Aspects of enumerative and categorical algebraic geometry

by

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Abstract

In this thesis, we study some aspects of algebraic geometry that have had a significant influx of ideas from physics. The first part focuses on the Eynard-Orantin topological recursion and its variants as a theory of enumerative geometry. We investigate the conjectural relationship between the topological recursion and quantum curves in the case of elliptic curves. We show that the perturbative wave-function is not the solution to a quantum curve, while the non-perturbative one is (up to certain order in $\hbar$).

We define the formalism of Higher Airy Structures (HAS), which quantized higher order Lagrangians in a symplectic vector space. By showing that the Bouchard-Eynard topological recursion is a HAS, we find a necessary and sufficient condition on the spectral curve for producing symmetric correlators $\omega_{g,n}$. We construct numerous other examples, some of which are (sometimes conjecturally) related to FJRW theory or open (r-spin) intersection theory.

In the second part of this thesis, we focus on the study of derived categories and its connections to birational geometry; in particular, we are interested in a conjecture of Bondal and Orlov about flops. Using the presentation of a flop as a variation of geometric invariant theory (VGIT) problem, Ballard, Diemer and Favero proposed a Fourier-Mukai kernel and conjectured that it induces a derived equivalence. We verify this conjecture in the case of Grassmann flops first. Then, we tackle the case of singular VGIT problems using semi-free commutative dg-algebra resolutions, and prove that we obtain derived equivalences of dg-schemes, under some conditions.

Preface

We mention that all of the work done in this thesis is either submitted for publication or published. All of the chapters in this thesis are nearly identical to the submitted versions. The only changes are corrections of typos and minor mistakes.

Chapter 3 is joint work with V. Bouchard and T. Dauphinee, and has been published as *'Quantizing Weierstrass'* in *Communications in Number Theory and Physics, Volume 12 (2018), Number 2*, by *International Press of Boston, Inc.* The appendix to the article can be found in Appendix A. Equation (3.131) in the published version was incorrectly written as evaluating the expression at $z = 0$ and has been corrected here as being evaluated at $z_1 = z_2 = \cdots = z_j = 0$. The notation $W_{g,n}$ for the correlators in the published version has been changed to $\omega_{g,n}$ in this thesis for the sake of consistency in notation. For an online version of the article, please visit
Quantizing Weierstrass.

Chapter 4 is joint work with G. Borot, V. Bouchard, T. Creutzig and D. Noschenko, and has been submitted for publication. The appendix to the article can be found in Appendix B. For an preprint version of the article, please visit
Higher Airy structures, W algebras and topological recursion.

Chapter 6 is joint work with M. Ballard, D. Favero, P. McFaddin and R. Vandermolen, and has been submitted for publication. For a preprint version of the article, please visit
Kernels for Grassmann Flops.

Chapter 7 is joint work with D. Favero, and has been submitted for publication. For a preprint version of the article, please visit
Windows for cdgas.

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Contents

Chapter 1

Introduction

Ideas from theoretical physics, particularly string theory, have been extremely influential in steering the course of mathematical research in the past few decades. This influx of ideas, sometimes referred to as physical mathematics [Moo], has reinvigorated existing mathematical fields as well as contributed to the development of completely new fields of research. Predictions made by physicists about rather hard-to-compute (but purely) mathematical quantities, especially in algebraic geometry, has forced the mathematics community to take note of these developments. A couple of striking examples are the Kontsevich-Witten theorem [Wit91, Kon92b] and the rational curve counts of [CdlOGP91]. This thesis is primarily concerned with some facets of algebraic geometry that have been impacted significantly by these physical developments.

Algebraic geometry is a very classical field of research that one can describe, very broadly, as studying the geometry arising from zeroes of polynomial equations. As easy as that might sound, explicit calculations of interesting objects can become rather technical and cumbersome. Mathematical ideas originating from physical dualities have been shown, over the last few decades, to be capable of computing some of these extremely complicated algebro-geometric invariants. In fact, these ideas extend far beyond new techniques for computations; one could reasonably argue that physical mathematics has implications for the very nature of algebraic geometry itself.

To illustrate some of these aspects, particularly the ones that are relevant to this thesis, let us consider the mirror symmetry conjecture in string theory. Mirror symmetry is a duality between two physical theories – the 'A-model' defined on a symplectic manifold Y and the 'B-model' defined on a complex algebraic variety X. The conjecture is about the existence of the mirror model to a given A/B model, and it predicts various geometric relations among the mirror spaces X and Y. The simplest prediction is that the Hodge diamonds of X and Y are mirror to each other, but the conjecture ties X and Y together even closer still. Kontsevich's attempt to rephrase this relationship mathematically using the language of derived categories is called the homological mirror symmetry proposal [Kon95].

As our interest is primarily on the B-model, let us sketch the B-side physics heuristics. A crucial feature of the B-model is that it is independent of the symplectic structure of X. Mathematically, the B-model is the (as yet undefined in general) category of B-branes of the string theory, and we should view the B-model as a category 'over' a space that parametrizes the symplectic structure of X, often referred to as the stringy Kähler moduli space (SKMS). The SKMS has certain 'large-radius' points, near which this category of B-branes is identified with the *derived category* of the algebraic variety X.

Near different large-radius points on the SKMS, the associated algebraic variety changes, but these varieties are birational. In fact, they are different geometric invariant theory (GIT) quotients in the sense of Mumford [Mum65a]. As the B-model is independent of the symplectic structure, we may move along a path between two special points in the SKMS and "parallel transport" the categories to get an equivalence of derived categories. This physics expectation fits in nicely with the conjectures of Bondal and Orlov [BO95] that birational varieties related by flops have equivalent derived categories. For an introduction to derived categories and a review of the Bondal-Orlov conjecture, we refer the reader to Chapter 5. For a detailed explanation of the physics concepts, which is particularly addressed towards the idea of *windows* appearing in this thesis, see [EHKR17, HHP09].

Building upon the work on variation of GIT quotients of [HHP09, Seg11, BFK19, HL15], Ballard, Diemer and Favero [BDF17] proposed a method to resolve the Bondal-Orlov conjectures, called the Q-construction. Using their construction, the authors provide a Fourier-Mukai kernel that conjecturally induces the derived equivalence between smooth flops. In Chapter 6, we study a certain class of flops known as Grassmann flops and verify that the Q-construction provides a derived equivalence for the Grassmann flop (over an arbitrary field of characteristic zero), which was proved using other methods by [DS14, BLVdB16].

In Chapter 7, we further investigate the proposal of [BDF17] in the case of GIT problems on singular schemes. This involves resolving the singular scheme by an appropriate semi-free commutative differential graded algebra (cdga), and then applying the Q-construction to it. More generally, we study the Q-construction for semi-free dg-schemes, develop a theory of windows in this setting, and show, under some conditions, that it induces a derived equivalence between the different GIT quotient dg-schemes.

Mirror symmetry has concrete predictions for enumerative algebraic geometry (or counting problems), and this thesis also deals with some of these enumerative aspects. The A-model enumerative invariants are difficult to compute as they involve intersection theory on complicated moduli spaces. Typically, it is easier to compute them on the B-side. The *Eynard-Orantin topological recursion* [EO07, EO08, CEO06] (and various generalizations of it) is an enumerative theory on the B-model side which can be used to compute A-model invariants in cases including (r-spin) intersection numbers on $\overline{\mathcal{M}}_{g,n}$ [DBNO+19, EO08], open intersection numbers [Saf16], Gromov-Witten invariants of toric Calabi-Yau threefolds [BKMnP09, Mn08] and various Hurwitz numbers [BMn08, BHSLM14, DBKO+15, DLN16] (see [Eyn14b] for a review of some of the applications).

The topological recursion is conjectured to be closely related to certain non-commutative deformations of plane curves called quantum curves. In Chapter 3, we study the topological recursion in the context of elliptic curves, and attempt to understand a conjecture of Borot and Eynard [BE12a] in relation to these quantum curves. We check that a non-perturbative construction of the wave-function using the topological recursion for a particular elliptic curve, provides the WKB-solution to a quantum curve up to a certain order in $\hbar$.

In Chapter 4, we define an algebraic framework called higher Airy structures (generalizing the notion of quantum Airy structures defined by Kontsevich and Soibelman [KS18]) and use it to prove that the Bouchard-Eynard topological recursion [BE13] constructs highest weight vectors to representations of certain vertex algebras called $\mathcal{W}$-algebras (see [Ara17] for an introduction to vertex algebras). We construct Airy structures (again, based on $\mathcal{W}$-algebras) that appear in other contexts such as (r-spin) open intersection theory [PST14, BCT18] and FJRW theory [FJR13]. The enumerative interpretations of some of these Airy structures are conjectural or unknown (see Chapter 8 for more details).

Although the aspects of algebraic geometry treated in this thesis are intimately connected through the physics heuristics described above, they have distinct mathematical flavors – categorical and enumerative. Hence, this thesis is also structured as two different parts. Part I deals with the enumerative aspects in relation to the topological recursion, while Part II deals with the categorical aspects, more precisely the study of derived categories and the relations to birational geometry.

Part I

Topological recursion and enumerative geometry

Chapter 2

Background and Introduction

Chekhov, Eynard and Orantin [EO07, EO09, CE06] found a recursive structure in the solution to various random matrix models, observed that it appears in other contexts in algebraic geometry, and dubbed it the 'topological recursion'. A few examples where the topological recursion makes an appearance are (r-spin) intersection theory on the moduli space of curves, semi-simple cohomological field theories (CohFTs), Gromov-Witten theory on toric Calabi-Yau threefolds and the (weighted) projective line, Hurwitz theory, random matrix theory and knot theory (see [Eyn14b] for a review of some of these applications).

The topological recursion is a purely algebro-geometric formalism which takes an algebraic curve (with some additional data to be defined later) as input, and outputs symmetric differential forms on copies of the curve. These differential forms are indexed by two positive integers g and n, which can be thought of as the genus of a Riemann surface and the number of marked points on it respectively. Then, the name 'topological recursion' refers to the fact that these differential forms $\omega_{g,n}$ are constructed recursively on the integer $(2g + n - 2)$, which is the negative of the Euler characteristic of a Riemann surface of genus g, with n punctures. Using an appropriate choice of the curve and a basis of differential forms, these $\omega_{g,n}$ can be identified as generating functions for the enumerative invariants of interest.

The topological recursion has many variants, generalizations and reinterpretations. In this chapter, we will define the topological recursion (and some associated notions) in the level of generality that suffices for our purposes in this thesis.

2.1 Spectral Curve

The version of the topological recursion that we are mostly interested in the generalization due to Bouchard and Eynard [BE13] following [BHL+14], and this is the one that we present here. In [BE13], it was called the global topological recursion and has been since known as the Bouchard-Eynard topological recursion, but in this chapter, we refer to it simply as the topological recursion.

The input data for the topological recursion is a spectral curve.

Definition 2.1.1 (Spectral curve). A *spectral curve* is the four-tuple $(\Sigma, x, \omega_{0,1}, \omega_{0,2})$, where,

- Σ is a Riemann surface;
- $x : \Sigma \to \mathbb{P}^1$ is a meromorphic function on Σ;
- $\omega_{0,1}$ is a meromorphic differential form on Σ;
- $\omega_{0,2}$ is a meromorphic bi-differential on Σ^2 that has a double pole on the diagonal with bi-residue 1 and no other poles.

For clarity, this means that $\omega_{0,2}$ takes the following form. When z_1 and z_2 are local coordinates near a point p on Σ,

$$\omega_{0,2}(z_1, z_2) = \left(\frac{1}{(z_1 - z_2)^2} + \cdots \right) dz_1 dz_2,$$

where the $\cdots$ represent terms that are regular in z_1 and z_2.

Let us introduce some more notation pertaining to the above definition of a spectral curve. The map

$$x : \Sigma \to \mathbb{P}^1,$$

realizes Σ as a branched covering of $\mathbb{P}^1$. We denote the degree of this map by d. The set of ramification points of this branched covering which are also zeroes of dx is denoted by R, i.e.,

$$R := \{ a \in \Sigma \,|\, dx(a) = 0 \}.$$

Assume that this set R is finite and has cardinality c.

Remark 2.1.2. In certain applications of the topological recursion, one is also required to include the poles of x in the set R. For instance, in the setting of quantum curves, which will be discussed in Chapter 3, this is essential. For a more detailed discussion about the inclusion of the poles of x, the reader may look at [BE17].

A few remarks are in order, in order to clarify the definition of a spectral curve (Definition 2.1.1), and compare it to the variations that appear in the literature on the topological recursion.

Remark 2.1.3 (Local spectral curve)**.** The Riemann surface Σ is not necessarily compact. For instance, for the purposes of the 'local' topological recursion (as considered in [DBOSS14] when all the ramification points in R are simple or as considered in Chapter 4 when the ramification points are allowed to be higher order), Σ is chosen to be the disjoint union of a number of disks. If the Riemann surface Σ is chosen to be copies of a disk, the spectral curve is called a *local spectral curve*.

Remark 2.1.4 (Global spectral curve)**.** On the other hand, for certain applications, such as constructing quantum curves, we require the Riemann surface to be connected and compact; this is sometimes referred to as a *global spectral curve*. Moreover, we often require that the Riemann surface is Torelli-marked, i.e., we fix a symplectic basis for $H_1(\Sigma, \mathbb{Z})$, and we will also require that the bilinear differential $\omega_{0,2}$ is normalized on A-cycles. The conjectured connection between the topological recursion and quantum curves is discussed briefly in Section 2.3.

Remark 2.1.5. As a follow-up to Remark 2.1.3 and Remark 2.1.4, we note that a global spectral curve always gives rise to a local spectral curve. The converse is not true in general, and certain aspects of this question in the context of topological recursion and semi-simple cohomological field theories (CohFTs) are addressed by [DBNO+19].

Consider a ramification point a_α in R, where the ramification index is r_α. In a neighborhood U_α of this ramification point $a_\alpha \in R$, we can define a local coordinate ζ_α such that

$$x\Big|_{U_\alpha}(\zeta_\alpha) = \frac{\zeta_\alpha^{r_\alpha}}{r_\alpha} + x(a_\alpha)\,. \tag{2.1}$$

In this neighborhood U_α, we can also expand $\omega_{0,1}$ as

$$\omega_{0,1} = \sum_{l>0} \tau_l^\alpha \, \zeta_\alpha^{l-1} \, \mathrm{d}\zeta_\alpha\,,$$

and for future use, we define

$$s_\alpha := \min\left\{ l > 0 \quad | \quad \tau_l^\alpha \neq 0 \;\text{ and }\; r_\alpha \nmid l \right\}.$$

Remark 2.1.6. Often in the literature, the definition of a spectral curve contains a meromorphic function $y : \Sigma \to \mathbb{P}^1$ instead of the differential form $\omega_{0,1}$. This data is equivalent; the two quantities are related as follows

$$y = \frac{\omega_{0,1}}{dx}.$$

We prefer to use the differential form $\omega_{0,1}$ over y as the topological recursion formula (2.2) directly takes the $\omega_{0,1}$ as input.

Remark 2.1.7. The special case where the map $x : \Sigma \to \mathbb{P}^1$ is simply ramified was the one considered in the original formalism of Eynard and Orantin [EO07]. In other words, this means that r_α is 2 for all the ramification points a_α in R. This restriction was first lifted in [BHL+14].

Finally, we also need the notion of an admissible spectral curve as defined in Chapter 4 (Definition 4.5.8).

Definition 2.1.8 (Admissible spectral curve)**.** We say that a spectral curve $(\Sigma, x, \omega_{0,1}, \omega_{0,2})$ is *admissible* if we have the condition

$$1 \leq s_\alpha \leq r_\alpha + 1 \text{ and } r_\alpha = \pm 1 \bmod s_\alpha,$$

for every ramification point a_α in R. The choice of sign $(\pm)$ may depend on the a_α.

We will discuss the implications of this admissibility condition at some length after we define the topological recursion (see Remark 2.2.11).

2.2 The Topological recursion formula

Throughout this section, we consider a fixed admissible spectral curve $(\Sigma, x, \omega_{0,1}, \omega_{0,2})$, where $x : \Sigma \to \mathbb{P}^1$ is a branched covering of degree d. We need to introduce some more notation first.

Notation 2.2.1. Recall that the branched covering $x : \Sigma \to \mathbb{P}^1$ has degree d. We introduce

$$\tau' : \Sigma \setminus R \to \Sigma^{\times(d-1)},$$

which is defined as

$$\tau'(z) := x^{-1}\left(x(z)\right) \setminus \{z\}.$$

Now, we define some combinatorial notation in order to simplify the expressions that appear later on.

Notation 2.2.2. Consider a finite set A. A set $\mathbf{L} = \{L_1, L_2, \cdots, L_l\}$ where the L_i are non-empty subsets of A is called a *set partition* of A, if the L_i are pairwise disjoint and the union $\cup_i L_i$ is A. Such a set partition is denoted by $\mathbf{L} \vdash A$.

Notation 2.2.3. Consider two finite sets A and B, and a set partition $\mathbf{L} \vdash A$. Denote the power set of B by $\mathcal{P}(B)$. A *partition of B indexed by* $\mathbf{L}$ is a map

$$F : \mathbf{L} \to \mathcal{P}(B),$$

where the $\{F(L)_{L\in\mathbf{L}}\}$ are (possibly empty) pair-wise disjoint subsets of B, such that their union is B, i.e., $\cup_{L\in\mathbf{L}} = B$.

The goal of the topological recursion is to construct symmetric meromorphic n-differential forms $\omega_{g,n}$ called *correlators* (for $g \geq 0$ and $n \geq 1$ and $2g+n-2>0$) on the Riemann surface Σ with possible poles at the ramification points R, i.e., we want

$$\omega_{g,n} \in H^0(\Sigma^{\times n}, K_\Sigma(*R)^{\boxtimes n})^{S_n}; \qquad 2g+n-2>0\,,$$

where, K_Σ is the canonical bundle of Σ and S_n is the symmetric group on n letters that acts by permuting the copies of the curve. The $(*R)$ means that we are twisting by some power of the divisor R.

The correlators $\omega_{g,n}$ where $2g+n-2>0$ are called *stable*, while $\omega_{0,1}$ and $\omega_{0,2}$ are called *unstable*. Recall that the data of a spectral curve defines the unstable correlators $\omega_{0,1}$ and $\omega_{0,2}$. Given a sequence $\omega_{g,n}$ of differential forms for $g \geq 0$ and $n \geq 1$, we define the following object.

Definition 2.2.4. Let A and B be finite sets of coordinates on Σ with cardinality i and $n-1$ respectively. We define

$$\mathcal{R}^{(i)}_{g,n}(A|B) := \sum_{\mathbf{L} \vdash A} \sum_{\substack{h\,:\,\mathbf{L}\to\mathbb{N} \\ i+\sum_{L\in\mathbf{L}} h_L = g+|\mathbf{L}|}} \sideset{}{'}\sum_{\boldsymbol{\mu} \vdash_{\mathbf{L}} B} \left(\prod_{L\in\mathbf{L}} \omega_{h_L,|L|+|\mu_L|}(L,\mu_L) \right),$$

where the prime that appears in the sum signifies that any term that contains an $\omega_{0,1}$ is excluded from the sum.

Let us clarify the above definition with a few examples.

Example 2.2.5. When $g=1, n=1$, we have

$$\mathcal{R}^{(2)}_{1,1}(a_1,a_2|\emptyset) := \omega_{0,2}(a_1,a_2)\,.$$

We also note that the second sum in Definition 2.2.4 forces

$$\mathcal{R}^{(i)}_{1,1}(A|\emptyset) = 0 \qquad \text{when} \qquad i>2\,.$$

Example 2.2.6. When $g=0, n=3$, we have

$$\mathcal{R}^{(2)}_{0,3}(a_1,a_2|b_1,b_2) := \omega_{0,2}(a_1,b_2)\omega_{0,2}(a_2,b_1) + \omega_{0,2}(a_1,b_1)\omega_{0,2}(a_2,b_2)\,.$$

As in the previous example, the second sum in Definition 2.2.4 forces

$$\mathcal{R}^{(i)}_{0,3}(A|B) = 0 \qquad \text{when} \qquad i>2\,.$$

Example 2.2.7. When $g=1, n=2$, we have

$$\mathcal{R}^{(2)}_{1,2}(a_1,a_2|b_1) := \omega_{0,2}(a_1,b_2)\omega_{1,1}(a_2) + \omega_{0,2}(a_2,b_2)\omega_{1,1}(a_1) + \omega_{0,3}(a_1,a_2,b_1)\,,$$

and

$$\mathcal{R}^{(3)}_{1,2}(a_1,a_2,a_3|b_1) := \\ \omega_{0,2}(a_1,a_2)\omega_{0,2}(a_3,b_1) + \omega_{0,2}(a_1,a_3)\omega_{0,2}(a_2,b_1) + \omega_{0,2}(a_2,a_3)\omega_{0,2}(a_1,b_1)\,.$$

Again, the second sum in Definition 2.2.4 forces

$$\mathcal{R}^{(i)}_{1,2}(A|B) = 0 \qquad \text{when} \qquad i>3\,.$$

Finally, we are ready to present the *topological recursion* as defined by [BE17, BE13].

Definition 2.2.8 (Topological recursion)**.** Consider an admissible spectral curve $(\Sigma, x, \omega_{0,1}, \omega_{0,2})$. Choose a base-point $b \in \Sigma$, and let $\mathbf{z} = (z_2, \ldots, z_n)$. The *correlators* $\omega_{g,n}$ are recursively defined by the *topological recursion* as

$$\begin{aligned}\omega_{g,n}(z_1, \mathbf{z}) := \sum_{a \in R} \operatorname{Res}_{z=a} \left(\int_b^z (\omega_{0,2})(\cdot, z_1) \right) \times \\ \sum_{i=1}^{d} \sum_{\substack{\beta(z) \subseteq \tau'(z), \\ |\beta(z)|=i}} \left(\prod_{\beta_j(z) \in \beta(z)} \frac{(-1)^{i+1}}{(\omega_{0,1}(z) - \omega_{0,1}(\beta_j(z)))} \right) \mathcal{R}^{(i+1)}_{g,n}(z, \beta(z)|\mathbf{z}) \,, \quad (2.2)\end{aligned}$$

A few remarks about this definition are in order.

Remark 2.2.9 (Graphical interpretation)**.** Definition 2.2.8 has a graphical interpretation in terms of a pair-of-pants decomposition. The RHS of equation (2.2) is obtained by summing over all possible ways of decomposing a Riemann surface of genus g and n marked points using $(i+1)$-holed pairs-of-pants. For more details on this graphical interpretation, and illustrative pictures, see [BHL$^+$14].

Remark 2.2.10 (Recursiveness)**.** It is easy to see that the formula is recursive in $(2g + n - 2)$, i.e., the correlators $\omega_{h,m}$ that appear on the RHS of equation (2.2) are such that

$$2h + m - 2 < 2g + n - 2.$$

This is the origin of the name ‘topological recursion’.

Remark 2.2.11 (Admissibility and symmetry)**.** The careful reader will have observed that the formula (2.2) is not symmetric in all the z_i. The coordinate z_1 plays a special role and hence the symmetry of the correlators $\omega_{g,n}$ is far from obvious. In order to ensure the symmetry of the correlators we have to choose an admissible spectral curve in the definition.

When $r_\alpha = 2$ for all ramification points a_α in R, this symmetry was proved in the original paper of [EO07]. In the case of spectral curves with arbitrary ramification, this is one of the main results of Chapter 4 (see Theorem 5). Using the formalism of higher Airy structures that we define, we show that the topological recursion on an admissible spectral curve constructs the unique solution to a set of $\mathcal{W}$-algebra constraints and thereby deduce the symmetry of the correlators $\omega_{g,n}$. In fact, we show that the condition of admissibility is *necessary and sufficient* to ensure the symmetry of the correlators.

Remark 2.2.12 (Local vs global)**.** Definition 2.2.8 is often referred to as the *global* topological recursion. The reason for this terminology is that the $\tau'(z)$ in equation (2.2) are the globally defined deck transformations. A consequence of using the $\tau'(z)$ in the definition of the topological recursion, is that the expression on the RHS of equation (2.2) is globally defined on Σ.

In the original definition of [EO07] (for simple ramification) and [BHL+14] (for arbitrary ramification), the locally defined exchange of sheets around a ramification point was used in place of the global deck transformations. This results in expressions that are only locally defined in neighborhoods of the ramification points. Definition 2.2.8 first appeared in [BE13], where the authors also prove that their global formulation is equivalent to the local one.

We note that the global formulation is better suited for understanding various properties of the topological recursion. For instance, this global version is essential in the context of quantum curves as demonstrated in [BE17]. Quantum curves are also the topic of Chapter 3, and hence we use the global formulation there. On the other hand, the local version is much better suited to computations as the $\tau'(z)$ typically do not have analytic expressions. Hence, whenever we need to do explicit computations of the correlators we use the local version.

Finally, we also note that in Chapter 4, we use the local version of the topological recursion. The reason is that our topic of interest in Chapter 4, Airy structures, only 'sees' the local (symplectic) data near the ramification points. In addition, we need to carry out explicit computations in order to relate the topological recursion to $\mathcal{W}$-algebra modules, and the local version is better suited for this purpose.

Remark 2.2.13. These $\omega_{g,n}$ enjoy a plethora of special properties (see [EO07, BE13] for a discussion). One of these properties that is crucial for us is that the stable correlators $\omega_{g,n}$ only have poles at the ramification points R. Typically, one chooses a basis of meromorphic differentials with poles on R and expands them in order to extract the enumerative geometric information.

We also define the *free energies* F_g for $g \in \mathbb{Z}_{\geq 2}$.

Definition 2.2.14. [Free energy] Consider an admissible spectral curve $(\Sigma, x, \omega_{0,1}, \omega_{0,2})$, a fixed base point b, and the stable correlators $\omega_{g,n}$ constructed by the topological recursion. Then the *free energies* F_g in $\mathbb{C}$ are defined as

$$F_g := \frac{1}{2g-2} \sum_{a \in R} \mathrm{Res}_{z=a} \left(\int_b^z \omega_{0,1}(\cdot) \right) \omega_{g,1}(z); \qquad g = 2, 3, \cdots .$$

Remark 2.2.15. Although Definitions 2.2.8 and 2.2.14 appear to depend on the choice of a base-point b it is easy to prove that the correlation functions $\omega_{g,n}$ and the free energies F_g do not depend on the chosen base point. A proof can be found in [BE17, Section 3.1].

2.3 Quantum curves

A main object of interest for us is the notion of quantum curves and wave-functions for quantum curves. The concept of quantum curves was first introduced in the physics literature, and constructing wave-functions for quantum curves is an active area of research in string theory. Among the various applications of the topological recursion is the connection to quantum curves [EO07, BE17, BE15, EMn11]. A elementary review of the story can be found in [Nor16][1]. Conjecturally, the topological recursion on a spectral curve can be used to construct the wave-function for a quantum curve whose 'classical limit' is the algebraic curve underlying the spectral curve that we started with.

In this section, we will define the notion of quantum curves briefly, and refer the reader to Section 3.1 for more details. As discussed in Remark 2.1.4,

[1] A word of caution is that the conjecture stated in [Nor16] is not valid for higher genus spectral curves.

we use a Torelli-marked compact Riemann surface Σ as part of the definition of a spectral curve. We will use the compact Riemann surface and the associated algebraic curve interchangeably in this section. As we have two meromorphic functions x and y (see Remark 2.1.6 for the definition of y) on a compact Riemann surface Σ, they must satisfy a polynomial equation that we denote by

$$P(x, y) = 0.$$

We note that this polynomial will have degree d in y (as x is a branched covering of degree d).

To 'quantize'[2] the curve, we map the commutative variables (x, y) to non-commutative differential operators

$$\hat{x} = x, \qquad \hat{y} = \hbar \frac{d}{dx},$$

satisfying the commutation relation

$$[\hat{y}, \hat{x}] = \hbar.$$

Here, $\hbar$ is a formal variable. This turns the polynomial $P(x, y)$ into a rank d linear differential operator. However, we note that this procedure is far from unique due to ordering ambiguities. Note that

$$yx - xy = 0 \qquad \text{but} \qquad \hat{y}\hat{x} - \hat{x}\hat{y} = \hbar,$$

and hence, we may add any function $f(xy - yx)$ to $P(x, y)$ and obtain a different quantization. This motivates the following definition using a normal-ordering procedure.

Definition 2.3.1 (Quantum curve)**.** A *quantum curve* $\hat{P}$ of a spectral curve $(\Sigma, x, \omega_{0,1}, \omega_{0,2})$ is a rank d linear differential operator in x, such that, after normal ordering (that is bringing all the $\hat{x}$'s to the left of the $\hat{y}$'s), it takes the form

$$\hat{P}(\hat{x}, \hat{y}; \hbar) = P(\hat{x}, \hat{y}) + \sum_{n \geq 1} \hbar^n P_n(\hat{x}, \hat{y}), \tag{2.3}$$

[2]This is a standard procedure in quantum mechanics and this is the origin of the name quantization.

where the leading order term $P(\hat{x},\hat{y})$ recovers the polynomial equation of the original spectral curve (normal ordered), and the $P_n(\hat{x},\hat{y})$ are differential operators in x of rank at most $d-1$.

In order to describe the connection to the topological recursion, we need to define the notion of a wave-function.

Definition 2.3.2 (Wave-function)**.** Fix a spectral curve $(\Sigma, x, \omega_{0,1}, \omega_{0,2})$, and choose a quantization $\hat{P}(\hat{x},\hat{y};\hbar)$. The *wave-function* for this quantum curve $\hat{P}(\hat{x},\hat{y})$ is a function $\psi(z)$ of the form

$$\psi(z) = \exp\left(\sum_{k=0}^{\infty} \hbar^{k-1} S_k(z)\right),$$

where $S_k(z)$ for $k \geq 0$ are meromorphic functions[3] on Σ, and satisfies the relation

$$\hat{P}(\hat{x},\hat{y};\hbar)\psi(z) = 0.$$

The above equation is interpreted using the WKB-method.

The interpretation of the statement that $\hat{P}(\hat{x},\hat{y};\hbar)\psi(z) = 0$ using the WKB method is the following. We take the expression $\psi^{-1}(z)\hat{P}(\hat{x},\hat{y};\hbar)\psi(z)$ and expand it in powers of $\hbar$; then the coefficients of $\hbar^i$ for all $i \geq 0$ vanish identically. Let us illustrate this using a simple example.

Example 2.3.3. Consider the quantum curve $\hat{P} = \hbar^2 \frac{d^2}{dx^2} - f(x)$, which is a quantization of the curve defined by the polynomial $P(x,y) = y^2 - f(x)$. Consider a wave-function $\psi(z)$. Then, we have

$$\psi(z)^{-1}\left(\hbar^2 \frac{d^2}{dx^2} - f(x)\right)\psi(z) = \left(\left(\frac{dS_0}{dx}\right)^2 - f(x)\right)\hbar^0 + \\ \left(\frac{d^2 S_0}{dx^2} + 2S_0\frac{dS_1}{dx}\right)\hbar^1 + \left(\frac{d^2 S_1}{dx^2} + \left(\frac{dS_1}{dx}\right)^2 + 2\frac{dS_1}{dx}\frac{dS_0}{dx}\right)\hbar^2 + O(\hbar^3).$$

[3]The function S_0 may be multi-valued on Σ and hence defined only on a cover of Σ.

Then, we interpret the equation $\hat{P}\psi(z) = 0$ as

$$\begin{aligned}\left(\frac{dS_0}{dx}\right)^2 - f(x) &= 0\,,\\ \frac{d^2S_0}{dx^2} + 2S_0\frac{dS_1}{dx} &= 0\,,\\ \frac{d^2S_1}{dx^2} + \left(\frac{dS_1}{dx}\right)^2 + 2\frac{dS_1}{dx}\frac{dS_0}{dx} &= 0\,,\end{aligned}$$

and so on for every order in $\hbar$.

In fact as suggested by the above example, the $\left\{\frac{dS_k}{dx}\right\}$ form a triangular system and hence, by solving this system, the wave-function to a quantum curve is essentially unique.

2.3.1 Relation between the topological recursion and quantum curves

Very briefly, the conjecture that the topological recursion constructs the wave-function to a quantization of the algebraic curve underlying the spectral curve goes as follows. Given a spectral curve, the topological recursion constructs the free energies F_g, which can be assembled into a generating function known as the *perturbative* partition function. In general, this perturbative partition function does not have modularity properties. The authors of [EMn11, Eyn09] add corrections to the perturbative partition function, using appropriate combinations of theta functions to cancel these non-modularities, and define a modular-invariant *non-perturbative partition function*.

The authors of [BE12a] study this non-perturbative partition function further, and conjecture that it is a *tau*-function, in the sense that it satisfies the Hirota bilinear equations. Assuming that the conjecture is true, one can then construct a *wave-function* associated to the partition function as its Schlesinger transform. It is then expected from the theory of Baker-Akhiezer functions that this wave-function should be the WKB asymptotic solution of a quantization of the spectral curve. This was first proposed in [BE12a] and further investigated in the context of knots in [BE15].

If the spectral curve has genus zero, the non-perturbative corrections vanish, and the non-perturbative and perturbative versions coincide, and hence the problem is dramatically simplified. At the time of writing and publication of Chapter 3, the only known results were for genus zero curves. (As of the writing of this thesis, the most 'uniform' results are [BE17] which proves the conjecture for a large class of genus zero curves, and [MO19, EG19] which prove the conjecture for (certain) hyper-elliptic curves.)

Chapter 3 was the first paper to explore the conjecture about non-perturbative wave-functions for higher genus curves. We studied both the perturbative and non-perturbative wave-functions for elliptic curves (satisfying a quantization condition), showed that the perturbative wave-function is annihilated by a differential operator which is not a quantum curve, and checked up to order $O(\hbar^5)$ that the non-perturbative wave-function is the solution to a quantum curve. This was the first result to unequivocally demonstrate the need for a non-perturbative definition of the wave-function. As a side-result of the analysis that we carried out, we also obtained an infinite sequence of previously unknown relations involving A-cycle integrals of elliptic functions and quasi-modular forms.

Remark 2.3.4. Due to an unfortunate choice of terminology the notion of a partition function referred to in this section (which is a generating function of the free energies F_g constructed by the topological recursion), has no relation to the notion of a partition function defined in the next section in the context of Airy structures.

2.4 Airy structures

Kontsevich and Soibelman [KS18] reformulated the topological recursion as an example of a broader algebraic framework called *quantum Airy structures*. They associate a partition function to the quantization of a Lagrangian subvariety in a symplectic vector space. We limit ourselves to a very brief introduction of this notion as Chapter 4 is a detailed study of a generalization of this notion called *higher Airy structures*.

Let V be a vector space over a field k of characteristic zero, with basis $\{x_i\}_{i \in I}$, where I is a finite or countably infinite index set. Via canonical quantization on the symplectic vector space $T^*V = V \oplus V^*$, we get a Lie algebra generated by $\{x_i, \hbar\partial_{x_i}\}_{i\in I}$ as a $k[\![\hbar]\!]$-module[4] with Lie bracket

$$[\hbar\partial_{x_i}, x_j] = \hbar\delta_{ij},$$

known as the Weyl algebra, denoted by $\mathcal{D}_V^\hbar$. We equip the Weyl algebra with the grading $\deg x_i = 1$, $\deg \hbar\partial_{x_i} = 1$ and $\deg \hbar = 2$.

Here is the notion of a quantum Airy structure as defined by [KS18].

Definition 2.4.1 (Quantum Airy structure)**.** A *quantum Airy structure* is the data of at most quadratic (in the above grading) polynomials $\{H_i\}_{i\in I}$ in $\mathcal{D}_V^\hbar$, satisfying the following conditions

1. The polynomials H_i are of the form

$$H_i = \hbar\partial_{x_i} - P_i,$$

where the P_i are of degree 2.

2. The subspace $\oplus_{i\in I} kH_i$ is a Lie sub-algebra of $\mathcal{D}_V^\hbar$.

Then, the authors of [KS18] prove the existence and uniqueness of a WKB-type partition function that is annihilated by all the H_i.

Theorem 2.4.2. *There exists a unique formal solution (called the partition function) Z of the form*

$$Z = \exp\left(\sum_{g\geq 0, n\geq 1}\sum_{i_1,\cdots,i_n\in I} \hbar^{g-1}F_{g,n}(i_1,\cdots,i_n)x^{i_1}\cdots x^{i_n}\right)$$

to the system of equations

$$H_i\, Z = 0 \qquad \textit{for} \qquad i \in I,$$

where the $F_{g,n}(i_1,\cdots,i_n) \in k$ are invariant under the permutation of the indices and $F_{0,2}(i_1,i_2) = 0 = F_{0,1}(i_1)$ for all i_1 and i_2.

[4] $\hbar$ is a formal variable.

Then [ABCO17, KS18] show that (the Eynard-Orantin version of) the topological recursion is realized as an example of an Airy structure in the following manner. Consider a spectral curve $(\Sigma, x, \omega_{0,1}, \omega_{0,2})$ with p simple ramification points. Then, take p copies of the Virasoro algebra[5] with central charge 1. Using the initial data $\omega_{0,1}$ and $\omega_{0,2}$, we can construct a representation of this algebra such that the positive half of the generators satisfy the conditions in Definition 2.4.1 to be a quantum Airy structure. Then the $F_{g,n}$ constructed by Theorem 2.4.2 are the coefficients that one obtains by expanding the $\omega_{g,n}$ constructed by the topological recursion on a basis of meromorphic differentials with poles at the ramification points (see Remark 2.2.13).

2.4.1 Higher Airy Structures

Chapter 4 defines a generalization of the notion of quantum Airy structures, which we call higher quantum Airy structures (or higher Airy structures for convenience). We drop the requirement of quadraticity for the differential operators in Definition 2.4.1 (i.e., the condition $\deg P_i = 2$), and make appropriate modifications to the definition (see Definition 4.2.6). This allows us to handle the case of topological recursion on spectral curves with an arbitrary ramification profile and other interesting enumerative problems. Some of the enumerative geometry partition functions that we can compute using the formalism of higher Airy structures include the topological recursion (in full generality), FJRW theory of type ADE and open intersection theory.

Using the notion of higher Airy structures, we also obtain numerous partition functions which have no known enumerative interpretation. Conjecturally, these include r-spin open intersection theory and intersection theory on certain moduli spaces that generalize Witten's r-spin moduli space, which were first considered by Chiodo in [Chi08]. For some comments on these and other

[5] The Virasoro algebra is an infinite dimensional Lie algebra generated by L_i with $i \in \mathbb{Z}$ satisfying the following commutation relations

$$[L_i, L_j] = (i-j)L_{i+j} + c\frac{i^3 - i}{12}\delta_{i+j,0},$$

where $c \in k$ is called the central charge.

work in progress see Chapter 8.

Let us briefly describe our strategy for constructing higher Airy Structures (this is explained in Section 4.1.2). We choose a $\mathcal{W}$-algebra (typically at the 'self-dual' level), which can be identified as a sub-algebra of a free-field algebra, such as a Heisenberg or free-fermion algebra. Then, we consider an automorphism, say σ, of this free-field algebra that leaves the $\mathcal{W}$-algebra invariant. This automorphism naturally defines a σ-twisted module of the free-field algebra and upon restriction, this defines a(n untwisted) $\mathcal{W}$-algebra module. We pick a sub-algebra of modes of this $\mathcal{W}$-algebra module that satisfies the Lie sub-algebra condition in the definition of higher Airy structures (see eq (4.8) in Definition 4.2.6). Finally, we may have to do a conjugation of the modes via a 'dilaton shift' in order to bring it into the form of a higher Airy structure (see eq (4.7) in Definition 4.2.6).

Perhaps the hardest, and consequently the most interesting, aspect of this story is to identify the enumerative geometric meaning of the partition function associated to the constructed higher Airy structure. We carry this out for some of the examples we construct, but a majority of them are still awaiting an enumerative interpretation. One of the consequences of our construction to the theory of the topological recursion is the definition of admissibility 2.1.8 and its relation to the symmetry of the correlators $\omega_{g,n}$ as explained in Remark 2.2.11.

Chapter 3

Quantizing Weierstrass

3.1 Introduction

The starting point of the Eynard-Orantin topological recursion [EO07, EO08] is a spectral curve. For the purposes of this paper, we can think of a spectral curve as an irreducible algebraic curve $\{P(x, y) = 0\} \subset \mathbb{C}^2$. Then, the topological recursion recursively constructs an infinite sequence of symmetric meromorphic differentials $\omega_{g,n}$, $g \geq 0$ and $n \geq 1$, on the spectral curve. Depending on the choice of spectral curve, these differentials turn out to be generating functions for many different types of enumerative invariants, such as Gromov-Witten invariants, Hurwitz numbers, knot invariants, etc.[1] (See for instance [BE15, BEMS11, BKMnP09, BMn08, BHSLM14, DFM11, DLN16, DBOSS14, DBKO$^+$15, DBMN$^+$17, EMS11, EO07, EO08, EO15, FLZ16a, FLZ20, FLZ16b, GJKS15, Mn08, MSS13].)

The Eynard-Orantin topological recursion originated in the context of matrix models [CEO06, Eyn04, EO07, EO08]. But given its rather universal enumerative geometric interpretation, it has now a life of its own. However, it is still interesting to explore its roots, and see whether matrix model theory suggests further connections to *a priori* unrelated mathematical structures. Those may lead to unexpected results in the various geometric contexts.

One such connection that is suggested by matrix models relates the Eynard-Orantin topological recursion to WKB asymptotic solutions of differential

[1] To be precise, for most of these applications the definition of the spectral curve must be generalized slightly.

equations. In matrix models one can construct an object ψ called the wave-function of the theory. On general grounds, it is then expected that there exists a "quantization" $\hat{P}(\hat{x}, \hat{y}; \hbar)$ of the spectral curve that annihilates ψ; this quantization of the spectral curve is generally called a *quantum curve*.

But what do we mean by quantization here? Assume that the spectral curve $P(x, y) = 0$ has degree d in y. Let $\hat{P}(\hat{x}, \hat{y}; \hbar)$ be a polynomial in $\hat{x}$ and $\hat{y}$ of degree d in $\hat{y}$, with coefficients that are possibly power series in $\hbar$. We let $\hat{x}$ and $\hat{y}$ be quantizations of the variables x and y:

$$\hat{x} = x, \qquad \hat{y} = \hbar \frac{d}{dx}, \tag{3.1}$$

such that $[\hat{y}, \hat{x}] = \hbar$. This makes $\hat{P}(\hat{x}, \hat{y}; \hbar)$ into a rank d differential operator in x, with coefficients that are polynomial in x and possibly power series in $\hbar$. We then say that $\hat{P}(\hat{x}, \hat{y}; \hbar)$ is a *quantum curve* (of the original spectral curve) if, after normal ordering (that is bringing all the $\hat{x}$'s to the left of the $\hat{y}$'s), it takes the form

$$\hat{P}(\hat{x}, \hat{y}; \hbar) = P(\hat{x}, \hat{y}) + \sum_{n \geq 1} \hbar^n P_n(\hat{x}, \hat{y}), \tag{3.2}$$

where the leading order term $P(\hat{x}, \hat{y})$ recovers the original spectral curve (normal ordered), and the $P_n(\hat{x}, \hat{y})$ are differential operators in x of rank at most $d - 1$.

With this definition, the expectation from matrix models is that the wave-function ψ should be the WKB asymptotic solution of the differential equation

$$\hat{P}(\hat{x}, \hat{y}; \hbar)\psi = 0, \tag{3.3}$$

for some quantum curve $\hat{P}(\hat{x}, \hat{y}; \hbar)$. This expectation follows from determinantal formulae in matrix models [BE09a, BE09b].

The question of existence of quantum curves can be explored without reference to matrix models. Indeed, one can construct a wave-function ψ in a natural way from the meromorphic differentials $\omega_{g,n}$ obtained from the topological recursion, without reference to any underlying matrix model. The question is then: for arbitrary spectral curves, does there exist a quantization that kills the wave function? And if so, can we construct this quantum curve explicitly from the topological recursion?

It should be noted here that this connection also has a deep relation with integrable systems. As explained in [BE12b], one can think of the wave-function as the Schlesinger transform of the partition function of the theory. If one assumes that the partition function is a τ-function, *i.e.* that it satisfies the Hirota equations, then it follows that the wave-function should be annihilated by a quantization of the spectral curve. However, it is not known whether the partition function constructed from the topological recursion is a τ-function in general.

An answer to the question above about the existence of quantum curves was provided in [BE17] for a very large class of genus zero spectral curves. More precisely, using the global topological recursion constructed in [BE13, BHL$^+$14], it was proven that there exists a quantization that kills the wave-function for all spectral curves whose Newton polygons have no interior point and that are smooth as affine curves. For any such spectral curve, the quantum curve was reconstructed explicitly from the topological recursion. In fact, the quantum curve is not unique; one obtains different quantum curves (corresponding to different choices of ordering) depending on how one integrates the $\omega_{g,n}$ to construct the wave-function. More details can be found in [BE17].

The goal of this paper is to study the relation between the topological recursion and quantum curves for genus one spectral curves. More precisely, we will focus on the family of spectral curves given by the Weierstrass equation

$$y^2 = 4x^3 - g_2(\tau)x - g_3(\tau). \tag{3.4}$$

We can apply the Eynard-Orantin topological recursion to this family of curves; the meromorphic differentials $\omega_{g,n}$ are elliptic and quasi-modular, while the free energies F_g obtained from the recursion are quasi-modular forms. An interesting open question is whether these $\omega_{g,n}$ and F_g have an enumerative interpretation for some geometric problem. We do not have an answer to this question. Nevertheless, this spectral curve is an interesting playground to study the connection between the topological recursion and quantum curves for spectral curves of genus greater than zero, since everything can be calculated very explicitly in terms of Weierstrass $\wp$ and $\wp'$ functions and Eisenstein series.

We initially study the wave-function ψ constructed as in [BE17]; this is known as the *perturbative wave-function*. For the Weierstrass spectral curve, it is obtained from the $\omega_{g,n}$ through the standard equation

$$\psi(z) = \exp\left(\frac{1}{\hbar}\sum_{2g-1+n\geq 0}\frac{\hbar^{2g+n-1}}{n!}\int_0^z\cdots\int_0^z\Big(\omega_{g,n}(z_1,\ldots,z_n)\right.$$
$$\left.-\delta_{g,0}\delta_{n,2}\frac{dx(z_1)dx(z_2)}{(x(z_1)-x(z_2))^2}\Big)\right). \quad (3.5)$$

For this $\psi(z)$ we follow the steps in [BE17], suitably generalized for our genus one curve, and construct a rank two differential operator that annihilates ψ; however, it is not a quantum curve as defined above. But this was to be expected; from matrix model theory, when the spectral curve has genus greater than zero, the right wave-function to consider is not the perturbative wave-function. Rather, it needs to be corrected non-perturbatively.

A non-perturbative partition function was defined in [Eyn09, EMn11] from the topological recursion directly, without reference to matrix models. The idea was to make the partition function modular invariant, which requires non-perturbative corrections. From the non-perturbative partition function one can define a non-perturbative wave-function as the Schlesinger transform [BE15, BE12b]. This non-perturbative wave-function is the object that should be annihilated by a quantum curve.

We study the non-perturbative wave-function for the Weierstrass spectral curve. To obtain a well defined power series expansion in $\hbar$, it turns out that a quantization condition must be satisfied. The simplest elliptic curve that satisfies the quantization condition is:

$$y^2 = 4(x^3-1),$$

which is of course a very special elliptic curve — for instance, its j-invariant vanishes. Focusing on this spectral curve, through extensive symbolic calculations on Mathematica we calculate the wave-function up to order 5 in $\hbar$. Remarkably, while the $\omega_{g,n}$ become extremely complicated, they somehow combine into very nice and simple elliptic functions in the non-perturbative wave-function. Using these calculations we are able to construct a quantum curve that annihilates the non-perturbative wave-function up to order $\hbar^5$. Perhaps unexpectedly, the quantization of the spectral curve includes non-trivial $\hbar$ corrections — in fact probably an infinite number of such corrections. Nonetheless, the quantum curve is a true quantization of the spectral curve according to the definition above, as suggested by matrix model arguments. Therefore we obtain a proof of the existence of the quantum curve for the non-perturbative wave-function up to order $\hbar^5$, for this particular elliptic curve.

Going back to the perturbative wave-function, we perform the calculation of the differential operator in two different ways. It turns out that equivalence of the two approaches implies an infinite sequence of identities for A-cycle integrals of elliptic functions with quasi-modular properties. In particular, an infinite sub-sequence relates A-cycle integrals of elliptic functions to quasi-modular forms. In this paper, we write down explicitly only the first few identities, but it would certainly be interesting to study whether these identities are interesting from the point of view of elliptic functions and quasi-modular forms. For instance, they may be related to the results on quasi-modular forms obtained in [GM20]. We hope to report on that in the near future.

Outline

In Section 3.2 we review background material on elliptic functions and quasi-modular forms that will be needed in this paper. We also define the Eynard-Orantin topological recursion. In Section 3.3, we generalize the approach of [BE17] to construct a rank two differential operator that annihilates the perturbative wave-function. In Section 3.4, we construct the same differential operator using a different approach, via the Riemann bilinear identity. We explore the connection and equivalence between the two approaches, which leads to the proof of an infinite sequence of identities for elliptic functions and quasi-modular forms. We briefly explore the first few of these identities in

Section 3.5. Then in Section 3.6 we study the non-perturbative wave-function and construct a quantum curve up to order $\hbar^5$. We conclude in Section 3.7, with open questions and research avenues. Finally, we record in Appendix A.1 the first few $\omega_{g,n}$ constructed from the Weierstrass spectral curve, and in Appendix A.2 we provide an independent proof of the simplest identity that we obtained for elliptic functions, without reference to the topological recursion.

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3.2 Topological recursion on Weierstrass spectral curve

3.2.1 Elliptic functions and modular forms

We start by defining standard objects that will be useful in this paper.

Let $\mathcal{H} = \{\tau \in \mathbb{C} | \Im(\tau) > 0\}$ be the upper half-plane, and define the lattice $\Lambda = \mathbb{Z} + \mathbb{Z}\tau$. The quotient $\mathbb{C}/\Lambda$ is topologically a torus. Functions on $\mathbb{C}/\Lambda$ are given by doubly periodic functions, known as elliptic functions.

The Weierstrass function $\wp(z;\tau)$ is an example of an elliptic function. It is defined by

$$\wp(z;\tau) = \frac{1}{z^2} + \sum_{\omega\in\Lambda^*}\left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\right), \tag{3.6}$$

where $\Lambda^* = \Lambda \setminus \{0\}$. The Weierstrass function $\wp'(z;\tau)$ is the derivative of $\wp(z;\tau)$ with respect to z; it is given by

$$\wp'(z;\tau) = -2\sum_{\omega\in\Lambda}\frac{1}{(z-\omega)^3}. \tag{3.7}$$

We also define the Weierstrass function $\zeta(z;\tau)$ by

$$\zeta(z;\tau) = \frac{1}{z} + \sum_{\omega\in\Lambda^*} \left(\frac{1}{z-\omega} + \frac{1}{\omega} + \frac{z}{\omega^2} \right). \tag{3.8}$$

This function is not elliptic; rather, it is quasi-elliptic, since $\zeta(z+1;\tau) = \zeta(z;\tau) + 2\zeta(1/2;\tau)$ and $\zeta(z+\tau;\tau) = \zeta(z;\tau) + 2\zeta(\tau/2;\tau)$. It is clear that

$$\wp(z;\tau) = -\frac{d}{dz}\zeta(z;\tau). \tag{3.9}$$

The Eisenstein series $G_{2n}(\tau)$, for $n \geq 2$, are defined by the uniformly convergent series

$$G_{2n}(\tau) = \sum_{\omega\in\Lambda^*} \frac{1}{\omega^{2n}}. \tag{3.10}$$

They are weight $2n$ modular forms, which means that they transform as

$$G_{2n}\left(\gamma\tau\right) = (c\tau+d)^{2n} G_{2n}(\tau), \quad \forall\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}), \tag{3.11}$$

with

$$\gamma\tau = \frac{a\tau+b}{c\tau+d}. \tag{3.12}$$

We can extend this definition to $n=1$, but then the series is not absolutely convergent anymore, so the order of summation matters. We define the second Eisenstein series $G_2(\tau)$

$$G_2(\tau) = \sum_{m\neq 0} \frac{1}{m^2} + \sum_{n\neq 0}\sum_{m\in\mathbb{Z}} \frac{1}{(m+n\tau)^2}. \tag{3.13}$$

Because of the non-absolute convergence, we cannot change the order of summation, and it can be shown that it implies that $G_2(\tau)$ is not a modular form, but is rather a quasi-modular form of weight 2. This means that it transforms with a shift, as

$$G_2\left(\gamma\tau\right) = (c\tau+d)^2 G_2(\tau) - 2\pi i c(c\tau+d). \tag{3.14}$$

We define the invariants

$$g_2(\tau) = 60G_4(\tau), \qquad g_3(\tau) = 140G_6(\tau). \tag{3.15}$$

It is well known that the Weierstrass functions satisfy the equation

$$\wp'(z;\tau)^2 = 4\wp(z;\tau)^3 - g_2(\tau)\wp(z;\tau) - g_3(\tau). \tag{3.16}$$

In other words, $x = \wp(z;\tau)$ and $y = \wp'(z;\tau)$ parameterize the Weierstrass curve

$$y^3 = 4x^3 - g_2(\tau)x - g_3(\tau). \tag{3.17}$$

The Weierstrass $\wp'(z;\tau)$ has three simple zeros at the half-periods

$$w_1 = \frac{1}{2}, \qquad w_2 = \frac{\tau}{2}, \qquad w_3 = -\frac{1}{2}(1+\tau). \tag{3.18}$$

As is customary, we denote by

$$e_1 = \wp(w_1;\tau), \qquad e_2 = \wp(w_2;\tau), \qquad e_3 = \wp(w_3;\tau) \tag{3.19}$$

the value of the $\wp(z;\tau)$ function at the half-periods. We introduce the discriminant

$$\Delta(\tau) = g_2(\tau)^3 - 27g_3(\tau)^2 = 16(e_1-e_2)^2(e_2-e_3)^2(e_3-e_1)^2, \tag{3.20}$$

which is a modular form of weight 12.

The Weierstrass $\wp(z;\tau)$ function has a double pole at $z=0$. Its expansion near $z=0$ has a nice form; it is given by

$$\wp(z;\tau) = \frac{1}{z^2} + \sum_{k=1}^{\infty}(2k+1)G_{2k+2}(\tau)z^{2k}. \tag{3.21}$$

Following, for instance, [MT10], let us define a new function $P_2(z;\tau)$ by including the $k=0$ term in the sum above:

$$P_2(z;\tau) = \wp(z;\tau) + G_2(\tau). \tag{3.22}$$

Of course $P_2(z;\tau)$ is not modular anymore, it is quasi-modular, because of $G_2(\tau)$. It is straightforward to show that it can be rewritten as

$$P_2(z;\tau) = (2\pi i)^2 \sum_{n\in\mathbb{Z}^*} \frac{nq_z^n}{1-q^n}, \tag{3.23}$$

where $q_z = \exp(2\pi i z)$ and $q = \exp(2\pi i\tau)$.

We also introduce $P_1(z;\tau)$ such that $\frac{dP_1(z;\tau)}{dz} = P_2(z;\tau)$. It follows that

$$P_1(z;\tau) = 2\pi i \sum_{n\in\mathbb{Z}^*} \frac{q_z^n}{1-q^n} + A, \tag{3.24}$$

for some constant A. We fix A such that $P_1(-z;\tau) = -P_1(z;\tau)$. It follows that $A = \pi i$, that is,

$$P_1(z;\tau) = 2\pi i \left(\sum_{n\in\mathbb{Z}^*} \frac{q_z^n}{1-q^n} + \frac{1}{2} \right). \tag{3.25}$$

In terms of standard elliptic functions, we get

$$P_1(z;\tau) = -\zeta(z;\tau) + G_2(\tau)z. \tag{3.26}$$

$P_1(z;\tau)$ is not elliptic anymore, but its transformation properties can be calculated. It is straightforward to show that

$$P_1(z+1;\tau) = P_1(z), \qquad P_1(z+\tau;\tau) = P_1(z) + 2\pi i. \tag{3.27}$$

The second transformation is of course what makes it not quite elliptic.

3.2.2 Spectral curve

To define the topological recursion we need to introduce the notion of spectral curve.

Definition 3.2.1. A *spectral curve* is a triple (Σ, x, y) where Σ is a Torelli marked genus $\hat{g}$ compact Riemann surface[2] and x and y are meromorphic functions on Σ, such that the zeros of dx do not coincide with the zeros of dy.

In this paper we will focus on one particular family of spectral curves, which we call the Weierstrass spectral curve.

Definition 3.2.2. The *Weierstrass spectral curve* is defined by the triple (Σ, x, y), where, $\Sigma = \mathbb{C}/\Lambda$ with lattice $\Lambda = \mathbb{Z} \oplus \tau\mathbb{Z}$, $x = \wp(z;\tau)$ and $y = \wp'(z;\tau)$. Then the meromorphic functions x and y identically satisfy the Weierstrass equation

$$y^2 = 4x^3 - g_2(\tau)x - g_3(\tau). \tag{3.28}$$

[2]A Torelli marked compact Riemann surface Σ is a genus $\hat{g}$ Riemann surface Σ with a choice of symplectic basis of cycles $(A_1, \ldots, A_{\hat{g}}, B_1, \ldots, B_{\hat{g}}) \in H_1(\Sigma, \mathbb{Z})$.

This is of course a genus one spectral curve, since Σ is a torus. In fact, it is a family of curves, parameterized by the complex modulus τ.

As usual in topological recursion we are interested in the branched covering $\pi : \Sigma \to \mathbb{P}^1$ given by the meromorphic function x. For the Weierstrass spectral curve, π is a double cover. The deck transformation that exchanges the two sheets is simply given by $z \mapsto \tau(z) = -z$, since $\wp(z;\tau)$ is an even function in z.

We denote by R the set of ramification points of π, which is given by the zeros of dx and the poles of x of order ≥ 2. For the Weierstrass spectral curve, since

$$dx = \wp'(z;\tau)dz, \tag{3.29}$$

the zeros of dx are given by the half-periods w_i, $i = 1, 2, 3$ introduced earlier. Moreover, $x = \wp(z;\tau)$ has a double pole at $z = 0$. Therefore $R = \{w_1, w_2, w_3, 0\}$.

3.2.3 Geometric objects

For the topological recursion we also need the following objects that are canonically defined on a genus $\hat{g}$ compact Riemann surface Σ with a symplectic basis of cycles for $H_1(\Sigma, \mathbb{Z})$.

Definition 3.2.3. Let $a, b \in \Sigma$. The *canonical differential of the third kind* $\omega^{a-b}(z)$ is a meromorphic one-form on Σ such that:

- it is holomorphic away from $z = a$ and $z = b$;
- it has a simple pole at $z = a$ with residue $+1$;
- it has a simple pole at $z = b$ with residue -1;
- it is normalized on A-cycles:

$$\oint_{z \in A_i} \omega^{a-b}(z) = 0, \qquad \text{for } i = 1, \ldots, \hat{g}. \tag{3.30}$$

Definition 3.2.4. The *canonical bilinear differential of the second kind* $B(z_1, z_2)$ is the unique bilinear differential on Σ^2 satisfying the conditions:

- It is symmetric, $B(z_1, z_2) = B(z_2, z_1)$;
- It has its only pole, which is double, along the diagonal $z_1 = z_2$, with leading order term (in any local coordinate z)

$$B(z_1, z_2) \underset{z_1 \to z_2}{\to} \frac{dz_1 dz_2}{(z_1 - z_2)^2} + \ldots; \tag{3.31}$$

- It is normalized on A-cycles:

$$\oint_{z_1 \in A_i} B(z_1, z_2) = 0, \qquad \text{for } i = 1, \ldots, \hat{g}. \tag{3.32}$$

Remark 3.2.5. It follows from the definition that

$$B(z_1, z_2) = d_1 \omega^{z_1 - b}(z_2). \tag{3.33}$$

Equivalently,

$$\omega^{a-b}(z) = \int_{z_1=b}^{a} B(z_1, z), \tag{3.34}$$

where the integral is taken over the unique homology chain with boundary $[a] - [b]$ that doesn't intersect the homology basis.

It is not too difficult to identify what these objects are on the Weierstrass spectral curve. Recall that $\Sigma = \mathbb{C}/\Lambda$ with lattice $\Lambda = \mathbb{Z} \oplus \tau\mathbb{Z}$. We fix the A-cycle to be given by $z \in [0, 1)$,[3] and the B-cycle to be given by $z \in [0, \tau)$. The canonical bilinear differential of the second kind is given by

$$B(z_1, z_2) = P_2(z_1 - z_2; \tau) dz_1 dz_2, \tag{3.35}$$

where $P_2(z; \tau)$ was introduced in (3.22). First, it is symmetric, since $P_2(z; \tau)$ is an even function of z. Second, it is clear from (3.21) that it has a double pole on the diagonal with the right leading behavior. It has no other poles. As for normalization, one can check that it is normalized on the A-cycle.

The canonical differential of the third kind is then given by

$$\omega^{a-b}(z) = \left(P_1(z - b; \tau) - P_1(z - a; \tau)\right) dz. \tag{3.36}$$

[3]To be precise, to integrate elliptic functions over A-cycles we need to shift it by $i\epsilon$ to avoid poles on the contour. For the same reason, a similar shift by a purely real ϵ must be done when evaluating B-cycle integrals.

3.2.4 Topological recursion

We now introduce the topological recursion formalism, which was first proposed in [CEO06, Eyn04, EO07, EO08]. For clarity of presentation, we will only introduce the formalism in the context of the Weierstrass spectral curve.

Let (Σ, x, y) be a spectral curve. The topological recursion constructs an infinite tower of symmetric meromorphic differentials $\omega_{g,n}(z_1, \ldots, z_n)$, known as *correlation functions*, on Σ^n. We now consider the special case where (Σ, x, y) is the Weierstrass spectral curve.

Definition 3.2.6 (Topological recursion)**.** We first define the initial conditions

$$\omega_{0,1}(z) = y(z)dx(z) = \wp'(z;\tau)^2 dz, \tag{3.37}$$

$$\omega_{0,2}(z_1, z_2) = B(z_1, z_2) = P_2(z_1 - z_2; \tau) dz_1 dz_2. \tag{3.38}$$

Let $\mathbf{z} = \{z_1, \ldots, z_n\} \in \Sigma^n$. For $n \geq 0$, $g \geq 0$ and $2g - 2 + n \geq 0$, we uniquely construct symmetric meromorphic differentials $\omega_{g,n}$ on Σ^n with poles along R via the *Eynard-Orantin topological recursion*:

$$\omega_{g,n+1}(z_0, \mathbf{z}) = \sum_{a \in R} \operatorname*{Res}_{z=a} K(z_0; z) \mathcal{R}^{(2)} \omega_{g,n+1}(z, -z; \mathbf{z}), \tag{3.39}$$

where the recursion kernel is given by

$$K(z_0; z) = \frac{\omega^{z-\alpha}(z_0)}{(y(z) - y(-z))dx(z)} = \frac{(P_1(z_0 - \alpha; \tau) - P_1(z_0 - z; \tau))\, dz_0}{2\wp'(z;\tau)^2 dz}, \tag{3.40}$$

with α an arbitrary base point on Σ (it can be checked that the definition is independent of the choice of α, as a long as it is generic). The recursive structure is given by

$$\begin{aligned} &\mathcal{R}^{(2)} \omega_{g,n+1}(z, -z; \mathbf{z}) \\ &\quad = \omega_{g-1,n+2}(z, -z, \mathbf{z}) + \sum_{g_1 + g_2 = g} \sideset{}{'}\sum_{I \cup J = \mathbf{z}} \omega_{g_1, |I|+1}(z, I) \omega_{g_2, |J|+1}(-z, J). \end{aligned} \tag{3.41}$$

In the second sum we are summing over all disjoint subsets $I, J \subset \mathbf{z}$ whose union is $\mathbf{z}$, and the prime means that we exclude the cases $(g_1, |I|) = (0, 0)$ and $(g_2, |J|) = (0, 0)$.

We can compute the first few correlation functions explicitly for the Weierstrass spectral curve. Those are presented in Appendix A.

For later reference we also introduce the free energies $F_g := \omega_{g,0}$, $g \geq 2$, for the spectral curve. Those are obtained via the auxiliary equation

$$F_g = \frac{1}{2-2g} \sum_{a \in R} \operatorname*{Res}_{z=a} \phi(z) \omega_{g,1}(z), \tag{3.42}$$

with $\phi(z) = \int y(z)dx(z)$ an arbitrary anti-derivative of the one-form ydx.

3.2.5 Quantum curve

The purpose of this paper is to related the meromorphic differentials $\omega_{g,n}$ constructed from the topological recursion to the WKB asymptotic solution of a differential operator, known as a quantum curve. The connection will be explored in more detail later on in this paper, but for clarity and further reference let us define here what we mean by quantum curve.

Consider a spectral curve (Σ, x, y). Let $P(x, y) = 0$ be the minimal irreducible polynomial equation satisfied by x and y. Assume that it has degree d in y.

Define the quantization of the variables x and y as:

$$\hat{x} = x, \qquad \hat{y} = \hbar \frac{d}{dx}, \tag{3.43}$$

such that $[\hat{y}, \hat{x}] = \hbar$.

Definition 3.2.7 (Quantum curve)**.** Let $\hat{P}(\hat{x}, \hat{y}; \hbar)$ be a polynomial in $\hat{x}$ and $\hat{y}$ of degree d in $\hat{y}$, with coefficients that are possibly power series in $\hbar$. $\hat{P}(\hat{x}, \hat{y}; \hbar)$ is a rank d differential operator in x, with polynomial coefficients in x that are possibly power series in $\hbar$. We say that $\hat{P}(\hat{x}, \hat{y}; \hbar)$ is a *quantum curve* (of the original spectral curve) if, after normal ordering (that is bringing all the $\hat{x}$'s to the left of the $\hat{y}$'s), it takes the form

$$\hat{P}(\hat{x}, \hat{y}; \hbar) = P(\hat{x}, \hat{y}) + \sum_{n \geq 1} \hbar^n P_n(\hat{x}, \hat{y}), \tag{3.44}$$

where the leading order term $P(\hat{x}, \hat{y})$ recovers the original spectral curve (normal ordered), and the $P_n(\hat{x}, \hat{y})$ are differential operators in x of rank at most $d - 1$.

We note here that the requirement that the $P_n(\hat{x}, \hat{y})$ have rank at most $d-1$ is equivalent to requiring that the coefficient of the highest degree $\hat{y}^d$ term in $\hat{P}(\hat{x}, \hat{y}; \hbar)$ does not depend on $\hbar$.

As an example that will be particularly relevant later on, consider the elliptic spectral curve $y^2 = 4(x^3 - 1)$. According to the definition above, a quantum curve for this spectral curve must take the form

$$\hat{P}(\hat{x}, \hat{y}; \hbar) = \hbar^2 \frac{\mathrm{d}^2}{\mathrm{d}x^2} - 4(x^3 - 1) + \sum_{i \geq 2} \hbar^i A_i(x) \frac{\mathrm{d}}{\mathrm{d}x} + \sum_{j \geq 1} \hbar^j B_j(x), \tag{3.45}$$

with the $A_i(x)$ and $B_j(x)$ polynomials in x.

3.3 Perturbative wave-function: first approach

In this section we approach the problem of constructing the quantum curve for the Weierstrass spectral curve naively. We apply the method of [BE17] directly with a few modifications. The idea here is to construct the wave-function as in [BE17], which is what we will call the "perturbative wave-function", and then show that it is annihilated by a rank two differential operator. However, this differential operator is not a quantum curve, according to Definition 3.2.7. But this is because, as we will see in section 3.6, and as is already expected from matrix models (see for instance [BE15]), the perturbative wave-function is not the right object to look at. For spectral curves of genus ≥ 1, one should use the non-perturbative wave-function to construct the quantum curve. We will study that in more detail in section 3.6.

Coming back to the goal of this section, recall that in [BE17] quantum curves were obtained for all spectral curves whose Newton polygons have no interior point and that are smooth as affine curves. Of course, the Weierstrass spectral curve does not fall into that class, since its Newton polygon has an interior point (it has genus one). However, the main results of [BE17] can be adapted for this particular case, which is what we do in this section.

In this section we borrow heavily on the notation and calculations of [BE17], even though the calculations are much simpler in the case at hand. The reader may want to refer to this paper for more detail.

3.3.1 Reconstructing loop equations

The first step in [BE17] is to reconstruct some sort of "loop equations" from the topological recursion. This is the content of Lemma 4.7 in [BE17]. Let us recall the notation. Here we focus on the Weierstrass spectral curve with the branched covering π. Since π is a double cover and the deck transformation is given by $z \mapsto -z$, the objects introduced in [BE17] simplify drastically.

We first define

$$\begin{aligned} &Q^{(2)}_{g,n+1}(z;\mathbf{z}) \\ &\quad = \omega_{g-1,n+2}(z,-z,\mathbf{z}) + \sum_{g_1+g_2=g} \sum_{I\cup J=\mathbf{z}} \omega_{g_1,|I|+1}(z,I)\omega_{g_2,|J|+1}(-z,J). \end{aligned} \tag{3.46}$$

This is just like the recursive structure $\mathcal{R}^{(2)}\omega_{g,n+1}(z,-z;\mathbf{z})$ introduced in (3.41), but with the $(g_1,|I|)=(0,0)$ and $(g_2,|J|)=(0,0)$ terms included.

In our context, Lemma 4.7 of [BE17] becomes the following statement:

Lemma 3.3.1. *For $2g-2+n \geq 0$, the meromorphic one-forms (in z)*

$$dz\left(\frac{Q^{(2)}_{g,n+1}(z;\mathbf{z})}{dx(z)^2}\right). \tag{3.47}$$

can only have poles (in z) at $z = \pm z_i$, $i = 1,\ldots,n$.

Proof. The proof of Lemma 4.7 presented in [BE17] only requires the spectral curve to be smooth as an affine curve; it does not require the property that the Newton polygon has no interior point. Since the Weierstrass spectral curve is generically smooth as an affine curve, the proof goes through untouched. □

The next step in the approach of [BE17] is to prove Lemma 4.8, Lemma 4.9 and Theorem 4.12. Here the proofs need to be modified, and the results will differ. So let us go through these statements carefully.

The first lemma is

Lemma 3.3.2. *For the Weierstrass spectral curve,*

$$\frac{Q^{(2)}_{0,1}(z)}{dx(z)^2} = -4x(z)^3 + g_2(\tau)x(z) + g_3(\tau). \tag{3.48}$$

Proof. This is straightforward since

$$\frac{Q_{0,1}^{(2)}(z)}{dx(z)^2} = \frac{\omega_{0,1}(z)\omega_{0,1}(-z)}{dx(z)^2} = y(z)y(-z) = -y(z)^2. \tag{3.49}$$

□

The second lemma is a little more involved:

Lemma 3.3.3. *For the Weierstrass spectral curve,*

$$\frac{Q_{0,2}^{(2)}(z;z_1)}{dx(z)^2} = -d_{z_1}\left(\frac{1}{(x(z)-x(z_1))}\frac{\omega_{0,1}(z_1)}{dx(z_1)}\right) - 2P_2(z_1;\tau)dz_1. \tag{3.50}$$

Proof. Here the proof from [BE17] needs to be modified, so let us do it carefully. We have:

$$\begin{aligned}\frac{Q_{0,2}^{(2)}(z;z_1)}{dx(z)^2} =& \frac{B(z,z_1)}{dx(z)}\frac{\omega_{0,1}(-z)}{dx(z)} + \frac{B(-z,z_1)}{dx(z)}\frac{\omega_{0,1}(z)}{dx(z)} \\ =& -\frac{B(z,z_1)}{dx(z)}\frac{\omega_{0,1}(z)}{dx(z)} - \frac{B(-z,z_1)}{dx(z)}\frac{\omega_{0,1}(-z)}{dx(z)} \\ =& -\operatorname*{Res}_{z'=z}\frac{B(z',z_1)}{x(z')-x(z)}\frac{\omega_{0,1}(z')}{dx(z')} - \operatorname*{Res}_{z'=-z}\frac{B(z',z_1)}{x(z')-x(z)}\frac{\omega_{0,1}(z')}{dx(z')}.\end{aligned} \tag{3.51}$$

Now the expression

$$\frac{B(z',z_1)}{x(z')-x(z)}\frac{\omega_{0,1}(z')}{dx(z')} \tag{3.52}$$

is a meromorphic one-form in z' on the compact Riemann surface Σ. Therefore, the sum of its residues must vanish. Its only poles are at $z'=z_1$, $z'=\pm z$, and at the pole of $\frac{\omega_{0,1}(z')}{dx(z')} = y(z')$, that is, at $z'=0$.[4] Thus we get

$$\begin{aligned}\frac{Q_{0,2}^{(2)}(z;z_1)}{dx(z)^2} =& -\operatorname*{Res}_{z'=z_1}\frac{B(z',z_1)}{x(z)-x(z')}\frac{\omega_{0,1}(z')}{dx(z')} - \operatorname*{Res}_{z'=0}\frac{B(z',z_1)}{x(z)-x(z')}\frac{\omega_{0,1}(z')}{dx(z')} \\ =& -d_{z_1}\left(\frac{1}{x(z)-x(z_1)}\frac{\omega_{0,1}(z_1)}{dx(z_1)}\right) - 2P_2(z_1;\tau)dz_1,\end{aligned} \tag{3.53}$$

where we used the fact that $\wp(z;\tau) \sim \frac{1}{z^2}$ near $z=0$, $\wp'(z;\tau) \sim -\frac{2}{z^3}$, and $B(z_1,z_2) = P_2(z_1-z_2;\tau)dz_1dz_2$. □

[4] This is where the proof differs from [BE17]. When the Newton polygon has no interior point, the only poles are at $z'=z_1$ and at $z'=\tau_i(z)$ where $\tau_i(z)$ indexes the sheets of the branched covering π.

Remark 3.3.4. Note that by replacing both the left-hand-side and right-hand-side of Lemma 3.3.3 in terms of elliptic functions, one can show that the statement above reduces to the well known identity:

$$\begin{aligned} &P_2(z-z_1;\tau)+P_2(z+z_1;\tau) \\ &\qquad = \frac{\wp''(z_1;\tau)}{\wp(z;\tau)-\wp(z_1;\tau)} + \frac{\wp'(z_1;\tau)^2}{(\wp(z;\tau)-\wp(z_1;\tau))^2} + 2P_2(z_1;\tau). \end{aligned} \tag{3.54}$$

And finally, the main result that replaces Theorem 4.12 of [BE17] is the following theorem:

Proposition 3.3.5. *For the Weierstrass spectral curve, for $2g-2+n \geq 0$,*

$$\frac{Q^{(2)}_{g,n+1}(z;\mathbf{z})}{dx(z)^2} = -\sum_{i=1}^{n} d_{z_i}\left(\frac{1}{x(z)-x(z_i)}\frac{\omega_{g,n}(\mathbf{z})}{dx(z_i)}\right) - 2\left(\frac{\omega_{g,n+1}(z',\mathbf{z})}{dz'}\right)_{z'=0}. \tag{3.55}$$

For $(g,n)=(0,1)$,

$$\frac{Q^{(2)}_{0,2}(z;z_1)}{dx(z)^2} = -d_{z_1}\left(\frac{1}{(x(z)-x(z_1))}\frac{\omega_{0,1}(z_1)}{dx(z_1)}\right) - 2P_2(z_1;\tau)dz_1, \tag{3.56}$$

while for $(g,n)=(0,0)$,

$$\frac{Q^{(2)}_{0,1}(z)}{dx(z)^2} = -4x(z)^3 + g_2(\tau)x(z) + g_3(\tau). \tag{3.57}$$

Proof. The cases $(g,n)=(0,0)$ and $(g,n)=(0,1)$ were proven in the two previous lemmas. So let us focus on $2g-2+n \geq 0$.

First, notice that we can write

$$\begin{aligned} \frac{Q^{(2)}_{g,n+1}(z;\mathbf{z})}{dx(z)^2} =& \frac{y(z)dz}{dx(z)}\frac{Q^{(2)}_{g,n+1}(z;\mathbf{z})}{dx(z)^2} \\ =& \frac{1}{2}\left(-\frac{y(-z)dz}{dx(z)}\frac{Q^{(2)}_{g,n+1}(-z;\mathbf{z})}{dx(z)^2} + \frac{y(z)dz}{dx(z)}\frac{Q^{(2)}_{g,n+1}(z;\mathbf{z})}{dx(z)^2}\right) \\ =& \frac{1}{2}\left(\operatorname*{Res}_{z'=z}\frac{y(z')dz'}{x(z')-x(z)}\frac{Q^{(2)}_{g,n+1}(z';\mathbf{z})}{dx(z')^2}\right. \\ & \left. + \operatorname*{Res}_{z'=-z}\frac{y(z')dz'}{x(z')-x(z)}\frac{Q^{(2)}_{g,n+1}(z';\mathbf{z})}{dx(z')^2}\right). \end{aligned} \tag{3.58}$$

The expression

$$\frac{y(z')dz'}{x(z')-x(z)}\frac{Q^{(2)}_{g,n+1}(z';\mathbf{z})}{dx(z')^2} \tag{3.59}$$

is a meromorphic one-form in z' on a compact Riemann surface, hence the sum of its residues must be zero. But recall that the one-forms

$$dz'\left(\frac{Q^{(2)}_{g,n+1}(z';\mathbf{z})}{dx(z')^2}\right) \tag{3.60}$$

can only have poles (in z') at $z'=\pm z_i$, $i=1,\dots,n$. Therefore, the expression

$$\frac{y(z')dz'}{x(z')-x(z)}\frac{Q^{(2)}_{g,n+1}(z';\mathbf{z})}{dx(z')^2} \tag{3.61}$$

can only have poles (in z') at $z'=\pm z$, $z'=\pm z_i$, $i=1,\dots,n$, and at the pole of $y(z')$, that is, at $z'=0$.[5] Thus we get

$$\begin{aligned}\frac{Q^{(2)}_{g,n+1}(z;\mathbf{z})}{dx(z)^2} =&\frac{1}{2}\left(\sum_{i=1}^{n}\operatorname*{Res}_{z'=\pm z_i}\frac{y(z')dz'}{x(z)-x(z')}\frac{Q^{(2)}_{g,n+1}(z';\mathbf{z})}{dx(z')^2}\right.\\
&\left.+\operatorname*{Res}_{z'=0}\frac{y(z')dz'}{x(z)-x(z')}\frac{Q^{(2)}_{g,n+1}(z';\mathbf{z})}{dx(z')^2}\right)\\
=&\frac{1}{2}\sum_{i=1}^{n}\left(\operatorname*{Res}_{z'=z_i}\frac{y(z')dz'}{x(z)-x(z')}\frac{B(z',z_i)\omega_{g,n}(-z',\mathbf{z}\setminus\{z_i\})}{dx(z')^2}\right.\\
&\left.+\operatorname*{Res}_{z'=-z_i}\frac{y(z')dz'}{x(z)-x(z')}\frac{B(-z',z_i)\omega_{g,n}(z',\mathbf{z}\setminus\{z_i\})}{dx(z')^2}\right)\\
&+\left(\frac{Q^{(2)}_{g,n+1}(z';\mathbf{z})}{dx(z')^2}\right)_{z'=0}\\
=&-\sum_{i=1}^{n}d_{z_i}\left(\frac{y(z_i)dz_i}{x(z)-x(z_i)}\frac{\omega_{g,n}(\mathbf{z})}{dx(z_i)^2}\right)-2\left(\frac{\omega_{g,n+1}(z';\mathbf{z})}{dz'}\right)_{z'=0}\\
=&-\sum_{i=1}^{n}d_{z_i}\left(\frac{1}{x(z)-x(z_i)}\frac{\omega_{g,n}(\mathbf{z})}{dx(z_i)}\right)-2\left(\frac{\omega_{g,n+1}(z';\mathbf{z})}{dz'}\right)_{z'=0}.\end{aligned} \tag{3.62}$$

Here we used the fact that as $z'\to 0$, we have that

$$\frac{Q^{(2)}_{g,n+1}(z';\mathbf{z})}{dx(z')^2}\to\frac{\omega_{0,1}(z')\omega_{g,n+1}(-z';\mathbf{z})+\omega_{0,1}(-z')\omega_{g,n+1}(z';\mathbf{z})}{dx(z')^2}, \tag{3.63}$$

since all other terms vanish because $z'=0$ is a pole of $dx(z')$.

□

[5] Again, this last pole at $z'=0$ does not occur when the spectral curve is such that its Newton polygon has no interior point. This is what makes the Weierstrass spectral curve different from the curves studied in [BE17].

The following corollary then follows directly from the definition of $Q^{(2)}_{g,n+1}(z;\mathbf{z})$ and the fact that

$$\omega_{0,2}(z,z_1)+\omega_{0,2}(-z,z_1)=\frac{dx(z)dx(z_1)}{(x(z)-x(z_1))^2}. \tag{3.64}$$

Corollary 3.3.6. *For $2g-2+n\geq 0$,*

$$\begin{aligned}-\frac{\omega_{g-1,n+2}(-z,z,\mathbf{z})}{dx(z)^2}+\sum_{g_1+g_2=g}\sum_{I\cup J=\mathbf{z}}\frac{\omega_{g_1,|I|+1}(-z,I)}{dx(z)}\frac{\omega_{g_2,|J|+1}(-z,J)}{dx(z)}\\ -\sum_{i=1}^{n}\Bigg(\frac{dx(z_i)}{(x(z)-x(z_i))^2}\frac{\omega_{g,n}(-z,\mathbf{z}\setminus\{z_i\})}{dx(z)}\\ -d_{z_i}\left(\frac{1}{x(z)-x(z_i)}\frac{\omega_{g,n}(-z_i,\mathbf{z}\setminus\{z_i\})}{dx(z_i)}\right)\Bigg)+2\left(\frac{\omega_{g,n+1}(-z',\mathbf{z})}{dz'}\right)_{z'=0}=0.\end{aligned} \tag{3.65}$$

For $(g,n)=(0,1)$,

$$\begin{aligned}2\frac{\omega_{0,2}(-z,z_1)}{dx(z)}\frac{\omega_{0,1}(-z)}{dx(z)}-\frac{dx(z_1)}{(x(z)-x(z_1))^2}\frac{\omega_{0,1}(-z)}{dx(z)}\\ +d_{z_1}\left(\frac{1}{(x(z)-x(z_1))}\frac{\omega_{0,1}(-z_1)}{dx(z_1)}\right)-2P_2(z_1;\tau)dz_1=0,\end{aligned} \tag{3.66}$$

while for $(g,n)=(0,0)$,

$$2\frac{\omega_{0,1}(-z)\omega_{0,1}(-z)}{dx(z)^2}-4x(z)^3+g_2(\tau)x(z)+g_3(\tau)=0. \tag{3.67}$$

3.3.2 Integration

The next step is to integrate the equation above. Following the notation in [BE17], we choose the integration divisor to be $D=[z]-[0]$, since $z=0$ is the only pole of $x(z)$. While $z=0$ is in R, it is easy to show that the correlation functions do not have poles at $z=0$, therefore the integrals will converge.

Definition 3.3.7. We define

$$G_{g,n+1}(z;\mathbf{z})=\int_0^{z_1}\cdots\int_0^{z_n}\omega_{g,n+1}(-z,z_1,\dots,z_n). \tag{3.68}$$

Note that we are integrating in each variable $z_1,\dots,z_n$, with base point 0, but we are not integrating in the variable z.

Now we can integrate Corollary 3.3.6 in $z_1,\dots,z_n$. We get:

Lemma 3.3.8. *For $2g-2+n \geq 0$,*

$$-\left(\frac{\partial}{\partial x(z_{n+1})} \frac{G_{g-1,n+2}(z;\mathbf{z},z_{n+1})}{dx(z)}\right)_{z_{n+1}=z} + \sum_{g_1+g_2=g} \sum_{I \cup J = \mathbf{z}} \frac{G_{g_1,|I|+1}(z;I)}{dx(z)} \frac{G_{g_2,|J|+1}(z;J)}{dx(z)} - \sum_{i=1}^{n} \left(\frac{1}{x(z_i)-x(z)} \left(\frac{G_{g,n}(z_i;\mathbf{z}\setminus\{z_i\})}{dx(z_i)} - \frac{G_{g,n}(z;\mathbf{z}\setminus\{z_i\})}{dx(z)} \right) \right) + 2\left(\frac{G_{g,n+1}(z';\mathbf{z})}{dz'}\right)_{z'=0} = 0. \quad (3.69)$$

For $(g,n) = (0,1)$,

$$2\frac{G_{0,2}(z;z_1)}{dx(z)} \frac{G_{0,1}(z)}{dx(z)} - \frac{1}{x(z_1)-x(z)} \left(\frac{G_{0,1}(z_1)}{dx(z_1)} - \frac{G_{0,1}(z)}{dx(z)} \right) - 2P_1(z_1;\tau) = 0, \quad (3.70)$$

while for $(g,n) = (0,0)$,

$$2\frac{G_{0,1}(z)G_{0,1}(z)}{dx(z)^2} - 4x(z)^3 + g_2(\tau)x(z) + g_3(\tau) = 0. \quad (3.71)$$

Proof. The integration is straightforward; all we need to do is be careful with the base point 0.

For $2g-2+n \geq 0$, integrating the terms inside the summation $\sum_{i=1}^{n}$ gives rise to a term of the form

$$\sum_{i=1}^{n} \lim_{z_i=0} \left(\frac{1}{x(z)-x(z_i)} \frac{G_{g,n}(z;\mathbf{z}\setminus\{z_i\})}{dx(z)} - \frac{1}{x(z)-x(z_i)} \frac{G_{g,n}(z_i;\mathbf{z}\setminus\{z_i\})}{dx(z_i)} \right). \quad (3.72)$$

The first term clearly vanishes since $z_i = 0$ is a pole of $x(z_i)$. As for the second term, it also vanishes, because $G_{g,n}(z_i;\mathbf{z}\setminus\{z_i\})$ cannot have a pole at $z_i = 0$. Hence we get the expression in the Lemma.

For $(g,n) = (0,1)$, integrating in z_1 gives rise to a term of the form

$$\lim_{z_1=0} \left[\frac{1}{x(z_1)-x(z)} \left(\frac{G_{0,1}(z_1)}{dx(z_1)} - \frac{G_{0,1}(z)}{dx(z)} \right) + 2P_1(z_1;\tau) \right] = \lim_{z_1=0} \left[\frac{1}{x(z_1)-x(z)} \left(-y(z_1)+y(z)\right) + 2P_1(z_1;\tau) \right]. \quad (3.73)$$

But since, as $z_1 \to 0$,

$$x(z_1) = \wp(z_1;\tau) \to \frac{1}{z_1^2}, \qquad y(z_1) = \wp'(z_1;\tau) \to -\frac{2}{z_1^3}, \qquad P_1(z_1;\tau) \to -\frac{1}{z_1}, \tag{3.74}$$

we see that the limit actually vanishes. Thus we get expression in the Lemma.

As for $(g,n) = (0,0)$, we are not integrating so the expression is obvious.

□

3.3.3 Principal specialization

Then we "principal specialize" by setting $z_1 = \ldots = z_n = z$. We define

$$\hat{G}_{g,n+1}(z';z) = G_{g,n+1}(z';z,\ldots,z). \tag{3.75}$$

We get:

Lemma 3.3.9. *For $2g-2+n \geq 0$,*

$$\begin{aligned} &-\frac{1}{n+1}\left(\frac{d}{dx(z)}\frac{\hat{G}_{g-1,n+2}(z';z)}{dx(z')}\right)_{z'=z} \\ &\qquad + \sum_{g_1+g_2=g}\sum_{m=0}^{n}\frac{n!}{m!(n-m)!}\frac{\hat{G}_{g_1,m+1}(z;z)}{dx(z)}\frac{\hat{G}_{g_2,n-m+1}(z;z)}{dx(z)} \\ &\qquad - n\left(\frac{d}{dx(z')}\frac{\hat{G}_{g,n}(z';z)}{dx(z')}\right)_{z'=z} + 2\left(\frac{\hat{G}_{g,n+1}(z';z)}{dz'}\right)_{z'=0} = 0. \end{aligned} \tag{3.76}$$

For $(g,n) = (0,1)$,

$$2\frac{\hat{G}_{0,2}(z;z)}{dx(z)}\frac{\hat{G}_{0,1}(z)}{dx(z)} - \frac{d}{dx(z)}\left(\frac{\hat{G}_{0,1}(z)}{dx(z)}\right) - 2P_1(z;\tau) = 0, \tag{3.77}$$

while for $(g,n) = (0,0)$,

$$2\frac{\hat{G}_{0,1}(z)\hat{G}_{0,1}(z)}{dx(z)^2} - 4x(z)^3 + g_2(\tau)x(z) + g_3(\tau) = 0. \tag{3.78}$$

Proof. This is straightforward, the only terms that need to be treated carefully are those that give rise to the derivatives. □

Finally, we sum over g and n. More precisely, we define

$$\xi_1(z';z) = -\sum_{g,n=0}^{\infty}\frac{\hbar^{2g+n}}{n!}\frac{\hat{G}_{g,n+1}(z';z)}{dx(z')}. \tag{3.79}$$

Multiplying the equations in Lemma 3.3.9 by $\frac{\hbar^{2g+n}}{n!}$, and summing over g and n, we get:

Lemma 3.3.10.

$$\hbar\frac{d}{dx(z)}\xi_1(z;z)+\xi_1(z;z)^2-4x(z)^3+g_2(\tau)x(z)+g_3(\tau)$$
$$-2\hbar P_1(z;\tau)+2\sum_{2g-2+n\geq 0}\frac{\hbar^{2g+n}}{n!}\left(\frac{\hat{G}_{g,n+1}(z';z)}{dz'}\right)_{z'=0}=0. \quad (3.80)$$

□

3.3.4 Differential operator

As in [BE17] we introduce the perturbative wave-function:

$$\psi(z)=\exp\left(\frac{1}{\hbar}\sum_{2g-1+n\geq 0}\frac{\hbar^{2g+n-1}}{n!}\int_0^z\cdots\int_0^z\left(\omega_{g,n}(z_1,\ldots,z_n)\right.\right.$$
$$\left.\left.-\delta_{g,0}\delta_{n,2}\frac{dx(z_1)dx(z_2)}{(x(z_1)-x(z_2))^2}\right)\right), \quad (3.81)$$

and we define

$$\psi_1(z';z)=\psi(z)\xi_1(z';z). \quad (3.82)$$

Then it is easy to show that

$$\psi_1(z;z)=\hbar\frac{d}{dx}\psi(z). \quad (3.83)$$

(see Lemma 5.10 in [BE17].) It follows that we can rewrite (3.80) as:

Theorem 3.3.11.

$$\left[\hbar^2\frac{d^2}{dx^2}-4x(z)^3+g_2(\tau)x(z)+g_3(\tau)-2\hbar P_1(z;\tau)\right.$$
$$\left.+2\sum_{2g-2+n\geq 0}\frac{\hbar^{2g+n}}{n!}\left(\frac{\hat{G}_{g,n+1}(z';z)}{dz'}\right)_{z'=0}\right]\psi(z)=0. \quad (3.84)$$

Proof. We start with (3.80), multiply by $\psi(z)$, to get

$$\hbar\psi(z)\frac{d}{dx}\xi_1(z;z)+\psi_1(z;z)\xi_1(z;z)+(-4x(z)^3+g_2(\tau)x(z)+g_3(\tau))\psi(z)$$
$$-2\hbar P_1(z;\tau)\psi(z)+2\psi(z)\sum_{2g-2+n\geq 0}\frac{\hbar^{2g+n}}{n!}\left(\frac{\hat{G}_{g,n+1}(z';z)}{dz'}\right)_{z'=0}=0. \quad (3.85)$$

But

$$\begin{aligned}\hbar\psi(z)\frac{d}{dx}\xi_1(z;z) =&\hbar\frac{d}{dx}\psi_1(z;z) - \xi_1(z;z)\hbar\frac{d}{dx}\psi(z)\\ =&\hbar^2\frac{d^2}{dx^2}\psi(z) - \xi_1(z;z)\psi_1(z;z),\end{aligned} \tag{3.86}$$

thus the equation becomes

$$\begin{aligned}&\hbar^2\frac{d^2}{dx^2}\psi(z) + (-4x(z)^3 + g_2(\tau)x(z) + g_3(\tau))\psi(z)\\ &\quad - 2\hbar P_1(z;\tau)\psi(z) + 2\psi(z)\sum_{2g-2+n\geq 0}\frac{\hbar^{2g+n}}{n!}\left(\frac{\hat{G}_{g,n+1}(z';z)}{dz'}\right)_{z'=0} = 0.\end{aligned} \tag{3.87}$$

$\square$

Theorem 3.3.11 gives a rank two differential operator that annihilates the perturbative wave-function $\psi(z)$. However, this is not a quantum curve, according to Definition 3.2.7. It has an infinite series of $\hbar$ corrections, and those corrections are not polynomials in x; in fact they are not even functions of x. They also have poles at the ramification points of the branched covering π.

What we have constructed is a rank two differential operator that kills the standard perturbative wave-function (3.81), but it is not a quantum curve. However, it may be possible to define a new wave-function, which can be obtained from ψ, and that is annihilated by a proper quantum curve. To achieve this, we need to define the non-perturbative wave-function, which we will do in section 3.6.

Remark 3.3.12. We remark here that we checked numerically on Mathematica that Theorem 3.3.11 is indeed satisfied for the first few orders in $\hbar$.

3.4 Perturbative wave-function: second approach

In this section we study a second approach to obtain the rank two differential operator that kills the perturbative wave-function. We will see that we obtain a differential operator that looks quite different *a priori* from the differential operator obtained in the previous section. But we can prove that the two are equivalent. In fact, this equivalent gives rise to an infinite tower of identities for cycle integrals of elliptic functions.

We start with the topological recursion (Equation 3.39):

$$\omega_{g,n+1}(z_0,\mathbf{z}) = \mathrm{d}z_0 \sum_{a\in R} \operatorname*{Res}_{z=a} \left(\int_\alpha^z P_2(z'-z_0;\tau)\mathrm{d}z'\right) \frac{\mathcal{R}^{(2)}\omega_{g,n+1}(z,-z;\mathbf{z})}{2\wp'(z;\tau)^2\mathrm{d}z}. \tag{3.88}$$

We now wish to express the sum over residues around poles in R in terms of residues of the other poles of the integrand. If the integrand was a well defined meromorphic differential in z over the compact Riemann surface Σ, then the sum of its residues at all poles would have to vanish. However, it is not a well defined meromorphic differential in z; because of the line integral from α to z, it is only defined in the fundamental domain. Thus what we need to use is the Riemann bilinear identity.

3.4.1 Riemann bilinear identity

The integral form of the Riemann bilinear identity can be stated as follows:

$$\sum_{\text{all poles } b \text{ of } u\eta} \operatorname*{Res}_{z=b} u\eta = \frac{1}{2\pi i}\sum_{j=1}^{g}\left(\oint_{B_j}\omega\oint_{A_j}\eta - \oint_{B_j}\eta\oint_{A_j}\omega\right),$$

where η is a meromorphic differential on the compact Riemann surface Σ of genus g, and (A_j,B_j), $j=1,\dots,g$ is a symplectic basis of cycles. Moreover,

$$u(z) = \int_\alpha^z \omega, \tag{3.89}$$

where ω is a residueless meromorphic differential, α is an arbitrary base point, and the line integral is taken in the fundamental domain.

