LOCALLY PIECEWISE AFFINE FUNCTIONS

by

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Abstract

Piecewise affine functions as defined in [1] and denoted by the set S are those functions in $C(\mathbb{R}^m)$ that agree with a finite number of affine functions. In this thesis, we extend their study by introducing the set of locally piecewise affine functions denoted by S_{lp}. Unlike piecewise affine functions, a locally piecewise affine function could possibly agree with an infinite number of affine functions on $\mathbb{R}^m$. We discuss the relationship between the two sets under the umbrella of order theory. In order to define the set of locally piecewise affine functions we first define piecewise affine functions on arbitrary subsets of $\mathbb{R}^m$ and discuss the conditions that guarantee the natural extension of a piecewise affine function on arbitrary sets to a piecewise affine function on the whole space. We then define the set of locally piecewise affine functions and discuss how the properties of piecewise affine functions that have been studied previously [1] can be extended to the new set.

The literature of vector lattices contains the study of the equivalence or lack thereof of three main definitions for order convergence. However, this problem has not been studied in $C(\mathbb{R}^m)$. In this thesis we utilize the results by Anderson and Mathews [2] to study this problem. In doing so, we investigate if $C(\mathbb{R}^m)$ possesses the countable sup property which allows us to show that for bounded nets, two main definitions of order convergence in the literature coincide.

We also study S and S_{lp} as sublattices of $C(\mathbb{R}^m)$ and we show that both S and subsequently S_{lp} are order dense minorizing sublattices in $C(\mathbb{R}^m)$. We then study the relationship betwen S and S_{lp} by introducing the definition of locally finite sets of functions. This definition allows us to show that any locally finite set of functions in S has a supremum and an infimum both of which are in S_{lp}. In addition, we show that any function in S_{lp} can be expressed as the difference of the supremums of two locally finite sequences of functions in S.

The Stone-Weierstrass theorem can be directly applied to show that piecewise affine functions can uniformly approximate continuous functions on compact sets. However, piecewise affine functions cannot be used to uniformly approximate functions in $C(\mathbb{R}^m)$. In this thesis, we show that the set of locally piecewise affine functions can be used to uniformly approximate continuous functions in $C(\mathbb{R}^m)$.

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Chapter 1

Introduction

1.1 Thesis Objective and Outline

The concepts of ordered vector spaces and lattices have been shown to be important to many applications in engineering and economics. In particular, multivariate piecewise affine functions in the context of order vector spaces have been shown to be important tools in economic theory [3]. Aliprantis and Tourky [1] present a comprehensive study of the space of multivariate piecewise affine functions in $C(\mathbb{R}^m)$ in which they show that the subspace of piecewise affine function is equivalent to the sublattice generated by the affine functions. Their work, however, is limited to multivariate piecewise affine functions with finite number of components. While multivariate piecewise affine functions with finite number of components can uniformly approximate any function in $C(K)$ where K is a compact subset of $\mathbb{R}^m$, however, such functions cannot be used to uniformly approximate many functions in $C(\mathbb{R}^m)$.

The objective of this thesis is to extend the work by Aliprantis and Tourky [1] to define a new class of multivariate piecewise affine functions with infinite components that agree with the multivariate piecewise affine functions defined by Aliprantis and Tourky [1] on compact sets. We call these functions locally piecewise affine functions. We then study the relationship between the two classes of functions in the context of order convergence.

In the first chapter of this thesis, we present the basic structure of partially ordered sets followed by the structure of ordered vector spaces and lattices. We only present the tools needed for our analysis of the space $C(\mathbb{R}^m)$.

In the second chapter of this thesis, we introduce the sets of one dimensional and multivariate piecewise affine functions as presented in Chapter 7 of the book by Aliprantis and Tourky [1].

In the third chapter of this thesis, we introduce the set of locally piecewise affine functions. In addition we present sufficient conditions on a subset $U \subset \mathbb{R}^m$ that guarantees that a piecewise affine function on U can be naturally extended to a piecewise affine function in $C(\mathbb{R}^m)$. This result is useful on its own and in the study of the relationship between the piecewise affine functions and the locally piecewise affine functions. We then discuss the properties of the locally piecewise affine functions and show that in most cases, they are similar to the properties of piecewise affine functions.

In the fourth chapter of this thesis, we study how piecewise affine functions and locally piecewise affine functions approximate functions in $C(\mathbb{R}^m)$. In doing so, we first present the various definitions of order convergence in partially ordered sets and vector lattices and study their agreement or lack theroff on the lattice $C(\mathbb{R}^m)$. We then show our original result that two definitions of order convergence agree on $C(\mathbb{R}^m)$. We then present three different definitions for "order dense" subspaces and study under these definitions how piecewise affine functions and locally piecewise affine function approximate $C(\mathbb{R}^m)$. We then present a new class of sets that we call: "locally finite sets of functions". Using this definition we show that a locally piecewise affine function can be written as the difference between the supremums of two locally finite sequences of piecewise affine functions. Finally, we present our original result that locally piecewise affine functions uniformly approximate functions in $C(\mathbb{R}^m)$.

1.2 Order Structure

In this section we present the traditional definitions of order structure and ordered sets. We follow a convention similar to that presented in Aliprantis and Tourky [1].

Definition 1.2.1: **Partial Order Relation:** Let S be a set. A relation "$\leq$" is called a partial order if it satisfies the following three properties:

1. Reflexivity: $\forall x \in S : x \leq x$.

2. Antisymmetry: $\forall x, y \in S : x \leq y$ and $y \leq x \Rightarrow x = y$
3. Transitivity: $\forall x, y, z \in S : x \leq y$ and $y \leq z \Rightarrow x \leq z$

In this case, S is called a partially ordered set. If $\forall x, y \in S : x \leq y$ or $y \leq x$, then the relationship is called a total order and S is called a totally ordered set. We write

- $x \geq y$ if $y \leq x$
- $x < y$ if $x \leq y$ and $x \neq y$
- $y > x$ if $x < y$

Definition 1.2.2: : **Directed Set:** A partially (or totally) ordered set Γ is called an upward-directed set if $\forall \alpha, \beta \in \Gamma : \exists \gamma \in \Gamma$ such that $\alpha \leq \gamma$ and $\beta \leq \gamma$. Similarly, Γ is called a downward-directed set if $\forall \alpha, \beta \in \Gamma : \exists \gamma \in \Gamma$ such that $\alpha \geq \gamma$ and $\beta \geq \gamma$. For example: $\mathbb{N}$ with the natural order is an (upward- and downward-) directed set. Directed sets, can be used to generalize the concept of sequences as follows:

Definition 1.2.3: **Net:** Let Γ be an upward-directed set, X be another set. A function $x : \Gamma \to X$ is called a net and is written: $\{x_\alpha\}_{\alpha \in \Gamma}$ or just $\{x_\alpha\}$. If X is endowed with a partial (or total) order, then a net is increasing if $\forall \alpha, \beta, \gamma \in \Gamma$ such that $\alpha \leq \gamma$ and $\beta \leq \gamma$ then $x_\alpha \leq x_\gamma$ and $x_\beta \leq x_\gamma$. We write $x_\alpha \uparrow$ to indicate an increasing net. Similarly, a net is decreasing if $\forall \alpha, \beta, \gamma \in \Gamma$ such that $\alpha \leq \gamma$ and $\beta \leq \gamma$ then $x_\alpha \geq x_\gamma$ and $x_\beta \geq x_\gamma$. We write $x_\alpha \downarrow$ to indicate a decreasing net. For example, a sequence $\{x_n\}_{n=1}^\infty \subset \mathbb{R}$ is a net $x : \mathbb{N} \to \mathbb{R}$.

Definition 1.2.4: **Supremum and Infimum:** Let S be a partially ordered set. Let $A \subset S$. If $\exists a \in S$ such that $\forall x \in A : x \leq a$, then a is an upper bound of A. If $a \in A$ then, a is also the greatest element of A. Simiarly, if $\exists b \in S$ such that $\forall x \in A : x \geq b$, then b is a lower bound of A. If $b \in A$ then, b is also the least element of A. Let $B \subset S$. If the set of upper bounds of B has a least element, then this least element is called the supremum (or the least upper bound) of B. If it exists, we write $\sup B$ to indicate the supremum of B. Similarly, if the set of lower bounds of B has a greatest element, then this greatest element is called the infimum (or the greatest lower bound) of B. If it exists, we write $\inf B$ to indicate the infimum of B.

If $x, y \in S$, we write $x \vee y$ to indicate $\sup\{x, y\}$ and $x \wedge y$ to indicate $\inf\{x, y\}$ if they exist.

If an increasing net $\{x_\alpha\}$ has a supremum a, then we write: $x_\alpha \uparrow a$. Similarly, if a decreasing net $\{x_\alpha\}$ has an infimum b, then we write: $x_\alpha \downarrow b$.

Definition 1.2.5: **Lattice:** A partially ordered set S is called a lattice if $\forall x, y \in S : x \vee y$ and $x \wedge y$ exist. If $A \subset S$, we define $A^\wedge = \{x_1 \wedge x_2 \wedge \cdots \wedge x_n | n \in \mathbb{N}, \forall i \leq n : x_i \in A\}$, $A^\vee = \{x_1 \vee x_2 \vee \cdots \vee x_n | n \in \mathbb{N}, \forall i \leq n : x_i \in A\}$. It should be noted that $A^{\vee\wedge} \neq A^{\wedge\vee}$. For example, consider the lattice $S = \{a, b, x, y, z, e, f\}$ shown in **Figure** 1.1. A straight arrow between two elements indicate that they are related and define the order. For example, $a < f$, a is not related to b, and $a \vee b = f$. Let $A = \{a, b, d, e\}$. Then: $A^{\vee\wedge} = \{a, b, c, d, e, f, g, h, z\}$ while $A^{\wedge\vee} = \{a, b, d, e, f, g, h, z\}$.

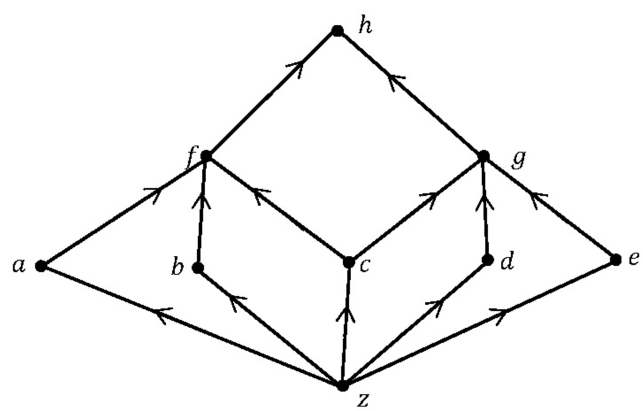

Figure 1.1: $S = \{a, b, c, d, e, f, g, h, z\}$ is a lattice, $A = \{a, b, d, e\}$. $A^{\vee\wedge} \neq A^{\wedge\vee}$

Definition 1.2.6: **Order Completeness:** A partially ordered set S is called order complete, if every non-empty bounded above subset of S has a supremum. This is equivalent to the statement that every non-empty bounded below subset of S has an infimum. To see this consider an order complete partially ordered set S. Let $T \subset S$ be a bounded below set. Consider the set $T^\downarrow$ of lower bounds of T, namely $T^\downarrow = \{t | t \leq T\}$. $T^\downarrow$ is bounded above, therefore, it has a supremum, say a. Therefore, a is the least element in the set of upper bounds of $T^\downarrow$ which is denoted by $T^{\downarrow\uparrow}$. But $T \subset T^{\downarrow\uparrow}$ therefore, $a \leq T$. Therefore, $a \in T^\downarrow$ is the greatest element. Therefore, $\inf T = a$.

In this thesis we are dealing with piecewise affine functions as subsets of the space of continuous functions $C(\mathbb{R}^n)$ which is NOT order complete. To see this consider the example of the sequence of functions $\{f_n\}_{n=1}^{\infty} \subset C(\mathbb{R})$ defined as:

$$f_n(t) = \begin{cases} 0, & t \leq 0 \\ \sqrt[n]{t}, & 0 \leq t \leq 1 \\ 1, & 1 \leq t \end{cases}$$

Clearly, f_n is increasing, however, on the positive part of $\mathbb{R}$: f_n is bounded above by 1 while it is bounded above by 0 on the negative part of $\mathbb{R}$. There is no continuous function that would be the least upper bound of f_n.

1.3 Vector Lattices

In this section we introduce the structure of vector lattices along with some important vector lattices operations and properties.

Definition 1.3.1: **Ordered Vector Space:** A vector space E is said to be an ordered vector space if it is a partially (or totally) ordered set such that

- $\forall x, y, z \in E : x \leq y \Rightarrow x + z \leq y + z$.
- $\forall x, y \in E$ and $\forall \lambda \in \mathbb{R}^+ : x \leq y \Rightarrow \lambda x \leq \lambda y$.

Definition 1.3.2: **Vector Lattice and its Subspaces:** An ordered vector space E that is also a lattice is called a vector lattice. $A \subset E$ is a lattice subspace if it is a vector subspace and a lattice under the order induced from E. A is a sublattice if it is closed under the lattice operations.

Definition 1.3.3: **Archimedean Vector Lattice:** A vector lattice E is Archimedean if $\forall x \in E$ then: the set $\{nx | n \in \mathbb{N}\}$ is bounded above implies $x \leq 0$. An equivalent definition is that E is Archimedean if $\forall x \in E^+$: if $\exists y \in E^+$ such that $\forall n \in \mathbb{N} : nx \leq y$ then $x = 0$. Roughly speaking the Archimedean property is that of having no infinitely large or infinitely small elements.

Proposition 1.3.4: *Let E be a vector lattice, then we have the following identities $\forall x, y, z \in E, \forall \lambda \in \mathbb{R}^+$:*

1. $-(x \wedge y) = (-x) \vee (-y)$ *and* $-(x \vee y) = (-x) \wedge (-y)$.
2. $x + y = x \vee y + x \wedge y$.
3. $x + y \wedge z = (x + y) \wedge (x + z)$ *and* $x + y \vee z = (x + y) \vee (x + z)$.
4. $\lambda(x \vee y) = (\lambda x) \vee (\lambda y)$ *and* $\lambda(x \wedge y) = (\lambda x) \wedge (\lambda y)$.
5. $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ *and* $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$.

Proof. The first identity can be shown as follows: $x \wedge y \leq x \Rightarrow -x \leq -(x \wedge y)$. Similarly, $x \wedge y \leq y \Rightarrow -y \leq -(x \wedge y)$. Therefore, $-(x \wedge y)$ is an upper bound for both $-x$ and $-y$. Therefore, $(-x) \vee (-y) \leq -(x \wedge y)$.

To show the opposite inequality: $-x \leq (-x) \vee (-y) \Rightarrow -((-x) \vee (-y)) \leq x$ and $-y \leq (-x) \vee (-y) \Rightarrow -((-x) \vee (-y)) \leq y$. Therefore, $-((-x) \vee (-y))$ is a lower bound for both x and y. Therefore, $-((-x) \vee (-y)) \leq x \wedge y \Rightarrow -(x \wedge y) \leq (-x) \vee (-y)$. Therefore, $-(x \wedge y) = (-x) \vee (-y)$. The identity $-(x \vee y) = (-x) \wedge (-y)$ can be shown similarly.

The third identity can be shown as follows: $y \wedge z \leq y \Rightarrow x + y \wedge z \leq x + y$ and $y \wedge z \leq z \Rightarrow x + y \wedge z \leq x + z$. Therefore, $x + y \wedge z \leq (x+y) \wedge (x+z)$. For the opposite inequality: $(x+y) \wedge (x+z) \leq x+y \Rightarrow -x + (x+y) \wedge (x+z) \leq y$ and $(x + y) \wedge (x + z) \leq x + z \Rightarrow -x + (x + y) \wedge (x + z) \leq z$. Therefore, $-x + (x + y) \wedge (x + z) \leq y \wedge z \Rightarrow (x + y) \wedge (x + z) \leq x + y \wedge z$. Therefore, $x + y \wedge z = (x + y) \wedge (x + z)$. The identity $x + y \vee z = (x + y) \vee (x + z)$ can be shown similarly.

The second identity can be shown using both the first and the third as follows: Using the third identity we have: $-x + (x \wedge y) = (-x + x) \wedge (-x + y) = 0 \wedge (y - x)$. Using the first and then the third identity we have: $y - (x \vee y) = y + (-x) \wedge (-y) = (y - x) \wedge (y - y) = (y - x) \wedge 0$. Therefore, $-x + (x \wedge y) = y - (x \vee y)$. Therefore, $x + y = x \vee y + x \wedge y$.

The fourth identity is straightforward: $\lambda(x \wedge y) \leq \lambda x$ and $\lambda(x \wedge y) \leq \lambda y$. Therefore, $\lambda(x \wedge y) \leq (\lambda x) \wedge (\lambda y)$. For the opposite inequality we have: $(\lambda x) \wedge (\lambda y) \leq \lambda x \Rightarrow \frac{1}{\lambda}((\lambda x) \wedge (\lambda y)) \leq x$ and similarly, $\frac{1}{\lambda}((\lambda x) \wedge (\lambda y)) \leq y$. Therefore, $\frac{1}{\lambda}((\lambda x) \wedge (\lambda y)) \leq x \wedge y \Rightarrow (\lambda x) \wedge (\lambda y) \leq \lambda(x \wedge y)$. Therefore, $\lambda(x \wedge y) = (\lambda x) \wedge (\lambda y)$. The identity $\lambda(x \vee y) = (\lambda x) \vee (\lambda y)$ can be shown similarly.

For the fifth identity, one inequality is true for every lattice as follows:

$x \wedge y \leq x$ and $x \wedge y \leq y \leq y \vee z$. Similarly, $x \wedge z \leq x$ and $x \wedge z \leq z \leq y \vee z$. Therefore, $(x \wedge y) \vee (x \wedge z) \leq x \wedge (y \vee z)$. The opposite inequality is true only for vector lattices: Using the second identity above $y = x \vee y + x \wedge y - x \leq x \vee y \vee z + (x \wedge y) \vee (x \wedge z) - x$ and $z = x \vee z + x \wedge z - x \leq x \vee y \vee z + (x \wedge y) \vee (x \wedge z) - x$. Therefore, $y \vee z \leq x \vee y \vee z + (x \wedge y) \vee (x \wedge z) - x \Rightarrow x + y \vee z - x \vee y \vee z \leq (x \wedge y) \vee (x \wedge z) \Rightarrow x \wedge (y \vee z) \leq (x \wedge y) \vee (x \wedge z)$. Therefore, $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$. The other identity $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$ can be shown similarly. ■

Proposition 1.3.5: *Let E be a vector lattice, $A \subset E$, and $x \in E$. Therefore,*

1. $x + \sup A = \sup\{x + a | a \in A\}$ *and* $x + \inf A = \inf\{x + a | a \in A\}$.
2. $x \wedge \sup A = \sup\{x \wedge a | a \in A\}$ *and* $x \vee \inf A = \inf\{x \vee a | a \in A\}$.

In the first identity, the existence of the supremum or infimum on either side, guarantees the existence of the supremum or infimum on the opposite side. In the second identity, the existence of the supremum or infimum on the left side guarantees the existence of the supremum or infimum on the right side.

Proof. For the first identity, assume $a_0 = \sup A$ exists. Therefore, $\forall x \in E, \forall a \in A : a \leq a_0 \Rightarrow x + a \leq x + a_0$. Therefore, $x + a_0$ is an upper bound for $\{x + a | a \in A\}$. To show that it is the least upper bound, assume t to be another bound, i.e., $\forall a \in A : t \geq a + x$. Therefore, $t - x \geq a \Rightarrow t - x \geq a_0 \Rightarrow t \geq x + a_0$. Therefore, $x + a_0 = \sup\{x + a | a \in A\}$. Now assume $a_1 = \sup\{x + a | a \in A\}$ exists. Therefore, $\forall x \in E$, $\forall a \in A : x + a \leq a_1 \Rightarrow a \leq a_1 - x$. Therefore, $a_1 - x$ is an upper bound for A. To show that it is the least upper bound, let t to be another bound, i.e., $\forall a \in A : t \geq a$. Therefore, $t + x \geq a + x \Rightarrow t + x \geq a_1 \Rightarrow t \geq a_1 - x$. Therefore, $x + \sup A = a_1$. The same proof applies to the identity: $x + \inf A = \inf\{x + a | a \in A\}$.

