter 5

Non-arbitrage under Stopping with Random Time

This chapter is dedicated to the problem of how an arbitrage-free semimartingale model is affected when stopped at a random horizon. We focus on weaker non-arbitrage concept: No-Unbounded-Profit-with-Bounded-Risk (called NUPBR hereafter) concept, which is also known in the literature as the first kind of non-arbitrage. For this non-arbitrage notion, we obtain two principal results. The first result lies in describing the pairs of market model and random time for which the resulting stopped model fulfills NUPBR condition. The second main result characterizes the random time models that preserve the NUPBR property after stopping for any market model. These results are elaborated in a very general market model, and also some particular and practical models.

Mathematically speaking, we consider a general semimartingale model S satisfying the NUPBR property under the "public information" and an arbitrary random time τ and we investigate the following problems:

$$\text{For which pairs } (S,\tau)\text{, does the NUPBR property hold for } S^\tau? \qquad \textbf{(Prob(5.I))}$$

In Theorem 5.6 we characterize the pairs of initial market S and of random time τ, for which **(Prob(5.I))** has a positive answer.
Our second main question consists of

For which τ, is NUPBR preserved for any S after stopping at τ? **(Prob(5.II))**

To deepen our understanding of the precise interplay between S (initial market model) and τ (the random time model), we address these two principal questions separately in the case of quasi-left-continuous models, and then in the case of thin processes with predictable jumps. Afterwards, we combine the two cases and state the results for the most general framework. The results for the quasi-left-continuous models are Theorem 5.2 and Proposition 5.3, where the questions **(Prob(5.I))** and **(Prob(5.II))** are fully answered respectively. For the case of thin processes with predictable jumps, our main result is Theorem 5.5. Then, the general case follows by splitting the process S into a quasi-left-continuous process and a thin process with predictable jumps.

This problem was studied in the literature (see [63]) for particular case of continuous filtration, under the hypothesis that τ avoids $\mathbb{F}$-stopping times and that the market is complete.

5.1 Notations and Preliminary Results on NUPBR

We consider a stochastic basis $(\Omega, \mathcal{A}, \mathbb{F} = (\mathcal{F}_t)_{t\geq 0}, P)$, where $\mathbb{F}$ is a filtration satisfying the usual hypotheses (i.e., right continuity and completeness) and represents the flow of public information through time with $\mathcal{F}_\infty := \bigvee_{t\geq 0} \mathcal{F}_t \subseteq \mathcal{A}$. On this basis, we consider an arbitrary but fixed d-dimensional càdlàg $\mathbb{F}$-semimartingale S that represents the discounted price processes of d-risky assets; while the non-risky asset's price is assumed to be constant one.

Beside the initial model $(\Omega, \mathcal{A}, \mathbb{F}, P, S)$, we consider a random time τ, i.e., a non-negative $\mathcal{A}$-measurable random variable. To this random time, we associate the process D and the filtration $\mathbb{G}$ given by

$$D := I_{[\![\tau,+\infty[\![}, \quad \mathbb{G} = (\mathcal{G}_t)_{t\geq 0}\,, \quad \mathcal{G}_t = \bigcap_{s>t}\Big(\mathcal{F}_s \vee \sigma(D_u, u \leq s)\Big).$$

The filtration $\mathbb{G}$ is the smallest right-continuous filtration which contains $\mathbb{F}$ and makes τ a stopping time. In the probabilistic literature, $\mathbb{G}$ is called the progressive enlargement of $\mathbb{F}$ with τ. In addition to $\mathbb{G}$ and D, we associate to τ two important $\mathbb{F}$-supermartingales given by

$$Z_t := P\left(\tau > t \;\middle|\; \mathcal{F}_t\right) \text{ and } \quad \widetilde{Z}_t := P\Big(\tau \geq t \;\Big|\; \mathcal{F}_t\Big)\,. \tag{5.1}$$

The supermartingale Z is right-continuous with left limits and coincides with the $\mathbb{F}$-optional projection of $I_{]\!]0,\tau[\![}$, while $\widetilde{Z}$ admits right limits and left limits only and is the $\mathbb{F}$-optional projection of $I_{]\!]0,\tau]\!]}$. The decomposition of Z leads to an important $\mathbb{F}$-martingale m, given by

$$m := Z + D^{o,\mathbb{F}}, \tag{5.2}$$

where $D^{o,\mathbb{F}}$ is the $\mathbb{F}$-dual optional projection of D (See [83] for more details).

In what follows, $\mathbb{H}$ is a generic filtration satisfying the usual hypotheses and Q is a probability measure on the filtered probability space $(\Omega, \mathbb{H})$. The set of martingales for the filtration $\mathbb{H}$ under Q is denoted by $\mathcal{M}(\mathbb{H}, Q)$. When $Q = P$, we simply denote $\mathcal{M}(\mathbb{H})$. As usual, $\mathcal{A}^+(\mathbb{H})$ denotes the set of increasing, right-continuous, $\mathbb{H}$-adapted and integrable processes.

If $\mathcal{C}(\mathbb{H})$ is a class of $\mathbb{H}$ adapted processes, we denote by $\mathcal{C}_0(\mathbb{H})$ the set of processes $X \in \mathcal{C}(\mathbb{H})$ with $X_0 = 0$, and by $\mathcal{C}_{loc}$ the set of processes X such that there exists a sequence $(T_n)_{n\geq 1}$ of $\mathbb{H}$-stopping times that increases to $+\infty$ and the stopped processes X^{T_n} belong to $\mathcal{C}(\mathbb{H})$. We put $\mathcal{C}_{0,loc}(\mathbb{H}) = \mathcal{C}_0(\mathbb{H}) \cap \mathcal{C}_{loc}(\mathbb{H})$.

For a process K with $\mathbb{H}$-locally integrable variation, we denote by $K^{o,\mathbb{H}}$ its dual optional projection. The dual predictable projection of K (also called the $\mathbb{H}$-predictable dual projection) is denoted $K^{p,\mathbb{H}}$. For a process X, we denote ${}^{o,\mathbb{H}}X$ (resp. ${}^{p,\mathbb{H}}X$) its optional (resp. predictable) projection with respect to $\mathbb{H}$.
For an $\mathbb{H}$- semi-martingale Y, the set $L(Y,\mathbb{H})$ is the set of $\mathbb{H}$ predictable processes integrable w.r.t. Y and for $H \in L(Y,\mathbb{H})$, we denote $H \centerdot Y_t := \int_0^t H_s dY_s$.

To distinguish the effect of filtrations, we will denote $\langle .,. \rangle^{\mathbb{F}}$, or $\langle .,. \rangle^{\mathbb{G}}$ the sharp bracket (predictable covariation process) calculated in the filtration $\mathbb{F}$ or $\mathbb{G}$, if confusion may rise. We recall that, for general semi-martingales X and Y, the sharp bracket is (if it exists) the dual predictable projection of the covariation process $[X,Y]$.

Below, we recall the non-arbitrage notion that will be addressed in this chapter.

Definition 5.1: An $\mathbb{H}$-semimartingale X satisfies the *No-Unbounded-Profit-with-Bounded-Risk* condition under $(\mathbb{H},Q)$ (hereafter called NUPBR$(\mathbb{H},Q)$) if for any $T \in (0,+\infty)$ the set

$$\mathcal{K}_T(X,\mathbb{H}) := \Big\{ (H \centerdot S)_T \ \Big| \quad H \in L(X,\mathbb{H}), \ \text{and} \ H \centerdot X \geq -1 \ \Big\}$$

is bounded in probability under Q. When $Q \sim P$, we simply write, with an abuse of language, X satisfies NUPBR$(\mathbb{H})$.

Remark 5.1: (i) It is important to notice that this definition for NUPBR condition appeared first in [95] (up to our knowledge), and it differs when the time horizon is infinite from that of the literature given in Delbaen and Schachermayer [47], Kabanov [86] and Karatzas and Kardaras [90] (see Definition 2.17 of Subsection 2.4 in Chapter 2). It is obvious that, when the horizon is deterministic and finite, the current NUPBR condition coincides with that of the literature. We could name the current NUPBR as NUPBR_{loc}, but for the sake of simplifying notation, we opted for the usual terminology.

(ii) In general, when the horizon is infinite, the NUPBR condition of the literature implies the NUPBR condition defined above. However, the reverse implication may not hold in general. In fact if we consider $S_t = \exp(W_t + t),\ t \geq 0$, then it is clear that S satisfies our NUPBR($\mathbb{H}$), while the NUPBR($\mathbb{H}$) of the literature is violated. To see this last claim, it is enough to remark that

$$\lim_{t \to +\infty} (S_t - 1) = +\infty \quad P-a.s. \quad S^t - 1 = H \centerdot S \geq -1 \quad H := I_{]\!]0,t]\!]}.$$

The following proposition slightly generalizes Takaoka's results obtained for a finite horizon (see Theorem 2.6 in [137]) to our NUPBR context.

Proposition 5.1: *Let X be an $\mathbb{H}$-semimartingale. Then the following assertions are equivalent.*

(a) *X satisfies NUPBR($\mathbb{H}$).*

(b) *There exist a positive $\mathbb{H}$-local martingale, Y and an $\mathbb{H}$-predictable process θ satisfying $0 < \theta \leq 1$ and $Y(\theta \centerdot X)$ is a local martingale.*

Proof. The proof of the implication (b)$\Rightarrow$ (a) is based on [137] and is omitted. Thus, we focus on proving the reverse implication and suppose that assertion (a) holds. Therefore, a direct application of Theorem 2.6 in [137] to each $(S_{t\wedge n})_{t\geq 0}$, we obtain the existence of a positive $\mathbb{H}$-local martingale $Y^{(n)}$ and an $\mathbb{H}$-predictable process θ_n such that $0 < \theta_n \leq 1$ and $Y^{(n)}(\theta_n \centerdot S^n)$ is a local martingale. Then, it is obvious that the process

$$N := \sum_{n=1}^{+\infty} I_{]\!]n-1,n]\!]} (Y^{(n)}_{-})^{-1} \centerdot Y^{(n)}$$

is a local martingale and $Y := \mathcal{E}(N) > 0$. On the other hand, the $\mathbb{H}$-predictable process $\theta := \sum_{n\geq 1} I_{]\!]n-1,n]\!]}\theta_n$ satisfies $0 < \theta \leq 1$ and $Y(\theta \cdot S)$ is a local martingale. This ends the proof of the proposition. $\square$

For any $\mathbb{H}$-semimartingale X, the local martingales fulfilling the assertion (b) of Proposition 5.1 are called σ-martingale densities for X. The set of these σ-martingale densities will be denoted throughout the paper by

$$\mathcal{L}(\mathbb{H}, X) := \left\{ Y \in \mathcal{M}_{loc}(\mathbb{H}) \middle| \; Y > 0, \; \exists \theta \in \mathcal{P}(\mathbb{H}), \, 0 < \theta \leq 1, \; Y(\theta \cdot X) \in \mathcal{M}_{loc}(\mathbb{H}) \right\} \tag{5.3}$$

where, as usual, $\mathcal{P}(\mathbb{H})$ stands for predictable processes. We state, without proof, an obvious lemma.

Lemma 5.1: *For any $\mathbb{H}$-semimartingale X and any $Y \in \mathcal{L}(\mathbb{H}, X)$, one has* ${}^{p,\mathbb{H}}(Y|\Delta X|) < \infty$ *and* ${}^{p,\mathbb{H}}(Y\Delta X) = 0$

Remark 5.2: Proposition 5.1 implies that for any process X and any **finite** stopping time σ, the two concepts of NUPBR($\mathbb{H}$) (the current concept and the one of the literature) coincide for X^σ.

Below, we prove that, under infinite horizon, the current NUPBR condition is stable under localization, while the NUPBR condition defined in the literature fails.

Proposition 5.2: *Let X be an $\mathbb{H}$ adapted process. Then, the following assertions are equivalent.*

(a) *There exists a sequence $(T_n)_{n\geq 1}$ of $\mathbb{H}$-stopping times that increases to $+\infty$, such that for each $n \geq 1$, there exists a probability Q_n on $(\Omega, \mathbb{H}_{T_n})$ such that $Q_n \sim P$ and X^{T_n} satisfies NUPBR($\mathbb{H}$) under Q_n.*

(b) *X satisfies NUPBR($\mathbb{H}$).*

(c) *There exists an $\mathbb{H}$-predictable process ϕ, such that $0 < \phi \leq 1$ and $(\phi \cdot X)$ satisfies NUPBR($\mathbb{H}$).*

Proof. The proof for (a)$\Longleftrightarrow$(b) follows from the stability of NUPBR condition for a finite horizon under localization which is due to [137] (see also [37] for further discussion about this issue), and the fact that the NUPBR condition is stable under any equivalent probability change.

The proof of (b)$\Rightarrow$(c) is trivial and is omitted. To prove the reverse, we assume that (c) holds. Then Proposition 5.1 implies the existence of an $\mathbb{H}$-predictable process ψ such that $0 < \psi \leq 1$ and a positive $\mathbb{H}$-local martingale $Z = \mathcal{E}(N)$ such that $Z(\psi\phi \centerdot X)$ is a local martingale. Since $\psi\phi$ is predictable and $0 < \psi\phi \leq 1$, we deduce that S satisfies NUPBR($\mathbb{H}$). This ends the proof of the proposition. □

Lemma 5.2: *Let X be an $\mathbb{H}$-predictable process with finite variation. Then X satisfies NUPBR($\mathbb{H}$) if and only if $X \equiv X_0$ (i.e. the process X is constant).*

Proof. It is obvious that if $X \equiv X_0$, then X satisfies NUPBR($\mathbb{H}$). Suppose that X satisfies NUPBR($\mathbb{H}$). Consider a positive $\mathbb{H}$-local martingale Y, and an $\mathbb{H}$-predictable process θ such that $0 < \theta \leq 1$ and $Y(\theta \centerdot X)$ is a local martingale. Let $(T_n)_{n\geq 1}$ be a sequence of $\mathbb{H}$-stopping times that increases to $+\infty$ such that Y^{T_n} and $Y^{T_n}(\theta \centerdot X)^{T_n}$ are true martingales. Then, for each $n \geq 1$, define $Q_n := (Y_{T_n}/Y_0) \centerdot P$. Since X is predictable, then $(\theta \centerdot X)^{T_n}$ is also predictable with finite variation and is a Q_n-martingale. Thus, we deduce that $(\theta \centerdot X)^{T_n} \equiv 0$ for each $n \geq 1$. Therefore, we deduce that X is constant (since $X^{T_n} - X_0 = \theta^{-1} \centerdot (\theta \centerdot X)^{T_n} \equiv 0$). This ends the proof of the lemma. □

Now, let us characterize the σ-martingale density in terms of characteristics of semimartingale, see also [107].

Theorem 5.1: *Let X be an $\mathbb{H}$-semi-martingale with predictable triplet (b, c, ν). Then X satisfies NUPBR($\mathbb{H}$) with the σ-martingale density $\mathcal{E}(N)$ if and only if the following two properties hold:*

(a) *The integral*

$$\int \Big| (x - h(x) + xf(x)) \Big| F(dx) + \int |x| M^P_\mu \left(|g| \;\Big|\; \widetilde{\mathcal{P}}(\mathbb{H}) \right) < +\infty, \quad P \otimes A - a.s \quad \textit{and} \tag{5.4}$$

(b) *The following equality holds:*

$$b + \beta c + \int (x - h(x) + xf(x)) \, F(dx) \equiv 0, \quad P \otimes A - a.s. \tag{5.5}$$

where (β, f, g, N') *are the Jacod's parameters of* N *with respect to* X *(see (2.18)),* F *is the kernel and* A *is an* $\mathbb{H}$*-predictable process associated with compensator* ν *(see (2.11)).*

Proof. Thanks to Itô formula, we deduce that ZX is a σ-martingale if and only if there exists an $\mathbb{H}$-predictable process $0 < \phi \leq 1$ such that $\phi \centerdot X + \phi \centerdot [X, N]$ is a local martingale. From the Jacod Decomposition (2.18), we derive that

$$\begin{aligned}
\phi \centerdot X + \phi \centerdot [X, N] &= \phi \centerdot X + \phi \centerdot [X^c, N^c] + \sum \phi \Delta X \Delta N \\
&= X_0 + \phi \centerdot X^c + \phi h \star (\mu - \nu) + \phi (x - h) \star \mu + \phi b \centerdot A + \phi \beta c \centerdot A \\
&\quad + \sum \phi \Delta X \left(f(\Delta X) + g(\Delta X) \right) I_{\{\Delta X \neq 0\}} \\
&= X_0 + \phi \centerdot X^c + \phi h \star (\mu - \nu) + \phi b \centerdot A \\
&\quad + \phi \beta c \centerdot A + \phi [x - h(x) + x \left(f(x) + g(x) \right)] \star \mu.
\end{aligned}$$

Therefore, $\phi \centerdot X + \phi \centerdot [X, N]$ is a local martingale if and only if the assertions (a) and (b) are satisfied. □

5.2 Main Results and their Interpretations

This section presents our main results and their immediate consequences. To this end, we start specifying our mathematical setting and the economic concepts.

5.2.1 The Quasi-Left-Continuous Processes

In this subsection, we present our two main results on the NUPBR condition under stopping at τ for quasi-left-continuous processes. The first result consists of characterizing the pairs (S,τ) of market and random time models, for which S^τ fulfills the NUPBR condition. The second result focuses on determining the models of random times τ such that, for any semi-martingale S enjoying NUPBR($\mathbb{F}$), the stopped process S^τ enjoys NUPBR($\mathbb{G}$) .

The following theorem gives a characterization of $\mathbb{F}$-quasi-left-continuous processes that satisfy NUPBR($\mathbb{G}$) after stopping with τ. Recall that μ, ν and M^P_μ are defined by (2.9), (2.11) and (2.10) in Subsection 2.3 respectively. The proof of this theorem will be given in Subsection 5.5.1, while its statement is based on the following $\mathbb{F}$-semimartingale

$$S^{(0)} := xI_{\{\psi=0<Z_-\}} \star \mu, \qquad \text{where} \qquad \psi := M^P_\mu \left(I_{\{\widetilde{Z}>0\}} \middle| \widetilde{\mathcal{P}}(\mathbb{F}) \right). \tag{5.6}$$

Theorem 5.2: *Suppose that S is $\mathbb{F}$-quasi-left-continuous. Then, the following assertions are equivalent.*

(a) *S^τ satisfies NUPBR($\mathbb{G}$).*

(b) *For any $\delta > 0$, the process*

$$I_{\{Z_-\geq\delta\}} \cdot \left(S - S^{(0)}\right) \qquad \textit{satisfies NUPBR}(\mathbb{F}). \tag{5.7}$$

(c) *For any $n \geq 1$, the process $(S - S^{(0)})^{\sigma_n}$ satisfies NUPBR($\mathbb{F}$), where*

$$\sigma_n := \inf\{t \geq 0 \ : \ Z_t < 1/n\}.$$

Remark 5.3: 1) From assertion (c) one can understand that the NUPBR($\mathbb{G}$) property for S^τ can be verified by checking whether $S - S^{(0)}$ satisfies NUPBR($\mathbb{F}$) up to $\sigma_\infty := \sup_n \sigma_n$. This is also equivalent to NUPBR($\mathbb{F}$) of the same process on the predictable sets $\{Z_- \geq \delta\}$, $\delta > 0$.

2) The functionals ψ and $Z_- + f_m := M^P_\mu(\widetilde{Z}|\widetilde{\mathcal{P}}(\mathbb{F}))$ satisfy

$$\{\psi = 0\} = \{Z_- + f_m = 0\} \subset \{\widetilde{Z} = 0\}, \quad M^P_\mu - a.e. \tag{5.8}$$

Indeed, due to $\widetilde{Z} \leq I_{\{\widetilde{Z}>0\}}$, we have

$$0 \leq Z_- + f_m = M^P_\mu\left(\widetilde{Z}|\ \widetilde{\mathcal{P}}(\mathbb{F})\right) \leq \psi.$$

Thus, we get $\{\psi = 0\} \subset \{Z_- + f_m = 0\} \subset \{\widetilde{Z} = 0\}$ $M^P_\mu - a.e.$ on the one hand. On the other hand, the reverse inclusion follows from

$$0 = M^P_\mu\left(I_{\{Z_-+f_m=0\}}I_{\{\widetilde{Z}=0\}}\right) = M^P_\mu\left(I_{\{Z_-+f_m=0\}}\psi\right).$$

3) As a result of remark 2) above and $\{\widetilde{Z} = 0 < Z_-\} \subset [\![\sigma_\infty]\!]$, we deduce that $S^{(0)}$ is a càdlàg $\mathbb{F}$-adapted process with finite variation that takes the form of

$$S^{(0)} := \Delta S_{\sigma_\infty} I_{\{\widetilde{Z}_{\sigma_\infty}=0=\psi(\sigma_\infty,\Delta S_{\sigma_\infty})\ \&\ Z_{\sigma_\infty-}>0\}} I_{[\![\sigma_\infty,+\infty[\![}.$$

This proves the claim stated before Theorem 5.2 about the process $S^{(0)}$.

The following corollary is useful for studying the problem (**Prob(5.II)**), and it describes examples of $\mathbb{F}$-quasi-left-continuous model S that fulfills (5.6) as well.

Corollary 5.2.1: Suppose that S is $\mathbb{F}$-quasi-left-continuous and satisfies NUPBR($\mathbb{F}$). Then, the following assertions hold.

(a) If $\left(S, S^{(0)}\right)$ satisfies NUPBR($\mathbb{F}$), then S^τ satisfies NUPBR($\mathbb{G}$).

(b) If $S^{(0)} \equiv 0$, then the process S^τ satisfies NUPBR($\mathbb{G}$).

(c) If $\{\Delta S \neq 0\} \cap \{\widetilde{Z} = 0 < Z_-\} = \emptyset$, then S^τ satisfies NUPBR($\mathbb{G}$).

(d) If $\widetilde{Z} > 0$ (equivalently $Z > 0$ or $Z_- > 0$), then S^τ satisfies NUPBR($\mathbb{G}$).

Proof. (a) Suppose that $\left(S, S^{(0)}\right)$ satisfies NUPBR($\mathbb{F}$). Then, it is obvious that $S - S^{(0)}$ satisfies NUPBR($\mathbb{F}$), and assertion (a) follows from Theorem 5.2.
(b) Since S satisfies NUPBR($\mathbb{F}$) and $S^{(0)} \equiv 0$, then $\left(S, S^{(0)}\right) \equiv (S, 0)$ satisfies NUPBR($\mathbb{F}$), and assertion (b) follows from assertion (a).
(c) It is easy to see that $\{\Delta S \neq 0\} \cap \{\widetilde{Z} = 0 < Z_-\} = \emptyset$ implies that $S^{(0)} \equiv 0$ (due to (5.8)). Hence, assertions (c) and (d) follow from assertion (b), and the proof of the corollary is completed. □

Remark 5.4: It is worth mentioning that $X - Y$ may satisfy NUPBR($\mathbb{H}$), while (X, Y) may fail NUPBR($\mathbb{H}$). For a non trivial example, consider $X_t = B_t + \lambda t$ and $Y_t = N_t$, where B is a standard Brownian motion and N is the Poisson process with intensity λ.

We now give an answer to the second problem **(Prob(5.II))** for the quasi-left-continuous semimartingales. Later on (in Theorem 5.7) we will generalize this result.

Theorem 5.3: *The following assertions are equivalent:*

(a) *The thin set* $\left\{\widetilde{Z} = 0 \ \&\ Z_- > 0\right\}$ *is accessible.*

(b) *For any (bounded) S that is $\mathbb{F}$-quasi-left-continuous and satisfies NUPBR($\mathbb{F}$), the process S^τ satisfies NUPBR($\mathbb{G}$).*

Proof. The implication (a)⇒(b) follows from Corollary 5.2.1–(c), since we have

$$\{\Delta S \neq 0\} \cap \{\widetilde{Z} = 0 < Z_-\} = \emptyset.$$

We now focus on proving the reverse implication. To this end, we suppose that assertion (b) holds, and we consider an $\mathbb{F}$-stopping time σ such that $[\![\sigma]\!] \subset \{\widetilde{Z} = 0 < Z_-\}$. It is known that σ can be decomposed into a totally inaccessible part σ^i and an accessible part σ^a such that $\sigma = \sigma^i \wedge \sigma^a$. Consider the quasi-left-continuous $\mathbb{F}$-martingale

$$M = V - \widetilde{V} \in \mathcal{M}_{0,loc}(\mathbb{F})$$

where $V := I_{[\![\sigma^i,+\infty[\![}$ and $\widetilde{V} := (V)^{p,\mathbb{F}}$. It is known from [53, paragraph 14, Chapter XX], that

$$\{\widetilde{Z} = 0\} \text{ and } \{Z_- = 0\} \text{ are disjoint from }]\!]0, \tau]\!]\,. \tag{5.9}$$

This implies that $\tau < \sigma \leq \sigma^i \quad P - a.s.$. Hence, we get

$$M^\tau = -\widetilde{V}^\tau \quad \text{is } \mathbb{G}\text{-predictable.} \tag{5.10}$$

Since M^τ satisfies NUPBR($\mathbb{G}$), then we conclude that this process is null (i.e. $\widetilde{V}^\tau = 0$) due to Lemma 5.2. Thus, we get

$$0 = E\left(\widetilde{V}_\tau\right) = E\left(\int_0^{+\infty} Z_{s-} d\widetilde{V}_s\right) = E\left(Z_{\sigma^i-} I_{\{\sigma^i < +\infty\}}\right),$$

or equivalently $Z_{\sigma^i-} I_{\{\sigma^i<+\infty\}} = 0 \quad P-a.s.$ This is possible only if $\sigma^i = +\infty \quad P-a.s.$ since on $\{\sigma^i < +\infty\} \subset \{\sigma = \sigma^i < +\infty\}$ we have $Z_{\sigma^i-} = Z_{\sigma-} > 0$. This proves that σ is an accessible stopping time. Since $\{\widetilde{Z} = 0 < Z_-\}$ is an optional thin set, assertion (a) follows immediately. This ends the proof of the proposition.

□

5.2.2 Thin Processes with Predictable Jump Times

In this subsection, we outline the main results on the NUPBR condition for the stopped accessible parts of $\mathbb{F}$-semimartingales with a random time. This boils down to consider thin semimartingales with predictable jump times only. We start by addressing the question **(Prob(5.I))** in the case of single jump process with predictable jump time.

Theorem 5.4: *Consider an $\mathbb{F}$-predictable stopping time T and an $\mathcal{F}_T$-measurable random variable ξ such that $E(|\xi|\big|\ \mathcal{F}_{T-}) < +\infty \ P - a.s.$.*

If $S := \xi I_{\{Z_{T-}>0\}} I_{[\![T,+\infty[\![}$, then the following assertions are equivalent:

(a) *S^τ satisfies NUPBR($\mathbb{G}$),*

(b) *The process* $\widetilde{S} := \xi I_{\{\widetilde{Z}_T>0\}} I_{[\![T,+\infty[\![} = I_{\{\widetilde{Z}>0\}} \centerdot S$ *satisfies NUPBR*($\mathbb{F}$).

(c) *There exists a probability measure on* $(\Omega, \mathcal{F}_T)$*, denoted by* Q_T*, such that* Q_T *is absolutely continuous with respect to* P*, and* S *satisfies NUPBR*$(\mathbb{F}, Q_T)$.

The proof of this theorem is long and requires intermediary results that are interesting in themselves. Thus, this proof will be given later in Section 5.5.

Remark 5.5: 1) The importance of Theorem 5.4 goes beyond its vital role, as a building block for the more general result. In fact, Theorem 5.4 provides two different characterizations for NUPBR($\mathbb{G}$) of S^τ. The first characterization is expressed in term of NUPBR($\mathbb{F}$) of S under absolute continuous change of measure, while the second characterization uses transformation of S without any change of measure. Furthermore, Theorem 5.4 can be easily extended to the case of countably many ordered predictable jump times $T_0 = 0 \leq T_1 \leq T_2 \leq ...$ with $\sup_n T_n = +\infty$ $P-a.s.$.

2) In Theorem 5.4, the choice of S having the form $S := \xi I_{\{Z_{T-}>0\}} I_{[\![T,+\infty[\![}$ is not restrictive. This can be understood from the fact that any single jump process S can be decomposed as follows

$$S := \xi I_{[\![T,+\infty[\![} = \xi I_{\{Z_{T-}>0\}} I_{[\![T,+\infty[\![} + \xi I_{\{Z_{T-}=0\}} I_{[\![T,+\infty[\![} =: \overline{S} + \widehat{S}.$$

Thanks to $\{T \leq \tau\} \subset \{Z_{T-} > 0\}$, we have $\widehat{S}^\tau = \xi I_{\{Z_{T-}=0\}} I_{\{T\leq\tau\}} I_{[\![T,+\infty[\![} \equiv 0$ which is (obviously) a $\mathbb{G}$-martingale. Thus, the only part of S that requires careful attention is $\overline{S} := \xi I_{\{Z_{T-}>0\}} I_{[\![T,+\infty[\![}$.

The following result is a complete answer to **(Prob(5.II))** in the case of predictable single jump processes.

Proposition 5.3: *Let* T *be an* $\mathbb{F}$*-predictable stopping time. Then, the following assertions are equivalent:*

(a) *On* $\{T < +\infty\}$, *we have*

$$\left\{\widetilde{Z}_T = 0\right\} \subset \left\{Z_{T-} = 0\right\}. \tag{5.11}$$

(b) *For any* $M := \xi I_{[\![T,+\infty[\![}$ *where* $\xi \in L^\infty(\mathcal{F}_T)$ *such that* $E(\xi|\mathcal{F}_{T-}) = 0$, M^τ *satisfies NUPBR*($\mathbb{G}$).

Proof. We start by proving (a) $\Rightarrow$ (b). Suppose that (5.11) holds; due to the above Remark 5.5-(2), we can restrict our attention to the case when $M := \xi I_{\{Z_{T-}>0\}} I_{[\![T,+\infty[\![}$, where $\xi \in L^\infty(\mathcal{F}_T)$ such that $E(\xi|\mathcal{F}_{T-}) = 0$. Since assertion (a) is equivalent to $[\![T]\!] \cap \{\widetilde{Z} = 0 \ \& \ Z_- > 0\} = \emptyset$, we deduce that

$$\widetilde{M} := \xi I_{\{\widetilde{Z}_T>0\}} I_{\{Z_{T-}>0\}} I_{[\![T,+\infty[\![} = M \qquad \text{is an } \mathbb{F}\text{-martingale.}$$

Therefore, a direct application of Theorem 5.4 (to M) allows us to conclude that M^τ satisfies the NUPBR($\mathbb{G}$). This ends the proof of (a)$\Rightarrow$ (b). To prove the reverse implication, we suppose that assertion (b) holds and consider

$$M := \xi I_{[\![T,+\infty[\![}, \qquad \text{where } \ \xi := I_{\{\widetilde{Z}_T=0\}} - P(\widetilde{Z}_T = 0|\mathcal{F}_{T-}).$$

From (5.9), we obtain $\{T \leq \tau\} \subset \{\widetilde{Z}_T > 0\} \subset \{Z_{T-} > 0\}$ which implies that

$$M^\tau = -P(\widetilde{Z}_T = 0|\mathcal{F}_{T-}) I_{\{T\leq\tau\}} I_{[\![T,+\infty[\![} \ \text{ is } \mathbb{G}\text{-predictable.}$$

Therefore, M^τ satisfies NUPBR($\mathbb{G}$) if and only if it is a constant process equal to $M_0 = 0$ (see Lemma 5.2). This is equivalent to

$$0 = E\Big[P(\widetilde{Z}_T = 0|\mathcal{F}_{T-}) I_{\{T\leq\tau\}} I_{[\![T,+\infty[\![}\Big] = E\left(Z_{T-} I_{\{\widetilde{Z}_T=0 \ \& \ T<+\infty\}}\right).$$

It is obvious that this equality is equivalent to (5.11), and assertion (a) follows. This ends the proof of the theorem. □

We now state the following version of Theorem 5.4, which provides, as already said, an answer to **(Prob(5.I))** in the case where there are countable many arbitrary predictable jumps. The proof of this theorem will be given in Subsection 5.5.3.

Theorem 5.5: *Let S be a thin process with predictable jump times only and satisfying NUPBR($\mathbb{F}$). Then, the following assertions are equivalent.*

(a) *The process S^τ satisfies NUPBR($\mathbb{G}$).*

(b) *For any $\delta > 0$, there exists a positive $\mathbb{F}$-local martingale, Y, such that ${}^{p,\mathbb{F}}\left(Y|\Delta S|\right) < +\infty$ and*

$$ {}^{p,\mathbb{F}}\left(Y\Delta S I_{\{\widetilde{Z}>0\ \&\ Z_-\geq\delta\}}\right) = 0. \tag{5.12}$$

Remark 5.6: 1) Suppose that S is a thin process with predictable jumps only, satisfying NUPBR($\mathbb{F}$), and that $\{\widetilde{Z} = 0\ \&\ Z_- > 0\} \cap \{\Delta S \neq 0\} = \emptyset$ holds. Then, S^τ satisfies NUPBR($\mathbb{G}$). This follows immediately from Theorem 5.5 by using $Y \in \mathcal{L}(S,\mathbb{F})$ and Lemma 5.1.

2) Similarly to Proposition 5.3, we can easily prove that the thin set $\{\widetilde{Z} = 0\ \&\ Z_- > 0\}$ is totally inaccessible if and only if X^τ satisfies NUPBR($\mathbb{G}$) for any thin process X with predictable jumps only satisfying NUPBR($\mathbb{F}$).

5.2.3 The General Framework

Throughout the paper, with any $\mathbb{H}$-semimartingale, X, we associate a sequence of ($\mathbb{H}$)-predictable stopping times $(T_n^X)_{n\geq1}$ that exhaust the accessible jump times of X. Furthermore, we can decompose X as follows.

$$X = X^{(qc)} + X^{(a)},\ X^{(a)} := I_{\Gamma_X} \centerdot X,\ X^{(qc)} := X - X^{(a)},\ \Gamma_X := \bigcup_{n=1}^{\infty} [\![T_n^X]\!]. \tag{5.13}$$

The process $X^{(a)}$ (the accessible part of X) is a thin process with predictable jumps only, while $X^{(qc)}$ is an $\mathbb{H}$-quasi-left-continuous process (the quasi-left-continuous part of X).

Lemma 5.3: *Let X be an $\mathbb{H}$-semimartingale. Then X satisfies NUPBR($\mathbb{H}$) if and only if $X^{(a)}$ and $X^{(qc)}$ satisfy NUPBR($\mathbb{H}$).*

Proof. Thanks to Proposition 5.1, X satisfies NUPBR($\mathbb{H}$) if and only if there exist an $\mathbb{H}$-predictable real-valued process $\phi > 0$ and a positive $\mathbb{H}$-local martingale Y such that $Y(\phi \centerdot X)$ is an $\mathbb{H}$- local martingale. Then, it is obvious that $Y(\phi I_{\Gamma_X} \centerdot X)$ and $Y(\phi I_{{\Gamma_X}^c} \centerdot X)$ are both $\mathbb{H}$-local martingales. This proves that $X^{(a)}$ and $X^{(qc)}$ both satisfy NUPBR($\mathbb{H}$).
Conversely, if $X^{(a)}$ and $X^{(qc)}$ satisfy NUPBR($\mathbb{H}$), then there exist two $\mathbb{H}$-predictable real-valued processes $\phi_1, \phi_2 > 0$ and two positive $\mathbb{H}$-local martingales $D_1 = \mathcal{E}(N_1), D_2 = \mathcal{E}(N_2)$ such that $D_1(\phi_1 \centerdot (I_{\Gamma_X} \centerdot S))$ and $D_2(\phi_2 \centerdot (I_{{\Gamma_X}^c} \centerdot X))$ are both $\mathbb{H}$-local martingales. Remark that there is no loss of generality in assuming $N_1 = I_{\Gamma_X} \centerdot N_1$ and $N_2 = I_{{\Gamma_X}^c} \centerdot N_2$. Put

$$N := I_{\Gamma_X} \centerdot N_1 + I_{{\Gamma_X}^c} \centerdot N_2 \quad \text{and} \quad \psi := \phi_1 I_{\Gamma_X} + \phi_2 I_{{\Gamma_X}^c}.$$

Obviously, $\mathcal{E}(N) > 0$, $\mathcal{E}(N)$ and $\mathcal{E}(N)(\psi \centerdot S)$ are $\mathbb{H}$-local martingales, ψ is $\mathbb{H}$-predictable and $0 < \psi \leq 1$. This ends the proof of the lemma. □

Below, we state the answer to question **(Prob(5.I))**, in this general framework, which is a consequence of Theorems 5.2 and 5.4, due to Lemma 5.3.

Theorem 5.6: *Suppose that S satisfies NUPBR($\mathbb{F}$). Then, the following assertions are equivalent.*

(a) *The process S^τ satisfies NUPBR($\mathbb{G}$).*

(b) *For any $\delta > 0$, the process*

$$I_{\{Z_- \geq \delta\}} \centerdot (S^{(qc)} - S^{(qc,0)}) := I_{\{Z_- \geq \delta\}} \centerdot (S^{(qc)} - I_{\Gamma^c} \centerdot S^{(0)})$$

satisfies NUPBR($\mathbb{F}$), and there exists a positive $\mathbb{F}$-local martingale, Y, such that ${}^{p,\mathbb{F}}(Y|\Delta S|) < +\infty$ *and*

$${}^{p,\mathbb{F}}\left(Y \Delta S I_{\{\widetilde{Z}>0 \ \& \ Z_- \geq \delta\}}\right) = 0.$$

Proof. Due to Lemma 5.3, it is obvious that S^τ satisfies NUPBR($\mathbb{G}$) if and only if both $(S^{(qc)})^\tau$ and $(S^{(a)})^\tau$ satisfy NUPBR($\mathbb{G}$). Thus, using both Theorems 5.2 and 5.5, we deduce that this last fact is true if and only if for any $\delta > 0$, the process $I_{\{Z_- \geq \delta\}} \centerdot (S^{(qc)} - I_{\Gamma^c} \centerdot S^{(0)})$ satisfies NUPBR($\mathbb{F}$) and there exists a positive $\mathbb{F}$-local martingale Y such that

$${}^{p,\mathbb{F}}(Y|\Delta S|) = {}^{p,\mathbb{F}}\left(Y|\Delta S^{(a)}|\right) < +\infty \quad \text{and}$$

$${}^{p,\mathbb{F}}\left(Y \Delta S I_{\{\widetilde{Z}>0, \ Z_- \geq \delta\}}\right) = {}^{p,\mathbb{F}}\left(Y \Delta S^{(a)} I_{\{\widetilde{Z}>0, \ Z_- \geq \delta\}}\right) = 0.$$

This ends the proof of the theorem. □

Corollary 5.6.1: The following assertions hold.

(a) If either m is continuous or Z is positive (equivalently $\widetilde{Z} > 0$ or $Z_- > 0$), S^τ satisfies NUPBR($\mathbb{G}$) whenever S satisfies NUPBR($\mathbb{F}$).

(b) If S satisfies NUPBR($\mathbb{F}$) and $\{\Delta S \neq 0\} \cap \{\widetilde{Z} = 0 < Z_-\} = \emptyset$, then S^τ satisfies NUPBR($\mathbb{G}$).

(c) If S is continuous and satisfies NUPBR($\mathbb{F}$), then for any random time τ, S^τ satisfies NUPBR($\mathbb{G}$).

Proof. 1) The proof of assertion (a) of the corollary follows easily from Theorem 5.6. Indeed, in the two cases, one has $\{\widetilde{Z} = 0 < Z_-\} = \emptyset$ which implies that

$\{\widetilde{Z} = 0, \delta \leq Z_-\} = \emptyset$ and $S^{(qc,0)} \equiv 0$ (due to (5.8)). Then, due to Lemma 5.1, It suffices to take $Y \in \mathcal{L}(S, \mathbb{F})$ —since this set is non-empty— and apply Theorem 5.6. 2) it is obvious that assertion (c) follows from assertion (b). To prove this latter, it is enough to remark that $\{\Delta S \neq 0\} \cap \{\widetilde{Z} = 0, \delta \leq Z_-\} = \emptyset$ implies that

$$I_{\{Z_- \geq \delta\}} \centerdot S^{(qc,0)} \equiv 0 \quad \text{and} \quad \Delta S I_{\{\widetilde{Z}=0, \delta \leq Z_-\}} = 0.$$

Thus, again, it is enough to take $Y \in \mathcal{L}(S, \mathbb{F})$ (i.e. Y is the σ-martingale density of S) and apply Theorem 5.6. This ends the proof of the corollary. □

Remark 5.7: Any of the two assertions of the above corollary generalizes the main result of Fontana et al. [63], obtained under some restrictive assumptions on the random time τ and the market model as well.

Below, we provide a general answer to question **(Prob(5.II))**, as a consequence of Theorems 5.2 and 5.5 and Proposition 5.3.

Theorem 5.7: *The following assertions are equivalent:*

(a) *The thin set* $\left\{\widetilde{Z} = 0 \ \& \ Z_- > 0\right\}$ *is evanescent.*

(b) *For any (bounded) X satisfying NUPBR($\mathbb{F}$), X^τ satisfies NUPBR($\mathbb{G}$).*

Proof. Suppose that assertion (a) holds, and consider a process X satisfying NUPBR($\mathbb{F}$). Then, $X^{(qc,0)} := I_{\Gamma_X^c} \centerdot X^{(0)} \equiv 0$, where $X^{(0)}$ is defined as in (5.6). Hence $I_{\{Z_- \geq \delta\}} \centerdot \left(X^{(qc)} - I_{\Gamma_X^c} \centerdot X^{(0)}\right)$ satisfies NUPBR($\mathbb{F}$) for any $\delta > 0$, and NUPBR($\mathbb{G}$) property of $(X^{(qc)})^\tau$ follows immediately from Theorem 5.2 on the one hand. On the other hand, it is easy to see that $X^{(a)}$ fulfills the condition (5.12) with $Y \equiv 1$. Thus, thanks to Theorem 5.5 (applied to the thin process $X^{(a)}$ satisfying NUPBR($\mathbb{F}$)), we conclude that $(X^{(a)})^\tau$ satisfies NUPBR($\mathbb{G}$). Thus, due to Lemma 5.3, the proof of (a)⇒(b) is completed.

We now suppose that assertion (b) holds. On the one hand, from Proposition 5.3, we deduce that $\{\widetilde{Z} = 0 < Z_-\}$ is accessible and can be covered with the graphs of $\mathbb{F}$-predictable stopping times $(T_n)_{n\geq 1}$. On the other hand, a direct application of Proposition 5.3 to all single predictable jump $\mathbb{F}$-martingales, we obtain $\{\widetilde{Z} = 0 < Z_-\} \cap [\![T]\!] = \emptyset$ for any $\mathbb{F}$-predictable stopping time T. Therefore, we get

$$\{\widetilde{Z} = 0 < Z_-\} = \bigcup_{n=1}^{\infty} \Big(\{\widetilde{Z} = 0 < Z_-\} \cap [\![T_n]\!]\Big) = \emptyset.$$

This proves assertion (a), and the proof of the theorem is completed. □

5.3 Stochastics from–and–for Informational Non-Arbitrage

In this section, we develop new stochastic results that will play a key role in the proofs and/or the statements of the main results outlined in the previous section. The first subsection compares the $\mathbb{G}$-compensators and the $\mathbb{F}$-compensators, while the second subsection studies $\mathbb{G}$-localization and $\mathbb{F}$-localization. Section 5.3.3 is constructing a $\mathbb{G}$-local martingale that is vital in the explicit construction of deflators. We recall that $Z_- + \Delta m = \widetilde{Z}$ (see [84]).

Lemma 5.4: *Let Z and $\widetilde{Z}$ be the two supermartingales given by (5.1).*

(a) *The three sets $\{\widetilde{Z} = 0\}$, $\{Z = 0\}$ and $\{Z_- = 0\}$ have the same début which is an $\mathbb{F}$-stopping time that we denote by*

$$\widehat{R} := \inf\{t \geq 0 \big| \; Z_{t-} = 0\}. \tag{5.14}$$

(b) *The following $\mathbb{F}$-stopping times*

$$\widehat{R}_0 := \begin{cases} \widehat{R} & \text{on } \{Z_{\widehat{R}-} = 0\} \\ +\infty & \text{otherwise} \end{cases} \quad \text{and} \quad \widetilde{R}_0 := \begin{cases} \widehat{R} & \text{on } \{\widetilde{Z}_{\widehat{R}} = 0\} \\ +\infty & \text{otherwise} \end{cases}$$

are such that $\widehat{R}_0$ is a $\mathbb{F}$-predictable stopping time, and

$$\tau \leq \widehat{R}, \qquad \tau < \widetilde{R}_0, \qquad P-a.s. \tag{5.15}$$

(c) *The $\mathbb{G}$-predictable process*

$$H_t := (Z_{t-})^{-1}\, I_{[\![0,\tau]\!]}(t), \tag{5.16}$$

is $\mathbb{G}$-locally bounded.

Proof. From [53, paragraph 14, Chapter XX], for any random time τ, the sets $\{\widetilde{Z} = 0\}$ and $\{Z_- = 0\}$ are disjoint from $]\!]0,\tau]\!]$ and have the same lower bound $\widehat{R}$, the smallest $\mathbb{F}$-stopping time greater than τ. Thus, we also conclude that $\{Z = 0\}$ is disjoint from $]\!]0,\tau[\![$. This leads to assertion (a). The process $X := Z^{-1} I_{]\!]0,\tau[\![}$ being a càdlàg $\mathbb{G}$-supermartingale [141], its left limit is locally bounded. Then, due to

$$(Z_-)^{-1} I_{]\!]0,\tau]\!]} = X_-,$$

the local boundedness of H follows. This ends the proof of the lemma. $\square$

5.3.1 Projection and Compensator under $\mathbb{G}$ and $\mathbb{F}$

The main results of this subsection are summarized in Lemmas 5.5 and 5.6, where we address the question of how to compute $\mathbb{G}$-dual predictable projections in term of $\mathbb{F}$-dual predictable projections and vice versa. These results are based essentially on the following standard result on progressive enlargement of filtration (we refer the reader to [53, 83] for proofs).

Proposition 5.4: *Let M be an $\mathbb{F}$-local martingale. Then, for any random time τ, the process $\widehat{M}$ given by*

$$\widehat{M}_t := M_{t\wedge\tau} - \int_0^{t\wedge\tau} \frac{d\langle M, m\rangle^{\mathbb{F}}_s}{Z_{s-}} \tag{5.17}$$

is a $\mathbb{G}$-local martingale, where m is defined in (5.2).

Remark 5.8: Throughout this chapter, it is worthy to keep in mind that the process

$\widehat{X}$ would be defined via (5.17) for any $\mathbb{F}$-local martingale X.

In the following lemma, we express the $\mathbb{G}$–dual predictable projection of an $\mathbb{F}$-locally integrable variation process in terms of an $\mathbb{F}$–dual predictable projection, and $\mathbb{G}$-predictable projection in terms of $\mathbb{F}$-predictable projection.

Lemma 5.5: *The following assertions hold.*

(a) *For any $\mathbb{F}$-adapted process V with locally integrable variation, we have*

$$(V^\tau)^{p,\mathbb{G}} = (Z_-)^{-1} I_{]\!]0,\tau]\!]} \bullet \big(\widetilde{Z} \bullet V\big)^{p,\mathbb{F}}. \tag{5.18}$$

(b) *For any $\mathbb{F}$-local martingale M, we have, on $[\![0,\tau]\!]$*

$${}^{p,\mathbb{G}}\left(\frac{\Delta M}{\widetilde{Z}}\right) = \frac{{}^{p,\mathbb{F}}\Big(\Delta M I_{\{\widetilde{Z}>0\}}\Big)}{Z_-}, \quad and \quad {}^{p,\mathbb{G}}\left(\frac{1}{\widetilde{Z}}\right) = \frac{{}^{p,\mathbb{F}}\Big(I_{\{\widetilde{Z}>0\}}\Big)}{Z_-}. \tag{5.19}$$

(c) *For any quasi-left-continuous $\mathbb{F}$-local martingale M, we have, on $[\![0,\tau]\!]$*

$${}^{p,\mathbb{G}}\left(\frac{\Delta M}{\widetilde{Z}}\right) = 0, \quad and \quad {}^{p,\mathbb{G}}\left(\frac{1}{Z_- + \Delta m^{(qc)}}\right) = \frac{1}{Z_-}, \tag{5.20}$$

where $m^{(qc)}$ is the quasi-left-continuous $\mathbb{F}$-martingale defined via (5.13).

Proof. (a) By considering $M = V - V^{p,\mathbb{F}}$ and $\widehat{M}$ defined via (5.17), we obtain

$$V^\tau = I_{]\!]0,\tau]\!]} \bullet V^{p,\mathbb{F}} + \widehat{M} + \frac{1}{Z_-} I_{]\!]0,\tau]\!]} \bullet \langle V, m\rangle^{\mathbb{F}} = \widehat{M} + I_{]\!]0,\tau]\!]} \bullet V^{p,\mathbb{F}} + \frac{1}{Z_-} I_{]\!]0,\tau]\!]} \bullet (\Delta m \bullet V)^{p,\mathbb{F}},$$

which proves assertion (a).

(b) Let M be an $\mathbb{F}$-local martingale, then, for any positive integers (n,k) the process $V^{(n,k)} := \sum \frac{\Delta M}{\widetilde{Z}} I_{\{|\Delta M| \geq k^{-1},\ \widetilde{Z} \geq n^{-1}\}}$ has locally integrable variation. Then, by using the known equality ${}^{p,\mathbb{G}}(\Delta V) = \Delta(V^{p,\mathbb{G}})$ (see Theorem 76 in pages 149–150 of [54] or Theorem 5.27 in page 150 of [71]), and applying assertion (a) to the process $V^{(n,k)}$, we get, on $\rrbracket 0,\tau\rrbracket$

$$ {}^{p,\mathbb{G}}\left(\frac{\Delta M}{\widetilde{Z}} I_{\{|\Delta M|\geq k^{-1},\ \widetilde{Z}\geq n^{-1}\}}\right) = \frac{1}{Z_-}\ {}^{p,\mathbb{F}}\left(\Delta M I_{\{|\Delta M|\geq k^{-1},\ \widetilde{Z}\geq n^{-1}\}}\right). $$

Since M is a local martingale, by stopping we can exchange limits with projections in both sides. Then by letting n and k go to infinity, and using the fact that $\widetilde{Z} > 0$ on $\rrbracket 0,\tau\rrbracket$, we deduce that

$$ {}^{p,\mathbb{G}}\left(\frac{\Delta M}{\widetilde{Z}}\right) = \frac{1}{Z_-}\ {}^{p,\mathbb{F}}\left(\Delta M I_{\{\widetilde{Z}>0\}}\right). $$

This proves the first equality in (5.19), while the second equality follows from $\widetilde{Z} = \Delta m + Z_-$:

$$ \begin{aligned} Z_-\ {}^{p,\mathbb{G}}\left(\widetilde{Z}^{-1}\right) &= {}^{p,\mathbb{G}}\left((\widetilde{Z}-\Delta m)/\widetilde{Z}\right) = 1 - {}^{p,\mathbb{G}}\left(\Delta m/\widetilde{Z}\right) \\ &= 1-(Z_-)^{-1}\ {}^{p,\mathbb{F}}\left(\Delta m I_{\{\widetilde{Z}>0\}}\right) = 1 - {}^{p,\mathbb{F}}\left(I_{\{\widetilde{Z}=0\}}\right) = {}^{p,\mathbb{F}}\left(I_{\{\widetilde{Z}>0\}}\right). \end{aligned} $$

In the above string of equalities, the third equality follows from the first equality in (5.19), while the fourth equality is due to ${}^{p,\mathbb{F}}(\Delta m) = 0$ and $\Delta m I_{\{\widetilde{Z}=0\}} = -Z_- I_{\{\widetilde{Z}=0\}}$. This ends the proof of assertion (b).

(c) If M is a quasi-left-continuous $\mathbb{F}$-local martingale, then ${}^{p,\mathbb{F}}\left(\Delta M I_{\{\widetilde{Z}>0\}}\right) = 0$, and the first property of the assertion (c) follows. Applying the first property to $M = m^{(qc)}$ and using that, on $\rrbracket 0,\tau\rrbracket$, one has $\Delta m^{(qc)}\left(Z_- + \Delta m\right)^{-1} = \Delta m^{(qc)}\left(Z_- + \Delta m^{(qc)}\right)^{-1}$, we obtain

$$ \frac{1}{Z_-}\ {}^{p,\mathbb{G}}\left(\frac{Z_-}{Z_- + \Delta m^{(qc)}}\right) = \frac{1}{Z_-}\left(1 - {}^{p,\mathbb{G}}\left(\frac{\Delta m^{(qc)}}{Z_- + \Delta m^{(qc)}}\right)\right) = \frac{1}{Z_-}. $$

This proves assertion (c), and the proof of the lemma is achieved. □

The next lemma proves that $\widetilde{Z}^{-1} I_{\rrbracket 0,\tau\rrbracket}$ is Lebesgue-Stieltjes-integrable with respect to any process that is $\mathbb{F}$-adapted with $\mathbb{F}$-locally integrable variation. Using this fact, the lemma addresses the question of how an $\mathbb{F}$-compensator stopped at τ can be written in terms of a $\mathbb{G}$-compensator, and constitutes a *sort of* converse result to Lemma 5.5–(a).

Lemma 5.6: *Let V be an $\mathbb{F}$-adapted càdlàg process. Then the following hold.*

(a) *If V belongs to $\mathcal{A}^+_{loc}(\mathbb{F})$ (respectively $V \in \mathcal{A}^+(\mathbb{F})$), then the process*

$$U := \widetilde{Z}^{-1} I_{]\!]0,\tau]\!]} \cdot V, \tag{5.21}$$

belongs to $\mathcal{A}^+_{loc}(\mathbb{G})$ (respectively to $\mathcal{A}^+(\mathbb{G})$).

(b) *If V has $\mathbb{F}$-locally integrable variation, then the process U is well defined, its variation is $\mathbb{G}$-locally integrable, and its $\mathbb{G}$-dual predictable projection is given by*

$$U^{p,\mathbb{G}} = \left(\frac{1}{\widetilde{Z}} I_{]\!]0,\tau]\!]} \cdot V\right)^{p,\mathbb{G}} = \frac{1}{Z_-} I_{]\!]0,\tau]\!]} \cdot \left(I_{\{\widetilde{Z}>0\}} \cdot V\right)^{p,\mathbb{F}}. \tag{5.22}$$

In particular, if $supp(V) \subset \{\widetilde{Z} > 0\}$, then, on $]\!]0,\tau]\!]$, one has $V^{p,\mathbb{F}} = Z_- \cdot U^{p,\mathbb{G}}$.

Proof. (a) Suppose that $V \in \mathcal{A}^+_{loc}(\mathbb{F})$. First, remark that, due to the fact that $\widetilde{Z}$ is positive on $]\!]0,\tau]\!]$, U is well defined. Let $(\vartheta_n)_{n\geq 1}$ be a sequence of $\mathbb{F}$-stopping times that increases to $+\infty$ such that $E(V_{\vartheta_n}) < +\infty$. Then, if $E(U_{\vartheta_n}) \leq E(V_{\vartheta_n})$, assertion (a) follows. Thus, we calculate

$$\begin{aligned} E(U_{\vartheta_n}) &= E\left(\int_0^{\vartheta_n} I_{\{0<t\leq\tau\}} \frac{1}{\widetilde{Z}_t} dV_t\right) = E\left(\int_0^{\vartheta_n} \frac{P(\tau \geq t|\mathcal{F}_t)}{\widetilde{Z}_t} I_{\{\widetilde{Z}_t>0\}} dV_t\right) \\ &\leq E(V_{\vartheta_n}). \end{aligned}$$

The last inequality is obtained due to $\widetilde{Z}_t := P(\tau \geq t|\mathcal{F}_t)$. This ends the proof of assertion (a) of the lemma.