We can apply the Riemann bilinear identity to (3.88). First, we have that

$$\begin{aligned}\sum_{\text{all poles } b} \operatorname*{Res}_{z=b} &\left(\int_\alpha^z P_2(z'-z_0;\tau)\mathrm{d}z'\right)\frac{\mathcal{R}^{(2)}\omega_{g,n+1}(z,-z;\mathbf{z})}{2\wp'(z;\tau)^2\mathrm{d}z} \\ &= \frac{1}{2\pi i}\left(\oint_B P_2(z-z_0;\tau)\,\mathrm{d}z\oint_A \frac{\mathcal{R}^{(2)}\omega_{g,n+1}(z,-z;\mathbf{z})}{2\wp'(z;\tau)^2\mathrm{d}z}\right. \\ &\qquad\left. - \oint_A P_2(z-z_0;\tau)\,\mathrm{d}z\oint_B \frac{\mathcal{R}^{(2)}\omega_{g,n+1}(z,-z;\mathbf{z})}{2\wp'(z;\tau)^2\mathrm{d}z}\right).\end{aligned} \tag{3.90}$$

We note that:

$$\oint_A P_2(z - z_0; \tau)\ \mathrm{d}z = 0, \qquad \oint_B P_2(z - z_0; \tau)\ \mathrm{d}z = 2\pi i, \tag{3.91}$$

thus

$$\sum_{\text{all poles } b} \operatorname*{Res}_{z=b} \left(\int_\alpha^z P_2(z' - z_0; \tau)\mathrm{d}z' \right) \frac{\mathcal{R}^{(2)}\omega_{g,n+1}(z, -z; \mathbf{z})}{2\wp'(z;\tau)^2\mathrm{d}z} = B_{g,n+1}(\mathbf{z}). \tag{3.92}$$

where we defined

$$B_{g,n+1}(\mathbf{z}) := \oint_A \frac{\mathcal{R}^{(2)}\omega_{g,n+1}(z, -z; \mathbf{z})}{2\wp'(z;\tau)^2\mathrm{d}z}.$$

But the poles b can be separated into poles in R and poles that are not in R, which means that we can rewrite 3.88 as:

$$\begin{aligned} \frac{\omega_{g,n+1}(z_0, \mathbf{z})}{dz_0} &= B_{g,n+1}(\mathbf{z}) \\ &\quad - \sum_{a \notin R} \operatorname*{Res}_{z=a} \left(P_1(z - z_0; \tau) - P_1(\alpha - z_0; \tau)\right) \frac{\mathcal{R}^{(2)}\omega_{g,n+1}(z, -z; \mathbf{z})}{2\wp'(z;\tau)^2\mathrm{d}z} \end{aligned} \tag{3.93}$$

Given this equation we now seek to write it in a form (nearly) identical to equation (3.65). To do this we must first calculate the residues.

3.4.2 Calculating the residues

Now since, $P_1(z - z_0) \to -\frac{1}{z - z_0}$ as $z \to z_0$, we see that there is a simple pole at $z = z_0$. However there is also a collection of poles at each of the marked points z_j (with $j = \{1, \cdots, n\}$) coming from the recursive structure. To see this more clearly, let us examine equation (3.41) more closely:

$$\begin{aligned} \mathcal{R}^{(2)}\omega_{g,n+1}(z, -z; \mathbf{z}) &= \omega_{g-1,n+2}(z, -z, \mathbf{z}) \\ &\quad + \sum_{\text{stable}} \omega_{g_1,|I|+1}(z, I)\omega_{g_2,|J|+1}(-z, J) \\ &\quad + \sum_{j=1}^{n} \omega_{0,2}(z, z_j)\omega_{g,n}(-z, \mathbf{z}/z_j) + \omega_{0,2}(-z, z_j)\omega_{g,n}(z, \mathbf{z}/z_j). \end{aligned}$$

The "stable" sum term excludes the cases where either $(g_1, |I|)$ or $(g_2, |J|)$ is equal to $(0, 0)$ or $(0, 1)$. From here we note that as $z \to \pm z_j$, $\omega_{0,2}(\pm z, z_j) \to$

$\pm\frac{\mathrm{d}z\mathrm{d}z_j}{(z\mp z_j)^2}$, hence there are second order poles at each of the points $\pm z_j$ (with $j=\{1,\cdots,n\}$). Now we can proceed to calculate the residues:

$$\text{Residue at } z_0 = -\frac{\mathrm{d}z_0}{2\wp'(z_0;\tau)^2}\left(\frac{\omega_{g-1,n+2}(z_0,-z_0,\mathbf{z})}{\mathrm{d}z_0^2}\right.$$
$$\left.+\sum_{g_1+g_2=g}\sum_{I\cup J=\mathbf{z}}^{\prime}\frac{\omega_{g_1,|I|+1}(z_0,I)}{\mathrm{d}z_0}\frac{\omega_{g_2,|J|+1}(-z_0,J)}{\mathrm{d}z_0}\right) \tag{3.94}$$

$$\text{Residue at } z_i = \mathrm{d}_{z_i}\left(\frac{P_1(z_i-z_0;\tau)-P_1(\alpha-z_0;\tau)}{2\wp'(z_i,\tau)^2\mathrm{d}z_i}\omega_{g,n}(-z_i,\mathbf{z}/z_i)\right)\mathrm{d}z_0 \tag{3.95}$$

$$\text{Residue at } (-z_i) = \mathrm{d}_{z_i}\left(\frac{P_1(z_i+z_0;\tau)+P_1(\alpha-z_0;\tau)}{2\wp'(z_i,\tau)^2\mathrm{d}z_i}\omega_{g,n}(-z_i,\mathbf{z}/z_i)\right)\mathrm{d}z_0 \tag{3.96}$$

Summing all of these contributions, dividing both sides by $\mathrm{d}z_0$ and rearranging the expression we arrive at:

Proposition 3.4.1. *For $2g-2+n\geq 0$,*

$$-\frac{\omega_{g-1,n+2}(-z_0,z_0,\mathbf{z})}{dx(z_0)^2}+\sum_{g_1+g_2=g}\sum_{I\cup J=\mathbf{z}}\frac{\omega_{g_1,|I|+1}(-z_0,I)}{dx(z_0)}\frac{\omega_{g_2,|J|+1}(-z_0,J)}{dx(z_0)}$$
$$-\sum_{i=1}^{n}\left(\frac{dx(z_i)}{(x(z_0)-x(z_i))^2}\frac{\omega_{g,n}(-z_0,\mathbf{z}\setminus\{z_i\})}{dx(z_0)}\right.$$
$$\left.-d_{z_i}\left(\frac{1}{x(z_0)-x(z_i)}\frac{\omega_{g,n}(-z_i,\mathbf{z}\setminus\{z_i\})}{dx(z_i)}\right)\right)$$
$$+\sum_{i=1}^{n}d_{z_i}\left(\frac{2P_1(z_i;\tau)}{\wp'(z_i;\tau)}\frac{\omega_{g,n}(-z_i,\mathbf{z}\setminus\{z_i\})}{dx(z_i)}\right)-2B_{g,n+1}(\mathbf{z})=0. \tag{3.97}$$

For $(g,n)=(0,1)$,

$$2\frac{\omega_{0,2}(-z_0,z_1)}{dx(z_0)}\frac{\omega_{0,1}(-z_0)}{dx(z_0)}-\frac{dx(z_1)}{(x(z_0)-x(z_1))^2}\frac{\omega_{0,1}(-z_0)}{dx(z_0)}$$
$$+d_{z_1}\left(\frac{1}{(x(z_0)-x(z_1))}\frac{\omega_{0,1}(-z_1)}{dx(z_1)}\right)+d_{z_1}\left(\frac{2P_1(z_1;\tau)}{\wp'(z_1;\tau)}\frac{\omega_{0.1}(-z_1)}{\wp'(z_1);\tau}\right)=0, \tag{3.98}$$

while for $(g,n)=(0,0)$,

$$2\frac{\omega_{0,1}(-z_0)\omega_{0,1}(-z_0)}{dx(z)^2}-4x(z)^3+g_2(\tau)x(z)+g_3(\tau)=0. \tag{3.99}$$

This is to compare with Corollary 3.3.6 obtained in the previous section.

3.4.3 Differential operator

From Proposition 3.4.1 we want to obtain a differential operator that annihilates the perturbative wave-function. We follow the procedure outlined in the previous section. We arrive at the following differential equation:

Theorem 3.4.2.

$$\Big[\hbar^2 \frac{d^2}{dx^2} - 2\hbar^2 \frac{P_1(z;\tau)}{\wp'(z;\tau)}\frac{d}{dx} - 4x(z)^3 + g_2(\tau)x(z) + g_3(\tau) \\ -2\sum_{2g-2+n\geq 0} \frac{\hbar^{2g+n}}{n!}\left(\int_0^z \cdots \int_0^z B_{g,n+1}(\mathbf{z})\right)\Big]\psi(z) = 0. \quad (3.100)$$

This is to be contrasted with the differential operator that was obtained in Theorem 3.3.11. The perturbative wave-function ψ is the same for both Theorems. It is then expected that the two differential equations should be equivalent, even though they look quite different *a priori*.

3.4.4 Connection with the calculation of the previous section

In the previous section, we calculated a differential equation satisfied by ψ; in this section we also computed a differential equation satisfied by ψ, which looks different *a priori*. Let us now show that they are the same.

First, let us compare Proposition 3.4.1 with Corollary 3.3.6 of the previous section. In particular, for equation (3.97) to be equivalent to equation (3.65), the following identity must hold:

Corollary 3.4.3. *For the Weierstrass spectral curve, for* $2g-2+n \geq 0$,

$$B_{g,n+1}(\mathbf{z}) = -\left(\frac{\omega_{g,n+1}(-z_0,\mathbf{z})}{dz_0}\right)_{z_0=0} + \sum_{i=1}^{n} d_{z_i}\left(\frac{P_1(z_i;\tau)}{\wp'(z_i;\tau)}\frac{\omega_{g,n}(-z_i,\mathbf{z}\setminus\{z_i\})}{dx(z_i)}\right). \quad (3.101)$$

This is a non-trivial identity between elliptic functions; in fact, it gives an infinite tower of expressions for cycle integrals of elliptic functions. We will come back to that in the next subsection. But we note here that we can also prove Corollary 3.4.3 directly.

We start with (3.93), which we rewrite as

$$B_{g,n+1}(\mathbf{z}) = -\frac{\omega_{g,n+1}(-z_0,\mathbf{z})}{dz_0} + \sum_{a\notin R}\operatorname*{Res}_{z=a}\left(P_1(z-z_0;\tau)-P_1(\alpha-z_0;\tau)\right)\frac{\mathcal{R}^{(2)}\omega_{g,n+1}(z,-z;\mathbf{z})}{2\wp'(z;\tau)^2\mathrm{d}z}. \quad (3.102)$$

We take the limit as $z_0 \to 0$. We obtain

$$B_{g,n+1}(\mathbf{z}) = -\left(\frac{\omega_{g,n+1}(-z_0,\mathbf{z})}{dz_0}\right)_{z_0=0} + \sum_{a\notin R}\operatorname*{Res}_{z=a}\left(P_1(z;\tau)-P_1(\alpha;\tau)\right)\frac{\mathcal{R}^{(2)}\omega_{g,n+1}(z,-z;\mathbf{z})}{2\wp'(z;\tau)^2\mathrm{d}z}. \quad (3.103)$$

The only poles in the sum over $a \notin R$ are at $z = \pm z_i$, for $i = 1,\dots,n$. The residue at $z = z_i$ gives rise to a term of the form

$$d_{z_i}\left(\frac{P_1(z_i;\tau)-P_1(\alpha;\tau)}{2\wp'(z_i;\tau)}\frac{\omega_{g,n}(-z_i,\mathbf{z}\setminus\{z_i\})}{dx(z_i)}\right), \quad (3.104)$$

while the residue at $z = -z_i$ gives rise to a term of the form

$$d_{z_i}\left(\frac{P_1(z_i;\tau)+P_1(\alpha;\tau)}{2\wp'(z_i;\tau)}\frac{\omega_{g,n}(-z_i,\mathbf{z}\setminus\{z_i\})}{dx(z_i)}\right). \quad (3.105)$$

Putting those together, we obtain

$$B_{g,n+1}(\mathbf{z}) = -\left(\frac{\omega_{g,n+1}(-z_0,\mathbf{z})}{dz_0}\right)_{z_0=0} + \sum_{i=1}^{n} d_{z_i}\left(\frac{P_1(z_i;\tau)}{\wp'(z_i;\tau)}\frac{\omega_{g,n}(-z_i,\mathbf{z}\setminus\{z_i\})}{dx(z_i)}\right), \quad (3.106)$$

which is precisely (3.101).

We can also compare Theorem 3.4.2 with Theorem 3.3.11 of the previous section. These give two differential equations satisfied by the perturbative wave-function ψ. For Theorems 3.3.11 and 3.4.2 to be equivalent, we need the following equation for the wave-function ψ to be satisfied:

Corollary 3.4.4.

$$\left[-\hbar P_1(z;\tau) + \sum_{2g-2+n\geq 0}\frac{\hbar^{2g+n}}{n!}\left(\frac{\hat{G}_{g,n+1}(z';z)}{dz'}\right)_{z'=0}\right]\psi = -\left[\sum_{2g-2+n\geq 0}\frac{\hbar^{2g+n}}{n!}\left(\int_0^z\cdots\int_0^z B_{g,n+1}(\mathbf{z})\right) + \hbar^2\frac{P_1(z;\tau)}{\wp'(z;\tau)}\frac{d}{dx}\right]\psi. \quad (3.107)$$

It turns out that we can indeed show directly that this is the case. We start with (3.101), which is valid for $2g-2+n \geq 0$:

$$B_{g,n+1}(\mathbf{z}) = -\left(\frac{\omega_{g,n+1}(-z',\mathbf{z})}{dz'}\right)_{z'=0} + \sum_{i=1}^{n} d_{z_i}\left(\frac{P_1(z_i;\tau)}{\wp'(z_i;\tau)}\frac{\omega_{g,n}(-z_i,\mathbf{z}\setminus\{z_i\})}{dx(z_i)}\right). \tag{3.108}$$

We integrate in $z_1,\ldots,z_n$ from 0 to z, multiply by $\frac{\hbar^{2g+n}}{n!}$, and sum over g and n from $2g-2+n \geq 0$. We get:

$$\begin{aligned}&\sum_{2g-2+n\geq 0}\frac{\hbar^{2g+n}}{n!}\left(\int_0^z\cdots\int_0^z B_{g,n+1}(\mathbf{z})\right) = \\ &-\sum_{2g-2+n\geq 0}\frac{\hbar^{2g+n}}{n!}\left(\frac{\hat{G}_{g,n+1}(z';z)}{dz'}\right)_{z'=0} + \sum_{2g-2+n\geq 0}\frac{\hbar^{2g+n}}{(n-1)!}\frac{P_1(z;\tau)}{\wp'(z;\tau)^2}\frac{\hat{G}_{g,n}(z;z)}{dz}.\end{aligned} \tag{3.109}$$

Then, using (3.83), we notice that

$$\begin{aligned}\hbar\frac{d}{dx}\psi(z) =& \psi_1(z;z) \\ =& \left[\wp'(z;\tau) - \frac{1}{\wp'(z;\tau)}\sum_{2g-2+n\geq -1}\frac{\hbar^{2g+n}}{n!}\frac{\hat{G}_{g,n+1}(z;z)}{dz}\right]\psi(z).\end{aligned} \tag{3.110}$$

Redefining the index in the sum and multiplying the equation by $\hbar$, we get

$$\hbar^2\frac{d}{dx}\psi(z) = \left[\hbar\wp'(z;\tau) - \frac{1}{\wp'(z;\tau)}\sum_{2g-2+n\geq 0}\frac{\hbar^{2g+n}}{(n-1)!}\frac{\hat{G}_{g,n}(z;z)}{dz}\right]\psi(z). \tag{3.111}$$

Going back to (3.109), we multiply by $-\psi(z)$ and use the above to rewrite it as

$$\begin{aligned}-&\left[\sum_{2g-2+n\geq 0}\frac{\hbar^{2g+n}}{n!}\left(\int_0^z\cdots\int_0^z B_{g,n+1}(\mathbf{z})\right) + \hbar^2\frac{P_1(z;\tau)}{\wp'(z;\tau)}\frac{d}{dx}\right]\psi(z) \\ &= \left[-\hbar P_1(z;\tau) + \sum_{2g-2+n\geq 0}\frac{\hbar^{2g+n}}{n!}\left(\frac{\hat{G}_{g,n+1}(z';z)}{dz'}\right)_{z'=0}\right]\psi(z),\end{aligned}$$

which is precisely (3.107).

3.5 Identities for elliptic functions

In this section we go back to Corollary 3.4.3 and explore its consequences for elliptic functions. We see that (3.101) gives rise to an infinite sequence of identities for cycle integrals of elliptic functions. Let us have a look at these identities for the first few levels in $2g-2+n$.

Let us start at the first level, namely $2g-2+n=0$. We start with $(g,n)=(1,0)$. In this case (3.101) becomes

$$B_{1,1} = -\left(\frac{\omega_{1,1}(-z_0)}{dz_0}\right)_{z_0=0}. \tag{3.112}$$

By definition

$$B_{1,1} = -\oint_A \frac{P_2(2z;\tau)}{2\wp'(z;\tau)^2}dz, \tag{3.113}$$

while from Appendix A, after a few simplifications, we obtain

$$\left(\frac{\omega_{1,1}(-z_0)}{dz_0}\right)_{z_0=0} = -\frac{G_4(\tau)(G_2(\tau)^2-5G_4(\tau))}{60(20G_4(\tau)^3-49G_6(\tau)^2)}. \tag{3.114}$$

The identity is then

Corollary 3.5.1.

$$\oint_A \frac{P_2(2z;\tau)}{\wp'(z;\tau)^2}dz = \frac{G_4(\tau)(5G_4(\tau)-G_2(\tau)^2)}{30(20G_4(\tau)^3-49G_6(\tau)^2)}. \tag{3.115}$$

In other words, we obtain an explicit expression for the A-cycle integral of the elliptic function $\frac{P_2(2z;\tau)}{\wp'(z;\tau)^2}$ in terms of quasi-modular forms. To emphasize the non-triviality of this expression, we provide an independent proof of this Corollary in Appendix B directly from the theory of elliptic functions.

Let us now study the other case at level $2g-2+n=0$, namely $(g,n)=(0,3)$. In this case (3.101) becomes:

$$\begin{aligned}\frac{B_{0,3}(z_1,z_2)}{dz_1dz_2} = -\left(\frac{\omega_{0,3}(-z_0,\mathbf{z})}{dz_0dz_1dz_2}\right)_{z_0=0} &- \frac{d}{dz_1}\left(\frac{P_1(z_1;\tau)}{\wp'(z_1;\tau)^2}P_2(z_1+z_2;\tau)\right)\\ &- \frac{d}{dz_2}\left(\frac{P_1(z_2;\tau)}{\wp'(z_2)^2}P_2(z_1+z_2;\tau)\right).\end{aligned} \tag{3.116}$$

But by definition

$$\begin{aligned}\frac{B_{0,3}(z_1,z_2)}{dz_1dz_2} = -\oint_A &\frac{P_2(z-z_1;\tau)P_2(z+z_2;\tau)}{2\wp'(z;\tau)^2}dz\\ &- \oint_A \frac{P_2(z+z_1;\tau)P_2(z-z_2;\tau)}{2\wp'(z;\tau)^2}dz.\end{aligned} \tag{3.117}$$

Moreover, the result for $\omega_{0,3}(z_0, z_1, z_2)$ in Appendix A reads

$$\begin{aligned}&\left(\frac{\omega_{0,3}(-z_0, z_1, z_2)}{dz_0 dz_1 dz_2}\right)_{z_0=0}\\&= -\frac{12}{\Delta}\sum_{i=1}^{3}(20G_4(\tau) - e_i^2)(e_i + G_2(\tau))P_2(z_1 - \omega_i;\tau)P_2(z_2 - \omega_i;\tau).\end{aligned} \tag{3.118}$$

Therefore, the identity becomes

Corollary 3.5.2.

$$\begin{aligned}&\oint_A \frac{P_2(z - z_1;\tau)P_2(z + z_2;\tau)}{2\wp'(z;\tau)^2}dz + \oint_A \frac{P_2(z + z_1;\tau)P_2(z - z_2;\tau)}{2\wp'(z;\tau)^2}dz\\&\quad= \frac{d}{dz_1}\left(\frac{P_1(z_1;\tau)}{\wp'(z_1;\tau)^2}P_2(z_1 + z_2;\tau)\right) + \frac{d}{dz_2}\left(\frac{P_1(z_2;\tau)}{\wp'(z_2;\tau)^2}P_2(z_1 + z_2;\tau)\right)\\&\quad - \frac{12}{\Delta}\sum_{i=1}^{3}(20G_4(\tau) - e_i^2)(e_i + G_2(\tau))P_2(z_1 - \omega_i;\tau)P_2(z_2 - \omega_i;\tau).\end{aligned} \tag{3.119}$$

We can continue producing such identities at higher levels of $2g-2+n \geq 0$. In particular, for all cases with $n = 1$, we obtain identities relating cycle integrals of elliptic functions to quasi-modular forms. It would be interesting to study whether these identities are of interest from the point of view of elliptic functions and quasi-modular forms. For instance, they may be related to the cycle integrals studied in [GM20].

3.6 Non-perturbative wave-function and quantum curve

In each of the previous two sections, we obtained a differential operator that annihilates the perturbative wave-function. However, these differential operators were not quantum curves, according to Definition 3.2.7.

In this section we will switch gears and study the *non-perturbative topological recursion* formalism described in [BE15, BE12b, Eyn09, EMn11]. It is expected from matrix models that this non-perturbative wave-function should be annihilated by quantum curve. This is what we study in this section.

3.6.1 Non-perturbative wave-function

Let us now introduce a non-perturbative wave-function, along the lines of [BE15, BE12b, Eyn09, EMn11].

One of the major motivations for the definition of the non-perturbative partition function in [BE15, BE12b, Eyn09, EMn11] is to construct a τ-function for an arbitrary algebraic curve. τ-functions in classical integrable systems are functions that satisfy Hirota bilinear equations. The Hirota equations are also equivalent to a self-replication property of the kernel. Either of these statements imply that there exists a system of differential equations, which we can use to get the quantum curve. In [BE12b], it is conjectured that this non-perturbative partition function is indeed a τ-function.

Notation

Before we write down the expression for the non-perturbative partition function, we need to define some fundamental objects that we will use.

We define a Jacobi theta function (called θ_{11} in some references):[6]

$$\theta(z|\tau) = \sum_{n\in\mathbb{Z}} e^{i\pi(n+1/2)^2\tau+2\pi i(z+1/2)(n+1/2)} \tag{3.120}$$

where $z \in \mathbb{C}$ and $\tau \in \mathbb{H}$. We also define:

$$\zeta_\hbar(\tau) = \frac{1}{2\pi i\hbar}\left(\oint_{\mathcal{B}} y\mathrm{d}x - \tau\oint_{\mathcal{A}} y\mathrm{d}x\right) \tag{3.121}$$

and introduce the following notation:

$$\begin{aligned}\theta(\tau) &= \theta(\zeta_\hbar(\tau)|\tau) \\ \theta_\bullet(z|\tau) &= \theta(\zeta_\hbar(\tau)+z|\tau)\end{aligned} \tag{3.122}$$

Then we define the perturbative partition function:

$$Z_{\text{pert}}(\tau) = \exp\left(\frac{1}{\hbar^2}\sum_{k\geq 0}\hbar^k F_k(\tau)\right), \tag{3.123}$$

[6]We make a choice of characteristics here in defining our theta function; it may be interesting to study other choices of characteristics.

where the F_k's are the free energies of the spectral curve defined in (3.42) (F_0 and F_1 can be defined independently; we refer the reader to [BE12b, EO07, EO08] for more details) . However, this partition function is non-modular, which is not what we expect from a "true" partition function coming from a quantum field theory. In order to construct the non-perturbative partition function (conjectured to be a τ-function), we multiply the perturbative partition function by certain combinations of theta functions and their derivatives, which exactly cancel out the non-modularity (proved in [Eyn09, EMn11]). This also ensures that the non-perturbative partition function is background independent.

Non-perturbative partition function and wave-function

The non-perturbative partition function introduced in [Eyn09, EMn11] is defined as a functional on the spectral curve:

$$\begin{aligned} Z_{\mathrm{NP}}(\tau) &= \exp\left(\frac{1}{\hbar^2}\sum_{k\geq 0}\hbar^k F_k(\tau)\right) \\ &\times\left\{\sum_{r\geq 0}\frac{1}{r!}\sum_{\substack{h_j,d_j\geq 0\\ 2h_j+d_j-2>0}}\hbar^{\sum 2h_j+d_j-2}\prod_{j=1}^{r}\left(\frac{F_{h_j}^{(d_j)}(\tau)}{(2\pi i)^{d_j}d_j!}\right)\nabla^{(\sum_j d_j)}\theta(\tau)\right\} \end{aligned} \tag{3.124}$$

where

$$F_h^{(d)}(\tau) = \frac{1}{n!}\frac{1}{(2\pi i)^d d!}\oint_{\mathcal{B}}\cdots\oint_{\mathcal{B}}\omega_{h,d}(z_1,\ldots,z_d), \tag{3.125}$$

and $\omega_{h,d}(z_1,\ldots,z_d)$ are the meromorphic differentials constructed from the spectral curve. Here, $\nabla\theta(\tau) = \left(\frac{\mathrm{d}}{\mathrm{d}z}\theta(z|\tau)\right)|_{z=0}$.

From the non-perturbative partition function, one can define a non-perturbative wave-function, following [BE15, BE12b]. What we will call non-perturbative wave-function in this paper, and denote by ψ_{NP}, is the $(1|1)$-kernel of [BE15, BE12b]. It is defined as a "Schlesinger" transformation of the non-perturbative partition function:

$$\psi_{\mathrm{NP}}(p_1,p_2) = \frac{\mathcal{T}_\hbar[y\mathrm{d}x \to y\mathrm{d}x + \hbar\omega^{p_2-p_1}]}{\mathcal{T}_\hbar[y\mathrm{d}x]}, \tag{3.126}$$

where $\omega^{p_2-p_1}$ was defined in (3.34) (we removed the p-dependence for clarity). In the following, for the Weierstrass spectral curve we will choose the base point $p_1 = 0$, and consider the wave-function as a function of $p_2 = z$. So we write

$$\psi_{\mathrm{NP}}(z) = \frac{\mathcal{T}_\hbar[y\mathrm{d}x \to y\mathrm{d}x + \hbar\omega^{z-0}]}{\mathcal{T}_\hbar[y\mathrm{d}x]}. \tag{3.127}$$

Graphical interpretation

It turns out that ψ_{NP} has a nice graphical interpretation in terms of connected graphs satisfying certain properties (see [BE15] for more details). We define $S_k(z)$s, $k \geq 0$ as follows:

$$\psi_{\mathrm{NP}}(z) = \exp\left(\frac{1}{\hbar}\sum_{k\geq 0}\hbar^k S_k(z)\right). \tag{3.128}$$

Then, we can write a general expression for the S_ks. For $k \geq 2$,

$$\begin{aligned} S_k(z) = \sum_{\substack{h_j,n_j,d_j\geq 0 \\ \sum_{j\geq 0}(2h_j+n_j+d_j-2)=k-1}} & \frac{1}{j!}\left(\prod_j \quad G_{n_j}^{h_j,d_j}(z)\right) \\ & \times\left(V_\bullet^{(d_1,d_2,\cdots,d_j)} - \delta_{(\sum_j n_j),0}V^{(d_1)}V^{(d_2)}\cdots V^{(d_j)}\right), \end{aligned} \tag{3.129}$$

where we used the notation

$$G_n^{h,d}(z) = \frac{1}{n!}\frac{1}{(2\pi i)^d d!}\int_0^z\cdots\int_0^z\oint_{\mathcal{B}}\cdots\oint_{\mathcal{B}}\omega_{h,n+d}(z_1,\ldots,z_{n+d}) \tag{3.130}$$

$$V_\bullet^{(d_1,\cdots,d_j)} = \frac{\partial}{\partial z_1}\cdots\frac{\partial}{\partial z_j}\log\left[E\left(\exp\sum z_i\nabla^{d_i}\right)\right]|_{z_1=z_2=\cdots=z_j=0}. \tag{3.131}$$

Here, E is defined as $E(\nabla^{d_i}) = \frac{\nabla^{d_i}\theta_\bullet(z|\tau)}{\theta_\bullet(z|\tau)}$, with $\nabla = \frac{\mathrm{d}}{\mathrm{d}z}$. The undotted $V^{(\cdots)}$s are given by the same expressions but in terms of undotted theta functions. We note as well that by connectedness we have

$$V_\bullet^{(d_1,d_2)} = V_\bullet^{(d_1+d_2)} - V_\bullet^{(d_1)}V_\bullet^{(d_2)}, \tag{3.132}$$

$$\begin{aligned} V_\bullet^{(d_1,d_2,d_3)} = {} & V_\bullet^{(d_1+d_2+d_3)} - V_\bullet^{(d_1+d_2)}V_\bullet^{(d_3)} - V_\bullet^{(d_2+d_3)}V_\bullet^{(d_1)} - V_\bullet^{(d_3+d_1)}V_\bullet^{(d_2)} \\ & + 2V_\bullet^{(d_1)}V_\bullet^{(d_2)}V_\bullet^{(d_3)}, \end{aligned} \tag{3.133}$$

and so on.

S_0 and S_1 are defined differently. For the Weierstrass spectral curve, they are simply given by

$$S_0 = \int_0^z \wp'(z)^2 \mathrm{d}z, \tag{3.134}$$

$$S_1 = -\int_0^z \frac{\wp''(z)}{2\wp'(z)} \mathrm{d}z. \tag{3.135}$$

Coming back to (3.129), we can write down the graphical expansion explicitly. The expansion was written down in [BE15]; we rewrite it here for reference.[7]

$$\begin{aligned} S_2(z) = {} & G_3^{0,0}(z) + G_1^{1,0}(z) + G_2^{0,1} V_\bullet^{(1)} + G_0^{1,1}(z) \left(V_\bullet^{(1)} - V^{(1)}\right) \\ & + G_1^{0,2}(z) V_\bullet^{(2)} + G_0^{0,3}(z)(V_\bullet^{(3)} - V^{(3)}), \end{aligned} \tag{3.136}$$

$$\begin{aligned} S_3(z) = {} & G_4^{0,0}(z) + G_2^{1,0}(z) + G_3^{0,1}(z) V_\bullet^{(1)} + G_1^{1,1}(z) V_\bullet^{(1)} + G_2^{0,2}(z) V_\bullet^{(2)} \\ & + G_0^{1,2}(z)(V_\bullet^{(2)} - V^{(2)}) + G_1^{0,3}(z) V_\bullet^{(3)} + G_0^{0,4}(z)(V_\bullet^{(4)} - V^{(4)}) \\ & + \frac{1}{2}(G_2^{0,1}(z))^2 V_\bullet^{(1,1)} + G_2^{0,1}(z) G_0^{1,1}(z) V_\bullet^{(1,1)} \\ & + \frac{1}{2}(G_0^{1,1}(z))^2 (V_\bullet^{(1,1)} - (V^{(2)})^2) + G_2^{0,1}(z) G_1^{0,2}(z) V_\bullet^{(1,2)} \\ & + G_2^{0,1}(z) G_0^{0,3}(z) V_\bullet^{(1,3)} + G_0^{0,3}(z) G_0^{1,1}(z)(V_\bullet^{(1,3)} - V^{(1)} V^{(3)}) \\ & + \frac{1}{2} G_1^{0,2}(z) V_\bullet^{(2,2)} + G_1^{0,2}(z) G_0^{0,3}(z) V_\bullet^{(2,3)} \\ & + G_1^{0,2}(z) G_0^{1,1}(z) V_\bullet^{(1,2)} + \frac{1}{2}(G_0^{0,3}(z))^2 (V_\bullet^{(3,3)} - (V^{(3)})^2), \end{aligned} \tag{3.137}$$

[7] Note that a few typos in the expressions of [BE15] were corrected here.

$$
\begin{aligned}
S_4(z) = {} & G_5^{0,0}(z) + G_3^{1,0}(z) + G_1^{2,0}(z) + G_4^{0,1}(z)V_\bullet^{(1)} + G_2^{1,1}(z)V_\bullet^{(1)} \\
& + G_1^{1,2}(z)V_\bullet^{(2)} + G_2^{0,3}(z)V_\bullet^{(3)} + G_0^{1,3}(z)(-V^{(3)} + V_\bullet^{(3)}) + G_1^{0,4}(z)V_\bullet^{(4)} \\
& + G_0^{0,5}(z)(-V^{(5)} + V^{(5)}{}_\bullet) + G_2^{0,1}(z)G_3^{0,1}(z)V_\bullet^{(1,1)} + G_3^{0,1}(z)G_0^{1,1}(z)V_\bullet^{(1,1)} \\
& + G_2^{0,1}(z)G_1^{1,1}(z)V_\bullet^{(1,1)} + G_0^{1,1}(z)G_1^{1,1}(z)V_\bullet^{(1,1)} + G_3^{0,1}(z)G_1^{0,2}(z)V_\bullet^{(1,2)} \\
& + G_2^{0,1}(z)G_2^{0,2}(z)V_\bullet^{(1,2)} + G_2^{0,2}(z)G_0^{1,1}(z)V_\bullet^{(1,2)} + G_1^{0,2}(z)G_1^{1,1}(z)V_\bullet^{(1,2)} \\
& + G_2^{0,1}(z)G_0^{1,2}(z)V_\bullet^{(1,2)} + G_0^{1,1}(z)G_0^{1,2}(z)(-V^{(1)}V^{(2)} + V_\bullet^{(1,2)}) \\
& + G_3^{0,1}(z)G_0^{0,3}(z)V_\bullet^{(1,3)} + G_2^{0,1}(z)G_1^{0,3}(z)V_\bullet^{(1,3)} + G_1^{0,3}(z)G_0^{1,1}(z)V_\bullet^{(1,3)} \\
& + G_0^{0,3}(z)G_1^{1,1}(z)V_\bullet^{(1,3)} + G_2^{0,1}(z)G_0^{0,4}(z)V_\bullet^{(1,4)} + G_3^{0,2}(z)V_\bullet^{(2)} \\
& + G_0^{0,4}(z)G_0^{1,1}(z)(-V^{(1)}V^{(4)} + V_\bullet^{(1,4)}) + G_1^{0,2}(z)G_2^{0,2}(z)V_\bullet^{(2,2)} \\
& + G_2^{0,2}(z)G_0^{0,3}(z)V_\bullet^{(2,3)} + G_1^{0,2}(z)G_1^{0,3}(z)V_\bullet^{(2,3)} + G_0^{2,1}(z)(-V^{(1)} + V_\bullet^{(1)}) \\
& + G_1^{0,2}(z)G_0^{0,4}(z)V_\bullet^{(2,4)} + G_0^{0,3}(z)G_1^{0,3}(z)V_\bullet^{(3,3)} + G_1^{0,2}(z)G_0^{1,2}(z)V_\bullet^{(2,2)} \\
& + \frac{1}{6}G_2^{0,1}(z)^3V_\bullet^{(1,1,1)} + \frac{1}{2}G_2^{0,1}(z)^2G_0^{1,1}(z)V_\bullet^{(1,1,1)} \\
& + \frac{1}{6}G_0^{1,1}(z)^3(-(V^{(1)})^3 + V_\bullet^{(1,1,1)}) + \frac{1}{2}G_2^{0,1}(z)^2G_1^{0,2}(z)V_\bullet^{(1,1,2)} \\
& + G_2^{0,1}(z)G_1^{0,2}(z)G_0^{1,1}(z)V_\bullet^{(1,1,2)} + \frac{1}{2}G_1^{0,2}(z)G_0^{1,1}(z)^2V_\bullet^{(1,1,2)} \\
& + \frac{1}{2}G_2^{0,1}(z)^2G_0^{0,3}(z)V_\bullet^{(1,1,3)} + G_2^{0,1}(z)G_0^{0,3}(z)G_0^{1,1}(z)V_\bullet^{(1,1,3)} \\
& + \frac{1}{2}G_0^{0,3}(z)G_0^{1,1}(z)^2(-(V^{(1)})^2V^{(3)} + V_\bullet^{(1,1,3)}) + \frac{1}{2}G_2^{0,1}(z)G_1^{0,2}(z)^2V_\bullet^{(1,2,2)} \\
& + \frac{1}{2}G_1^{0,2}(z)^2G_0^{1,1}(z)V_\bullet^{(1,2,2)} + G_2^{0,1}(z)G_1^{0,2}(z)G_0^{0,3}(z)V_\bullet^{(1,2,3)} \\
& + G_1^{0,2}(z)G_0^{0,3}(z)G_0^{1,1}(z)V_\bullet^{(1,2,3)} + \frac{1}{2}G_2^{0,1}(z)G_0^{0,3}(z)^2V_\bullet^{(1,3,3)} \\
& + \frac{1}{2}G_0^{0,3}(z)^2G_0^{1,1}(z)(-V^{(1)}(V^{(3)})^2 + V_\bullet^{(1,3,3)}) + \frac{1}{6}G_1^{0,2}(z)^3V_\bullet^{(2,2,2)} \\
& + \frac{1}{2}G_1^{0,2}(z)^2G_0^{0,3}(z)V_\bullet^{(2,2,3)} + \frac{1}{2}G_1^{0,2}(z)G_0^{0,3}(z)^2V_\bullet^{(2,3,3)} \\
& + \frac{1}{6}G_0^{0,3}(z)^3(-(V^{(3)})^3 + V_\bullet^{(3,3,3)}) + G_0^{0,3}(z)G_0^{1,2}(z)(-V^{(2)}V^{(3)} + V_\bullet^{(2,3)}) \\
& + G_0^{0,3}(z)G_0^{0,4}(z)(-V^{(3)}V^{(4)} + V_\bullet^{(3,4)}) + \frac{1}{2}G_2^{0,1}(z)G_0^{1,1}(z)^2V_\bullet^{(1,1,1)}
\end{aligned}
\tag{3.138}
$$

Let us remark here that the theta functions vanish at argument 0, and hence the undotted $V^{(d)}$'s are not defined. However, they only appear along with terms $G_n^{h,d}$ with $n = 0$ (which are constant in z). These terms only change the wave-function ψ by an overall (possibly $\hbar$-dependent) constant, and hence do not pose an issue in the following.

3.6.2 Quantization condition

In the previous subsection, we defined a non-perturbative wave-function ψ_{NP}. We also studied its graphical interpretation in the form

$$\psi_{\mathrm{NP}}(z) = \exp\left(\frac{1}{\hbar}\sum_{k\geq 0}\hbar^k S_k(z)\right). \tag{3.139}$$

This was considered as a formal asymptotic series in $\hbar$. But in general, the $S_k(z)$ will also depend on $\hbar$; they however will not have power series expansions in $\hbar$. Thus, for this expansion to be useful for us, the $S_k(z)$ should be independent of $\hbar$. This can be referred to as a *quantization condition* for the spectral curve.[8]

The problem comes from

$$\zeta_\hbar(\tau) = \frac{1}{2\pi i\hbar}\left(\oint_{\mathcal{A}} y\mathrm{d}x - \tau\oint_{\mathcal{B}} y\mathrm{d}x\right), \tag{3.140}$$

which enters into the definition of $\theta_\bullet(z|\tau)$. In general, it depends on $\hbar$. The simplest way to ensure that the S_k's do not depend on $\hbar$ is to set $\zeta_\hbar = 0$.[9] Let us work out what this means for the Weierstrass spectral curve. First,

$$\begin{aligned}\oint_{\mathcal{B}} y\mathrm{d}x &= \oint_{\mathcal{B}} \wp'(z;\tau)^2\mathrm{d}z \\ &= -\int_0^\tau \wp''(z;\tau)\wp(z;\tau)\mathrm{d}z \\ &= -\int_0^\tau (6\wp(z;\tau)^2 - \frac{g_2(\tau)}{2})\wp(z;\tau)\mathrm{d}z \\ &= \left(-\frac{3}{5}g_3(\tau)z + \frac{2}{5}g_2(\tau)\zeta(z;\tau)\right)_0^\tau \\ &= -\frac{3}{5}g_3(\tau)\tau + \frac{2}{5}g_2(\tau)(2\pi i + \tau G_2(\tau)).\end{aligned} \tag{3.141}$$

[8] The quantization condition for spectral curves was explored in [Guk05, GSk12] and subsequently in [BE15] for spectral curves in $\mathbb{C}^* \times \mathbb{C}^*$, in the context of knot theory and the AJ conjecture. In this context, the quantization condition has a beautiful interpretation in terms of algebraic K-theory. See [GSk12] and also [BE15] for more details.

[9] We note here that this is not the only way however; see [BE15, GSk12] for more details. It would be interesting to investigate more general elliptic spectral curves that satisfy the quantization condition.

Second,

$$\oint_{\mathcal{A}} y \mathrm{d}x = \left(-\frac{3}{5} g_3(\tau) z + \frac{2}{5} g_2(\tau) \zeta(z;\tau) \right)_0^1 \tag{3.142}$$

$$= -\frac{3}{5} g_3(\tau) + \frac{2}{5} g_2(\tau) G_2(\tau). \tag{3.143}$$

Therefore, we get that

$$\zeta_\hbar \equiv \frac{1}{\hbar} \left(\frac{2}{5} g_2(\tau) \right). \tag{3.144}$$

This tells us that for $\zeta_\hbar = 0$, we should set $g_2(\tau) = 0$, which fixes the isomorphism class of the elliptic curve (i.e., fixes τ in the fundamental domain). Without loss of generality we can also choose $g_3(\tau) = 4$ to get the curve in the form $y^2 = 4(x^3 - 1)$. For this curve, the values of $x = \wp(z;(\tau))$ at the half-periods are the third roots of unity: $1, \omega, \omega^2$.

This elliptic curve is of course very special. It corresponds to the curve with $\tau = \exp\left(\frac{2\pi i}{3}\right)$. Its j-invariant vanishes. It also has complex multiplication. In fact, after rescaling $y \to 2y$, it becomes the curve 144A1 in Cremona's classification. It would be interesting to investigate what role these special properties of the elliptic curve play in the non-perturbative setting.

For the rest of this section we focus on that particular elliptic curve, hence we will remove the explicit τ dependence, since τ is now fixed. The formulae that we will derive for the non-perturbative S_k's are only valid for this particular elliptic curve.

3.6.3 Quantum curve

The authors of [BE12b] conjecture that the non-perturbative partition function (3.124) is a τ-function, *i.e.* that it satisfies the Hirota equations. Assuming this conjecture, they argue that there should be a quantum curve $\hat{P}(\hat{x}, \hat{y}; \hbar)$ of the spectral curve that kills the non-perturbative wave-function:

$$\hat{P}(\hat{x}, \hat{y}; \hbar) \psi_{\mathrm{NP}}(z) = 0. \tag{3.145}$$

We refer the reader to [BE12b] for the details of the argument. The goal of this subsection is to study whether this conjecture is true for the Weierstrass spectral curve $y^2 = 4(x^3 - 1)$.

To be more precise, for the case of the Weierstrass spectral curve the conjecture can be formulated as follows:

Conjecture 3.6.1. *Consider the Weierstrass spectral curve* $P(x,y) = y^2 - 4(x^3-1) = 0$*. Then there is a unique quantum curve* $\hat{P}(\hat{x},\hat{y};\hbar)$ *of the form*

$$\hat{P}(\hat{x},\hat{y};\hbar) = \hbar^2 \frac{d^2}{dx^2} - 4(x^3-1) + \sum_{i\geq 1} \hbar^{2i} A_{2i}(x) \frac{d}{dx} + \sum_{j\geq 1} \hbar^{2j} B_{2j}(x), \quad (3.146)$$

where the $A_i(x)$ *and* $B_j(x)$ *are polynomials of* x*, which kills the non-perturbative wave-function constructed by equation* (3.126)*:*

$$\hat{P}(\hat{x},\hat{y};\hbar)\psi_{NP}(z) = 0. \quad (3.147)$$

Note that only even powers of $\hbar$ *appear in the quantum curve. Moreover, we conjecture that the* $A_{2i}(x)$ *are polynomials of degree at most* $i-2$*, and the* $B_{2j}(x)$ *are polynomials of degree at most* j*.*

The general form of a quantum curve for the spectral curve $y^2 = 4(x^3-1)$ was given in (3.45). The fact that only even powers of $\hbar$ should appear is clear. It is easy to see that the S_k's transform as $S_k(-z) = (-1)^{k+1} S_k(z)$ from the transformation properties of the $\omega_{g,n}$ and the $V_\bullet^{(\cdots)}$s. The WKB expansion of (3.147) is

$$S''_{k-1} + \sum_{l=0}^{k} S'_l S'_{k-l} + B_k + \sum_{l=0}^{k} S'_{k-l} A_{l+1} = 0. \quad (3.148)$$

As $A_i(z) = A_i(-z)$ and $B_j(z) = B_j(-z)$, we see that $A_{2i+1} = B_{2i+1} = 0$ for all $i \in \mathbb{Z}$.

Uniqueness of the quantum curve also follows directly from the WKB expansion above, which uniquely defines the $A_{2i}(x)$ and $B_{2j}(x)$.

As for the degree of $A_{2i}(x)$ and $B_{2j}(x)$, the conjectured bound is easily obtained by looking at the behavior of the S_k's at $z=0$ (*i.e.*, the double pole of x). For all $k \geq 1$, S_k is of order $3-k$ at $z=0$ (positive order meaning a zero of degree $3-k$, negative order meaning a pole of order $|3-k|$). As for $k=0$, S'_0 has a pole of order 3. Now, (3.148) (for a specified k) ensures that B_{2k} cannot have a pole of order greater than $2k$, while A_{2k} cannot have a pole of order greater than $2k-4$, which justifies the bound stated in the conjecture, since x has a double pole at $z=0$.

What remains to be proven however is that the non-perturbative wave-function is annihilated by a quantum curve at all. At the moment we do not have a complete proof of Conjecture 3.6.1. However, we computed correlation functions $\omega_{g,n}$ for the Weirstrass curve up to level $2g-2+n=3$ — see Appendix A. Focusing on the particular curve $y^2 = 4(x^3-1)$, we then calculated the relevant cycle integrals to construct the S_k's for the non-perturbative wave-function ψ_{NP}. Using Mathematica, we were then able to verify Conjecture 3.6.1 to order $\hbar^5$:

Theorem 3.6.2. *The quantum curve to order $\hbar^5$ is*

$$\hat{P}(\hat{x},\hat{y};\hbar) = \hbar^2 \frac{d^2}{dx^2} - 4(x^3-1) + \hbar^4 \frac{1}{2^6 3^2}\frac{d}{dx} + \hbar^2 \frac{x}{2^2 3} + \hbar^4 \frac{x^2}{2^8 3^3} + O(\hbar^5) \tag{3.149}$$

In particular, it satisfies the requirements of Conjecture 3.6.1.

Proof. The proof is computational. Using the correlation functions calculate in Appendix A, restricting to the curve with $g_2=0$ and $g_3=4$, and evaluating the relevant cycle integrals on Mathematica, we obtain the following expressions for the non-perturbative S_k's defined in the previous subsection:

$$\begin{aligned}
S_0'(z) &= \wp'(z) \\
S_1'(z) &= -\frac{3\wp^2(z)}{\wp'(z)^2} \\
S_2'(z) &= -\frac{\wp(z)}{24\wp'(z)} - \frac{21\wp(z)}{8\wp'(z)^3} - \frac{45\wp(z)}{2\wp'(z)^5} \\
S_3'(z) &= -\frac{1}{1152} - \frac{1}{24\wp'(z)^2} - \frac{109}{16\wp'(z)^4} - \frac{243}{2\wp'(z)^6} - \frac{405}{\wp'(z)^8} \\
S_4'(z) &= -\frac{\wp(z)^2}{13824\wp'(z)} - \frac{\wp(z)^2}{1152\wp'(z)^3} - \frac{31\wp(z)^2}{64\wp'(z)^5} - \frac{13641\wp(z)^2}{128\wp'(z)^7} - \frac{41769\wp(z)^2}{16\wp'(z)^9} \\
&\quad - \frac{89505\wp(z)^2}{8\wp'(z)^{11}}
\end{aligned}$$

Here primes refer to differentiation with respect to $x=\wp(z)$. Using those results it is straightforward to check that the non-perturbative wave-function ψ_{NP} is annihilated by the differential operator above, up to order $\hbar^5$. ☐

We note here that the $S'_k(z)$ that we calculated are rational functions of $\wp(z)$ and $\wp'(z)$, but we also remind the reader that these results are only valid for the particular Weierstrass spectral curve $y^2 = 4(x^3 - 1)$. For other Weierstrass curves, the non-perturbative S_k's will generally depend on $\hbar$, and the expressions above will certainly not be valid. It is not clear to us whether we can reconstruct the WKB expansion of a quantization of the general Weierstrass spectral curve from the non-perturbative topological recursion.

3.7 Conclusion and open questions

In this paper we studied how to quantize the Weierstrass spectral curve via the Eynard-Orantin topological recursion. More precisely, we investigated whether there exists a quantization of the Weierstrass spectral curve that kills the wave-function constructed from the topological recursion.

We first studied the naive question of whether there is such a quantization that kills the perturbative wave-function, as is the case for genus zero spectral curves. Not surprisingly, we did obtain a differential operator that annihilates the wave-function, but it is not a quantum curve according to Definition 3.2.7. Nevertheless, we obtained this differential operator using two different approaches, and as a side result we produced an infinite tower of identities for cycle integrals of elliptic functions.

We then constructed a non-perturbative wave-function, which is a better candidate for a quantum curve, as expected from matrix models. By direct computations on Mathematica, we showed that indeed, up to order $\hbar^5$, the non-perturbative wave-function is killed by a non-trivial quantization of the Weierstrass spectral curve. However, we could only construct the non-perturbative wave-function if the quantization condition was satisfied; for this we focused on the simple elliptic curve $y^2 = 4(x^3 - 1)$.

There are many open questions that should be further studied. To name a few:

- Conjecture 3.6.1 remains to be proven for the spectral curve $y^2 = 4(x^3 - 1)$. It is somehow expected to be true from the point of view of matrix models, but it would be very nice to have a formal proof of this conjecture. It may be possible to use our results from Sections 3.3 and 3.4 about the perturbative wave-function to construct a proof of the conjecture for the non-perturbative wave-function.

- In the non-perturbative case, we restricted ourselves to the elliptic curve $y^2 = 4(x^3 - 1)$ to ensure that the non-perturbative wave-function has an expansion in $\hbar$. It would be very nice to see whether we can get rid of this constraint and study more general Weierstrass curves in the non-perturbative setting.

- Via the non-perturbative approach, we obtained a proper quantization of the Weierstrass spectral curve. However, it is a rather non-trivial one, since it has an infinite number of $\hbar$ corrections. A more natural quantization would consist in

$$\hat{P}(\hat{x}, \hat{y}) = \hbar^2 \frac{d^2}{dx^2} - 4(x^3 - 1), \tag{3.150}$$

  that is, without $\hbar$ corrections. It would certainly be very interesting to study whether the WKB asymptotic solution to this equation can somehow be reconstructed, non-perturbatively, from the Eynard-Orantin topological recursion.

- In this paper we focused on the Weierstrass spectral curve. But of course it would be very interesting to study larger classes of spectral curves, both at genus one and at higher genus, in the spirit of [BE17] for genus zero curves. The approach of [BE17] for the perturbative wave-function can certainly be generalized to higher genus curves, as we did in section 3.3 (the general expressions become rather messy quickly though). But the most interesting question would be to study the non-perturbative wave-function.

- Coming back to the Weierstrass spectral curve, in Appendix A we calculated many correlation functions produced by the topological recursion. Generally speaking, in most applications of the topological recursion those correlation functions are generating functions for some interesting enumerative invariants. It is unclear at the moment whether there is such an interpretation for the correlation functions produced by the Weierstrass spectral curve. This is certainly worth investigating.

- Finally, we obtained in section 3.5 an infinite sequence of identities for cycle integrals of elliptic functions. A natural question is whether those are interesting from the point of view of elliptic functions and quasi-modular forms. In particular, they may be related to the results on cycle integrals of elliptic functions obtained in [GM20]. Moreover, the manipulations done in this paper are quite general, and would probably lead to analogous relations for higher genus curves, which would certainly be interesting to investigate. We hope to report on that in the near future.

Chapter 4

Higher Airy structures, $\mathcal{W}$-algebras and topological recursion

4.1 Introduction

4.1.1 Motivation

Virasoro constraints are ubiquitous in enumerative geometry. The general statement goes as follows. Given a particular enumerative geometric context, such as intersection theory on the moduli spaces of curves, or Gromov-Witten theory of a given target space, an interesting object of study is the generating series F for connected descendant invariants and the corresponding generating series $Z = e^F$ for disconnected invariants. The statement of Virasoro constraints is that Z satisfies a collection of differential equations of the form $H_k Z = 0$, where the H_ks are differential operators (in the formal coordinates of the generating series F) that form a representation of a subalgebra of the Virasoro algebra. In this context, the starting point is a given enumerative theory, and the goal is to show that the generating series Z satisfies Virasoro constraints.

An interesting question is whether the sequence of events can be reversed. Can "Virasoro-like constraints" be formulated abstractly such that there always exists a unique solution to the collection of differential equations, in the form of the exponential of a generating series? One may understand the recent of work of Kontsevich and Soibelman [KS18] (see also [ABCO17]) as providing an answer to this question, in the form of "quantum Airy structures".

Let V be a vector space of dimension D (which may be countably infinite) over $\mathbb{C}$. Using the notation $I = \{1, \ldots, D\}$, let $(x_i)_{i \in I}$ be linear coordinates on V^*, and denote by

$$\mathcal{D}^{\hbar}_{T^*V} \cong \mathbb{C}[\![\hbar, (x_l)_{l \in I}, (\hbar \partial_{x_l})_{l \in I}]\!]$$

the completed algebra of differential operators on V. We introduce a grading on $\mathcal{D}^{\hbar}_{T^*V}$ by assigning:

$$\deg x_l = \deg \hbar \partial_{x_l} = 1 \,, \qquad \deg \hbar = 2 \,.$$

Then a quantum Airy structure is a collection of differential operators $(H_k)_{k \in I}$ of the form

$$H_k = \hbar \partial_{x_k} - P_k \,, \tag{4.1}$$

where $P_k \in \mathcal{D}^{\hbar}_{T^*V}$ is homogeneous of degree 2, such that the H_k generate a graded Lie subalgebra of $\mathcal{D}^{\hbar}_{T^*V}$. That is, there exists scalars $c^m_{k,l}$ such that

$$[H_k, H_l] = \hbar \sum_{m \in I} c^m_{k,l} H_m \,. \tag{4.2}$$

The crucial theorem proved in [KS18] is that for any quantum Airy structure, there exists a unique solution Z to the collection of differential constraints $H_k Z = 0$, $k \in I$, of the following form:

$$Z = \exp\left(\sum_{\substack{g \geq 0,\ n \geq 1 \\ 2g-2+n>0}} \frac{\hbar^{g-1}}{n!} \sum_{\alpha \in I^n} F_{g,n}[\alpha] x_{\alpha_1} \cdots x_{\alpha_n} \right) . \tag{4.3}$$

It does not say what kind of enumerative invariants the coefficients $F_{g,n}[\alpha]$ are; this depends on the choice of quantum Airy structure. But the existence and uniqueness of a solution to the differential constraints is guaranteed.

There are two key features in the definition of quantum Airy structures that are responsible for existence and uniqueness of a solution. The first one is the particular form of the differential operators H_i, which implies that the differential constraints $H_i Z = 0$ translate into a recursive system for the coefficients $F_{g,n}[\alpha]$. The second is the subalgebra property, which, together with the form of the operators, ensures the existence of a solution.

While quantum Airy structures may be understood as an abstract construction of Virasoro-like constraints, they were first introduced in [KS18] as generalizations of the topological recursion of Chekhov, Eynard and Orantin [EO07, EO09]. The Chekhov-Eynard-Orantin topological recursion appears rather different from quantum Airy structures or Virasoro constraints *a priori*. It starts from the geometry of a spectral curve $\mathcal{C}$, and constructs an infinite sequence of meromorphic symmetric differentials on $\mathcal{C}^n$ through a period computation. But it turns out that for any admissible spectral curve with simple ramification, the Chekhov-Eynard-Orantin topological recursion can be recast into a quantum Airy structure. In other words, its recursive structure is equivalent to the collection of differential constraints of a quantum Airy structure.

Thus, quantum Airy structures provide a clear conceptual framework behind the Chekhov-Eynard-Orantin topological recursion, and a generalization thereof. In particular, it clarifies the relations between topological recursion, symplectic geometry and deformation quantization. It also incorporates earlier observations of Kazarian about the role of symplectic loop spaces and polarizations in the theory of [EO09] (see also [KZ15]) and provides a simpler approach to the relation with the Givental group action and semi-simple cohomological field theories established in [DBOSS14].

Natural generalizations of Virasoro constraints that appear in enumerative geometry are $\mathcal{W}$ constraints. They are known to be satisfied in some contexts, such as intersection theory on the moduli space of curves with r-spin structures, and certain Fan-Jarvis-Ruan theories. $\mathcal{W}$ constraints are similar in nature to Virasoro constraints. They consist of a collection of differential constraints $H_i Z = 0$ for a generating series of disconnected invariants, but where the H_is form a representation of a subalgebra of a $\mathcal{W}$ algebra. Recall that $\mathcal{W}$ algebras are non-linear extensions of the Virasoro algebra, which arise in conformal field theory when the theory contains chiral primary fields of conformal weight > 2. $\mathcal{W}$ algebras always contain the Virasoro algebra as a subalgebra.

In this paper we provide an answer to the question: Can "$\mathcal{W}$-like constraints" be formulated abstractly such that there always exists a unique solution to the collection of differential equations, and that this solution has the form of an exponential of a generating series?

The answer takes the form of "higher quantum Airy structures". We use the same conceptual framework as for quantum Airy structures, but we relax the two conditions on the differential operators. We consider differential operators $(H_k)_{k\in I}$ of the same form as in (4.1), but with $P_k \in \mathcal{D}^{\hbar}_{T^*V}$ a sum of terms of degree ≥ 2. The subalgebra condition is replaced by the requirement that the left $\mathcal{D}^{\hbar}_{T^*V}$-ideal generated by the H_k is a graded Lie subalgebra of $\mathcal{D}^{\hbar}_{T^*V}$. Concretely, this means that (4.2) is replaced by:

$$[H_k, H_l] = \hbar \sum_{m\in I} g^m_{k,l}\, H_m\,, \qquad g^m_{k,l} \in \mathcal{D}^{\hbar}_{T^*V}\,.$$

Under these conditions, Kontsevich and Soibelman (in [KS18]) already proved the existence and uniqueness of a solution to the collection of differential constraints $H_k Z = 0$ of the same form as (4.3). The goal of this paper is to construct many examples of higher quantum Airy structures from $\mathcal{W}$ algebras and discuss their enumerative meaning.

4.1.2 Main results

Let us now describe the main results of the paper briefly. First, we construct various types of $\mathcal{W}$ constraints. Second, we show that the Bouchard-Eynard topological recursion of [BE13, BE17, BHL+14] is equivalent to a previously constructed class of $\mathcal{W}$ constraints. We also proved along the way a property about the modes of the $\mathcal{W}(\mathfrak{gl}_{N+1})$ algebra at the self-dual level, which was essential to our construction of $\mathcal{W}$ constraints.

Higher quantum Airy structures from $\mathcal{W}$ constraints

Our general recipe to produce higher quantum Airy structures from modules of $\mathcal{W}$ algebras goes as follows. The starting ingredients are a Lie algebra $\mathfrak{g}$ and an element σ of the Weyl group of $\mathfrak{g}$. We then consider the principal $\mathcal{W}$-algebra of $\mathfrak{g}$ at the self-dual level $k = -h^{\vee} + 1$ ($h^{\vee}$ is the dual Coxeter number of $\mathfrak{g}$). This vertex operator algebra is denoted by $\mathcal{W}(\mathfrak{g})$ and we realize it as a subalgebra of the Heisenberg vertex operator algebra associated to the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. Then:

1. We construct a σ-twisted module $\mathcal{T}$ of the Heisenberg vertex operator algebra.

2. Upon restriction to the $\mathcal{W}(\mathfrak{g})$ algebra, we obtain an untwisted module. We realize the modes of the generators of the $\mathcal{W}(\mathfrak{g})$ algebra as differential operators acting on the space of formal series in countably many variables.

3. We pick a subset of modes generating a left ideal which is a graded Lie subalgebra of the algebra of modes. These modes fulfill the second (and hardest to check) condition to be a higher quantum Airy structure.

4. If possible, we conjugate these modes (dilaton shift) to bring them in the form of a higher quantum Airy structure.

Remarkably, following this simple recipe we can construct a large variety of higher quantum Airy structures, including many that have interesting enumerative interpretations. Our general construction reproduces some of the $\mathcal{W}$ constraints that have already appeared in the literature, but most of the higher quantum Airy structures that we obtain are new.

We also note that our construction relies on certain explicit strong generators of the $\mathcal{W}$ algebras that are known in the literature. We discuss this in detail in Section 4.3.2.

$\mathcal{W}(\mathfrak{gl}_{N+1})$ **higher quantum Airy structures: first class.** For clarity let $r := N+1$. Our first set of examples starts with $\mathfrak{g} = \mathfrak{gl}_r$ and $\sigma = (1\,2\ \cdots\ r)$ – the Coxeter element of the Weyl group $\mathfrak{S}_r$. Theorem 4.4.9 is the main result of this construction, which can be summarized as follows:

Theorem 1. Let $r \geq 2$ and $s \in \{1, \ldots, r+1\}$ be such that $r = r's - \epsilon$ with $\epsilon \in \{\pm 1\}$ and r' integer. Let

$$\mathfrak{d}^i = i-1-\left\lfloor \frac{s(i-1)}{r} \right\rfloor, \qquad \tilde{S}_s = \left\{(i,k) \ \middle|\ i \in \{1, \ldots, r\} \quad \text{and} \quad k \geq \mathfrak{d}^i + \delta_{i,1}\right\}.$$

There exists an (explicit) quantum r-Airy structure on $V = \bigoplus_{l>0} \mathbb{C}\langle x_l \rangle$ based on a representation of the subset of modes $(W^i_k)_{(i,k)\in\tilde{S}_s}$ of the $\mathcal{W}(\mathfrak{gl}_r)$ algebra generators with central charge r in $\mathcal{D}^{\hbar}_{T^*V}$. We use $Z_{(r,s)}$ to denote its partition function. Its coefficients for $2g-2+n=1$ are

$$F_{0,3}[l_1,l_2,l_3] = \epsilon r' l_1 l_2 l_3\, \delta_{l_1+l_2+l_3,s}, \qquad F_{1,1}[l] = \frac{r^2-1}{24}\, \delta_{l,s}.$$

The case $s = r+1$ (for all r) was studied by Bakalov and Milanov [BM08, BM13, Mil16]. $Z_{(r,r+1)}$ is a generating series for intersection numbers on the moduli space of curves with r-spin structure, as explained in Section 4.6.1. Other choices of s however are new. As explained in the proof of Theorem 4.4.9, the condition that s be coprime with r arises for the dilaton shift to yield differential operators of the right form for a higher quantum Airy structure. The condition that $r = \pm 1 \bmod s$ is necessary and sufficient for the left ideal generated by the subset of modes to be a graded Lie subalgebra (see Theorem 9). We will come back to this statement, and state the precise result in Section 4.1.2.

The enumerative meaning of the cases corresponding to these general values of s is particularly intriguing. The partition function $Z_{(2,1)}$ corresponds to the Brézin-Gross-Witten tau function of the KdV hierarchy [BG80, GW80]. Further, Norbury constructed in [Nor17] a cohomology class on the moduli space of curves such that the partition function $Z_{(2,1)}$ generates its descendant invariants. It is then natural to ask whether similar results exist for $r > 2$, and for the various allowed values of s. It would be interesting to find an enumerative interpretation for all $Z_{(r,s)}$ since they are the building blocks for the Givental-like decomposition proved in Theorem 7 below for the Bouchard-Eynard topological recursion. For instance, we can ask: does $Z_{(r,1)}$ coincide with the r-Brézin-Gross-Witten tau function? And if it does, what is the analog of the Norbury class such that $Z_{(r,1)}$ becomes the generating series of its descendant invariants? In Section 4.6.2, we explore these questions in greater detail.

Propositions 4.4.12 and 4.4.13 provide straightforward generalizations of the above construction, by allowing direct sums and conjugations of the quantum Airy structures of Theorem 1. This easy observation will be necessary to compare the $\mathcal{W}$ constraints with the Bouchard-Eynard topological recursion.

$\mathcal{W}(\mathfrak{gl}_{N+1})$ higher quantum Airy structures: second class. For our second important class of examples, we keep $\mathfrak{g} = \mathfrak{gl}_r$, with $r = N + 1$, but replace the Coxeter element of the Weyl group by an arbitrary automorphism σ. Although part of our construction is general, we only complete the program in the case $\sigma = (1 \cdots r-1)$. Theorem 4.4.16 realizes these modules as higher quantum Airy structures with half-integer powers of $\hbar$ (which we call "crosscapped") and can be summarized as follows.

Theorem 2. Let $r \geq 3$ and $s \in \{1, \ldots, r\}$ dividing r. Let

$$\mathfrak{d}^i = i - 1 - \left\lfloor \frac{s(i-1)}{r-1} \right\rfloor, \quad \hat{S}_s = \left\{ (i,k) \quad \middle| \quad i \in \{1, \ldots, r\} \quad \text{and} \quad k \geq \mathfrak{d}^i + \delta_{i,1} + \delta_{i,r} \right\}.$$

There exists an (explicit) 1-parameter family of crosscapped quantum r-Airy structures on $V = \bigoplus_{p>0} \mathbb{C}\langle x_p^1 \rangle \oplus \mathbb{C}\langle x_p^2 \rangle$ based on a representation of the subset of modes $(W_k^i)_{(i,k) \in \hat{S}_s}$ of the $\mathcal{W}(\mathfrak{gl}_r)$ algebra generators with central charge r into $\mathcal{D}_{T^*V}^{\hbar^{1/2}}$.

In Section 4.6.3, we speculate that the enumerative geometry interpretation of these quantum Airy structures lies in the open intersection theory developed by Pandharipande, Solomon and Tessler [PST14, ST]. Indeed, for $(r, s) = (3, 3)$ we can identify them with the $\mathcal{W}(\mathfrak{sl}_3)$ constraints derived by Alexandrov in [Ale17] for the partition function of the open intersection theory on the moduli space of bordered Riemann surfaces. For higher r, do we recover the tau function of the extended $(r-1)$-KdV hierarchy constructed by Bertola and Yang [BY15]? Can it be understood in terms of the open $(r-1)$-spin intersection theory of [BCT18]?

It would be interesting to classify the automorphisms σ that can lead to higher quantum Airy structures and the corresponding structures themselves, as we did when σ is a r or $(r-1)$-cycle.

$\mathcal{W}(\mathfrak{so}_{2N})$ higher quantum Airy structures. Another class of examples is obtained by choosing the Lie algebra $\mathfrak{g} = \mathfrak{so}_{2N}$ and the Coxeter element σ of the Weyl group, which has order $r = 2(N-1)$. The resulting higher quantum Airy structures are presented in Theorem 4.4.20, summarized here:

Theorem 3. Let $N \geq 3$, that is $r = 2(N-1) \geq 4$, and $s = 1$ or $r+1$. Let

$$\mathfrak{d}^i = \delta_{s,1}(i-1)\,, \qquad \tilde{S}_s = \big\{(i,k) \quad \big| \quad i \in \{2,4,\ldots,2N-2\} \cup \{N\} \quad \text{and} \quad k \geq \mathfrak{d}^i\big\}\,.$$

There exists an (explicit) quantum r-Airy structure on $V = \bigoplus_{p>0} \mathbb{C}\langle x_{2p+1}\rangle \oplus \mathbb{C}\langle \tilde{x}_{2p+1}\rangle$ based on a representation of the subset of modes $(W^i_k)_{(i,k)\in\tilde{S}_s}$ of the $\mathcal{W}(\mathfrak{so}_{2N})$ algebra generators with central charge N in $\mathcal{D}^{\hbar}_{T^*V}$.

Here, for any r we get two higher quantum Airy structures (where $s=1$ and $s=r+1$), corresponding to the well-known subalgebra of modes of Proposition 4.3.13 and 4.3.14. For now, we do not have a construction for more general values of s for $\mathfrak{so}_{2N}$, as in Theorem 1 for $\mathfrak{gl}_{N+1}$ (equivalently, the analog of Theorem 8 for $\mathfrak{so}_{2N}$). The enumerative meaning of these higher quantum Airy structures is discussed in Section 4.6.4, in terms of Fan-Jarvis-Ruan theory [FJR13].

Exceptional higher quantum Airy structures. We construct two higher quantum Airy structures starting with the exceptional Lie algebras $\mathfrak{g} = \mathfrak{e}_N$ with $N \in \{6,7,8\}$ and using the Coxeter element (of order denoted by r) as the automorphism σ. Our main result here is Theorem 4.4.24, which we summarize as follows.

Theorem 4. Let $\mathbb{D} = \{d_1,\ldots,d_N\}$ the set of Dynkin exponents of $\mathfrak{e}_N$ (see Section 4.4.4). Let $s \in \{1, r+1\}$ and denote

$$\tilde{S}_s := \big\{(i,k) \quad \big| \quad i \in \{1,\ldots,N\} \text{ and } k \geq \mathfrak{d}^i\big\} \qquad \mathfrak{d}^i = \begin{cases} 0 & \text{if } s = r+1 \\ d_i - 1 & \text{if } s = 1 \end{cases}\,.$$

There exists a quantum r-Airy structure on $V = \bigoplus_{p\in\mathbb{D}+r\mathbb{N}} \mathbb{C}\langle x_p\rangle$ based on a representation of the subset of modes $(W^i_k)_{(i,k)\in\tilde{S}_s}$ of the $\mathcal{W}(\mathfrak{e}_N)$ algebra generators with central charge N into $\mathcal{D}^{\hbar}_{T^*V}$.

This Airy structure is as (un)explicit as the generators of the $\mathcal{W}(\mathfrak{e}_N)$ algebra, see Theorem 4.4.21. For $s = r+1$ it is not new: its partition function coincides with the Fan-Jarvis-Ruan invariants of E-type (see Section 4.6.4) and it was already known that it is uniquely determined by $\mathcal{W}$ constraints, see Section 4.6.4 for references. We have a new case $s=1$ whose enumerative geometry interpretation is currently unknown. For simple but non simply-laced

Lie algebras, according to a private communication of Di Yang, $\mathcal{W}$ constraints cannot be brought to the form (4.1) and therefore cannot yield higher quantum Airy structures.

Higher quantum Airy structures from topological recursion

The Chekhov-Eynard-Orantin topological recursion [EO07, EO09] associates, to the data of a spectral curve $\mathcal{S} = (\mathcal{C}, x, y, \omega_{0,2})$ satisfying certain conditions, a sequence of meromorphic differentials $(\omega_{g,n})_{2g-2+n>0}$ that generate enumerative invariants. It was shown in [KS18, ABCO17] that for a given $\mathcal{S}$ with simple ramification, the topological recursion is equivalent to a quantum Airy structure that has countable dimension and whose Lie algebra is isomorphic to a direct sum of subalgebras of the Virasoro algebra. The $F_{g,n}$s for $2g-2+n > 0$ encode the coefficients of decomposition of the meromorphic n-differentials $\omega_{g,n}$s of [EO09] on a suitable basis of meromorphic 1-forms. The choice of polarization in the construction of the quantum Airy structure is determined by $\omega_{0,2}$, which is part of the data of the spectral curve. This dictionary was established in detail in [KS18, ABCO17].

The original formulation of the Chekhov-Eynard-Orantin topological recursion requires the branched cover $x : \mathcal{C} \to \mathbb{C}$ to have simple ramification points only, *i.e.* $\mathrm{d}x$ has simple zeroes. This restriction on the order of the ramification points was lifted in [BE13, BE17, BHL$^+$14]. For arbitrary spectral curves, the combinatorial structure of the topological recursion becomes a little more involved; it is now known in the literature as the Bouchard-Eynard topological recursion.

In Section 4.5 we extend the dictionary between topological recursion and Airy structures to arbitrary spectral curves without any restriction on the order of ramifications. Our main results (Theorems 4.5.30 and Theorem 4.5.32 in the text) can be summarized as follows.

For each ramification point p_α, denote r_α the order of ramification at p_α, and introduce a local coordinate ζ such that $x(z) - x(p_\alpha) = \frac{\zeta^{r_\alpha}(z)}{r_\alpha}$. Let us consider the series expansion (denoted with $\equiv$) near the ramification points

$$y(z) \equiv \sum_{l>0} F_{0,1}\left[\begin{smallmatrix} \alpha \\ -l \end{smallmatrix}\right], \zeta(z)^{l-r_\alpha} \qquad z \to p_\alpha \,,$$

and introduce

$$s_\alpha := \min\left\{l > 0 \quad \middle| \quad F_{0,1}\left[\begin{smallmatrix} \alpha \\ -l \end{smallmatrix}\right] \neq 0 \;\text{ and }\; r_\alpha \nmid l\right\}.$$

The statement of Theorem 4.5.32 can be summarized as follows:

Theorem 5. The Bouchard-Eynard topological recursion is well defined (*i.e.* produces symmetric $\omega_{g,n}$) if and only if $r_\alpha = \pm 1 \mod s_\alpha$ for all α (the $\pm$ could depend on α). When this condition is not satisfied, the lack of symmetry is apparent in $\omega_{0,3}$.

Definition 6. We say that the spectral curve is admissible when $r_\alpha = \pm 1 \mod s_\alpha$ for all ramification points p_α.

Now let us consider the series expansion

$$\omega_{0,2}(z_1, z_2) \equiv \left(\frac{\delta_{\alpha_1,\alpha_2}}{(\zeta(z_1) - \zeta(z_2))^2} + \sum_{l_1,l_2>0} \phi^{\alpha_1,\alpha_2}_{l_1,l_2}\, \zeta(z_1)^{l_1-1}\zeta(z_2)^{l_2-1}\right) \mathrm{d}\zeta(z_1) \otimes \mathrm{d}\zeta(z_2)$$

as $\qquad z_j \to p_{\alpha_j}$, and introduce for $l > 0$ the meromorphic 1-forms on $\mathcal{C}$

$$\xi^\alpha_l(z) := \mathrm{Res}_{z'=p_\alpha} \left(\int_{p_\alpha}^{z'} \omega_{0,2}(\cdot, z)\right) \frac{\mathrm{d}\zeta(z')}{\zeta(z')^{l+1}}\,.$$

Theorem 4.5.30 relates the Bouchard-Eynard topological recursion to higher quantum Airy structures, as summarized below:

Theorem 7. For any admissible spectral curve, the $\omega_{g,n}$ computed by the Bouchard-Eynard topological recursion can be decomposed as finite sums

$$\omega_{g,n}(z_1, \ldots, z_n) = \sum_{\substack{\alpha_1,\ldots,\alpha_n \\ l_1,\ldots,l_n>0}} F_{g,n}\left[\begin{smallmatrix} \alpha_1 & \cdots & \alpha_n \\ l_1 & \cdots & l_n \end{smallmatrix}\right] \bigotimes_{j=1}^{n} \mathrm{d}\xi^{\alpha_j}_{l_j}(z_j)\,,$$

and the generating series

$$Z = \exp\left(\sum_{\substack{g\geq 0,\ n\geq 1 \\ 2g-2+n>0}} \frac{\hbar^{g-1}}{n!} \sum_{\substack{\alpha_1,\ldots,\alpha_n \\ l_1,\ldots,l_n>0}} F_{g,n}\left[\begin{smallmatrix} \alpha_1 & \cdots & \alpha_n \\ l_1 & \cdots & l_n \end{smallmatrix}\right] \prod_{j=1}^{n} x^{\alpha_j}_{l_j}\right)$$

is the partition function of a higher quantum Airy structure based on an (explicit) representation of a subset of modes of the $\bigoplus_\alpha \mathcal{W}(\mathfrak{gl}_{r_\alpha})$ algebra generators as differential operators.

More precisely, Z satisfies a Givental-like decomposition:

$$Z = \exp\left(\sum_{\alpha,l} \frac{F_{0,1}\left[\begin{smallmatrix}\alpha \\ -l\end{smallmatrix}\right] + \delta_{l,s_\alpha}}{l}\, \partial_{x_l^\alpha} + \frac{\hbar}{2} \sum_{\substack{\alpha_1,\alpha_2 \\ l_1,l_2>0}} \frac{\phi_{l_1,l_2}^{\alpha_1,\alpha_2}}{l_1\, l_2}\, \partial_{x_{l_1}^{\alpha_1}} \partial_{x_{l_2}^{\alpha_2}} \right) \prod_\alpha Z_{(r_\alpha,s_\alpha)}\big((x_l^\alpha)_{l>0}\big)\,, \tag{4.4}$$

where the $Z_{(r,s)}$s are the partition functions of the quantum Airy structures described in Theorem 1.

The formula (4.4) is a Givental-like decomposition for the Bouchard-Eynard topological recursion. If $r_\alpha = 2$, Theorem 7 was obtained in [Eyn14a] for $s_\alpha = 3$ and in [CN19] when s_α can take any of the admissible values 1 or 3.

Let us comment on our approach as we do not construct the higher quantum Airy structures directly from the topological recursion as in [ABCO17]. Rather, we start with the notion of "higher abstract loop equations" of Definition 4.5.21 which generalizes the one of [BEO15, BS17] to arbitrary ramifications. We prove in Appendix B.3 that if a solution to the higher abstract loop equations exists, then it is uniquely given by the Bouchard-Eynard topological recursion. Thus, it is just as good to take the higher abstract loop equations as starting point. But there is a fundamental reason why we start with the loop equations instead of the topological recursion. It is not too difficult to construct differential operators that produce a recursive structure equivalent to the Bouchard-Eynard topological recursion; but proving the graded Lie subalgebra condition required for existence of a common solution of these differential operators (*i.e.* the symmetry of the $F_{g,n}$) appears to be quite difficult. While if we start with loop equations, we observe that the resulting differential operators can be identified directly with those coming from modules over $\mathcal{W}(\mathfrak{gl}_r)$ algebras; therefore we can use Theorems 8 and 9 to prove the graded Lie subalgebra condition. In other words, the loop equations make the algebraic structure of the corresponding higher quantum Airy structure explicit, at the expense of obscuring the recursive structure of the original system of equations.

The identification with higher quantum Airy structures constructed from $\mathcal{W}(\mathfrak{gl}_r)$ algebras has a number of interesting consequences. We are not aware of a direct proof that the Bouchard-Eynard topological recursion produces symmetric differentials for arbitrary spectral curves. An indirect argument exists for spectral curves that appear as limits of family of curves with simple ramification [BE17], but it is not clear which spectral curves precisely satisfy this condition. A consequence of our identification between loop equations and higher quantum Airy structures is that for any admissible spectral curves, a solution to the loop equations exist. It must then be given uniquely by the Bouchard-Eynard topological recursion. It then follows that for all admissible spectral curves the Bouchard-Eynard topological recursion produces symmetric differentials (the announced Theorem 5, which is Theorem 4.5.32 in the text).

What is particularly intriguing though is the cases that fail. The admissibility in Definition 6 is a constraint on the local behavior of $\omega_{0,1} = y\mathrm{d}x$. While the condition that s is coprime with r is easy to understand from the geometry of spectral curves (it says that $\mathcal{C}$ is locally irreducible at its ramification points), the condition that $r = \pm 1 \mod s$ is rather unexpected and its geometric meaning is mysterious for us. Nonetheless, when it is not satisfied, we show in Proposition B.2.2 that the Bouchard-Eynard topological recursion *does not* in fact produce symmetric differentials. The simplest such case is $(r, s) = (7, 5)$. Consequently, we can deduce that the left ideal generated by the appropriate set of modes of the $\mathcal{W}(\mathfrak{gl}_r)$ algebra cannot be a graded Lie subalgebra of the algebra of modes, and that the collection of differential operators is not a higher quantum Airy structure.

For $r = 3$, Safnuk, in [Saf16], recast the $\mathcal{W}$ constraints of [Ale17] for open intersection theory into a period computation, which turns out to be an unusual modification of the topological recursion on the spectral curve $x = y^2/2$. It would be interesting – but beyond the scope of this article – to generalize Safnuk's result and realize the $F_{g,n}$ of Theorem 2 as a period computation. The same question could be asked if quantum Airy structures are found for other automorphisms $\sigma \in \mathfrak{S}_r$. It amounts to asking what is the appropriate modification of the topological recursion to treat reducible spectral curves, and whether there will be new conditions of admissibility (like the one we find in Theorem 5). This level of generality may enlighten the geometric meaning of those admissibility constraints.

Results on $\mathcal{W}$ algebras

As a side result of our construction, we prove a certain curious property of the algebra of modes of the $\mathcal{W}(\mathfrak{gl}_r)$ algebra at the self dual level. Propositions 4.3.13, 4.3.14 and Theorem 4.3.16 can be combined into the following result.

Theorem 8. Let $r \geq 2$ and $\lambda_1 \geq \cdots \geq \lambda_p \geq 1$ such that $\sum_{i=1}^{p} \lambda_i = r$. For $i \in \{1, \ldots, r\}$ denote

$$\lambda(i) := \min \Big\{ m > 0 \quad \Big| \quad \sum_{j=1}^{m} \lambda_j \geq i \Big\},$$

and

$$S_\lambda = \big\{ (i,k) \quad \big| \quad i \in \{1, \ldots, r\} \quad \text{and} \quad k \geq i - \lambda(i) \big\} \,.$$

The left ideal generated by the modes W_k^i of the $\mathcal{W}(\mathfrak{gl}_r)$ algebra of central charge r indexed by $(i,k) \in S_\lambda$ forms a Lie subalgebra of the algebra $\mathcal{A}$ of modes, *i.e.* there exists $g_{(k_1,i_1),(k_2,i_2)}^{(k_3,i_3)} \in \mathcal{A}$ such that

$$\forall (i_1,k_1),(i_2,k_2) \in S_\lambda \,, \qquad [W_{k_1}^{i_1}, W_{k_2}^{i_2}] = \sum_{(i_3,k_3) \in S_\lambda} g_{(i_1,k_1),(i_2,k_2)}^{(i_3,k_3)} \, W_{k_3}^{i_3} \,.$$

The case $\lambda = (1, \ldots, 1)$, which corresponds to $k \geq 0$, gives a well-known Lie subalgebra. It is the one generated by the modes annihilating the vacuum vector. The other cases however seem new. Some of these Lie subalgebras are used to prove Theorem 1, thanks to an arithmetic correspondence established in Proposition B.2.1 in Appendix B.2, which is summarized here:

Proposition 9. For any $s \in \{1, \ldots, r+1\}$ such that $r = r's + r''$ with $r'' \in \{1, s-1\}$, we have the equality $S_\lambda = \tilde{S}_s$ between the set of modes appearing in Theorem 1 and Theorem 8 for the choice

$$\lambda_1 = \cdots = \lambda_{r''} = r' + 1, \qquad \lambda_{r''+1} = \cdots = \lambda_s = r'$$

If $r \neq \pm 1 \bmod s$, there exists a unique sequence $(\lambda_j)_j$ such that $S_\lambda = \tilde{S}_s$ but it is not weakly decreasing and the left ideal generated by the modes $(W^i_k)_{(i,k)\in\tilde{S}_s}$ does not form a subalgebra of $\mathcal{A}$.

The proof of Theorem 8 relies on the construction of a highest weight module whose highest weight vector is annihilated by the modes indexed by S_λ. The existence of such a highest weight module is perhaps unexpected; it relies heavily on our realization of the $\mathcal{W}(\mathfrak{gl}_r)$ algebra as a subalgebra of the Heisenberg vertex operator algebra and on certain embeddings of $\mathfrak{gl}_{\lambda_1} \oplus \cdots \oplus \mathfrak{gl}_{\lambda_p}$ into $\mathfrak{gl}_r$. It would be worth investigating this construction further, and see whether it can be generalized to $\mathcal{W}$ algebra of other types. In particular, this would yield generalizations of Theorem 3. Note that it is important in the proof for $\mathfrak{gl}_r$ that $(\lambda_j)_j$ be a weakly decreasing sequence and this is confirmed by the counterexamples mentioned in the last claim in Theorem 9.

4.1.3 Outline

We start in Section 4.2 by defining higher quantum Airy structures. We first propose in Section 4.2.1 a basis-independent definition, starting from the point of view of quantization of classical higher Airy structures, as in [KS18]. In Section 4.2.2 we revisit higher quantum Airy structures using bases. We calculate the explicit recursive system satisfied by the coefficients $F_{g,n}$. We also prove a reduction statement to get rid of linear differential operators in higher quantum Airy structures. We introduce crosscapped Airy structures in Section 4.2.3, which are related to generating functions in open intersection theory.

In Section 4.3 we first introduce the background on vertex operators algebras (Section 4.3.1) and $\mathcal{W}(\mathfrak{g})$ algebras (Section 4.3.2) that will be needed for the construction of our first type of higher quantum Airy structures. We construct in Section 4.3.3 a number of left ideals for the algebra of modes that are graded Lie subalgebras (see Propositions 4.3.13, 4.3.14 and Theorem 4.3.16).

We then review the concept of twisted modules for vertex operators algebras in Section 4.3.4, in preparation for the next section.

Our construction of higher quantum Airy structures as modules of $\mathcal{W}(\mathfrak{g})$ algebras is proposed in Section 4.4. The first class of $\mathcal{W}(\mathfrak{gl}_r)$ higher quantum Airy structures, with the automorphism σ given by the Coxeter element of the Weyl group, is explored in Section 4.4.1. The second class of $\mathcal{W}(\mathfrak{gl}_r)$ higher quantum Airy structures for arbitrary automorphisms σ is studied in Section 4.4.2. We introduce the $\mathcal{W}(\mathfrak{so}_{2r})$ higher quantum Airy structures in Section 4.4.3, and the $\mathcal{W}(\mathfrak{e}_r)$ higher quantum Airy structures in Section 4.4.4.

Section 4.5 is devoted to the reconstruction of the higher quantum Airy structures associated to the Bouchard-Eynard topological recursion on arbitrary admissible spectral curves. We study the geometry of local spectral curves in Section 4.5.1, and describe the relation with the standard notion of (global) spectral curves. We introduce the Bouchard-Eynard topological recursion and higher abstract loop equations in Section 4.5.2. We then prove that the higher abstract loop equations for local spectral curves with one component are equivalent to $\mathcal{W}(\mathfrak{gl}_r)$ higher quantum Airy structures in Section 4.5.3. The general result for local spectral curves with several components is obtained in Section 4.5.4.

Finally, Section 4.6 reviews the known and conjectural enumerative geometric interpretations of the higher quantum Airy structures that we construct; we also attempt to summarize the rich history of existing results in this area. We discuss the (closed) r-spin intersection theory (Section 4.6.1), higher analogs of the Brézin-Gross-Witten theory (Section 4.6.2), open r-spin intersection theory (Section 4.6.3), and Fan-Jarvis-Ruan theories (Section 4.6.4).

We conclude with three appendices. In Appendix B.1 we prove, by elementary means, various properties of certain sums over roots of unity that play an important role in our construction of the $\mathcal{W}(\mathfrak{gl}_r)$ quantum Airy structures. In Appendix B.2, we show that the graded Lie subalgebra property is only satisfied for values of (r, s) such that $r = \pm 1 \mod s$. When $r = \pm 1 \mod s$, we show that we get a subalgebra of the intermediate type described in Theorem 9; and when $r \neq \pm 1 \mod s$, we prove that there is no symmetric solution to the system of differential equations, and hence the left ideal generated by the set of modes cannot be a graded Lie subalgebra. The proof consists of an explicit computation and check of (lack of) symmetry for $F_{0,3}$ by elementary – but still lengthy – arithmetics. We also compute explicitly $F_{1,1}$ in Appendix B.2.3. Lastly, in Appendix B.3 we prove that if a solution to the higher abstract loop equations that respects the polarization exists, then it is uniquely constructed by the Bouchard-Eynard topological recursion.

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4.2 Higher quantum Airy structures

4.2.1 A conceptual approach

In this section, we provide a conceptual introduction to the concept of higher Airy structures, starting from the point of view of quantization of higher classical Airy structures. We propose a basis-free definition of ∞-Airy structures, and its finite counterpart r-Airy structures. We offer a basis-dependent and computational approach to higher Airy structures in Section 4.2.2. Readers who are mostly interested in the computational aspects of Airy structures may prefer to skip directly to Section 4.2.2.

Classical picture

Let $\mathbf{k}$ be a field of characteristic zero and W be a finite-dimensional symplectic $\mathbf{k}$-vector space equipped with the symplectic form Ω. $\mathbf{k}[\![W]\!]$ is the completion of the graded ring of polynomial functions on W. It is a Poisson algebra. The projection from $\mathbf{k}[\![W]\!]$ onto its subspace of degree i is denoted π_i. In fact, we can consider π_1 as a linear map $\mathbf{k}[\![W]\!] \to W$ since the subspace of linear functions on W is naturally isomorphic to W^* and can be identified with W itself via the pairing Ω.

Definition 4.2.1. A *classical ∞-Airy structure* on (W, Ω) is the data of a $\mathbf{k}$-vector space V together with a linear map $\lambda : V \to \mathbf{k}[\![W]\!]$ such that

(i) $\pi_0 \circ \lambda = 0$.

(ii) $\mathcal{I} = \pi_1 \circ \lambda : V \to W$ is a linear embedding of V as a Lagrangian subspace of W.

(iii) The $\mathbf{k}[\![W]\!]$-ideal generated by $\operatorname{Im}\lambda$ is a Poisson subalgebra of $\mathbf{k}[\![W]\!]$.

If $\operatorname{Im}\lambda$ is a subspace of the space $\mathbf{k}_r[W]$ of polynomial functions of degree at most r for some given integer $r \geq 2$, we will call it a *classical r-Airy structure*.

For $r = 2$, *(iii)* is equivalent to requiring that $\operatorname{Im}\lambda$ is a Poisson subalgebra and we recover the Airy structures studied in [KS18, ABCO17]. Definition 4.2.1 formally corresponds to $r = \infty$.

The quantization problem

The Poisson algebra $\mathbb{C}[\![W]\!]$ can be quantized by forming the Weyl algebra $\mathcal{D}_W^{\hbar}$. We define it as the completion of the graded associative algebra over $\mathbb{C}[\![\hbar]\!]$ of non-commutative polynomials in elements of W modulo the relations $[w, w'] = \hbar\Omega(w, w')$ for any $w, w' \in W$. The grading is defined by $\deg W = 1$ and $\deg \hbar = 2$.

Definition 4.2.2. A subspace $\hat{\Lambda} \subset \mathcal{D}_W^{\hbar}$ is a *graded Lie subalgebra* if $[L, L'] \in \hbar \cdot \hat{\Lambda}$ for any $L, L' \in \hat{\Lambda}$.

We have a linear map $\mathrm{cl} : \mathcal{D}_W^\hbar \to \mathbb{C}[\![W]\!]$ which is a reduction to $\hbar = 0$ and is called the *classical limit.* It is such that

$$\mathrm{cl}\left(\frac{1}{\hbar}[L, L']\right) = \{\mathrm{cl}(L), \mathrm{cl}(L')\}\,.$$

Obviously, if $\hat{\Lambda}$ is a graded Lie subalgebra, then $\mathrm{cl}(\hat{\Lambda})$ is a Poisson subalgebra in $\mathbb{C}[\![W]\!]$. Conversely, given a Poisson subalgebra $\Lambda \subset \mathbb{C}[\![W]\!]$, we may ask whether it can be quantized, *i.e.* whether there exists a graded Lie subalgebra $\hat{\Lambda} \subseteq \mathcal{D}_W^\hbar$ such that $\mathrm{cl}(\hat{\Lambda}) = \Lambda$. In this article, we will study the quantization of classical ∞-Airy structures in the following sense.

Definition 4.2.3. A quantum ∞-Airy structure on V is a linear map $\hat{\lambda} : V \to \mathcal{D}_W^\hbar$ such that $\mathrm{cl} \circ \hat{\lambda}$ is a classical ∞-Airy structure and the left ideal $\hat{\Lambda} = \mathcal{D}_W^\hbar \cdot \mathrm{Im}\,\hat{\lambda}$ is a graded Lie subalgebra. As before, if $\mathrm{Im}\,\hat{\lambda}$ is a subspace of the space of elements of degree at most r in $\mathcal{D}_W^\hbar$, we will call it a *quantum r-Airy structure* instead.

For $r = 2$, the condition that $\mathcal{D}_W^\hbar \cdot \mathrm{Im}\,\hat{\lambda}$ is a graded Lie subalgebra is equivalent to $\mathrm{Im}\,\hat{\lambda}$ being a graded Lie subalgebra, but this is no longer true for $r > 2$.

The partition function

The most important fact about quantum Airy structures is that they determine a partition function via a topological recursion once a polarization is chosen.

Definition 4.2.4. A *polarization* of a symplectic vector space (W, Ω) is a decomposition $W = V \oplus V'$ such that V and V' are Lagrangian.

If we are given a Lagrangian subspace V of W, a polarization is a choice of a transverse Lagrangian subspace V'. In this case, we say that the *polarization is adapted to V*.

If we choose a polarization, the symplectic pairing gives a canonical identification $V^* \cong V'$ and, therefore, an isomorphism $W \cong T^*V$ of symplectic vector spaces. Here $T^*V = V \oplus V^*$ is equipped with the natural symplectic form defined for $v \in V$ and $\phi \in V^*$ by $\Omega(v, \phi) = \phi(v)$. Therefore, the $\mathbb{C}[\![\hbar]\!]$ algebra $\mathcal{D}^\hbar_W$ acts faithfully on the space of functions on the formal neighborhood of 0 in V

$$\mathrm{Fun}^\hbar_V = \prod_{d \geq 0} \mathrm{Sym}^d(V^*)[\![\hbar]\!] \,.$$

Elements $v \in V$ act on $f \in \mathrm{Fun}^\hbar_V$ by derivation and $x \in V^*$ by multiplication by linear functions

$$v \cdot f = \hbar \partial_v f, \qquad x \cdot f = xf \,.$$

and the commutation relations in $\mathcal{D}^\hbar_W$ are represented by the Leibniz rule.

The space $\mathrm{Fun}^\hbar_V$ is in fact a graded associative algebra, where the grading is specified again by $\deg V^* = 1$ and $\deg \hbar = 2$. Besides, the action of $\mathcal{D}^\hbar_W$ respects the grading

$$(\mathcal{D}^\hbar_W)_d \cdot (\mathrm{Fun}^\hbar_V)_{d'} \subseteq (\mathrm{Fun}^\hbar_V)_{d+d'} \,.$$

Thanks to this grading, for any elements $L \in \mathcal{D}^\hbar_W$ and $F \in \mathrm{Fun}^\hbar_V$ it is possible to make sense of $e^{-F/\hbar}(L \cdot e^{F/\hbar})$ as an element of $\mathrm{Fun}^\hbar_V$.