The second identity can be shown similarly. Assume $a_0 = \sup A$ exists. Then, $\forall x \in E, \forall a \in A : x \wedge a \leq x \wedge a_0$. Therefore, $x \wedge a_0$ is an upper bound for $\{x \wedge a | a \in A\}$. To show that it is the least upper bound, let t be another upper bound. I.e., $\forall a \in A : t \geq x \wedge a = x + a - x \vee a \geq -x \vee a_0 + x + a \Rightarrow t + x \vee a_0 - x \geq a \Rightarrow t + x \vee a_0 - x \geq a_0 \Rightarrow t \geq x + a_0 - x \vee a_0 = x \wedge a_0$. Therefore, $x \wedge a_0 = \sup\{x \wedge a | a \in A\}$. It is worth mentioning that the existence of $\sup\{x \wedge a | a \in A\}$ does not guarantee the existence of $\sup A$.

For example, considering $\mathbb{R}^2$ with its conventional order. Let $x = (1, 0)$ and $A = \{(0, n) | n \in \mathbb{N} \setminus \{0\}\}$. Then, $\sup\{x \wedge a | a \in A\} = (0, 0)$ while $\sup A$ does not exist. The same proof and argument apply to the identity: $x \vee \inf A = \inf\{x \vee a | a \in A\}$. ■

Proposition 1.3.6: *Let E be a vector lattice, $A \subset E$ is a subspace. Then, $A^{\vee\wedge} = A^{\wedge\vee}$. $A^{\vee\wedge}$ is the sublattice generated by A, namely the smallest sublattice containing A.*

Proof. Clearly, $A \subset A^{\vee}$. Therefore, $A^{\wedge} \subset A^{\vee\wedge}$. Let $y \in A^{\wedge\vee}$. Therefore, $\exists m \in \mathbb{N}$ and $\forall i \leq m : \exists n_i \in \mathbb{N}$ such that

$$\begin{aligned} y =& (a_{11} \wedge a_{12} \wedge \cdots \wedge a_{1n_1}) \vee (a_{21} \wedge a_{22} \wedge \cdots \wedge a_{2n_2}) \vee \cdots \\ & \vee (a_{m1} \wedge a_{m2} \wedge \cdots \wedge a_{mn_m}) \\ =& \bigvee_{i=1}^{m} \left(\bigwedge_{j=1}^{n_i} a_{ij} \right) \end{aligned}$$

The identities in **Proposition** 1.3.4 can be used to show that

$$\begin{aligned} y =& (a_{11} \vee a_{21} \vee \cdots \vee a_{m1}) \wedge (a_{11} \vee a_{21} \vee \cdots \vee a_{m2}) \wedge \cdots \wedge (a_{1n_1} \vee a_{2n_2} \vee \cdots \\ & \vee a_{mn_m}) \\ =& \bigwedge_{t_1=1}^{n_1} \bigwedge_{t_2=1}^{n_2} \cdots \bigwedge_{t_m=1}^{n_m} (a_{1t_1} \vee a_{2t_2} \vee \cdots \vee a_{mt_m}) \end{aligned}$$

Therefore, $y \in A^{\vee\wedge}$ and $A^{\wedge\vee} \subset A^{\vee\wedge}$ Similarly, $A^{\vee\wedge} \subset A^{\wedge\vee}$. Therefore, $A^{\vee\wedge} = A^{\wedge\vee}$.

Let $B \subset E$ be a sublattice such that $A \subset B$. Clearly, $\forall y \in A^{\vee\wedge} : y \in B$. Therefore, $A^{\vee\wedge} \subset B$. Therefore, $A^{\vee\wedge}$ is the smallest sublattice containing A. ■

Definition 1.3.7: **Positive Part, Negative Part and Modulus:** Let E be a vector lattice and $x \in E$. Then, the positive part, negative part, and modulus of x respectively are: $x^+ = x \vee 0$, $x^- = (-x) \vee 0$, and $|x| = x \vee (-x)$. **Proposition** 1.3.4 and **Definition** 1.3.7 can be used to show that the operations $\forall x, y \in E : x \vee y, x \wedge y, |x|, x^+, x^-$ can be expressed via each other. Therefore, it is enough to show that one operation is well defined to show that an ordered vector space is a lattice. Notice as well that $x = x + 0 = x \vee 0 + x \wedge 0 = x^+ - (-x) \vee 0 = x^+ - x^-$.

The vector space $C(\mathbb{R}^m)$ is an ordered set with the partial order defined as $\forall f, g \in C(\mathbb{R}^m) : f \leq g \Leftrightarrow \forall x \in \mathbb{R}^m : f(x) \leq g(x)$. Clearly, this order satisfies the properties of ordered vector spaces. In addition, $C(\mathbb{R}^m)$ is a lattice since the function:

$$g(x) = \max\{f(x), 0\}$$

satisfies $g = f \vee 0$ and $g \in C(\mathbb{R}^m)$. In addition, $C(\mathbb{R}^m)$ is Archimedean since if $h, k \in C(\mathbb{R}^m)^+$ are such that $\forall n \in \mathbb{N} : nh \leq k$, then $\forall x \in \mathbb{R}^m$: $nh(x) \leq k(x)$. Since $h(x)$ and $k(x)$ are positive real numbers, then $h(x) = 0$ therefore, $h \equiv 0$.

Chapter 2

Piecewise Affine Functions

This chapter is based on Chapter 7 of the book by Aliprantis and Tourky [1].

2.1 One Dimensional Piecewise Affine Functions

Definition 2.1.1: **Affine Functions:** $f : \mathbb{R} \to \mathbb{R}$ is called an affine function if $\exists b, m \in \mathbb{R}$ such that $\forall t \in \mathbb{R} : f(t) = mt + b$.

Definition 2.1.2: **Piecewise Affine Functions:** $f : \mathbb{R} \to \mathbb{R}$ is called a piecewise affine function if $\exists \{a_i\}_{i=0}^{n} \subset \mathbb{R}$, $\{(m_i, b_i)\}_{i=0}^{n+1} \subset \mathbb{R}^2$, with $n \in \mathbb{N}$, and $\forall 0 < i \leq n : a_{i-1} < a_i$ such that:

$$f(t) = \begin{cases} m_0 t + b_0, & t \leq a_0 \\ m_i t + b_i, & a_{i-1} \leq t \leq a_i \\ m_{n+1} t + b_{n+1}, & a_n \leq t \end{cases}$$

The sets of parameters $\{a_i\}_{i=0}^{n}$ and $\{(m_i, b_i)\}_{i=0}^{n+1}$ are called the **representation** of f while the functions $\{f_i(t) = m_i t + b_i\}_{i=0}^{n+1}$ are called the **components** of f.

Similarly, the function $f : [a_0, a_n] \to \mathbb{R}$ is called a piecewise affine function if $\exists \{a_i\}_{i=0}^{n} \subset \mathbb{R}$, $\{(m_i, b_i)\}_{i=1}^{n} \subset \mathbb{R}^2$, with $n \in \mathbb{N}$, and $\forall 0 < i \leq n : a_{i-1} < a_i$

such that:

$$f(t) = m_i t + b_i, \qquad a_{i-1} \leq t \leq a_{i+1}$$

Remarks:

- The way picewise affine functions were defined ensures their continuity.
- The spaces of piecewise affine functions defined as:

$$S_{[a,b]} = \{f : [a,b] \to \mathbb{R} | f \text{ is a piecewise affine function}\}$$
$$S = \{f : \mathbb{R} \to \mathbb{R} | f \text{ is a piecewise affine function}\}$$

  are linear vector spaces and $S_{[a,b]} \subset C[a,b]$ where $C[a,b]$ is the space of continuous functions on the interval $[a,b]$. Similarly, $S \subset C(\mathbb{R})$.
- S and $S_{[a,b]}$ are vector lattices. This can directly deduced from the fact that if we have $f_1(t) = m_1 t + b_1$ and $f_2(t) = m_2 t + b_2$ defined on the interval $[a_{i-1}, a_i]$, then $f_1 \wedge f_2$ is a piecewise affine function defined on the same interval.
- Let $V = \{g : \mathbb{R} \to \mathbb{R} | g \text{ is an affine function}\}^{\wedge\vee}$ and $V_{[a,b]} = \{g : [a,b] \to \mathbb{R} | g \text{ is an affine function}\}^{\wedge\vee}$. Clearly, $V \subset S$ and $V_{[a,b]} \subset S_{[a,b]}$. We will later show that $V = S$ and $V_{[a,b]} = S_{[a,b]}$.
- When it is clear from the context, we will use the symbol S to represent both S and $S_{[a,b]}$ and similarly will be the case for the symbol V.

Lemma 2.1.3: *If $f : \mathbb{R} \to \mathbb{R}$ is a piecewise affine function, then $\forall a \leq b \in \mathbb{R}$ the restriction $f|_{[a,b]}$ is a piecewise affine function. Also, if $f : [a,b] \to \mathbb{R}$ is a piecewise affine function, then it can be extended (in many ways) to a piecewise affine function $f : \mathbb{R} \to \mathbb{R}$.*

Proof. This is straightforward. See **Lemma 3.1.3**. ■

Lemma 2.1.4: *Real functions on compact sets can be uniformly approximated with piecewise affine functions. Namely, S is uniformly dense in $C[a,b]$.*

Proof. Since the unit function $f \equiv 1$ and the function $g(t) = t$ which separates points in $[a,b]$ are both elements of S, then by the lattice version of the Stone-Weierstrass approximation theorem (see **Theorem** 11.3 page 88 in [4]), S is uniformly dense $C[a,b]$.

We can also show this directly using the uniform continuity of functions in $C[a,b]$. Let $f \in C[a,b]$. Fix ε, then $\exists \delta$ such that $\forall x \in [a,b]$, $\forall y \in B(x,\delta)$ we have $f(y) \in B(f(x),\varepsilon)$. Divide the interval $[a,b]$ into n subintervals with end points $a = a_0 < a_1 < \cdots < a_n = b$ such that $\forall 0 < i \leq n : a_i - a_{i-1} < \delta$. Let $f_i(t) = f(a_{i-1}) + \frac{f(a_i)-f(a_{i-1})}{a_i - a_{i-1}}(t - a_{i-1})$. Let

$$g(t) = f_i(t) \qquad a_{i-1} < t \leq a_i$$

Clearly, $g \in S$. Let $t \in [a_{i-1}, a_i]$, therefore, $|g(t) - f(t)| = |f_i(t) - f(t)| \leq |f_i(t) - f_i(a_{i-1})| + |f_i(a_{i-1}) - f(t)| \leq |f_i(a_i) - f_i(a_{i-1})| + |f_i(a_{i-1}) - f(t)| \leq \varepsilon + \varepsilon = 2\varepsilon$. Therefore, $\sup_t |g(t) - f(t)| \leq 2\varepsilon$. ■

Lemma 2.1.5: *$S_{[a,b]} \subset V_{[a,b]}$. Moreover, $\forall f \in S_{[a,b]} : f(t) = c + \sum_{i=1}^n c_i(f_i(t) \vee 0) = c + \sum_{i=1}^n c_i(t - a_{i-1})^+$ where $\forall i \leq n : c_i, c \in \mathbb{R}$, $a = a_0 < a_1 < \cdots < a_n = b$, and $f_i(t) = t - a_{i-1} \in V$.*

Proof. Fix $f \in S_{[a,b]}$ and let $\{a_i\}_{i=0}^n$ and $\{(m_i, b_i)\}_{i=1}^n$ be the representation of f as in **Definition** 2.1.2. Set $c = b_1 + m_1 a_0, c_1 = m_1, \forall i > 1 : c_i = m_i - m_{i-1}$. Let

$$g(t) = c + \sum_{i=1}^{n} c_i(t - a_{i-1})^+$$

Note that by the continuity of f we have: $m_i a_i + b_i = m_{i+1} a_i + b_{i+1}$. Then: For $a_0 \leq t \leq a_1$:

$$g(t) = b_1 + m_1 a_0 + m_1(t - a_0) = b_1 + m_1 t = f(t)$$

For $a_{i-1} \leq t \leq a_i$:

$$\begin{aligned} g(t) &= b_1 + m_1 a_0 + \sum_{j=1}^{j=i} c_j (t - a_{j-1})^+ \\ &= b_1 + m_1 t + \sum_{j=2}^{j=i} (m_j - m_{j-1})(t - a_{j-1}) \\ &= b_1 + m_i t + \sum_{j=2}^{j=i} (m_j - m_{j-1})(-a_{j-1}) \\ &= b_1 + m_i t + \sum_{j=2}^{j=i} (b_j - b_{j-1}) \\ &= b_i + m_i t = f(t) \end{aligned}$$

Therefore, $g \equiv f$. This shows that $f \in S_{[a,b]}$ has the representation $f(t) = c + \sum_{i=1}^{n} c_i (t - a_{i-1})^+$. Therefore, f can be written as the finite sum of sups of functions in $V_{[a,b]}$, i.e., $f \in V_{[a,b]}$. Therefore, $S_{[a,b]} \subset V_{[a,b]}$. ■

Lemma 2.1.6: *$S \subset V$. Moreover, $\forall f \in S : f(t) = b_0 + m_0 t + \sum_{i=1}^{n+1} c_i (f_i(t) \vee 0) = b_0 + m_0 t + \sum_{i=1}^{n+1} c_i (t - a_{i-1})^+$ where $\forall 0 < i \leq n+1 : c_i, b_0, m_0 \in \mathbb{R}$, $a_0 < a_1 < \cdots a_n \in \mathbb{R}$, and $f_i(t) = t - a_{i-1} \in V$.*

Proof. Fix $f \in S$ and let $\{a_i\}_{i=0}^{n}$ and $\{(m_i, b_i)\}_{i=0}^{n+1}$ be its representation as in **Definition** 2.1.2. Set $\forall i \geq 1 : c_i = m_i - m_{i-1}$. Let

$$g(t) = b_0 + m_0 t + \sum_{i=1}^{n+1} c_i (t - a_{i-1})^+$$

Note that by the continuity of f we have: $m_i a_i + b_i = m_{i+1} a_i + b_{i+1}$. Then: For $t \leq a_0$:

$$g(t) = b_0 + m_0 t = f(t)$$

For $a_{i-1} \leq t \leq a_i$:

$$\begin{aligned} g(t) &= b_0 + m_0 t + \sum_{j=1}^{j=i} c_j (t - a_{j-1})^+ \\ &= b_0 + m_0 t + \sum_{j=1}^{j=i} (m_j - m_{j-1})(t - a_{j-1}) \\ &= b_0 + m_i t + \sum_{j=1}^{j=i} (m_j - m_{j-1})(-a_{j-1}) \\ &= b_0 + m_i t + \sum_{j=1}^{j=i} (b_j - b_{j-1}) \\ &= b_i + m_i t = f(t) \end{aligned}$$

For $a_n \leq t$:

$$\begin{aligned} g(t) &= b_0 + m_0 t + \sum_{j=1}^{j=n+1} c_j (t - a_{j-1})^+ \\ &= b_0 + m_0 t + \sum_{j=1}^{j=n+1} (m_j - m_{j-1})(t - a_{j-1}) \\ &= b_0 + m_{n+1} t + \sum_{j=1}^{j=n+1} (m_j - m_{j-1})(-a_{j-1}) \\ &= b_0 + m_{n+1} t + \sum_{j=1}^{j=n+1} (b_j - b_{j-1}) \\ &= b_{n+1} + m_{n+1} t = f(t) \end{aligned}$$

Therefore, $g \equiv f$. This shows that $f \in S$ has the representation $f(t) = b_0 + m_0 t + \sum_{i=1}^{n+1} c_i (t - a_{i-1})^+$. Therefore, f can be written as the finite sum of sups of functions in V, i.e., $f \in V$. Therefore, $S \subset V$. ■

Corollary 2.1.7: $S_{[a,b]} = V[a,b] = \text{span}\,\{1, t, (t-\alpha)^+ | \alpha \in [a,b]\} \subset C[a,b]$ and $S = V = \text{span}\,\{1, t, (t-\alpha)^+ | \alpha \in \mathbb{R}\} \subset C(\mathbb{R})$.

Proof. This is a direct consequence of **Lemma** 2.1.5 and **Lemma** 2.1.6. ■

Lemma 2.1.8: *Let* $f \in S_{[a,b]}$ *and* $m = \frac{f(b)-f(a)}{b-a}$. *Then* $\exists i \leq n, t_0 \in [a_{i-1}, a_i]$, *and* $m_i \geq m$ *such that* $f(t_0) = f(a) + m(t_0 - a)$.

Proof. Assume not, then $\forall m_i \geq m$, $\forall t \in [a_{i-1}, a_i] : f(t) \neq f(a) + m(t - a)$.

For $t \in [a_0, a_1]$: Since $f(a) = f(a) + m(a - a)$, then $m_1 < m$ and $\forall t \in (a_0, a_1] : f(t) = b_1 + m_1 a + m_1(t - a) < f(a) + m(t - a)$ and in particular $f(a_1) < f(a) + m(a_1 - a)$.

For $t \in [a_1, a_2]$:

$$\begin{aligned} f(t) &= b_2 + m_2 t = b_2 + m_2(t - a_1) + m_2(a_1) \\ &= m_2(t - a_1) + f(a_1) \end{aligned} \tag{2.1}$$

If $m_2 < m$ then: $f(t) = m_2(t - a_1) + f(a_1) < m(t - a_1) + f(a) + m(a_1 - a) = m(t - a) + f(a)$. And if $m_2 \geq m$, then by assumption and using the intermediate value theorem, $f(t) < f(a) + m(t - a)$ and in particular $f(a_2) < f(a) + m(a_2 - a)$.

Proceeding inductively, $f(b) < f(a) + m(b - a)$ which is a contradiction. ■

Corollary 2.1.9: Let $f \in S_{[a,b]}$. Let $m = \frac{f(b)-f(a)}{b-a}$. Then $\exists i \leq n$ and m_i such that $f(a) \geq m_i a + b_i$ and $f(b) \leq m_i b + b_i$.

Proof. Let i and $t_0 \in [a_{i-1}, a_i]$ be as in **Lemma** 2.1.8, then:

$$f(t_0) = f(a) + m(t_0 - a) = m_i(t_0) + b_i$$

Since $f(a) = f(b) - m(b - a)$, we also have:

$$f(t_0) = f(b) + m(t_0 - b) = m_i(t_0) + b_i$$

Since $t_0 \geq a$ and $m \leq m_i$ we have:

$$\begin{aligned} f(a) = f(t_0) - m(t_0 - a) &\geq f(t_0) - m_i(t_0 - a) \\ &\geq m_i(t_0) + b_i - m_i(t_0 - a) \\ &\geq b_i + m_i a \end{aligned}$$

And since Since $t_0 \leq b$ and $m \leq m_i$ we have:

$$\begin{aligned} f(b) = f(t_0) - m(t_0 - b) &\leq f(t_0) - m_i(t_0 - b) \\ &\leq m_i(t_0) + b_i - m_i(t_0 - b) \\ &\leq b_i + m_i b \end{aligned}$$

∎

2.2 Multivariate Piecewise Affine Functions

In this section we introduct the multivariate affine and piecewise affine functions. We follow a convention similar to **Section** 2.1.