(b) Suppose that $V \in \mathcal{A}_{loc}(\mathbb{F})$, and denote by $W := V^+ + V^-$ its variation. Then

$W \in \mathcal{A}^+_{loc}(\mathbb{F})$, and a direct application of the first assertion implies that

$$\left(\widetilde{Z}\right)^{-1} I_{]\!]0,\tau]\!]} \centerdot W \in \mathcal{A}^+_{loc}(\mathbb{G}).$$

As a result, we deduce that U given by (5.21) for the case of $V = V^+ - V^-$ is well defined and has variation equal to $\left(\widetilde{Z}\right)^{-1} I_{]\!]0,\tau]\!]} \centerdot W$ which is $\mathbb{G}$-locally integrable. By setting $U_n := I_{]\!]0,\tau]\!]} \centerdot \left(\widetilde{Z}^{-1} I_{\{\widetilde{Z}\geq 1/n\}} \centerdot V\right)$, we derive, due to (5.18),

$$(U_n)^{p,\mathbb{G}} = \frac{1}{Z_-} I_{]\!]0,\tau]\!]} \centerdot \left(I_{\{\widetilde{Z}\geq 1/n\}} \centerdot V\right)^{p,\mathbb{F}}.$$

Hence, since $U^{p,\mathbb{G}} = \lim_{n\longrightarrow+\infty} (U_n)^{p,\mathbb{G}}$, by taking the limit in the above equality, (5.22) follows immediately, and the lemma is proved. □

5.3.2 $\mathbb{G}$-Localization versus $\mathbb{F}$-Localization

Lemma 5.7: *Let $H^{\mathbb{G}}$ be a $\widetilde{\mathcal{P}}(\mathbb{G})$-measurable functional. Then, the following hold.*

(a) *There exist an $\widetilde{\mathcal{P}}(\mathbb{F})$-measurable functional $H^{\mathbb{F}}$ and a $\mathcal{B}(\mathbb{R}_+)\otimes\widetilde{\mathcal{P}}(\mathbb{F})$-measurable functionals $K^{\mathbb{F}} : \mathbb{R}_+ \times \mathbb{R}_+ \times \Omega \times \mathbb{R}^d \to \mathbb{R}$ such that*

$$H^{\mathbb{G}}(\omega, t, x) = H^{\mathbb{F}}(\omega, t, x) I_{]\!]0,\tau]\!]} + K^{\mathbb{F}}(\tau(\omega), t, \omega, x) I_{]\!]\tau,+\infty]\!]}. \tag{5.23}$$

(b) *If furthermore $H^{\mathbb{G}} > 0$ (respectively $H^{\mathbb{G}} \leq 1$), then we can choose $H^{\mathbb{F}} > 0$ (respectively $H^{\mathbb{F}} \leq 1$) such that*

$$H^{\mathbb{G}}(\omega, t, x) I_{]\!]0,\tau]\!]} = H^{\mathbb{F}}(\omega, t, x) I_{]\!]0,\tau]\!]}.$$

(c) *If $L^{\mathbb{G}}$ is an $\widetilde{\mathcal{O}}(\mathbb{G})$-measurable functional, then there exist a $\widetilde{\mathcal{O}}(\mathbb{G})$-measurable functional $L^{(1)}(t, \omega, x)$, a $\widetilde{\mathcal{P}}_{prog}(\mathbb{F})$-measurable functional $L^{(2)}(t, \omega, x)$ and $\widetilde{\mathcal{P}}_{prog}(\mathbb{F})\otimes \mathcal{B}(\mathbb{R}^d)$-measurable functional, $L^{(3)}(t, \omega, x, v)$, such that*

$$L^{\mathbb{G}}(t, \omega, x) = L^{(1)}(t, \omega, x) I_{]\!]0,\tau[\![} + L^{(1)}(t, \omega, x) I_{[\![\tau]\!]} + L^{(3)}(t, \omega, x, \tau) I_{]\!]\tau,+\infty[\![} \tag{5.24}$$

where $\mathcal{P}_{prog}(\mathbb{F})$ *is the* $\mathbb{F}$*-progressive* σ*-field on* $\Omega\times\mathbb{R}^+$*, and* $\widetilde{\mathcal{P}}_{prog}(\mathbb{F}) := \mathcal{P}_{prog}(\mathbb{F})\otimes\mathcal{B}(\mathbb{R}^d)$*. If furthermore,* $0 < L^{\mathbb{G}}$ *(respectively* $L^{\mathbb{G}} \leq 1$*), then all* $L^{(i)}$ *can be chosen such that* $0 < L^{(i)}$ *(respectively* $L^{(i)} \leq 1$*),* $i = 1, 2, 3$*.*

(d) *For any* $\mathbb{F}$*-stopping time,* T*, and any positive* $\mathcal{G}_T$*-measurable random variable* $Y^{\mathbb{G}}$*, there exist two positive* $\mathcal{F}_T$*-measurable random variables,* $Y^{(1)}$ *and* $Y^{(2)}$*, satisfying*

$$Y^{\mathbb{G}} I_{\{T\leq\tau\}} = Y^{(1)} I_{\{T<\tau\}} + Y^{(2)} I_{\{\tau=T\}}. \tag{5.25}$$

Proof. (a) We are mimicking the approach of Jeulin [83]. Notice that $\widetilde{\mathcal{P}}(\mathbb{G})$ is generated by the process of the form

$$h^{\mathbb{G}}(\omega, t, x) := \alpha\left(s\wedge\tau(\omega)\right) f_s(\omega) g(x) I_{\{s<t\}},$$

where α and g are Borel measurable functions and $f_s \in \mathcal{F}_s$. Now, by taking

$$h^{\mathbb{F}}(\omega, t, x) := \alpha\left(s\right) f_s(\omega) g(x) I_{\{s<t\}} \in \widetilde{\mathcal{P}}(\mathbb{F}) \quad \text{and}$$
$$k^{\mathbb{F}}(y, t, \omega, x) := \alpha\left(y\wedge s\right) f_s(\omega) g(x) I_{\{s<t\}} \in \mathcal{B}(R_+)\otimes\widetilde{\mathcal{P}}(\mathbb{F})$$

we have

$$h^{\mathbb{G}}(\omega, t, x) = h^{\mathbb{F}} I_{[\![0,\tau[\![} + k^{\mathbb{F}}(\tau(\omega), t, \omega, x) I_{[\![\tau,+\infty[\![}.$$

Therefore, (5.23) follows immediately.

(b) To prove positivity of $H^{\mathbb{F}}$ when $H^{\mathbb{G}} > 0$, we consider

$$\overline{H}^{\mathbb{F}} := (H^{\mathbb{F}})^+ + I_{\{H^{\mathbb{F}}=0\}} > 0,$$

and we remark that due to (5.23), we have $]\!]0,\tau]\!] \subset \{H^{\mathbb{G}} = H^{\mathbb{F}}\} \subset \{H^{\mathbb{F}} > 0\}$. Thus, we get

$$H^{\mathbb{G}} I_{]\!]0,\tau]\!]} = \overline{H}^{\mathbb{F}} I_{]\!]0,\tau]\!]}.$$

Similarly, we consider $H^{\mathbb{F}} \wedge 1$, and we deduce that if $H^{\mathbb{G}}$ is upper-bounded by one, the process $H^{\mathbb{F}}$ can also be chosen to not exceed one. This ends the proof of assertion (b).

(c) For the proof of assertion (c), we mimic Jeulin (see [83]) in his proof of Proposition (5,3)–(b). In fact, it is clear that the σ-field $\widetilde{\mathcal{O}}(\mathbb{H})$ is generated by the functionals $H((t,\omega,x) := g(x) I_{[\![\sigma_1,\sigma_2]\!]}$ where σ_i are $\mathbb{G}$-stopping times such that $\sigma_1 < \sigma_2$ on $\{\sigma_1 < +\infty\}$. Then, it is easy to remark that

$$H(t,\omega,x) = \limsup_{s\longrightarrow t} \overline{H}(s,\omega,x), \qquad \text{where } \ \overline{H}(s,\omega,x) := g(x) I_{[\![\sigma_1,\sigma_2[\![}(s).$$

Thus a direct application of assertion (a) to the functional $\overline{H}$, we obtain the existence of an $\widetilde{\mathcal{P}}(\mathbb{F})$-measurable functional, $J(t,\omega,x)$, and a $\mathcal{B}(\mathbb{R}_+) \otimes \widetilde{\mathcal{P}}(\mathbb{F})$-measurable functional, $K(t,\omega,x,v)$, such that

$$\overline{H}(s,\omega,x) = J(t,\omega,x) I_{]\!]0,\tau]\!]} + K(t,\omega,x,\tau) I_{]\!]\tau,+\infty]\!]}.$$

Put

$$\overline{J}(t,\omega,x) := \limsup_{s\longrightarrow t} J(s,\omega,x), \quad \overline{K}(t,\omega,x,v) := \limsup_{s\longrightarrow t} K(s,\omega,x,v),$$

and

$$W(t,\omega,x,v) := \sup\left\{ \limsup_{s\rightarrow t, s<t} J(s,\omega,x), \quad \limsup_{s\rightarrow t, s>t} K(s,\omega,x,v) \right\}.$$

Then, due to [52] (see also Lemma (4,1) in Jeulin [83]), $\overline{J}$ is $\widetilde{\mathcal{O}}(\mathbb{F})$-measurable, $\overline{K}$ is $\widetilde{\mathcal{O}}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}^+)$-measurable and $W(t,\omega,x,v)$ is $\widetilde{\mathcal{P}}_{prog}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}^+)$-measurable. As a result, we deduce (due to Lemma 5.8 below), that $\overline{W}(t,\omega,x) = W(t,\omega,x,t)$ is

$\widetilde{\mathcal{P}}_{prog}(\mathbb{F})$-measurable, and we have

$$H(t,\omega,x) = \overline{J}(t,\omega,x)I_{]\!]0,\tau[\![} + \overline{W}(t,\omega,x)I_{[\![\tau]\!]} + \overline{K}(t,\omega,x,\tau)I_{]\!]\tau,+\infty]\!]}.$$

Thus, the proof of the first part of the assertion (c) follows from the class monotone theorem. The proof of the positivity and the upper boundedness follows the same arguments as in the proof of assertion (b).

(d) The proof of assertion (d) is a direct application of assertion (c) combined with the fact that for any $\mathbb{F}$-progressively measurable process Y and any $\mathbb{F}$-stopping time T, we have $Y_T I_{\{T<+\infty\}}$ is $\mathcal{F}_T$-measurable. For this last fact, we refer the reader to Theorem 64 in [54]. This ends the proof of the lemma. □

Below, we state a simple and useful lemma that generalizes a result of [38].

Lemma 5.8: *If $X(t,\omega,x,v)$ is an $\widetilde{\mathcal{O}}(\mathbb{H})\otimes\mathcal{B}(\mathbb{R}^+)$-measurable functional, then $\overline{X}(t,\omega,x) := X(t,\omega,x,t)$ is $\widetilde{\mathcal{O}}(\mathbb{H})$-measurable.*

Proof. The proof of this lemma is immediate from a combination of the class monotone theorem, and the proof of the lemma for the generators of $\widetilde{\mathcal{O}}(\mathbb{H})\otimes\mathcal{B}(\mathbb{R}^+)$ having the form of $X(t,\omega,x,v) = H(t,\omega,x)k(v)$. Here H is $\widetilde{\mathcal{O}}(\mathbb{H})$-measurable and k is $\mathcal{B}(\mathbb{R}^+)$-measurable. For these generators, we have $\overline{X}(t,\omega,x) = H(t,\omega,x)k(t)$ which is obviously $\widetilde{\mathcal{O}}(\mathbb{H})$-measurable. □

In the following, we state and prove our main results of this subsection.

Proposition 5.5: *For any $\alpha>0$, the following assertions hold:*

(a) *Let h be a $\widetilde{\mathcal{P}}(\mathbb{H})$-measurable functional. Then, $\sqrt{(h-1)^2\star\mu}\in\mathcal{A}^+_{loc}(\mathbb{H})$ iff*

$$(h-1)^2 I_{\{|h-1|\leq\alpha\}}\star\mu \ \textit{and}\ \ |h-1|I_{\{|h-1|>\alpha\}}\star\mu \ \ \textit{belong to}\ \ \mathcal{A}^+_{loc}(\mathbb{H}).$$

(b) *Let $(\sigma_n^{\mathbb{G}})_n$ be a sequence of $\mathbb{G}$-stopping times that increases to infinity. Then,* *there exists a nondecreasing sequence of* $\mathbb{F}$*-stopping times,* $(\sigma_n^{\mathbb{F}})_{n\geq 1}$*, satisfying the following properties*

$$\sigma_n^{\mathbb{G}} \wedge \tau = \sigma_n^{\mathbb{F}} \wedge \tau, \quad \sigma_\infty := \sup_n \sigma_n^{\mathbb{F}} \geq \widehat{R} \quad P-a.s., \tag{5.26}$$

$$\text{and} \qquad Z_{\sigma_\infty -} = 0 \quad P-a.s. \quad \text{on} \quad \Sigma \cap (\sigma_\infty < +\infty), \tag{5.27}$$

where $\Sigma := \bigcap_{n\geq 1} (\sigma_n^{\mathbb{F}} < \sigma_\infty)$.

(c) *Let* V *be an* $\mathbb{F}$*-predictable and non-decreasing process. Then,* $V^\tau \in \mathcal{A}_{loc}^+(\mathbb{G})$ *if and only if* $I_{\{Z_- \geq \delta\}} \centerdot V \in \mathcal{A}_{loc}^+(\mathbb{F})$ *for any* $\delta > 0$.

(d) *Let* h *be a nonnegative and* $\widetilde{\mathcal{P}}(\mathbb{F})$*-measurable functional. Then,* $hI_{]\!]0,\tau]\!]} \star \mu \in \mathcal{A}_{loc}^+(\mathbb{G})$ *if and only if for all* $\delta > 0$, $hI_{\{Z_- \geq \delta\}} \star \mu^1 \in \mathcal{A}_{loc}^+(\mathbb{F})$*, where* $\mu^1 := \widetilde{Z}$ *centerdot*μ.

(e) *Let* f *be positive and* $\widetilde{\mathcal{P}}(\mathbb{F})$*-measurable, and* $\mu^1 := \widetilde{Z} \centerdot \mu$*. Then* $\sqrt{(f-1)^2 I_{]\!]0,\tau]\!]} \star \mu} \in \mathcal{A}_{loc}^+(\mathbb{G})$ *iff* $\sqrt{(f-1)^2 I_{\{Z_- \geq \delta\}} \star \mu^1} \in \mathcal{A}_{loc}^+(\mathbb{F})$*, for all* $\delta > 0$.

Proof. (a) Put $W := (h-1)^2 \star \mu = W_1 + W_2$, where $W_1 := (h-1)^2 I_{\{|h-1|\leq\alpha\}} \star \mu$, $W_2 := (h-1)^2 I_{\{|h-1|>\alpha\}} \star \mu$ and $W_2' := |h-1| I_{\{|h-1|>\alpha\}} \star \mu$. Note that

$$\sqrt{W} \leq \sqrt{W_1} + \sqrt{W_2} \leq \sqrt{W_1} + W_2'.$$

Therefore $\sqrt{W_1}, W_2' \in \mathcal{A}_{loc}^+$ imply $\sqrt{W}$ is locally integrable.
Conversely, if $\sqrt{W} \in \mathcal{A}_{loc}^+$, $\sqrt{W_1}$ and $\sqrt{W_2}$ are both locally integrable. Since W_1 is locally bounded and has finite variation, W_1 is locally integrable. In the following, we focus on the proof of the local integrability of W_2'. Denote

$$\tau_n := \inf\{t \geq 0 : \ V_t > n\}, \quad V := W_2.$$

It is easy to see that τ_n increases to infinity and $V_- \leq n$ on the set $]\!]0, \tau_n]\!]$. On the set $\{\Delta V > 0\}$, we have $\Delta V \geq \alpha^2$. By using the elementary inequality $\sqrt{1+\frac{n}{\alpha^2}} - \sqrt{\frac{n}{\alpha^2}} \leq$

$\sqrt{1+x}-\sqrt{x}\leq 1$, when $0\leq x\leq \frac{n}{\alpha^2}$, we have

$$\sqrt{V_-+\Delta V}-\sqrt{V_-}\geq \beta_n\sqrt{\Delta V}\quad \text{on }]\!]0,\tau_n]\!],\ \text{where}\ \ \beta_n:=\sqrt{1+\frac{n}{\alpha^2}}-\sqrt{\frac{n}{\alpha^2}},$$

and

$$\left(W_2'\right)^{\tau_n}=\left(\sum\sqrt{\Delta V}\right)^{\tau_n}\leq\frac{1}{\beta_n}\left(\sum\Delta\sqrt{V}\right)^{\tau_n}=\frac{1}{\beta_n}\left(\sqrt{W_2}\right)^{\tau_n}\in\mathcal{A}^+_{loc}(\mathbb{H})$$

Therefore $W_2'\in(\mathcal{A}^+_{loc}(\mathbb{H}))_{loc}=\mathcal{A}^+_{loc}(\mathbb{H})$.

(b) Due to Jeulin [83], there exists a sequence of $\mathbb{F}$-stopping times $(\sigma_n^{\mathbb{F}})_n$ such that

$$\sigma_n^{\mathbb{G}}\wedge\tau=\sigma_n^{\mathbb{F}}\wedge\tau. \tag{5.28}$$

By putting $\sigma_n:=\sup_{k\leq n}\sigma_k^{\mathbb{F}}$, we shall prove that

$$\sigma_n^{\mathbb{G}}\wedge\tau=\sigma_n\wedge\tau, \tag{5.29}$$

or equivalently $\{\sigma_n^{\mathbb{F}}\wedge\tau<\sigma_n\wedge\tau\}$ is negligible. Due to (5.28) and $\sigma_n^{\mathbb{G}}$ is nondecreasing, we derive

$$\{\sigma_n^{\mathbb{F}}<\tau\}=\{\sigma_n^{\mathbb{G}}<\tau\}\subset\bigcap_{i=1}^{n}\{\sigma_i^{\mathbb{G}}=\sigma_i^{\mathbb{F}}\}\subset\{\sigma_n^{\mathbb{F}}=\sigma_n\}.$$

This implies that,

$$\{\sigma_n^{\mathbb{F}}\wedge\tau<\sigma_n\wedge\tau\}=\{\sigma_n^{\mathbb{F}}<\tau,\ \&\ \sigma_n^{\mathbb{F}}<\sigma_n\}=\emptyset,$$

and the proof of (5.29) is completed. Without loss of generality we assume that the sequence $\sigma_n^{\mathbb{F}}$ is nondecreasing. By taking limit in (5.28), we obtain $\tau=\sigma_\infty\wedge\tau, P-$a.s. which is equivalent to $\sigma_\infty\geq\tau, P-$a.s. Since $\widehat{R}$ is the smallest $\mathbb{F}$-stopping time greater or equal than τ almost surely, we obtain, $\sigma_\infty \geq \widehat{R} \geq \tau \;\; P-a.s.$. This achieves the proof of (5.26).

On the set Σ, it is easy to show that

$$I_{[\![0,\sigma_n^{\mathbb{F}}]\!]} \longrightarrow I_{[\![0,\sigma_\infty^{\mathbb{F}}[\![}, \quad \text{when } n \text{ goes to } +\infty.$$

Then, thanks again to (5.28) (by taking $\mathbb{F}$-predictable projection and let n go to infinity afterwards), we obtain

$$Z_- = Z_- I_{[\![0,\sigma_\infty^{\mathbb{F}}[\![}, \qquad \text{on } \Sigma. \tag{5.30}$$

Hence, (5.27) follows immediately, and the proof of assertion (b) is completed.

(c) Suppose that $hI_{[\![0,\tau]\!]} \star \mu \in \mathcal{A}^+_{loc}(\mathbb{G})$. Then, there exists a sequence of $\mathbb{G}$-stopping times $(\sigma_n^{\mathbb{G}})_{n\geq 1}$ increasing to infinity such that $hI_{[\![0,\tau]\!]} \star \mu^{\sigma_n^{\mathbb{G}}}$ is integrable. Consider $(\sigma_n)_{n\geq 1}$ a sequence of $\mathbb{F}$-stopping times satisfying (5.26)–(5.27) (its existence is guaranteed by assertion (b)). Therefore, for any fixed $\delta > 0$

$$W^n := M^P_\mu\left(\widetilde{Z}|\widetilde{\mathcal{P}}\right) I_{\{Z_- \geq \delta\}} h \star \nu^{\sigma_n} \in \mathcal{A}^+(\mathbb{F}), \tag{5.31}$$

or equivalently, this process is càdlàg predictable with finite values. Thus, it is obvious that the proof of assertion (iii) will follow immediately if we prove that the $\mathbb{F}$-predictable and nondecreasing process

$$W := M^P_\mu\left(\widetilde{Z}|\widetilde{\mathcal{P}}\right) I_{\{Z_- \geq \delta\}} h \star \nu \quad \text{is càdlàg with finite values.} \tag{5.32}$$

To prove this last fact, we consider the random time τ^δ defined by

$$\tau^\delta := \sup\{t \geq 0 : Z_{t-} \geq \delta\}.$$

Then, it is clear that $I_{]\!]\tau^\delta,+\infty[\![}\centerdot W\equiv 0$ and

$$\tau^\delta\leq \widehat{R}\leq\sigma_\infty\quad\text{and}\quad Z_{\tau^\delta-}\geq\delta\qquad P\text{–}a.s.\qquad\text{on }\{\tau^\delta<+\infty\}.$$

The proof of (5.32) will be achieved by considering three sets, namely $\{\sigma_\infty=\infty\}$, $\Sigma\cap\{\sigma_\infty<+\infty\}$, and $\Sigma^c\cap\{\sigma_\infty<+\infty\}$. It is obvious that (5.32) holds on $\{\sigma_\infty=\infty\}$. Due to (5.27), we deduce that $\tau^\delta<\sigma_\infty, P-$a.s. on $\Sigma\cap\{\sigma_\infty<+\infty\}$. Since W is supported on $[\![0,\tau^\delta]\!]$, then (5.32) follows immediately on the set $\Sigma\cap\{\sigma_\infty<+\infty\}$. Finally, on the set

$$\Sigma^c\cap\{\sigma_\infty<+\infty\}=\left(\bigcup_{n\geq 1}\{\sigma_n=\sigma_\infty\}\right)\cap\{\sigma_\infty<+\infty\},$$

the sequence σ_n increases stationarily to σ_∞, and thus (5.32) holds on this set. This completes the proof of (5.32), and hence $hI_{\{Z_-\geq\delta\}}\star(\widetilde{Z}\centerdot\mu)$ is locally integrable, for any $\delta>0$.

Conversely, if $hI_{\{Z_-\geq\delta\}}\widetilde{Z}\star\mu\in\mathcal{A}^+_{loc}(\mathbb{F})$, there exists a sequence of $\mathbb{F}$-stopping times $(\tau_n)_{n\geq1}$ that increases to infinity and $\left(hI_{\{Z_-\geq\delta\}}\widetilde{Z}\star\mu\right)^{\tau_n}\in\mathcal{A}^+(\mathbb{F})$. Then, we have

$$E\left[hI_{\{Z_-\geq\delta\}}I_{[\![0,\tau]\!]}\star\mu(\tau_n)\right]=E\left[hI_{\{Z_-\geq\delta\}}\widetilde{Z}\star\mu(\tau_n)\right]<+\infty.\tag{5.33}$$

This proves that $hI_{\{Z_-\geq\delta\}}I_{[\![0,\tau]\!]}\star\mu$ is $\mathbb{G}$-locally integrable, for any $\delta>0$. Since $(Z_-)^{-1}I_{[\![0,\tau]\!]}$ is $\mathbb{G}$-locally bounded, then there exists a family of $\mathbb{G}$-stopping times $(\tau_\delta)_{\delta>0}$ that increases to infinity when δ decreases to zero, and

$$[\![0,\tau\wedge\tau_\delta]\!]\subset\{Z_-\geq\delta\}.$$

This implies that the process $\left(hI_{[\![0,\tau]\!]}\star\mu\right)^{\tau_\delta}$ is $\mathbb{G}$-locally integrable, and hence assertion (c) follows immediately.

(d) The proof of assertion (d) follows from combining assertions (a) and (b). This ends the proof of the proposition. □

5.3.3 An Important $\mathbb{G}$-local martingale

In this subsection, we introduce a $\mathbb{G}$-local martingale that will be crucial for the construction of the deflator.

Lemma 5.9: *The following nondecreasing process*

$$V_t^{\mathbb{G}} := \sum_{0\leq u\leq t} {}^{p,\mathbb{F}}\left(I_{\{\widetilde{Z}=0\}}\right)_u I_{\{u\leq\tau\}} \tag{5.34}$$

is $\mathbb{G}$-predictable, càdlàg, and locally bounded.

Proof. The $\mathbb{G}$-predictability of $V^{\mathbb{G}}$ being obvious, it remains to prove that this process is $\mathbb{G}$-locally bounded. Since $Z_-^{-1}I_{]\!]0,\tau]\!]}$ is $\mathbb{G}$-locally bounded, then there exists a sequence of $\mathbb{G}$-stopping times $(\tau_n^{\mathbb{G}})_{n\geq1}$ increasing to infinity such that

$$\left(\frac{1}{Z_-}I_{]\!]0,\tau]\!]}\right)^{\tau_n^{\mathbb{G}}} \leq n+1.$$

Consider a sequence of $\mathbb{F}$-stopping times $(\sigma_n)_{n\geq1}$ that increases to infinity such that $\langle m,m\rangle_{\sigma_n} \leq n+1$. Then, for any nonnegative $\mathbb{F}$-predictable process H which is bounded by $C>0$, we calculate that

$$\begin{aligned}(H\centerdot V^{\mathbb{G}})_{\sigma_n\wedge\tau_n^{\mathbb{G}}} &= \sum_{0\leq u\leq\sigma_n\wedge\tau_n^{\mathbb{G}}} H_u\ {}^{p,\mathbb{F}}\left(I_{\{\widetilde{Z}=0\}}\right)_u I_{\{u\leq\tau\}}I_{\{Z_{u-}\geq\frac{1}{n+1}\}}\\ &\leq \sum_{0\leq u\leq\sigma_n} H_u\ {}^{p,\mathbb{F}}\left(I_{\{\Delta m\leq-\frac{1}{n+1}\}}\right)_u\\ &\leq (n+1)^2 H\centerdot\langle m,m\rangle_{\sigma_n} \leq C(n+1)^3.\end{aligned}$$

This ends the proof of the proposition. □

The important $\mathbb{G}$-local martingale will result from an optional integral which will play the role of deflator for a class of processes.

Proposition 5.6: *Consider the $\mathbb{F}$-martingale m in (5.2) and $\widehat{m}$ defined via (5.17). The following process K given by*

$$K := \frac{Z_-^2}{Z_-^2 + \Delta\langle m\rangle^{\mathbb{F}}} \frac{1}{\widetilde{Z}} I_{]\!]0,\tau]\!]}, \tag{5.35}$$

belongs to the space ${}^oL^1_{loc}(\widehat{m}, \mathbb{G})$ defined in 2.12. Furthermore, the $\mathbb{G}$-local martingale

$$L := -K \odot \widehat{m}, \tag{5.36}$$

satisfies the following

(a) $\mathcal{E}(L) > 0$ *(or equivalently* $1 + \Delta L > 0$*).*

(b) *For any $M \in \mathcal{M}_{0,loc}(\mathbb{F})$ and $\widehat{M}$ defined via (5.17), we have*

$$[L, \widehat{M}] \in \mathcal{A}_{loc}(\mathbb{G}) \quad \left(i.e.\ \langle L, \widehat{M}\rangle^{\mathbb{G}} \quad exists\right). \tag{5.37}$$

Proof. The proof contains three steps. In the first step, we prove that the optional integral L in (5.36) is well-defined. For the second step, we prove (a). The last step is focusing on (b).

Step 1: For the sake of simplicity in notations, throughout this proof, we will use $\kappa := Z_-^2 + \Delta\langle m\rangle^{\mathbb{F}}$. We start by calculating on $]\!]0,\tau]\!]$, making use of Lemma 5.5.

$$\begin{aligned} &K\Delta\widehat{m} - {}^{p,\mathbb{G}}(K\Delta\widehat{m}) = \frac{I_{]\!]0,\tau]\!]} Z_-^2 \Delta\widehat{m}}{\kappa\widetilde{Z}} - {}^{p,\mathbb{G}}\left(\frac{I_{]\!]0,\tau]\!]}\, Z_-^2}{\kappa\widetilde{Z}}\,\Delta\widehat{m}\right) \\ &= \frac{(Z_-^2\Delta m - Z_-\Delta\langle m\rangle^{\mathbb{F}})}{\kappa\,\widetilde{Z}} + \frac{{}^{p,\mathbb{F}}(I_{\{\widetilde{Z}>0\}}\,\Delta\langle m\rangle^{\mathbb{F}})}{\kappa} - \frac{{}^{p,\mathbb{F}}(\Delta m I_{\{\widetilde{Z}>0\}})\, Z_-}{\kappa} \\ &= \frac{\Delta m}{\widetilde{Z}} I_{]\!]0,\tau]\!]} - {}^{p,\mathbb{F}}\left(I_{\{\widetilde{Z}=0\}}\right) I_{]\!]0,\tau]\!]} =: \Delta V - \Delta V^{\mathbb{G}}. \end{aligned} \tag{5.38}$$

Here, $V^{\mathbb{G}}$, defined in (5.34), is nondecreasing, càdlàg and $\mathbb{G}$-locally bounded (see Proposition 5.9). Hence, we immediately deduce that $\sum(\Delta V^{\mathbb{G}})^2 = \Delta V^{\mathbb{G}} \centerdot V^{\mathbb{G}}$ is locally bounded, and in the rest of this part we focus on proving $\sqrt{\sum(\Delta V)^2} \in \mathcal{A}^+_{loc}(\mathbb{G})$. To this end, we consider $\delta \in (0,1)$, and define $C := \{\Delta m < -\delta Z_-\}$ and C^c its complement in $\Omega \otimes [0,+\infty[$. Then we obtain

$$\begin{aligned}\sqrt{\sum(\Delta V)^2} &\leq \left(\sum \frac{(\Delta m)^2}{\widetilde{Z}^2} I_C I_{]\!]0,\tau]\!]}\right)^{1/2} + \left(\sum \frac{(\Delta m)^2}{\widetilde{Z}^2} I_{C^c} I_{]\!]0,\tau]\!]}\right)^{1/2}\\ &\leq \sum \frac{|\Delta m|}{\widetilde{Z}} I_C I_{]\!]0,\tau]\!]} + \frac{1}{1-\delta}\left(I_{]\!]0,\tau]\!]}\frac{1}{Z_-^2} \centerdot [m]\right)^{1/2} =: V_1 + V_2.\end{aligned}$$

The last inequality above is due to $\sqrt{\sum(\Delta X)^2} \leq \sum |\Delta X|$ and $\widetilde{Z} \geq Z_-(1-\delta)$ on C^c. Using the fact that $(Z_-)^{-1} I_{]\!]0,\tau]\!]}$ is $\mathbb{G}$-locally bounded and that m is an $\mathbb{F}$-locally bounded martingale, it follows that V_2 is $\mathbb{G}$-locally bounded. Hence, we focus on proving the $\mathbb{G}$-local integrability of V_1.

Consider a sequence of $\mathbb{G}$-stopping times $(\vartheta_n)_{n\geq 1}$ that increases to $+\infty$ and

$$\left((Z_-)^{-1} I_{]\!]0,\tau]\!]}\right)^{\vartheta_n} \leq n.$$

Also consider an $\mathbb{F}$-localizing sequence of stopping times, $(\tau_n)_{n\geq 1}$, for the process $V_3 := \sum \frac{(\Delta m)^2}{1+|\Delta m|}$. Then, it is easy to prove

$$U_n := \sum |\Delta m| I_{\{\Delta m < -\delta/n\}} \leq \frac{n+\delta}{\delta} V_3,$$

and conclude that $(U_n)^{\tau_n} \in \mathcal{A}^+(\mathbb{F})$. Therefore, due to

$$\begin{aligned}C \cap]\!]0,\tau]\!] \cap [\![0,\vartheta_n]\!] &= \{\Delta m < -\delta Z_-\} \cap]\!]0,\vartheta_n]\!] \cap]\!]0,\tau]\!]\\ &\subset]\!]0,\tau]\!] \cap]\!]0,\vartheta_n]\!] \cap \{\Delta m < -\frac{\delta}{n}\},\end{aligned}$$

we derive

$$(V_1)^{\vartheta_n \wedge \tau_n} \leq \left(\widetilde{Z}\right)^{-1} I_{]\!]0,\tau]\!]} \centerdot (U_n)^{\tau_n}.$$

Since $(U_n)^{\tau_n}$ is $\mathbb{F}$-adapted, nondecreasing and integrable, then due to Lemma 5.6, we deduce that the process $V_1^{\vartheta_n\wedge\tau_n}$ is nondecreasing, $\mathbb{G}$-adapted and integrable. Since $\vartheta_n \wedge \tau_n$ increases to $+\infty$, we conclude that the process V_1 is $\mathbb{G}$-locally integrable. This completes the proof of $K \in {}^{o}L^1_{loc}(\widehat{m}, \mathbb{G})$, and the process L (given via (5.36)) is a $\mathbb{G}$-local martingale.

Step 2: We now prove assertions (a). Due to (5.38), we have, on $]\!]0,\tau]\!]$,

$$-\Delta L = K\Delta\widehat{m} - {}^{p,\mathbb{G}}(K\Delta\widehat{m}) = 1 - Z_-\left(\widetilde{Z}\right)^{-1} - {}^{p,\mathbb{F}}\left(I_{\{\widetilde{Z}=0\}}\right).$$

Thus, we deduce that $1 + \Delta L > 0$, and assertion (a) is proved.

Step 3: In the rest of this proof, we will prove (5.37). To this end, let $M \in \mathcal{M}_{0,loc}(\mathbb{F})$. The formula (5.37) is equivalent to

$$K \centerdot [\widehat{m}, \widehat{M}] \in \mathcal{A}_{loc}(\mathbb{G}) \text{ (or equivalently } V_4 := \frac{1}{\widetilde{Z}} I_{]\!]0,\tau]\!]} \centerdot [\widehat{m}, \widehat{M}] \in \mathcal{A}_{loc}(\mathbb{G})),$$

for any $M \in \mathcal{M}_{0,loc}(\mathbb{F})$. Then, it is easy to check that

$$\begin{aligned} V_4 &= \frac{Z_-}{\widetilde{Z}} I_{]\!]0,\tau]\!]} \centerdot [\widehat{m}, \widehat{M}] = \frac{1}{\widetilde{Z}} I_{]\!]0,\tau]\!]} \centerdot [m, \widehat{M}] - \frac{1}{Z_-\,\widetilde{Z}} I_{]\!]0,\tau]\!]} \centerdot [\langle m\rangle^{\mathbb{F}}, \widehat{M}] \\ &= \frac{1}{\widetilde{Z}} I_{]\!]0,\tau]\!]} \centerdot [m, M] - \frac{1}{Z_-\,\widetilde{Z}} I_{]\!]0,\tau]\!]} \centerdot [m, \langle M, m\rangle^{\mathbb{F}}] \\ &\quad - \frac{1}{Z_-\widetilde{Z}} I_{]\!]0,\tau]\!]} \centerdot [\langle m\rangle^{\mathbb{F}}, M] + \frac{1}{Z_-^2\,\widetilde{Z}} I_{]\!]0,\tau]\!]} \centerdot [\langle m\rangle^{\mathbb{F}}, \langle M, m\rangle^{\mathbb{F}}]. \end{aligned}$$

Since m is an $\mathbb{F}$-locally bounded local martingale, all the processes

$$[m, M],\ [m, \langle M, m\rangle^{\mathbb{F}}],\ [\langle m\rangle^{\mathbb{F}}, M],\ \text{and } [\langle m\rangle^{\mathbb{F}}, \langle M, m\rangle^{\mathbb{F}}]$$

belong to $\mathcal{A}_{loc}(\mathbb{F})$. Thus, by combining this fact with Lemma 5.6 and the $\mathbb{G}$-local boundedness of $Z_-^{-p} I_{]\!]0,\tau]\!]}$ for any $p > 0$, it follows that $V_4 \in \mathcal{A}_{loc}(\mathbb{G})$. This ends the proof of the proposition. □

5.4 Explicit Deflators

This section describes some classes of $\mathbb{F}$-quasi-left-continuous local martingales for which the NUPBR is preserved after stopping with τ. For these stopped processes, we describe explicitly their local martingale densities in Theorems 5.8–5.9 with an increasing degree of generality. We recall that $m^{(qc)}$ was defined in (5.13) and L was defined in Proposition 5.6.

Theorem 5.8: *Suppose that S is a quasi-left-continuous $\mathbb{F}$-local martingale. If S and τ satisfy*

$$\{\Delta S \neq 0\} \cap \{Z_- > 0\} \cap \{\widetilde{Z} = 0\} = \emptyset, \tag{5.39}$$

then the following equivalent assertions hold

(a) $\mathcal{E}\left(L\right) S^{\tau}$ *is a $\mathbb{G}$-local martingale.*

(b) $\mathcal{E}\left(I_{\{\widetilde{Z}=0<Z_-\}} \odot m^{(qc)}\right) S$ *is an $\mathbb{F}$-local martingale.*

Proof. We start by giving some useful observations. Since S is $\mathbb{F}$-quasi-left-continuous, on the one hand we deduce that (Γ_m is defined in (5.13))

$$\langle S, m\rangle^{\mathbb{F}} = \langle S, m^{(qc)}\rangle^{\mathbb{F}} = \langle S, I_{\Gamma_m^c} \centerdot m\rangle^{\mathbb{F}}. \tag{5.40}$$

On the other hand, we note that assertion (a) is equivalent to $\mathcal{E}(L^{(qc)})S^{\tau}$ is a $\mathbb{G}$-local martingale, where $L^{(qc)}$ is the quasi-left-continuous local martingale part of L given by $L^{(qc)} := I_{\Gamma_m^c} \centerdot L = -K \odot \widehat{m}^{(qc)}$. Here K is given in Proposition 5.6 and

$$\widehat{m}^{(qc)} := I_{]\!]0,\tau]\!]} \centerdot m^{(qc)} - (Z_-)^{-1} I_{]\!]0,\tau]\!]} \centerdot \langle m^{(qc)}\rangle^{\mathbb{F}}.$$

It is easy to check that (5.39) is equivalent to

$$I_{\{Z_->0\ \&\ \widetilde{Z}=0\}} \cdot [S,m] = 0. \tag{5.41}$$

We now compute $-\langle L^{(qc)}, \widehat{S}\rangle^{\mathbb{G}}$, where $\widehat{S}$ is defined via (5.17). Due to the quasi-left-continuity of S and that of $m^{(qc)}$, the two processes $\langle S,m\rangle^{\mathbb{F}}$ and $\langle m^{(qc)}\rangle^{\mathbb{F}}$ are continuous and $[S, m^{(qc)}] = [S,m]$. Hence, we obtain

$$\begin{aligned} K \cdot [\widehat{S}, \widehat{m}^{(qc)}] &= K \cdot [S, \widehat{m}^{(qc)}] - K\Delta \widehat{m}^{(qc)} (Z_-)^{-1} \cdot \langle S,m\rangle^{\mathbb{F}} \\ &= (\widetilde{Z})^{-1} I_{]\!]0,\tau]\!]} \cdot [S, m^{(qc)}] = (\widetilde{Z})^{-1} I_{]\!]0,\tau]\!]} \cdot [S,m]. \end{aligned}$$

It follows that

$$\begin{aligned} -\langle L^{(qc)}, \widehat{S}\rangle^{\mathbb{G}} &= \Big(K \cdot [\widehat{S}, \widehat{m}^{(qc)}]\Big)^{p,\mathbb{G}} = \Big((\widetilde{Z})^{-1} I_{]\!]0,\tau]\!]} \cdot [S,m]\Big)^{p,\mathbb{G}} \\ &= (Z_-)^{-1} I_{]\!]0,\tau]\!]} \cdot \Big(I_{\{\widetilde{Z}>0\}} \cdot [S,m]\Big)^{p,\mathbb{F}} \\ &= (Z_-)^{-1} I_{]\!]0,\tau]\!]} \cdot \langle S,m\rangle^{\mathbb{F}} - (Z_-)^{-1} I_{]\!]0,\tau]\!]} \cdot \Big(I_{\{\widetilde{Z}=0<Z_-\}} \cdot [S,m]\Big)^{p,\mathbb{F}} \\ &= (Z_-)^{-1} I_{]\!]0,\tau]\!]} \cdot \langle S,m\rangle^{\mathbb{F}} + (Z_-)^{-1} I_{]\!]0,\tau]\!]} \cdot \langle S, -I_{\{\widetilde{Z}=0<Z_-\}} \odot m^{(qc)}\rangle^{\mathbb{F}}. \end{aligned} \tag{5.42}$$

The first and the last equality follow from Proposition 2.3 applied to $L^{(qc)}$ and $-I_{\{\widetilde{Z}=0<Z_-\}} \odot m^{(qc)}$ respectively. The second and the third equalities are due to (5.40) and (5.22) respectively.

Now, we prove the theorem. Thanks to (5.42), it is obvious that assertion (a) is equivalent to $\langle S, -I_{\{\widetilde{Z}=0<Z_-\}} \odot m^{(qc)}\rangle^{\mathbb{F}} \equiv 0$ which in turn is equivalent to assertion (b). This ends the proof of the equivalence between (a) and (b).

It is also clear that the condition (5.39) or equivalently (5.41) implies that assertion (b), due to $\langle I_{\{\widetilde{Z}=0<Z_-\}} \odot m^{(qc)}, S\rangle^{\mathbb{F}} = \Big(I_{\{\widetilde{Z}=0<Z_-\}} \cdot [m,S]\Big)^{p,\mathbb{F}} \equiv 0.$ ☐

Remark 5.9: Suppose that S is a quasi-left-continuous $\mathbb{F}$-local martingale and let

$\widetilde{R}_0$ be defined in Lemma 5.4–(b). Then, $\mathcal{E}(L)\,\overline{S}^{\tau}$ is a $\mathbb{G}$-local martingale, where

$$\overline{S} := S^{\widetilde{R}_0-} + \left(\Delta S_{\widetilde{R}_0} I_{[\![\widetilde{R}_0,+\infty[\![}\right)^{p,\mathbb{F}}. \tag{5.43}$$

Indeed, writing

$$\overline{S} := S^{\widetilde{R}_0} - \Delta S_{\widetilde{R}_0} I_{[\![\widetilde{R}_0,+\infty[\![} + \left(\Delta S_{\widetilde{R}_0} I_{[\![\widetilde{R}_0,+\infty[\![}\right)^{p,\mathbb{F}}$$

it is easy to see that the condition (5.39) is satisfied for $\overline{S}$.

Corollary 5.8.1: If S is quasi-left continuous and satisfies NUPBR($\mathbb{F}$) and $\{\Delta S \neq 0\} \cap \{Z_- > 0\} \cap \{\widetilde{Z} = 0\} = \emptyset$, then S^{τ} satisfies NUPBR($\mathbb{G}$).

Proof. This follows from Proposition 5.2, Theorem 5.8 and the fact that, if Q is equivalent to P, then we have

$$\{Z_- > 0\} \cap \{\widetilde{Z} = 0\} = \{Z^Q_- > 0\} \cap \{\widetilde{Z}^Q = 0\}.$$

Here $Z^Q_t = Q(\tau > t|\mathcal{F}_t)$ and $\widetilde{Z}^Q_t = Q(\tau \geq t|\mathcal{F}_t)$. This last claim is a direct application of the optional and predictable selection measurable theorems, see Theorems 84 and 85 (or apply Theorem 86 directly) in [54]. □

In order to generalize the previous result, we need to introduce more notations and recall other notations and some results. For the random measure μ, we associate its predictable compensator random measure ν. A direct application of Jacod representation theorem, to the martingale m, leads to the existence of a local martingale $m^{\perp}$ as well as a $\widetilde{\mathcal{P}}(\mathbb{F})$-measurable functional f_m, a process $\beta_m \in L(S^c, \mathbb{F})$ and an $\widetilde{\mathcal{O}}(\mathbb{F})$-measurable functional g_m such that $f_m \in \mathcal{G}^1_{loc}(\mu, \mathbb{F})$, $g_m \in \mathcal{H}^1_{loc}(\mu, \mathbb{F})$ and $\beta_m \in L(S^c)$ such that

$$m = \beta_m \centerdot S^c + f_m \star (\mu - \nu) + g_m \star \mu + m^{\perp}. \tag{5.44}$$

Due to the quasi-left-continuity of S, $\mathcal{G}^1_{loc}(\mu,\mathbb{F})$ (respectively $\mathcal{H}^1_{loc}(\mu,\mathbb{F})$) is the set of all $\widetilde{\mathcal{P}}(\mathbb{F})$-measurable functions (respectively all $\widetilde{\mathcal{O}}(\mathbb{F})$-measurable functions) W such that

$$\sqrt{W^2 \star \mu} \in \mathcal{A}^+_{loc}(\mathbb{H}).$$

we introduce $\mu^{\mathbb{G}} := I_{[\![0,\tau]\!]} \star \mu$ and its $\mathbb{G}$-compensator measure

$$\nu^{\mathbb{G}}(dt,dx) := (1 + f_m(x)/Z_{t-}) I_{[\![0,\tau]\!]}(t)\nu(dt,dx). \tag{5.45}$$

Below, we state our general result that extends the previous theorem.

Theorem 5.9: *Suppose that S is an $\mathbb{F}$-quasi-left-continuous local martingale. Consider $S^{(0)}$, ψ, and L defined in (5.6) and (5.36) respectively. If $\left(S, S^{(0)}\right)$ is an $\mathbb{F}$-local martingale, then $\mathcal{E}\left(L + L^{(1)}\right) S^\tau$ is a $\mathbb{G}$-local martingale, where*

$$L^{(1)} := g_1 \star (\mu^{\mathbb{G}} - \nu^{\mathbb{G}}), \quad \textit{and} \quad g_1 := \frac{1-\psi}{1 + f_m/Z_-} I_{\{\psi>0\}}, \tag{5.46}$$

and $\mathcal{E}\left(L + L^{(1)}\right) > 0$, i.e. $1 + \Delta L + \Delta L^{(1)} > 0$.

Proof. We start by recalling from (5.8) that $\{\psi = 0\} = \{Z_- + f_m = 0\}$ $M^P_\mu - a.e.$. Thus the functional g_1 is a well defined non-negative $\widetilde{\mathcal{P}}(\mathbb{F})-$measurable functional. The proof of the theorem will be completed in two steps. In the first step we prove that the process $L^{(1)}$ is a well defined local martingale, while in the second step we prove the main statement of the theorem.

1) Herein, we prove that the integral $g_1 \star \left(\mu^{\mathbb{G}} - \nu^{\mathbb{G}}\right)$ is well-defined. To this end, it is enough to prove that $g_1 \star \mu^{\mathbb{G}} \in \mathcal{A}^+(\mathbb{G})$. Therefore, remark that

$$(1-\psi) I_{\{0<Z_-\}} = M^P_\mu \left(I_{\{\widetilde{Z}=0<Z_-\}} | \widetilde{\mathcal{P}}(\mathbb{F}) \right) = M^P_\mu \left(I_{[\![\widetilde{R}_0]\!]} | \widetilde{\mathcal{P}}(\mathbb{F}) \right) I_{\{0<Z_-\}},$$

and calculate

$$\begin{aligned} E\left(g_1\star\mu^{\mathbb{G}}(\infty)\right) &= E\left(g_1\widetilde{Z}\star\mu(\infty)\right)\\ &\leq E\left(I_{[\![\widetilde{R}_0]\!]}\star\mu(\infty)\right) = P\left(\Delta S_{\widetilde{R}_0}\neq 0\ \&\ \widetilde{R}_0<+\infty\right)\leq 1. \end{aligned}$$

Thus, the process $L^{(1)}$ is a well defined $\mathbb{G}$-martingale.

2) In this part, we prove that $\mathcal{E}\left(L+L^{(1)}\right)S^{\tau}$ is a $\mathbb{G}$-local martingale. To this end, it is enough to prove that $\langle S^{\tau},L+L^{(1)}\rangle^{\mathbb{G}}$ exists and

$$S^{\tau}+\left\langle S^{\tau},L+g_1\star\left(\mu^{\mathbb{G}}-\nu^{\mathbb{G}}\right)\right\rangle^{\mathbb{G}}\quad \text{is a } \mathbb{G}\text{-local martingale.} \tag{5.47}$$

Recall that

$$L=-\frac{Z_-^2}{Z_-^2+\Delta\langle m\rangle^{\mathbb{F}}}\,\frac{1}{\widetilde{Z}}\,I_{]\!]0,\tau]\!]}\odot\widehat{m},$$

and hence $\langle S^{\tau},L\rangle^{\mathbb{G}}$ exists due to Proposition 5.6–(b). By stopping, there is no loss of generality in assuming that S is a true martingale. Then, using similar calculation as in the first part 1), we can easily prove that

$$E\left[|x|g_1\star\mu^{\mathbb{G}}(\infty)\right]\leq E\left(|\Delta S_{\widetilde{R}_0}|I_{\{\widetilde{R}_0<+\infty\}}\right)<+\infty.$$

This proves that $\left\langle S^{\tau},L+L^{(1)}\right\rangle^{\mathbb{G}}$ exists. Now, we calculate and simplify the expression in (5.47) as follows.

$$\begin{aligned} S^{\tau}+\left\langle S^{\tau},L+g_1\star\left(\mu^{\mathbb{G}}-\nu^{\mathbb{G}}\right)\right\rangle^{\mathbb{G}} &= \widehat{S}+\frac{1}{Z_-}I_{]\!]0,\tau]\!]}\centerdot\langle S,m\rangle^{\mathbb{F}}+\langle S^{\tau},L\rangle^{\mathbb{G}}+xg_1\star\nu^{\mathbb{G}}\\ &= \widehat{S}+\frac{1}{Z_-}I_{]\!]0,\tau]\!]}\centerdot\langle S,m\rangle-\frac{1}{Z_-}I_{]\!]0,\tau]\!]}\centerdot\left(I_{\{\widetilde{Z}>0\}}\centerdot[S,m]\right)^{p,\mathbb{F}}+xg_1\star\nu^{\mathbb{G}}\\ &= \widehat{S}+\frac{1}{Z_-}I_{]\!]0,\tau]\!]}\centerdot\left(I_{\{\widetilde{Z}=0\}}\centerdot[S,m]\right)^{p,\mathbb{F}}+xM^P_{\mu}\left(I_{\{\widetilde{Z}=0<Z_-\}}|\widetilde{\mathcal{P}}(\mathbb{F})\right)I_{\{Z_-+f_m>0\}}I_{]\!]0,\tau]\!]}\star\nu\\ &= \widehat{S}-xM^P_{\mu}\left(I_{\{\widetilde{Z}=0<Z_-\}}|\widetilde{\mathcal{P}}(\mathbb{F})\right)I_{\{\psi=0\}}I_{]\!]0,\tau]\!]}\star\nu=\widehat{S}\in\mathcal{M}_{loc}(\mathbb{G}). \end{aligned}$$

The second equality is due to (5.42), while the last equality follows directly form the fact that $S^{(0)}$ is an $\mathbb{F}$-local martingale (which is equivalent to $xI_{\{\psi=0<Z_-\}}\star\nu \equiv 0$) and $M^P_\mu\left(I_{\{\widetilde{Z}=0<Z_-\}}|\widetilde{\mathcal{P}}(\mathbb{F})\right) = I_{\{0<Z_-\}}(1-\psi)$. This ends the proof of the theorem. ☐

Remark 5.10: 1) Both Theorems 5.8-5.9 provide methods that build-up explicitly σ-martingale density for X^τ, whenever X is an $\mathbb{F}$-quasi-left-continuous process that is a local martingale under a locally equivalent probability measure and is fulfilling the assumptions of the theorems respectively.

2) The extension of Theorem 5.8 to the general case where S is an $\mathbb{F}$-local martingale (not necessarily quasi-left-continuous) boils down to find a predictable process Φ such that Φ is locally bounded, $\Phi \geq -1$, $\{\Phi > 1\}$ is thin and $Y^{(1)} := \mathcal{E}(\Phi\centerdot L)$ will be the martingale density for S^τ. Finding the process Φ will be easy to guess when we will address the case of thin semimartingale. However the proof of $Y^{(1)}$ is a local martingale density for S^τ is very technical. The extension of Theorem 5.9 to the case of arbitrary $\mathbb{F}$-local martingale S requires additional careful modification of the functional g_1 so that $1 + \Phi(\Delta L) + \Delta L^{(1)}$ remains positive. While both extensions remain very feasible, we opted to not overload the paper with technicalities.

5.5 Proofs of Three Main Theorems

This section is devoted to the proofs of Theorems 5.2, 5.4 and 5.5. They are quite long, since some integrability results have to be proved. For the reader's convenience, we recall the canonical decomposition of S by

$$S = S_0 + S^c + h\star(\mu-\nu) + b\centerdot A + (x-h)\star\mu,$$

where h defined as $h(x) := xI_{\{|x|\leq 1\}}$ is the truncation function. The canonical decomposition of S^τ under $\mathbb{G}$ is given by

$$S^\tau = S_0 + \widehat{S^c} + h\star(\mu^{\mathbb{G}} - \nu^{\mathbb{G}}) + \frac{c\beta_m}{Z_-} I_{]\!]0,\tau]\!]} \centerdot A + h\frac{f_m}{Z_-} I_{]\!]0,\tau]\!]} \star \nu + b \centerdot A^\tau + (x-h)\star\mu^{\mathbb{G}}$$

where $\mu^{\mathbb{G}}$ and $\nu^{\mathbb{G}}$ and (β_m, f_m) are given in (5.45) and (5.44) respectively and $\widehat{S^c}$ is defined via (5.17).

5.5.1 Proof of Theorem 5.2

The proof of Theorem 5.2 will be completed in four steps. The first step provides an equivalent formulation to assertion (a) using the filtration $\mathbb{F}$ instead. In the second step, we prove (a)$\Rightarrow$(b), while the reverse implication is proved in the third step. The proof of (b)$\Longleftrightarrow$ (c) is given in the last step.