Note that any $F \in \mathrm{Fun}^\hbar_V$ can be uniquely decomposed as

$$F = \sum_{g,n \geq 0} \frac{\hbar^g}{n!} F_{g,n}, \qquad F_{g,n} \in \mathrm{Sym}^n(V^*) \,.$$

Theorem 4.2.5 ([KS18], Theorem 2.4.2)**.** *Let $\hat{\lambda} : V \to \mathcal{D}^\hbar_W$ be a quantum ∞-Airy structure on V and choose a polarization of (W, Ω) adapted to the Lagrangian subspace $\mathcal{I}(V)$ of W (as given by the corresponding classical ∞-Airy structure). There exists a unique $F \in \mathrm{Fun}^\hbar_V$ such that*

- *(i)* $e^{-F/\hbar}(\hat{\lambda}(v) \cdot e^{F/\hbar}) = 0$ *for any* $v \in V$*,*
- *(ii)* $F_{g,0} = 0$ *for any* $g \geq 0$*,*
- *(iii)* $F_{0,1} = 0$ *and* $F_{0,2} = 0$*.*

The $F_{g,n}$ are usually called "amplitudes" or "correlation functions" or "free energies", and $Z = e^{F/\hbar}$ is called the "partition function". Conditions $(i) - (ii) - (iii)$ imply that the amplitudes are uniquely determined by a recursion on $2g - 2 + n > 0$. The recursive formula is spelled out in Corollary 4.2.16. The main feature to remember about this formula is that its terms are in correspondence with equivalence classes of excisions of embedded $S \mapsto \Sigma_{g,n}$ of smooth surfaces S of genus h with k ordered boundaries into a smooth surface $\Sigma_{g,n}$ of genus g with n ordered boundaries, such that the first boundary component of S coincides with the first boundary component of $\Sigma_{g,n}$. Here, two embeddings $S \mapsto \Sigma_{g,n}$ and $S' \mapsto \Sigma_{g,n}$ are considered equivalent if they are related by a diffeomorphism of $\Sigma_{g,n}$ preserving the ordering of the boundary components of $\Sigma_{g,n}$. Therefore, the number of equivalence classes is finite and they are characterized by the topology of $\Sigma_{g,n} - S$. This justifies the name *topological recursion*. In the special case of $(r = 2)$-Airy structures, the only terms appearing correspond to excisions of pairs of pants, *i.e.* $(h, k) = (0, 3)$. The topological recursion modeled on the excision of surfaces other than pairs of pants first appeared in [BHL+14].

Finite vs. countable dimension

We will encounter vector spaces V which are countable products of finite-dimensional vector spaces.

$$V = \prod_{p \geq 0} V_p \,.$$

This situation can be handled without difficulty in our discussions, by *defining* tensorial constructions relying on the unambiguous finite-dimensional tensorial constructions. For instance, we agree that the dual is

$$V^* = \bigoplus_{p \geq 0} V_p^* \,,$$

where $\bigoplus$ is the direct sum as opposed to the direct product $\prod$. Then, the cotangent space $T^*V = V \oplus V^*$ has a well-defined symplectic pairing. We define the tensor product as

$$V \otimes V' = \prod_{p \geq 0} (V \otimes V')_p, \qquad (V \otimes V')_p = \bigoplus_{q=0}^{p} V_q \otimes V'_{p-q} \,.$$

Classical versus quantum Airy structures

Due to Theorem 4.2.5, quantum Airy structures can be considered as initial data for the topological recursion. As many known examples show, the $F_{g,n}$s often have an interpretation in enumerative geometry or topological field theory, *i.e.* count surfaces of genus g with n punctures/boundaries in various instances. Another trend of applications (for $r = 2$) concerns the computation of WKB expansions of sections of holomorphic bundles on curves annihilated by a flat $\hbar$-connection. The beauty of the theory is that all these problems fit in the same universal scheme of the topological recursion. On the other hand, it is not an obvious task to construct quantum Airy structures.

Reversing the usual path from a problem to its solution, we think that it is worth searching for other constructions of quantum Airy structures as they would probably provide solutions to interesting geometric problems. In particular, it is appealing to look for constructions directly from the symplectic or Kähler geometry of manifolds and their Lagrangians. We certainly have the possible applications to the moduli space of flat connections on curves in mind.

We now point out that symplectic geometry easily gives rise to *classical* ∞-Airy structures. Consider for instance a real symplectic manifold (X, Ω_X), which can be assumed to be real-analytic without any loss of generality according to [KL00]. Let L_0 and L be two real-analytic Lagrangian subvarieties, which intersect and are tangent at a point $p \in M$. Take $W = T_pX$ with the symplectic form induced by $\Omega := \Omega_X|_p$, and $V = T_pL_0$. A suitable choice of Darboux coordinates gives an analytic isomorphism $f : U_X \to U_W$ from a neighborhood $U_X \subseteq X$ of p to a neighborhood $U_W \subseteq W$ of 0, preserving the symplectic structure, such that $f(p) = 0$ and $f(L_0 \cap U_X) = V \cap U_W$. By the inverse function theorem, upon taking smaller Us, there exists a linear map λ_{an} from V to the space of real-analytic functions on U_W, realizing L locally as the zero-locus of λ_{an}

$$f(L \cap U_X) = \{w \in U_W \quad | \quad \forall v \in V, \quad \lambda_{\mathrm{an}}(v)(w) = 0\}\,.$$

$\mathrm{Im}\,\lambda_{\mathrm{an}}$ generates the ideal of the ring of real-analytic functions on U_W which vanish on $f(L \cap U_X)$. In fact, since L is Lagrangian, this ideal is a Poisson subalgebra. The condition $T_p\Lambda_0 = T_pL$ implies that the linear map $V \to W$, which associates to $v \in V$ the unique $t_v \in W$ such that $\Omega(t_v, \cdot) = \mathrm{d}_0\lambda_{\mathrm{an}}(v)$, namely the differential of λ_{an} at $0 \in W$, is a Lagrangian embedding. Consequently, we obtain a classical ∞-Airy structure by taking $\lambda(v)$ to be the formal Taylor series at 0 of $\lambda_{\mathrm{an}}(v)$.

In the less frequent situation where there are Darboux coordinates such that L is cut out (locally around p) by polynomial equations of degree $\leq r$, we obtain a classical r-Airy structure.

It is not easy to exhibit a classical 2-Airy structure λ. In fact, this amounts (see [ABCO17]) to finding a collection of functions $(\lambda_i)_{i\in I}$ of the form

$$\lambda_i = y_i - \sum_{a,b\in I} \big(\tfrac{1}{2}A^i_{a,b}x_ax_b + B^i_{a,b}x_ay_b + \tfrac{1}{2}C^i_{a,b}y_ay_b\big), \qquad \{\lambda_i, \lambda_j\} = \sum_{a\in I}(B^i_{j,a} - B^j_{i,a})\lambda_a\,, \tag{4.5}$$

where $(x_i)_{i\in I}$ is a basis of linear coordinates on V, $(y_i)_{i\in I}$ the dual coordinates on V^* such that $\{x_i, y_j\} = \delta_{i,j}$ and $(A^*_{**}, B^*_{**}, C^*_{**})$ are scalars. The Poisson commutation relations impose an overdetermined system of linear and quadratic constraints on these scalars. However, once a classical 2-Airy structure has been found, it is fairly easy to describe its possible quantizations. Indeed, such a quantization λ_i must be of the form

$$\hat{\lambda}_i = \hbar\partial_{x_i} - \sum_{a,b\in I} \big(\tfrac{1}{2}A^i_{a,b}x_ax_b + B^i_{a,b}x_a\,\hbar\partial_{x_b} + \tfrac{1}{2}C^i_{a,b}\,\hbar^2\partial_{x_a}\partial_{x_b}\big) - \hbar\, D^i\,,$$

such that

$$[\hat{\lambda}_i, \hat{\lambda}_j] = \sum_{a\in I} \hbar(B^i_{j,a} - B^j_{i,a})\hat{\lambda}_a\,,$$

for some scalars D^*. The Lie algebra commutation relations are in fact equivalent to affine constraints for D^*. Note that the "quantum correction" D^* arises naturally from the ambiguity in the ordering of x and $\hbar\partial_x$ to quantize the B-terms in (4.5).

The previous example suggests that the difficulty in finding classical r-Airy structures decreases with r, and disappears for $r = \infty$. On the contrary, the difficulty of quantizing a given classical r-Airy structure is absent for $r = 2$, but increases with r. Indeed, one has to introduce an increasing number of quantum corrections which, for $r > 2$, must satisfy non-linear constraints in order to lift the Poisson subalgebra condition to a graded Lie subalgebra condition.

4.2.2 A computational approach

The basis-free definitions of quantum higher Airy structures given in Section 4.2.1 clarify the geometric context of our work. We are going to restart from scratch and give a roughly equivalent presentation of the setup using bases. It can be read independently of Section 4.2.1, some readers may find these more basic definitions easier to grasp, it facilitates the exposition and is closer to the notations of [KS18].

Basis-dependent definition

Let V be a $\mathbb{C}$-vector space[1]. We are going to assume that V has finite dimension D, but there is no difficulty in adapting it to the case of countably infinite dimension as in Section 4.2.1. Denoting $I = \{1, \ldots, D\}$, let $(y_l)_{l \in I}$ be a basis of V and $(x_l)_{l \in I}$ be the dual basis. We can think of ys as linear coordinates on V^* and xs as linear coordinates on V^*. Then $W = V \oplus V^*$ is equipped with the Poisson bracket

$$\forall l, m \in I, \qquad \{x_l, y_m\} = \delta_{l,m}, \qquad \{x_l, x_m\} = \{y_l, y_m\} = 0\,.$$

We identify $\mathcal{D}_W^\hbar \cong \mathbf{k}[\![\hbar, (x_l)_{l \in I}, (\hbar \partial_{x_l})_{l \in I}]\!]$ with the completed algebra of differential operators on V. We define an algebra grading by assigning

$$\deg x_l = \deg \hbar \partial_{x_l} = 1, \qquad \deg \hbar = 2\,. \tag{4.6}$$

Definition 4.2.6. A *higher quantum Airy structure* on V is a family of differential operators $(H_k)_{k \in I}$ of the form

$$H_k = \hbar \partial_{x_k} - P_k, \tag{4.7}$$

where $P_k \in \mathcal{D}_W^\hbar$ is a sum of terms of degree ≥ 2. Moreover, we require that the left $\mathcal{D}_W^\hbar$-ideal generated by the H_ks forms a graded Lie subalgebra, *i.e.* there

[1] The paper would be equally valid over a field of characteristic 0.

exists $g^{k_3}_{k_1,k_2} \in \mathcal{D}^{\hbar}_W$ such that

$$[H_{k_1}, H_{k_2}] = \hbar \sum_{k_3 \in I} g^{k_3}_{k_1,k_2} H_{k_3}. \tag{4.8}$$

This definition is a basis-dependent definition that should be compared with the basis-free Definition 4.2.3. As introduced there, we may define a *quantum r-Airy structure* as a higher quantum Airy structure such that all P_k only have terms of degree $\leq r$.

Remark 4.2.7. In the particular case where all the P_k are homogeneous of degree equal to 2, the $g^{k_3}_{k_1,k_2}$ must be scalars, and the H_k generate a graded Lie subalgebra. We then recover the standard definition of quantum Airy structures in [KS18].

Remark 4.2.8. It is easy to see the two distinctive properties of higher quantum Airy structures from the basis-dependent definition

1. The operators H_k have a very specific form. There are exactly D operators, and they all start with a linear term of the form $\hbar\partial_{x_k}$. This precise form is what is responsible for the uniqueness of the solution to the constraints $H_k \cdot Z = 0$, as we will see computationally by calculating the resulting topological recursion.

2. The operators satisfy the subalgebra property (4.8), which is crucial to ensure that a solution to the constraints $H_k \cdot Z = 0$ exists.

We can write down an explicit decomposition of the differential operators H_i in monomials. To simplify notation, and anticipating further interpretations, we introduce the operators

$$J_l = \hbar\partial_{x_l}, \qquad J_{-l} = l x_l \qquad l \in I. \tag{4.9}$$

We define a new index set $\mathcal{I} = \{-D, \ldots, -1, 1, \ldots, D\}$.

Let d_k (possibly ∞) be the maximal degree in H_k. We can decompose

$$H_k = J_k - \sum_{m=2}^{d_k} \sum_{\substack{\ell, j \geq 0 \\ \ell + 2j = m}} \frac{\hbar^j}{\ell!} \sum_{\alpha \in \mathcal{I}^\ell} C^{(j)}[k|\alpha] \!:\! J_{\alpha_1} \cdots J_{\alpha_\ell} \!:, \tag{4.10}$$

where $:\cdots:$ denotes normal ordering, *i.e.* all the J_i with negative is are on the left. The coefficients $C^{(j)}[k|\alpha]$ are fully symmetric under permutations of $\alpha = (\alpha_1, \ldots, \alpha_\ell)$. By convention, the product $: J_{\alpha_1} \cdots J_{\alpha_\ell} :$ is replaced by 1 when $\ell = 0$.

Remark 4.2.9. Note that in the quantization framework introduced in Section 4.2.1, the terms with $j = 0$ correspond to the quantization of classical terms with normal ordering, while the terms with $j > 0$ arise as quantum ordering ambiguities.

Example 4.2.10. To clarify the notation, let us compare with the notation used in [ABCO17] for quantum Airy structures, where $d_k = 2$ for all $k \in I$. In this case we have

$$\begin{aligned} H_k &= J_k - \tfrac{1}{2} \sum_{\alpha_1,\alpha_2 \in \mathcal{I}} C^{(0)}[k|\alpha_1,\alpha_2] : J_{\alpha_1} J_{\alpha_2} : + \hbar\, C^{(1)}[k|\emptyset] \\ &= \hbar\partial_{x_k} - \Bigg(\tfrac{1}{2} \sum_{\alpha_1,\alpha_2 \in I} C^{(0)}[k|-\alpha_1,-\alpha_2]\, \alpha_1\alpha_2 x_{\alpha_1} x_{\alpha_2} \\ &\quad + \sum_{\alpha_1,\alpha_2 \in I} C^{(0)}[k|-\alpha_1,\alpha_2]\, \alpha_1 x_{\alpha_1}\, \hbar\partial_{x_{\alpha_2}} \\ &\quad + \tfrac{1}{2} \sum_{\alpha_1,\alpha_2 \in I} C^{(0)}[k|\alpha_1,\alpha_2]\, \hbar^2 \partial_{x_{\alpha_1}} \partial_{x_{\alpha_2}} + \hbar C^{(1)}[k|\emptyset] \Bigg) . \end{aligned} \tag{4.11}$$

In the notation of [ABCO17], we recognize the tensors

$$\begin{aligned} C^{(0)}[k|-\alpha_1,-\alpha_2] &= \frac{A^k_{\alpha_1,\alpha_2}}{\alpha_1\alpha_2} , \\ C^{(0)}[k|-\alpha_1,\alpha_2] &= \frac{B^k_{\alpha_1,\alpha_2}}{\alpha_1} , \\ C^{(0)}[k|\alpha_1,\alpha_2] &= C^k_{\alpha_1,\alpha_2} , \\ C^{(1)}[k|\emptyset] &= D^k . \end{aligned}$$

To a higher quantum Airy structure, we can associate a partition function due to the following key result of Kontsevich and Soibelman (see Theorem 4.2.5).

Theorem 4.2.11. *[KS18, Theorem 2.4.2] Given a higher quantum Airy structure $(H_k)_{k\in I}$, the system of equations*

$$\forall k \in I, \qquad H_k \cdot Z = 0 ,$$

has a unique solution of the form

$$Z = \exp\left(\sum_{\substack{g\geq 0,\ n\geq 1\\ 2g-2+n>0}} \frac{\hbar^{g-1}}{n!} F_{g,n}\right), \qquad F_{g,n} \in \mathrm{Sym}^n V^* . \tag{4.12}$$

The existence of partition functions associated with higher quantum Airy structures and the fact that they often have enumerative geometric interpretations (see Section 4.6) is essentially the reason why quantum Airy structures are interesting. We can decompose

$$F_{g,n} = \sum_{\alpha \in I^n} F_{g,n}[\alpha] x_{\alpha_1} \cdots x_{\alpha_n} ,$$

where $F_{g,n}[\alpha]$ is fully symmetric under permutations of $\alpha = (\alpha_1, \ldots, \alpha_n)$, and see the $F_{g,n}$ as generating series for the coefficients $F_{g,n}[\alpha]$, which are expected to have an interesting interpretation in enumerative geometry. By applying the differential operators H_k on Z, we can obtain the $F_{g,n}[\alpha]$ by induction on $2g - 2 + n > 0$, as we now show explicitly.

Recursive system

The set of constraints $H_k \cdot Z = 0$ can be turned into a recursive system for the $F_{g,n}[\alpha]$. Due to the specific form of the differential operators H_k (see Remark 4.2.8), this recursive system is always triangular. And because it is known that a solution to the constraints exists (see Theorem 4.2.11), it follows that the recursive system uniquely determines this solution.

Let us explicitly write down the recursive system satisfied by the $F_{g,n}[\alpha]$. Given a formal series f in $\hbar$, we introduce the notation $[\hbar^g]f$ to denote the coefficient of f of order g in $\hbar$.

Definition 4.2.12. For $\alpha \in \mathcal{I}^i$ and $\beta \in I^{n-1}$, we define

$$\Xi^{(i)}_{g,n}[\alpha|\beta] = [\hbar^g]\, \partial_{x_{\beta_1}} \cdots \partial_{x_{\beta_{n-1}}} \big(Z^{-1} : J_{\alpha_1} \cdots J_{\alpha_i} : Z\big)\Big|_{x=0} . \tag{4.13}$$

Notice that $\Xi^{(0)}_{g,n}[\emptyset|\beta] = \delta_{g,0}\delta_{n,1}$.

Then we have the following result.

Lemma 4.2.13. *The system of equations*

$$\forall k \in I, \qquad H_k \cdot Z = 0\,,$$

implies the following system of equations

$$\Xi^{(1)}_{g,n}[k|\beta] = \sum_{m=2}^{d_k} \sum_{\substack{\ell,j\geq 0 \\ \ell+2j=m}} \frac{1}{\ell!} \sum_{\alpha\in\mathcal{I}^\ell} C^{(j)}[k|\alpha]\,\Xi^{(\ell)}_{g-j,n}[\alpha|\beta]\,, \tag{4.14}$$

for all $\beta \in I^{n-1}$, $n \geq 0$, *all* $g \geq 0$, *and all* $k \in I$.

Proof. Apply the differential operator $\partial_{x_{\beta_2}} \cdots \partial_{x_{\beta_n}}$ to $Z^{-1}H_k \cdot Z = 0$, set all $(x_l)_{l\in I}$ to zero and pick the coefficient of order g in $\hbar$. □

We need some more notation. The coefficients $F_{g,n}[\alpha]$ were defined for $2g-2+n>0$ and $\alpha \in I^n$ in (4.12). We extend this definition to $\alpha \in \mathcal{I}^n$, by setting $F_{g,n}[\alpha] = 0$ whenever one of the α_l is negative. For $2g-2+n=0$, we introduce

$$F_{0,2}[\alpha_1,\alpha_2] = |\alpha_1|\delta_{\alpha_1,-\alpha_2}\,. \tag{4.15}$$

Let $\alpha \in \mathcal{I}^i$ and $\beta \in I^{n-1}$. The notation $\boldsymbol{\lambda} \vdash \alpha$ means that $\boldsymbol{\lambda}$ is a set partition of α, *i.e.* a set of $|\boldsymbol{\lambda}|$ non-empty subsets of α which are pairwise disjoint and whose union is α. We denote the elements (sets) of the partition $\boldsymbol{\lambda}$ generically by λ. A partition of β indexed by $\boldsymbol{\lambda}$ is a map $\boldsymbol{\mu} : \boldsymbol{\lambda} \to \mathfrak{P}(\beta)$ such that $(\mu_\lambda)_{\lambda\in\boldsymbol{\lambda}}$ are possibly empty, pairwise disjoint subsets of β whose union is β. We summarize this notion with the notation $\boldsymbol{\mu} \vdash_{\boldsymbol{\lambda}} \beta$.

Then we have the following result.

Lemma 4.2.14. *Let* $i,n \geq 1$. *For* $\alpha \in \mathcal{I}^i$ *and* $\beta \in I^{n-1}$, *we have*

$$\Xi^{(i)}_{g,n}[\alpha|\beta] = \sum_{\boldsymbol{\lambda}\vdash\alpha} \sum_{\substack{h\,:\,\boldsymbol{\lambda}\to\mathbb{N} \\ i+\sum_{\lambda\in\boldsymbol{\lambda}} h_\lambda = g+|\boldsymbol{\lambda}|}} \sum_{\boldsymbol{\mu}\vdash_{\boldsymbol{\lambda}}\beta}'' \left(\prod_{\lambda\in\boldsymbol{\lambda}} F_{h_\lambda,|\lambda|+|\mu_\lambda|}[\lambda,\mu_\lambda] \right), \tag{4.16}$$

where the double prime over the summation symbol means that terms with $h_\lambda = 0$, $|\mu_\lambda| = 0$ *and* $|\lambda| \leq 2$ *are excluded from the sum. In other words,* $F_{0,1}$ *does not appear in the sum, and* $F_{0,2}$ *only appears with* $|\lambda| = 1$ *and* $|\mu_\lambda| = 1$.

Proof. For $\alpha \in I^i$, *i.e.* all $\alpha_l > 0$, the identity is straightforward. It involves $F_{0,2}$ only via positive indices, therefore such terms are zero. When some of the α_l are negative, we remember that $J_{\alpha_l} = |\alpha_l| x_{|\alpha_l|}$. Thus one of the β_m must be $\beta_m = |\alpha_m|$, otherwise by definition (see (4.13)) the contribution would be zero. We can include these cases by introducing coefficients $F_{0,2}[\alpha_l, \beta_m]$ that are equal to $|\alpha_l|$ when $\beta_m = -\alpha_l$, and zero otherwise. This is precisely how we defined the $F_{0,2}$ coefficients in (4.15). Thus the formula remains valid with these cases included, as long as the condition enforced by the double primed summation is there. □

Example 4.2.15. To clarify the notation, let us write down explicitly what this expression looks like for $i = 1, 2, 3$.

$$\begin{aligned} \Xi^{(1)}_{g,n}[\alpha_1|\beta] &= F_{g,n}[\alpha_1, \beta]\,, \\ \Xi^{(2)}_{g,n}[\alpha_1, \alpha_2|\beta] &= F_{g-1,n+1}[\alpha_1, \alpha_2, \beta] + \sum''_{\substack{h_1+h_2=g \\ \mu_1 \sqcup \mu_2 = \beta}} F_{h_1, 1+|\mu_1|}[\alpha_1, \mu_1] F_{h_2, 1+|\mu_2|}[\alpha_2, \mu_2]\,. \end{aligned}$$

Note that the second line is not valid for $(g, n) = (1, 1)$, in which case $\Xi^{(2)}_{1,1}[\alpha_1, \alpha_2|\emptyset] = 0$ because of the double prime condition in the summation (*i.e.* $F_{0,2}[\alpha_1, \alpha_2]$ cannot appear).

Further,

$$\begin{aligned} &\Xi^{(3)}_{g,n}[\alpha_1, \alpha_2, \alpha_3|\beta] \\ = \;& F_{g-2,n+2}[\alpha_1, \alpha_2, \alpha_3, \beta] + \sum''_{\substack{h_1+h_2=g-1 \\ \mu_1 \sqcup \mu_2 = \beta}} \big(F_{h_1, 1+|\mu_1|}[\alpha_1, \mu_1] F_{h_2, 2+|\mu_2|}[\alpha_2, \alpha_3, \mu_2] \\ & + F_{h_1, 1+|\mu_1|}[\alpha_2, \mu_1] F_{h_2, 2+|\mu_2|}[\alpha_1, \alpha_3, \mu_2] + F_{h_1, 1+|\mu_1|}[\alpha_3, \mu_1] F_{h_2, 2+|\mu_2|}[\alpha_1, \alpha_2, \mu_2] \big) \\ & + \sum''_{\substack{h_1+h_2+h_3=g \\ \mu_1 \sqcup \mu_2 \sqcup \mu_3 = \beta}} F_{h_1, 1+|\mu_1|}[\alpha_1, \mu_1] F_{h_2, 1+|\mu_2|}[\alpha_2, \mu_2] F_{h_3, 1+|\mu_3|}[\alpha_3, \mu_3]\,. \end{aligned}$$

Substituting (4.16) back into (4.14), we get the following formula for the coefficients $F_{g,n}[\alpha]$.

Corollary 4.2.16. *For all $\beta \in I^{n-1}$ we have*

$$F_{g,n}[k,\beta] = \sum_{\substack{\ell,j\geq 0 \\ 2\leq \ell+2j\leq d_k}} \frac{1}{\ell!} \sum_{\alpha\in\mathcal{I}^{\ell}} C^{(j)}[k|\alpha] \sum_{\boldsymbol{\lambda}\vdash\alpha} \sum_{\substack{h\,:\,\boldsymbol{\lambda}\to\mathbb{N} \\ \ell+j+\sum_{\lambda\in\boldsymbol{\lambda}} h_\lambda = g+|\boldsymbol{\lambda}|}} \sideset{}{''}\sum_{\boldsymbol{\mu}\vdash_{\boldsymbol{\lambda}}\beta} \bigg(\prod_{\lambda\in\boldsymbol{\lambda}} F_{h_\lambda,|\lambda|+|\mu_\lambda|}[\lambda,\mu_\lambda] \bigg). \tag{4.17}$$

Let us now argue that Corollary 4.2.16 is a recursive system for the $F_{g,n}[\alpha]$. For each term in the right-hand side, using the constraints under the sums we get

$$\sum_{\lambda} \big(2h_\lambda - 2 + |\lambda| + |\mu_\lambda|\big) = 2\big(g + |\boldsymbol{\lambda}| - (\ell+j)\big) - 2|\boldsymbol{\lambda}| + \ell + n - 1 = (2g-2+n) + (1-\ell-2j)\,.$$

Since we have $\ell + 2j \geq 2$, we deduce that

$$\sum_{\lambda} \big(2h_\lambda - 2 + |\lambda| + |\mu_\lambda|\big) < 2g - 2 + n\,. \tag{4.18}$$

Since the $F_{0,1}$ terms are absent, all terms in the left-hand side of the inequality are non-negative, hence $2h_\lambda - 2 + |\lambda| + |\mu_\lambda| < 2g - 2 + n$ for each $\lambda \in \boldsymbol{\lambda}$. In other words, (4.17) is a recursion on $2g - 2 + n \geq 0$ determining uniquely $F_{g,n}$ starting from the value of $F_{0,2}$ given by (4.15). For instance, the formula gives for $2g - 2 + n = 1$

$$F_{0,3}[k,\beta_1,\beta_2] = \beta_1\beta_2\, C^{(0)}[k| - \beta_1, -\beta_2]\,, \tag{4.19}$$

$$F_{1,1}[k] = C^{(1)}[k|\emptyset]\,, \tag{4.20}$$

and for $2g - 2 + n = 2$

$$\begin{aligned} F_{0,4}[k,\beta_1,\beta_2,\beta_3] = {} & \beta_1\beta_2\beta_3\, C^{(0)}[k| - \beta_1, -\beta_2, -\beta_3] \\ & + \sum_{\alpha\in I} \big(\beta_1\, C^{(0)}[k| - \beta_1, \alpha]\, F_{0,3}[\alpha,\beta_2,\beta_3] \\ & + \beta_2\, C^{(0)}[k| - \beta_2, \alpha]\, F_{0,3}[\alpha,\beta_1,\beta_3] \\ & + \beta_3\, C^{(0)}[k| - \beta_3, \alpha]\, F_{0,3}[\alpha,\beta_1,\beta_2]\big)\,, \end{aligned} \tag{4.21}$$

and

$$\begin{aligned} F_{1,2}[k,\beta] = {} & \sum_{\alpha\in I} \beta\, C^{(0)}[k| - \beta, \alpha]\, F_{1,1}[\alpha] + \sum_{\alpha_1,\alpha_2\in I} \tfrac{1}{2}\, C^{(0)}[k|\alpha_1,\alpha_2]\, F_{0,3}[\beta,\alpha_1,\alpha_2] \\ & + \beta\, C^{(1)}[k| - \beta]\,. \end{aligned} \tag{4.22}$$

Remark 4.2.17. While (4.17) is recursive, it does not treat k and $\beta_1, \ldots, \beta_{n-1}$ in a symmetric fashion. In other words, it is not clear from (4.17) that the $F_{g,n}[k, \beta]$ thus constructed are fully symmetric. It could happen that no symmetric solution to (4.17) exists. That is, the recursive system does not justify the existence part of Theorem 4.2.11; it does however imply uniqueness if a solution exists. In fact symmetry cannot hold for general coefficients Cs. The graded subalgebra property of $(H_k)_{k \in I}$ – which implies nonlinear relations between the Cs – is essential in proving the existence of a solution Z to the constraints, which is equivalent to proving the existence of a symmetric solution to (4.17).

Reduction

In general a higher quantum Airy structure $(H_k)_{k \in I}$ may involve linear differential operators. In this section we argue that we can essentially get rid of the linear differential operators. Note that this section is not essential for the rest of the paper.

Let $(H_k)_{k \in I}$ be a higher quantum Airy structure. Assume that $I_{\mathrm{lin}} \subset I$ is such that $H_k = J_k$ for all $k \in I_{\mathrm{lin}}$. For any $k \in I$, we introduce the reduced differential operator $H_k|_{\mathrm{red}}$, which is obtained from H_k by formally setting $J_m = 0$ (in the normal-ordered expression for H_k) whenever $|m| \in I_{\mathrm{lin}}$. Note that $H_i|_{\mathrm{red}} = 0$ for all $i \in I_{\mathrm{lin}}$. We can think of the $H_k|_{\mathrm{red}}$ as differential operators on V or on its subspace

$$V_{\mathrm{red}} = \Big\{ x \in V \quad | \quad \forall m \in I_{\mathrm{lin}}, \;\; x_m = 0 \Big\}.$$

Lemma 4.2.18. *There exists a unique solution to the differential constraints $H_k|_{\mathrm{red}} Z = 0$. Moreover, the partition function Z, considered as a formal function on V, coincides with the unique solution to the differential constraints $H_k Z = 0$.*

In other words, if we are interested in calculating Z, we can forget about the linear differential constraints H_i for $i \in I_{\mathrm{lin}}$, and instead solve the reduced differential constraints $H_k|_{\mathrm{red}} Z = 0$ on V_{red}.

Proof. Let $\mathcal{J}$ be the left ideal generated by the H_k, and let Z be the unique solution to the differential constraints $H_k Z = 0$. It is straightforward to show inductively that for all $k \in I$, $H_k|_{\mathrm{red}} \in \mathcal{J}$. Thus, $H_k|_{\mathrm{red}} Z = 0$, and hence Z is also a solution to the reduced differential constraints.

To show that it is unique, we look at the form of the differential operators. First, we know that $H_k Z = J_k Z = 0$ for all $k \in I_{\mathrm{lin}}$, so Z does not depend on those x_k. It follows that Z depends on the same number of variables as the number of non-zero $H_k|_{\mathrm{red}}$. Moreover, it is clear that the non-zero $H_k|_{\mathrm{red}}$ satisfy the degree 1 condition of quantum higher Airy structures with respect to these variables. Together those imply that the differential constraints $H_k|_{\mathrm{red}} Z = 0$ uniquely reconstruct the coefficients $F_{g,n}[\alpha]$ of the partition function by topological recursion. It follows that the solution is unique. $\square$

What we have proven is that there always exists a unique solution to the reduced differential constraints $H_k|_{\mathrm{red}} Z = 0$, and that this solution coincides with the unique partition function of the higher quantum Airy structure $(H_k)_{k \in I}$. It is tempting to conclude that the $H_k|_{\mathrm{red}}$s thus also form a higher quantum Airy structure. But to claim that we would need to show that the left ideal generated by the reduced $H_k|_{\mathrm{red}}$ is a graded Lie subalgebra of the algebra of differential operators on V_{red}. While we expect this to be true and we prove it in a special case (Lemma 4.4.11), we do not have a complete proof of this fact currently.

4.2.3 Crosscapped Airy structures

A variant of the topological recursion involving $F_{g,n}$ for half-integer g is required in applications to large size expansions in β-matrix integrals [CE06] and to open intersection theory [Saf16, Ale17]. We can include this variant in the formalism of Airy structures by allowing half-integer powers of $\hbar$, *i.e.* a formal variable $\hbar^{1/2}$ of degree 1, as follows.

Definition 4.2.19. A *crosscapped higher quantum Airy structure* on a vector space V equipped with a basis of linear coordinates $(x_k)_{k \in I}$ is a family of differential operators indexed by $k \in I$ of the form $H_k = \hbar\,\partial_{x_k} - P_k$ where the terms in $P_k \in \mathcal{D}^{\hbar^{1/2}}_{T^*V}$ have degree ≥ 2. Moreover, we require that the left $\mathcal{D}^{\hbar^{1/2}}_{T^*V}$-ideal generated by the Hs forms a graded Lie subalgebra *i.e.* there exists $g^{k_3}_{k_1,k_2} \in \mathcal{D}^{\hbar^{1/2}}_{T^*V}$ such that

$$\forall k_1, k_2 \in I, \qquad [H_{k_1}, H_{k_2}] = \sum_{k_3 \in I} g^{k_3}_{k_1,k_2} H_{k_3}\,.$$

The degree condition means that we have a decomposition

$$H_k = J_k - \sum_{m \geq 2} \sum_{\substack{\ell, \jmath \geq 0 \\ \ell + \jmath = m}} \frac{\hbar^{\jmath/2}}{\ell!} \sum_{\alpha \in \mathcal{I}^\ell} C^{(\jmath/2)}[k|\alpha] \; : J_{\alpha_1} \cdots J_{\alpha_\ell} : .$$

Proposition 4.2.20. *Given a crosscapped higher quantum Airy structure* $(H_k)_{k \in I}$*, the system of equations*

$$\forall k \in I, \qquad H_k \cdot Z = 0\,, \tag{4.23}$$

has a unique solution of the form

$$Z = \exp\left(\sum_{\substack{g \in \mathbb{N}/2,\ n \geq 1 \\ 2g-2+n > 0}} \frac{\hbar^{g-1}}{n!} F_{g,n} \right), \qquad F_{g,n} \in \mathrm{Sym}^n(V^*)\,. \tag{4.24}$$

given by the recursive system (4.17) *where one allows half-integer genera.*

Proof. The proof of existence is a small adaptation of the proof of [KS18] and therefore omitted. To prove uniqueness, we repeat the arguments of Section 4.2.2 to show that (4.23) computes the $F_{g,n}$ inductively on $2g-2+n > 0$. In fact, this recursive system takes the form (4.17) except that j, g and h_λ can be nonnegative integers or half-integers (but note that $2g-2+n$ is always an integer). The condition (4.18) is still valid and implies, as there are no F_{g_0,n_0} with $2g_0 - 2 + n_0 < 0$ in (4.24), that this recursive system determines uniquely all $F_{g,n}$ from the value of $F_{0,2}$ specified by the convention (4.15). ☐

It is perhaps instructive to write down the value of $F_{g,n}$ given by the recursion. In fact this also gives for $g = 0$ a recursion on n, therefore the formula for the $F_{0,n}$ are the same as those of Section 4.2.2. Notice that $F_{1/2,1}$ is absent from (4.24). With $2g-2+n=1$, we have a new term $F_{1/2,2}$ while the formulae (4.19)-(4.20) remain unchanged

$$\begin{aligned} F_{0,3}[k,\beta_1,\beta_2] &= \beta_1\beta_2\, C^{(0)}[k|-\beta_1,-\beta_2]\,, \\ F_{1/2,2}[k,\beta] &= \beta\, C^{(1/2)}[k|-\beta]\,, \\ F_{1,1}[k] &= C^{(1)}[k|\emptyset]\,. \end{aligned}$$

With $2g-2+n=2$, $F_{1,2}$ receives a new contribution compared to (4.22) and we have two new terms with half-integer genus

$$\begin{aligned} F_{0,4}[k,\beta_1,\beta_2,\beta_3] &= \beta_1\beta_2\beta_3\, C^{(0)}[k|-\beta_1,-\beta_2,-\beta_3] \\ &+\sum_{\alpha\in I}\big(\beta_1\, C^{(0)}[k|-\beta_1,\alpha]\, F_{0,3}[\alpha,\beta_2,\beta_3] \\ &+\beta_2\, C^{(0)}[k|-\beta_2,\alpha]\, F_{0,3}[\alpha,\beta_1,\beta_3] \\ &+\beta_3\, C^{(0)}[k|-\beta_3,\alpha]\, F_{0,3}[\alpha,\beta_1,\beta_2]\big)\,, \end{aligned}$$

$$\begin{aligned} F_{1/2,3}[k,\beta_1,\beta_2] &= \beta_1\beta_2\, C^{(1/2)}[k|-\beta_1,-\beta_2]+\sum_{\alpha\in I} C^{(1/2)}[k|\alpha]\, F_{0,3}[\alpha,\beta_1,\beta_2] \\ &+\sum_{\alpha\in I}\big(\beta_1\, C^{(0)}[k|-\beta_1,\alpha]\, F_{1/2,2}[\alpha,\beta_2] \\ &+\beta_2\, C^{(0)}[k|-\beta_2,\alpha]\, F_{1/2,2}[\alpha,\beta_1]\big)\,, \end{aligned}$$

$$\begin{aligned} F_{1,2}[k,\beta] &= \beta\, C^{(1)}[k|-\beta]+\sum_{\alpha\in I}\beta\, C^{(0)}[k|-\beta,\alpha]\, F_{1,1}[\alpha] \\ &+\sum_{\alpha_1,\alpha_2\in I}\tfrac{1}{2}\, C^{(0)}[k|\alpha_1,\alpha_2]\, F_{0,3}[\beta,\alpha_1,\alpha_2]\,, \\ F_{3/2,1}[k] &= C^{(3/2)}[k|\emptyset]+\sum_{\alpha\in I}\big(C^{(1)}[k|\alpha]\, F_{1/2,1}[\alpha]+C^{(1/2)}[k|\alpha]\, F_{1,1}[\alpha]\big) \\ &+\sum_{\alpha_1,\alpha_2\in I}\tfrac{1}{2}\, C^{(0)}[k|\alpha_1,\alpha_2]\, F_{1/2,2}[\alpha_1,\alpha_2]\,. \end{aligned}$$

4.3 $\mathcal{W}$ algebras and twisted modules

Our main construction of higher quantum Airy structures will take the form of $\mathcal{W}$ constraints for some particular modules of $\mathcal{W}$ algebras. $\mathcal{W}$ algebras are vertex operator algebras (VOAs), and hence we introduce some terminology and notation about VOAs and modules over them.

We are primarily interested in Heisenberg VOAs and $\mathcal{W}$ algebras in this paper. From a conformal field theory point of view, $\mathcal{W}$ algebras arise as the algebra of modes when the CFT includes chiral primary fields of conformal weight > 2. Algebraically, they are certain "non-linear" extensions of the Virasoro algebra; the first examples were constructed in [Zam85].

To obtain higher quantum Airy structures we need to construct particular modules for these VOAs. Those will always be obtained by restriction of twisted modules of Heisenberg VOAs to $\mathcal{W}$ algebras. In order to construct a twisted module, we essentially construct fields that have fractional power expansions in formal variables. From the point of view of conformal field theories, these correspond to choosing a branch in the orbifold VOA.

In this section we introduce VOAs and twisted modules. Along the way we construct a number of interesting left ideals for the algebra of modes of $\mathcal{W}$ algebras that are graded Lie subalgebras. This will prove crucial in the next section to construct higher quantum Airy structures.

4.3.1 Vertex operator algebras

There are many references on this topic. We mostly follow the presentation of [BM13, Doy08, FLM89, FBZ01].

Definition 4.3.1. A *vertex operator algebra* (VOA) is a quadruple $(V, Y, |0\rangle, |w\rangle)$ such that

- V is a $\mathbb{Z}$-graded vector space (the *space of states*) $V = \oplus_{l\in\mathbb{Z}} V_l$ such that $V_l = 0$ for l sufficiently negative and $\dim V_l < \infty$ for all $l \in \mathbb{Z}$. If $|v\rangle \in V_l$, we say that the conformal weight of $|v\rangle$ is l.

- Y is a linear map (the *state-field correspondence*)

$$Y(\cdot, z) \;:\; \begin{array}{ccl} V & \longrightarrow & \mathrm{End}(V)[\![z, z^{-1}]\!] \\ |v\rangle & \longmapsto & Y(|v\rangle, z) = \sum_{l\in\mathbb{Z}} v_l z^{-l-1} \end{array} .$$

$Y(|v\rangle, z)$ is called the *vertex operator* (or *field*) associated to the state $|v\rangle$, and v_n its *modes*.

- $|0\rangle \in V$ is the *vacuum state*, which satisfies the vacuum property

$$Y(|0\rangle, z) = \mathrm{id}_V$$

and the creation property

$$\forall\, |v\rangle \in V, \qquad Y(|v\rangle, z)\,|0\rangle - |v\rangle \in zV[\![z]\!]\,.$$

- $|w\rangle \in V$ is the *conformal state*, which satisfies the truncation condition

$$\forall\, |v\rangle \in V, \qquad v_l\,|w\rangle = 0 \qquad \text{for } l \in \mathbb{Z} \text{ sufficiently positive,}$$

and the Virasoro algebra condition, which can be stated as follows. Let ω_n be the modes of $Y(|w\rangle, z)$, and define $L_l = \omega_{l+1}$. Then

$$[L_l, L_m] = (l-m)L_{l+m} + \mathfrak{c}\,\frac{l^3 - l}{12}\delta_{l+m,0}\mathrm{id}_V,$$

where $\mathfrak{c} \in \mathbb{C}$ is the central charge. Further, if $|v\rangle$ is homogeneous of conformal weight n, then $L_0\,|v\rangle = n\,|v\rangle$ and we have the derivation property

$$\forall\, |u\rangle \in V, \qquad Y\left(L_{-1}\,|u\rangle, z\right) = \frac{\mathrm{d}}{\mathrm{d}z}Y(|u\rangle, z).$$

- Finally, we have the axiom of *locality*. $\big(Y(|v\rangle, z)\big)_{v\in V}$ is a local family of fields; *i.e.*, for $|u\rangle, |v\rangle \in V$,

$$(z_1 - z_2)^{N_{u,v}}[Y(|u\rangle, z_1), Y(|v\rangle, z_2)] = 0 \qquad \text{for some } N_{u,v} \in \mathbb{Z}_+,$$

Although innocuous looking, this axiom gives the vertex operator algebra much of its structure. In particular, this is equivalent to the Jacobi identity/Borcherds identity.

We will often drop the $Y, |0\rangle$ and $|w\rangle$ in the definition of a VOA and merely denote it by the underlying space of states V. We note that the mode L_0 keeps track of the conformal weight of the states.

As the vertex algebra is (usually) non-commutative, we define the notion of normal ordering

Definition 4.3.2. We define the *normally ordered product* of two fields $Y(|u\rangle, z)$ and $Y(|v\rangle, z)$ as the following

$$:Y(|u\rangle, z)Y(|v\rangle, w):=Y(|u\rangle, z)_-Y(|v\rangle, w)+Y(|v\rangle, w)Y(|u\rangle, z)_+, \tag{4.25}$$

where we defined $Y(|w\rangle, z)_+ := \sum_{l>0} w_l z^{-l-1}$ and $Y(|w\rangle, z)_- := \sum_{l\leq 0} w_l z^{-l-1}$.

4.3.2 $\mathcal{W}(\mathfrak{g})$ algebras

There are various equivalent constructions of $\mathcal{W}$ algebras. They are defined as the semi-infinite cohomology of affine vertex algebras of level $k \in \mathbb{C}$ [FF90] associated to a Lie algebra $\mathfrak{g}$. For generic k, they are isomorphic to certain intersections of kernels of screening operators on free field/Heisenberg algebras [FF90, FBZ01, Gen17], and for the principal $\mathcal{W}$ algebras of simply-laced type there is also a coset realization [ACL18]. Both the coset and screening realizations admit a certain limit where the $\mathcal{W}$ algebra is described as an orbifold by the compact Lie group G of the Lie algebra $\mathfrak{g}$. This is the situation we are interested in. In this case, the $\mathcal{W}$ algebra is a subalgebra of the Heisenberg vertex algebra of rank equal to the rank of $\mathfrak{g}$. For $\mathcal{W}$ algebras of type $\mathfrak{gl}_{N+1}$, we can also use the quantum Miura transformation, which gives us explicit generators.

We now construct our first example of a VOA, the Heisenberg VOA. Then we explain the construction of $\mathcal{W}$ algebras as subalgebras of the Heisenberg VOA.

Heisenberg vertex operator algebras

Let L be a lattice of finite rank equipped with a symmetric non-degenerate bilinear form

$$\langle\cdot,\cdot\rangle \, : \, L \times L \to \mathbb{Z}$$

Define $\mathfrak{h} := L \otimes_{\mathbb{Z}} \mathbb{C}$. The bilinear form on L induces a bilinear form on $\mathfrak{h}$. We define the *Heisenberg Lie algebra* $\hat{\mathfrak{h}}$ as the affine Lie algebra

$$\hat{\mathfrak{h}} = \left(\bigoplus_{l\in\mathbb{Z}} \mathfrak{h} \otimes t^l \right) \oplus \mathbb{C}K\,, \tag{4.26}$$

with Lie bracket relations

$$[\xi_l, \eta_m] = \langle \xi, \eta \rangle \, l\delta_{l+m,0}K, \qquad \xi, \eta \in \mathfrak{h}, \quad l, m \in \mathbb{Z}\,, \tag{4.27}$$

$$[K, \hat{\mathfrak{h}}] = 0\,, \tag{4.28}$$

where we introduced the notation $\xi_l := \xi \otimes t^l$, $l \in \mathbb{Z}$ for any $\xi \in \mathfrak{h}$.

We define the Weyl algebra $\mathcal{H}_L$ as the universal enveloping algebra of $\hat{\mathfrak{h}}$ quotiented by the relation $K = 1$. We also define a class of modules over $\mathcal{H}_L$ called Fock modules as follows. For any $\lambda \in \mathfrak{h}$, define the Fock module $\mathcal{S}_\lambda$ as the $\mathcal{H}_L$-module generated by the vector $|\lambda\rangle$, such that for any $\xi \in \mathfrak{h}$,

$$\forall l > 0, \qquad \xi_l \,|\lambda\rangle = 0\,, \qquad \text{and} \qquad \xi_0 \,|\lambda\rangle = \langle \xi, \lambda \rangle \,|\lambda\rangle\ . \tag{4.29}$$

If we define $\mathcal{H}_L^-$ as the subalgebra of $\mathcal{H}_L$ generated by the negative elements $\{\xi_l \mid \xi \in \mathfrak{h},\ l < 0\}$, we have the isomorphism $\mathcal{S}_\lambda \cong \mathrm{Sym}(\mathcal{H}_L^-)\,|\lambda\rangle$ as vector spaces.

The Fock module $\mathcal{S}_0 \cong \mathrm{Sym}(\mathcal{H}_L^-)\,|0\rangle$ admits a vertex operator algebra structure, by which we mean that we can find a quadruple $(\mathcal{S}_0, Y, |0\rangle, |w\rangle)$, that satisfies the axioms of Definition 4.3.1. The vacuum state is $|0\rangle$. The state-field correspondence $Y(\cdot, z) : \mathcal{S}_0 \to \mathrm{End}(\mathcal{S}_0)[\![z, z^{-1}]\!]$ is defined as

$$Y(|0\rangle, z) = \mathrm{id}_{\mathcal{S}_0}\,, \tag{4.30}$$

$$\forall \xi \in \mathfrak{h}, \qquad Y(\xi_{-1}\,|0\rangle, z) = \sum_{l\in\mathbb{Z}} \xi_l z^{-l-1}\,. \tag{4.31}$$

States of the form $\xi^1_{-k_1} \cdots \xi^n_{-k_n}\,|0\rangle$ where $k_i > 0$ clearly span $\mathcal{S}_0$, and the state-field correspondence is defined as

$$Y(\xi^1_{-k_1} \cdots \xi^n_{-k_n}\,|0\rangle, z) = :\frac{1}{(k_1-1)!}\frac{\mathrm{d}^{k_1-1}}{\mathrm{d}z^{k_1-1}} Y(\xi^1_{-1}\,|0\rangle, z) \cdots \frac{1}{(k_n-1)!}\frac{\mathrm{d}^{k_n-1}}{\mathrm{d}z^{k_n-1}} Y(\xi^n_{-1}\,|0\rangle, z):. \tag{4.32}$$

Finally, if we pick an orthonormal basis $\bar{\xi}^1, \ldots, \bar{\xi}^d$ for $\mathfrak{h}$, we define the conformal vector $|\omega\rangle$ as

$$|\omega\rangle = \frac{1}{2} \sum_{i=1}^{d} \bar{\xi}^i_{-1} \bar{\xi}^i_{-1} |0\rangle \ . \tag{4.33}$$

Its modes form a Virasoro algebra with central charge $\mathfrak{c} = \dim \mathfrak{h} = \operatorname{rank} L$. It can be checked that those satisfy the axioms of a VOA.

Definition 4.3.3. We denote the *Heisenberg vertex operator algebra* associated to $\mathfrak{h}$ by $\mathcal{S}_0$.

Lattice vertex operator algebras

From the previous section, one can naturally define the lattice vertex operator algebra associated to L, which contains the Heisenberg VOA as a sub-VOA.

The underlying vector space of the lattice VOA is $\mathcal{V}_L := \bigoplus_{\lambda \in L} \mathcal{S}_\lambda$ (recall that $\mathcal{S}_\lambda$ are the Fock modules defined in the previous section). In particular $\mathcal{S}_0 \subset \mathcal{V}_L$, and we define the vacuum state $|0\rangle$ and the conformal state $|\omega\rangle$ as the ones for the Heisenberg VOA $\mathcal{S}_0$. The state-field correspondence defined earlier (4.30) also holds. It suffices to define the state-field correspondence for the states $|\lambda\rangle$. (The general prescription is obtained by taking normally ordered products as in (4.32).) We have[2]

$$\mathbf{V}_\lambda(z) := Y(|\lambda\rangle, z) = U_\lambda z^{\lambda_0} \exp\left(-\sum_{l<0} \frac{\lambda_l}{l} z^{-l}\right) \exp\left(-\sum_{l>0} \frac{\lambda_l}{l} z^{-l}\right),$$

where U_λ is a shift operator

$$U_\lambda |\nu\rangle = c_{\lambda,\nu} |\nu + \lambda\rangle \qquad \text{and} \qquad [U_\lambda, \lambda_n] = 0,\ n \neq 0\,,$$

and $c_{\lambda,\nu} \in \mathbb{C}^\times$ is a (essentially) unique 2-cocycle. We will also denote the state $|\lambda\rangle$ by e^λ.

Definition 4.3.4. We denote the *lattice vertex operator algebra* associated to the even lattice L by $\mathcal{V}_L$.

[2] $V_\lambda(z)$ is the standard notation for these operators, here we use bold letters not to confuse them with vector spaces of VOAs also denoted V elsewhere in the text.

If $L = Q$ is the root lattice of a simple simply-laced Lie algebra $\mathfrak{g}$ then $\mathcal{V}_Q$ is isomorphic to the simple affine vertex algebra of $\mathfrak{g}$ at level one and is also denoted by $L_1(\mathfrak{g})$.

The $\mathcal{W}(\mathfrak{g})$ algebras

A standard introduction to $\mathcal{W}$ algebras is [Ara17]. Let $\mathfrak{g}$ be a simple finite-dimensional Lie algebra. Then to each embedding of $\mathfrak{sl}_2$ in $\mathfrak{g}$ one can associate the $\mathcal{W}$ algebra of $\mathfrak{g}$ at level $k \in \mathbb{C}$ via quantum Hamiltonian reduction from the affine vertex algebra of $\mathfrak{g}$ at level k. The best-known case is the one of the principal embedding of $\mathfrak{sl}_2$ in $\mathfrak{g}$, which we will simply denote by $\mathcal{W}^k(\mathfrak{g})$. Let now $\mathfrak{g}$ be simply-laced. In this case the principal $\mathcal{W}$ algebra can also be realized as a coset [ACL18, Main Theorem 2], that is, for generic k

$$\mathcal{W}^\ell(\mathfrak{g}) \cong (V_k(\mathfrak{g}) \otimes L_1(\mathfrak{g}))^{\mathfrak{g}[t]} , \qquad \ell = -h^\vee + \frac{k + h^\vee}{k + h^\vee + 1} ,$$

with $h^\vee$ the dual Coxeter number of $\mathfrak{g}$, and $V_k(\mathfrak{g})$ the universal affine vertex algebra of $\mathfrak{g}$ at level k and $L_1(\mathfrak{g})$ its simple quotient at level one. Let G be the compact Lie group whose Lie algebra is $\mathfrak{g}$. In the limit $k \to \infty$ this coset becomes just the G-orbifold of the lattice vertex algebra [CL19] and this is the case we are interested in

$$\mathcal{W}(\mathfrak{g}) := \mathcal{W}^{-h^\vee+1}(\mathfrak{g}) \cong L_1(\mathfrak{g})^G .$$

$\mathcal{W}^\ell(\mathfrak{g})$ and in particular $\mathcal{W}(\mathfrak{g})$ is strongly generated by elements W^i of conformal weights $d_i + 1$, where the d_i are the Dynkin exponents of $\mathfrak{g}$, see for example [FBZ01, Theorem 15.1.9]. For generic level it is also freely generated by these fields and the orbifold limit is always a generic point of a deformable family of vertex algebras by [CL19].

Remark 4.3.5. In summary, the principal $\mathcal{W}$ algebras form a one-parameter family of vertex algebras and we are interested in a very special point, namely the level for which the $\mathcal{W}$ algebra can be realized as a G-orbifold inside the lattice vertex algebra (for this $\mathfrak{g}$ needs to be simply-laced). This level is special for a second reason. $\mathcal{W}$ algebras enjoy Feigin-Frenkel duality [FF91] and our level is the self-dual case, *i.e.* $\mathcal{W}(\mathfrak{g})$ is its own Feigin-Frenkel dual.

For completeness we recall the definition of strong generators for a vertex operator algebra:

Definition 4.3.6. A vertex operator algebra V is said to be *strongly generated* by elements $(\gamma^i)_{i=1}^n$ in V if the underlying vector space V is spanned by

$$\gamma^1_{-k_1} \cdots \gamma^n_{-k_n} |0\rangle \ , \qquad \text{where } k_i > 0\,.$$

In addition, V is said to be *freely generated* if the above spanning set is a basis for the underlying vector space V.

Remark 4.3.7. If we know the state-field correspondence for the set of strong generators of a vertex operator algebra V, say $\gamma^i = \gamma^i_{-1} |0\rangle$, we can use the strong reconstruction theorem [FBZ01, Theorem 4.4.1] to determine the state-field correspondence for the states $\gamma^1_{-k_1} \cdots \gamma^n_{-k_n} |0\rangle$ where $k_i > 0$

$$Y(\gamma^1_{-k_1} \cdots \gamma^n_{-k_n} |0\rangle\,, z) = :\frac{1}{(k_1-1)!}\frac{\mathrm{d}^{k_1-1}}{\mathrm{d}z^{k_1-1}} Y(\gamma^1_{-1} |0\rangle\,, z) \cdots \frac{1}{(k_n-1)!}\frac{\mathrm{d}^{k_n-1}}{\mathrm{d}z^{k_n-1}} Y(\gamma^n_{-1} |0\rangle\,, z):. \tag{4.34}$$

Hence, we can interpret strong generation as the statement that all fields of the VOA can be obtained as linear combinations of normally ordered products of the fields $Y(\gamma^i, z)$ where $i \in \{1, \dots, n\}$ and their derivatives.

Examples

Let us now study some examples of $\mathcal{W}(\mathfrak{g})$ algebras.

Example 4.3.8. The algebra $\mathcal{W}(\mathfrak{sl}_2)$ is isomorphic to the Virasoro vertex algebra with central charge $\mathfrak{c} = 1$. It is well known that this VOA is strongly generated by a single vector of conformal weight 2.

Strictly speaking, we only defined $\mathcal{W}$ algebras for simple and simply-laced Lie algebras. It is straightforward to construct $\mathcal{W}$ algebras for direct sums of those. In particular, in the following we will study the algebra $\mathcal{W}(\mathfrak{gl}_{N+1}) := \mathcal{W}(\mathfrak{sl}_{N+1}) \otimes \mathcal{S}_0$, defined as the tensor product of $\mathcal{W}(\mathfrak{sl}_{N+1})$ and a rank one Heisenberg vertex algebra $\mathcal{S}_0$.

Example 4.3.9. The Lie algebra $\mathfrak{gl}_{N+1}$ is the algebra of $(N+1)\times(N+1)$ matrices over $\mathbb{C}$. Its Cartan subalgebra $\mathfrak{h}$ can be described as the subspace of diagonal matrices. We equip it with the basis $(\chi^i)_{i=0}^{N+1}$ where χ^i is the matrix element that has a 1 in the $(i+1)$th place on the diagonal and 0 elsewhere. The algebra $\mathcal{W}(\mathfrak{gl}_{N+1})$ with central charge $\mathfrak{c}=N+1$ is strongly freely generated by the following $N+1$ vectors in the Heisenberg VOA $\mathcal{S}_0$ associated to $\mathfrak{h}$

$$e_i(\chi^0_{-1},\ldots,\chi^N_{-1})\,|0\rangle \qquad i\in\{1,\ldots,N+1\}\,, \tag{4.35}$$

where the e_i denotes the i-th elementary symmetric polynomial. The proof of this statement follows immediately from the Miura transformation, see [AM17, Corollary 2.2] where we take the limit $\alpha\to 0$. The result is originally due to [FL88].

Example 4.3.10. The Lie algebra $D_N=\mathfrak{so}_{2N}$ is the Lie algebra of orthogonal $2N\times 2N$ matrices over $\mathbb{C}$. The roots of $\mathfrak{so}_{2N}$ can be described as $\pm\chi^i\pm\chi^j$ where $(\chi^i)_{i=1}^N$ is an orthonormal basis for the Cartan subalgebra $\mathbb{C}^N$. The following vectors in $\mathcal{S}_0$ strongly generate the algebra $\mathcal{W}(\mathfrak{so}_{2N})$ with central charge $\mathfrak{c}=N$.

$$\nu^d=\left(\sum_{i=1}^N \mathrm{e}^{\chi^i}_{-d}\mathrm{e}^{-\chi^i}_{-1}+\mathrm{e}^{-\chi^i}_{-d}\mathrm{e}^{\chi^i}_{-1}\right)|0\rangle \qquad d\in\{2,4,6,\ldots,2N-2\}\,, \tag{4.36}$$

$$\tilde{\nu}^N=\chi^1_{-1}\chi^2_{-1}\,\cdots\,\chi^N_{-1}\,|0\rangle\,. \tag{4.37}$$

The conformal weight of these vectors are $2,4,\ldots,2N-2$ and N, which are indeed the Dynkin exponents of $\mathfrak{so}_{2N}$. This statement follows from the results of [ACL18, CL19], *i.e.* from the description of $\mathcal{W}(\mathfrak{so}_{2N})$ as SO_{2N}-orbifold of the lattice vertex algebra of $\mathfrak{so}_{2N}$.

Remark 4.3.11. We note the important fact that $\mathcal{W}(\mathfrak{g})$ is invariant under G and hence under the action of the Weyl group of $\mathfrak{g}$. This remark will be fundamental, in our construction of higher Airy structures as $\mathcal{W}(\mathfrak{g})$-modules in Section 4.4.

4.3.3 The graded Lie subalgebra property

In this section we construct a number of left ideals for the algebra of modes of $\mathcal{W}$ algebras that are graded Lie subalgebras. This will be essential for the construction of higher quantum Airy structures from modules of $\mathcal{W}$ algebras in the next section.

Graded Lie subalgebras and left ideals

Let V be a vertex operator algebra with finitely many strong generators $V^1, \ldots, V^n$ and let $\mathcal{A}$ be its associative algebra of modes. We fix an order in the set of modes and by $L(\mathcal{A})$ we mean possibly infinite sums of ordered monomials in $\mathcal{A}$ of bounded degree and conformal weight, *i.e.* there exists a d and h such that each monomial appearing in the sum has at most degree d and conformal weight h. We assume that every polynomial in the modes is in $L(\mathcal{A})$ and so equipped with the commutator of modes $L(\mathcal{A})$ becomes a Lie algebra. This Lie algebra is graded by conformal weight, that is L_0-eigenvalue.

Our goal is to find Lie subalgebras of $L(\mathcal{A})$. We consider certain subsets S of the modes of the strong generators and show that the left ideal $\mathcal{A}.S$ generated by the modes in S is a graded Lie subalgebra of $L(\mathcal{A})$.

We can make this subalgebra property more explicit by introducing an ordering on the set of all modes (*i.e*, the underlying set of $\mathcal{A}$). We define an ordering such that a mode in S is always greater than a mode not in S. We say that elements of the ideal $\mathcal{A}.S$ are good with respect to S. In particular, γ is good if the right-most term of every ordered monomial of γ (expressed in terms of the strong generators) is in S.

The following lemma is clear.

Lemma 4.3.12. *The left ideal generated by the modes in S is a graded Lie subalgebra of $L(\mathcal{A})$ if and only if for any two modes $X, Y \in S$, one has that $[X, Y] \in L(\mathcal{A})$ is good with respect to S.*

The following subsections give examples in an increasing order of complexity. However the idea of construction is always the same. We are looking for a suitable module $\mathcal{M}_\lambda$ generated by a highest weight vector $|\lambda\rangle$ and such that this highest weight vector is annihilated by a mode if and only if this mode is in the set S of interest (*i.e.*, it is a good mode). It then remains to show that the commutator of two modes in S is still good and essentially this amounts to showing that a basis of $\mathcal{M}_\lambda$ is given by all the ordered monomials that are not good. We start with the case where $\mathcal{M}_\lambda$ is the vacuum of our vertex algebra.

The vacuum subalgebra $\mathcal{A}_{\geq 0}$

Our first subalgebra is the left ideal generated by all modes of the strong generators of a $\mathcal{W}$ algebra that kill the vacuum state $|0\rangle$.

Proposition 4.3.13. *Consider a vertex operator algebra V freely strongly generated by homogeneous states $\gamma^i \in V$ indexed by $i \in \mathcal{I}$ (where $\mathcal{I}$ is a finite set), with respective conformal weights $\Delta_i \in \mathbb{Z}$. Let $\mathcal{A}$ denote the associative algebra of modes of V. Consider the left $\mathcal{A}$-ideal $\mathcal{A}_{\geq 0}$ generated by γ^i_k for $i \in \mathcal{I}$ and $k \geq 0$. Then, $\mathcal{A}_{\geq 0}$ is a graded Lie subalgebra of $L(\mathcal{A})$. Equivalently, when $k, k' \geq 0$,*

$$[\gamma^i_k, \gamma^{i'}_{k'}] = \sum_{j=1}^{n} \sum_{p \geq 0} f^{(j,l)}_{(i,k),(i',k')} \gamma^j_p \tag{4.38}$$

for some $f^{(l,j)}_{(i,k),(i',k')} \in \mathcal{A}$.

Proof. We have the following commutation relations which follow from the locality axiom/Borcherds identity [FBZ01, Section 3.3.6]

$$[\gamma^i_k, \gamma^{i'}_{k'}] = \sum_{m \geq 0} \binom{k}{m} (\gamma^i_m \gamma^{i'})_{k+k'-m}\,, \tag{4.39}$$

where $k, k' \geq 0$.

The assumption on strong generation implies that we can express each $(\gamma^i_m \gamma^{i'})_{k+k'-m}$ as a finite linear combination of normal ordered monomials in the generators. Let us look at one of these normally ordered terms

$$\gamma^{b_1}_{p_1} \gamma^{b_2}_{p_2} \cdots \gamma^{b_L}_{p_L}. \tag{4.40}$$

This monomial could either annihilate the vacuum state $|0\rangle$ or not. Let us first consider the case where it does. The normal ordering prescription implies that the term furthest to the right, i.e. $\gamma^{b_L}_{p_L}$ annihilates the vacuum. In that case, we are done, as $\gamma^{b_L}_{p_L}$ is an element of $\mathcal{A}_{\geq 0}$.

Now, let us assume that the term (4.40) does not annihilate the vacuum $|0\rangle$. Then $p_L < 0$, and due to the normal ordering prescription, this implies that all the modes appearing in (4.40) are negative modes. We know that $\gamma^i_k |0\rangle = 0 = \gamma^{i'}_{k'} |0\rangle$ and hence $[\gamma^i_k, \gamma^{i'}_{k'}] |0\rangle = 0$. This means that $\gamma^{b_1}_{p_1}\gamma^{b_2}_{p_2} \cdots \gamma^{b_L}_{p_L} |0\rangle$ must cancel with some other terms (which are also normally ordered products of negative modes) in the sum on the right-hand side of (4.39) after acting on the vacuum state $|0\rangle$. However, this contradicts the assumption of free generation (which is that vectors of the form $\gamma^{b_1}_{p_1} \cdots \gamma^{b_L}_{p_L} |0\rangle$ where $p_i < 0$ form a basis for V), and hence cannot occur. □

The subalgebra $\mathcal{A}_\Delta$

We can now construct another interesting left ideal that is a graded Lie subalgebra of the Lie algebra $L(\mathcal{A})$. In this case, we consider all modes γ^i_k of the generators of a $\mathcal{W}$ algebra for $k \geq \Delta_i - 1$, where Δ_i is the conformal weight of γ^i. The construction is rather straightforward.

Proposition 4.3.14. *Consider a vertex operator algebra V strongly generated by homogeneous states $\gamma^i \in V$ indexed by $i \in \mathcal{I}$ where $\mathcal{I}$ is a finite set, with respective conformal weights $\Delta_i \in \mathbb{Z}$. Let $\mathcal{A}$ denote the associative algebra of modes. Then, the $\mathcal{A}$-ideal $\mathcal{A}_\Delta$ generated by γ^i_k for $i \in \mathcal{I}$ and $k \geq \Delta_i - 1$ is a graded Lie subalgebra of $L(\mathcal{A})$. Equivalently, for $k \geq \Delta_i - 1$ and $k' \geq \Delta_{i'} - 1$, we have*

$$[\gamma^i_k, \gamma^{i'}_{k'}] = \sum_{j=1}^{n} \sum_{p \geq \Delta_j - 1} f^{(j,l)}_{(i,k),(i',k')} \gamma^j_p \,. \tag{4.41}$$

Proof. Using the strong generation assumption, we can express the commutator (4.41) as sums of normally ordered monomials of the form

$$\gamma^{b_1}_{p_1}\gamma^{b_2}_{p_2} \cdots \gamma^{b_L}_{p_L}, \tag{4.42}$$

where $\sum_{i=1}^{L}(p_i - b_i + 1) = (k - \Delta_i + 1) + (k' - \Delta_{i'} + 1)$ due to the conformal weight condition. As $k \geq \Delta_i - 1$ and $k' \geq \Delta_{i'} - 1$, we get

$$\sum_{i=1}^{L} p_i \geq \sum_{i=1}^{L} (b_i - 1), \tag{4.43}$$

and hence at least one of the $p_i \geq b_i - 1$. Due to the normal ordering procedure, the last mode on the right $\gamma^{b_L}_{p_L}$ will have this property. This gives the statement of the Lemma. □

The intermediate subalgebras

In fact we can construct many more subalgebras as intermediate cases interpolating between $\mathcal{A}_{\geq 0}$ and $\mathcal{A}_{\Delta}$ for the $\mathcal{W}(\mathfrak{gl}_{N+1})$ algebras that we described in Example 4.3.9. The particular form of the strong generators, namely as elementary symmetric polynomials, is crucial for the construction.

In this subsection, we use a different convention for mode expansion of a field as we find it more convenient. We shift the index of the modes by the conformal weight, *i.e.*, when $|v\rangle$ has conformal weight Δ_v

$$Y(|v\rangle\,, z) = \sum_{l\in\mathbb{Z}} \mathsf{v}_l\, z^{-l-\Delta_v}\,. \tag{4.44}$$

The correspondence between the two ways of indexing is $\mathsf{v}_l = v_{l+\Delta_v-1}$.

Let us start with the setup. We aim to find one subalgebra in $\mathcal{W}(\mathfrak{gl}_{N+1})$ for each partition of $r := N+1$. So let $\lambda = (\lambda_1, \ldots, \lambda_p)$ be a fixed partition of r, that is the λ_i are positive integers such that $r = \lambda_1 + \lambda_2 + \cdots + \lambda_p$ and we order them by size, i.e. $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_p \geq 1$. Such a partition defines good modes as follows.

Definition 4.3.15. We say that W^a_{-m} is λ-good if $\lambda(a) - m > 0$ where

$$\lambda(a) := \min\{\, s \,|\, \lambda_1 + \cdots + \lambda_s \geq a\,\}\,.$$

Now fix a λ-order on $\{1, \ldots, r\} \times \mathbb{Z}$ with the following properties

1. $(a, -m) > (b, -n)$ if W^a_{-m} is λ-good but W^b_{-n} is not λ-good.
2. $(a, -m) > (b, -n)$ if W^a_{-m} and W^b_{-n} are λ-good and both $m, n \geq 0$ and $a < b$.
3. $(a, -m) > (b, -n)$ if W^a_{-m} and W^b_{-n} are λ-good and both $m, n \geq 0$ and $a = b$ and $m < n$.

Let $I = \{(a_1, -m_1) \geq (a_2, -m_2) \geq \ldots \geq (a_\ell, -m_\ell)\}$ be an ordered set. Then we say that

$$\mathsf{W}_I := \mathsf{W}^{a_\ell}_{-m_\ell} \cdots \mathsf{W}^{a_2}_{-m_2} \mathsf{W}^{a_1}_{-m_1}$$

is an ordered element of the universal enveloping algebra of modes. We define the λ-degree of a mode to be

$$\deg_\lambda(\mathsf{W}^a_{-m}) = \begin{cases} 2a-1 & \text{if } \lambda(a) \neq m \\ 2a & \text{if } \lambda(a) = m \end{cases},$$

and extend this definition to ordered monomials as the sum of the λ-degrees of the terms. The λ-degree of a ordered polynomial is then the maximal λ-degree of its ordered summands. Note that since W^a_{-m} is a polynomial of degree a in the modes of the Heisenberg vertex algebra it follows immediately that the λ-degree of any commutator $[\mathsf{W}^a_{-m}, \mathsf{W}^b_{-n}]$ is strictly smaller than $\deg_\lambda(\mathsf{W}^a_{-m}) + \deg_\lambda(\mathsf{W}^b_{-n})$.

We will call a λ-order simply an order whenever it is clear which λ we are using.

Theorem 4.3.16. *Let $\mathcal{A}$ be the mode algebra of $\mathcal{W}(\mathfrak{gl}_r)$ and λ a partition of r, then the algebra of λ-good modes forms a graded Lie subalgebra of the Lie algebra of modes. In addition, there exists a $\mathcal{W}(\mathfrak{gl}_r)$-module $\mathcal{M}_\lambda$ generated by a highest weight vector $|\lambda\rangle$ such that $\mathsf{W}^a_{-m}|\lambda\rangle = 0$ if and only if W^a_{-m} is a λ-good mode.*

Proof. We first consider the partition $\lambda = (r)$ of r. The corresponding λ-good modes are all non-negative modes $(\mathsf{W}^a_{-m})_{m\leq 0}$. Let ν be a generic weight of the rank r Heisenberg vertex algebra $\mathcal{S}_0$ so that via the embedding of $\mathcal{W}(\mathfrak{gl}_r)$ in $\mathcal{S}_0$ the Fock module $\mathcal{S}_\nu$ also becomes a $\mathcal{W}(\mathfrak{gl}_r)$-module. For generic weight ν this is a simple $\mathcal{W}(\mathfrak{gl}_r)$-module and so ordered words in the negative modes acting on the highest weight vector $|v_\nu\rangle$ of $\mathcal{S}_\nu$ form a basis of S_ν

$$\mathcal{S}_\nu = \operatorname{span}_\mathbb{C}\big(\mathsf{W}^{a_\ell}_{-m_\ell} \cdots \mathsf{W}^{a_1}_{-m_1}|v_\nu\rangle \;\big|\; (a_1, -m_1) \geq (a_2, -m_2) \geq \cdots \geq (a_\ell, -m_\ell), \\ m_l > 0 \text{ for } l = 1, \dots, \ell\big).$$

We now consider the vector space $\mathcal{M}$ with above graded PBW-basis but consider the weight as a variable so that $\mathcal{M}$ can be analytically continued to a module of $\mathcal{W}(\mathfrak{gl}_r)$ over the polynomial ring in r variables $\nu_1, \dots, \nu_r$. Here the ν_i are the eigenvalues of the zero-modes of N strong generators of the Heisenberg vertex algebra. Then specializing to any weight ν defines a new module $\mathcal{M}_\nu$. At generic ν this module will be simple while at special non-generic points it will be indecomposable but reducible. We generically have $\mathcal{M}_\nu \cong \mathcal{S}_\nu$ but for example $\mathcal{M}_0 \ncong \mathcal{S}_0$. Denote the highest weight vector of $\mathcal{M}_0$ by $|0\rangle$. By construction $\mathsf{W}^a_{-m}|0\rangle = 0$ if and only if W^a_{-m} is a λ-good mode. In order to prove that these λ-good modes form a graded Lie subalgebra of the algebra of modes we have to show that for any two λ-good modes W^a_{-m} and W^b_{-n} the commutator $[\mathsf{W}^a_{-m}, \mathsf{W}^b_{-n}]$ is an ordered polynomial in the modes and the right most term in each summand is λ-good. Consider an ordered set $I = \{(a_1, -m_1) \geq (a_2, -m_2) \geq \cdots \geq (a_\ell, -m_\ell)\}$ so that

$$\mathsf{W}_I := \mathsf{W}^{a_\ell}_{-m_\ell} \cdots \mathsf{W}^{a_2}_{-m_2} \mathsf{W}^{a_1}_{-m_1}$$

is an ordered element of the universal enveloping algebra of modes. We call W_I a λ-good monomial if $\mathsf{W}^{a_1}_{-m_1}$ is λ-good and say that the index set I is λ-good. The PBW-basis on $\mathcal{M}_0$ is then given by all $\mathsf{W}_I|0\rangle$ such that I is not a λ-good index set. It follows that

$$[\mathsf{W}^a_{-m}, \mathsf{W}^b_{-n}] = \sum_I c_I \mathsf{W}_I = \sum_{I\ \lambda\text{-good}} c_I \mathsf{W}_I + \sum_{I\ \text{not}\ \lambda\text{-good}} c_I W_I \,.$$

Acting on $|0\rangle$ and since all λ-good modes annihilate $|0\rangle$ we have

$$0 = \sum_{I\ \text{not}\ \lambda\text{-good}} c_I \mathsf{W}_I |0\rangle \ .$$

Since the $\mathsf{W}_I|0\rangle$ with I not a λ-good index set form a basis of $\mathcal{M}_0$ it follows that $c_I = 0$ for I not a λ-good index set. We thus have proven the claim for the partition $\lambda = (r)$. We note that this is precisely the result proved in Proposition 4.3.14.

The general case is not much different and can be reduced to this case. We prove it by induction for r. The base case $r = 1$ is trivial and just a special case of what we have just proven, since $\mathcal{W}(\mathfrak{gl}_1)$ is the rank one Heisenberg vertex algebra and the only partition of 1 is $\lambda = (1)$.

Let $r > 1$. The induction hypothesis is that the statement of the Theorem is true for all $r' < r$, i.e. for all partitions μ of $\mathcal{W}(\mathfrak{gl}_{r'})$ and in addition we require the existence of a $\mathcal{W}(\mathfrak{gl}_{r'})$-module $\mathcal{M}_\mu$ generated by a highest weight vector $|\mu\rangle$ that is annihilated by all μ-good modes and the $\mathsf{W}_I\,|\mu\rangle$ with I not μ-good form a basis of $\mathcal{M}_\mu$. With this notation the module $\mathcal{M}_0$ is also denoted by $\mathcal{M}_{(r')}$ and the highest weight vector $|0\rangle$ is denoted by $|(r')\rangle$.

Let $\lambda = (\lambda_1, \dots, \lambda_p)$ be a fixed partition of r, that is the λ_i are positive integers such that $N = \lambda_1 + \lambda_2 + \cdots + \lambda_p$ and we order them by size, i.e. $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_p \geq 1$. Further let $r' = r - \lambda_p$ so that $\mu = (\lambda_1, \dots, \lambda_{\ell-1})$ is a partition of r'. We consider the embedding $\mathcal{W}(\mathfrak{gl}_r)$ in $\mathcal{W}(\mathfrak{gl}_{r'}) \otimes \mathcal{W}(\mathfrak{gl}_{\lambda_p})$ and the module $\mathcal{M}_\mu \otimes \mathcal{M}_{(\lambda_p)}$. We want to prove that via this embedding as $\mathcal{W}(\mathfrak{gl}_r)$-modules $\mathcal{M}_\lambda \cong \mathcal{M}_\mu \otimes \mathcal{M}_{(\lambda_p)}$.

We denote the strong generators of $\mathcal{W}(\mathfrak{gl}_N)$ by $(\mathsf{W}^a)_{a=1}^r$, and the ones of $\mathcal{W}(\mathfrak{gl}_{r'}) \otimes \mathcal{W}(\mathfrak{gl}_{\lambda_p})$ by $(\mathsf{Z}^b)_{b=1}^{r'}$ and $(\mathsf{Y}^c)_{c=1}^{\lambda_p}$. Then due to the realization of the strong generators of the $\mathcal{W}$ algebras in terms of normally ordered elementary symmetric polynomials of Heisenberg vertex algebra fields we immediately have that

$$\mathsf{W}^a(z) = \mathsf{Z}^a(z) + \sum_{d=1}^{a-1} :\mathsf{Z}^{a-d}(z)\mathsf{Y}^d(z): + \mathsf{Y}^a(z)\,,$$

where we note that many of these terms on the right may not appear. For instance $\mathsf{Z}^a(z) = 0$ for $a > r'$ and $\mathsf{Y}^a(z) = 0$ for $a > \lambda_p$. Hence

$$\mathsf{W}^a_{-m} = \mathsf{Z}^a_{-m} \otimes 1 + \sum_{d=1}^{a-1} \sum_{n \in \mathbb{Z}} \mathsf{Z}^{a-d}_{n-m} \otimes \mathsf{Y}^d_{-n} + 1 \otimes \mathsf{Y}^a_{-m}\,,$$

and of course $\mathsf{Z}^a_{-m} = 0$ for $a > r'$ and $\mathsf{Y}^a = 0_{-m}$ for $a > \lambda_p$. The μ-degree of $\mathcal{W}(\mathfrak{gl}_{r'})$ lifts to a degree map on $\mathcal{W}(\mathfrak{gl}_{r'}) \otimes \mathcal{W}(\mathfrak{gl}_{\lambda_p})$ by saying that the Y^a_{-m} all have μ-degree zero. Let $|\lambda\rangle := |\mu\rangle \otimes |(\lambda_p)\rangle$. Then a straightforward verification tells us that

$$\mathsf{W}^a_{-m}\,|\lambda\rangle = 0 \qquad \text{if and only if } \mathsf{W}^a_{-m} \text{ is } \lambda\text{-good}\,.$$

In particular, if W^a_{-m} is not λ-good then its leading degree summand is $\mathsf{Z}^a_{-m} \otimes 1$ if $a \leq r'$ and $\mathsf{Z}^{r'}_{-\lambda(a)} \otimes \mathsf{Y}^{a-r'}_{-m+\lambda(a)}$ if $a > r'$. In either case the leading degree summand does not annihilate $|\lambda\rangle$ which is equivalent to saying that $\mathsf{Z}^a_{-m} \otimes 1$ if $a \leq r'$ is not μ-good and $\mathsf{Z}^{r'}_{-\lambda(a)}$ and $\mathsf{Y}^{a-r'}_{-m+\lambda(a)}$ are neither μ-good respectively (λ_p)-good if $a > r'$. Let $I = \{(a_1, -m_1) \geq (a_2, -m_2) \geq \cdots \geq (a_\ell, -m_\ell)\}$ be an ordered set with ordered monomial $\mathsf{W}_I := \mathsf{W}^{a_\ell}_{-m_\ell} \cdots \mathsf{W}^{a_2}_{-m_2} \mathsf{W}^{a_1}_{-m_1}$. Let s satisfy

$s = \ell$ if $a_\ell \leq r'$, $s = 0$ if $a_1 > r'$ and otherwise defined such that $a_s \leq r'$ but $a_{s+1} > r'$. It follows that the projection of W_I on leading μ-degree, which we denote by X_I, is

$$\mathsf{X}_I = \mathsf{Z}^{r'}_{-\lambda(a_\ell)} \cdots \mathsf{Z}^{r'}_{-\lambda(a_{s+1})} \mathsf{Z}^{a_s}_{-m_s} \cdots \mathsf{Z}^{a_2}_{-m_2} \mathsf{Z}^{a_1}_{-m_1} \otimes \mathsf{Y}^{a_\ell - r'}_{-m_\ell + \lambda(a_\ell)} \cdots \mathsf{Y}^{a_{s+1}-r'}_{-m_{s+1}+\lambda(a_{s+1})} .$$

Looking back at our requirements on the order of modes we see that the first factor is μ-ordered and the second one is (λ_p)-ordered. Consider a polynomial of type

$$\sum_{I \text{ not } \lambda\text{-good}} c_I \mathsf{W}_I .$$

Assume that it annihilates $|\lambda\rangle$. In particular, the leading μ-degree summands annihilate $|\lambda\rangle$ and hence

$$\begin{aligned} 0 &= \sum_{I \text{ not } \lambda\text{-good}} c_I \mathsf{X}_I \,|\lambda\rangle \\ &= \sum_{I \text{ not } \lambda\text{-good}} c_I \; \mathsf{Z}^{r'}_{-\lambda(a_\ell)} \cdots \mathsf{Z}^{r'}_{-\lambda(a_{s+1})} \mathsf{Z}^{a_s}_{-m_s} \cdots \mathsf{Z}^{a_2}_{-m_2} \mathsf{Z}^{a_1}_{-m_1} \,|\mu\rangle \\ &\qquad \otimes \mathsf{Y}^{a_\ell - r'}_{-m_\ell + \lambda(a_\ell)} \cdots \mathsf{Y}^{a_{s+1}-r'}_{-m_{s+1}+\lambda(a_{s+1})} \,|(\lambda_p)\rangle \ . \end{aligned}$$

By the induction hypothesis, non-good monomials acting on the highest weight vector form a basis of $\mathcal{M}_\mu$ respectively $\mathcal{M}_{(\lambda_p)}$ and hence all $c_I = 0$. We thus have constructed the claimed module $\mathcal{M}_\lambda$.

It is now easy to show that for any two λ-good modes W^a_{-m} and W^b_{-n} the commutator $[\mathsf{W}^a_{-m}, \mathsf{W}^b_{-n}]$ is an ordered polynomial in the modes and the right-most term in each summand is λ-good. We have

$$[\mathsf{W}^a_{-m}, \mathsf{W}^b_{-n}] = \sum_{I \ \lambda\text{-good}} c_I \mathsf{W}_I + \sum_{I \text{ not } \lambda\text{-good}} c_I \mathsf{W}_I .$$

Acting on $|\lambda\rangle$ and since all λ-good modes annihilate $|\lambda\rangle$ we have

$$0 = \sum_{I \text{ not } \lambda\text{-good}} c_I \mathsf{W}_I \,|\lambda\rangle \ .$$

Since we just proved that all the $\mathsf{W}_I\,|\lambda\rangle$ where I is not a λ-good index set form a basis of $\mathcal{M}_\lambda$, it follows that $c_I = 0$ for I not a λ-good index set. This finishes the proof of the Theorem. $\square$

4.3.4 Twisted modules

In preparation for the construction of higher quantum Airy structures in the next section, we now introduce twisted modules for the Heisenberg VOAs. Those will restrict to interesting modules for the $\mathcal{W}$ algebras realized as subalgebras of the Heisenberg VOAs.

Definitions

Let us define automorphisms of vertex operator algebras.

Definition 4.3.17. An automorphism σ, of finite order r, of a vertex operator algebra V is an automorphism $\sigma : V \to V$ on the (vector) space of states, with $\sigma^r = \mathrm{id}_V$, which preserves the vacuum state $|0\rangle$ and the conformal state $|w\rangle$, and such that for any $|v\rangle \in V$ it acts as

$$\sigma Y(|v\rangle, z)\sigma^{-1} = Y(\sigma |v\rangle, z)\,.$$

Given such an automorphism, we define the notion of a twisted module.

Definition 4.3.18. A *$\mathbb{Z}$-graded σ-twisted V-module W* is a $\mathbb{Z}$-graded vector space $W = \bigoplus_{l\in\mathbb{Z}} W_l$ such that $W_l = 0$ for l sufficiently negative and $\dim W_l < \infty$ for all $l \in \mathbb{Z}$, with a linear map

$$Y_\sigma(\cdot, z) : \begin{array}{ccl} V & \longrightarrow & \mathrm{End}(W)[\![z^{1/r}, z^{-1/r}]\!] \\ |v\rangle & \longmapsto & Y_\sigma(|v\rangle, z) = \sum_{l\in\frac{1}{r}\mathbb{Z}} v_l z^{-l-1} \end{array}\,.$$

We require that the vacuum property, creation property and the Virasoro algebra condition hold for W and $Y_\sigma(\cdot, z)$. In addition, we require the following conditions.

- The *monodromy* around $z = 0$ is given by the action of σ, namely if $\sigma|v\rangle = e^{2\mathbf{i}\pi q/r}|v\rangle$ we have

$$v_a = 0 \text{ unless } a \in q/r + \mathbb{Z}\,. \tag{4.45}$$

- $\big(Y_W(|v\rangle, z)\big)_{v\in V}$ is a *local family of fields*; *i.e.* for $|u\rangle, |v\rangle \in V$,

$$(z_1 - z_2)^{N_{u,v}}[Y_\sigma(|u\rangle, z_1), Y_\sigma(|v\rangle, z_2)] = 0 \qquad \text{for some } N_{u,v} \in \mathbb{Z}_+\,. \tag{4.46}$$

- We have a *product formula*

$$\frac{1}{k!}\frac{\mathrm{d}^k}{\mathrm{d}z_1^k}\Big\{(z_1-z_2)^N Y_\sigma(|u\rangle\,,z_1)Y_\sigma(|v\rangle\,,z_2)\,|w\rangle\Big\}\Big|_{z_1=z_2=z} = Y\big(u_{N-1-k}\,|v\rangle\,,z\big)\,|w\rangle \tag{4.47}$$

  for all $|u\rangle\,,|v\rangle \in V$, $|w\rangle \in W$ and where $N = N_{u,v}$ is chosen from the locality axiom.

In the above definition if we set $\sigma = \mathrm{id}$, we get the usual notion of (untwisted) modules. The idea of twisted modules is to introduce fields that have expansions in fractional powers of z. In physics this formalizes the notion of orbifold CFTs. Intuitively, we are working on the branched covering $z = \zeta^r$, by rewriting the fields as expansions in ζ (or fractional powers of z). However, we have to be careful about the normal ordering in this context. In physics terms, the operator product expansion (OPE) of the fields changes, and the product formula (4.47) captures this precisely. This product formula (and easy corollaries) will be very useful in our $\mathcal{W}$ algebra computations.

Remark 4.3.19. Note that since $\sigma(|\omega\rangle) = |\omega\rangle$, the conformal field has a mode expansion

$$Y_W(|\omega\rangle\,,z) = \sum_{l\in\mathbb{Z}} L_l\,z^{-l-1}. \tag{4.48}$$

with only integer powers of z.

Twisted modules of the Heisenberg VOA

Given an automorphism σ of $\mathfrak{h}$, we define a σ-twisted $\mathcal{S}_0$-module as follows. We define the σ-twisted Heisenberg Lie algebra $\hat{\mathfrak{h}}_\sigma$ and define a $\hat{\mathfrak{h}}_\sigma$-module called the twisted Fock module, denoted $\mathcal{T}$. The latter carries the structure of a σ-twisted module over the Heisenberg vertex operator algebra $\mathcal{S}_0$.

Here is the detailed construction. Let σ be an automorphism of the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ of finite order r

$$\langle\sigma(\xi),\sigma(\eta)\rangle = \langle\xi,\eta\rangle\,, \qquad \sigma^r = \mathrm{id}_{\mathfrak{h}}\,.$$

Any such automorphism lifts to an automorphism of $\mathcal{S}_0$ which we also denote by σ. We note that $\mathfrak{h}$ admits an orthonormal basis of eigenstates for the action of σ.

We extend the automorphism σ to $\mathfrak{h}[\![t^{1/r}, t^{-1/r}]\!] \oplus \mathbb{C}K$ as follows. Given $\xi \in \mathfrak{h}$, we use the notation $\xi_n := \xi \otimes t^n$ where $n \in \frac{1}{r}\mathbb{Z}$ as before. The action of σ is then

$$\sigma(\xi_l) = \sigma(\xi) \otimes e^{2\mathbf{i}\pi l} t^l, \qquad \sigma(K) = K, \qquad l \in \tfrac{1}{r}\mathbb{Z}.$$

The σ-twisted Heisenberg algebra is the subspace of σ-invariant elements

$$\hat{\mathfrak{h}}_\sigma := \left(\mathfrak{h}[\![t^{1/r}, t^{-1/r}]\!] \oplus \mathbb{C}K\right)^\sigma.$$

The algebra $\hat{\mathfrak{h}}_\sigma$ is generated by the elements ξ_l such that ξ is diagonal under the action of σ, and the central element K, with the following Lie bracket relations

$$[\xi_l, \eta_m] = l\delta_{l+m,0} \langle \xi, \eta \rangle K, \qquad [K, \hat{\mathfrak{h}}_\sigma] = 0. \tag{4.49}$$

We also introduce its negative part

$$\hat{\mathfrak{h}}_\sigma^- = \bigoplus_{l \in \frac{1}{r}\mathbb{Z}_{<0}} \mathfrak{h}_\sigma \otimes t^l.$$

Definition 4.3.20. Let $\mathcal{T} = \mathrm{Sym}(\mathfrak{h}_\sigma^-) |0\rangle$ be the $\hat{\mathfrak{h}}_\sigma$-module such that $K|0\rangle = |0\rangle$ and $\xi_l |0\rangle = 0$ for $\xi \in \mathfrak{h}$ and $l > 0$.

We would like to give $\mathcal{T}$ the structure of a σ-twisted module of the Heisenberg VOA $\mathcal{S}_0$ as follows. Let $\xi \in \mathfrak{h}$ be a diagonal element, *i.e.* $\sigma(\xi) = e^{-2\mathbf{i}\pi p}\xi$ for some $p \in \{0, \frac{1}{r}, \ldots, \frac{r-1}{r}\}$. Then the state-field correspondence for the module is defined as follows

$$\begin{aligned} Y_\sigma(|0\rangle, z) &= \mathrm{id}_{\mathcal{T}}, \\ Y_\sigma(\xi_{-1}|0\rangle, z) &= \sum_{n \in p+\mathbb{Z}} \xi_n z^{-n-1}. \end{aligned} \tag{4.50}$$

It is easy to check that this gives $\mathcal{T}$ the structure of a σ-twisted module over $\mathcal{S}_0$.

Remark 4.3.21. The state-field correspondence for general elements in $\mathcal{T}$ can be computed using the state-field correspondence for the states $\xi_{-1}|0\rangle$ (4.50) and the product formula for twisted modules (4.47).

4.3.5 Introducing $\hbar$

From now on, it is convenient to rescale the Killing form by some formal parameter $\hbar^{1/2}$, and base change to the field[3] $\mathbb{C}_{\hbar^{1/2}} := \mathbb{C}[\hbar^{-1/2}, \hbar^{1/2}]\!]$. In other words, we have a new Heisenberg VOA (still denoted $\mathcal{S}_0$) in which the commutation relations read

$$[\xi_l, \eta_m] = \hbar\, l\, \langle \xi, \eta \rangle\, \delta_{l+m,0}\,. \tag{4.51}$$

The reason to write $\hbar^{1/2}$ instead of $\hbar$ is to match with the convention (4.3) adopted in [KS18, ABCO17] for the partition functions of quantum Airy structures. The construction of Section 4.3.4 can still be applied to define a σ-twisted module again denoted $\mathcal{T}$. The only notable modification compared to the previous sections is that in Propositions 4.3.13 and 4.3.14 a factor of $\hbar$ appears in the right-hand side of the commutation relations, and that in the reconstruction of the state-field correspondence, one should include a factor of $\hbar^{1/2}$ per each ∂_z. In particular, the Lie subalgebras constructed in Section 4.3.3 become graded Lie subalgebras.

In all our examples except Section 4.4.2, only integer powers of $\hbar$ will remain the end of the day and we could effectively work with $\mathbb{C}_{\hbar} \subset \mathbb{C}_{\hbar^{1/2}}$.

[3]Here and in the following, we use the notation $\mathbb{C}[x^{-1}, x]\!]$ to denote Laurent series in x.

4.4 Higher quantum Airy structures from $\mathcal{W}$ algebras

This section gives a general prescription to produce higher quantum Airy structures starting with a Lie algebra $\mathfrak{g}$ and an element σ of the Weyl group of $\mathfrak{g}$.

1. We construct a σ-twisted module $\mathcal{T}$ of the Heisenberg VOA associated to the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$.

2. Upon restriction to the $\mathcal{W}$ algebra $\mathcal{W}(\mathfrak{g})$ (which is a sub-VOA of the Heisenberg VOA), the module becomes untwisted. The underlying vector space of $\mathcal{T}$ is the space of formal series in countably many variables, and elements of $\mathcal{W}(\mathfrak{g})$ act as differential operators (of order at most $\mathrm{rank}(\mathfrak{g})$) in those variables.

3. In Section 4.3.3, we constructed a number of ideals that are graded Lie subalgebras of the Lie algebra of modes. We pick one of these subalgebras from the algebra of modes of the $\mathcal{W}$ algebra module $\mathcal{T}$. These modes fulfill the second (and hardest to check) condition to be a higher quantum Airy structure.

4. A further conjugation of these modes (a.k.a dilaton shift) allows us to realize the first condition about degree 1 terms, thereby producing quantum $\mathrm{rank}(\mathfrak{g})$-Airy structures.

We apply this program in detail for $\mathfrak{gl}_{N+1}$ (type A_N) and $\mathfrak{so}_{2N}$ (type D_N) for different choices of the Weyl group element σ.