Definition 2.2.1: **Affine Functions:** $f : \mathbb{R}^m \to \mathbb{R}$ is called an affine function if $\exists v \in \mathbb{R}^m$ and $b \in \mathbb{R}$ such that $\forall x \in \mathbb{R}^m : f(x) = v \cdot x + b$. We will denote $A = \{f | f \text{ is an affine function}\}$ and $V = A^{\vee\wedge}$. $V \subset C(\mathbb{R}^m)$ where $C(\mathbb{R}^m)$ is the space of continuous functions defined on $\mathbb{R}^m$.

Lemma 2.2.2: *Let $f, g \in A$. $f \equiv g$ if and only if there is a nonempty open subset $U \subset \mathbb{R}^m$ such that $f|_U = g|_U$.*

Proof. Assume $f(x) = v_f \cdot x + b_f$ and $g(x) = v_g \cdot x + b_g$. Let U be nonempty and open subset of $\mathbb{R}^m$ and $f|_U = g|_U$. Then, $\exists \delta > 0$ and $x_0 \in U$ such that $\forall y \in B(x_0, \delta) : f(y) = g(y) \Rightarrow (v_f - v_g) \cdot y + (b_f - b_g) = 0$. If $v_f = v_g$, then $b_f = b_g$ and $f \equiv g$. Otherwise, let $y_1 = x_0 + \frac{\delta(v_f - v_g)}{2\|v_f - v_g\|}$. Since $y_1 \in B(x_0, \delta)$ we have: $(v_f - v_g) \cdot x_0 + b_f - b_g + (v_f - v_g) \cdot \frac{\delta(v_f - v_g)}{2\|v_f - v_g\|} = 0$ Therefore: $(v_f - v_g) \cdot \frac{\delta(v_f - v_g)}{2\|v_f - v_g\|} = 0 \Rightarrow \|v_f - v_g\| = 0 \Rightarrow v_f = v_g \to b_f = b_g \Rightarrow f \equiv g$.

∎

Multivariate piecewise affine functions can be defined in a manner similar to one dimensional piecewise affine functions. Here we will introduce this definition after which we will show that the definition ensures the continuity of the piecewise affine functions. Then, we will introduce a theorem that shows that piecewise affine functions can be defined as continous functions that agree with a finite number of affine functions.

Definition 2.2.3: **Piecewise Affine Functions:** $f : \mathbb{R}^m \to \mathbb{R}$ is called a piecewise affine function if $\exists n \in \mathbb{N}$, distinct affine functions $\{f_i\}_{i=1}^n \subset A$, and subsets $\{S_i \subset \mathbb{R}^m\}_{i=1}^n$ such that:

1. $\forall i : \text{Int}(S_i) \neq \emptyset$ and $S_i = \overline{\text{Int}(S_i)}$
2. $\forall i \neq j : \text{Int}(S_i) \cap \text{Int}(S_j) = \emptyset$
3. $\bigcup_{i=1}^n S_i = \mathbb{R}^m$
4. $f|_{S_i} \equiv f_i|_{S_i}$

The sets $\{f_i\}_{i=1}^n, \{S_i\}_{i=1}^n, \{(S_i, f_i)\}_{i=1}^n$ are called the **components**, **regions**, and **characteristic pairs** of f, respectively. We denote

$$S = \{f : \mathbb{R}^m \to \mathbb{R} | f \text{ is a piecewise affine function}\}$$

Lemma 2.2.4: $S \subset C(\mathbb{R}^m)$.

Proof. We will argue by contradiction. Let $f \in S$ but is not continuous, then $\exists x_n \xrightarrow{n\to\infty} x$ while $f(x_n) \nrightarrow f(x)$. Passing to a subsequence, $\exists \varepsilon > 0$ such that $\forall n : |f(x_n) - f(x)| > \varepsilon$. Since the number of regions is finite, there is a further subsequence y_n such that $\forall n : y_n \subset S_i$ for some i. Since S_i is closed, $y_n \to y = x \in S_i$ and $\varepsilon < |f(y_n) - f(x)|$. However, $f|_{S_i} = f_i|_{S_i}$ and f_i is a continous function by definition which is a contradiction. Therefore, f is continuous. ■

Lemma 2.2.5: *Let $f \in C(\mathbb{R}^m)$ such that $\exists n \in \mathbb{N}$ and a subset $\{f_i\}_{i=1}^n \subset A$ such that $\forall x \in \mathbb{R}^m : \exists i \leq n$ with $f(x) = f_i(x)$. Then, $\forall V \subset \mathbb{R}^m$ that is open and nonempty, $\exists W \subset V$ that is open and nonempty such that $\forall y \in W$: $f(y) = f_j(y)$ for some j.*

Proof. Assume V as in the statement of the lemma. We will argue by contradiction, i.e., $\exists x_1 \in V$ such that $f(x_1) \neq f_1(x_1)$. Pick $\delta = \frac{|f(x_1) - f_1(x_1)|}{3}$. Therefore, $B(f(x_1), \delta) \cap B(f_1(x_1), \delta) = \emptyset$. Let $V_1 = f^{-1}(B(f(x_1), \delta)) \cap f_1^{-1}(B(f_1(x_1), \delta)) \cap V$. Since f, f_1 are continuous then V_1 is open. In addition, $x_1 \in V_1$ so, V_1 is not empty and $\forall x \in V_1 : f(x) \neq f_1(x)$. By repeating the argument, $\exists x_2 \in V_1$ such that $f(x_2) \neq f_2(x_2)$ and we can similarly construct $V_2 \subset V_1 \subset V$ that is open and not empty and $\forall x \in V_2 : f(x) \neq f_2(x_2)$.

Therefore, by induction $\exists V_n \subset V$ such that $\forall i, \forall x \in V_n : f(x) \neq f_i(x)$ which is a contradiction.

Alternatively, the above argument can be rewritten as follows. Let V be as in the statement of the lemma. Set $V_1 = V$. If $f|_{V_1} = f_1|_{V_1}$ then the lemma is true. Otherwise, set $V_2 = V_1 \setminus \{x|f(x) \neq f_1(x)\}$. V_2 is nonempty and is open. If $f|_{V_2} = f_2|_{V_2}$ then the lemma is true. Otherwise, set $V_3 = V_2 \setminus \{x|f(x) \neq f_2(x)\}$ which is again nonempty and open. Continuing in this manner, if none of the sets $i \leq V_i$ satisfy the lemma, then let $V_{n+1} = V_n \setminus \{x|f(x) \neq f_n(x)\}$ which is nonempty and open. f is different from $f_1, f_2, \cdots, f_n$ on $V_{n=1}$ which is a contradiction. ■

Theorem 2.2.6: *$f \in S \Leftrightarrow f$ is continuous and agrees with a finite number of affine function. I.e., $\exists n \in \mathbb{N}$ and a subset $\{f_i\}_{i=1}^n \subset A$ such that $\forall x \in \mathbb{R}^m$: $f(x) = f_i(x)$ for some i.*

Proof. By **Definition**2.2.3 and **Lemma** 2.2.4 a piecewise affine function is continuous and agrees with a finite number of affine functions.

For the opposite direction, and using **Lemma** 2.2.5 $\forall V \subset_{open} \mathbb{R}^m$ $\exists$ a non-empty open subset $W \subset V$ such that $f(x) = f_j(x)$ for some j. Let $O_i = \bigcup\{U|U \subset_{open} \mathbb{R}^m, f_i|_U \equiv f|_U\}$. If $O_i = \emptyset$ for some i, then remove O_i and f_i from the list and renumber and note that by **Lemma** 2.2.5 and setting $V = \mathbb{R}^m$, there is at least one $O_i \neq \emptyset$. Set $S_i = \overline{O_i}$. We will show that $\{S_i\}_{i=1}^n$ satisfy the conditions of **Definition** 2.2.3 of the regions of a piecewise affine function. Condition 4 is satified since both f and f_i are continuous and $S_i = \overline{O_i}$, therefore: $f|_{S_i} \equiv f_i|_{S_i}$. For condition 1 we have $O_i \subset \text{Int}(S_i)$ and since $f|_{\text{Int}(S_i)} \equiv f_i|_{\text{Int}(S_i)}$ and $\text{Int}(S_i)$ is open, then $\text{Int}(S_i) \subset O_i$. Therefore, $O_i = \text{Int}(S_i) \neq \emptyset$ and $\overline{O_i} = \overline{\text{Int}(S_i)} = S_i$. Condition 2 is satisfied as follows: Let $U = O_i \cap O_j$. Since O_i and O_j are open, then U is open and $f_i|_U \equiv f_j|_U$. Then, by **Lemma** 2.2.2 $f_i \equiv f_j$. Finally, for condition 3 we have $\bigcup_{i=1}^n S_i \subset_{closed} \mathbb{R}^m$. Therefore, $V = \mathbb{R}^m \setminus \bigcup_{i=1}^n S_i$ is open and by **Lemma** 2.2.5, $\exists W \subset_{open} V$ such that $f|_W \equiv f_j|_W$ for some j which contradicts the maximality of O_j. Therefore, $\bigcup_{i=1}^n S_i = \mathbb{R}^m$.

■

Corollary 2.2.7: S is a vector lattice.

Proof. Assume $f, g \in S$. Therefore, f is continuous and agrees with $\{f_i\}_{i=1}^n \subset A$, and g is continuous and agrees with $\{g_i\}_{i=1}^k \subset A$. Therefore, $f \vee g$ is continuous and agrees with $\{f_i\}_{i=1}^n \cup \{g_i\}_{i=1}^k \subset A$, i.e., $f \vee g \in S$. Therefore, S is a vector lattice. ■

Corollary 2.2.8: $V \subset S$.

Proof. Since S is a vector lattice and $A \subset S$ then $V = A^{\vee\wedge} \subset S$. ■

Definition 2.2.9: **Hyperplane:** A hyperplane is a subset $H \subset \mathbb{R}^m$ of the form $H = \{x \in \mathbb{R}^m | v \cdot x + b = 0\}$ where $0 \neq v \in \mathbb{R}^m$ and $b \in \mathbb{R}$. Clearly H is closed and has a Lebesque measure zero. Also, H has an orientation since H divides $\mathbb{R}^m$ into H and the positive part $H_1 = \{x \in \mathbb{R}^m | v \cdot x + b > 0\}$ and the negative part $H_2 = \{x \in \mathbb{R}^m | v \cdot x + b < 0\}$. This orientation is reversible since $H = \tilde{H} = \{x \in \mathbb{R}^m | -v \cdot x - b = 0\}$ while $\tilde{H}_1 = H_2$ and $\tilde{H}_2 = H_1$.

Lemma 2.2.10: *The set where two distinct affine functions agree is either empty or a hyperplane.*

Proof. Let $f \neq g \in A$ such that $\forall x \in \mathbb{R}^m : f(x) = v_f \cdot x + b_f$ and $g(x) = v_g \cdot x + b_g$ with $v_f, v_g \in \mathbb{R}^m$ and $b_f, b_g \in \mathbb{R}$. Set $H_{fg} = \{x | f(x) = g(x)\} = \{x | (v_f - v_g) \cdot x + b_f - b_g = 0\}$. If $v_f = v_g$ then $H_{fg} = \emptyset$. Otherwise, H_{fg} is a hyperplane. ■

Lemma 2.2.11: *The boundaries of the regions of a piecewise affine function are subsets of hyperplanes and the regions only intersect on their boundaries. In other words: Let $f \in S$, then $\forall i : \partial S_i = \bigcup_{j \neq i} S_i \cap S_j$.*

Proof. Using **Lemma** 2.2.10, and since the boundaries of the regions are the locations where affine functions agree, therefore, the boundaries are subsets of hyperplanes. Let $x \in \partial S_i$, then $\forall k \in \mathbb{N} : B(x, \frac{1}{k}) \cap \mathbb{R}^m \setminus S_i \neq \emptyset$. $\forall k$ pick $x_k \in B(x, \frac{1}{k}) \cap \mathbb{R}^m \setminus S_i$. Since $x_k \to x$ and $\{x_k\}_{k=1}^\infty \subset \bigcup_{r \neq i} S_r$ there is a subsequence and an index $j \neq i$ such that $\{x_k\}_{k=1}^\infty \subset S_j$. Since $x_k \to x$ and S_j is closed, we have $x \in S_j$. Therefore, $x \in S_i \cap S_j$.

For the other inclusion, let $x \in S_i \cap S_j, i \neq j$. We will argue by conatrdiction by assuming that $x \notin \partial S_i$ but $x \in \mathrm{Int}(S_i)$. Therefore, $\exists \delta > 0$ such that $B(x, \delta) \subset \mathrm{Int}(S_i)$. Since $\mathrm{Int}(S_i) \cap \mathrm{Int}(S_j) = \emptyset$, then $x \in \partial S_j$, therefore,

$B(x, \delta) \cap \text{Int}(S_j) \neq \emptyset \Rightarrow \text{Int}(S_i) \cap \text{Int}(S_j) \neq \emptyset$ which is a contradiction. Therefore, $x \in \partial S_i$. ∎

Lemma 2.2.12: *The characteristic pairs of piecewise affine functions are uniquely defined up to reordering.*

Proof. Let $f \subset S$ and let $\{(S_i, f_i)\}_{i=1}^n$ and $\{(\tilde{S}_j, \tilde{f}_j)\}_{j=1}^k$ be two sets of characteristic pairs for f. Fix i and consider $\text{Int}(S_i)$. By **Lemma** 2.2.5 $\exists W \subset_{open} \text{Int}(S_i)$ such that $f_W \equiv \tilde{f}_j|_W \equiv f_i|_W$ for some $j \leq k$. By **Lemma** 2.2.2 $f_i = \tilde{f}_j$ and therefore $S_i = \tilde{S}_j$. Since the components are distinct, then the characteristic pairs are uniquely defined up to reordering. ∎

Corollary 2.2.13: **Definition** 2.1.2 and **Definition** 2.2.3 are equivalent for one dimensional piecewise affine functions.

Proof. Let f be a one dimensional piecewise affine function defined according to **Definition** 2.1.2. Then, the sets $S_1 = (-\infty, a_0]$, $\forall 2 \leq i \leq n+1 : S_i = [a_{i-2}, a_{i-1}]$, and $S_{n+2} = [a_n, \infty)$ satisfy the conditions of **Definition** 2.2.3, with $n+2$ components. If the components are not distinct, i.e. $\exists i < j$ such that $f_i \equiv f_j$, then by renumbering the components and setting $S_i = S_i \cup S_j$ then the resulting renumbered components are distinct.

For the opposite direction, let f be a one dimensional piecewise affine function defined according to **Definition** 2.2.3. By **Lemma** 2.2.11 $\partial S_i = \bigcup_{j \neq i} S_i \cap S_j \subset \bigcup_{j \neq i} \{x | f_i(x) = f_j(x)\}$. However, the set $\bigcup_{j \neq i} \{x | f_i(x) = f_j(x)\}$ is a finite set of points as it is the union of a finite number of hyperplanes in $\mathbb{R}$ and each hyperplane in $\mathbb{R}$ is a set of one point. Therefore, ∂S_i is a finite set. Since $\text{Int}(S_i) \neq \emptyset$, therefore, $\text{Int}(S_i)$ is a union of pairwise disjoint open intervals whose end points are in ∂S_i. Therefore, f is a piecewise affine function according to **Definition** 2.1.2.

∎

Cells Induced by Hyperplanes formed by the Components of Piecewise Affine Functions: Let $k \in \mathbb{N}$ and $\{H_e\}_{e=1}^k$ be a family of hyperplanes such that $\forall e \leq k : H_e = \{x \in \mathbb{R}^m | v_e \cdot x + b_e = 0\}$ where $0 \neq v_e \in \mathbb{R}^m$ and $b_e \in \mathbb{R}$. Let H_{e1} and H_{e2} be the positive and negative parts according to **Definition** 2.2.9 respectively. Consider the set $D = \{+1, -1, 0\}^k$. Let $\sigma : \mathbb{R}^m \to D$ be such that $\forall x \in \mathbb{R}^m : \sigma(x) = \{\text{sgn}\,(v_e \cdot x + b_e)\}_{e=1}^k$. Denote $M = \text{Range}(\sigma)$. Denote a vector $T = \{T_e\}_{e=1}^k \in M$ satisfying $\forall e : T_e \neq 0$ by a **tope** of M. Clearly, the set of topes is finite because D is finite. Additionally, the set of topes is not empty because: $\bigcup_{e=1}^k H_e \neq \mathbb{R}^m$, therefore, $\exists x \in \mathbb{R}^m \setminus \bigcup_{e=1}^k H_e$ and therefore $\sigma(x)$ is a tope of M. Let J be the cardinality of the topes of M and let $\{T_j\}_{j=1}^J$ be an enumeration. The sets $K_j = \{x \in \mathbb{R}^m | \sigma(x) = T_j\}$ define **cells** in $\mathbb{R}^m$ induced by the family of hyperplanes independent of the orientation. Each cell K_j is not empty, convex and open because it is the intersection of a finite number of open and convex sets of the form H_{e1} and H_{e2}. In addition, $\bigcup_{j=1}^J K_j = \mathbb{R}^m \setminus \bigcup_{e=1}^k H_e$. Since the number of cells J is finite, then $\overline{\bigcup_{j=1}^J K_j} = \bigcup_{j=1}^J \overline{K_j} = \mathbb{R}^m$.

Let $\{f_i\}_{i=1}^n \subset A$. By **Lemma** 2.2.10, the sets where these functions agree with each other are either empty or hyperplanes. Let $H_{lm} = \{x \in \mathbb{R}^m | f_l(x) = f_m(x)\}$ and let k be the cardinality of $E = \{(l, m) | H_{lm} \neq \emptyset\}$. As defined above, let J be an enumeration of the topes and let $\{K_j\}_{j=1}^J$ be the cells induced by the family of hyperplanes $\{H_e\}_{e \in E}$. These cells, by definition, are locations where the affine functions do not agree.

Lemma 2.2.14: *Let $f \in S$. Let $\{K_h\}_{h=1}^J$ be the cells induced by the components of $\{f_i\}_{i=1}^n$. Then, $\forall i, h, \forall x \in K_h$:*

1. *If $f(x) = f_i(x)$, then $\forall y \in K_h : f(y) = f_i(y)$.*
2. *If $f(x) < f_i(x)$, then $\forall y \in K_h : f(y) < f_i(y)$.*
3. *If $f(x) > f_i(x)$, then $\forall y \in K_h : f(y) > f_i(y)$.*

Moreover, $\forall K_h : \exists! i_h$ such that $\forall x \in K_h : f(x) = f_{i_h}(x)$.

Proof. The moreoever part is a trivial outcome of the statement and the fact that the components are unique and do not agree on any cell K_h.

Let $x \in K_h$ be such that $f(x) = f_i(x)$ for some i. **Claim**: $\exists \delta > 0$ such that $\forall z \in B(x, \delta) \subset K_h : f(z) = f_i(z)$. **Proof**: Let $j \neq i$, therefore $f_j(x) \neq f_i(x) = f(x)$. Let $\varepsilon = \frac{|f_j(x) - f(x)|}{3}$. Therefore, $B(f_j(x), \varepsilon) \cap B(f(x), \varepsilon) = \emptyset$. Since f, f_j are continuous, then $f^{-1}(B(f(x), \varepsilon)) \cap K_h \cap f_j^{-1}(B(f_j(x), \varepsilon))$ is open and not empty. Therefore, $\exists \delta_j$ such that $\forall z \in B(x, \delta_j) : f(x) \neq f_j(x)$. Repeating $\forall j \neq i$ and setting $\delta = \min_{j \neq i} \delta_j$, the claim is proved.