Step 1: Formulation of assertion (a): Thanks to Proposition 5.1, S^τ satisfies NUPBR($\mathbb{G}$) if and only if there exist a $\mathbb{G}$-local martingale $N^{\mathbb{G}}$ with $1+\Delta N^{\mathbb{G}} > 0$ and a $\mathbb{G}$-predictable process $\phi^{\mathbb{G}}$ such that $0 < \phi^{\mathbb{G}} \leq 1$ and $\mathcal{E}\left(N^{\mathbb{G}}\right)\left(\phi^{\mathbb{G}} \centerdot S^\tau\right)$ is a $\mathbb{G}$-local martingale. We can reduce our attention to processes $N^{\mathbb{G}}$ having the form of (see Theorem 5.1)

$$N^{\mathbb{G}} = \beta^{\mathbb{G}} \centerdot \widehat{S^c} + (f^{\mathbb{G}} - 1)\star(\mu^{\mathbb{G}} - \nu^{\mathbb{G}}),$$

where $\beta^{\mathbb{G}} \in L(\widehat{S^c}, \mathbb{G})$ and $f^{\mathbb{G}}$ is positive such that $(f^{\mathbb{G}} - 1) \in \mathcal{G}^1_{loc}(\mu^{\mathbb{G}}, \mathbb{G})$.
Then, one notes that $\mathcal{E}\left(N^{\mathbb{G}}\right)\left(\phi^{\mathbb{G}} \centerdot S^\tau\right)$ is a $\mathbb{G}$-local martingale if and only if $\phi^{\mathbb{G}} \centerdot S^\tau + [\phi^{\mathbb{G}} \centerdot S^\tau, N^{\mathbb{G}}]$ is a $\mathbb{G}$-local martingale, which in turn, is equivalent to

$$\phi^{\mathbb{G}} |x f^{\mathbb{G}}(x) - h(x)| \left(1 + \frac{f_m(x)}{Z_-}\right) I_{[\![0,\tau]\!]} \star \nu \in \mathcal{A}^+_{loc}(\mathbb{G}), \tag{5.48}$$

and $P \otimes A - a.e.$ on $[\![0,\tau]\!]$, ($F_{\omega,t}(dx)$ is the transit kernel)

$$b + c(\frac{\beta_m}{Z_-} + \beta^{\mathbb{G}}) + \int \left[(xf^{\mathbb{G}}(x) - h(x)) \left(1 + \frac{f_m(x)}{Z_-}\right) - h(x)\frac{f_m(x)}{Z_-} \right] F(dx) = 0. \tag{5.49}$$

From Lemma 5.7, there exist $\phi^{\mathbb{F}}$ and $\beta^{\mathbb{F}}$ two $\mathbb{F}$-predictable processes and a positive $\widetilde{\mathcal{P}}(\mathbb{F})$-measurable functional, $f^{\mathbb{F}}$, such that $0 < \phi^{\mathbb{F}} \leq 1$,

$$\beta^{\mathbb{F}} = \beta^{\mathbb{G}},\ \phi^{\mathbb{F}} = \phi^{\mathbb{G}},\ f^{\mathbb{F}} = f^{\mathbb{G}} \text{ on } [\![0,\tau]\!]. \tag{5.50}$$

In virtue of these and taking account integrability conditions given in Proposition 5.5, we deduce that (5.48)–(5.49) imply that, on $\{Z_- \geq \delta\}$, we have

$$W^{\mathbb{F}} := \int |(xf^{\mathbb{F}}(x) - h(x))| \left(1 + \frac{f_m(x)}{Z_-}\right) F(dx) < +\infty \quad P \otimes A - a.e, \tag{5.51}$$

and $P \otimes A$-a.e. on $\{Z_- \geq \delta\}$, we have

$$b + c\left(\beta^{\mathbb{F}} + \frac{\beta_m}{Z_-}\right) - \int h(x) I_{\{\psi=0\}} F(dx) + \int \left[xf^{\mathbb{F}}(x)(1 + \frac{f_m(x)}{Z_-}) - h(x) \right] I_{\{\psi>0\}} F(dx) = 0. \tag{5.52}$$

Due to (5.51), this latter equality follows immediately by taking the $\mathbb{F}$-predictable projection of (5.49) after inserting (5.50).

Step 2: Proof of (a) $\Rightarrow$ (b). Suppose that S^τ satisfies NUPBR($\mathbb{G}$), hence (5.51)–(5.52) hold. To prove that $I_{\{Z \geq \delta\}} \centerdot (S - S^{(0)})$ satisfies NUPBR($\mathbb{F}$), we consider

$$\beta := \left(\frac{\beta_m}{Z_-} + \beta^{\mathbb{F}}\right) I_{\{Z_- \geq \delta\}} \text{ and } f = f^{\mathbb{F}} \left(1 + \frac{f_m}{Z_-}\right) I_{\{Z_- \geq \delta\ \&\ \psi>0\}} + I_{\{0 \leq Z_- < \delta \text{ or } \psi=0\}}. \tag{5.53}$$

If $\beta \in L(S^c, \mathbb{F})$ and $(f-1) \in \mathcal{G}^1_{loc}(\mu, \mathbb{F})$, we conclude that

$$N := \beta \centerdot S^c + (f-1) \star (\mu - \nu). \tag{5.54}$$

is a well defined $\mathbb{F}$-local martingale. Therefore, by choosing $\phi = \left(1 + W^{\mathbb{F}} I_{\{Z_- \geq \delta\}}\right)^{-1}$, using (5.52), and applying Itô's formula for $\mathcal{E}(N)\left(\phi I_{\{Z_- \geq \delta\}} \centerdot (S - S^{(0)})\right)$, we deduce that this process is a local martingale. Hence, $I_{\{Z_- \geq \delta\}} \centerdot (S - S^{(0)})$ satisfies NUPBR($\mathbb{F}$), and the proof of (a)⇒(b) is completed.

Now we focus on proving $\beta \in L(S^c)$ and $(f-1) \in \mathcal{G}^1_{loc}(\mu, \mathbb{F})$ (or equivalently $\sqrt{(f-1)^2 \star \mu} \in \mathcal{A}^+_{loc}(\mathbb{F})$). Since $\beta_m \in L(S^c)$, then it is obvious that $\frac{\beta_m}{Z_-} I_{\{Z_- \geq \delta\}} \in L(S^c)$ on the one hand. On the other hand, $(\beta^{\mathbb{F}})^T c \beta^{\mathbb{F}} I_{\{0 \leq Z_- < \delta\}} \centerdot A \in \mathcal{A}^+_{loc}(\mathbb{F})$ due to $(\beta^{\mathbb{F}})^T c \beta^{\mathbb{F}} \centerdot A^\tau = (\beta^{\mathbb{G}})^T c \beta^{\mathbb{G}} \centerdot A^\tau \in \mathcal{A}^+_{loc}(\mathbb{G})$ and Proposition 5.5–(c). This completes the proof of $\beta \in L(S^c)$.

Now, we focus on proving $(f-1) \in \mathcal{G}^1_{loc}(\mu, \mathbb{F})$. Since S is quasi-left-continuous, this is equivalent to prove $\sqrt{(f-1)^2 \star \mu} \in \mathcal{A}^+_{loc}(\mathbb{F})$. Thanks to Proposition 5.5 and $\sqrt{(f^{\mathbb{F}} - 1)^2 \star \mu^{\mathbb{G}}} = \sqrt{(f^{\mathbb{G}} - 1)^2 \star \mu^{\mathbb{G}}} \in \mathcal{A}^+_{loc}(\mathbb{G})$, we deduce that

$$(f^{\mathbb{F}} - 1)^2 I_{\{|f^{\mathbb{F}} - 1| \leq \alpha\}} \widetilde{Z} I_{\{Z_- \geq \delta\}} \star \mu \text{ and } |f^{\mathbb{F}} - 1| I_{\{|f^{\mathbb{F}} - 1| > \alpha\}} \widetilde{Z} I_{\{Z_- \geq \delta\}} \star \mu \in \mathcal{A}^+_{loc}(\mathbb{F}). \tag{5.55}$$

By stopping, there is no loss of generality in assuming that these two processes and $[m, m]$ are integrable. By putting $\Sigma_0 := \{Z_- \geq \delta \;\&\; \psi > 0\}$, then we get

$$f - 1 = \left(f^{\mathbb{F}} - 1\right)\left(1 + \frac{f_m}{Z_-}\right) I_{\Sigma_0} + \frac{f_m}{Z_-} I_{\Sigma_0} =: h_1 + h_2. \tag{5.56}$$

Therefore, we derive that

$$\begin{aligned} E\left[h_1^2 I_{\{|f^{\mathbb{F}} - 1| \leq \alpha\}} \star \mu_\infty\right] &\leq \delta^{-2} E\left[\left(f^{\mathbb{F}} - 1\right)^2 (Z_- + f_m)^2 I_{\{|f^{\mathbb{F}} - 1| \leq \alpha\}} I_{\{Z_- \geq \delta\}} \star \mu_\infty\right] \\ &\leq \delta^{-2} E\left[\left(f^{\mathbb{F}} - 1\right)^2 \widetilde{Z} I_{\{|f^{\mathbb{F}} - 1| \leq \alpha\}} I_{\{Z_- \geq \delta\}} \star \mu_\infty\right] < +\infty, \end{aligned}$$

and

$$E\left[|h_1| I_{\{|f^{\mathbb{F}}-1|>\alpha\}} \star \mu_\infty\right] \leq \delta^{-1} E\left[|f^{\mathbb{F}}-1|\ |Z_- + f_m| I_{\{|f^{\mathbb{F}}-1|>\alpha\}} I_{\{Z_-\geq\delta\}} \star \mu_\infty\right]$$
$$= \delta^{-1} E\left[|f^{\mathbb{F}}-1| \widetilde{Z} I_{\{|f^{\mathbb{F}}-1|>\alpha\}} I_{\{Z_-\geq\delta\}} \star \mu_\infty\right] < +\infty.$$

By combining the above two inequalities, we conclude that $\left(h_1^2 \star \mu\right)^{1/2} \in \mathcal{A}^+_{loc}(\mathbb{F})$. It is easy to see that $\left(h_2^2 \star \mu\right)^{1/2} \in \mathcal{A}^+_{loc}(\mathbb{F})$ which follows from

$$E\left[h_2^2 \star \mu_\infty\right] \leq \delta^{-2} E\left[f_m^2 \star \mu_\infty\right] \leq \delta^2 E\left[(\Delta m)^2 \star \mu_\infty\right] \leq \delta^{-2} E\left[m, m\right]_\infty < +\infty.$$

Step 3: Proof of (b) $\Rightarrow$ (a). Suppose that for any $\delta > 0$, the process $I_{\{Z_-\geq\delta\}} \centerdot \left(S - S^{(0)}\right)$ satisfies NUPBR($\mathbb{F}$). Then, there exist an $\mathbb{F}$-local martingale $N^{\mathbb{F}}$ and an $\mathbb{F}$-predictable process ϕ such that $0 < \phi \leq 1$ and $\mathcal{E}\left(N^{\mathbb{F}}\right)\left(\phi I_{\{Z_-\geq\delta\}} \centerdot \left(S - S^{(0)}\right)\right)$ is an $\mathbb{F}$-local martingale. Again, thanks to Theorem 5.1, we can restrict our attention to the case

$$N^{\mathbb{F}} := \beta^{\mathbb{F}} \centerdot S^c + (f^{\mathbb{F}} - 1) \star (\mu - \nu), \tag{5.57}$$

where $\beta^{\mathbb{F}} \in L(S^c)$ and $f^{\mathbb{F}}$ is positive such that $(f^{\mathbb{F}} - 1) \in \mathcal{G}^1_{loc}(\mu, \mathbb{F})$.
Thanks to Itô's formula, the fact that $\mathcal{E}\left(N^{\mathbb{F}}\right)\left(\phi I_{\{Z_-\geq\delta\}} \centerdot \left(S - S^{(0)}\right)\right)$ is an $\mathbb{F}$-local martingale implies that on $\{Z_- \geq \delta\}$

$$k^{\mathbb{F}} := \int |x f^{\mathbb{F}}(x) I_{\{\psi(x)>0\}} - h(x)| F(dx) < +\infty \quad P \otimes A - a.e. \tag{5.58}$$

and $P \otimes A$-a.e. on $\{Z_- \geq \delta\}$, we have

$$b - \int h(x) I_{\{\psi=0\}} F(dx) + c\beta^{\mathbb{F}} + \int \left[x f^{\mathbb{F}}(x) - h(x)\right] I_{\{\psi>0\}} F(dx) = 0. \tag{5.59}$$

Consider

$$\beta^{\mathbb{G}} := \left(\beta^{\mathbb{F}} - \frac{\beta_m}{Z_-}\right) I_{]\!]0,\tau]\!]} \quad \text{and} \quad f^{\mathbb{G}} := \frac{f^{\mathbb{F}}}{1 + f_m/Z_-} I_{\{\psi>0\}} I_{]\!]0,\tau]\!]} + I_{\{\psi=0\}\cup]\!]\tau,+\infty[\![}, \tag{5.60}$$

and assume that

$$\beta^{\mathbb{G}} \in L(\widehat{S^c}) \quad \text{and} \quad (f^{\mathbb{G}} - 1) \in \mathcal{G}^1_{loc}(\mu^{\mathbb{G}}). \tag{5.61}$$

Then, necessarily $N^{\mathbb{G}} := \beta^{\mathbb{G}} \centerdot \widehat{S^c} + (f^{\mathbb{G}} - 1) \star (\mu^{\mathbb{G}} - \nu^{\mathbb{G}})$ is a well defined $\mathbb{G}$-local martingale satisfying $\mathcal{E}(N^{\mathbb{G}}) > 0$. Furthermore, due to (5.59) and to $\{\psi = 0\} = \{Z_- + f_m = 0\}$ (see(5.8)), on $]\!]0,\tau]\!]$ we obtain

$$b + c\left(\beta^{\mathbb{G}} + \frac{\beta_m}{Z_-}\right) + \int \left(x f^{\mathbb{G}}\left(1 + \frac{f_m}{Z_-}\right) - h(x)\right) F(dx) = 0. \tag{5.62}$$

Then, by taking $\phi^{\mathbb{G}} := \left(1 + k^{\mathbb{F}} I_{\{Z_- \geq \delta\}}\right)^{-1}$, and applying Itô's formula for $(\phi^{\mathbb{G}} I_{\{Z_-\geq\delta\}} \centerdot S^\tau)\mathcal{E}(N^{\mathbb{G}})$, we conclude that this process is a $\mathbb{G}$-local martingale due to (5.62). Thus, $I_{\{Z_-\geq\delta\}} \centerdot S^\tau$ satisfies NUPBR($\mathbb{G}$) as long as (5.61) is fulfilled.
Since $Z_-^{-1} I_{]\!]0,\tau]\!]}$ is $\mathbb{G}$-locally bounded, then there exists a family of $\mathbb{G}$-stopping times $(\tau_\delta)_{\delta>0}$ such that $[\![0,\tau_\delta]\!] \subset \{Z_- \geq \delta\}$ (or equivalently $I_{\{Z_-\geq\delta\}} \centerdot S^{\tau\wedge\tau_\delta} = S^{\tau\wedge\tau_\delta}$) and τ_δ increases to infinity when δ goes to zero. Thus, using Proposition 5.2, we deduce that S^τ satisfies NUPBR($\mathbb{G}$). This achieves the proof of (b)$\Rightarrow$(a) under (5.61).

To prove that (5.61) holds true, we remark in a first step that $Z_-^{-1} I_{]\!]0,\tau]\!]}$ is $\mathbb{G}$-locally bounded and both β_m and $\beta^{\mathbb{F}}$ belong to $L(S^c)$. This, easily, implies that $\beta^{\mathbb{G}} \in L(\widehat{S^c})$. Now, we prove that $\sqrt{(f^{\mathbb{G}} - 1)^2 \star \mu^{\mathbb{G}}} \in \mathcal{A}^+_{loc}(\mathbb{G})$. Since $\sqrt{(f^{\mathbb{F}} - 1)^2 \star \mu} \in \mathcal{A}^+_{loc}(\mathbb{F})$, Proposition 5.5 allows us again to deduce that

$$(f^{\mathbb{F}} - 1)^2 I_{\{|f^{\mathbb{F}}-1|\leq\alpha\}} \star \mu \in \mathcal{A}^+_{loc}(\mathbb{F}) \text{ and } |f^{\mathbb{F}} - 1| I_{\{|f^{\mathbb{F}}-1|>\alpha\}} \star \mu \in \mathcal{A}^+_{loc}(\mathbb{F}). \tag{5.63}$$

Without loss of generality, we assume that these two processes and $[m, m]$ are integrable. Put

$$f^{\mathbb{G}} - 1 = I_{\{\psi>0\}} I_{]\!]0,\tau]\!]} \frac{Z_-(f^{\mathbb{F}} - 1)}{f_m + Z_-} - I_{\{\psi>0\}} I_{]\!]0,\tau]\!]} \frac{f_m}{f_m + Z_-} := f_1 + f_2. \tag{5.64}$$

Then, we calculate

$$E\left(f_1^2 I_{\{f_m+Z_->\delta/2\}\cap\{|f^{\mathbb{F}}-1|\leq\alpha\}} \star \mu^{\mathbb{G}}_\infty\right) \leq \left(\tfrac{2}{\delta}\right)^2 E\big[(f^{\mathbb{F}} - 1)^2 I_{\{|f^{\mathbb{F}}-1|\leq\alpha\}} \star \mu_\infty\big] < +\infty.$$

and

$$\begin{aligned} & E\sqrt{f_1^2 I_{\{f_m+Z_-\leq\delta/2\}\cap\{|f^{\mathbb{F}}-1|\leq\alpha\}\cap\{Z_-\geq\delta\}} \star \mu^{\mathbb{G}}_\infty} \\ & \leq \alpha E\big(I_{\{f_m+Z_-\leq\delta/2\}\cap\{Z_-\geq\delta\}} (Z_- + f_m)^{-1} \star \mu^{\mathbb{G}}(\infty)\big) \\ & \leq E\big(I_{\{|f_m|\geq\delta/2\}} \star \mu(\infty)\big) \leq \frac{4\alpha}{\delta^2} E[m,m]_\infty < +\infty. \end{aligned}$$

This proves that $\sqrt{f_1^2 I_{\{|f^{\mathbb{F}}-1|\leq\alpha\}} \star \mu^{\mathbb{G}}} \in \mathcal{A}^+_{loc}(\mathbb{G})$. Similarly, we calculate

$$\begin{aligned} E\sqrt{f_1^2 I_{\{|f^{\mathbb{F}}-1|>\alpha\}} \star \mu^{\mathbb{G}}_\infty} & \leq E\big(|f_1| I_{\{|f^{\mathbb{F}}-1|>\alpha\}} \star \mu^{\mathbb{G}}_\infty\big) \leq E\Big(\frac{|f^{\mathbb{F}} - 1|}{1 + f_m/Z_-} I_{\{|f^{\mathbb{F}}-1|>\alpha\}} \star \mu^{\mathbb{G}}_\infty\Big) \\ & \leq E\big(|f^{\mathbb{F}} - 1| I_{\{|f^{\mathbb{F}}-1|>\alpha\}} \star \mu_\infty\big) < +\infty. \end{aligned}$$

Thus, by combining all the remarks obtained above, we conclude that $\sqrt{f_1^2 \star \mu^{\mathbb{G}}}$ is $\mathbb{G}$-locally integrable. For the functional f_2, we proceed as follows. We calculate

$$E\big(f_2^2 I_{\{f_m+Z_->\delta/2\}} \star \mu^{\mathbb{G}}_\infty\big) \leq (2/\delta)^2 E\big(f_m^2 \star \mu_\infty\big) \leq (2/\delta)^2 E[m,m]_\infty < +\infty,$$

and

$$\begin{aligned} E\sqrt{f_2^2 I_{\{f_m+Z_-\leq\delta/2\}\cap\{Z_-\geq\delta\}} \star \mu^{\mathbb{G}}_\infty} & \leq E\big(|f_m| I_{\{|f_m|\geq\delta/2\}} \star \mu(\infty)\big) \\ & \leq (2/\delta) E\big(f_m^2 \star \mu(\infty)\big) \leq (2/\delta) E[m,m]_\infty < +\infty. \end{aligned}$$

This proves that $\sqrt{f_2^2 \star \mu^{\mathbb{G}}}$ is $\mathbb{G}$-locally integrable. Therefore, we conclude that

(5.61) is valid, and the proof of (b)⇒(a) is completed.

Step 3: Proof of (b) $\Longleftrightarrow$ (c). For any $\delta > 0$ and any $n \in \mathbb{N}$, we denote

$$\sigma_\infty := \inf\{t \geq 0 : \quad Z_t = 0\}, \qquad \tau_\delta := \sup\{t\ : Z_{t-} \geq \delta\}.$$

Then, due to $]\!]\sigma_\infty, +\infty[\![\subset \{Z_- = 0\} \subset \{Z_- < \delta\}$, we deduce

$$\sigma_{1/\delta} \leq \tau_\delta \leq \sigma_\infty \quad \text{ and } \quad Z_{\tau_\delta -} \geq \delta > 0 \qquad P-a.s. \ \text{ on } \ \{\tau_\delta < \infty\}\,.$$

Furthermore, setting $\Sigma := \bigcap_{n\geq 1}(\sigma_n < \sigma_\infty)$, we have

$$\text{on } \ \Sigma \cap \{\sigma_\infty < \infty\} \quad Z_{\sigma_\infty -} = 0, \ \text{ and } \quad \tau_\delta < \sigma_\infty \quad P-a.s.$$

We introduce the semimartingale $X := S - S^{(0)}$. For any $\delta > 0$, and any H predictable such that $H_\delta := HI_{\{Z_- \geq \delta\}} \in L(X)$ and $H_\delta \centerdot X \geq -1$, due to Theorem 23 of [54] (page 346 in the French version),

$$(H_\delta \centerdot X)_T = (H_\delta \centerdot X)_{T\wedge\tau_\delta}, \quad \text{and on } \{\theta \geq \tau_\delta\} \ \ (H_\delta \centerdot X)_T = (H_\delta \centerdot X)_{T\wedge\theta}.$$

Then, for any $T \in (0, +\infty)$, we calculate the following

$$P((H_\delta \centerdot X)_T > c) = P((H_\delta \centerdot X)_T > c \ \&\ \sigma_n \geq \tau_\delta) + P((H_\delta \centerdot X)_T > c \ \&\ \sigma_n < \tau_\delta)$$

$$\leq 2 \sup_{\phi \in L(X^{\sigma_n}) : \phi\centerdot X^{\sigma_n} \geq -1} P((\phi \centerdot X)_{\sigma_n \wedge T} > c) + P(\sigma_n < \tau_\delta \wedge T). \tag{5.65}$$

It is easy to prove that $P(\sigma_n < \tau_\delta \wedge T) \longrightarrow 0$ as n goes to infinity. This can be seen due to the fact that on Σ, we have, on the one hand, $\tau_\delta \wedge T < \sigma_\infty$ (by differentiating the two cases whether σ_∞ is finite or not). On the other hand, the event $(\sigma_n < \sigma_\infty)$

increases to Σ with n. Thus, by combining these, we obtain the following

$$\begin{aligned} P(\sigma_n < \tau_\delta \wedge T) &= P((\sigma_n < \tau_\delta \wedge T) \cap \Sigma) + P((\sigma_n < \tau_\delta \wedge T) \cap \Sigma^c) \\ &\leq P(\sigma_n < \tau_\delta \wedge T < \sigma_\infty) + P((\sigma_n < \sigma_\infty) \cap \Sigma^c) \longrightarrow 0. \end{aligned} \tag{5.66}$$

Now suppose that for each $n \geq 1$, the process $(S - S^{(0)})^{\sigma_n}$ satisfies NUPBR($\mathbb{F}$). Then a combination of (5.65) and (5.66) implies that for any $\delta > 0$, the process $I_{\{Z_- \geq \delta\}} \centerdot X := I_{\{Z_- \geq \delta\}} \centerdot (S - S^{(0)})$ satisfies NUPBR($\mathbb{F}$), and the proof of (c)$\Rightarrow$ (b) is completed. The proof of the reverse implcation is obvious due to the fact that

$$[\![0, \sigma_n]\!] \subset \{Z_- \geq 1/n\} \subset \{Z_- \geq \delta\}, \quad \text{for} \quad n \leq \delta^{-1},$$

which implies that $(I_{\{Z_- \geq \delta\}} \centerdot X)^{\sigma_n} = X^{\sigma_n}$. This ends the proof of (b) $\Longleftrightarrow$(c), and the proof of the theorem is achieved.

5.5.2 Intermediate Result

The proofs of Theorems 5.4 and 5.5 rely on the following intermediatory result about single jump $\mathbb{F}$-martingales, which is interesting in itself.

Lemma 5.10: *Let T be a finite $\mathbb{F}$-predictable stopping time. Then the following holds.*

$$\{T \leq \tau\} \ \subset \ \{\widetilde{Z}_T > 0\} \ \subset \ \{Z_{T-} > 0\} = \ \Gamma(T) := \left\{P\left(\widetilde{Z}_T > 0 \middle| \mathcal{F}_{T-}\right) > 0\right\}.$$

Proof. It is enough to prove the non-trivial equality $\{Z_{T-} > 0\} = \Gamma(T)$. Indeed, due to $E\left(P(\widetilde{Z}_T > 0|\mathcal{F}_{T-}) I_{\{Z_{T-}=0\}}\right) = P(\widetilde{Z}_T > 0 = Z_{T-}) = 0$, we get $\Gamma(T) \subset \{Z_{T-} > 0\}$. On the other hand, due to $E\left(Z_{T-} I_{\Gamma(T)^c}\right) = E\left(\widetilde{Z}_T \ I_{\Gamma(T)^c}\right) \leq E\left(I_{\{\widetilde{Z}_T > 0\}} \ I_{\Gamma(T)^c}\right) = 0$, we obtain $\{Z_{T-} > 0\} \subset \Gamma(T)$. This ends the proof of the lemma. $\square$

Lemma 5.11: *Let R be an equivalent probability to P. Then the following hold.*

$$\{\widetilde{Z}=0\}=\{\widetilde{Z}^R=0\},\quad \textit{and}\quad \{Z_-=0\}=\{Z^R_-=0\},$$

where $\widetilde{Z}^R_t := R(\tau\geq t|\mathcal{F}_t)$ and Z^R_- is the $(R,\mathbb{F})$-predictable projection of $\widetilde{Z}^R$.

Proof. For any $\mathbb{F}$-stopping time σ and any $\mathbb{F}$-predictable stopping time T, due to

$$E\left[\widetilde{Z}_\sigma I_{\{\widetilde{Z}^R_\sigma=0\}}\right]=E\left[I_{\{\tau\geq\sigma\}}I_{\{\widetilde{Z}^R_\sigma=0\}}\right]=0\ \text{ and}$$
$$E\left[Z_{T-}I_{\{Z^R_{T-}=0\}}\right]=E\left[I_{\{\tau\geq T\}}I_{\{\widetilde{Z}^R_{T-}=0\}}\right]=0,$$

we obtain $\{\widetilde{Z}^R=0\}\subset\{\widetilde{Z}=0\}$ and $\{Z^R_-=0\}\subset\{Z_-=0\}$. The symmetric roles of R and P complete the proof of the lemma. □

Proposition 5.7: *Let M be an $\mathbb{F}$-martingale given by $M:=\xi I_{[\![T,+\infty[\![}$, where T is an $\mathbb{F}$-predictable stopping time, and ξ is an $\mathcal{F}_T$-measurable random variable. Then the following assertions are equivalent.*

(a) *M is an $\mathbb{F}$-martingale under Q_T given by*

$$\frac{dQ_T}{dP}:=\frac{I_{\{\widetilde{Z}_T>0\}\cap\Gamma(T)}}{P(\widetilde{Z}_T>0\big|\ \mathcal{F}_{T-})}+I_{\Gamma^c(T)},\qquad \Gamma(T):=\{P(\widetilde{Z}_T>0|\mathcal{F}_{T-})>0\}.\quad(5.67)$$

(b) *On the set $\{T<+\infty\}$, we have*

$$E\left(M_T I_{\{\widetilde{Z}_T=0<Z_{T-}\}}\big|\ \mathcal{F}_{T-}\right)=0,\quad P-a.s.\qquad(5.68)$$

(c) *M^τ is a $\mathbb{G}$-martingale under $Q^{\mathbb{G}}_T:=\left(U^{\mathbb{G}}(T)/E(U^{\mathbb{G}}(T)\big|\ \mathcal{G}_{T-})\right)\cdot P$ where*

$$U^{\mathbb{G}}(T):=I_{\{T>\tau\}}+I_{\{T\leq\tau\}}\frac{Z_{T-}}{\widetilde{Z}_T}>0.\qquad(5.69)$$

Proof. The proof will be achieved in two steps. In Step 1, we prove (a)⇔(b); while Step 2 focuses on (a)⇔(c). First, we remark that the probability $Q^{\mathbb{G}}_T$ in assertion

(c) is well defined since $E\left(L_b^{\mathbb{G}}(T)\middle|\ \mathcal{G}_{T-}\right) = I_{\{T>\tau\}} + P(\widetilde{Z}_T > 0|\mathcal{F}_{T-})I_{\{T\leq\tau\}} > 0$ due to Lemma 5.10.

Step 1: Here, we prove (a)$\Longleftrightarrow$(b). To this end, we calculate that

$$E^{Q_T}(\xi\big|\mathcal{F}_{T-}) = I_{\Gamma(T)}E(\xi I_{\{\widetilde{Z}_T>0\}}\big|\mathcal{F}_{T-})\Big(P(\widetilde{Z}_T > 0\big|\mathcal{F}_{T-})\Big)^{-1}$$
$$= -I_{\Gamma(T)}E(\xi I_{\{\widetilde{Z}_T=0\}}\big|\mathcal{F}_{T-})\left(P(\widetilde{Z}_T > 0\big|\mathcal{F}_{T-})\right)^{-1}.$$

Therefore, we conclude that assertion (a) (or equivalently $E^{Q_T}(\xi|\mathcal{F}_{T-}) = 0$) is equivalent to (5.68). This ends the proof of (a) $\Longleftrightarrow$ (b).

Step 2: To prove (a)$\Longleftrightarrow$(c), we calculate that

$$E\left(L_b^{\mathbb{G}}(T)\middle|\ \mathcal{G}_{T-}\right)\ E^{Q_T^{\mathbb{G}}}\left(\xi I_{\{T\leq\tau\}}|\mathcal{G}_{T-}\right) = E\left(\frac{Z_{T-}}{\widetilde{Z}_T}\xi I_{\{T\leq\tau\}}|\mathcal{G}_{T-}\right)$$
$$= E\left(\xi I_{\{\widetilde{Z}_T>0\}}|\mathcal{F}_{T-}\right)I_{\{T\leq\tau\}} = E^{Q_T}\left(\xi|\mathcal{F}_{T-}\right)P\left(\widetilde{Z}_T > 0\middle|\mathcal{F}_{T-}\right)I_{\{T\leq\tau\}}.$$

This equality proves that M^τ is a martingale under $(\mathbb{G}, Q_T^{\mathbb{G}})$ –where $Q_T^{\mathbb{G}}$ is defined via (5.69)– if and only if M is a martingale under $(\mathbb{F}, Q_T)$, and the proof of (a)$\Longleftrightarrow$(c) is completed. This ends the proof of the proposition. □

5.5.3 Proof of Theorem 5.4

For the reader convenience, in order to prove Theorem 5.4, we state a more precise version of the theorem, in which we describe explicitly the choice for the probability measure Q_T.

Theorem 5.10: *Suppose that the assumptions of Theorem 5.4 are in force. Then, the assertions* (a) *and* (b) *of Theorem 5.4 are equivalent to the following assertions.*

(d) *S satisfies NUPBR*$(\mathbb{F}, \widetilde{Q}_T)$*, where* $\widetilde{Q}_T$ *is*

$$\widetilde{Q}_T := \left(\frac{\widetilde{Z}_T}{Z_{T-}} I_{\{Z_{T-}>0\}} + I_{\{Z_{T-}=0\}} \right) \cdot P. \tag{5.70}$$

(e) *S satisfies NUPBR*$(\mathbb{F}, Q_T)$*, where* Q_T *is defined in (5.67).*

Proof. The proof of this theorem will be achieved by proving (d) $\Longleftrightarrow$ (e) $\Longleftrightarrow$ (b) and (b) $\Rightarrow$ (a) $\Rightarrow$ (d). These will be carried out in four steps.

Step 1: In this step, we prove (d) $\Longleftrightarrow$ (e). By putting $\Gamma(T) := \left\{ P\left(\widetilde{Z}_T > 0 \middle| \mathcal{F}_{T-}\right) > 0 \right\}$ and using Lemma 5.10, we deduce that

$$Q_T = \left(\frac{I_{\{\widetilde{Z}_T>0\}}}{P(\widetilde{Z}_T > 0 \big| \, \mathcal{F}_{T-})} + I_{\Gamma(T)^c} \right) \cdot P, \quad \widetilde{Q}_T := \left(\frac{\widetilde{Z}_T}{Z_{T-}} I_{\{\widetilde{Z}_T>0\}} + I_{\{Z_{T-}=0\}} \right) \cdot P,$$

where Q_T and $\widetilde{Q}_T$ are defined in (5.67) and (5.70) respectively. Then it is easy to see that Q_T and $\widetilde{Q}_T$ are equivalent. This achieves this first step.

Step 2: This step proves (e)$\Longleftrightarrow$ (b). Assume that (e) holds. Then, there exists a positive and $\mathcal{F}_T$–measurable random variable, Y, such that $P-a.s.$ on $\{T < +\infty\}$, we have

$$E^{Q_T}(Y|\mathcal{F}_{T-}) = 1, \quad E^{Q_T}(Y|\xi||\mathcal{F}_{T-}) < +\infty, \quad \& \quad E^{Q_T}(Y\xi I_{\{Z_{T-}>0\}}|\mathcal{F}_{T-}) = 0.$$

Since $Y > 0$ on $\{\widetilde{Z}_T > 0\}$, by putting

$$Y_1 := Y I_{\{\widetilde{Z}_T>0\}} + I_{\{\widetilde{Z}_T=0\}} \text{ and } \widetilde{Y}_1 := \frac{Y_1}{E[Y_1|\mathcal{F}_{T-}]},$$

it is easy to check that $Y_1 > 0$, $\widetilde{Y}_1 > 0$,

$$E\left[\widetilde{Y}_1|\mathcal{F}_{T-}\right] = 1 \text{ and } E\left[\widetilde{Y}_1\xi I_{\{\widetilde{Z}_T>0\}}|\mathcal{F}_{T-}\right] = \frac{E\left[Y\xi I_{\{\widetilde{Z}_T>0\}}|\mathcal{F}_{T-}\right]}{E[Y_1|\mathcal{F}_{T-}]} = 0.$$

Therefore, $\widetilde{S}$ is a martingale under $R_1 := \widetilde{Y}_1 \cdot P \sim P$, and hence $\widetilde{S}$ satisfies NUPBR($\mathbb{F}$). This proves assertion (b).

To prove the reverse sense, we suppose assertion (b) holds. Then, there exists $0 < Y \in L^0(\mathcal{F}_T)$, such that $E[Y|\xi|I_{\{\widetilde{Z}_T>0\}}|\mathcal{F}_{T-}] < +\infty$, $E[Y|\mathcal{F}_{T-}] = 1$ and $E[Y\xi I_{\{\widetilde{Z}_T>0\}}|\mathcal{F}_{T-}] = 0$. Then, consider

$$Y_2 := \frac{YI_{\{\widetilde{Z}_T>0\}}P(\widetilde{Z}_T > 0|\mathcal{F}_{T-})}{E[YI_{\{\widetilde{Z}_T>0\}}|\mathcal{F}_{T-}]} + I_{\Gamma_1^c}, \quad \text{where } \Gamma_1 := \left\{E\left(YI_{\{\widetilde{Z}_T>0\}}|\mathcal{F}_{T-}\right) > 0\right\}.$$

Then it is easy to verify that $Y_2 > 0, \quad Q_T - a.s.$,

$$E^{Q_T}\left(Y_2|\mathcal{F}_{T-}\right) = 1, \quad \text{and} \quad E^{Q_T}\left(Y_2\xi I_{\{Z_{T-}>0\}}|\mathcal{F}_{T-}\right) = \frac{E\left[Y\xi I_{\{\widetilde{Z}_T>0\}}|\mathcal{F}_{T-}\right]}{E[YI_{\{\widetilde{Z}_T>0\}}|\mathcal{F}_{T-}]}I_{\Gamma_1} = 0.$$

This proves assertion (e), and the proof of (e)$\Longleftrightarrow$(b) is achieved.

Step 3: Herein, we prove (a) $\Rightarrow$ (d). Suppose that S^τ satisfies NUPBR($\mathbb{G}$). Then there exists a positive $\mathcal{G}_T$-measurable random variable $Y^{\mathbb{G}}$ such that on $\{T < +\infty\}$, we have

$$E[Y^{\mathbb{G}}I_{\{T\le\tau\}}|\mathcal{G}_{T-}] = I_{\{T\le\tau\}} \quad \text{and} \quad E[\xi Y^{\mathbb{G}}I_{\{T\le\tau\}}|\mathcal{G}_{T-}] = 0. \tag{5.71}$$

Due to Lemma 5.7–(d), we deduce the existence of two positive $\mathcal{F}_T$-measurable variables $Y^{(1)}$ and $Y^{(2)}$ such that

$$Y^{\mathbb{G}}I_{\{T\le\tau\}} = Y^{(1)}I_{\{T<\tau\}} + Y^{(2)}I_{\{\tau=T\}}. \tag{5.72}$$

By inserting (5.72) into the first equation in (5.71), we get

$$I_{\{T\leq\tau\}} = E[Y^{\mathbb{G}} I_{\{T\leq\tau\}}|\mathcal{G}_{T-}] = E(Z_T Y^{(1)} + (\widetilde{Z}_T - Z_T)Y^{(2)}|\mathcal{F}_{T-})\frac{1}{Z_{T-}} I_{\{T\leq\tau\}}.$$

Therefore, $E(Z_T Y^{(1)} + (\widetilde{Z}_T - Z_T)Y^{(2)}|\mathcal{F}_{T-})Z_{T-}^{-1} I_{\{Z_{T-}>0\}} = I_{\{Z_{T-}>0\}}$. Thus by putting

$$\widetilde{Y} := \left[\frac{Z_T}{\widetilde{Z}_T} Y^{(1)} + (1 - \frac{Z_T}{\widetilde{Z}_T})Y^{(2)}\right] I_{\{\widetilde{Z}_T>0\}} + I_{\{Z_{T-}=0\}} > 0,\ \widetilde{Q}_T - a.s.,$$

we derive

$$E^{\widetilde{Q}_T}(\widetilde{Y}|\mathcal{F}_{T-}) = E(Z_T Y^{(1)} + (\widetilde{Z}_T - Z_T)Y^{(2)}|\mathcal{F}_{T-})Z_{T-}^{-1} I_{\{Z_{T-}>0\}} + I_{\{Z_{T-}=0\}} = 1.$$

Similarly, by plugging (5.72) into the second equation of (5.71), we obtain

$$E^{\widetilde{Q}_T}(\xi I_{\{\widetilde{Z}_T>0\}}\widetilde{Y}|\mathcal{F}_{T-}) = E\left(\left(Z_T Y^{(1)} + (\widetilde{Z}_T - Z_T)Y^{(2)}\right)\xi\big|\mathcal{F}_{T-}\right)\frac{I_{\{\widetilde{Z}_T>0\}}}{Z_{T-}} = 0.$$

Then, we conclude that S satisfies the NUPBR$(\widetilde{Q}_T, \mathbb{F})$. This ends the proof of (a)⇒(d).

Step 4: This last step proves (b)$\Longrightarrow$(a). Suppose that $\widetilde{S}$ satisfies the NUPBR($\mathbb{F}$). Then, there exists $Y \in L^1(\mathcal{F}_T)$ such that on $\{T < +\infty\}$ we have

$$E[Y|\mathcal{F}_{T-}] = 1,\quad Y > 0,\quad E[Y|\xi| I_{\{\widetilde{Z}_T>0\}}|\mathcal{F}_{T-}] < +\infty,\ \text{and},\ E[Y\xi I_{\{\widetilde{Z}_T>0\}}|\mathcal{F}_{T-}] = 0.$$

Then by considering $R := Y \cdot P \sim P$ and using Lemma 5.11 (precisely $\{\widetilde{Z} = 0\} = \{\widetilde{Z}^R = 0\}$), we get

$$E^R\big[\widetilde{S}_T\big|\mathcal{F}_{T-}\big] = E^R\big[\xi I_{\{\widetilde{Z}_T>0\}}\big|\mathcal{F}_{T-}\big] = E^R\big[\xi I_{\{\widetilde{Z}^R_T>0\}}\big|\mathcal{F}_{T-}\big] = 0.$$

Therefore, by applying Proposition 5.7 to $M := \widetilde{S}$ under $R \sim P$ (it is easy to see that the condition (5.68) in Proposition 5.7 holds for $(\widetilde{S}, R)$, i.e. $E^R(\widetilde{S}_T I_{\{\widetilde{Z}^R_T=0\}}|\mathcal{F}_{T-}) = 0$). Then we conclude that $\widetilde{S}^\tau = S^\tau$ satisfies NUPBR$(R, \mathbb{F})$ and NUPBR$(P, \mathbb{F})$, due to $R \sim P$. This ends the fourth step and the proof of the theorem is completed. $\square$

5.5.4 Proof of Theorem 5.5

To highlight the precise difficulty in proving Theorem 5.5, we remark that on $\{T < +\infty\}$,

$$\frac{U^{\mathbb{G}}(T)}{E(U^{\mathbb{G}}(T)\big|\, \mathcal{G}_{T-})} = \frac{1 + \Delta L_T - \Delta V^{\mathbb{G}}_T}{1 - \Delta V^{\mathbb{G}}_T} \neq 1 + \Delta L_T = \frac{\mathcal{E}(L)_T}{\mathcal{E}(L)_{T-}}.$$

where $U^{\mathbb{G}}(T)$ is defined in (5.69). This highlights one of the main difficulties that we will face when we will formulate the results for possible many predictable jumps that might not be ordered. Simply, it might not be possible to piece up

$$U^{\mathbb{G}}(T_n) = 1 - \frac{\Delta m_{T_n}}{\widetilde{Z}_{T_n}} I_{\{T_n \leq \tau\}}, \quad n \geq 1$$

to form a positive $\mathbb{G}$-local martingale density for the process $(I_{\cup [\![T_n]\!]} \centerdot S)^\tau$.

Thus, in virtue of the above, the key idea behind the proof of Theorem 5.5 lies in connecting the NUPBR condition with the existence of a positive supermartingale (instead) that is a deflator for the market model under consideration.

Definition 5.2: Consider an $\mathbb{H}$-semimartingale X. Then, X is said to admit an $\mathbb{H}$-deflator if there exists a positive $\mathbb{H}$-supermartingale Y such that $Y(\theta \centerdot X)$ is a supermartingale, for any $\theta \in L(X, \mathbb{H})$ such that $\theta \centerdot X \geq -1$.

For supermartingale deflators, we reader the reader to Rokhlin [121]. Again, the above definition differs from that of the literature when the horizon is infinite, while it is the same as the one of the literature when the horizon is finite (even random). Below, we slightly generalize [121] to our context.

Lemma 5.12: *Let X be an $\mathbb{H}$-semimartingale. Then, the following assertions are equivalent.*

(a) *X admits an $\mathbb{H}$-deflator.*

(b) *X satisfies NUPBR($\mathbb{H}$).*

Proof. The proof of this lemma is straightforward, and is omitted. □

Now, we start giving the proof of Theorem 5.5.

Proof. of Theorem 5.5 The proof of the theorem will given in two steps, where we prove (b)⇒(a) and the reverse implication respectively. For the sake of simplifying the overall proof of the theorem, we remark that

$$\{\widetilde{Z}_T^Q = 0\} = \{\widetilde{Z}_T = 0\}, \quad \text{for any } Q \sim P \text{ and any } \mathbb{F}\text{-stopping time } T, \tag{5.73}$$

where $\widetilde{Z}_t^Q := Q[\tau \geq t | \mathcal{F}_t]$. This equality follows from

$$E\left[\widetilde{Z}_T I_{\{\widetilde{Z}_T^Q = 0\}}\right] = E\left[I_{\{\tau \geq T\}} I_{\{\widetilde{Z}_T^Q = 0\}}\right] = 0,$$

(which implies $\{\widetilde{Z}^Q = 0\} \subset \{\widetilde{Z} = 0\}$) and the symmetric role of Q and P.

Step 1: Here, we prove (b)⇒ (a). Suppose that assertion (b) holds, and consider a sequence of $\mathbb{F}$-stopping times $(\tau_n)_n$ that increases to infinity such that Y^{τ_n} is an $\mathbb{F}$-martingale. Then, setting $Q_n := Y_{\tau_n}/Y_0 \centerdot P$, and using (5.73) and Proposition 5.2, we deduce that there is no loss of generality in assuming $Y \equiv 1$. Condition (5.68) in Proposition 5.7 holds for $\Delta S_{T_n} I_{\{\widetilde{Z}_{T_n} > 0\}}$ and $\Delta S_{T_n} I_{\{\widetilde{Z}_{T_n} > 0\}} I_{[\![T_n, +\infty[\![}$. Therefore, using the notation $V^{\mathbb{G}}$ and L defined in (5.34) and (5.36), for each n, $(1 + \Delta L_{T_n} - \Delta V^{\mathbb{G}}_{T_n}) \Delta S_{T_n} I_{\{T_n \leq \tau\}} I_{[\![T_n, +\infty[\![}$ is a $\mathbb{G}$-martingale. Then, a direct application of Yor's exponential formula, we get that, for any $\theta \in L(S^\tau, \mathbb{G})$

$$\mathcal{E}\left(I_\Gamma \centerdot L - I_\Gamma \centerdot V^{\mathbb{G}}\right) \mathcal{E}\left(\theta I_\Gamma \centerdot S^\tau\right) = \mathcal{E}\left(X\right)$$

where $\Gamma = \cup_{n\geq 1}[\![T_n]\!]$ and

$$X := I_\Gamma \centerdot L - I_\Gamma \centerdot V^{\mathbb{G}} + \sum_{n\geq 1} \theta_{T_n} \left(1 + \Delta L_{T_n} - \Delta V^{\mathbb{G}}_{T_n}\right) \Delta S_{T_n} I_{\{T_n \leq \tau\}} I_{[\![T_n,+\infty[\![}.$$

Consider now the $\mathbb{G}$-predictable process

$$\phi = \sum_{n\geq 1} \xi_n I_{[\![T_n]\!]\cap[\![0,\tau]\!]} + I_{\Gamma^c \cup]\!]\tau+\infty[\![}, \qquad \text{where}$$

$$\xi_n := \frac{2^{-n}\left(1 + \mathcal{E}(X)_{T_n-}\right)^{-1}}{\left(1 + E\left[|\Delta L_{T_n}| \,\middle|\mathcal{G}_{T_n-}\right] + \Delta V^{\mathbb{G}}_{T_n-} + E\left[|\theta_{T_n} \frac{Z_{T_n-}}{\widetilde{Z}_{T_n}} I_{\{T_n\leq\tau\}} \Delta S_{T_n}| \,\middle|\mathcal{G}_{T_n-}\right]\right)}.$$

Then, it is easy to verify that $0 < \phi \leq 1$ and $E\left(|\phi \centerdot \mathcal{E}(X)|_{var}(+\infty)\right) \leq \sum_{n\geq 1} 2^{-n} = 1$. Hence, $\phi\centerdot\mathcal{E}(X) \in \mathcal{A}(\mathbb{G})$. Since, $\Delta L_{T_n} I_{[\![T_n,+\infty[\![}$ and $(1+\Delta L_{T_n} - \Delta V^{\mathbb{G}}_{T_n})\Delta S_{T_n} I_{\{T_n\leq\tau\}} I_{[\![T_n,+\infty[\![}$ are $\mathbb{G}$-martingales, we derive

$$(\phi \centerdot \mathcal{E}(X))^{p,\mathbb{G}} = \sum_{n\geq 1} \phi_{T_n} \mathcal{E}_{T_n-}(X) E(\Delta X_{T_n} | \mathcal{G}_{T_n-}) I_{[\![T_n,+\infty[\![} = -\phi\mathcal{E}_-(X) \centerdot V^{\mathbb{G}} \leq 0.$$

This proves that $\mathcal{E}(X)$ is a positive σ-supermartingale[1] as long as $\theta\Delta S^\tau \geq -1$. Thus, thanks to Kallsen [89], we conclude that it is a supermartingale and $\left(I_{\{Z_-\geq\delta\}} \centerdot S\right)^\tau$ admits a $\mathbb{G}$-deflator. Then, thanks to Lemma 5.12, we deduce that $\left(I_{\{Z_-\geq\delta\}} \centerdot S\right)^\tau$ satisfies NUPBR($\mathbb{G}$). Remark that, due to the $\mathbb{G}$-local boundedness of $(Z_-)^{-1} I_{[\![0,\tau]\!]}$, there exists a family of $\mathbb{G}$-stopping times $\tau_\delta,\ \delta > 0$ such that τ_δ converges almost surely to infinity when δ goes zero and

$$[\![0, \tau \wedge \tau_\delta]\!] \subset \{Z_- \geq \delta\} \cap [\![0, \tau]\!].$$

This implies that $S^{\tau\wedge\tau_\delta}$ satisfies NUPBR($\mathbb{G}$), and the assertion (a) follows from Proposition 5.2 (by taking $Q_n = P$ for all $n \geq 1$). This ends the proof of (b)$\Rightarrow$(a).

[1] Recall that a process X is said to be a σ-supermartingale if it is a semimartingale and there exists a predictable process ϕ such that $0 < \phi \leq 1$ and $\phi \centerdot X$ is a supermartingale

Step 2: In this step, we focus on (a)⇒(b). Suppose that S^τ satisfies NUPBR($\mathbb{G}$). Then, there exists a σ-martingale density under $\mathbb{G}$, for $I_{\{Z_-\geq\delta\}} \centerdot S^\tau$, $(\delta > 0)$, that we denote by $D^{\mathbb{G}}$. Then, from a direct application of Jacod's representation Theorem 2.7 and Theorem 5.1, we deduce the existence of a positive $\widetilde{\mathcal{P}}(\mathbb{G})$-measurable functional, $f^{\mathbb{G}}$, such that $D^{\mathbb{G}} := \mathcal{E}(N^{\mathbb{G}}) > 0$, with

$$N^{\mathbb{G}} := w^{\mathbb{G}} \star (\mu^{\mathbb{G}} - \nu^{\mathbb{G}}),\ w^{\mathbb{G}} := f^{\mathbb{G}} - 1 + \frac{\widehat{f^{\mathbb{G}}} - a^{\mathbb{G}}}{1 - a^{\mathbb{G}}} I_{\{a^{\mathbb{G}}<1\}},$$

where $\nu^{\mathbb{G}}$ was defined in (5.45), and, introducing f_m defined in (5.44)

$$xf^{\mathbb{G}} I_{\{Z_-\geq\delta\}} \star \nu^{\mathbb{G}} = xf^{\mathbb{G}} \left(1 + \frac{f_m}{Z_-}\right) I_{]\!]0,\tau]\!]} I_{\{Z_-\geq\delta\}} \star \nu \equiv 0. \tag{5.74}$$

Thanks to Lemma 5.7, we conclude to the existence of a positive $\widetilde{\mathcal{P}}(\mathbb{F})$-measurable functional, f, such that $f^{\mathbb{G}} I_{]\!]0,\tau]\!]} = f I_{]\!]0,\tau]\!]}$. Thus (5.74) becomes

$$U^{\mathbb{G}} := xf \left(1 + \frac{f_m}{Z_-}\right) I_{]\!]0,\tau]\!]} I_{\{Z_->\delta\}} \star \nu \equiv 0.$$

Introduce the following notations

$$\begin{aligned} &\mu_0 := I_{\{\widetilde{Z}>0\ \&\ Z_-\geq\delta\}} \centerdot \mu,\quad \nu_0 := h_0 I_{\{Z_-\geq\delta\}} \centerdot \nu,\ h_0 := M^P_\mu \left(I_{\{\widetilde{Z}>0\}} | \widetilde{\mathcal{P}}\right), \\ &g := \frac{f(1+\frac{f_m}{Z_-})}{h_0} I_{\{h_0>0\}} + I_{\{h_0=0\}},\quad a_0(t) := \nu_0(\{t\}, \mathbb{R}^d), \end{aligned} \tag{5.75}$$

and assume that

$$\sqrt{(g-1)^2 \star \mu_0} \in \mathcal{A}^+_{loc}(\mathbb{F}). \tag{5.76}$$

Then, thanks to Lemma 2.6, we deduce that $W := (g-1)/(1-a^0+\widehat{g}) \in \mathcal{G}^1_{loc}(\mu_0, \mathbb{F})$, and the local martingales

$$N^0 := \frac{g-1}{1-a^0+\widehat{g}} \star (\mu_0 - \nu_0),\quad Y^0 := \mathcal{E}(N^0), \tag{5.77}$$

are well defined satisfying $1+\Delta N^0>0$, $[N^0,S]\in\mathcal{A}(\mathbb{F})$, and on $\{Z_->\delta\}$ we have

$$\begin{aligned}\frac{{}^{p,\mathbb{F}}\left(Y^0\Delta SI_{\{\widetilde{Z}>0\}}\right)}{Y^0_-} &= {}^{p,\mathbb{F}}\left((1+\Delta N^0)\Delta SI_{\{\widetilde{Z}>0\}}\right) = {}^{p,\mathbb{F}}\left(\frac{g}{1-a^0+\widehat{g}}\Delta SI_{\{\widetilde{Z}>0\}}\right)\\ &= \Delta\frac{gxh_0}{1-a^0+\widehat{g}}\star\nu = \Delta\frac{xf(1+f_m/Z_-)}{1-a^0+\widehat{g}}\star\nu = Z_-^{-1}\frac{{}^{p,\mathbb{F}}\left(\Delta U^{\mathbb{G}}\right)}{1-a^0+\widehat{g}}\equiv 0.\end{aligned}$$

This proves that assertion (b) holds under the assumption (5.76).

The remaining part of the proof will show that this assumption always holds. To this end, we start by noticing that on the set $\{h_0>0\}$,

$$\begin{aligned} g-1 &= \frac{f(1+\frac{f_m}{Z_-})}{h_0}-1 = \frac{(f-1)(1+\frac{f_m}{Z_-})}{h_0}+\frac{f_m}{Z_-h_0}+\frac{M^P_\mu\left(I_{\{\widetilde{Z}=0\}}|\widetilde{\mathcal{P}}\right)}{h_0}\\ &:= g_1+g_2+g_3.\end{aligned}$$

Since $\left((f-1)^2I_{]\!]0,\tau]\!]}\star\mu\right)^{1/2}\in\mathcal{A}^+_{loc}(\mathbb{G})$, then due to Proposition 5.5–(e)

$$\sqrt{(f-1)^2I_{\{Z_-\geq\delta\}}\star(\widetilde{Z}\centerdot\mu)}\in\mathcal{A}^+_{loc}(\mathbb{F}),\quad \text{for any}\quad \delta>0.$$

Then, a direct application of Proposition 5.5–(a), for any $\delta>0$, we have

$$(f-1)^2I_{\{|f-1|\leq\alpha\ \&\ Z_-\geq\delta\}}\star(\widetilde{Z}\centerdot\mu),\quad |f-1|I_{\{|f-1|>\alpha\ \&\ Z_-\geq\delta\}}\star(\widetilde{Z}\centerdot\mu)\in\mathcal{A}^+_{loc}(\mathbb{F}).$$

By stopping, without loss of generality, we assume these two processes and $[m,m]$ belong to $\mathcal{A}^+(\mathbb{F})$. Remark that $Z_-+f_m=M^P_\mu\left(\widetilde{Z}|\widetilde{\mathcal{P}}\right)\leq M^P_\mu\left(I_{\{\widetilde{Z}>0\}}|\widetilde{\mathcal{P}}\right)=h_0$ that follows from $\widetilde{Z} \leq I_{\{\widetilde{Z}>0\}}$. Therefore, we derive

$$\begin{aligned}
E\left[g_1^2 I_{\{|f-1|\leq\alpha\}} \star \mu_0(\infty)\right] &= E\left[\frac{(f-1)^2(1+\frac{f_m}{Z_-})^2}{h_0^2} I_{\{|f-1|\leq\alpha\}} \star \mu_0(\infty)\right] \\
&= E\left[\frac{(f-1)^2(1+\frac{f_m}{Z_-})^2}{h_0^2} I_{\{|f-1|\leq\alpha\}} \star \nu_0(\infty)\right] \\
&\leq \delta^{-2} E\left[(f-1)^2(Z_- + f_m) I_{\{|f-1|\leq\alpha \ \& \ Z_-\geq\delta\}} \star \nu(\infty)\right] \\
&= \delta^{-2} E\left[(f-1)^2 I_{\{|f-1|\leq\alpha\}} \star (\widetilde{Z} I_{\{Z_-\geq\delta\}} \centerdot \mu)(\infty)\right] < +\infty,
\end{aligned}$$

and

$$\begin{aligned}
E\left[g_1 I_{\{|f-1|>\alpha\}} \star \mu_0(\infty)\right] &= E\left[\frac{|f-1|(1+\frac{f_m}{Z_-})}{h_0} I_{\{|f-1|>\alpha\}} \star \mu_0(\infty)\right] \\
&= E\left[|f-1|(1+\frac{f_m}{Z_-}) I_{\{|f-1|>\alpha\}} I_{\{Z_-\geq\delta\}} \star \nu(\infty)\right] \\
&\leq \delta^{-1} E\left[|f-1| I_{\{|f-1|>\alpha\}} \star (\widetilde{Z} I_{\{Z_-\geq\delta\}} \centerdot \mu)(\infty)\right] < +\infty.
\end{aligned}$$

Here μ_0 and ν_0 are defined in (5.75). Therefore, again by Proposition 5.5–(a), we conclude that $\sqrt{g_1^2 \star \mu_0} \in \mathcal{A}^+_{loc}(\mathbb{F})$.