4.4.1 The $\mathcal{W}(\mathfrak{gl}_{N+1})$ Airy structures

The twisted module $\mathcal{T}$ for the Heisenberg VOA

Recall Example 4.3.9. The Cartan subalgebra $\mathfrak{h} \subset \mathfrak{gl}_{N+1}$ has a basis given by χ^i where $i \in \{0, \ldots, N\}$, with the following bilinear form

$$\langle \chi^i, \chi^j \rangle = \delta_{i,j}\,.$$

We shall focus on the automorphism σ of the Cartan subalgebra $\mathfrak{h}$ induced by the Coxeter element of the Weyl group $\mathfrak{S}_{N+1}$, namely

$$\chi^0 \longrightarrow \chi^1 \longrightarrow \cdots \longrightarrow \chi^N \longrightarrow \chi^0.$$

This automorphism has order

$$r = N+1\,.$$

We define a primitive r-th root of unity $\theta := e^{2\mathbf{i}\pi/r}$, which will appear throughout the section. Applying a discrete Fourier transform, we can define a basis $(v^a)_{a=0}^{r-1}$ of $\mathfrak{h}$ that is diagonal under the action of σ

$$v^a := \sum_{j=0}^{N} \theta^{-aj}\chi^j \qquad a \in \{0,\ldots,r-1\}\,. \tag{4.52}$$

Then we indeed have $\sigma(v^a) = \theta^a v^a$. Note that

$$\langle v^a, v^b\rangle = r\delta_{r|a+b}\,. \tag{4.53}$$

where the notation $\delta_{r|k}$ means 1 if k is divisible by r, and 0 otherwise. We observe that $v^a \otimes t^{-a/r-k-1}$ is invariant under σ for $k \in \mathbb{Z}$. Hence we can represent the $\mathcal{S}_0(\mathfrak{gl}_{N+1})$-twisted module

$$\mathcal{T}(\mathfrak{gl}_{N+1}) \cong \mathbb{C}_{\hbar^{1/2}}[x_1, x_2, x_3, \ldots]\,,$$

with the fields

$$v^a(z) := Y_\sigma\big(v^a_{-1}\,|0\rangle\,, z\big) = \sum_{k\in a/r+\mathbb{Z}} J_{rk}z^{-k-1}\,.$$

We also recall the differential operators defined in (4.9)

$$\forall l>0, \qquad J_l = \hbar\partial_{x_l}, \qquad J_{-l} = lx_l\,.$$

J_0 has not been defined before: we set it equal to a scalar $J_0 = Q$. The differential operators J_l satisfy the expected bracket relations

$$[J_{rk}, J_{rk'}] = \hbar\, r\delta_{r|a+b}\, k\delta_{k+k',0} = \hbar\,\langle v^a, v^b\rangle k\delta_{k+k',0} \qquad \text{for } k\in a/r{+}\mathbb{Z} \text{ and } k'\in a'/r{+}\mathbb{Z}\,,$$

therefore we do have an equivalent description of the twisted module introduced in Section 4.3.4. We also stress that the normal ordering of the modes carries over to this realization as the standard normal ordering on differential operators, with derivatives on the right and multiplication by variables on the left.

Via restriction, we can now consider $\mathcal{T}$ as a module for the subvertex algebra $\mathcal{W}(\mathfrak{gl}_{N+1}) \subset \mathcal{S}_0(\mathfrak{gl}_{N+1})$.

Remark 4.4.1. Even though $\mathcal{T}$ is not a twisted module for the subvertex algebra $\mathcal{W}(\mathfrak{gl}_{N+1})$, we will slightly abuse notation and still refer to the fields associated to the generators of $\mathcal{W}(\mathfrak{gl}_{N+1})$ as "twist fields". We will use the notation

$$\xi(z) := Y_\sigma\big(\xi_{-1}\,|0\rangle\,, z\big) \tag{4.54}$$

for the twist fields, where σ is the automorphism of the Heisenberg VOA used to construct the twisted module.

Computing the twist fields of the generators of $\mathcal{W}(\mathfrak{gl}_{N+1})$

From Example 4.3.9, we know that the elementary symmetric polynomials

$$e_i(\chi^0_{-1}, \ldots, \chi^N_{-1})\,|0\rangle \in \mathcal{S}_0 \qquad i \in \{1, \ldots, N+1\}$$

are a set of strong generators for $\mathcal{W}(\mathfrak{gl}_{N+1})$, and we are going to compute the modes of their twist fields.

Let us introduce some notation and prove some essential lemmas now. We first want to express the twist field corresponding to the state $v^{a_1}_{-1} v^{a_2}_{-1} \cdots v^{a_i}_{-1}\,|0\rangle$ in terms of the twist fields $v^l(z)$. If $A = (a_j)_{j=1}^i$ is a finite sequence, we use $\mathcal{P}(A)$ to denote the set of unordered, pairwise disjoint subsequences of length 2 of A. If $B \in \mathcal{P}(A)$, we use $|B|$ to denote the number of pairs appearing in B, and $A \setminus B$ the subsequence of A where one has removed the elements that appear in the pairs appearing in B. Of course $|B| \leq \lfloor i/2 \rfloor$. For instance, the elements B of $\mathcal{P}(a_1, a_2, a_3, a_4)$ such that $|B| = 2$ are

$$\{(a_1, a_2), (a_3, a_4)\}\,, \qquad \{(a_1, a_3), (a_2, a_4)\}\,, \qquad \{(a_1, a_4), (a_2, a_3)\}\,.$$

Lemma 4.4.2. *Let* $A = (a_k)_{k=0}^i$ *where* $a_k \in \{0, \dots, N\}$. *The twist field* $Y_\sigma(v_{-1}^{a_1} v_{-1}^{a_2} \cdots v_{-1}^{a_i} |0\rangle, z)$ *can be expressed as the following normally ordered product*

$$Y_\sigma\big(v_{-1}^{a_1} v_{-1}^{a_2} \cdots v_{-1}^{a_i} |0\rangle, z\big) = \sum_{B \in \mathcal{P}(A)} (\hbar z^{-2})^{|B|} \prod_{\{b_1, b_2\} \in B} \frac{b_1 b_2\, \delta_{b_1+b_2,r}}{2r} : \prod_{l \in A \setminus B} v^l(z) : . \tag{4.55}$$

Proof. This is an application of the product formula (4.47). If $i = 2$ in the above expression, we choose $N = 2$ in the product formula to get

$$Y_\sigma(v_{-1}^{a_1} v_{-1}^{a_2} |0\rangle, z) = \frac{1}{2} \frac{\mathrm{d}^2}{\mathrm{d}z_1^2} \Big\{ (z_1 - z_2)^2 v^{a_1}(z_1) v^{a_2}(z_2) \Big\} \Big|_{z_1 = z_2 = z}. \tag{4.56}$$

We compute the OPE of the twist fields, *i.e.* we express their product in terms of the normally ordered products

$$\begin{aligned} v^{a_1}(z_1) v^{a_2}(z_2) &= \,:v^{a_1}(z_1) v^{a_2}(z_2):\, + \sum_{\substack{k_1 \in a_1/r + \mathbb{Z} \\ k_2 \in a_2/r + \mathbb{Z}}} [v_{k_1}^{a_1}, v_{k_2}^{a_2}]\, z_1^{-k_1-1} z_2^{-k_2-1}\, \delta_{k_1>0} \delta_{k_2<0} \\ &= \,:v^{a_1}(z_1) v^{a_2}(z_2):\, + \sum_{\substack{k_1 \in a_1/r + \mathbb{Z} \\ k_1 > 0}} \hbar k_1 \langle v^{a_1}, v^{a_2} \rangle\, z_1^{-k_1-1} z_2^{k_1-1}, \end{aligned}$$

and the scalar product is given by (4.53). Notice that we can extend the sum to $k_1 = 0$. Let us write $k_1 = a_1/r + k_1'$ for $k_1' \in \mathbb{Z}$. The condition $k_1 \geq 0$ is then equivalent to $k_1' \geq 0$. We therefore obtain

$$\begin{aligned} v^{a_1}(z_1) v^{a_2}(z_2) &= \,:v^{a_1}(z_1) v^{a_2}(z_2):\, + \sum_{k_1' \geq 0} \hbar (\delta_{a_1,0} \delta_{a_2,0} + \delta_{a_1+a_2,r}) r (a_1/r + k_1')\, z_1^{-a_1/r-1-k_1'} z_2^{a_1/r+k_1'-1} \\ &= \,:v^{a_1}(z_1) v^{a_2}(z_2):\, + \hbar (\delta_{a_1,0} \delta_{a_2,0} + \delta_{a_1+a_2,r}) r\, \partial_{z_2} \left(\frac{(z_2/z_1)^{a_1/r}}{z_1 - z_2} \right). \end{aligned}$$

Inserting this result into (4.56), we get

$$\begin{aligned} Y_\sigma(v_{-1}^{a_1} v_{-1}^{a_2} |0\rangle, z) &= \,:v^{a_1}(z) v^{a_2}(z):\, + \hbar (\delta_{a_1,0} \delta_{a_2,0} + \delta_{a_1+a_2,r}) \frac{a_1(r - a_1)}{2rz^2} \\ &= \,:v^{a_1}(z) v^{a_2}(z):\, + \hbar\, \delta_{a_1+a_2,r} \frac{a_1 a_2}{2rz^2}. \end{aligned}$$

The general formula follows from an easy induction argument. $\square$

Definition 4.4.3. We introduce certain sums over r-th roots of unity which we encounter throughout our computations

$$\Psi^{(j)}(a_{2j+1},\dots,a_i) := \frac{1}{i!}\sum_{\substack{m_1,\dots,m_i=0\\ m_l\neq m_{l'}}}^{r-1}\left(\prod_{l'=1}^{j}\frac{\theta^{m_{2l'-1}+m_{2l'}}}{(\theta^{m_{2l'}}-\theta^{m_{2l'-1}})^2}\prod_{l=2j+1}^{i}\theta^{-m_l a_l}\right). \tag{4.57}$$

In the special case where $j=0$, we drop the (0) *i.e.* $\Psi(a_1,\dots,a_i) := \Psi^{(0)}(a_1,\dots,a_i)$.

Note that we prove several properties of these sums over roots of unity in Appendix B.1.

Definition 4.4.4. We introduce the twist fields

$$W^i(z) := r^{i-1}Y_\sigma\big(e_i(\chi^0_{-1},\dots,\chi^N_{-1})\,|0\rangle\,,z\big)\,,\qquad i\in\{1,\dots,r\}\,.$$

The scalar prefactor r^{i-1} is just a convenient normalization. Let us express the twist fields in terms of the Heisenberg twist fields $v^l(z)$.

Proposition 4.4.5. *We have for any* $i\in\{1,\dots,r\}$

$$W^i(z) = \frac{1}{r}\sum_{\substack{a_{2j+1},\dots,a_i=0\\ j\leq\lfloor i/2\rfloor}}^{r-1}\frac{i!}{2^j\,j!(i-2j)!}\Psi^{(j)}(a_{2j+1},\dots,a_i)(\hbar z^{-2})^j\ :\prod_{l=2j+1}^{i}v^{a_l}(z):.$$

Proof. We express the elementary symmetric polynomials $e_i(\chi^0,\dots,\chi^N)$ in terms of the basis $(v^i)_{i=0}^{r-1}$ of $\mathfrak{h}$. Inverting (4.52), we get

$$\chi^i = \frac{1}{r}\sum_{a=0}^{r-1}\theta^{-ia}v^a,$$

and plugging it into the expression for the elementary symmetric polynomials gives

$$e_i(\chi^0,\dots,\chi^N) = \frac{1}{r^i}\sum_{a_1,\dots,a_i=0}^{r-1}\Psi(-a_1,\dots,-a_i)\,v^{a_1}v^{a_2}\cdots v^{a_i}\,.$$

We observe that $\Psi(-a_1,\dots,-a_i)=\Psi(a_1,\dots,a_i)$. Now, we use Lemma 4.4.2

to compute the twist fields associated to $e_i(\chi^0_{-1},\ldots,\chi^N_{-1})\,|0\rangle$.

$$W^i(z) = \frac{1}{r}\sum_{a_1,\ldots,a_i=0}^{r-1} \Psi(a_1,\ldots,a_i) Y_\sigma\big(v^{a_1}_{-1}v^{a_2}_{-1}\cdots v^{a_i}_{-1}\,|0\rangle\,,z\big)$$
$$= \frac{1}{r}\sum_{a_1,\ldots,a_i=0}^{r-1} \Psi(a_1,\ldots,a_i) \sum_{B\in\mathcal{P}(a_1,\ldots,a_i)} (\hbar z^{-2})^{|B|} \prod_{\{b_1,b_2\}\in B} \frac{b_1 b_2\,\delta_{b_1+b_2,r}}{2r} \,:\prod_{l\in A\setminus B} v^l(z):. \tag{4.58}$$

We would like to separate the sums over the $2j$ indices appearing in the pairs and the others, for $j\in\{0,\ldots,\lfloor i/2\rfloor\}$. As A is an ordered set, we first need to identify the subset $J\subseteq\{1,\ldots,i\}$ of cardinality $|J|=2j$ which correspond the indices of elements of A that appear in B. For fixed j, there are $\frac{i!}{(2j)!(i-2j)!}$ such Js and the corresponding terms in the sum (4.58) are all equal. For instance, they are equal to the case $\{1,\ldots,2j\}$. B now corresponds to a choice of a pairing between elements of J. The sum over the values $a_k\in\{0,\ldots,r-1\}$ for $k\in J$ will not depend on the choice of pairing B. There are $(2j-1)!!$ such pairings. It is enough to consider the single pairing $B=\{(1,2),(3,4),\ldots,(2j-1,2j)\}$ provided we multiply our sums by

$$\frac{i!}{(2j)!(i-2j)!}\cdot(2j-1)!! = \frac{i!}{2^j j!(i-2j)!}\,.$$

Consequently,

$$W^i(z) = \frac{1}{r}\sum_{a_1,\ldots,a_i=0}^{r-1}\sum_{j=0}^{\lfloor i/2\rfloor} \frac{i!\,(\hbar z^{-2})^j}{2^j\,j!(i-2j)!}\,\Psi(a_1,\ldots,a_i)\prod_{l'=1}^{j}\frac{a_{2l'-1}a_{2l'}\,\delta_{a_{2l'-1}+a_{2l'},r}}{2r}\,:\prod_{l=2j+1}^{i} v^{a_l}:.$$

The claim follows by performing the sum over $a_1,\ldots,a_{2j}$ using Lemma B.1.1 proved in Appendix B.1. □

Definition 4.4.6. We define the modes W^i_k of the twist field $W^i(z)$ as

$$W^i(z) = \sum_{k\in\mathbb{Z}} W^i_k z^{-k-1}\,,$$

We observe that the expression for the modes W^i_k only involve integer powers of $\hbar$

We extract the expression for the modes from Proposition 4.4.5.

Corollary 4.4.7. *We have*

$$W_k^i = \frac{1}{r}\sum_{j=0}^{\lfloor i/2\rfloor} \frac{i!\,\hbar^j}{2^j\, j!(i-2j)!} \sum_{\substack{p_{2j+1},\dots,p_i\in\mathbb{Z}\\ \sum_l p_l = r(k-i+1)}} \Psi^{(j)}(p_{2j+1},p_{2j+2},\dots,p_i)\ :\prod_{l=2j+1}^{i} J_{p_l}:, \tag{4.59}$$

where for cases such that $j=i/2$ the condition $\sum_l p_l = r(k-i+1)$ is understood as the Kronecker delta condition $\delta_{k,i-1}$.

Proof. We start with Proposition 4.4.5 and compute the residue

$$\begin{aligned} W_k^i &= \frac{1}{r}\sum_{j=0}^{\lfloor i/2\rfloor}\sum_{a_{2j+1},\dots,a_i=0}^{r-1} \frac{i!\,\hbar^j}{2^j\, j!(i-2j)!}\Psi^{(j)}(a_{2j+1},\dots,a_i)\,\mathrm{Res}_{z=0}\left(\mathrm{d}z\, z^{k-2j}\ :\prod_{l=2j+1}^{i} v^{a_l}(z):\right)\\ &= \frac{1}{r}\sum_{j=0}^{\lfloor i/2\rfloor}\sum_{a_{2j+1},\dots,a_i=0}^{r-1} \frac{i!\,\hbar^j}{2^j\, j!(i-2j)!} \sum_{\substack{k_l\in a_l/r+\mathbb{Z}\\ \sum_l k_l = k-i+1}} \Psi^{(j)}(rk_{2j+1},\dots,rk_i)\ :\prod_{l=2j+1}^{i} J_{rb_l}:. \end{aligned}$$

To get to the second line, we used that $\Psi^{(j)}$ is a r-periodic function of each of its arguments, because they appear as powers of r-th roots of unity. Summing over $a_{2j+1},\dots,a_i$ amounts to summing over $p_l = rk_l\in\mathbb{Z}$ with the only constraint $\sum_l p_\beta = r(k-i+1)$. Note that in the case where $j=i/2$, the condition $\sum_l p_l = r(k-i+1)$ becomes the delta condition that $k=i-1$. ☐

It is easy to compute the $\Psi^{(j)}(a_{2j+1},\dots,a_i)$ for low values of i (see Lemma B.1.5). For instance, we have the linear and quadratic operators for $r\geq 2$

$$W_k^1 = J_{kr}\,, \tag{4.60}$$

$$W_k^2 = \frac{1}{2}\sum_{\substack{p_1,p_2\in\mathbb{Z}\\ p_1+p_2=r(k-1)}}\left(r\delta_{r|p_1}\delta_{r|p_2}-1\right)\ :J_{p_1}J_{p_2}: - \frac{(r^2-1)\hbar}{24}\delta_{k,1}\,. \tag{4.61}$$

For $r\geq 3$ we have the cubic operator

$$\begin{aligned} W_k^3 = {}& \frac{1}{6}\sum_{\substack{p_1,p_2,p_3\in\mathbb{Z}\\ p_1+p_2+p_3=r(k-2)}}\left(r^2\delta_{r|p_1}\delta_{r|p_2}\delta_{r|p_3} - r\delta_{r|p_1} - r\delta_{r|p_2} - r\delta_{r|p_3}+2\right)\ :J_{p_1}J_{p_2}J_{p_3}:\\ & -\frac{(r-2)(r^2-1)\hbar}{24}\,J_{r(k-2)}\,, \end{aligned} \tag{4.62}$$

and so on.

The higher quantum Airy structures

We are ready to prove one of our main results. As noted in Example 4.3.9, we know that the $\mathcal{W}(\mathfrak{gl}_{N+1})$ vertex algebra with central charge $\mathfrak{c} = N+1$ is strongly freely generated by the states $e_i(\chi^0_{-1}, \ldots, \chi^N_{-1})\,|0\rangle$. Thus we can use the construction of Section 4.3.3 to obtain a number of left ideals for the algebra of modes of the twist fields $W^i(z)$ that are graded Lie subalgebras. This gives us the second condition that is required to obtain a higher quantum Airy structure. For the first condition, we need to modify the modes W^i_k so as to create a term of degree 1 of the form J_p for some $p > 0$ – which acts as a derivation on $\mathcal{T}(\mathfrak{gl}_{N+1})$. This can be achieved via the following operation.

Definition 4.4.8. We define the *dilaton shift* as a conjugation of the differential operators W^i_k

$$H^i_k := \hat{T}_s W^i_k \hat{T}_s^{-1}, \qquad \hat{T}_s := \exp\Big(-\frac{J_s}{s\hbar}\Big)\,. \tag{4.63}$$

We note here that by the Baker-Campbell-Hausdorff formula, conjugating by $\hat{T}_s$ is equivalent to shifting $J_{-s} \to J_{-s} - 1$ in the modes W^i_k.

We then construct the following class of higher quantum Airy structures

Theorem 4.4.9. *Let $r \geq 2$, and $s \in \{1, \ldots, r+1\}$ be such that $r = \pm 1$ mod s. Let*

$$\mathfrak{d}^i := i - 1 - \Big\lfloor \frac{s(i-1)}{r} \Big\rfloor\,.$$

Assume $J_0 = Q = 0$. The family of differential operators

$$H^i_k = \hat{T}_s W^i_k \hat{T}_s^{-1} \qquad i \in \{1, \ldots, r\}, \qquad k \geq \mathfrak{d}^i + \delta_{i,1}\,, \tag{4.64}$$

forms a quantum r-Airy structure on the vector space $V = \bigoplus_{p>0} \mathbb{C}\langle x_p\rangle$ equipped with the basis of linear coordinates $(x_p)_{p>0}$.

Proof. We note that the W^i_k defined in (4.59) is a differential operator on $\mathbb{C}_{\hbar^{1/2}}[\![x_1, x_2, x_3, \ldots]\!]$ which is a linear combination of terms of degree $i + 2j$ for $j \in \{0, \ldots, \lfloor i/2 \rfloor\}$ using the notion of degree introduced in (4.6). We need to check the two conditions of Definition 4.2.6 for the differential operators H^i_k.

First, we note that the algebra of modes of a VOA-module has the same Lie algebraic structure as the modes of the VOA itself. Further, conjugating by $\hat{T}_s$ does not change the algebra of the modes. Then, the graded Lie subalgebra condition for higher quantum Airy structures follows directly from Section 4.3.3. In the case $s = r+1$, the indicated W_k^i form the graded Lie subalgebra $\mathcal{A}_{\geq 0}$; in the case $s = 1$, they form the graded Lie subalgebra $\mathcal{A}_{\geq 0}$; and for the remaining values of $1 < s < r$ such that $r \pm 1 = 0 \mod s$, we prove in Appendix B.2 that we get a partition of r (as in Section 4.3.3 – see Theorem 4.3.16), and they form a graded Lie subalgebra $\mathcal{A}_{(r,s)}$. The only subtlety here is that in these subalgebras, the mode $W_0^1 = Q$ is always present; since it is a scalar, to be part of a higher quantum Airy structure we must set $Q = 0$.

To check the second condition about the form of the operators H_k^i, we need to identify the terms of degree at most 1 in H_k^i. We take $s \in \mathbb{Z}$ arbitrary to start with. We first examine the terms of degree 0. A term $:\prod_{l=2j+1}^{i} J_{p_l}:$ will contribute if and only if $j = 0$ and $p_l = -s$ for all l. The constraint on the sum of p imposes $rk + is = 0$, which is not possible if $s < 0$ since we always have at least $k \geq 0$ and $i \geq 1$. Hence, we assume $s > 0$ in the remainder of the proof.

We turn to the terms of degree 1. Clearly, since $J_0 = Q = 0$, a term $:\prod_{l=2j+1}^{i} J_{p_l}:$ will contribute if and only if $j = 0$ and there is some l_0 such that for any $l \neq l_0$ we have $p_l = -s$. The constraint on the sum of ps imposes $p_{l_0} = rk + (s-r)(i-1)$. We therefore obtain using the r-periodicity of Ψ in each argument

$$H_k^i = \frac{i}{r}\,(-1)^{i-1}\,\Psi\big(\underbrace{-s,-s,\ldots,-s}_{i-1\text{ times}},(i-1)s\big)\;J_{rk+(s-r)(i-1)} + O(2)\,, \qquad (4.65)$$

where $O(2)$ indicates terms of degree ≥ 2. The prefactor involving Ψ is evaluated in Lemmas B.1.3 and B.1.4 and shown to be never zero. In particular, for s coprime to r, we get

$$H_k^i = J_{rk+(s-r)(i-1)} + O(2)\,.$$

Let us introduce the set $\mathcal{I}_{r,s} = \{(i,k) \mid 1 \leq i \leq r \text{ and } k \geq \mathfrak{d}^i + \delta_{i,1}\}$ and the map

$$\Pi_s : \begin{array}{ccc} \mathcal{I} & \longrightarrow & \mathbb{Z} \\ (i,k) & \longmapsto & rk + (s-r)(i-1) \end{array} . \tag{4.66}$$

We obtain a higher quantum Airy structure if Π_s is a bijection onto $\mathbb{Z}_+$, *i.e.* if each $J_p = \hbar\partial_{x_p}$ with $p > 0$ appears exactly in one operator H^i_k for $(i,k) \in \mathcal{I}_{r,s}$. It is easy to see that the non-empty fibers of Π_s have cardinality $d = \gcd(r,s)$. In other words, when r and s are not coprime, the same $\hbar\partial_{x_p}$ will appear as degree one term in two different operators H^k_i, which cannot happen in higher quantum Airy structure. Let us now assume that r and s are coprime, so that Π_s is injective. We can rewrite $rk + (s-r)(i-1) > 0$ as the condition $k > i - 1 - \frac{s}{r}(i-1)$. For $i = 1$, this is $k \geq 1$. For $i \geq 2$, since s is coprime with r and $2 \leq i \leq r$, it follows that $k > i - 1 - \frac{s}{r}(i-1)$ if and only if $k \geq i - 1 - \lfloor \frac{s(i-1)}{r} \rfloor$. Therefore $\Pi_s(\mathcal{I}_{r,s}) = \mathbb{Z}_+$.

□

From the last paragraph of the proof, we see that the first condition to be a higher quantum Airy structure restricts the allowed values of s to be positive integers that are coprime to r. The second condition to be a higher Airy structure, or equivalently the subalgebras of modes that we identified in Section 4.3.3, imposes the stronger constraint that $r = \pm 1 \mod s$.

Remark 4.4.10. For completeness, we compute $F_{0,3}$ and $F_{1,1}$ for all these higher quantum Airy structures in Appendix B.2. In fact, we do a little bit more; we calculate $F_{0,3}$ for all choices of s that are coprime with r. The result is that $F_{0,3}$ is indeed well defined and symmetric for $r = \pm 1 \mod s$, as expected; however, it cannot be symmetric when $r \neq \pm 1 \mod s$ (see Proposition B.2.2). In other words, when $r \neq \pm 1 \mod s$, the H^i_k cannot form a higher quantum Airy structure, since a solution Z to the differential constraints $H^i_k Z = 0$ does not exist. Given that for any s coprime with r the H_i have the right form to be a higher quantum Airy structure, it follows that the left ideal generated by the H_i is a graded Lie subalgebra if and only if $r = \pm 1 \mod s$.

Reduction to $\mathfrak{sl}_{N+1}$

The quantum r-Airy structures of Theorem 4.4.9 always contain $H^1_k = J_{kr}$ for $k > 0$. Hence their partition function Z is independent of the variables x_{kr} for $k > 0$. Let us define the reduced operators by the formula

$$\begin{aligned} W^i_k|_{\mathrm{red}} &= W^i_k|_{J_{kr}=0\ k\in\mathbb{Z}} \\ &= \frac{1}{r}\sum_{j=0}^{\lfloor i/2\rfloor} \frac{i!\,\hbar^j}{2^j\,j!\,(i-2j)!} \sum_{\substack{p_{2j+1},\ldots,p_i\in\mathbb{Z}\setminus r\mathbb{Z} \\ \sum_l p_l = r(k-i+1)}} \Psi^{(j)}(p_{2j+1},\ldots,p_i) :\prod_{l=2j+1}^{i} J_{p_l}: \end{aligned} \tag{4.67}$$

As the dilaton shift in Theorem 4.4.9 does not affect the modes indexed by k divisible by r, we also have

$$H^i_k|_{\mathrm{red}} = \hat{T}_s W^i_k|_{\mathrm{red}} \hat{T}_s^{-1} = H^i_k|_{J_{pr}=0\ p\in\mathbb{Z}}\,.$$

Although we do not know a general reason for $H^i_k|_{\mathrm{red}}$ to be a quantum Airy structure itself, for this particular case we can check that it is indeed the case. We also reprove Lemma 4.2.18 in this particular case.

Lemma 4.4.11. *Let us consider a quantum r-Airy structure from Theorem 4.4.9. Its partition function is equivalently characterized by the constraints $J_{kr}\cdot Z = 0$ for any $k > 0$ and*

$$J_{kr}\cdot Z = 0\ \ k>0, \qquad \text{and} \qquad H^i_k|_{\mathrm{red}}\cdot Z = 0, \qquad i\in\{2,\ldots,r\}, \qquad k \geq \mathfrak{d}^i + \delta_{i,1}\,. \tag{4.68}$$

Moreover, the family of operators $H^i_k|_{\mathrm{red}}$ indexed by $i \in \{2,\ldots,r\}$ and $k \geq \mathfrak{d}^i + \delta_{i,1}$ forms a quantum r-Airy structure on the vector space with basis of linear coordinates $(x_p)_{p\in\mathbb{N}\setminus r\mathbb{N}}$.

Proof. As a preliminary, we are going to show that H^i_k can be expressed solely in terms of the reduced operators. Since the dilaton shift does not affect the modes $(J_{kr})_{k\in\mathbb{Z}}$ it is enough to prove this property for W^i_k instead of H^i_k, and the result will follow by conjugation. We can always decompose

$$W^i_k = \sum_{\substack{\ell,m\geq 0 \\ \ell+m\leq i}} \sum_{\substack{a_1,\ldots,a_\ell>0 \\ b_1,\ldots,b_m>0}} J_{-ra_1}\cdots J_{-ra_\ell}\,\Upsilon^i_{k,\mathbf{a},\mathbf{b}}\,J_{rb_1}\cdots J_{rb_m}\,.$$

where the $\Upsilon^i_{k,\mathbf{a},\mathbf{b}}$ do not involve the modes J_{rp} for $l \in \mathbb{Z}$. Using the expressions (4.59) for the operators W^i_k, and using the r-periodicity of $\Psi^{(j)}$ with respect to any of its entries, we get:

$$\begin{aligned}\Upsilon^i_{k,\mathbf{a},\mathbf{b}} &= \frac{1}{r} \sum_{j=0}^{\lfloor (i-\ell-m)/2 \rfloor} \sum_{\substack{p_{2j+1},\ldots,p_{i-\ell-m} \in \mathbb{Z}\setminus r\mathbb{Z} \\ \sum_l p_l = r(k-i+1+\sum_l a_l - \sum_{l'} b_{l'})}} \frac{i!\,\hbar^j}{2^j\, j!(i-\ell-m-2j)!} \\ &\qquad \times \Psi^{(j)}(p_{2j+1},\ldots,p_{i-\ell-m},\underbrace{0,\ldots,0}_{\ell+m \text{ times}}) : \prod_{l=2j+1}^{i-\ell-m} J_{p_l} : .\end{aligned}$$

We also used that $J_{-m_l r}$ are always on the left (resp. $J_{n_{l'} r}$ are on the right) of a normal ordered expression, so we can remove them outside the normal ordering. The $\Psi^{(j)}$ with the 0s in them is evaluated using the Lemma B.1.2 proved in the Appendix to get

$$\begin{aligned}\Upsilon^i_{k,\mathbf{a},\mathbf{b}} &= \frac{1}{r} \frac{(r-i+\ell+m)!}{(r-i)!} \frac{(i-\ell-m)!}{i!} \\ &\quad \times \sum_{j=0}^{\lfloor (i-\ell-m)/2 \rfloor} \sum_{\substack{p_{2j+1},\ldots,p_{i-\ell-m} \in \mathbb{Z}\setminus r\mathbb{Z} \\ \sum_l p_l = r(k-i+1+\sum_l m_l - \sum_{l'} n_{l'})}} \frac{i!\,\Psi^{(j)}(p_{2j+1},\ldots,p_{i-\ell-m})}{2^j\, j!(i-\ell-m-2j)!} : \prod_{l=2j+1}^{i-\ell-m} J_{p_l} : ,\end{aligned}$$

and therefore

$$H^i_k = \sum_{\substack{\ell,m \geq 0 \\ \ell+m \leq i}} \sum_{\substack{a_1,\ldots,a_\ell > 0 \\ b_1,\ldots,b_m > 0}} \frac{(r-i+\ell+m)!}{(r-i)!} J_{-ra_1} \cdots J_{-ra_\ell} H^{i-\ell-m}_{k+\sum_l (a_l-1) - \sum_{l'}(b_{l'}+1)}\big|_{\mathrm{red}} J_{rb_1} \cdots J_{rb_m} . \tag{4.69}$$

Now consider the constraints $H^i_k \cdot Z = 0$ for $i \in \{1,\ldots,r\}$ and $k \geq \mathfrak{d}^i + \delta_{i,1}$. They contain $H^1_k \cdot Z = J_{kr} \cdot Z = 0$ for all $k > 0$ so the partition function is independent of x_{rm} for $m > 0$. As a result, for $i \geq 2$ and $k \geq \mathfrak{d}^i$, the coefficient of $x_{rb_1} \cdots x_{rb_m}$ in $H^i_k \cdot Z$ is proportional to $\Upsilon^i_{k,\mathbf{a},\emptyset} \cdot Z$ therefore to $H^{i-\ell}_{k+\sum_l (a_l-1)}|_{\mathrm{red}}$. Since $m_l > 0$, we get the family of constraints

$$H^i_k|_{\mathrm{red}} \cdot Z = 0, \qquad i \in \{2,\ldots,r\}, \qquad k \geq \mathfrak{d}^i . \tag{4.70}$$

Conversely, the constraints (4.70) together with $J_{rk} \cdot Z = 0$ for $k > 0$ imply, by reconstructing the linear combinations (4.69), that $H^i_k \cdot Z = 0$ for $i \in \{1,\ldots,r\}$ and $k \geq \mathfrak{d}^i + \delta_{i,1}$.

For the last statement, let

$$V = \bigoplus_{p>0} \mathbb{C}\langle x_p \rangle \qquad V_{\mathrm{red}} = \bigoplus_{p\in\mathbb{N}\setminus r\mathbb{N}} \mathbb{C}\langle x_p \rangle \,,$$

and consider the Weyl algebras $\mathcal{D}^{\hbar}_{T^*V_{\mathrm{red}}} \subset \mathcal{D}^{\hbar}_{T^*V}$ of differential operators on V_{red} and V. Let $\mathcal{J}^{+}$ be the graded subalgebra of $\mathcal{D}^{\hbar}_{T^*V}$ generated by the J_{pr} for $\pm p > 0$. We have a canonical decomposition

$$\mathcal{D}^{\hbar}_{T^*V} = \mathcal{J}^{-}\mathcal{D}^{\hbar}_{T^*V_{\mathrm{red}}}\mathcal{J}^{+} \,,$$

and a natural projection $\rho : \mathcal{D}^{\hbar}_{T^*V} \to \mathcal{D}^{\hbar}_{T^*V_{\mathrm{red}}}$. By definition

$$H^{i}_{k}|_{\mathrm{red}} = \rho(H^{i}_{k}) \,.$$

We denote $\mathcal{H}_{\mathrm{red}}$ – respectively $\mathcal{H}$ – the subspace spanned by $H^{i}_{k}|_{\mathrm{red}}$ for $i \in \{2, \ldots, r\}$, respectively $i \in \{1, \ldots, r\}$) – and $k \geq \mathfrak{d}^{i} + \delta_{i,1}$ over the field $\mathbb{C}[\![\hbar]\!]$. The graded Lie subalgebra condition for these H^{i}_{k} translates into

$$[\mathcal{H}, \mathcal{H}] = \hbar\, \mathcal{D}^{\hbar}_{T^*V} \cdot \mathcal{H} \,.$$

Let us apply the projection ρ to this equation. We get on the right-hand side $\hbar\, \mathcal{D}^{\hbar}_{T^*V} \cdot \mathcal{H}_{\mathrm{red}}$. On the left-hand side, we have to take into account that if $j_1^{\pm}, j_2^{\pm} \in \mathcal{J}^{\pm}$ and $h_1, h_2 \in \mathcal{D}^{\hbar}_{T^*V_{\mathrm{red}}}$

$$\big[j_1^{+}h_1j_1^{-}, j_2^{+}h_2j_2^{-}\big] - j_1^{+}\,[j_1^{-}, j_2^{+}]\, h_1h_2j_2^{-} - j_2^{+}[j_1^{+}, j_2^{-}]h_2h_1j_1^{-} = j_1^{+}j_2^{+}[h_1, h_2]j_1^{-}j_2^{-} \,.$$

After applying ρ we find a result of the form

$$\rho\big[j_1^{+}h_1j_1^{-}, j_2^{+}h_2j_2^{-}\big] - \hbar\, c\, h_1h_2 + \hbar\, c'\, h_1h_2 = [h_1, h_2] \,,$$

for some $c, c' \in \mathbb{C}[\![\hbar]\!]$. Therefore

$$[\mathcal{H}_{\mathrm{red}}, \mathcal{H}_{\mathrm{red}}] = \hbar\, \mathcal{D}_{T^*V_{\mathrm{red}}}\mathcal{H}_{\mathrm{red}} \,,$$

which proves that the ideal generated by $\mathcal{H}_{\mathrm{red}}$ is a graded Lie subalgebra. As it is already clear that for any $p \in \mathbb{N} \setminus r\mathbb{N}$ there exists a unique (k, i) such that $H^{i}_{k}|_{\mathrm{red}} = \hbar\, \partial_{x_p} + O(2)$, this proves the claim. $\square$

Arbitrary dilaton shifts and changes of polarization

In this subsection, we will construct deformations of the quantum r-Airy structures of Theorem 4.4.9, by exploring more general conjugations. Although this may seem superfluous, these examples will appear naturally in the next section when we study higher quantum Airy structures coming from general spectral curves.

We first introduce more general dilaton shifts. We would like to conjugate the modes W_k^i in (4.59) by an operator of the form

$$\hat{T} := \exp\left(\frac{1}{\hbar}\sum_{l>0}\frac{\tau_l}{l}J_l\right),$$

where τ_l are scalars. This simultaneously shifts $J_{-l} \to J_{-l} + \tau_l$ for all $l > 0$.

Proposition 4.4.12. *Let $r \geq 2$. Denote*

$$s := \min\{l > 0 \mid \tau_l \neq 0 \text{ and } r \nmid l\}, \qquad \mathfrak{d}^i := i - 1 - \left\lfloor \frac{s(i-1)}{r} \right\rfloor$$

and assume that $1 \leq s \leq r+1$ and $r = \pm 1 \mod s$. The family of differential operators

$$H_k^i = (-\tau_s)^{1-i}\,\hat{T}W_k^i\hat{T}^{-1} \qquad i \in \{1,\ldots,r\}, \qquad k \geq \mathfrak{d}^i + \delta_{i,1}, \tag{4.71}$$

forms a quantum r-Airy structure up to a change of basis of linear coordinates.

Proof. We need to show that the two conditions in Definition 4.2.6 are satisfied. Since $\tau_s \neq 0$, we define $\tilde{\tau}_q$ as

$$\tau_q = \tau_s(\delta_{s,q} + \tilde{\tau}_q),$$

so that $\tilde{\tau}_q = 0$ for $q \leq s$. We compute as in the proof of Theorem 4.4.9 that (up to rescaling by constants)

$$H_k^i = \frac{i}{r}(-1)^{i-1} \sum_{\substack{p\in\mathbb{Z},\ q_2,\ldots,q_i \geq s \\ p = r(k-i+1)+\sum_{l=2}^{i} q_l}} \Psi(-q_2,\ldots,-q_i,p)\left[\prod_{l=2}^{i}(\delta_{s,q_l} + \tilde{\tau}_{q_l})\right] J_p + O(2).$$

Therefore we can write

$$H_k^i = \sum_{p\geq s} L_{\Pi_s(i,k),p}\, J_p + O(2),$$

where Π_s was defined in (4.66) and is a bijection between the set of indices (i,k) considered in (4.71) and the set of positive integers. $(L_{a,b})_{a,b>0}$ is an upper triangular matrix with diagonal entries 1 (this value comes from Lemma B.1.4 for r and s coprime). Let us perform the change of basis on linear coordinates

$$y_b = \sum_{m\geq 0}(-1)^m \sum_{b=a_0>a_1>\ldots>a_{m-1}>a_m>0} \left[\prod_{l=0}^{m-1} L_{a_l,a_{l+1}}\right] x_{a_m}\,.$$

For any $b>0$ the right-hand side is well defined as it is a finite sum (using the upper-triangularity of L). By construction we have

$$H_k^i = \sum_{p\geq s} L_{\Pi_s(i,k),p}\,\hbar\partial_{x_p} + O(2) = \frac{\partial}{\partial y_{\Pi_s(i,k)}} + O(2)\,.$$

Notice that these expressions make sense using the prescriptions for vector spaces of countable dimension described in Section 4.2.1, *i.e.* ∂_{y_p} are elements of V and linear coordinates are elements of the dual. We therefore have checked the first condition of Definition 4.2.6.

The graded Lie subalgebra condition which holds for the operators of Theorem 4.4.9 is preserved after conjugation. Hence we obtain a higher quantum Airy structure. □

Another conjugation that will appear in the next section is the change of polarization. We would like to conjugate our modes with an operator of the form

$$\hat{\Phi} := \exp\left(\frac{1}{2\hbar}\sum_{l,m>0}\frac{\phi_{l,m}}{l\,m}\,J_l J_m\right),$$

where $\phi_{l,m}=\phi_{m,l}$ are scalars. Using the Baker-Campbell-Hausdorff formula, we see that it shifts the modes as

$$\forall a>0, \qquad J_{-a} \longrightarrow J_{-a} + \sum_{l>0}\frac{\phi_{a,l}}{l}J_l\,. \tag{4.72}$$

and leaves J_a invariant if $a>0$.

Proposition 4.4.13. *Under the same conditions as in Proposition 4.4.12, the family of differential operators*

$$H_k^i = (-\tau_s)^{1-i}\,\hat{\Phi}\,\hat{T}W_k^i\hat{T}^{-1}\,\hat{\Phi}^{-1}, \qquad i\in\{1,\ldots,r\}, \qquad k\geq \mathfrak{d}^i+\delta_{i,1}\,,$$

forms a quantum r-Airy structure up to a change of basis of linear coordinates.

Proof. The graded Lie subalgebra condition is stable under conjugation. We are going to argue that

$$\hat{\Phi}\,(\hat{T}W_k^i\hat{T}^{-1})\,\hat{\Phi}^{-1} = (\hat{T}W_k^i\hat{T}^{-1}) + O(2)\,. \tag{4.73}$$

This will automatically imply that the $(-\tau_s)^{1-i}\,\hat{\Phi}\hat{T}W_k^i\hat{T}^{-1}\hat{\Phi}^{-1}$ satisfy the first condition in Definition 4.2.6, hence form a quantum r-Airy structure.

We observe that the operation (4.72) respects the degree. It replaces J_{-l}s by J_ms. If the result is not normal ordered anymore, normal ordering creates a new term where two Js are replaced by a $\hbar$ (which is still of the same degree). As there is no term of degree 1 of the form J_{-l} with $l > 0$ in H_k^i we get the claimed (4.73). □

4.4.2 $\mathcal{W}(\mathfrak{gl}_{N+1})$ Airy structures for other automorphisms

The twisted module

We come back to the $\mathcal{W}(\mathfrak{gl}_{N+1})$ algebra, but now we construct twisted modules for an arbitrary automorphism σ, consisting of $d \geq 2$ disjoint cycles of order $r_1, \ldots, r_d$ which sum to $r := N+1$. We relabel the basis elements of $\mathfrak{h}$

$$\chi^{\mu,i} := \chi^{i-1+\sum_{\nu<\mu} r_\nu} \qquad \mu \in \{1,\ldots,d\}, \qquad i \in \{1,\ldots,r_\mu\}\,,$$

such that

$$\sigma(\chi^{\mu,i}) = \chi^{\mu,i+1 \bmod r_\mu}\,.$$

We then introduce the basis of eigenvectors indexed by $\mu \in \{1,\ldots,d\}$ and $a \in \{0,\ldots,r_\mu - 1\}$

$$v^{\mu,a} = \sum_{j=0}^{r_\mu-1} \theta_{r_\mu}^{-aj}\,\chi^{\mu,j}, \qquad \theta_{r_\mu} = e^{2\mathbf{i}\pi/r_\mu}\,.$$

which are diagonal under the σ action

$$\sigma(v^{\mu,a}) = \theta_{r_\mu}^a v^{\mu,a}, \qquad \langle v^{\mu,a}, v^{\nu,b}\rangle = \delta_{\mu,\nu}\,r_\mu\,\delta_{r_\mu|a+b}\,.$$

Hence we can represent the $\mathcal{S}_0(\mathfrak{gl}_{N+1})$-twisted module

$$\mathcal{T}(\mathfrak{gl}_{N+1}) \cong \mathbb{C}_{\hbar^{1/2}}[x_1^1, x_1^2, \ldots, x_1^d, x_2^1, x_2^2, \ldots, x_2^d, x_3^1, \ldots]\,,$$

with the fields

$$v^{\mu,a}(z) = \sum_{k \in a/r_\mu + \mathbb{Z}} J^\mu_{kr_\mu}\, z^{-k-1}\,,$$

and

$$J^\mu_l = \begin{cases} \hbar\, \partial_{x^\mu_l} & l > 0 \\ Q^\mu & l = 0 \\ -l\, x^\mu_{-l} & l < 0 \end{cases}\,.$$

where Q^μ now are arbitrary scalars. Upon restriction, it becomes an untwisted $\mathcal{W}(\mathfrak{gl}_{N+1})$-module.

Remark 4.4.14. In contrast to Section 4.4.1, we have the freedom to take $J^\mu_0 = Q^\mu \neq 0$ in our construction of Airy structures, provided Q^μ is equal to 0 modulo terms of positive degree. Scalars with this property in $\mathbb{C}_{\hbar^{1/2}}$ must be $O(2)$. Note that J_l for $l \neq 0$ are elements of $\mathcal{D}^\hbar_{T^*V}$ of degree 1. It is therefore natural to replace the base field with $\mathbb{C}_{\hbar^{1/2}}$ to allow scalars of degree 1. So we construct crosscapped Airy structures as defined in Section 4.2.3 rather than usual Airy structures.

We are going to compute the modes W^i_k of the twist fields associated to the strong generators of $\mathcal{W}(\mathfrak{gl}_{N+1})$. The result is expressed in terms of the modes $W^{\mu,i}_k$ of the $\mathcal{W}(\mathfrak{gl}_{r_\mu})$-module constructed in Section 4.4.1 via twisting by the Coxeter element of $\mathfrak{gl}_{r_\mu}$, which are according to Corollary 4.59

$$W^{\mu,i}_k = \frac{1}{r_\mu} \sum_{j=0}^{\lfloor i/2 \rfloor} \frac{i!\, \hbar^j}{2^j j! (i-2j)!} \sum_{\substack{p_{2j+1},\dots,p_i \in \mathbb{Z} \\ \sum_l p_l = r_\mu(k-i+1)}} \Psi^{(j)}_{r_\mu}(p_{2j+1}, p_{2j+2}, \dots, p_i)\ :\prod_{l=2j+1}^{i} J^\mu_{p_l}:\,,$$

where we have use the notation $\Psi^{(j)}_{r_\mu}$ for $\Psi^{(j)}$ to insist that we choose the r_μ-th roots of unity for its definition.

Lemma 4.4.15. *We have*

$$W^i_k = r^{i-1} \sum_{M \subseteq \{1,\dots,d\}} \sum_{\substack{1 \leq i_\mu \leq r_\mu\ \mu \in M \\ \sum_\mu i_\mu = i}} \sum_{\substack{\mathbf{k} \in \mathbb{Z}^M \\ \sum_\mu k_\mu = k+1-|M|}} \prod_{\mu \in M} \frac{1}{r_\mu^{i_\mu - 1}}\, W^{\mu, i_\mu}_{k_\mu}\,. \tag{4.74}$$

Proof. We express the strong generators of the $\mathcal{W}(\mathfrak{gl}_{N+1})$ by grouping the basis elements that belong to the same cycle σ together

$$e_i(\chi^0, \dots, \chi^{r-1}) = \sum_{\substack{i_1,\dots,i_d \geq 0 \\ \sum_\mu i_\mu = i}} \prod_{\mu=1}^{d} e_{i_\mu}(\chi^{\mu,1}, \dots, \chi^{\mu, r_\mu})\,.$$

Now we compute the fields associated to these generators in our twisted module. We note that the modes corresponding to different μ commute, and for each μ we recognize (up to the factor $r^{i_\mu-1}$) the fields associated with the $\mathfrak{gl}_{r_\mu}$ generators in Definitions 4.4.4-4.4.6. Therefore

$$W^i(z) = r^{i-1} \sum_{\substack{i_1,\ldots,i_d \geq 0 \\ \sum_\mu i_\mu = i}} \prod_{\mu=1}^{d} \frac{1}{r_\mu^{i_\mu-1}} \, W^{\mu,i_\mu}(z)\,,$$

where by convention $W^{\mu,0}(z) = 1$. Collecting the coefficient of z^{-k-1} entails the claim. $\square$

Higher quantum Airy structures

It seems rather tedious to find all the dilaton shifts of (4.74) that could lead to higher quantum Airy structures. Instead, we focus on the case σ is a cycle of length $r-1$, that is

$$r_1 = r-1, \qquad r_2 = 1\,.$$

According to Lemma 4.4.15, we have

$$W_k^1 = J_{r_1 k}^1 + J_k^2\,, \tag{4.75}$$

$$(r_1/r)^{i-1}\, W_k^i = W_k^{1,i} + \sum_{\substack{k_1,k_2\in\mathbb{Z} \\ k_1+k_2=k-1}} r_1\, W_{k_1}^{1,i-1}\, J_{k_2}^2 \qquad i \in \{2,\ldots,r_1\}\,, \tag{4.76}$$

$$(r_1/r)^{r-1}\, W_k^r = \sum_{\substack{k_1,k_2\in\mathbb{Z} \\ k_1+k_2=k-1}} r_1\, W_{k_1}^{1,r_1} J_{k_2}^2\,. \tag{4.77}$$

Let $s \in \{1,\ldots,r_1+1\}$ with s coprime with r_1. We perform a dilaton shift $J_{-s}^1 \to J_{-s}^1 - 1$, *i.e.* we define

$$H_k^i = (r_1/r)^{i-1} \exp\left(-\frac{J_s}{\hbar s}\right) W_k^i \exp\left(\frac{J_s}{\hbar s}\right). \tag{4.78}$$

Theorem 4.4.16. *Let $s \in \{1,\ldots,r\}$ such that $s|r$, and let*

$$\mathfrak{d}^i := i - 1 - \left\lfloor \frac{s(i-1)}{r-1} \right\rfloor.$$

Let $q \in \mathbb{C}$ and assume that $Q^1 = \hbar^{1/2} q = -Q^2$. The family of operators

$$\begin{array}{rcll} H_k^1 + H_{r-1-s+k}^r & = & J_{(r-1)k}^1 + O(2) & k \geq 1 \\ H_k^i & = & J_{(r-1)(k-i+1)+s(i-1)}^1 + O(2) & k \geq \mathfrak{d}^i, \qquad i \in \{2,\ldots,r-1\} \\ -H_k^r & = & J_{k+s-r+1}^2 + O(2) & k \geq (r-s) \end{array}.$$

forms a crosscapped higher quantum Airy structure on $\bigoplus_{p>0} \left(\mathbb{C}\langle x_p^1\rangle \oplus \mathbb{C}\langle x_p^2\rangle\right)$.

Proof. We first check the subalgebra property. In the case $s = r_1 + 1 = r$, the indicated W_k^i form the graded Lie subalgebra $\mathcal{A}_{\geq 0}$; in the case $s = 1$, they form the graded Lie subalgebra $\mathcal{A}_{\geq 0}$. For the remaining values of $1 < s < r_1$, we need to find for what values of (s, r_1) do we get a partition of r (as in Theorem 4.3.16). Using Proposition B.2.1, replacing $r \to r_1$, we know that the modes with $i \in \{1, \ldots, r_1\}$ generate an ideal that is a graded Lie subalgebra if $r_1 = \pm 1 \mod s$. What we need to check is that the left ideal generated by adding the modes $W_k^{r_1+1}$ with $k \geq r_1 + 1 - s$ is still a graded Lie subalgebra. For this to be the case, we need to show that the enlarged set of modes still correspond to a partition.

For $W_k^{r_1}$ and $s < r_1$, the condition is $k \geq r_1 - 1 - s + \lceil \frac{s}{r_1} \rceil = (r_1 - 1) - (s - 1)$. Moreover, for $W_k^{r_1+1}$ the condition that we want is $k \geq r_1 - (s-1)$. In the notation of Theorem 4.3.16, this means that we are adding one to the last part of the partition corresponding to the subalgebra generated by the modes with $1 \leq i \leq r_1$. This will remain an ordered partition only if all other parts of the original partition are at least one larger than the last part. Looking again at Proposition B.2.1, we see that this will be the case precisely when $r_1 = r's + r''$ with $r'' = s - 1$. In other words, $r_1 = -1 \mod s$, or equivalently $s|r$. Therefore we conclude that the left ideal generated by the modes W_k^i, with $i \in \{1, \ldots, r_1 + 1\}$ and satisfying the condition above, is a subalgebra if and only if $s|r$.

For all these cases, as the W_k^i satisfy the graded Lie subalgebra condition, so do the H_i^k with the same indices. Now as usual we need to be careful with zero modes. For $i = 1$ it is clear that

$$H_k^1 = W_k^1 = J_{r_1 k}^1 + J_k^2 \,. \tag{4.79}$$

The graded Lie subalgebras contain the mode H_0^1 which is equal to the scalar $Q^1 + Q^2$. We must assume $Q^1 + Q^2 = 0$ if we desire to have a higher quantum Airy structure. Due to the condition on the degrees, we fix $Q^1 = \hbar^{1/2} q = -Q^2$ for some $q \in \mathbb{C}$. We can drop the zero differential operator H_0^1 and deduce that H^i_k for $i \in \{1, \ldots, r\}$ and $k \geq \mathfrak{d}^i + \delta_{i,1} + \delta_{i,r}$ still satisfy the graded Lie subalgebra condition.

It remains to check that the degree 1 condition holds. We start by computing the result of the shift $J_{-s} \to J_{-s} - 1$ in $W^{1,i}_k$. Compared to Section 4.4.1, the fact that $Q^1 = \hbar^{1/2} q$ gives an extra term

$$\exp(-\partial_{x^1_s}) r^{-(i-1)} W^{1,i}_k \exp(\partial_{x^1_s}) = -\frac{\delta_{i,r_1}\, \delta_{r_1 s + r_1(k-i+1),0}}{r_1} + \frac{\hbar^{1/2} q}{r_1} \delta_{k,0} \delta_{i,1} + J^1_{r_1(k-i+1)+s(i-1)} + O(2)\,.$$

In particular for $i < r_1$ the degree 0 term (the first term) vanishes. We now consider $i \in \{2, \ldots, r_1\}$ and compute H^i_k modulo $O(2)$. We get from (4.76)

$$H^i_k = -\frac{\delta_{i,r_1}\, \delta_{s+k-i+1,0}}{r_1} + \; J^1_{r_1(k-i+1)+s(i-1)} + O(2)\,. \tag{4.80}$$

For $s \in \{1, \ldots r_1 + 1\}$ coprime with r_1 and $k \geq \mathfrak{d}^i$ we see that the degree 0 term in (4.80) is absent. Under these conditions, we have

$$H^i_k = \; J^1_{r_1 k + (s-r_1)(i-1)} + O(2)\,,$$

which involves a J^1_p with $p > 0$. Finally, we compute H^r_k modulo $O(2)$ from (4.77) and find

$$H^r_k = -J^2_{k+s-r_1} + O(2)\,.$$

For $s \in \{1, \ldots r_1 + 1\}$ coprime with r_1 and $k \geq r_1 + 1 - s$ we see that $k + s - r_1 \geq 1$, and hence the J appearing there is a derivation. We then see that

$$\begin{array}{rcll} H^1_k + H^r_{r_1 - s + k} & = & J^1_{r_1 k} + O(2) & k \geq 1 \\ H^i_k & = & J^1_{r_1(k-i+1)+s(i-1)} + O(2) & k \geq \mathfrak{d}^i, \qquad i \in \{2, \ldots, r_1\} \\ -H^r_k & = & J^2_{k+s-r_1} + O(2) & k \geq r_1 + 1 - s \end{array} \;.$$

forms a quantum r-Airy structure, which is the claim. ☐

We will discuss the enumerative geometry interpretation (through open intersection theory) of the associated partition function in Section 4.6.3. Note that we can easily formulate and prove an analog of Proposition 4.4.13 to describe more general higher quantum Airy structures obtained from the ones of Proposition 4.4.16 by further dilaton shifts and changes of polarization.

4.4.3 The $\mathcal{W}(\mathfrak{so}_{2N})$ Airy structures

The twisted module $\mathcal{T}$

Recall Example 4.3.10. The roots of the Lie algebra of D_N type can be described as $(\pm\chi_i \pm \chi_j)$ where χ_i is an orthonormal basis for $\mathbb{C}^N$. Let σ be the Coxeter element of the Weyl group, defined by the following action

$$\chi_1 \to \chi_2 \to \cdots \to \chi_{N-1} \to -\chi_1 \to -\chi_2 \to \cdots -\chi_{N-1} \to \chi_1 \qquad \text{and} \qquad \chi_N \to -\chi_N \,.$$

This element has order $r = 2(N-1)$. We define a basis $(v^1, v^3, v^5, \ldots, v^{r-1}, \tilde{v})$ of $\mathfrak{h}$ that is diagonal under the action of σ as follows

$$\begin{aligned} v^a &= \sqrt{2}\sum_{j=0}^{\frac{r}{2}-1} \theta^{-aj}\chi_{j+1} \qquad a \in \{1,3,5,\ldots,r-1\}\,, \\ \tilde{v} &= \sqrt{2}\,\chi_N\,, \end{aligned} \tag{4.81}$$

It has the property $\sigma(v^a) = \theta^a v^a$ and $\sigma(\tilde{v}) = -\tilde{v}$, and the inner product for any $a, b \in \{1, 3, \ldots, r-1\}$

$$\langle v^a, v^b \rangle = r\delta_{r|a+b}\,, \qquad \langle \tilde{v}, v^a \rangle = 0\,, \qquad \langle \tilde{v}, \tilde{v} \rangle = 2\,. \tag{4.82}$$

We can therefore define

$$\mathcal{T}(\mathfrak{so}_{2N}) = \mathbb{C}_{\hbar^{1/2}}[\![x_1, x_3, x_5, \ldots, \tilde{x}_1, \tilde{x}_3, \ldots]\!]$$

with

$$v^a(z) := \sum_{k \in a/r + \mathbb{Z}} J_{rk} z^{-k-1}, \qquad \tilde{v}(z) = \sum_{k \in 1/2+\mathbb{Z}} \widetilde{J}_{2k}\, z^{-k-1}\,, \tag{4.83}$$

where we take for any odd $l > 0$

$$J_l = \hbar\partial_{x_l}, \qquad J_{-l} = l x_l, \qquad \widetilde{J}_l = \hbar\partial_{\tilde{x}_l}, \qquad \widetilde{J}_{-l} = l\tilde{x}_l\,.$$

The evaluation of the pairing (4.82) shows that these assignments reproduce the desired commutation relations of the Heisenberg algebra. Thus $\mathcal{T}(\mathfrak{so}_{2N})$ is a twisted $\mathcal{S}_0$-module, which we restrict to obtain a $\mathcal{W}(\mathfrak{so}_{2N})$-module.

Twist fields for the generators

$\mathcal{W}(\mathfrak{so}_{2N})$ is freely and strongly generated by ν^d for $d \in \{2, 4, 6, \ldots, 2N-2\}$ and $\tilde{\nu}^N$. The corresponding fields can be computed using the following lemma.

Lemma 4.4.17. *[BM13, Lemma 3.7] For all $d \geq 1$, $i \in \{1, \ldots, N\}$ and $\varepsilon \in \{-1, 1\}$, we have*

$$Y_\sigma(\mathrm{e}^{\varepsilon\chi^i}_{-d}\mathrm{e}^{-\varepsilon\chi^i}_{-1}, z) = \sum_{\ell=0}^{d} \frac{\hbar^{\ell/2}}{z^\ell}\, c_i^{(\ell)}\, S_{d-\ell}(\varepsilon\chi^i, z)\,, \tag{4.84}$$

where for any $v \in \mathbb{C}^N$

$$S_n(v, z) := :\frac{1}{n!}\left(\hbar^{1/2}\partial_z + v(z)\right)^n 1:$$

are the Faà di Bruno polynomials, and cs are scalars such that $c_i^{(0)} = 1$.

The main ingredient of the proof in [BM13] is the product formula (4.47). Our version only differs by specialization to the orthonormal basis vectors χ^i, and inserting suitable powers of $\hbar$. These merely keep track of the conformal weights. For instance, we would like ∂_z to have degree 1 and hence we add a $\hbar^{1/2}$. The $\hbar^{\ell/2}$ comes from the product formula.

Remark 4.4.18. $Y(\nu^d, z)$ is a field of conformal weight d. The $\hbar$ grading keeps track of this weight. In particular, the half-integer powers of $\hbar$ will vanish in $Y_\sigma(\nu^d, z)$. This is easy to see using that ν^d is invariant under $\chi^i \to -\chi^i$.

We define the modes of the twist fields $Y(\nu^d, z)$ and $Y(\tilde{\nu}^N, z)$ by

$$Y_\sigma(\nu^d, z) = \sum_{k\in\mathbb{Z}} W_k^d z^{-k-1}\,, \qquad Y_\sigma(\tilde{\nu}^N, z) = \sum_{k\in\mathbb{Z}} \widetilde{W}_k^N z^{-k-1}\,.$$

Higher quantum Airy structures

We can then apply Propositions 4.3.13 and 4.3.14 to get graded Lie subalgebras by considering the ideals generated by certain subsets of the modes. As in the $\mathfrak{gl}_{N+1}$ case, we need to perform a dilaton shift in order to get a higher quantum Airy structure. In this section we will only consider the subalgebras of Propositions 4.3.13 and 4.3.14, since we have not constructed the more general intermediate subalgebras analogous to Theorem 4.3.16 for $\mathcal{W}(\mathfrak{so}_{2N})$.

Definition 4.4.19. The dilaton shifted modes of the module $\mathcal{T}(\mathfrak{so}_{2N})$ are defined as follows

$$\begin{aligned} H_k^d &= \gamma_d \hat{T}_s W_k^d \hat{T}_s^{-1} \qquad && d \in \{2, 4, \ldots, 2(N-1)\}\,, \\ \widetilde{H}_k^N &= \tilde{\gamma}_{N,s} \hat{T}_s \widetilde{W}_k^N \hat{T}_s^{-1}\,, \end{aligned}$$

with constants

$$\gamma_d = d^{-1} 2^{d/2} (N-1)^{d-1}, \qquad \tilde{\gamma}_{N,s} = (-1)^{s(N-2)/2} 2^{(N-1)/2} (N-1)^{N-1}\,,$$

and we recall that $\hat{T}_s = \exp\big(-\frac{J_{-s}}{\hbar s}\big)$.

For $s = 1$ or $r+1$, we obtain in this way quantum r-Airy structures.

Theorem 4.4.20. *Let $N \geq 3$, that is $r = 2(N-1) \geq 4$. The family of differential operators*

$$\begin{aligned} H_k^d &= \gamma_d \hat{T}_{r+1} W_k^d \hat{T}_{r+1}^{-1} \qquad && d \in \{2, 4, \ldots, 2(N-1)\}, \quad k \geq 0\,, \\ \widetilde{H}_k^N &= \tilde{\gamma}_{N,r+1} \hat{T}_{r+1} \widetilde{W}_k^N \hat{T}_{r+1}^{-1} \qquad && k \geq 0 \end{aligned}$$

forms a quantum r-Airy structure on the vector space $V = \bigoplus_{p>0} \mathbb{C}\langle x_{2p+1}\rangle \oplus \mathbb{C}\langle \tilde{x}_{2p+1}\rangle$. The same is true for the family of differential operators

$$\begin{aligned} H_k^d &= \gamma_d \hat{T}_1 W_k^d \hat{T}_1^{-1} \qquad && d \in \{2, 4, \ldots, 2(N-1)\}, \qquad k \geq d-1 \\ \widetilde{H}_k^N &= \tilde{\gamma}_{N,1} \hat{T}_1 \widetilde{W}_k^N \hat{T}_1^{-1} \qquad && k \geq N-1\,. \end{aligned}$$

Proof. We first look at the modes of $Y_\sigma(\nu^d, z)$ for $d \in \{2, 4, \ldots, 2N-2\}$ and study the terms that can contribute in degree ≤ 1 after a dilaton shift $J_{-s} \to J_{-s} + 1$ for some $s \in 2\mathbb{Z}+1$. From Lemma 4.4.17, we see that

$$Y_\sigma(\nu^d, z) = \sum_{i=1}^{N-1} (\chi^i(z))^d + O(\hbar)\,.$$

Now, we implement the change of basis (4.36). The inverse change of basis is

$$\chi^i = \frac{1}{\sqrt{2}(N-1)} \sum_{\substack{1 \leq a \leq r-1 \\ a \text{ odd}}} \theta^{a(i-1)}\, v^a, \qquad \chi^N = \frac{\tilde{v}}{\sqrt{2}}\,.$$

We note that the modes W_k^d of ν^d are homogeneous differential operators of degree d. The mode $J_{-s} = x_s/s$ that appears in the dilaton shift is only present in a single $v^s(z)$ as coefficient of $z^{s/r-1}$. So

$$\hat{T}_s^{-1}\chi^i(z)\hat{T}_s = \frac{\theta^{-(i-1)s}\, z^{s/r-1}}{\sqrt{2}(N-1)} + \chi^i(z)\,.$$

Consequently, the terms of degree 0 and 1 will be

$$\begin{aligned}
&\frac{d\, H_k^d}{2^{d/2}(N-1)^{d-1}} + O(2) \\
= \;& [z^{-(k+1)}]\left(\frac{z^{(s/r-1)d}\,\delta_{r|s}}{2^{d/2}(N-1)^{d-1}} + \frac{d}{2^{(d-1)/2}(N-1)^{d-1}}\sum_{i=1}^{N-1}\theta^{-(i-1)s(d-1)} z^{(s/r-1)(d-1)}\,\chi^i(z)\right) \\
= \;& [z^{-(k+1)}]\left(\frac{d}{2^{d/2}(N-1)^{d}}\sum_{i=1}^{N-1}\sum_{\substack{1\leq a\leq r-1\\ a\text{ odd}}}\sum_{m\in\mathbb{Z}}\theta^{(i-1)(a-s(d-1))}\,J_{a+rm}\, z^{-a/r-m-1}\, z^{(s/r-1)(d-1)}\right) \\
= \;& \frac{d\, J_{r(k-d+1)+(d-1)s}}{2^{d/2}(N-1)^{d-1}}\,,
\end{aligned}$$

In the third line, we dropped the term $\delta_{r|s}$ because s is odd and r is even. So there is no degree 0 term and we obtain $H_k^d = J_{r(k-d+1)+(d-1)s} + O(2)$. This explains the choice of the constant prefactor in the definition of H_k^d.

Now, we consider the modes of the twist field $Y_\sigma(\tilde{\nu}^N, z)$. We use a similar argument as above, using the product formula (4.47) instead of Lemma 4.4.17

$$\begin{aligned}
\frac{(-1)^{-s(N-2)/2}\,\widetilde{H}_k^i}{2^{(N-1)/2}(N-1)^{N-1}} + O(2) \;&=\; [z^{-(k+1)}]\left(\prod_{i=1}^{N-1}\frac{\theta^{-s(i-1)}\, z^{s/r-1}}{\sqrt{2}(N-1)}\right)\sum_{m\in 1/2+\mathbb{Z}}\widetilde{J}_{2m}\, z^{-m-1} + O(2) \\
&=\; \frac{(-1)^{-s(N-2)/2}}{2^{(N-1)/2}(N-1)^{N-1}}\,\widetilde{J}_{2k-s+r}\,,
\end{aligned}$$

hence $\widetilde{H}_k^N = \widetilde{J}_{2k-s+r} + O(2)$.

First, we consider the graded Lie subalgebra $\mathcal{A}_{\geq 0}$ corresponding to the modes $k \geq 0$ (see Proposition 4.3.13). If we want these modes to contain each J_p and $\widetilde{J}_p$ (for $p > 0$ odd) exactly once, we must choose $s = r+1$, and we do obtain a quantum r-Airy structure.

Second, if we focus on the graded Lie subalgebra $\mathcal{A}_\Delta$ corresponding to the modes H_k^d with $k \geq d-1$ and $\widetilde{H}_k^N$ with $k \geq N-1$ (see Proposition 4.3.14), the same condition forces us to choose $s = 1$ and we obtain a quantum r-Airy structure in this case as well.

□

As usual, we note that we can easily formulate and prove an analog of Proposition 4.4.13 to describe more general higher quantum Airy structures obtained from the ones of Theorem 4.4.20 by further dilaton shifts and changes of polarization.

4.4.4 The exceptional types

The twisted module

We consider the simple complex Lie algebra $\mathfrak{e}_N$ with $N \in \{6,7,8\}$, together with a Coxeter element of the Weyl group σ. Its order is denoted by r. σ acts on the Cartan subalgebra with simple eigenvalues $(\theta_r^{d_a-1})_{a=1}^N$. We can always order $2 \leq d_1 < \cdots < d_N \leq r$, and we have $d_{N+1-a} + d_a = r+2$ for any $a \in \{1,\ldots,N\}$. Let $\mathbb{D} = \{d_1-1,\ldots,d_N-1\}$

N	r	$d_1,\ldots,d_N$
6	12	$2,5,6,8,9,12$
7	18	$2,6,8,10,12,14,18$
8	30	$2,8,12,14,18,20,24,30$

We can find a basis of eigenvectors $(v^a)_{a=1}^N$ such that

$$\sigma(v^a) = \theta_r^{d_a-1}\, v^a, \qquad \langle v^a, v^b\rangle = r\delta_{a+b,r}\,.$$

We obtain a σ-twisted module for the Heisenberg algebra

$$\mathcal{T}(\mathfrak{e}_N) = \mathbb{C}_{\hbar^{1/2}}[\![(x_l)_{l\in\mathbb{D}+r\mathbb{N}}]\!]$$

by assigning

$$v^a(z) = \sum_{p\in(d_a-1)/r+\mathbb{Z}} J_{rp}\, z^{-p-1}\,,$$

where $J_l = \hbar\partial_{x_l}$ for $l>0$, $J_0 = 0$ and $J_l = -l\,x_{-l}$ for $l<0$. Via restriction, we get an untwisted $\mathcal{W}(\mathfrak{e}_N)$-module over $\mathbb{C}_{\hbar^{1/2}}$.

The generators and their twist fields

The VOA $\mathcal{W}(\mathfrak{e}_N)$ is strongly and freely generated by elements w_i of conformal weight d_i for $i\in\{1,\ldots,N\}$. Although there is no canonical choice making the generators particularly simple, we will rely on the following structural result.

Theorem 4.4.21. *[FF96] and [LYZ13, Lemma 3.4] One can choose generators of the form*[4]

$$w^i = (v^N_{-1})^{d_i-1} v^{N+1-i}_{-1} + \sum_{m=2}^{d_i} (v^N_{-1})^{d_i-m}\, P_{i,m}(v^1_{-1},\ldots,v^{N-1}_{-1}) + \tilde{w}^i\,,$$

where $P_{i,d}$ is a homogeneous polynomial of degree $d_i - d$ and $\tilde{w}_i$ belongs to the left ideal generated by the χ_{-n} for $\chi \in \mathfrak{h}$ and $n \geq 2$.

The fields associated to w_i are denoted

$$Y_\sigma(w^i, z) = \sum_{k\in\mathbb{Z}} W^i_k\, z^{-k-1}\,.$$

Remark 4.4.22. As $\mathfrak{e}_N$ is a subalgebra of $\mathfrak{so}_{2N}$, and the generators of the $\mathcal{W}(\mathfrak{so}_{2N})$ algebra after $\hbar$-rescaling indicated in Section 4.3.5 only involve integer powers of $\hbar$ (see Remark 4.4.18), the same must be true for the generators of $\mathcal{W}(\mathfrak{e}_N)$.

Definition 4.4.23. The dilaton shifted modes of $\mathcal{T}(\mathfrak{e}_N)$ are defined as

$$H^i_k := \hat{T}_s^{-1} W^i_k \hat{T}_s, \qquad \hat{T}_s = \exp\left(-\frac{J_s}{\hbar s}\right) \qquad i \in \{1,\ldots,N\}, \qquad k \in \mathbb{Z}\,.$$

For $s = 1$ or $r + 1$ we obtain in this way quantum r-Airy structures. The one with $s = r + 1$ was anticipated in [LYZ13], while the one for $s = 1$ is new.

Theorem 4.4.24. *Let $\epsilon \in \{0,1\}$ and denote $s := 1 + \epsilon r$. The family of differential constraints*

$$H^i_k \qquad i \in \{1,\ldots,N\}, \qquad k \geq (1-\epsilon)(d_i - 1)$$

forms a quantum Airy structure on the vector space $V = \bigoplus_{i=1}^N \bigoplus_{l>0} \mathbb{C}\langle x_{rl+d_{i-1}}\rangle$.

Proof. The formulas (4.34)-(4.47) show that we have for $k_1,\ldots,k_n > 0$ and $\chi^{i_1},\ldots,\chi^{i_n} \in \mathfrak{h}$

$$Y_\sigma\big(\chi^{i_1}_{-k_1}\cdots\chi^{i_n}_{-k_n}\,|0\rangle\,, z\big) = \big(\delta_{\sum_j (k_j-1),0} + \hbar^{1/2}\delta_{\sum_j (k_j-1),1}\big) :\prod_{j=1}^n \frac{\mathrm{d}^{k_j-1}}{\mathrm{d}z^{k_j-1}}\chi^{i_1}(z): + O(2)\,,$$

[4] The discrepancy with the notations in [LYZ13] comes from the fact that we use a dual basis.

where the $O(2)$ arises from terms involving $\hbar^m$ with $m \geq 1$. Following Theorem 4.4.21 the mode W_k^i is a sum of three parts, which we study them up to $O(2)$. The first part is

$$\sum_{p_1,\ldots,p_{d_i}\in\mathbb{Z}} \delta\Big(-(k+2-d_i)+\sum_{j=1}^{d_i}(p_j+1)\Big) : J_{d_i-1+rp_1}\, J_{r-1+rp_2}\cdots\, J_{r-1+rp_{d_i}}\, J_{d_i-1+rp_{d_i}} :+O(2)\,,$$

where $\delta(n) := \delta_{n,0}$ is the Kronecker delta. The second part is a sum, over $m \in \{2,\ldots,d_i\}$ and $a_1,\ldots,a_m \in \{1,\ldots,N-1\}$ of an expression which up to $O(2)$ is proportional to

$$\sum_{p_1,\ldots,p_{d_i}\in\mathbb{Z}} \delta\Big(-(k+1+m-d_i)-\frac{d_i}{r}+\sum_{j=1}^{m}\frac{d_{a_j}}{r}+\sum_{j=1}^{d_i}(p_j+1)\Big)$$
$$\times : J_{d_{a_1}-1+rp_1}\cdots\, J_{d_{a_m}-1+rp_m}\, J_{r-1+rp_{m+1}}\cdots\, J_{r-1+rp_{d_i}} :.$$

The third part is a sum, over $\ell \in \{1,\ldots,d_i-1\}$, $b_1,\ldots,b_{\ell-1} > 0$ and $a_1,\ldots,a_q \in \{1,\ldots,N\}$ such that if we set $b_q := 2$ we have $\sum_{j=1}^{q} b_j = d_i$, of an expression which up to $O(2)$ is proportional to

$$\hbar^{(d_i-\ell)/2}\sum_{p_1,\ldots,p_\ell\in\mathbb{Z}} \delta\Big(-(k+1)+\sum_{j=1}^{\ell}\frac{d_{a_j}-1}{r}+p_j+b_j\Big)\prod_{j=1}^{\ell}\frac{\Gamma\big(\frac{d_{a_j}-1}{r}+p_j+b_j\big)}{\Gamma\big(\frac{d_{a_j}-1}{r}+p_j+1\big)} :\prod_{j=1}^{\ell} J_{d_{a_j}-1+rp_j}:+O(2)\,.$$

Due to the power of $\hbar$ in prefactor, up to $O(2)$, we only have to take into account the terms where $\ell = d_i - 1$ and $b_1 = \cdots = b_{d_i-2} = 1$.

For a fixed $\epsilon \in \{0,1\}$ we set $s = 1+\epsilon r$ and apply the shift $J_{-s} \to J_{-s}+1$ to obtain H_k^i. We want to identify the terms of degree 0 and 1. In degree 0, there is no contribution from the first and second parts, because they do not come from a monomial of the form $(v^N(z))^q$ for some $q > 0$, and since the third part has at least a power of $\hbar^{1/2}$ it does not contribute either. We now turn to degree 1 terms. The first part contributes for $p_2 = \cdots = p_{d_i} = -(1+\epsilon)$ and yields

$$J_{s(d_i-1)+r(k+1-d_i)} + O(2)\,.$$

The second part remains $O(2)$ since it is at least quadratic in the non-shifted J_l. The third part could contribute when $\ell = d_i - 1$, $b_1 = \cdots = b_{d_i-2} = 1$ and $(a_j, p_j) = (N, -1-\epsilon)$ for all $j \in \{1,\ldots,d_i-1\}$. But the Kronecker delta would impose that $r|(d_i - 1)$ which is never possible. Hence

$$H^i_k = J_{\Pi_s(i,k)} + O(2), \qquad \Pi_s(i,k) := s(d_i - 1) + r(k + 1 - d_i)\,.$$

Let us denote

$$\tilde{S}_s = \Big\{(i,k) \quad \Big| \quad i \in \{1,\ldots,N\} \text{ and } k \geq (1-\epsilon)(d_i - 1)\Big\} \qquad \text{with } s = 1 + r\epsilon\,.$$

If $s = r + 1$, we know from Lemma 4.3.13 that the subset of modes W^i_k indexed by $(i,k) \in \tilde{S}_{r+1}$ generate a graded Lie subalgebra $\mathcal{A}_{\geq 0}$. Therefore, so do the corresponding H^i_k. As we remark that Π_{r+1} is a bijection between $\tilde{S}_{r+1}$ and $\mathbb{D} + r\mathbb{N}$, for each independent linear coordinate $(x_p)_{p \in \mathbb{D}+r\mathbb{N}}$ on V we have a unique operator H^i_k containing the derivation $\hbar\partial_{x_p}$ as its degree 1 term. Therefore we have obtained a quantum Airy structure. For $s = 1$ we reach a similar conclusion if we use Lemma 4.3.14 and the graded Lie subalgebra $\mathcal{A}_\Delta$ instead.

In fact, due to Remark 4.4.22 we could have ignored the third part of the generators right from the start, as it could only be a $O(2)$.

$\square$

4.5 Higher quantum Airy structures from higher abstract loop equations

In this section, we show that the Bouchard-Eynard topological recursion of [BE13, BE17, BHL+14] for admissible spectral curves with arbitrary ramification yields higher quantum Airy structures. More precisely, we study higher abstract loop equations (whose unique, polarized solution is constructed by the Bouchard-Eynard topological recursion) and construct the associated higher quantum Airy structures. We then show that they coincide with the general dilaton-shifted, polarization changed modules for direct sums of $\mathcal{W}(\mathfrak{gl}_r)$ algebras. The precise dilaton shift and change of polarization involved here is dictated by the local expansion of the spectral curve data around the ramification points.

4.5.1 Geometry of local spectral curves

Local spectral curve with one component

Let us first introduce the notion of local spectral curves. We start with the complex vector space

$$V_z = \left\{\omega \in \mathbb{C}[z^{-1}, z]\!]\,\mathrm{d}z \quad \middle| \quad \mathrm{Res}_{z=0}\ \omega(z) = 0\right\}, \tag{4.85}$$

equipped with the symplectic pairing

$$\Omega_z(\mathrm{d}f_1, \mathrm{d}f_2) = \mathrm{Res}_{z=0}\ f_1(z)\,\mathrm{d}f_2(z)\,. \tag{4.86}$$

Let V_z^+ be the Lagrangian subspace $V_z^+ = \mathbb{C}[\![z]\!]\mathrm{d}z \subset V_z$. Let us define a basis $(\mathrm{d}\xi_l)_{l>0}$ for V_z^+ with $\mathrm{d}\xi_l(z) = z^{l-1}\mathrm{d}z$.

Definition 4.5.1. A local spectral curve with one component consists of the data of the symplectic space V_z over $\mathbb{C}$, with a Lagrangian subspace V_z^+ and

- An integer $r \geq 2$. We use it to consider the group action $G \times V_z \to V_z$ with $G = \mathbb{Z}/r\mathbb{Z}$ and $r \geq 2$, such that the generator ρ of G acts as

$$\rho \cdot \mathrm{d}f(z) \longmapsto \mathrm{d}f(\theta z)\,,$$

where $\theta = e^{2\mathbf{i}\pi/r}$ is a primitive r-th root of unity.

- A one-form $\omega_{0,1} \in V_z^+$. We write its expansion as

$$\omega_{0,1}(z) = \sum_{l>0}^{\infty} \tau_l \, \mathrm{d}\xi_l(z) \,.$$

- A choice of polarization, *i.e.* a Lagrangian subspace $V_z^- \subset V_z$ complementary to V_z^+, with basis $(\mathrm{d}\xi_l)_{l<0}$ such that

$$\forall l, m \in \mathbb{Z}_{\neq 0} \setminus \{0\}, \qquad \Omega_z(\mathrm{d}\xi_l, \mathrm{d}\xi_m) = \frac{1}{l}\, \delta_{l+m,0} \,.$$

Definition 4.5.2. We say that a local spectral curve is *admissible*[5] if

$$s := \min \big\{ l > 0 \quad | \quad \tau_l \neq 0 \ \text{ and } \ r \nmid l \big\}$$

satisfies $1 \leq s \leq r+1$ and $r = \pm 1 \bmod s$. Notice that this congruence implies that r and s are coprime. If $s = r+1$, we say the spectral curve is *regular*, while it is *irregular* if $s < r$.

Remark 4.5.3. In the standard topological recursion formalism of Chekhov-Eynard-Orantin [EO07], one would need to choose $r = 2$. This requirement was dropped in the Bouchard-Eynard topological recursion [BE17, BHL+14].

Note that the basis $\mathrm{d}\xi_l(z)$ for V_z^+ is an eigenvector (with eigenvalue θ^l) for the action of the generator of G, but it may not be the case for the polarization basis $\mathrm{d}\xi_{-l}(z)$.

The choice of polarization can be encoded in terms of a formal bidifferential. For $l > 0$, we can write

$$\mathrm{d}\xi_{-l}(z) = \frac{\mathrm{d}z}{z^{l+1}} + \sum_{m>0} \frac{\phi_{l,m}}{l}\, \mathrm{d}\xi_m(z) \,,$$

for some coefficients $\phi_{l,m}$. The requirement that V_z^- is Lagrangian, namely

$$\forall l, m > 0, \qquad \Omega_z(\mathrm{d}\xi_{-l}(z), \mathrm{d}\xi_{-m}(z)) = 0$$

imposes that

$$\phi_{l,m} = \phi_{m,l} \,.$$

[5]This admissibility requirement will become clear when we construct the associated higher quantum Airy structures. Remarkably, it turns out that the Bouchard-Eynard topological recursion constructs symmetric differentials only for admissible spectral curves.

We then introduce the formal bidifferential

$$\omega_{0,2}(z_1,z_2) = \frac{\mathrm{d}z_1 \otimes \mathrm{d}z_2}{(z_1-z_2)^2} + \sum_{l,m>0} \phi_{l,m} \mathrm{d}\xi_l(z_1) \otimes \mathrm{d}\xi_m(z_2)\,. \tag{4.87}$$

which is symmetric under exchange of z_1 and z_2. This is not an element in $V_{z_1} \otimes V_{z_2}$ but

$$\omega_{0,2}(z_1,z_2) - \frac{\mathrm{d}z_1 \otimes \mathrm{d}z_2}{(z_1-z_2)^2} \in V_{z_1}^+ \otimes V_{z_2}^+\,.$$

In any case, for any $l > 0$ we can write

$$\begin{aligned} \mathrm{d}\xi_{-l}(z') &= \Omega_z\left(\omega_{0,2}(z,z'), \frac{\mathrm{d}z}{z^{l+1}}\right) \\ &= \mathrm{Res}_{z=0}\left(\int_1^z \omega_{0,2}(\cdot,z')\right)\frac{\mathrm{d}z}{z^{l+1}}\,, \end{aligned}$$

where in the first line the symplectic pairing acts on the variable z. In other words, for $|z_1| < |z_2|$, we can write the expansion

$$\omega_{0,2}(z_1,z_2) \underset{|z_1|<|z_2|}{\approx} \sum_{l>0} l\,\mathrm{d}\xi_l(z_1) \otimes \mathrm{d}\xi_{-l}(z_2)\,.$$

Definition 4.5.4. We call *standard polarization* the choice of $V_z^- = z^{-1}\mathbb{C}[z^{-1}]\mathrm{d}z$, with basis $\mathrm{d}\xi_{-l}(z) = z^{-l-1}\mathrm{d}z$. In this case, the bidifferential is simply

$$\omega_{0,2}(z_1,z_2) = \frac{\mathrm{d}z_1 \otimes \mathrm{d}z_2}{(z_1-z_2)^2}\,.$$

Projection map

We will also need to define a few important maps associated to the group action. Pick an element $\mathrm{d}f \in V_z$, and denote by $G \cdot \mathrm{d}f$ the orbit of $\mathrm{d}f$, that is

$$\begin{aligned} G \cdot \mathrm{d}f(z) &= \left\{ g \cdot \mathrm{d}f(z) \;\middle|\; g \in G \right\} \\ &= \left\{ \mathrm{d}f(z), \mathrm{d}f(\theta z), \ldots, \mathrm{d}f(\theta^{r-1}z) \right\}. \end{aligned}$$

There is a natural averaging map

$$\begin{aligned} \mu_z :\quad V_z &\longrightarrow V_z \\ \mathrm{d}f(z) &\longmapsto \sum_{\mathrm{d}g \in G\cdot \mathrm{d}f} \mathrm{d}g(z) = \sum_{k=0}^{r-1} \mathrm{d}f(\theta^k z)\,. \end{aligned}$$

We want to extend this map to tensor products of V_z. Let us consider first the case of $V_{z_1} \otimes V_{z_2}$. There is a natural group action of $G \times G$ on $V_{z_1} \otimes V_{z_2}$, with each factor of G acting individually on V_{z_1} and V_{z_2} respectively, and a diagonal group action of G. Let us denote by $G^{(2)} := (G \times G) \setminus G$ the set $G \times G$ minus the diagonal embedding of G. For $\omega(z_1, z_2) \in V_{z_1} \otimes V_{z_2}$, let

$$G^{(2)} \cdot \omega = \left\{ g \cdot \omega \quad \middle| \quad g \in G^2 \right\},$$

and define the averaging map

$$\begin{aligned} \mu_{z_1,z_2} \ : \quad V_{z_1} \otimes V_{z_2} \ &\longrightarrow \ V_{z_1} \otimes V_{z_2} \\ \omega(z_1, z_2) &\longmapsto \sum_{\lambda \in G^{(2)} \cdot \omega} \lambda(z_1, z_2) = \frac{1}{2} \sum_{\substack{m_1, m_2 = 0 \\ m_1 \neq m_2}}^{r-1} \omega(\theta^{m_1} z_1, \theta^{m_2} z_2) \,. \end{aligned}$$

We also define a specialization map

$$\begin{aligned} \sigma_{z_1,z_2|t} \ : \quad V_{z_1} \otimes V_{z_2} \ &\longrightarrow \mathbb{C}[t^{-1}, t]] \\ \omega(z_1, z_2) &\longmapsto \frac{\omega(t,t)}{(\mathrm{d}\xi_r(t))^2} \,. \end{aligned}$$

It is then easy to see that the composition, which we call "projection map",

$$P_{z_1,z_2|t} := \sigma_{z_1,z_2|z} \circ \mu_{z_1,z_2} \ : V_{z_1} \otimes V_{z_2} \longrightarrow \mathbb{C}[t^{-r}, t^r]]$$

maps elements of $V_{z_1} \otimes V_{z_2}$ to Laurent series that are invariant under the group action G.

The generalization to more than two variables straightforward.

Definition 4.5.5. Let $\mathbf{z} = (z_l)_{l=1}^{i}$. For $i \in \{1, \ldots, r\}$ we define the averaging map

$$\begin{aligned} \mu_{\mathbf{z}} \ : \quad \bigotimes_{l=1}^{i} V_{z_l} \ &\longrightarrow \ \bigotimes_{l=1}^{i} V_{z_l} \\ \omega(\mathbf{z}) \ &\longmapsto \frac{1}{i!} \sum_{\substack{m_1, \ldots, m_i = 0 \\ m_l \neq m_{l'}}}^{r-1} \omega(\theta^{m_1} z_1, \ldots, \theta^{m_i} z_i) \,, \end{aligned}$$

and the specialization map

$$
\begin{aligned}
\sigma_{\mathbf{z}|t} \ : \quad \bigotimes_{l=1}^{i} V_{z_l} & \longrightarrow \ \mathbb{C}[t^{-1}, t]\!] \\
\omega(\mathbf{z}) & \longmapsto \frac{\omega(t, \ldots, t)}{(\mathrm{d}\xi_r(t))^i} \, .
\end{aligned}
$$

We define the projection map

$$
P_{\mathbf{z}|t} := \sigma_{\mathbf{z}|t} \circ \mu_{\mathbf{z}} \ : \ \bigotimes_{l=1}^{i} V_{z_l} \longrightarrow \mathbb{C}[t^{-r}, t^r]\!] \, ,
$$

which maps elements of $\bigotimes_{l=1}^{i} V_{z_l}$ to Laurent series that are invariant under the group action.

We would like to extend this map to objects that involve the formal bidifferential $\omega_{0,2}$ defined in (4.87). $\omega_{0,2}(z_1, z_2)$ does not live in $V_{z_1} \otimes V_{z_2}$, and in fact the specialization map $\sigma_{z_1,z_2|t}$ is not well defined on $\omega_{0,2}(z_1, z_2)$ due to the pole on the diagonal. However, it is easy to see that the projection map $P_{z_1,z_2|t}$ is well defined on $\omega_{0,2}(z_1, z_2)$. Similarly, the projection map $P_{\mathbf{z}|t}$ is well defined on objects that may involve factors of $\omega_{0,2}(z_l, z_{l'})$ for $l \neq l'$.

Remark 4.5.6. We notice that the projection map $P_{\mathbf{z}|t}$ is invariant under permutations of $z_1, \ldots, z_i$.

Local spectral curves with c components

We can generalize the notion of local spectral curves by, roughly speaking, taking c copies of the symplectic space V_z. Let V_z be as in (4.85). For $c \geq 1$, we define the larger symplectic space

$$
\mathcal{V}_z = \mathbb{C}^c \otimes V_z \, . \tag{4.88}
$$

We equip $\mathbb{C}^c$ with a scalar product $\cdot$ such that the standard basis $(e_i)_{i=1}^c$ is orthonormal, namely $e_i \cdot e_j = \delta_{i,j}$. We define the symplectic pairing on $\mathcal{V}_z$ as being given by

$$
\Omega_z(u_1 \otimes \mathrm{d}f_1, u_2 \otimes \mathrm{d}f_2) = u_1 \cdot u_2 \ \mathrm{Res}_{z=0} \, f_1(z)\mathrm{d}f_2(z) \, , \tag{4.89}
$$

where $u_1, u_2 \in \mathbb{C}^\ell$. For $\alpha \in \{1, \ldots, c\}$, we write $\mathcal{V}_z^{(\alpha)} = e_\alpha \otimes V_z \subseteq \mathcal{V}_z$.

Let us define the Lagrangian subspace $\mathcal{V}_z^+ = \mathbb{C}^c \otimes \mathbb{C}[\![z]\!]\mathrm{d}z \subset \mathcal{V}_z$, with basis $\mathrm{d}\xi_{\alpha,l}(z)$ with $l > 0$ and $\alpha \in \{1, \ldots, c\}$ and given by

$$\mathrm{d}\xi_{\alpha,l}(z) := e_\alpha \otimes \mathrm{d}\xi_l(z) = e_\alpha \otimes z^{l-1}\mathrm{d}z\,. \tag{4.90}$$

Definition 4.5.7. A local spectral curve with c components consists of the data of the symplectic space $\mathcal{V}_z$ together with its Lagrangian subspace $\mathcal{V}_z^+$, and

- A family of integers $r_\alpha \geq 2$ for $\alpha \in \{1, \ldots, c\}$. We use them to consider the group action $G \times \mathcal{V}_z \to \mathcal{V}_z$, with $G = G_1 \times \ldots \times G_\ell$ and $G_\alpha = \mathbb{Z}/r_\alpha\mathbb{Z}$. It is such that the generator ρ_α of G_α acts only on $\mathcal{V}_z^{(\alpha)}$ as

$$\rho_\alpha \cdot \mathrm{d}\xi_{\alpha,l}(z) = \mathrm{d}\xi_{\alpha,l}(\theta_\alpha z)\,, \tag{4.91}$$

  where θ_α is a primitive r_α-th root of unity.

- A one-form $\omega_{0,1} \in \mathcal{V}_z^+$. We write its expansion as

$$\omega_{0,1}(z) = \sum_{\alpha=1}^{c} \sum_{l>0} \tau_l^\alpha \, \mathrm{d}\xi_{\alpha,l}(z)\,.$$

- A choice of polarization, that is, a choice of Lagrangian subspace $\mathcal{V}_z^- \subset \mathcal{V}_z$ complementary to $\mathcal{V}_z^+$, with basis $\mathrm{d}\xi_{\alpha,-l}(z)$, with $l > 0$ and $\alpha \in \{1, \ldots, c\}$ such that

$$\forall \alpha, \beta \in \{1, \ldots, c\}, \quad \forall l, m \in \mathbb{Z} \qquad \Omega_z(\mathrm{d}\xi_{\alpha,l}, \mathrm{d}\xi_{\beta,m}) = \frac{1}{l}\delta_{\alpha,\beta}\delta_{l+m,0}\,.$$

Definition 4.5.8. We say that a spectral curve is *admissible* if for each $\alpha \in \{1, \ldots, c\}$,

$$s_\alpha := \min\big\{ l > 0 \quad | \quad \tau_l^\alpha \neq 0 \ \text{ and } \ r_\alpha \nmid l \big\}$$

satisfies $1 \leq s_\alpha \leq r_\alpha + 1$ and $r_\alpha = \pm 1 \bmod s_\alpha$ (the sign could depend on α). We say that the spectral curve is *regular at* α if $s_\alpha = r_\alpha + 1$, while we say that it is *irregular at* α if $s_\alpha < r_\alpha$.