We will argue by contradiction, let $y \in K_h, x \neq y$ be such that $f(y) \neq f_i(y)$. Since K_h is convex, the interval $\{x + t(y-x) | 0 \leq t \leq 1\} \subset K_h$. Let $t_0 = \inf_t\{x + t(y-x) | f(x + t(y-x)) \neq f_i(x + t(y-x))\}$. Clearly $0 < t_0 < 1$ (If $t_0 = 1$, then by continuity, $f(y) = f_i(y)$ which is a contradiction while $t_0 > 0$ is a result of the claim above). Therefore, $\exists j \neq i$ such that $f_j(x + t_0(y-x)) = f(x + t_0(y-x))$. By applying the claim to f_j, $\exists \delta > 0$ such that f agrees with f_j on $B(x + t_0(y-x), \delta)$. Therefore, $f\left(x + (t_0 - \frac{\delta}{3\|y-x\|})(y-x)\right) = f_j\left(x + (t_0 - \frac{\delta}{3\|y-x\|})(y-x)\right)$ which contradicts that t_0 is the infimum. Therefore, $\forall y \in K_h : f(y) = f_i(y)$.

Let $x \in K_h$ be such that $f(x) > f_i(x)$. By the first part of the proof, $\exists j \neq i$ such that $\forall y \in K_h : f_j(y) = f(y)$. Since $f_j(x) > f_i(x)$, therefore, $K_h \subset \{z | f_j(z) > f_i(z)\}$. Therefore, $\forall y \in K_h : f_j(y) = f(y) > f_i(y)$. The same holds if $f(x) < f_i(x)$.

■

Theorem 2.2.15: *Let $f \in S$ with n distinct components. Let $\{K_h\}_{h=1}^J$ be the cells generated by the components. Let $O_i = \bigcup\{K_h | f|_{K_h} \equiv f_i|_{K_h}\}$. Let $S_i = \overline{O_i}$. Then, $\{(S_i, f_i)\}_{i=1}^n$ are exactly the characteristic pairs of f.*

Proof. First notice that if O_i is empty, then we can remove f_i and renumber the components. We will show that $\{S_i\}_{i=1}^n$ satisfy the conditions of **Definition** 2.2.3. Indeed, the fourth condition is satisfied by the continuity of f and f_i so $f|_{S_i} \equiv f_i|_{S_i}$. Using the result of **Lemma** 2.2.14: $\forall h : \exists! i_h$ such that $f|_{K_h} \equiv f_{i_h}|_{K_h}$. For condition 3, $\bigcup_{i=1}^n S_i = \bigcup_{h=1}^J \overline{K_h} = \mathbb{R}^m$. For condition 1, $O_i \neq \emptyset$ and therefore $\text{Int}(S_i) \neq \emptyset$. Also, O_i is open so $O_i \subset \text{Int}(S_i)$, therefore, $S_i \subset \overline{O_i} \subset \overline{\text{Int}(S_i)} \subset S_i \Rightarrow \overline{\text{Int}(S_i)} = S_i$. For condition 2, if $x \in \text{Int}(S_i) \cap \text{Int}(S_j)$, then $f(x) = f_i(x) = f_j(x)$ and $\exists \delta_i$ and δ_j such that $\forall y \in B(x, \min\{\delta_i, \delta_j\}) : f(y) = f_i(y) = f_j(y)$. Therefore, by **Lemma** 2.2.2 $f_i \equiv f_j$ and $S_i = S_j$. By **Lemma** 2.2.12 $\{(S_i, f_i)\}_{i=1}^n$ are exactly the characteristic pairs of f.

■

Lemma 2.2.16: *Let $f \in S$ with n distinct components. Then, $\forall a, b \in \mathbb{R}^m, \exists i \leq n$ such that $f_i(a) \leq f(a)$ and $f_i(b) \geq f(b)$.*

Proof. Let $h : [0,1] \to \mathbb{R}^m$ be such that $\forall t \in [0,1] : h(t) = a + t(b-a)$. Let $g = f \circ h$, i.e., $g(t) = f(a + t(b-a))$. $\forall i : g_i \equiv f_i \circ h$ is an affine function, indeed: $f_i(a + t(b-a)) = v_i \cdot a + t((b-a) \cdot v_i) + b_i = m_i t + c_i$ where $m_i = (b-a) \cdot v_i$ and $c_i = b_i + v_i \cdot a$. Therefore, $g : [0,1] \to \mathbb{R}$ is a continuous function and agrees with the affine functions $\{g_i\}_{i=1}^n$. Then, by **Theorem** 2.2.6 and **Corollary** 2.2.13, g is a piecewise affine function. By **Corollary** 2.1.9 $\exists i \leq n$ such that $g_i(0) = f_i(a) \leq g(0) = f(a)$ and $g_i(1) = f_i(b) \geq g(1) = f(b)$.

■

Theorem 2.2.17: *$S \subset V$.*

Proof. Let $f \in S$. Let $\{K_h\}_{h=1}^J$ be the cells generated by the components $\{f_i\}_{i=1}^n$ of f. Using **Lemma** 2.2.14, we can define the set of indices of the components that are larger than or equal to f on the cell K_h. I.e.,

$$\forall h \leq J : E_h = \{i \leq n | f|_{K_h} \leq f_i|_{K_h}\}$$

For each h define the function $g_h = \bigwedge_{i \in E_h} f_i$. Then, $\forall x \in K_h : g_h(x) \geq f(x)$. Using **Lemma** 2.2.14, there exists a unique index i_h in E_h such that $f_{i_h}(x) = f(x)$, therefore, $g_h(x) = f(x)$.

Fix h. Fix $x \in K_h$ and $y \in K_k$ with $k \neq h$. By **Lemma** 2.2.16: $\exists i$ such that $f_i(x) \leq f(x)$ and $f_i(y) \geq f(y)$. In particular, $i \in E_k$, therefore $g_k \leq f_i$. We also have $g_k(x) \leq f_i(x) \leq f(x) = g_h(x)$. This is true $\forall k \neq h$, therefore, $f(x) = \bigvee_{k \leq J} g_k(x)$.

Therefore, $\forall z \in \bigcup_{h \leq J} K_h$:

$$f(z) = \bigvee_{h \leq J} \bigwedge_{i \in E_h} f_i(z)$$

Since f is continuous and $\overline{\bigcup_{h \leq J} K_h} = \mathbb{R}^m$, therefore the equality holds $\forall z \in \mathbb{R}^m$. Therefore, $f \in A^{\vee\wedge} = V$.

■

Theorem 2.2.18: *S=V.*

Proof. Using **Theorem** 2.2.17 and **Corollary** 2.2.8, then, $S = V$. ∎

Chapter 3

Locally Piecewise Affine Functions

3.1 Motivation and Definitions

The motivation for this chapter is to introduce a new class of piecewise affine functions that have infinitely many distinct components since **Definition** 2.2.3 is restricted to functions with finitely many components. In addition, **Theorem** 2.2.6 cannot be naturally extended to functions with countably many components. Moreover, it is not clear when a piecewise affunction on a subset of $\mathbb{R}^m$ can be extended to a piecewise affine function on $\mathbb{R}^m$. As per the previous chapter, the affine functions are denoted by $A = \{f : \mathbb{R}^m \to \mathbb{R} | f \text{ is an affine function}\}$ and $V = A^{\vee\wedge}$, in addition the space of multivariate piecewise affine functions is denoted by S.

Definition 3.1.1: **Piecewise Affine Functions on Arbitrary Subsets:** Let $U \subset \mathbb{R}^m$ such that it is the union of at most finite number of connected sets. $f : U \to \mathbb{R}$ is called a piecewise affine function if f is continuous and f agrees with $\{f_i\}_{i=1}^n \subset A$ on U.

Definition 3.1.2: **Locally Piecewise Affine Function:** $f : \mathbb{R}^m \to \mathbb{R}$ is called a locally piecewise affine function if $\forall x \in \mathbb{R}^m : \exists$ an open neighbourhood U of x such that U is the union of at most finite number of connected sets and $f : U \to \mathbb{R}$ is piecewise affine. We denote the space of locally piecewise affine functions by:

$$S_{lp} = \{f : \mathbb{R}^m \to \mathbb{R} | f \text{ is locally piecewise affine}\}$$

Remarks:

- $\forall f \in S_{lp} : f$ is locally continuous, therefore continuous. I.e., $S_{lp} \subset C(\mathbb{R}^m)$.
- $S \subset S_{lp}$.
- **Definition** 3.1.1 is restricted to subsets that are the union of at most finite number of connected sets to exclude examples similar to the function: $f : \mathbb{N} \subset \mathbb{R} \to \mathbb{R}$ defined as:

$$f(n) = \begin{cases} n, & n \text{ is even} \\ 2n, & n \text{ is odd} \end{cases}$$

  f as defined agrees with 2 affine functions, but it cannot be naturally extended to a piecewise affine function on $\mathbb{R}$.
- Another example of a piecewise affine function as per **Definition** 3.1.1 that cannot be extended to a piecewise affine function on $\mathbb{R}$ is the function $f : [0, 2] \setminus \{1\} \to \mathbb{R}$ defined as:

$$f(t) = \begin{cases} 1, & t \in [0, 1) \\ 2, & t \in (1, 2] \end{cases}$$

Here we list some lemmas that offer some conditions on the form of the proper subsets $U \subset \mathbb{R}^m$ that allow the natural extension of a piecewise affine function on U to a piecewise affine function on $\mathbb{R}$.

Lemma 3.1.3: *Let $f : [a, b] \to \mathbb{R}$ be a continuous function that agrees with a finite number of affine functions on $[a, b]$. Then, $f \in S_{[a,b]}$ and can be extended (in possibly many ways) to a function $\tilde{f} \in S$.*

Proof. Assume $f(a) = f_1(a)$ and $f(b) = f_n(b)$ where $f_1, f_2 \in A$. Let

$$\tilde{f} = \begin{cases} f_1(t), & t \leq a \\ f(t), & a \leq t \leq b \\ f_n(t), & b \leq t \end{cases}$$

$\tilde{f} : \mathbb{R} \to \mathbb{R}$ is a continuous function that agrees with a finite number of affine functions. Therefore by **Theorem** 2.2.6 and **Corollary** 2.2.13, $\tilde{f} \in S$ and its restriction on $[a, b]$ is $f \in S_{[a,b]}$. ■

Lemma 3.1.4: *Let $U \subset \mathbb{R}^m$ be closed and convex and let $f : U \to \mathbb{R}$ be a piecewise affine function. Then, f can be extended (possibly in many ways) to a piecewise affine function $\tilde{f} \in S$.*

Proof. Assume $\mathrm{Int}(U) \neq \emptyset$. U is closed and convex, then $\overline{\mathrm{Int}(U)} = U$. Also, if $\mathrm{Int}(U) = \emptyset$, then U can be considered as a convex subset with non-empty interior of $\mathbb{R}^l$ for some $l < m$ and the lemma can then be applied. A piecewise affine function on $\mathbb{R}^l$ can be trivially extended to $\mathbb{R}^m$. So, without loss of generality $\mathrm{Int}(U) \neq \emptyset$.

Consider the cells $\{K_h \subset \mathbb{R}^m\}_{h=1}^{J}$ generated by the distinct components $\{f_i\}_{i=1}^{n} \subset A$. It is possible that the components of f do not agree anywhere and the set of cells is empty. This is possible because some of the components might not be actually used in f. In that case, set $J = 1$ and $K_1 = \mathbb{R}^m$. In any case, there is at least one cell K_i such that $\mathrm{Int}(U) \cap K_i \neq \emptyset$. Let $E = \{i \leq J : K_i \cap \mathrm{Int}(U) \neq \emptyset\}$.

Claim 1: $\forall i \in E$, fix $x \in K_i \cap \mathrm{Int}(U)$: (1) Assume $f(x) = f_j(x)$ for some j, then $\forall y \in K_i \cap \mathrm{Int}(U) : f(y) = f_j(y)$, (2) Assume $f(x) < f_j(x)$ for some j, then $\forall y \in K_i \cap \mathrm{Int}(U) : f(y) < f_j(y)$, (3) Assume $f(x) > f_j(x)$ for some j, then $\forall y \in K_i \cap \mathrm{Int}(U) : f(y) > f_j(y)$.

Proof: (1) U and K_i are convex, therefore, $K_i \cap \mathrm{Int}(U)$ is convex. Therefore, the line $\{x + \alpha(y - x) | 0 \leq \alpha \leq 1\} \subset (K_i \cap \mathrm{Int}(U))$. We will argue by contradiction. Assume that $f(x) = f_j(x)$ and $f(y) \neq f_j(y)$. Consider the function $g : [0, 1] \to \mathbb{R}$ defined by $g(t) = f(x + t(y - x))$. g is piecewise affine with at least two distinct components g_1 and g_2, then, $\exists 0 \leq \beta \leq 1$ such that

$g_1(\beta) = g_2(\beta)$. Therefore, $x + \beta(y - x) \notin K_i$ which is a contradiction. (2) and (3) are straightforward as in **Lemma** 2.2.14.

Claim 2: $\forall a, b \in U, \exists i \leq n$ such that $f_i(a) \leq f(a)$ and $f_i(b) \geq f(b)$.

Proof: As in **Lemma** 2.2.16, let $t \in [0, 1]$ and let $h : [0, 1] \to \mathbb{R}^m$ be such that $\forall t \in [0, 1] : h(t) = a + t(b - a)$. Let $g = f \circ h$, i.e., $g(t) = f(a + t(b - a))$. Since U is convex, then g is well defined. The result follows as in **Lemma** 2.2.16.

The rest of the proof follows as in **Theorem** 2.2.17. Define $\forall i \leq J : \tilde{K}_i = K_i \cap \text{Int}(U)$ and let $E_i = \{j | \forall x \in \tilde{K}_i : f(x) \leq f_j(x)\}$. Define $g_i = \bigwedge_{j \in E_i} f_j$. Then, $\forall x \in \tilde{K}_i : g_i(x) = f(x)$. Using claims 1 and 2 above and following the proof in **Theorem** 2.2.17 we have: $\forall k \neq i, \forall x \in \tilde{K}_i : g_k(x) \leq g_i(x)$. Therefore, $\forall z \in \bigcup_{h \leq J} \tilde{K}_h$:

$$f(z) = \bigvee_{h \leq J} \bigwedge_{i \in E_h} f_i(z)$$

Since f is continuous and $\overline{\bigcup_{h \leq J} \tilde{K}_h} = U$ then the equality holds $\forall z \in U$. Let $\tilde{f} = f$ then $\tilde{f} \in S$ and is a natural extention for f to $\mathbb{R}^m$. ■

Conjecture

Let $U \subset \mathbb{R}^m$ be open, bounded, connected and $\partial U = \overline{U} \setminus \text{Int}(\overline{U})$. Let $f : U \to \mathbb{R}$ be a piecewise affine function. Then, f can be extended possibly in many ways to a piecewise affine function $\tilde{f} \in S$.

3.2 Properties of S_{lp}

Lemma 3.2.1: *$f \in S_{lp}$ if and only if $\forall \delta > 0, \forall x \in \mathbb{R}^m, f : B(x, \delta) \to \mathbb{R}$ is piecewise affine.*

Proof. Assume $f \in S_{lp}$. Fix $x \in \mathbb{R}^m$ and $\delta > 0$. Therefore, $\overline{B(x, \delta)}$ is compact. $\forall y \in B(x, \delta) : \exists U_y \subset_{open} \mathbb{R}^m$ such that $f : U_y \to \mathbb{R}$ is continuous and agrees with a finite number n_y of affine functions on U_y. The open neighbourhoods

U_y form an open cover for $\overline{B(x,\delta)}$ so it admits a finite subcover U_{y_i} with $i \leq m \in \mathbb{N}$. Therefore, $f : \bigcup_{i=1}^{m} U_{y_i} \to \mathbb{R}$ is continuous and agrees with a finite number of affine functions on $\bigcup_{i=1}^{m} U_{y_i} \supset \overline{B(x,\delta)}$, therefore, $f : \overline{B(x,\delta)} \to \mathbb{R}$ is piecewise affine.

The opposite direction is straightforward from **Definition** 3.1.2.

■

Lemma 3.2.2: *$f \in S_{lp}$ if and only if $\forall C \subset_{compact} \mathbb{R}^m$ and C is the union of at most finite number of connected sets: $f : C \to \mathbb{R}$ is piecewise affine.*

Proof. For one direction, assume $f \in S_{lp}$. Consider a compact set C that is the union of at most finite number of connected sets. Then $\forall x \in C : \exists V_x \subset_{open} \mathbb{R}^m$ such that $f : V_x \to \mathbb{R}$ is continuous and f agrees with a finite number n_x of affine functions on V_x. The open neighbourhoods V_x form an open cover for C. Since C is compact, C admits a finite subcover V_{x_i} with $i \leq k \in \mathbb{N}$ (k is the cardinality of the subcover). $C \subset \bigcup_{i=1}^{k} V_{x_i}$. $f : \bigcup_{i=1}^{k} V_{x_i} \to \mathbb{R}$ is continuous and agrees with a finite number of affine functions on $\bigcup_{i=1}^{k} V_{x_i}$ therefore $f : C \to \mathbb{R}$ is piecewise affine.

The opposite direction is straightforward from **Lemma** 3.2.1.

■

Lemma 3.2.3: *S_{lp} is a vector lattice.*

Proof. Clearly S_{lp} is a vector space. Assume $f, g \in S_{lp}$. Consider $x \in \mathbb{R}^m$ and $\delta > 0$. By **Lemma** 3.2.1, f and g are continuous and agree with $\{f_i\}_{i=1}^{n} \subset A$ and $\{g_j\}_{j=1}^{m} \subset A$ on $\overline{B(x,\delta)}$. Therefore, $f \vee g$ is continuous and agrees with $\{f_i\}_{i=1}^{n} \cup \{g_j\}_{j=1}^{m} \subset A$ on $\overline{B(x,\delta)}$. Then, by **Lemma** 3.2.1, $f \vee g \in S_{lp}$. ■

Lemma 3.2.4: *Let $f \in S_{lp}$. Then $\forall V \subset_{open} \mathbb{R}^m, \exists W \subset_{open} V, W \neq \emptyset$ such that $\forall x \in W : f(x) = f_j(x)$ where $f_j \in A$.*

Proof. Let $V \subset_{open} \mathbb{R}^m$. Let $x \in V$. Then, $\exists \delta > 0$ and $\overline{B(x,\delta)} \subset V$ such that $f : \overline{B(x,\delta)} \to \mathbb{R}$ is piecewise affine. Let $g \equiv f|_{\overline{B(x,\delta)}}$. Since $\overline{B(x,\delta)}$ is closed and convex and using **Lemma** 3.1.4, $\exists \tilde{g} \in S$ such that $\tilde{g}|_{\overline{B(x,\delta)}} \equiv g|_{\overline{B(x,\delta)}} \equiv f|_{\overline{B(x,\delta)}}$. Applying **Lemma** 2.2.5 to $\tilde{g}$ and $B(x,\delta) \subset V$, the result is obtained. ■

Lemma 3.2.5: *$f \in S_{lp}$ if and only if $\exists$ distinct affine functions $\{f_i\}_{i=1}^{\infty}$ and subsets $\{S_i \subset \mathbb{R}^m\}_{i=1}^{\infty}$ such that:*

1. $\forall i : \text{Int}(S_i) \neq \emptyset$. $\overline{\text{Int}(S_i)} = S_i$.
2. $\forall \delta > 0 : \forall x \in \mathbb{R}^m : E_{x,\delta} = \{j \in \mathbb{N} : S_j \cap B(x, \delta) \neq \emptyset\}$ *is finite.*
3. $\forall i \neq j : \text{Int}(S_i) \cap \text{Int}(S_j) = \emptyset$.
4. $\bigcup_{i=1}^{\infty} S_i = \mathbb{R}^m$.
5. $f|_{S_i} \equiv f_i|_{S_i}$.

Similar to the previous definitions, The sets $\{S_i\}_{i=1}^{\infty}$ are called the regions of f and $\{f_i\}_{i=1}^{\infty}$ are called the components of f. The pairs $\{(S_i, f_i)\}_{i=1}^{\infty}$ are called the characteristic pairs of f.