Notice that $g_2 + g_3 = \frac{M^P_\mu\left(\Delta m I_{\{\widetilde{Z}>0\}}|\widetilde{\mathcal{P}}\right)}{Z_- h_0}$, and due to Lemma 2.4, we derive

$$\begin{aligned}
E\left[(g_2+g_3)^2 \star \mu_0(\infty)\right] &= E\left[\frac{M^P_\mu\left(\Delta m I_{\{\widetilde{Z}>0\}}|\widetilde{\mathcal{P}}\right)^2}{Z_-^2 h_0^2} \star \mu_0(\infty)\right] \\
&\leq E\left[\frac{M^P_\mu\left((\Delta m)^2|\widetilde{\mathcal{P}}\right) M^P_\mu\left(I_{\{\widetilde{Z}>0\}}|\widetilde{\mathcal{P}}\right)}{Z_-^2 h_0^2} \star \mu_0(\infty)\right] \\
&= E\left[\frac{M^P_\mu\left((\Delta m)^2|\widetilde{\mathcal{P}}\right)}{Z_-^2} I_{\{Z_-\geq\delta\}} \star \mu(\infty)\right] \\
&\leq \delta^{-2} E\left[[m,m]_\infty\right] < +\infty.
\end{aligned}$$

Hence, we conclude that $\sqrt{(g-1)^2 \star \mu_0} \in \mathcal{A}^+_{loc}(\mathbb{F})$. This ends the proof of (5.76), and the proof of the theorem is completed. □

5.6 Discrete Time Market Models

In this section, we study discrete time market models. That is the case when there are only finite number of trading times $n = 1, 2, ..., N$. We suppose given a stochastic basis $(\Omega, \mathbb{A}, \mathbb{F} := (\mathcal{F}_n)_{0\leq n\leq N}, \mathbb{P})$ on which we consider a real-valued $\mathbb{F}$-adapted stochastic process $S = (S_n)_{0\leq n\leq N}$ which represents the discounted risky assets. For the sake of simplicity, we assume S is an $\mathbb{F}$-martingale in this section. We say a process X is said to satisfy the non-arbitrage condition under the filtration $\mathbb{H} := (\mathcal{H}_n)_{0\leq n\leq N}$ (hereafter, NA($\mathbb{H}$)) if

$$\text{for any predictable process } H := (H_n)_{0\leq n\leq N},\ (i.e.\ H_n \in \mathcal{H}_{n-1}) \text{ such that}$$
$$\sum_{1\leq n\leq N} H_n \Delta X_n \geq 0,\ \mathbb{P}-a.s., \quad \text{we have } \sum_{1\leq n\leq N} H_n \Delta X_n \equiv 0,\ \mathbb{P}-a.s. \tag{5.78}$$

The process H can be interpreted as the trading strategy that one holds dynamically through time. Loosely speaking, the non-arbitrage condition means there is no possibility that one can make profit out of nothing and without risk. The equivalence between the non-arbitrage condition and equivalent martingale measure is essentially due to the work of Dalang, Morton and Willinger [44] (see also different approaches in Schachermayer [127] and Rogers [120]).

Theorem 5.11 (Dalang-Morton-Willinger): *The process X satisfies the non-arbitrage condition if and only there exists an equivalent martingale measure. In this case, the equivalent martingale measure $\mathbb{Q}$ can be chosen to have a uniformly bounded density $d\mathbb{Q}/d\mathbb{P}$.*

Comparing with continuous time models, NA($\mathbb{H}$), NUPBR($\mathbb{H}$), NFLVR($\mathbb{H}$) are equivalent in discrete time. We remark that optional stochastic integral does not play a crucial rôle here; one can use conditional expectation instead.

In this section, we shall prove that the non-arbitrage property is preserved (under one mild condition) when the market is stopped at the random horizon τ. Furthermore, we gave the necessary and sufficient conditions (on τ or the stopping times in (5.80)) to guarantee the stability of the non-arbitrage condition for any market S^τ.

Below, we define some notations related to the random time τ that would be fixed throughout the discrete time framework. For any random time τ, we associate the following two Azéma supermartingales

$$Z_n := P[\tau > n|\mathcal{F}_n] \quad \text{and} \quad \widetilde{Z}_n := P[\tau \geq n|\mathcal{F}_n], \tag{5.79}$$

and the $\mathbb{F}$-stopping times

$$\begin{aligned} R_1 := \inf\{n \geq 1 : Z_n = 0\}, \quad R_2 := \inf\{n \geq 1 : Z_{n-1} = 0\} \\ \text{and } R_3 := \inf\{n \geq 1 : \widetilde{Z}_n = 0\}. \end{aligned} \tag{5.80}$$

To incorporate the information from the random time τ, we enlarge the filtration $\mathbb{F}$ to include τ and obtain the filtration $\mathbb{G} = (\mathcal{G}_n)_{1\leq n\leq N}$, where

$$\mathcal{G}_n := \mathcal{F}_n \vee \sigma(\tau \leq n). \tag{5.81}$$

The progressively enlarged filtration $\mathbb{G}$ is the smallest one that contains $\mathbb{F}$ and makes τ a stopping time.

Lemma 5.13: *For any random time τ and the associated stopping times in (5.80), the following properties hold:*

(a) *The inclusions hold: $\{Z_{n-1} = 0\} \subset \{\widetilde{Z}_n = 0\} \subset \{Z_n = 0\}$ for all n.*

(b) *R_2 is an $\mathbb{F}$-predictable stopping time and $R_1 \leq R_3 \leq R_2$.*

(c) *$\tau \leq R_1$, Z_{n-1} and $\widetilde{Z}_n$ are both positive when $n \leq \tau$.*

Proof. (a) Notice that

$$E\left[\widetilde{Z}_n \mathbb{1}_{\{Z_{n-1}=0\}}\right] = E\left[Z_{n-1}\mathbb{1}_{\{Z_{n-1}=0\}}\right] = 0. \tag{5.82}$$

Hence, $\{Z_{n-1} = 0\} \subset \{\widetilde{Z}_n = 0\}$. Due to $Z_n \leq \widetilde{Z}_n$, we have $\{\widetilde{Z}_n = 0\} \subset \{Z_n = 0\}$.

(b) We observe that $\{R_2 \leq n\} = \{Z_{n-1} = 0\} \in \mathcal{F}_{n-1}$. Therefore R_2 is predictable. The inequalities $R_1 \leq R_3 \leq R_2$ follow immediately from (a).

(c) Notice that

$$\begin{aligned} E\left[\ \mathbb{I}_{\{n\leq\tau\}}\mathbb{I}_{\{Z_{n-1}=0\}}\right] &= E\left[Z_{n-1}\mathbb{I}_{\{Z_{n-1}=0\}}\right] = 0, \text{ and} \\ E\left[\ \mathbb{I}_{\{n\leq\tau\}}\mathbb{I}_{\{\widetilde{Z}_n=0\}}\right] &= E\left[\widetilde{Z}_n\mathbb{I}_{\{\widetilde{Z}_n=0\}}\right] = 0. \end{aligned}$$

Therefore, Z_{n-1} and $\widetilde{Z}_n$ are strictly positive on the set $\{n \leq \tau\}$. □

Remark 5.11: It was proved, in Dellacherie and Meyer [53], that these three sets $\{Z_- = 0\}$, $\{Z = 0\}$ and $\{\widetilde{Z} = 0\}$ have the same début in continuous time setting that discrete time does not share. This fact is no longer true in the current discrete time context. The difference is due to the fact that the filtration $\mathbb{F}$ is no longer right-continuous in this discrete time framework.

Lemma 5.14: *The Azéma supermartingale Z_n has the following decomposition:*

$$Z_n = m_n - A_n, \quad m_n := P[\tau > n|\mathcal{F}_n] + \sum_{0\leq k\leq n} P[\tau = k|\mathcal{F}_k], \ A_n := \sum_{0\leq k\leq n} P[\tau = k|\mathcal{F}_k]. \tag{5.83}$$

where m_n is an $\mathbb{F}$-martingale and A is an $\mathbb{F}$-adapted increasing process.

Proof. It is enough to prove $(m_n)_{n\geq 0}$ is an $\mathbb{F}$-martingale. To this end, we derive

$$\begin{aligned} E\Big[m_{n+1}\Big|\mathcal{F}_n\Big] &= P\Big[\tau > n+1\Big|\mathcal{F}_n\Big] + \sum_{0\leq k\leq n+1} E\Big[P[\tau = k|\mathcal{F}_k]\Big|\mathcal{F}_n\Big] \\ &= P\Big[\tau > n+1\Big|\mathcal{F}_n\Big] + \sum_{0\leq k\leq n} P\Big[\tau = k|\mathcal{F}_k\Big] + P\Big[\tau = n+1\Big|\mathcal{F}_n\Big] \\ &= P[\tau > n|\mathcal{F}_n] + \sum_{0\leq k\leq n} P[\tau = k|\mathcal{F}_k] = m_n. \end{aligned}$$

This ends the proof of the lemma. □

The following lemma describes the connection between conditional expectations under $\mathbb{F}$ and $\mathbb{G}$. For its proof, we consult Jeulin [83].

Lemma 5.15: *Let Y be an integrable and $\mathbb{A}$-measurable random variable. Then, the following properties hold:*

(a) *On the set $\{n < \tau\}$, the conditional expectation under $\mathcal{G}_n$ is given by*

$$E\left[Y|\mathcal{G}_n\right]\mathbb{1}_{\{\tau>n\}} = E\left[Y\mathbb{1}_{\{\tau>n\}}|\mathcal{F}_n\right]\frac{1}{Z_n}\mathbb{1}_{\{\tau>n\}}. \tag{5.84}$$

(b) *On the set $\{n \leq \tau\}$, the conditional expectation under $\mathcal{G}_{n-1}$ is given by*

$$E\left[Y|\mathcal{G}_{n-1}\right]\mathbb{1}_{\{\tau\geq n\}} = E\left[Y\mathbb{1}_{\{\tau\geq n\}}|\mathcal{F}_{n-1}\right]\frac{1}{Z_{n-1}}\mathbb{1}_{\{\tau\geq n\}}. \tag{5.85}$$

Moreover, if Y is $\mathcal{F}_n$-measurable, we have

$$E\left[Y|\mathcal{G}_{n-1}\right]\mathbb{1}_{\{\tau\geq n\}} = E\left[Y\widetilde{Z}_n|\mathcal{F}_{n-1}\right]\frac{1}{Z_{n-1}}\mathbb{1}_{\{\tau\geq n\}}. \tag{5.86}$$

The following theorem characterizes the relationship between $\mathbb{F}$-martingales and $\mathbb{G}$-martingales. For the continuous time case, we refer the reader to Jeulin [83].

Theorem 5.12: *Let M be an $\mathbb{F}$-martingale and τ be an random time. Then the* *following process*

$$M_n^G := M_{n\wedge\tau} - \sum_{1\le k\le n} \frac{1}{Z_{k-1}} \mathbb{1}_{\{\tau\ge k\}} E\left[(M_k - M_{k-1})\widetilde{Z}_k | \mathcal{F}_{k-1}\right], \tag{5.87}$$

is a $\mathbb{G}$*-martingale.*

Proof. Although it can be derived from Jeulin [83], we opt to give a direct proof here. It is easy to see that

$$M_{(n+1)\wedge\tau} = M_{n\wedge\tau} + (M_{n+1} - M_n)\mathbb{1}_{\{\tau\ge n+1\}}. \tag{5.88}$$

Then, we get

$$\begin{aligned}
E\Big[M_{n+1}^G \Big| \mathcal{G}_n\Big] &= E\Big[M_{(n+1)\wedge\tau} - \sum_{1\le k\le n+1} \frac{1}{Z_{k-1}} \mathbb{1}_{\{\tau\ge k\}} E\left[(M_k - M_{k-1})\widetilde{Z}_k | \mathcal{F}_{k-1}\right] \Big| \mathcal{G}_n\Big] \\
&= M_{n\wedge\tau} + E\Big[(M_{n+1} - M_n)\mathbb{1}_{\{\tau\ge n+1\}} \Big| \mathcal{G}_n\Big] \\
&\quad - \sum_{1\le k\le n} \frac{1}{Z_{k-1}} \mathbb{1}_{\{\tau\ge k\}} E\left[(M_k - M_{k-1})\widetilde{Z}_k | \mathcal{F}_{k-1}\right] \\
&\quad - \frac{1}{Z_n} \mathbb{1}_{\{\tau\ge n+1\}} E\left[(M_{n+1} - M_n)\widetilde{Z}_{n+1} | \mathcal{F}_n\right] \\
&= M_{n\wedge\tau} - \sum_{1\le k\le n} \frac{1}{Z_{k-1}} \mathbb{1}_{\{\tau\ge k\}} E\left[(M_k - M_{k-1})\widetilde{Z}_k | \mathcal{F}_{k-1}\right] = M_n^G,
\end{aligned}$$

where we use the following fact that is due to Lemma 5.15

$$E\Big[(M_{n+1} - M_n)\mathbb{1}_{\{\tau\ge n+1\}} \Big| \mathcal{G}_n\Big] = \frac{1}{Z_n} \mathbb{1}_{\{\tau\ge n+1\}} E\left[(M_{n+1} - M_n)\widetilde{Z}_{n+1} | \mathcal{F}_n\right].$$

This ends the proof of theorem. □

In the following proposition, we construct a $\mathbb{G}$-martingale that would serve as the martingale density for a class of $\mathbb{G}$-semi-martingales.

Proposition 5.8: *The following process*

$$N_n^G := -\sum_{1\le k\le n} \mathbb{1}_{\{\tau\ge k\}} E[\mathbb{1}_{\{\widetilde{Z}_k>0\}}|\mathcal{F}_{k-1}] + \sum_{1\le k\le n} \frac{Z_{k-1}}{\widetilde{Z}_k}\mathbb{1}_{\{\tau\ge k\}} \tag{5.89}$$

is a $\mathbb{G}$*-martingale such that* $1+\Delta N_n^G > 0$ *for all* $n\ge 1$.

Proof. First, we prove that N^G is a $\mathbb{G}$-martingale. To this end, due to Lemma 5.15– precisely (5.86)– we have

$$E\left[\frac{Z_n}{\widetilde{Z}_{n+1}}\mathbb{1}_{\{\tau\ge n+1\}}\Big|\mathcal{G}_n\right] = E\left[\frac{Z_n}{\widetilde{Z}_{n+1}}\mathbb{1}_{\{\widetilde{Z}_{n+1}>0\}}\Big|\mathcal{G}_n\right]\mathbb{1}_{\{\tau\ge n+1\}} = \mathbb{1}_{\{\tau\ge n+1\}}E[\mathbb{1}_{\{\widetilde{Z}_{n+1}>0\}}|\mathcal{F}_n].$$

Thus, as a result, we get

$$\begin{aligned}
E\left[N_{n+1}^G|\mathcal{G}_n\right] &= E\left[-\sum_{1\le k\le n+1}\mathbb{1}_{\{\tau\ge k\}}E[\mathbb{1}_{\{\widetilde{Z}_k>0\}}|\mathcal{F}_{k-1}] + \sum_{1\le k\le n+1}\frac{Z_{k-1}}{\widetilde{Z}_k}\mathbb{1}_{\{\tau\ge k\}}\Big|\mathcal{G}_n\right]\\
&= -\sum_{1\le k\le n+1}\mathbb{1}_{\{\tau\ge k\}}E[\mathbb{1}_{\{\widetilde{Z}_k>0\}}|\mathcal{F}_{k-1}] + \sum_{1\le k\le n}\frac{Z_{k-1}}{\widetilde{Z}_k}\mathbb{1}_{\{\tau\ge k\}} + E\left[\frac{Z_n}{\widetilde{Z}_{n+1}}\mathbb{1}_{\{\tau\ge n+1\}}\Big|\mathcal{G}_n\right]\\
&= -\sum_{1\le k\le n+1}\mathbb{1}_{\{\tau\ge k\}}E[\mathbb{1}_{\{\widetilde{Z}_k>0\}}|\mathcal{F}_{k-1}] + \sum_{1\le k\le n}\frac{Z_{k-1}}{\widetilde{Z}_k}\mathbb{1}_{\{\tau\ge k\}} + \mathbb{1}_{\{\tau\ge n+1\}}E[\mathbb{1}_{\{\widetilde{Z}_{n+1}>0\}}|\mathcal{F}_n]\\
&= -\sum_{1\le k\le n}\mathbb{1}_{\{\tau\ge k\}}E[\mathbb{1}_{\{\widetilde{Z}_k>0\}}|\mathcal{F}_{k-1}] + \sum_{1\le k\le n}\frac{Z_{k-1}}{\widetilde{Z}_k}\mathbb{1}_{\{\tau\ge k\}} = N_n^G.
\end{aligned} \tag{5.90}$$

Secondly, we check the integrability of N^G. Indeed,

$$E[|N_n^G|] \le n + \sum_{1\le k\le n} E\left[\frac{Z_{k-1}}{\widetilde{Z}_k}\mathbb{1}_{\{\tau\ge k\}}\right] = n + \sum_{1\le k\le n} E\left[Z_{k-1}\mathbb{1}_{\{\widetilde{Z}_k>0\}}\right] \le 2n.$$

Finally, we show that $1+\Delta N_n^G > 0$. Indeed

$$1+\Delta N_n^G = 1 - \mathbb{1}_{\{\tau\ge n\}}E[\mathbb{1}_{\{\widetilde{Z}_n>0\}}|\mathcal{F}_{n-1}] + \frac{Z_{n-1}}{\widetilde{Z}_n}\mathbb{1}_{\{\tau\ge n\}} \ge \mathbb{1}_{\{\tau<n\}} + \frac{Z_{n-1}}{\widetilde{Z}_n}\mathbb{1}_{\{\tau\ge n\}} > 0.$$

This completes the proof of the proposition. ∎

Remark 5.12: It is worthy to notice that $\sum_{1\leq k\leq n} \mathbb{1}_{\{\tau\geq k\}} E[\mathbb{1}_{\{\widetilde{Z}_k>0\}}|\mathcal{F}_{k-1}]$ is the $\mathbb{G}$-compensator of the $\mathbb{G}$-adapted increasing process $\sum_{1\leq k\leq n} Z_{k-1}/\widetilde{Z}_k \mathbb{1}_{\{\tau\geq k\}}$.

Lemma 5.16: *The stochastic exponential $\mathcal{E}(N^G)$ of N^G takes the form of*

$$\mathcal{E}(N^G)_n = \prod_{1\leq k\leq n} (1+\Delta N^G_k). \tag{5.91}$$

Proof. It is straightforward from the calculation of the stochastic exponential. □

Now, we are ready to state our first main theorem for this section.

Theorem 5.13: *Consider any random time τ and the $\mathbb{F}$-martingale S. Denote the probability measure $\mathbb{Q}\sim\mathbb{P}$ with density $D_n := \mathcal{E}(Y)_n$ where*

$$\Delta Y_n := \widetilde{Z}_n \mathbb{1}_{\{Z_{n-1}>0\}} E\left[\mathbb{1}_{\{\widetilde{Z}_n=0\}}|\mathcal{F}_{n-1}\right] - Z_{n-1}\mathbb{1}_{\{\widetilde{Z}_n=0\}},\ Y_0 = 0. \tag{5.92}$$

Then the following are equivalent:

(a) *S is an $(\mathbb{F},\mathbb{Q})$-martingale;*

(b) *S is orthogonal to D and Y;*

(c) *$\mathcal{E}(N^G)_n S_{n\wedge\tau}$ is a $\mathbb{G}$-martingale.*

As a consequence, all the above three equivalent conditions imply that:

(d) *S^τ satisfies NA$(\mathbb{G},\mathbb{P})$ and NA$(\mathbb{G},\mathbb{Q})$.*

Proof. First, we remark that the probability measure $\mathbb{Q}$ is well defined and is equivalent to $\mathbb{P}$. Indeed, it is easy to check that $(Y_n)_{n\geq 1}$ is an $\mathbb{F}$-martingale and

$$1+\Delta Y_n = \widetilde{Z}_n \mathbb{1}_{\{Z_{n-1}>0\}} E\left[\mathbb{1}_{\{\widetilde{Z}_n=0\}}|\mathcal{F}_{n-1}\right] + \mathbb{1}_{\{\widetilde{Z}_n>0\}} + (1-Z_{n-1})\mathbb{1}_{\{\widetilde{Z}_n=0\}} > 0,$$

where we used the fact that on the set $\{\widetilde{Z}_n > 0\}$, $1+\Delta Y_n \geq 1$ and $\{\widetilde{Z}_n = 0\} \subset \{Z_{n-1} < 1\}$, which is due to $\{Z_{n-1} = 1\} \subset \{\widetilde{Z}_n = 1\}$. Therefore, D is a strictly positive martingale.

The equivalence between (a) and (b) is obvious. In the following, we will focus on proving the equivalence between (a) and (c). Recall that

$$N_n^G = -\sum_{1\leq k\leq n} \mathbb{1}_{\{\tau\geq k\}} E[\mathbb{1}_{\{\widetilde{Z}_k>0\}}|\mathcal{F}_{k-1}] + \sum_{1\leq k\leq n} \frac{Z_{k-1}}{\widetilde{Z}_k}\mathbb{1}_{\{\tau\geq k\}}. \tag{5.93}$$

Due to Lemma 5.15, we deduce that

$$\begin{aligned} E\Big[\frac{\Delta S_k}{\widetilde{Z}_k}\mathbb{1}_{\{\tau\geq k\}}|\mathcal{G}_{k-1}\Big] &= \frac{\mathbb{1}_{\{\tau\geq k\}}}{Z_{k-1}} E\Big[\Delta S_k \mathbb{1}_{\{\widetilde{Z}_k>0\}}|\mathcal{F}_{k-1}\Big], \\ E\Big[\Delta S_k \mathbb{1}_{\{\tau\geq k\}}|\mathcal{G}_{k-1}\Big] &= \frac{\mathbb{1}_{\{\tau\geq k\}}}{Z_{k-1}} E\Big[\Delta S_k \widetilde{Z}_k|\mathcal{F}_{k-1}\Big]. \end{aligned} \tag{5.94}$$

Therefore, we derive

$$\begin{aligned} & E\left[\mathcal{E}(N^G)_{n+1} S_{(n+1)\wedge\tau}\Big|\mathcal{G}_{n+1}\right] \\ &= \mathcal{E}(N^G)_n E\Big[(1+\Delta N^G_{n+1}) S_{(n+1)\wedge\tau}\Big|\mathcal{G}_n\Big] \\ &= \mathcal{E}(N^G)_n E\Big[S_{n\wedge\tau} + \Delta S_{n+1}\mathbb{1}_{\{n+1\leq\tau\}} + \Delta N^{\mathbb{G}}_{n+1} S_{n\wedge\tau} + \Delta S_{n+1}\Delta N^{\mathbb{G}}_{n+1}\mathbb{1}_{\{n+1\leq\tau\}}\Big|\mathcal{G}_n\Big] \\ &= \mathcal{E}(N^G)_n \left\{S_{n\wedge\tau} + E\Big[\Delta S_{n+1}\widetilde{Z}_{n+1} E\left[\mathbb{1}_{\{\widetilde{Z}_{n+1}=0\}}|\mathcal{F}_n\right]\Big|\mathcal{F}_n\Big] \frac{\mathbb{1}_{\{n+1\leq\tau\}}}{Z_n}\right\} \\ &\quad - \mathcal{E}(N^G)_n \left\{\mathbb{1}_{\{n+1\leq\tau\}} E\Big[\Delta S_{n+1}\mathbb{1}_{\{\widetilde{Z}_{n+1}=0\}}\Big|\mathcal{F}_n\Big]\right\} \\ &= \mathcal{E}(N^G)_n S_{n\wedge\tau} + \mathcal{E}(N^G)_n \left\{E\left[\Delta S_{n+1}\left\{\widetilde{Z}_{n+1} E\left[\mathbb{1}_{\{\widetilde{Z}_{n+1}=0\}}|\mathcal{F}_n\right] - Z_n \mathbb{1}_{\{\widetilde{Z}_{n+1}=0\}}\right\}\Big|\mathcal{F}_n\right]\right\} \frac{\mathbb{1}_{\{n+1\leq\tau\}}}{Z_n} \\ &= \mathcal{E}(N^G)_n S_{n\wedge\tau} + \mathcal{E}(N^G)_n E^{\mathbb{Q}}\left[\Delta S_{n+1}\Big|\mathcal{F}_n\right] \frac{\mathbb{1}_{\{n+1\leq\tau\}}}{Z_n}. \end{aligned}$$

Thus, (a) implies (c). Conversely, if (c) holds, we have

$$E^{\mathbb{Q}}\left[\Delta S_{n+1}|\mathcal{F}_n\right] \frac{\mathbb{1}_{\{n+1\leq\tau\}}}{Z_n} = 0, \text{ and } \; E^{\mathbb{Q}}\left[\Delta S_{n+1}|\mathcal{F}_n\right]\mathbb{1}_{\{Z_n>0\}} = 0.$$

Notice that $E^{\mathbb{Q}}\left[\Delta S_{n+1}|\mathcal{F}_n\right]\mathbb{1}_{\{Z_n=0\}} = 0,$ for all n . Thus, we conclude that $E^{\mathbb{Q}}\left[\Delta S_{n+1}|\mathcal{F}_n\right] = 0$, for all n. This completes the proof of the theorem. □

Remark 5.13: We observe from Theorem 5.13 that even though Y is an $\mathbb{F}$-martingale, the stopped process $Y_{n\wedge\tau} = \sum_{k\leq n} \widetilde{Z}_k E\left[\mathbb{1}_{\{\widetilde{Z}_k=0\}}|\mathcal{F}_{k-1}\right]\mathbb{1}_{\{k\leq\tau\}}$ does not satisfy NA($\mathbb{G}$) since it is a $\mathbb{G}$-increasing process. This also sheds some light on the importance of the conditions in Theorem 5.13.

Remark 5.14: It is worthy to notice that, in general, for an $\mathbb{F}$-martingale M, if M^τ satisfies NA($\mathbb{G}$), we can not conclude M is orthogonal to Y. To prove this claim, we consider the projection of Y with respect to m as follows

$$\Delta Y_n = H_n \Delta m_n + \Delta \overline{m}_n,$$

where $H_n \in \mathcal{F}_{n-1}$ and $\overline{m}$ is an $\mathbb{F}$-martingale, orthogonal to m. If Y is not null, $\overline{m}$ is not identical zero. By Theorem 5.12, it is easy to see that $\overline{m}^\tau$ is a $\mathbb{G}$-martingale. However, $\overline{m}$ can not be orthogonal to Y unless Y is null. Indeed, we can calculate explicitly the $\mathbb{F}$-martingale $\overline{m}$ that stays a $\mathbb{G}$-martingale.

$$\Delta\overline{m}_n = \Delta Y_n - H_n\Delta m_n = \Delta Y_n - \frac{E\left[\Delta Y_n \Delta m_n \mid \mathcal{F}_{n-1}\right]}{E\left[(\Delta m_n)^2 \mid \mathcal{F}_{n-1}\right]}\Delta m_n.$$

Corollary 5.13.1: Let M be an $\mathbb{F}$-martingale. If for all n,

$$\{\widetilde{Z}_n = 0\} = \{Z_{n-1} = 0\}. \tag{5.95}$$

Then the following properties hold:

(a) $(M_{n\wedge\tau})_{n\geq 1}$ satisfies NA($\mathbb{G}$);

(b) $\left(\mathcal{E}(N^G)_n M_{n\wedge\tau}\right)_{n\geq 1}$ is a $\mathbb{G}$-martingale, where N^G is given by (5.89) in Proposition 5.8;

(c) The probability measure $\mathbb{Q}$, given in (5.92) coincides with $= \mathbb{P}$.

In particular, the above three properties hold when $Z_n > 0$ for all $n \geq 1$.

Below, we state our second main theorem of this section, where we give the necessary and sufficient conditions on the random time τ (or equivalently the stopping times in (5.80)) to guarantee the no-arbitrage M^τ for any $\mathbb{F}$-martingale M.

Theorem 5.14: *Consider a random time τ and the associated stopping times defined in (5.80). Then the following are equivalent:*

(a) *For any $\mathbb{F}$-martingale M, the stopped process M^τ satisfies $NA(\mathbb{G})$.*

(b) $\{\widetilde{Z}_n = 0\} = \{Z_{n-1} = 0\}$ *for all n.*

(c) $R_1 + 1 = R_2 = R_3$.

(d) R_3 *is an $\mathbb{F}$-predictable stopping time.*

(e) *The probability $\mathbb{Q}$, defined via (5.92), coincides with $= \mathbb{P}$.*

Proof. The proof of the theorem would be achieved after four steps. In the first step, we prove (b)⇔(c). The second step focuses on (c)⇔(d). The third step deals with (b)⇔(e). In the last step, we will prove (a) ⇔ (b).

Step 1: The equivalence between (b) and (c) is obvious. Indeed, if (b) holds, it is trivial that $R_2 = R_3$. Conversely, if (c) holds, we derive that

$$E\left(Z_{n-1} I_{\{\widetilde{Z}_n=0\}}\right) = E\left(Z_{n-1} I_{\{\widetilde{Z}_n=0\}} I_{\{n\geq R_3\}}\right) = E\left(Z_{n-1} I_{\{\widetilde{Z}_n=0\}} I_{\{n\geq R_2\}}\right) = 0.$$

Hence, we conclude that $\{\widetilde{Z}_n = 0\} \subset \{Z_{n-1} = 0\}$ for all n.

Step 2: Herein, we will prove (c)⇔(d). If (c) holds, it is easy to see that R_3 is an $\mathbb{F}$-predictable stopping time due to $\{R_3 = n\} = \{R_1 = n-1\} \in \mathcal{F}_{n-1}$. Conversely, by the predictability of R_3, we have $0 = E[\widetilde{Z}_{R_3}] = E[Z_{R_3-1}]$; hence $Z_{R_3-1} = 0$ and $R_3 = R_2$.

Step 3: This step will prove (b)⇔(e). If (b) holds, apparently, $Y = 0$ and $\mathbb{Q} = \mathbb{P}$. Conversely, if (e) holds, $\Delta Y_n = 0$ for all n. Hence, $\widetilde{Z}_n \mathbb{1}_{\{Z_{n-1}>0\}} E\left[\mathbb{1}_{\{\widetilde{Z}_n=0\}} | \mathcal{F}_{n-1}\right] = Z_{n-1} \mathbb{1}_{\{\widetilde{Z}_n=0\}} = 0$ and $\{\widetilde{Z}_n = 0\} = \{Z_{n-1} = 0\}$ for all n.

Step 4: In this step, we focus on the proof of the equivalence between (a) and (b). Proof of (a)⇒(b): Suppose that for any $\mathbb{F}$-martingale M, the stopped process M^τ satisfies NA($\mathbb{G}$). Consider

$$V_n := 1\!\!1_{\{R_3>n\}} \quad \text{and} \quad \widetilde{V}_n := \sum_{1\leq k\leq n} \{E[V_k|\mathcal{F}_{k-1}] - V_{k-1}\}\,. \tag{5.96}$$

It is easy to see that $M_n := V_n - \widetilde{V}_n$ is an $\mathbb{F}$-martingale. Therefore $M_{n\wedge\tau} = 1 - \widetilde{V}_{n\wedge\tau}$ satisfies NA($\mathbb{G}$). Then there exists an equivalent probability $\mathbb{Q}_1 \sim \mathbb{P}$ such that $\widetilde{V}_{n\wedge\tau}$ is a $(\mathbb{G},\mathbb{Q}_1)$-martingale. Therefore $\widetilde{V}_{n\wedge\tau} \equiv 0$. Hence, we have

$$\begin{aligned}
0 = E\left[\widetilde{V}_{n\wedge\tau}\right] &= E\left[\sum_{1\leq k\leq n} Z_{k-1}\Delta\widetilde{V}_k\right]\\
&= \sum_{1\leq k\leq n} E\left[Z_{k-1}\left(E\left[1\!\!1_{\{R_3>k\}}|\mathcal{F}_{k-1}\right] - 1\!\!1_{\{R_3>k-1\}}\right)\right]\\
&= \sum_{1\leq k\leq n} E\left[Z_{k-1}1\!\!1_{\{R_3>k\}}\right] - E\left[Z_{k-1}1\!\!1_{\{R_3>k-1\}}\right]\\
&= -\sum_{1\leq k\leq n} E\left[Z_{k-1}1\!\!1_{\{R_3=k\}}\right] = -\sum_{1\leq k\leq n} E\left[Z_{k-1}1\!\!1_{\{\widetilde{Z}_k=0\}}\prod_{1\leq i\leq k} 1\!\!1_{\{\widetilde{Z}_{i-1}>0\}}\right]\\
&= -\sum_{1\leq k\leq n} E\left[Z_{k-1}1\!\!1_{\{Z_{k-1}>0\}}1\!\!1_{\{\widetilde{Z}_k=0\}}\prod_{1\leq i\leq k} 1\!\!1_{\{\widetilde{Z}_{i-1}>0\}}\right]\\
&= -\sum_{1\leq k\leq n} E\left[Z_{k-1}1\!\!1_{\{Z_{k-1}>0\}}1\!\!1_{\{\widetilde{Z}_k=0\}}\right] = -\sum_{1\leq k\leq n} E\left[Z_{k-1}1\!\!1_{\{\widetilde{Z}_k=0\}}\right].
\end{aligned}$$

In the last equality we used the fact that $\{Z_k > 0\} \subset \{\widetilde{Z}_k > 0\} \subset \{\widetilde{Z}_{k-1} > 0\}$. Therefore, for all n, $\{\widetilde{Z}_n = 0\} \subset \{Z_{n-1} = 0\}$ and $R_3 \geq R_2$.

The proof of (b)⇒(a) follows immediately from Theorem 5.13 or Corollary 5.13.1. This ends the proof of the theorem. □

The following is a sort of surprising corollary

Corollary 5.14.1: Consider a two period model $(\Omega, \mathcal{A} = \mathcal{F}_2, \mathbb{F} := (\mathcal{F}_n)_{n=0,1,2}, \mathbb{P})$ with an $\mathcal{A}$-measurable random time τ. For any $\mathbb{F}$-martingale M, the stopped process M^τ satisfies NA($\mathbb{G}$) if and only if τ is an $\mathbb{F}$-stopping time.

Proof. If τ is an $\mathbb{F}$-stopping time, it is trivial that M^τ satisfies NA($\mathbb{G}$) for any $\mathbb{F}$-martingale M. Conversely, for the random time τ, denote $\Omega_2 := \{\tau = 2\}$, $\Omega_1 := \{\tau = 1\}$ and $\Omega_1 \cup \Omega_2 = \Omega$. By the definitions of Z and $\widetilde{Z}$, we derive that

$$\widetilde{Z}_0 = 1, \widetilde{Z}_1 = 1, \widetilde{Z}_2 = I_{\Omega_2}, \text{ and } Z_0 = 1, Z_1 = \mathbb{P}(\Omega_2|\mathcal{F}_1), Z_2 = 0.$$

If for any $\mathbb{F}$-martingale M, the stopped process M^τ satisfies NA($\mathbb{G}$), by Theorem 5.14, we know that $\{\widetilde{Z}_2 = 0\} = \Omega_1 = \{Z_1 = 0\} \in \mathcal{F}_1$ and τ is an $\mathbb{F}$-stopping time. □

The following theorem is a sort of reverse of Theorem 5.16 and 5.14 and shows what we can conclude if X^τ satisfies NA($\mathbb{G}$) for some intergrable process X.

Theorem 5.15: *Let X be an arbitrary $\mathbb{F}$-adapted integrable process with the decomposition $X_n = X_0 + M_n + A_n$, where $\Delta A_n = E[\Delta X_n|\mathcal{F}_{n-1}]$ and $\Delta M_n = \Delta X_n - \Delta A_n$. If X^τ satisfies NA($\mathbb{G}$), then the following inclusion holds on the set $\{n \leq \tau\}$*

$$\left\{Z_{n-1}E\left[(\Delta X_n)^2\,\widetilde{Z}_n|\mathcal{F}_{n-1}\right] = E^2\left[\Delta X_n\widetilde{Z}_n|\mathcal{F}_{n-1}\right]\right\} \subset \left\{E\left[\Delta X_n\widetilde{Z}_n|\mathcal{F}_{n-1}\right] = 0\right\}. \tag{5.97}$$

Proof. Let $\mathbb{Q}^{\mathbb{G}}$ be the equivalent martingale measure for X^τ with the density process $D_n := \mathcal{E}\left(K^{\mathbb{G}}\right)_n$. Thanks to Itô's formula, this is equivalent to

$$X_{n\wedge\tau} + \sum_{1\leq k\leq n} E\left[\Delta K_k^{\mathbb{G}}\Delta X_k|\mathcal{G}_{k-1}\right]\mathbb{1}_{\{k\leq\tau\}}, \tag{5.98}$$

is a $\mathbb{G}$-martingale. Consider the projection of $K^{\mathbb{G}}$ with respect to $M^{\mathbb{G}}$ given by

$$\Delta K_n^{\mathbb{G}} = H_n^{\mathbb{G}}\Delta M_n^{\mathbb{G}} + \Delta\overline{K}_n, \tag{5.99}$$

where $M^{\mathbb{G}}$ is defined via (5.87) and $\overline{K}$ is a $\mathbb{G}$-martingale, orthogonal to $M^{\mathbb{G}}$. Notice that

$$X^{\tau} = X_0 + M^{\mathbb{G}} + \sum_{1 \le k \le n} \frac{1}{Z_{k-1}} E\left[\Delta M_k \Delta m_k | \mathcal{F}_{k-1}\right] 1\!\!1_{\{k \le \tau\}} + A^{\tau}, \tag{5.100}$$

and $E\left[\Delta \overline{K}_k \Delta X_k | \mathcal{G}_{k-1}\right] = 0$ for all k. Hence, the condition (5.98) is equivalent to for all n

$$\Delta A_n + \frac{1}{Z_{n-1}} E\left[\Delta M_n \Delta m_n | \mathcal{F}_{n-1}\right] + H_n^{\mathbb{G}} E\left[\Delta M_n^{\mathbb{G}} \Delta X_n | \mathcal{G}_{n-1}\right] \equiv 0, \text{ on the set } \{n \le \tau\}. \tag{5.101}$$

In the following, we will transfer the condition (5.101) to the form in terms of the public information $\mathbb{F}$. To this end, we calculate on the set $\{n \le \tau\}$ that

$$\begin{aligned} E\left[\Delta M_n^{\mathbb{G}} \Delta X_n \middle| \mathcal{G}_{n-1}\right] &= E\left[\left(\Delta X_n - \Delta A_n - \frac{1}{Z_{n-1}} E\left[\Delta X_n \Delta m_n | \mathcal{F}_{n-1}\right]\right) \Delta X_n \middle| \mathcal{G}_{n-1}\right] \\ &= E\left[\left(\Delta X_n - E[\Delta X_n | \mathcal{F}_{n-1}] - \frac{1}{Z_{n-1}} E\left[\Delta X_n \Delta m_n \middle| \mathcal{F}_{n-1}\right]\right) \Delta X_n \middle| \mathcal{G}_{n-1}\right] \\ &= E\left[\left(\Delta X_n - \frac{1}{Z_{n-1}} E\left[\Delta X_n \widetilde{Z}_n | \mathcal{F}_{n-1}\right]\right) \Delta X_n \middle| \mathcal{G}_{n-1}\right] \\ &= \frac{1}{Z_{n-1}} E\left[(\Delta X_n)^2 \widetilde{Z}_n | \mathcal{F}_{n-1}\right] - \frac{1}{Z_{n-1}^2} E^2\left[\Delta X_n \widetilde{Z}_n | \mathcal{F}_{n-1}\right], \end{aligned}$$

and

$$\Delta A_n + \frac{1}{Z_{n-1}} E\left[\Delta M_n \Delta m_n | \mathcal{F}_{n-1}\right] = \frac{1}{Z_{n-1}} E\left[\Delta X_n \widetilde{Z}_n | \mathcal{F}_{n-1}\right].$$

Therefore,

$$H_n^{\mathbb{G}} = \frac{-Z_{n-1} E\left[\Delta X_n \widetilde{Z}_n | \mathcal{F}_{n-1}\right]}{Z_{n-1} E\left[(\Delta X_n)^2 \widetilde{Z}_n | \mathcal{F}_{n-1}\right] - E^2\left[\Delta X_n \widetilde{Z}_n | \mathcal{F}_{n-1}\right]} 1\!\!1_{\{n \le \tau\}}.$$

and

$$\left\{Z_{n-1} E\left[(\Delta X_n)^2 \widetilde{Z}_n | \mathcal{F}_{n-1}\right] = E^2\left[\Delta X_n \widetilde{Z}_n | \mathcal{F}_{n-1}\right]\right\} \subset \left\{E\left[\Delta X_n \widetilde{Z}_n | \mathcal{F}_{n-1}\right] = 0\right\}.$$

This ends the proof of the theorem. □

5.7 Lévy Market Model

In this section, we shall study Lévy market model. Suppose that the traded financial asset is an exponential of a Lévy process given by $S = S_0 \exp(X)$, where

$$X_t = \gamma t + \sigma W_t + \int_0^t \int_{|x|\le 1} x\widetilde{N}(dt,dx) + \int_0^t \int_{|x|\ge 1} xN(dt,dx), \tag{5.102}$$

$\widetilde{N}(dt,dx) = N(dt,dx) - \nu_X(dx,dt)$ and $\nu_X(dx,dt) := F_X(dx)dt$. Here, γ and σ are real numbers ($\sigma > 0$); $W = (W_t)_{t\ge 0}$ represents a Brownian motion; $N(dt,dx)$ is a random measure on $[0,T]\otimes\mathbb{R}\backslash\{0\}$, called Poisson random measure; $\widetilde{N}(dt,dx)$ is the compensated Poisson measure with the intensity measure $F_X(dx)dt$, where $F_X(dx)$ is called the Lévy measure defined on $\mathbb{R}\backslash\{0\}$, satisfying

$$\int_{\mathbb{R}\backslash\{0\}} \left(|x|^2 \wedge 1\right) F_X(dx) < +\infty. \tag{5.103}$$

For more details about Lévy processes, we refer the reader to [126].

The relationship between the random measure μ and its compensator ν of S and $\mu_X(dt,dx) := N(dt,dx)$ is given by

$$\begin{aligned} &\mu(dt,dx) = \Phi(\mu_X(dt,dx)) = \text{the image of } \mu_X \text{ by the mapping } \Phi, \quad \text{and} \\ &\nu(dt,dx) = F(dx)dt, \quad F := \Phi(F_X) = \text{the image of } F_X \text{ by } \Phi, \end{aligned} \tag{5.104}$$

where $\Phi(\omega,t,\cdot):\ \mathbb{R}\longrightarrow\mathbb{R}:\ \Phi(\omega,t,x) = S_{t-}(\omega)\left(e^x - 1 - x\right)$. To give more details for the calculation, for any nonnegative $\widetilde{\mathcal{P}}(\mathbb{F})$-measurable functional, f, we have

$$\int f(t,\omega,x)F(\omega,t,dx) = \int f(t,\omega,S_{t-}(\omega)(e^x - 1 - x))F_X(dx).$$

Although the Lévy market is a particular case of quasi-left-continuous semi-martingale treated in Subsection 5.2.1, we want to give a practical condition, under which we will prove the validity of the NUPBR after stopping with τ. This condition is

$$\nu^{\mathbb{G}} \sim \nu, \quad \text{on} \quad [\![0,\tau]\!]. \tag{5.105}$$

It is easy to prove that this condition is equivalent to the condition where $\nu_X^{\mathbb{G}} \sim \nu_X$ on $[\![0,\tau]\!]$.

In the same spirit as Theorem 2.18, any local martingale Y in this model can be decomposed as follows

$$Y = \beta \centerdot S^c + f \star (\mu - \nu) + g \star \mu + Y^{\perp}. \tag{5.106}$$

Here, $(\beta, f, g, Y^{\perp})$ is the Jacod parameters of Y with respect to S .
Denote (β_m, f_m, g_m, m') be the Jacod parameters of m with respect to $(S, \mathbb{F}, \mathbb{P})$ as

$$m = \beta_m \centerdot S^c + f_m \star (\mu - \nu) + g_m \star \mu + m'. \tag{5.107}$$

The Jacod parameters of m would be fixed throughout this section.
Below, we recall the random measure and its compensator under the enlarged filtration $\mathbb{G}$ for Lévy market.

Proposition 5.9: *Consider the Lévy Market S, on $[\![0,\tau]\!]$, we have*

(a) *The compensator of μ in the filtration $\mathbb{G}$ is given by*

$$\nu^{\mathbb{G}} := \left(I_{[\![0,\tau]\!]} \centerdot \mu\right)^{p,\mathbb{G}} = \left(1 + \frac{f_m}{Z_-}\right) I_{[\![0,\tau]\!]} \centerdot \nu. \tag{5.108}$$

(b) *The canonical representation of* S^τ *is given by*

$$S^\tau = S_0 + \widehat{S^c} + h \star (\mu^{\mathbb{G}} - \nu^{\mathbb{G}}) + (x-h)\star\mu^{\mathbb{G}} + \widehat{B}, \tag{5.109}$$

where $\widehat{S^c}$ *is defined via (5.17) and* $\widehat{B} := I_{[\![0,\tau]\!]} \centerdot B + \frac{1}{Z_-} I_{[\![0,\tau]\!]} \centerdot \langle S^c, m\rangle^{\mathbb{F}} + h\frac{f_m}{Z_-} I_{[\![0,\tau]\!]} \star \nu.$

The following proposition proposes a local martingale density for S^τ.

Proposition 5.10: *The following process* $\mathcal{E}(N^{\mathbb{G}})$ *is well defined, being a positive* $\mathbb{G}$*-local martingale.*

$$\mathcal{E}(N^{\mathbb{G}}) := \mathcal{E}\left(-\frac{1}{Z_-}\beta_m I_{[\![0,\tau]\!]} \centerdot \widehat{S^c} - \frac{f_m}{f_m + Z_-} I_{[\![0,\tau]\!]} \star \left(\mu^{\mathbb{G}} - \nu^{\mathbb{G}}\right)\right), \tag{5.110}$$

where $\widehat{S^c}$ *is defined via (5.17).*

Proof. First, we show that the random integral $\frac{f_m}{f_m+Z_-} I_{[\![0,\tau]\!]} \star \left(\mu^{\mathbb{G}} - \nu^{\mathbb{G}}\right)$ is well defined. Note that

$$\begin{aligned} E\left[I_{[\![0,\tau]\!]} I_{\left\{M^P_\mu[\widetilde{Z}|\widetilde{\mathcal{P}}(\mathbb{F})]=0\right\}} \star \mu\right] &= E\left[\widetilde{Z} I_{\left\{M^P_\mu[\widetilde{Z}|\widetilde{\mathcal{P}}(\mathbb{F})]=0\right\}} \star \mu\right] \\ &= E\left[M^P_\mu\left[\widetilde{Z}|\widetilde{\mathcal{P}}(\mathbb{F})\right] I_{\left\{M^P_\mu[\widetilde{Z}|\widetilde{\mathcal{P}}(\mathbb{F})]=0\right\}} \star \mu\right] = 0. \end{aligned}$$

Therefore, $Z_- + f_m = M^P_\mu[\widetilde{Z}|\widetilde{\mathcal{P}}(\mathbb{F})] > 0, M^P_\mu$-a.s. on the set $[\![0,\tau]\!]$. Next, we shall prove that

$$\frac{f_m}{f_m + Z_-} I_{[\![0,\tau]\!]} \in \mathcal{G}^1_{loc}(\mu^{\mathbb{G}}).$$

To this end, we consider

$$\delta \in (0,1), \quad \Gamma := \{\Delta S \neq 0\} \text{ and } \Gamma^c \text{ its complement in } \Omega\otimes[0,+\infty[.$$

We calculate that

$$\begin{aligned} V &:= \sqrt{\sum_{0\le u\le \cdot} \left(\frac{f_m(\Delta S_u)}{Z_- + f_m(\Delta S_u)} I_\Gamma I_{[\![0,\tau]\!]}\right)^2} \\ &\le \sqrt{\sum_{0\le u\le \cdot} \left(\frac{f_m(\Delta S_u)}{Z_- + f_m(\Delta S_u)} I_{[\![0,\tau]\!]} I_\Gamma I_{\{f_m < -\delta Z_-\}}\right)^2} \\ &\quad + \sqrt{\sum_{0\le u\le \cdot} \left(\frac{f_m(\Delta S_u)}{Z_- + f_m(\Delta S_u)} I_{[\![0,\tau]\!]} I_\Gamma I_{\{f_m \ge -\delta Z_-\}}\right)^2} \\ &:= V_1 + V_2. \end{aligned} \tag{5.111}$$

Due to the $\mathbb{G}$-local boundedness of $1/Z_- I_{[\![0,\tau]\!]}$, we obtain that

$$\begin{aligned} V_2(t) &\le \frac{1}{1-\delta} \sqrt{\sum_{0\le u\le t} \left(\frac{f_m(\Delta S_u)}{Z_{u-}} I_{[\![0,\tau]\!]} I_\Gamma I_{\{f_m \ge -\delta Z_{u-}\}}\right)^2} \\ &\le \frac{1}{1-\delta} \sqrt{\sum_{0\le u\le t} \left(\frac{f_m^2(\Delta S_u)}{Z_{u-}^2} I_{[\![0,\tau]\!]} I_\Gamma\right)} = \frac{1}{1-\delta} \sqrt{\frac{f_m^2 I_{[\![0,\tau]\!]}}{Z_-^2} \star \mu} \in \mathcal{A}_{loc}^+(\mathbb{G}), \end{aligned}$$

where we used the fact that $f_m \in \mathcal{G}_{loc}^2(\mu, \mathbb{F})$, i.e. $f_m^2 \star \mu \in \mathcal{A}_{loc}^+(\mathbb{F})$. Again due to the $\mathbb{G}$-local boundedness of $1/Z_- I_{[\![0,\tau]\!]}$, and $f_m^2 \star \mu \in \mathcal{A}_{loc}^+(\mathbb{F})$, we deduce the existence of a sequence of $\mathbb{G}$-stopping times $(T_n)_{n\ge1}$ that increases to infinity and a sequence of $\mathbb{F}$-stopping times, $(\sigma_n)_{n\ge1}$, that increases to infinity such that

$$(Z_-)^{-1} I_{[\![0,\tau]\!]} \le n \quad \text{on} \quad [\![0, T_n]\!] \quad \text{and} \quad E f_m^2 \star \nu(\sigma_n) < +\infty.$$

Then, we derive that

$$
\begin{aligned}
E[V_1(T_n\wedge\sigma_n)] &\le E\left[\frac{|f_m|}{f_m+Z_-}I_{[\![0,\tau]\!]}I_{\{f_m<-\delta Z_-\}}\star\mu_{T_n\wedge\sigma_n}\right]\\
&\le E\left[\frac{|f_m|}{f_m+Z_-}I_{[\![0,\tau]\!]}I_{\{f_m<-\delta/n\}}\star\mu_{\sigma_n}\right]\\
&\le E\left[\frac{|f_m|}{f_m+Z_-}\widetilde{Z}I_{\{f_m+Z_->0\}}I_{\{f_m<-\delta/n\}}\star\mu_{\sigma_n}\right]\\
&= E\Big[|f_m|I_{\{\delta/n<-f_m<Z_-\}}\star\mu_{\sigma_n}\Big]\\
&\le \frac{n}{\delta}E[f_m^2\star\mu_{\sigma_n}]<+\infty.
\end{aligned}
$$

This proves that the process V is $\mathbb{G}$-locally integrable.

Secondly, the positivity of D is obvious. Indeed on $[\![0,\tau]\!]$, we have

$$
1+\Delta N_t = 1-\frac{f_m(\Delta S_t)}{Z_{t-}+f_m(\Delta S_t)}I_{\{\Delta S_t\neq 0\}} = I_{\{\Delta S_t=0\}}+\frac{Z_{t-}}{Z_{t-}+f_m(\Delta S_t)}I_{\{\Delta S_t\neq 0\}}>0.
$$

This ends the proof of the proposition. □

Theorem 5.16: *Let S be the Lévy market satisfying NUPBR($\mathbb{F}$) and τ be a random time satisfying the condition (5.105). Then S^τ satisfies NUPBR($\mathbb{G}$).*

Proof. Since S satisfies NUPBR($\mathbb{F}$), there exists a local martingale density D^S. Let $(\sigma_n)_{n\ge1}$ be the localizing sequence of D^S and $D^S S$. Put $Q^n := D^S_{\sigma_n}\centerdot P\sim P$. By change of probability, we could work under Q^n, $Z^Q_t := Q^n(\tau>t|\mathcal{F}_t)$ and $\widetilde{Z}^Q_t := Q^n(\tau\ge t|\mathcal{F}_t)$. Therefore, without loss of generality, we could assume S is an $\mathbb{F}$-local martingale. We shall prove that $\mathcal{E}(N^{\mathbb{G}})$ is a local martingale density of S^τ. Recalling from Proposition 5.9, we have

$$
\begin{aligned}
S^\tau \;&=\; I_{[\![0,\tau]\!]}\centerdot\widehat{S^c}+x\star(\mu^{\mathbb{G}}-\nu^{\mathbb{G}})+xI_{[\![0,\tau]\!]}\star(\nu^{\mathbb{G}}-\nu)+\frac{I_{[\![0,\tau]\!]}}{Z_-}\centerdot\langle S^c,m\rangle^{\mathbb{F}},\\
N^{\mathbb{G}} &:= -\frac{1}{Z_-}\beta_m I_{[\![0,\tau]\!]}\centerdot\widehat{S^c}-\frac{f_m}{f_m+Z_-}I_{[\![0,\tau]\!]}\star\left(\mu^{\mathbb{G}}-\nu^{\mathbb{G}}\right) := N^{c,\mathbb{G}}+N^{d,\mathbb{G}}.
\end{aligned}
$$

Then, we calculate that

$$
\begin{aligned}
S^{\tau}+\left[S^{\tau},N^{\mathbb{G}}\right] &= S^{\tau}+\left[(S^{\tau})^{c},N^{c,\mathbb{G}}\right]+\sum\Delta S^{\tau}\Delta N^{\mathbb{G}}\\
&= S_0+I_{[\![0,\tau]\!]}\centerdot\widehat{S^c}+x\star(\mu^{\mathbb{G}}-\nu^{\mathbb{G}})+\frac{xf_m}{Z_-}I_{[\![0,\tau]\!]}\star\nu+\frac{I_{[\![0,\tau]\!]}}{Z_-}\beta_m c\centerdot A\\
&\quad -\frac{I_{[\![0,\tau]\!]}}{Z_-}\beta_m c\centerdot A-\frac{xf_m}{f_m+Z_-}I_{[\![0,\tau]\!]}\star\mu \qquad (5.112)\\
&= S_0+I_{[\![0,\tau]\!]}\centerdot\widehat{S^c}+x\star(\mu^{\mathbb{G}}-\nu^{\mathbb{G}})+\frac{xf_m}{Z_-}I_{[\![0,\tau]\!]}\star\nu-\frac{xf_m}{f_m+Z_-}I_{[\![0,\tau]\!]}\star\mu.
\end{aligned}
$$

It remains to show $\frac{xf_m}{f_m+Z_-}I_{[\![0,\tau]\!]}\star\mu\in\mathcal{A}_{loc}(\mathbb{G})$. Let $(T_n)_{n\geq 1}$ be the localizing sequence of $[S,m]$. To this end, we calculate that

$$
\begin{aligned}
E\left[\frac{|xf_m|}{f_m+Z_-}I_{[\![0,\tau]\!]}\star\mu_{T_n}\right] &= E\left[\frac{|xf_m|}{f_m+Z_-}I_{\{f_m+Z_->0\}}\widetilde{Z}\star\mu_{T_n}\right]\\
&\leq E\left[|xf_m|\star\mu_{T_n}\right]\leq E\left[Var([S,m])_{T_n}\right]<+\infty.
\end{aligned}
$$

Notice that $\frac{xf_m}{Z_-}I_{[\![0,\tau]\!]}\star\nu$ is the $\mathbb{G}$-compensator of $\frac{xf_m}{f_m+Z_-}I_{[\![0,\tau]\!]}\star\mu$. From (5.112), we conclude that $S^{\tau}+\left[S^{\tau},N^{\mathbb{G}}\right]$ is a $\mathbb{G}$-local martingale. Thanks to Itô formula, $\mathcal{E}(N^{\mathbb{G}})S^{\tau}$ is a local martingale if and only if $S^{\tau}+[S^{\tau},N^{\mathbb{G}}]$ is a local martingale. This ends the proof of the theorem. □

Corollary 5.16.1: Let Y be a compensated Poisson process and τ be a random time satisfying the condition (5.105). Then, Y^{τ} satisfies NUPBR($\mathbb{G}$).