As before, the choice of polarization is nicely encoded in terms of a formal bidifferential. For $l > 0$ and $\alpha \in \{1, \ldots, c\}$, we can write

$$\mathrm{d}\xi_{\alpha,-l}(z) = \frac{e_\alpha \otimes \mathrm{d}z}{z^{l+1}} + \sum_{\beta=1}^{c} \sum_{m>0} \frac{\phi_{l,m}^{\alpha,\beta}}{l} \, \mathrm{d}\xi_{\beta,m}(z) \, ,$$

for some coefficients $\phi_{l,m}^{\alpha,\beta}$. The requirement that $\mathcal{V}_z^-$ is Lagrangian imposes the symmetry $\phi_{l,m}^{\alpha,\beta} = \phi_{m,l}^{\beta,\alpha}$. We define the formal bidifferential

$$\omega_{0,2}(z_1, z_2) = \sum_{\alpha=1}^{c} \frac{(e_\alpha \otimes \mathrm{d}z_1) \otimes (e_\alpha \otimes \mathrm{d}z_2)}{(z_1 - z_2)^2} + \sum_{\alpha,\beta=1}^{c} \sum_{l,m>0} \phi_{l,m}^{\alpha,\beta} \, \mathrm{d}\xi_{\alpha,l}(z_1) \otimes \mathrm{d}\xi_{\beta,m}(z_2) \, . \tag{4.92}$$

As before, $\omega_{0,2}(z_1, z_2)$ is not in $\mathcal{V}_{z_1} \otimes \mathcal{V}_{z_2}$, but

$$\left(\omega_{0,2}(z_1, z_2) - \sum_{\alpha=1}^{c} \frac{(e_\alpha \otimes \mathrm{d}z_1) \otimes (e_\alpha \otimes \mathrm{d}z_2)}{(z_1 - z_2)^2} \right) \in \mathcal{V}_{z_1}^+ \otimes \mathcal{V}_{z_2}^+ \, .$$

Then, for $l > 0$ and $\alpha \in \{1, \ldots, c\}$, we can write

$$\mathrm{d}\xi_{\alpha,-l}(z') = \Omega_z \left(\omega_{0,2}(z, z'), e_\alpha \otimes \frac{\mathrm{d}z}{z^{l+1}} \right) \, . \tag{4.93}$$

In other words, for $|z_1| < |z_2|$,

$$\omega_{0,2}(z_1, z_2) \underset{|z_1|<|z_2|}{\approx} \sum_{\alpha=1}^{c} \sum_{l>0} l \, \mathrm{d}\xi_{\alpha,l}(z_1) \otimes \mathrm{d}\xi_{\alpha,-l}(z_2) \, .$$

Definition 4.5.9. We call *standard polarization* the choice of $\mathcal{V}_z^- = \mathbb{C}^c \otimes z^{-1}\mathbb{C}[z^{-1}]\mathrm{d}z$, with basis $\mathrm{d}\xi_{\alpha,-l}(z) = e_\alpha \otimes z^{-l-1}\mathrm{d}z$ for $l > 0$. In this case, the bidifferential is simply

$$\omega_{0,2}(z_1, z_2) = \sum_{\alpha=1}^{c} \frac{(e_\alpha \otimes \mathrm{d}z_1) \otimes (e_\alpha \otimes \mathrm{d}z_2)}{(z_1 - z_2)^2} \, .$$

Projection map

We can also generalize the construction of the averaging, specialization and projection maps for each G_α.

Definition 4.5.10. Let $\mathbf{z} = (z_l)_{l=1}^{i}$. For $\alpha \in \{1, \ldots, c\}$ and $i \in \{1, \ldots, r_\alpha\}$, we define the averaging map

$$\begin{aligned}
\mu_{\mathbf{z}}^{(\alpha)} \ : \qquad & \bigotimes_{l=1}^{i} \mathcal{V}_{z_l} \longrightarrow \bigotimes_{l=1}^{i} \mathcal{V}_{z_l}^{(\alpha)} \\
& \left(\bigotimes_{l=1}^{i} e_{\alpha_l} \right) \otimes \omega(\mathbf{z}) \longmapsto \left(\prod_{l=1}^{i} \delta_{\alpha_l,\alpha} \right) e_\alpha^{\otimes i} \otimes \left(\frac{1}{i!} \sum_{\substack{m_1,\ldots,m_i=0 \\ m_l \neq m_{l'}}}^{r-1} \omega(\theta_\alpha^{m_1} z_1, \ldots, \theta_\alpha^{m_i} z_i) \right),
\end{aligned}$$

and the specialization map

$$\begin{aligned}
\sigma_{\mathbf{z}|t}^{(\alpha)} \ : \quad & \bigotimes_{l=1}^{i} \mathcal{V}_{z_l}^{(\alpha)} \longrightarrow \mathbb{C}[t^{-1}, t]\!] \\
& e_\alpha^{\otimes i} \otimes \omega(\mathbf{z}) \longmapsto \frac{\omega(t, \ldots, t)}{(\mathrm{d}\xi_{\alpha,r_\alpha}(t))^i} .
\end{aligned}$$

We define the projection map

$$P_{\mathbf{z}|t}^{(\alpha)} := \sigma_{\mathbf{z}|t}^{(\alpha)} \circ \mu_{\mathbf{z}}^{(\alpha)} \ : \ \bigotimes_{l=1}^{i} \mathcal{V}_{z_l} \longrightarrow \mathbb{C}[t^{-r_\alpha}, t^{r_\alpha}]\!] ,$$

which maps elements of $\bigotimes_{l=1}^{i} \mathcal{V}_{z_l}$ to Laurent series that are invariant under the group action G_α.

As before, we observe that this projection map is well defined on $\omega_{0,2}(z_1, z_2)$. It is also invariant under permutations of $z_1, \ldots, z_i$.

Relation with global spectral curves

The topological recursion of Chekhov-Eynard-Orantin [EO07], and the Bouchard-Eynard topological recursion [BE13, BE17, BHL+14], were not presented in terms of local spectral curves. Let us now briefly show that the notion of spectral curves used in these papers – which we here call "global" – is a special case of the local spectral curves defined above.

Definition 4.5.11. A *global spectral curve* is a quadruple $(\mathcal{C}, x, y, B)$, where

- $\mathcal{C}$ is a Riemann surface;
- x is a meromorphic function on $\mathcal{C}$. We denote by $R \subset \mathcal{C}$ the set of ramification points of x that are zeros of $\mathrm{d}x$. For any $p_\alpha \in R$, we let r_α be its order. We assume R is finite;

- y is a meromorphic function on $\mathcal{C}$.;
- B is a meromorphic symmetric bidifferential on $\mathcal{C} \times \mathcal{C}$, whose only singularity is a double pole on the diagonal with biresidue 1.

Definition 4.5.12. We say that a global spectral curve is *admissible* if for each $p_\alpha \in R$, either y has a pole of order $r_\alpha - s_\alpha$ with $s_\alpha \in \{1, \ldots, r_\alpha - 1\}$ and $r_\alpha = \pm 1 \mod s_\alpha$ (in which case we say that the curve is *irregular* at p_α)[6], or $y(p_\alpha)$ is finite and $\mathrm{d}y(p_\alpha) \neq 0$ (in which case we say that the curve is *regular* at p_α).

To recover the structure of a local spectral curve, we first need to construct the symplectic space $\mathcal{V}$. Here, we consider the space of meromorphic residueless one-forms on $\mathcal{C}$ with poles only on R, which takes the structure of $\mathcal{V}$ in (4.88) after expanding in local coordinates near the critical points.

More precisely, we replace $\mathcal{C}$ by the union of small disks U_α around the $p_\alpha \in R$. On each U_α, we define a local coordinate ζ such that

$$x\big|_{U_\alpha}(\zeta) = \frac{\zeta^{r_\alpha}}{r_\alpha} + x(p_\alpha)\,.$$

Then we can think of one-forms on $\mathcal{C}$ with poles on R as the sum of their formal Laurent expansions on the U_α in terms of the local coordinates ζ, and $\mathcal{V}$ then exhibits the structure in (4.88) with $z \to \zeta$.

The choice of one-form $\omega_{0,1} \in \mathcal{V}$ is naturally given by $\omega_{0,1} = y\mathrm{d}x$, after expanding locally on the U_α in terms of ζ. The admissibility condition matches with the analogous condition in the definition of local spectral curves.

The group actions $G_\alpha \times \mathcal{V} \to \mathcal{V}$ indexed by $\alpha \in \{1, \ldots, c\}$ with $G_\alpha = \mathbb{Z}/r_\alpha\mathbb{Z}$, are naturally given by the deck transformations $\zeta \mapsto \theta_\alpha \zeta$ in the local coordinates on each U_α.

Finally, the choice of polarization is given by the choice of bidifferential B. Indeed, if we define, for $\alpha \in \{1, \ldots, c\}$ and $l > 0$, the one-forms $\mathrm{d}\xi_{\alpha,-l}(z)$ on

[6] Note that it implies that s_α is coprime with r_α, which is the condition for the plane curve

$$\left\{(\tilde{x}, \tilde{y}) \in \mathbb{C}^2 \;\middle|\; \exists q \in \mathcal{C} \quad (x(q), y(q)) = (\tilde{x}, \tilde{y})\right\}$$

to be irreducible locally at $q = p_\alpha$.

$\mathcal{C}$ by

$$\mathrm{d}\xi_{\alpha,-l}(z') = \mathrm{Res}_{z=p_\alpha}\left(\int_{p_\alpha}^{z} B(\cdot, z')\right)\frac{\mathrm{d}\zeta(z)}{\zeta(z)^{l+1}}\,,$$

we see that this has the same form as (4.93) after expanding $B(\cdot, z')$ in local coordinates such that the expansion near $z' \to p_\beta$ gives the coefficient of e_β.

Example 4.5.13. Perhaps the simplest regular spectral curve is the so-called *r-Airy curve*, which corresponds to the choice

$$\mathcal{C} = \mathbb{C}\,, \qquad x = \frac{z^r}{r}\,, \qquad y = -z\,, \qquad B(z_1, z_2) = \frac{\mathrm{d}z_1 \otimes \mathrm{d}z_2}{(z_1 - z_2)^2}\,.$$

We observe that the corresponding local spectral curve has standard polarization.

Example 4.5.14. There is a natural family of irregular spectral curves, which we will call the (r, s) spectral curves. They are indexed by an integer $s \in \{1, \ldots, r-1\}$ such that $r = \pm 1 \mod s$,

$$\mathcal{C} = \mathbb{C}\,, \qquad x = \frac{z^r}{r}\,, \qquad y = -\frac{1}{z^{r-s}}\,, \qquad B(z_1, z_2) = \frac{\mathrm{d}z_1 \otimes \mathrm{d}z_2}{(z_1 - z_2)^2}\,.$$

We call the extreme case $s = 1$ the r-Bessel curve. The corresponding local spectral curves again have standard polarization.

Example 4.5.15. In fact, the more general spectral curve given by

$$\mathcal{C} = \mathbb{C}\,, \qquad x = \frac{z^r}{r}\,, \qquad y = \sum_{l>0}^{\infty} \tau_l z^{l-r}\,, \qquad B(z_1, z_2) = \frac{\mathrm{d}z_1 \otimes \mathrm{d}z_2}{(z_1 - z_2)^2}\,,$$

with the condition that the first non-zero term (apart maybe from τ_r) in the expansion of y is τ_s with $s \in \{1, \ldots, r+1\}$ such that $r = \pm 1 \mod s$, corresponds to the general local spectral curve with one component in standard polarization. The most general local spectral curve with one component in arbitrary polarization is associated with the bidifferential of the form

$$B(z_1, z_2) = \frac{\mathrm{d}z_1 \otimes \mathrm{d}z_2}{(z_1 - z_2)^2} + \sum_{l,m>0} \phi_{l,m}\, z_1^{l-1} z_2^{m-1} \mathrm{d}z_1 \otimes \mathrm{d}z_2\,.$$

Global spectral curves are particular examples of local spectral curves. However, local spectral curves are slightly more general. Since modules for

$\mathcal{W}$ algebras naturally give rise to the more general structure of local spectral curves, in the following we will reformulate the Bouchard-Eynard topological recursion and higher abstract loop equations in the language of local spectral curves.

4.5.2 Bouchard-Eynard topological recursion and higher abstract loop equations

Let us now review the construction of the Bouchard-Eynard topological recursion[7] of [BE13, BE17, BHL+14] and its relation with the higher analog of the abstract loop equations of [BS17]. We will reformulate everything now in the slightly more general language of local spectral curves. It should be clear to the reader that the global formulation of [BE13, BE17, BHL+14] is a particular case of the local formulation given below.

Notation and definitions

Consider an admissible local spectral curve with c components, as in Definition 4.5.7, with symplectic space $\mathcal{V}_z = \mathbb{C}^c \otimes V_z$. The aim of topological recursion is to construct a sequence of "multilinear differentials"

$$\omega_{g,n} \in \bigotimes_{j=1}^{n} \mathcal{V}_{z_j}^{-} \qquad g \geq 0 \text{ and } n \geq 1 \text{ such that } 2g-2+n > 0\,,$$

which are invariant under the natural action of the permutation group $\mathfrak{S}_n$. In terms of the choice of polarization basis $\mathrm{d}\xi_{\alpha,-l}(z)$ indexed by $\alpha \in \{1,\ldots,c\}$ and $l > 0$ for $\mathcal{V}^-(z)$, the $\omega_{g,n}$ have an expansion of the form

$$\omega_{g,n}(\mathbf{z}) = \sum_{\alpha_1,\ldots,\alpha_n=1}^{c} \sum_{l_1,\ldots,l_n>0} F_{g,n}\left[\begin{smallmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_n \\ l_1 & l_2 & \cdots & l_n \end{smallmatrix}\right] \bigotimes_{m=1}^{n} \mathrm{d}\xi_{\alpha_m,-l_m}(z_m)\,, \tag{4.94}$$

with $\mathbf{z} = (z_1,\ldots,z_n)$. The $F_{g,n}\left[\begin{smallmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_n \\ l_1 & l_2 & \cdots & l_n \end{smallmatrix}\right]$ are scalar coefficients, which are symmetric under the action of the permutation group $\mathfrak{S}_n$. In general they encode interesting enumerative invariants, see Section 4.6.

To define topological recursion we also need to define $\omega_{0,1}$ and $\omega_{0,2}$. We let $\omega_{0,1}$ be the one-form $\omega_{0,1} \in \mathcal{V}_z^+$ given in the data of a local spectral curve, that

[7] In these references it was called "generalized topological recursion".

is,

$$\omega_{0,1}(z) = \sum_{\alpha=1}^{c}\sum_{l>0}\tau_l^{\alpha}\mathrm{d}\xi_{\alpha,l}(z)\,. \tag{4.95}$$

We will also need to define formal Laurent series $y_\alpha(z)$ as follows

$$y_\alpha(z) = \sum_{l>0}\tau_l^{\alpha} z^{l-r_\alpha}\,,$$

so that we have the expansion when $z \to p_\alpha$

$$\omega_{0,1}(z) = \sum_{\alpha=1}^{c} y_\alpha(z)\mathrm{d}\xi_{\alpha,r_\alpha}(z)\,.$$

As for $\omega_{0,2}$, we take it to be the bidifferential that encapsulates the choice of polarization

$$\omega_{0,2}(z_1,z_2) = \sum_{\alpha=1}^{c}\frac{(e_\alpha\otimes \mathrm{d}z_1)\otimes(e_\alpha\otimes\mathrm{d}z_2)}{(z_1-z_2)^2} + \sum_{\alpha,\beta=1}^{c}\sum_{l,m>0}\phi_{l,m}^{\alpha,\beta}\mathrm{d}\xi_{\alpha,l}(z_1)\mathrm{d}\xi_{\beta,m}(z_2)\,. \tag{4.96}$$

To define the Bouchard-Eynard topological recursion, we use the notation in Section 4.2.2, which we recall here. Let A be a set of cardinality i, and B a set of cardinality $n-1$. The notation $\mathbf{L}\vdash A$ means that $\mathbf{L}$ is a set partition of A, *i.e.* a set of $|\mathbf{L}|$ non-empty subsets of A which are pairwise disjoint and whose union is A. We denote generically by L the elements (sets) of the partition $\mathbf{L}$. A partition of B indexed by $\mathbf{L}$ is a map $M : \mathbf{L}\to\mathfrak{P}(B)$ such that $(M_L)_{L\in\mathbf{L}}$ are possibly empty, pairwise disjoint subsets of B whose union is B. We summarize this notion with the notation $\mathbf{M}\vdash_{\mathbf{L}} B$.

Just as in Lemma 4.2.14, we define the objects

Definition 4.5.16. Let A and B be finite sets of coordinates with cardinality i and $n-1$ respectively. Then we define

$$\mathcal{E}_{g,n}^{(i)}(A|B) = \sum_{\mathbf{L}\vdash A}\sum_{\substack{h\,:\,\mathbf{L}\to\mathbb{N}\\ i+\sum_{L\in\mathbf{L}}h_L = g+|\mathbf{L}|}}\sum_{\boldsymbol{\mu}\vdash_{\mathbf{L}}B}\left(\bigotimes_{L\in\mathbf{L}}\omega_{h_L,|L|+|\mu_L|}(L,\mu_L)\right),$$

In the tensor product here and below it is assumed that the corresponding tensor factors are put in the place respecting the natural order in $(\mathbb{C}^c)^i\otimes(\mathbb{C}^c)^{\otimes(n-1)}$ associated with A and B coordinates. We also define

$$\mathcal{R}_{g,n}^{(i)}(A|B) = \sum_{\mathbf{L}\vdash A}\sum_{\substack{h\,:\,\mathbf{L}\to\mathbb{N}\\ i+\sum_{L\in\mathbf{L}}h_L = g+|\mathbf{L}|}}\sideset{}{'}\sum_{\boldsymbol{\mu}\vdash_{\mathbf{L}}B}\left(\bigotimes_{L\in\mathbf{L}}\omega_{h_L,|L|+|\mu_L|}(L,\mu_L)\right),$$

where the prime over the summation symbol means that terms that include $\omega_{0,1}$ are excluded from the sum. Finally, for future use, we define

$$\widetilde{\mathcal{R}}^{(i)}_{g,n}(A|B) = \sum_{\mathbf{L}\vdash A} \sum_{\substack{h:\mathbf{L}\to\mathbb{N} \\ i+\sum_{L\in\mathbf{L}} h_L = g+|\mathbf{L}|}} \sideset{}{''}\sum_{\boldsymbol{\mu}\vdash_{\mathbf{L}} B} \left(\bigotimes_{L\in\mathbf{L}} \omega_{h_L,|L|+|\mu_L|}(L,\mu_L) \right),$$

where the double prime over the summation symbol means that terms with $h_L = 0$, $|\mu_L| = 0$ and $|L| \leq 2$ are excluded from the sum. In other words, $\omega_{0,1}$ does not appear in the sum, and $\omega_{0,2}$ only appears when one of the entry comes from A and the other one from B.

We first give two useful combinatorial lemmas relating these three objects. First, we want to relate $\mathcal{E}^{(i)}_{g,n}(A|B)$ and $\mathcal{R}^{(i)}_{g,n}(A|B)$ by extracting the $\omega_{0,1}$ contributions.

Lemma 4.5.17. *[BE13, Lemma 3.18] For all $g, i \geq 0$ and $n \geq 1$*

$$\mathcal{E}^{(i)}_{g,n}(A|B) = \sum_{j=0}^{i} \sum_{\substack{\gamma\subseteq A \\ |\gamma|=j}} \left(\bigotimes_{l=1}^{i} \omega_{0,1}(\gamma_l) \right) \mathcal{R}^{(i-j)}_{g,n}(A\setminus\gamma|B).$$

Let us now relate $\mathcal{R}^{(i)}_{g,n}(A|B)$ and $\widetilde{\mathcal{R}}^{(i)}_{g,n}(A|B)$ by extracting contributions from $\omega_{0,2}$ with both entries coming from A. We get by straightforward combinatorics

Lemma 4.5.18.

$$\mathcal{R}^{(i)}_{g,n}(A|B) = \sum_{\substack{\ell,j\geq 0 \\ \ell+2j=i}} \sum_{\substack{\gamma\subseteq A \\ |\gamma|=2j}} \left(\sum_{\substack{\mathbf{L}\vdash\gamma \\ \forall L\in\mathbf{L}\ |L|=2}} \bigotimes_{L\in\mathbf{L}} \omega_{0,2}(L) \right) \widetilde{\mathcal{R}}^{(\ell)}_{g-j,n}(A\setminus\gamma|B),$$

Bouchard-Eynard topological recursion

We can now state the Bouchard-Eynard topological recursion formula, which recursively constructs the correlators $\omega_{g,n}$ [BE13, BE17, BHL+14].

Definition 4.5.19. Let $\mathbf{z} = (z_2, \ldots, z_n)$. The correlators $\omega_{g,n}$ are recursively defined by the *Bouchard-Eynard topological recursion*

$$\omega_{g,n}(z_1, \mathbf{z}) = \sum_{\alpha=1}^{c} \mathrm{Res}_{t=0} \left(\int_0^t (e_\alpha \cdot \omega_{0,2})(\cdot, z_1) \right) \sum_{i=1}^{r_\alpha - 1} \frac{(-1)^{i+1}}{i!} \times \sum_{\substack{m_1, \ldots, m_i = 1 \\ m_l \neq m_{l'}}}^{r_\alpha - 1} \left(\prod_{l=1}^{i} \frac{1}{(y_\alpha(t) - y_\alpha(\theta_\alpha^{m_l} t))} \right) \frac{e_\alpha^{\otimes(i+1)} \cdot \mathcal{R}_{g,n}^{(i+1)}(t, \theta_\alpha^{m_1} t, \ldots, \theta_\alpha^{m_i} t | \mathbf{z})}{(\mathrm{d}\xi_{\alpha, r_\alpha}(t))^i}, \tag{4.97}$$

where the scalar product with $e_\alpha^{\otimes(i+1)}$ only acts on the first $(i+1)$-tensor factors in $\mathcal{R}_{g,n}^{(i+1)}$.

Remark 4.5.20. Note that z_1 plays a special role in the higher topological recursion formula. It is *a priori* not obvious that the correlators $\omega_{g,n}$ constructed by (4.97) are fully symmetric. Symmetry was argued in [BE13] indirectly, only for spectral curves that arise as limits of families of curves with simple ramification points. It is however not clear to us which spectral curves precisely satisfy this condition. A proof of symmetry directly from the Bouchard-Eynard recursion formula is at the moment not known. As we will see, our identification of this recursive formula with higher quantum Airy structures in fact implies symmetry of the correlators for all admissible spectral curves (Definitions 4.5.7 and 4.5.8) as a corollary.

Higher abstract loop equations

Instead of extracting the higher quantum Airy structures corresponding to the Bouchard-Eynard topological recursion directly from the recursion formula, in this section we will rather take as starting point the higher abstract loop equations. As we will see, the loop equations give rise directly to the $\mathcal{W}(\mathfrak{gl}_r)$ quantum Airy structures constructed in Section 4.4.

Let us consider as usual a local spectral curve with c components.

Definition 4.5.21. Let $\mathbf{z} = (z_1, \ldots, z_n)$ and $\mathbf{w} = (w_1, \ldots, w_i)$. We call *higher abstract loop equations* the statement that, for all $g \geq 0$, $n \geq 1$, $2g-2+n > 0$, $\alpha \in \{1, \ldots, c\}$ and $i \in \{1, \ldots, r_i\}$,

$$P_{\mathbf{w}|t}^{(\alpha)} \left(\mathcal{E}_{g,n}^{(i)}(\mathbf{w}|\mathbf{z}) \right) \in t^{-r_\alpha \mathfrak{d}_\alpha^i} \, \mathbb{C}[\![t^{r_\alpha}]\!] \otimes \mathcal{V}_{z_2}^- \otimes \ldots \otimes \mathcal{V}_{z_n}^-, \tag{4.98}$$

where

$$\mathfrak{d}^i_\alpha := i - 1 - \left\lfloor \frac{s_\alpha (i-1)}{r_\alpha} \right\rfloor .$$

The key information here is that it is a formal series in t^{r_α} with either no negative terms if the spectral curve is regular at α ($\mathfrak{d}^i_\alpha = 0$), or starting at $t^{-r_\alpha \mathfrak{d}^i_\alpha}$ if the spectral curve is irregular at α. It is also necessarily G_α-invariant by construction, which is the reason why it is a series in t^{r_α}.

While the higher abstract loop equations do not appear to be recursive *a priori*, one can show that if a solution that respects the polarization (that is, such that $\omega_{g,n} \in \mathcal{V}^-_{z_2} \otimes \ldots \otimes \mathcal{V}^-_{z_n}$ for all $2g - 2 + n > 0$) exists, then it is uniquely constructed by the Bouchard-Eynard topological recursion of the previous subsection. The proof of this statement follows arguments similar to those presented in [BS17] for $r = 2$, and in [BKLS16, BE17] for general r. For completeness, we provide a proof in Appendix B.3 (Proposition B.3.1). Existence of a solution is however not obvious, but it will follow for admissible spectral curves as a corollary of the results of this section.

Our goal for the rest of this section is to show that solving higher abstract loop equations is equivalent to calculating the partition function of a higher quantum Airy structure, more precisely of the form of the $\mathcal{W}(\mathfrak{gl}_r)$ Airy structures constructed in the previous section. To do so, we will recast the loop equations in the form of the recursive structure in Section 4.2.2, and construct the corresponding differential operators. We then show that those are the same as the ones obtained from the $\mathcal{W}(\mathfrak{gl}_r)$ modules of the previous section.

Remark 4.5.22. We could have started with the Bouchard-Eynard topological recursion of Definition 4.5.19 instead of the higher abstract loop equations, and recast them as being obtained by the action of a sequence of differential operators acting on a partition function Z. This would have been more in line with what was done in [ABCO17, KS18]. However, to show that the differential system thus obtained is a higher quantum Airy structure, one would then need to show that the left ideal generated by the differential operators is a graded Lie subalgebra. This appears to be very difficult to prove in general. By starting with the higher abstract loop equations, we circumvent this obstacle, since we can identify the differential operators that we obtain with the $\mathcal{W}(\mathfrak{gl}_r)$ modules constructed in the previous section, and use the ideals constructed in Section 4.3.3 to prove the subalgebra property.

4.5.3 Local spectral curves with one component

Let us now focus on local spectral curves with one component for clarity. We will start with the higher abstract loop equations and reconstruct the constraints it gives on the coefficients of $\omega_{g,n}$ in the form of a higher quantum Airy structure. We will then identify them with the $\mathcal{W}(\mathfrak{gl}_r)$ higher quantum Airy structure of Proposition 4.4.13.

Reconstructing the higher quantum Airy structure

Proposition 4.5.23. *For any local spectral curve with one component, the higher abstract loop equations for* $\omega_{g,n} \in \mathrm{Sym}^n(V_z^-)$ *with* $2g-2+n>0$ *are equivalent to a system of differential equations*

$$\forall i \in \{1,\ldots,r\},\ \ \forall k \geq \mathfrak{d}^i + \delta_{i,1}, \qquad \mathcal{H}^i_k \cdot Z = 0\,,$$

with $\mathfrak{d}^i = i-1-\lfloor \frac{s(i-1)}{r} \rfloor$, *for the partition function*

$$Z = \exp\left(\sum_{\substack{g\geq 0,\ n\geq 1 \\ 2g-2+n>0}} \frac{\hbar^{g-1}}{n!} \sum_{\mathbf{b}\in(\mathbb{Z}_{>0})^n} F_{g,n}[\mathbf{b}]\, x_{b_1}\cdots x_{b_n} \right),$$

constructed from the coefficients of the expansion in (4.94). *The differential operators read*

$$\mathcal{H}^i_k = \sum_{m=1}^{i} \sum_{\substack{\ell,j\geq 0 \\ \ell+2j=m}} \frac{\hbar^j}{\ell!} \sum_{\mathbf{a}\in(\mathbb{Z}_{\neq 0})^\ell} D^{(j)}_i[k|\mathbf{a}] :J_{a_1}\cdots J_{a_\ell}:,$$

where

$$D^{(j)}_i[k|\mathbf{a}] = \frac{1}{(i-\ell-2j)!} \sum_{a_{\ell+1},\ldots,a_{i-2j}\in\mathbb{Z}_{\neq 0}} \left(\prod_{l=\ell+1}^{i-2j} F_{0,1}[a_l] \right) C^{(j)}[k|\mathbf{a},a_{\ell+1},\ldots,a_{i-2j}] \quad (4.99)$$

$$C^{(j)}[k|\mathbf{a}] = \frac{(\ell+2j)!}{j!\,2^j} \mathop{\mathrm{Res}}_{t=0} \left(\mathrm{d}\xi_{r(k+1)}(t) P_{\mathbf{w}|t} \Big(\bigotimes_{l=1}^{\ell} \mathrm{d}\xi_{-a_l}(w_l) \bigotimes_{l'=1}^{j} \omega_{0,2}(w_{\ell+2l'-1}, w_{\ell+2l'}) \Big) \right) \quad (4.100)$$

Proof. For local spectral curves with one component, the higher abstract loop equation is the statement that

$$P_{\mathbf{w}|t}\left(\mathcal{E}^{(i)}_{g,n}(\mathbf{w}|\mathbf{z})\right) \in t^{-r\mathfrak{d}^i}\mathbb{C}[\![t^r]\!] \otimes V^-_{z_2} \otimes \ldots \otimes V^-_{z_n}\,,$$

for $i \in \{1,\ldots,r\}$ and $\mathfrak{d}^i = i-1-\lfloor \frac{s(i-1)}{r} \rfloor$. This is equivalent to requiring that

$$\left[\mathrm{Res}_{t=0}\, \mathrm{d}\xi_{r(k+1)}(t)\, P_{\mathbf{w}|t}\left(\mathcal{E}^{(i)}_{g,n}(\mathbf{w}|\mathbf{z})\right)\right]_{\mathbf{b}} = 0\,,$$

for all $k \geq \mathfrak{d}^i$, $\mathbf{b} = (b_2,\ldots,b_n) \in (\mathbb{Z}_{>0})^n$ and $2g-2+n>0$. Here we introduced the notation $[\cdots]_{\mathbf{b}}$ which extracts the coefficient of the basis vector $\otimes_{l=2}^n \mathrm{d}\xi_{-b_l}(z_l)$ in $\otimes_{l=2}^n V^-_{z_l}$.

Let us first evaluate

$$\left[\mathrm{Res}_{t=0}\, \mathrm{d}\xi_{r(k+1)}(t)\left(P_{\mathbf{w}|t}\left(\widetilde{\mathcal{R}}^{(i)}_{g,n}(\mathbf{w}|\mathbf{z})\right)\right)\right]_{\mathbf{b}}\,,$$

with $\widetilde{\mathcal{R}}^{(i)}_{g,n}(\mathbf{w}|\mathbf{z})$ defined in Definition 4.5.16.

Lemma 4.5.24.

$$\left[\mathrm{Res}_{t=0}\, \mathrm{d}\xi_{r(k+1)}(t)\, P_{\mathbf{w}|t}\left(\widetilde{\mathcal{R}}^{(i)}_{g,n}(\mathbf{w}|\mathbf{z})\right)\right]_{\mathbf{b}} = \frac{1}{i!}\sum_{\mathbf{a}\in(\mathbb{Z}_{\neq 0})^i} C^{(0)}[k|\mathbf{a}]\, \Xi^{(i)}_{g,n}[\mathbf{a}|\mathbf{b}]\,,$$

where $\Xi^{(i)}_{g,n}[\mathbf{a}|\mathbf{b}]$ *was defined in* (4.16)*. Here, we introduced the coefficients*

$$C^{(0)}[k|\mathbf{a}] = i!\,\mathrm{Res}_{t=0}\left(\mathrm{d}\xi_{r(k+1)}(t)P_{\mathbf{w}|t}\Big(\bigotimes_{l=1}^{i}\mathrm{d}\xi_{-a_l}(w_l)\Big)\right)\,,$$

with $\mathbf{a} = (a_1,\ldots,a_i)$.

Proof. Recall from Definition 4.5.16 that

$$\widetilde{\mathcal{R}}^{(i)}_{g,n}(A|B) = \sum_{\mathbf{L}\vdash A}\ \sum_{\substack{h\,:\,\mathbf{L}\to\mathbb{N}\\ i+\sum_{L\in\mathbf{L}} h_L = g+|\mathbf{L}|}}\ \sideset{}{''}\sum_{\boldsymbol{\mu}\vdash_{\mathbf{L}} B}\left(\bigotimes_{L\in\mathbf{L}}\omega_{h_L,|L|+|\mu_L|}(L,\mu_L)\right),$$

where the double prime over the summation symbol means that terms with $h_L = 0$, $|\mu_L| = 0$ and $|L| \leq 2$ are excluded from the sum. For $2g-2+n>0$, we have an expansion

$$\begin{aligned}\omega_{g,n}(\mathbf{z}) &= \sum_{\mathbf{a}\in(\mathbb{Z}_{>0})^n} F[\mathbf{a}]\bigotimes_{l=1}^n \mathrm{d}\xi_{-a_l}(z_l)\\ &=: \sum_{\mathbf{a}\in(\mathbb{Z}_{\neq 0})^n} F[\mathbf{a}]\bigotimes_{l=1}^n \mathrm{d}\xi_{-a_l}(z_l)\,,\end{aligned}$$

where in the second line we extended the summation to all non-zero integers by setting the coefficients to be zero whenever one of the a_is is negative.

For $\omega_{0,2}$, since we are not including contribution where both entries come from A, after projection with $P_{\mathbf{w}|t}$ we know one of the entry of $\omega_{0,2}$ must be found among the w_ls, and thus project to z, while the second must be found in the z_ls. Thus we can use the following expansion for $\omega_{0,2}$,

$$\omega_{0,2}(w_1, z_2) \underset{|w_1|<|z_2|}{\approx} \sum_{a>0} a \, \mathrm{d}\xi_a(w_1) \otimes \mathrm{d}\xi_{-a}(z_2) \,,$$

and a similar one when the role of 1 and 2 is exchanged. In both situations we can do the replacement

$$\omega_{0,2}(w_1, z_2) \longleftarrow \sum_{\mathbf{a} \in (\mathbb{Z}_{\neq 0})^2} F_{0,2}[\mathbf{a}] \, \mathrm{d}\xi_{-a_1}(z_1) \otimes \mathrm{d}\xi_{-a_2}(z_2) \,,$$

where we introduced the coefficients $F_{0,2}[a_1, a_2] = |a_1| \delta_{a_1+a_2,0}$. With this notation, and recalling (4.16), we can write

$$\widetilde{\mathcal{R}}^{(i)}_{g,n}(\mathbf{w}|\mathbf{z}) = \sum_{\substack{\mathbf{a} \in (\mathbb{Z}_{\neq 0})^i \\ \mathbf{b} \in (\mathbb{Z}_{\neq 0})^{n-1}}} \Xi^{(i)}_{g,n}[\mathbf{a}|\mathbf{b}] \bigotimes_{l=1}^{i} \mathrm{d}\xi_{-a_l}(w_l) \bigotimes_{m=2}^{n} \mathrm{d}\xi_{-b_m}(z_m) \,.$$

It thus follows that

$$\begin{aligned} &\left[\mathrm{Res}_{t=0} \, \mathrm{d}\xi_{r(k+1)}(t) \, P_{\mathbf{w}|t} \left(\widetilde{\mathcal{R}}^{(i)}_{g,n}(\mathbf{w}|\mathbf{z}) \right) \right]_{\mathbf{b}} \\ &= \sum_{\mathbf{a} \in (\mathbb{Z}_{\neq 0})^i} \Xi^{(i)}_{g,n}[\mathbf{a}|\mathbf{b}] \, \mathrm{Res}_{t=0} \left(\mathrm{d}\xi_{r(k+1)}(t) \, P_{\mathbf{w}|t} \Big(\bigotimes_{l=1}^{i} \mathrm{d}\xi_{-a_l}(w_l) \Big) \right) \,, \end{aligned}$$

and the lemma is proven. $\square$

Let us now re-introduce contributions from $\omega_{0,2}$ with the two entries coming from $\mathbf{w}$.

Lemma 4.5.25.

$$\left[\mathrm{Res}_{t=0} \, \mathrm{d}\xi_{r(k+1)}(t) \, P_{\mathbf{w}|t} \left(\mathcal{R}^{(i)}_{g,n}(\mathbf{w}|\mathbf{z}) \right) \right]_{\mathbf{j}} = \sum_{\substack{\ell, j \geq 0 \\ \ell + 2j = i}} \frac{1}{\ell!} \sum_{\mathbf{a} \in (\mathbb{Z}_{\neq 0})^{\ell}} C^{(j)}[k|\mathbf{a}] \, \Xi^{(\ell)}_{g-j,n}[\mathbf{a}|\mathbf{b}] \,,$$

with the coefficients defined as

$$C^{(j)}[k|\mathbf{a}] = \frac{(\ell+2j)!}{j! \, 2^j} \mathrm{Res}_{t=0} \left(\mathrm{d}\xi_{r(k+1)}(t) \, P_{\mathbf{w}|t} \Big(\bigotimes_{l=1}^{\ell} \mathrm{d}\xi_{-a_l}(w_l) \bigotimes_{m=1}^{j} \omega_{0,2}(w_{\ell+2m-1}, w_{\ell+2m}) \Big) \right) \,. \tag{4.101}$$

Proof. Recall from Lemma 4.5.18 that

$$\mathcal{R}^{(i)}_{g,n}(A|B) = \sum_{\substack{\ell,j\geq 0\\ \ell+2j=i}} \sum_{\substack{\gamma\subseteq A\\ |\gamma|=2j}} \Bigg(\sum_{\substack{\mathbf{L}\vdash\gamma\\ \forall L\in\mathbf{L}\ |L|=2}} \bigotimes_{L\in\mathbf{L}} \omega_{0,2}(L) \Bigg) \otimes \widetilde{\mathcal{R}}^{(\ell)}_{g-j,n}(A\setminus\gamma|B)\,.$$

Thus

$$P_{\mathbf{w}|t}\left(\mathcal{R}^{(i)}_{g,n}(\mathbf{w}|\mathbf{z})\right) = \sum_{\substack{\ell,j\geq 0\\ \ell+2j=i}} P_{\mathbf{w}|t}\left(\sum_{\substack{\gamma\subseteq \mathbf{w}\\ |\gamma|=2j}} \Bigg(\sum_{\substack{\mathbf{L}\vdash\gamma\\ \forall L\in\mathbf{L}\ |L|=2}} \bigotimes_{L\in\mathbf{L}} \omega_{0,2}(L) \Bigg) \otimes \widetilde{\mathcal{R}}^{(\ell)}_{g-j,n}(\mathbf{w}\setminus\gamma|\mathbf{z}) \right)\,.$$

Since $P_{\mathbf{w}|t}$ is invariant under permutations of the w_ls, the order of the w_ls in this expression does not matter. So all terms that only differ by permutations of the w_ls will give the same result after acting with the projection operator. So we can order the w_ls once and for all. We simply need to count the number of terms for a given j. We first need to pick a subsequence γ of $\mathbf{w}$ of length $2j$: there are $\frac{i!}{(2j)!(i-2j)!}$ ways to do so. Then, we need to pick a set partition $\mathbf{L}$ of γ with parts that all have cardinality two. The number of ways of doing so is $(2j-1)\cdot(2j-3)\cdots 1 = \frac{(2j)!}{2^j\, j!}$. Thus we end up with

$$P_{\mathbf{w}|t}\left(\mathcal{R}^{(i)}_{g,n}(\mathbf{w}|\mathbf{z})\right) = \sum_{\substack{\ell,j\geq 0\\ \ell+2j=i}} \frac{i!}{\ell! j! 2^j} P_{\mathbf{w}|t}\left(\widetilde{\mathcal{R}}^{(\ell)}_{g-j,n}(w_1,\ldots,w_\ell|\mathbf{z}) \otimes \bigotimes_{l'=1}^{j} \omega_{0,2}(w_{\ell+2l'-1},w_{\ell+2l'}) \right)\,.$$

As before, we can write

$$\widetilde{\mathcal{R}}^{(\ell)}_{g-j,n}(w_1,\ldots,w_\ell|\mathbf{z}) = \sum_{\substack{\mathbf{a}\in(\mathbb{Z}_{\neq 0})^\ell\\ \mathbf{b}\in(\mathbb{Z}_{\neq 0})^{n-1}}} \Xi^{(\ell)}_{g-j,n}[\mathbf{a}|\mathbf{b}] \bigotimes_{l=1}^{\ell} \mathrm{d}\xi_{-a_l}(w_l) \bigotimes_{m=2}^{n} \mathrm{d}\xi_{-b_m}(z_m)\,.$$

Therefore,

$$\begin{aligned}\left[\mathrm{Res}_{t=0}\,\mathrm{d}\xi_{r(k+1)}(t)\left(P_{\mathbf{w}|t}\left(\mathcal{R}^{(i)}_{g,n}(\mathbf{w}|\mathbf{z})\right)\right)\right]_{\mathbf{b}} &= \sum_{\substack{\ell,j\geq 0\\ \ell+2j=i}} \frac{i!}{\ell! j! 2^j} \sum_{\mathbf{a}\in(\mathbb{Z}_{\neq 0})^i} \Xi^{(i)}_{g-j,n}[\mathbf{a}|\mathbf{b}] \\ \times\,\mathrm{Res}_{t=0}\left(\mathrm{d}\xi_{r(k+1)}(t) P_{\mathbf{w}|t}\left(\bigotimes_{l=1}^{\ell} \mathrm{d}\xi_{-a_l}(w_l) \right) \otimes \bigotimes_{l'=1}^{j} \omega_{0,2}(w_{\ell+2l'-1},w_{\ell+2l'}) \right)\,,\end{aligned}$$

and the lemma is proven. □

Finally we need to re-introduce the contributions from $\omega_{0,1}$. Recall that we can write

$$\omega_{0,1}(z) = \sum_{a>0} \tau_a \, \mathrm{d}\xi_a(z) =: \sum_{a \in \mathbb{Z}_{\neq 0}} F_{0,1}[a] \, \mathrm{d}\xi_{-a}(z) \, , \tag{4.102}$$

where we defined the coefficients to be $F_{0,1}[a] = 0$ and $F_{0,1}[-a] = \tau_a$ for all $a > 0$. Then

Lemma 4.5.26.

$$\left[\mathrm{Res}_{t=0} \, \mathrm{d}\xi_{r(k+1)}(t) \left(P_{\mathbf{w}|t}\left(\mathcal{E}^{(i)}_{g,n}(\mathbf{w}|\mathbf{z})\right)\right)\right]_{\mathbf{b}} = \sum_{m=1}^{i} \sum_{\substack{\ell, j \geq 0 \\ \ell + 2j = m}} \frac{1}{\ell!} \sum_{\mathbf{a} \in (\mathbb{Z}_{\neq 0})^{\ell}} D^{(j)}_i[k|\mathbf{a}] \, \Xi^{(\ell)}_{g-j,n}[\mathbf{a}|\mathbf{b}] \, ,$$

where we defined the coefficients

$$D^{(j)}_i[k|\mathbf{a}] := \frac{1}{(i-\ell-2j)!} \sum_{a_{\ell+1}, \ldots, a_{i-2j} \in \mathbb{Z}_{\neq 0}} \left(\prod_{l=\ell+1}^{i-2j} F_{0,1}[a_l]\right) C^{(j)}[k|\mathbf{a}, a_{\ell+1}, \ldots, a_{i-2j}] \, . \tag{4.103}$$

Proof. Recall from Lemma 4.5.17 that

$$\mathcal{E}^{(i)}_{g,n}(A|B) = \sum_{m=0}^{i} \sum_{\substack{\gamma \subseteq A \\ |\gamma| = m}} \left(\prod_{l=1}^{i} \omega_{0,1}(\gamma_l)\right) \otimes \mathcal{R}^{(i-m)}_{g,n}(A \setminus \gamma | B) \, .$$

Note that for $2g - 2 + n > 0$, the term with $m = i$ does not contribute, so we can terminate the sum over m at $i - 1$. Thus

$$P_{\mathbf{w}|t}\left(\mathcal{E}^{(i)}_{g,n}(\mathbf{w}|\mathbf{z})\right) = \sum_{m=0}^{i-1} P_{\mathbf{w}|t}\left(\sum_{\substack{\gamma \subseteq \mathbf{w} \\ |\gamma| = m}} \left(\bigotimes_{l=1}^{m} \omega_{0,1}(\gamma_l)\right) \otimes \mathcal{R}^{(i-m)}_{g,n}(\mathbf{w} \setminus \gamma | \mathbf{z})\right) \, .$$

As before, we use the argument that the projection operator is invariant under permutations of the w_ls to re-order the entries in the argument. Thus all terms contribute the same. The number of terms is the number of ways to choose m elements in $\mathbf{w}$, which is given by $\frac{i!}{m!(i-m)!}$. So we get

$$P_{\mathbf{w}|t}\left(\mathcal{E}^{(i)}_{g,n}(\mathbf{w}|\mathbf{z})\right) = \sum_{m=0}^{i-1} \frac{i!}{m!(i-m)!} P_{\mathbf{w}|t}\left(\mathcal{R}^{(i-m)}_{g,n}(w_1, \ldots, w_{i-m}|\mathbf{z}) \otimes \bigotimes_{l=i-m+1}^{i} \omega_{0,1}(w_l)\right) \, .$$

Using Lemma 4.5.25 we know that

$$P_{\mathbf{w}|t}\left(\mathcal{R}^{(i-m)}_{g,n}(w_1,\ldots,w_{i-m}|\mathbf{z})\bigotimes_{l=i-m+1}^{i}\omega_{0,1}(w_l)\right)$$
$$=\sum_{\substack{\ell,j\geq 0\\ \ell+2j=i-m}}\frac{(i-m)!}{\ell!\,j!\,2^j}P_{\mathbf{w}|t}\left(\widetilde{\mathcal{R}}^{(\ell)}_{g-j,n}(w_1,\ldots,w_\ell|\mathbf{z})\bigotimes_{l'=1}^{j}\omega_{0,2}(w_{\ell+2l'-1},w_{\ell+2l'})\bigotimes_{l=i-m+1}^{i}\omega_{0,1}(w_l)\right).$$

We now have as usual

$$\widetilde{\mathcal{R}}^{(\ell)}_{g-j,n}(w_1,\ldots,w_\ell|\mathbf{z})=\sum_{\substack{\mathbf{a}\in(\mathbb{Z}_{\neq 0})^\ell\\ \mathbf{b}\in(\mathbb{Z}_{\neq 0})^{n-1}}}\Xi^{(\ell)}_{g-j,n}[\mathbf{a}|\mathbf{b}]\bigotimes_{l=1}^{\ell}\mathrm{d}\xi_{-a_l}(w_l)\bigotimes_{m=2}^{n}\mathrm{d}\xi_{-b_m}(z_m).$$

Therefore,

$$\left[\mathrm{Res}_{t=0}\,\mathrm{d}\xi_{r(k+1)}(t)\,P_{\mathbf{w}|t}\left(\mathcal{E}^{(i)}_{g,n}(\mathbf{w}|\mathbf{z})\right)\right]_{\mathbf{b}}=\sum_{m=0}^{i-1}\sum_{\substack{\ell,j\geq 0\\ \ell+2j=i-m}}\frac{i!}{m!\,\ell!\,j!\,2^j}\sum_{\mathbf{a}\in(\mathbb{Z}_{\neq 0})^\ell}\Xi^{(\ell)}_{g-j,n}[\mathbf{a}|\mathbf{b}]$$
$$\times\mathrm{Res}_{t=0}\left(\mathrm{d}\xi_{r(k+1)}(t)P_{\mathbf{w}|t}\left(\bigotimes_{l=1}^{\ell}\mathrm{d}\xi_{-a_l}(w_l)\bigotimes_{l''=\ell+1}^{\ell+m}\omega_{0,1}(w_{l''})\bigotimes_{l'=1}^{j}\omega_{0,2}(w_{\ell+m+2l'-1},w_{\ell+m+2l'})\right)\right).$$

Expanding $\omega_{0,1}$ as in (4.102) we can write

$$\mathrm{Res}_{t=0}\left(\mathrm{d}\xi_{r(k+1)}(t)P_{\mathbf{w}|t}\Big(\bigotimes_{l=1}^{\ell}\mathrm{d}\xi_{-a_l}(w_l)\bigotimes_{l''=\ell+1}^{\ell+m}\omega_{0,1}(w_{l''})\bigotimes_{l'=1}^{j}\omega_{0,2}(w_{\ell+m+2l'-1},w_{\ell+m+2l'})\Big)\right)$$
$$=\sum_{\substack{a_{\ell+1},\ldots,a_{\ell+m}\\ \in\mathbb{Z}_{\neq 0}}}\left(\prod_{l''=\ell+1}^{\ell+m}F_{0,1}[a_{l''}]\right)\mathrm{Res}_{t=0}\left(\mathrm{d}\xi_{r(k+1)}(t)\,P_{\mathbf{w}|t}\Big(\bigotimes_{l=1}^{\ell+m}\mathrm{d}\xi_{-a_l}(w_l)\bigotimes_{l'=1}^{j}\omega_{0,2}(w_{\ell+m+2l'-1},w_{\ell+m+2l'})\Big)\right)$$
$$=\sum_{\substack{a_{\ell+1},\ldots,a_{\ell+m}\\ \in\mathbb{Z}_{\neq 0}}}\left(\prod_{l''=\ell+1}^{\ell+m}F_{0,1}[a_{l''}]\right)\frac{j!\,2^j}{(\ell+m+2j)!}\,C^{(j)}[k|\mathbf{a},a_{\ell+1},\ldots,a_{\ell+m}],$$

with $\mathbf{a}\in(\mathbb{Z}_{\neq 0})^\ell$ by definition of $C^{(j)}$ in (4.101). After introducing a new index $m'=i-m$ to rewrite the sum over $m\in\{0,\ldots,i-1\}$ as a sum over $m'\in\{1,\ldots,i\}$, we obtain

$$\left[\mathrm{Res}_{t=0}\,\mathrm{d}\xi_{r(k+1)}(t)\,P_{\mathbf{w}|t}\left(\mathcal{E}^{(i)}_{g,n}(\mathbf{w}|\mathbf{z})\right)\right]_{\mathbf{b}}=\sum_{m'=1}^{i}\sum_{\substack{\ell,j\geq 0\\ \ell+2j=m'}}\frac{1}{(i-m')!\,\ell!}$$
$$\times\sum_{\mathbf{a}\in(\mathbb{Z}_{\neq 0})^\ell}\Xi^{(\ell)}_{g-j,n}[\mathbf{a}|\mathbf{b}]\sum_{a_{\ell+1},\ldots,a_{i-2j}\in\mathbb{Z}_{\neq 0}}\left(\prod_{l''=\ell+1}^{i-2j}F_{0,1}[a_{l''}]\right)C^{(j)}[k|\mathbf{a},a_{\ell+1},\ldots,a_{i-2j}].$$

We recognize the coefficients $D_i^{(j)}$ introduced in (4.103) and the lemma is proven. $\square$

We can now finish the proof of Theorem 4.5.23. From Lemmas 4.5.24, 4.5.25 and 4.5.26, we find that the higher abstract loop equations for local spectral curves with one component hold if and only if

$$\sum_{m=1}^{i} \sum_{\substack{\ell,j\geq 0 \\ \ell+2j=m}} \frac{1}{\ell!} \sum_{\mathbf{a}\in(\mathbb{Z}_{\neq 0})^\ell} D_i^{(j)}[k|\mathbf{a}]\, \Xi_{g-j,n}^{(\ell)}[\mathbf{a}|\mathbf{b}] = 0\,,$$

for $i \in \{1,\ldots,r\}$, $k \geq \mathfrak{d}^i$ and $\mathbf{b} \in (\mathbb{Z}_{>0})^n$. Just as in Lemma 4.2.13 this is equivalent to the claimed system of differential equations. $\square$

Identification with the $\mathcal{W}(\mathfrak{gl}_r)$ quantum Airy structure

We now relate the differential operators appearing in Proposition 4.5.23 with the $\mathcal{W}(\mathfrak{gl}_r)$ quantum Airy structure of Proposition 4.4.13.

Theorem 4.5.27. *Under the conditions of Propositions 4.5.23, we have for any $i \in \{1,\ldots,r\}$ and $k \geq \mathfrak{d}^i + \delta_{i,1}$ the identification*

$$\mathcal{H}_k^i = -r\,\hat{\Phi}\hat{T}W_k^i\hat{T}^{-1}\,\hat{\Phi}^{-1}\,.$$

Here, W_k^i are defined in (4.59) *and we use the dilaton shift and change of polarization defined in terms of the coefficients of expansion* (4.95)-(4.96) *of $\omega_{0,1}$ and $\omega_{0,2}$*

$$\hat{T} = \exp\left(\frac{1}{\hbar}\sum_{a>0}\frac{F_{0,1}[-a]}{a}\,J_a\right)\,, \qquad \hat{\Phi} = \exp\left(\frac{1}{2\hbar}\sum_{l,m>0}\frac{\phi_{l,m}}{l\,m}J_lJ_m\right)\,.$$

In particular, for admissible spectral curves, where $1 \leq s \leq r+1$ and $r = \pm 1 \bmod s$, the $\mathcal{H}_k^i$ form a $\mathcal{W}(\mathfrak{gl}_r)$ higher quantum Airy structure as in Proposition 4.4.13. The coefficients $F_{g,n}$ of the partition function of this $\mathcal{W}(\mathfrak{gl}_r)$ higher quantum Airy structure in the basis $(x_l)_{l>0}$ coincide with the coefficients of the expansion (4.94) *of the unique $\omega_{g,n} \in (V_z^-)^{\otimes n}$ solution to the higher abstract loop equations* (4.98).

Proof. We first concentrate on the case of standard polarization, that is

$$\omega_{0,2}(z_1,z_2) = \frac{\mathrm{d}z_1 \otimes \mathrm{d}z_2}{(z_1-z_2)^2}\,.$$

We are going to evaluate the coefficients $C^{(j)}[k|\mathbf{a}]$ and $D_i^{(j)}[k|\mathbf{a}]$ and recognize the coefficients of the higher quantum Airy structure of Proposition 4.4.12. Recall in particular from Definition 4.4.3 the sums $\Psi^{(j)}(a_1,\ldots,a_i)$ over r-th roots of unity.

Lemma 4.5.28. *For a local spectral curve with one component in standard polarization,*

$$C^{(j)}[k|\mathbf{a}] = \frac{(\ell+2j)!}{j!\,2^j}\,\Psi^{(j)}(\mathbf{a})\,\delta_{r(\ell+2j-k-1)+\sum_{l=1}^{\ell} a_l,0}\,,$$

with $\mathbf{a}\in(\mathbb{Z}_{\neq 0})^{\ell}$.

Proof. Recall that

$$C^{(j)}[k|\mathbf{a}] = \frac{(\ell+2j)!}{j!\,2^j}\,\mathrm{Res}_{t=0}\left(\mathrm{d}\xi_{r(k+1)}(t)P_{\mathbf{w}|t}\left(\bigotimes_{l=1}^{\ell}\mathrm{d}\xi_{-a_l}(w_l)\bigotimes_{l'=1}^{j}\omega_{0,2}(w_{\ell+2l'-1},w_{\ell+2l'})\right)\right)\,.$$

For a curve in standard polarization, this simplifies to

$$C^{(j)}[k|\mathbf{m}] = \frac{(\ell+2j)!}{j!\,2^j}\,\mathrm{Res}_{t=0}\left(t^{r(k+1)-1}\mathrm{d}t\,P_{\mathbf{w}|t}\left(\prod_{l=1}^{\ell}w_l^{-a_l-1}\prod_{l'=1}^{j}\frac{1}{(w_{\ell+2l'-1}-w_{\ell+2l'})^2}\bigotimes_{l''=1}^{i}\mathrm{d}w_{l''}\right)\right)\,.$$

By definition of the projection operator and using Definition 4.4.3 for $\Psi^{(j)}$, we can write

$$\begin{aligned} C^{(j)}[k|\mathbf{a}] &= \frac{(\ell+2j)!}{j!\,2^j}\,\Psi^{(j)}(\mathbf{a})\,\mathrm{Res}_{t=0}\left(t^{rk+r-1-\sum_{l=1}^{\ell}a_l-r(\ell+2j)}\,\mathrm{d}t\right) \\ &= \frac{(\ell+2j)!}{j!\,2^j}\,\Psi^{(j)}(\mathbf{a})\,\delta_{r(\ell+2j-k-1)+\sum_{l=1}^{\ell}a_l,0}\,. \end{aligned}$$

∎

We can then calculate the coefficients $D_i^{(j)}[k|\mathbf{a}]$.

Lemma 4.5.29. *For a local spectral curve with one component in standard polarization,*

$$\begin{aligned} D_i^{(j)}[k|\mathbf{a}] &= \frac{i!}{(i-\ell-2j)!j!\,2^j} \\ &\quad\times\sum_{a_{\ell+1},\ldots,a_{i-2j}\in\mathbb{Z}_{\neq 0}}\left(\prod_{l=\ell+1}^{i-2j}F_{0,1}[a_l]\right)\Psi^{(j)}(\mathbf{a},a_{\ell+1},\ldots,a_{i-2j})\,\delta_{r(i-k-1)+\sum_{l=1}^{i-2j}a_l,0}\,, \end{aligned}$$

with $\mathbf{a} = (a_1, \ldots, a_\ell) \in (\mathbb{Z}_{\neq 0})^\ell$.

Proof. Recall that

$$D_i^{(j)}[k|\mathbf{a}] := \frac{1}{(i-\ell-2j)!} \sum_{a_{\ell+1},\ldots,a_{i-2j} \in \mathbb{Z}_{\neq 0}} \left(\prod_{l=\ell+1}^{i-2j} F_{0,1}[a_l] \right) C^{(j)}[k|\mathbf{a}, a_{\ell+1}, \ldots, a_{i-2j}] \,.$$

Thus, using the previous Lemma,

$$\begin{aligned} D_i^{(j)}[k|\mathbf{a}] \quad &= \quad \frac{i!}{(i-\ell-2j)!j!\,2^j} \\ &\qquad \times \sum_{a_{\ell+1},\ldots,a_{i-2j} \in \mathbb{Z}_{\neq 0}} \left(\prod_{l=\ell+1}^{i-2j} F_{0,1}[a_l] \right) \Psi^{(j)}(\mathbf{a}, a_{\ell+1}, \ldots, a_{i-2j}) \delta_{r(i-k-1)+\sum_{l=1}^{i-2j} a_l, 0} \,. \end{aligned}$$

□

Combining Proposition 4.5.23 with the two previous lemmas, we find that

$$\begin{aligned} \mathcal{H}_k^i \quad &= \quad \sum_{m=1}^{i} \sum_{\substack{\ell,j \geq 0 \\ \ell+2j=m}} \frac{\hbar^j}{\ell!} \frac{i!}{(i-\ell-2j)!j!\,2^j} \\ &\qquad \times \sum_{\mathbf{a} \in (\mathbb{Z}_{\neq 0})^{i-2j}} \left(\prod_{l=\ell+1}^{i-2j} F_{0,1}[a_l] \right) \Psi^{(j)}(\mathbf{a})\, \delta_{r(i-k-1)+\sum_{l'=1}^{i-2j} m_{l'}, 0} :J_{a_1} \cdots\, J_{a_\ell}: . \end{aligned}$$

This can be simplified by writing the sum in a more symmetric way. Instead of extracting $(a_{\ell+1}, \ldots, a_{i-2j})$, we can sum over all ways of extracting $i-2j-\ell$ as out of the $i-2j$ ones. We then need to multiply by the factor $\frac{(i-2j-\ell)!\ell!}{(i-2j)!}$ to avoid overcounting. We get

$$\mathcal{H}_k^i = \sum_{m=1}^{i} \sum_{\substack{\ell,j \geq 0 \\ \ell+2j=m}} \hbar^j \frac{i!}{(i-2j)!j!\,2^j} \sum_{\substack{\mathbf{a} \in (\mathbb{Z}_{\neq 0})^{i-2j} \\ \sum_{l=1}^{i-2j} m_l = r(k-i+1)}} \Psi^{(j)}(\mathbf{a}) \left(\sum_{\substack{\mathbf{c} \subseteq \mathbf{a} \\ |\mathbf{c}|=i-m}} \prod_{l'=1}^{i-m} F_{0,1}[c_{l'}] \right) :J_{a_1} \cdots\, J_{a_\ell}: ,$$

where in the cases that $i = 2j$ the condition $\sum_l a_l = r(k-i+1)$ is understood as the delta condition $\delta_{k,i-1}$.

Let us now compare to the dilaton-shifted W_k^is. Recall that conjugation by $\hat{T}$ is equivalent to the shift $J_{-a} \longrightarrow J_{-a} + F_{0,1}[-a]$. It results in that the coefficient of $:J_{a_1} \cdots J_{a_\ell}:$ in H_k^i should be the sum of all possible ways of starting with a term of the form $:J_{a_1} \cdots J_{a_{\ell'}}:$ with $\ell' > \ell$ in W_k^i, and replacing the extra J_as by $F_{0,1}[a]$. This is exactly what the formula for H_k^i does, up to a global prefactor $-r$.

We now turn to the general polarization, *i.e.* we have a basis for V_z^-

$$\mathrm{d}\xi_{-l}(z) = \frac{\mathrm{d}z}{z^{l+1}} + \sum_{m>0} \frac{\phi_{l,m}}{l} \mathrm{d}\xi_m(z) \,,$$

for some symmetric coefficients $\phi_{l,m}$, and a formal bidifferential

$$\omega_{0,2}(z_1, z_2) = \frac{\mathrm{d}z_1 \otimes \mathrm{d}z_2}{(z_1 - z_2)^2} + \sum_{l,m>0} \phi_{l,m} \mathrm{d}\xi_l(z_1) \otimes \mathrm{d}\xi_m(z_2) \,.$$

Looking at the definition of $D_i^{(j)}[k|\mathbf{a}]$, (4.99) and following the same argument as for the case of standard polarization, it is clear that the relation between the $D_i^{(j)}[k|\mathbf{a}]$ and the $C^{(j)}[k|\mathbf{a}]$ is given by the dilaton shift $J_{-a} \to J_{-a} + F_{0,1}[-a]$. Thus all we need to check here is the effect of the change of polarization. Recall the conjugation by $\hat{\Phi}$ amounts to the shift

$$\forall a > 0, \qquad J_{-a} \longrightarrow J_{-a} + \sum_{l>0} \frac{\phi_{a,l}}{l} J_l \,. \tag{4.104}$$

Suppose that we start with the W_k^i and do a polarization conjugation. Let us denote the coefficients of W_k^i by $C_{\mathrm{st}}^{(j)}[k|\mathbf{a}]$. Every factor of J_{-a} with $a > 0$ gets shifted as above. If we turn this around, this means that if we are calculating the coefficient of a term in the conjugated operator $\hat{\Phi} W_k^i \hat{\Phi}^{-1}$ that has a factor of J_m with $m > 0$, say $C^{(j)}[k|\ldots, m, \ldots]$, then this coefficient will get a contribution of the form

$$C_{\mathrm{st}}^{(j)}[k|\ldots, m, \ldots] + \sum_{l>0} \frac{\phi_{l,m}}{m} C_{\mathrm{st}}^{(j)}[k|\ldots, -l, \ldots] \,.$$

Now, since $C^{(j)}[k|\ldots, m, \ldots]$ comes with a factor of $\mathrm{d}\xi_{-m}(w)$ in the residue definition, we see that this change of coefficients is implemented by doing a change of basis

$$\mathrm{d}\xi_{-m}(w) \mapsto \mathrm{d}\xi_{-m}(w) + \sum_{l>0} \frac{\phi_{m,l}}{m} \mathrm{d}\xi_l(w) \,,$$

which is precisely what a change of polarization does. More precisely, if we start with the standard polarization, for which $\mathrm{d}\xi_{-m}(w) = w^{-m-1}\mathrm{d}w$, then this shift implements the basis definition for a general polarization

$$\mathrm{d}\xi_{-m}(w) = w^{-m-1}\mathrm{d}w + \sum_{l>0} \frac{\phi_{m,l}}{m}\, \mathrm{d}\xi_l(w)$$

in the definition of the coefficients (4.100). However, one needs to be careful. After shifting as in (4.104), the Js may not be normal ordered anymore. Normal ordering thus will produce extra contributions. This will happen whenever we are shifting the first factor in expressions of the form $J_{-a}J_{-b}$. After the shift, this becomes

$$J_{-a}J_{-b} \longrightarrow J_{-a}J_{-b} + \sum_{l>0} \frac{\phi_{a,l}}{l}\, J_l J_{-b}\,,$$

which is not normal ordered anymore. Normal ordering produces an extra contribution

$$J_{-a}J_{-b} \longrightarrow J_{-a}J_{-b} + \sum_{l>0} \frac{\phi_{a,l}}{l}\, {:}J_l J_{-b}{:} + \hbar\phi_{b,a}\,.$$

This means that the coefficient of, say, $C^{(j)}[k|\ldots]$ with $j \geq 1$, will get an extra contribution of the form $\phi_{b,a}C^{(j-1)}[k|\ldots,-b,-a]$. But since $C^{(j-1)}[k|\ldots,-b,-a]$ comes with a factor of $\mathrm{d}\xi_b(w_1)\otimes\mathrm{d}\xi_a(w_2)$ in the residue definition, these extra contributions are precisely accounted for by replacing the bidifferential $\frac{\mathrm{d}z_1\otimes\mathrm{d}z_2}{(z_1-z_2)^2}$ in standard polarization by the new bidifferential

$$\omega_{0,2}(z_1,z_2) = \frac{\mathrm{d}z_1\otimes\mathrm{d}z_2}{(z_1-z_2)^2} + \sum_{l,m>0} \phi_{l,m}\mathrm{d}\xi_l(z_1)\otimes\mathrm{d}\xi_m(z_2)$$

in the definition of the coefficients (4.100). We conclude that the definition of the coefficients (4.100) precisely implements a change of polarization from standard polarization to arbitrary polarization, hence the resulting operators are the result of conjugation by $\hat{\Phi}$. Combining with conjugation by $\hat{T}$ (note that $\hat{T}$ and $\hat{\Phi}$ commute), we obtain the statement of the theorem in full generality. □

4.5.4 Local spectral curves with several components

We now give the general result for local spectral curves with c components. Let $R = \{1,\ldots,c\}$, and define $W^i_{\alpha,k}$ to be copies indexed by $\alpha \in R$ of the differential operators (4.59) representing the modes of the generators of the $\mathcal{W}(\mathfrak{gl}_{r_\alpha})$ algebra, involving variables $(x_{\alpha,a})_{a>0}$.

Theorem 4.5.30. *For any spectral curve with c components as defined in Definition 4.5.7, the higher abstract loop equations for $\omega_{g,n} \in \mathrm{Sym}^n(\mathcal{V}_z^-)$ with $2g-2+n>0$ is equivalent to a system of differential equations*

$$\forall \alpha \in R\,, \quad i \in \{1,\ldots,r_\alpha\}\,, \quad \forall k \geq \mathfrak{d}^i_\alpha + \delta_{i,1}\,, \qquad \mathcal{H}^i_{\alpha,k}\cdot Z = 0\,,$$

where $\mathfrak{d}^i_\alpha = i - 1 - \left\lfloor \frac{s_\alpha(i-1)}{r_\alpha}\right\rfloor$, for the partition function

$$Z = \exp\left(\sum_{\substack{g\geq 0,\ n\geq 1\\ 2g-2+n>0}} \frac{\hbar^{g-1}}{n!} \sum_{\boldsymbol{\alpha}\in R^n} \sum_{\mathbf{b}\in(\mathbb{Z}_{>0})^n} F_{g,n}\left[\begin{smallmatrix}\boldsymbol{\alpha}\\ \mathbf{b}\end{smallmatrix}\right] x_{\alpha_1,b_1}\cdots x_{\alpha_n,b_n}\right)\,,$$

constructed from the coefficients of the expansion in (4.94). *The differential operators read*

$$\mathcal{H}^i_{\alpha,k} = -r_\alpha\,\hat{\Phi}\hat{T}W^i_{\alpha,k}\hat{T}^{-1}\hat{\Phi}^{-1}\,,$$

where the dilaton shift and change of polarization are given by

$$\begin{aligned}\hat{T} &= \exp\left(\frac{1}{\hbar}\sum_{\alpha=1}^{c}\sum_{a>0}\frac{F_{0,1}\left[\begin{smallmatrix}-\alpha\\ a\end{smallmatrix}\right]}{a}J_{\alpha,a}\right)\,,\\ \hat{\Phi} &= \exp\left(\frac{1}{2\hbar}\sum_{\alpha,\beta=1}^{c}\sum_{l,m>0}\frac{\phi^{\alpha,\beta}_{l,m}}{l\,m}\,J_{\alpha,l}J_{\beta,m}\right)\,.\end{aligned}$$

In particular, for admissible spectral curves, where $1 \leq s_\alpha \leq r_\alpha + 1$ and $r_\alpha = \pm 1 \bmod s_\alpha$, these $\mathcal{H}^i_{\alpha,k}$ form a higher quantum Airy structure, isomorphic to those for the $\bigoplus_\alpha \mathcal{W}(\mathfrak{gl}_{r_\alpha})$ algebra.

Proof. First, we consider the case of standard polarization, that is

$$\omega_{0,2}(z_1,z_2) = \sum_{\alpha=1}^{c}\frac{(e_\alpha\otimes \mathrm{d}z_1)\otimes(e_\alpha\otimes\mathrm{d}z_2)}{(z_1-z_2)^2}\,. \tag{4.105}$$

Since the $\mathrm{d}\xi_{\alpha,l}(z) \in \mathcal{V}_z^{(\alpha)}$ for all $l \in \mathbb{Z}_{\neq 1}$, we can really think of a local spectral curve with c components in standard polarization as c spectral curves with one component. The result then follows directly from Theorem 4.5.27.

For general polarization, the proof follows the exact same lines as for spectral curves with one component, except that we have to keep track of multi-indices. We therefore omit it. ∎

Remark 4.5.31. It is straightforward to reformulate Theorem 4.5.30 as a Givental-like decomposition formula for the partition function Z. This is presented in the statement of Theorem 7 in the introduction.

One direct consequence of the identification between higher abstract loop equations and higher quantum Airy structures is the symmetry of the meromorphic differentials constructed by the Bouchard-Eynard topological recursion.

Theorem 4.5.32. *For arbitrary admissible local spectral curves, as defined in Definitions 4.5.7 and 4.5.8, the Bouchard-Eynard topological recursion from Definition 4.5.19 produces symmetric differentials $\omega_{g,n}$.*

Proof. This follows directly from Appendix B.3. There, we show that if a polarized solution to the higher abstract loop equations exists, then it is uniquely constructed by the Bouchard-Eynard topological recursion. Thus a polarized solution exists if and only if the Bouchard-Eynard topological recursion produces symmetric differentials. But Theorem 4.5.30 implies that a polarized solution to the higher abstract loop equations does indeed exist for arbitrary admissible local spectral curves, and hence the Bouchard-Eynard must produce symmetric $\omega_{g,n}$. ☐

Remark 4.5.33. It should be emphasized here that the admissibility condition on s_α and r_α (see Definition 4.5.8) is crucial. In fact, unexpectedly, when this condition is not satisfied, the Bouchard-Eynard topological recursion does not produce symmetric differentials. This is proven in Proposition B.2.2. Indeed, for choices of s_α and r_α that are coprime but such that $r_\alpha \neq \pm 1 \mod s_\alpha$, our identification between the structure of the higher abstract loop equations and the differential equations produced by the $\mathcal{H}^i_{\alpha,k}$ is still valid. The question is whether there exists a solution to the differential constraints $\mathcal{H}^i_{\alpha,k} \cdot Z = 0$, or, equivalently, a polarized solution to the higher abstract loop equations. It turns out that the answer is no. It is argued in Proposition B.2.2 that there cannot be a symmetric solution to the differential constraints. Correspondingly, this means that the Bouchard-Eynard topological recursion cannot produce symmetric differentials in these cases[8], otherwise it would construct a polarized solution to the higher abstract loop equations. In the context of higher quantum Airy structures, this implies that for those choices of s_α and r_α the left ideal generated by the $\mathcal{H}^i_{\alpha,k}$ is not a graded Lie subalgebra.

[8]This can also be checked symbolically on Mathematica for various examples of spectral curves with $\gcd(r,s) = 1$ that do not meet the admissibility requirement, the simplest ones being

$$(r,s) = (7,5), (8,5), (9,7), (10,7), (11,7), (11,8), \text{ etc.}$$

4.6 $\mathcal{W}$ constraints and enumerative geometry

In this section, we review the currently known relations between $\mathcal{W}$ constraints and enumerative geometry. We view their possible extension to new cases as a motivation to study higher quantum Airy structures, and formulate new questions raised by our work. The leitmotiv is that for each instance of generating series appearing in one of the following situations

(i) intersection numbers of interesting classes on $\overline{\mathcal{M}}_{g,n}$,

(ii) tau functions of integrable hierarchies,

(iii) matrix integrals,

(iv) higher quantum Airy structures and their partition functions,

(v) differential constraints and partition function obtained from periods on a spectral curve,

one can ask for an equivalent description in the four other contexts.

4.6.1 r-spin intersection numbers

The Witten r-spin partition function is one of the only example completely understood from the five points of view. Its construction was sketched by Witten in [Wit93], where he proposed several conjectures which have been resolved since then.

(i) - Enumerative geometry

Let r, g, n be nonnegative integers such that $r \geq 2$, $n \geq 1$ and $2g - 2 + n > 0$. Let $i_1, \ldots, i_n \in \mathbb{Z}$ such that $2g - 2 - \sum_{l=1}^{n}(i_l - 1) \in r\mathbb{Z}$. An r-spin structure on a smooth curve $\mathcal{C}$ with punctures $p_1, \ldots, p_n$ is the data of a line bundle L and an isomorphism

$$L^{\otimes r} \simeq K\Big(- \sum_{l=1}^{n}(i_l - 1)p_l\Big)\,,$$

where K is the cotangent line bundle. Jarvis [Jar00] constructed the compactified moduli stack of (isomorphism classes of) r-spin structures $\overline{\mathcal{M}}_{g,n}(r;\mathbf{i})$.

Polishchuk and Vaintrob [Pol04, PV01], and later Chiodo [Chi06] by a different method, constructed a Chow cohomology class $\Omega_{g,n}(r, \mathbf{i})$ of pure dimension in $\overline{\mathcal{M}}_{g,n}(r; \mathbf{i})$ which has the basic properties expected by [Wit93] and called it the Witten r-spin class.

One can then introduce the r-spin partition function

$$Z_{r\text{spin}} = \exp\left\{ \sum_{2g-2+n>0} \frac{\hbar^{g-1}}{n!} \sum_{\substack{1 \leq i_1, \ldots, i_n \leq r \\ k_1, \ldots, k_n \geq 0}} \left(\int_{\overline{\mathcal{M}}_{g,n}(r;\mathbf{i})} \Omega_{g,n}(r; \mathbf{i}) \prod_{l=1}^{n} \psi_l^{k_l} \right) \prod_{l=1}^{n} t_{k_l}^{i_l} \right\} \tag{4.106}$$

on the formal variables t_k^i indexed by $k \geq 0$ and $i \in \{1, \ldots, r\}$. In fact, $\Omega_{g,n}(r; \mathbf{i})$ is zero when one of the i_l is equal to r, so one could restrict to $i \in \{1, \ldots, r-1\}$.

In particular for $r = 2$, we only have to consider the value $i = 1$ and we obtain the usual intersection numbers of ψ-classes on the moduli space of curves

$$Z_{2\text{spin}} = \exp\left\{ \sum_{2g-2+n>0} \frac{\hbar^{g-1}}{n!} \sum_{k_1, \ldots, k_n \geq 0} \left(\int_{\overline{\mathcal{M}}_{g,n}} \prod_{l=1}^{n} \psi_l^{k_l} \right) \prod_{l=1}^{n} t_{k_l} \right\}. \tag{4.107}$$

(ii) - Integrability

$Z_{r\text{spin}}$ is a tau function of the r-KdV (also called r-Gelfand-Dickey) hierarchy with respect to the times

$$x_{rk+i} = \frac{(-1)^k \, t_k^i}{\mathbf{i}\sqrt{r} \prod_{m=0}^{k} \left(m + \frac{i-1}{r}\right)}. \tag{4.108}$$

For $r = 2$, this is a famous theorem of Kontsevich [Kon92a]. For general r, it was established via a less direct path. Adler and van Moerbeke proved in [AvM92] the existence of a unique tau function $Z_{r\text{tau}}$ of the r-KdV hierarchy solving the string equation (it is unique up to a constant prefactor, which can be set equal to 1 and will not be mentioned anymore). Givental showed in [Giv03] that the total descendant potential of the A_{r-1}-singularity has the same property, and therefore coincides with $Z_{r\text{tau}}$. Later, Faber, Shadrin and Zvonkine established in [FSZ10] that $Z_{r\text{spin}}$ is equal to the total descendant potential of the A_{r-1}-singularity, as application of their proof that the Givental group preserves the notion of cohomological field theories in all genera. Therefore $Z_{r\text{spin}} = Z_{r\text{tau}}$.

(iii) - Formal matrix integrals

In [AvM92] it is proved that $Z_{r\text{tau}}$ admits a matrix model representation as follows. To be more precise, let $\mathcal{H}_N$ be the space of hermitian matrices of size N, and $Y \in \mathcal{H}_N$. We first introduce the formal matrix integral

$$Z^{(N)}_{r\text{tau}} = \frac{\int_{\mathcal{H}_N^{\text{formal}}} \mathrm{d}M\, e^{\hbar^{-1/2}\mathrm{Tr}[-V(Y+M)+V(M)+YV'(M)]}}{\int_{\mathcal{H}_N}^{\text{formal}} \mathrm{d}M\, e^{-\hbar^{-1/2}\,\mathrm{Tr}\,V_2(Y,M)}}\,, \tag{4.109}$$

where

$$V(M) = \frac{\mathbf{i}\sqrt{r}}{r+1}\, M^{r+1}, \qquad V_2(Y,M) = \frac{\mathbf{i}\sqrt{r}}{2}\sum_{m=0}^{r-1} Y^m M Y^{r-1-m} M\,.$$

It is possible to define the $N \to \infty$ limit of (4.109) as a formal series in the variables

$$x_k = \frac{\hbar^{1/2}}{k}\,\mathrm{Tr}\,Y^{-k}\,, \qquad k > 0\,,$$

which takes the form

$$Z_{r\text{tau}} = \exp\Bigg(\sum_{2g-2+n>0} \frac{\hbar^{g-1}}{n!} \sum_{k_1,\ldots,k_n \geq 0} F^{r\text{tau}}_{g,n}(k_1,\ldots,k_n) \prod_{l=1}^{n} x_{k_l}\Bigg)\,.$$

(iv) - $\mathcal{W}$ constraints

Another side result of [AvM92] says that $Z_{r\text{tau}}$ satisfies $\mathcal{W}(\mathfrak{gl}_r)$ constraints determining it uniquely. As a matter of fact, these constraints coincide with the differential operators of Theorem 4.4.9 with $s = r+1$, and $Z_{r\text{tau}} = Z_{(r,r+1)}$ the partition function of this quantum r-Airy structure.

(v) - Periods

The $\mathcal{W}$ constraints for the total descendant potential of the A_{r-1} singularity were expressed in terms of period computations by Bakalov and Milanov [BM08]. Milanov [Mil16] later established their equivalence with the Bouchard-Eynard topological recursion on the r-Airy spectral curve $y = -z$,

$x = \frac{z^r}{r}$. In our context, the theorem of Milanov is equivalent to the identification of the higher quantum Airy structure of Theorem 4.4.9 with $s = r + 1$ with the Bouchard-Eynard topological recursion as in Theorem 4.5.27, for the particular case of the r-Airy spectral curve $y = -z$, $x = \frac{z^r}{r}$.

4.6.2 Brézin-Gross-Witten theory

Consider the formal matrix integral introduced in [BG80, GW80]

$$Z^{(N)}_{r\mathrm{BGW}} = \frac{\int^{\mathrm{formal}}_{\mathcal{H}_N} \mathrm{d}M\,(\det M)^{-N}\, e^{\hbar^{-1/2}\,\mathrm{Tr}[YM + \frac{M^{1-r}}{r-1}]}}{\mathrm{Det}^{-1/2}(-Q_Y/2\pi\hbar^{1/2})\,(\det Y)^{N/r}\, e^{\hbar^{-1/2}\mathrm{Tr}[\frac{r}{r-1}\,Y^{1-1/r}]}}\,, \tag{4.110}$$

where Q_Y is the Hessian of $M \mapsto \mathrm{Tr}\,\frac{M^{r-1}}{r-1}$ at the point $Y^{-1/r}$, seen as an endomorphism in $\mathcal{H}_N$. It is possible to define the large N limit of (4.110) as a formal series $Z_{r\mathrm{BGW}}$ in the times $x_k = \frac{\hbar^{1/2}}{k}\,\mathrm{Tr}\,Y^{-k/r}$, which takes the form

$$Z_{r\mathrm{BGW}} = \exp\left(\sum_{2g-2+n>0} \frac{\hbar^{g-1}}{n!} \sum_{k_1,\ldots,k_n \geq 0} F^{r\mathrm{BGW}}_{g,n}(k_1,\ldots,k_n) \prod_{l=1}^{n} x_{k_l}\right).$$

The work of [MMS95] proves that $Z_{r\mathrm{BGW}}$ is a tau function of the KdV hierarchy with respect to the times $x_k = \frac{1}{k}\,\mathrm{Tr}\,Y^{-k/r}$.

If we focus on $Z_{2\mathrm{BGW}}$, [MMS95] proves that it satisfies Virasoro constraints (see also [Ale16]). These constraints are equivalent to the statement that

$$\omega^{2\mathrm{BGW}}_{g,n}(z_1,\ldots,z_n) = \sum_{k_1,\ldots,k_n \geq 0} F^{2\mathrm{BGW}}_{g,n}(k_1,\ldots,k_n) \prod_{l=1}^{n} \frac{(2k_l+1)!!\,\mathrm{d}z_l}{z_l^{2k_l+2}}$$

is computed by the Chekhov-Eynard-Orantin topological recursion for the Bessel spectral curve [ND16]

$$x(z) = \frac{z^2}{2}, \qquad y(z) = -\frac{1}{z}, \qquad \omega_{0,2}(z_1,z_2) = \frac{\mathrm{d}z_1 \mathrm{d}z_2}{(z_1 - z_2)^2}\,.$$

This is also equivalent [ABCO17] to saying that $Z_{2\mathrm{BGW}}$ is the partition function of a quantum Airy structure. Here it corresponds to the $\mathcal{W}(\mathfrak{sl}_2)$ quantum Airy structure associated with the dilaton shift $x_1 \to x_1 - 1$, *i.e.* after reduction of (4.64) to $x_{2m} = 0$ for $m > 0$ (see Section 4.2.2). In the notation of Theorem 1 this means $Z_{2\mathrm{BGW}} = Z_{(2,1)}$.

Norbury [Nor17] constructed a cohomology class $\Theta_{g,n} \in H^{4g-4+2n}(\overline{\mathcal{M}}_{g,n})$ such that

$$F_{g,n}^{2\mathrm{BGW}}(k_1,\ldots,k_n) = \int_{\overline{\mathcal{M}}_{g,n}} \Theta_{g,n} \prod_{l=1}^{n} \psi_l^{k_l}\,. \qquad (4.111)$$

His construction starts with the moduli space of spin structures $\overline{\mathcal{M}}_{g,n}^{(2)} = \mathcal{M}_{g,n}(2,(0,\ldots,0))$. One constructs a vector bundle $E_{g,n}$ over $\mathcal{M}_{g,n}^{(2)}$ whose fiber at a smooth point is $H^1(L^\vee)^\vee$ where $\vee$ indicates the dual. In abstract terms, one looks at the universal curve $\pi : \mathcal{C} \to \mathcal{M}_{g,n}^{(2)}$ and the bundle of the universal spin structure $\mathcal{E}$ over $\mathcal{C}$, and take $E_{g,n} = R^1\pi_*\mathcal{E}_{g,n}^\vee$. This is a bundle of rank $2g-2+n$ and one can consider its Euler class and push it forward through the forgetful map $p : \overline{\mathcal{M}}_{g,n}^{(2)} \longrightarrow \overline{\mathcal{M}}_{g,n}$. This is up to a normalization the desired class

$$\Theta_{g,n} = (-2)^n\, p_* c_{2g-2+n}(E_{g,n})\,,$$

where c_i is the i-th Chern class.

Therefore, we have a description of $Z_{2\mathrm{BGW}}$ from the five points of view. It is natural to ask if the same understanding can be obtained for the partition function of the $\mathcal{W}(\mathfrak{sl}_r)$ quantum Airy structure associated with the dilaton shift $x_1 \to x_1 - 1$. By Theorems 4.5.23 and 4.5.27 we know that it is computed by the Bouchard-Eynard topological recursion on the r-Bessel curve

$$x(z) = \frac{z^r}{r}, \qquad y(z) = -\frac{1}{z^{r-1}}, \qquad \omega_{0,2}(z_1,z_2) = \frac{\mathrm{d}z_1\mathrm{d}z_2}{(z_1-z_2)^2}\,.$$

Thus we already have the link between (iv) and (v). $Z_{r\mathrm{BGW}}$ is a natural candidate for its matrix integral/tau function representation. It is indeed known that $Z_{r\mathrm{BGW}}$ satisfies a set of $\mathcal{W}$ constraints. However they are not easy to write down explicitly, and should be compared to (4.64) to complete the identification. Di Yang and Chunhui Zhou informed us that they matched the two, hence $Z_{r\mathrm{BGW}} = Z_{(r,1)}$. As for the link with enumerative geometry,

Question 4.6.1. *Can one generalize Norbury's construction and get a class $\Theta_{g,n}^{(r)} \in H^\bullet(\overline{\mathcal{M}}_{g,n})$ such that the generating series of its intersection with ψ-classes is $Z_{r\mathrm{BGW}}$?*

We can ask the same question for the higher quantum Airy structured based on the $\mathcal{W}(\mathfrak{gl}_r)$-module obtained by twisting by a Coxeter element and the dilaton shift $x_s \to x_s - \frac{1}{s}$ for $s \in \{2, \ldots, r-1\}$ coprime with r and such that $r = \pm 1 \mod s$. We know that its partition function is computed by the topological recursion for the (r,s)-spectral curve

$$x(z) = \frac{z^r}{r}, \qquad y(z) = -\frac{1}{z^{r-s}}, \qquad \omega_{0,2}(z_1, z_2) = \frac{\mathrm{d}z_1 \mathrm{d}z_2}{(z_1 - z_2)^2}\,.$$

Question 4.6.2. *Is there a matrix model description of the partition function for the (r,s)-spectral curve? Can one find a $\Theta_{g,n}^{(r,s)} \in H^\bullet(\overline{\mathcal{M}}_{g,n})$ whose intersection with ψ-classes is encoded by the $F_{g,n}$ (or $\omega_{g,n}$)?*

4.6.3 Open intersection theory

Open intersection theory studies the enumerative geometry of bordered Riemann surfaces with marked points on the boundaries and in the interior, possibly carrying r-spin structures. Pandharipande, Solomon and Tessler first proposed in [PST14] an appropriate construction of this moduli space and associated numerical invariants in genus 0 for $r = 2$ (*i.e.* in absence of spin structures). The definition to all genera for $r = 2$ was announced in [ST], some details of which already appeared in [Tes15, Section 2]. The case $r \geq 2$ for genus 0 was settled by [BCT18] together with conjectures about the integrability property of the (yet not constructed) partition function at all genera. We refer to those articles for precise statements about the state of the art, and will continue so to speak as if all the desired constructions had already been established.

Let us focus on $r = 2$ to start with. One considers, for $2g-2+m+2n > 0$, the moduli space $\mathcal{M}_{g,n,m}$ of bordered Riemann surfaces with m marked points on the boundary, n marked points in the interior, such that the genus of the double is g. This moduli space is a real orbifold of dimension $3g-3+m+2n$, which admits a compactification $\overline{\mathcal{M}}_{g,n,m}$. There exists cotangent line bundles $\mathbb{L}_l$ at the interior punctures p_l for which relative orientations and boundary conditions can be constructed. Therefore they admit relative Euler classes and one can define a partition function for open intersection numbers as follows

$$Z_{\text{open}} = Z_{\text{closed}} \cdot \exp\left\{ \sum_{2g-2+m+2n>0} \frac{\hbar^{(g-1)/2}}{m!n!} \sum_{k_1,\ldots,k_n\geq 0} \left(\int_{\overline{\mathcal{M}}_{g,n,m}} e\Big(\bigoplus_{l=1}^{n} \mathbb{L}_l^{k_l}\Big)\right) \prod_{l=1}^{n} t^{\mathrm{B}}_{k_l}\, (s^{\mathrm{B}}_0)^m \right\}, \tag{4.112}$$

where $Z_{\text{closed}} = Z_{\text{2spin}}$ is (4.107). It was conjectured in [PST14] that Z_{open} is a tau function of the open KdV hierarchy. Buryak and Tessler proved this conjecture [BT17] based on a combinatorial model for the moduli space of bordered Riemann surface developed in [Tes15], while [Bur15] showed the equivalence between the open KdV equations with suitable initial data and a set of open Virasoro constraints (previously conjectured by [PST14]). In fact, Z_{open} is the specialization at $s^{\mathrm{B}}_k = 0$ for $k > 0$ of a partition function $Z^{\text{ext}}_{\text{open}}$ depending on all variables $(t^{\mathrm{B}}_k, s^{\mathrm{B}}_k)_{k>0}$ and which is a tau function for a larger integrable hierarchy [Bur16], later identified [Ale15] with the modified KdV hierarchy. According to [ST], the variables s^{B}_k for $k \geq 2$ have an enumerative interpretation as insertions of boundary descendant classes, paralleling the fact that t^{B}_k are coupled to insertions of ψ^k.

Alexandrov studied the formal hermitian matrix integral, called the Kontsevich-Penner matrix model

$$Z^{\text{ext}}_{\text{open},(N)}(q) = \frac{(\det Y)^q \int^{\text{formal}}_{\mathcal{H}_N} \mathrm{d}M\, (\det M)^{-q}\, e^{\hbar^{-1/2}\,\mathrm{Tr}[-\frac{M^3}{6}+\frac{Y^2M}{2}]}}{\int \mathrm{d}M\, e^{\hbar^{-1/2}\,\mathrm{Tr}[-\frac{YM^2}{2}+\mathrm{Tr}\frac{Y^3}{3}]}}\,, \tag{4.113}$$

It is possible to define the $N \to \infty$ limit of (4.113) as a formal series in $t_k = \frac{\hbar^{1/2}}{k}\,\mathrm{Tr}\,Y^{-k}$ for $k > 0$ which has the form

$$Z^{\text{ext}}_{\text{open}}(q) = \exp\left(\sum_{\substack{2h-2+n>0 \\ h\in\mathbb{N}/2}} \frac{\hbar^{h-1}}{n!} \sum_{k_1,\ldots,k_n\geq 0} F^{\text{open,ext}}_{h,n}(q;k_1,\ldots,k_n) \prod_{l=1}^{n} t_{k_l} \right).$$

He proves in [Ale15] that for $q = 1$ it is a tau function of the modified KdV hierarchy [Dic99] which coincides with the open extended partition function of Buryak with identification

$$t^{\mathrm{B}}_k = (2k+1)!!\, t_{2k+1}\,, \qquad s^{\mathrm{B}}_k = 2^{k+1}(k+1)!\, t_{2k+2}\,.$$

For general q Alexandrov derives in [Ale17] a set of $\mathcal{W}(\mathfrak{sl}_3)$ constraints annihilating $Z^{\text{ext}}_{\text{open}}(q)$. Safnuk gave in [Saf16] an equivalent description in terms of periods on the spectral curve $x = \frac{z^2}{2}$, $y = -z$, which is an unusual modification of the Bouchard-Eynard topological recursion [BE13] that involves half-integer genera. Our work in fact reproduces Alexandrov's constraints.

Proposition 4.6.3. *$Z^{\mathrm{ext}}_{\mathrm{open}}(q)$ is the partition function of the higher Airy structure of Proposition 4.4.16 with $r = 3$ and $s = 3$ considered as a function of $t_{2k} = \frac{1}{2}x^1_{2k} - x^2_k$ and $t_{2k+1} = x^1_{2k+1}$ for $k > 0$ with the identification $J^1_0 = \hbar^{1/2}q$.*

Proof. To make the comparison, we need to perform a reduction from $\mathcal{W}(\mathfrak{gl}_3)$ to $\mathcal{W}(\mathfrak{sl}_3)$. In order to do this, we define a set of independent (commuting) Heisenberg fields:

$$\begin{aligned} J^+_k &:= J^1_{2k} + J^2_k, \\ J^-_k &:= \sqrt{\frac{2}{3}}\left(\frac{1}{2}J^1_{2k} - J^2_k\right) \qquad \forall k \in \mathbb{Z} \end{aligned}$$

The normalization factor $1/2$ ensures that J^+ and J^- commute, while the factor $\sqrt{\frac{2}{3}}$ ensures that $[J^-_k, J^-_l] = k\delta_{l+k,0}$. Here we will also use the notation

$$J_{2k+1} := J^1_{2k+1}.$$

Now, we will rewrite the operators of (4.78) in the modes$\{J^+_k, J^-_k, J^1_{2k+1}\}$, and then formally set $W^1_k = J^+_k$ equal to 0 for $k \geq 0$. Then we get the following operators:

$$\begin{aligned} H^2_k &= J_{2k+1} - \sum_{a+b=k-1}\left(J^-_a J^-_b + \frac{1}{2}J_{2a-1}J_{2b+1}\right) - \frac{\hbar}{8}\delta_{k,1}\,, \\ -\sqrt{\frac{3}{2}}H^3_k &= J^-_{k+1} - \sum_{b+c=k-1} J_{2b+1}J^-_c + \sum_{a+b+c=k-2}\left(J_{2a-1}J_{2b+1}J^-_c - \frac{2}{3}J^-_a J^-_b J^-_c\right) + \frac{\hbar}{4}J^-_{k-2}\,. \end{aligned}$$

With the identifications

$$J^-_k = \begin{cases} \hbar\sqrt{\frac{3}{2}}\partial_{t_{2k}} & k > 0 \\ \hbar^{1/2}\sqrt{\frac{3}{2}}q & k = 0 \\ -\sqrt{\frac{2}{3}}kt_{-2k} & k < 0 \end{cases}, \qquad J_{2k+1} = \begin{cases} \hbar\partial_{t_{2k+1}} & k \geq 0 \\ (1-2k)t_{1-2k} & k < 0 \end{cases},$$

and $N = q$ we recognize $H^2_k = -2\widehat{\mathcal{L}}^N_k$ and $H^3_k = -4\widehat{\mathcal{M}}^N_k$ in the notation of [Ale17]. Our uniqueness result for the partition function (4.23) gives the statement of the Proposition. ∎

It is natural to speculate about higher r.

Question 4.6.4. *Consider the partition function associated with the higher quantum Airy structure based on $\mathcal{W}(\mathfrak{gl}_r)$ with automorphism $\sigma = (1 \ \cdots \ r-1)$ and dilaton shift $x_r \to x_r - \frac{1}{r}$ (see Proposition 4.4.16 with $s = r$).*

- *for $q = 1$, does it coincide with the tau function of the extended open $(r-1)$-KdV hierarchy constructed by Bertola and Yang [BY15]?*
- *for $q = 1$, can it be expressed in terms of the generating function of (extended) open $(r-1)$-spin intersection theory as constructed (in genus 0) by [BCT18]?*
- *for arbitrary q, does it have a formal matrix integral representation generalizing* (4.112)*?*

The two last questions can also be asked for the dilaton shifts $x_s \to x_s - \frac{1}{s}$ with $s \in \{1, \ldots, r-1\}$ such that $s|r$. In particular, for $s = 1$ this should give an open r-spin generalization of Norbury's class.