Proof. For one direction: assume that f satisfies the conditions listed in the lemma. If $x \in \text{Int}(S_i)$ for some i then $\exists \delta_x$ such that $V = B(x, \delta_x) \subset S_i$ and $f_V \equiv f_i|_V$ and thus V is an open neighbourhood of x with f being continuous on V and agreeing with a finite number (only one) of affine functions. If $x \in \partial S_i$ for some i, then, by condition 2: Fix $\delta > 0 \Rightarrow B(x, \delta) \subset \bigcup_{j \in E_{x,\delta}} S_j$. We need to show that f is continuous on $B(x, \delta)$ and that f agrees with a finite number of affine functions. Indeed, since $E_{x,\delta}$ is finite, then f agrees with a finite number of affine functions. Additionally, arguing by contradiction we will show that f is continuous at x. Consider a sequence $\{x_n\}_{n=1}^{\infty} \subset B(x, \delta)$ such that $x_n \to x$ and $f(x_n) \nrightarrow f(x)$. Since $E_{x,\delta}$ is finite, then, we can pass to a subsequence $\{x_n\}_{n=1}^{\infty} \subset S_j$ for some j satisfying $|f(x_n) - f(x)| > \varepsilon$ for some $\varepsilon > 0$. But since S_j is closed and f_j is continuous, then $f(x_n) = f_j(x_n) \to f_j(x) = f(x)$ which is a contradiction. Therefore, f is continuous at x. Since $x \in \partial S_i$ was chosen arbitrarily, therefore, f is continuous on $B(x, \delta) \cap \left(\bigcup_{j \in E_{x,\delta}} \partial S_j\right)$. From the first part, f is also continuous on $B(x, \delta) \cap \left(\bigcup_{j \in E_{x,\delta}} \text{Int}(S_j)\right)$. Therefore, f is continuous on $B(x, \delta)$ and agrees with a finite number of affine functions. Therefore, f is locally piecewise affine.

For the opposite direction, assume that $f \in S_{lp}$, and recall that $\mathbb{R}^m = \bigcup_{m=1}^{\infty} B(0, m)$. Using **Lemma** 3.2.1: $\forall m : f$ is piecewise affine on $\overline{B(0, m)}$, and therefore, $\exists$ a finite number $n_m \in \mathbb{N}$ of affine functions $\{f_j\}_{j=1}^{n_m}$ with which f agrees. By collecting the distinct affine functions associated with each open ball $B(0,m)$ then f agrees with a countable (infinite) set of distinct affine functions $\{f_i\}_{i=1}^{\infty}$. Let $O_i = \bigcup\{U | U \subset_{open} \mathbb{R}^m, f_i|_U \equiv f|_U\}$. If $O_i = \emptyset$ for some i then remove O_i and f_i from the list and renumber. Note that by setting $V = \mathbb{R}^m$ in **Lemma** 3.2.4, then at least one $O_i \neq \emptyset$. Thus, $\forall i : O_i \neq \emptyset$ and $\{f_i\}_{i=1}^{\infty}$ are distinct. Let $S_i = \overline{O_i}$. We will show that $\{S_i\}_{i=1}^{\infty}$ satisfy the conditions in the lemma. Condition 5 is satisfied trivially: Since $f|_{O_i} \equiv f_i|_{O_i}$ and both f and f_i are continuous, then, $f|_{S_i} \equiv f_i|_{S_i}$. For condition 1: Since $O_i \subset \text{Int}(S_i)$, therefore, $\text{Int}(S_i) \neq \emptyset$. Also since f agrees with f_i on $\text{Int}(S_i)$ which is open we have: $\text{Int}(S_i) \subset O_i$. Therefore, $O_i = \text{Int}(S_i) \neq \emptyset$ and $\overline{O_i} = \overline{\text{Int}(S_i)} = S_i$. Condition 3 is satisfied as follows: Let $U = O_i \cap O_j$. Since O_i and O_j are open, then U is open and $f_i|_U \equiv f_j|_U$. Then, by **Lemma** 2.2.2, $f_i \equiv f_j$. Condition 2 is satisfied as follows: Let $\delta > 0, x \in \mathbb{R}^m$. By **Lemma** 3.2.1 f is piecewise affine on $\overline{B(x,\delta)}$. Therefore, $f|_{B(x,\delta)}$ agrees with a finite number of affine functions. Therefore, $E_{x,\delta} = \{j \in \mathbb{N} : S_j \cap B(x,\delta) \neq \emptyset\}$ is finite. Condition 4 is satisfied as follows: We will argue by contradiction. Assume $\exists x \in \mathbb{R}^m \setminus \bigcup_{i=1}^{\infty} S_i$. Let $\delta > 0$. By condition 2, $E_{x,\delta}$ is finite, therefore, $V = B(x,\delta) \setminus \bigcup_{j \in E_{x,\delta}} S_j$ is open and by **Lemma** 3.2.4, $\exists W \subset_{open} V$ such that $\forall y \in W : f(y) = f_k(y)$ for some affine function f_k. Therefore, $x \in W \subset S_k$ which is a contradiction. Therefore, $\bigcup_{i=1}^{\infty} S_i = \mathbb{R}^m$. ■

Lemma 3.2.6: *The characteristic pairs of locally piecewise affine functions are uniquely defined up to reordering.*

Proof. Let $f \subset S_{lp}$ and let $\{(S_i, f_i)\}_{i=1}^{\infty}$ and $\{(\tilde{S}_j, \tilde{f}_j)\}_{j=1}^{\infty}$ be two characteristic pairs for f. Fix i and consider $\text{Int}(S_i)$. By **Lemma** 3.2.4 $\exists W \subset_{open} \text{Int}(S_i)$ such that $f_W \equiv \tilde{f}_j|_W \equiv f_i|_W$ for some j. By **Lemma** 2.2.2 $f_i = \tilde{f}_j$ and therefore $S_i = \tilde{S}_j$. Since the components are distinct, then the characteristic pairs are uniquely defined up to reordering. ■

Lemma 3.2.7: *The boundaries of the regions of a locally piecewise affine function are subsets of hyperplanes and the regions only intersect on their boundaries. In other words: Let $f \in S_{lp}$, then $\forall i : \partial S_i = \bigcup_{j \neq i} S_i \cap S_j$.*

Proof. Using **Lemma** 2.2.10, and since the boundaries of the regions are the locations where affine functions agree, therefore, the boundaries are subsets of hyperplanes. Let $x \in \partial S_i$, then $x \in S_i \cap \overline{(\mathbb{R}^m \setminus S_i)}$, then $\forall k \in \mathbb{N}$: $B(x, \frac{1}{k}) \cap \mathbb{R}^m \setminus S_i \neq \emptyset$. $\forall k$ pick $x_k \in B(x, \frac{1}{k}) \cap \mathbb{R}^m \setminus S_i$. By **Lemma** 3.2.5, $E_{x,1}$ is finite. Since $x_k \to x$ and $\{x_k\}_{k=1}^{\infty} \subset \bigcup_{r \in E_{x,1} \setminus \{i\}} S_r$ there is a subsequence and an index $j \neq i$ such that $\{x_k\}_{k=1}^{\infty} \subset S_j$. Since $x_k \to x$ and S_j is closed, we have $x \in S_j$. Therefore, $x \in S_i \cap S_j$.

For the other inclusion, let $x \in S_i \cap S_j, i \neq j$. We will argue by conatrdiction by assuming that $x \notin \partial S_i$ but $x \in \text{Int}(S_i)$. Therefore, $\exists \delta > 0$ such that $B(x, \delta) \subset \text{Int}(S_i)$. Since $\text{Int}(S_i) \cap \text{Int}(S_j) = \emptyset$, then $x \in \partial S_j$, therefore, $B(x, \delta) \cap \text{Int}(S_j) \neq \emptyset \Rightarrow \text{Int}(S_i) \cap \text{Int}(S_j) \neq \emptyset$ which is a contradiction. Therefore, $x \in \partial S_i$. ■

Cells Induced by Hyperplanes formed by the Components of Locally Piecewise Affine Functions: As shown in the previous chapter, there are finite number of cells induced by hyperplanes formed by the components of a piecewise affine function. In the case of locally piecewise affine functions, there are infinitely many cells and these cells are not necessarily open. Let $\{H_e\}_{e=1}^{\infty}$ be a family of countably (infinite) many hyperplanes such that $\forall e : H_e = \{x \in \mathbb{R}^m | v_e \cdot x + b_e = 0\}$ where $v_e \in \mathbb{R}^m$ and $b_e \in \mathbb{R}$. Let H_{e1} and H_{e2} be the positive and negative parts according to **Definition** 2.2.9 respectively. Consider the set $D = \{+1, -1, 0\}^{\infty}$. Let $\sigma : \mathbb{R}^m \to D$ be such that $\forall x \in \mathbb{R}^m : \sigma(x) = \{\text{sgn}(v_e \cdot x + b_e)\}_{e=1}^{\infty}$. Denote $M = \text{Range}(\sigma)$. Denote a vector $T = \{T_e\}_{e=1}^{\infty} \in M$ satisfying $\forall e : T_e \neq 0$ by a **tope** of M. The set of topes is not empty because each hyperplane is nowhere dense. Therefore, $\bigcup_{e=1}^{\infty} H_e$ is a meagre set and thus, using Baire Category Theorem, has empty interior implying that $\bigcup_{e=1}^{\infty} H_e \neq \mathbb{R}^m$. Therefore, $\exists x \in \mathbb{R}^m \setminus \bigcup_{e=1}^{\infty} H_e$ and $\sigma(x)$ is a tope of M. Let M_T be the set of topes of M. Unlike the finite case, the set of topes is not necessarily finite. Since the set of hyperplanes is countably infinite, then, the cardinality of M_T is at most equal to the cardinality of the real numbers and by chosing an adequate ordering of the topes, then M_T can be viewed as a subset of $\mathbb{R}$. Let $\{T_r\}_{r \in M_T}$ be a representation of the topes. The sets $K_r = \{x \in \mathbb{R}^m | \sigma(x) = T_r\}$ define **cells** in $\mathbb{R}^m$ induced by the family of hyperplanes independent of the orientation. Each cell K_r is not empty, convex but unlike the finite case, it is not necessarily open because it is the intersection of countably infinite open and convex sets of the form H_{e1} and H_{e2}. In addition, $\bigcup_{r\in M_T} K_r = \mathbb{R}^m \setminus \bigcup_{e=1}^{\infty} H_e$ and $\overline{\bigcup_{r\in M_T} K_r} = \mathbb{R}^m$. Let $\{f_i\}_{i=1}^{\infty} \subset A$. By **Lemma** 2.2.10, the sets were these functions interest are either empty or are hyperplanes. Let $H_{lm} = \{x \in \mathbb{R}^m | f_l(x) = f_m(x)\}$ and let $E = \{(l,m)|H_{lm} \neq \emptyset\}$. Note that E is countable since if $f_1, f_2, f_3, ...$ are the affine functions, then the set made of the elements $H_{1,2}, H_{1,3}, H_{2,3}, H_{1,4}, H_{2,4}, H_{3,4}, H_{1,5}, H_{2,5}, H_{3,5}, H_{4,5}, ...$ is also countable. Define a representation for the topes and let $\{K_r\}_{r\in M_T}$ be the cells induced by the family of hyperplanes $\{H_e\}_{e=1}^{\infty}$ as defined above. These cells, by definition, are locations where the affine functions do not agree.

Lemma 3.2.8: *Let $f \in S_{lp}$. Let $\{K_r\}_{r\in M_T}$ be the cells induced by the components $\{f_i\}_{i=1}^{\infty}$. Then, $\forall i \leq \infty, \forall r \in M_T, \forall x \in K_r$:*

1. *If $f(x) = f_i(x)$, then $\forall y \in K_r : f(y) = f_i(y)$.*
2. *If $f(x) < f_i(x)$, then $\forall y \in K_r : f(y) < f_i(y)$.*
3. *If $f(x) > f_i(x)$, then $\forall y \in K_r : f(y) > f_i(y)$.*

Moreover, $\forall K_r : \exists! i_r$ such that $\forall x \in K_r : f(x) = f_{i_r}(x)$.

Proof. The moreoever part is a trivial outcome of the statement and the fact that the components are unique and do not agree on any cell K_r.

Let $x, y \in K_r$ be such that $f(x) = f_i(x)$ and $f(y) = f_j(y)$ for some $i \neq j$. Unlike **Lemma** 2.2.14, K_r is not necessarily open. Since the cell K_r is a set where none of the functions agree, then $f_i(x) \neq f_j(x)$ and $f_i(y) \neq f_j(y)$. Consider the set $C = \{x + \alpha(x-y)|0 \leq \alpha \leq 1\} \subset K_r$. C is compact and connected. By **Lemma** 3.2.2, $f|_C$ agrees with a finite number of affine functions. Without loss of generality, we can assume that $f|_C$ agrees with the two distinct functions f_i and f_j, otherwise, we can pick another point y close enough to x. Consider the function $g : [0,1] \to \mathbb{R}$ defined as $g(t) = f(x + t(x-y))$. $g(t)$ is a piecewise affine function with two distinct components $g_1(t) = f_i(x+t(x-y))$ and $g_2(t) = f_j(x+t(x-y))$ such that $f(x) = g_1(0) = f_i(x) \neq g_2(0) = f_j(x)$ and $g_1(1) = f_i(y) \neq g_2(1) = f_j(y) = f(y)$. Therefore, $\exists 0 < t < 1$ such that $g_1(t) = g_2(t) = f_i(x + t(x-y)) = f_j(x + t(x-y))$ which contradicts the fact that $\forall x \in K_r, \forall i \neq j : f_i(x) \neq f_j(x)$.

Let $x \in K_r$ be such that $f(x) > f_i(x)$. By the first part of the proof,

$\exists j \neq i$ such that $\forall y \in K_r : f_j(y) = f(y)$. Since $f_j(x) > f_i(x)$, therefore, $K_r \subset \{z | f_j(z) > f_i(z)\}$. Therefore, $\forall y \in K_r : f_j(y) = f(y) > f_i(x)$. The same holds if $f(x) < f_i(x)$. ■

Theorem 3.2.9: *Let $f \in S_{lp}$ with components $\{f_i\}_{i=1}^{\infty}$. Let $\{K_r\}_{r \in M_T}$ be the cells generated by the components. Let $A_i = \bigcup\{K_r | \forall x \in K_r : f(x) = f_i(x)\}$. Let $O_i = \text{Int}(\overline{A_i})$. Then, $\{(\overline{O_i}, f_i)\}_{i=1}^{\infty}$ are exactly the characteristic pairs of f.*

Proof. First notice that if O_i is empty, then we can remove f_i and renumber the components. By setting $V = \mathbb{R}^m$ in **Lemma** 3.2.4 then $\exists \delta > 0$, $x \in \mathbb{R}^m$, and f_i such that $f|_{B(x,\delta)} \equiv f_i|_{B(x,\delta)}$. Therefore, $B(x, \delta) \subset A_i \cup \bigcup_{e=1}^{\infty} H_e$ and since $\bigcup_{e=1}^{\infty} H_e$ is a meagre set, therefore, $O_i \neq \emptyset$. We need to show that $\overline{O_i}$ as defined here satisfy the five conditions of **Lemma** 3.2.5, implying that $S_i = \overline{O_i}$.

Condition 5 is straightforward since f and f_i are continuous, so $f|_{\overline{O_i}} \equiv f_i|_{\overline{O_i}}$. For condition 1, by construction $O_i \neq \emptyset$, therefore $\forall i : \text{Int}(\overline{O_i}) \neq \emptyset$. Since O_i is open, then $O_i \subset \text{Int}(\overline{O_i})$ and therefore, $\overline{O_i} \subset \overline{\text{Int}(\overline{O_i})} \subset \overline{O_i}$. For condition 2, since f is a locally piecewise affine function, then f agrees with a finite number of components say $\{f_k\}_{k=1}^{m}$ on every compact set $\overline{B(x, \delta)}$ and the corresponding sets $\{A_k\}_{k=1}^{m}$ are dense in $\overline{B(x, \delta)}$. Therefore, $E_{x,\delta} = \{j \in \mathbb{N} | \overline{O_i} \cap B(x, \delta) \neq \emptyset\}$ is finite and not empty. For condition 3, notice that the set $A = \text{Int}(\overline{O_i}) \cap \text{Int}(\overline{O_j})$ is open. Let $x \in A$. Therefore, $\exists \delta > 0$ such that $\forall y \in B(x, \delta) : f(y) = f_i(y) = f_j(y)$. By **Lemma** 2.2.2, therefore, $f_i \equiv f_j$ and $\overline{O_i} = \overline{O_j}$. For condition 4, we will argue by contradiction. Assume that $\exists x$ such that $\forall i : x \notin \overline{O_i}$ which means $\forall i : \exists \delta_i$ such that $B(x, \delta_i) \cap O_i = \emptyset$. Pick $\delta > 0$ then $E_{x,\delta}$ is finite and clearly not empty. Therefore, $\exists$ at most $m \in \mathbb{N}$ with $m > 0$ such that $\forall i \leq m : B(x, \delta) \cap O_i \neq \emptyset$. Pick $\delta_0 < \min_{1 \leq i \leq m}\{\delta_i\}$ then $\forall j \in \mathbb{N} : B(x, \delta_0) \cap O_j = \emptyset$. By setting $V = B(x, \delta_0)$ in **Lemma** 3.2.4, $\exists \delta_s > 0, \delta_s \leq \delta_0$, $y \in B(x, \delta)$, and f_i such that $f|_{B(y,\delta_s)} \equiv f_i|_{B(y,\delta_s)}$. Therefore, $B(y, \delta_s) \subset A_i \cup \bigcup_{e=1}^{\infty} H_e$, therefore, $B(x, \delta) \cap O_i = B(x, \delta) \cap \text{Int}(\overline{A_i}) \neq \emptyset$ which is a contradiction. ■

Lemma 3.2.10: *Let $f \in S_{lp}$ with components $\{f_i\}_{i=1}^{\infty}$, then $\forall a, b \in \mathbb{R}^m, \exists i$ such that $f_i(a) \leq f(a)$ and $f_i(b) \geq f(b)$.*

Proof. Let $h : [0,1] \to \mathbb{R}^m$ be such that $\forall t \in [0,1] : h(t) = a + t(b-a)$. Let $g = f \circ h$, i.e., $g(t) = f(a + t(b-a))$. Since the set $C = \{a + t(b - a))|0 \leq t \leq 1\}$ is compact, therefore, $f : C \to \mathbb{R}$ is a piecewise affine function (continuous and agrees with a finite number of affine functions $\{f_i\}_{i=1}^k$ on C). Therefore, $\forall i : g_i \equiv f_i \circ h$ is an affine function, indeed: $f_i(a+t(b-a)) = v_i \cdot a + t((b-a)\cdot v_i) + b_i = m_i t + c_i$ where $m_i = (b-a)\cdot v_i$ and $c_i = b_i + v_i \cdot a$. Therefore, $g : [0,1] \to \mathbb{R}$ is a continuous function and agrees with the affine functions $\{g_i\}_{i=1}^n$. Then, by **Theorem** 2.2.6 and **Corollary** 2.2.13, g is a piecewise affine function. By **Corollary** 2.1.9 $\exists i \leq n$ such that $g_i(0) = f_i(a) \leq g(0) = f(a)$ and $g_i(1) = f_i(b) \geq g(1) = f(b)$.

■

Remark:

In **Theorem** 2.2.17, it was shown that piecewise affine functions are elements of the lattice generated by the components. Extending this theorem to locally piecewise affine functions requires taking infinite pointwise supremums and infimums of affine functions. In general, taking infinite pointwise sups and infs is not guaranteed to give continuous functions since $C(\mathbb{R}^m)$ is not Dedekind complete! At the same time, it is trivial to show that any continuous function is the pointwise supremum and/or infimum of affine functions. Therefore, no useful extension to **Theorem** 2.2.17 was found for locally piecewise affine functions.