Proof. The proof follows analogy with previous Theorem and the $\mathbb{G}$-local martingale density is given by $N^{\mathbb{G}}:=-\frac{f_m}{f_m+Z_-}I_{[\![0,\tau]\!]}\star(\mu^{\mathbb{G}}-\nu^{\mathbb{G}})$ since the continuous martingale part is null. For the simplicity of jump measure of Poisson process, we provide a direct easy proof of the integrability of $N^{\mathbb{G}}$. Let $(T_n)_{n\geq 1}$ be the localizing sequence of $1 \star \mu$. To this end, we calculate that

$$\begin{aligned} E\left[\frac{|f_m|}{f_m+Z_-} I_{[\![0,\tau]\!]} \star \mu^{\mathbb{G}}_{T_n}\right] &= E\left[\widetilde{Z}\frac{|f_m|}{f_m+Z_-} I_{\{f_m+Z_->0\}} \star \mu_{T_n}\right] \\ &= E\left[|f_m| I_{\{f_m+Z_->0\}} \star \nu_{T_n}\right] \\ &\leq 4E\left[1 \star \nu_{T_n}\right] < +\infty. \end{aligned}$$

This ends the proof of the Corollary. □

Conclusions:

In this chapter, we obtained two principal results. The first result lies in describing the pairs of market model and random time for which the resulting stopped model fulfills NUPBR condition. The second main result characterizes the random time models that preserve the NUPBR property after stopping for any market model. These results are elaborated in a very general market model, and also discrete time and Lévy market models. The analysis that drives these results is based on new stochastic developments in semimartingale theory with progressive enlargement. Furthermore, we construct explicit martingale densities (deflators) for some classes of local martingales when stopped at random time.

In the next chapter, we will investigate NUPBR on the stochastic interval $]\!]\tau, +\infty[\![$.

Chapter 6

Non-arbitrage for a Class of Honest Times

This chapter completes the study undertaken in Chapter 5 about non-arbitrage in informational market. Throughout the chapter, the financial market will be represented by a d-dimensional semimartingale S, (d-risky assets), and a non-risky asset (assumed to be constant one) on the stochastic basis $(\Omega, \mathcal{G}, \mathbb{F}, P)$, where $\mathbb{F} := (\mathcal{F}_t)_{t\geq 0}$ is a filtration satisfying the usual hypotheses (i.e., right continuity and completeness) and represents the flow of public information with $\mathcal{F}_\infty := \bigvee_{t\geq 0} \mathcal{F}_t \subseteq \mathcal{G}$.

In this chapter $(\Omega, \mathcal{G}, \mathbb{F}, S, \mathbb{P})$ represents the initial model (the defaultable-free model) to which we add a fixed random time denoted by τ. In Chapter 5, we addressed the arbitrage theory for the sub-model $(\Omega, \mathbb{G}, S^\tau)$, where $\mathbb{G}$ is the enlarged filtration that contains $\mathbb{F}$ and the information from τ. Therefore, in virtue of the obtained results, our main focus is the arbitrage of the sub-model $(\Omega, \mathbb{G}, S - S^\tau)$. It is known in the literature that a process satisfies the No-Unbounded-Profit-with-Bounded-Risk (called NUPBR hereafter) property only if this process is a semimartingale. Thus, the first question that arises before any arbitrage inquiry is whether the model $S-S^\tau$ is a $\mathbb{G}$-semimartingale. This is the main reason why we focus on random times that are honest times, when dealing with the part "after" τ. It is known that honest times (see [83]) preserve the semimartingale structures. It becomes also necessary via the work of Imkeller [75] (see also Fontana et al. [63]), that for honest times avoiding stopping times in a Brownian filtration, the NUPBR property is violated after τ. Thus, the first challenging question is:

Are there honest times for which the NUPBR holds for some models?

The positive answer to this question constitutes our first original contribution of this chapter. This class of honest times will be described in (6.4) and for this class of honest times, we will address two important problems.

$$\text{For which pairs } (S,\tau), \text{ the model } (\Omega, \mathbb{G}, S - S^{\tau}) \text{ is arbitrage free?} \quad \mathbf{(Prob(6.I))}$$

The investigating of this problem will lead us to a deep understanding of the precise parts from both the random time and the initial market model that allow arbitrage to occur. The second problem focuses on the model of τ such that

$$(\Omega, \mathbb{G}, S - S^{\tau}) \text{ is arbitrage free for any arbitrage free } S? \quad \mathbf{(Prob(6.II))}$$

This will allow us to single out the class of honest times that preserve the non-arbitrage considered in the thesis (i.e. NUPBR).

This chapter is organized as follows. In the following section (Section 6.1), we present our main results, their immediate consequences, and/or their economic and financial interpretations. These results are formulated for particular as well as general framework. Section 6.2 is devoted to new stochastic developments vital for the proof of the main results, while Section 6.3 deals with the derivation of explicit local martingale deflators. The last section (Section 6.4) focuses on proving the main theorems of Section 6.1.

6.1 The Main Results

In what follows, $\mathbb{H}$ denotes a filtration satisfying the usual hypotheses. For an $\mathbb{H}$-semi-martingale Y, the set $\mathcal{L}(Y,\mathbb{H})$ is the set of $\mathbb{H}$ predictable processes integrable w.r.t. Y and for $H \in \mathcal{L}(Y,\mathbb{H})$, we denote $H \centerdot Y_t := \int_0^t H_s dY_s$.

We recall the notion of non-arbitrage that is addressed in this Chapter.

Definition 6.1: An $\mathbb{H}$-semimartingale S satisfies the *No-Unbounded-Profit-with-Bounded-Risk* condition under $(\mathbb{H}, Q)$ (hereafter called NUPBR($\mathbb{H}, Q$)) if for any finite deterministic horizon T, the set

$$\mathcal{K}_T(S) := \Big\{ (H \centerdot S)_T \;\Big|\; H \in \mathcal{L}(S,\mathbb{H}), \text{ and } H \centerdot S \geq -1 \Big\}$$

is bounded in probability under Q. When $Q \sim P$, we simply write NUPBR($\mathbb{H}$)) and write S satisfies NUPBR($\mathbb{H}$) instead of S satisfies the NUPBR($\mathbb{H}$) property.

For more details about this type of non-arbitrage condition and its relationship to the literature, we refer to Definition 2.17 of Subsection 2.4 in Chapter 2.

Beside the defaultable-free model represented by $(\Omega, \mathbb{F}, P, S)$, we consider a finite random time $\tau : \Omega \to \mathbb{R}_+$. To this random time, we associate the process D and the filtration $\mathbb{G}$ given by

$$D := I_{[\![\tau,+\infty[\![}, \quad \mathbb{G} = (\mathcal{G}_t)_{t\geq 0}\,, \quad \mathcal{G}_t = \bigcap_{s>t} \Big(\mathcal{F}_s \vee \sigma(D_u, u \leq s) \Big).$$

Recall two important $\mathbb{F}$-supermartingales given by

$$Z_t := P\left(\tau > t \;\middle|\; \mathcal{F}_t\right) \text{ and } \quad \widetilde{Z}_t := P\left(\tau \geq t \;\middle|\; \mathcal{F}_t\right). \tag{6.1}$$

The decomposition of Z leads to an important $\mathbb{F}$-martingale that we denote by m, given by

$$m := Z + D^{o,\mathbb{F}}, \tag{6.2}$$

where $D^{o,\mathbb{F}}$ is the $\mathbb{F}$-dual optional projection of $D = I_{[\![\tau,\infty[\![}$.

Throughout the chapter, we will assume that

$$\tau \quad \text{is an honest time.} \tag{6.3}$$

For the reader's convenience, we recall the definition of honest time.

Definition 6.2: A random time σ is honest, if for any t, there exists an $\mathcal{F}_t$-measurable r.v. σ_t such that $\sigma I_{\{\sigma<t\}} = \sigma_t I_{\{\sigma<t\}}$.

We refer to Jeulin [83, Chapter 5] and Barlow [17] for more information about honest times. For the case of quasi-left-continuous processes, we restrict our study to the following subclass of honest times.

$$\tau \text{ is an honest time satisfying} \quad Z_\tau < 1, \qquad P-a.s. \tag{6.4}$$

It is clear that any stopping time satisfies (6.4). Furthermore, in [3], the authors provided many examples that are not stopping times satisfying (6.4).
We end this section by recalling the following lemma, obtained in Chapter 5.

Lemma 6.1: *Let X be an $\mathbb{H}$-predictable process with finite variation. Then X satisfies NUPBR($\mathbb{H}$) if and only if $X \equiv X_0$ (i.e. the process X is constant).*

6.1.1 The Case of Quasi-left-continuous Processes

In this subsection, we answer (**Prob(6.I)**) and (**Prob(6.II)**) for the case of $\mathbb{F}$-quasi-left-continuous processes.

We start investigating (**Prob(6.I)**) and characterizing processes S and honest times τ such that $S - S^\tau$ satisfies NUPBR($\mathbb{G}$). To this end, we recall μ, ν are given in (2.9) and (2.11), $\widetilde{\mathcal{P}}(\mathbb{H}) := \mathcal{P}(\mathbb{H}) \otimes \mathcal{B}(\mathbb{R}^d)$, $Z_- + \Delta m = \widetilde{Z}$ and we state the following.

Lemma 6.2: *The process $S^{(1)}$ given by*

$$S^{(1)} := x f_m I_{\{Z_- < 1 = Z_- + f_m\}} \star \mu, \qquad \textit{where} \quad f_m := M^P_\mu \left(\Delta m \;\middle|\; \widetilde{\mathcal{P}}(\mathbb{F}) \right) \tag{6.5}$$

is an $\mathbb{F}$-semimartingale.

Proof. We remark that $f_m^2 \star \mu$ is a càdlàg locally bounded and nondecreasing process and for any $0 \leq u < v$, we have

$$\mathrm{Var}(S^{(1)})_v - \mathrm{Var}(S^{(1)})_u \leq \sqrt{[S,S]_v - [S,S]_u} \sqrt{f_m^2 I_{]u,v]} \star \mu}.$$

Therefore, $S^{(1)}$ is a càdlàg process with locally integrable variation. $\square$

Theorem 6.1: *Suppose that S is an $\mathbb{F}$-quasi-left-continuous semimartingale, and τ is a finite honest time satisfying (6.4). If the process $(1 - Z_-) \centerdot S - S^{(1)}$ satisfies NUPBR($\mathbb{F}$), then the process $S - S^\tau$ satisfies NUPBR($\mathbb{G}$).*

Proof. The proof of this theorem will be given in Subsection 6.4.1. $\square$

Corollary 6.1.1: Suppose that S is $\mathbb{F}$-quasi-left-continuous and satisfies NUPBR($\mathbb{F}$). Then, the following assertions hold.

(a) If $(S, S^{(1)})$ satisfies NUPBR($\mathbb{F}$), then $S - S^\tau$ satisfies NUPBR($\mathbb{G}$).

(b) If $S^{(1)} \equiv 0$, then $S - S^\tau$ satisfies NUPBR($\mathbb{G}$).

(c) If $\{\Delta S \neq 0\} \cap \{\widetilde{Z} = 1 > Z_-\} = \emptyset$, then $S - S^\tau$ satisfies NUPBR($\mathbb{G}$).

(d) If S is continuous, then $S - S^\tau$ satisfies NUPBR($\mathbb{G}$).

Proof. (a) Suppose that $(S, S^{(1)})$ satisfies NUPBR($\mathbb{F}$). Then, the process $\left((1 - Z_-) \centerdot S - S^{(1)}\right) = (S, S^{(1)})$ satisfies NUPBR($\mathbb{F}$). Then assertion (a) follows immediately from Theorem 6.1.

(b) Suppose that $S^{(1)} \equiv 0$. Then, $(S, S^{(1)})$ satisfies NUPBR($\mathbb{F}$) and assertion (b) follows directly from assertion (a).

(c) Suppose that $\{\Delta S \neq 0\} \cap \{\widetilde{Z} = 1 > Z_-\} = \emptyset$. Then, due to

$$M^P_\mu\left((1-\widetilde{Z})I_{\{Z_-+f_m=1\}}\right) = M^P_\mu\left[(1 - M^P_\mu\left(\widetilde{Z}|\widetilde{\mathcal{P}}(\mathbb{F})\right))I_{\{Z_-+f_m=1\}}\right]$$

$$= M^P_\mu\left((1 - Z_- - f_m)I_{\{Z_-+f_m=1\}}\right) = 0.$$

we deduce that $\{\Delta S \neq 0\} \cap \{1 = Z_- + f_m\} \subset \{\Delta S \neq 0\} \cap \{\widetilde{Z} = 1\}$. Then, we get $S^{(1)} \equiv 0$. Hence assertion (c) follows from assertion (b).

(d) When S is continuous, then $\{\Delta S \neq 0\} \cap \{\widetilde{Z} = 1 > Z_-\} \subset \{\Delta S \neq 0\} = \emptyset$. Hence assertion (d) follows from assertion (c). This ends the proof of the corollary. $\square$

In the following proposition, we answer (**Prob(6.II)**) for quasi-left-continuous processes S.

Proposition 6.1: *The following assertions are equivalent.*

(a) *The thin set $\{\widetilde{Z} = 1 > Z_-\}$ is accessible.*

(b) *Every (bounded) $\mathbb{F}$-quasi-left-continuous martingale X, $X - X^\tau$ satisfies NUPBR($\mathbb{G}$).*

Proof. We start by proving that (a)$\Rightarrow$(b). Suppose that the thin set $\{\widetilde{Z} = 1 > Z_-\}$ is accessible. Consider an $\mathbb{F}$-quasi-left-continuous martingale X, and let $X^{(1)}$ be the process associated with X as in (6.5). Therefore, since X is quasi-left-continuous, we have $\{\Delta X \neq 0\} \cap \{\widetilde{Z} = 1 > Z_-\} = \emptyset$. Hence, we get $X^{(1)} \equiv 0$ and by Theorem 6.1, we deduce that $X - X^\tau$ satisfies NUPBR($\mathbb{G}$). This completes the proof of (a)$\Rightarrow$(b). To prove the reverse, we assume that assertion (b) holds and consider a sequence of stopping times $(T_n)_{n\geq 1}$ that exhausts the thin set $\{\widetilde{Z} = 1 \ \& \ Z_- < 1\}$ (i.e $\left\{\widetilde{Z} = 1 \ \& \ Z_- < 1\right\} = \bigcup_{n=1}^{+\infty} [\![T_n]\!]$). Then, each T_n (that we denote by T, in the rest of the proof, for the sake of simplicity) can be decomposed into a totally inaccessible part T^i and an accessible part T^a as $T = T^i \wedge T^a$. Consider the following quasi-left-continuous processes

$$V := I_{[\![T^i,+\infty[\![}, \quad \text{and} \quad M := V - V^{p,\mathbb{F}} =: V - \widetilde{V}.$$

Then, since $\{T^i < +\infty\} \subset \{\widetilde{Z}_{T^i} = 1\}$, we deduce that $\{T^i < +\infty\} \subset \{\tau \geq T^i\}$ and

$$I_{]\!]\tau,+\infty[\![} \centerdot M = -I_{]\!]\tau,+\infty[\![} \centerdot \widetilde{V} \quad \text{ is } \mathbb{G}\text{-predictable.}$$

Then, the finite variation process $I_{]\!]\tau,+\infty[\![} \centerdot M$ satisfies NUPBR($\mathbb{G}$) if and only if it is null, or equivalently

$$0 = E\left(I_{]\!]\tau,+\infty[\![} \centerdot \widetilde{V}_\infty\right) = E\left(\int_0^\infty (1 - Z_{s-}) d\widetilde{V}_s\right) = E\left((1 - Z_{T^i-}) I_{\{T^i<+\infty\}}\right).$$

Therefore, we conclude that $T^i = +\infty,\ P - a.s.$, and the stopping time T is an accessible stopping time. This ends the proof of the proposition. $\square$

6.1.2 Thin Processes with Predictable Jump Times

This subsection addresses (**Prob(6.I)**) and (**Prob(6.II)**) for the case where the process S is a single jump process with predictable jump time. The results of this framework can be easily generalized to the case of finite number of jumps. It is important to highlight the fact that, in this subsection, we work with the whole class of honest times and we do **not** assume the condition (6.4) on τ.

Below, we state our first main result in this context, that answers (6.1). We give a characterisation of processes S and honest times τ such that $S - S^\tau$ satisfies NUPBR($\mathbb{G}$).

Theorem 6.2: *Suppose that τ is an honest time. Consider an $\mathbb{F}$-predictable stopping time T and an $\mathcal{F}_T$-measurable r.v. ξ such that $E(|\xi| \big| \mathcal{F}_{T-}) < +\infty$ $P - a.s.$. If*

$S := \xi I_{\{Z_{T-}<1\}} I_{[\![T,+\infty[\![}$, then the following assertions are equivalent.

(a) *$S - S^{\tau}$ satisfies NUPBR($\mathbb{G}$).*

(b) *$\widetilde{S} := \xi I_{\{\widetilde{Z}_T<1\}} I_{[\![T,+\infty[\![}$ satisfies NUPBR($\mathbb{F}$).*

(c) *There exists a probability measure Q_T on $(\Omega, \mathcal{F}_T)$, absolutely continuous with respect to P such that S satisfies NUPBR($\mathbb{F}, Q_T$).*

The proof of this theorem is long and requires intermediary results. Thus, we postpone the proof to Subsection 6.4.2.

Remark 6.1: Theorem 6.2 provides two equivalent (and conceptually different) characterisations for the condition that $S - S^{\tau}$ satisfies NUPBR($\mathbb{G}$). One of these characterisations uses the NUPBR($\mathbb{F}$) property under P for a transformation of S, while the other characterisation is essentially based on the NUPBR($\mathbb{F}$) property for S under an absolutely continuous probability measure.

The following theorem answers (**Prob(6.II)**) for single predictable jump martingales.

Theorem 6.3: *Consider an $\mathbb{F}$-predictable stopping time T, and an honest time τ. Then, the following are equivalent*

(a) *On $\{T < +\infty\}$, we have*

$$\left\{\widetilde{Z}_T = 1\right\} \subset \{Z_{T-} = 1\}. \tag{6.6}$$

(b) *For any $\xi \in L^{\infty}(\mathcal{F}_T)$ such that $E(\xi \big| \mathcal{F}_{T-}) = 0$, the process $M - M^{\tau}$ satisfies NUPBR($\mathbb{G}$), where $M := \xi I_{[\![T,+\infty[\![}$.*

Proof. Suppose that assertion (a) holds, and consider $\xi \in L^{\infty}(\mathcal{F}_T)$ such that $E(\xi \mid \mathcal{F}_{T-}) = 0, \quad P-a.s.$ on $\{T < +\infty\}$. Splitting M as $M = I_{\{}Z_{T-} < 1\}\xi I_{[\![T,+\infty[\![} + I_{\{}Z_{T-} < 1\}\xi I_{[\![T,+\infty[\![} := M^1 + M^2$, and noting that $M^2 - (M^2)^{\tau} = 0$, one can restrict our attention to the case where $M = M^1$. Then, it is obvious that (6.6) is equivalent to

$\{\widetilde{Z}_T < 1\} = \{Z_{T-} < 1\}$, and hence

$$\widetilde{M} := I_{\{\widetilde{Z}_T<1\}} M = M \quad \text{is an } \mathbb{F}\text{-martingale.}$$

Thus, assertion (b) follows from a direct application of Theorem 6.2 to M. This ends the proof of (a)$\Rightarrow$ (b).

To prove the converse, we assume that assertion (b) holds, and we consider the $\mathcal{F}_T$-measurable and bounded r.v. $\xi := (I_{\{\widetilde{Z}_T=1\}} - P(\widetilde{Z}_T = 1|\mathcal{F}_{T-}))I_{\{T<+\infty\}}$ and the bounded $\mathbb{F}$-martingale $M := \xi I_{[\![T,+\infty[\![}$. Then, on the one hand, $M - M^\tau$ satisfies NUPBR($\mathbb{G}$). On the other hand, due to $\{T > \tau\} \subset \{\widetilde{Z}_T < 1\}$, the finite variation process

$$M - M^\tau = -P(\widetilde{Z}_T = 1|\mathcal{F}_{T-})I_{\{T>\tau\}}I_{[\![T,+\infty[\![} \ \text{ is } \mathbb{G} - \text{predictable.}$$

Thus, this finite variation process is null, or equivalently $\{Z_{T-} < 1\} \subset \{\widetilde{Z}_T < 1\}$ $P - a.s.$ on $\{T < +\infty\}$. This proves assertion (a), and the proof of the theorem is completed. □

We now extend the main result of single predictable jump to the semimartingale with countable predictable jumps that answers the question (**Prob(6.II)**). This theorem constitutes with Theorem 6.1 the building blocks for the general result that addresses (**Prob(6.II)**).

Theorem 6.4: *Suppose that τ is an honest time and S is a thin semimartingale with predictable jumps only (i.e., $S^{(qc)} \equiv 0$). If there exists a positive $\mathbb{F}$-local martingale Y, such that*

$$^{p,\mathbb{F}}\left(Y|\Delta S|\right) < +\infty \quad \textit{and} \quad ^{p,\mathbb{F}}\left(Y\Delta S I_{\{\widetilde{Z}<1 \ \& \ Z_-<1\}}\right) = 0, \tag{6.7}$$

then the process $S - S^\tau$ satisfies NUPBR($\mathbb{G}$).

The proof of this theorem is long and is based on the supermartingale deflator concept that we will define later on. Thus, this proof is postponed to Subsection 6.4.3.

6.1.3 The General Framework

With any càdlàg $\mathbb{F}$-semimartingale, X, we associate a sequence of $\mathbb{F}$-predictable stopping times $(T^n_X)_{n\geq 1}$ that exhaust all the accessible jumps of X. Throughout the chapter, we use the following notation

$$\Gamma_X := \bigcup_{n=1}^{+\infty} [\![T^n_X]\!], \qquad X^{(a)} := I_{\Gamma_X} \centerdot X, \quad X^{(qc)} := I_{\Gamma^c_X} \centerdot X := X - X^{(a)}. \tag{6.8}$$

We now state the first main result of this subsection, where we characterize pairs (S, τ) such that $S - S^\tau$ satisfies NUPBR ($\mathbb{G}$).

Theorem 6.5: *Suppose that τ satisfies (6.4). Consider $S^{(1)}$ given by (6.5). If*

(a) *The process $(1 - Z_-) \centerdot S^{(qc)} - I_{\Gamma^c} \centerdot S^{(1)}$ satisfies NUPBR($\mathbb{F}$), and*

(b) *There exists a positive $\mathbb{F}$-local martingale Y satisfying*

$$^{p,\mathbb{F}}\left(Y|\Delta S|\right) < +\infty \quad and \quad ^{p,\mathbb{F}}\left(Y \Delta S I_{\{\widetilde{Z}<1\}}\right) = 0 \quad on \quad \{Z_- < 1\},$$

then the process $S - S^\tau$ satisfies NUPBR($\mathbb{G}$).

Proof. The proof will be detailed in Section 6.4. □

As a direct consequence of Theorem 6.5, we obtain the following

Corollary 6.5.1: Suppose that τ satisfies (6.4), then the following hold.

(a) If $\{\widetilde{Z} = 1 > Z_-\} \cap \{\Delta S \neq 0\} = \emptyset$, then $S - S^\tau$ satisfies the NUPBR($\mathbb{G}$).

(b) If either m is continuous or Z is positive, then $S - S^\tau$ satisfies the NUPBR($\mathbb{G}$) for any S satisfying NUPBR($\mathbb{F}$).

(c) If S is continuous and satisfies NUPBR($\mathbb{F}$), then $S - S^\tau$ satisfies the NUPBR($\mathbb{G}$).

Proof. Apparently, assertions (b) and (c) follow from assertion (a). In the remaining part of the proof, we focus on proving assertion (a). If $\{\widetilde{Z} = 1 > Z_-\} \cap \{\Delta S \neq 0\} = \emptyset$, we get $S^{(1)} \equiv 0$. By taking Y any σ-martingale density of S (it exists since S satisfies NUPBR), the two conditions (a) and (b) of Theorem 6.5 hold. Hence $S - S^\tau$ satisfies the NUPBR($\mathbb{G}$). □

We conclude this section with our second main general result that answers (**Prob(6.II)**) in full generality and characterizes honest times τ that preserves the NUPBR.

Theorem 6.6: *Suppose that τ satisfies (6.4). Then, the following are equivalent.*

(a) *For any S satisfying NUPBR($\mathbb{F}$), the process $S - S^\tau$ satisfies NUPBR($\mathbb{G}$).*

(b) *The thin set $\{\widetilde{Z} = 1 \ \&\ Z_- < 1\}$ is evanescent.*

Proof. The implication (b) ⇒ (a) is a trivial consequence of Theorem 6.5 by taking Y any σ-martingale density for S (since S satisfies the NUPBR($\mathbb{F}$) and $S^{(1)} = 0$). To prove the reverse, we assume that assertion (a) holds, and consider $V := I_{[\![T,+\infty[\![}$ and $M = V - V^{p,\mathbb{F}}$, where T is any $\mathbb{F}$-stopping time such that $[\![T]\!] \subset \{\widetilde{Z} = 1 > Z_-\}$. Since $\{T > \tau\} \subset \{\widetilde{Z}_T < 1\}$, we derive

$$M - M^\tau = SI_{\{T>\tau\}}I_{[\![T,+\infty[\![} = -P\left(\widetilde{Z}_T = 1 \middle| \mathcal{F}_{T-}\right) I_{\{T>\tau\}}I_{[\![T,+\infty[\![},$$

and conclude that the finite variation process $M - M^\tau$ is $\mathbb{G}$-predictable. Thus, $M - M^\tau$ satisfies NUPBR($\mathbb{G}$) if and only if it is a null process. This is equivalent to

$$0 = E\left(P\left(\widetilde{Z}_T = 1 \middle| \mathcal{F}_{T-}\right) I_{\{+\infty>T>\tau\}}\right) = E\left((1 - Z_{T-})I_{\{\widetilde{Z}_T=1\}}I_{\{T<+\infty\}}\right).$$

As a result, we get $T = +\infty,\ P - a.s.$ Therefore, assertion (b) follows immediately from a combination of this and the fact that $\{\widetilde{Z} = 1 > Z_-\} \subset \{\Delta m \neq 0\}$ is a thin set. This ends the proof of the theorem. □

6.2 New Stochastic Developments

This section provides new stochastic results that constitute the key stochastic tools for the proof of the main results announced in the previous section. These results complete those elaborated in Section 5.3 of Chapter 5, where we addressed the same problems for the part up to an arbitrary random time τ. This section contains two subsections. The first subsection gives the relationship between the dual predictable projections under $\mathbb{F}$ and $\mathbb{G}$, while the second subsection defines two useful $\mathbb{G}$-semimartingales.

The results of this section are based on the following well known lemma.

Lemma 6.3: *Let M be an $\mathbb{F}$-local martingale, and τ be an honest time. Then the process $\widehat{M}^{(a)}$, defined as*

$$\widehat{M}^{(a)}_t := M_t - M_{t\wedge\tau} + \int_{\tau\wedge t}^{t} \frac{d\langle M, m\rangle^{\mathbb{F}}_s}{1 - Z_{s-}}. \tag{6.9}$$

is a $\mathbb{G}$-local martingale.

Proof. This lemma is a standard result on progressive enlargement of filtration, and we refer the reader to Jeulin [83] and Barlow [17] for proofs. □

Remark 6.2: Throughout this chapter, the process $\widehat{X}^{(a)}$ will be defined via (6.9) for any $\mathbb{F}$-local martingale X.

6.2.1 Comparing Dual Predictable Projections under $\mathbb{G}$ and $\mathbb{F}$

In the following, we start our study by writing the $\mathbb{G}$-compensators/projections in terms of $\mathbb{F}$-compensators/projections respectively.

Lemma 6.4: *Suppose that (6.4) holds, and denote*

$$J := 1 - Z_{-} - \Delta m = 1 - \widetilde{Z}\,. \tag{6.10}$$

Then the following assertions hold.

(a) *The $\mathbb{G}$-predictable process* $(1-Z_-)^{-1}I_{]\!]\tau,+\infty[\![}$ *is $\mathbb{G}$-locally bounded.*

(b) *For any $\mathbb{F}$-adapted process V, with locally integrable variation we have*

$$I_{]\!]\tau,+\infty[\![}\cdot V^{p,\mathbb{G}} = I_{]\!]\tau,+\infty[\![}\frac{1}{1-Z_-}\cdot (J\cdot V)^{p,\mathbb{F}}, \tag{6.11}$$

and on $]\!]\tau,+\infty[\![$

$${}^{p,\mathbb{G}}(\Delta V) = \frac{1}{1-Z_-}\ {}^{p,\mathbb{F}}(J\Delta V)\ . \tag{6.12}$$

(d) *For any $\mathbb{F}$-local martingale M, one has, on* $]\!]\tau,+\infty[\![$

$${}^{p,\mathbb{G}}\left(\frac{\Delta M}{1-\widetilde{Z}}\right) = \frac{{}^{p,\mathbb{F}}\left(\Delta M I_{\{\widetilde{Z}<1\}}\right)}{1-Z_-}, \quad \textit{and} \quad {}^{p,\mathbb{G}}\left(\frac{1}{J}\right) = \frac{{}^{p,\mathbb{F}}\left(I_{\{\widetilde{Z}<1\}}\right)}{1-Z_-}. \tag{6.13}$$

(e) *For any quasi-left-continuous $\mathbb{F}$-local martingale M, one has, on* $]\!]\tau,+\infty[\![$

$${}^{p,\mathbb{G}}\left(\frac{\Delta M}{J}\right) = 0, \quad \textit{and} \quad {}^{p,\mathbb{G}}\left(\frac{1}{1-Z_- - \Delta m^{(qc)}}\right) = \frac{1}{1-Z_-}, \tag{6.14}$$

where $m^{(qc)}$ is the quasi-left-continuous part (see (6.8)) of m, defined in (6.2).

Proof. The proof of the lemma will be achieved in four steps.

1) Herein, we prove assertion (a). It is known (see Chapter XX of [53]) that $Z=\widetilde{Z}$ on $]\!]\tau,+\infty[\![$, and

$$]\!]\tau,+\infty[\![\subset \{Z_- < 1\}\cap\{\widetilde{Z}<1\} = \{Z_-<1\}\cap\{Z<1\}\ .$$

Then, due to our specific assumption (6.4), we deduce that $[\![\tau,+\infty[\![\subset\{Z<1\}$, and hence the process

$$X := (1-Z)^{-1}I_{[\![\tau,+\infty[\![},$$

is càdlàg $\mathbb{G}$-adapted. Combining these with $]\!]\tau,+\infty[\![\subset\{Z_-<1\}$, we easily prove that $\sup_{0\le u\le t} X_u < +\infty,\ P-a.s.$ This is equivalent to say that X is pre-locally bounded and hence its left limit process $X_- = (1-Z_-)^{-1} I_{]\!]\tau,+\infty[\![}$ is locally bounded. This proves assertion (a).

2) This part proves assertion (b). From Lemma 6.3

$$I_{]\!]\tau,+\infty[\![} \centerdot V - I_{]\!]\tau,+\infty[\![} \centerdot V^{p,\mathbb{F}} + I_{]\!]\tau,+\infty[\![}(1-Z_-)^{-1} \centerdot \langle V, m\rangle^{\mathbb{F}}$$

is a $\mathbb{G}$-local martingale. Hence

$$\begin{aligned}
\left(I_{]\!]\tau,+\infty[\![} \centerdot V\right)^{p,\mathbb{G}} &= I_{]\!]\tau,+\infty[\![} \centerdot V^{p,\mathbb{F}} - I_{]\!]\tau,+\infty[\![}(1-Z_-)^{-1} \centerdot \langle V, m\rangle^{\mathbb{F}}, \\
&= I_{]\!]\tau,+\infty[\![} \centerdot V^{p,\mathbb{F}} - I_{]\!]\tau,+\infty[\![}(1-Z_-)^{-1} \centerdot (\Delta m \centerdot V)^{p,\mathbb{F}}, \\
&= I_{]\!]\tau,+\infty[\![}(1-Z_-)^{-1} \centerdot \Big((1-Z_- - \Delta m) \centerdot V\Big)^{p,\mathbb{F}},
\end{aligned}$$

where the second equality follows from $[V, m] = \Delta m \centerdot V$ (see [78]). This ends the proof of (6.11). The equality (6.12) follows immediately from (6.11) by taking the jumps in both sides.

3) Now, we prove assertion (c). By applying (6.11) for $V_{\epsilon,\delta} \in \mathcal{A}_{loc}(\mathbb{F})$ given by

$$V_{\epsilon,\delta} := \sum \frac{\Delta M}{J} I_{\{|\Delta M|\geq \epsilon,\ J\geq\delta\}},$$

we get, on $]\!]\tau,+\infty[\![$,

$${}^{p,\mathbb{G}}\left(\frac{\Delta M}{J} I_{\{|\Delta M|\geq \epsilon,\ J\geq\delta\}}\right) = (1-Z_-)^{-1}\ {}^{p,\mathbb{F}}\left(\Delta M I_{\{|\Delta M|\geq \epsilon,\ J\geq\delta\}}\right).$$

Then, by letting ϵ and δ go to zero, on $]\!]\tau,+\infty[\![$ we obtain

$${}^{p,\mathbb{G}}\left(\frac{\Delta M}{J}\right) = (1-Z_-)^{-1}\ {}^{p,\mathbb{F}}\left(\Delta M I_{\{J>0\}}\right).$$

This proves the first equality in (6.13). To prove the remaining equality in (6.13), we write that, on $]\!]\tau,+\infty[\![$,

$$\begin{aligned}
{}^{p,\mathbb{G}}\left(\frac{1}{J}\right) &= (1-Z_-)^{-1} + (1-Z_-)^{-1}\ {}^{p,\mathbb{G}}\left(\frac{\Delta m}{J}\right) \\
&= (1-Z_-)^{-1} + (1-Z_-)^{-2}\ {}^{p,\mathbb{F}}\left(\Delta m I_{\{J>0\}}\right) \\
&= (1-Z_-)^{-1} - (1-Z_-)^{-1}\ {}^{p,\mathbb{F}}\left(I_{\{J=0\}}\right) = (1-Z_-)^{-1}\ {}^{p,\mathbb{F}}\left(I_{\{\widetilde{Z}<1\}}\right),
\end{aligned}$$

where the second equality follows from the first equality in (6.13) and the third equality from the fact that ${}^{p,\mathbb{F}}(\Delta m) = 0$. This ends the proof of assertion (c).

4) The proof the first equality in (6.14) follows immediately from assertion (c) and the fact that the thin process ${}^{p,\mathbb{F}}\left(\Delta M I_{\{J>0\}}\right)$ may take nonzero values on countably many predictable stopping times only, on which ΔM already vanishes. A direct application of this first equality implies that on $\{Z_- < 1\}$, we have

$${}^{p,\mathbb{F}}\left(\frac{1}{1-Z_- - \Delta m^{(qc)}}\right) = \frac{1}{1-Z_-} + \frac{1}{1-Z_-}\ {}^{p,\mathbb{F}}\left(\frac{\Delta m^{(qc)}}{1-Z_- - \Delta m^{(qc)}}\right) = \frac{1}{1-Z_-}.$$

This completes the proof of (6.14) as well as the proof of the lemma. ☐

The next lemma focuses on the integrability of the process $J^{-1}I_{]\!]\tau,+\infty[\![}$ with respect to any process with $\mathbb{F}$-locally integrable variation. As a result, we complete our comparison of $\mathbb{G}$ and $\mathbb{F}$ compensators.

Lemma 6.5: *Let V be a càdlàg $\mathbb{F}$-adapted process and*

$$U := (1-Z)^{-1}\, I_{]\!]\tau,+\infty[\![} \centerdot V = J^{-1} I_{]\!]\tau,+\infty[\![} \centerdot V\,. \tag{6.15}$$

Then, the following assertions hold.

(a) *If V is nondecreasing and locally integrable (respectively integrable), then U is $\mathbb{G}$-locally integrable (respectively $\mathbb{G}$-integrable).*

(b) *If $V \in \mathcal{A}_{loc}(\mathbb{F})$, then U is a well defined process, $U \in \mathcal{A}_{loc}(\mathbb{G})$ and*

$$(U)^{p,\mathbb{G}} = I_{]\!]\tau,+\infty[\![}(1-Z_-)^{-1} \centerdot (I_{\{\widetilde{Z}<1\}} \centerdot V)^{p,\mathbb{F}}. \tag{6.16}$$

Proof. (a) Let $(\vartheta_n)_{n\geq 1}$ be a sequence of $\mathbb{F}$-stopping times that increases to $+\infty$ such that

$$E\left(V_{\vartheta_n}\right) < +\infty.$$

Then assertion (a) follows immediately if we prove $E\left(U_{\vartheta_n}\right) \leq E\left(V_{\vartheta_n}\right)$. To this end, we use the fact that $Z = \widetilde{Z}$ after τ to obtain

$$\begin{aligned} E\left(U_{\vartheta_n}\right) &= E\left(\int_0^{\vartheta_n} I_{\{t>\tau\}} \frac{1}{1-\widetilde{Z}_t} dV_t\right) = E\left(\int_0^{\vartheta_n} \frac{P(\tau < t|\mathcal{F}_t)}{1-\widetilde{Z}_t} I_{\{\widetilde{Z}_t<1\}} dV_t\right) \\ &= E\left(\int_0^{\vartheta_n} I_{\{\widetilde{Z}_s<1\}} dV_s\right) \leq E\left(V_{\vartheta_n}\right). \end{aligned}$$

Since ϑ_n increases to $+\infty$, we get $U \in \mathcal{A}^+_{loc}(\mathbb{G})$. This proves assertion (a).
(b) Suppose that $V \in \mathcal{A}_{loc}(\mathbb{F})$. Then, $\mathrm{Var}(V) = V^+ + V^- \in \mathcal{A}^+_{loc}(\mathbb{F})$, and due to assertion (a), we deduce that U has a $\mathbb{G}$-local integrable variation that coincides with

$$\mathrm{Var}(U) = (1-Z)^{-1} I_{]\!]\tau,+\infty[\![} \centerdot \mathrm{Var}(V).$$

For any $n \geq 1$, introduce

$$U_n := (1-Z)^{-1} I_{]\!]\tau,+\infty[\![} I_{\{\widetilde{Z}\leq 1-\frac{1}{n}\}} \centerdot V = \left(1-\widetilde{Z}\right)^{-1} I_{]\!]\tau,+\infty[\![} I_{\{\widetilde{Z}\leq 1-\frac{1}{n}\}} \centerdot V.$$

Then, thanks to (6.11), we derive

$$U^{p,\mathbb{G}} = \lim_{n\longrightarrow+\infty} (U_n)^{p,\mathbb{G}} = \lim_{n\longrightarrow+\infty} (1-Z_-)^{-1} I_{]\!]\tau,+\infty[\![} \centerdot \left(I_{\{\widetilde{Z}\leq 1-\frac{1}{n}\}} \centerdot V\right)^{p,\mathbb{F}}.$$

This clearly implies (6.16), and the proof of the lemma is completed. □

6.2.2 An Important $\mathbb{G}$-Local Martingale

In this subsection, we introduce a $\mathbb{G}$-local martingales that will play the role of deflator for a class of $\mathbb{F}$-local martingale. The construction of this local martingale relies on the following.

Proposition 6.2: *Let τ be an honest time satisfying (6.4). Then, the following assertions hold.*

(a) *The nondecreasing process*

$$W_t^{\mathbb{G}} := \sum_{0<u\leq t} {}^{p,\mathbb{F}}\left(I_{\{\widetilde{Z}=1\}}\right)_u I_{\{u>\tau\}}, \tag{6.17}$$

is $\mathbb{G}$-predictable, càdlàg, and locally bounded.

(b) *The nonnegative and $\mathbb{G}$-predictable process $\left(1-\Delta W^{\mathbb{G}}\right)^{-1}$ is locally bounded.*

Proof. a) Since the process $(1-Z_-)^{-1}I_{]\!]\tau,+\infty[\![}$ is $\mathbb{G}$-locally bounded under assumption (6.4), then there exists a sequence of $\mathbb{G}$-stopping times $(\tau_n)_{n\geq 1}$ that increases to infinity such that

$$\left((1-Z_-)^{-1}I_{]\!]\tau,+\infty]\!]}\right)^{\tau_n} \leq n.$$

Consider a sequence of $\mathbb{F}$-stopping times, $(\sigma_n)_{n\geq 1}$, that increases to infinity and $\langle m,m\rangle_{\sigma_n} \leq n$. Then, for any nonnegative $\mathbb{F}$-predictable process H which is bounded by $C>0$, we derive

$$\begin{aligned}
(H\bullet W^{\mathbb{G}})_{\sigma_n\wedge\tau_n} &\leq \sum_{0\leq u\leq\sigma_n} H_u\ {}^{p,\mathbb{F}}\left(I_{\{\widetilde{Z}=1>Z_-\}}\right)_u I_{\{1-Z_{u-}\geq\frac{1}{n}\}}\\
&= \sum_{0\leq u\leq\sigma_n} H_u\ {}^{p,\mathbb{F}}\left(I_{\{\Delta m=1-Z_-\geq\frac{1}{n}\}}I_{\{Z_-\leq 1-\frac{1}{n}\}}\right)_u\\
&\leq \sum_{0\leq u\leq\sigma_n} H_u\ {}^{p,\mathbb{F}}\left(I_{\{\Delta m\geq\frac{1}{n}\}}\right)_u \leq (n)^2 H\bullet\langle m,m\rangle_{\sigma_n} \leq C(n)^3.
\end{aligned}$$

This ends the proof of the assertion (a).

(b) We first calculate

$$1-\Delta W^{\mathbb{G}} = 1- {}^{p,\mathbb{F}}\left(I_{\{\widetilde{Z}=1\}}\right) I_{]\!]\tau,+\infty[\![} = I_{]\!]0,\tau]\!]} + {}^{p,\mathbb{F}}\left(I_{\{\widetilde{Z}<1\}}\right) I_{]\!]\tau,+\infty[\![} \geq I_{]\!]0,\tau]\!]} + (1-Z_-) I_{]\!]\tau,+\infty[\![}.$$

The last inequality follows from $1-\widetilde{Z} \leq I_{\{\widetilde{Z}<1\}}$. Thus, we deduce that

$$(1-\Delta W^{\mathbb{G}})^{-1} \leq I_{]\!]0,\tau]\!]} + (1-Z_-)^{-1} I_{]\!]\tau,+\infty[\![},$$

which is locally bounded due to Lemma 6.4–(a). This proves assertion (b), and the proof of the proposition is completed. □

For the construction of the local martingale deflator, we will use, as in Chapter 5, the optional integral.

Proposition 6.3: *Suppose that τ satisfies (6.4) and consider the $\mathbb{F}$-martingale m in (6.2) and $\widehat{m}^{(a)}$ defined via (6.9). Then, the process $K^{(a)}$ given by*

$$K^{(a)} := \frac{(1-Z_-)^2}{(1-Z_-)^2 + \Delta\langle m\rangle^{\mathbb{F}}} \frac{1}{1-\widetilde{Z}} I_{]\!]\tau,+\infty[\![}, \tag{6.18}$$

belongs to ${}^{o}L^1_{loc}(\widehat{m}^{(a)}, \mathbb{G})$ defined in Definition 2.12.

Furthermore, the associated $\mathbb{G}$-local martingale,

$$L^{\mathbb{G}} := K^{(a)} \odot \widehat{m}^{(a)}, \tag{6.19}$$

satisfies the following

(a) $\mathcal{E}(L^{\mathbb{G}}) > 0$ *(or equivalently $1+\Delta L^{\mathbb{G}} > 0$) and* $I_{]\!]0,\tau]\!]} \cdot L^{\mathbb{G}} = 0$.

(b) *Consider any $M \in \mathcal{M}_{0,loc}(\mathbb{F})$ and $\widehat{M}^{(a)}$ defined in (6.9). Then,*

$$[L^{\mathbb{G}}, \widehat{M}^{(a)}] \in \mathcal{A}_{loc}(\mathbb{G}) \quad \left(i.e.\ \langle L^{\mathbb{G}}, \widehat{M}^{(a)}\rangle^{\mathbb{G}} \text{ exists}\right). \tag{6.20}$$

Proof. This proof contains two steps. In the first step, we prove the optional integral

$L^{\mathbb{G}}$ in (6.19) is well-defined; while for the second step, we prove the remaining assertions.

Step 1: Thanks to (6.13), we derive

$$0 \leq {}^{p,\mathbb{G}}\left(K^{(a)}\right) \leq \frac{(1-Z_-)}{(1-Z_-)^2 + \Delta\langle m\rangle^{\mathbb{F}}} I_{]\!]\tau,+\infty[\![} \leq \frac{1}{1-Z_-}.$$

Thus, we conclude that $K \in {}^{o}L^1_{loc}(\widehat{m}^{(a)}, \mathbb{G})$ if and only if

$$\left(\sum\left(\frac{1}{J(1-Z_-)}\Delta\widehat{m}^{(a)} - {}^{p,\mathbb{G}}\left(\frac{1}{J(1-Z_-)}\Delta\widehat{m}^{(a)}\right)\right)^2\right)^{1/2} \in \mathcal{A}^+_{loc}(\mathbb{G}). \tag{6.21}$$

To prove this fact, on $]\!]\tau,+\infty[\![$, we calculate, making use of Lemma 6.4

$$\begin{aligned}\frac{\Delta\widehat{m}^{(a)}}{J(1-Z_-)} - {}^{p,\mathbb{G}}\left(\frac{\Delta\widehat{m}^{(a)}}{J(1-Z_-)}\right) &= \frac{\Delta m}{(1-Z_-)J} + (1-Z_-)^{-2}\frac{1}{J}\Delta\langle m\rangle^{\mathbb{F}} \\ &\quad -(1-Z_-)^{-3}\,\Delta\langle m\rangle^{\mathbb{F}}\ {}^{p,\mathbb{F}}(I_{\{J>0\}}) - (1-Z_-)^{-2}\ {}^{p,\mathbb{F}}(\Delta m I_{\{J>0\}}) \\ &= \frac{\Delta m}{1-\widetilde{Z}} + {}^{p,\mathbb{F}}\left(I_{\{\widetilde{Z}=1\}}\right).\end{aligned} \tag{6.22}$$

Therefore, in virtue of Proposition 6.2, the proof of (6.21) follows from

$$\sqrt{\sum\left(\frac{\Delta m}{1-\widetilde{Z}}\right)^2 I_{]\!]\tau,+\infty[\![}} \in \mathcal{A}^+_{loc}(\mathbb{G}). \tag{6.23}$$

To prove this property, we put

$$\Sigma := \{\Delta m \leq (1-\delta)(1-Z_-)\} \quad \text{and} \quad W_1 := \sum\left(\frac{\Delta m}{1-\widetilde{Z}}\right)^2 I_{]\!]\tau,+\infty[\![}.$$

Remark that $I_{\Sigma}\cdot W_1 \leq \delta^{-2}(1-Z_-)^{-1} I_{]\!]\tau,+\infty[\![}\cdot[m,m] \in \mathcal{A}^+_{loc}(\mathbb{G})$ and

$$\sqrt{I_{\Sigma^c} I_{\{Z_-\leq\alpha\}}\cdot W} \leq \sum\frac{|\Delta m|}{1-\widetilde{Z}} I_{]\!]\tau,+\infty[\![} I_{\{\Delta m>(1-\delta)\alpha\}} \in \mathcal{A}^+_{loc}(\mathbb{G}),$$

for any $\alpha \in (0,1)$. Thus, for any $\alpha \in (0,1)$, the process $\sqrt{I_{\{Z_-\leq\alpha\}} \cdot W_1} \in \mathcal{A}^+_{loc}(\mathbb{G})$. Since the process $(1-Z_-)^{-1}I_{]\!]\tau,+\infty[\![}$ is $\mathbb{G}$-locally bounded, then there exists a sequence of $\mathbb{G}$-stopping times, $(\sigma_n)_{n\geq 1}$, that increases to infinity such that on $]\!]\tau,+\infty[\![\cap[\![0,\sigma_n]\!]$, we have $Z_- \leq 1-1/(n+1)$. This implies that for each n, we have $\sqrt{W_1^{\sigma_n}} \in \mathcal{A}^+_{loc}(\mathbb{G})$, and hence $\sqrt{W_1} \in \left(\mathcal{A}^+_{loc}(\mathbb{G})\right)_{loc} = \mathcal{A}^+_{loc}(\mathbb{G})$. This proves (6.23).

Step 2: Assertion (a) follows from $I_{]\!]0,\tau]\!]} \centerdot \widehat{m}^{(a)} = 0$ and (6.22) implies that

$$1 + \Delta L^{\mathbb{G}} = 1 + \frac{\Delta m}{1-\widetilde{Z}} I_{]\!]\tau,+\infty[\![} + {}^{p,\mathbb{F}}\left(I_{\{\widetilde{Z}=1\}}\right) I_{]\!]\tau,+\infty[\![} > 0.$$

We now prove (6.20). Due to Proposition 2.3 and the $\mathbb{G}$-local boundedness of $(1-Z_-)^{-2}I_{]\!]\tau,+\infty[\![}$ (see Lemma 6.4(a)), we deduce that $[L^{\mathbb{G}}, \widehat{M}^{(a)}] \in \mathcal{A}_{loc}(\mathbb{G})$ if and only if $J^{-1} \centerdot [\widehat{m}, \widehat{M}^{(a)}] \in \mathcal{A}_{loc}(\mathbb{G})$. We calculate this quantity

$$J^{-1} \centerdot [\widehat{m}^{(a)}, \widehat{M}^{(a)}] = J^{-1}I_{]\!]\tau,+\infty[\![} \centerdot [m,M] + J^{-1}(1-Z_-)^{-1}\Delta m I_{]\!]\tau,+\infty[\![} \centerdot \langle m, M\rangle^{\mathbb{F}}$$

$$+\frac{\Delta M}{J(1-Z_-)} I_{]\!]\tau,+\infty[\![} \centerdot \langle m\rangle^{\mathbb{F}} + \frac{\Delta \langle m\rangle^{\mathbb{F}}}{J(1-Z_-)^2} I_{]\!]\tau,+\infty[\![} \centerdot \langle m, M\rangle^{\mathbb{F}}.$$

On the one hand,

$$(1-Z_-)^{-1}\Delta m I_{]\!]\tau,+\infty[\![},\ \Delta\langle m\rangle^{\mathbb{F}}(1-Z_-)^{-2}I_{]\!]\tau,+\infty[\![} \text{ and } (1-Z_-)^{-1}I_{]\!]\tau,+\infty[\![}$$

are $\mathbb{G}$-locally bounded. On the other hand, since $[m,M]$, $\Delta M \centerdot \langle m\rangle^{\mathbb{F}}$ and $\langle m, M\rangle^{\mathbb{F}}$ belong to $\mathcal{A}_{loc}(\mathbb{F})$, thanks to Lemma 6.5–(a) we conclude that

$$J^{-1}I_{]\!]\tau,+\infty[\![} \centerdot [m,M],\ J^{-1}I_{]\!]\tau,+\infty[\![} \centerdot \left(\Delta M \centerdot \langle m\rangle^{\mathbb{F}}\right) \text{ and } J^{-1}I_{]\!]\tau,+\infty[\![} \centerdot \langle m, M\rangle^{\mathbb{F}}$$

belong to $\mathcal{A}_{loc}(\mathbb{G})$. Therefore, this achieves the proof of (6.20), and the proof of the proposition is completed. $\square$

6.3 Explicit Deflators

We start this section by constructing explicitly local martingale densities (deflators) under $\mathbb{G}$ for a class of $\mathbb{F}$-local martingales. This will be achieved in Subsections 6.3.1 and 6.4.1, where the cases of quasi-left-continuous and thin with predictable jumps are addressed respectively.

6.3.1 The Quasi-Left-Continuous case

Theorem 6.7: *Let τ be an honest time satisfying (6.4), and $L^{\mathbb{G}}$ be the $\mathbb{G}$-local martingale defined by (6.19). Let M be a quasi-left-continuous $\mathbb{F}$-local martingale such that $\{\widetilde{Z} = 1 > Z_- \ \&\ \Delta M \neq 0\}$ is evanescent. Then, the following two assertions are equivalent and hold.*

(a) $\mathcal{E}(L^{\mathbb{G}})\,(M - M^{\tau})$ *is a $\mathbb{G}$-local martingale.*

(b) $\mathcal{E}\left(I_{\{\widetilde{Z}=1>Z_-\}} \odot m\right) M$ *is an $\mathbb{F}$-local martingale.*

Proof. Suppose that $\{\widetilde{Z} = 1 > Z_- \ \&\ \Delta M \neq 0\}$ is evanescent. Then

$$\sum I_{\{\widetilde{Z}=1>Z_-\}} \Delta M(1 - Z_-) = I_{\{\widetilde{Z}=1>Z_-\}} \centerdot [M, m] \equiv 0,$$

and due to the quasi-left-continuity of M and Proposition 2.3-(b), we derive

$$\langle I_{\{\widetilde{Z}=1>Z_-\}} \odot m, M\rangle^{\mathbb{F}} = \left(I_{\{\widetilde{Z}=1>Z_-\}} \centerdot [M, m]\right)^{p,\mathbb{F}} = 0,$$

which implies assertion (b). To achieve the proof of the theorem, we will prove that assertions (a) and (b) are equivalent.

Notice that the quasi-left-continuity of M implies

$$[\langle m\rangle, M] = [m, \langle m, M\rangle] = [\langle m\rangle, \langle m, M\rangle] = 0.$$

Hence,

$$K^{(a)} \centerdot [\widehat{m}^{(a)}, \widehat{M}^{(a)}] = K^{(a)} \centerdot [m, M] = \frac{1}{1-\widetilde{Z}} I_{]\!]\tau,+\infty[\![} \centerdot [m, M], \tag{6.24}$$

where $K^{(a)}$ is defined in (6.18). Then, by combining (6.24), Proposition 2.3-(b) and (6.11), we calculate

$$\begin{aligned}\langle L^{\mathbb{G}}, \widehat{M}^{(a)} \rangle^{\mathbb{G}} &= \left(K^{(a)} \centerdot [\widehat{m}^{(a)}, \widehat{M}^{(a)}]\right)^{p,\mathbb{G}} = \left((1-\widetilde{Z})^{-1} I_{]\!]\tau,+\infty[\![} \centerdot [m, M]\right)^{p,\mathbb{G}} \\ &= \frac{I_{]\!]\tau,+\infty[\![}}{1-Z_-} \centerdot \langle m, M\rangle^{\mathbb{F}} - \frac{I_{]\!]\tau,+\infty[\![}}{1-Z_-} \centerdot \left(I_{\{\widetilde{Z}=1>Z_-\}} \centerdot [M, m]\right)^{p,\mathbb{F}} \\ &= \frac{I_{]\!]\tau,+\infty[\![}}{1-Z_-} \centerdot \langle m, M\rangle^{\mathbb{F}} - \frac{I_{]\!]\tau,+\infty[\![}}{1-Z_-} \centerdot \langle I_{\{\widetilde{Z}=1>Z_-\}} \odot m, M\rangle^{\mathbb{F}}.\end{aligned}$$

Then, from the above equality, the equivalence between the two assertions (a) and (b) follows immediately. □

As an immediate consequence of this theorem, we describe a class of $\mathbb{F}$-quasi-left-continuous processes for which the NUPBR is preserved for the part after τ.