4.6.4 Fan-Jarvis-Ruan theories

Let $\mathbf{W} \in \mathbb{C}[x_1, \ldots, x_t]$ be a quasi-homogeneous polynomial, *i.e.* there exist positive integers $d, \nu_1, \ldots, \nu_t$ for which

$$\forall \lambda \in \mathbb{C}^*, \qquad \mathbf{W}(\lambda^{\nu_1} x_1, \ldots, \lambda^{\nu_t} x_t) = \lambda^d \mathbf{W}(x_1, \ldots, x_t) \,.$$

Assume that $\mathbf{W}$ is non-degenerate, *i.e.* $\mathrm{W} = 0$ has an isolated singularity at 0 and $\tilde{\nu}_i = \nu_i / d$ are uniquely determined by $\mathbf{W}$. Then one can always assume that $d, \nu_1, \ldots, \nu_t$ are minimal. This assumption implies that the group of diagonal symmetries of $\mathbf{W}$

$$\Gamma = \Big\{ \alpha \in (\mathbb{C}^*)^t \quad | \quad \mathbf{W}(\alpha_1 x_1, \ldots, \alpha_t x_t) = \mathbf{W}(x_1, \ldots, x_t) \Big\}$$

is finite. Let us decompose $\mathbf{W}$ into monomials

$$\mathbf{W}(x_1, \ldots, x_t) = \sum_{\mu} c_\mu \prod_{u=1}^{t} x_u^{\mu_u} \,.$$

Fan, Jarvis and Ruan [FJR08] constructed a compactified moduli stack of twisted spin curves, which describe isomorphism classes of orbifold curves $\mathcal{C}$ equipped with n punctures $p_1, \ldots, p_n$ and line bundles $L_1, \ldots, L_t$ together with isomorphisms

$$\phi_\mu \, : \, \bigotimes_{u=1}^{t} L_u^{\otimes \mu_u} \longrightarrow K_{\mathcal{C}}\Big(\sum_{i=1}^{n} p_i\Big)\,.$$

They also describe a virtual fundamental class on this moduli stack. After pushforward to $\overline{\mathcal{M}}_{g,n}$, it yields a cohomological field theory with Frobenius algebra given by the Jacobi ring

$$\mathrm{Jac}(\mathrm{W}) = \frac{\mathbb{C}[x_1, \ldots, x_t]}{\big\langle \partial_1 \mathbf{W}, \ldots, \partial_t \mathbf{W} \big\rangle}\,.$$

Polishchuk and Vaintrob gave another, more algebraic construction of this cohomological field theory from the category of matrix factorizations of $\mathbf{W}$ [PV12, PV16]. We denote $Z^{\mathbf{W}}$ the generating series of its intersection numbers with the ψ-classes.

The most fundamental examples of such hypersurfaces $\{\mathbf{W} = 0\}$ are given by the simple singularities, of type ADE. Their total descendant potential are tau functions of an integrable hierarchy [GM05, FGM10], and satisfy $\mathcal{W}$ constraints [BM13]. These constraints coincide with the $\mathcal{W}(\mathfrak{g})$ quantum Airy structure associated with the dilaton shift $x_{r+1} \to x_{r+1} - \frac{1}{r+1}$ described in Section 4.4. The uniqueness of their solution was proved in [LYZ13]. The r-spin partition function of Section 4.6.1 corresponds to the A_{r-1} case. The total descendant potential of the ADE singularity $\{\mathbf{W}_{\mathrm{ADE}} = 0\}$ in fact coincides with the Fan-Jarvis-Ruan partition function $Z^{\mathbf{W}_{\mathrm{ADE}}}$ [FJR13].

Question 4.6.5. *For any non degenerate quasihomogeneous polynomial in n variables $\mathbf{W}$, can one write down a higher quantum Airy structure (based on $\mathcal{W}$ algebras) associated with $\mathbf{W}$ whose partition function encodes the correlators of the cohomological field theory constructed by Fan-Jarvis-Ruan?*

Question 4.6.6. *Can the partition functions in Theorem 4.4.20 for D_N-type and Theorem 4.4.24 for E_N-type in the case $s = 1$ receive an interpretation in terms of Fan-Jarvis-Ruan theories ?*

Part II

Derived categories in algebraic geometry

Chapter 5

Background and Introduction

Derived categories were invented by Grothendieck and Verdier [Ver96] as a formalism for homological algebra. Since the advent of string theory and the pioneering work of Bondal and Orlov [BO95], they have been recognized as a rich invariant of algebraic varieties. In the context of algebraic geometry, we are interested in the derived category of (quasi-)coherent sheaves on schemes (or stacks).

In a nutshell, the idea of the derived category is the following. Starting with an abelian category $\mathcal{A}$ (typically the category of coherent or quasi-coherent sheaves on a space X), we construct the category of complexes $\mathrm{Kom}(\mathcal{A})$, and then invert the quasi-isomorphisms to obtain the derived category $D(\mathcal{A})$. In other words, the objects of $D(\mathcal{A})$ are complexes in $\mathcal{A}$, and the morphisms are morphisms of complexes with formal inverses added for every quasi-isomorphism.

In this thesis, we are primarily interested in understanding the transformation of derived categories under birational morphisms known as flops using the theory of Variation of Geometric Invariant Theory quotients (VGIT) as developed by [Seg11, BFK19, HL15]. Let us introduce the relevant objects now.

5.1 The derived category

We will give a sketch of the construction of the derived category here (for more details, see [KS06, Huy06]). Although we will often be interested in the derived categories of stacks, we restrict to the easy case of a scheme X in this section. Denote the abelian category of coherent sheaves on X by $\mathrm{Coh}(X)$ and the category of (chain) complexes of coherent sheaves by $\mathrm{Kom}(X)$. As a brief reminder, a complex $A^\bullet$ in $\mathrm{Kom}(X)$ is a diagram

$$\cdots \xrightarrow{d^{i-2}} A^{i-1} \xrightarrow{d^{i-1}} A^i \xrightarrow{d^i} A^{i+1} \xrightarrow{d^{i+1}} \cdots,$$

where the A^i are coherent sheaves on X and d^i are morphisms between them such that $d^i \circ d^{i-1} = 0$. The morphisms between chain complexes, say $f : A^\bullet \to B^\bullet$, are defined as commutative diagrams of the form

$$\begin{tikzcd}
\cdots \arrow[r, "d^{i-2}"] & A^{i-1} \arrow[r, "d^{i-1}"] \arrow[d, "f^{i-1}"] & A^i \arrow[r, "d^i"] \arrow[d, "f^i"] & A^{i+1} \arrow[r, "d^{i+1}"] \arrow[d, "f^{i+1}"] & \cdots \\
\cdots \arrow[r, "d^{i-2}"] & B^{i-1} \arrow[r, "d^{i-1}"] & B^i \arrow[r, "d^i"] & B^{i+1} \arrow[r, "d^{i+1}"] & \cdots
\end{tikzcd}$$

Remark 5.1.1. It is straightforward to check that the category $\mathrm{Kom}(X)$ is abelian.

Let us define quasi-isomorphisms which will be essential for defining derived categories.

Definition 5.1.2 (Quasi-isomorphisms)**.** A morphism of complexes $f : A^\bullet \to B^\bullet$ is called a *quasi-isomorphism* (or qis) if the induced maps on cohomology are isomorphisms, i.e. $H^i(f) : H^i(A^\bullet) \to H^i(B^\bullet)$ is an isomorphism $\forall i \in \mathbb{Z}$.

We wish to invert all quasi-isomorphisms (see Definition 5.1.2) in order to get the derived category; this is typically accomplished in two steps. The reason for this two-step procedure is that it allows one to get a better handle on the morphisms (and consequently, the triangulated structure) of the derived category. The first step is to construct the homotopy category $K(X)$.

We define the notion of a homotopy.

Definition 5.1.3 (Homotopy of morphisms)**.** Two morphisms of complexes $f, g : A^\bullet \to B^\bullet$ are said to be homotopic (denoted $f \sim g$) if there exists morphisms $h^i : A^i \to B^{i-1}$, $i \in \mathbb{Z}$ such that

$$f^i - g^i = h^{i+1} \circ d^i_A + d^{i-1}_B \circ h^i$$

$$\begin{tikzcd}
\cdots \arrow[r, "d^{i-2}"] & A^{i-1} \arrow[r, "d^{i-1}"] \arrow[d] \arrow[dl, "h^{i-1}"'] & A^{i} \arrow[r, "d^{i}"] \arrow[d] \arrow[dl, "h^{i}"'] & A^{i+1} \arrow[r, "d^{i+1}"] \arrow[d] \arrow[dl, "h^{i+1}"'] & \cdots \\
\cdots \arrow[r, "d^{i-2}"] & B^{i-1} \arrow[r, "d^{i-1}"] & B^{i} \arrow[r, "d^{i}"] & B^{i+1} \arrow[r, "d^{i+1}"] & \cdots
\end{tikzcd}$$

Consider two morphisms $f : A^\bullet \to B^\bullet$ and $g : B^\bullet \to A^\bullet$, such that

$$f \circ g \sim Id_B, \qquad \text{and} \qquad g \circ f \sim Id_A.$$

Then, we can check that the maps f and g are quasi-isomorphisms. The idea of the homotopy category is to invert this class of quasi-isomorphisms.

Definition 5.1.4 (Homotopy category)**.** The *homotopy category* of complexes on X, denoted $K(X)$, is the category whose objects are complexes and morphisms are morphisms of complexes defined up to homotopy, i.e.

$$\mathrm{Hom}_{K(X)}(A^\bullet, B^\bullet) = \mathrm{Hom}_{\mathrm{Kom}(X)}(A^\bullet, B^\bullet)/\sim .$$

The verification that this category is well defined is fairly straightforward.

Finally, we are ready to introduce the derived category $D(X)$. The definition is rather technical, so we give a sketch and refer the reader to [Huy06] for the details. The class of quasi-isomorphisms form a *localizing class* in $K(X)$ (but not in $\mathrm{Kom}(X)$), and hence we can construct the derived category as the localization category

$$D(X) := K(X)[\mathrm{qis}^{-1}].$$

Localization is a general procedure for categories, where one adds a formal inverse for all morphisms in the localizing class.

The following universal property clarifies the above construction.

Proposition 5.1.5. *[Huy06, Theorem 2.10] On a scheme X, there exists a category $D(X)$, called the derived category of X, and a functor*

$$Q : \mathrm{Kom}(X) \to D(X),$$

such that

- *If $f : A^\bullet \to B^\bullet$ is a quasi-isomorphism, then $Q(f)$ is an isomorphism in $D(X)$;*

- *Any functor F : $\mathrm{Kom}(X) \to \mathcal{D}$ satisfying the above condition factors uniquely as follows*

$$\begin{tikzcd} \mathrm{Kom}(X) \arrow[rr, "Q"] \arrow[dr, "F"'] & & D(X) \arrow[dl, dotted, "\exists! G"] \\ & \mathcal{D} & \end{tikzcd}$$

Remark 5.1.6. Often, we will be interested in the bounded version of this category, denoted by $D^b(X)$, whose objects are bounded complexes of coherent sheaves on X.

The derived category is naturally a triangulated category (see [Huy06, Chapter 1] for a review) and we would like to understand functors between them that preserve this triangulated structure. A huge class of such functors are provided by Fourier-Mukai transforms.

Definition 5.1.7. Consider two smooth schemes X and Y over a field k. Consider an object $E \in D(X \times Y)$. We have natural projections $p : X \times Y \to X$ and $q : X \times Y \to Y$.

$$\begin{tikzcd} & X \times Y \arrow[dl, "p"'] \arrow[dr, "q"] & \\ X & & Y \end{tikzcd}$$

The functor $\Phi_E : D(X) \longrightarrow D(Y)$ defined as

$$F \longmapsto q_*(E \otimes p^* F)$$

is called the *Fourier-Mukai transform* induced by E. E is called the *Fourier-Mukai kernel.* The functors $p^*, q_*, \otimes$ are all implicitly derived.

A couple of illustrative examples are the following.

Example 5.1.8. The identity functor $\mathrm{Id} : D(X) \to D(X)$ is the Fourier-Mukai transform of $\mathcal{O}_\Delta$, where Δ is the image of the diagonal morphism $X \to X \times X$.

Example 5.1.9. Given a morphism, $f : X \to Y$, the pushforward functor $f_* : D(X) \to D(Y)$ is the Fourier-Mukai transform of $\mathcal{O}_{\Gamma_f}$, where $\Gamma_f \subset X \times Y$ is the graph of f.

Instead if we view the graph Γ_f as an object of $D(Y \times X)$, and use it as a Fourier-Mukai kernel, we get the pullback functor $f^* : D(Y) \to D(X)$.

Indeed, we expect all "reasonable" geometric functors to be Fourier-Mukai transforms, thanks to a famous representability result of Orlov, which states that all triangulated fully faithful functors between smooth projective varieties are of Fourier-Mukai type [Orl97] (and an enhancement of it to dg-categories due to Toën [Toë07]).

Definition 5.1.10 (Derived equivalence)**.** If the derived categories of two schemes, X and Y, are equivalent, we say that the schemes X and Y are *D-equivalent* or *derived equivalent*.

Notice that by Orlov's representability theorem, all derived equivalences are Fourier-Mukai transforms.

5.2 Derived categories and birational geometry

The idea of understanding derived categories as invariants of algebraic varieties was first explored by Bondal and Orlov [BO95]. One of the very interesting aspects of this problem is the relationship between the derived category and the birational equivalence class of a variety. The results of Orlov [Orl03] that show that the canonical ring and Kodaira dimension (which are birational invariants) are invariants of the derived category (for smooth projective varieties), is an indication of a close connection between the two.

In general, neither is a stronger invariant than the other, i.e., there are derived equivalent varieties which are not birational (for an example, abelian varieties and their duals [Muk87]) and birational varieties that are not derived equivalent (for example, blow-ups [Huy06]). This suggests a natural question – Can one describe the class of varieties for which birational equivalence implies derived equivalence and vice versa? A conjectural answer in one direction was proposed by [BO95].

Conjecture 5.2.1 (Bondal-Orlov)**.** *Consider two smooth projective varieties X and Y over $\mathbb{C}$ related by a flop. Then there is an equivalence of categories*

$$D^b(X) \cong D^b(Y).$$

A flop is an elementary birational transformation that preserves the canonical class. Here are the precise definitions of flips and flops as in [Tha96], where they are referred to as D-flips and D-flops respectively.

Definition 5.2.2 (Flips and flops)**.** Let X^-, X^+ and X_0 be varieties over a field k such that

- There is a small contraction $f : X^- \to X_0$, i.e. f is a proper birational transformation that is an isomorphism in codimension ≥ 1;
- X^- is equipped with a $\mathbb{Q}$-Cartier divisor D, such that $\mathcal{O}(-D)$ is f-ample;
- There is a small contraction $g : X^+ \to X_0$, with induced birational map $h : X^- \dashrightarrow X^+$ such that $\mathcal{O}(h_*D)$ is $\mathbb{Q}$-Cartier and g-ample.

We represent this via the following diagram.

$$\begin{tikzcd} X^- \arrow[rr, dotted, "h"] \arrow[dr, "f"'] & & X^+ \arrow[dl, "g"] \\ & X_0 & \end{tikzcd}$$

Then, X^+ is the *flip* of X^- over X_0. A *flop* is a flip where the maps f and g are crepant (i.e. they preserve the canonical class).

Remark 5.2.3. This conjecture fits well with the expectations from homological mirror symmetry. Indeed, as alluded to in the introduction (Chapter 1), these flops can be thought of as arising from paths in the Stringy Kähler Moduli Space.

There is a rather large set of examples where this conjecture is known to be true – flops in dimension three [Bri02], stratified Mukai flops [Cau12], standard flops [BO95], Abuaf flop [Seg11] and Grassmann flops [DS14]. We also note that Kawamata refined the Bondal-Orlov Conjecture 5.2.1 to the famous K-equivalence implies D-equivalence conjecture in [Kaw02]. Roughly, this conjecture states that any birational transformation that 'does not change' the canonical bundle leaves the derived category invariant. The author also studies the converse question, i.e., whether D- equivalence implies K-equivalence, and proves a partial result.

There have been a variety of different approaches to tackle this conjecture (see [VdB04, BFK19, Bri02, HL15] for various approaches and [Kaw18] for a review of them). We are interested in the method known as Variation of Geometric Invariant Theory Quotients (VGIT) or grade-restriction windows (referred to as windows in short). This was introduced by the physicists [HHP09], introduced into the mathematics literature by [Seg11], and then generalized substantially by [BFK19, HL15]. The technique can be applied to understand the relationship between the derived categories of different GIT quotients.

An observation of Reid (Proposition 1.9 in [Tha96]) allows us to reinterpret a flop as the data of a variety X, with a $\mathbb{G}_m$-action on it, such that the varieties related by a flop, say X^+ and X^-, are realized as two different GIT quotients of $\mathbb{G}_m$ acting on X. Hence, we can try to use the method of windows to tackle the Bondal-Orlov conjecture on flops. Moreover, as mentioned in the introduction, Chapter 1, VGIT problems appear naturally in the context of string theory in physics, and hence are interesting in their own right.

5.3 VGIT and windows

This is a very brief description of (a special case of) the method of windows as developed by Ballard-Favero-Katzarkov [BFK19]. The general setup treats the case where we have a linearly reductive group G acting on a variety X. We make some restrictive simplifications here for the purposes of exposition. Firstly, we will assume that the group G is $\mathbb{G}_m$. In the general setting, one picks one-parameters subgroups of an arbitrary group G and then varies the stability under these one-parameter subgroups in order to understand the transformation of the derived category. Secondly, we will also assume that the variety X is affine; this suffices for us to tackle the case of local models of flops.

As discussed in the previous paragraph, we consider an affine scheme $X = \mathrm{Spec}(T)$ (over a fixed commutative ring k) with a $\mathbb{G}_m$-action on it. The $\mathbb{G}_m$-action essentially makes R a $\mathbb{Z}$-graded k-algebra. Then our varieties (or possibly Deligne-Mumford stacks) of interest are the two different GIT quotients, denoted $X/\!\!/_+$ and $X/\!\!/_-$, which are defined as follows.

Definition 5.3.1. Define I^+ and I^- as the ideals of T generated by positive and negatively graded elements of T respectively. We define the GIT quotients $X/\!\!/_\pm$ as

$$X^\pm := X \setminus V(I^\pm), \qquad X/\!\!/_\pm := \left[X^\pm/\mathbb{G}_m\right],$$

where the [ ] brackets denote we take the quotient as a stack.

Then, the construction of [BFK19, HL15] proceeds by constructing window functors and window subcategories. We have the following definitions.

Definition 5.3.2 (Windows)**.** A (positive/negative) *window functor* is any fully faithful functor

$$\Phi_\pm : D^b(X/\!\!/_\pm) \to D^b([X/\mathbb{G}_m]),$$

and the essential image of this functor is called a (positive/negative) *window subcategory* $\mathbb{W}^\pm$.

In order to compare the different GIT quotients, we use the following notion of wall-crossing functors.

Definition 5.3.3 (Wall-crossing functors)**.** With the setup as before, we consider the following open immersion,

$$j_- : X^- \longrightarrow X.$$

Then we can define the composition

$$\Phi_{\mathrm{wc}} := j_{-}^{*} \circ \Phi_{+} : D^b(X/\!\!/_{+}) \to D^b(X/\!\!/_{-}),$$

known as a *wall-crossing* functor. We obtain a wall-crossing functor in the opposite direction by swapping the $+$ and $-$ in the above definition.

Now, in principle, one can compare the derived categories of the different GIT quotients.

However, one of the major obstacles in this theory is that, apart from the simplest class of VGIT problems called elementary wall-crossings in [BFK19], there is no uniform construction of a window functor $\Phi_{\pm}$, or consequently, a window subcategory $\mathbb{W}^{\pm}$. Moreover, the theory does not provide an explicit Fourier-Mukai kernel for the window functors or the wall-crossing functors. As an attempt to remedy this, Ballard, Diemer and Favero [BDF17] recently proposed a method to construct Fourier-Mukai kernels for window functors, called the Q-construction, which we shall briefly review next.

5.3.1 The Q-construction

Recall that we have a $\mathbb{G}_m$-action on an affine scheme $X = \operatorname{Spec} T$. We denote the action and projection maps by σ and π respectively (and by abuse of notation, use σ and π to denote the co-action and co-projection maps respectively)

$$\pi : T \longrightarrow T \otimes k[\mathbb{G}_m], \qquad \sigma : T \longrightarrow T \otimes k[\mathbb{G}_m].$$

We define the notion of a partial compactification of the group action as follows.

Definition 5.3.4 ([BDF17, Definition 3.1.1]). Consider a linearly reductive group G acting on a scheme X. Let $\tilde{X}$ be a scheme equipped with a $G \times G$-action such that it fits in the following $G \times G$-equivariant[1] commutative diagram

$$\begin{tikzcd} & \tilde{X} \arrow[d, "p\times s"] \\ G \times X \arrow[ur, hook, "i"] \arrow[r, "\pi\times\sigma"] & X \times X, \end{tikzcd}$$

[1] We equip $G \times X$ with a $G \times G$-action such that the map $\pi \times \sigma : G \times X \to X \times X$ is $G \times G$-equivariant.

where i is an open immersion. Then $\tilde{X}$ along with the maps i, p, s is called a *partial compactification of the action of G on X*.

In the case of $\mathbb{G}_m$-actions on an affine scheme $X = \operatorname{Spec} T$, a partial compactification was constructed in [BDF17]. They partially compactified the $\mathbb{G}_m$ to an $\mathbb{A}^1$ (in a non-trivial way!). When X is *smooth*, the authors then considered the object

$$Q := (p \times s)_* \mathcal{O}_{\tilde{X}}. \tag{5.1}$$

Alternatively, in the case of $\mathbb{G}_m$-actions on affine schemes, we can directly define the object Q as follows.

Definition 5.3.5 (The object Q)**.** Given a $\mathbb{Z}$-graded k-algebra T, define the object $Q(T)$ as

$$Q(T) := \langle \pi(T), \sigma(T), u \rangle \subseteq T[u, u^{-1}],$$

i.e., the k-subalgebra generated by u and the images of T under the co-action and co-projection maps.

The object Q defined via equation (5.1) is the sheaf associated to the module $Q(T)$ (Definition 5.3.5), and is a $\mathbb{G}_m \times \mathbb{G}_m$-equivariant quasi-coherent sheaf on $X \times X$. Then, we consider the objects

$$Q_\pm := Q|_{(X^\pm \times X)}.$$

Then, [BDF17] proves that the Fourier-Mukai functors associated to the objects $Q_\pm$ are window functors.

Theorem 5.3.6 ([BDF17])**.** *When X is a smooth affine scheme with a $\mathbb{G}_m$-action, the functors*

$$\Phi_{Q_\pm} : D^b(X /\!\!/_\pm) \to D^b([X/\mathbb{G}_m])$$

are fully faithful, and the essential image is the window subcategory as defined in [BFK19, HL15].

There are a few natural questions arising out of this construction. One can ask if partial compactifications, and consequently window functors exist for other groups G and not just $\mathbb{G}_m$. Chapter 6 addresses this question for general linear group $GL(V)$, in the context of Grassmann flops. Another natural question is to extend this construction of [BDF17] to the case of singular schemes. The authors propose that the right construction in that setting is to find a simplicial resolution of the smooth ring, and then apply the Q-construction to it. In Chapter 7, we use the philosophy of the monoidal Dold-Kan correspondence to study this further using dg-resolutions. Let us elaborate on these generalizations a little bit.

5.3.2 Grassmann flop

Grassmann flops were studied by Donovan and Segal [DS14] using window techniques. First, we review the GIT presentation of Grassmann flops. Consider the affine scheme

$$X := \mathrm{Hom}(V, W) \times \mathrm{Hom}(W', V),$$

where V, W, W' are vector spaces, over a field k, with dimensions $d < d_W, d_{W'}$ respectively. Consider the $GL(V)$-action on X,

$$A.(B, C) = (AB, CA^{-1}),$$

for A in $GL(V)$ and (B, C) in $\mathrm{Hom}(V, W) \times \mathrm{Hom}(W', V)$.

We define the positive semi-stable locus X^+ as the subvariety of X obtained by removing the non-full rank matrices in $\mathrm{Hom}(V, W)$, and the negative semi-stable locus X^- as the subvariety of X obtained by removing the non-full rank matrices in $\mathrm{Hom}(W', V)$. We define two GIT quotients $X/\!\!/_+$ and $X/\!\!/_-$ as the stack quotients of the positive or negative semistable locus by the $GL(V)$-action described above

$$X/\!\!/_\pm := [X^\pm / GL(V)],$$

and we have isomorphisms

$$X/\!\!/_+ \cong \mathrm{tot}_{Gr(d,W)}(S^{\oplus d_{W'}}), \qquad X/\!\!/_- \cong \mathrm{tot}_{Gr(d,W')}(S^{\oplus d_W}),$$

where S is the tautological sub bundle, and 'tot' means the total space of the sheaf.

We get the following birational map

$$X/\!\!/_+ \dashleftarrow\dashrightarrow X/\!\!/_-$$

through this GIT construction. In the case when $d_W = d_{W'}$, this birational map (which is not the identity map!) is called the *Grassmann flop*. In the case when $d = 1$, this is the well-known Atiyah/standard flop.

In Chapter 6, we define a partial compactification for this $GL(V)$-action, construct an object Q, and use it to define window functors over an arbitrary field k of characteristic 0. We prove that the window subcategory is generated by objects in Kapranov's collection [Kap88]. In the case of the Grassmann flop, we then show that the wall-crossing functor is an equivalence of categories. This extends the work of Donovan and Segal to an arbitrary field of characteristic zero.

We also show that the Fourier-Mukai kernel of the wall-crossing functor is the same as the one discovered in [BLVdB16]. The fact that we have an explicit description of the kernel allows us to use a result of Orlov [Orl02] to prove derived equivalences for twisted flops (for example, flops arising from generalized Severi-Brauer varieties over a field of characteristic zero).

5.3.3 Beyond the smooth case

In the context of VGIT problems arising from flops, the spaces that one encounters are often singular. Hence it becomes necessary to understand window functors involving singular schemes in order to tackle the Bondal-Orlov/Kawamata conjectures mentioned earlier. In the case of singular schemes X, the naive definition of Q does not produce a fully faithful functor as in Theorem 5.3.6. The authors of [BDF17] propose a certain 'derived' Q-construction for $\mathbb{G}_m$-actions on singular affine schemes, using derived algebraic geometry techniques.

The idea of the derived Q-construction is to resolve the singular scheme X by a simplicial scheme, and then apply the Q-construction. Motivated by the monoidal Dold-Kan correspondence, in Chapter 7, we propose to 'resolve' the scheme X by a semi-free dg-scheme (as defined by [CFK01, Ric10]) , say $\mathbf{X}$, and then apply the Q-construction. In fact, we work in a more general context and develop a theory of windows for dg-schemes equipped with a $\mathbb{G}_m$-action. Let us describe this theory and our results briefly.

We start with an affine dg-scheme

$$\mathbf{X} := (X, \mathcal{R}),$$

where $\mathcal{R}$ is a semi-free commutative dg-$\mathcal{O}_X$-module on an scheme $X = \operatorname{Spec} T$ (obtained from a semi-free cdga[2] R over T), equipped with a $\mathbb{G}_m$-action. We define the object Q as a $\mathbb{G}_m \times \mathbb{G}_m$-equivariant sheaf on $\mathbf{X} \times \mathbf{X}$ in a fashion very similar to Definition 5.3.5. We obtain window functors by restricting Q to the natural (dg-)semi-stable loci and then prove that these define fully faithful functors between derived categories of dg-schemes.

In general, we are unable to identify the window subcategories, but we manage this under some conditions. If the cdga R is generated as a T-algebra by non-positive elements, then we identify the window subcategory explicitly and prove wall-crossing statements about the dg-GIT quotients.

The main application of our formalism is to the setting of singular VGIT problems arising from flops. In this case, we resolve the $\mathbb{Z}$-graded singular ring $k[X]$, by a $\mathbb{Z}$-graded semi-free cdga R, using the Koszul-Tate resolution (which is a generalization of the Koszul resolution for complete intersections) [Tat57]. Then, if the non-positivity condition that we discussed before holds, we obtain the derived equivalence statement for flops. For instance, we recover the local case of the equivalence for the Mukai flop studied in [Nam03, Kaw06, Mor18, ADM19], using VGIT and window techniques.

[2] A semi-free cdga is one that is a polynomial algebra upon forgetting the differential.

Chapter 6

Kernels for Grassmann flops

6.1 Introduction

Derived categories, once viewed as a mere technical book-keeping device, have flourished as a topic of investigation as volumes of literature have exposed their geometric nature. Derived categories of coherent sheaves on algebraic varieties bind algebraic geometry to commutative algebra, representation theory, symplectic geometry, and theoretical physics in deep and surprising ways.

These bindings come in the form of fully-faithful functors or, better yet, equivalences relating different varieties or categories. An obvious and central question: what is a reasonably robust and general source for such functors? Experience in algebraic geometry tells us that moduli spaces are often a good place to look but beyond this source the examples of fully-faithful functors are more idiosyncratic.

Recently, a new construction, in the context of group actions, was introduced in [BDF17]. There it was called the Q-construction. Given a variety X with a $\mathbb{G}_m$-action, the authors constructed an idempotent kernel on the equivariant derived category $\mathsf{D}^{\mathsf{b}}([X/\mathbb{G}_m])$. The kernel Q, being the identity on its essential image, fully-faithfully identifies an interesting component of the derived category $\mathsf{D}^{\mathsf{b}}([X/\mathbb{G}_m])$. In fact, it always gives a two-term semi-orthogonal decomposition. This construction has some natural extensions.

Following Drinfeld [Dri13], we can recognize it as a piece of more general story. The inclusion $\mathbb{G}_m \subset \mathbb{A}^1$ can be viewed as a partial compactification of $\mathbb{G}_m$ as a monoid in schemes. The fibers of the multiplication map $\mathbb{A}^1 \times \mathbb{A}^1 \to \mathbb{A}^1$

are a family of $\mathbb{G}_m$ orbits which degenerate over $0 \in \mathbb{A}^1$. From Drinfeld's perspective, the idempotent kernel constructed in [BDF17] is the structure sheaf of a variety that parametrizes such degenerations in X.

This viewpoint allows for an immediate generalization: we can replace $\mathbb{A}^1$ with M where M is monoidal scheme. If we have a variety with an action of the units of M, we can produce a kernel for X. In this paper, we study the monoidal scheme $\mathrm{End}(V)$ for V a finite dimensional vector space and the natural action of the units $\mathrm{GL}(V)$ on the vector space $\mathrm{Hom}(V, W) \times \mathrm{Hom}(W, V)$ for W another finite dimensional vector space. We show that the idempotent property still holds:

Theorem 1. There exists a morphism of kernels $Q \to \Delta$ inducing an isomorphism of $Q \circ Q \to Q$, where $\circ$ denotes convolution of kernels and Δ is the kernel of the identity.

The robustness of the equivariant setting is indicated by the fact that any flip between normal varieties arises as a variation of GIT problem for a $\mathbb{G}_m$ action on a variety. We remind the reader of the following conjecture of Bondal and Orlov [BO95] extended by Kawamata [Kaw02]:

Conjecture (Bondal-Orlov 1995)**.** Assume that Z and Z' are smooth complex varieties. If Z and Z' are related by a flop, then there is a $\mathbb{C}$-linear triangulated equivalence of their bounded derived categories of coherent sheaves

$$\mathsf{D}^{\mathsf{b}}(Z) \cong \mathsf{D}^{\mathsf{b}}(Z').$$

As an application of the Q-construction, in [BDF17], the first author, Diemer, and the third author gave a means of constructing a kernel on $Z \times Z'$ for any D-flip (Z, Z'). For flops of smooth projective varieties, the associated integral transform is conjectured to be the desired equivalence of Bondal and Orlov. This provides a single unified, though still conjectural, approach to constructing equivalences from flops.

In the setting of this paper, variation of GIT leads to a flop first studied by Buchweitz, Leuschke, and Van den Bergh in [BLVdB16], and further studied by Donovan and Segal [DS14] in the case of an algebraically-closed base field of characteristic zero. Donovan and Segal labeled it a Grassmann flop since it comes from contracting the zero section of a vector bundle over a Grassmannian. Furthermore, they exhibited equivalences for Grassmann flops using a set of representations identified by Kapranov [Kap88]. While the authors in [BLVdB16][Theorem D] show for k an (arbitrary) field of characteristic zero. The structure sheaf of the fiber product of a twisted Grassmann flop diagram

$$\begin{tikzcd} X^+ \arrow[rr, dashed, leftrightarrow] \arrow[dr] & & X^- \arrow[dl] \\ & X_0 & \end{tikzcd}$$

induces an equivalence $\mathsf{D}^{\mathsf{b}}(X^+) \cong \mathsf{D}^{\mathsf{b}}(X^-)$, i.e. $\mathcal{O}_{X^+ \times_{X_0} X^-}$ is a Fourier-Mukai kernel.

As an application of our construction, we show that the kernel of [BLVdB16] is equivalent to our new Q-construction.

Theorem 2. Let $\hat{Q} := Q|_{X^+ \times X^-}$ then

$$\hat{Q} \cong \mathcal{O}_{X^+ \times_{X_0} X^-}.$$

This equivalence illustrates the utility of our generalization, yet perhaps the most astounding utility is the ability to give a geometric approach to "windows", whose use in equivariant derived categories is evident from works such as, for example [HL15, BFK19, DS14, vVdB17]. Identifying windows involves some choices or special conditions. The next result indicates that windows come simply from the choice of the monoid M compactifying G.

Theorem 3. Let $Q_+ := Q|_{X^+ \times X}$. Then

- The functor
$$\Phi_{Q_+} : \mathsf{D}^{\mathsf{b}}(X^+) \to \mathsf{D}^{\mathsf{b}}\left([Z/\operatorname{GL}(V)]\right)$$
is fully-faithful.

- The restriction map $j^* : \mathsf{D}^{\mathsf{b}}\left([Z/\operatorname{GL}(V)]\right) \to \mathsf{D}^{\mathsf{b}}(X^+)$ is a left inverse to Φ_{Q_+}.

- Kapranov's representations form a set of generators for the essential image of Φ_{Q_+}.

Theorem 3, in particular, provides a completely geometric explanation for the appearance of Kapranov's representations. Our method of monoid compactification can therefore be seen as part of a program to produce canonical windows for quotients via linearly reductive groups.

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6.1.2 Notation

Throughout, k denotes a field of characteristic zero. So that no confusion arises we consider $0 \in \mathbb{N}$. Further, for $\ell \in \mathbb{N}$, $\ell \neq 0$, we let $[\ell] := \{1, ..., \ell\}$. We let Vec_k denote the category of finite-dimensional k-vector spaces and k-linear transformations, and for a k-algebra S we denote Aff_S the category of affine $\mathrm{Spec}(S)$-schemes and morphisms of schemes. We utilize standard results in Geometric Invariant Theory and attempt to align our notation with that of [Mum65b]. All schemes considered here are k-schemes, and we denote the global sections of the structure sheaf of a k-scheme Z as $k[Z]$. The word *point* will always mean *k-point*.

6.2 Background

Throughout, fix a d-dimensional k-vector space V.

Definition 6.2.1. For any k-vector space W we can naturally associate a scheme over $\mathrm{Spec}(k)$ defined as the spectrum of the symmetric algebra of the dual space $W^\vee$, that is

$$\mathrm{Spec}\left(\mathrm{Sym}\left(W^\vee\right)\right)$$

we will refer to this scheme as the **geometric bundle** of W over $\operatorname{Spec}(k)$. In general for a k-scheme X and a locally free $\mathcal{O}_X$-module M we will refer to the relative spectrum of the symmetric sheaf of algebras of $M^\vee$ as the corresponding geometric bundle over X.

With the above definition in mind we will denote by $\mathrm{GL}(V)$ the linear algebraic group

$$\mathrm{GL}(V) := \operatorname{Spec}\left(k\left[C, (\det(C))^{-1}\right]\right),$$

where $C = (c_{ij})_{i,j\in[d]}$ is a collection of indeterminates. We use $\det(C)$ to denote the polynomial

$$\det(C) := \sum_{\delta\in\mathfrak{S}_n} (-1)^{\operatorname{sgn}(\delta)} \left(\prod_{i\in[d]} c_{i\delta(i)}\right).$$

Next, recall that for a k-scheme Z, an action of $\mathrm{GL}(V)$ on Z is defined by a morphism of schemes

$$\sigma_Z : \mathrm{GL}(V) \times_k Z \to Z.$$

If Z and Y are k-schemes with $\mathrm{GL}(V)$-action given by σ_Z and σ_Y, we say that a morphism $f : Z \to Y$ is $\mathrm{GL}(V)$-equivariant whenever the following diagram commutes:

$$\begin{tikzcd}
\mathrm{GL}(V) \times_k Z \arrow[r, "1_{\mathrm{GL}(V)}\times_k f"] \arrow[d, "\sigma_Z"'] & \mathrm{GL}(V) \times_k Y \arrow[d, "\sigma_Y"] \\
Z \arrow[r, "f"] & Y
\end{tikzcd}$$

Most of this work deals with categories whose objects carry a $\mathrm{GL}(V)$-action and $\mathrm{GL}(V)$-equivariant morphisms. To denote such a restriction, we simply use the superscript $\mathrm{GL}(V)$. For example, let R be a commutative ring with a $\mathrm{GL}(V)$-action. We denote the category of modules with an inherited $\mathrm{GL}(V)$-action and $\mathrm{GL}(V)$-equivariant morphisms as $\operatorname{Mod}^{\mathrm{GL}(V)}(R)$.

Next we will establish further notational conventions. Given a collection $A = \{a_{ij}\}_{i\in[m],j\in[d]}$ of indeterminates, for any subsets $J \subseteq [d]$, $I \subseteq [m]$, we let

$A_{I,J}$ denote the collection of variables

$$A_{I,J} := (a_{ij})_{i\in I,\, j\in J}.$$

For $I \subseteq [m]$ with $|I| = d$, we may list this set in increasing order, and denote the corresponding ordered set by $I := \{\ell_1, .., \ell_d\}$. We then write

$$\det\left(A_{I,[d]}\right) := \sum_{\sigma\in\mathfrak{S}_d} \operatorname{sgn}(\sigma) \left(\prod_{1\le i\le d} A_{\ell_i,\sigma(i)}\right).$$

Given two collections $A = \{a_{ij}\}_{i\in[m],j\in[d]}$ and $B = \{b_{ij}\}_{i\in[d],j\in[m]}$, we use BA to denote the collection of polynomials

$$\left\{\sum_{\ell=1}^{m} b_{i\ell}a_{\ell j}\right\}_{i\in[d],j\in[d]}.$$

Lastly, given two k-algebras R and S and elements $r \in R$ and $s \in S$, we let r^L and s^R denote the elements $r \otimes 1$ and $1 \otimes s$ in $R \otimes_k S$.

6.3 The kernel

Recall that in [BDF17] the authors exhibit a kernel using a partial compactification of a certain $\mathbb{G}_m$-action. We follow a similar line of reasoning, and begin by defining our main categories of interest. Let W and W' be arbitrary finite dimensional k-vector spaces. Next, consider the vector spaces

$$\operatorname{Hom}_{\mathsf{Vec}_k}(V, W) \oplus \operatorname{Hom}_{\mathsf{Vec}_k}(W', V),$$

which carry a natural action, for $\varsigma \in \mathrm{GL}(V)$ (a point), $\varphi \in \operatorname{Hom}_{\mathsf{Vec}_k}(V, W)$, and

$\vartheta \in \operatorname{Hom}_{\mathsf{Vec}_k}(W', V)$ as $\varsigma \cdot (\varphi \circ \varsigma, \varsigma^{-1} \circ \vartheta)$. The category of all such geometric bundles of the above type will be denoted as $\mathsf{AM}_k^{\mathrm{GL}(V)}$, whose morphisms are morphisms of schemes which are $\mathrm{GL}(V)$-equivariant relative to the above action. Let us provide a few more specifics concerning the action of $\mathrm{GL}(V)$ on arbitrary objects of $\mathsf{AM}_k^{\mathrm{GL}(V)}$. For such an object Z, the induced action

$$\sigma_Z : \mathrm{GL}(V) \times_k Z \to Z$$

is equivalent to the co-action as Hopf algebra modules. Choosing bases for V, W and W', we may write

$$Z = \operatorname{Spec}\left(k\left[\{a_{ij}\}_{i\in[d],j\in[m_0]}, \{b_{ij}\}_{i\in[m_1],j\in[d]}\right]\right),$$

where $m_1 = \dim W$ and $m_0 = \dim W'$. Letting $B := (b_{ij})$ and $A := (a_{ij})$, we have

$$Z = \operatorname{Spec}\left(k\left[A, B\right]\right). \tag{6.1}$$

The co-action on the global sections

$$\sigma_Z^\sharp : k[Z] \to k\left[\mathrm{GL}(V)\right] \otimes_k k[Z],$$

is defined on the generators as

$$\begin{aligned} b_{ij} &\mapsto (\det(C))^{-1} \sum_{r=1}^{d} \operatorname{Adj}(C)_{rj} \otimes_k b_{ir} \\ a_{ij} &\mapsto \sum_{r=1}^{d} c_{ir} \otimes_k a_{rj}, \end{aligned}$$

where $\operatorname{Adj}(C)$ is the adjoint matrix of C.

To build the readers intuition further let $\varphi \in \operatorname{Hom}(V, W)$, and choose a basis of V and W such that $\varphi = (\varphi_{ij})$ with $i \in [m_0]$ and $j \in [d]$, and $\varsigma \in \mathrm{GL}(V)$ such that under the chose of basis for V then $\varsigma = (\varsigma_{s\ell})$ for $s, \ell \in [d]$. Therefore the action of $\mathrm{GL}(V)$ on $\operatorname{Hom}(V, W)$ can be defined as

$$\begin{aligned} (\varsigma, \varphi) &\mapsto \varphi \circ \varsigma \\ \varphi_{ij} &\mapsto \sum_{\ell=1}^{d} \varphi_{i\ell}\varsigma_{\ell j} \end{aligned}$$

Thus we can define a co-action on the dual $(\operatorname{Hom}(V, W))^\vee$ by:

$$\varphi_{ji} = (\varphi_{ij})^\vee \mapsto \left(\sum_{\ell=1}^{d} \varphi_{i\ell}\varsigma_{\ell j}\right)^\vee = \sum_{\ell=1}^{d} \varsigma_{j\ell}\varphi_{\ell i}$$

Therefore if we identify $(\operatorname{Hom}(V, W))^\vee$ with $\operatorname{Sym}^1\left((\operatorname{Hom}(V, W)^\vee\right)$ we get an action on $\operatorname{Spec}\left(\operatorname{Sym}\left((\operatorname{Hom}(V, W))^\vee\right)\right)$, i.e. the geometric bundle of $\operatorname{Hom}(V, W)$ over $\operatorname{Spec}(k)$. With a similar calculation on the regular functions of $\operatorname{Hom}(W', V)$ we derive part of the action on Z described above.

Furthermore, the projection

$$\pi_Z : \mathrm{GL}(V) \times_k Z \to Z$$

induces the map

$$\pi_Z^\sharp : k[Z] \to k\,[\mathrm{GL}(V)] \otimes_k k[Z].$$

Further, let $\mathsf{HP}_k^{\mathrm{GL}(V)}$ denote the full subcategory of $\mathsf{AM}_k^{\mathrm{GL}(V)}$ consisting of objects of the form

$$\mathrm{Hom}(V, W) \oplus \mathrm{Hom}(W', V)$$

such that $\dim(W), \dim(W') \geq \dim(V)$.

6.3.1 The functor

Before turning attention to our functor $\mathbf{Q}$, we introduce the functor Δ, which gives the kernel of the identity functor.

Notation 6.3.1. If $Z = \mathrm{Spec}(R)$ is an element of $\mathsf{AM}_k^{\mathrm{GL}(V)}$, we define the following scheme

$$\Delta_Z := Z \times_k \mathrm{GL}(V).$$

The assignment $Z \mapsto \Delta_Z$ defines a functor $\Delta : \mathsf{AM}_k^{\mathrm{GL}(V)} \to \mathsf{Aff}_k^{\mathrm{GL}(V) \times_k \mathrm{GL}(V)}$.

We aim to use Δ_Z to produce the Fourier-Mukai kernel for the identity functor on the bounded $\mathrm{GL}(V)$-equivariant derived category $\mathsf{D}^{\mathsf{b}}(\mathrm{Qcoh}^{\mathrm{GL}(V)} Z)$ by associating to it an object of $\mathsf{D}^{\mathsf{b}}(\mathrm{Qcoh}^{\mathrm{GL}(V) \times_k \mathrm{GL}(V)} Z \times_k Z)$. To achieve this, consider the morphism

$$\pi_Z \times_k \sigma_Z : Z \times_k \mathrm{GL}(V) \to Z \times_k Z.$$

We define a sheaf of modules over $Z \times_k Z$ associated to Δ_Z as

$$\widetilde{\Delta}_Z := (\pi_Z \times_k \sigma_Z)_* \mathcal{O}_{\Delta_Z},$$

where $\mathcal{O}_{\Delta_Z}$ denotes the structure sheaf of the affine scheme Δ_Z. It remains to define an action that realizes this sheaf as a $(\mathrm{GL}(V) \times_k \mathrm{GL}(V))$-equivariant sheaf over $Z \times_k Z$, which is provided in the next lemma.

Lemma 6.3.2. *For $Z = \mathrm{Spec}(R)$ an object of $\mathsf{AM}_k^{\mathrm{GL}(V)}$, the scheme $\mathrm{GL}(V) \times_k Z = \Delta_Z$ has a natural $(\mathrm{GL}(V) \times_k \mathrm{GL}(V))$-action*

$$\sigma_{\Delta_Z} : \mathrm{GL}(V) \times_k \mathrm{GL}(V) \times_k \Delta_Z \to \Delta_Z$$

uniquely determined by the co-action

$$\sigma^\sharp_{\Delta_Z} : k[\Delta_Z] \to k\left[\mathrm{GL}(V)\right] \otimes_k k\left[\mathrm{GL}(V)\right] \otimes_k k[\Delta_Z]$$

defined by

$$\begin{aligned} \sigma^\sharp_{\Delta_Z}(1 \otimes r) &= \left(\iota_1 \otimes 1_{\Delta_Z}\right) \circ \sigma^\sharp_Z(r) \\ \sigma^\sharp_{\Delta_Z}(t \otimes 1) &= \left(\left(1 \otimes \mu^\sharp\right) \circ \left(\beta^\sharp \otimes 1\right) \circ s^\sharp \circ \mu^\sharp(t)\right) \otimes 1_R. \end{aligned}$$

Here $r \in R$, $t \in k[\mathrm{GL}(V)]$ and $\iota_1 : k\left[\mathrm{GL}(V)\right] \to k\left[\mathrm{GL}(V)\right] \otimes_k k\left[\mathrm{GL}(V)\right]$ is the natural inclusion into the first component; $\beta^\sharp : k\left[\mathrm{GL}(V)\right] \to k\left[\mathrm{GL}(V)\right]$ is the co-inverse,
$\mu^\sharp : k\left[\mathrm{GL}(V)\right] \to k\left[\mathrm{GL}(V)\right] \otimes k\left[\mathrm{GL}(V)\right]$ is the group co-multiplication and $s^\sharp : k\left[\mathrm{GL}(V)\right] \otimes k\left[\mathrm{GL}(V)\right] \to k\left[\mathrm{GL}(V)\right] \otimes k\left[\mathrm{GL}(V)\right]$ switches the factors in the tensor product.

Moreover, the map $\pi_Z \times_k \sigma_Z : \mathrm{GL}(V) \times_k Z \to Z \times_k Z$ is equivariant with respect to this $\mathrm{GL}(V) \times_k \mathrm{GL}(V)$ action.

The proof is a straight-forward diagram chase and is left to the reader.

Lemma 6.3.3. *Let $Z = \mathrm{Spec}(R)$ be an object of $\mathsf{AM}_k^{\mathrm{GL}(V)}$ and M an object of $\mathrm{Mod}^{\mathrm{GL}(V)}(R)$. Then there is a $k[\mathrm{GL}(V)]$-co-module isomorphism*

$$\left(k[\mathrm{GL}(V)] \otimes_k M\right)^{\mathrm{GL}(V)} \cong M,$$

where $k[\mathrm{GL}(V)] \otimes_k M$ is given the left $k[\mathrm{GL}(V)]$-co-action as a $k[\Delta_Z]$-module.

Proof. Note that there is a natural morphism

$$M \to \left(k[\mathrm{GL}(V)] \otimes_k M\right)^{\mathrm{GL}(V)}$$

given by the equivariant structure of M. Since the extension $\overline{k}/k$ is faithfully-flat, it suffices to show that this map is an isomorphism over $\overline{k}$. Assume that $\overline{k} = k$.

By the Peter-Weyl Theorem, there is a decomposition $k[\mathrm{GL}(V)] = \bigoplus S_i \otimes S_i^\vee$, where S_i runs over every irreducible representation of $\mathrm{GL}(V)$. Furthermore, since $\mathrm{GL}(V)$ is linearly reductive, we have a decomposition $M = \bigoplus M_i$ into irreducible components. Thus, we have

$$\Big(k[\mathrm{GL}(V)]\otimes_k M\Big)^{\mathrm{GL}(V)} \cong \bigoplus S_i \otimes (S_i^\vee \otimes M_i)^{\mathrm{GL}(V)} \cong \bigoplus S_i \otimes \mathrm{Hom}_k^{\mathrm{GL}(V)}(S_i, M_i),$$

and our result follows from Schur's Lemma. □

Lemma 6.3.4. *For any object Z of $\mathsf{AM}_k^{\mathrm{GL}(V)}$, the object*

$$\widetilde{\Delta}_Z \in \mathsf{D}^{\mathsf{b}}(\mathrm{Qcoh}^{\mathrm{GL}(V)\times_k \mathrm{GL}(V)}(Z \times_k Z))$$

is the Fourier-Mukai kernel of the identity functor on $\mathsf{D}^{\mathsf{b}}(\mathrm{Qcoh}^{\mathrm{GL}(V)} Z)$.

Proof. First note that since Δ_Z is flat via either module structure and the Reynolds operator is flat, it is sufficient to prove this on the level of R-modules. For an R-module M, the integral transform associated to $\widetilde{\Delta}_Z$ is given by

$$\Phi_{\widetilde{\Delta}_Z}(\widetilde{M}) := \left[\mathbf{R}\pi_{2*}\left(\widetilde{\Delta}_Z \otimes^{\mathbf{L}} \mathbf{L}\pi_1^*\widetilde{M}\right)\right]^{\mathrm{GL}(V)},$$

where π_i are the natural $\mathrm{GL}(V)$-equivariant projections $Z \times_k Z \to Z$ (see [BFK14, Section 2] for background). Our desired result is a consequence of the following calculation:

$$\begin{aligned}
\Phi_{\widetilde{\Delta}_Z}(\widetilde{M}) &= \left[\mathbf{R}\pi_{2*}\left[((\pi_Z \times_k \sigma_Z)_*\mathcal{O}_{\Delta_Z}) \otimes^{\mathbf{L}}_{\mathcal{O}_{Z\times_k Z}} (\mathbf{L}\pi_1^*\widetilde{M})\right]\right]^{\mathrm{GL}(V)} \\
&\cong \left[\pi_{2^*}(\pi_Z \times_k \sigma_Z)_*\left[\mathcal{O}_{\Delta_Z} \otimes_{\mathcal{O}_{\Delta_Z}} \left((\pi \times_k \sigma_Z)^*\pi_1^*\widetilde{M}\right)\right]\right]^{\mathrm{GL}(V)} \\
&\cong \left[\sigma_{Z*}\pi_Z^*\widetilde{M}\right]^{\mathrm{GL}(V)} \\
&\cong \left[\mathcal{O}_{\Delta_Z} \otimes_{\mathcal{O}_Z} \widetilde{M}\right]^{\mathrm{GL}(V)} \\
&\cong \left[\left(\mathcal{O}_{\mathrm{GL}(V)} \otimes_k \mathcal{O}_Z\right) \otimes_{\mathcal{O}_Z} \widetilde{M}\right]^{\mathrm{GL}(V)} \\
&\cong \left[\mathcal{O}_{\mathrm{GL}(V)} \otimes_k \widetilde{M}\right]^{\mathrm{GL}(V)} \\
&\cong \widetilde{M},
\end{aligned}$$

where the first isomorphism follows from the projection formula, and the last follows from Lemma 6.3.3. Furthermore, on the second isomorphism we may forego the process of deriving these functors as they are either exact or remain an adapted class (as discussed above). □

We now define the natural generalization of the functor Q from [BDF17, Defn 2.1.6].

Definition 6.3.5. Given an object $Z = \mathrm{Spec}(R)$ of $\mathsf{AM}_k^{\mathrm{GL}(V)}$, define

$$Q_Z := \left(\pi_Y^\sharp(R), \sigma_Y^\sharp(R), C\right) \subseteq k\left[\mathrm{GL}(V) \times_k Z\right].$$

that is the k-subalgebra of $k[\mathrm{GL}(V) \times Z]$ generated by the images of $\sigma_Z^\sharp, \pi_Z^\sharp$ and the image of the inclusion $k[\mathrm{End}(V)] \hookrightarrow k[\mathrm{GL}(V) \times_k Z]$. For ease of notation we denote $\mathbf{Q}_Z := \mathrm{Spec}(Q_Z)$.

Remark 6.3.6. Similar to the functor Q in [BDF17, Def 2.1.6] our definition provides a partial compactification of the action of $\mathrm{GL}(V)$ on Z. For ease of reference we recall the definition of a partial compactification next.

Definition 6.3.7. Let G be an algebraic group and Z a k-scheme with G-action. Let $\tilde{Z}$ be a k-scheme together an action of $G \times_k G$ which is equipped with a $(G \times_k G)$-equivariant open immersion

$$i : G \times_k Z \hookrightarrow \widetilde{Z},$$

as well as a $(G \times_k G)$-equivariant morphism

$$(p, s) : \widetilde{Z} \to Z \times_k Z$$

such that the following diagram commutes

$$\begin{tikzcd} & \widetilde{Z} \arrow[d, shift right, "p"'] \arrow[d, shift left, "s"] \\ G \times_k Z \arrow[ur, "i"] \arrow[r, shift left, "\pi"] \arrow[r, shift right, "\sigma"'] & Z \end{tikzcd}$$

where σ is the action of G on Z and π is the projection to Z. In this case, we refer to $\widetilde{Z}$, with the maps p, s, i, as a *partial compactification of the action of G on Z*.

Example 6.3.8. If $\dim V = 1$, the category $\mathsf{AM}_k^{\mathrm{GL}(V)} = \mathsf{AM}_k^{\mathbb{G}_m}$ is a subcategory of $\mathsf{CR}_k^{\mathbb{G}_m}$ as studied in [BDF17]. In this case, the definition of Q given here recovers that found in loc. cit.

Lemma 6.3.9. *Let $Z = \operatorname{Spec}(R)$ be an object of $\mathsf{AM}_k^{\mathrm{GL}(V)}$. Then there are morphisms*

$$\mathbf{Q}_Z \underset{s}{\overset{p}{\rightrightarrows}} Z.$$

which precompose with the open immersion $\Delta_Z \to \mathbf{Q}_Z$ to give the morphisms π_Z and σ_Z.

Proof. By definition, the maps $\pi_Z^\sharp$ and $\sigma_Z^\sharp$ both have images which lie in Q_Z. □

Lemma 6.3.10. *For any object Z of $\mathsf{AM}_k^{\mathrm{GL}(V)}$, we have an isomorphism*

$$Q_Z \cong k\left[A^L, B^L, A^R, B^R, C\right] / \left(B^L - B^R C, A^R - C A^L\right) \cong k[A^L, B^R, C]. \tag{6.2}$$

Proof. We provide the reader with an easily verifiable isomorphism defined on the generators by

$$\begin{aligned} c_{ij} \mapsto c_{ij}, \quad \pi_Z^\sharp(a_{ij}) \mapsto a_{ij}^L, \quad \pi_Z^\sharp(b_{ij}) \mapsto b_{ij}^L, \\ \sigma_Z^\sharp(a_{ij}) \mapsto a_{ij}^R, \quad \sigma_Z^\sharp(b_{ij}) \mapsto b_{ij}^R. \end{aligned}$$

□

Remark 6.3.11. It follows from Equation (6.2) that $\mathbf{Q}_Z$ is isomorphic to the closed subvariety of $(Z \times_k Z) \times_k \operatorname{End}(V)$, consisting of the following points

$$\{(\psi_1, \psi_2, \psi_3, \psi_4, \varphi) | \, \psi_1 = \psi_3 \circ \varphi, \ \psi_4 = \varphi \circ \psi_2\}$$

Lemma 6.3.12. *For any object Z of $\mathsf{AM}_k^{\mathrm{GL}(V)}$, the scheme $\mathbf{Q}_Z$ admits a $(\mathrm{GL}(V) \times_k \mathrm{GL}(V))$-action, denoted $\sigma_{\mathbf{Q}_Z}$, which is uniquely defined by the co-action*

$$\sigma_{\mathbf{Q}_Z}^\sharp : k[A^L, B^R, C] \to k[D^L, (\det D^L)^{-1}] \otimes k[D^R, (\det D^R)^{-1}] \otimes k[A^L, B^R, C],$$

which maps the generators

$$\begin{aligned} b_{ij}^R &\mapsto \left(\det(D^R)\right)^{-1} \sum_{r=1}^n \operatorname{Adj}(D^R)_{rj} \otimes_k b_{ir}^R, \\ a_{ij}^L &\mapsto \sum_{r=1}^n {d^L}_{ir} \otimes_k a_{rj}^L, \\ c_{ij} &\mapsto \left(\det(D^L)\right)^{-1} \sum_{r=1}^n \operatorname{Adj}(D^L)_{sj} \otimes_k {d^R}_{ir} \otimes_k c_{rs}, \end{aligned}$$

where $\mathrm{Adj}(D)$ *is the adjoint of the matrix* D*.*

Proof. This follows by restricting the action of $(\mathrm{GL}(V)\times_k \mathrm{GL}(V))$ on Δ_Z that was defined in Lemma 6.3.2. $\square$

The next lemma gives explicit descriptions of the two module structures that Q_Z possesses.

Lemma 6.3.13. *For* $Z = \mathrm{Spec}(k[A,B])$ *an object of* $\mathsf{AM}_k^{\mathrm{GL}(V)}$*, we have the following two* $k[A,B]$*-module structures on* Q_Z *given by* $p^\sharp$ *and* $s^\sharp$*, respectively:*

$$\begin{aligned} p^\sharp : k[A,B] &\to k[A^L, B^R, C] \\ B &\mapsto B^R C \\ A &\mapsto A^L \\ s^\sharp : k[A,B] &\to k[A^L, B^R, C] \\ B &\mapsto B^R \\ A &\mapsto C A^L \end{aligned}$$

Proof. These are just the maps induced by the description of Q_Z from Lemma 6.3.10 under the identification

$$Q_Z = k[A^L, B^R, C].$$

$\square$

Proposition 6.3.14. *For any object* Z *of* $\mathsf{AM}_k^{\mathrm{GL}(V)}$ *the assignment* $Z \mapsto \mathbf{Q}_Z$ *defines a functor* $\mathbf{Q} : \mathsf{AM}_k^{\mathrm{GL}(V)} \to \mathsf{Aff}_{k[\mathrm{End}(V)]}^{\mathrm{GL}(V)\times \mathrm{GL}(V)}$.

Proof. Let $X = \mathrm{Spec}(R)$ and $Y = \mathrm{Spec}(S)$ be objects of $\mathsf{AM}_k^{\mathrm{GL}(V)}$ with $f \in \mathrm{Hom}_{\mathsf{AM}_k^{\mathrm{GL}(V)}}(X,Y)$. Note that $\mathbf{Q}f : \mathbf{Q}_X \to \mathbf{Q}_Y$ is defined as the restriction of $f\otimes 1 : X \times_k \mathrm{GL}(V) \to Y \times_k \mathrm{GL}(V)$, which is well defined since f is assumed to be $\mathrm{GL}(V)$-equivariant. With this description it is readily verified that indeed $\mathbf{Q}$ is functorial. $\square$

Remark 6.3.15. It follows immediately from the definition that Δ is a subfunctor of $\mathbf{Q}$. Furthermore, this definition easily extends to any affine variety with a GL(V)-action; yet this level of generalization is outside the scope of this paper. We note that our choice of subcategory $\mathsf{AM}_k^{\mathrm{GL}(V)} \subset \mathsf{Aff}_k^{\mathrm{GL}(V)}$, is intended to give an appropriate generalization of the varieties considered in [DS14] while not having to encounter any unnecessary technical difficulties in the statements of this preliminary section.

Now, we prove some properties of Q that will be used in Section 6.3.3 to prove the fullness of a Fourier-Mukai transform constructed using Q.

Lemma 6.3.16. *For an object $Z = \mathrm{Spec}(R)$ of $\mathsf{AM}_k^{\mathrm{GL}(V)}$, we have*

$$\mathrm{Tor}_i({}_pQ_Z,\, {}_sQ_Z) = 0$$

for all $i > 0$, where the subscripts preceding Q_Z denote the R-module structures given by $p^\sharp$ or $s^\sharp$, respectively.

Proof. Let $R := k[A, B]$ as in Equation (6.1). By Lemma 6.3.13, we have

$$\begin{aligned} {}_pQ_Z &\cong k[A, B, B', C] \,/(B - B'C) \\ {}_sQ_Z &\cong k[A, B, A', C] \,/(A - CA') \end{aligned}$$

Let us compute $Q_Z \;{}_s\otimes_p^{\mathbf{L}} Q_Z$ using the above expressions:

$$\begin{aligned} Q_Z \;{}_s\otimes_p^{\mathbf{L}} Q_Z &= k[A, B, A', C] \,/(A - CA') \otimes_{k[A,B]}^{\mathbf{L}} k[A, B, B', C] \,/(B - B'C) \\ &\cong k[A, B, A', C] \,/(A - CA') \otimes_{k[A,B]} \mathcal{K}_{k[A,B,B',C]}(B - B'C) \\ &\cong \mathcal{K}_{k[A,B,A',B',C_1,C_2]/(A - C_2A')}(B - B'C_1) \\ &\cong \mathcal{K}_{k[B,A',B',C_1,C_2]}(B - B'C_1), \end{aligned}$$

where we resolved the regular sequence $(B - B'C)$ by the Koszul complex, denoted by $\mathcal{K}$, on the second line.

Finally, we see that the sequence $(B - B'C_1)$ is still regular in the ring $k[B, A', B', C_1, C_2]$ and hence all the higher homologies vanish. $\square$

Notation 6.3.17. Similar to an observation of $k[\Delta_Z]$, the ring Q_Z is naturally associated to a sheaf of modules over Z, with its module structure defined via

s or p. We may thus realize Q_Z as a sheaf of modules over $Z \times_k Z$, and we denote this module by

$$\widehat{Q}_Z := (p \times_k s)_* \mathcal{O}_{\mathbf{Q}_Z}.$$

We will use the same notation in the derived setting (see Section 6.3.2, particularly Remark 6.3.27). Furthermore, as Δ_Z is an open subset of $\mathbf{Q}_Z$ we will denote the natural open immersion as

$$\eta : \mathrm{GL}(V) \times_k Z \to \mathbf{Q}_Z.$$

We now specialize to the case where $Z = \mathrm{Hom}(V, W) \oplus \mathrm{Hom}(W', V)$ is an arbitrary object of $\mathsf{HP}_k^{\mathrm{GL}(V)}$, so that $\dim W, \dim W' \geq \dim V = d$. We also recall that we denote $\dim W = m_0$ and $\dim W' = m_1$. Now consider the two open sets

$$\begin{aligned} U^+ &:= \Big(\mathrm{Hom}(V, W) \setminus \{\varphi : \mathrm{rank}(\varphi) \leq (d-1)\} \Big) \oplus \mathrm{Hom}(W', V) \\ U^- &:= \mathrm{Hom}(V, W) \oplus \Big(\mathrm{Hom}(W', V) \setminus \{\vartheta : \mathrm{rank}(\vartheta) \leq (d-1)\} \Big). \end{aligned}$$

It will be useful to denote the following open covers of these quasi-affine sets. Let

$$U^+ = \bigcup_{J \subseteq [m_0],\ |J| = d} U_J^+, \tag{6.3}$$

$$U^- = \bigcup_{I \subseteq [m_1],\ |I| = d} U_I^-, \tag{6.4}$$

where

$$\begin{aligned} U_I^+ &:= \mathrm{Spec}\left(k \left[A, B, \left(\det(A_{[d],I})\right)^{-1} \right] \right) \\ U_J^- &:= \mathrm{Spec}\left(k \left[A, B, \left(\det(B_{J,[d]})\right)^{-1} \right] \right) \end{aligned}$$

and (for example) $\det\left(A_{[d],I}\right)$ denotes the $(d \times d)$ minor of A consisting of the rows indexed by I. Therefore, we have the following affine open covers:

$$U^+ \times_{Z /\!\!/ 0} U^- = \bigcup_{I \subseteq [m_0],\ J \subseteq [m_1],\ |I| = |J| = d} U_I^+ \times_{Z /\!\!/ 0} U_J^-, \tag{6.5}$$

$$U^+ \times_k U^- = \bigcup_{I \subseteq [m_0],\ J \subset [m_1],\ |I|=|J|=d} U_I^+ \times_k U_J^-, \tag{6.6}$$

where $Z /\!\!/ 0 := \operatorname{Spec}(k[A, B]^{\mathrm{GL}(V)})$ denotes the invariant theoretic quotient of Z.

Lemma 6.3.18. *Let Z be an object of $\mathsf{AM}_k^{\mathrm{GL}(V)}$. There is an isomorphism*

$$k\left[Z \times_{Z/\!\!/0} Z\right] \cong k\left[A^L, B^L, A^R, B^R\right] / \left(B^L A^L - B^R A^R\right),$$

where the generators and relations are as in Definition 6.3.5.

Proof. From Weyl's fundamental theorems for the action of $\mathrm{GL}(V)$ (for example see [KP96, Chapter 2.1] or the original text [Wey46]) we have

$$Z/\!\!/0 = \{D \in \operatorname{Hom}_k(W, W) \mid \operatorname{rank} D \leq \dim V\}.$$

The map $Z \to Z/\!\!/0$ is thus given by the homomorphism

$$\begin{aligned} k[Z/\!\!/0] &\to k[A, B] \\ D &\mapsto BA. \end{aligned}$$

Hence,

$$k\left[Z \times_{Z/\!\!/0} Z\right] = k[A^L, B^L] \otimes_{k[Z/\!\!/0]} k[A^R, B^R] \cong k[A^L, B^L, A^R, B^R]/(B^L A^L - B^R A^R).$$

□

Lemma 6.3.19. *There exists a morphism*

$$\kappa := p^\# \otimes s^\# : k[A^L, B^L] \otimes_{k[Z/\!\!/0]} k[A^R, B^R] \to Q_Z.$$

Proof. This follows since $p^\#$ and $s^\#$ are equal on $k[Z/\!\!/0]$, by definition. □

Lemma 6.3.20. *With the conventions above we have the following containment of ideals in the ring $k[A^L, B^L, A^R, B^R, C]$:*

$$\left(B^L A^L - B^R A^R\right) \subset \left(B^L - B^R C, A^R - C A^L\right).$$

Proof. This follows from

$$(B^L - B^R C)A^L + B^R(CA^L - A^R) = B^L A^L - B^R A^R.$$

$\square$

Proposition 6.3.21. *Let $Z = \mathrm{Spec}(k[A,B])$ be an object of $\mathsf{HP}_k^{\mathrm{GL}(V)}$ and Q_Z as in Equation (6.2). Let $\widehat{Q}_Z|_{U^+ \times_k U^-}$ be the restriction of $\widehat{Q}_Z$ to the open subset $U^+ \times_k U^- \subset Z \times_k Z$. Then κ restricts to an isomorphism*

$$\kappa|_{U^+ \times_k U^-} : \widehat{Q}_Z\Big|_{U^+ \times_k U^-} \xrightarrow{\sim} \mathcal{O}_{U^+ \times_{Z/\!\!/0} U^-}.$$

Proof. We look affine-locally using the covers of Equations 6.5 and 6.6. We need only show that under the above localization the map $\kappa : k[Z \times_{Z/\!\!/0} Z] \to Q_Z$ becomes an isomorphism. For surjectivity, it suffices to show that there is an element (we find two such) which map to C. Indeed, we have

$$\begin{aligned} \Big((B^R)_{J[d]}\Big)^{-1} A^L &\mapsto C \\ B^R \Big((A^L)_{[d]I}\Big)^{-1} &\mapsto C, \end{aligned}$$

easily verified by the relations $B^L - B^R C$ and $A^R - CA^L$ in $Q(k[A,B])$ given in Definition 6.3.5.

For injectivity, it suffices to check that under this localization we have the containment

$$\left(B^L - B^R C, A^R - CA^L\right) \subset \left(B^L A^L - B^R A^R\right),$$

since the opposite containment is Lemma 6.3.21. To see this, simply note that by multiplying by the appropriate elements in the above identification, we have

$$(B^L)_{J[d]} = (B^R)_{J[d]} C \qquad \text{and} \qquad (A^R)_{[d]I} = C(A^L)_{[d]I}.$$

Hence, multiplying by the appropriate units in our localization, we have

$$\left(B^L - B^R C, A^R - CA^L\right) = \left((B^R)_{J[d]}\left(A^R - CA^L\right), \left(B^L - B^R C\right)(A^L)_{[d]I}\right).$$

For example, by Equation (6.3.1), we have

$$(B^R)_{J[d]}\left(A^R - CA^L\right) = \left((B^R)_{J[d]}A^R - (B^L)_{J[d]}A^L\right) \in \left(B^L A^L - B^R A^R\right),$$

while the other relation follows similarly. This gives our desired isomorphism. □

Consider the restriction $Q|_{U^+\times U^-}$. By descent, we have a corresponding object P on the quotient $Z^+ \times Z^-$.

Theorem 6.3.22. *For an object Z of $\mathsf{HP}_k^{\mathrm{GL}(V)}$ we have an isomorphism*

$$P \cong \mathcal{O}_{Z^+\times_{Z_0} Z^-}$$

Proof. This follows immediately by passing to the quotient in Proposition 6.3.21. □

We now examine a useful invariant when studying kernels in the next subsection. Note that for Z, an object of $\mathsf{AM}_k^{\mathrm{GL}(V)}$, the tensor product $Q_Z\ {}_s\!\otimes_p Q_Z$ is equipped with a natural $\mathrm{GL}(V)^{\times 4}$-action. This induces a $\mathrm{GL}(V)^{\times 3}$-action, which we denote

$$\sigma_3 : \mathrm{GL}(V)^{\times 3} \times_k \mathbf{Q}_Z^{\times 2} \to \mathbf{Q}_Z^{\times 2}$$

and is defined as the product of the following compositions

$$\begin{tikzcd}
 & \mathrm{GL}(V)^{\times 3} \times_k \mathbf{Q}_Z^{\times 2} \arrow[dl, "\pi_{1,2,4}"'] \arrow[dr, "\pi_{2,3,5}"] & \\
\mathrm{GL}(V)^{\times 2} \times_k \mathbf{Q}_Z \arrow[d, "\sigma_{\mathbf{Q}_Z}"'] & & \mathrm{GL}(V)^{\times 2} \times_k \mathbf{Q}_Z \arrow[d, "\sigma_{\mathbf{Q}_Z}"] \\
\mathbf{Q}_Z & & \mathbf{Q}_Z
\end{tikzcd}$$

Here $\pi_{i,j,k} : \mathrm{GL}(V)^{\times 3} \times_k \mathbf{Q}_Z^{\times 2} \to \mathrm{GL}(V)^{\times 2} \times_k \mathbf{Q}_Z$ is the projection onto the i^{th}, j^{th} and k^{th} components. For any ring T with $\mathrm{GL}(V)^{\times 3}$-action, we will denote the invariant subring associated to the action corresponding to the middle component of $\mathrm{GL}(V)^{\times 3}$ by $T^{\bowtie}$. The notation $(-)^{\bowtie}$ is suggestive of pinching a module in the middle. Since taking invariants is functorial for equivariant morphisms, we obtain the following:

Lemma 6.3.23. *The following diagram commutes*

$$\begin{tikzcd}
 & (Q_Z \, {}_p\!\otimes_\sigma \Delta_Z)^{\bowtie} \arrow[dr, "\sim"] & \\
(Q_Z \, {}_p\!\otimes_s Q_Z)^{\bowtie} \arrow[ur, "(1\otimes\eta)^{\bowtie}"] \arrow[dr, "(\eta\otimes 1)^{\bowtie}"'] & & Q_Z \\
 & (\Delta_Z \, {}_\pi\!\otimes_s Q_Z)^{\bowtie} \arrow[ur, "\sim"'] &
\end{tikzcd}$$

Furthermore, the morphism $\rho_Z : (Q_Z \, {}_p\!\otimes_s Q_Z)^{\bowtie} \to Q_Z$ is an isomorphism.

Proof. First recall that we have a presentation from Lemma 6.3.13 of ${}_sQ_Z$ and ${}_pQ_Z$, which for ease of calculation we set the following simplified notation, with the hope that no confusion arises:

$$\begin{aligned}
{}_pQ_Z &\cong k[A, B^L, B^R, C]/\left(B^L - B^R C\right) \cong k[A, B^R, C] \\
&:= k[A, B, C] \\
{}_sQ_Z &\cong k[A^L, A^R, B, C]/\left(A^R - CA^L\right) \cong k[A^L, B, C] \\
&:= k[A, B, C]
\end{aligned}$$

Further we recall the notational preference that for k-algebras R, S and $r \in R$, $s \in S$ that the following pure tensors will be denoted: $r \otimes 1 := r^L$ and $1 \otimes s := s^R$. With these conventions we have the following presentations of rings:

$$\begin{aligned}
Q_Z \, {}_p\!\otimes_s Q_Z &\cong k[A^L, A^R, B^L, B^R, C^L, C^R] \big/ \left(B^L C^L - B^R, A^L - C^R A^R\right) \\
&\cong k[A^R, B^L, C^L, C^R] \\
k[\Delta_Z] \, {}_\pi\!\otimes_s Q_Z &\cong k[A^L, A^R, B^L, B^R, C^L, C^R, \det(C^L)^{-1}] \big/ \left(B^L - B^R, A^L - C^R A^R\right) \\
&\cong k[A^R, B^L, C^L, C^R, \det(C^L)^{-1}] \\
Q_Z \, {}_p\!\otimes_\sigma k[\Delta_Z] &\cong k[A^L, A^R, B^L, B^R, C^L, C^R, \det(C^R)^{-1}] \Big/ \left(B^L - B^R \left(C^R\right)^{-1}, A^L - C^R A^R\right) \\
&\cong k[A^R, B^L, C^L, C^R, \det(C^R)^{-1}]
\end{aligned}$$

Hence, commutativity of the above diagram is clear. Furthermore, one verifies that we have an isomorphism $k[\Delta_Z] \, {}_\pi\!\otimes_s Q_Z \cong Q_Z \, {}_p\!\otimes_\sigma k[\Delta_Z]$, and thus

$$(k[\Delta_Z] \, {}_\pi\!\otimes_s Q_Z)^{\bowtie} \cong (Q_Z \, {}_p\!\otimes_\sigma k[\Delta_Z])^{\bowtie}.$$

It is clear that the maps on the right-hand side of the diagram are isomorphisms since $k[\Delta_Z]$ is the kernel of the identity by Lemma 6.3.4. We claim that

$$(Q_Z\,{}_p\otimes_s Q_Z)^{\bowtie} = k[A^R, B^L, C^L \cdot C^R]$$
$$(Q_Z\,{}_p\otimes_\sigma k[\Delta_Z])^{\bowtie} = k[A^R, B^L, C^L \cdot C^R]$$

from which it follows that these rings are isomorphic. This claim is simply Weyl's Theorem for the invariants of $k[V \otimes V^\vee]$.

□

6.3.2 The integral kernel

We now use $\mathbf{Q}$ to construct Fourier-Mukai kernels. We begin by recalling the following from [BDF17, Definition 3.1.4].

Definition 6.3.24. Let $\widetilde{Z}$ be a partial compactification of an action $\sigma : G \times_k Z \to Z$, with maps p, s, and i as above. We define the *boundary* of $\widetilde{Z}$ to be

$$\partial^s_{\widetilde{Z}} := \widetilde{Z} \setminus i\,(G \times_k Z)\,,$$

the *s-unstable locus* to be

$$Z^{\mathrm{us}}_s := s\left(\partial_{\widetilde{Z}}\right),$$

and the *s-semistable locus* to be

$$Z^{\mathrm{ss}}_s := Z \setminus Z^{\mathrm{us}}.$$

One similarly defines the *p-unstable* and *p-semistable* loci.

Remark 6.3.25. It follows from [BDF17, Example 3.1.10] that for an object Z of $\mathsf{HP}_k^{\mathrm{GL}(V)}$, the s-semistable locus Z^{ss}_s coincides with U^+ from Equation (6.3). Similarly, the p-semistable locus Z^{ss}_p coincides with U^- from Equation (6.4).

Definition 6.3.26. For an object Z of $\mathsf{HP}_k^{\mathrm{GL}(V)}$, we let

$$\widehat{Q}_Z := (p \times s)_* \mathcal{O}_{\mathbf{Q}_Z} \in \mathsf{D}^{\mathsf{b}}\left(\mathrm{Qcoh}^{\mathrm{GL}(V)\times_k \mathrm{GL}(V)}\, Z \times_k Z\right),$$

where the pushforward is understood to be derived. We denote by $\widehat{Q}^+_Z$ the quasi-coherent sheaf on $Z^{\mathrm{ss}}_s \times_k Z$ realized by restricting $\widehat{Q}_Z$ from $Z \times_k Z$. That is,

$$\widehat{Q}^+_Z = (j \times 1_Z)^* \widehat{Q}_Z,$$

where $j : Z_s^{\mathrm{ss}} \to Z$ is the inclusion. Finally, taking $\widehat{Q}_Z^+$ as the Fourier-Mukai kernel, we have the functor

$$\Phi_{\widehat{Q}_Z^+} : \mathsf{D}^{\mathsf{b}}\left(\mathrm{Qcoh}^{\mathrm{GL}(V)} Z_s^{\mathrm{ss}}\right) \to \mathsf{D}^{\mathsf{b}}\left(\mathrm{Qcoh}^{\mathrm{GL}(V)} Z\right). \tag{6.7}$$

Remark 6.3.27. Since the functor $(p \times s)_*$ is exact, $\widehat{Q}_Z$ is just the $\mathrm{GL}(V)$-linearized sheaf associated to Q_Z with its (p, s)-bimodule structure given in Lemma 6.3.2. This justifies our use of $\widehat{Q}_Z$ in Notation 6.3.17.

Lemma 6.3.28. *Let Z be an object of $\mathsf{HP}_k^{\mathrm{GL}(V)}$. Then $\Phi_{\widehat{Q}_Z^+}$ is faithful.*

Proof. Our proof follows from the fact that the functor

$$i^* : \mathsf{D}^{\mathsf{b}}(\mathrm{Qcoh}_{\mathrm{GL}(V)}(Z_s^{ss})) \to \mathsf{D}^{\mathsf{b}}(\mathrm{Qcoh}_{\mathrm{GL}(V)}(Z))$$

is the left inverse of $\Phi_{\widehat{Q}_Z^+}$. To see this, note that for any maximal minor m of B, we have $R_m \otimes_s Q_Z \cong k[\mathrm{GL}(V)] \otimes_k R = k[\Delta_Z]$. Indeed, inverting a minor on the left amounts to inverting the determinant of C. Since Δ_Z is the kernel of the identity, we obtain the desired result. □

The fullness of this functor depends on certain localization properties, which are the focus of the next section.

6.3.3 Bousfield localizations

This section recalls Bousfield (co)-localizations which will be used to establish fullness of the functor $\Phi_{\widehat{Q}^+}$ from Equation (6.7). We recall that the existence of a Bousfield triangle produces a semi-orthogonal decomposition, and we show that the essential image of our functor is an inclusion into one of these pieces. We refer the reader to [Kra10] for a more detailed treatment of these concepts. While the proofs of the statements refer to [BDF17] we recall all of the statements here for ease of reference.

Definition 6.3.29. Let $\mathcal{T}$ be a triangulated category. A *Bousfield localization* is an exact endofunctor $L : \mathcal{T} \to \mathcal{T}$ equipped with a natural transformation $\delta : 1_{\mathcal{T}} \to L$ such that:

a) $L\delta = \delta L$ and

b) $L\delta : L \to L^2$ is invertible.

A *Bousfield co-localization* is given by an endofunctor $C : \mathcal{T} \to \mathcal{T}$ equipped with a natural transformation $\epsilon : C \to 1_{\mathcal{T}}$ such that:

a) $C\epsilon = \epsilon C$ and

b) $C\epsilon : C^2 \to C$ is invertible.

Definition 6.3.30. Assume there are natural transformations of endofunctors

$$C \xrightarrow{\epsilon} 1_{\mathcal{T}} \xrightarrow{\delta} L$$

of a triangulated category $\mathcal{T}$ such that

$$Cx \xrightarrow{\epsilon_{Cx}} x \xrightarrow{\delta_x} Lx$$

is an exact triangle for any object x of $\mathcal{T}$. Then we refer to $C \to 1_{\mathcal{T}} \to L$ as a *Bousfield triangle* for $\mathcal{T}$ when any of the following equivalent conditions are satisfied:

1) L is a Bousfield localization and $C(\epsilon_x) = \epsilon_{C_x}$

2) C is a Bousfield co-localization and $L(\delta_x) = \delta_{L_x}$

3) L is a Bousfield localization and C is a Bousfield co-localization.

For a proof that the above properties are indeed equivalent, we refer the reader to [BDF17, Definition 3.33]. Denoting $S := k[\Delta_Z]/Q_Z$, we have morphisms

$$Q_Z \xrightarrow{\eta^\sharp} k[\Delta_Z] \to S \to Q_Z[1]$$

in $\mathsf{D}^{\mathsf{b}}(\mathrm{Mod}^{\mathrm{GL}(V)}\, k[Z])$, where $\eta^\sharp$ is the morphism induced by η as in Equation 6.3.17. This yields an exact triangle. Furthermore, if we let $\hat{\eta} : \Phi_{\widehat{Q}_Z} \to 1$ denote the morphism induced by $\eta^\sharp$, we see that for any x in $\mathsf{D}^{\mathsf{b}}(\mathrm{Qcoh}^{\mathrm{GL}(V)}\, Z)$ the following is also exact:

$$\Phi_{\widehat{Q}_Z}(x) \to x \to \Phi_{\widetilde{S}}(x)$$

With these observations in mind, we present one of the main results of this section.

Proposition 6.3.31. *Let Z be an object of $\mathsf{AM}_k^{\mathrm{GL}(V)}$. Then the triangle of functors*

$$\Phi_{\widehat{Q}_Z} \xrightarrow{\hat{\eta}} 1 \to \Phi_{\widetilde{S}}.$$

is a Bousfield triangle.

Proof. This follows identically as in [BDF17, Lemma 3.3.6], by Lemma 6.3.16 and Lemma 6.3.23. □

We are now ready to prove that $\Phi_{\widehat{Q}+}$ is full. Let $J_+ := j_* \circ j^*$, where $j : Z_s^{\mathrm{ss}} \to Z$ is the natural inclusion, and let Γ_+ be the local cohomology.

Proposition 6.3.32. *Let Z be an object of $\mathsf{HP}_k^{\mathrm{GL}(V)}$. There is a semi-orthogonal decomposition*

$$\mathsf{D}(\mathrm{Qcoh}^{\mathrm{GL}(V)} Z) = \langle \mathrm{Im}\, \Phi_{\widetilde{S}}, \mathrm{Im}\, \Phi_Q \circ \Gamma_+, \mathrm{Im}\, \Phi_{\widehat{Q}+} \rangle,$$

where Im *denotes the essential image. Furthermore, $\Phi_{\widehat{Q}+}$ is fully-faithful.*

Proof. This follows identically to the proof of Proposition 3.3.9 in [BDF17] □

Letting $j' : Z_p^{ss} \to Z$ be the inclusion and Γ_+ its local cohomology, we have the following dual statement.

Proposition 6.3.33. *Let Z be an object of $\mathsf{HP}_k^{\mathrm{GL}(V)}$. There is a semi-orthogonal decomposition*

$$\mathsf{D}(\mathrm{Qcoh}^{\mathrm{GL}(V)} Z) = \langle \mathrm{Im}\, \Phi_{\widetilde{S}}, \mathrm{Im}\, \Phi_Q \circ \Gamma_-, \mathrm{Im}\, \Phi_{\widehat{Q}-} \rangle,$$

where Im *denotes the essential image. Furthermore, $\Phi_{\widehat{Q}-}$ is fully-faithful.*

6.4 A geometric resolution

For this section, we will denote Z as the scheme

$$\mathrm{Hom}(V, W) \oplus \mathrm{Hom}(W', V).$$

Having established that $\Phi_{\widehat{Q}+}$ is fully faithful, the remaining objective of this work is to examine the essential image of the functor $\Phi_{\widehat{Q}+}$. We will show that this image is generated by an exceptional collection first discovered by Kapranov in [Kap88]. The method which we use is based on the underlying techniques of the well known 'geometric technique' of Kempf (see e.g. [Wey03]).

6.4.1 A sketch of Kempf

The objective of the method of Kempf is to provide a free resolution of special modules by pulling back to a trivial geometric bundle over a projective variety.

Consider an algebraic variety Y. The total space of the sheaf $\mathcal{O}_Y^{\oplus n}$ is the scheme $Y \times \mathbb{A}^n$. Now let X be the total space of a locally free sheaf $\mathcal{F} \subset \mathcal{O}_Y^{\oplus n}$ on Y. Let π denote the projection $Y \times \mathbb{A}^n \to Y$.

We have the exact sequence of locally free sheaves on $Y \times \mathbb{A}^n$

$$0 \longrightarrow \pi^*\mathcal{F} \longrightarrow \pi^*\mathcal{O}_Y^{\oplus n} \xrightarrow{f} \pi^*\mathcal{T} \longrightarrow 0,$$

where $\mathcal{T}$ is the quotient sheaf.

Consider the section $s := f \circ \mathrm{taut} : \mathcal{O}_{Y \times \mathbb{A}^n} \to \pi^*\mathcal{T}$, where taut denotes the tautological section of $\pi^*\mathcal{O}_Y^{\oplus n}$ on $Y \times \mathbb{A}^n$. Then, we have the following statement.

Proposition 6.4.1. *With the above notation, a locally free resolution of the sheaf $\mathcal{O}_X$ as a $\mathcal{O}_{Y \times \mathbb{A}^n}$-module is given by the Koszul complex*

$$\mathcal{K}(s)_\bullet : 0 \to \bigwedge^{\mathrm{rnk}(\mathcal{T})} (\pi^*\mathcal{T}^\vee) \to \ldots \to \bigwedge^{2} (\pi^*\mathcal{T}^\vee) \to \pi^*\mathcal{T}^\vee \to \mathcal{O}_{Y \times \mathbb{A}^n}$$

Proof. On the vanishing locus $Z(s)$, the tautological section taut factors through $\pi^*\mathcal{F}$. Hence, the vanishing locus is the total space of the sheaf $\mathcal{F}$, which is X. We see that the section is regular as the codimension of $Z(s)$ equals the rank of the sheaf $\pi^*\mathcal{T}$; and the Koszul complex resolves $\mathcal{O}_X$. For more details, see [Wey03, Proposition 3.3.2]. □

6.4.2 The resolution

Now we are ready to present a resolution which will open a window to view $\mathrm{Im}(\Phi_{\widehat{Q}^+})$. First recall that we set $\dim(V) := d$. We define $\mathbf{Q}_Z^+$ as the base change:

$$\begin{tikzcd}
\mathbf{Q}_Z^+ \arrow[r] \arrow[d] & \mathbf{Q}_Z \arrow[d, "p\times s"] \\
Z_s^{\mathrm{ss}} \times Z \arrow[r, hook] & Z \times Z
\end{tikzcd}$$

Let $\mathcal{S}$ be the tautological bundle on $\mathrm{Gr}(d,W)$ i.e. the locally free sheaf on $\mathrm{Gr}(d,W) = [\mathrm{Hom}(V,W)_s^{\mathrm{ss}}/\,\mathrm{GL}(V)^L]$ corresponding to the $\mathrm{GL}(V)^L$-representation V. Then, we have the Euler sequence for the Grassmannian $\mathrm{Gr}(d,W)$:

$$0 \to \mathcal{S} \to W \to \mathcal{Q} \to 0.$$

Consider the pullback of the above sequence to $\mathrm{Gr}(d,W)\times\mathrm{Hom}(W',V)$ along q and apply $\mathscr{H}om(t^*V,-)$, where $q : \mathrm{Gr}(d,W)\times\mathrm{Hom}(W',V) \to \mathrm{Gr}(d,W)$ and $t : \mathrm{Gr}(d,W)\times\mathrm{Hom}(W',V) \to \mathrm{Hom}(W',V)$ are projections

$$0 \to \mathscr{H}om(t^*V, q^*\mathcal{S}) \xrightarrow{\rho} \mathscr{H}om(t^*V, q^*W) \xrightarrow{\Xi} \mathscr{H}om(t^*V, q^*\mathcal{Q}) \to 0. \tag{6.8}$$

Let us denote $\mathcal{T} := \mathscr{H}om(t^*V, q^*\mathcal{Q})$. We denote the total space of the locally free sheaf $\mathscr{H}om(A,B)$ as $\mathbf{Hom}(A,B)$. From the discussion in the previous subsection, we get the following result:

Lemma 6.4.2. *The following Koszul complex is a free resolution for $\mathcal{O}_{\mathbf{Hom}(t^*V,q^*\mathcal{S})}$ as an $\mathcal{O}_{\mathrm{Gr}(d,W)\times Z}$-module.*

$$\mathcal{K}(s)_\bullet : \bigwedge^{d(m-d)} \pi^*\mathcal{T}^\vee \to \ldots \to \bigwedge^2 \pi^*\mathcal{T}^\vee \to \pi^*\mathcal{T}^\vee \to \mathcal{O}_{\mathrm{Gr}(d,W)\times Z} \tag{6.9}$$

where $\pi : \mathrm{Gr}(d,W)\times\mathrm{Hom}(V,W)\times\mathrm{Hom}(W',V) \to \mathrm{Gr}(d,W)\times\mathrm{Hom}(W',V)$ is the projection morphism.

Proof. We choose $Y = \mathrm{Gr}(d,W)\times\mathrm{Hom}(W',V)$, and $\mathcal{F} = \mathscr{H}om(t^*V, q^*\mathcal{S})$, and apply Proposition 6.4.1. Notice that the total space of $\mathscr{H}om(t^*V, q^*W)$ on $\mathrm{Gr}(d,W)\times\mathrm{Hom}(W',V)$ is $\mathrm{Gr}(d,W)\times Z$. □

Now, we can identify $[\mathbf{Q}_Z^+/\,\mathrm{GL}(V)^L]$ as the total space $\mathbf{Hom}(t^*V, q^*\mathcal{S})$.

Lemma 6.4.3. *The quotient space $[\mathbf{Q}_Z^+/\,\mathrm{GL}(V)^L]$ is $\mathrm{GL}(V)^R$-equivariantly isomorphic to the total space $\mathbf{Hom}(t^*V, q^*\mathcal{S})$ as schemes over $\mathrm{Gr}(d,W)\times\mathrm{Hom}(W',V)$.*

Proof. Recall from Equation (6.2), that $\mathbf{Q}_Z$ is associated to the module

$$k[A^L, B^R, C]$$

Geometrically, we may view $\mathbf{Q}_Z$ as the total space of the locally free sheaf $\mathrm{End}(V)$ over $\operatorname{Spec} k[A^L, B^R]$. Once we base change to the semistable locus and take the quotient with respect to the $\mathrm{GL}(V)^L$ action, we get that $[\mathbf{Q}_Z^+/\operatorname{GL}(V)^L]$ is isomorphic to the total space

$$\mathbf{Hom}(t^*V, q^*\mathcal{S}) \to \mathrm{Gr}(d, W) \times \mathrm{Hom}(W', V).$$

Moreover, the inclusion, $\mathscr{H}om(t^*V, q^*\mathcal{S}) \to \mathscr{H}om(t^*V, q^*W)$ realizes it as a subspace of the total space $\mathbf{Hom}(t^*V, q^*W)$ over $\mathrm{Gr}(d, W) \times \mathrm{Hom}(W', V)$ which is $Z \times \mathrm{Gr}(d, W)$.

This inclusion $\mathscr{H}om(t^*V, q^*\mathcal{S}) \to \mathscr{H}om(t^*V, q^*W)$ is induced by the ring homomorphism

$$\begin{gathered} k[A^L, A^R, B^R] \to k[A^L, B^R, C] \\ A^L \mapsto A^L \\ A^R \mapsto CA^L \\ B^R \mapsto B^R. \end{gathered}$$

which is equivariant with respect to the remaining $\mathrm{GL}(V)^R$-action. $\square$

We denote $\pi_1 : [Z_s^{\mathrm{ss}}/\operatorname{GL}(V)^L] \to [\mathrm{Hom}(V, W)_s^{\mathrm{ss}}/\operatorname{GL}(V)^L]$ as the projection. Putting Lemma 6.4.2 and Lemma 6.4.3 together, we get a resolution of the sheaf $(\pi_1 \times \mathrm{Id}_Z)_* \widehat{Q}_Z^+$.

Corollary 6.4.4. *The Koszul complex (6.9) is a locally free resolution of the sheaf* $(\pi_1 \times \mathrm{Id}_Z)_* \widehat{Q}_Z^+$ *of* $\mathcal{O}_{\mathrm{Gr}(d,W)\times Z}$*-modules.*

Remark 6.4.5. We note that we could also have constructed a locally free resolution of $\widehat{Q}_Z^+$ on $Z^{\mathrm{ss}} \times Z$ by the same method, and this will also lead to a similar proof as in the remainder of this paper.

6.5 Analyzing the integral transform

In this section, we show that the kernel $\widehat{Q}_Z^+$ induces a derived equivalence for a Grassmann flop. We begin by showing that the essential image of this functor coincides with the 'window' description studied by Donovan and Segal in [DS14, Section 3.1]. Recall that Z is the scheme

$$\mathrm{Hom}(V, W) \oplus \mathrm{Hom}(W', V)$$

with $\dim(W) =: m, \dim(W') =: m' \geq d := \dim(V)$. Specifically, we will show that the image of $\Phi_{\widehat{Q}_Z^+}$ is generated by a collection of vector bundles corresponding to representations identified by Kapranov [Kap88].

Let us recall Kapranov's collection. Consider the standard $\mathrm{GL}(V)$ representation V, where $GL(V)$ acts by left multiplication. Consider the Schur modules of V associated to a Young diagram (or equivalently, partition) α, and denote them by $L_\alpha V$. Kapranov's collection is defined by

$$\mathfrak{K}_{d,m} := \left\{ L_\alpha V \,\middle|\, \alpha \in \text{Young diagrams of height} \leq m-d \text{ and width} \leq d \right\}.$$

We also consider pull backs of these representations to $\mathrm{Gr}(d, W)$ along the structure morphism. As V pulls back to the tautological bundle $\mathcal{S}$, the Schur functors $L_\alpha V$ pull back to $L_\alpha \mathcal{S}$ and these are the locally free sheaves considered by Kapranov. By abuse of notation, we will consider $\mathfrak{K}_{d,m}$ as a collection of locally free sheaves on $\mathrm{Hom}(V, W) \oplus \mathrm{Hom}(W', V)$ or $\mathrm{Hom}(W', V)$ (again, by pulling back along the structure morphism). Note that when $k = \mathbb{C}$, this is exactly the dual of the zero$^{\text{th}}$ window W_0 from [DS14, Section 3.1].