Chapter 4

Order Convergence and Order Structures for Piecewise Affine Functions

In the first part of this chapter, the various definitions of order convergence that exist in the literature are investigated. Then, it is shown that two definitions agree on the space of continuous functions $C(\mathbb{R}^m)$. In the second part, the order structure of S and S_{lp} as subspaces of $C(\mathbb{R}^m)$ and with respect to each other is investigated.

4.1 Order Convergence

Definition 4.1.1: **Order Convergence I in Partially Ordered Sets:** Let E be a partially ordered set and let $\{f_\alpha\} \subset E$ be a net, we say that f_α converges in order (I) to $f \in E$ and write $f_\alpha \stackrel{o1}{\longrightarrow} f$ if $\exists \{h_\beta\}, \{g_\gamma\} \subset E$ such that $h_\beta \downarrow f$, $g_\gamma \uparrow f$, and $\forall \beta, \gamma : \exists \alpha_0(\beta, \gamma)$ such that $\forall \alpha \geq \alpha_0$ we have $g_\gamma \leq f_\alpha \leq h_\beta$.

Proposition 4.1.2: **Order Convergence I in Vector Lattices:** *Let E be a vector lattice and let $\{f_\alpha\} \subset E$ be a net, then $f_\alpha \stackrel{o1}{\longrightarrow} f$ if and only if $\exists \{k_\beta\} \subset E$ such that $k_\beta \downarrow 0$ and $\forall \beta : \exists \alpha_0(\beta)$ such that $\forall \alpha \geq \alpha_0$ we have $|f_\alpha - f| \leq k_\beta$.*

Proof. For the first direction, assume $f_\alpha \xrightarrow{o1} f$. Therefore, $\exists\{h_\beta\},\{g_\gamma\} \subset E$ such that $h_\beta \downarrow f$ and $g_\gamma \uparrow f$ and $\forall \beta,\gamma : \exists \alpha_0(\beta,\gamma)$ such that $\forall \alpha \geq \alpha_0$ we have $g_\gamma \leq f_\alpha \leq h_\beta$. Let $x_\beta = (h_\beta - f)$, $y_\gamma = (f - g_\gamma)$, and $k_{\beta,\gamma} = x_\beta \vee y_\gamma$. Clearly, $0 \leq k_{\beta,\gamma}$ and $\forall \beta : k_{\beta,\gamma} \downarrow$ and $\forall \gamma : k_{\beta,\gamma} \downarrow$. Claim: $0 = \inf_{\beta,\gamma}\{x_\beta \vee y_\gamma\}$. Proof: Using the vector lattice identities: $\forall \beta,\gamma : 0 \leq x_\beta \vee y_\gamma$, $-(x_\beta \wedge y_\gamma) \leq 0$, and $x_\beta + y_\gamma = x_\beta \vee y_\gamma + x_\beta \wedge y_\gamma$. Let t be another lower bound of $x_\beta \vee y_\gamma$, i.e., $\forall \beta,\gamma : t \leq x_\beta \vee y_\gamma \leq x_\beta + y_\gamma - x_\beta \wedge y_\gamma \leq x_\beta + y_\gamma$. Therefore, $\forall \beta,\gamma : t - x_\beta \leq y_\gamma \Rightarrow t - x_\beta \leq 0 \Rightarrow t \leq x_\beta \Rightarrow t \leq 0$ and the claim is proved.

Fix β,γ, then $\forall \alpha \geq \alpha_0 : (f_\alpha - f) \leq x_\beta$ and $(f - f_\alpha) \leq y_\gamma$, therefore, $|f_\alpha - f| \leq k_{\beta,\gamma}$. We can combine the net with indices β and γ such that we have one net with index $\lambda = (\beta,\gamma)$ with the natural partial order of $\mathbb{R}^2$. Therefore, $k_\lambda \downarrow 0$ and the first direction is proved.

For the opposite direction, assume $\exists\{k_\beta\} \subset E$ such that $k_\beta \downarrow 0$ and $\forall \beta : \exists \alpha_0(\beta)$ such that $\forall \alpha \geq \alpha_0$ we have $|f_\alpha - f| \leq k_\beta$. Let $h_\beta = k_\beta + f$ and $g_\beta = -k_\beta + f$ we have $\forall \alpha \geq \alpha_0(\beta)$:$f_\alpha - f \leq k_\beta \Rightarrow f_\alpha \leq k_\beta + f = h_\beta$ and $f - f_\alpha \leq k_\beta \Rightarrow -k_\beta + f \leq f_\alpha$. Therefore, $g_\beta \leq f_\alpha \leq h_\beta$ with $g_\beta = -k_\beta + f \uparrow f$ and $h_\beta = k_\beta + f \downarrow f$. Therefore, $f_\alpha \xrightarrow{o1} f$.

■

Remark:

It is important to note that we have two different index sets for f_α and k_β so that convergence wouldn't be altered if we add more terms at the beginning of the net f_α [5]. See [5] for examples showing that **Definition** 4.1.1 allows for adding more terms at the beggining of the net f_α without altering convergence. The following is another "stronger" definition for order convergence according to [6]:

Definition 4.1.3: **Order Convergence II in Partially Ordered Sets:** Let E be a partially ordered set and let $\{f_\alpha\} \subset E$ be a net, we say that f_α converges in order (II) to $f \in E$ and write $f_\alpha \xrightarrow{o2} f$ if $\exists\{h_\alpha\},\{g_\alpha\} \subset E$ such that $h_\alpha \downarrow f$, $g_\alpha \uparrow f$, and $\forall \alpha$ we have $g_\alpha \leq f_\alpha \leq h_\alpha$.

Proposition 4.1.4: **Order Convergence II in Vector Lattices:** *Let E be a vector lattice and let $\{f_\alpha\} \subset E$ be a net, then $f_\alpha \xrightarrow{o2} f$ if and only if*

$\exists\{k_\alpha\} \subset E$ such that $k_\alpha \downarrow 0$ and $\forall\alpha$ we have $|f_\alpha - f| \leq k_\alpha$.

Proof. The proof is analogous to that of **proposition** 4.1.2. ∎

Clearly from the definitions, order convergence II leads to order convergence I. In addition, [7] showed that there are two conditions that guarantee that order convergence I and order convergence II are equivalent. The first condition is that f_α has to be bounded and the second is that the set E has to be Dedekind complete. The importance of the boundedness of the net f_α is crucial as can be shown when considering $E = C(\mathbb{R})$. Consider the sets $A_1 = \{(x, 0)|x \in \mathbb{N}\}$ and $A_2 = \{(0, y)|y \in \mathbb{N}\}$. Consider the set $A = A_1 \cup A_2$ with the dictionary (lexicographical) order. A is a directed set. Consider the net $\{f_{(x,y)\in A}\} \subset C(\mathbb{R})$ such that $f_{(x,y)} \equiv y$. Clearly, $f_{(x,y)}$ does not order converge (II) since the beginning of the net is unbounded. However, the net order converges (I) to the function $f \equiv 0$. Some authors [5] and [2] use the following modified version of order convergence II to allow for the exclusion of the "beginning" of a net:

Definition 4.1.5: **Modified Order Convergence II in Partially Ordered Sets:** Let E be a partially ordered set and let $\{f_\alpha\} \subset E$ be a net, we say that f_α converges in order (mII) to $f \in E$ and write $f_\alpha \xrightarrow{mo2} f$ if $\exists\{h_\alpha\}, \{g_\alpha\} \subset E$ such that $h_\alpha \downarrow f$, $g_\alpha \uparrow f$, and $\exists\alpha_0$ such that $\forall\alpha \geq \alpha_0$ we have $g_\alpha \leq f_\alpha \leq h_\alpha$.

Proposition 4.1.6: **Modified Order Convergence II in Vector Lattices:** *Let E be a vector lattice and let $\{f_\alpha\} \subset E$ be a net, then $f_\alpha \xrightarrow{mo2} f$ if and only if $\exists\{k_\alpha\} \subset E$ such that $k_\alpha \downarrow 0$ and $\exists\alpha_0$ such that $\forall\alpha \geq \alpha_0$ we have $|f_\alpha - f| \leq k_\alpha$.*

Proof. The proof is analogous to that of **proposition** 4.1.2. ∎

Note that for bounded nets, modified order convergence (II) is equivalent to order convergence (II). [5] proved that if $f_\alpha \xrightarrow{o1} f$ in a vector lattice E, then $f_\alpha \xrightarrow{mo2} f$ in E^δ where E^δ is the order completion of E. Notice as well that in the above example, $f_{(x,y)} \xrightarrow{mo2} 0$.

There are examples given by [5] and [2] that show that modified order convergence (II) and order convergence (I) are not equivalent, i.e., even for bounded nets, order convergence (II) and order convergence (I) are not necessarily equivalent. However, [2] proved that order convergence (I) and modified order convergence (II) are equivalent when the partially ordered set E possess a special property which they named: "Property B". A more modern term is: "Countable Sup Property" as defined by [8] for vector lattices. We will use the modern term for the definitions.

Definition 4.1.7: **Countable Sup Property in Partially Ordered Sets:** Let E be a partially ordered set. E possesses the countable sup property if and only if:(a) if $\emptyset \neq M \subset E$ is increasing, with $\sup M = y$, then, $\exists J \subset M$ such that J is increasing, countable and $\sup J = y$, and (b) if $\emptyset \neq M \subset E$ is decreasing, with $\inf M = y$, then, $\exists J \subset M$ such that J is decreasing, countable and $\inf J = y$.

Proposition 4.1.8: **Countable Sup Property in Vector Lattices:** *Let E be a vector lattice. E possesses the countable sup property if and only if for every net $\{x_\alpha\} \subset E$ with $x_\alpha \uparrow x \in E$, there exists a subsequence $\{x_n\}_{n=1}^{\infty} \subset E$ such that $x_n \uparrow x$.*

Proof. The proof is straightforward using the definitions. ■

Note that the set $\mathbb{R}$ as a vector lattice has the countable sup property. If $\{x_\alpha\} \uparrow x \in \mathbb{R}$, then $\forall n \in \mathbb{N} : \exists x_n \in \{x_\alpha\}$ such that $x - \frac{1}{n} < x_n \leq x$ and $x_n \geq x_{n-1}$. The following Theorem (Theorem 8.22 in [9]) is important in identifying spaces that have the countable sup property.

Theorem 4.1.9: *If a vector lattice E admits a strictly positive linear functional, then E is Archimedean and has the countable sup property.*

Proof. Let f^* be a strictly positive linear functional on E. Let $a, b \in E$ be such that $a \neq 0$ and $\forall n : 0 \leq b \leq \frac{a}{n}$. Therefore, $0 \leq f^*(b) \leq \frac{f^*(a)}{n}$. Since $\mathbb{R}$ is Archimedean, therefore, $f^*(b) = 0$ and since f^* is strictly positive, we have $b = 0$. Therefore, E is Archimedean.

Let $\{x_\alpha\} \subset E$ be an increasing net with $x_\alpha \uparrow x \in E$. Since f^* is strictly positive, we have $f^*(x_\alpha) \uparrow$ and bounded by $f^*(x)$. Since $\{f^*(x_\alpha)\}$ is a bounded increasing net in $\mathbb{R}$, then it has a least upper bound. Let $s = \sup_\alpha\{f^*(x_\alpha)\}$. Since $\mathbb{R}$ has the countable sup property, then $\exists\{x_n\}_{n=1}^\infty$ such that $f^*(x_n) \uparrow s$ and $x_n \uparrow$. Note that $\forall\alpha, \forall n : f^*(x_\alpha) \vee f^*(x_n) \le f^*(x_\alpha \vee x_n)$ and since $x_\alpha \vee x_n \subset \{x_\alpha\}$ we have $\sup_n \{f^*(x_\alpha \vee x_n)\} = s$. We claim that $x_n \uparrow x$. Indeed, let $x_n \le g \in E$. Therefore, $f^*(x_n) \le f^*(g)$. Also, $\forall\alpha : 0 = f^*(0) \le f^*((x_\alpha - g) \vee 0) \le f^*((x_\alpha - x_n) \vee 0) = f^*(x_\alpha \vee x_n) - f^*(x_n) \xrightarrow{n\to\infty} s - s = 0$. Since f^* is strictly positive, therefore, $(x_\alpha - g) \vee 0 = 0$. i.e., $x_\alpha \le g$. Therefore, g is also an upper bound for $\{x_\alpha\}$, therefore, $g \ge x$. Therefore, $x_n \uparrow x$. ■

Corollary 4.1.10: Let $x_1 < x_2 \in \mathbb{R}$. Then, $C([x_1, x_2])$ possesses the countable sup property.

Proof. Since the Rieman integral operator is a strictly positive functional on $C([x_1, x_2])$, then by **Theorem** 4.1.9, $C([x_1, x_2])$ possesses the countable sup property. ■

Now that we have shown that $\forall x_1 < x_2 \in \mathbb{R} : C([x_1, x_2])$ possesses the countable sup property, we wish to extend this to show that $C(\mathbb{R})$ also possesses the countable sup property.

Lemma 4.1.11: *Let* $\{f_\alpha\}_{\alpha\in\Gamma} \subset C(\mathbb{R}^m)$ *be an increasing net with* $\sup_\alpha \{f_\alpha\} = f \in C(\mathbb{R}^m)$. *Let* $n \in \mathbb{N}$ *and consider the interval* $I_n = [-n, n]$ *and* $K = (I_n)^m$. *The restriction of the continuous functions* $\{f_\alpha\}$ *on* $C(K)$ *satisfy* $\sup_\alpha\{f_\alpha|_K\} = f|_K$ *where* $\sup_\alpha\{f_\alpha|_K\} \in C(K)$.

Proof. We will argue by contradiction. Assume $\exists g \in C(K)$ such that $\sup_\alpha\{f_\alpha|_K\} = g|_K$ and $g|_K < f|_K$. If $g|_{\partial K} = f|_{\partial K}$ then the function:

$$\tilde{g}(x) = \begin{cases} g(x), & x \in K \\ f(x), & Otherwise \end{cases}$$

is such that $\tilde{g} \in C(\mathbb{R}^m)$, $\tilde{g} < f$ and is an upper bound for $\{f_\alpha\}$ which is a contradiction.

Assume then that $\exists x_0 \in \partial K$ such that $g(x_0) < f(x_0)$. Let $\varepsilon = f(x_0) - g(x_0)$. Since both g and f are continuous, $\exists\delta_0 > 0$ such that $\forall x \in B(x_0, \delta_0)\cap$

$K : |f(x) - f(x_0)| \leq \varepsilon/4$, $|g(x) - g(x_0)| \leq \varepsilon/4$, and therefore, $f(x) > g(x)$. Since $x_0 \in \partial K$, then $B(x_0, \delta_0) \cap \text{Int}(K) \neq \emptyset$, therefore, $\exists \delta_1 > 0, x_1 \in B(x_0, \delta_0) \cap \text{Int}(K)$ such that $B(x_1, \delta_1) \subset B(x_0, \delta_0) \cap \text{Int}(K)$. Notice that $\forall x \in B(x_1, \delta_1), \forall \alpha : f(x) > g(x) > f_\alpha(x)$. Consider the function:

$$\tilde{g}(x) = \begin{cases} g(x) - \frac{\|x - x_1\|}{\delta_1}(g(x) - f(x)) & x \in B(x_1, \delta_1) \\ f(x), & Otherwise \end{cases}$$

$\tilde{g} \in C(\mathbb{R})$, $\tilde{g} < f$ and is an upper bound for $\{f_\alpha\}$ which is a contradiction.

■

Theorem 4.1.12: *$C(\mathbb{R})$ possesses the countable sup property.*

Proof. Let $\{f_\alpha\} \subset C(\mathbb{R})$ be a net such that $f_\alpha \uparrow f \in C(\mathbb{R})$. By **Lemma** 4.1.11 $\sup_\alpha \{f_\alpha|_{I_n}\} = f|_{I_n}$. By **Corollary** 4.1.10, there exists a countable sequence $\{f_{n_m}|_{I_n}\}_{m=1}^\infty$ such that $f_{n_m}|_{I_n} \uparrow f|_{I_n}$. Since $\mathbb{R}$ is a countable union $\mathbb{R} = \bigcup_{n=1}^\infty I_n$ then we can construct a sequence of functions $\{f_{\beta_i}\}_{i=1}^\infty$ such that $f_{\beta_i} \uparrow f$ as follows: Let $f_{\beta_1} = f_{1_1}$. Let f_{β_2} be such that $f_{\beta_2} \geq \{f_{1_2}, f_{2_1}, f_{\beta_1}\}$. Let f_{β_n} be such that $f_{\beta_n} \geq \{f_{1_n}, f_{2_{n-1}}, f_{3_{n-2}}, \ldots, f_{n_1}, f_{\beta_{n-1}}\}, \ldots$ etc. Clearly, $f_{\beta_i} \uparrow f$ otherwise, there is another continuous function $f_2 < f$ such that it is an upper bound of f_{β_i}. However, for every interval $I_n = [-n, n] : f_{\beta_i}|_{I_n} \uparrow f_{I_n}$ which contradicts that $f_{\beta_i} \leq f_2 < f$.

■

Corollary 4.1.13: Let $\{f_\alpha\} \subset C(\mathbb{R})$ be a net and $f \in \mathbb{C}(\mathbb{R})$. Then $f_\alpha \xrightarrow{o1} f \Leftrightarrow f_\alpha \xrightarrow{mo2} f$.

Proof. By **Theorem** 4.1.12, $C(\mathbb{R})$ possesses the countable sup property. Therefore, using the main result of [2], order convergence (I) and modified order convergence (II) are equivalent on $C(\mathbb{R})$. ■

Corollary 4.1.14: Let $\{f_\alpha\} \subset C(\mathbb{R})$ be a bounded net and $f \in \mathbb{C}(\mathbb{R})$. Then $f_\alpha \xrightarrow{o1} f \Leftrightarrow f_\alpha \xrightarrow{o2} f$.

Proof. By **Theorem** 4.1.12, $C(\mathbb{R})$ possesses the countable sup property. Therefore, using the main result of [2], order convergence I and modified order convergence (II) are equivalent on $C(\mathbb{R})$, but since the net is bounded, then, modified order convergence (II) is equivalent to order convergence (I). ■

Remark:

Corollary 4.1.10 can be extended naturally to $C(I_1 \times I_2 \times I_3 \times \cdots \times I_m)$ where $\forall i \leq m : I_i = [x_{i1}, x_{i2}] \subset \mathbb{R}$. In addition, **Theorem** 4.1.12 and **Corollaries** 4.1.13 and 4.1.14 can be extended naturally to $C(\mathbb{R}^m)$ by considering the compact sets $K = I_n \times I_n \times \cdots \times I_n = (I_n)^m$ with $I_n = [-n, n]$.

4.2 Order Structure for S and S_{lp}

In this section we present three different definitions available in the literature for order dense subspaces in a vector lattice. Since we are concerned with $C(\mathbb{R}^m)$, the definitions given here, when applicable, will utilize the modified order convergence (II) in vector lattices as per **Definition** 4.1.6. We will then investigate S and S_{lp} with respect to each other and as as subspaces of $C(\mathbb{R}^m)$.

Definition 4.2.1: **Toplogically Order Dense:** Let $A \subset E$ where E is a vector lattice. A is topologically order dense in E if $\forall x \in E : \exists \{y_\alpha\} \subset A$ such that $y_\alpha \xrightarrow{mo2} x$.

Definition 4.2.2: **Order Dense Minorizing Sublattice:** Let A be a sublattice of a vector lattice E. A is an order dense minorizong sublattice in E if $\forall x \in E^+, \exists y \in A$ such that $0 < y \leq x$.