Corollary 6.7.1: Suppose that τ satisfies (6.4), and that S is $\mathbb{F}$-quasi-left-continuous and satisfies NUPBR($\mathbb{F}$) and $\{\Delta S \neq 0\} \cap \{\widetilde{Z} = 1 < 1\}$ is evanescent. Then, $S - S^{\tau}$ satisfies NUPBR($\mathbb{G}$).

Proof. The proof follows immediately from a combination of Theorem 6.7 and the fact that

$$\{\widetilde{Z} = 1 < Z_-\} = \{\widetilde{Z}^Q = 1 < Z^Q_-\}, \quad \text{for any } Q \sim P, \tag{6.25}$$

where $\widetilde{Z}^Q_t := Q(\tau \geq t \big| \mathcal{F}_t)$ and $Z^Q_t := Q(\tau > t | \mathcal{F}_t)$. This last fact is an immediate application of Theorem 86 of [54] by taking $X = I_{\{\widetilde{Z}=0\}}$ & $Y = I_{\{\widetilde{Z}^Q=0\}}$ and $X = I_{\{Z_-=0\}}$ & $Y = I_{\{Z^Q_-=0\}}$. □

6.3.2 The Case of Thin Processes with Predictable Jumps

The construction of deflators for thin $\mathbb{F}$-local martingales requires the following results that are interesting in themselves.

Lemma 6.6: *Let T be a finite $\mathbb{F}$-predictable stopping time. Then the following holds.*

$$\{T>\tau\}\ \subset\ \{\widetilde{Z}_T<1\}\ \subset\ \{Z_{T-}<1\}=\ \Gamma(T):=\left\{P\left(\widetilde{Z}_T<1\middle|\mathcal{F}_{T-}\right)>0\right\}.$$

Proof. It is enough to prove the non-trivial equality $\{Z_{T-}<1\}=\Gamma(T)$. Indeed, due to $E\left(P(\widetilde{Z}_T<1|\mathcal{F}_{T-})I_{\{Z_{T-}=1\}}\right)=P(\widetilde{Z}_T<1=Z_{T-})=0$, we get $\Gamma(T)\subset\{Z_{T-}<1\}$. On the other hand, due to

$$E\left((1-Z_{T-})I_{\Gamma(T)^c}\right)=E\left((1-\widetilde{Z}_T)\ I_{\Gamma(T)^c}\right)\le E\left(I_{\{\widetilde{Z}_T<1\}}\ I_{\Gamma(T)^c}\right)=0,$$

we obtain $\{Z_{T-}<1\}\subset\Gamma(T)$. This ends the proof of the lemma. □

Lemma 6.7: *Let R be an equivalent probability to P. Then the following hold.*

$$\{\widetilde{Z}=1\}=\{\widetilde{Z}^R=1\},\quad \textit{and}\quad \{Z_-=1\}=\{Z^R_-=1\},$$

where $\widetilde{Z}^R_t:=R(\tau\ge t|\mathcal{F}_t)$ and Z^R_- is the $(R,\mathbb{F})$-predictable projection of $\widetilde{Z}^R$.

Proof. For any $\mathbb{F}$-stopping time σ and any $\mathbb{F}$-predictable stopping time T, due to

$$E\left[(1-\widetilde{Z}_\sigma)I_{\{\widetilde{Z}^R_\sigma=1\}}\right]=E\left[I_{\{\tau<\sigma\}}I_{\{\widetilde{Z}^R_\sigma=1\}}\right]=0,\ \text{and}$$
$$E\left[(1-Z_{T-})I_{\{Z^R_{T-}=1\}}\right]=E\left[I_{\{\tau<T\}}I_{\{\widetilde{Z}^R_{T-}=1\}}\right]=0,$$

we obtain $\{\widetilde{Z}^R=1\}\subset\{\widetilde{Z}=1\}$ and $\{Z^R_-=1\}\subset\{Z_-=1\}$. The symmetric roles of R and P complete the proof of the lemma. □

Theorem 6.8: *Let τ be an $\mathbb{F}$-honest time. Consider an $\mathbb{F}$-predictable stopping time T and an $\mathcal{F}_T$-measurable random variable ξ such that $E[|\xi||\mathcal{F}_{T-}]<+\infty, P$-a.s.*

Define $M := \xi I_{\{Z_{T-}<1\}} I_{[\![T,+\infty[\![}$,

$$\frac{dQ_T^{\mathbb{F}}}{dP} := D^{\mathbb{F}} := \frac{I_{\{\widetilde{Z}_T<1 \ \& \ P(\widetilde{Z}_T<1|\mathcal{F}_{T-})>0\}}}{P(\widetilde{Z}_T<1|\mathcal{F}_{T-})} + I_{\{P(\widetilde{Z}_T<1|\mathcal{F}_{T-})=0\}}, \quad \textit{and}$$
$$\frac{dQ_T^{\mathbb{G}}}{dP} := D^{\mathbb{G}} := \frac{1-Z_{T-}}{(1-\widetilde{Z}_T)P(\widetilde{Z}_T<1|\mathcal{F}_{T-})} I_{\{T>\tau\}} + I_{\{T\leq\tau\}}. \tag{6.26}$$

Then the following assertions are equivalent.

(a) M *is a* $(Q_T^{\mathbb{F}}, \mathbb{F})$*-martingale.*

(b) *On* $\{Z_{T-} < 1\}$*, we have*

$$E\left(\xi I_{\{\widetilde{Z}_T<1\}} \mid \mathcal{F}_{T-}\right) = 0, \qquad P-a.s. \tag{6.27}$$

(c) $(M - M^\tau)$ *is a* $(Q_T^{\mathbb{G}}, \mathbb{G})$*-martingale.*

Proof. The proof of the theorem will be achieved in two steps. In Step 1, we prove (a)$\Longleftrightarrow$(b); while Step 2 proves (b)$\Longleftrightarrow$(c). First we remark that the two probability measures $Q_T^{\mathbb{F}}$ and $Q_T^{\mathbb{G}}$ in (6.26) are well defined due to Lemma 6.6.

Step 1: Herein, we will prove (a)$\Longleftrightarrow$(b). Thanks to $E[D^{\mathbb{F}}|\mathcal{F}_{T-}] = 1$, we obtain

$$E^{Q_T^{\mathbb{F}}}[\xi I_{\{Z_{T-}<1\}}|\mathcal{F}_{T-}] = E\left[D^{\mathbb{F}} \xi I_{\{Z_{T-}<1\}}|\mathcal{F}_{T-}\right] = \frac{E\left[\xi I_{\{\widetilde{Z}_T<1\}}|\mathcal{F}_{T-}\right]}{P(\widetilde{Z}_T<1|\mathcal{F}_{T-})} I_{\{Z_{T-}<1\}}.$$

Therefore, the equivalence between assertions (a) and (b) follows from a combination of this equality and the fact that M is a $(Q_T^{\mathbb{F}}, \mathbb{F})$-martingale if and only if $E^{Q_T^{\mathbb{F}}}(M_T \mid \mathcal{F}_{T-}) = 0$ on $\{T < +\infty\}$.

Step 2: This step will prove (b)$\Longleftrightarrow$ (c). To this end, we first notice that $M - M^\tau$ is $(Q_T^{\mathbb{G}}, \mathbb{G})$-martingale if and only if $E^{Q_T^{\mathbb{G}}}[\xi I_{\{Z_{T-}<1 \ \& \ T>\tau\}}|\mathcal{G}_{T-}] = 0$ on $\{T < +\infty\}$.

Furthermore, due to Lemma 6.6 and $E[D^{\mathbb{G}}|\mathcal{G}_{T-}]=1$, we have

$$\begin{aligned}
E^{Q_T^{\mathbb{G}}}[\xi I_{\{Z_{T-}<1\ \&\ T>\tau\}}|\mathcal{G}_{T-}] &= E\left[D^{\mathbb{G}}\xi I_{\{Z_{T-}<1\}}I_{\{T>\tau\}}|\mathcal{G}_{T-}\right]\\
&= E\left[\frac{\xi I_{\{T>\tau\}}}{1-\widetilde{Z}_T}\Big|\mathcal{G}_{T-}\right]\frac{1-Z_{T-}}{P(\widetilde{Z}_T<1|\mathcal{F}_{T-})}I_{\{Z_{T-}<1\ \&\ T>\tau\}}\\
&= \frac{E\left[\xi I_{\{\widetilde{Z}_T<1\}}|\mathcal{F}_{T-}\right]}{P(\widetilde{Z}_T<1|\mathcal{F}_{T-})}I_{\{Z_{T-}<1\}}I_{\{T>\tau\}},
\end{aligned}\tag{6.28}$$

where the last equality in (6.28) follows from the fact that

$$E\left(H\mid\mathcal{G}_{T-}\right)I_{\{T>\tau\}}=E\left(H(1-\widetilde{Z}_T)\mid\mathcal{F}_{T-}\right)\frac{I_{\{T>\tau\}}}{1-Z_{T-}},$$

for any random variable H for which the above conditional expectations exist (see Proposition 5.3 of [83] or Proposition 2.7 in Chapter 2). Therefore, if assertion (b) holds, then assertion (c) follows immediately from (6.28). Conversely, if assertion (c) holds, then $E^{Q_T^{\mathbb{G}}}[\xi I_{\{Z_{T-}<1\}}I_{\{T>\tau\}}|\mathcal{G}_{T-}]=0$. Thus, a combination of this with (6.28) leads to $E\left[\xi I_{\{\widetilde{Z}_T<1\}}|\mathcal{F}_{T-}\right](1-Z_{T-})=0$. This proves assertion (b), and the proof of the theorem is completed. ☐

Remark 6.3: It is important to notice that

$$D^{\mathbb{G}}(T):=\left(1+\frac{\Delta m_T}{1-\widetilde{Z}_T}\right)I_{\{T>\tau\}}+I_{\{T\le\tau\}}\neq 1+\Delta\widetilde{N}=\frac{\mathcal{E}_T\left(L^{\mathbb{G}}\right)}{\mathcal{E}_{T-}\left(L^{\mathbb{G}}\right)}.$$

This explains one of the main difficulties that we will face when dealing with countable many predictable jumps. In fact, it might not be possible to piece-up the sequence $\left(D^{\mathbb{G}}(T_n),\ n\ge 1\right)$ into a local martingale density when the stopping times are not ordered. This explains the fact that the proof of the general case needs a different idea and method.

Theorem 6.9: *Let M be a thin $\mathbb{F}$-local martingale (that is $M^{(qc)}\equiv 0$) such that*

$\{\Delta M \neq 0\} \cap \{\widetilde{Z} = 1 > Z_-\} = \emptyset$. *Then*

$$\mathcal{E}\left((1-\Delta W^{\mathbb{G}})^{-1} \centerdot L^{\mathbb{G}}\right)(M - M^\tau) \quad \textit{is a } \mathbb{G}\textit{-local martingale.} \tag{6.29}$$

Proof. Thanks to Itô's formula, it is immediate that (6.29) is equivalent to

$$X^{\mathbb{G}} := M - M^\tau + [(1-\Delta W^{\mathbb{G}})^{-1} \centerdot L^{\mathbb{G}}, M - M^\tau] \quad \text{is a } \mathbb{G}\text{-local martingale.} \tag{6.30}$$

It is obvious that $X^{\mathbb{G}}$ is a $\mathbb{G}$-special semimartingale since $\langle L^{\mathbb{G}}, M - M^\tau\rangle^{\mathbb{F}}$ exits (see Proposition 6.3). Hence it is enough to prove that $X^{\mathbb{G}}$ is a σ-martingale under $\mathbb{G}$. Let $(T_n)_{n\geq 1}$ be the sequence of predictable stopping times that exhaust the jumps of M. Then, thanks to Theorem 6.7, since $\{\Delta M \neq 0\} \cap \{\widetilde{Z} = 1 > Z_-\} = \emptyset$, we conclude that for any $n \geq 1$,

$$M^n := (S - S^\tau)_{T_n}\left(1 + \frac{\Delta L^{\mathbb{G}}_{T_n}}{1 - \Delta W^{\mathbb{G}}_{T_n}}\right) I_{[\![T_n,+\infty[\![} \quad \text{is a } \mathbb{G}\text{-martingale.} \tag{6.31}$$

Consider the following $\mathbb{G}$-predictable and positive process that is bounded by one

$$\Phi := \sum_{n\geq 1} \frac{2^{-n}}{1+\xi_n} I_{[\![T_n]\!]\cap]\!]\tau,+\infty[\![} + I_{\Gamma^c\cap[\![0,\tau]\!]},\ \xi_n := E\left[|\Delta S_{T_n}|(1 + \frac{|\Delta L^{\mathbb{G}}_{T_n}|}{1-\Delta W^{\mathbb{G}}_{T_n}})\big|\ \mathcal{G}_{T_n-}\right] I_{\{T_n<+\infty\}},$$

where $\Gamma := \cup_{n\geq 1}[\![T_n]\!]$. Then, it is easy to check that $\Phi \centerdot X^{\mathbb{G}}$ satisfies $E\left[\mathrm{Var}(\Phi \centerdot X^{\mathbb{G}})(\infty)\right] \leq 1$. Furthermore, due to (6.31), we calculate

$$\left(\Phi \centerdot X^{\mathbb{G}}\right)^{p,\mathbb{G}} = \sum_{n\geq 1} (\Phi \centerdot M^n)^{p,\mathbb{G}} \equiv 0.$$

This proves that $\Phi \centerdot X^{\mathbb{G}}$ is a $\mathbb{G}$-martingale. As a result, $X^{\mathbb{G}}$ is a $\mathbb{G}$-local martingale. This ends the proof of the theorem. □

Corollary 6.9.1: Suppose that S is thin, $\{\Delta S \neq 0\} \cap \{\widetilde{Z} = 1 > Z_-\} = \emptyset$, and S satisfies the NUPBR($\mathbb{F}$). Then $S - S^\tau$ satisfies the NUPBR($\mathbb{G}$).

Proof. Since S satisfies the NUPBR($\mathbb{F}$), then there exist a $\mathbb{F}$-predictable process ϕ, a sequence of $\mathbb{F}$-stopping times $(T_n)_{n\geq 1}$ that increases to infinity, and a probability measure $Q_n \sim P$ on $(\Omega, \mathcal{F}_{T_n})$ such that

$$0 < \phi \leq 1, \qquad \phi \centerdot S^{T_n} \in \mathcal{M}_{0,loc}(Q,\mathbb{F}).$$

Then, a combination of $\{\Delta S \neq 0\} \cap \{\widetilde{Z} = 1 > Z_-\} = \emptyset$ and (6.39) leads to

$$\{\Delta(\phi \centerdot S^{T_n}) \neq 0\} \cap \{\widetilde{Z}^{Q_n} = 1 > Z_-^{Q_n}\} = \emptyset.$$

Therefore, by applying directly Theorem 6.9 to $\phi \centerdot S^{T_n}$ under Q_n, we conclude that $\phi \centerdot S^{T_n} - (\phi \centerdot S^{T_n})^\tau$ has a local martingale density under $(Q_n, \mathbb{G})$. Thus, this implies that $S^{T_n} - S^{T_n \wedge \tau}$ satisfies the NUPBR($\mathbb{G}, Q_n$). Thus, the corollary follows immediately from Proposition 5.2. This ends the proof of the corollary. $\square$

6.4 Proofs of Three Main Theorems

In this section, we will prove three theorems that were not proved in Section 6.1, namely Theorems 6.1, 6.2 and 6.5. To this end, we introduce some notations that will be used throughout the rest of the chapter. Recall that μ is the random measure of the jumps of S, ν is its random measure compensator and the functional f_m is defined in (6.5). Here we put

$$\mu_{\mathbb{G}}(dt, dx) := I_{]\!]\tau,+\infty[\![}(t)\mu(dt,dx), \quad \nu_{\mathbb{G}}(dt,dx) := I_{]\!]\tau,+\infty[\![}(t)\left(1 - \frac{f_m(x,t)}{1 - Z_{t-}}\right)\nu(dt,dx). \tag{6.32}$$

It is easy to check that $\nu_{\mathbb{G}}$ is the random measure compensator under $\mathbb{G}$ of $\mu_{\mathbb{G}}$. The canonical decomposition of $S - S^\tau$ under $\mathbb{G}$ is given by

$$S - S^\tau = \widehat{S^c}^{(a)} + h\star(\mu_{\mathbb{G}} - \nu_{\mathbb{G}}) + bI_{]\!]\tau,+\infty[\![} \centerdot A - \frac{c\beta_m}{1 - Z_-} I_{]\!]\tau,+\infty[\![} \centerdot A - h\frac{f_m}{1 - Z_-} I_{]\!]\tau,+\infty[\![}\star\nu + (x - h)\star\mu_{\mathbb{G}},$$

where $\widehat{S^c}^{(a)}$ is defined by (6.9).

6.4.1 Proof of Theorem 6.1

The proof of the theorem will be achieved in two steps. Put

$$\overline{S} := (1 - Z_-) \centerdot S - S^{(1)} \qquad \text{and} \qquad \Gamma(1) := \{Z_- < 1 = Z_- + f_m\}.$$

Step 1: Suppose that $\overline{S}$ satisfies the NUPBR($\mathbb{F}$). Then, there exist an $\mathbb{F}$-local martingale $N^{\mathbb{F}}$ and an $\mathbb{F}$-predictable process ϕ such that $0 < \phi \leq 1$ and $\mathcal{E}\left(N^{\mathbb{F}}\right)\left(\phi \centerdot \overline{S}\right)$ is an $\mathbb{F}$-local martingale. Again, thanks to Theorem 5.1, we can restrict our attention to the case

$$N^{\mathbb{F}} := \beta^{\mathbb{F}} \centerdot S^c + (f^{\mathbb{F}} - 1) \star (\mu - \nu), \tag{6.33}$$

where $\beta^{\mathbb{F}} \in L(S^c)$ and $f^{\mathbb{F}}$ is a positive $\widetilde{\mathcal{P}}(\mathbb{F})$-measurable functional.

Thanks to Itô's formula, the fact that $\mathcal{E}\left(N^{\mathbb{F}}\right)\left(\phi \centerdot \overline{S}\right)$ is an $\mathbb{F}$-local martingale implies

$$k^{\mathbb{F}} := \int |x\,(1 - Z_- - f_m)\, f - h(x)| F(dx) < +\infty \qquad P \otimes A - a.e. \tag{6.34}$$

and we have

$$(1 - Z_-)(b + c\beta) + \int \left(x\,(1 - Z_- - f_m)\, f - h(x)\right) F(dx) = 0, \;\; P \otimes A - a.e.. \tag{6.35}$$

Now, we will construct a σ-martingale density for $S - S^{\tau}$ as follows. Consider

$$\beta^{\mathbb{G}} := \left(\beta + \frac{\beta_m}{1 - Z_-}\right) I_{]\!]\tau,+\infty[\![}, \qquad f^{\mathbb{G}} := \frac{f}{1 - f_m(x)/(1 - Z_-)} I_{]\!]\tau,+\infty[\![} + I_{]\!]0,\tau]\!]},$$

and assume that

$$\beta^{\mathbb{G}} \in L(\widehat{S^c}^{(a)}, \mathbb{G}) \quad \text{and} \quad (f^{\mathbb{G}} - 1) \in \mathcal{G}^1_{loc}(\mu_{\mathbb{G}}, \mathbb{G}). \tag{6.36}$$

Then, using Itô's formula and (6.34)—(6.35) afterwards, we can easily prove that $\left(\phi^{\mathbb{G}} \centerdot (S - S^{\tau})\right) \mathcal{E}(N^{\mathbb{G}})$ is a $\mathbb{G}$-local martingale, where

$$\phi^{\mathbb{G}} := \left(1 + \int ||xf(x)I_{\Gamma^c} - h(x)|F(dx)I_{]\!]\tau,+\infty[\![}\right)^{-1}.$$

Step 2: In this step, we will prove that (6.36) holds. Since $\beta^T c\beta \centerdot A$ and $\beta_m^T c\beta_m \centerdot A$ belong to $\mathcal{A}^+_{loc}(\mathbb{F}) \subset \mathcal{A}^+_{loc}(\mathbb{G})$ and $(1 - Z_-)^{-1} I_{]\!]\tau,+\infty[\![}$ is $\mathbb{G}$-locally bounded, we deduce that $\beta^{\mathbb{G}} \in L(\widehat{S^c}^{(a)}, \mathbb{G})$. To prove the second property of (6.36), we start by calculating on $]\!]\tau, +\infty[\![$,

$$f^{\mathbb{G}} - 1 = \frac{f-1}{1 - f_m/(1-Z_-)} + \frac{f_m}{1 - Z_- - f_m} =: g_1 + g_2.$$

Since $\sqrt{(f-1)^2 \star \mu} \in \mathcal{A}^+_{loc}(\mathbb{F})$ and due to Proposition 5.5 in Chapter 5, we deduce

$$\left[(f-1)^2 I_{\{|f-1|\leq\alpha\}} + |f-1| I_{\{|f-1|>\alpha\}}\right] \star \mu \in \mathcal{A}^+_{loc}(\mathbb{F}).$$

Without loss of generality we will assume that above two processes and $f_m^2 \star \mu$ are integrable. Since $(1 - Z_-)^{-1} I_{]\!]\tau,+\infty[\![}$ is $\mathbb{G}$-locally bounded, there exists a a sequence of $\mathbb{G}$-stopping times $(\tau_n)_{n\geq 1}$ that increase to infinity and $Z_-^{\tau_n} \leq 1 - 1/n$ on $]\!]\tau, +\infty[\![$. Then, we put $\Gamma_n(\alpha) := \{|f - 1| \leq \alpha \ \& \ 1 - Z_- - f_m \geq 1/(2n)\}$, and we calculate

$$E\left[g_1^2 I_{\Gamma_n(\alpha)} \star \mu_{\mathbb{G}}(\infty)\right] = E\left[g_1^2 I_{\Gamma_n(\alpha)} \star \nu_{\mathbb{G}}(\infty)\right] \leq 2nE\left[(f-1)^2 I_{\{|f-1|\leq\alpha\}} \star \nu(\infty)\right] < +\infty$$

and

$$E\left[\sqrt{g_1^2 I_{\Gamma_n(\alpha)} \star \mu_{\mathbb{G}}(\tau_n)}\right] \leq E\left[|g_1| I_{\Gamma_n(\alpha)} \star \nu_{\mathbb{G}}(\tau_n)\right]$$

$$\leq \alpha E\left[I_{\{|f_m|>\frac{1}{2n}\}} \star \nu(\infty)\right] \leq 4n^2 \alpha E\left[f_m^2 \star \nu(\infty)\right] < +\infty.$$

Also we calculate

$$E\sqrt{g_1^2 I_{\{|f-1|>\alpha\}}\star\mu_{\mathbb{G}}(\infty)}\leq E\left[|g_1|I_{\{|f-1|\leq\alpha\}}\star\nu_{\mathbb{G}}(\infty)\right]$$
$$\leq \alpha E\left[|f-1|I_{\{|f-1|\leq\alpha\}}\star\nu(\infty)\right]<+\infty.$$

This proves that $\sqrt{g_1^2\star\mu_{\mathbb{G}}}\in\mathcal{A}^+_{loc}(\mathbb{G})$. Now, similarly, we prove that $\sqrt{g_2^2\star\mu_{\mathbb{G}}}\in\mathcal{A}^+_{loc}(\mathbb{G})$.

$$E\left[g_2^2 I_{\{|f_m|\leq\alpha\ \&\ 1-Z_--f_m\geq\frac{1}{2n}\}}\star\mu_{\mathbb{G}}(\infty)\right]=E\left[g_2^2 I_{\{|f_m|\leq\alpha\ \&\ 1-Z_--f_m\geq\frac{1}{2n}\}}\star\nu_{\mathbb{G}}(\infty)\right]$$
$$\leq 2nE\left[(f_m)^2 I_{\{|f_m|\leq\alpha\}}\star\nu(\infty)\right]<+\infty$$

and

$$E\sqrt{g_2^2 I_{\{|f_m|\leq\alpha\ \&\ 1-Z_--f_m<\frac{1}{2n}\}}\star\mu_{\mathbb{G}}(\tau_n)}\leq E\left[|g_2|I_{\Sigma_n}\star\mu_{\mathbb{G}}(\tau_n)\right]$$
$$\leq E\left[|f_m|I_{\{|f_m|>\frac{1}{2n}\}}\star\nu(\infty)\right]$$
$$\leq 2nE\left[f_m^2\star\nu(\infty)\right]<+\infty,$$

where $\Sigma_n:=\{1-Z_-\geq 1/n\ \&\ 1-Z_--f_m\geq 1/(2n)\}$ and

$$E\sqrt{g_2^2 I_{\{|f_m|>\alpha\}}\star\mu_{\mathbb{G}}(\infty)}\leq E\left[|g_2|I_{\{|f_m|>\alpha\}}\star\nu_{\mathbb{G}}(\infty)\right]\leq\alpha E\left[f_m^2\star\mu(\infty)\right]<+\infty.$$

Then, a direct application of Proposition 5.5 of Chapter 5 to these processes ends the proof of this implication, and the proof of the theorem is completed. □

6.4.2 Proof of Theorem 6.2

For the reader's convenience, we state and prove a very detailed version of Theorem 6.2, where we provide explicit forms for the absolute continuous probability measure Q_T mentioned in Theorem 6.2–(c).

Theorem 6.10: *Suppose that the assumptions of Theorem 6.2 hold. Then, the* *assertions* (a) *and* (b) *of Theorem 6.2 are equivalent to the following assertions* (d) *and* (e)*:*

(d) S *satisfies NUPBR*$(\mathbb{F}, \widetilde{Q}^{\mathbb{F}}_T)$*, where*

$$\widetilde{Q}^{\mathbb{F}}_T := \Big(\frac{1-\widetilde{Z}_T}{1-Z_{T-}} I_{\{Z_{T-}<1\}} + I_{\{Z_{T-}=1\}}\Big) \cdot P. \tag{6.37}$$

(e) S *satisfies NUPBR*$(\mathbb{F}, Q^{\mathbb{F}}_T)$*, where* $Q^{\mathbb{F}}_T$ *is given by (6.26).*

Proof. The proof consists of four steps. The first step addresses (d)⇔(e), the second step focuses on (e)⇒(b), while the third step and the fourth step deal with (b)⇒ (a) and (a)⇒ (d) respectively.

Step 1: Herein, we will prove (d)⇔(e). Due to $\{Z_{T-} = 1\} = \{P(\widetilde{Z}_T < 1|\mathcal{F}_{T-}) = 0\} \subset \{\widetilde{Z}_T = 1\}$ (see Lemma 6.6), it is easy to see that $\widetilde{Q}^{\mathbb{F}}_T \sim Q^{\mathbb{F}}_T \ll P$. Therefore, the equivalence between (d) and (e) follows immediately.

Step 2: This step focuses on proving (e)⇒(b). Suppose (e) holds. Then, there exists an $\mathcal{F}_T$-measurable random variable $Y_T > 0$, $Q^{\mathbb{F}}_T$-a.s. such that $E^{Q^{\mathbb{F}}_T}[S_T Y_T|\mathcal{F}_{T-}] = 0$, or equivalently

$$E[\xi Y_T I_{\{\widetilde{Z}_T<1\}}|\mathcal{F}_{T-}] I_{\{Z_{T-}<1\}} = 0 \ \text{ and } \ E[\xi Y_T|\mathcal{F}_{T-}] I_{\{Z_{T-}=1\}} = 0. \tag{6.38}$$

Since, on the set $\{Z_{T-} = 1\}$, $\widetilde{S} \equiv 0$, it is enough to focus on the part corresponding to $\{Z_{T-} < 1\}$. Put

$$\widetilde{Y}_T := Y_T I_{\{\widetilde{Z}_T<1\}} + I_{\{\widetilde{Z}_T=1\}} \quad \text{and} \ \ Q_1 := \widetilde{Y}_T / E(\widetilde{Y}_T|\mathcal{F}_{T-}) \cdot P \sim P.$$

Then, from (6.38), we derive that $E^{Q_1}[\widetilde{S}_T|\mathcal{F}_{T-}] = E^{Q_1}[\xi I_{\{\widetilde{Z}_T<1\}}|\mathcal{F}_{T-}] = 0$. Therefore, $\widetilde{S}$ is a $(Q_1, \mathbb{F})$-martingale and assertion (b) follows.

Step 3: This step proves (b)⇒ (a). Suppose that $\widetilde{S}$ satisfies NUPBR($\mathbb{F}$). Then, there exists an $\mathcal{F}_T$-measurable $Y_3 > 0$ such that $E[Y_3\xi I_{\{\widetilde{Z}_T<1\}}|\mathcal{F}_{T-}] = 0$. Put $Q_3 := Y_3/E(Y_3|\mathcal{F}_{T-}) \cdot P \sim P$ and remark that $\{\widetilde{Z}_T < 1\} = \{\widetilde{Z}^{Q_3}_T < 1\}$, where $\widetilde{Z}^{Q_3}_t := Q_3(\tau \geq t|\mathcal{F}_t)$. Therefore, a direct application of Theorem 6.8 under Q_3, we conclude that $S - S^\tau = \widetilde{S} - \widetilde{S}^\tau$ satisfies NUPBR($\mathbb{G}$), and assertion (a) holds.

Step 4: This last step will prove (a)⇒ (d). Suppose $S - S^\tau$ satisfies NUPBR($\mathbb{G}$). There exists a positive $\mathcal{G}_T$-measurable $Y^{\mathbb{G}}$ such that $E[\xi Y^{\mathbb{G}} I_{\{T>\tau\}}|\mathcal{G}_{T-}] = 0$. Then, thanks to Jeulin [83] (Proposition 5.3), we deduce the existence of a positive and $\mathcal{F}_T$-measurable $Y^{\mathbb{F}}$ such that $Y^{\mathbb{G}} I_{\{T>\tau\}} = Y^{\mathbb{F}} I_{\{T>\tau\}}$. Then, we calculate

$$\begin{aligned} 0 = E[\xi Y^{\mathbb{G}} I_{\{T>\tau\}}|\mathcal{G}_{T-}] &= E[\xi Y^{\mathbb{F}}(1-\widetilde{Z}_T)|\mathcal{F}_{T-}] \frac{I_{\{T>\tau\}}}{1-Z_{T-}} \\ &= E^{\widetilde{Q}^{\mathbb{F}}_T}\left(\xi Y^{\mathbb{F}}\big|\ \mathcal{F}_{T-}\right) I_{\{T>\tau\}} = E^{\widetilde{Q}^{\mathbb{F}}_T}\left(\xi \widetilde{Y}^{\mathbb{F}}\big|\ \mathcal{F}_{T-}\right) I_{\{T>\tau\}}, \end{aligned}$$

where $\widetilde{Y}^{\mathbb{F}} := Y^{\mathbb{F}} I_{\{Z_{T-}<1\}} + I_{\{Z_{T-}=1\}}$. Therefore, by taking conditional expectation and using the fact that $S_T I_{\{Z_{T-}=0\}} = 0$, we obtain

$$(1-Z_{T-})E^{\widetilde{Q}^{\mathbb{F}}_T}[\xi \widetilde{Y}^{\mathbb{F}}\big|\mathcal{F}_{T-}] = 0, \text{ or equivalently} \quad E^{\widetilde{Q}^{\mathbb{F}}_T}[S_T \widetilde{Y}^{\mathbb{F}}\big|\mathcal{F}_{T-}] = 0 \quad P-a.s.$$

By using the equality $E(Y^{\mathbb{F}} I_{\{T>\tau\}}|\mathcal{G}_{T-}) = E(Y^{\mathbb{G}} I_{\{T>\tau\}}|\mathcal{G}_{T-}) = I_{\{T>\tau\}}$, we deduce that $E^{\widetilde{Q}^{\mathbb{F}}_T}[\widetilde{Y}^{\mathbb{F}}\big|\mathcal{F}_{T-}] = 1$. This proves that S satisfies NUPBR($\mathbb{F}, \widetilde{Q}^{\mathbb{F}}_T$) and the proof of the theorem is achieved. □

6.4.3 Proof of Theorem 6.5

We start by outlining a number of remarks that simplify tremendously the proof. Due to Lemma 6.7, on $\{T < +\infty\}$, we have

$$\{\widetilde{Z}^Q_T = 1\} = \{\widetilde{Z}_T = 1\} \quad \text{for any } Q \sim P \text{ and any } \mathbb{F}-\text{stopping time } T, \tag{6.39}$$

where $\widetilde{Z}_t^Q := E^Q[I_{\{\tau \geq t\}} | \mathcal{F}_t]$.

On the other hand, assertion (b) is equivalent to the existence of a positive $\mathbb{F}$-local martingale Y such that

$$ {}^{p,\mathbb{F}}\left(Y \Delta S^a I_{\{\widetilde{Z}<1\}} \right) = 0. $$

These two remarks imply that without loss of generality we will assume that $S = S^a$ in the rest of the proof.

Consider a sequence of $\mathbb{F}$-stopping times $(\sigma_n)_{n\geq 1}$ such that Y^{σ_n} is a martingale, and put $Q_n := (Y_{\sigma_n}/Y_0) \cdot P$. Therefore, thanks to Proposition 5.2, it is enough to prove that assertion (a) holds true under Q_n for S^{σ_n}. Therefore, without loss of generality, we assume $Y \equiv 1$. Put

$$ X^n = \Delta S_{T_n} I_{\{\widetilde{Z}_{T_n}<1\}} \quad \text{and} \quad M^n = X^n I_{[\![T_n,+\infty[\![}. $$

Hence, the condition (6.27) in Theorem 6.8 is trivially satisfied for X^n and M^n. Thus, we deduce that for each n,

$$ (1 + \Delta L^{\mathbb{G}}{}_{T_n} - \Delta W^{\mathbb{G}}_{T_n}) \Delta S_{T_n} I_{\{T_n>\tau\}} I_{[\![T_n,+\infty[\![}, $$

is a $\mathbb{G}$-martingale. Then, for each $\theta \in L(S - S^\tau)$ such that $\theta^T \Delta S > -1$, Yor's formula implies that

$$ \mathcal{E}\left(I_\Gamma \centerdot L^{\mathbb{G}} - I_\Gamma \cdot W^{\mathbb{G}} \right) \mathcal{E}\left(\theta I_\Gamma \centerdot (S - S^\tau) \right) = \mathcal{E}\left(W^{(1)} \right), $$

where

$$ W^{(1)} := I_\Gamma \centerdot (L^{\mathbb{G}} - W^{\mathbb{G}}) + \sum_{n\geq 1} \theta_{T_n} \left(1 + \Delta L^{\mathbb{G}}{}_{T_n} - \Delta W^{\mathbb{G}}_{T_n} \right) \Delta S_{T_n} I_{\{T_n>\tau\}} I_{[\![T_n,+\infty[\![}, $$

and $L^{\mathbb{G}(a)}$ and $V^{(a)}$ are defined in (6.19) and (6.17) respectively. Put

$$\phi = \sum_{n\geq 1} \frac{2^{-n}\left(1+\mathcal{E}(W^{(1)})_{T_n-}\right)^{-1}}{(1+\Xi_n)} I_{]\!]\tau,+\infty[\![\cap[\![T_n]\!]} + I_{\Gamma^c\cup[\![0,\tau]\!]} \quad \text{and}$$

$$\Xi_n := E\left[|\Delta L^{\mathbb{G}}{}_{T_n}| \,\Big|\mathcal{G}_{T_n-}\right] + \Delta W^{\mathbb{G}}_{T_n} + E\left[\frac{|\theta_{T_n}\Delta S_{T_n}|(1-Z_{T_n-})}{1-\widetilde{Z}_{T_n}}\Big|\mathcal{G}_{T_n-}\right] I_{\{T_n<+\infty\}}.$$

Then, it is easy to check that ϕ is $\mathbb{G}$-predictable, $0<\phi\leq 1$, and $E\left(|\phi \centerdot \mathcal{E}(W^{(1)})|_{var}(+\infty)\right) \leq \sum_{n\geq 1} 2^{-n} = 1$. Hence $\phi \centerdot \mathcal{E}(W^{(1)}) \in \mathcal{A}(\mathbb{G})$, and using the fact that $\Delta L^{\mathbb{G}}_{T_n} I_{[\![T_n,+\infty[\![}$ and $(1+\Delta L^{\mathbb{G}}_{T_n} - \Delta W^{\mathbb{G}}_{T_n})\Delta S_{T_n} I_{\{T_n>\tau\}} I_{[\![T_n,+\infty[\![}$ are $\mathbb{G}$-martingales, we get

$$\left(\phi \centerdot \mathcal{E}(W^{(1)})\right)^{p,\mathbb{G}} = \sum_{n\geq 1} \phi_{T_n} E\left(\Delta W^{(1)}_{T_n}|\mathcal{G}_{T_n-}\right) I_{[\![T_n,+\infty[\![} = -\phi\mathcal{E}_-(W^{(1)}) \centerdot W^{\mathbb{G}}$$

is a non-increasing process. This proves that $\mathcal{E}(W^{(1)})$ is a positive σ-supermartingale, and hence it is a supermartingale due to Kallsen [89]. Thus, $S - S^{\tau}$ admits a deflator. □

6.5 Discrete Time Market Models

Similar as Section 5.6 of Chapter 5, in the current section, we consider discrete time market models. That is the case where there are only finite number of trading times $n = 1, 2, ..., N$. In this context, we shall investigate the stability of non-arbitrage after an honest time τ. For τ, we associate the following stopping times:

$$\sigma_1 := \inf\{n \geq 1 : Z_n < 1\}, \quad \sigma_2 := \inf\{n \geq 1 : Z_{n-1} < 1\} \text{ and } \sigma_3 := \inf\{n \geq 1 : \widetilde{Z}_n < 1\}. \tag{6.40}$$

First, we remark that in discrete time we do not assume $Z_\tau < 1$ considered in the previous sections.

Lemma 6.8: *For an honest time τ and the associated stopping times in (6.40), the* *following properties hold for any $n \leq N$:*

(a) $\{\widetilde{Z}_n < 1\} \subset \{Z_{n-1} < 1\}$ *and* $\{\widetilde{Z}_n < 1\} \subset \{Z_n < 1\}$.

(b) σ_2 *is an $\mathbb{F}$-predictable stopping time and* $\sigma_2 \leq \sigma_3$ *and* $\sigma_1 \leq \sigma_3$.

(c) *On $\{\tau < n\}$, we have*

$$\tau \geq \sigma_1, \quad Z_{n-1} < 1, \ \textit{and}\ \widetilde{Z}_n < 1, \quad P-a.s..$$

Proof. (a) Notice that

$$E\left[\left(1-\widetilde{Z}_n\right)1\!\!1_{\{Z_{n-1}=1\}}\right] = E\left[(1-Z_{n-1})\,1\!\!1_{\{Z_{n-1}=1\}}\right] = 0.$$

Hence, $\{Z_{n-1}=1\} \subset \{\widetilde{Z}_n = 1\}$. Due to $Z_n \leq \widetilde{Z}_n$, we have $\{\widetilde{Z}_n < 1\} \subset \{Z_n < 1\}$.
(b) Since $\{\sigma_2 \leq n\} = \{Z_{n-1} < 1\} \in \mathcal{F}_{n-1}$, we conclude that σ_2 is predictable. The inequalities $\sigma_2 \leq \sigma_3$ and $\sigma_1 \leq \sigma_3$ follow immediately from (a).
(c) Notice that

$$\begin{aligned} E\left[\ \mathbb{I}_{\{n>\tau\}}\mathbb{I}_{\{Z_{n-1}=1\}}\right] &= E\left[(1-Z_{n-1})\mathbb{I}_{\{Z_{n-1}=1\}}\right] = 0, \ \text{and} \\ E\left[\ \mathbb{I}_{\{n>\tau\}}\mathbb{I}_{\{\widetilde{Z}_n=1\}}\right] &= E\left[(1-\widetilde{Z}_n)\mathbb{I}_{\{\widetilde{Z}_n=1\}}\right] = 0. \end{aligned}$$

Therefore, $Z_{n-1} < 1$ and $\widetilde{Z}_n < 1$ on the set $\{n > \tau\}$. This ends the proof of the lemma. □

The following lemma, (already stated and proved in Jeulin [83]), describes the connection between conditional expectations under $\mathbb{F}$ and $\mathbb{G}$.

Lemma 6.9: *Let Y be an integrable $\mathbb{A}$-measurable random variable. Then, the following properties hold:*

(a) *On the set $\{n > \tau\}$, the conditional expectation under $\mathcal{G}_n$ is given by*

$$E\left[Y|\mathcal{G}_n\right]1\!\!1_{\{\tau<n\}} = E\left[Y1\!\!1_{\{\tau<n\}}|\mathcal{F}_n\right]\frac{1}{1-\widetilde{Z}_n}1\!\!1_{\{\tau<n\}}. \tag{6.41}$$

(b) *On the set $\{n > \tau\}$, the conditional expectation under $\mathcal{G}_{n-1}$ is given by*

$$E\left[Y|\mathcal{G}_{n-1}\right]\mathbb{1}_{\{\tau<n\}} = E\left[Y\mathbb{1}_{\{\tau<n\}}|\mathcal{F}_{n-1}\right]\frac{1}{1-Z_{n-1}}\mathbb{1}_{\{\tau<n\}}. \tag{6.42}$$

Moreover, if Y is $\mathcal{F}_n$-measurable, we have

$$E\left[Y|\mathcal{G}_{n-1}\right]\mathbb{1}_{\{\tau<n\}} = E\left[Y(1-\widetilde{Z}_n)|\mathcal{F}_{n-1}\right]\frac{1}{1-Z_{n-1}}\mathbb{1}_{\{\tau<n\}}. \tag{6.43}$$

The following theorem characterizes the relationship between $\mathbb{F}$-martingales and $\mathbb{G}$-martingales on the stochastic interval $]\!]\tau, +\infty[\![$. For the continuous time case, we consult Jeulin [83].

Theorem 6.11: *Let M be an $\mathbb{F}$-martingale and τ be an honest time. Then the following process*

$$M_n^{(a)} := M_{n\vee\tau} - M_\tau - \sum_{1\le k\le n}\frac{1}{1-Z_{k-1}}\mathbb{1}_{\{\tau<k\}}E\left[(M_k - M_{k-1})(1-\widetilde{Z}_k)|\mathcal{F}_{k-1}\right],$$

is a $\mathbb{G}$-martingale.

Proof. Although it can be derived from Jeulin [83], here we give a direct proof here. It is easy to see that

$$M_{(n+1)\vee\tau} = M_{n\vee\tau} + (M_{n+1} - M_n)\mathbb{1}_{\{\tau<n+1\}}. \tag{6.44}$$

Then, we calculate

$$
\begin{aligned}
E\Big[M_{n+1}^{(a)}\Big|\mathcal{G}_n\Big] &= E\Big[M_{(n+1)\vee\tau} - M_\tau - \sum_{1\le k\le n+1}\frac{1}{1-Z_{k-1}}\mathbb{1}_{\{\tau<k\}}E\left[(M_k - M_{k-1})(1-\widetilde{Z}_k)|\mathcal{F}_{k-1}\right]\Big|\mathcal{G}_n\Big] \\
&= M_{n\vee\tau} - M_\tau + E\Big[(M_{n+1}-M_n)\mathbb{1}_{\{\tau<n+1\}}\Big|\mathcal{G}_n\Big] \\
&\quad - \sum_{1\le k\le n}\frac{1}{1-Z_{k-1}}\mathbb{1}_{\{\tau<k\}}E\left[(M_k - M_{k-1})(1-\widetilde{Z}_k)|\mathcal{F}_{k-1}\right] \\
&\quad - \frac{1}{1-Z_n}\mathbb{1}_{\{\tau<n+1\}}E\left[(M_{n+1}-M_n)(1-\widetilde{Z}_{n+1})|\mathcal{F}_n\right] \\
&= M_{n\vee\tau} - M_\tau + \sum_{1\le k\le n}\frac{1}{1-Z_{k-1}}\mathbb{1}_{\{\tau<k\}}E\left[(M_k - M_{k-1})(1-\widetilde{Z}_k|)\mathcal{F}_{k-1}\right] = M_n^{(a)},
\end{aligned}
$$

where we used the following equality the follows from Lemma 6.9

$$
E\Big[(M_{n+1}-M_n)\mathbb{1}_{\{\tau<n+1\}}\Big|\mathcal{G}_n\Big] = \frac{1}{1-Z_n}\mathbb{1}_{\{\tau<n+1\}}E\left[(M_{n+1}-M_n)(1-\widetilde{Z}_{n+1})|\mathcal{F}_n\right].
$$

This ends the proof of the theorem. □

The following proposition constructs a $\mathbb{G}$-martingale density for a class of $\mathbb{G}$-semi-martingales.

Proposition 6.4:

$$
N_n^{(a)} := -\sum_{1\le k\le n}\mathbb{1}_{\{\tau<k\}}E[\mathbb{1}_{\{\widetilde{Z}_k<1\}}|\mathcal{F}_{k-1}] + \sum_{1\le k\le n}\frac{1-Z_{k-1}}{1-\widetilde{Z}_k}\mathbb{1}_{\{\tau<k\}} \tag{6.45}
$$

is a $\mathbb{G}$*-martingale such that* $1+\Delta N_n^{(a)} > 0$.

Proof. First, we prove that $N^{(a)}$ is a $\mathbb{G}$-martingale. To this end, we calculate

$$
\begin{aligned}
E\left[N^{(a)}_{n+1}|\mathcal{G}_n\right] &= E\left[-\sum_{1\le k\le n+1} 1\!\!1_{\{\tau<k\}}E[1\!\!1_{\{\widetilde{Z}_k<1\}}|\mathcal{F}_{k-1}] + \sum_{1\le k\le n+1} \frac{1-Z_{k-1}}{1-\widetilde{Z}_k}1\!\!1_{\{\tau<k\}}\Big|\mathcal{G}_n\right] \\
&= -\sum_{1\le k\le n+1} 1\!\!1_{\{\tau<k\}}E[1\!\!1_{\{\widetilde{Z}_k<1\}}|\mathcal{F}_{k-1}] + \sum_{1\le k\le n} \frac{1-Z_{k-1}}{1-\widetilde{Z}_k}1\!\!1_{\{\tau<k\}} + E\left[\frac{1-Z_n}{1-\widetilde{Z}_{n+1}}1\!\!1_{\{\tau<n+1\}}\Big|\mathcal{G}_n\right] \\
&= -\sum_{1\le k\le n+1} 1\!\!1_{\{\tau<k\}}E[1\!\!1_{\{\widetilde{Z}_k<1\}}|\mathcal{F}_{k-1}] + \sum_{1\le k\le n} \frac{1-Z_{k-1}}{1-\widetilde{Z}_k}1\!\!1_{\{\tau<k\}} + 1\!\!1_{\{\tau<n+1\}}E[1\!\!1_{\{\widetilde{Z}_{n+1}<1\}}|\mathcal{F}_n] \\
&= -\sum_{1\le k\le n} 1\!\!1_{\{\tau<k\}}E[1\!\!1_{\{\widetilde{Z}_k<1\}}|\mathcal{F}_{k-1}] + \sum_{1\le k\le n} \frac{1-Z_{k-1}}{1-\widetilde{Z}_k}1\!\!1_{\{\tau<k\}} = N^{(a)}_n. \qquad (6.46)
\end{aligned}
$$

The third equality above is obtained due to

$$
E\left[\frac{1-Z_n}{1-\widetilde{Z}_{n+1}}1\!\!1_{\{\tau<n+1\}}\Big|\mathcal{G}_n\right] = 1\!\!1_{\{\tau<n+1\}}E[1\!\!1_{\{\widetilde{Z}_{n+1}<1\}}|\mathcal{F}_n], \qquad (6.47)
$$

which follows from Lemma 6.9 –precisely (6.43). Next, we show that $1+\Delta N^{(a)}_n > 0$. Indeed

$$
1+\Delta N^{(a)}_n = 1 - 1\!\!1_{\{\tau<n\}}E[1\!\!1_{\{\widetilde{Z}_n<1\}}|\mathcal{F}_{n-1}] + \frac{1-Z_{n-1}}{1-\widetilde{Z}_n}1\!\!1_{\{\tau<n\}} \ge 1\!\!1_{\{\tau\ge n\}} + \frac{1-Z_{n-1}}{1-\widetilde{Z}_n}1\!\!1_{\{\tau<n\}} > 0.
$$

The integrability of $N^{(a)}$ follows from the fact that $E|N^{(a)}_n| \le 2n$. This completes the proof of the proposition. ☐

Below, we state the first main theorem of this section.

Theorem 6.12: *Consider an honest time τ and an $\mathbb{F}$-martingale S. Denote the probability measure $\mathbb{Q}^{(a)} \sim \mathbb{P}$ with density $D^{(a)}_n := \mathcal{E}(Y^{(a)})_n$ where*

$$
\Delta Y^{(a)}_n := (1-\widetilde{Z}_n)1\!\!1_{\{Z_{n-1}<1\}}E\left[1\!\!1_{\{\widetilde{Z}_n=1\}}|\mathcal{F}_{n-1}\right] - (1-Z_{n-1})1\!\!1_{\{\widetilde{Z}_n=1\}},\ Y^{(a)}_0 = 0. \qquad (6.48)
$$

Then the following are equivalent:

(a) *S is a $(\mathbb{Q}^{(a)}, \mathbb{F})$-martingale;*

(b) *S is orthogonal to both $D^{(a)}$ and $Y^{(a)}$;*

(c) *$\mathcal{E}(N^G)_n(S_n - S_{n\wedge\tau})$ is a $\mathbb{G}$-martingale.*

As a consequence, all the above three equivalent conditions imply

(d) *$S - S^\tau$ satisfies $NA(\mathbb{G}, \mathbb{P})$ and $NA(\mathbb{G}, \mathbb{Q}^{(a)})$.*

Proof. First, we remark that $Y^{(a)}$ is an $\mathbb{F}$-martingale and $1 + \Delta Y^{(a)} > 0$. Indeed,

$$1 + \Delta Y_n^{(a)} = (1 - \widetilde{Z}_n)\mathbb{1}_{\{Z_{n-1}<1\}} E\left[\mathbb{1}_{\{\widetilde{Z}_n=1\}}|\mathcal{F}_{n-1}\right] + \mathbb{1}_{\{\widetilde{Z}_n<1\}} + Z_{n-1}\mathbb{1}_{\{\widetilde{Z}_n=1\}} > 0,$$

where we used the fact that on the set $\{\widetilde{Z}_n < 1\}$, $1 + \Delta Y_n^{(a)} \geq 1$ and the inclusion $\{\widetilde{Z}_n = 1\} \subset \{Z_{n-1} > 0\}$, since $\{Z_{n-1} = 0\} \subset \{\widetilde{Z}_n = 0\}$. Therefore, $D^{(a)}$ is a strictly positive martingale.

The equivalence between (a) and (b) is obvious. In the following, we will prove the equivalence between (a) and (c). Recall that

$$N_n^{(a)} = -\sum_{1\leq k\leq n} \mathbb{1}_{\{\tau<k\}} E[\mathbb{1}_{\{\widetilde{Z}_k<1\}}|\mathcal{F}_{k-1}] + \sum_{1\leq k\leq n} \frac{1 - Z_{k-1}}{1 - \widetilde{Z}_k}\mathbb{1}_{\{\tau<k\}}. \tag{6.49}$$

Due to Lemma 5.15 , we deduce that

$$\begin{aligned} E\Big[\frac{\Delta S_k}{1 - \widetilde{Z}_k}\mathbb{1}_{\{\tau<k\}}|\mathcal{G}_{k-1}\Big] &= \frac{\mathbb{1}_{\{\tau<k\}}}{1 - Z_{k-1}} E\Big[\Delta S_k \mathbb{1}_{\{\widetilde{Z}_k<1\}}|\mathcal{F}_{k-1}\Big], \\ E\Big[\Delta S_k \mathbb{1}_{\{\tau<k\}}|\mathcal{G}_{k-1}\Big] &= \frac{\mathbb{1}_{\{\tau<k\}}}{1 - Z_{k-1}} E\Big[\Delta S_k(1 - \widetilde{Z}_k)|\mathcal{F}_{k-1}\Big]. \end{aligned} \tag{6.50}$$

Notice that

$$S_{n+1} - S_{(n+1)\wedge\tau} = S_n - S_{n\wedge\tau} + \Delta S_{n+1}\mathbb{1}_{\{n+1>\tau\}}, \tag{6.51}$$

and hence we get

$$
\begin{aligned}
& E\left[\mathcal{E}(N^{(a)})_{n+1}S_{(n+1)\wedge\tau}|\mathcal{G}_{n+1}\right] \\
& = \mathcal{E}(N^{(a)})_n E\left[(1+\Delta N^{(a)}_{n+1})S_{(n+1)\wedge\tau}|\mathcal{G}_n\right] \\
& = \mathcal{E}(N^{(a)})_n E\left[S_n - S_{n\wedge\tau} + \Delta S_{n+1}1\!\!1_{\{n+1>\tau\}} + \Delta S_{n+1}\Delta N^{\mathbb{G}}_{n+1}1\!\!1_{\{n+1>\tau\}}|\mathcal{G}_n\right] \\
& = \mathcal{E}(N^{(a)})_n \left\{S_n - S_{n\wedge\tau} + E\left[\Delta S_{n+1}(1-\widetilde{Z}_{n+1})E\left[1\!\!1_{\{\widetilde{Z}_{n+1}=1\}}|\mathcal{F}_n\right]|\mathcal{F}_n\right]\frac{1\!\!1_{\{n+1>\tau\}}}{1-Z_n}\right\} \\
& -\mathcal{E}(N^{(a)})_n \left\{1\!\!1_{\{n+1>\tau\}}E\left[\Delta S_{n+1}1\!\!1_{\{\widetilde{Z}_{n+1}=1\}}|\mathcal{F}_n\right]\right\} \\
& = \mathcal{E}(N^{(a)})_n \left(S_n - S_{n\wedge\tau}\right) \\
& +\mathcal{E}(N^{(a)})_n \left\{E\left[\Delta S_{n+1}\left\{(1-\widetilde{Z}_{n+1})E\left[1\!\!1_{\{\widetilde{Z}_{n+1}=1\}}|\mathcal{F}_n\right] - (1-Z_n)1\!\!1_{\{\widetilde{Z}_{n+1}=1\}}\right\}|\mathcal{F}_n\right]\right\}\frac{1\!\!1_{\{n+1>\tau\}}}{1-Z_n} \\
& = \mathcal{E}(N^{(a)})_n(S_n - S_{n\wedge\tau}) + \mathcal{E}(N^{(a)})_n E^{\mathbb{Q}^{(a)}}\left[\Delta S_{n+1}|\mathcal{F}_n\right]\frac{1\!\!1_{\{n+1>\tau\}}}{1-Z_n}.
\end{aligned}
$$

Therefore (a) implies (c). Conversely, if (c) holds, we have

$$
E^{\mathbb{Q}^{(a)}}\left[\Delta S_{n+1}|\mathcal{F}_n\right]\frac{1\!\!1_{\{n+1>\tau\}}}{1-Z_n} = 0, \text{ and } \ E^{\mathbb{Q}^{(a)}}\left[\Delta S_{n+1}|\mathcal{F}_n\right]1\!\!1_{\{Z_n<1\}} = 0.
$$

Notice that

$$
E^{\mathbb{Q}^{(a)}}\left[\Delta S_{n+1}|\mathcal{F}_n\right]1\!\!1_{\{Z_n=1\}} = 0, \text{ for all } n\ .
$$

Thus, we conclude that $E^{\mathbb{Q}^{(a)}}\left[\Delta S_{n+1}|\mathcal{F}_n\right] = 0$, for all n. This ends the proof of the theorem. □

Remark 6.4: Similarly, we observe from Theorem 6.12 that even though $Y^{(a)}$ is an $\mathbb{F}$-martingale, the process $Y^{(a)}_n - Y^{(a)}_{n\wedge\tau} = \sum_{k\le n}(1-\widetilde{Z}_k)E\left[1\!\!1_{\{\widetilde{Z}_k=1\}}|\mathcal{F}_k\right]1\!\!1_{\{k>\tau\}}$ does not satisfy NA($\mathbb{G}$) since it is a $\mathbb{G}$-increasing process. This also highlights the importance of the conditions in Theorem 6.12.