It is the objective of this section to show that the thick triangulated subcategory generated by elements of $\mathfrak{K}_{d,m}$ is equivalent to $\mathrm{Im}\left(\Phi_{\widehat{Q}_Z^+}\right)$. We show one containment in Proposition 6.5.1, which relies on the work of Section 6.4.

6.5.1 Windows from a resolution

Consider the projection $\pi_1 : Z_s^{\mathrm{ss}} \to \mathrm{Hom}(V, W)_s^{\mathrm{ss}}$. To demonstrate that the image of $\Phi_{\widehat{Q}_Z^+}$ is contained in $\langle \mathfrak{K}_{d,m} \rangle$, we exhibit a particular $\mathrm{GL}(V)^L \times \mathrm{GL}(V)^R$-equivariant resolution $\mathcal{K}_\bullet$ of $(\mathrm{Id}_Z \times \pi_1)_* \widehat{Q}_Z^+$ over $\mathrm{Hom}(V, W)_s^{\mathrm{ss}} \times$

Z. Equivalently, this is a $\mathrm{GL}(V)^R$-equivariant resolution of $(\mathrm{Id}_Z \times \pi_1)_* \widehat{Q}_Z^+$ over $\mathrm{Gr}(d,W) \times Z$. The resolution obtained in equation (6.4.2) in Section 6.4.2 is the one we are looking for and resolves the functor $\Phi_{\widehat{Q}_Z^+} \circ \pi_1^*$.

In this subsection, we will show that the components $\mathcal{K}^i$ of the resolution have a filtration whose associated graded pieces are of the form $J \boxtimes K$ with $K \in \mathsf{K}_{d,m}$. This decomposition of the Fourier-Mukai transform $\Phi_{\widehat{Q}_Z^+} \circ \pi_1^*$ yields a functorial way to describe $\Phi_{\widehat{Q}_Z^+} \circ \pi_1^*(M)$ using objects of $\mathsf{K}_{d,m}$ for all objects $\pi_1^*(M) \in \mathsf{D}^{\mathsf{b}}([Z_s^{\mathrm{ss}} / \mathrm{GL}(V)])$. As such objects generate $\mathsf{D}^{\mathsf{b}}([Z_s^{\mathrm{ss}} / \mathrm{GL}(V)])$ this is enough to conclude the goal of this section, $\mathrm{Im}\left(\Phi_{\widehat{Q}_Z^+}\right) \subseteq \mathsf{K}_{d,m}$.

Proposition 6.5.1. *With notation as above, we have*

$$\mathrm{Im}\left(\Phi_{\widehat{Q}_Z^+}\right) \subseteq \left\langle \mathsf{K}_{d,m} \right\rangle,$$

where $\left\langle \mathsf{K}_{d,m} \right\rangle$ *is the thick triangulated subcategory generated by elements in* $\mathsf{K}_{d,m}$.

Proof. By Corollary 6.4.4, we have a quasi-isomorphism with the Koszul complex

$$\mathcal{K}_\bullet \cong (\mathrm{Id}_Z \times \pi_1)_* \widehat{Q}_Z^+$$

The components of the Koszul complex are $\bigwedge^l \pi^* \mathscr{H}om(t^*V, q^*\mathcal{Q})^\vee$ for $0 \leq l \leq d$. We can appeal to the Cauchy Formula, e.g. [Wey03, Theorem 2.3.2(a)], to get a filtration on $\bigwedge^i t^*\pi^* \mathscr{H}om(t^*V, q^*\mathcal{Q})^\vee$ whose associated graded pieces are

$$\pi^* \left(\bigoplus_{|\lambda|=i} L_\lambda V \boxtimes L_{\lambda'} \mathcal{Q}^\vee \right).$$

Thus, each term in the Koszul complex can be generated using iterated exact sequences from the locally free sheaves

$$\pi^* \left(L_\lambda V \boxtimes L_{\lambda'} \mathcal{Q}^\vee \right).$$

These components, in turn, generate $\widehat{Q}_Z^+$. Hence, for all M, $\Phi_{\widehat{Q}_Z^+}(\pi_1^* M)$ is generated by objects of the form

$$\Phi_{\pi^*(L_\lambda V \boxtimes L_\lambda \mathcal{Q}^\vee)}(\pi_1^* M) = \mathbf{R}\Gamma(M \otimes L_\lambda \mathcal{Q}^\vee) \otimes_k L_\lambda V$$

all of which lie in $\mathsf{K}_{d,m}$. Now, since π_1 is an affine map, $\mathsf{D}^{\mathsf{b}}([Z^{\mathrm{ss}} / \mathrm{GL}(V)])$ is generated by the essential image of π_1^*. The result follows. $\square$

6.5.2 Truncation operator

In this section we will see that $\Phi_{\widehat{Q}_Z^+}$ has a useful description on $\mathrm{GL}(V)$-representations. Yet before we go deeper into the representation theory we define a truncation operator over our field k of characteristic zero.

Definition 6.5.2. Let $M \in \mathrm{Mod}^{\mathrm{GL}(V)}(k[\mathrm{Hom}(V,W)])$, we define the *truncation operator* as follows

$$M_{\geq 0} := \Big(M \otimes k[\mathrm{End}(V)]\Big)^{\mathrm{GL}(V)}$$

Recall, further that there is a $\mathrm{GL}(V) \times_k \mathrm{GL}(V)$-module decomposition

$$k[\mathrm{End}(V)] \cong \bigoplus N_i^\vee \otimes_k N_i, \tag{6.10}$$

where we sum over all irreducible representations of $\mathrm{GL}(V)$ with all positive weights [Pro07], these representations are also referred to as *polynomial representations*. Since $\mathrm{GL}(V)$ is linearly reductive over a field of characteristic zero, we may decompose any $\mathrm{GL}(V)$-module M as $M \cong \bigoplus M_i$, where M_i is irreducible and we have the following description of the truncation operator 6.5.2:

Lemma 6.5.3. *Let $M \in \mathrm{Mod}^{\mathrm{GL}(V)}(k[\mathrm{Hom}(V,W)])$; then decompose M over k into irreducibles as*

$$M = \bigoplus_{M_i \textit{ irreducible}} M_i. \tag{6.11}$$

Then the truncation operator may be described as follows

$$M_{\geq 0} = \bigoplus_{\substack{M_i \textit{ irreducible} \\ \textit{and polynomial}}} M_i$$

□

Lemma 6.5.4. *For any $M \in \mathrm{Mod}^{\mathrm{GL}(V)}(k[\mathrm{Hom}(V,W)])$, $M_{\geq 0}$ is a $k[\mathrm{Hom}(V,W)]$-submodule of M and $(\_)_{\geq 0}$ is exact.*

Proof. The exactness of the functor follows since $\mathrm{GL}(V)$ is linearly reductive and thus our operator is just a projection. That $M_{\geq 0}$ is a $k[\mathrm{Hom}(V,W)]$-submodule follows since $k[\mathrm{Hom}(V,W)]_{\geq 0} = k[\mathrm{Hom}(V,W)]$ since $k[\mathrm{Hom}(V,W)]$ is a polynomial representation. □

To deliver a cleaner picture we define some more notation $Y' := \mathrm{Hom}(V, W)$. For the remainder of this subsection we will exploit the commutativity of the following diagram.

$$\begin{tikzcd} U_Z^+ \arrow[r, hook, "j"] \arrow[d, "q_1|_{U_Z^+}"'] & \mathrm{Hom}(V,W) \oplus \mathrm{Hom}(W',V) \arrow[d, "q_1"] \\ U_{Y'}^+ \arrow[r, hook, "i"'] & \mathrm{Hom}(V,W) \end{tikzcd}$$

Lemma 6.5.5. *Let* $M \in \mathrm{Mod}^{\mathrm{GL}(V)}(k[\mathrm{Hom}(V,W)])$ *then*

$$\Phi_{Q_{Y'}}(M) = M_{\geq 0}$$

Proof. The coaction map defines a morphism

$$M_{\geq 0} \to (k[\mathrm{End}(V)] \otimes M_{\geq 0})^{\mathrm{GL}(V)} \hookrightarrow (k[\mathrm{End}(V)] \otimes M)^{\mathrm{GL}(V)},$$

which we claim is an isomorphism. Notice that the coaction map lands in $k[\mathrm{End}(V)] \subset k[\mathrm{GL}(V)]$ as $M_{\geq 0}$ is a polynomial representation. To check that this map is an isomorphism, we may base change to $\overline{k}$ (which is faithfully flat over k). Hence, assume that $k = \overline{k}$.

Using equation (6.10) and Lemma 6.5.3, we get

$$\begin{aligned} (k[\mathrm{End}(V)] \otimes M)^{\mathrm{GL}(V)} &\cong \bigoplus N_j \otimes (N_j^\vee \otimes M_i)^{\mathrm{GL}(V)} \\ &\cong M_{\geq 0} \end{aligned}$$

where we are considering the left $\mathrm{GL}(V)$ invariant submodule and the second line follows from Schur's Lemma.

Finally, by Lemma 6.3.10 we have $Q_{Y'} \cong k[A] \otimes k[\mathrm{End}(V)]$, and we get

$$\begin{aligned} (Q_{Y'} \otimes M)^{\mathrm{GL}(V)} &\cong k[A] \otimes (k[\mathrm{End}(V)] \otimes M)^{\mathrm{GL}(V)} \\ &\cong M_{\geq 0}. \end{aligned}$$

□

Lemma 6.5.6. *We have an isomorphism*

$$(q_1 \times \mathrm{Id})_{*\ s}(Q_Z)_p \cong (\mathrm{Id} \times q_1)^*{}_s(Q_{Y'})_p$$

as objects of $\mathrm{Mod}^{\mathrm{GL}(V) \times \mathrm{GL}(V)}(Y' \times Z)$.

Proof. This follows from the following calculation.

$$\begin{aligned}
{}_sQ_Z &\cong k[A^L, B^R, C] \\
&\cong k[A^R, B^R] \otimes_{k[A]} k[A^L, C] \\
&\cong Z \otimes_{k[Y']} Q_{Y'},
\end{aligned}$$

where the first isomorphism follows from Lemma 6.3.13 and in the second line, $k[A]$ acts on the left by going to A^R and on the right by going CA^L. $\square$

Corollary 6.5.7. *Let $M \in \mathrm{Mod}^{\mathrm{GL}(V)}(k[Y'])$, then*

$$\Phi_{Q_Z}(q_1^* M) \cong q_1^* \Phi_{Q_{Y'}}(M)$$

Proof. This follows from Lemma 6.5.6 which says that it is true at the level of the Fourier-Mukai kernels. $\square$

Lemma 6.5.8. *For $L_\alpha V \in ᛕ_{d,m}$ we have that*

$$(\mathbf{R}i_*\mathbf{L}i^* L_\alpha V)_{\geq 0} \cong L_\alpha V$$

Proof. To see this we will denote the irreducible components as $(\_)_\beta$ where β is the highest weight corresponding to the isotypical piece, and by $\beta \geq 0$ we denote weights correspond to polynomial representations.

$$\begin{aligned}
(\mathbf{R}i_*\mathbf{L}i^* L_\alpha V)_{\geq 0} &= \bigoplus_{\beta \geq 0} (\mathbf{R}i_*\mathbf{L}i^* L_\alpha V)_\beta \\
&\cong \bigoplus_{\beta \geq 0} (\mathbf{R}i_*\mathbf{L}i^* L_\alpha V \otimes L_\beta V^\vee)^{\mathrm{GL}(V)} \\
&\cong \bigoplus_{\beta \geq 0} (\mathbf{R}i_*\mathbf{L}i^* (L_\alpha V \otimes L_\beta V^\vee))^{\mathrm{GL}(V)} \\
&\cong \bigoplus_{\beta \geq 0} \mathbf{R}\Gamma(\mathrm{Gr}(d, W), L_\alpha S \otimes L_\beta S^\vee) \\
&\cong \bigoplus_{\beta \geq 0} \Gamma(\mathrm{Gr}(d, W), L_\alpha S \otimes L_\beta S^\vee) \qquad (6.12) \\
&\cong \bigoplus_{\beta \geq 0} \Gamma(\mathrm{Hom}(V, W), L_\alpha V \otimes L_\beta V^\vee)^{\mathrm{GL}(V)} \qquad (6.13) \\
&\cong \bigoplus_{\beta \geq 0} (\mathrm{Sym}(\mathrm{Hom}(W, V)) \otimes L_\alpha V \otimes L_\beta V^\vee)^{\mathrm{GL}(V)} \\
&\cong \mathrm{Sym}(\mathrm{Hom}(W, V)) \otimes L_\alpha V \qquad (6.14) \\
&\cong \mathcal{O}_{\mathrm{Hom}(V,W)} \otimes L_\alpha V
\end{aligned}$$

Equation (6.12) follows from [Kap88, Lemma 3.2.a] (this uses the assumption that $L_\alpha V \in \mathfrak{K}_{d,m}$ and the fact that the weights of the irreducible summands of $L_\alpha V \otimes L_\beta V^\vee$ are all strictly larger than $-(m-d)$.) Equation (6.13) follows as $\mathrm{Gr}(d,W)$ has co-dimension greater than 2 in the global quotient stack $[\mathrm{Hom}(V,W)/\operatorname{GL}(V)]$. Equation (6.14) follows from Schur's Lemma and the fact that all representations in $\mathrm{Sym}(\mathrm{Hom}(W,V)) \otimes L_\alpha V$ are polynomial (this uses the fact that $L_\alpha V$ is polynomial). $\square$

Proposition 6.5.9. *If $L_\alpha V \in \mathfrak{K}_{d,m}$ then*

$$\Phi_{Q_Z^+}(L_\alpha V) \cong L_\alpha V$$

Proof. This result follows from another calculation,

$$\begin{aligned} \Phi_{Q_Z^+}(L_\alpha V) &\cong \Phi_Q(\mathbf{R}j_*\mathbf{L}j^*L_\alpha V) \\ &\cong \pi^*\Phi_{Q_{Y'}}(\mathbf{R}i_*\mathbf{L}i^*L_\alpha V) \\ &\cong \pi^*\Big(\mathbf{R}i_*\mathbf{L}i^*L_\alpha V\Big)_{\geq 0}, \end{aligned}$$

where the second line follows from Corollary 6.5.7 and the last line by Lemma 6.5.5. Hence our result follows from Lemma 6.5.8. $\square$

Corollary 6.5.10. $\operatorname{Im}\Phi_{\widehat{Q}_Z^+} = \langle \mathfrak{K}_{d,m}\rangle$.

Proof. This is an immediate consequence of Proposition 6.5.1 and Lemma 6.5.9. $\square$

Note that we have a similar equality for $\Phi_{\widehat{Q}^-}$.

Corollary 6.5.11. $\operatorname{Im}\Phi_{\widehat{Q}_Z^-} = \langle \big(\mathfrak{K}_{d,m'}\big)^\vee\rangle = \langle \mathfrak{K}_{d,m'} \otimes \det(V^*)^{m'-d}\rangle$.

Proof. We can switch the roles of W and W' by taking transposes. This is anti-equivariant, i.e., equivariant up to inversion in $\operatorname{GL}(V)$. Consequently, we replace all representations with their duals which gives the first equality. The second is a standard identity. $\square$

6.5.3 The equivalence

Finally, we combine things to provide Fourier-Mukai equivalences for (twisted) Grassmann flops. As usual, let $\daleth$ be an (arbitrary) field of characteristic zero.

We recall that P is the object obtained by the restriction of Q_Z^+ to $Z^+ \times Z^-$.

Theorem 6.5.12. *Assume* $\dim W' \geq \dim W$. *The wall crossing functor*

$$\Phi_P : \mathsf{D}^{\mathsf{b}}(Z^+) \to \mathsf{D}^{\mathsf{b}}(Z^-)$$

is fully-faithful. If $\dim W' = \dim W$, *it is an equivalence.*

Proof. Proposition 6.3.32 tells us that $\Phi_{\widehat{Q}_Z^+}$ is fully-faithful. Thus, we reduce to checking that j_-^* is fully-faithful on the image of $\Phi_{\widehat{Q}_Z^+}$. Also, from Proposition 6.3.33, we know that j_-^* is fully-faithful on the image of $\Phi_{\widehat{Q}_Z^-}$.

From Corollaries 6.5.10 and 6.5.11, we see that

$$\operatorname{Im} \Phi_{\widehat{Q}_Z^+} \subseteq \operatorname{Im} \Phi_{\widehat{Q}_Z^-} \otimes \det(V^*)^{d-m'}.$$

Since restriction commutes with tensoring with a line bundle, if j_-^* is fully-faithful on a full subcategory $\mathcal{C}$ then it is also on $\mathcal{C} \otimes \mathcal{L}$ for any line bundle $\mathcal{L}$. Now Corollaries 6.5.10 and 6.5.11 show j_-^* must be fully-faithful on the image of $\Phi_{\widehat{Q}_Z^+}$.

If $\dim W' = \dim W$, then both varieties are Calabi-Yau. As Calabi-Yau's can have no nontrivial admissible subcategories our fully-faithful functor must be an equivalence. $\square$

Remark 6.5.13. If $\overline{k} = k$, once one knows that

$$\operatorname{Im} \Phi_{\widehat{Q}_Z^+} = \langle \overline{K}_{d,m} \rangle$$

one can conclude Theorem 6.5.12 using [DS14, Proposition 3.6]. But, the technology presented here makes for a simple direct proof.

Remark 6.5.14. In general, if we have two smooth projective varieties X and Y over k, then the existence of an equivalence

$$\mathsf{D}^{\mathsf{b}}(X_{\overline{k}}) \cong \mathsf{D}^{\mathsf{b}}(Y_{\overline{k}})$$

does not guarantee the existence of an equivalence

$$\mathsf{D}^{\mathsf{b}}(X) \cong \mathsf{D}^{\mathsf{b}}(Y).$$

A simple class of counter-examples is Severi-Brauer varieties.

One needs, at least, a kernel over k which base changes to furnish the equivalence to appeal to [Orl02, Lemma 2.12]. Without providing a kernel for general k for the equivalence in [DS14], the results in loc.cit. cannot be used to deduce equivalences over arbitrary fields of characteristic zero.

One can go even further. We give the following definition

Definition 6.5.15. We say

$$\begin{tikzcd} Y^+ \arrow[rr, dashed, leftrightarrow] \arrow[dr] & & Y^- \arrow[dl] \\ & Y_0 & \end{tikzcd}$$

is a *twisted Grassmann flop* if the base change to the separable closure of k

$$\begin{tikzcd} Y^+_{k^{\mathrm{sep}}} \arrow[rr, dashed, leftrightarrow] \arrow[dr] & & Y^-_{k^{\mathrm{sep}}} \arrow[dl] \\ & (Y_0)_{k^{\mathrm{sep}}} & \end{tikzcd}$$

is isomorphic to a Grassmann flop.

Example 6.5.16. Let A be a central simple k-algebra of degree n. For $0 < l < n$, the l-th generalized Severi-Brauer variety of A $\mathrm{SB}_l(A)$ is the variety parameterizing right ideals of dimension ln in A. Such a variety is a twisted form of $\mathrm{Gr}(l,n)$, i.e.,

$$\mathrm{SB}_l(A)_{k^{\mathrm{sep}}} \cong \mathrm{Gr}(l,n)_{k^{\mathrm{sep}}}.$$

On $\mathrm{SB}_l(A)$, the tautological vector bundle $\mathcal{T}$, whose fibers are the ideals, base changes to $\mathrm{Hom}(W, \mathcal{S})$. Let T denote the associate geometric vector bundle. The map

$$\mathrm{SB}_l(A) \to \operatorname{Spec}\Gamma(T, \mathcal{O}_T)$$

contracts the zero section and base changes to $X^+ \to X_0$. One can then take two copies of $\operatorname{Spec}\Gamma(T, \mathcal{O}_T)$ and identify them with the involution that base changes to transposition the linear maps. The resulting diagram is a(n honestly) twisted Grassmann flop.

We also have equivalences for twisted Grassmann flops in characteristic zero.

Corollary 6.5.17. *Assume* $\operatorname{char} k = 0$*. If we have a twisted Grassmann flop, then there is an equivalence*

$$\mathsf{D}^{\mathsf{b}}(Y^+) \to \mathsf{D}^{\mathsf{b}}(Y^-).$$

Proof. Theorem 6.3.22 says that the structure sheaf of the fiber product $Y^+_{k^{\mathrm{sep}}} \times_{(Y_0)_{k^{\mathrm{sep}}}} Y^-_{k^{\mathrm{sep}}}$ is a Fourier-Mukai kernel. Applying [Orl02, Lemma 2.12] shows that the $Y^+ \times_{Y_0} Y^-$ is also a Fourier-Mukai kernel. □

Chapter 7

Windows for cdgas

7.1 Introduction

Flops are some of the most elementary birational transformations. A deep relationship between birational geometry and derived categories originated in the work of Bondal and Orlov [BO95]. They conjectured that flops have equivalent derived categories; Kawamata refined and generalized this further in his famous K-equivalence implies D-equivalence conjecture [Kaw02].

There are various solutions to this problem for explicit types of flops; for example, Bondal and Orlov [BO95] for the standard/Atiyah flop, Bridgeland [Bri02] for flops in dimension 3, and Namikawa and Kawamata for Mukai flops [Nam03, Kaw06]. However there is no agreed upon solution to tackle this problem in full generality.

One of the various successful techniques in addressing this problem, known as "grade-restriction windows", was introduced by Segal [Seg11] based on ideas from string theory revealed by physicists Herbst, Hori, and Page [HHP09]. Segal's work and the introduction of this technique quickly lead to significant generalizations by Ballard, Favero and Katzarkov [BFK14] and Halpern-Leistner [HL15] i.e. grade-restriction windows can be used to prove Kawamata's D-equivalence implies K-equivalence conjecture for certain Variation of Geometric Invariant Theory(VGIT) problems, even in non-abelian cases.

Due to a result of Reid (see [Tha96, Proposition 1.7]), one can reduce the problem of flops to a VGIT problem. In more detail, given a flop

$$Y_1 \dashrightarrow Y_2,$$

one can find a scheme Y with a $\mathbb{G}_m$-action, such that Y_1 and Y_2 are realized as two different GIT quotients. Hence, we may try to prove Bondal-Orlov–Kawamata's conjecture in general using VGIT. In the known cases, the idea is as follows.

Consider a scheme Y equipped with an action of a linearly reductive group G. We denote the GIT quotient with respect to an G-equivariant ample line bundle as $Y/\!\!/_G$. The *window functor* is a fully faithful functor

$$\Phi : D(Y/\!\!/_G) \longrightarrow D(\mathrm{QCoh}^G Y).$$

The essential image of this functor is referred to as the *window subcategory* or as the window in short. We can also consider a different GIT quotient and construct a different window functor. The proof of the equivalence between the derived categories of the different GIT quotients works by comparing the two window subcategories in $D(\mathrm{QCoh}^G Y)$.

Though it is not constructed this way in [BFK14, HL15], the window functor Φ is expected to be of geometric origin. Namely, it should be defined by a Fourier-Mukai kernel P which is an object in $D(Y/\!\!/_{\mathbb{G}_m} \times [Y/\mathbb{G}_m])$, such that

$$\Phi(F) = \pi_{2*}(P \otimes \pi_1^* F),$$

where π_1 and π_2 are projections from $Y/\!\!/_{\mathbb{G}_m} \times [Y/\mathbb{G}_m]$ to the first and second factor respectively. Recently, Ballard, Diemer, and Favero [BDF17] proposed a consistent way to produce homologically well behaved Fourier-Mukai kernels for these window functors in the case of $\mathbb{G}_m$-actions, called the *Q-construction*. This was extended to GL_n-actions for Grassmannian flops in [BCF+19].

One of the notable properties is that this Q behaves well with respect to base change. We note here that even in examples of flops where the derived equivalences are known to hold, there is not always a construction of the Fourier-Mukai kernel that induces this equivalence.

In [BDF17], the authors recovered grade-restriction windows for smooth affine schemes X with a $\mathbb{G}_m$-action via an explicit construction of a kernel. However, birational geometry demands we study more than smooth schemes. Indeed, even in the setting of smooth flops, the scheme X is often singular (for example, the Mukai flop). In the case of singular affine schemes Ballard, Diemer, and Favero proposed a *derived Q-construction*, which uses techniques from derived algebraic geometry. Furthermore, they conjecture that this derived Q-construction provides a Fourier-Mukai equivalence for flops (i.e. they propose an explicit method to prove Bondal-Orlov–Kawamata's conjecture).

In order to tackle this problem of VGIT for singular schemes, the idea is to resolve the singular scheme, by a smooth dg-scheme and then apply the Q-construction. Instead of working in the simplicial setting (as [BDF17] does), we use the philosophy of the monoidal Dold-Kan correspondence to work in the dg setting instead.

7.1.1 Windows

In this paper, we develop a theory of variation of GIT quotients arising from semi-free commutative differential graded algebras (cdga). This uses the language of dg-schemes as developed in [CFK01, Ric10, BR12, MR15]. We prefer to use this over the more involved machinery of derived algebraic geometry (DAG) as it is explicit and better suited (at least for us) for computations.

Let us start with a $\mathbb{Z}$-graded semi-free (i.e., free upon forgetting the differential) cdga R over a smooth finitely-generated k-algebra T, where k is an arbitrary field. We will also assume that the homological degree zero part is $R^0 = T$. This data gives a dg-scheme with a $\mathbb{G}_m$-action (for more details about dg-schemes, see Section 7.2.2),

$$\mathbf{X} := (X, \mathcal{R}).$$

Here, $X = \operatorname{Spec} T$, and $\mathcal{R}$ is the cdga R considered as a dg-$\mathcal{O}_X$-module on X. Let us denote the ideal generated by all the positively graded elements in T by J^+, and the associated semi-stable locus as

$$X^+ := X - V(J^+).$$

We can also consider the restriction of the sheaf of cdgas $\mathcal{R}$ to the semi-stable locus, and denote it by $\mathcal{R}^+$. Then, we define the positive semi-stable locus of the dg-scheme $\mathbf{X}$ as

$$\mathbf{X}^+ := (X^+, \mathcal{R}^+).$$

In Section 7.3.2, we define a Fourier-Mukai kernel Q_+ and following [BDF17], we get a fully-faithful functor Φ_{Q_+}.

Proposition 7.1.1 (Lemmas 7.3.14 and 7.4.8)**.** *There is a fully-faithful (window) functor*

$$\Phi_{Q_+} : D(\mathrm{QCoh}^{\mathbb{G}_m}(\mathbf{X}^+)) \longrightarrow D(\mathrm{QCoh}^{\mathbb{G}_m}(\mathbf{X})).$$

In order to identify the essential image of the window functor (known as a window) explicitly, we need to impose a condition on the internal degree of the homological generators of R over T.

Theorem 7.1.2 (Theorem 7.4.13)**.** *Assume that R is generated as a T-algebra by non-positive elements. The functor*

$$\Phi_{Q_+} : \mathrm{Perf}^{\mathbb{G}_m}(\mathbf{X}^+) \longrightarrow \mathbb{W}^+_{\mathbf{X}}$$

is an equivalence of categories where $\mathbb{W}^+_{\mathbf{X}}$ is étale locally the full subcategory generated by $\mathcal{R}(i)$ for i in a prescribed range (see Definition 7.4.9).

Comparison with the literature

Let us now restrict to the setting of singular VGIT problems (which we describe in more detail in Section 7.1.3). Then, the cdga R is chosen to be a semi-free dg-resolution of a singular ring. Under certain assumptions called Properties $L(+)$ and A, [HL15] also defines a window subcategory and proves that the derived category of the GIT quotient is equivalent to this window (via different methods).

We note that our conditions on the degree in Theorem 7.1.2 need not satisfy Property A or Property L^+ of [HL15] (see Example 7.4.5). In fact, these conditions are roughly complementary to the ones of [HL15]. Moreover, in Example 7.4.6, where L^+ and A are satisfied (but our Theorem 7.1.2 does not hold), we check that the essential image of our window functor Φ_{Q_+} is the same as the window described in [HL15]. This suggests that the Fourier-Mukai kernel that we study in this paper is indeed the right one (for example, to solve the Bondal-Orlov-Kawamata conjectures). We hope that a more refined definition of the window will allow us to remove the restrictions in Theorem 7.1.2.

7.1.2 Wall-crossing for cdgas

Using the window functor and the explicit description of the window from the previous section, we can compare the derived categories of the positive and negative GIT quotients. This allows us to obtain the following results about wall-crossings.

By considering the ideal J^- generated by the negatively graded elements of T, we may define the negative semi-stable locus of the dg-scheme $\mathbf{X}$ analogously to the previous section, i.e.,

$$\mathbf{X}^- := (X^-, \mathcal{R}^-).$$

We can also prove the analogue of the window statement (Theorem 7.1.2) for $\mathbf{X}^-$ if we assume that the cdga R is generated by non-negatively graded elements. We can combine these results to understand the wall-crossings between $\mathbf{X}^+$ and $\mathbf{X}^-$.

Let $\mu_\pm$ be the sum of the weights of the conormal bundle of $\operatorname{Spec} T/J^\pm$ in X. Let $j_\pm : \mathbf{X}^\pm \hookrightarrow \mathbf{X}$ be the inclusion of dg-schemes.

Theorem 7.1.3 (Theorem 7.4.15)**.** *Assume that the dg-algebra generators of R over T have internal degree zero. When, $\mu_+ + \mu_- = 0$, the wall crossing functor*

$$\Phi_+^{\mathrm{wc}} := j_-^* \circ (- \otimes \mathcal{O}(-\mu_+ - 1)) \circ \Phi_{Q_+} : \operatorname{Perf}^{\mathbb{G}_m}(\mathbf{X}^+) \longrightarrow \operatorname{Perf}^{\mathbb{G}_m}(\mathbf{X}^-)$$

is an equivalence of categories. The inverse functor is

$$\Phi_-^{\mathrm{wc}} := j_+^* \circ (- \otimes \mathcal{O}(-\mu_- + 1)) \circ \Phi_{Q_-}$$

We also study the case where $\mu_+ + \mu_- \neq 0$ and X is isomorphic to an affine space over an arbitrary Noetherian ring k. Consider the ring

$$T = k[\mathbf{x}, \mathbf{y}],$$

where we use the shorthand notation $\mathbf{x}$ to mean $x_1, x_2, \cdots, x_l$ and $\mathbf{y}$ to mean $y_1, y_2, \cdots, y_m$ with internal $\mathbb{Z}$-grading $\deg x_i > 0$ and $\deg y_i < 0$.

In this setting, we have the following result.

Theorem 7.1.4 (Theorem 7.4.17)**.** *Let k be an arbitrary Noetherian ring. Assume that the algebra generators of R over $T = k[\mathbf{x}, \mathbf{y}]$ have internal degree zero. Consider the (derived) fixed locus $R^{\mathbb{G}_m} := R/(\mathbf{x}, \mathbf{y})$.*

1. *When, $\mu_+ + \mu_- > 0$, we have the following semi-orthogonal decomposition*

$$\operatorname{Perf}^{\mathbb{G}_m} \mathbf{X}^+ \cong \langle \operatorname{Perf}(R^{\mathbb{G}_m})_{\mu_+ + \mu_-}, \cdots, \operatorname{Perf}(R^{\mathbb{G}_m})_2, \operatorname{Perf}(R^{\mathbb{G}_m})_1, \Phi_-^{wc}(\operatorname{Perf}^{\mathbb{G}_m} \mathbf{X}^-) \rangle.$$

2. *When, $\mu_+ + \mu_- < 0$, we have the following semi-orthogonal decomposition*

$$\operatorname{Perf}^{\mathbb{G}_m} \mathbf{X}^- \cong \langle \operatorname{Perf}(R^{\mathbb{G}_m})_{\mu_+ + \mu_-}, \cdots, \operatorname{Perf}(R^{\mathbb{G}_m})_{-2}, \operatorname{Perf}(R^{\mathbb{G}_m})_{-1}, \Phi_+^{wc}(\operatorname{Perf}^{\mathbb{G}_m} \mathbf{X}^+) \rangle.$$

The main application of the above results that we are interested in is the setting of VGIT problems arising from singular schemes (in particular, those obtained as a VGIT presentation of a flop).

7.1.3 Applications to singular VGIT problems

Consider a (possibly singular) subscheme $Y = \operatorname{Spec} S$ of a smooth affine scheme $X = \operatorname{Spec} T$ of finite type over a field k, which is equipped with a $\mathbb{G}_m$-action,

$$Y = \operatorname{Spec} S \hookrightarrow X = \operatorname{Spec} T.$$

Then we resolve the $\mathbb{Z}$-graded singular ring S by a $\mathbb{Z}$-graded semi-free cdga R, using the Koszul-Tate resolution (which is a generalization of the Koszul resolution for complete intersections) [Tat57],

$$R \simeq S.$$

In particular, R has only finitely many generators in each homological degree, but could have infinitely many generators in total. Note that the degree zero part $R^0 = T$. We introduce the following notation for the GIT quotients of Y.

$$Y/\!\!/_{\pm} := [Y - V(J^{\pm})/\mathbb{G}_m].$$

Now we may use Theorem 7.1.3 to prove the following derived equivalences.

Corollary 7.1.5 (Corollary 7.4.18)**.** *Assume that the dg-algebra generators of R over T have internal degree zero. When $\mu_+ + \mu_- = 0$, the wall crossing functor*

$$\Phi^{\mathrm{wc}} : \mathrm{Perf}(Y/\!\!/_+) \longrightarrow \mathrm{Perf}(Y/\!\!/_-)$$

is an equivalence of categories. In particular, if the semi-stable loci are smooth (for example, the setting of smooth flops),

$$\Phi^{\mathrm{wc}} : D^b(Y/\!\!/_+) \longrightarrow D^b(Y/\!\!/_-)$$

is an equivalence of categories.

As a special case we recover the local case of the equivalence in [Nam03, Kaw06, Mor18, ADM19], using VGIT and window techniques.

Theorem 7.1.6 (Corollary 7.4.19)**.** *For the local model of the Mukai flop over a fixed commutative Noetherian ring k, the wall crossing functor*

$$\Phi^{wc} : D^b(Y/\!\!/_+) \to D^b(Y/\!\!/_-)$$

is an equivalence of categories.

While this is a well-known result over $\mathbb{C}$, our proof holds over an arbitrary Noetherian ring k.

7.1.4 Outline of the paper

In Section 7.2, we first set up notation and introduce the language of dg-schemes. We discuss the derived category of equivariant dg-schemes and the existence of derived functors in Section 7.2.2. We define the category of perfect complexes in Section 7.2.3 which is the category we are interested in, and in Section 7.2.4, we introduce the Variation of GIT quotients problem that we study in this paper.

In Section 7.3, we define the object Q which is the main object of this paper, and study various properties of it. Section 7.3.1 is a brief reminder of the construction of Q in the setting of (non-dg) rings, and then we define it for semi-free cdgas in Section 7.3.2. We study various properties of Q in Section 7.3.3, which will be important to understand the Fourier-Mukai transforms induced by it. In Section 7.3.4, we find conditions for fully-faithfulness of the window functor defined using Q.

In Section 7.4, we study the window functors and wall-crossing functors defined using Q as the Fourier-Mukai kernel. We focus on the case where the underlying scheme is affine space in Section 7.4.1 and the general case of a smooth affine scheme in Section 7.4.2. Then, we study the induced wall-crossing functors in Section 7.4.3 and finally discuss applications to VGIT problems arising from flops in Section 7.4.4.

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7.2 Setup

Let us set notation and recall relevant definitions and results about dg-schemes following [CFK01, Ric10, MR15, BR12].

7.2.1 Notation

Throughout, k will denote a fixed commutative Noetherian ring over which all objects are defined. In this paper, we will only work with commutative differential graded algebras concentrated in non-positive degrees, and henceforth we will simply refer to one as a cdga. Bold letters, e.g. $\mathbf{X}$, will denote a dg-scheme, and the underlying ordinary scheme will be denote by unbolded letters, e.g. X. We will assume that all our schemes are separated and Noetherian of finite dimension.

Often, our objects (cdgas, for example) will have a $\mathbb{Z}$-grading coming from a $\mathbb{G}_m$-action (on dg-schemes, for example) which we will refer to as the *internal* grading, as opposed to the homological grading coming from the cdga structure. In order to denote the homological grading, we will use upper indices, whereas we use lower indices for the internal grading.

We say that the graded vector space $k(i)$ has weight i. For a graded module M, the degree 0 piece of the shifted module $M(i)$ is the degree i piece of M, notated M_i so that $(M(i))_j = M_{i+j}$.

7.2.2 dg-schemes

The data of a dg-scheme $\mathbf{X}$ is the pair

$$\mathbf{X} = (X, \mathcal{A}),$$

where X is a scheme and $\mathcal{A}$ is a non-positively graded, commutative dg $\mathcal{O}_X$-algebra, such that $\mathcal{A}^i$ is a quasi-coherent $\mathcal{O}_X$-module for any $i \in \mathbb{Z}_{\leq 0}$. We denote the homological graded pieces of $\mathcal{A}$ by $\mathcal{A}^i$.

Definition 7.2.1. Let $\mathbf{X} = (X, \mathcal{A})$ be a dg-scheme. A *quasi-coherent* dg-sheaf $\mathcal{F}$ on $\mathbf{X}$ is a $\mathcal{A}$-dg-module such that $\mathcal{F}^i$ is a quasi-coherent $\mathcal{O}_X$-module for any $i \in \mathbb{Z}$.

We define the derived category of quasi-coherent dg-sheaves by $D(\operatorname{QCoh}\mathbf{X})$. The derived category is defined in the usual way as the localization of the homotopy category with respect to the class of quasi-isomorphisms. This has the structure of a triangulated category. Note that in the case of ordinary schemes, i.e., if $\mathcal{A} = \mathcal{O}_X$, the category $D(\operatorname{QCoh}\mathbf{X}) \cong D(\operatorname{QCoh} X)$.

In [BR12], it is shown that there are enough K-flat and K-injective objects in $\operatorname{QCoh}(\mathbf{X})$, and hence one can use these to define all right derived functors, and left derived tensor products and pullbacks. They also prove the adjunction between derived pushforwards and pullbacks, the projection formula and a base change formula. It is worth noting that in the context of dg-schemes, flatness is not required for the base-change formula; instead we use the derived fiber product.

In the $\mathbb{G}_m$-equivariant setting, i.e., when we have a scheme X equipped with a $\mathbb{G}_m$-action and $\mathcal{A}$ is equivariant with respect to this $\mathbb{G}_m$-action, some of the appropriate generalizations are defined by [MR15] under the following technical assumptions:

1. For any $\mathcal{F}$ in $\mathrm{QCoh}^{\mathbb{G}_m}(X)$, there exists a P in $\mathrm{QCoh}^{\mathbb{G}_m}(X)$ which is flat over $\mathcal{O}_X$ and a surjection $P \twoheadrightarrow \mathcal{F}$ in $\mathrm{QCoh}^{\mathbb{G}_m}(X)$.

2. Assume that $\mathcal{A}$ is locally free over $\mathcal{A}^0$, $\mathcal{A}^0$ is locally finitely generated as an $\mathcal{O}_X$-algebra, and finally that $\mathcal{A}$ is K-flat as a $\mathbb{G}_m$-equivariant A^0-dg-module.

In this paper, we will always work over X quasi-affine or affine, and hence the first condition is automatically satisfied. We will only consider dg-schemes such that $\mathcal{A}^0 = \mathcal{O}_X$ and that $\mathcal{A}$ is semi-free over $\mathcal{O}_X$, and hence the second condition is always satisfied as well.

We note that [MR15] imposes a slightly stronger technical assumption which is that $\mathcal{A}$ is locally free of finite rank over $\mathcal{A}^0$. However, this assumption is unnecessary in the proof of [MR15, Proposition 2.8]. Moreover, the proof of [Ric10, Theorem 1.3.6] carries over to this setting as well. Hence we have the following statement.

Lemma 7.2.2. *Consider a $\mathbb{G}_m$-equivariant dg-scheme $\mathbf{X}$ satisfying the assumptions above. For any object $\mathcal{F}$ in $\mathrm{QCoh}^{\mathbb{G}_m}(\mathbf{X})$, there exists a K-injective equivariant resolution $\mathcal{F} \to \mathcal{I}$.*

Proof. First we use [MR15, Proposition 2.8] to find a resolution for bounded below $\mathcal{A}$-dg-modules; this resolution has flabby graded components. Then, the proof in [Ric10, Theorem 1.3.6] carries over to this setting. As a brief reminder, we truncate the dg-sheaf $\mathcal{F}$, resolve the truncations by K-injectives and then take the inverse limit over the truncations. In the proof of [Ric10, Theorem 1.3.6], we take $\mathfrak{B} = \mathrm{QCoh}^{\mathbb{G}_m}(X)$ and consider equivariant open covers of X. $\square$

Using the above lemma, we can define all right derived functors. In order to define left derived inverse images and tensor products, we use [MR15, Lemma 2.7] which proves the existence of K-flat resolutions. Using these K-injective and K-flat resolutions it is easy to prove the projection formula and the adjunction between pushforwards and pullbacks by adapting the arguments of [BR12] to our setting, and we will use these in this paper.

7.2.3 Perfect objects

Let us give an ad hoc definition of the category $\mathrm{Perf}^{\mathbb{G}_m}\mathbf{X}$ where $\mathbf{X} = (X, \mathcal{A})$ with the assumptions as before, and X is quasi-projective. Consider an ample line bundle $\mathcal{L}$ on X.

Definition 7.2.3. The thick triangulated subcategory of $D(\mathrm{QCoh}^{\mathbb{G}_m}\mathbf{X})$ generated by $\mathcal{L}^{\otimes j} \otimes_{\mathcal{O}_X} \mathcal{A}(i)$ for all $i, j \in \mathbb{Z}$ is called the *category of perfect dg-sheaves*, denoted $\mathrm{Perf}^{\mathbb{G}_m}\mathbf{X}$.

Remark 7.2.4. We expect that the 'right' definition of the category of perfect dg-sheaves on $\mathbf{X} = (X, \mathcal{A})$, i.e. the triangulated category generated by dg-sheaves that are quasi-isomorphic to a finite locally semi-free $\mathcal{A}$-module (i.e., finite locally free after forgetting the differential) is equivalent to our ad-ho definition in this context. In particular, it is easy to show that if we consider an ordinary scheme (if $\mathcal{A} = \mathcal{O}$) equipped with an ample line bundle then the definitions coincide.

7.2.4 GIT problem

We start with a $\mathbb{Z}$-graded semi-free[1] commutative differential graded algebra (cdga), say R. We denote the homological degree zero piece of R by $T := R^0$, and we require that T is a finitely generated smooth ring over k. We do not

[1]Recall that a semi-free cdga R is one that is free when considered as a commutative algebra by forgetting the differential.

require any finiteness conditions on R. To this data, we associate the affine dg-scheme

$$X := \operatorname{Spec} T, \qquad \mathbf{X} := (X, \mathcal{R}),$$

where $\mathcal{R}$ is the dg-$\mathcal{O}_X$-module associated to the cdga R. The $\mathbb{Z}$-grading on $T \to R$ is equivalent to a $\mathbb{G}_m$-action on $\mathbf{X}$.

We denote the semi-stable loci (which are dg-schemes) as

$$\mathbf{X}^{\pm} := (X^{\pm}, \mathcal{R}^{\pm}),$$

where we define

$$X^{\pm} := X - V(J^{\pm}), \qquad \mathcal{R}^{\pm} = \mathcal{R}|_{X^{\pm}}.$$

Here $J^{\pm}$ denotes the ideal generated by all the strictly positive/negative-ly graded elements. Then we define two Geometric Invariant Theory (GIT) quotients $\mathbf{X}/\!\!/_{\pm}$ of $\mathbf{X}$ with respect to the $\mathbb{G}_m$-action. The stacks

$$X/\!\!/_{\pm} := [X^{\pm}/\mathbb{G}_m],$$

with the sheaf of cdgas obtained by the descent of $\mathcal{R}^{\pm}$ to $X/\!\!/_{\pm}$ are the GIT quotient "dg-stacks"

$$\mathbf{X}/\!\!/_{\pm} = (X/\!\!/_{\pm}, \mathcal{R}^{\pm}).$$

Remark 7.2.5. We do not define or introduce dg-stacks in general as the ones we consider are global quotients of dg-schemes, and we are only concerned about the corresponding derived categories. We define the derived category of the global quotient dg-stack merely as the corresponding equivariant derived category of the dg-scheme.

7.3 Defining Q

We will first recall the Q-construction briefly in the setting of $\mathbb{Z}$-graded (not dg-)rings, and then extend this definition to cdgas. In this section, we continue to use the notations of the previous section. In particular, we remind the reader that T is a $\mathbb{Z}$-graded smooth ring finitely generated over k, R is a $\mathbb{Z}$-graded semi-free cdga over T and $\mathbf{X}$ is an affine dg-scheme equipped with a $\mathbb{G}_m$-action.

7.3.1 Q for smooth rings

This section is a brief reminder of the construction of [BDF17]. The $\mathbb{Z}$-grading on the ring T is equivalent to a $\mathbb{G}_m$-action on $X = \operatorname{Spec} T$. Viewed in this manner, we say that T is equipped with the co-projection and co-action morphisms, which we denote by π and σ respectively. The morphisms act as

$$\begin{aligned} \pi : & T \to T[u, u^{-1}] & \qquad \sigma : & T \to T[u, u^{-1}] \\ & t \mapsto t & & t \mapsto t u^{\deg t}, \end{aligned} \tag{7.1}$$

for any homogeneous element t in T. We assign a $\mathbb{Z} \times \mathbb{Z}$-grading to the equivariant diagonal of $\operatorname{Spec} T$, which is

$$k[\mathbb{G}_m \times X] = T[u, u^{-1}],$$

such that the morphisms π and σ equivariant. The grading for homogeneous t in T is defined as

$$\deg \pi(t) = (\deg t, 0), \;\; \deg \sigma(t) = (0, \deg t) \;\text{ and }\; \deg u = (-1, 1).$$

Given the smooth ring T, we define

$$Q(T) := \langle \pi(T), \sigma(T), u \rangle \subseteq T[u, u^{-1}],$$

as the k-subalgebra of $T[u, u^{-1}]$ generated by the image of the co-action and co-projection maps, and u. It suffices to keep the image of the negative elements under the co-action map along with $T[u]$,

$$Q(T) = \langle \bigoplus_{i<0} T^i u^i, T[u] \rangle$$

where T^i denotes the i-th graded piece of T. The co-projection and co-action maps factor through $Q(T)$, giving the maps p and s,

$$T \underset{s}{\overset{p}{\rightrightarrows}} Q(T) \hookrightarrow \Delta(T). \tag{7.2}$$

The $T \otimes T$-module $Q(T)$ also inherits the $\mathbb{Z} \times \mathbb{Z}$ grading from $T[u, u^{-1}]$. Hence, we can consider $Q(T)$ as an element of $D(\operatorname{mod}^{\mathbb{G}_m \times \mathbb{G}_m}(T \otimes T))$ using the $p \otimes s$ module structure.

In order to clarify the construction above, let us consider the example where X is an affine space. This example will play an important role in the paper.

Example 7.3.1. Let us consider the ring $T = k[\mathbf{x}, \mathbf{y}]$, where we are using the shorthand notation $\mathbf{x}$ to mean $x_1, \cdots, x_l$, and $\mathbf{y}$ to mean $y_1, \cdots, y_m$. We assign the $\mathbb{Z}$-grading

$$\deg x_i = a_i > 0 \qquad \deg y_i = b_i < 0$$

to T. Then,

$$Q(T) = k[\mathbf{x}, \mathbf{z}, u],$$

with the $p, s : T \to Q(T)$ maps given by

$$\begin{aligned} p(x_i) &= x_i & s(x_i) &= u^{a_i} x_i \\ p(y_i) &= u^{-b_i} z_i & s(y_i) &= z_i \end{aligned}$$

Remark 7.3.2. For the sake of notational simplicity, we have not added any x_i or y_j of internal degree zero. The Q construction does not affect such degree zero generators and hence we may view it as a part of the Noetherian ring k.

There is a geometric motivation for this definition of Q, and we refer the reader to [BDF17] for more details. The idea is to define Q as a partial compactification of the $\mathbb{G}_m$-action on X, generalizing a construction of [Dri13]. In *loc. cit.*, the authors prove various properties of this object Q; one that is worth mentioning here is that the assignment $Q : \mathrm{CR}_k^{\mathbb{G}_m} \to \mathrm{CR}_{k[u]}^{\mathbb{G}_m \times \mathbb{G}_m}$ is functorial. We also note that the equivariant injective map $Q(T) \hookrightarrow \Delta(T)$ provides a natural transformation between the Fourier-Mukai functors $\Phi_{Q(T)} \to \Phi_{\Delta(T)} = \mathrm{Id}_X$.

7.3.2 Q for semi-free cdgas

Now we define Q for semi-free cdgas, which is the case of interest in this paper. Recall that the dg-scheme $\mathbf{X} = (\operatorname{Spec} T, \mathcal{R})$ is equipped with a $\mathbb{G}_m$-action. This means that we have the projection and action maps, which we denote by π and σ respectively,

$$\mathbb{G}_m \times \mathbf{X} \underset{\pi}{\overset{\sigma}{\rightrightarrows}} \mathbf{X}.$$

We want to view the cdga associated to $\mathbb{G}_m \times \mathbf{X}$ as a dg-sheaf on $\mathbf{X} \times \mathbf{X}$ using the morphism $\pi \times \sigma$. More precisely, we define a semi-free cdga $\Delta(R)$ (which should be thought of as the equivariant diagonal in the dg-setting)

$$\Delta(R) := R[u, u^{-1}],$$

which is equipped with a $\mathbb{Z} \times \mathbb{Z}$-grading (which, as before, is chosen in order to make the action and projection maps equivariant). The grading for homogeneous r in R is defined as

$$\deg \pi(r) = (\deg r, 0), \;\; \deg \sigma(r) = (0, \deg r) \;\text{ and }\; \deg u = (-1, 1).$$

In order to get the dg-structure, we note that the elements, u and u^{-1} are in homological degree zero, and hence are killed by the differential.

We also have the co-action and co-projection dg-morphisms (which we denote by π and σ by abuse of notation)

$$\begin{aligned} \pi : & R \to R[u, u^{-1}] & \qquad \sigma : & R \to R[u, u^{-1}] \\ & r \mapsto r & & r \mapsto r u^{\deg r}, \end{aligned} \tag{7.3}$$

for homogeneous r in R.

Consider $\Delta(R)$ as a $R \otimes R$-module with the module structure $\pi \otimes \sigma$. Then, we have the associated quasi-coherent dg-sheaf, also denoted $\Delta(R)$, on $\mathbf{X} \times \mathbf{X}$. Taking into account the $\mathbb{Z} \times \mathbb{Z}$-grading we view it as an element of $D(\mathrm{QCoh}^{\mathbb{G}_m \times \mathbb{G}_m} \mathbf{X} \times \mathbf{X})$.

Lemma 7.3.3. *The object $\Delta(R) \in D(\mathrm{QCoh}^{\mathbb{G}_m \times \mathbb{G}_m} \mathbf{X} \times \mathbf{X})$ is a Fourier-Mukai kernel of the identity functor on $D(\mathrm{QCoh}^{\mathbb{G}_m} \mathbf{X})$.*

Proof. As all the objects are affine, we may work on the level of cdgas. Then, keeping track of the equivariant structures carefully, we have the following sequence of isomorphisms:

$$\begin{aligned} \Phi_{\Delta(R)}(A) &\cong (\sigma_* \pi^* A)^{\mathbb{G}_m} \\ &\cong (A[u, u^{-1}])_{(0,*)} \\ &\cong \oplus A^i u^i, \end{aligned}$$

where the R-module structure is

$$r.(au^i) = (r.a) u^{i + \deg r}.$$

Similarly, for the inverse functor,

$$\begin{aligned} \Phi_{\Delta(R)}(A) &\cong (\pi_* \sigma^* A)^{\mathbb{G}_m} \\ &\cong (A[u, u^{-1}])_{(*,0)} \\ &\cong \oplus A^i u^{-i}, \end{aligned}$$

where the R-module structure is

$$r.(au^i) = (r.a)u^{i-\deg r}.$$

It is clear that $\oplus A^i u^i$ and $\oplus A^i u^{-i}$ are isomorphic to A with the original R-module structure, and hence we are done.

$\square$

Now, we define an analogue of the Q-construction of [BDF17] for the semi-free cdga R.

Definition 7.3.4. The object $Q(R)$ is defined as the following $\mathbb{Z} \times \mathbb{Z}$-graded cdga

$$Q(R) = \langle \pi(R), \sigma(R), u \rangle \subseteq R[u, u^{-1}]$$

generated as a k-subalgebra of $R[u, u^{-1}]$. The dg-structure and the $\mathbb{Z} \times \mathbb{Z}$-grading are inherited from the object $\Delta(R) = R[u, u^{-1}]$.

The co-action and co-projection maps to $\Delta(R)$ factor through $Q(R)$, giving the maps p and s,

$$R \underset{s}{\overset{p}{\rightrightarrows}} Q(R) \hookrightarrow \Delta(R).$$

The maps p, s extend to $\Delta(R)$ to give the co-projection π and co-action σ maps on R respectively.

Let us clarify these notions by looking at an extension of Example 7.3.1.

Example 7.3.5. Let $T = k[\mathbf{x}, \mathbf{y}]$ as in Example 7.3.1. Consider a homogeneous regular sequence of length n, say $(h_1, h_2, \cdots, h_{n_1+n_2})$ in T. Assume further, without loss of generality, that the internal degree of h_i is d_i, and that

$$d_i \geq 0 \text{ for } i \in [0, n_1], \qquad d_i < 0 \text{ for } i \in [n_1 + 1, n_1 + n_2].$$

Then, we define the dg-algebra R to be the Koszul resolution on this regular sequence.

$$R = k[\mathbf{x}, \mathbf{y}, e_1, \cdots, e_{n_1}, f_1, \cdots, f_{n_2}], \qquad de_i = h_i, \; df_j = h_{n_1+j}.$$

Then we can compute $Q(R)$ to be

$$Q(R) = k[\mathbf{x}, \mathbf{z}, e_1, \cdots, e_{n_1}, g_1, \cdots, g_{n_2}],$$

where the maps $p, s : R \to Q(R)$ are given by

$$\begin{aligned} p(x_i) &= x_i & s(x_i) &= u^{a_i} x_i \\ p(y_i) &= u^{-b_i} z_i & s(y_i) &= z_i \\ p(e_i) &= e_i & s(e_i) &= u^{d_i} e_i \\ p(f_i) &= u^{-d_{n_1+i}} g_i & s(f_i) &= g_i \end{aligned}$$

Let us return to the general setting now. The maps p, s give $Q(R)$ the structure of a $\mathbb{Z} \times \mathbb{Z}$-graded $R \otimes R$ dg-module with the $p \otimes s$-module structure. We will view the associated dg-sheaf, also denoted $Q(R)$, as an element of $D(\mathrm{QCoh}^{\mathbb{G}_m \times \mathbb{G}_m} \mathbf{X} \times \mathbf{X})$.

Explicit description of $Q(R)$

Let us introduce some more notation and describe $Q(R)$ explicitly. As R is a semi-free cdga over T, let us choose a set of (possibly infinite) homogeneous (in the internal grading) algebra generators. We denote the positive generators by e_i and the negative generators by f_i, where i takes values in a possibly countably infinite set. We will use the short hand notation $\mathbf{e}$ ($\mathbf{f}$) to refer to the set of all e_i (f_i). Using this notation,

$$R = T[\mathbf{e}, \mathbf{f}].$$

Defining $g_i := u^{\deg f_i} f_i$, we can express $Q(R)$ explicitly as

$$Q(R) = Q(T)[\mathbf{e}, \mathbf{g}], \tag{7.4}$$

with the p and s dg-module structures given by

$$\begin{aligned} p :& R \to Q(R) & s :& R \to Q(R) \\ & t \mapsto p_T(t) & & t \mapsto s_T(t) \\ & e_i \mapsto e_i & & e_i \mapsto u^{\deg e_i} e_i \\ & f_i \mapsto u^{-\deg f_i} g_i & & f_i \mapsto g_i, \end{aligned} \tag{7.5}$$

where the maps p_T and s_T are the maps defined in equation 7.2, but we have added the subscript T for clarity.

We also note the bi-degrees of the elements in $Q(R)$ for convenience:

$$\text{bi-deg}\, e_i = (\deg e_i, 0) \qquad \text{bi-deg}\, g_i = (0, \deg f_i),$$

and the degrees of the elements in $Q(T)$ are as defined in Section 7.3.1.

7.3.3 Some properties of $Q(R)$

Let us denote the $\mathbb{G}_m \times \mathbb{G}_m$-equivariant inclusion of $Q(R)$ into $\Delta(R)$ by η,

$$\eta : Q(R) \hookrightarrow \Delta(R) = R[u, u^{-1}].$$

Let us study the relation of $Q(R)$ to $\Delta(R)$ further. In particular, we show that they become isomorphic if we localize by an element in T of non-zero internal degree.

Lemma 7.3.6. *Consider $Q(R)$ as a $R \otimes R$ dg-module with the $p \otimes s$ structure. Let t in T be a homogeneous element, with the corresponding localization map $R_t = T_t[\mathbf{e}, \mathbf{f}] \to R = T[\mathbf{e}, \mathbf{f}]$. If $\deg t > 0$,*

$$1 \otimes_s \eta : R_t \otimes_s Q(R) \to R_t \otimes_s \Delta(R)$$

is an isomorphism. If $\deg t < 0$,

$$1 \otimes_p \eta : R_t \otimes_p Q(R) \to R_t \otimes_p \Delta(R)$$

is an isomorphism.

Proof. The morphisms are injective as R_t is flat over R.

To check that the morphisms are surjective, we just need to check that we get u^{-1} in the image. In the first case, $t^{-1} \otimes u^{\deg t - 1} t$ maps to u^{-1}; in the second case, $t^{-1} \otimes s(t) u^{-\deg t - 1}$ maps to u^{-1}.

□

We need to study some properties of the object $Q(R) \,{}_s{\otimes}_p\, Q(R)$ here as it will play a role when we discuss fully faithfulness of the window functor in the next subsection. The object $Q(R) \,{}_s{\otimes}_p\, Q(R)$ inherits a $\mathbb{G}_m^{\times 3}$-action, where the $\mathbb{Z}^3$ grading is as follows:

$$\deg q \otimes 1 = (a, b, 0) \qquad \deg 1 \otimes q = (0, a, b)$$

if $q \in Q(R)$ is a homogeneous element of degree (a, b).

We would like to understand what happens when we take middle degree invariants; we denote this by $(M)_0$ where M is $\mathbb{Z}^3$-graded.

Lemma 7.3.7. *The following diagram commutes*

$$\begin{tikzcd}
 & (Q(R)\ {}_s{\otimes}_\pi\ \Delta)_0 \arrow[dr, "\sim"] & \\
(Q(R)\ {}_s{\otimes}_p\ Q(R))_0 \arrow[ur] \arrow[dr] & & Q(R) \\
 & (\Delta\ {}_\sigma{\otimes}_p\ Q(R))_0 \arrow[ur, "\sim"'] &
\end{tikzcd}$$

Proof. Consider the upper right arrow

$$\begin{aligned}
(Q(R)\ {}_s{\otimes}_\pi\ \Delta)_0 &= Q(R)[v, v^{-1}]_0 \\
&\cong (\langle \bigoplus_{i \le 0} R_i u^i, R[u] \rangle [v, v^{-1}])_0 \\
&\cong \langle \bigoplus_{i \le 0} R_i u^i v^i, R[uv] \rangle \\
&\cong Q(R).
\end{aligned}$$

In order to take the middle degree zero invariants, we used that $\deg u = (-1, 1, 0)$ and $\deg v = (0, -1, 1)$. Similarly, one can prove the same for the bottom row, and the commutativity is clear. □

We consider the following morphism,

$$\rho : (Q(R)\ {}_s{\otimes}^L_p\ Q(R))_0 \to (Q(R)\ {}_s{\otimes}_p\ Q(R))_0 \to Q(R) \tag{7.6}$$

where the first morphism is the map from the left derived functor to the (underived) functor, and the second morphism is the one constructed in Lemma 7.3.6. Fully faithfulness of the window functor is related to properties of the morphism ρ as we will see in Section 7.3.4.

Our goal is to construct a Fourier-Mukai kernel for the window functors

$$D(\mathrm{QCoh}^{\mathbb{G}_m} \mathbf{X}^+) \longrightarrow D\left(\mathrm{QCoh}^{\mathbb{G}_m} \mathbf{X}\right).$$

Using the $\mathbb{G}_m$-equivariant morphism of dg-schemes

$$j : \mathbf{X}^+ \to \mathbf{X},$$

we define

$$Q_+ := (j \times \mathrm{Id})^* Q(R),$$

and consider it as an object of $D(\mathrm{Qcoh}^{\mathbb{G}_m\times\mathbb{G}_m}\mathbf{X}^+\times\mathbf{X})$. We have dropped the R in Q_+ in the interest of notational convenience. We will focus on the positive GIT quotient, but the arguments are analogous for the negative GIT quotient.

7.3.4 Fully faithfulness

In this section, we find sufficient conditions for the Fourier-Mukai functor Φ_{Q_+} to be fully faithful. Checking faithfulness is easy and follows the arguments of [BDF17].

Lemma 7.3.8. *The composition $j^*\circ\Phi_{Q_+}$ is naturally isomorphic to the identity. In particular, the functor*

$$\Phi_{Q_+}:D(\mathrm{Qcoh}^{\mathbb{G}_m}\mathbf{X}^+)\longrightarrow D\left(\mathrm{QCoh}^{\mathbb{G}_m}\mathbf{X}\right),$$

is faithful.

Proof. We want to show that $j^*\circ\Phi_{Q_+}$ is the identity functor. On the level of the kernels, it suffices to show that the morphism

$$(j\times j)^*Q\to(j\times j)^*\Delta(R)$$

is an isomorphism. We can do this locally on the $\mathbb{G}_m$-invariant affine cover of $X^+\times X^+$, obtained by inverting the positive elements $t\in J^+$ in $T=k[X]$. This is precisely the content of the first part of Lemma 7.3.6. $\square$

Fullness of the functor is more involved and is best phrased in the language of Bousfield localizations. Let us recall some of the definitions and results that we need about Bousfield (co)-localizations. The existence of a Bousfield triangle produces a semi-orthogonal decomposition, and we show that the essential image of our functor is equivalent to one part of the semi orthogonal decomposition.

Definition 7.3.9. Let $\mathcal{T}$ be a triangulated category. A *Bousfield localization* is an exact endofunctor $L:\mathcal{T}\to\mathcal{T}$ equipped with a natural transformation $\delta:1_{\mathcal{T}}\to L$ such that:

a) $L\delta = \delta L$ and

b) $L\delta : L \to L^2$ is invertible.

A *Bousfield co-localization* is given by an endofunctor $C : \mathcal{T} \to \mathcal{T}$ equipped with a natural transformation $\epsilon : C \to 1_{\mathcal{T}}$ such that:

a) $C\epsilon = \epsilon C$ and

b) $C\epsilon : C^2 \to C$ is invertible.

Definition 7.3.10. Assume there are natural transformations of endofunctors

$$C \xrightarrow{\epsilon} 1_{\mathcal{T}} \xrightarrow{\delta} L$$

of a triangulated category $\mathcal{T}$ such that

$$Cx \xrightarrow{\epsilon_{Cx}} x \xrightarrow{\delta_x} Lx$$

is an exact triangle for any object x of $\mathcal{T}$. Then we refer to $C \to 1_{\mathcal{T}} \to L$ as a *Bousfield triangle* for $\mathcal{T}$ when any of the following equivalent conditions are satisfied:

1) L is a Bousfield localization and $C(\epsilon_x) = \epsilon_{C_x}$

2) C is a Bousfield co-localization and $L(\delta_x) = \delta_{L_x}$

3) L is a Bousfield localization and C is a Bousfield co-localization.

For a proof that the above properties are indeed equivalent, we refer the reader to [BDF17, Definition 3.33].

Remark 7.3.11. Any Fourier-Mukai functor Φ_P with a morphism $\Delta \to P$ satisfies the condition a) to be a Bousfield localization. It is easy to see that $\Phi_P(\delta_A) = \delta_{\Phi_P(A)}$. Analogously, any Fourier-Mukai functor $\Phi_{P'}$ with a morphism $P' \to \Delta$ satisfies the condition a) to be a Bousfield co-localization.

Lemma 7.3.12. *[Property P] The triangle of functors*

$$\Phi_{Q(R)} \longrightarrow \mathrm{Id} \longrightarrow \Phi_{\mathrm{cone}(\eta)}$$

is a Bousfield triangle if the morphism

$$\rho : (Q(R) \,{}_s\otimes_p^L Q(R))_0 \to Q(R),$$

is an isomorphism.

Proof. The proof follows the arguments of the proof of [BDF17, Lemma 3.3.6].

We have a morphism $Q(R) \to \Delta(R)$, and a morphism $\Delta \to \mathrm{cone}(\eta)(R)$. Hence (using Remark 7.3.11), we only need to check the second condition for $\Phi_{Q(R)}$ to be Bousfield co-localization. This condition translates to

$$(Q(R) \,{}_s\otimes_p^L Q(R))_0 \cong Q(R).$$

$\square$

If $Q(R)$ satisfies the condition of Lemma 7.3.12, we will say that $Q(R)$ satisfies *Property P*.

We have another Bousfield triangle given as follows.

Lemma 7.3.13. *The following triangle is a Bousfield triangle*

$$\Phi_{\mathrm{cone}(\gamma)[\text{-}1]} \longrightarrow \mathrm{Id} \longrightarrow j_* \circ j^*,$$

where γ is the morphism $\Delta \to (\mathrm{Id} \times j)_*(\mathrm{Id} \times j)^*\Delta$.

Proof. First, we check that j_*j^* is a Bousfield localization. We have the pair of adjoint functors $j^* \dashv j_*$. We apply base change to the following Cartesian square

$$\begin{tikzcd} \mathbf{X}^+ \arrow[r, "\mathrm{Id}"] \arrow[d, "\mathrm{Id}"] & \mathbf{X}^+ \arrow[d, "j"] \\ \mathbf{X}^+ \arrow[r, "j"] & \mathbf{X}, \end{tikzcd}$$

to see that the co-unit map

$$j^*j_* \xrightarrow{\sim} \mathrm{Id},$$

is an isomorphism. Now, we can show that the functor j_*j^* equipped with the unit $\mathrm{Id} \to j_*j^*$ is a Bousfield localization (using base change for example). This is similar to the arguments of [HR17, Example 1.2].

Finally, we take the cone of the unit natural transformation to get the desired Bousfield triangle

$$\Phi_{\text{cone}(\gamma)[\text{-}1]} \to \text{Id} \to j_* \circ j^*.$$

Notice that we may do this on the level of the Fourier-Mukai kernels as we have explicit kernels. $\square$

For convenience, we define $C := \Phi_{\text{cone}(\gamma)[\text{-}1]}$.

Lemma 7.3.14. *If $Q(R)$ satisfies property P, there is a weak semi-orthogonal decomposition*

$$D(\text{QCoh}^{\mathbb{G}_m} \mathbf{X}) = \langle \operatorname{Im} \Phi_{\text{cone}(\eta)}, \operatorname{Im} \Phi_{Q(R)} \circ C,\ \operatorname{Im} \Phi_{Q_+} \rangle,$$

where Im *denotes the essential image. Furthermore, the functor*

$$\Phi_{Q_+} : D(\text{Qcoh}^{\mathbb{G}_m} \mathbf{X}^+) \to D\left(\text{Qcoh}^{\mathbb{G}_m} \mathbf{X}\right)$$

is fully faithful.

Proof. We want to use [BDF17, Lemma 3.3.5] to prove this statement. This says that if $C_1 \to \text{Id} \to L_1$ and $C_2 \to \text{Id} \to L_2$ are Bousfield triangles in a triangulated category $\mathcal{T}$ such that $L_1 C_2 \to L_1$ is an isomorphism, there is a weak semi-orthogonal decomposition

$$\mathcal{T} = \langle \operatorname{Im} C_2 \circ L_1, \operatorname{Im} C_2 \circ C_1, \operatorname{Im} L_2 \rangle.$$

This induces a fully-faithful functor

$$F : \mathcal{T} / \operatorname{Im} C_1 \to \mathcal{T}.$$

We shall take the two Bousfield triangles from Lemma 7.3.13 and Lemmas 7.3.12 to be the triangles 1 and 2 above respectively. Hence we need to show that

$$j_* \circ j^* \circ \Phi_{Q(R)} = j_* \circ j^*.$$

The Fourier-Mukai transform $j^* \circ \Phi_{Q(R)}$ is the one induced by the kernel $(\text{Id} \times j)^* Q(R)$. Note that the $R \otimes R$ module structure on $Q(R)$ is $p \otimes s$. By

Lemma 7.3.6 applied to an open cover of $X \times X^+$, $(\mathrm{Id} \times j)^* Q(R)$ is isomorphic to $(\mathrm{Id} \times j)^* \Delta(R)$, as we are inverting a positive element with the s-module structure. Noting that $\Phi_{(\mathrm{Id} \times j)^* \Delta(R)} = j^*$ proves the claim.

The functor F mentioned above is the functor Φ_{Q_+}. $\square$

Remark 7.3.15. We mention here that [BDF17] proves a semi-orthogonal decomposition result for the homotopy category of the category of spectra in simplicial graded modules over R (viewed as a simplicial graded commutative ring), which is morally equivalent to a part of the semi-orthogonal decomposition (Lemma 7.3.14) that we prove. The relevant result is [BDF17, Proposition 5.4.7] for the interested reader.

7.4 Windows and Wall-crossings

In this section, we study the object Q_+ defined earlier as a Fourier-Mukai kernel for the window functor,

$$\Phi_{Q_+} : \mathrm{Perf}^{\mathbb{G}_m}(\mathbf{X}^+) \longrightarrow \mathrm{Perf}^{\mathbb{G}_m}(\mathbf{X}).$$

We use the various properties of Q proved in the previous section to prove that the functor is always fully faithful. Then, we identify the essential image of this functor explicitly under certain assumptions on out dg-scheme $\mathbf{X}$, which proves our first main theorem, Theorem 7.1.3. Then, we look at wall-crossing functors between the GIT quotients and prove that it is an equivalence in some cases, and this gives our second main result, Theorem 7.1.4. We also study the cases where the wall-crossing functor is not an equivalence, but only fully-faithful. Then, under certain conditions on $\mathbf{X}$, we find a semi-orthogonal decomposition for a GIT quotient, where one of the pieces of the SOD is the derived category of the other GIT quotient.

For ease of exposition, we first tackle the case that X is affine space (i.e., Example 7.3.1) in Section 7.4.1, as the calculations are more explicit. The case of general smooth affine schemes X will reduce to this using the Luna slice theorem in Section 7.4.2. Wall-crossings are studied in Section 7.4.3. Finally, we also look at applications to flops and the Bondal-Orlov conjecture, and prove the derived equivalence result for the Mukai flop in Section 7.4.4.

7.4.1 Windows over affine space

Here, we consider the case where the underlying scheme of the dg-scheme $\mathbf{X}$ is affine space. On the level of the rings, this means that we will choose (as in Example 7.3.1, 7.3.5)

$$T = k[\mathbf{x}, \mathbf{y}],$$

where we use the shorthand notation $\mathbf{x}$ to mean $x_1, x_2, \cdots, x_l$ and $\mathbf{y}$ to mean $y_1, y_2, \cdots, y_m$ with internal $\mathbb{Z}$-grading $\deg x_i = a_i > 0$ and $\deg y_i = b_i < 0$. We will repeatedly use the above convention for bold letters for notational convenience. We also define the following two integers:

$$\mu_+ := \sum a_i, \qquad \mu_- := \sum b_i.$$

Lemma 7.4.1. *When $T = k[\mathbf{x}, \mathbf{y}]$ with the notations above, the object $Q(R)$ satisfies property P, i.e., the morphism*

$$\rho : \left(Q(R) \;{}_s\!\otimes_p^L Q(R)\right)_0 \to Q(R),$$

from equation (7.6) *is an isomorphism.*

Proof. In order to compute the derived tensor product, we need to find a semi-free resolution of $Q(R)$. We claim that the following cdga K is a semi-free resolution of $Q(R)$

$$K := (R \otimes R[u, \boldsymbol{\kappa}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\nu}], d),$$

such that

$$d\boldsymbol{\kappa} = (\mathbf{x}^2 - u^{\deg \mathbf{x}}\mathbf{x}^1),\ d\boldsymbol{\lambda} = (\mathbf{e}^2 - u^{\deg \mathbf{e}}\mathbf{e}^1),\ d\boldsymbol{\mu} = (\mathbf{y}^1 - u^{-\deg \mathbf{y}}\mathbf{y}^2),\ d\boldsymbol{\nu} = (\mathbf{f}^1 - u^{-\deg \mathbf{f}}\mathbf{f}^2).$$

In order to make the notation compact, we are using $\boldsymbol{\kappa}$ to denote a set of elements (as many as the number of x_is); and similarly for $\boldsymbol{\lambda}$, $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$. The superscripts (1 or 2) refer to which copy of R the variable belongs to; for example $\mathbf{x}^1$ denotes $\mathbf{x}^1 \in R \otimes R$ (we apologize for this unfortunate choice of notation)

It is easy to see that the morphism

$$K \to Q$$

sending $\boldsymbol{\kappa}, \boldsymbol{\lambda}, \boldsymbol{\mu}$ and $\boldsymbol{\nu}$ to zero is a quasi-isomorphism.

Now, we explicitly calculate $\big(Q(R)\ {}_s\!\otimes_p^L Q(R)\big)$:

$$\begin{aligned}\big(Q(R)\ {}_s\!\otimes_p^L Q(R)\big) &= T \otimes T[\boldsymbol{\kappa}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\nu}, u]\ {}_s\!\otimes_p k[\mathbf{x}, \mathbf{z}, \mathbf{e}, \mathbf{g}, v] \\ &\cong k[\mathbf{x}, \mathbf{x}^1, \mathbf{e}, \mathbf{e}^1, \mathbf{z}, \mathbf{y}^1, \mathbf{g}, \mathbf{f}^1, \boldsymbol{\kappa}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\nu}, u, v],\end{aligned}$$

where the dg structure remains unchanged on all the elements expect $\kappa, \lambda, \mu, \nu$ on which it is given by

$$d\boldsymbol{\kappa} = (\mathbf{x} - u^{\deg \mathbf{x}}\mathbf{x}^1),\ d\boldsymbol{\lambda} = (\mathbf{e} - u^{\deg \mathbf{e}}\mathbf{e}^1),\ d\boldsymbol{\mu} = (\mathbf{y}^1 - (uv)^{-\deg \mathbf{y}}\mathbf{z}),\ d\boldsymbol{\nu} = (\mathbf{f}^1 - (uv)^{-\deg \mathbf{f}}\mathbf{g}).$$

Now, we see that this cdga is quasi-isomorphic (by “solving out” the above relations) to

$$k[\mathbf{x}^1, \mathbf{e}^1, \mathbf{z}, \mathbf{g}, u, v],$$

where the tri-degree of the generators are as follows

$$\mathrm{tri}-\deg x_i^1 = (\deg x_i, 0, 0)\ \ \mathrm{tri}-\deg e_i^1 = (\deg e_i, 0, 0)\ \ \mathrm{tri}-\deg z_i = (0, 0, \deg y_i),$$

$$\mathrm{tri}-\deg g_i = (0, 0, \deg f_i),\ \ \mathrm{tri}-\deg u = (-1, 1, 0),\ \ \mathrm{tri}-\deg v = (0, -1, 1).$$

Taking middle degree 0 invariants we get

$$\begin{aligned}\big(Q(R)\ {}_s\!\otimes_p^L Q(R)\big)_0 &\cong k[\mathbf{x}^1, \mathbf{e}^1, \mathbf{z}, \mathbf{g}, uv] \\ &\cong Q(R).\end{aligned}$$

If we carefully trace through the series of isomorphisms in the proof, we see that it is induced by the morphism ρ. □

Lemma 7.4.2. *The category of perfect objects* $\mathrm{Perf}^{\mathbb{G}_m}(\mathbf{X}^+)$ *is generated by* $j^*\mathcal{R}(i)$ *for* $i \in (-\mu_+, 0]$.

Proof. The structure sheaf $\mathcal{O}_{X^+}$ is an ample line bundle as X^+ is quasi-affine. By Definition 7.2.3, the category is generated by $j^*\mathcal{R}(i)$ where $i \in \mathbb{Z}$. In fact, we can restrict to a smaller set of generators, $j^*\mathcal{R}(i)$ where $i \in (-\mu_+, 0]$ due to the following argument.

Consider the Koszul complex $\mathcal{K}_\bullet(x_1, x_2, \cdots, x_n)$ on X

$$0 \longrightarrow \mathcal{O}_X(-\mu_+) \longrightarrow \cdots \longrightarrow \bigoplus_i \mathcal{O}_X(-a_i). \xrightarrow{(x_1, x_2, \cdots, x_n)} \mathcal{O}_X \longrightarrow 0$$

This is acyclic when restricted to X^+ and the twists in all intermediate terms lie in $(-\mu_+, 0]$. Now, we may apply $\otimes_{\mathcal{O}_X^+} j^*\mathcal{R}$ to see that $j^*\mathcal{R}(-\mu_+)$ is generated as claimed. We can twist the Koszul complex by $\mathcal{R}(p)$ where $p < 0$ to show that $j^*\mathcal{R}(i)$ for $i \leq -\mu_+$ is generated as claimed. Similarly, we twist by $\mathcal{R}(q)$ where $q > 0$, to show that $j^*\mathcal{R}(i)$ for $i > 0$ is generated as claimed. □

Now, we would like to find a set of generators for the essential image of Φ_{Q_+}. For this result we need to make the assumption that all the positive generators e_i vanish, i.e., R is semi-free over T, where all the (homological) generators f_i of R over T are of non-positive internal degree.

Lemma 7.4.3. *Assume that the degrees of the generators f_i of $T \to R$ are non-positive, i.e., $\deg f_i \leq 0$. Then, the image of $\Phi_{Q_+} : \mathrm{Perf}^{\mathbb{G}_m}(\mathbf{X}^+) \longrightarrow \mathrm{Perf}^{\mathbb{G}_m}(\mathbf{X})$ is the full subcategory of $\mathrm{Perf}^{\mathbb{G}_m}(\mathbf{X})$ generated by $\mathcal{R}(i)$ where $i \in (-\mu_+, 0]$. Moreover, we have*

$$\Phi_{Q_+} \circ j^*(\mathcal{R}(i)) = \mathcal{R}(i) \textit{ where } i \in (-\mu_+, 0].$$

Proof. It suffices to find the image of Φ_{Q_+} on the generators obtained in Lemma 7.4.2 – $j^*\mathcal{R}(i)$ for $i \in (-\mu_+, 0]$. We have the following sequence of isomorphisms.

$$\begin{aligned}
\Phi_{Q_+}(j^*\mathcal{R}(i)) &= (R\pi_{2*}(Q_{+\ p} \otimes \pi_1^* j^*\mathcal{R}(i)))^{\mathbb{G}_m} \\
&= (R\pi_{2*}Q_+(i,0))_{(0,*)} \\
&= (\pi_{2*}(Q_+ \otimes \mathcal{C}_{X^+ \times X})(i,0))_{(0,*)},
\end{aligned}$$

where $\mathcal{C}_{X^+\times X}$ denotes the Čech resolution obtained by inverting the x_i. We claim that we can use the Čech resolution to compute the derived pushforward $R\pi_{2*}$ due to the following argument.

The morphism $\pi_2 : \mathbf{X}^+ \times \mathbf{X} \to \mathbf{X}$ factors as

$$\mathbf{X}^+ \times \mathbf{X} \xrightarrow{for} (X^+ \times X, \mathcal{O}_{X^+} \boxtimes \mathcal{R}) \xrightarrow{\pi^0} (X, \mathcal{R}) = \mathbf{X},$$

where the first morphism is the forgetful morphism. To compute $R\pi^0_*$, we can use the Čech resolution and then to compute $R\pi_{2*}$, we use the composition of derived functors.

Notice that we are tensoring over the p module structure and hence the result of the cohomology computation will have the s module structure.

We claim that the morphism

$$\pi_{2*}(Q^b_+(i,0)) \to \pi_{2*}(Q^b_+ \otimes \mathcal{C}_{X^+\times X}(i,0)). \tag{7.7}$$

is an isomorphism for all $b \leq 0$ after taking internal degree $(0,*)$ invariants. Here, the upper index b denotes the homological degree. First, let us compute the right hand side using a standard Čech cohomology computation. Notice that

$$H^c(\pi_{2*}(Q^b_+ \otimes \mathcal{C}_{X^+\times X}(i,0))) = 0 \qquad c \neq 0, l.$$

The $c = l$ term is generated by

$$\prod_{i=1}^{l} x_i^{-p_i} k[\mathbf{z}, u] \prod_{\sum \operatorname{hdeg} f_i = b} g_i,$$

where $p_i > 0$ for all $0 \leq i \leq l$. Here, we use hdeg to denote the homological degree, and where all the a_i are strictly positive. Upon taking degree $(i,*)$ invariants, this term vanishes as $i \in (-\mu_+, 0]$. The left hand side of (7.7) is just a module and by taking its cohomology, we just mean taking degree 0 invariants. This can be computed directly as

$$\pi_{2*}(Q^b)_{(i,*)} \cong u^{-i} k[u^{\deg \mathbf{x}}.\mathbf{x}, \mathbf{z}] \prod_{\sum \operatorname{hdeg} f_i = b} g_i \cong s(\mathcal{R})^b(i)$$

Finally, we apply H^0 and $(0,*)$ invariants to equation (7.7) to get

$$s(\mathcal{R})^b(i) \to H^0(\pi_{2*}(Q^b_+ \otimes \mathcal{C}_{X^+\times X}))_{(i,*)} \cong u^{-i} k[u^{\deg \mathbf{x}}.\mathbf{x}, \mathbf{z}] \prod_{\sum \operatorname{hdeg} f_i = b} g_i,$$

which is clearly an isomorphism and this proves our claim.

Now, this claim implies that

$$\pi_{2*}(Q_+ \otimes \mathcal{C}_{X^+ \times X}))_{(i,*)} \cong \mathcal{R}(i),$$

with the s-module structure and hence we are done.

$\square$

We can do the same analysis for $\Phi_{Q_-} : \operatorname{Perf}^{\mathbb{G}_m}(\mathbf{X}^-) \longrightarrow \operatorname{Perf}^{\mathbb{G}_m}(\mathbf{X})$ where $Q_- := Q|_{X^- \times X}$. Again, we need to make the assumption that all the negative generators f_i vanish, i.e., R is semi-free over T, where all the generators e_i are of non-negative degree.

Lemma 7.4.4. *Assuming that the degrees of the generators e_i of $T \to R$ are non-positive, i.e.,* $\deg e_i \geq 0$*, the image of* $\Phi_{Q_-} : \operatorname{Perf}^{\mathbb{G}_m}(\mathbf{X}^-) \longrightarrow \operatorname{Perf}^{\mathbb{G}_m}(\mathbf{X})$ *is the full subcategory of* $\operatorname{Perf}^{\mathbb{G}_m}(\mathbf{X})$ *generated by* $\mathcal{R}(i)$ *where* $i \in [0, -\mu_-)$*.*

Proof. This holds by symmetry. $\square$

Now, we look at an example of a hypersurface in affine space, in order to compare our conditions on the degree of e_i with Property $(L+)$ and Property (A) of [HL15].

Example 7.4.5. We consider the hypersurface case of Example 7.3.5. This is also studied in [HL15, Example 2.5]. However, as a warning to the reader, the convention on weights in *loc. cit.* appears to be the opposite of our work. Consider the cdga $R = k[\mathbf{x}, \mathbf{y}, f]$, where $df = h$. The internal $\mathbb{Z}$-grading is given by

$$\deg x_i > 0, \qquad \deg y_i < 0,$$

as before. If we have the condition

$$\deg h \leq 0,$$

Lemma 7.4.3 identifies the window subcategory explicitly. By comparison, Property $(L+)$ and Property (A) of [HL15] can be satisfied either by imposing $\deg h \geq 0$ or by requiring that $h \mod \mathbf{x}$ is non-zero and linear in at least one of the y_i. Hence our condition on the degree appears to be almost complementary to the one imposed in *loc. cit.*

Now, let us look at a specific case of Example 7.4.5 with $\deg h \leq 0$.

Example 7.4.6. We consider the cdga $R = k[x_1, x_2, f]$ with $df = x_1x_2$ and internal $\mathbb{Z}$-grading $\deg x_1 = \deg x_2 = 1$. Note that $\deg f = 2$ in this example, and hence Lemma 7.4.3 does not apply. However, we wish to understand the essential image of the window functor Φ_{Q_+} in this example. It is easy to do so by direct computation. As we wish to compare our window to [HL15], we also use the isomorphism

$$\mathrm{Perf}^{\mathbb{G}_m}(R) \cong \mathrm{Perf}^{\mathbb{G}_m}(k[x_1, x_2]/(x_1x_2)). \tag{7.8}$$

It suffices to find the image of $j^*\mathcal{R}$ as $j^*\mathcal{R}(-1) \cong j^*\mathcal{R}$. So, we compute $\Phi_{Q_+}(j^*\mathcal{R})$ as in Lemma 7.4.3 to get

$$\Phi_{Q_+}(j^*\mathcal{R}) = (Q_+ \otimes \mathcal{C}_{X^+ \times X})_{(0,*)} = \left(k[x_1, x_2, x_1^{-1}, e, u] \oplus k[x_1, x_2, x_2^{-1}, e, u]\right)_{(0,*)}. \tag{7.9}$$

To get this, observe that the other term of the Čech complex is

$$k[x_1, x_2, x_1^{-1}, x_2^{-1}, e, u] \cong 0$$

as $de = x_1x_2$ is inverted.

Under the isomorphism 7.8, we can compute the right hand side of equation 7.9 to get

$$\begin{aligned}\left(k[x_1, x_2, x_1^{-1}, u]/(x_1x_2) \oplus k[x_1, x_2, x_2^{-1}, u]/(x_1x_2)\right)_{(0,*)} &\cong \left(k[x_1, x_1^{-1}, u] \oplus k[x_2, x_2^{-1}, u]\right)_{(0,*)} \\ &\cong k[ux_1] \oplus k[ux_2].\end{aligned}$$

Now, one can check that the window subcategory as defined in [HL15] is generated by $k[x_1]$ and $k[x_2]$.

7.4.2 Windows in general

Now, we consider the general case (for us), where T is a finitely generated smooth $\mathbb{Z}$-graded ring over a *field* k. We need to restrict to a field k in order to use the Luna slice theorem. We recall the following result from [BDF17] for (non-dg) k-algebras.

Lemma 7.4.7. *For any smooth finitely generated $\mathbb{Z}$-graded k-algebra T, $Q(T)$ satisfies Property P i.e.,*

$$\rho_T : \left(Q(T) \otimes^L Q(T)\right)_0 \cong Q(T)$$

Proof. The proof uses the Luna slice theorem to reduce to the case of affine space. See [BDF17, Lemma 4.2.5, Proposition 4.2.6]. □

Lemma 7.4.8. *When T is a finitely generated smooth $\mathbb{Z}$-graded ring over a field k, the object $Q(R)$ satisfies property P, i.e., the morphism*

$$\rho : \left(Q(R) \; {}_s\!\otimes_p^L Q(R)\right)_0 \to Q(R),$$

is an isomorphism.

Proof. Consider a semi-free resolution K of $Q(T)$ as a $T \otimes T$-module. This provides a semi-free resolution $K[\mathbf{e}, \mathbf{g}]$ of $Q(R)$ and we can compute the derived tensor product,

$$\begin{aligned} Q(R) \; {}_s\!\otimes_p^L Q(R) &\cong (K[\mathbf{e}, \mathbf{g}] \; {}_s\!\otimes_p Q(T)[\mathbf{e}', \mathbf{g}']) \\ &\cong K \otimes_{{}^sT^p} Q(T)[\mathbf{e}, \mathbf{g}'] \\ &\cong Q(T) \otimes^L_{{}^sT^p} Q(T)[\mathbf{e}, \mathbf{g}']. \end{aligned}$$

Hence

$$\begin{aligned} \left(Q(R) \; {}_s\!\otimes_p^L Q(R)\right)_0 &\cong \left(Q(T) \otimes^L_{{}^sT^p} Q(T)\right)_0 [\mathbf{e}, \mathbf{g}'] && \text{since } \mathbf{e}, \mathbf{g}' \text{ have middle degree } 0 \\ &\cong Q(T)[\mathbf{e}, \mathbf{g}'] && \text{by Lemma 7.4.7} \\ &\cong Q(R). \end{aligned}$$

Tracing through the chain of maps, we see that the map inducing the isomorphism is ρ.

□

Defining windows

Let us recall the definition of windows in the setting of ordinary schemes as in [BDF17] now. Let T be a $\mathbb{Z}$-graded smooth ring over a field k. Let $\mu_\pm$ be the sum of the weights of the conormal bundle of $\operatorname{Spec} T/J^{\pm}$ in $\operatorname{Spec} T = X$, where $J^{\pm}$ is the ideal generated by all the positive/negative-ly graded elements. For simplicity, we assume that the fixed locus $V(J^{+}) \cap V(J^{-})$ is connected. If not, we need to define different $\mu_{\pm}$ for different connected components.

Then we define the grade restriction window $\mathbb{W}_X^+$ to be the full subcategory of $\mathrm{D}^b(\mathrm{QCoh}^{\mathbb{G}_m} X)$ generated by objects A such that for any fixed point $y \in X$, and some affine étale neighborhood $V = \operatorname{Spec} S$ in $X = \operatorname{Spec} T$, the restriction

$$A \otimes_T S$$

is generated by $S(i)$ for $i \in (-\mu_+, 0]$. Notice that this definition of the window matches the one from Lemma 7.4.3 in the case where X is affine space.

In our setting of dg-schemes we will view the scheme $X = \operatorname{Spec} T$ as a dg-scheme $(X, \mathcal{O}_X)$, but by abuse of notation denote it by just X. We use j' for the following inclusion:

$$j' : (X^+, \mathcal{O}_{X^+}) \hookrightarrow (X, \mathcal{O}_X).$$

For the dg-scheme $\mathbf{X} = (X, \mathcal{R})$ that we are interested in, we will define the window $\mathbb{W}_{\mathbf{X}}^+$ as the pullback of the window $\mathbb{W}_X^+$ under the forgetful morphism. Just as in the case of ordinary schemes, we assume that the fixed locus is connected for simplicity. In more detail, we have the following.

Definition 7.4.9. Let

$$f : \mathbf{X} = (X, \mathcal{R}) \to X = (X, \mathcal{O}_X)$$

be the forgetful morphism. The *grade restriction window* $\mathbb{W}_{\mathbf{X}}^+$ to be the full subcategory of $\mathrm{Perf}^{\mathbb{G}_m}(\mathbf{X})$ generated by objects f^*A with $A \in \mathrm{Perf}^{\mathbb{G}_m}(X)$ such that for any fixed point $y \in X$, and some affine étale neighborhood $V = \operatorname{Spec} S$ in $X = \operatorname{Spec} T$, the restriction

$$A \otimes_T S$$

is generated by $S(i)$ for $i \in (-\mu_+, 0]$ i.e.

$$\mathbb{W}_{\mathbf{X}}^+ := \langle f^*A, A \text{ in } \mathbb{W}_X^+ \rangle.$$

We also introduce the forgetful morphism on the semi-stable locus

$$f^+ : \mathbf{X}^+ \to X^+,$$

and the forgetful morphism

$$\alpha := f^+ \times f : \mathbf{X}^+ \times \mathbf{X} \to X^+ \times X$$

Let us reiterate that we will denote the dg-scheme $(X, \mathcal{O}_X)$ by just X, as opposed to the dg-scheme $\mathbf{X} = (X, \mathcal{R})$. The following commutative diagrams summarize the notation:

$$\begin{tikzcd}
& \mathbf{X}^+ \times \mathbf{X} \arrow[r, "\alpha"] \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2", near end] & X^+ \times X \arrow[dl, "\pi_1'"', near end] \arrow[dr, "\pi_2'"] & \\
\mathbf{X}^+ \arrow[r, "f^+"] & X^+ & \mathbf{X} \arrow[r, "f"] & X,
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathbf{X}^+ \arrow[r, hook, "j"] \arrow[d, "f^+"] & \mathbf{X} \arrow[d, "f"] \\
X^+ \arrow[r, hook, "j'"] & X.
\end{tikzcd}$$

The following result about base change for Q under the forgetful morphism $f : \mathbf{X} \to X$ will help us compute the window $\mathbf{W}^+_{\mathbf{X}}$.

Lemma 7.4.10. *Assuming that the degrees of the generators f_i of $T \to R$ are non-positive, i.e.,* $\deg f_i \leq 0$*, the following is an isomorphism of functors*

$$\Phi_{Q(R)} \circ f^* \cong f^* \circ \Phi_{Q(T)},$$

where $Q(R)$ and $Q(T)$ are considered with the $p \otimes s$-module structure.

Proof. We claim that the following morphism of $\mathbb{Z} \times \mathbb{Z}$-graded $R \otimes R$-dg-modules

$$\begin{gathered}
Q(T)_s \otimes_T R \to Q(T)[\mathbf{g}] \\
q \otimes 1 \mapsto q \\
1 \otimes f_i \mapsto g_i \\
1 \otimes t \mapsto s(t),
\end{gathered} \tag{7.10}$$

where $q \in Q(T)$, $f_i \in R$ and $t \in T \subset R$, is an isomorphism.

If we forget the dg-structure, it is clear that the map is surjective and injective, and respects the grading when we view R in $Q(T)_s{\otimes_T} R$ with the grading $(0,*)$. The morphism also respects the dg-structure; if df_i is in T,

$$d(1\otimes f_i) = 1\otimes df_i = s(df_i)\otimes 1 \mapsto s(df_i) = dg_i.$$

This proves the isomorphism

$$Q(T)_s{\otimes_T} R \cong Q(R).$$

Now, let M be an object of $D(X)$. Then

$$\begin{aligned}
\Phi_{Q(R)}\circ f^*(M) &= \pi_{2*}((M\otimes_T R)\otimes_R {}_pQ(R))^{\mathbb{G}_m}\\
&= \pi_{2*}((M\otimes_T R)\otimes_R {}_p(Q(T)_s{\otimes_T} R))^{\mathbb{G}_m}\\
&= \pi_{2*}((M\otimes_T {}_pQ(T)_s)\otimes_T R)^{\mathbb{G}_m}\\
&= \pi_{2*}((M\otimes_T {}_pQ(T)))^{\mathbb{G}_m}{}_s{\otimes_T} R\\
&= f^*\circ\Phi_{Q(T)}(M)
\end{aligned}$$

To get the second to last line of the above chain of isomorphisms, we use that R has grading $(0,*)$ and that the $\mathbb{G}_m$-invariants is with respect to the first grading. □

We recall one of the main results of [BDF17].