Definition 4.2.3: **Interval Order Dense:** Let $A \subset E$ where E is a vector lattice. A is interval order dense in E if $\forall x < y \in E, \exists z \in A$ such that $x < z < y$. This definition is motivated by the fact that an order on a partially ordered set E is termed a "dense order" if $\forall x < y \in E : \exists z \in E$ such that $x < z < y$.

Remarks:

- S is not interval order dense in $C(\mathbb{R})$. Consider the functions $f_1(x) = \sin x$ and $f_2(x) = 0.9 \sin x$. Obviously, it is not possible to fit a piecewise affine function between the two curves due to the periodic nature of the functions.
- S_{lp} is not interval order dense in $C(\mathbb{R})$. Consider any two functions $f_1 < f_2$ that are nonlinear and equal on $\mathbb{R} \setminus I$, where I is a compact interval and $f_1|_I < f_2|_I$. Then, it is not possible to fit a locally finite piecewise affine function between the two curves on the portion where $f_1 = f_2$.

Lemma 4.2.4: *Interval order dense is a stronger condition than order dense minorizing sublattice. In other words, let A be a sublattice of a vector lattice E. Assume A is an interval order dense set, then A is an order dense minorizing sublattice in E.*

Proof. Assume A is an interval order dense set. Therefore, $\forall 0 < x \in E^+ : \exists z \in A$ such that $0 < z < x$. Therefore, A is an order dense minorizing sublattice. ∎

Lemma 4.2.5: *(From problem 6: page 66. in [7]): Assume E is an Archimedean vector lattice. A is a sublattice. A is an order dense minorizing sublattice in E if and only if $\forall x \in E^+, \{y \in A | 0 < y \leq x\} \uparrow x$.*

Proof. : Assume $\forall x \in E^+, \{y \in A | 0 < y \leq x\} \uparrow x$. Therefore, $\exists y_0 \in A$ such that $0 < y_0 \leq x$. Therefore, A is an order dense minorizong sublattice. For the opposite direction, assume A is an order dense minorizong sublattice in E. Let $C = \{y \in A | 0 < y \leq x\}$. Therefore, $x \geq C$ and $C \neq \emptyset$. Assume that x is not the least upper bound of C, therefore, $\exists z < x$ such that $z \geq C$. Let $u = x - z$. Note that $x > u = x - z > 0$. Therefore, $\exists v \in A$ such that $0 < v \leq u = x - z < x$. Therefore, $v \in C$ and $v \leq z$. Therefore, $v + v \leq u + z = x$. The proof follows by induction. First fix n and assume $nv \leq x$. Therefore, $nv \in C$ and $nv \leq z$. We also have $v \leq u$. Therefore, $v + nv \leq u + z \leq x \Rightarrow (1 + n)v \leq x$. Therefore, $\forall n : 0 < nv \leq x$ and since E is Archimedean, $v = 0$ which is a contradiction. ∎

Lemma 4.2.6: *In Archimedean spaces: Order dense minorizong sublattice is a stronger condition than topologically order dense. In other words, let A be a sublattice in an Archimedean vector lattice E. Assume A is an order dense minorizing sublattice of E, then A is topologically order dense in E.*

Proof. Assume A is an order dense minozirong sublattice of E. By **Lemma** 4.2.5, $\forall 0 < x \in E^+$ the set $C = \{z \in A | 0 < z \leq x\} \uparrow x$. Therefore, $\exists\{y_\alpha\} \subset A$ such that $y_\alpha \xrightarrow{mo2} x$. Now let $x \in E$. Therefore, $x = x^+ - x^-$. Therefore, $\exists\{y_\alpha\} \subset A$ and $\exists\{z_\beta\} \subset A$ such that $\{y_\alpha\} \uparrow x^+$ and $\{z_\beta\} \uparrow x^-$. Therefore, $|y_\alpha - z_\beta - x^+ + x^-| \leq |y_\alpha - x^+| + |z_\beta - x^-| \downarrow 0$, therefore, $y_\alpha - z_\beta \xrightarrow{mo2} x$.

■

Remark

In general, an order dense minorizong sublattice is not equivalent to a toplogically order dense sublattice. As a counter example, consider the space $A = C[0,1] + G$ where $G = \{f : [0,1] \to \mathbb{R} | \{x | f(x) \neq 0\}$ is finite$\}$ (G is the set of real valued functions on $[0,1]$ with finite support), then $C[0,1]$ is a topologically order dense sublattice in A. However, it is not order dense minorizing since the function

$$g(t) = \begin{cases} 1, & t = \frac{1}{2} \\ 0, & Otherwise \end{cases}$$

is such that $g \in A^+$ but $\exists$ no function $f \in C[0,1]^+$ such that $f \leq g$. This last example shows that, in general, a topologically order dense sublattice is not necessarily an order dense minorizong sublattice. However, an interesting question is to check whether a toplogically order dense sublattice in $C(\mathbb{R}^m)$ is necessarily an order dense minorizinong sublattice. While we were not able to answer this question, here are some results regarding order convergence and topologically order dense sublattices in $C(\mathbb{R}^m)$.

Lemma 4.2.7: *Let $D \subset C(\mathbb{R}^m)$ be a topologically order dense sublattice. Then, the set $A = \{x | \forall f \in D : f(x) = 0\}$ is closed and nowhere dense.*

Proof. A is closed is straightforward from the continuity of the functions in D since if $x_n \xrightarrow{n\to\infty} x$ where $x_n \in A$, then $\forall f \in D : f(x_n) = 0 \xrightarrow{n\to\infty} f(x)$. Therefore, $f(x) = 0$. Therefore, $x \in A$. To show that it is nowhere dense, we will argue by contradiction. Assume that $\mathrm{Int}(A) \neq \emptyset$. Therefore, $\exists \delta > 0$ and $x \in \mathbb{R}^m$ such that $B(x, \delta) \in A$. Consider the function:

$$g(t) = \begin{cases} 1 - \frac{\|t-x\|}{\delta}, & t \in B(x,\delta) \\ 0, & Otherwise \end{cases}$$

Then, g is continuous. However, there is no sequence (or net) in A that would converge in order (modified order converge II) to g since $\forall f \in D$: $|f|_{B(x,\delta)} - g|_{B(x,\delta)}| = g|_{B(x,\delta)}$. ■

Lemma 4.2.8: *Let $f_\alpha \downarrow 0$ in $C(\mathbb{R}^m)$, be such that $\exists x \in \mathbb{R}^m$ such that $\inf_\alpha f_\alpha(x) > 0$, then the function $g(y) = \inf_\alpha f_\alpha(y)$ is not continuous at x.*

Proof. Since $f_\alpha \downarrow 0$, then, $\forall h \in C(\mathbb{R}^m)^+ : \exists x_h \in \mathbb{R}^m$ and β such that $f_\beta(x_h) \leq h(x_h)$. Let $\varepsilon = g(x)$. Consider the sequence of functions:

$$g_n(t) = \begin{cases} \frac{\varepsilon}{n}\left(1 - \frac{\|t-x\|}{\frac{1}{n}}\right), & t \in B(x, \frac{1}{n}) \\ 0, & Otherwise \end{cases}$$

then, $\forall n \in \mathbb{N} : \exists x_n$ and β_n such that $f_{\beta_n}(x_n) \leq g_n(x_n) \leq \varepsilon/n$. Therefore, $x_n \xrightarrow{n\to\infty} x$, however, $g(x_n) = \inf_\alpha f_\alpha(x_n) \leq f_{\beta_n}(x_n) \xrightarrow{n\to\infty} 0 \neq g(x) = \inf_\alpha f_\alpha(x) = \varepsilon$.

■

Conjectures

- Let $B \subset C(\mathbb{R}^m)$ be a sublattice. Then, B is order dense minorizing if and only if B is topologically order dense.
- Let $f_n \downarrow 0$ in $C(\mathbb{R}^m)$, then the set $A = \{x \in \mathbb{R}^m | \inf_n f_n(x) > 0\}$ does not contain an open ball.

Lemma 4.2.9: *S is an order dense minorizing sublattice in $C(\mathbb{R})$.*

Proof. Let $f \in C(\mathbb{R})$ be such that $f > 0$. Therefore, $\exists x \in \mathbb{R}$ such that $f(x) = a > 0$. Let $\varepsilon = \frac{a}{2}$, then $\exists \delta > 0$ such that $\forall y$ with $|x - y| \leq \delta$ we have $f(y) \geq \frac{a}{2}$. Consider the compact interval $K = [x - \delta, x + \delta]$. Set $x_1 = x - \delta$ and $x_2 = x + \delta$. Consider the function

$$g(h) = \begin{cases} \frac{a}{2(x-x_2)}(h - x_2) & x \leq h \leq x_2 \\ \frac{a}{2(x-x_1)}(h - x_1) & x_1 \leq h \leq x \\ 0, & Otherwise \end{cases}$$

Then, $g \in S$, $g > 0$, and $g < f$. Therefore, S is an order dense minorizing sublattice in $C(\mathbb{R})$. ■

Lemma 4.2.10: *S is an order dense minorizing sublattice in $C(\mathbb{R}^m)$.*

Proof. Let $f \in C(\mathbb{R}^m)$ be such that $f > 0$. Therefore, $\exists x \in \mathbb{R}^m$ such that $f(x) = a > 0$. Let $\varepsilon = \frac{a}{2}$, then $\exists \delta > 0$ such that $\forall y \in B(x, \delta)$ we have $f(y) \geq \frac{a}{2}$. Without loss of generality, $x = 0$. Let $\{e_i\}_{i=1}^{m}$ be the standard orthonormal basis set for $\mathbb{R}^m$. Let $\forall i \leq m : f_i(y) = \frac{a}{4}\left(1 - \frac{4}{\delta}(e_i \cdot y)\right)$ and $g_i(y) = \frac{a}{4}\left(1 + \frac{4}{\delta}(e_i \cdot y)\right)$. Consider the function $g = \bigwedge_i ((f_i \vee 0) \wedge (g_i \vee 0))$. Let $C = \{y | |y_i| \leq \frac{\delta}{4}\}$. $C \subset B(0, \delta)$ and $\forall y \notin C : g(y) = 0 \leq f(y)$. Also, $\forall y \in C : g(y) \leq \frac{a}{4} < f(y)$. Therefore, $0 < g < f$ and $g \in S$. Therefore, S is an order dense minorizong sublattice in $C(\mathbb{R}^m)$. ■

Corollary 4.2.11: S is topologically order dense in $C(\mathbb{R})$.

Proof. This is straight forward by **Lemmas** 4.2.6 and 4.2.9 and since $C(\mathbb{R})$ is Archimedean. ■

Corollary 4.2.12: S is topologically order dense in $C(\mathbb{R}^m)$.

Proof. This is straight forward by **Lemmas** 4.2.6 and 4.2.10 and since $C(\mathbb{R}^m)$ is Archimedean.

■

Lemma 4.2.9 and **Corollary** 4.2.11 are implied by **Lemma** 4.2.10 and **Corollary** 4.2.12 but they were presented here since the ideas of the higher dimensional space are clearer in the one dimensional case. Also, since $S \subset S_{lp}$, then the results apply to S_{lp}, i.e., S_{lp} is topologically order dense in $C(\mathbb{R}^m)$.

Another approach to show that S is topologically order dense in $C(K)$ where $K \subset \mathbb{R}^m$ relies on the Stone-Weierstrass approximation theorem and is presented here as well.

Lemma 4.2.13: *Real functions on compact sets can be uniformly approximated with piecewise affine functions. In other words, let $K \subset \mathbb{R}^m$ be compact, S is uniformly dense in $C(K)$.*

Proof. $g \equiv 1 \in S$. Let $a \neq b \in K$. Let $f(x) = (a-b) \cdot x$ be an affine function defined on K. $f(a) - f(b) = (a-b) \cdot (a-b) \neq 0$. Therefore, S separates points in K. Using the lattice version of the Stone-Weierstrass approximation theorem (see **Theorem** 11.3 page 88 in [4]), S is uniformly dense in $C(K)$. ∎

Lemma 4.2.14: *Let $K \subset \mathbb{R}^m$, K is compact. Then, S is topologically order dense in $C(K)$.*

Proof. By **Lemma** 4.2.13, S is uniformly dense in $C(K)$, therefore, $\forall f \in C(K) : \exists f_n \in S$ such that $f_n \to f$ using the *sup* norm. By passing to a subsequence, $\forall x \in K : |f_m(x) - f(x)| \leq \frac{1}{m}$. Let $g \equiv 1 \in S$. Therefore, $|f_m - f| \leq \frac{1}{m} g$. Let $h_m = \frac{1}{m} g$. Clearly, $h_m \downarrow 0$, therefore, $f_m \xrightarrow{mo2} f$. ∎

Remarks:

One cannot uniformly approximate continuous (bounded or not) functions in $C(\mathbb{R})$ using piecewise affine functions. For example, consider the function $f \in C(\mathbb{R}) : f(x) = \sin x$. Because piecewise affine functions have finite number of components, $f(x)$ cannot be uniformly approximated using piecewise affine functions. However, we will show that locally piecewise affine functions can approximate any function in $C(\mathbb{R})$ which is also a different approach to show that S_{lp} is topologically order dense in $C(\mathbb{R})$. Note that the classical Stone-Weierstrass theorem cannot be used for $C(\mathbb{R})$

or for $C_b(\mathbb{R}) = \{f \in C(\mathbb{R}) | f \text{ is bounded}\}$. Clearly, S separates points in $\mathbb{R}$ and contains the unit function, yet it cannot approximate the function $f(x) = \sin x$, i.e., it is not uniformly dense in the Banach space $C_b(\mathbb{R})$.

Lemma 4.2.15: *Functions in $C(\mathbb{R})$ can be uniformly approximated by functions in S_{lp}. In other words, $\forall \varepsilon, \forall f \in C(\mathbb{R}) : \exists g \in S_{lp}$ such that* $\sup |f - g| \leq \varepsilon$.

Proof. Let $f \in C(\mathbb{R})$. Fix $\varepsilon > 0$. Consider $I_n = [n-1, n] \in \mathbb{R}, n \in \mathbb{Z}$. Since f is continuous and I_n is compact, therefore, $\exists \delta_n$ such that $\forall x \in I_n : \forall y \in [x - \delta_n, x + \delta_n] : |f(y) - f(x)| < \frac{\varepsilon}{2}$. Let $M_n = \lceil \frac{1}{\delta_n} \rceil$, $x_0 = n - 1$, $x_i = x_0 + \frac{i}{M_n}$, $1 \leq i \leq M_n$. Define the function $g_n : I_n \to \mathbb{R}$:

$$g_n(x) = \frac{f(x_{i+1})(x_{i+1} - x)}{(x_{i+1} - x_i)} + \frac{f(x_i)(x - x_i)}{(x_{i+1} - x_i)}, \qquad x_i \leq x \leq x_{i+1}$$

Clearly, g_n is a piecewise affine function with a finite number of components and $\sup |(f - g_n)|_{I_n}| \leq \varepsilon$. Let $g : \bigcup_{n=-\infty}^{\infty} I_n \to \mathbb{R}$ such that $g(x) = g_n(x)$ whenever $x \in I_n$. Since, $\forall n : g_n(n) = g_{n+1}(n)$ and $g_{n-1}(n-1) = g_n(n-1)$. Therefore, the function g is well defined and is continuous. Also, clearly for every compact set C that is the union of at most finite number of connected components we have $g : C \to \mathbb{R}$ is piecewise affine, therefore, using **Lemma** 3.2.2, $g \in S_{lp}$ and $\sup |f - g| \leq \varepsilon$. ■

Lemma 4.2.16: *S_{lp} is topologically order dense in $C(\mathbb{R})$.*

Proof. Let $f \in C(\mathbb{R})$. By **Lemma** 4.2.15, $\forall m \in \mathbb{N} : \exists f_m \in S_{lp}$ such that $\forall x \in \mathbb{R} : |f_m(x) - f(x)| \leq \frac{1}{m}$. Let $g \equiv 1 \in S_{lp}$. Therefore, $|f_m - f| \leq \frac{1}{m} g$. Let $h_m = \frac{1}{m} g$. Clearly, $h_m \downarrow 0$, therefore, $f_m \xrightarrow{mo2} f$.

Alternatively, **Corollary** 4.2.11 can be used to prove the lemma. Since $S \subset S_{lp}$ and using **Corollary** 4.2.11, then, S_{lp} is topologically order dense in $C(\mathbb{R})$. ■

In the following part, we will attempt to describe S_{lp} as the order closure of a class of sets termed: "locally finite" in S.

Definition 4.2.17: **Locally Finite Set of Functions:** Let $F \subset C(\mathbb{R}^m)$. F is termed a locally finite set of functions if $\forall A \subset \mathbb{R}^m$ where A is compact, the set $E_A = \{f \in F | \exists x \in A : f(x) \neq 0\}$ is finite. If F is a sequence, then F is termed a locally finite sequence of functions. Clearly, any subset of a locally finite set of functions is another locally finite set of functions. Additionally, if $F \subset C(\mathbb{R}^m)$ is a locally finite set of functions, then $F^{\vee}$, $F^{\wedge}$ and consequently $F^{\vee\wedge}$ are all locally finite sets of functions.

Lemma 4.2.18: *Let $F \subset C(\mathbb{R}^m)$ be a locally finite set of functions. Then, $f_a = \sup F$ and $f_b = \inf F$ are well defined and $f_a, f_b \in C(\mathbb{R}^m)$.*

Proof. Assume F is at least countably infinite, otherwise the statement is trivial. Let $A, B \subset \mathbb{R}^m$ be compact. Since E_A is finite, then $\exists g, h \in F$ such that $g(A) = 0$ and $h(B) = 0$. Let $f_A = \left(\bigvee_{f \in E_A} f|_A\right) \vee 0 \in C(A)$. Similarly $f_B \in C(B)$. In addition, the function

$$f_{AB}(x) = \begin{cases} f_A(x) & x \in A \\ f_B(x) & x \in B \end{cases}$$

is continuous since if $x \in A \cap B$, then $f_A(x) = f_B(x)$. Since $\mathbb{R}^m$ can be expressed as $\mathbb{R}^m = \bigcup_{i=1}^{\infty} A_i$ where A_i is compact, then, the function $g : \mathbb{R}^m \to \mathbb{R}$ such that $\forall i, \forall x \in A_i : g(x) = f_{A_i}(x)$ is well defined and continuous. In addition, $\forall i, \forall x \in A_i : f_a(x) = g(x)$. Therefore, $f_a(x) \in C(\mathbb{R}^m)$. The same applies to f_b. ■

Lemma 4.2.19: *Let $F \subset S \subset C(\mathbb{R}^m)$ be a locally finite set of functions. Then, $f_a = \sup F$ and $f_b = \inf F$ are such that $f_a, f_b \in S_{lp}$.*

Proof. By **Lemma** 4.2.18, f_a and f_b are continuous function. Since on any compact set, f_a and f_b agree with a finite number of affine functions, then using **Lemma** 3.2.2, f_a and $f_b \in S_{lp}$. ■

Lemma 4.2.20: *Given a hyperplane $H_1 = \{x | a_1 \cdot x + b_1 = 0\}$ where $a_1 \in \mathbb{R}^m, b_1 \in \mathbb{R}$ and an affine function $f_1(x) = v_1 \cdot x + t_1$, $v_1 \in \mathbb{R}^m$ and $t_1 \in \mathbb{R}$. Then, an affine function $f(x) = v \cdot x + t$, with $v \in \mathbb{R}^m$ and $t \in \mathbb{R}$ is such that $f|_{H_1} = f_1|_{H_1}$ if and only if $\exists \alpha \in \mathbb{R}$ such that $v = v_1 - \alpha a_1$ and $t = t_1 - \alpha b_1$. In that case f is uniquely determined by α.*

Proof. One direction is straightforward. If $\exists \alpha \in \mathbb{R}$ such that $v = v_1 - \alpha a_1$ and $t = t_1 - \alpha b_1$. Therefore, $\forall x \in H_1 : f(x) - f_1(x) = (v - v_1) \cdot x + t - t_1 = -\alpha(a_1 \cdot x + b_1) = 0$. Therefore, $f|_{H_1} = f_1|_{H_1}$.