Corollary 6.12.1: For any $\mathbb{F}$-martingale M, if for all n

$$
\{\widetilde{Z}_n = 1\} = \{Z_{n-1} = 1\}. \tag{6.52}
$$

Then the following properties hold:

(a) The process $M_n - M_{n\wedge\tau}$ satisfies NA($\mathbb{G}$);

(b) $\left(\mathcal{E}(N^{(a)})_n \left(M_n - M_{n\wedge\tau}\right)\right)_{n\geq 1}$ is a $\mathbb{G}$-martingale, where $N^{(a)}$ is given by (6.45) in Proposition 6.4;

(c) The probability measure $\mathbb{Q}^{(a)}$, given in (6.48), coincides with $\mathbb{P}$.

Below, we state our second main theorem in this section, where we give the necessary and sufficient conditions on the random time τ (or equivalently the stopping times in (6.40)) to guarantee that the process $M - M^\tau$ satisfies NA($\mathbb{G}$) for any $\mathbb{F}$-martingale M.

Theorem 6.13: *Consider an honest time τ and the associated stopping times defined in (6.40). Then the following are equivalent:*

(a) *For any $\mathbb{F}$-martingale M, the process $M_n - M_{n\wedge\tau}$ satisfies NA($\mathbb{G}$).*

(b) $\{\widetilde{Z}_n = 1\} = \{Z_{n-1} = 1\}$ *for all n.*

(c) $\sigma_1 + 1 = \sigma_2 = \sigma_3$.

(d) σ_3 *is an $\mathbb{F}$-predictable stopping time.*

(e) *The probability $\mathbb{Q}^{(a)}$, defined via (6.48)), coincides with $\mathbb{P}$.*

Proof. The proof of the theorem would be achieved after four steps. In the first step, we prove (b)⇔(c). The second step focuses on (b)⇔(d). The third step deals with (b)⇔(e). In the last step, we prove (a) ⇔ (b).

Step 1: The equivalence between (b) and (c) is obvious. Indeed, if (b) holds, it is trivial that $\sigma_2 = \sigma_3$. Conversely, if (c) holds, we derive that

$$E\left((1 - Z_{n-1})I_{\{\widetilde{Z}_n=1\}}\right) = E\left((1 - Z_{n-1})I_{\{\widetilde{Z}_n=1\}}I_{\{n<\sigma_3\}}\right) = E\left((1 - Z_{n-1})I_{\{\widetilde{Z}_n=1\}}I_{\{n<\sigma_2\}}\right) = 0.$$

Hence, we conclude that $\{\widetilde{Z}_n = 1\} \subset \{Z_{n-1} = 1\}$ for all n.

Step 2: Here, we will prove (b)$\Leftrightarrow$(d). If (b) holds, it is easy to see that σ_3 is an $\mathbb{F}$-predictable stopping time. Conversely, due to the predictability of σ_3 and

$$E\left[(1-Z_{n-1})I_{\{\widetilde{Z}_n=1\}}\right] = E\left[(1-Z_{n-1})I_{\{n<\sigma_3\}}\right] = E\left[(1-\widetilde{Z}_n)I_{\{n<\sigma_3\}}\right] = 0,$$

we conclude that $\{\widetilde{Z}_n = 1\} \subset \{Z_{n-1} = 1\}$ for all n.

Step 3: This step will prove (b)$\Leftrightarrow$(e). If (b) holds, apparently, $Y^{(a)} = 0$ and $\mathbb{Q}^{(a)} = \mathbb{P}$. Conversely, if (e) holds, $\Delta Y_n^{(a)} = 0$ for all n. Hence, $(1-\widetilde{Z}_n)\mathbb{1}_{\{Z_{n-1}<1\}}E\left[\mathbb{1}_{\{\widetilde{Z}_n=1\}}|\mathcal{F}_{n-1}\right] = (1-Z_{n-1})\mathbb{1}_{\{\widetilde{Z}_n=1\}} = 0$ and $\{\widetilde{Z}_n = 1\} = \{Z_{n-1} = 1\}$ for all n.

Step 4: In this step, we focus on the proof of the equivalence between (a) and (b). Suppose for any $\mathbb{F}$-martingale M, the stopped process M^τ satisfies NA($\mathbb{G}$). Consider the $\mathbb{F}$-martingale

$$M_n := \sum_{1\leq k\leq n}\left(\mathbb{1}_{\{\widetilde{Z}_k=1\}} - E\left[\mathbb{1}_{\{\widetilde{Z}_k=1\}}|\mathcal{F}_{k-1}\right]\right).$$

It is easy to see that $M_n - M_{n\wedge\tau} = -\sum_{1\leq k\leq n} E\left[\mathbb{1}_{\{\widetilde{Z}_k=1\}}|\mathcal{F}_{k-1}\right]\mathbb{1}_{\{\tau<k\}}$. Note that $M_n - M_{n\wedge\tau}$ is a $\mathbb{G}$ predictable decreasing process satisfying NA($\mathbb{G}$). Therefore it is null. Then, we deduce that

$$\begin{aligned}0 = E\left[M_n - M_{n\wedge\tau}\right] &= \sum_{1\leq k\leq n} E\left[E\left[\mathbb{1}_{\{\widetilde{Z}_k=1\}}|\mathcal{F}_{k-1}\right]\mathbb{1}_{\{\tau<k\}}\right]\\ &= \sum_{1\leq k\leq n} E\left[(1-Z_{k-1})\mathbb{1}_{\{\widetilde{Z}_k=1\}}\right].\end{aligned}$$

Hence, $\{\widetilde{Z}_k = 1\} \subset \{Z_{k-1} = 1\}$ for all k.

The reverse implication follows immediately from Theorem 6.12 or Corollary 6.12.1. This ends the proof of the theorem. $\square$

6.6 Lévy Market Model

In this section, we shall study Lévy market model on the stochastic interval $]\!]\tau, +\infty[\![$ as Section 5.7 of Chapter 5. Suppose that the traded financial asset is an exponential of a Lévy process given by $S = S_0 \exp(X)$, where

$$X_t = \gamma t + \sigma W_t + \int_0^t \int_{|x|\leq 1} x\widetilde{N}(dt, dx) + \int_0^t \int_{|x|\geq 1} xN(dt, dx), \tag{6.53}$$

$\widetilde{N}(dt, dx) = N(dt, dx) - \nu_X(dx, dt)$ and $\nu_X(dx, dt) := F_X(dx)dt$. Here, γ and σ are real numbers ($\sigma > 0$); $W = (W_t)_{t\geq 0}$ represents a Brownian motion; $N(dt, dx)$ is a random measure on $[0, T] \otimes \mathbb{R}\backslash\{0\}$, called Poisson random measure; $\widetilde{N}(dt, dx)$ is the compensated Poisson measure with the intensity measure $F_X(dx)dt$, where $F_X(dx)$ is called the Lévy measure defined on $\mathbb{R}\backslash\{0\}$, satisfying

$$\int_{\mathbb{R}\backslash\{0\}} \left(|x|^2 \wedge 1\right) F_X(dx) < +\infty. \tag{6.54}$$

Recall the random measure μ and its compensator ν are defined in (5.104) of Subsection 5.7 in Chapter 5.

In the same spirit as in Theorem 2.7, any local martingale Y can be decomposed as follows

$$Y = \beta \cdot S^c + f \star (\mu - \nu) + g \star \mu + Y^{\perp}. \tag{6.55}$$

Here, $(\beta, f, g, Y^{\perp})$ is the Jacod's components of Y with respect to S .

Let (β_m, f_m, g_m, m') be the Jacod's parameters of m with respect to $(S, \mathbb{F}, \mathbb{P})$ such that

$$m = \beta_m \cdot S^c + f_m \star (\mu - \nu) + g_m \star \mu + m'. \tag{6.56}$$

The Jacod parameters of m would be fixed throughout this section in which we assume that

$$Z_\tau < 1, \quad \text{and} \;\; \nu^{\mathbb{G}} \sim \nu \;\; \text{on} \;\;]\!]\tau, \infty[\![. \tag{6.57}$$

Below, we recall the compensator of the random measure μ after τ (see Proposition 2.8 and Theorem 2.12 in Chapter 2).

Proposition 6.5: *Consider the Lévy market S. On $]\!]\tau, +\infty[\![$, we have*

(a) *The compensator of μ in the filtration $\mathbb{G}$ is given by*

$$\nu^{\mathbb{G}} := \left(I_{]\!]\tau,\infty[\![} \cdot \mu\right)^{p,\mathbb{G}} = \left(1 - \frac{f_m}{1-Z_-}\right) I_{]\!]\tau,\infty[\![} \cdot \nu \tag{6.58}$$

(b) *The canonical representation of $I_{]\!]\tau,\infty[\![} \cdot S$ is given by*

$$I_{]\!]\tau,\infty[\![} \cdot S = S_0 + \widehat{S^c}^{(a)} + h \star (\mu^{\mathbb{G}} - \nu^{\mathbb{G}}) + (x-h) \star \mu^{\mathbb{G}} + \overline{B}, \tag{6.59}$$

where $\widehat{S^c}^{(a)}$ is defined via (6.9) and $\overline{B} = I_{]\!]\tau,\infty[\![} \cdot B - \frac{1}{1-Z_-} I_{]\!]\tau,\infty[\![} \cdot \langle S^c, m\rangle^{\mathbb{F}} - h\frac{f_m}{1-Z_-} I_{]\!]\tau,\infty[\![} \star \nu$.

Proposition 6.6: *Let τ be an honest time satisfying (6.57). The following process $D^{(a)}$ is well defined, being a positive $\mathbb{G}$-local martingale,*

$$D^{(a)} := \mathcal{E}(N^{(a)}) := \mathcal{E}\left(\frac{1}{1-Z_-}\beta_m I_{]\!]\tau,\infty[\![} \cdot \widehat{S^c}^{(a)} + \frac{f_m}{1-Z_- - f_m} I_{]\!]\tau,\infty[\![} \star \left(\mu^{\mathbb{G}} - \nu^{\mathbb{G}}\right)\right),$$

where $\widehat{S^c}^{(a)}$ is defined via (6.9) and $1 + \Delta N^{(a)} > 0$.

Proof. Note that

$$\begin{aligned} E\left[I_{]\!]\tau,+\infty[\![} I_{\left\{1-M^P_\mu[\widetilde{Z}|\widetilde{\mathcal{P}}(\mathbb{F})]=0\right\}} \star \mu\right] &= E\left[(1-\widetilde{Z}) I_{\left\{1-M^P_\mu[\widetilde{Z}|\widetilde{\mathcal{P}}(\mathbb{F})]=0\right\}} \star \mu\right] \\ &= E\left[(1 - M^P_\mu\left[\widetilde{Z}|\widetilde{\mathcal{P}}(\mathbb{F})\right]) I_{\left\{1-M^P_\mu[\widetilde{Z}|\widetilde{\mathcal{P}}(\mathbb{F})]=0\right\}} \star \mu\right] = 0. \end{aligned}$$

Therefore, $1-Z_- - f_m = 1 - M^P_\mu[\widetilde{Z}|\widetilde{\mathcal{P}}(\mathbb{F})] > 0$, M^P_μ-a.s. on $]\!]\tau, +\infty[\![$. Let us consider

$$\delta \in (0,1), \quad \Gamma := \{\Delta S \neq 0\} \text{ and } \Gamma^c \text{ its complement in } \Omega \otimes [0,+\infty[.$$

Then, we calculate

$$\begin{aligned}
V &:= \sqrt{\sum_{0\leq u\leq \cdot}\left(\frac{f_m(\Delta S_u)}{1-Z_- - f_m(\Delta S_u)} I_\Gamma I_{]\!]\tau,\infty[\![}\right)^2} \\
&\leq \sqrt{\sum_{0\leq u\leq \cdot}\left(\frac{f_m(\Delta S_u)}{1-Z_- - f_m(\Delta S_u)} I_{]\!]\tau,\infty[\![} I_\Gamma I_{\{f_m<\delta(1-Z_-)\}}\right)^2} \\
&\quad + \sqrt{\sum_{0\leq u\leq \cdot}\left(\frac{f_m(\Delta S_u)}{1-Z_- - f_m(\Delta S_u)} I_{]\!]\tau,\infty[\![} I_\Gamma I_{\{f_m\geq\delta(1-Z_-)\}}\right)^2} \\
&:= V_1 + V_2.
\end{aligned} \tag{6.60}$$

Due to the $\mathbb{G}$-local boundedness of $(1-Z_-)^{-1}I_{]\!]\tau,\infty[\![}$ and $f_m \in \mathcal{G}^2_{loc}(\mu,\mathbb{F})$ (i.e. $f_m^2 \star \mu \in \mathcal{A}^+_{loc}(\mathbb{F})$), we obtain that

$$\begin{aligned}
V_1(t) &\leq \frac{1}{1-\delta}\sqrt{\sum_{0\leq u\leq t}\left(\frac{f_m(\Delta S_u)}{1-Z_{u-}} I_{]\!]\tau,\infty[\![} I_\Gamma I_{\{f_m<\delta(1-Z_{u-})\}}\right)^2} \\
&\leq \frac{1}{1-\delta}\sqrt{\sum_{0\leq u\leq t}\left(\frac{f_m^2(\Delta S_u)}{(1-Z_{u-})^2} I_{]\!]\tau,\infty[\![} I_\Gamma\right)} \in \mathcal{A}^+_{loc}(\mathbb{G}).
\end{aligned}$$

Now, we focus on the proof the $\mathbb{G}$-local integrability of V_2. Again due to the $\mathbb{G}$-local boundedness of $(1-Z_-)^{-1}I_{]\!]\tau,\infty[\![}$ and $f_m^2 \star \mu \in \mathcal{A}^+_{loc}(\mathbb{F})$, we deduce the existence of a sequence of $\mathbb{G}$-stopping times $(T_n)_{n\geq 1}$ that increases to infinity and a sequence of $\mathbb{F}$-stopping times, $(\sigma_n)_{n\geq 1}$, that increases to infinity such that

$$(1-Z_-)^{-1}I_{]\!]\tau,+\infty[\![} \leq n \text{ on } [\![0,T_n]\!] \quad \text{and} \quad Ef_m^2 \star \nu(\sigma_n) < +\infty.$$

Then, we derive that

$$\begin{aligned} E\left[V_2(T_n\wedge\sigma_n)\right] &\leq E\left[\frac{|f_m|}{1-f_m-Z_-}I_{]\!]\tau,\infty[\![}I_{\{\delta(1-Z_-)\leq f_m<1-Z_-\}}\star\mu_{T_n\wedge\sigma_n}\right]\\ &\leq E\left[\frac{|f_m|}{1-f_m-Z_-}I_{]\!]\tau,\infty[\![}I_{\{\delta/n\leq f_m<1-Z_-\}}\star\mu_{\sigma_n}\right]\\ &\leq E\left[\frac{|f_m|}{1-f_m-Z_-}(1-\widetilde{Z})I_{\{1-f_m-Z_->0\}}I_{\{\delta/n\leq f_m<1-Z_-\}}\star\mu_{\sigma_n}\right]\\ &= E\Big[|f_m|I_{\{\delta/n\leq f_m<1-Z_-\}}\star\mu_{\sigma_n}\Big]\\ &\leq \frac{n}{\delta}E[f_m^2\star\mu_{\sigma_n}]<+\infty. \end{aligned}$$

As a result, V is $\mathbb{G}$-locally integrable.

Secondly, the positivity of $D^{(a)}$ is obvious. Indeed, on $]\!]\tau,+\infty[\![$

$$1+\Delta N_t^{(a)}=1+\frac{f_m(\Delta S_t)}{1-Z_{t-}-f_m(\Delta S_t)}I_{\{\Delta S_t\neq0\}}=I_{\{\Delta S_t=0\}}+\frac{1-Z_{t-}}{1-Z_{t-}-f_m(\Delta S_t)}I_{\{\Delta S_t\neq0\}}>0.$$

This ends the proof of the proposition. ☐

Theorem 6.14: *Let S be the Lévy market satisfying NUPBR($\mathbb{F}$) and τ be an honest time satisfying (6.57). Then, $S-S^\tau$ satisfies NUPBR($\mathbb{G}$).*

Proof. The proof of the theorem mimics the proof of Theorem 5.16 in Chapter 5. Here, we only need to show that

$$\sum\Delta S\Delta N^{(a)}I_{]\!]\tau,+\infty[\![}=x\frac{f_m}{1-Z_--f_m}I_{]\!]\tau,+\infty[\![}\star\mu$$

is locally integrable. Let $(T_n)_{n\geq1}$ be the localizing sequence of $[S,m]$. Then, we have

$$\begin{aligned} E\left[\frac{|xf_m|}{1-f_m-Z_-}I_{]\!]\tau,+\infty[\![}\star\mu_{T_n}\right] &= E\left[\frac{|xf_m|}{1-f_m-Z_-}I_{\{1-f_m-Z_->0\}}(1-\widetilde{Z})\star\mu_{T_n}\right]\\ &\leq E\left[|xf_m|\star\mu_{T_n}\right]\leq E\left[Var([S,m])_{T_n}\right]<+\infty. \end{aligned}$$

This ends the proof the theorem. ☐

Corollary 6.14.1: Let Y be a compensated Poisson process and τ be an honest time satisfying (6.57). Then $Y - Y^{\tau}$ satisfies NUPBR($\mathbb{G}$).

Proof. The proof is the same as the proof of Corollary 5.16.1 in Chapter 5 and its proof would be omitted. □

Conclusions:

In this chapter, we obtained two principal results in the same spirit as those in Chapter 5. The first result lies in describing the pairs of market model and honest time for which the model fulfills NUPBR condition after an honest time. The second main result characterizes the honest time models that preserve the NUPBR condition. These results are elaborated in a very general market model, and also discrete time and Lévy market models. Furthermore, we construct explicit martingale densities (deflators) for some classes of local martingales.

Chapter 7

Structural Models under Additional Information

This chapter is dedicated to the question of how an extra information will affect the stochastic structures of the market. The so-called "Structure Conditions" (SC) in the literature is imperative for the "local" existence of the Markowitz' optimal portfolio or the solution to the local risk minimization problem. Herein, we consider a semi-martingale market model (initial market model) fulfilling Structure Conditions, and an arbitrary random time τ that is not adapted to the flow of the "public" information. By adding additional uncertainty to the initial market model, via this random time τ, those structures may fail.

There are two mainstreams to combine the information coming from τ to the public information $\mathbb{F}$: The initial enlargement and progressive enlargement of the filtration $\mathbb{F}$ (see [83], [77], [141] and the references therein). Herein, we restrict our attention to adding the information from τ progressively to $\mathbb{F}$, which results in a progressively enlarged filtration $\mathbb{G}$.

Precisely, we are dedicated to investigate the following two questions:

For which pair (τ, S), does S satisfy SC($\mathbb{G}$) if S satisfies SC($\mathbb{F}$)? **(Prob(7.I))**

and

For which τ, SC($\mathbb{G}$) holds for any model satisfying SC($\mathbb{F}$)? **(Prob(7.II))**

To answer the two problems **(Prob(7.I))** and **(Prob(7.II))**, we split the time horizon $[\![0,+\infty[\![$ into two disjoint intervals $[\![0,\tau]\!]$ and $]\!]\tau,+\infty[\![$. In other words, we investigate the impact of τ on the structures of S by studying S^τ and $S-S^\tau$ separately.

Our analysis allowed us to conclude that under some mild assumptions on the market model and the random time, these structures will remain valid on the one hand. Furthermore, we provide two examples illustrating the importance of these assumptions. On the other hand, we describe the random time models for which these structure conditions are preserved for any market model. These results are elaborated separately for the two contexts of stopping with random time and incorporating totally a specific class of random times respectively.

Below, we recall the definition of *Structure Conditions* and two simple but useful lemmas on Structure Conditions.

Definition 7.1: Let X be an $\mathbb{H}$-adapted process. We say that X satisfies the *Structure Conditions* under $(\mathbb{H},\mathbb{Q})$ (hereafter, SC($\mathbb{H},\mathbb{Q}$)), if there exist $M \in \mathcal{M}^2_{0,loc}(\mathbb{Q},\mathbb{H})$ and $\lambda \in L^2_{loc}(M,\mathbb{H})$ such that

$$X = X_0 + M - \lambda \centerdot \langle M\rangle^{\mathbb{H}}. \tag{7.1}$$

When $\mathbb{Q}=\mathbb{P}$, we simply put SC($\mathbb{H}$).

Lemma 7.1: *Let V be an $\mathbb{H}$-predictable with finite variation process. Then, V satisfies SC($\mathbb{H}$) if and only if V is constant (i.e. $V_t \equiv V_0,\ t \geq 0$).*

Proof. If V satisfies SC($\mathbb{H}$), then there exist an $\mathbb{H}$-local martingale M^V and an $\mathbb{H}$-predictable process $\lambda^{\mathbb{H}} \in L^2_{loc}(M)$ such that $V = V_0 + M^V + \lambda^{\mathbb{H}} \centerdot \langle M^V, M^V \rangle^{\mathbb{H}}$. Therefore, M^V is an $\mathbb{H}$-predictable local martingale with finite variation. Hence M is null, and $V \equiv V_0$. This ends the proof of the lemma. □

The following lemma explains why one can split the study of the Structure Conditions of $(S, \mathbb{G})$ into two separate cases, namely $(S^\tau, \mathbb{G})$ and $(S - S^\tau, \mathbb{G})$.

Lemma 7.2: $(S, \mathbb{G})$ *satisfies the Structure Conditions if and only if both* $(S^\tau, \mathbb{G})$ *and* $(S - S^\tau, \mathbb{G})$ *do.*

Proof. The proof follows immediately from the definition. □

This chapter is organized as follows. Section 7.1 contains two subsections where we present the main results before τ and after τ. In Section 7.2, we develop the stochastic tools that would be crucial to prove the main theorems. The last section provides the proofs of the main theorems announced in Section 7.1.

7.1 The Main Results

In this section, we will summarize our main results in two subsections. The first subsection addresses the problems **(Prob(7.I))** and **(Prob(7.II))** under stopping with τ (i.e. we study S^τ instead), while the second subsection treats the case of τ being an honest time and focuses on $S - S^\tau$ instead. To elaborate our main results, we start by a stochastic basis $(\Omega, \mathcal{A}, \mathbb{F} = (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$, where $\mathbb{F}$ is a filtration satisfying the usual conditions of right continuity and completeness and represents the flow of "public" information over time. On this filtered probability space we consider a d-dimensional $\mathbb{F}$-adapted semimartingale, S, that models the tradable risky assets.

We assume that S is a special semimartingale with the Doob-Meyer decomposition

$$S = S_0 + M^S + A^S, \tag{7.2}$$

where M^S is a locally square integrable $\mathbb{F}$-local martingale and A^S is an $\mathbb{F}$-predictable finite variation process. Thus, $(\Omega, \mathcal{A}, \mathbb{F}, S, \mathbb{P})$ constitutes the initial market model.

In addition to this model, we consider additional information and/or uncertainty that is modelled by an $\mathcal{A}$-measurable random time $\tau : \Omega \to \mathbb{R}_+$ that is fixed from the beginning and for the entire chapter. To formulate this rigorously, we associate to τ the process D and the progressive enlargement of filtration $\mathbb{G}$ given by

$$D := I_{[\![\tau,+\infty[\![}, \quad \mathbb{G} = (\mathcal{G}_t)_{t\geq 0}\,, \text{ where } \mathcal{G}_t = \bigcap_{s>t}\Big(\mathcal{F}_s \vee \sigma(D_u, u \leq s)\Big). \tag{7.3}$$

Recall the two Azéma supermartingales given by

$$Z_t := P\left(\tau > t \;\middle|\; \mathcal{F}_t\right) \quad \text{and} \quad \widetilde{Z}_t := P\left(\tau \geq t \;\middle|\; \mathcal{F}_t\right). \tag{7.4}$$

The decomposition of Z leads to another important martingale m by

$$m := Z + D^{o,\mathbb{F}}, \tag{7.5}$$

where $D^{o,\mathbb{F}}$ is the $\mathbb{F}$-dual optional projection of $D = I_{[\![\tau,\infty[\![}$. Furthermore, we have $\widetilde{Z}_+ = Z$ and $\widetilde{Z} = Z_- + \Delta m$.

7.1.1 Structure Conditions under Stopping with Random Time

In this section, we will investigate and quantify the effect of stopping with τ on the Structure Conditions in two different ways. Below, we state the first main result of this subsection, which provides sufficient condition on τ and S for which the Structure Conditions are preserved after stopping with τ. This answers partially the problem **(Prob(7.I))**.

Theorem 7.1: *Consider any random time τ and suppose that S satisfies $SC(\mathbb{F})$ with the Doob-Meyer decomposition $S := S_0 + M^S + A^S$. If*

$$\{\Delta M^S \neq 0\} \cap \{\widetilde{Z} = 0\} \cap \{Z_- > 0\} = \emptyset, \tag{7.6}$$

then S^τ satisfies $SC(\mathbb{G})$.

Proof. The proof requires many intermediary results that are interesting in themselves. Therefore, this proof is delegated to Section 7.3. □

Our second main theorem of this subsection answers completely the problem **(Prob(7.II))**, and describes the random time models for which the Structure Conditions are preserved after stopping with τ.

Theorem 7.2: *Let τ be a random time. Then, the following are equivalent.*

(a) *The thin set $\{\widetilde{Z} = 0\} \cap \{Z_- > 0\}$ is evanescent.*

(b) *For any process X satisfying $SC(\mathbb{F})$, X^τ satisfies $SC(\mathbb{G})$.*

Proof. The proof of (a) $\Rightarrow$ (b) follows immediately from Theorem 7.1. To prove the reverse sense, we assume that assertion (b) holds. Remark that $\{\widetilde{Z} = 0\} \cap \{Z_- > 0\} \subset \{\Delta m \neq 0\}$, it is a thin set, and consider T a stopping time such that $[\![T]\!] \subset \{\widetilde{Z} = 0\} \cap \{Z_- > 0\}$. Then, M^τ satisfies SC($\mathbb{G}$), where

$$M = V - \widetilde{V} \in \mathcal{M}_0(\mathbb{F}), \qquad V := I_{[\![T,+\infty[\![} \text{ and } \widetilde{V} := (V)^{p,\mathbb{F}}. \tag{7.7}$$

Since $\tau < T,\ P-a.s.$ on $\{T < +\infty\}$ (due to $\widetilde{Z}_T = 0$ on $\{T < +\infty\}$), we deduce that

$$M^\tau = -(\widetilde{V})^\tau \text{ is } \mathbb{G}-\text{predictable and satisfies SC}(\mathbb{G}). \tag{7.8}$$

Hence, by combining (7.8) and Lemma 7.1, we conclude that M^τ is null (i.e. $(\widetilde{V})^\tau = 0$), or equivalently

$$0 = E\left(\widetilde{V}_\tau\right) = E\left(\int_0^{+\infty} Z_{s-} d\widetilde{V}_s\right) = E\left(Z_{T-} I_{\{T<+\infty\}}\right). \tag{7.9}$$

Thus $T = +\infty,\ P-a.s.$, and the thin set $\{\widetilde{Z} = 0\} \cap \{Z_- > 0\}$ is evanescent (see Proposition 2.18 on Page 20 in [78]). This ends the proof of the theorem. $\square$

Corollary 7.2.1: For any random time τ, if either m is continuous or Z is positive, then S^τ satisfies SC($\mathbb{G}$) for any process S that satisfies SC($\mathbb{F}$).

Proof. Under the condition either m is continuous or Z is positive, the set $\{\widetilde{Z} = 0\} \cap \{Z_- > 0\}$ is evanescent. Hence, the proof of the corollary follows immediately from Theorem 7.2. Below, we detail a direct proof for the case when m is continuous. In fact this direct proof contains the key ideas for the proof of Theorem 7.1, but with more delicate arguments due to the jumps.

Suppose that S satisfies SC($\mathbb{F}$) with the canonical decomposition $S = S_0 + M - \lambda \centerdot \langle M, M\rangle^{\mathbb{F}}$, where $M \in \mathcal{M}^2_{loc}(\mathbb{F})$ and $\lambda \in L^2_{loc}(M, \mathbb{F})$. Put the $\mathbb{G}$-locally square integrable local martingale (see Jeulin [83])

$$\widehat{M} := I_{[\![0,\tau]\!]} \centerdot M - (Z_-)^{-1} I_{[\![0,\tau]\!]} \centerdot \langle M, m\rangle^{\mathbb{F}}. \tag{7.10}$$

Then, the canonical decomposition of S^τ under $\mathbb{G}$ has the form of

$$S^\tau = S_0 + \widehat{M} - \lambda I_{[\![0,\tau]\!]} \centerdot \langle M, M\rangle^{\mathbb{F}} + (Z_-)^{-1} I_{[\![0,\tau]\!]} \centerdot \langle M, m\rangle^{\mathbb{F}}. \tag{7.11}$$

Thus, the proof will follow as long as we find a $\mathbb{G}$-predictable process $\widehat{\lambda} \in L^2_{loc}(\widehat{M}, \mathbb{G})$ such that

$$-\lambda I_{[\![0,\tau]\!]} \centerdot \langle M, M\rangle^{\mathbb{F}} + (Z_-)^{-1} I_{[\![0,\tau]\!]} \centerdot \langle M, m\rangle^{\mathbb{F}} = \widehat{\lambda} \centerdot \langle \widehat{M}\rangle^{\mathbb{G}}. \tag{7.12}$$

Indeed, since m is continuous, the Galtchouk-Kunita-Watanabe decomposition of m with respect to M under $\mathbb{F}$ implies the existence of an $\mathbb{F}$-predictable process $\beta_m \in L^2_{loc}(M)$ and a locally square integrable $\mathbb{F}$-local martingale $m^\perp$ such that

$$m = m_0 + \beta_m \centerdot M + m^\perp \qquad \text{and} \qquad \langle M, m^\perp\rangle^{\mathbb{F}} = 0. \tag{7.13}$$

Therefore,

$$\begin{aligned}-\lambda I_{[\![0,\tau]\!]} \centerdot \langle M, M\rangle^{\mathbb{F}} + \frac{1}{Z_-} I_{[\![0,\tau]\!]} \centerdot \langle M, m\rangle^{\mathbb{F}} &= -\lambda I_{[\![0,\tau]\!]} \centerdot \langle M, M\rangle^{\mathbb{F}} + \frac{\beta_m}{Z_-} I_{[\![0,\tau]\!]} \centerdot \langle M, M\rangle^{\mathbb{F}} \\ &= \left(-\lambda + \frac{\beta_m}{Z_-}\right) I_{[\![0,\tau]\!]} \centerdot \langle M, M\rangle^{\mathbb{F}}. \end{aligned} \tag{7.14}$$

It is easy to prove that $I_{[\![0,\tau]\!]} \centerdot \langle M\rangle^{\mathbb{F}} = I_{[\![0,\tau]\!]} \centerdot \langle M\rangle^{\mathbb{G}} = I_{[\![0,\tau]\!]} \centerdot \langle \widehat{M}\rangle^{\mathbb{G}}$, due to the continuity of m. Thus, we obtain

$$S^\tau = S_0 + \widehat{M} - \widehat{\lambda} \centerdot \langle \widehat{M}\rangle^{\mathbb{G}}, \tag{7.15}$$

where $\widehat{\lambda} := \left(\lambda - \frac{\beta_m}{Z_-}\right) I_{[\![0,\tau]\!]}$. It is obvious that $\widehat{\lambda} \in L^2_{loc}(\widehat{M})$ due to the local boundedness of $(Z_-)^{-1} I_{[\![0,\tau]\!]}$. This ends the proof of the corollary. □

One may wonder what could happen when the condition (7.6) fails. Below, we provide an example when $\{\widetilde{Z} = 0 < Z_-\}$ is nonempty, and S^τ fails to satisfy SC($\mathbb{G}$) (for the arbitrage opportunities in the example, we refer to Aksamit et al. [3]).

Proposition 7.1: *Suppose that the stochastic basis $(\Omega, \mathbb{A}, \mathbb{F} = (\mathcal{F}_t)_{t\geq 0}, P)$ supports a Poisson process N with intensity λ, and the stock price –denoted by X– is given by*

$$dX_t = X_{t-}\psi dM_t, \quad \textit{where}, \quad \psi > -1, \ \textit{and}\ \psi \neq 0, \qquad M_t = N_t - \lambda t.$$

If

$$\tau = k_1 T_1 + k_2 T_2,$$

where $T_i = \inf\{t \geq 0 : N_t \geq i\},\ i \geq 1,\quad k_1, k_2 > 0,$ *and* $k_1 + k_2 = 1,$

then X^τ *does not satisfy* $SC(\mathbb{G})$.

Proof. We recall from Aksamit et al. [3] that the Azéma supermartingale Z and m take the forms of:

$$Z = I_{[\![0,T_1[\![} + \phi^m I_{[\![T_1,T_2[\![}, \quad m = 1 - \phi^m I_{]\!]T_1,T_2]\!]} \centerdot M, \quad \text{where } \phi^m_t = e^{-\lambda \frac{k_1}{k_2}(t-T_1)}. \tag{7.16}$$

Then, it is easy to calculate that

$$\begin{aligned}
\frac{1}{Z_-} I_{[\![0,\tau]\!]} \centerdot \langle X, m\rangle_t &= \frac{-1}{Z_-} I_{]\!]T_1,\tau]\!]} X_- \psi \phi^m \centerdot \langle M\rangle_t \\
&= -\lambda \int_0^t \frac{1}{Z_{u-}} I_{]\!]T_1,\tau]\!]} X_{u-} \psi_u \phi^m_u du = -\lambda \int_0^t X_{u-} \psi_u I_{]\!]T_1,\tau]\!]} du
\end{aligned}$$

and

$$\begin{aligned}
\left\langle \widehat{X}, \widehat{X} \right\rangle^{\mathbb{G}}_t &= I_{[\![0,\tau]\!]} \centerdot \langle X, X\rangle_t + \frac{1}{Z_-} I_{[\![0,\tau]\!]} \centerdot \left(\sum \Delta m (\Delta X)^2 \right)^{p,\mathbb{F}}_t \\
&= \lambda \int_0^t X^2_{u-} \psi^2_u I_{[\![0,\tau]\!]} du - \lambda \int_0^t \frac{1}{Z_{u-}} X^2_{u-} \psi^2_u \phi^m_u I_{]\!]T_1,\tau]\!]} du \\
&= \lambda \int_0^t X^2_{u-} \psi^2_u I_{[\![0,T_1]\!]} du,
\end{aligned}$$

where $\widehat{X}$ is defined via (7.26). Hence, there is no $\mathbb{G}$-predictable process $\widehat{\lambda} \in L^2_{loc}(\widehat{X})$ satisfying

$$\frac{1}{Z_-} I_{[\![0,\tau]\!]} \centerdot \langle X, m\rangle = \widehat{\lambda} \centerdot \left\langle \widehat{X}, \widehat{X} \right\rangle^{\mathbb{G}}, \tag{7.17}$$

since $]\!]T_1, \tau]\!]$ and $[\![0, T_1]\!]$ are disjoint. This ends the proof of the proposition. □

7.1.2 Structure Conditions under a Class of Honest Times

In this section, we focus on answering the two problems **(Prob(7.I))** and **(Prob(7.II))** when we totally incorporate a random time. This can be achieved by splitting the whole half line into two stochastic intervals $]\!]0, \tau]\!]$ and $]\!]\tau, +\infty[\![$. The first part, i.e.

S^τ is already studied in the previous section. Thus, this section will concentrate on studying the Structure Conditions of S on the stochastic interval $]\!]\tau, +\infty[\![$. The first obstacle that one can face in this study is how far the (H')–hypothesis is preserved on this interval (i.e. any $\mathbb{F}$-semimartingale stays a $\mathbb{G}$-semimartingale)? To overcome this difficulty that is not our main focus in this chapter, we restrict our study to the important class of random times, called honest times. Below, we recall its definition.

Definition 7.2: A random time τ is called an honest time, if for any t, there exists an $\mathcal{F}_t$-measurable random variable τ_t such that $\tau I_{\{\tau<t\}} = \tau_t I_{\{\tau<t\}}$.

We refer to Jeulin [83, Chapter 4] for more information on honest times. Throughout this section, the random time τ is supposed to be honest and satisfies

$$Z_\tau < 1, \quad P - a.s. \tag{7.18}$$

Remark 7.1: This assumption is also crucial for the validity of No-Unbounded-Profit-with-Bounded-Risk after an honest time. We refer to Choulli et al [4] for more details on this subject.

Now, we state the two main theorems in this section. This answers partially the problem **(Prob(7.I))** and completely the problem **(Prob(7.II))**.

Theorem 7.3: *Let τ be an honest time satisfying $Z_\tau < 1, a.s.$. If S is a process satisfying $SC(\mathbb{F})$ and*

$$\{\Delta M^S \neq 0\} \cap \{\widetilde{Z} = 1 > Z_-\} = \emptyset, \tag{7.19}$$

then $S - S^\tau$ satisfies $SC(\mathbb{G})$.

Proof. The proof will be detailed in Section 7.3. □

Theorem 7.4: *Let τ be an $\mathbb{F}$-honest time satisfying $Z_\tau < 1, a.s.$. Then, the following are equivalent.*

(a) *The thin set* $\{\widetilde{Z} = 1 > Z_-\}$ *is evanescent.*

(b) *For any process* X *satisfying* $SC(\mathbb{F})$, $X - X^\tau$ *satisfies* $SC(\mathbb{G})$.

Proof. The proof of (a) $\Rightarrow$ (b) is a direct consequence of Theorem 7.3. To prove the reverse, we assume that assertion (b) holds, and follow similar steps as in the proof of Theorem 7.2. For the sake of completeness, we give the full details. Since $\{\widetilde{Z} = 1\} \cap \{Z_- < 1\} \subset \{\Delta m \neq 0\}$, it is a thin set. Let T be any stopping time such that $[\![T]\!] \subset \{\widetilde{Z} = 1\} \cap \{Z_- < 1\}$. Then, consider the following $\mathbb{F}$-martingale,

$$M = V - \widetilde{V} \in \mathcal{M}_0(\mathbb{F}), \text{ where } V := I_{[\![T,+\infty[\![} \text{ and } \widetilde{V} := (V)^{p,\mathbb{F}}. \tag{7.20}$$

Since $\{T > \tau\} \subset \{\widetilde{Z}_T < 1\}$ (see Jeulin [83] or Choulli et al. [4]) and $\widetilde{Z}_T = 1$, we deduce that $\tau \geq T,\ P - a.s.$, and

$$M - M^\tau = -I_{]\!]\tau,+\infty[\![} \centerdot \widetilde{V} \tag{7.21}$$

which is $\mathbb{G}$-predictable and satisfies SC($\mathbb{G}$). By combining (7.21) with Lemma 7.1, we conclude that $M - M^\tau$ is null (i.e. $I_{]\!]\tau,+\infty[\![} \centerdot \widetilde{V} = 0$). Thus, we get

$$0 = E\left(I_{]\!]\tau,+\infty[\![} \centerdot \widetilde{V}_\infty\right) = E\left(\int_0^{+\infty}(1 - Z_{s-})d\widetilde{V}_s\right) = E\left((1 - Z_{T-})I_{\{T<+\infty\}}\right), \tag{7.22}$$

or equivalently $(1-Z_{T-})I_{\{T<+\infty\}} = 0$ that implies that $T = +\infty,\ P-a.s.$. Therefore the thin set $\{\widetilde{Z} = 1\} \cap \{Z_- < 1\}$ is evanescent (see Proposition 2.18 on Page 20 in [78]). This ends the proof of the theorem. □

As a simple corollary, we have

Corollary 7.4.1: For any $\mathbb{F}$-honest time τ satisfying $Z_\tau < 1, P - a.s.$, if m is continuous, then $S - S^\tau$ satisfies SC($\mathbb{G}$) for any process S that satisfies SC($\mathbb{F}$).

Proof. It is enough to notice that under the condition m is continuous, the set $\{1 = \widetilde{Z} > Z_-\} \subset \{\Delta m \neq 0\}$ is empty. □

Example 7.1 Herein, we present an example for which $\{\Delta X \neq 0\} \cap \{1 = \widetilde{Z} > Z_-\} \neq \emptyset$, and $X - X^\tau$ fails to satisfy SC($\mathbb{G}$). We suppose given a Poisson process N, with intensity rate $\lambda > 0$, and the natural filtration $\mathbb{F}$. The stock price process is given by

$$dX_t = X_{t-}\sigma dM_t, \quad X_0 = 1, \qquad M_t := N_t - \lambda t,$$

or equivalently $X_t = \exp(-\lambda\sigma t + \ln(1+\sigma)N_t)$, where $\sigma > 0$. In what follows, we introduce the notations

$$a := -\frac{1}{\ln(1+\sigma)}\ln b, \quad 0 < b < 1, \quad \mu := \frac{\lambda\sigma}{\ln(1+\sigma)} \quad \text{and} \quad Y_t := \mu t - N_t.$$

We associate to the process Y its ruin probability, denoted by $\Psi(x)$ given by, for $x \geq 0$,

$$\Psi(x) = P(T^x < \infty), \qquad \text{with} \quad T^x = \inf\{t : x + Y_t < 0\}\,. \tag{7.23}$$

Proposition 7.2: *Consider the model and its notations in Example 7.1, and the following random time*

$$\tau := \sup\{t :\; X_t \geq b\} = \sup\{t : Y_t \leq a\}. \tag{7.24}$$

Then $X - X^\tau$ fails to satisfy SC($\mathbb{G}$).

Proof. We recall from Aksamit et al. [3] that the supermartingale Z and m are given by

$$\begin{aligned} Z_t &= P(\tau > t|\mathcal{F}_t) = \Psi(Y_t - a)I_{\{Y_t \geq a\}} + I_{\{Y_t < a\}} = 1 + I_{\{Y_t \geq a\}}\left(\Psi(Y_t - a) - 1\right), \\ \Delta m &= I_{\{Y_- > a+1\}}\left(\Psi(Y_- - a - 1) - 1\right)\Delta N - I_{\{Y_- > a\}}\left(\Psi(Y_- - a) - 1\right)\Delta N \\ &:= I_{\{Y_- > a+1\}}\phi_1 \Delta N - I_{\{Y_- > a\}}\phi_2 \Delta N, \end{aligned}$$

where Ψ is defined in (7.23). Then it is easy to calculate that

$$\frac{1}{1-Z_-} I_{]\!]\tau,+\infty[\![} \cdot \langle X, m\rangle_t = -\lambda\sigma \int_0^t \frac{X_{u-}}{\phi_2(u)} \left\{ I_{\{Y_{u-}>a+1\}}\phi_1(u) - I_{\{Y_{u-}>a\}}\phi_2(u) \right\} I_{]\!]\tau,+\infty[\![} du, \text{ and}$$

$$\begin{aligned} I_{]\!]\tau,+\infty[\![} \cdot \left\langle \widehat{X}, \widehat{X} \right\rangle_t^{\mathbb{G}} &= I_{]\!]\tau,+\infty[\![} \frac{1}{1-Z_-} \cdot \left((1-\widetilde{Z}) \cdot [X,X] \right)_t^{p,\mathbb{F}} \\ &= \lambda\sigma^2 \int_0^t \frac{\phi_1(u) X_{u-}^2}{\phi_2(u)} I_{\{Y_{u-}>a+1\}} I_{]\!]\tau,+\infty[\![} du, \end{aligned}$$

where $\widehat{X}$ is defined via (7.47). Notice that on the interval $\{a+1 \geq Y_- > a\}$,

$$\frac{1}{1-Z_-} I_{]\!]\tau,+\infty[\![} \cdot \langle X, m\rangle_t = \lambda\sigma \int_0^t X_{u-} I_{\{Y_{u-}>a\}} I_{]\!]\tau,+\infty[\![} du, \quad \text{while } I_{]\!]\tau,+\infty[\![} \cdot \left\langle \widehat{X}, \widehat{X} \right\rangle^{\mathbb{G}} = 0.$$

Hence, there is no $\mathbb{G}$-predictable process $\widehat{\lambda} \in L^2_{loc}(X)$ such that

$$\frac{1}{1-Z_-} I_{]\!]\tau,+\infty[\![} \cdot \langle X, m\rangle = I_{]\!]\tau,+\infty[\![} \widehat{\lambda} \cdot \left\langle \widehat{X}, \widehat{X} \right\rangle^{\mathbb{G}}.$$

Hence, $X - X^\tau$ fails to satisfy SC($\mathbb{G}$). □

7.2 The Key Stochastic Tools

In this section, we will provide the crucial stochastic tools for the proof of two main theorems announced in Section 7.1. This section contains three subsections. In subsection 7.2.1, we recall a Lazaro and Yor's result that we extend to the case of locally square integrable martingales. Then, we give the definition and important properties of the optional stochastic integral. In subsections 7.2.2–7.2.3, we provide innovative lemmas and propositions that play key roles in the proof of the two main theorems.

7.2.1 Lazaro–Yor's Representation

This subsection introduces and slightly extends two stochastic tools that are pillars in our analysis, namely the Lazaro-Yor's representation and the optional stochastic integral. The following extends the representation of Lazaro and Yor [45] to the "local and dynamic" framework.

Lemma 7.3: *Let M be a local martingale and $(Y^n_t), (Y_t)$ be two uniform integrable martingales such that Y^n_∞ converges to Y_∞ weakly in L^1. If Y^n admits representation as stochastic integrals with respect to M, i.e.*

$$Y^n_t = \int_0^t \phi^n_s dM_s. \tag{7.25}$$

Then there exists a predictable process ϕ such that $Y_t = \int_0^t \phi_s dM_s$.

Proof. For the proof we refer the reader to Lazaro and Yor [45] or Jacod [76]. □

To extend this lemma to the dynamic case, we first define the weak convergence in the space $\mathcal{M}^2_{loc}(\mathbb{H})$.

Definition 7.3: A sequence of elements of $\mathcal{M}^2_{loc}(\mathbb{H})$, $(Y^n)_{n\geq 1}$, is said to converge weakly in $\mathcal{M}^2_{loc}(\mathbb{H})$ if there exist $Y \in \mathcal{M}^2_{loc}(\mathbb{H})$, and a sequence of $\mathbb{H}$-stopping times that increases to infinity, $(\sigma_k)_{k\geq 1}$ such that for each $k \geq 1$, the sequence $(Y^n_{\sigma_k})_{n\geq 1}$ converges weakly to Y_{σ_k} in $L^2(\mathbb{P})$.

Below, we extend Lazaro-Yor's lemma to the dynamic and local framework, which will play important rôles in the proofs of the main results.

Lemma 7.4: *Let M be a locally square integrable local martingale, and $(\phi_n)_{n\geq 1}$ be a sequence of predictable processes that belong to $L^2_{loc}(M)$. If $(\phi_n \centerdot M)$ converges weakly in $\mathcal{M}^2_{loc}(\mathbb{H})$, then there exists $\phi \in L^2_{loc}(M)$ such that $\phi \centerdot M$ coincides with its limit.*

Proof. If $(\phi_n \centerdot M)$ converges weakly in $\mathcal{M}^2_{loc}(\mathbb{H})$ with the localizing sequence $(\sigma_k)_{k\geq 1}$, then there exists $Y \in \mathcal{M}^2_{loc}$ such that Y^{σ_k} and $(\phi_n \centerdot M)^{\sigma_k}$ are square integrable martingales and $(\phi_n \centerdot M)_{\sigma_k}$ converges weakly in $L^1(P)$ to Y_{σ_k}. Hence, a direct application of Lemma 7.3 implies the existence of a predictable process ψ^k that is

M^{σ_k}-integrable and the resulting integrable $(\psi^k \centerdot M^{\sigma_k})$ coincides with Y^{σ_k} (due to the uniqueness of the limit). Then, by putting

$$\phi := \sum_{k=1}^{+\infty} \psi^k I_{]\!]\sigma_{k-1},\sigma_k]\!]},$$

we can easily deduce that $\phi \in L^2_{loc}(M)$, as well as $Y = \phi \cdot M$. This ends the proof of the corollary. $\square$

7.2.2 The Key Stochastic Results for the Part up to Random Horizon

In this section, we will present some lemmas and propositions that are crucial for the proof of Theorem 7.1. First, let us point out that to prove $S^\tau \in \mathrm{SC}(\mathbb{G})$, it is essential to find a $\mathbb{G}$-predictable process $\phi^{\mathbb{G}} \in L^2_{loc}(\widehat{M}^S)$ such that $I_{[\![0,\tau]\!]} \centerdot \langle m, M^S\rangle = \phi^{\mathbb{G}} \centerdot \langle \widehat{M}^S\rangle^{\mathbb{G}}$. As one could predict, the difficulty lies in the existence and the locally square integrability of $\phi^{\mathbb{G}}$. Due to Jeulin [83], the (H')-hypothesis is preserved, i.e. any $\mathbb{F}$-semimartingale stays a $\mathbb{G}$-semimartingale on $[\![0,\tau]\!]$.

Lemma 7.5: *To any $\mathbb{F}$-local martingale M, we associate the process $\widehat{M}^{(b)}$ given by*

$$\widehat{M}^{(b)}_t := M_{t\wedge\tau} - \int_0^{t\wedge\tau} \frac{d\langle M, m\rangle^{\mathbb{F}}_s}{Z_{s-}}, \tag{7.26}$$

which is a $\mathbb{G}$-local martingale.

Proof. The proof of this lemma can be found in Jeulin [83]. $\square$

On the stochastic integral $[\![0,\tau]\!]$, it is worthy to keep in mind that for any $\mathbb{F}$-local martingale M, $\widehat{M}^{(b)}$ would be defined via (7.26) in what follows.

Below, we recall an important lemma due to Choulli et al. [4].

Lemma 7.6 ([4]): *The following assertions hold.*

(a) *For any $\mathbb{F}$-adapted process V with locally integrable variation, we have*

$$(V^{\tau})^{p,\mathbb{G}} = (Z_{-})^{-1} I_{[\![0,\tau]\!]} \centerdot \left(\widetilde{Z} \centerdot V\right)^{p,\mathbb{F}}. \tag{7.27}$$

(b) *For any $\mathbb{F}$-local martingale M, we have, on $[\![0,\tau]\!]$*

$${}^{p,\mathbb{G}}\left(\frac{\Delta M}{\widetilde{Z}}\right) = \frac{{}^{p,\mathbb{F}}\left(\Delta M I_{\{\widetilde{Z}>0\}}\right)}{Z_{-}}, \quad \textit{and} \quad {}^{p,\mathbb{G}}\left(\frac{1}{\widetilde{Z}}\right) = \frac{{}^{p,\mathbb{F}}\left(I_{\{\widetilde{Z}>0\}}\right)}{Z_{-}}. \tag{7.28}$$

Remark 7.2: To explain the main difficulty that one will encounter when proving the Structure Conditions for S^{τ}, we assume that S is an $\mathbb{F}$-local martingale. Then, due to Choulli et al. [4], the condition $\{\Delta S \neq 0\} \cap \{\widetilde{Z} = 0\} \cap \{Z_{-} > 0\} = \emptyset$ implies that

$$\mathcal{E}\left(\frac{-Z_{-}^{2}}{Z_{-}^{2} + \Delta\langle m\rangle} \frac{1}{\widetilde{Z}} I_{[\![0,\tau]\!]} \odot \widehat{m}^{(b)}\right) \tag{7.29}$$

is a local martingale density for S^{τ}. However, because of the term $1/\widetilde{Z}$, in general it is not locally square integrable. To overcome this difficulty, one applies the Galtchouk-Kunita-Watanabe decomposition of

$$H^{n} \odot \widehat{m}^{(b)} := \frac{1}{\widetilde{Z}} I_{\{\widetilde{Z}\geq 1/n\}} I_{[\![0,\tau]\!]} \odot \widehat{m}^{(b)}$$

with respect to $\widehat{M}^{(b)}$, and obtain

$$H^{n} \odot \widehat{m}^{(b)} = \phi^{n} \centerdot \widehat{M}^{(b)} + L^{n}, \text{ where } \phi^{n} \in L^{2}_{loc}(\widehat{M}^{(b)}), \text{ and } L^{n} \perp \widehat{M}^{(b)}.$$

Then, the main difficulty lies in proving the weak convergence of $\phi^{n} \centerdot \widehat{M}^{(b)}$, and by using Lemma 7.3 and Lemma 7.4 afterwards, we conclude that $\phi^{n} \centerdot \widehat{M}^{(b)}$ converges weakly in $\mathcal{M}^{2}_{loc}(\mathbb{G})$ to a locally square integrable local martingale having the form of $\Phi_1 \centerdot \widehat{M}^{(b)}$. Therefore, this will establish the connection between $\langle \widehat{M}^{(b)} \rangle^{\mathbb{G}}$ and $\langle m, M\rangle^{\mathbb{F}}$ on the stochastic interval $[\![0,\tau]\!]$.

We start with a proposition that is dealing with weakly convergence in $\mathcal{M}^2_{loc}$ that would be frequently used in what follows.

Proposition 7.3: *Let M, N be two $\mathbb{H}$-locally square integrable local martingales and H be an non-negative $\mathbb{H}$-optional process such that $H \bullet [M,M]$ and $H \bullet [N,N]$ have finite values. If $H^n := HI_{\{H\leq n\}}$, then the following assertions hold.*

(a) If $\sqrt{H} \in {}^oL^2_{loc}(M)$, then $\sqrt{H}I_{\{H\leq n\}} \odot M$ converges in $\mathcal{M}^2_{loc}(\mathbb{H})$ to $\sqrt{H} \odot M$.

(b) If $\sqrt{H} \in {}^oL^2_{loc}(M) \cap {}^oL^2_{loc}(N)$, then there exists a sequence of $\mathbb{H}$-stopping times $(\eta_k)_{k\geq 1}$ increasing to infinity such that for all $k \geq 1$,

$$\left\langle \sqrt{H}I_{\{H\leq n\}} \odot M, \sqrt{H}I_{\{H\leq n\}} \odot N \right\rangle_{\eta_k} \quad \text{converges in } \ L^1 \ \text{ to } \left\langle \sqrt{H} \odot M, \sqrt{H} \odot N \right\rangle_{\eta_k}.$$

(c) Suppose that $\sqrt{H} \in {}^oL^2_{loc}(Y) \cap {}^oL^2_{loc}(N)$, where $Y \in \{\psi \bullet M : \ \psi \in L^2_{loc}(M)\}$. Consider the Galtchouk-Kunita-Watanabe decomposition of $H^n \odot N$ with respect to M given by

$$H^n \odot N = \phi^n \bullet M + L^n, \quad \text{where } \phi^n \in L^2_{loc}(M), \text{ and } L^n \perp M. \qquad (7.30)$$

Then, $\phi^n \bullet M$ converges weakly in $\mathcal{M}^2_{loc}(\mathbb{H})$.