Lemma 7.4.11 ([BDF17, Theorem 4.2.9])**.** *When X is a smooth scheme over a field k, the following*

$$\Phi_{Q(T)_+} : D^b(X/\!\!/_+) \xrightleftharpoons{} \mathbb{W}^+_X : j'^*$$

is an equivalence of categories.

Finally, we show that our window functor Φ_{Q_+} lands in the window $\mathbb{W}^+_{\mathbf{X}}$.

Lemma 7.4.12. *Assuming that the degrees of the generators f_i of R as a T-algebra are non-positive, i.e., $\deg f_i \leq 0$, the image of $\Phi_{Q_+} : \mathrm{Perf}^{\mathbb{G}_m}(\mathbf{X}^+) \longrightarrow \mathrm{Perf}^{\mathbb{G}_m}(\mathbf{X})$ lies in the window subcategory $\mathbb{W}^+_{\mathbf{X}} \subset \mathrm{Perf}^{\mathbb{G}_m}(\mathbf{X})$.*

Proof. We have the following isomorphisms:

$$\begin{aligned} f^* \circ \Phi_{Q(T)} \circ j'_* &\cong \Phi_{Q(R)} \circ f^* \circ j'_* \\ &\cong \Phi_{Q(R)} \circ j_* \circ f^{+*} \\ &\cong \Phi_{Q(R)_+} \circ f^{+*}, \end{aligned}$$

where we used the isomorphism of Lemma 7.4.10 in the first line. The second line follows using the projection formula,

$$\begin{aligned} j_* \circ f^{+*}(\mathcal{F}) &= j_*(j'^* R \otimes \mathcal{F}) \\ &= R \otimes j'_* \mathcal{F} \\ &= f^* \circ j'_*(\mathcal{F}) \end{aligned}$$

This gives us the following isomorphism of functors.

$$\Phi_{Q(R)_+} \circ f^{+*} \cong f^* \circ \Phi_{Q(T)_+}.$$

Clearly, $\mathrm{Perf}^{\mathbb{G}_m}(\mathbf{X}^+)$ is generated by the essential image of f^{+*}. Hence we only need to find the image of Φ_{Q_+} on objects of the form $f^{+*} j'^* A$, where A is in $\mathbb{W}^+_X$.

Now, if we consider an object A in $\mathbb{W}^+_X$,

$$\begin{aligned} \Phi_{Q(R)_+}(f^{+*} j'^* A) &\cong f^* \circ \Phi_{Q(T)_+}(j'^* A) \\ &\cong f^* A \end{aligned}$$

where we used Lemma 7.4.11 in the last line.

This shows that the essential image of $\Phi_{Q(R)_+}$ lands in the window $\mathbb{W}^+_{\mathbf{X}}$. □

The previous statement can now be upgraded to the following theorem.

Theorem 7.4.13. *Assume that R is generated as a T-algebra by non-positive elements. The functor*

$$\Phi_{Q_+} : \mathrm{Perf}^{\mathbb{G}_m}(\mathbf{X}^+) \longrightarrow \mathbb{W}^+_{\mathbf{X}}$$

is an equivalence of categories with inverse functor j^*.

Proof. The fullness follows from Lemma 7.3.14 using Lemma 7.4.8. The faithfulness was proved in Lemma 7.3.8. Lemma 7.4.12 implies that a set of generators of $\mathbb{W}^+_{\mathbf{X}}$ lies in the essential image of Φ_{Q_+}. This shows that Φ_{Q_+} is an equivalence. The inverse functor is j^* by the proof of Lemma 7.4.12. □

Similarly, we can define the negative window functor and the negative window for the negative GIT quotient, when the generators e_i of $T \to R$ are non-negative.

Theorem 7.4.14. *Assume that R is generated as a T-algebra by non-negative elements. The functor*

$$\Phi_{Q_-} : \mathrm{Perf}^{\mathbb{G}_m}(\mathbf{X}^-) \longrightarrow \mathbb{W}^-_{\mathbf{X}}$$

is an equivalence of categories with inverse functor j_-^*, where $j_- : \mathbf{X}^- \hookrightarrow \mathbf{X}$ is the inclusion of dg-schemes.

Proof. This is true by symmetry. □

7.4.3 Wall-crossings

By combining Theorem 7.4.13 and Theorem 7.4.14, we can prove results about the wall-crossing functors when all the *generators e_i of $T \to R$ have degree* 0. Hence, we assume that the generators e_i of the semi-free cdga R over T have internal degree 0 in this section.

We consider the easiest case first. Recall that T is a $\mathbb{Z}$-graded smooth ring over k, and that $J^{\pm}$ are generated by the positively/negatively graded elements of T. Let $\mu_{\pm}$ be the sum of the weights of the conormal bundle of $\operatorname{Spec} T/J^{\pm}$ in $\operatorname{Spec} T$. Our first result is when $\mu_+ + \mu_- = 0$, often referred to as the Calabi-Yau condition.

Theorem 7.4.15. *Let R be a semi-free cdga equipped with a $\mathbb{G}_m$-action such that $T = R^0$ is smooth, and the generators e_i of R over T have internal degree* 0*. Let $\mu_{\pm}$ be the sum of the weights of the conormal bundle of $T/J^{\pm}$ in T. Let $j_- : \mathbf{X}^- \hookrightarrow \mathbf{X}$ be the inclusion of dg-schemes. When, $\mu_+ + \mu_- = 0$, the wall crossing functor*

$$\Phi^{\mathrm{wc}} := j_-^* \circ (- \otimes \mathcal{R}(\mu_+ - 1)) \circ \Phi_{Q_+} : \mathrm{Perf}^{\mathbb{G}_m}(\mathbf{X}^+) \longrightarrow \mathrm{Perf}^{\mathbb{G}_m}(\mathbf{X}^-)$$

is an equivalence of categories.

Proof. Theorem 7.4.13 shows that Φ_{Q_+} gives an equivalence of categories

$$\mathrm{Perf}^{\mathbb{G}_m}(\mathbf{X}^+) \cong \mathbb{W}^+_{\mathbf{X}}.$$

A similar analysis for Φ_{Q_-} shows that

$$\mathrm{Perf}^{\mathbb{G}_m}(\mathbf{X}^-) \cong \mathbb{W}^-_{\mathbf{X}}.$$

We have the condition $\mu_+ + \mu_- = 0$ which ensures that applying $(- \otimes \mathcal{R}(\mu_+ - 1))$ exchanges the positive and negative windows

$$\mathbb{W}^+_{\mathbf{X}} \otimes \mathcal{R}(\mu_+ - 1) \cong \mathbb{W}^-_{\mathbf{X}}.$$

As j_-^* is the inverse functor to Φ_{Q_-}, we have the result. $\square$

Now, we would like to understand the case when the windows are of different lengths, i.e., when $\mu_+ + \mu_- \neq 0$. We only consider the (easier) case when X is an affine space.

Affine space

Recall the setting of Section 7.4.1, where

$$T = k[\mathbf{x}, \mathbf{y}],$$

with internal $\mathbb{Z}$-grading $\deg x_i = a_i > 0$ and $\deg y_i = b_i < 0$. The assumption that the generators e_i of $R = T[\mathbf{e}]$ are of internal degree zero still stands.

For convenience, we also introduce the notation

$$\mathbb{W}_{[a,b]} = \langle R(a), R(a+1), \cdots, R(b) \rangle \subset \mathrm{Perf}^{\mathbb{G}_m}(\mathbf{X}), \qquad a < b \text{ in } \mathbb{Z}$$

Let us consider the case when $\mu_+ + \mu_- > 0$, i.e, the positive window $\mathbb{W}^+_{\mathbf{X}}$ is longer than the negative window $\mathbb{W}^-_{\mathbf{X}}$. Then, we have the following semi-orthogonal decomposition for the window.

Lemma 7.4.16. *Assume $b - a \geq \mu_+$. The following is a semi-orthogonal decomposition*

$$\mathbb{W}_{[a,b+1]} = \langle \mathrm{Perf}(R^{\mathbb{G}_m}), \mathbb{W}_{[a,b]} \rangle.$$

where $R^{\mathbb{G}_m} := R/(\mathbf{x}, \mathbf{y})$ is the dg fixed locus. Furthermore, $\mathrm{Perf}(R^{\mathbb{G}_m})$ *is the category generated by $R/\mathbf{x}(b+1)$.*

Proof. First, we show that the category $W_{[a,b+1]}$ is generated by $R/\mathbf{x}(b+1)$ and $W_{[a,b]}$. The only missing generator is $R(b+1)$, and we can get this by considering the following Koszul resolution

$$R \otimes \mathcal{K}^{\bullet}(x_1, \cdots, x_l)(b+1) \cong R/\mathbf{x}(b+1).$$

The top term of the resolution is $R(b+1)$ and the rest of the terms are in $W_{[a,b]}$. Hence $R(b+1)$ is generated by $\langle R/\mathbf{x}(b+1), W_{[a,b]}\rangle$.

Now, it suffices to verify the semi-orthogonal decomposition condition on the generators. Hence we need to check that

$$R\operatorname{Hom}(R(i), R/\mathbf{x}(b+1)) = 0 \qquad \text{for } a \leq i \leq b.$$

We have

$$R\operatorname{Hom}(R(i), R/\mathbf{x}(b+1)) = (R/\mathbf{x})_{(b+1-i)} = k[\mathbf{y}]_{(b+1-i)}$$

as $b+1-i > 0$ and hence, we are done.

We now compute the dg-endomorphism ring of $R/\mathbf{x}(i)$ as:

$$\begin{aligned} R\operatorname{Hom}(R/\mathbf{x}, R/\mathbf{x}) &= R\operatorname{Hom}(R \otimes \mathcal{K}(x_1, \cdots, x_l), R/\mathbf{x}) \\ &= \left(\bigwedge\nolimits^{\bullet} \bigoplus_{i=1}^{i} k(a_i) \otimes R/\mathbf{x} \right)_0 \\ &= R^{\mathbb{G}_m} \end{aligned}$$

The derived category of perfect dg-modules of the endomorphism ring $\mathrm{Perf}(R^{\mathbb{G}_m})$ is then equivalent to the category generated by $R/\mathbf{x}(b+1)$ [Kel06, Theorem 3.8b)] (note that $\langle R/\mathbf{x}(b+1)\rangle$ is idempotent-complete as it is generated by a compact object of the category $D(\mathbf{X})$ which is idempotent-complete as it admits countable co-products). ◻

Using this, we can get a semi orthogonal decomposition for $\mathrm{Perf}^{\mathbb{G}_m}(\mathbf{X}^+)$ when $\mu > 0$.

Theorem 7.4.17. *The following is a semi-orthogonal decomposition*

$$\mathrm{Perf}^{\mathbb{G}_m}(\mathbf{X}^+) = \langle \mathrm{Perf}(R^{\mathbb{G}_m})_{\mu}, \cdots, \mathrm{Perf}(R^{\mathbb{G}_m})_1, \Phi^{wc}(\mathrm{Perf}^{\mathbb{G}_m}(\mathbf{X}^+))\rangle$$

where $\mathrm{Perf}(R^{\mathbb{G}_m})_i \cong \mathrm{Perf}(R^{\mathbb{G}_m})$ *denotes the full subcategory generated by* $j_+^* R/\mathbf{x}(i)$.

Proof. First, we use Lemma 7.4.16 inductively to get the semi-orthogonal decomposition

$$W_{[-\mu_+ +1,0]} = \langle R/\mathbf{x}(\mu), \cdots, R/\mathbf{x}(2), R/\mathbf{x}(1), W_{[\mu_- +1,0]} \rangle,$$

in $\mathrm{Perf}^{\mathbb{G}_m}(\mathbf{X})$. Then, we pull back along $j_+ : \mathbf{X}^+ \hookrightarrow \mathbf{X}$ to get the result. $\square$

7.4.4 Applications to flops

As discussed in the introduction, one of the major applications of our results that we have in mind is to the Bondal-Orlov conjecture on derived equivalences of flops [BO95]. By an observation of Reid [Tha96, Proposition 1.7], a flop between smooth projective varieties can be realized as different GIT quotients of $\mathbb{G}_m$ acting on a scheme Y. Let us consider the local setting and assume that Y is affine. In general, Y is singular, and the idea is to resolve it as a semi-free cdga in order to construct the wall-crossing functor. We can prove the derived equivalence under certain conditions, using the results of the previous sections.

To be precise, consider a sub-variety Y of a smooth affine variety $X = \mathrm{Spec}\, T$, equipped with a $\mathbb{G}_m$-action. Now, consider the Koszul-Tate resolution of $k[Y] = T/I$ to get a semifree cdga R over T; in particular, the cdga R can be chosen such that there are only finitely many generators in each homological degree [Sta19, Lemma 23.6.9]. In general, we require an infinite number of dg-algebra generators. Then, we have

$$R = (T[e_1, e_2, \cdots], d), \qquad \text{such that} \qquad H^0(R) \simeq k[Y].$$

If Y is a complete intersection in X defined by functions $(h_1, \cdots, h_n)$, we can take the semi-free cdga R to be the Koszul resolution; in this case, the number of dg-algebra generators is finite, and the differential acts as

$$de_i = h_i, \qquad i \in [1, n].$$

In either case, if we assume that the internal degree of all the e_i is zero, we can use the results of the previous section, in particular Theorem 7.4.15, to obtain the following result.

Corollary 7.4.18. *Assume that the condition $\mu_+ + \mu_- = 0$ is satisfied. Also assume that the dg-algebra generators e_i in R have internal degree zero. Then the wall crossing functor*

$$\Phi^{\mathrm{wc}} : \mathrm{Perf}(Y /\!\!/_+) \longrightarrow \mathrm{Perf}(Y /\!\!/_-)$$

is an equivalence of categories.

Proof. This is a direct application of Theorem 7.4.15 to the setting of this section. We use the fact that quasi-isomorphisms of dg-schemes induce equivalences of derived categories, which is an immediate extension of [Ric10, Proposition 1.5.6] to the equivariant setting, i.e.,

$$D^{\mathbb{G}_m}(X^{\pm}, \mathcal{R}^{\pm}) \cong D^{\mathbb{G}_m}(Y^{\pm}, \mathcal{O})$$

as $\mathcal{R}$ is a dg-resolution of $k[Y]$. As mentioned in Remark 7.2.4, it is easy to see that our ad hoc definition of the category of perfect objects coincides with the standard one in the case of ordinary schemes, and hence we are done. □

Moreover, if we do not have the condition $\mu_+ + \mu_- = 0$, the results on wall crossings (Theorem 7.4.17) can be carried over to our setting in this section. These statements are straightforward to write down, but we leave it to the interested reader.

Mukai flop

In particular, this allows us to give a VGIT proof of the derived equivalence for the Mukai flop. Let us briefly remind the reader about the VGIT construction of the Mukai flop. Here, k will be an arbitrary Noetherian ring, and we recall Example 7.3.1. Consider the $\mathbb{Z}$-graded ring $T = k[x_1, \cdots, x_l, y_1, \cdots, y_l]$ with internal degree

$$\deg x_i = 1 \qquad \deg y_i = -1.$$

Consider the cdga

$$R = T[e], \qquad de = \sum_{i=1}^{l} x_i y_i,$$

which is quasi-isomorphic to the ring

$$S := k[x_1, \cdots, x_l, y_1, \cdots, y_l]/(\sum_{i=1}^{l} x_i y_i).$$

The (local model of the) Mukai flop is the birational transformation between the two GIT quotients

$$Y/\!\!/_{+} \dashleftarrow\dashrightarrow Y/\!\!/_{-},$$

where $Y = \operatorname{Spec} S$. As the e has internal degree zero, we can apply Corollary 7.4.18 to get the derived equivalence.

Corollary 7.4.19. *Consider the VGIT presentation of the Mukai flop as above. The wall crossing functor:*

$$\Phi^{\mathrm{wc}} : D^b(Y/\!\!/_{+}) \longrightarrow D^b(Y/\!\!/_{-})$$

is an equivalence.

Proof. This is a direct application of Corollary 7.4.18. We note that as the GIT quotients $Y/\!\!/_{\pm}$ are smooth, the category of perfect complexes is equivalent to the bounded derived category of coherent sheaves. □

Remark 7.4.20. This result is well known, however, all the existing proofs in the literature are over $\mathbb{C}$ [Nam03, ADM19, Mor18]. Our results hold over an arbitrary Noetherian ring k.

Chapter 8

Conclusion

In this chapter, we address some of the natural questions posed by this thesis, and developments in the field that have occurred since the time of completion of the papers that comprise this thesis. We point out the most interesting questions here, and strategies to tackle them whenever possible; some of these are being pursued actively by myself and various collaborators.

In Chapter 3, we demonstrated that the non-perturbative definition of the wave-function is crucial to understanding the relationship between quantum curves and topological recursion. We studied the simplest non-trivial case of the conjecture, where the algebraic curve is an elliptic curve satisfying a quantization condition. In this elliptic curve case, the conjecture has since been proved by [Iwa19]. The most recent result in this direction is [MO19, EG19], which proves the conjecture for any hyperelliptic curve. A somewhat unrelated question in the case of the topological recursion applied to elliptic curves, is whether the correlators which are elliptic functions and the free energies which are quasi-modular forms, admit an enumerative geometric interpretation.

In the context of higher Airy structures, we posed a number of questions throughout Chapter 4. One of the questions was about an intersection theory interpretation of the higher Airy structure associated to a spectral curve of type (r, s) (see Question 4.6.1). In the case where $s = r - 1$, we have the following conjecture. Consider the smooth proper moduli stack $\overline{\mathcal{M}}_{g,n}^{r,s}$ that parametrizes r-th roots of the bundle

$$\mathcal{K} = \omega_{\log}^{\otimes(s-r)}(-\sum_{i=1}^{n} m_i p_i)$$

of an r-stable[1] curve C with n-marked points $p_1, \cdots, p_n$. Consider the universal r-th root $\mathcal{S}$. Then, we define a cohomology class

$$\Theta_{g,n}^{(r,s)} := c_{\text{top}}(R\pi_*\mathcal{S})$$

, as the Euler class of the derived pushforward from the universal curve $\pi : \mathcal{C} \to \overline{\mathcal{M}}_{g,n}^{r,s}$ (which is a vector bundle).

Conjecture 8.0.1. *The partition function associated to the $(r, r-1)$-spectral curve is the intersection theory of the cohomology class $\Theta_{g,n}^{(r,r-1)}$, i.e.,*

$$F_{g,n}(ra_1 + m_1, \cdots, ra_n + m_n) = \int_{\overline{\mathcal{M}}_{g,n}^{r,s}} \Omega_{g,n}(m_1, \cdots, m_n)\psi_1^{a_1}\psi_2^{a_2} \cdots \psi_n^{a_n},$$

when $0 \leq m_i \leq r-1$.

In the general case, i.e., when $s \neq r-1$, it is easy to check that the $F_{g,n}$ do not calculate the intersection theory of the cohomology class $\Theta_{g,n}^{(r,s)}$. However, we still understand how to obtain the intersection theory of the class $\Theta_{g,n}^{(r,s)}$ using the topological recursion (by taking limits of certain families of spectral curves) and this is a problem that we are actively pursuing.

Question 8.0.2. *Is there a topological recursion (or $\mathcal{W}$-algebra constraints) that governs the intersection theory of the class $\Theta_{g,n}^{(r,s)}$ described above?*

What is the intersection theory interpretation of the $F_{g,n}$ constructed by the topological recursion on the admissible (r, s)-spectral curve when $1 \leq s \leq r-2$?

We can also use ideas similar to those described in the previous paragraph to try and understand the intersection theory associated to $\mathcal{W}$-algebra constraints in the case of D_N and E_N with $s = 1$. We do not spell out the details here, but this provides a conjectural answer to Question 4.6.6.

[1] This is a certain notion of stability required to obtain a proper moduli space, see [Chi08, Definition 2.1.1]

Question 8.0.3. *Do the partition functions constructed in Theorem 4.4.20 for D_N-type and Theorem 4.4.24 for E_N-type in the case $s = 1$ have an interpretation in terms of Fan-Jarvis-Ruan theories?*

In Chapter 6, we studied the Q-construction in the case of $\mathrm{GL}(V)$-actions, and proved the derived equivalence for the Grassmann flop. In general, one could study similar VGIT problems, where the wall-crossing functor is not an equivalence. Identifying the orthogonal component to the essential image of the wall-crossing functor would be a very interesting problem.

Question 8.0.4. *When $d_W \neq d_{W'}$, what is the orthogonal complement to the essential image of the wall-crossing functor that we construct in Theorem 6.5.12?*

By extending the techniques of Chapter 6 to the case of $\mathrm{GL}(V)$-action on arbitrary schemes, one could try to study other examples of flops. In the case of $\mathrm{GL}(V)$-actions on a singular scheme, we can combine the techniques of Chapter 6 and Chapter 7, we can resolve the singular scheme by a semi-free cdga, and then apply the generalized Q-construction in order to understand the window functors. In fact, we are working on applying these techniques in the context of the stratified Mukai flop as studied by [Cau12, Kaw06].

Question 8.0.5. *Does the derived Q-construction provide a Fourier-Mukai kernel for the derived equivalence of the stratified Mukai flop?*

The derived equivalence for the Abuaf flop was proved by Segal [Seg11] using tilting bundles. A VGIT presentation of the flop (using $\mathbb{G}_m$-actions) is defined using a non-complete intersection cut out by functions of non-zero degree, and hence our Theorem 7.4.15 is not directly applicable. However, we still hope that we will be able to identify the window explicitly in this example by combining our results with the ones of [HL15].

As noted in Chapter 7, the only major obstruction to verifying the conjectures of Bondal and Orlov is that we are unable to describe the window subcategory explicitly in general. By comparing with the construction and definition of [HL15], we hope to be able to find a general definition of the window. This is work in progress.

Question 8.0.6. *Can one find an explicit description of the window subcategory for the fully-faithful functor that was constructed in Proposition 7.1.1?*

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Appendix A

Explicit computations for Weierstrass curve

A.1 Correlation functions for Weierstrass curve

In this appendix we record the correlation functions constructed from the Eynard-Orantin topological recursion at the first few recursive levels. Those are needed to calculate the first few terms (S_2, S_3 and S_4) in the WKB expansion in section 3.6.

First, at level $2g-2+n=1$, we get:

$$\begin{aligned} &W_{0,3}(z_0,z_1,z_2) \\ &= \frac{12}{\Delta}\mathrm{d}z_0\mathrm{d}z_1\mathrm{d}z_2\sum_{i=1}^{3}(20G_4(\tau)-e_i^2)P_2(z_0-\omega_i)P_2(z_1-\omega_i)P_2(z_2-\omega_i), \end{aligned} \tag{A.1}$$

and

$$\begin{aligned} &W_{1,1}(z_0) \\ &= \frac{6}{\Delta}\mathrm{d}z_0\sum_{i=1}^{3}(20G_4(\tau)-e_i^2)\left((G_2(\tau)-e_i)P_2(z_0-\omega_i,\tau)+\frac{1}{4!}P_2^{(2)}(z_0-\omega_i,\tau)\right). \end{aligned} \tag{A.2}$$

At level $2g-2+n=2$, we get

$$W_{1,2}(z_0,z_1)=\frac{1}{\Delta^2}\sum_{i=1}^{3}9(20G_4-e_i^2)(-60G_4(e_i^2-20G_4)$$
$$(G_2+\wp(z_0-w_i))(G_2+\wp(z_1-w_i))-(e_i^2-20G_4)(6\wp(z_0-w_i)^3-30G_4\wp(z_0-w_i)$$
$$+\wp'(z_0-w_i)^2)(G_2+\wp(z_1-w_i))-6(e_i^2-20G_4)(5G_4-\wp(z_0-w_i)^2)$$
$$(-4G_2^2+8e_iG_2-\wp(z_1-w_i)^2+5G_4+(8e_i-4G_2)\wp(z_1-w_i))$$
$$+(G_2+\wp(z_0-w_i))(-8(e_{i+2}^2-20G_4)(G_2+\wp(z_1-w_{i+2}))(e_{i+1}+G_2)^2$$
$$-8(e_i^2-20G_4)(G_2^2+6G_4)(G_2+\wp(z_1-w_i))$$
$$-60(e_i^2+2G_4)(e_i^2-20G_4)(G_2+\wp(z_1-w_i))+2((4G_2(G_2-e_i)+G_4)(20G_4-e_i^2)$$
$$+(20G_4-e_{i+2}^2)(e_{i+1}^2+4(e_{i+1}+G_2)(G_2-e_{i+2})-5G_4)$$
$$+(20G_4-e_{i+1}^2)(e_{i+2}^2+4(G_2-e_{i+1})(e_{i+2}+G_2)-5G_4))(G_2+\wp(z_1-w_i))$$
$$-24(e_i-G_2)(e_i^2-20G_4)(5G_4-\wp(z_1-w_i)^2)$$
$$-8e_i(e_i^2-20G_4)(-12G_2^2+4e_iG_2-3\wp(z_1-w_i)^2+15G_4$$
$$+4(e_i-3G_2)\wp(z_1-w_i))-8(e_{i+2}+G_2)^2(e_{i+1}^2-20G_4)(G_2+\wp(z_1-w_{i+1}))$$
$$+(e_i^2-20G_4)(-6\wp(z_1-w_i)^3+30G_4\wp(z_1-w_i)-\wp'(z_1-w_i)^2))),$$

and

$$W_{0,4}(z_0,z_1,z_2,z_3)=\sum_{i=1}^{3}3\frac{144(20G_4-e_i^2)}{\Delta^2}(e_i^2-20G_4)$$
$$(5G_4-\wp(z_0-w_i)^2)(G_2+\wp(z_1-w_i))(G_2+\wp(z_2-w_i))(G_2+\wp(z_3-w_i))$$
$$+\frac{144(20G_4-e_i^2)}{\Delta^2}(G_2+\wp(z_0-w_i))(12e_i(e_i^2-20G_4)(G_2+\wp(z_1-w_i))$$
$$(G_2+\wp(z_2-w_i))(G_2+\wp(z_3-w_i))+3(e_i^2-20G_4)(5G_4-\wp(z_1-w_i)^2)$$
$$(G_2+\wp(z_2-w_i))(G_2+\wp(z_3-w_i))+3(e_i^2-20G_4)(G_2+\wp(z_1-w_i))$$
$$(5G_4-\wp(z_2-w_i)^2)(G_2+\wp(z_3-w_i))+(-G_2(e_i^2-20G_4)(G_2+\wp(z_1-w_i))$$
$$(G_2+\wp(z_2-w_i))-(e_{i+2}+G_2)(e_{i+1}^2-20G_4)(G_2+\wp(z_1-w_{i+1}))$$
$$(G_2+\wp(z_2-w_{i+1}))-(e_{i+1}+G_2)(e_{i+2}^2-20G_4)(G_2+\wp(z_1-w_{i+2}))$$
$$(G_2+\wp(z_2-w_{i+2})))(G_2+\wp(z_3-w_i))+3(e_i^2-20G_4)(G_2+\wp(z_1-w_i))$$
$$(G_2+\wp(z_2-w_i))(5G_4-\wp(z_3-w_i)^2)+(G_2+\wp(z_2-w_i))(-G_2(e_i^2-20G_4)$$
$$(G_2+\wp(z_1-w_i))(G_2+\wp(z_3-w_i))-(e_{i+2}+G_2)(e_{i+1}^2-20G_4)$$
$$(G_2+\wp(z_1-w_{i+1}))(G_2+\wp(z_3-w_{i+1}))-(e_{i+1}+G_2)(e_{i+2}^2-20G_4)$$
$$(G_2+\wp(z_1-w_{i+2}))(G_2+\wp(z_3-w_{i+2})))+(G_2+\wp(z_1-w_i))(-G_2(e_i^2-20G_4)$$
$$(G_2+\wp(z_2-w_i))(G_2+\wp(z_3-w_i))-(e_{i+2}+G_2)(e_{i+1}^2-20G_4)$$
$$(G_2+\wp(z_2-w_{i+1}))(G_2+\wp(z_3-w_{i+1}))-(e_{i+1}+G_2)(e_{i+2}^2-20G_4)$$
$$(G_2+\wp(z_2-w_{i+2}))(G_2+\wp(z_3-w_{i+2})))),$$

where the index i is defined mod 3.

We also calculated the correlation functions at level $2g-2+n=3$, namely $W_{0,5}(z_0,z_1,z_2,z_3,z_4)$, $W_{1,3}(z_0,z_1,z_2)$ and $W_{2,1}(z_0)$. The expressions are very long though so we will not include them here. They are available upon request.

A.2 An independent proof of Corollary 3.5.1

In this Appendix we provide an independent proof of Corollary 3.5.1 directly from the theory of elliptic functions. Recall that Corollary 3.5.1 states that:

$$\oint_A \frac{P_2(2z;\tau)}{\wp'(z;\tau)^2}dz = \frac{G_4(\tau)(5G_4(\tau) - G_2(\tau)^2)}{30(20G_4(\tau)^3 - 49G_6(\tau)^2)}. \tag{A.3}$$

Let us evaluate the period integral on the LHS explicitly and show that it is indeed equal to the quasi-modular form on the RHS. In this Appendix we will suppress the τ-dependence for brevity.

First we expand the integrand with a "double angle" identity:

$$P_2(2z) = G_2 - 2\wp(z) + \frac{1}{4}\left(\frac{\wp''(z)}{\wp'(z)}\right)^2 \tag{A.4}$$

Hence our original integral splits into the following three integrals:

$$\oint_A \frac{P_2(2z)}{\wp'(z)^2}dz = G_2 \oint_A \frac{\mathrm{d}z}{\wp'(z)^2} - 2\oint_A \frac{\wp(z)}{\wp'(z)^2}\mathrm{d}z + \frac{1}{4}\oint_A \frac{\wp''(z)^2}{\wp'(z)^4}\mathrm{d}z \tag{A.5}$$

Let us focus on the third constituent integral. Using integration by parts and the fact that $\wp'''(z) = 12\wp(z)\wp'(z)$ we see that it simplifies into a more familiar form:

$$\frac{1}{4}\oint_A \frac{\wp''(z)^2}{\wp'(z)^4}\mathrm{d}z = \frac{1}{4}\left\{-\frac{\wp''(z)}{3\wp'(z)^3}\bigg|_0^1 + 4\oint_A \frac{\wp(z)}{\wp'(z)^2}\right\} = \oint_A \frac{\wp(z)}{\wp'(z)^2} \tag{A.6}$$

Hence our original problem reduces to solving only two integrals:

$$\oint_A \frac{P_2(2z)}{\wp'(z)^2}dz = G_2 \oint_A \frac{\mathrm{d}z}{\wp'(z)^2} - \oint_A \frac{\wp(z)}{\wp'(z)^2}\mathrm{d}z \tag{A.7}$$

To solve both we need a very useful identity, which follows directly from the differential equation for the Weierstrass $\wp$-function (3.16) and the fact that $\frac{2}{3}(\wp''(z) - g_2) = 4\wp(z)^2 - g_2$:

$$\frac{1}{\wp'(z)^2} = \frac{1}{g_3}\left[\frac{2}{3}\frac{\wp(z)\left(\wp''(z) - g_2\right)}{\wp'(z)^2} - 1\right] \tag{A.8}$$

As it turns out, using integration by parts we can express these two integrals in terms of one another:

$$\oint_A \frac{\mathrm{d}z}{\wp'(z)^2} = \frac{1}{g_3}\left[-1 + \frac{2}{3}\oint_A \frac{\wp(z)\left(\wp''(z) - g_2\right)}{\wp'(z)^2}\right] \tag{A.9}$$

$$= -\frac{1}{3g_3}\left[1 + 2g_2 \oint_A \frac{\wp(z)}{\wp'(z)^2}\mathrm{d}z\right] \tag{A.10}$$

$$\oint_A \frac{\wp(z)}{\wp'(z)^2}\mathrm{d}z = \frac{1}{g_3}\left[\frac{2}{3}\oint_A \frac{\wp(z)^2\left(\wp''(z) - g_2\right)}{\wp'(z)^2} - \oint_A \wp(z)\mathrm{d}z\right] \tag{A.11}$$

$$= -\frac{1}{3g_3}\left[G_2 + \frac{g_2^2}{6}\oint_A \frac{\mathrm{d}z}{\wp'(z)^2}\right] \tag{A.12}$$

For the last equation, we used the fact that

$$\oint_A \wp(z)\mathrm{d}z = -G_2, \tag{A.13}$$

since

$$0 = \oint_A P_2(z)\mathrm{d}z = \oint_A (\wp(z) + G_2)\mathrm{d}z. \tag{A.14}$$

Solving the system of equations (A.9) and (A.11) results in the following explicit expressions (with $\Delta = g_2^3 - 27g_3^2$):

$$\oint_A \frac{\mathrm{d}z}{\wp'(z)^2} = \frac{18g_3 - 12G_2 g_2}{2\Delta} \tag{A.15}$$

$$\oint_A \frac{\wp(z)}{\wp'(z)^2}\mathrm{d}z = \frac{18G_2 g_3 - g_2^2}{2\Delta} \tag{A.16}$$

As a result we see that the original integral (A.7) is given by:

$$\oint_A \frac{P_2(2z)}{\wp'(z)^2}\mathrm{d}z = \frac{18G_2 g_3 - 12G_2^2 g_2}{2\Delta} - \frac{18G_2 g_3 - g_2^2}{2\Delta} = \frac{g_2(g_2 - 12G_2^2)}{2\Delta} \tag{A.17}$$

Making the substitutions $g_2 = 60G_4$ and $g_3 = 140G_6$ we arrive at the final expected result:

$$\oint_A \frac{P_2(2z)}{\wp'(z)^2}\mathrm{d}z = \frac{G_4(5G_4 - G_2^2)}{30(20G_4^3 - 49G_6^2)} \tag{A.18}$$

Appendix B

Appendix to Higher Airy Structures

B.1 Sums over roots of unity

We encountered in Definition 4.4.3 the following sums for $i \in \{1, \ldots, r\}$

$$\Psi(a_1, \ldots, a_i) = \frac{1}{i!} \sum_{\substack{m_1, \ldots, m_i = 0 \\ m_l \neq m_{l'}}}^{r-1} \prod_{l=1}^{i} \theta^{-m_l a_l}\,, \tag{B.1}$$

where $\theta = e^{2\mathbf{i}\pi/r}$ is a primitive r-th root of unity. This is the $j = 0$ case of the more general sum for $j \in \{0, \ldots, \lfloor i/2 \rfloor\}$

$$\Psi^{(j)}(a_{2j+1}, \ldots, a_i) = \frac{1}{i!} \sum_{\substack{m_1, \ldots, m_i = 0 \\ m_l \neq m_{l'}}}^{r-1} \left(\prod_{l'=1}^{\ell} \frac{\theta^{m_{2l'-1} + m_{2l'}}}{(\theta^{m_{2l'}} - \theta^{m_{2l'-1}})^2} \prod_{l=2j+1}^{i} \theta^{-m_l a_l} \right). \tag{B.2}$$

This appendix is devoted to the proof of several properties of these functions used in Section 4.4 and an explicit computation of $\Psi^{(0)}$.

Lemma B.1.1. *For any $j \in \{0, \ldots, \lfloor i/2 \rfloor\}$, we have*

$$\sum_{a_1, \ldots, a_{2j} = 0}^{r-1} \Psi(a_1, \ldots, a_i) \prod_{l'=1}^{j} \frac{a_{2l'-1} a_{2l'}\, \delta_{a_{2l'-1} + a_{2l'}, r}}{2r} = \Psi^{(j)}(a_{2\ell+1}, \ldots, a_i)\,. \tag{B.3}$$

Proof. The left hand side of (B.3) is equal to

$$\frac{1}{i!} \sum_{\substack{m_1, \ldots, m_i = 0 \\ m_l \neq m_{l'}}}^{r-1} \sum_{a_1, a_3, \ldots, a_{2j-1} = 0}^{r-1} \prod_{l'=1}^{j} \theta^{(m_{2l'} - m_{2l'-1}) a_{2l'-1}} \prod_{l=2j+1}^{i} \theta^{-m_l a_l}\,. \tag{B.4}$$

We compute the sum

$$\sum_{a=0}^{r-1} \frac{a(r-a)x^a}{2r} = \frac{x((r-1)x-(r+1))x^r + (r+1)x - (r-1)}{2r(x-1)^3}\,.$$

Setting $x = \theta^{m_1 - m_2}$ for distinct $m_1, m_2 \in \{0, \ldots, r-1\}$ gives

$$\sum_{a=0}^{r-1} \frac{a(r-a)\theta^{a(m_1-m_2)}}{2r} = \frac{\theta^{m_1-m_2}}{(\theta^{m_1-m_2}-1)^2} = \frac{\theta^{m_1+m_2}}{(\theta^{m_1}-\theta^{m_2})^2}\,.$$

Using this formula to perform the sum over $a_1, a_3, \ldots, a_{2j-1} = 0$ in (B.4) entails the claim. □

We can get rid of zero entries in $\Psi^{(j)}$ in a simple way.

Lemma B.1.2.

$$\Psi^{(j)}(a_{2j+1}, \ldots, a_{i-\ell}, \underbrace{0, \ldots, 0}_{\ell \text{ times}}) = \frac{(i-\ell)!}{i!}\,\frac{(r-i+\ell)!}{(r-i)!}\,\Psi^{(j)}(a_{2j+1}, \ldots, a_{i-\ell})\,.$$

Proof. In the sum (B.2) defining $\Psi^{(j)}(a_{2j+1}, \ldots, a_{i-\ell}, 0, \ldots, 0)$, the terms only depend on

$$\mathbf{m} = (m_1, \ldots, m_{i-\ell})$$

and the as. Therefore, we can perform the sum over the ordered j-tuple $(m_l)_{l=i-\ell+1}^{i}$ of pairwise disjoint integers in $\{0, \ldots, r-1\} \setminus \{m_1, \ldots, m_{i-\ell}\}$ and get a global factor of $\frac{(r-i+\ell)!}{(r-i)!}$. We then recognize $\Psi^{(j)}(a_{2j+1}, \ldots, a_{i-\ell})$ up to another global factor $\frac{(i-\ell)!}{i!}$. □

Lemma B.1.3. *We have* $i\,\Psi(\underbrace{r-1, \ldots, r-1}_{i-1 \text{ times}}, i-1) = (-1)^{i-1} r$.

Proof. Let us denote for $m \in \{0, \ldots, r-1\}$

$$\vec{\theta} = (1, \theta, \theta^2, \ldots, \theta^{r-1}), \qquad \vec{\theta}[m] = \vec{\theta} \setminus \theta^m\,.$$

Coming back to the definition of $\Psi = \Psi^{(0)}$, we can write

$$i\,\Psi\big(\underbrace{r-1, \ldots, r-1}_{i-1 \text{ times}}, i-1\big) = \sum_{m=0}^{r-1} \theta^{-m(i-1)} \Bigg(\sum_{\substack{L \subseteq \{0,\ldots,r-1\}\setminus\{m\} \\ |L| = i-1}} \prod_{l \in L} \theta^{-l(r-1)} \Bigg) = \frac{1}{i} \sum_{m=0}^{r-1} \theta^{-m(i-1)} e_{i-1}\big(\vec{\theta}[m]\big)\,, \tag{B.5}$$

where e_j is the j-th elementary symmetric polynomial. Since $-\theta$ are the simple roots of the polynomial $1+(-1)^r t^r$ we get

$$\sum_{k=0}^{r-1} e_k(\vec{\theta})t^k = \prod_{a=0}^{r-1}(1+t\theta^a) = 1+(-1)^r t^r\,,$$

from which we deduce that $e_k(\vec{\theta}) = \delta_{k,0}$ for $k \in \{1,\ldots,r-1\}$. On the other hand we have by inclusion-exclusion

$$e_k(\vec{\theta}) = \theta^m e_{k-1}\big(\vec{\theta}[m]\big) + e_k(\vec{\theta}[m])\,.$$

So we deduce by induction that

$$\forall k \in \{1,\ldots,r-1\}, \qquad e_k(\vec{\theta}[m]) = (-1)^k\theta^{mk}\,.$$

Inserting this result in (B.5) gives

$$i\,\Psi\big(\underbrace{r-1,\ldots,r-1}_{i-1\text{ times}}, i-1\big) = \sum_{m=0}^{r-1}(-1)^{i-1}\theta^{-m(i-1)}\theta^{m(i-1)} = (-1)^{i-1}r\,.$$

$\square$

Lemma B.1.4. *More generally for any $s>0$, let us introduce $d=\gcd(r,s)$ and $r' := r/d$. We have*

$$i\,\Psi\big(\underbrace{-s,\ldots,-s}_{i-1\text{ times}},(i-1)s\big) = r\,(-1)^{i-1+\lfloor\frac{i-1}{r'}\rfloor}\binom{d-1}{\lfloor\frac{i-1}{r'}\rfloor}$$

Proof. The strategy is similar. We have

$$\psi_{i,s} := i\,\Psi\big(\underbrace{-s,\ldots,-s}_{i-1\text{ times}},(i-1)s\big) = \sum_{m=0}^{r-1}\theta^{-m(i-1)s}\Bigg(\sum_{\substack{L\subseteq\{0,\ldots,r-1\}\setminus\{m\}\\ |L|=i-1}}\prod_{l\in L}\theta^{ls}\Bigg) = \sum_{m=0}^{r-1}\theta^{-m(i-1)s}e_{i-1}(\vec{\theta}^s[m])\,,$$

where $\vec{\theta}^s = (1^s,\theta^s,\ldots,\theta^{s(r-1)})$ and $\vec{\theta}^s[m]$ is the sequence $\vec{\theta}^s$ with θ^{ms} omitted. We write by inclusion-exclusion

$$e_i(\vec{\theta}^s) = \theta^{ms}e_{i-1}(\vec{\theta}^s[m]) + e_i(\vec{\theta}^s[m])\,,$$

with the convention $e_{-1}=0$. We multiply this identity by θ^{-ims} and sum over $m\in\{0,\ldots,r-1\}$ to find

$$r\delta_{r|is}\,e_i(\vec{\theta}^s) = \psi_{i,s} + \psi_{i+1,s}\,, \tag{B.6}$$

where $\delta_{a|b}$ is equal to 1 is $a|b$ and to 0 otherwise. With the value $\psi_{1,s} = r$ we will obtain by induction on i a formula for $\psi_{i,s}$, provided we can compute $e_i(\vec{\theta}^s)$. For this purpose we observe that $\vec{\theta}^s$ contains each $r' := r/d$-th root of unity, with multiplicity d. Thus

$$\sum_{i=0}^{r} e_i(\vec{\theta}^s)\, t^i = (1-(-t)^{r'})^d = \sum_{j=0}^{d} (-1)^{j(r'+1)} \binom{d}{j} t^{jr'} ,$$

therefore

$$r\delta_{r|is}\, e_i(\vec{\theta}^s) = r\delta_{is|r}\delta_{r'|i}\,(-1)^{i(1+1/r')}\binom{d}{i/r'} = r\delta_{r'|i}\,(-1)^{i/r'}\binom{d}{i/r'} . \tag{B.7}$$

Solving the recursion (B.6) with (B.7) as left-hand side yields

$$i\,\Psi\big(\underbrace{-s,\ldots,-s}_{i-1 \text{ times}},(i-1)s\big) = (-1)^{i-1} r \sum_{j=0}^{\lfloor (i-1)/r' \rfloor} (-1)^j \binom{d}{j} .$$

Recall that $d \geq 2$, and let us denote

$$\psi(k,d) := \sum_{j=0}^{k} (-1)^j \binom{d}{j}, \qquad k \in \{0,\ldots,d-1\} ,$$

so that

$$i\,\Psi\big(\underbrace{-s,\ldots,-s}_{i-1 \text{ times}},(i-1)s\big) = (-1)^{i-1} r\, \psi_{(-1)^{r'+1}}\big(\lfloor \tfrac{i-1}{r'} \rfloor, d\big) .$$

We observe that

$$\psi(k,d) - \psi(k-1,d) = (-1)^k \binom{d}{k} , \tag{B.8}$$

which is also valid for $k=0$ with the convention $\psi(-1,d)=0$. But Pascal's identity

$$\binom{d}{k} = \binom{d-1}{k} + \binom{d-1}{k-1} ,$$

implies that

$$\psi(k,d) = \psi(k,d-1) - \psi(k-1,d-1) , \tag{B.9}$$

with the convention that $\binom{d}{-1} = 0$. Combining (B.9) and (B.8) we find

$$\psi(k,d) = (-1)^k \binom{d-1}{k} .$$

$\square$

Now we focus on the evaluation of these sums for small values of i.

Lemma B.1.5. *For $r \geq 1$ we have $\Psi(a_1) = r\delta_{r|a_1}$. For $r \geq 2$ we have*

$$\begin{aligned} \Psi(a_1, a_2) &= \frac{1}{2}\big(r^2\delta_{r|a_1}\delta_{r|a_2} - r\delta_{r|a_1+a_2}\big)\,, \\ \Psi^{(1)}(\emptyset) &= -\frac{r(r^2-1)}{24}\,. \end{aligned}$$

For $r \geq 3$ we have

$$\begin{aligned} \Psi(a_1, a_2, a_3) &= \frac{1}{6}\Big(r^3\delta_{r|a_1}\delta_{r|a_2}\delta_{r|a_3} - r^2\delta_{r|a_1}\delta_{r|a_2+a_3} - r^2\delta_{r|a_2}\delta_{r|a_1+a_3} \\ &\qquad -r^2\delta_{r|a_3}\delta_{r|a_1+a_2} + 2r\delta_{r|a_1+a_2+a_3}\Big) \\ \Psi^{(1)}(a_3) &= -\frac{r(r-2)(r^2-1)}{72}\,\delta_{r|a_3}\,. \end{aligned}$$

Proof. We have for $i = 1$ and $j = 0$

$$\Psi(a_1) = \sum_{m=0}^{r-1} \theta^{-ma_1} = r\delta_{r|a_1}\,. \tag{B.10}$$

For $i = 2$, j can take the two values 0 or 1. We have

$$\begin{aligned} \Psi(a_1, a_2) &= \frac{1}{2}\sum_{\substack{m_1,m_2=0 \\ m_1\neq m_2}}^{r-1} \theta^{-m_1a_1-m_2a_2} = \frac{1}{2}\Bigg(\sum_{m_1,m_2=0}^{r-1} - \sum_{m_1=m_2=0}^{r-1}\Bigg)\theta^{-m_1a_1-m_2a_2} \\ &= \frac{1}{2}\big(r^2\delta_{r|a_1}\delta_{r|a_2} - r\delta_{r|a_1+a_2}\big)\,. \end{aligned}$$

Using Lemma B.1.1 and the formula we just proved, we compute

$$\Psi^{(1)}(\emptyset) = \sum_{a_1,a_2=0}^{r-1} \Psi(a_1, a_2)\,\frac{a_1a_2}{2r}\,\delta_{a_1+a_2,r} = -\frac{1}{4}\sum_{a_1=0}^{r-1} a_1(r-a_1) = -\frac{r(r^2-1)}{24}\,.$$

For $i = 3$, we apply the same strategy

$$\begin{aligned} \Psi(a_1, a_2, a_3) &= \frac{1}{6}\sum_{n=0}^{r-1}\theta^{-na_1}\Bigg(\sum_{\substack{m_2,m_3=0 \\ m_2\neq m_3 \\ m_\alpha\neq n}}^{r-1} \theta^{-m_2a_2-m_3a_3}\Bigg) \\ &= \frac{1}{6}\sum_{n=0}^{r-1}\Bigg(\theta^{-na_1}2\Psi_2(a_2,a_3) - \theta^{-n(a_1+a_2)}\sum_{\substack{m_3=0 \\ m_3\neq n}}^{r-1}\theta^{-m_3a_3} - \theta^{-n(a_1+a_3)}\sum_{\substack{m_2=0 \\ m_2\neq n}}^{r-1}\theta^{-m_2a_2}\Bigg) \\ &= \frac{1}{6}\big(r\delta_{r|a_1}\,2\Psi_2(a_2,a_3) - r^2\delta_{r|a_1+a_2}\delta_{r|a_3} - r^2\delta_{r|a_1+a_3}\delta_{r|a_2} + 2r\delta_{r|a_1+a_2+a_3}\big) \\ &= \frac{1}{6}\big(r^3\delta_{r|a_1}\delta_{r|a_2}\delta_{r|a_3} - r^2\delta_{r|a_1}\delta_{r|a_2+a_3} - r^2\delta_{r|a_2}\delta_{r|a_1+a_3} \\ &\qquad -r^2\delta_{r|a_3}\delta_{r|a_1+a_2} + 2r\delta_{r|a_1+a_2+a_3}\big)\,, \end{aligned} \tag{B.11}$$

after using the result just found for $i = 2$. Using Lemma B.1.1 we further compute

$$\Psi^{(1)}(a_3) = \sum_{a=0}^{r-1} \frac{a(r-a)}{2r}\, \Psi(a, r-a, a_3)\,.$$

The three first terms in (B.11) do not give any contribution as they force the prefactor $a(r-a)$ to vanish. We obtain for the two last terms

$$\Psi^{(1)}(a_3) = \frac{1}{6}\sum_{a=0}^{r-1} \frac{a(r-a)}{2r}\,(2r-r^2)\,\delta_{r|a_3} = -\frac{r(r-2)(r^2-1)}{72}\,\delta_{r|a_3}\,.$$

$\square$

We can give a general formula for $\Psi(a_1,\ldots,a_i)$, which involve the following notations. We denote $\mathbf{L} \vdash (a_1,\ldots,a_i)$ when $\mathbf{L}$ is an unordered set of $||\mathbf{L}||$ non-empty, pairwise disjoint subsequences of $(a_1,\ldots,a_n)$ whose concatenation is equal to $(a_1,\ldots,a_n)$. The length of a subsequence $L \in \mathbf{L}$ is denoted $|L|$. These notations agree with the ones used in Section 4.2.2.

Lemma B.1.6. *For general $i \in \{1,\ldots,r\}$ and $a_1,\ldots,a_r \in \mathbb{Z}$ we have the formula*

$$i!\,\Psi(a_1,\ldots,a_i) = \sum_{\mathbf{L}\vdash(a_1,\ldots,a_i)} r^{||\mathbf{L}||}\,(-1)^{i-||\mathbf{L}||}\prod_{L\in\mathbf{L}}(|L|-1)!\,\delta_{r|\sum_{l\in L}a_l}\,.$$

In particular, $\Psi(a_1,\ldots,a_i) \in r\mathbb{Z}$.

Proof. We continue with the strategy of the proof of Lemma B.1.5 and find by successive inclusion-exclusion

$$\begin{aligned}\Psi(a_1,\ldots,a_i) &= \sum_{n=0}^{r-1}\sum_{L\subseteq\{2,\ldots,i\}} \theta^{-n(a_1+\sum_{l\in L}a_l)}\,(-1)^{|L|}\,\frac{(i-|L|-1)!|L|!}{i!}\,\Psi\big((a_{l'})_{\alpha\notin(L\cup\{1\})}\big)\\ &= r\sum_{L\subseteq\{2,\ldots,i\}}(-1)^{|L|}\,\frac{(i-|L|-1)!|L|!}{i!}\,\Psi\big((a_{l'})_{\alpha\notin(L\cup\{1\})}\big)\,\delta_{r|a_1+\sum_{l\in L}a_l}\,,\end{aligned}$$

where for $L = \{2,\ldots,n\}$ there appears $\Psi(\emptyset)$ which is by convention equal to 1. This is a recursive formula for $i!\,\Psi(a_1,\ldots,a_i)$ on $i \in \{1,\ldots,r\}$, which is solved by the claimed formula. $\square$

Corollary B.1.7. *Let $i \in \{1, \ldots, r\}$ and $a \in \mathbb{Z}$ coprime with r. Then, for any $b \in \{0, \ldots, i-1\}$*

$$\Psi\big(\underbrace{0, \ldots, 0}_{b \text{ times}}, \underbrace{a, \ldots, a}_{i-b \text{ times}}\big) = \begin{cases} \delta_{i,r}(-1)^{r-1} & \text{if } b = 0 \\ 0 & \text{if } b \in \{1, \ldots, i-1\} \\ 1 & \text{if } b = i \end{cases} .$$

Proof. The case $b = i$ is obvious from the definition B.1. For $b = 0$, since a is coprime with r, we gave $\delta_{r|\sum_{l \in L} a_l} = \delta_{r|\,|L|a} = \delta_{r|\,|L|}$. Therefore, the only non-zero contribution in the formula of Lemma B.1.6 occurs when $i = r$ and for the unique term which corresponds to $\mathbf{L}$ being the partition consisting of a single set, *i.e.* $||\mathbf{L}|| = 1$. With $b \in \{1, \ldots, i-1\}$ we first get rid of the zeroes thanks to Lemma B.1.2, and use the previous result to find that the expression evaluates to 0. □

Corollary B.1.8. *For any $i \in \{1, \ldots, r\}$ and $j \in \{0, \ldots, \lfloor i/2 \rfloor\}$, $\Psi^{(j)}(a_{2j+1}, \ldots, a_i) \in \mathbb{Z}$ vanishes unless there exists $\mathbf{L} \vdash (a_{2j+1}, \ldots, a_i)$ such that the partial sums $\sum_{l \in L} a_l$ are divisible by r for any $L \in \mathbf{L}$.*

Proof. For $j > 0$, we insert the formula of Lemma B.1.6 in Lemma B.1.1

$$i!\, \Psi^{(j)}(a_{2j+1}, \ldots, a_i) = \sum_{a_1, \ldots, a_{2j} = 0}^{r-1} \sum_{\mathbf{L} \vdash (a_1, \ldots, a_i)} c_{\mathbf{L}}\, \delta_{r|\sum_{l \in L} a_l} \prod_{l'=1}^{j} \frac{a_{2l'-1} a_{2l'} \delta_{a_{2l'-1} + a_{2l'}, r}}{2r} ,$$

where

$$c_{\mathbf{L}} = r^{||\mathbf{L}||} (-1)^{i - ||\mathbf{L}||} \prod_{L \in \mathbf{L}} (|L| - 1)! \,.$$

We first focus on the sum over the first ordered pair (a_1, a_2) such that $a_1 + a_2 = r$, and meet two types of terms. If a_1 and a_2 are in the same subsequence L, we will have a contribution of the form

$$\sum_{a=0}^{r-1} \frac{a(r-a)}{2r}\, \delta_{r|b} = \frac{(r^2 - 1)}{12}\, \delta_{r|b} \,,$$

while if they are in two different subsequences, we rather have a contribution of the form

$$\sum_{a=0}^{r-1} \frac{a(r-a)}{2r}\, \delta_{r|b_1 + a} \delta_{r|b_2 - a} = \frac{\langle b_1 \rangle \langle b_2 \rangle}{2r}\, \delta_{r|b_1 + b_2} \,,$$

where $\langle b_1 \rangle$ is the unique integer in $\{0, \ldots, r-1\}$ such that $b_1 - \langle b_1 \rangle \in r\mathbb{Z}$. Considering successively the sums over the other pairs $(a_{2l'-1}, a_{2l'})$, we observe a similar phenomenon and obtain the claim. □

For instance, we obtain for $i = 4$ and $r \geq 4$ the following formulas

$$
\begin{aligned}
24\Psi(a_1, a_2, a_3, a_4) \;&=\; r\Big(r^3\delta_{r|a_1}\delta_{r|a_2}\delta_{r|a_3}\delta_{r|a_4} - r^2(\delta_{r|a_1+a_2}\delta_{r|a_3}\delta_{r|a_4} + \;\cdots) + r(\delta_{r|a_1+a_2}\delta_{r|a_3+a_4} + \;\cdots) \\
&\quad +2r(\delta_{r|a_1+a_2+a_3}\delta_{r|a_4} + \;\cdots) - 6\,\delta_{r|a_1+a_2+a_3+a_4}\Big)\,,
\end{aligned}
$$

where the $\cdots$ indicate other terms necessary to enforce symmetry under permutation of a_1, a_2, a_3, a_4. Furthermore, exploiting the method sketched in the proof of Corollary B.1.8, we find

$$
\begin{aligned}
24\,\Psi^{(1)}(a_3, a_4) \;&=\; -\,\frac{(r+1)r^2(r-1)(r-4)}{12}\,\delta_{r|a_3}\delta_{r|a_4} && \text{(B.12)} \\
&\quad + \left(\frac{(r+1)r(r-1)(r-6)}{12} + r\langle a_3\rangle\langle a_4\rangle\right)\delta_{r|a_3+a_4}\,, && \text{(B.13)} \\
24\,\Psi^{(2)}(\emptyset) \;&=\; \frac{(r+1)r(r-1)(r-2)(r-3)(5r+7)}{720}\,. && \text{(B.14)}
\end{aligned}
$$

It would be interesting to find a closed formula generalizing Lemma B.1.6 to all $j \in \{0, \ldots, \lfloor i/2 \rfloor\}$.

B.2 Characterization of admissible (r, s)

B.2.1 The values of s corresponding to the intermediate subalgebras

In Section 4.3.3, we showed that for any partition $\lambda_1 \geq \cdots \geq \lambda_p \geq 1$ such that $\sum_{j=1}^{p} \lambda_j = r$, the left ideal generated by the modes W_k^i of the $\mathcal{W}(\mathfrak{gl}_r)$ algebra indexed by $(i, k) \in S_\lambda$ is a graded Lie subalgebra. The set S_λ consists of the pairs (i, k) with $i \in \{1, \ldots, r\}$ and

$$k \geq i - \lambda(i), \qquad \lambda(i) := \min\Big\{m > 0 \;\Big|\; \sum_{j=1}^{m} \lambda_j \geq i\Big\}. \tag{B.15}$$

Besides, for any $s \in \{1, \ldots, r+1\}$ coprime with r, we showed in the proof of Theorem 4.64 that the family W_k^i indexed by $(i, k) \in \tilde{S}_s$ after dilaton shift $x_s \to x_s - \frac{1}{s}$ satisfies the degree 1 condition of Definition 4.2.6. The set $\tilde{S}_s$ consists of the pairs (i, k) with $i \in \{1, \ldots, r\}$ and

$$r(k - i + 1) + (i - 1)s \geq 0\,. \tag{B.16}$$

This section is devoted to a characterization of the values of s for which (B.16) can be equivalently described as (B.15). For us this implies that the W_k^i indexed by such $(i,k) \in \tilde{\mathcal{S}}_s$ form a higher quantum Airy structure (Theorem 4.64).

Proposition B.2.1. *Let $s \in \{1,\ldots,r+1\}$ coprime with $r \geq 2$. There exists a partition λ such that $\tilde{S}_s = S_\lambda$ if and only if $r = \pm 1 \bmod s$. In this case, we can decompose $r = r's + r''$ with $r'' \in \{1, s-1\}$ and the partition is given by*

$$\lambda_1 = \cdots = \lambda_{r''} = r' + 1, \qquad \lambda_{r''+1} = \cdots = \lambda_s = r'$$

if $s \neq r+1$, and by $\lambda = (1,\ldots,1)$ if $s = r+1$.

Proof. Equation B.16 is equivalent to $k \geq i - 1 - \lfloor \frac{(i-1)s}{r} \rfloor$ so we are asking for the characterization of s appearing as

$$\lambda(i) = 1 + \left\lfloor \frac{(i-1)s}{r} \right\rfloor,$$

where λ is a partition. In the case $s = r+1$, we have

$$\forall i \in \{1,\ldots,r\}, \qquad 1 + \left\lfloor \frac{(i-1)s}{r} \right\rfloor = i\,,$$

and it is clear that it arises with $\lambda = (1,\ldots,1)$. In the case $s = 1$, we have

$$\forall i \in \{1,\ldots,r\}, \qquad 1 + \left\lfloor \frac{(i-1)s}{r} \right\rfloor = 1\,,$$

and it is clear that it arises with $\lambda = (r)$. In the remaining of the proof we treat the cases $s \in \{2,\ldots,r-1\}$. Let us decompose

$$r = r's + r'', \qquad r'' \in \{1,\ldots,s-1\}, \qquad r' > 0\,.$$

We can assume that $r'' \neq 0$ since $\gcd(r,s) = 1$ and $r' > 0$ since $s \in \{2,\ldots,r-1\}$.

Assume that we are given a weakly increasing function $\mu : \{1,\ldots,r\} \to \mathbb{N}$ such that $\mu(1) = 1$ and $\mu(i+1) - \mu(i) \in \{0,1\}$. Let us write down the complete list of integers for which μ jumps, namely

$$1 \leq \kappa_1 < \ldots < \kappa_{p-1} < r, \qquad \mu(\kappa_j + 1) = \mu(\kappa_j) + 1\,, \tag{B.17}$$

and adopt the convention $\kappa_0 = 0$ and $\kappa_p = r$. If we set

$$\lambda_j = \kappa_j - \kappa_{j-1}, \qquad j \in \{1, \ldots, p\},$$

we get a p-tuple of positive integers such that $\sum_{j=1}^{p} \lambda_j = r$ and by construction

$$\mu(i) = \min \left\{ m \;\middle|\; \sum_{j=1}^{m} \lambda_j \geq i \right\}. \tag{B.18}$$

We however stress that $(\lambda_j)_{j=1}^{p}$ may not be weakly decreasing.

We apply this construction to

$$\mu(i) = 1 + \left\lfloor \frac{(i-1)s}{r} \right\rfloor, \tag{B.19}$$

which clearly satisfies $\mu(1) = 1$ and $\mu(i+1) - \mu(i) \in \{0, 1\}$. We compute from the definition $\mu(r) = s$ and comparing to (B.18) we conclude that $p = s$. To make the proof more transparent, we keep the letter p to indicate the length of the sequence $(\lambda_j)_j$. We are going to compute the sequence κ. Since $\mu(r'+1) = 1$ and $\mu(r'+2) = 2$ we deduce that $\kappa_1 = \lambda_1 = r'+1$. For any $j \in \{1, \ldots, p-1\}$ we can decompose

$$\kappa_j s = \beta_j r + \gamma_j, \qquad \gamma_j \in \{0, \ldots, s-1\}, \qquad \beta_j \in \mathbb{N}.$$

For instance we have $\gamma_1 = s - r''$. Notice that $(\kappa_j + r')s = (\beta_j + 1)r + \gamma_j - r''$. If $\gamma_j < r''$ we deduce

$$(\kappa_j + r')s < (\beta_j + 1)r \leq (\kappa_j + r' + 1)s,$$

and thus $\kappa_{j+1} = \kappa_j + r' + 1$ which implies $\lambda_{j+1} = r' + 1$ and $\gamma_{j+1} = \gamma_j + s - r''$. If $\gamma_j \geq r''$ we rather have

$$(\kappa_j + r' - 1)s < (\beta_j + 1)r \leq (\kappa_j + r')s,$$

and thus $\kappa_{j+1} = \kappa_j + r'$ which implies $\lambda_{j+1} = r'$ and $\gamma_{j+1} = \gamma_j - r''$. To summarize, we always have $\lambda_j \in \{r', r'+1\}$. We start with $\lambda_1 = r'+1$ and $\gamma_1 = s - r''$. Let $\ell > 0$ be the minimum integer such that $\lambda_{\ell+1} = r'$. It means that $\gamma_\ell \geq r''$. According to the previous rules, we have

$$\gamma_j = j(s - r''), \qquad j \in \{1, \ldots, \ell\},$$

and

$$\ell = \left\lceil \frac{r''}{s - r''} \right\rceil . \tag{B.20}$$

Assume that $(\lambda_j)_j$ is weakly decreasing. It is equivalent to the existence of $\ell \in \{1, \dots, p\}$ such that

$$\lambda_j = \begin{cases} r' + 1 & \text{if } j \leq \ell \,, \\ r' & \text{if } j > \ell \,. \end{cases} \tag{B.21}$$

We can compute

$$r = \sum_{j=1}^{p} \lambda_j = (r' + 1)\ell + (p - \ell)r' = pr' + \ell \,.$$

Remembering that $p = s$ it shows that $\ell = r''$. So (B.20) yields

$$r'' = \left\lceil \frac{r''}{s - r''} \right\rceil . \tag{B.22}$$

The latter is equivalent to

$$1 + (r'' - 1)(s - r'') \leq r'' \leq r''(s - r'') \,.$$

The upper bound always holds, while the lower bound can be rewritten as

$$(r'' - 1)(s - r'' - 1) \leq 0 \,.$$

So (B.22) is equivalent to $r'' = 1$ or $r'' = s - 1$. This shows that $r'' \in \{1, s-1\}$ is a necessary condition for λ to be of the form (B.21).

Conversely, if we assume that $r'' \in \{1, s - 1\}$, the equivalence we just stressed shows that (B.22) holds, so that ℓ defined in (B.20) is equal to r''. Then, for any $j \in \{\ell, \dots, s - 1\}$ we have

$$\gamma_\ell - (j - \ell)r'' = r''(s - j) \geq r'' \,,$$

hence $\gamma_j = \gamma_\ell - jr'' \geq r''$ and we must have $\lambda_{j+1} = r'$. This shows that $(\lambda_j)_j$ is of the form (B.21), in particular it is weakly decreasing. $\square$

B.2.2 Computation of $F_{0,3}$ and characterization of symmetry

We consider the modes W_k^i of the $\mathcal{W}(\mathfrak{gl}_r)$ algebra using the twist by the Coxeter element whose expression is given in (4.59). For $s \in \{1, \ldots, r+1\}$ coprime with r, we are going to evaluate

$$H_{k_1}^{i_1} = -\frac{1}{2} \sum_{q_2, q_3} C^{(0)}[q_1 | -q_2, -q_3]\, J_{-q_2} J_{-q_3} - \hbar\, C^{(1)}[q_1|\emptyset] + \cdots$$

where $\cdots$ includes monomials which are different from the ones emphasized here, and we remind that the correspondence between positive integers q and indices $(i,k) \in S_s$ for the mode W_k^i equal to $J_q + O(2)$ is

$$q = \Pi_s(i,k) = r(k - i + 1) + s(i_j - 1)\,, \tag{B.23}$$

and the set $\mathcal{S}_s$ consists precisely of those (i,k) that yields a positive $q = \Pi_s(i,k)$. The notation we use is $q_j = \Pi_s(i_j, k_j)$.

If $(H_{k_1}^{i_1})_{(i_1,k_1) \in S_s}$ forms a higher quantum Airy structure, then

$$F_{0,3}[q_1, q_2, q_3] = q_2 q_3\, C^{(0)}[q_1 | -q_2, -q_3] \tag{B.24}$$

must be invariant under permutation of (q_1, q_2, q_3). On the one hand, the representation theoretic arguments in Section 4.3.3 allowed us in Theorem 4.4.9 the conclusion that if $r = \pm 1 \bmod s$, $(H_k^i)_{(i,k) \in S_s}$ is indeed a higher quantum Airy structure, so (B.24) is *a priori* fully symmetric. Here we compute explicitly $F_{0,3}$ and indeed check that it is fully symmetric. On the other hand, when $r \neq \pm 1 \bmod s$, our explicit computation shows that the right-hand side of (B.24) is not fully symmetric. Therefore, the left ideal generated by the $(H_k^i)_{(i,k) \in S_s}$ is not a graded Lie subalgebra and the results of Section 4.3.3 are in this sense optimal.

Proposition B.2.2. *Let $s \in \{1, \ldots, r+1\}$ be coprime with r.*

- *If $r = r's + s - 1$ for $r' \geq 0$, we have $F_{0,3}[q_1, q_2, q_3] = (r' + 1) q_1 q_2 q_3\, \delta_{q_1+q_2+q_3, s}$.*
- *If $r = r's + 1$ for $r' \geq 0$, we have $F_{0,3}[q_1, q_2, q_3] = -r' q_1 q_2 q_3\, \delta_{q_1+q_2+q_3, s}$.*

- *In all other cases, there exists $q_1, q_2, q_3 > 0$ such that*

$$q_2 q_3\, C^{(0)}[q_1| - q_2, -q_3] \neq q_1 q_3\, C^{(0)}[q_2| - q_1, q_3]\,.$$

Proof. Starting from the expression (4.59) for the differential operators W^i_k, due to the dilaton shift we must get $(i-2)$ variables p_js equal to $-s$ and the two last ps must be equal to q_1 and q_2. So

$$H^{i_1}_{k_1} = \sum_{q_2,q_3>0} \frac{(-1)^{i_1-2} i_1(i_1-1)}{2r}\, \delta_{q_2+q_3+(i_1-2)s+r(k_1-i_1+1),0}\, \Psi(q_2, q_3, \underbrace{s, \ldots, s}_{i_1-2 \text{ times}})\, J_{-q_2} J_{-q_3} + \cdots$$

Remember that $H^{i_1}_{k_1} = J_{q_1} + O(2)$. So the Kronecker delta imposes

$$q_1 + q_2 + q_3 = s\,. \tag{B.25}$$

Since $q_j > 0$ for $j \in \{1, 2, 3\}$ and $1 \leq s \leq r+1$ this imposes $q_j < r$. Let us evaluate Ψ under this condition using Lemma B.1.6. Since s and r are coprime, r cannot divide sm for $m \in \{1, \ldots, i_1 - 2\}$ therefore the only partitions $\mathbf{L}$ that contribute are those for which each $L \in \mathbf{L}$ contains q_1 or q_2

$$\begin{aligned} \Psi(q_2, q_3, \underbrace{s, \ldots, s}_{i_1-2 \text{ times}}) &= \frac{(-1)^{i_1-1}\, r}{i_1}\, \delta_{r|q_2+q_3+(i_1-2)s} \\ &\quad + \sum_{\substack{m_2,m_3\geq 0 \\ m_2+m_3=i_1-2}} \frac{(i_1-2)!}{m_2! m_3!}\, \frac{(-1)^{i_1-2}\, r^2\, m_2! m_3!}{i_1!}\, \delta_{r|q_2+m_2 s} \delta_{r|q_3+m_3 s}\,. \end{aligned} \tag{B.26}$$

The extra combinatorial factor $\frac{(i_1-2)!}{m_1! m_2!}$ is the number of ways of splitting the sequence $(s, \ldots, s)$ of length $i_1 - 2$ into two subsequences of length m_1 and m_2. If there exists two m_2 and m_2' in $\{0, \ldots, i_1 - 2\}$ such that $r|q_2 + m_2 s$ and $r|q_2 + m_2' s$, then $r|(m_2 - m_2')$. Since $i_1 - 2 < r$ we must have $m_2 = m_2'$. Therefore, the sum over m_2 contains at most one term, and if it contains one term it is equal to 1. Under the condition (B.25), we have for any $m \in \mathbb{Z}$

$$-(k_1 - i_1 + 1)r = (q_2 + ms) + (q_3 + (i_1 - 2 - m)s)\,.$$

Therefore, we can omit the factor $\delta_{r|q_3+m_3 s}$ in (B.26). Finally, notice that we can write $q_2 + s(r - i_2 + 1) = r(k_2 - i_2 + 1 + s)$ with $r - i_2 + 1 \in \{1, \ldots, r\}$. So the existence of $b_2 > 0$ and $m_2 \geq 0$ such that $b_2 r = q_2 + s m_2$ with $m_2 \leq i_1 - 2$

is equivalent to $i_2 - 1 = r - m_2 \geq r - (i_1 - 2)$ that is $i_1 + i_2 \geq r + 3$. As a consequence

$$C^{(0)}[q_1| - q_2, -q_3] = \big((i_1 - 1) - r\,\epsilon(q_1, q_2, q_3)\big)\,\delta_{q_1+q_2+q_3,s}\,, \tag{B.27}$$

where $\epsilon(q_1, q_2, q_3) = 1$ if $q_1+q_2+q_3 = s$ and $i_1+i_2-3 \geq r$, and $\epsilon(q_1, q_2, q_3) = 0$ otherwise.

• If $s = 1$ we always have $F_{0,3} = 0$.

• Assume $s = r + 1$. It is not possible for $q_2 \leq s - 2 < r$ to be divisible by r so $\epsilon(q_1, q_2, q_3) = 0$. Likewise $q_1 < r$ so we must have $k_1 = 0$ and $i_1 - 1 = q_1$. Therefore

$$C^{(0)}[q_1| - q_2, -q_3] = q_1\,\delta_{q_1+q_2+q_3,s}\,,$$

and $F_{0,3}[q_1, q_2, q_3] = q_1 q_2 q_3\,\delta_{q_1+q_2+q_3,s}$, which is manifestly symmetric.

We now turn to the values $s \in \{2, \ldots, r - 1\}$.

• Assume $r = r's + s - 1$ with $r' > 0$. We multiply by q_1 and get

$$q_1 = q_1((r' + 1)s - r) = -rq_1 + (r' + 1)q_1 s\,.$$

Thus q_1 corresponds to $i_1 = (r' + 1)q_1 + 1$ and $k_1 = r'q_1$. In particular $i_1 - 1 = (r' + 1)q_1$. Assume there exists $b_2 > 0$ and $m_2 \in \{0, \ldots, i_1 - 2\}$ such that $rb_2 = q_2 + m_2 s$. Since $q_2 = -rq_2 + (r' + 1)q_2 s$, there must exist $l_2 \in \mathbb{Z}$ such that

$$b_2 = -q_2 + l_2 s, \qquad m_2 = -(r' + 1)q_2 + l_2 r\,.$$

Since $q_1 + q_2 + q_3 = s$ with $q_3 > 0$, we have $q_1 + q_2 \leq s - 1$, hence $i_1 - 2 \leq (r' + 1)(s - 1 - q_2)$. Together with $0 \leq m_2 \leq i_1 - 2$ it leads to the inequality

$$0 < (r'+1)q_1 \leq l_2 r \leq (r'+1)(s-1-q_2)+(r'+1)q_2 = (r'+1)(s-1) = r-r' < r\,,$$

which contradicts the existence of l_2. Hence there are no such m_2, meaning that $\epsilon(q_1, q_2, q_3) = 0$. Consequently

$$C^{(0)}[q_1| - q_2, -q_3] = (i_1 - 1)\,\delta_{q_1+q_2+q_3,s} = (r' + 1)q_1\,\delta_{q_1+q_2+q_3,s}\,,$$

and $F_{0,3}[q_1, q_2, q_3] = (r' + 1)q_1 q_2 q_3\,\delta_{q_1+q_2+q_3,s}$, which is manifestly symmetric.

- Assume $r = r's + 1$ with $r' \geq 0$. We see that

$$s - q_1 = s + q_1(r's - r) = -rq_1 + (r'q_1 + 1)s\,,$$

so q_1 corresponds to $i_1 - 1 = r'(s-q_1)+1 = r - r'q_1$ and $k_1 = (s-q_1)(r'-1)+1$. We deduce that $i_1 - 2 = r'(q_2 + q_3)$. Since $q_2 + r'q_2 s = rq_2$ is divisible by r, the choice $m = r'q_2$ satisfies $0 \leq m \leq i_1 - 2$ and therefore $\epsilon(q_1, q_2, q_3) = 1$. We deduce that

$$C^{(0)}[q_1| - q_2, -q_3] = (i_1 - 1 - r)\delta_{q_1+q_2+q_3,s} = -r'q_1\, \delta_{q_1+q_2+q_3,s}\,.$$

This implies that $F_{0,3}[q_1, q_2, q_3] = -r'q_1q_2q_3\, \delta_{q_1+q_2+q_3,s}$, which is manifestly symmetric.

- Now assume that $r'' \in \{2, \ldots, s-2\}$. We are going to show that assuming the symmetry

$$q_2 C^{(0)}[q_1| - q_2, -q_3] = q_1 C^{(0)}[q_2| - q_1, -q_3] \tag{B.28}$$

for any $q_1, q_2, q_3 > 0$ such that $q_1 + q_2 + q_3 = s$ leads to a contradiction. We first remark from the definition that $\epsilon(q_1, q_2, q_3) = \epsilon(q_2, q_1, q_3)$ in (B.27). We set $q_1 = 1$, which we decompose as usual $q_1 = (k_1 - i_1 + 1)r + (i_1 - 1)s$ for some $i_1 \in \{1, \ldots, r\}$. Choosing $q_2 = s - r''$ corresponds to $i_2 = r' + 2$ and we can take $q_3 = s - (q_1 + q_2) = r'' - 1 > 0$. Denoting $\varepsilon = \epsilon(1, s - r'', r'' - 1)$, the condition (B.28) implies

$$(s - r'')(1 - i_1 + \varepsilon r) = -1 - r' + \varepsilon r\,.$$

Choosing $q_2 = r''$ corresponds to $i_2 = r - r' + 1$ and we can take $q_3 = s - (q_1 + q_2) = s - r'' - 1 > 0$. Denoting $\varepsilon' = \epsilon(1, r'', s - r'' - 1)$ the condition (B.28) implies

$$r''(1 - i_1 + \varepsilon' r) = -r + r' + \varepsilon' r\,.$$

Summing the two equations gives

$$s(1 - i_1 + \varepsilon r) + 1 = r\big((\varepsilon + \varepsilon' - 1) + r''(\varepsilon - \varepsilon')\big)\,. \tag{B.29}$$

From the definition we have that

$$\varepsilon = \begin{cases} 1 & \text{if } i_1 \geq r - r' + 1, \\ 0 & \text{otherwise} \end{cases}, \qquad \varepsilon' = \begin{cases} 1 & \text{if } i_1 \geq r' + 2, \\ 0 & \text{otherwise} \end{cases}. \tag{B.30}$$

Since $r - r' + 1 - (r' + 2) = r'(s - 2) + r'' - 1 > 0$, we see that the only possible values of $(\varepsilon, \varepsilon')$ are $(1, 1)$, $(0, 0)$ and $(0, 1)$. In the two first cases, reducing (B.29) modulo s gives

$$r'' = \pm 1 \bmod s\,,$$

which is impossible since we assumed $2 \leq r'' \leq s - 2$ from the beginning. So we must be in the case $(\varepsilon, \varepsilon') = (0, 1)$, which means that

$$r' + 1 \leq i_1 - 1 \leq r - r' - 1\,. \tag{B.31}$$

The equation (B.29) then implies

$$1 = (i_1 - 1)s - rr''\,. \tag{B.32}$$

Now we express the symmetry

$$q_3 C^{(0)}[q_1| - q_3, -q_2] = q_1 C^{(0)}[q_3| - q_1, q_2]\,, \tag{B.33}$$

with the choice $(q_1, q_2, q_3) = (1, s - r'', r'' - 1)$. We can write

$$r'' - 1 = r - r's - \big((i_1 - 1)s - rr''\big) = \big(r - r' - (i_1 - 1)\big)s + r(r'' + 1 - s)\,.$$

Due to the upper bound in (B.31) we have $r - r' - (i_1 - 1) > 0$ therefore $i_3 - 1 = r - r' - (i_1 - 1)$. Notice that $i_1 + i_3 = 2 + r - r' < r + 3$ thus $\epsilon(q_1, q_3, q_2) = \epsilon(q_3, q_1, q_2) = 0$. Using (B.27), the equality (B.33) becomes

$$r' + (i_1 - 1) - r = (r'' - 1)(1 - i_1)\,,$$

that is $r''b = r - r'$. If we use this result when multiplying (B.32) by r'', we find that $r'' = rs - (r - r'') - rr''$. This implies $r'' = s - 1$, but this contradicts the starting assumption $r'' \leq s - 2$. In other words, we have proved that if $r'' \in \{2, \ldots, s - 2\}$, we cannot have the symmetry (B.27) for all $q_1, q_2, q_3 > 0$. □

B.2.3 Computation of $F_{1,1}$

Let us compute $F_{1,1}$ for the quantum Airy structure of Theorem 4.4.9 – that is, assuming $r = \pm 1 \bmod s$.

Lemma B.2.3. *We have* $F_{1,1}[q] = \frac{r^2-1}{24}\,\delta_{q,s}$.

Proof. We need to isolate the term

$$H_k^i = -\hbar\, F_{1,1}[\Pi_s(i,k)] + \cdots$$

where $F_{1,1}[q]$ is a scalar, $\Pi_s(i,k)$ is given in (B.23) and $\cdots$ represent the other monomials which we are not interested in. We recall that H_k^i is obtained from the generators W_k^i given in (4.59) after the shift $J_{-s} \to J_{-s} - 1$. The only contribution to this term comes from $j = 1$ and $p_\ell = -s$ for all $l \in \{3, \dots, i\}$ such that $-s(i-2) = r(k-i+1)$. Since r and s are coprime, such a contribution can only appear for $i = 2$ and $k = 1$, and in that case $\Pi_s(2,1) = s$. So we have

$$F_{1,1}[q] = -\frac{1}{r}\,\Psi^{(1)}(\emptyset)\,\delta_{q,s}\,,$$

which is equal to $\frac{r^2-1}{24}\,\delta_{q,s}$ using Lemma B.1.5 to evaluate $\Psi^{(1)}$. □

B.3 Proof of uniqueness of the solution to the higher abstract loop equations

In this appendix we show that if a solution to the higher abstract loop equations that respects the polarization exists, then it is uniquely constructed by the Bouchard-Eynard topological recursion of [BE13, BHL+14]. The argument follows along similar lines to what was presented in [BS17, BE17]. For clarity we will only present the proof here for local spectral curves with one component, but it is straightforward to generalize it to local spectral curves with ℓ components.

Consider a spectral curve with one component, as defined in Definition 4.5.1. Let

$$\omega_{0,1}(z) = \sum_{l>0} \tau_l\, \mathrm{d}\xi_l(z) = y(z)\mathrm{d}\xi_r(z)\,,$$

where $y(z) = \sum_{l>0}^{\infty} \tau_l z^{l-r}$. Let us assume that the spectral curve is admissible (Definition 4.5.2), that is, $1 \le s \le r+1$ and $r = \pm 1 \bmod s$ with

$$s := \min\left\{ l > 0 \quad | \quad \tau_l \neq 0 \;\text{ and }\; r \nmid l \right\}.$$

Let $\omega_{0,2}(z_1,z_2)$ be the formal bidifferential that encodes the choice of polarization

$$\omega_{0,2}(z_1,z_2)=\frac{\mathrm{d}z_1\otimes\mathrm{d}z_2}{(z_1-z_2)^2}+\sum_{l,m>0}\phi_{l,m}\,\mathrm{d}\xi_l(z_1)\otimes\mathrm{d}\xi_m(z_2)\,.$$

Then the following result holds.

Proposition B.3.1. *Fix an admissible*[1] *local spectral curve with one component. Assume there exists a sequence of formal symmetric differentials $\omega_{g,n}\in\bigotimes_{j=1}^n V_{z_j}^-$ satisfying the higher abstract loop equations*

$$P_{\mathbf{w}|t}\left(\mathcal{E}^{(i)}_{g,n}(\mathbf{w}|\mathbf{z})\right)\in t^{-r\mathfrak{d}^i}\,\mathbb{C}[\![t^r]\!]\otimes V_{z_2}^-\otimes\ldots\otimes V_{z_n}^-\,,$$

for all $g\geq 0$, $n\geq 1$, $2g-2+n>0$, and $i\in\{1,\ldots,r\}$, where $\mathbf{z}=(z_2,\ldots,z_n)$ and $\mathbf{w}=(w_1,\ldots,w_i)$. Here, $\mathfrak{d}^i=i-1-\left\lfloor\frac{s(i-1)}{r}\right\rfloor$ and $\mathcal{E}^{(i)}_{g,n}(A|B)$ was introduced in Definition 4.5.16, and the projection map $P_{\mathbf{w}|t}$ in Definition 4.5.5.

Then, this sequence of formal symmetric differentials is constructed by the Bouchard-Eynard topological recursion formula

$$\begin{aligned}\omega_{g,n}(z_1,\mathbf{z})=\mathrm{Res}_{t=0}&\left(\int_0^t\omega_{0,2}(\cdot,z_1)\right)\sum_{i=1}^{r-1}(-1)^{i+1}\frac{1}{i!}\\ &\times\sum_{\substack{a_1,\ldots,a_i=1\\ a_m\neq a_l}}^{r-1}\left(\prod_{m=1}^{i}\frac{1}{(y(t)-y(\theta^{a_m}t))}\right)\frac{\mathcal{R}^{(i+1)}_{g,n}(t,\theta^{a_1}t,\ldots,\theta^{a_i}t|\mathbf{z})}{(\mathrm{d}\xi_r(t))^k}\,.\end{aligned}\tag{B.34}$$

Proof. Suppose that there exists a sequence of $\omega_{g,n}\in\bigotimes_{j=1}^n V_{z_j}^-$ such that

$$P_{\mathbf{w}|t}\left(\mathcal{E}^{(i)}_{g,n}(\mathbf{w}|\mathbf{z})\right)\in\frac{1}{t^{r\mathfrak{d}^i}}\mathbb{C}[\![t^r]\!]\otimes V_{z_2}^-\otimes\ldots\otimes V_{z_n}^-\,.$$

Let us now argue that the expression

$$\frac{\mathrm{d}\xi_r(t)}{\prod_{m=1}^{r-1}(y(t)-y(\theta^m t))}\sum_{i=1}^{r}(-1)^i(y(t))^{r-i}P_{\mathbf{w}|t}\left(\mathcal{E}^{(i)}_{g,n}(\mathbf{w}|\mathbf{z})\right)\tag{B.35}$$

lives in $\mathbb{C}[\![t]\!]\mathrm{d}t\otimes V_{z_2}^-\otimes\ldots\otimes V_{z_n}^-$. On the one hand, if $y(t)\in\mathbb{C}[\![t]\!]$, then this is clear as long as $y'(0)\neq 0$, that is $\tau_{r+1}\neq 0$. This is the regular case. In the irregular case, let us suppose that $\tau_s\neq 0$ with $s\in\{1,\ldots,r-1\}$. Then

[1] To be precise, in the proof here we only need the requirement that s is coprime with r, we do not need the stronger admissibility requirement that $r=\pm 1\mod s$.

$y(t) \sim \tau_s\, t^{s-r}$ when $t \to 0$. Assume that s is coprime with r. We rewrite $\mathfrak{d}^i = i - 1 - \lfloor \frac{s(i-1)}{r} \rfloor = i - \lceil \frac{1+s(i-1)}{r} \rceil$. We have when $t \to 0$

$$\begin{aligned} \frac{\mathrm{d}\xi_r(t)}{\prod_{m=1}^{r-1}(y(t) - y(\theta^m t))} &\sim \tau_s^{1-r}\, t^{(r-1)(r-s+1)}\mathrm{d}t\,, \\ (y(t))^{r-i} P_{\mathbf{w}|t}\left(\mathcal{E}^{(i)}_{g,n}(\mathbf{w}|\mathbf{z})\right) &\sim \frac{\tau_s^{1-r}\, t^{(r-i)(s-r)}}{t^{r\left(i - \lceil \frac{1+s(i-1)}{r} \rceil\right)}} = \frac{\tau_s^{1-r}}{t^{\delta}}\,, \end{aligned}$$

with

$$\delta = ri - r\left\lceil \frac{1+s(i-1)}{r} \right\rceil - (r-i)(s-r) \leq (r-1)(r-s+1)\,,$$

and the result follows.

We can rewrite this statement as a residue condition

$$\mathrm{Res}_{t=0} \left(\int_0^t \omega_{0,2}(\cdot, z_1) \right) \frac{\mathrm{d}\xi_r(t)}{\prod_{m=1}^{r-1}(y(t) - y(\theta^m t))} \left(\sum_{i=1}^{r} (-1)^i\, y(t)^{r-i}\, P_{\mathbf{w}|t}\Big(\mathcal{E}^{(i)}_{g,n}(\mathbf{w}|\mathbf{z})\Big) \right) = 0\,.$$

Now let us manipulate this expression a little bit, by writing down the projection map explicitly. By definition,

$$\sum_{i=1}^{r} (-1)^i (y(t))^{r-i} P_{\mathbf{w}|t}\left(\mathcal{E}^{(i)}_{g,n}(\mathbf{w}|\mathbf{z})\right) = \sum_{i=1}^{r} \frac{(-1)^i}{i!}\, y(t)^{r-i} \sum_{\substack{a_1,\ldots,a_i=0 \\ a_m \neq a_l}}^{r-1} \frac{\mathcal{E}^{(i)}_{g,n}(\theta^{a_1}t, \ldots, \theta^{a_i}t|\mathbf{z})}{(\mathrm{d}\xi_r(t))^i}\,.$$

We can extract the $\omega_{0,1}$ contributions out of the $\mathcal{E}^{(i)}_{g,n}(\mathbf{w}|\mathbf{z})$ using Lemma 4.5.17.

We get

$$
\begin{aligned}
&\sum_{i=1}^{r}(-1)^i (y(t))^{r-i} P_{\mathbf{w}|t}\left(\mathcal{E}^{(i)}_{g,n}(\mathbf{w}|\mathbf{z})\right) \\
&= \sum_{i=1}^{r}(-1)^i \sum_{\substack{a_1,\dots,a_i=0\\ a_m\neq a_l}}^{r-1} \sum_{j=1}^{i} \frac{1}{j!(i-j)!} y(t)^{r-i} \left(\prod_{l=1}^{i-j} y(\theta^{a_{j+l}}t)\right) \frac{\mathcal{R}^{(j)}_{g,n}(\theta^{a_1}t,\dots,\theta^{a_j}t|\mathbf{z})}{(\mathrm{d}\xi_r(t))^j} \\
&= \sum_{\substack{a_1,\dots,a_r=0\\ a_m\neq a_l}}^{r-1} \sum_{i=1}^{r}(-1)^i \sum_{j=1}^{i} \frac{1}{j!(i-j)!(r-i)!} y(t)^{r-i} \left(\prod_{l=1}^{i-j} y(\theta^{a_{j+l}}t)\right) \frac{\mathcal{R}^{(j)}_{g,n}(\theta^{a_1}t,\dots,\theta^{a_j}t|\mathbf{z})}{(\mathrm{d}\xi_r(t))^j} \\
&= \sum_{\substack{a_1,\dots,a_r=0\\ a_m\neq a_l}}^{r-1} \sum_{\ell=1}^{r} \sum_{m=0}^{r-\ell} \frac{(-1)^{\ell+m}}{\ell!(r-\ell-m)!m!} y(t)^{r-\ell-m} \left(\prod_{l=1}^{m} y(\theta^{a_{\ell+l}}t)\right) \frac{\mathcal{R}^{(\ell)}_{g,n}(\theta^{a_1}t,\dots,\theta^{a_\ell}t|\mathbf{z})}{(\mathrm{d}\xi_r(t))^\ell} \\
&= \sum_{\substack{a_1,\dots,a_r=0\\ a_m\neq a_l}}^{r-1} \sum_{\ell=1}^{r} \frac{(-1)^{\ell}}{\ell!(r-\ell)!} \left(\prod_{l=1}^{r-\ell} (y(t)-y(\theta^{a_{\ell+l}}t))\right) \frac{\mathcal{R}^{(\ell)}_{g,n}(\theta^{a_1}t,\dots,\theta^{a_\ell}t|\mathbf{z})}{(\mathrm{d}\xi_r(t))^\ell} \\
&= \sum_{\substack{a_1,\dots,a_{r-1}=1\\ a_l\neq a_m}}^{r-1} \sum_{\ell=1}^{r} \frac{(-1)^{\ell}}{(\ell-1)!(r-\ell)!} \left(\prod_{l=1}^{r-\ell} (y(t)-y(\theta^{a_{\ell-1+l}}t))\right) \frac{\mathcal{R}^{(\ell)}_{g,n}(t,\theta^{a_1}t,\dots,\theta^{a_{\ell-1}}t|\mathbf{z})}{(\mathrm{d}\xi_r(t))^\ell}\,.
\end{aligned}
$$

Then

$$
\begin{aligned}
&\frac{\mathrm{d}\xi_r(t)}{\prod_{m=1}^{r-1}(y(t)-y(\theta^m t))} \sum_{i=1}^{r}(-1)^i (y(t))^{r-i} P_{\mathbf{w}|t}\left(\mathcal{E}^{(i)}_{g,n}(\mathbf{w}|\mathbf{z})\right) \\
&= \sum_{i=1}^{r} \frac{(-1)^i}{(i-1)!} \sum_{\substack{a_1,\dots,a_{i-1}=1\\ a_l\neq a_m}}^{r-1} \left(\prod_{l=1}^{i-1} \frac{1}{(y(t)-y(\theta^{a_l}t))}\right) \frac{\mathcal{R}^{(i)}_{g,n}(t,\theta^{a_1}t,\dots,\theta^{a_{i-1}}t|\mathbf{z})}{(\mathrm{d}\xi_r(t))^{i-1}}\,.
\end{aligned}
$$

Thus the higher abstract loop equations imply that

$$
\begin{aligned}
\mathrm{Res}_{t=0}\left(\int_0^t \omega_{0,2}(\cdot,z_1)\right) \sum_{i=1}^{r} \frac{(-1)^i}{(i-1)!} \sum_{\substack{a_1,\dots,a_{i-1}=1\\ a_l\neq a_m}}^{r-1} \left(\prod_{l=1}^{i-1} \frac{1}{(y(t)-y(\theta^{a_l}t))}\right)& \\
\times \frac{\mathcal{R}^{(i)}_{g,n}(t,\theta^{a_1}t,\dots,\theta^{a_{i-1}}t|\mathbf{z})}{(\mathrm{d}\xi_r(t))^{i-1}} &= 0\,.
\end{aligned}
$$

Now we can take out the term with $i=1$, which is equal to

$$
-\mathrm{Res}_{t=0}\left(\int_0^t \omega_{0,2}(\cdot,z_1)\right)\omega_{g,n}(t,\mathbf{z})\,.
$$

Assuming that $\omega_{g,n}\in\otimes_{l=1}^n V^-_{z_l}$, the residue simply replaces the terms with $\xi_{-l}(z)$ with the same terms but with $\xi_{-l}(z_1)$. Thus

$$
-\mathrm{Res}_{t=0}\left(\int_0^t \omega_{0,2}(\cdot,z_1)\right)\omega_{g,n}(t,\mathbf{z}) = -\omega_{g,n}(z_1,\mathbf{z})\,,
$$

and we get the topological recursion

$$\omega_{g,n}(z_1, \mathbf{z}) = \mathrm{Res}_{t=0} \left(\int_0^t \omega_{0,2}(\cdot, z_1) \right) \sum_{i=1}^{r-1} \frac{(-1)^{i+1}}{i!} \\ \times \sum_{\substack{a_1,\dots,a_i=1 \\ a_l \neq a_m}}^{r-1} \left(\prod_{l=1}^{i} \frac{1}{(y(t) - y(\theta^{a_l} t))} \right) \frac{\mathcal{R}_{g,n}^{(i+1)}(t, \theta^{a_1} t, \dots, \theta^{a_i} t | \mathbf{z})}{(\mathrm{d}\xi_r(t))^i},$$

which uniquely determines the $\omega_{g,n}$. □

Remark B.3.2. To prove existence of a (polarized) solution to the higher abstract loop equations, one could follow the proof above step-by-step in reverse. However, for this to work one would need to prove initially that the Bouchard-Eynard topological recursion produces symmetric $\omega_{g,n}$. This is known from [BE13] for spectral curves that appear as a limit of a family of curves with simple ramification points, by an indirect argument. It is however not clear to us which spectral curves satisfy this condition. Since in Section 4.5 we identify the higher abstract loop equations with higher quantum Airy structures for admissible spectral curves, it implies via Theorem 4.2.11 that the solution to the higher loop equations exists, in the case when $1 \leq s \leq r+1$ and $r = \pm 1 \mod s$. Hence we have a new, more direct (and perhaps more general) proof that the Bouchard-Eynard topological recursion produces symmetric $\omega_{g,n}$ for all admissible spectral curves, independently of the arguments of [BE13], as stated in Theorem 4.5.32.