Proof. (a) Denote $(\sigma_k)_{k\geq 1}$ the localizing sequence of $H \bullet [M,M]$. Due to Lemma 2.3, we calculate that

$$\begin{aligned}\left\| \left(\sqrt{H}I_{\{H\leq n\}} - \sqrt{H}\right) \odot M^{\sigma_k} \right\|^2_{\mathcal{M}^2(\mathbb{H})} &= E\left(\left\langle \left(\sqrt{H}I_{\{H\leq n\}} - \sqrt{H}\right) \odot M \right\rangle_{\sigma_k} \right) \\ &= E\left(\left\langle \left(\sqrt{H}I_{\{H> n\}}\right) \odot M \right\rangle_{\sigma_k} \right) \\ &\leq E\left(HI_{\{H>n\}} \bullet [M,M]_{\sigma_k} \right) \to 0, \text{as } n \to +\infty.\end{aligned}$$

(b) Without lose of generality, we could assume $\sqrt{H} \odot M$ and $\sqrt{H} \odot N$ are both square integrable. We derive from (a) and Kunita-Watanabe inequality that

$$\begin{aligned}
&E\left(\left|\left\langle \sqrt{H}I_{\{H\le n\}}\odot M, \sqrt{H}I_{\{H\le n\}}\odot N\right\rangle_{\infty} - \left\langle \sqrt{H}\odot M, \sqrt{H}\odot N\right\rangle_{\infty}\right|\right)\\
&\le E\left(\left|\left\langle \sqrt{H}I_{\{H\le n\}}\odot M, \sqrt{H}I_{\{H\le n\}}\odot N\right\rangle_{\infty} - \left\langle \sqrt{H}I_{\{H\le n\}}\odot M, \sqrt{H}\odot N\right\rangle_{\infty}\right|\right)\\
&+E\left(\left|\left\langle \sqrt{H}I_{\{H\le n\}}\odot M, \sqrt{H}\odot N\right\rangle_{\infty} - \left\langle \sqrt{H}\odot M, \sqrt{H}\odot N\right\rangle_{\infty}\right|\right)\\
&= E\left(\left|\left\langle \sqrt{H}I_{\{H\le n\}}\odot M, \sqrt{H}I_{\{H> n\}}\odot N\right\rangle_{\infty}\right|\right) + E\left(\left|\left\langle \sqrt{H}I_{\{H> n\}}\odot M, \sqrt{H}\odot N\right\rangle_{\infty}\right|\right)\\
&\le \sqrt{E\left(\left\langle \sqrt{H}I_{\{H\le n\}}\odot M\right\rangle_{\infty}\right)}\sqrt{E\left(\left\langle \sqrt{H}I_{\{H> n\}}\odot N\right\rangle_{\infty}\right)}\\
&+\sqrt{E\left(\left\langle \sqrt{H}I_{\{H> n\}}\odot M\right\rangle_{\infty}\right)}\sqrt{E\left(\left\langle \sqrt{H}\odot N\right\rangle_{\infty}\right)} \to 0, \text{ as } n\to+\infty.
\end{aligned}$$

(c) For any $\psi \in L^2_{loc}(M)$, using Lemma 2.3, we derive

$$\begin{aligned}
\langle \phi^n \centerdot M, \psi \centerdot M\rangle^{\mathbb{H}} &= \langle H^n \odot N, \psi \centerdot M\rangle^{\mathbb{H}} = \left(H^n \centerdot [N, \psi \centerdot M]\right)^{p,\mathbb{H}}\\
&= \left\langle \sqrt{H^n}\odot N, \psi\sqrt{H^n}\odot M\right\rangle^{\mathbb{H}} + \sum{}^{p,\mathbb{H}}\left(\sqrt{H^n}\Delta N\right)^{p,\mathbb{H}}\left(\psi\sqrt{H^n}\Delta M\right)\\
&:= V_1^n + V_2^n. \qquad (7.31)
\end{aligned}$$

Thanks to assertions (a) and (b), we deduce that both processes $\sqrt{H^n}\odot N$ and $\psi\sqrt{H^n}\odot M$ converge weakly in $\mathcal{M}^2_{loc}(\mathbb{H})$ to $\sqrt{H}\odot N$ and $\sqrt{H}\psi\odot M$, and

$$V_1^n \text{ converges locally in } L^1(P) \text{ to } \left\langle \sqrt{H}\odot N, \psi\sqrt{H}\odot M\right\rangle^{\mathbb{H}}. \qquad (7.32)$$

Due to Cauchy–Schwarz inequality, we derive

$$\begin{aligned}
|V_2^{n+l} - V_2^n| &= \Big| \sum {}^{p,\mathbb{H}}\Big(\sqrt{H^{n+l}}\Delta N\Big)\ {}^{p,\mathbb{H}}\Big(\psi\sqrt{H^{n+l}}\Delta M\Big) - \ {}^{p,\mathbb{H}}\Big(\sqrt{H^n}\Delta N\Big)\ {}^{p,\mathbb{H}}\Big(\psi\sqrt{H^n}\Delta M\Big)\Big| \\
&\le \Big| \sum {}^{p,\mathbb{H}}\Big(\sqrt{H^{n+l}}\Delta N\Big)\ \Big(\ {}^{p,\mathbb{H}}\Big(\psi\sqrt{H^{n+l}}\Delta M\Big) - \ {}^{p,\mathbb{H}}\Big(\psi\sqrt{H^n}\Delta M\Big)\Big)\Big| \\
&\quad + \Big| \sum \Big(\ {}^{p,\mathbb{H}}\Big(\sqrt{H^{n+l}}\Delta N\Big) - \ {}^{p,\mathbb{H}}\Big(\sqrt{H^n}\Delta N\Big)\Big)\ {}^{p,\mathbb{H}}\Big(\psi\sqrt{H^n}\Delta M\Big)\Big| \\
&= \Big| \sum {}^{p,\mathbb{H}}\Big(\sqrt{H^{n+l}}\Delta N\Big)\ {}^{p,\mathbb{H}}\Big(\psi\sqrt{H} I_{\{n<H\le n+l\}}\Delta M\Big)\Big| \\
&\quad + \Big| \sum \ {}^{p,\mathbb{H}}\Big(\sqrt{H} I_{\{n<H\le n+l\}}\Delta N\Big)\ {}^{p,\mathbb{H}}\Big(\psi\sqrt{H^n}\Delta M\Big)\Big| \\
&\le \sqrt{(H \centerdot [N,N])^{p,\mathbb{H}}}\ \sqrt{\left(H I_{\{n<H\le n+l\}} \centerdot [\psi \centerdot M]\right)^{p,\mathbb{H}}} \\
&\quad + \sqrt{\left(I_{\{n<H\le n+l\}} H \centerdot [N,N]\right)^{p,\mathbb{H}}}\ \sqrt{(H \centerdot [\psi \centerdot M])^{p,\mathbb{H}}}.
\end{aligned}$$

An application of the Lebesgue Dominating convergence theorem implies the local convergence in $L^1(P)$ of the process V_2^n. This proves that $\langle \phi^n \centerdot M, \psi \centerdot M\rangle^{\mathbb{H}} = \langle H^n \odot N, \psi \centerdot M\rangle^{\mathbb{H}}$ converges locally in $L^1(P)$. Then, for any $K \in \mathcal{M}^2_{loc}(\mathbb{H})$, we have

$$K = \theta^K \cdot M + N^K, \quad \text{where} \quad \theta^K \in L^2_{loc}(M) \quad \text{and} \quad \langle M, N^K\rangle^{\mathbb{H}} = 0.$$

Therefore, $\langle \phi^n \centerdot M, K\rangle^{\mathbb{H}} = \left\langle \phi^n \centerdot M, \theta^K \centerdot M\right\rangle^{\mathbb{H}}$ converges locally in $L^1(\mathbb{P})$ and $\phi^n \centerdot M$ converges weakly in $\mathcal{M}^2_{loc}(\mathbb{H})$. This completes the proof of the proposition. $\square$

The following proposition proves that $(\widetilde{Z})^{-\frac{1}{2}} I_{[\![0,\tau]\!]}$ is locally square integrable with respect to a class of $\mathbb{G}$-local martingales.

Proposition 7.4: *If M is an $\mathbb{F}$-locally square integrable local martingale, then $I_{[\![0,\tau]\!]}(\widetilde{Z})^{-1} \centerdot [\widehat{M}^{(b)}, \widehat{M}^{(b)}] \in \mathcal{A}^+_{loc}(\mathbb{G})$. As a result, $I_{[\![0,\tau]\!]}(\widetilde{Z})^{-\frac{1}{2}}$ belongs to ${}^{o}L^2_{loc}(\widehat{M}^{(b)}, \mathbb{G})$, where $\widehat{M}^{(b)}$ is given by (7.26).*

Proof. Since $M \in \mathcal{M}^2_{loc}(\mathbb{F})$ and m is bounded, then there exists a sequence of $\mathbb{F}$-stopping times, $(T_k)_{k\ge 1}$, that increases to $+\infty$ such that

$$\langle M\rangle^{\mathbb{F}}_{T_k} + Var(\langle M, m\rangle^{\mathbb{F}})_{T_k} \le k, \quad P - a.s. \tag{7.33}$$

Since $Z_-^{-1} I_{[\![0,\tau]\!]}$ is locally bounded, then there exists a sequence of $\mathbb{G}$-stopping times, $(\tau_k)_{k\geq 1}$, that increases to $+\infty$ and

$$\sup_{0\leq t\leq \tau_k} \left((Z_{t-})^{-1} I_{\{t\leq\tau\}} \right) \leq k, \quad P-a.s. \tag{7.34}$$

Consider

$$\sigma_k := \tau_k \wedge T_k, \quad k\geq 1, \quad \text{and} \quad W := I_{[\![0,\tau]\!]}(Z_-)^{-1} \bullet \langle M, m\rangle^{\mathbb{F}}. \tag{7.35}$$

Then it is clear that $(\sigma_k)_{k\geq 1}$ is a localizing sequence for the process W, and due to Lemma 7.6-(b), we obtain

$$\begin{aligned} E\left\{ \frac{I_{[\![0,\tau]\!]}}{\widetilde{Z}} \bullet [\widehat{M}^{(b)}]_{\sigma_k} \right\} &\leq 2E\left\{ \frac{I_{[\![0,\tau]\!]}}{\widetilde{Z}} \bullet [M]_{T_k} \right\} + 2E\left\{ \frac{I_{[\![0,\tau]\!]}}{\widetilde{Z}} \bullet [W]_{\sigma_k} \right\} \\ &\leq 2E\left\{ I_{\{\widetilde{Z}>0\}} \bullet [M]_{T_k} \right\} + 2E\left\{ \frac{I_{[\![0,\tau]\!]}}{Z_-} \bullet [W]_{\sigma_k} \right\} \\ &\leq 2E[M]_{T_k} + 2kE[W]_{\sigma_k} < +\infty. \end{aligned}$$

Hence, $\widetilde{Z}^{-1} I_{[\![0,\tau]\!]} \bullet [\widehat{M}^{(b)}]$ is locally integrable and $I_{[\![0,\tau]\!]}(\widetilde{Z})^{-1/2} \in {}^{o}L^2_{loc}(\widehat{M}^{(b)}, \mathbb{G})$ (see He et al. [71]). □

Throughout the rest of this subsection, we will stick to the following notations

$$h^n := I_{\{\widetilde{Z}\geq \frac{1}{n}\}}, \quad H^n := Z_- \left(\widetilde{Z}\right)^{-1} h^n, \quad n\geq 1. \tag{7.36}$$

As an application of Proposition 7.3 and Proposition 7.4, in the following, we prove that $\phi^n \bullet \widehat{M}^{(b)}$ converges weakly in $\mathcal{M}^2_{loc}(\mathbb{G})$, where $H^n \odot \widehat{m}^{(b)} = \phi^n \bullet \widehat{M}^{(b)} + L^n$, $\phi^n \in L^2_{loc}(\widehat{M}^{(b)})$, and $L^n \perp \widehat{M}^{(b)}$.

Proposition 7.5: *Let $M \in \mathcal{M}^2_{loc}(\mathbb{F})$ and $\widehat{M}^{(b)}$ is given by (7.26). Then the following hold:*

(a) $\left(I_{[\![0,\tau]\!]}(\widetilde{Z})^{-1/2} I_{\{\widetilde{Z}\geq 1/n\}} \odot \widehat{M}^{(b)} \right)$ *converges in* $\mathcal{M}^2_{loc}(\mathbb{G})$ *to* $\left(I_{[\![0,\tau]\!]}(\widetilde{Z})^{-1/2} \odot \widehat{M}^{(b)} \right)$.

(b) *For any $K \in \mathcal{M}^2_{loc}(\mathbb{F})$, there exists a sequence of $\mathbb{G}$-stopping times, $(\eta_k)_{k\geq 1}$ that increases to $+\infty$, and for each $k \geq 1$*

$$\left\langle I_{[\![0,\tau]\!]}(\widetilde{Z})^{-1/2} I_{\{\widetilde{Z}\geq 1/n\}} \odot \widehat{M}^{(b)}, I_{[\![0,\tau]\!]}(\widetilde{Z})^{-1/2} I_{\{\widetilde{Z}\geq 1/n\}} \odot \widehat{K}^{(b)} \right\rangle^{\mathbb{G}}_{\eta_k}$$

converges in $L^1(P)$ to

$$\left\langle I_{[\![0,\tau]\!]}(\widetilde{Z})^{-1/2} \odot \widehat{M}^{(b)}, I_{[\![0,\tau]\!]}(\widetilde{Z})^{-1/2} \odot \widehat{K}^{(b)} \right\rangle^{\mathbb{G}}_{\eta_k},$$

where $\widehat{K}^{(b)}$ is defined via (7.26).

(c) *Consider the Galtchouk-Kunita-Watanabe decomposition of $H^n \odot \widehat{m}^{(b)}$ with respect to $\widehat{M}^{(b)}$*

$$H^n \odot \widehat{m}^{(b)} = \phi^n \centerdot \widehat{M}^{(b)} + L^n, \text{ where } \phi^n \in L^2_{loc}(\widehat{M}^{(b)}), \text{ and } L^n \perp \widehat{M}^{(b)}. \tag{7.37}$$

Then, $\phi^n \centerdot \widehat{M}^{(b)}$ converges weakly in $\mathcal{M}^2_{0,loc}(\mathbb{G})$ and

$$\Phi_1 := \lim_{n\to+\infty} \frac{d\langle H^n \odot \widehat{m}^{(b)}, \widehat{M}^{(b)}\rangle^{\mathbb{G}}}{d\langle \widehat{M}^{(b)}, \widehat{M}^{(b)}\rangle^{\mathbb{G}}} \in L^2_{loc}(\widehat{M}^{(b)}, \mathbb{G}). \tag{7.38}$$

Proof. It is a direct consequence of Proposition 7.3 and Proposition 7.4 by taking $H := I_{[\![0,\tau]\!]} Z_-(\widetilde{Z})^{-1}$, $H^n := Z_-\widetilde{Z}^{-1} I_{[\![0,\tau]\!]} I_{\{\widetilde{Z}\geq \frac{1}{n}\}}$, and $\mathbb{H} = \mathbb{G}$. To complete the proof, we just need to show (7.38). To this end, we apply Proposition 7.3 and Lemma 7.4 to $H^n \odot \widehat{m}^{(b)}$ to conclude that there exists $\Phi_1 \in L^2_{loc}(\widehat{M}^{(b)}, \mathbb{G})$ such that $\langle H^n \odot \widehat{m}^{(b)}, \widehat{M}^{(b)}\rangle^{\mathbb{G}}$ converges locally in L^1 to $\langle \Phi_1 \centerdot \widehat{M}^{(b)}, \widehat{M}^{(b)}\rangle^{\mathbb{G}}$ and

$$\Phi_1 := \lim_{n\to+\infty} \frac{d\langle H^n \odot \widehat{m}^{(b)}, \widehat{M}^{(b)}\rangle^{\mathbb{G}}}{d\langle \widehat{M}^{(b)}, \widehat{M}^{(b)}\rangle^{\mathbb{G}}}. \tag{7.39}$$

This completes the proof of the proposition. □

As explained in Remark 7.2, we will characterize the relationship between $I_{[\![0,\tau]\!]} \centerdot$

$\langle m, M\rangle^{\mathbb{F}}$ and $\langle \widehat{M}^{(b)}, \widehat{M}^{(b)}\rangle^{\mathbb{G}}$. This is the main focus of the following.

Proposition 7.6: *Let $M \in \mathcal{M}^2_{loc}(\mathbb{F})$ and $\widehat{M}^{(b)}$ be given in (7.26). If $\{\Delta M \neq 0\} \cap \{0 = \widetilde{Z} < Z_-\} = \emptyset$, we have*

$$\frac{1}{Z_-} I_{[\![0,\tau]\!]} \centerdot \langle m, M\rangle^{\mathbb{F}} = \widehat{\Phi_1} \centerdot \langle \widehat{M}^{(b)}\rangle^{\mathbb{G}}, \quad \text{and} \quad \widehat{\Phi_1} \in L^2_{loc}(\widehat{M}^{(b)}, \mathbb{G}), \tag{7.40}$$

where $\widehat{\Phi_1} := \Phi_1 \left({}^{p,\mathbb{F}}\Big(I_{\{\widetilde{Z}>0\}}\Big)\big(Z_-^2 + \Delta\langle m\rangle^{\mathbb{F}}\big)\right)^{-1} Z_- I_{[\![0,\tau]\!]}$ and Φ_1 is given in (7.38).

Proof. By Proposition 7.5, we know that

$$\lim_n \langle H^n \odot \widehat{m}^{(b)}, \widehat{M}^{(b)}\rangle^{\mathbb{G}} = \Phi_1 \centerdot \langle \widehat{M}^{(b)}, \widehat{M}^{(b)}\rangle^{\mathbb{G}}, \tag{7.41}$$

where H^n and $\widehat{m}^{(b)}$ are defined in (7.36) and (7.26) respectively. Now it remains to describe explicitly the limit $\lim_n \langle H^n \odot \widehat{m}^{(b)}, \widehat{M}^{(b)}\rangle^{\mathbb{G}}$. To this end, we first calculate

$$\begin{aligned}\frac{1}{Z_-} \centerdot [\widehat{m}^{(b)},\ \widehat{M}^{(b)}] = \frac{1}{Z_-} I_{[\![0,\tau]\!]} \centerdot [m, M] - I_{[\![0,\tau]\!]} \frac{\Delta M}{Z_-^2} \centerdot \langle m\rangle^{\mathbb{F}} - \frac{\Delta m}{Z_-^2} I_{[\![0,\tau]\!]} \centerdot \langle M, m\rangle^{\mathbb{F}} \\ + \frac{1}{Z_-^3} I_{[\![0,\tau]\!]} \Delta\langle m\rangle^{\mathbb{F}} \centerdot \langle M, m\rangle^{\mathbb{F}}.\end{aligned} \tag{7.42}$$

Then, by integrating H^n on both sides above, and using the properties of optional integration (see Proposition 2.3 and Proposition 7.3), we obtain

$$\begin{aligned}&\frac{1}{Z_-} I_{[\![0,\tau]\!]} \centerdot (h^n \centerdot [m, M])^{p,\mathbb{F}} = \frac{1}{Z_-} I_{[\![0,\tau]\!]} \centerdot (H^n \centerdot [m, M])^{p,\mathbb{G}} \\ &= \Big(\frac{1}{Z_-} H^n \centerdot [\widehat{m}^{(b)},\ \widehat{M}^{(b)}]\Big)^{p,\mathbb{G}} + I_{[\![0,\tau]\!]} \frac{{}^{p,\mathbb{G}}\Big(H^n \Delta M\Big)}{Z_-^2} \centerdot \langle m\rangle^{\mathbb{F}} \\ &+ I_{[\![0,\tau]\!]} \frac{{}^{p,\mathbb{G}}\Big(H^n \Delta m\Big)}{Z_-^2} \centerdot \langle M, m\rangle^{\mathbb{F}} - I_{[\![0,\tau]\!]} \frac{\Delta\langle M, m\rangle^{\mathbb{F}}}{Z_-^3}\ {}^{p,\mathbb{G}}\Big(H^n\Big) \centerdot \langle m\rangle^{\mathbb{F}}.\end{aligned} \tag{7.43}$$

Due to $\{\Delta M \neq 0\} \cap \{0 = \widetilde{Z} < Z_-\} = \emptyset$, we get

$$ {}^{p,\mathbb{F}}\left(\Delta M I_{\{\widetilde{Z}>0\}}\right) = {}^{p,\mathbb{F}}(\Delta M) = 0. $$

By taking the limit, we derive

$$ \begin{aligned} \lim_n \frac{I_{[\![0,\tau]\!]}}{Z_-} \bullet (h^n \bullet [m,M])^{p,\mathbb{F}} &= \frac{I_{[\![0,\tau]\!]}}{Z_-} \bullet \left(I_{\{\widetilde{Z}>0\}} \bullet [m,M]\right)^{p,\mathbb{F}} = \frac{I_{[\![0,\tau]\!]}}{Z_-} \bullet \langle m, M\rangle^{\mathbb{F}}, \\ \lim_n I_{[\![0,\tau]\!]} \frac{{}^{p,\mathbb{G}}\left(H^n \Delta M\right)}{Z_-^2} \bullet \langle m\rangle^{\mathbb{F}} &= I_{[\![0,\tau]\!]} \frac{-{}^{p,\mathbb{F}}\left(\Delta M I_{\{\widetilde{Z}=0\}}\right)}{Z_-^2} \bullet \langle m\rangle^{\mathbb{F}} = 0, \end{aligned} \tag{7.44} $$

and

$$ \begin{aligned} \lim_n {}^{p,\mathbb{G}}(H^n) I_{[\![0,\tau]\!]} &= {}^{p,\mathbb{F}}\left(I_{\{\widetilde{Z}>0\}}\right) I_{[\![0,\tau]\!]}, \\ \lim_n I_{[\![0,\tau]\!]} \frac{{}^{p,\mathbb{G}}\left(H^n \Delta m\right)}{Z_-^2} &= I_{[\![0,\tau]\!]} \frac{{}^{p,\mathbb{F}}\left(\Delta m I_{\{\widetilde{Z}>0\}}\right)}{Z_-^2} = I_{[\![0,\tau]\!]} \frac{{}^{p,\mathbb{F}}\left(I_{\{\widetilde{Z}=0\}}\right)}{Z_-}, \end{aligned} \tag{7.45} $$

where in (7.43)-(7.45) we used Lemma 7.6. Then, by combining the above equalities and (7.38), we conclude that

$$ \Phi_1 \bullet \langle \widehat{M}^{(b)}\rangle^{\mathbb{G}} = \lim_n \left(H^n \bullet [\widehat{m}^{(b)},\ \widehat{M}^{(b)}]\right)^{p,\mathbb{G}} = I_{[\![0,\tau]\!]}\ {}^{p,\mathbb{F}}\left(I_{\{\widetilde{Z}>0\}}\right)\left(1 + \frac{\Delta\langle m\rangle^{\mathbb{F}}}{Z_-^2}\right) \bullet \langle M, m\rangle^{\mathbb{F}}. $$

The proof of the proposition is completed due to the $\mathbb{G}$-local boundedness of $\left({}^{p,\mathbb{F}}\left(I_{\{\widetilde{Z}>0\}}\right)\right)^{-1} I_{[\![0,\tau]\!]}$ (see Lemma 7.7 below). □

Lemma 7.7: *The following process*

$$ V^{(b)} := \left({}^{p,\mathbb{F}}\left(I_{\{\widetilde{Z}>0\}}\right)\right)^{-1} I_{[\![0,\tau]\!]} \tag{7.46} $$

is $\mathbb{G}$-predictable and locally bounded.

Proof. It is enough to notice that $\widetilde{Z} \leq I_{\{\widetilde{Z}>0\}}$ and the process $(Z_-)^{-1} I_{[\![0,\tau]\!]}$ is $\mathbb{G}$-locally bounded. □

7.2.3 The Key Stochastic Results for the Part after an Honest Time

In this section, we will present some lemmas and propositions that are useful for the proof of Theorem 7.3.

Lemma 7.8: *For any $\mathbb{F}$-local martingale M, we associate $\widehat{M}^{(a)}$ given by,*

$$\widehat{M}^{(a)} := I_{]\!]\tau,+\infty[\![} \centerdot M + I_{]\!]\tau,+\infty[\![} (1 - Z_-)^{-1} \centerdot \langle M, m\rangle^{\mathbb{F}}, \tag{7.47}$$

which is a $\mathbb{G}$-local martingale.

Proof. The proof can be found in [17], [53] and [83]. □

Similarly as in the part on $[\![0,\tau]\!]$, we will keep the notation $\widehat{M}^{(a)}$ defined in (7.47) for any $\mathbb{F}$-local martingale M.

Below, we recall an important lemma due to Choulli et al. [4].

Lemma 7.9: *Suppose that $Z_\tau < 1$. Then the following assertions hold.*

(a) *The process $(1 - Z_-)^{-1} I_{]\!]\tau,+\infty[\![}$ is a $\mathbb{G}$-locally bounded and predictable process.*

(b) *For any $\mathbb{F}$-adapted process with locally integrable variation, V, we have*

$$I_{]\!]\tau,+\infty[\![} \centerdot V^{p,\mathbb{G}} = I_{]\!]\tau,+\infty[\![} (1 - Z_-)^{-1} \centerdot \left((1 - \widetilde{Z}) \centerdot V\right)^{p,\mathbb{F}}. \tag{7.48}$$

(c) *For any process V as in* (b)*, the $\mathbb{G}$-predictable projection of ΔV, is given on $]\!]\tau,+\infty[\![$ by*

$${}^{p,\mathbb{G}}(\Delta V) = (1 - Z_-)^{-1} \ {}^{p,\mathbb{F}}\left((1 - \widetilde{Z})\Delta V\right). \tag{7.49}$$

(d) *For any $\mathbb{F}$-local martingale, on $]\!]\tau,+\infty[\![$, we have*

$${}^{p,\mathbb{G}}\left(\frac{\Delta M}{1 - \widetilde{Z}}\right) = \frac{{}^{p,\mathbb{F}}\left(\Delta M I_{\{\widetilde{Z}<1\}}\right)}{1 - Z_-}, \quad \textit{and} \quad {}^{p,\mathbb{G}}\left(\frac{1}{1 - \widetilde{Z}}\right) = \frac{{}^{p,\mathbb{F}}\left(I_{\{\widetilde{Z}<1\}}\right)}{1 - Z_-}. \tag{7.50}$$

The following proposition proves that $(1-\widetilde{Z})^{-\frac{1}{2}} I_{]\!]\tau,+\infty[\![}$ is locally square integrable with respect to a class of $\mathbb{G}$-local martingales.

Proposition 7.7: *Let M be an $\mathbb{F}$-locally square integrable local martingale, then*

$$I_{]\!]\tau,+\infty[\![}\left(1-\widetilde{Z}\right)^{-1} \centerdot [\widehat{M}^{(a)}] \in \mathcal{A}^{+}_{loc}(\mathbb{G}),$$

where $\widehat{M}^{(a)}$ is defined via (7.47). As a result, $I_{]\!]\tau,+\infty[\![}\left(1-\widetilde{Z}\right)^{-1/2} \in {}^{o}L^2_{loc}(\widehat{M}^{(a)},\mathbb{G})$.

Proof. Let us denote the localizing sequences of $W := I_{]\!]\tau,+\infty[\![}(1-Z_-)^{-1/2} \centerdot \langle M, m\rangle^{\mathbb{F}}$, $(1-Z_-)^{-1} I_{]\!]\tau,+\infty[\![}$ and $[M]$ by $(\sigma_n)_{n\geq 1}$, $(\tau_n)_{n\geq 1}$ and $(T_n)_{n\geq 1}$ respectively. Then, due to Lemma 7.9, we derive that

$$\begin{aligned} E\left\{\frac{I_{]\!]\tau,+\infty[\![}}{1-\widetilde{Z}} \centerdot [\widehat{M}^{(a)}]_{\sigma_n\wedge\tau_n\wedge T_n}\right\} &\leq 2E\left\{\frac{I_{]\!]\tau,+\infty[\![}}{1-\widetilde{Z}} \centerdot [M]_{T_n}\right\} + 2E\left\{\frac{I_{]\!]\tau,+\infty[\![}}{1-\widetilde{Z}} \centerdot [W]_{\sigma_n\wedge\tau_n}\right\} \\ &\leq 2E[M]_{T_n} + E\left\{\frac{2I_{]\!]\tau,+\infty[\![}}{1-Z_-} \centerdot [W]_{\sigma_n\wedge\tau_n}\right\} < +\infty. \end{aligned}$$

This ends the proof of the proposition. □

Throughout the rest of this subsection, we will use the following notations

$$k^n := I_{\{1-\widetilde{Z}\geq \frac{1}{n}\}}, \quad K^n := (1-Z_-)\left(1-\widetilde{Z}\right)^{-1} k^n, \quad n\geq 1. \tag{7.51}$$

As a counterpart of Proposition 7.5, we have on $]\!]\tau,+\infty[\![$:

Proposition 7.8: *Let $M \in \mathcal{M}^2_{loc}(\mathbb{F})$, $\widehat{M}^{(a)}$ defined by (7.47) and*

$$U^n := K_n\,(1-Z_-)^{-1}\, I_{]\!]\tau,+\infty[\![},\ n\geq 1.$$

Then, the following assertions hold.

(a) $\left(\sqrt{U^n}\odot \widehat{M}^{(a)}\right)$ *converges to* $\left(\frac{I_{]\!]\tau,+\infty[\![}}{\sqrt{1-\widetilde{Z}}}\odot \widehat{M}^{(a)}\right)$ *in* $\mathcal{M}^2_{loc}(\mathbb{G})$.

(b) *For any $L \in \mathcal{M}^2_{loc}(\mathbb{F})$, there exists a sequence of $\mathbb{G}$-stopping times $(\eta_k)_{k\geq 1}$*

increasing to infinity such that for all k, $\left\langle \sqrt{U^n} \odot \widehat{M}^{(a)}, \sqrt{U^n} \odot \widehat{L} \right\rangle_{\eta_k}^{\mathbb{G}}$ *converges in* $L^1(P)$ *to* $\left\langle \frac{I_{]\!]\tau,+\infty[\![}}{\sqrt{1-\widetilde{Z}}} \odot \widehat{M}^{(a)}, \frac{I_{]\!]\tau,+\infty[\![}}{\sqrt{1-\widetilde{Z}}} \odot \widehat{L} \right\rangle_{\eta_k}^{\mathbb{G}}$.

Proof. The proof is similar to the proof of Proposition 7.3. Indeed, it is enough to consider the $\mathbb{G}$-locally bounded process $(1-Z_-)^{-1} I_{]\!]\tau,+\infty[\![}$, and use the same techniques as in Proposition 7.3. □

Lemma 7.10: *Under the condition* $Z_\tau < 1$*, the following process*

$$V^{(a)} := \left({}^{p,\mathbb{F}} \left(I_{\{\widetilde{Z}<1\}} \right) \right)^{-1} I_{]\!]\tau,+\infty[\![} \tag{7.52}$$

is $\mathbb{G}$*-predictable and locally bounded.*

Proof. It is enough to notice that $1-\widetilde{Z} \le I_{\{\widetilde{Z}<1\}}$ and the process $(1-Z_-)^{-1} I_{]\!]\tau,+\infty[\![}$ is $\mathbb{G}$-locally bounded. □

Proposition 7.9: *Let* $M \in \mathcal{M}^2_{loc}(\mathbb{F})$ *and* $\widehat{M}^{(a)}$ *is given by (7.47). Then, the following hold.*

(a) *Consider the Galtchouk-Kunita-Watanabe decomposition of* $K^n \odot \widehat{m}^{(a)}$ *with respect to* $\widehat{M}^{(a)}$

$$K^n \odot \widehat{m}^{(a)} = \theta^n \centerdot \widehat{M}^{(a)} + L^n, \ \ \textit{where } \theta^n \in L^2_{loc}(\widehat{M}^{(a)}), \ \ \textit{and } L^n \perp \widehat{M}^{(a)}. \tag{7.53}$$

Then, $(\theta^n \centerdot \widehat{M}^{(a)})$ *converges weakly in* $\mathcal{M}^2_{loc}(\mathbb{G})$*.*

As a result, we have

$$\Phi_2 := \lim_{n\to+\infty} \frac{d\langle K^n \odot \widehat{m}^{(a)}, \widehat{M}^{(a)} \rangle^{\mathbb{G}}}{d\langle \widehat{M}^{(a)}, \widehat{M}^{(a)} \rangle^{\mathbb{G}}} \in L^2_{loc}(\widehat{M}^{(a)}, \mathbb{G}). \tag{7.54}$$

(b) *If* $\{\Delta M \neq 0\} \cap \{1 = \widetilde{Z} > Z_-\} = \emptyset$, *we have*

$$\frac{1}{1-Z_-} I_{]\!]\tau,+\infty[\![} \centerdot \langle m, M\rangle^{\mathbb{F}} = \widehat{\Phi_2} I_{[\![0,\tau]\!]} \centerdot \langle \widehat{M^{(a)}}\rangle^{\mathbb{G}}, \quad \text{and } \widehat{\Phi_2} \in L^2_{loc}(\widehat{M^{(a)}}), \tag{7.55}$$

where

$$\widehat{\Phi_2} := \Phi_2 \left({}^{p,\mathbb{F}}\!\left(I_{\{\widetilde{Z}<1\}}\right)\left(1 + \frac{\Delta\langle m\rangle^{\mathbb{F}}}{(1-Z_-)^2}\right)\right)^{-1} \frac{I_{]\!]\tau,+\infty[\![}}{1-Z_-}, \text{ and } \Phi_2 \text{ is given in (7.54).}$$

Proof. (a) It is a consequence of Proposition 7.3 and Lemma 7.4 by taking

$$K^n := (1-Z_-)\left(1-\widetilde{Z}\right)^{-1} I_{\{1-\widetilde{Z}\geq \frac{1}{n}\}}, \quad \text{and} \quad \mathbb{H} = \mathbb{G}. \tag{7.56}$$

(b) Now it remains to describe explicitly the limit in (7.54), that is $\lim_n \langle K^n \odot \widehat{m}^{(a)}, \widehat{M^{(a)}}\rangle^{\mathbb{G}}$. To this end, we first calculate

$$\begin{aligned} \frac{1}{1-Z_-} \centerdot [\widehat{m}^{(a)},\ \widehat{M^{(a)}}] &= \frac{1}{Z_-} I_{]\!]\tau,+\infty[\![} \centerdot [m, M] + I_{]\!]\tau,+\infty[\![} \frac{\Delta M}{(1-Z_-)^2} \centerdot \langle m\rangle^{\mathbb{F}} \\ &+ \frac{\Delta m}{(1-Z_-)^2} I_{]\!]\tau,+\infty[\![} \centerdot \langle M, m\rangle^{\mathbb{F}} + \frac{1}{(1-Z_-)^3} I_{]\!]\tau,+\infty[\![} \Delta\langle m\rangle^{\mathbb{F}} \centerdot \langle M, m\rangle^{\mathbb{F}}. \end{aligned} \tag{7.57}$$

Then by integrating K^n on both sides above, and using the properties of optional integration (see Proposition 2.3 and Proposition 7.9), we obtain

$$\begin{aligned} &\frac{1}{1-Z_-} I_{]\!]\tau,+\infty[\![} \centerdot (k^n \centerdot [m, M])^{p,\mathbb{F}} = \frac{1}{1-Z_-} I_{]\!]\tau,+\infty[\![} \centerdot (K^n \centerdot [m, M])^{p,\mathbb{G}} \\ &= \left(\frac{1}{1-Z_-} K^n \centerdot [\widehat{m}^{(a)},\ \widehat{M^{(a)}}]\right)^{p,\mathbb{G}} - I_{]\!]\tau,+\infty[\![} \frac{{}^{p,\mathbb{G}}\!\left(K^n \Delta M\right)}{(1-Z_-)^2} \centerdot \langle m\rangle^{\mathbb{F}} \\ &- I_{]\!]\tau,+\infty[\![} \frac{{}^{p,\mathbb{G}}\!\left(K^n \Delta m\right)}{(1-Z_-)^2} \centerdot \langle M, m\rangle^{\mathbb{F}} - I_{]\!]\tau,+\infty[\![} \frac{\Delta\langle M, m\rangle^{\mathbb{F}}}{(1-Z_-)^3}\, {}^{p,\mathbb{G}}\!\left(K^n\right) \centerdot \langle m\rangle^{\mathbb{F}}. \end{aligned} \tag{7.58}$$

Due to $\{\Delta M \neq 0\} \cap \{1 = \widetilde{Z} > Z_-\} = \emptyset$, we have

$$ {}^{p,\mathbb{F}}\!\left(\Delta M I_{\{\widetilde{Z}<1\}}\right) I_{\{Z_-<1\}} = {}^{p,\mathbb{F}}\left(\Delta M\right) I_{\{Z_-<1\}} = 0.$$

By taking the limit, we derive

$$\lim_n \frac{I_{]\!]\tau,+\infty[\![}}{1-Z_-} \centerdot (k^n \centerdot [m,M])^{p,\mathbb{F}} = \frac{I_{]\!]\tau,+\infty[\![}}{1-Z_-} \centerdot \Big(I_{\{\widetilde{Z}<1\}} \centerdot [m,M]\Big)^{p,\mathbb{F}} = \frac{I_{]\!]\tau,+\infty[\![}}{1-Z_-} \centerdot \langle m, M\rangle^{\mathbb{F}},$$

$$\lim_n I_{]\!]\tau,+\infty[\![} \frac{{}^{p,\mathbb{G}}\Big(K^n \Delta M\Big)}{(1-Z_-)^2} \centerdot \langle m\rangle^{\mathbb{F}} = I_{]\!]\tau,+\infty[\![} \frac{-{}^{p,\mathbb{F}}\Big(\Delta M I_{\{\widetilde{Z}=1\}}\Big)}{(1-Z_-)^2} \centerdot \langle m\rangle^{\mathbb{F}} = 0, \tag{7.59}$$

and

$$\lim_n {}^{p,\mathbb{G}}\left(K^n\right) I_{]\!]\tau,+\infty[\![} = {}^{p,\mathbb{F}}\Big(I_{\{\widetilde{Z}<1\}}\Big) I_{]\!]\tau,+\infty[\![},$$

$$\lim_n I_{]\!]\tau,+\infty[\![} \frac{{}^{p,\mathbb{G}}\Big(K^n \Delta m\Big)}{(1-Z_-)^2} = I_{]\!]\tau,+\infty[\![} \frac{{}^{p,\mathbb{F}}\Big(\Delta m I_{\{\widetilde{Z}<1\}}\Big)}{(1-Z_-)^2} = -I_{]\!]\tau,+\infty[\![} \frac{{}^{p,\mathbb{F}}\Big(I_{\{\widetilde{Z}=1\}}\Big)}{1-Z_-}. \tag{7.60}$$

Then, by combining (7.54), (7.58) and (7.59), we conclude that

$$\begin{aligned}
\Phi_2 \centerdot \langle \widehat{M}^{(a)}, \widehat{M}^{(a)}\rangle^{\mathbb{G}} &= \lim_n \Big(K^n \centerdot \Big[\widehat{m}^{(b)},\ \widehat{M}^{(a)}\Big]\Big)^{p,\mathbb{G}} \\
&= I_{]\!]\tau,+\infty[\![}\ {}^{p,\mathbb{F}}\Big(I_{\{\widetilde{Z}<1\}}\Big)\left(1 + \frac{\Delta\langle m\rangle^{\mathbb{F}}}{(1-Z_-)^2}\right) \centerdot \langle M, m\rangle^{\mathbb{F}}.
\end{aligned}$$

The proof of the proposition is completed. □

7.3 Proofs of Theorems 7.1 and 7.3

Now, we have prepared all the ingredients to prove the two main theorems of Section 7.1 (Theorems 7.1 and 7.3).

7.3.1 Proof of Theorem 7.1

Suppose that S satisfies Structure Conditions under $\mathbb{F}$. Then, there exist a locally square integrable $\mathbb{F}$-local martingale, M^S, and an $\mathbb{F}$-predictable process $\widehat{\lambda} \in L^2_{loc}(M^S, \mathbb{F})$ such that

$$S = S_0 + M^S + A^S = S_0 + M^S - \widehat{\lambda} \centerdot \langle M^S\rangle^{\mathbb{F}}. \tag{7.61}$$

For notational simplicity, we put $M = M^S$ and $\widehat{M}^{(b)} = \widehat{M^S}^{(b)}$, where $\widehat{M^S}^{(b)}$ is defined via (7.26). Then the $\mathbb{G}$-canonical decomposition of S^τ has the form of

$$S^\tau = S_0 + M^\tau - \widehat{\lambda} I_{[\![0,\tau]\!]} \bullet \langle M\rangle^{\mathbb{F}} =: S_0 + \widehat{M}^{(b)} + \frac{1}{Z_-} I_{[\![0,\tau]\!]} \bullet \langle M, m\rangle^{\mathbb{F}} - \widehat{\lambda} I_{[\![0,\tau]\!]} \bullet \langle M\rangle^{\mathbb{F}}.$$

Recall the notations in (7.36),

$$h^n := I_{\{\widetilde{Z} \geq \frac{1}{n}\}}, \quad H^n := Z_- \left(\widetilde{Z}\right)^{-1} h^n, \quad n \geq 1.$$

Consider the following locally square integrable $\mathbb{G}$-local martingale.

$$\widetilde{N}^{(b)} := \frac{1}{Z_-} \bullet \widehat{m}^{(b)} - \widehat{\lambda} \bullet \widehat{M}^{(b)}. \tag{7.62}$$

Notice that

$$\begin{aligned}
&\frac{1}{Z_-} I_{[\![0,\tau]\!]} \bullet [m, M] - \widehat{\lambda} I_{[\![0,\tau]\!]} \bullet [M] = [\widetilde{N}^{(b)},\ \widehat{M}^{(b)}] + \frac{\Delta m}{Z_-^2} I_{[\![0,\tau]\!]} \bullet \langle M, m\rangle^{\mathbb{F}} \\
&+ \frac{\Delta M}{Z_-^2} I_{[\![0,\tau]\!]} \bullet \langle m\rangle^{\mathbb{F}} - \frac{\Delta\langle M, m\rangle^{\mathbb{F}}}{Z_-^3} I_{[\![0,\tau]\!]} \bullet \langle m\rangle^{\mathbb{F}} - \frac{2\widehat{\lambda}}{Z_-} \Delta M \bullet \langle M, m\rangle^{\mathbb{F}} + \frac{\widehat{\lambda}}{Z_-^2} \Delta\langle M, m\rangle^{\mathbb{F}} \bullet \langle M, m\rangle^{\mathbb{F}}.
\end{aligned}$$

Then, by integrating both sides with H^n and combining the obtained equality with the properties of the optional integral (see Proposition 2.3 and Lemma 7.6), we derive that

$$
\begin{aligned}
&\frac{1}{Z_-}I_{[\![0,\tau]\!]}\centerdot(h^n\centerdot[m,M])^{p,\mathbb{F}}-\widehat{\lambda}I_{[\![0,\tau]\!]}\centerdot(h^n\centerdot[M])^{p,\mathbb{F}}\\
&=\frac{1}{Z_-}I_{[\![0,\tau]\!]}\centerdot(H^n\centerdot[m,M])^{p,\mathbb{G}}-\widehat{\lambda}I_{[\![0,\tau]\!]}\centerdot(H^n\centerdot[M])^{p,\mathbb{G}}\\
&=\Big(H^n\centerdot[\widetilde{N}^{(b)},\ \widehat{M}^{(b)}]\Big)^{p,\mathbb{G}}+\frac{{}^{p,\mathbb{G}}\Big(H^n\Delta M\Big)}{(Z_-)^2}I_{[\![0,\tau]\!]}\centerdot\langle m\rangle^{\mathbb{F}}-\frac{\Delta\langle M,m\rangle^{\mathbb{F}}\ {}^{p,\mathbb{G}}\Big(H^n\Big)}{(Z_-)^3}\ I_{[\![0,\tau]\!]}\centerdot\langle m\rangle^{\mathbb{F}}\\
&-\frac{2\widehat{\lambda}}{Z_-}\ {}^{p,\mathbb{G}}\Big(H^n\Delta M\Big)I_{[\![0,\tau]\!]}\centerdot\langle M,m\rangle^{\mathbb{F}}+\frac{\widehat{\lambda}\Delta\langle M,m\rangle^{\mathbb{F}}}{(Z_-)^2}\ {}^{p,\mathbb{G}}\Big(H^n\Big)I_{[\![0,\tau]\!]}\centerdot\langle M,m\rangle^{\mathbb{F}}\\
&+\frac{{}^{p,\mathbb{G}}\Big(H^n\Delta m\Big)}{(Z_-)^2}I_{[\![0,\tau]\!]}\centerdot\langle M,m\rangle^{\mathbb{F}}. \qquad (7.63)
\end{aligned}
$$

Then, the similar arguments as the limits in (7.44) lead to conclude that

$$
\lim_n\Big(H^n\centerdot[\widetilde{N}^{(b)},\ \widehat{M}^{(b)}]\Big)^{p,\mathbb{G}}=I_{[\![0,\tau]\!]}\frac{1}{Z_-}\centerdot\langle M,m\rangle^{\mathbb{F}}-\widehat{\lambda}I_{[\![0,\tau]\!]}\centerdot\langle M\rangle^{\mathbb{F}}+R^{(b)}\centerdot\langle M,m\rangle^{\mathbb{F}},
$$

where

$$
R^{(b)}:=\frac{1}{Z_-}\left(-\ {}^{p,\mathbb{F}}(I_{\{\widetilde{Z}=0\}})+\frac{\Delta\langle m\rangle^{\mathbb{F}}}{Z_-^2}\ {}^{p,\mathbb{F}}(I_{\{\widetilde{Z}>0\}})-\frac{\widehat{\lambda}\Delta\langle M,m\rangle^{\mathbb{F}}}{Z_-}\ {}^{p,\mathbb{F}}(I_{\{\widetilde{Z}>0\}})\right)I_{[\![0,\tau]\!]}.
$$

Again, by applying Proposition 7.3 to $H^n\odot\widetilde{N}^{(b)}$ and $\widehat{M}^{(b)}$, we conclude that there exists $\Phi^{(b)}\in L^2_{loc}(\widehat{M}^{(b)},\mathbb{G})$ such that

$$
\langle H^n\odot\widetilde{N}^{(b)},\widehat{M}^{(b)}\rangle^{\mathbb{G}}\ \text{ converges locally in } L^1 \text{ to }\ \langle\Phi^{(b)}\centerdot\widehat{M}^{(b)},\widehat{M}^{(b)}\rangle^{\mathbb{G}}.
$$

Recall that $I_{[\![0,\tau]\!]}\centerdot\langle M,m\rangle^{\mathbb{F}}=\widehat{\Phi_1}\centerdot\langle\widehat{M}^{(b)}\rangle^{\mathbb{G}}$ in (7.55). Thus, the uniqueness of the limit leads to

$$
I_{[\![0,\tau]\!]}\frac{1}{Z_-}\centerdot\langle M,m\rangle^{\mathbb{F}}-\widehat{\lambda}I_{[\![0,\tau]\!]}\centerdot\langle M\rangle^{\mathbb{F}}=\Big(\Phi^{(b)}-R^{(b)}\ \widehat{\Phi_1}\Big)\centerdot\langle\widehat{M}^{(b)}\rangle^{\mathbb{G}}. \qquad (7.64)
$$

Due to the locally boundedness of $Z_-^{-1}I_{[\![0,\tau]\!]}$, it is easy to see that

$$\widehat{\lambda}^{\mathbb{G}} := \left(\Phi^{(b)} - R^{(b)}\ \widehat{\Phi_1}\right) \in L^2_{loc}(\widehat{M}^{(b)}, \mathbb{G}), \tag{7.65}$$

and

$$S^\tau = S_0 + \widehat{M}^{(b)} + \widehat{\lambda}^{\mathbb{G}} \cdot \langle \widehat{M}^{(b)}, \widehat{M}^{(b)} \rangle^{\mathbb{G}}. \tag{7.66}$$

This proves the Structure Conditions for S^τ under $\mathbb{G}$, and the proof of the theorem is completed. □

7.3.2 Proof of Theorem 7.3

Suppose that S satisfies Structure Conditions under $\mathbb{F}$. Then, there exist a locally square integrable $\mathbb{F}$-local martingale, M^S, and an $\mathbb{F}$ predictable process $\widehat{\lambda} \in L^2_{loc}(M^S, \mathbb{F})$ such that

$$S = S_0 + M^S + A^S = S_0 + M^S - \widehat{\lambda} \cdot \langle M^S \rangle^{\mathbb{F}}. \tag{7.67}$$

For notational simplicity, we put $M = M^S$ and $\widehat{M}^{(a)} = \widehat{M^S}^{(a)}$, where $\widehat{M^S}^{(a)}$ is defined via (7.47). Then we get

$$\begin{aligned} I_{]\!]\tau,+\infty[\![} \cdot S &= I_{]\!]\tau,+\infty[\![} \cdot M - \widehat{\lambda} I_{]\!]\tau,+\infty[\![} \cdot \langle M \rangle^{\mathbb{F}} \\ &=: \widehat{M}^{(a)} - \frac{1}{1-Z_-} I_{]\!]\tau,+\infty[\![} \cdot \langle M, m \rangle^{\mathbb{F}} - \widehat{\lambda} I_{]\!]\tau,+\infty[\![} \cdot \langle M \rangle^{\mathbb{F}}. \end{aligned} \tag{7.68}$$

Recall the notations in (7.51),

$$k^n := I_{\{1-\widetilde{Z} \geq \frac{1}{n}\}}, \quad K^n := (1-Z_-)\left(1-\widetilde{Z}\right)^{-1} k^n, \quad n \geq 1.$$

Consider the following locally square integrable $\mathbb{G}$-local martingale.

$$\widetilde{N}^{(a)} := -\frac{1}{1-Z_-} \centerdot \widehat{m}^{(a)} - \widehat{\lambda} \centerdot \widehat{M}^{(a)}, \tag{7.69}$$

where $\widehat{m}^{(a)}$ is given by (7.47). Notice that

$$\begin{aligned}
-\frac{1}{1-Z_-} I_{]\!]\tau,+\infty[\![} \centerdot [m,M] - \widehat{\lambda} I_{]\!]\tau,+\infty[\![} \centerdot [M] = [\widetilde{N}^{(a)},\ \widehat{M}^{(a)}] + \frac{\Delta m}{(1-Z_-)^2} I_{]\!]\tau,+\infty[\![} \centerdot \langle M,m\rangle^{\mathbb{F}} \\
+\frac{\Delta M}{(1-Z_-)^2} I_{]\!]\tau,+\infty[\![} \centerdot \langle m\rangle^{\mathbb{F}} + \frac{\Delta\langle M,m\rangle^{\mathbb{F}}}{(1-Z_-)^3} I_{]\!]\tau,+\infty[\![} \centerdot \langle m\rangle^{\mathbb{F}} \\
+\frac{2\widehat{\lambda}}{1-Z_-} \Delta M \centerdot \langle M,m\rangle^{\mathbb{F}} + \frac{\widehat{\lambda}}{(1-Z_-)^2} \Delta\langle M,m\rangle^{\mathbb{F}} \centerdot \langle M,m\rangle^{\mathbb{F}}.
\end{aligned}$$

Then, by integrating both sides with K^n and combining the obtained equality with the properties of the optional integral (see Proposition 2.3 and Lemma 7.9), we derive

$$\begin{aligned}
&-\frac{1}{1-Z_-} I_{]\!]\tau,+\infty[\![} \centerdot (k^n \centerdot [m,M])^{p,\mathbb{F}} - \widehat{\lambda} I_{]\!]\tau,+\infty[\![} \centerdot (k^n \centerdot [M])^{p,\mathbb{F}} \\
&= -\frac{1}{1-Z_-} I_{]\!]\tau,+\infty[\![} \centerdot (K^n \centerdot [m,M])^{p,\mathbb{G}} - \widehat{\lambda} I_{]\!]\tau,+\infty[\![} \centerdot (K^n \centerdot [M])^{p,\mathbb{G}} \\
&= \Big(K^n \centerdot [\widetilde{N}^{(a)},\ \widehat{M}^{(a)}]\Big)^{p,\mathbb{G}} + \frac{{}^{p,\mathbb{G}}\Big(K^n \Delta M\Big)}{(1-Z_-)^2} I_{]\!]\tau,+\infty[\![} \centerdot \langle m\rangle^{\mathbb{F}} + \frac{\Delta\langle M,m\rangle^{\mathbb{F}}\ {}^{p,\mathbb{G}}\Big(K^n\Big)}{(1-Z_-)^3}\ I_{]\!]\tau,+\infty[\![} \centerdot \langle m\rangle^{\mathbb{F}} \\
&+\frac{2\widehat{\lambda}}{1-Z_-}\ {}^{p,\mathbb{G}}\Big(K^n \Delta M\Big) I_{]\!]\tau,+\infty[\![} \centerdot \langle M,m\rangle^{\mathbb{F}} + \frac{\widehat{\lambda}\Delta\langle M,m\rangle^{\mathbb{F}}}{(1-Z_-)^2}\ {}^{p,\mathbb{G}}\Big(K^n\Big) I_{]\!]\tau,+\infty[\![} \centerdot \langle M,m\rangle^{\mathbb{F}} \\
&+\frac{{}^{p,\mathbb{G}}\Big(K^n \Delta m\Big)}{(1-Z_-)^2} I_{]\!]\tau,+\infty[\![} \centerdot \langle M,m\rangle^{\mathbb{F}}.
\end{aligned} \tag{7.70}$$

Then, the similar arguments as in (7.59) lead to conclude that

$$\lim_n \Big(K^n \centerdot [\widetilde{N}^{(a)},\ \widehat{M}^{(a)}]\Big)^{p,\mathbb{G}} = I_{]\!]\tau,+\infty[\![} \frac{-1}{1-Z_-} \centerdot \langle M,m\rangle^{\mathbb{F}} - \widehat{\lambda} I_{]\!]\tau,+\infty[\![} \centerdot \langle M\rangle^{\mathbb{F}} + R^{(a)} \centerdot \langle M,m\rangle^{\mathbb{F}},$$

where

$$R^{(a)} := \frac{-1}{1-Z_-}\left(- {}^{p,\mathbb{F}}(I_{\{\widetilde{Z}=1\}}) + \frac{\Delta\langle m\rangle^{\mathbb{F}}}{(1-Z_-)^2}\ {}^{p,\mathbb{F}}\left(I_{\{\widetilde{Z}<1\}}\right) + \frac{\widehat{\lambda}\Delta\langle M, m\rangle^{\mathbb{F}}}{1-Z_-}\ {}^{p,\mathbb{F}}\left(I_{\{\widetilde{Z}<1\}}\right)\right) I_{]\!]\tau,+\infty[\![}.$$

Again, by applying Proposition 7.3 to $K^n \odot \widetilde{N}^{(a)}$ and $\widehat{M}^{(a)}$, we conclude that there exists $\Phi^{(a)} \in L^2_{loc}(\widehat{M}^{(a)}, \mathbb{G})$ such that

$$\langle K^n \odot \widetilde{N}^{(a)}, \widehat{M}^{(a)}\rangle^{\mathbb{G}} \text{ converges locally in } L^1 \text{ to } \langle \Phi^{(a)} \cdot \widehat{M}^{(a)}, \widehat{M}^{(a)}\rangle^{\mathbb{G}}.$$

Recall that $I_{]\!]\tau,\infty[\![} \cdot \langle M, m\rangle^{\mathbb{F}} = \widehat{\Phi_2} \cdot \langle \widehat{M}^{(a)}\rangle^{\mathbb{G}}$ in (7.55). Thus, the uniqueness of the limit leads to

$$I_{]\!]\tau,+\infty[\![}\frac{-1}{1-Z_-} \cdot \langle M, m\rangle^{\mathbb{F}} - \widehat{\lambda} I_{]\!]\tau,+\infty[\![} \cdot \langle M\rangle^{\mathbb{F}} = \left(\Phi^{(a)} - R^{(a)}\ \widehat{\Phi_2}\right) \cdot \langle \widehat{M}^{(a)}\rangle^{\mathbb{G}}. \quad (7.71)$$

Due to the locally boundedness of $(1-Z_-)^{-1} I_{]\!]\tau,+\infty[\![}$, it is easy to see that

$$\widehat{\lambda}^{\mathbb{G}} := \left(\Phi^{(a)} - R^{(a)}\ \widehat{\Phi_2}\right) \in L^2_{loc}(\widehat{M}^{(a)}, \mathbb{G}), \quad (7.72)$$

and satisfies

$$I_{]\!]\tau,+\infty[\![} \cdot S = \widehat{M}^{(a)} + \widehat{\lambda}^{\mathbb{G}} \cdot \langle \widehat{M}^{(a)}, \widehat{M}^{(a)}\rangle^{\mathbb{G}}. \quad (7.73)$$

This proves the Structure Conditions for $S - S^\tau$ under $\mathbb{G}$, and the proof of the theorem is completed. □

Conclusions:

In this chapter, we addressed the problem that how the Structure Conditions is affected by some extra information (characterized by a random time) that would be the knowledge only insider traders could get through time in progressive enlargement of filtration. Our main results are twofold. First of all, for a fixed market model, we proved that the Structure Conditions are preserved under a mild condition. Secondly, we singled out the necessary and sufficient conditions on the random time for which Structure Conditions are preserved in the enlarged filtration for any initial market model satisfying this structures. Two explicit examples were presented to illustrate the importance of the condition and the consequence of its failure.

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