For the opposite direction: assume that $f|_{H_1} = f_1|_{H_1}$. Therefore, $\forall x \in H_1 : (v - v_1) \cdot x + t - t_1 = 0$. We will argue by contradiction, i.e., assume that $v - v_1$ and a_1 are not linearly dependent. Let the orthogonal project of $v - v_1$ on a_1 by αa_1, therefore, $\exists \alpha \in \mathbb{R}$ and $u \in \mathbb{R}^m, u \neq 0$ such that $v - v_1 = \alpha a_1 + u$ and $u \cdot \alpha a_1 = 0 \neq (v - v_1) \cdot u$. Since $a_1 \cdot (x + u) + b_1 = a_1 \cdot x + b_1 = 0$, therefore $x + u \in H_1$. However, $(v - v_1) \cdot (x + u) + t - t_1 = (v - v_1) \cdot u \neq 0$ which contradicts that $f|_{H_1} = f_1|_{H_1}$. Therefore,$\exists \alpha \in \mathbb{R}$ such that $v = v_1 - \alpha a_1$. Showing that $t = t_1 - \alpha b_1$ is straightforward.

■

Lemma 4.2.21: *Given two hyperplanes* $H_1 = \{x | a_1 \cdot x + b_1 = 0\} \neq H_2 = \{x | a_2 \cdot x + b_2 = 0\}$ *where* $a_1, a_2 \in \mathbb{R}^m$ *and* $b_1, b_2 \in \mathbb{R}$. *Given two affine functions* $f_1(x) = v_1 \cdot x + t_1, v_1 \in \mathbb{R}^m, t_1 \in \mathbb{R}$ *and* $f_2(x) = v_2 \cdot x + t_2, v_2 \in \mathbb{R}^m, b_2 \in \mathbb{R}$. *Then, there exists an affine function* $f(x) = v \cdot x + t$ *such that* $f|_{H_1} = f_1|_{H_1}$ *and* $f|_{H_2} = f_2|_{H_2}$ *if and only if* $\exists \alpha_1, \alpha_2 \in \mathbb{R}$ *such that* $v_1 - v_2 = \alpha_1 a_1 + \alpha_2 a_2$ *and* $t_1 - t_2 = \alpha_1 b_1 + \alpha_2 b_2$. *In this case* $v = v_1 - \alpha_1 a_1 = v_2 + \alpha_2 a_2$ *and* $t = t_1 - \alpha_1 b_1 = t_2 + \alpha_2 b_2$. *In addition,* α_1 *and* α_2 *are unique.*

Proof. One direction is straightforward. Let $f(x) = v \cdot x + t$ be as described. Then, $\forall x \in H_1 : f(x) = (v_1 - \alpha_1 a_1) \cdot x + t_1 - \alpha_1 b_1 = f_1(x) - \alpha_1(a_1 \cdot x + b_1) = f_1(x)$. Similarly, $\forall x \in H_2 : f(x) = (v_2 + \alpha_2 a_2) \cdot x + t_2 + \alpha_2 b_2 = f_2(x) + \alpha_2(a_2 \cdot x + b_2) = f_2(x)$.

For the opposite direction, assume that there exists an affine function $f(x) = v \cdot x + t$ such that $f|_{H_1} = f_1|_{H_1}$ and $f|_{H_2} = f_2|_{H_2}$. By **Lemma** 4.2.20 $\exists \alpha_1, \alpha_2 \in \mathbb{R}$ such that $v = (v_1 - \alpha_1 a_1) = (v_2 + \alpha_2 a_2)$ and $t = t_1 - \alpha_1 b_1 = t_2 + \alpha_2 b_2$. Therefore, we reach the following equation that dictates the possible values for α_1 and α_2 so that the two conditions $f|_{H_1} = f_1|_{H_1}$ and $f|_{H_2} = f_2|_{H_2}$ are simultaniously satisfied:

$$\begin{pmatrix} v_1 - v_2 \\ t_1 - t_2 \end{pmatrix} = \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}$$

If the underlying space is one dimensional ($m = 1$), and since $H_1 \neq H_2$, therefore, $a_1 b_2 - a_2 b_1 \neq 0$ which means that there is a unique solution for α_1 and α_2. Therefore, we can always find f, α_1, and α_2 that satisfy the lemma. When $m > 1$, then the above equation is an overconstrained system of equations ($m+1$ equations) with only two unknowns and if there are two constants α_1 and α_2 satisfying the $m+1$ equations, then, they are unique.

■

Lemma 4.2.22: *Let $f \in S^+ \subset C(\mathbb{R}^m)$ with components $\{f_i\}_{i=1}^n$, then, $\forall \varepsilon > 0 : \exists k \in S$ such that $f|_K = k|_K, f|_{\mathbb{R}^m \setminus K} > k|_{\mathbb{R}^m \setminus K}$ and $k|_{\mathbb{R}^m \setminus (K \pm \varepsilon [0,1]^m)} \equiv 0$ where $K = [0,1]^m$.*

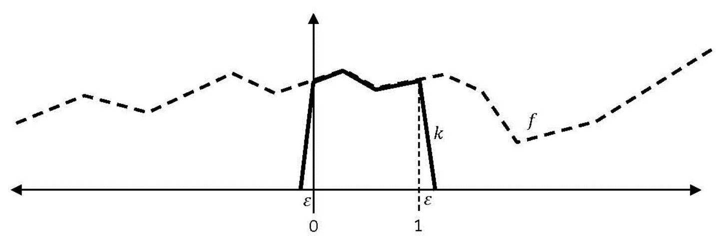

Figure 4.1: $f, k \in S^+ \subset C(\mathbb{R})$ satisfying **Lemma** 4.2.22

Proof. **Figure** 4.1 shows the construction for $m = 1$. Let $\{K_j\}_{j=1}^J$ be the finite number of cells generated by the components of f. $\forall j \leq J : K_j$ is open and convex. Let $L_j = K_j \cap \text{Int}(K)$. Therefore, L_j is open and convex. Fix j such that $L_j \neq \emptyset$. ∂L_j is formed by a finite set of hyperplanes (either the hyperplanes generated by $\{f_i\}_{i=1}^n$ or the boundaries of K) associated with a finite number of neighbouring cells. By **Lemma** 2.2.14, $\exists i \leq n$ such that $f|_{L_j} \equiv f_i|_{L_j}$. Assume $f_i(x) = v_i \cdot x + t_i, v_i \in \mathbb{R}^m, b_i \in \mathbb{R}$. Let $h = \max_{x \in \overline{L_j}}\{f(x)\}$. h is well defined since f is continous and $\overline{L_j}$ is compact. Let $h_j \equiv f_i - 4h$, i.e., $h_j(x) = v_i \cdot x + t_i - 4h$. Let $E_j = \{K_i | \overline{K_i} \cap \overline{L_j} \neq \emptyset\}$. E_j is finite. Let $\{H_i = \{x | a_i \cdot x + b_i = 0\}\}_{i=1}^l$ be the hyperplanes that form the boundaries of L_j such that $\forall i : L_j \subset \{x | a_i \cdot x + b_i < 0\}$. Let L_{j_ε} be a slightly larger cell whose boundaries are the hyperplanes $\{H_{i\varepsilon} = \{x | a_i \cdot x + b_i - \varepsilon \|a_i\| = 0\}\}_{i=1}^l$ where ε is chosen such that the following three conditions are satisfied $\overline{L_{j_\varepsilon}} \cap \bigcup_{K_k \notin E_j} \overline{K_k} = \emptyset$, $h_j|_{L_{j_\varepsilon}} < 0$, and $\varepsilon < \frac{1}{2}$. Since the number of cells is finite, then, $\varepsilon > 0$. Consider $H_1 = \{x | a_1 \cdot x + b_1 = 0, a_1 \in \mathbb{R}^m, b_1 \in \mathbb{R}\}$. Claim: $\text{dist}(H_1, H_{1\varepsilon}) = \varepsilon$. Proof: Let $x_1 \in H_1$. Let $u \in \mathbb{R}^m$ be such that $x_1 + u \in H_{1\varepsilon}$. Therefore: $a_1 \cdot (x_1 + u) + b_1 - \varepsilon \|a_1\| = a_1 \cdot u - \varepsilon \|a_1\| = 0$. Therefore, $\|u\| \geq \varepsilon$. Picking $u = \varepsilon \frac{a_1}{\|a_1\|}$ shows that $\inf_{x_2 \in H_{1\varepsilon}} \{\|x_1 - x_2\|\} = \varepsilon$. By applying **Lemma** 4.2.21 $\exists g_1 \in A$ such that $g_1|_{H_1} = f_i|_{H_1}$ and $g_1|_{H_{1\varepsilon}} = h_j|_{H_{1\varepsilon}}$ and in this case $g_1(x) = (v_i - \delta a_1) \cdot x + (t_i - \delta b_1)$ where $\delta = \frac{4h}{\varepsilon \|a_1\|}$. Similarly, we can construct $g_2, g_3, \cdots, g_l$ for the l hyperplanes that are boundaries of L_j. Let $\hat{g}_j = f_i \wedge g_1 \wedge g_2 \wedge \cdots \wedge g_l$. Claim 1: $f|_{L_j} \equiv \hat{g}_j|_{L_j}$. Proof: By construction: $\forall k \leq l : f_i|_{H_k} \equiv g_k|_{H_k}$ and $f_i|_{H_{k\varepsilon}} > h_j|_{H_{k\varepsilon}} \equiv g_k|_{H_{k\varepsilon}}$, therefore, $g_K|_{L_j} > f_i|_{L_j}$ and $\hat{g}_j|_{L_j} = f_i|_{L_j} = f|_{L_j}$. Claim 2: $\hat{g}_j|_{L_{j\varepsilon} \setminus L_j} < f|_{L_{j\varepsilon} \setminus L_j}$. Proof: Let $x \in L_{j\varepsilon} \setminus L_j$. Therefore, $x \in K_k$ for some $K_k \in E_j$ with $K_k \cap \partial L_{j\varepsilon} \neq \emptyset$. Let $C = K_k \cap L_{j\varepsilon}$. Since $\hat{g}_j|_{\partial L_{j\varepsilon}} < 0$ therefore, $\hat{g}_j|_C < f_k|_C = f|_C$ where f_k is the affine function associated with the neighbouring cell K_k. Claim 3: $\hat{g}_j|_{\mathbb{R}^m \setminus L_{j\varepsilon}} < 0$. Proof: This is straightforward from the fact that $\forall y \in \mathbb{R}^m \setminus L_{j\varepsilon} : \exists i \leq l$ such that $y \in \{x | a_i \cdot x + b_i - \varepsilon \|a_i\| > 0\}$ and $\exists x_1 \in H_i, x_2 \in H_{i\varepsilon}, \alpha > 0$ such that $y - x_2 = \alpha(x_2 - x_1)$. Assume $g_i(x) = v_g \cdot x + b_g$. We also have, $g_i(x_2) - g_i(x_1) = v_g \cdot (x_2 - x_1) < 0$. Therefore, $g_i(y) - g_i(x_2) = v_g \cdot (y - x_2) = \alpha v_g \cdot (x_2 - x_1) < 0$. Therefore, $g_i(y) < 0$. Therefore, $\hat{g}_j|_{\mathbb{R}^m \setminus L_{j\varepsilon}} < 0$. From the three claims above we have: $0 \vee \hat{g}_j \leq f \vee 0 = f$ where $0 \vee \hat{g}_j$ is equal to 0 outside of $L_{j\varepsilon}$. Repeating this construction for every j such that $L_j \neq \emptyset$ and setting $k = \left(\bigvee_{j, L_j \neq \emptyset} \hat{g}_j\right) \vee 0$ we have: $f|_K = k|_K$, $f|_{\mathbb{R}^m \setminus K} > k|_{\mathbb{R}^m \setminus K}$ and $\exists \varepsilon_k$ such that $k|_{\mathbb{R}^m \setminus (K \pm \varepsilon_k [0,1]^m)} = 0$.

■

Theorem 4.2.23: $\forall f \in S_{lp}^+ \subset C(\mathbb{R}^m) : f = \sup_i \{f_i\}$ *where* $\{f_i\}_{i=1}^{\infty}$ *is a locally finite sequence of functions in* S.

Proof. Consider the set $K = [0,1]^m$. By **Lemma** 4.2.22, there exists $k \in S$ such that $f|_K = k|_K$, $f|_{\mathbb{R}^m \setminus K} > k|_{\mathbb{R}^m \setminus K}$ and $\exists \varepsilon_k$ as small as possible and $k|_{\mathbb{R}^m \setminus (K \pm \varepsilon_k [0,1]^m)} = 0$.

We can now divide $\mathbb{R}^m$ into the countable union of translates of K such that $\mathbb{R}^m = \bigcup_{i=1}^{\infty} K_i$ where $K_i = K + v_i$ is a translate of K and $v_i = \{n_1, n_2, \cdots, n_m\} \in \mathbb{R}^m$ and $\forall j \leq m : n_j \in \mathbb{Z}$. For each compact set K_i we can find the function k_i such that k_i is equal to f on K_i and equals to

0 outside an ε neighbourhood of K_i. Therefore, $\{k_i\}_{i=1}^{\infty}$ is a locally finite sequence of functions in S and $f = \sup_i\{k_i\}$. ■

Corollary 4.2.24: $\forall f \in S_{lp} \subset C(\mathbb{R}^m) : f = \sup_i\{f_i^+\} - \sup_j\{f_j^-\}$ where $\{f_i^+\}_{i=1}^{\infty}$ and $\{f_j^-\}_{j=1}^{\infty}$ are two locally finite sequence of functions in S.

Proof. This is straightforward by applying **Theorem** 4.2.23 to $f = f^+ - f^-$. ■

Theorem 4.2.25: *S_{lp} uniformly approximate functions in $C(\mathbb{R}^m)$. In other words, $\forall \varepsilon > 0, \forall f \in C(\mathbb{R}^m) : \exists g \in S_{lp}$ such that $\sup|f - g| \leq \varepsilon$.*

Proof. Assume first that $f \in \mathbb{C}(\mathbb{R}^m)^+$. Fix $\varepsilon > 0$. Consider $K = [0,1]^m$. Consider $f|_K \subset C(K)$. By **Lemma** 4.2.13, $\exists f_K \in S$ such that $\forall x \in K : |f_K(x) - f(x)| \leq \frac{\varepsilon}{4}$. Note that by **Lemma** 3.1.4 f_K can be naturally extended to $\mathbb{R}^m$. By **Lemma** 4.2.22 $\exists f_{K\delta} \in S$ such that $f_K|_K = f_{K\delta}|_K, f_K|_{\mathbb{R}^m\setminus K} > f_{K\delta}|_{\mathbb{R}^m\setminus K}$ and $f_{K\delta}|_{\mathbb{R}^m\setminus(K\pm\delta[0,1]^m)} \equiv 0$ where δ can be chosen as small as possible. In this case δ is chosen as small as possible such that the variation of the function f to be approximated close to the boundary of K is smaller than $\frac{\varepsilon}{4}$. I.e., $\forall x \in \partial K, \forall y \in \{x\} \pm \delta * [0,1]^m : |f(x) - f(y)| \leq \frac{\varepsilon}{4}$. This is guaranteed by the fact that $\partial K \pm \delta * [0,1]^m$ is a compact set and f is continuous. This construction can be repeated for a translate of K say $H = K + \{1, 0, 0, \cdots, 0\}$, therefore, $\exists f_H \in S, f_{H\delta} \in S$ as above. We claim that the function $g = f_{H\delta} \vee f_{K\delta}$ is such that $\forall x \in K \cup H : |f(x) - g(x)| \leq \frac{\varepsilon}{2}$ (See **Figure** 4.2). Clearly, $\forall x \in (K \setminus (H \pm \delta * [0,1]^m)) : g(x) = f_{K\delta}(x)$ and therefore, $|f(x) - g(x)| \leq \frac{\varepsilon}{4}$. Similarly, $\forall x \in (H \setminus (K \pm \delta * [0,1]^m)) : g(x) = f_{H\delta}(x)$ and therefore, $|f(x) - g(x)| \leq \frac{\varepsilon}{4}$. Consider $x \in K \cap (H \pm \delta * [0,1]^m)$. If $g(x) = f_{K\delta}(x)$ then $|f(x) - g(x)| \leq \frac{\varepsilon}{4}$. However, if $g(x) = f_{H\delta}(x)$ then $\exists x_1 \in \partial K \cap \partial H$ such that $x \in \{x_1\} \pm \delta[0,1]^m$ and by the construction in **Lemma** 4.2.22 $g(x) \leq f_{H\delta}(x_1)$. Therefore, $f_{K\delta}(x) \leq g(x) \leq f_{H\delta}(x_1) \Rightarrow f(x) - \varepsilon/4 \leq f_{K\delta}(x) \leq g(x) \leq f_{H\delta}(x_1) \leq f(x_1) + \frac{\varepsilon}{4} \leq f(x) + \frac{\varepsilon}{4} + \frac{\varepsilon}{4}$. Therefore, $|f(x) - g(x)| \leq \frac{\varepsilon}{2}$. By assuming that $\mathbb{R}^m$ is equal to the countable union of translates of K, i.e., $\mathbb{R}^m = \bigcup_{i=1}^{\infty} K_i$ where $K_i = K + v_i$ is a translate of K and $v_i = \{n_1, n_2, \cdots, n_m\} \in \mathbb{R}^m$ and $\forall j \leq m : n_j \in \mathbb{Z}$. For each compact set K_i we can find the function f_{K_i} such that $f_{K_i} \in S$ and uniformly approximates f on K_i. In addition, f_{K_i} equals to 0 outside a δ neighbourhood of K_i. Therefore, $\{f_{K_i}\}_{i=1}^{\infty}$ is a locally finite sequence of functions in S and $g = \sup_i\{f_{K_i}\} \in S_{lp}$. In addition, g uniformly approximates f on every K_i.

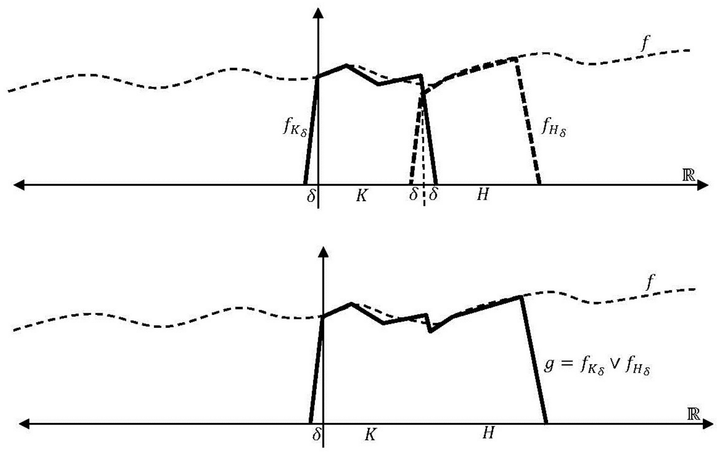

Figure 4.2: Construction of $f_{K\delta}, f_{H\delta}$ and $g = f_{K\delta} \vee f_{H\delta} \subset C(\mathbb{R})$

Let $f \in C(\mathbb{R}^m)$ then $f = f^+ - f^-$. Therefore, $\exists g^+, g^- \in S_{lp}^+$, $g = g^+ - g^- \in S_{lp}$ such that $\forall x \in \mathbb{R}^m : |f^+(x) - g^+(x)| \leq \varepsilon/2$ and $|f^-(x) - g^-(x)| \leq \varepsilon/2$, therefore, $|f(x) - g(x)| \leq |f^+(x) - g^+(x)| + |f^-(x) - g^-(x)| \leq \varepsilon$. ■

Chapter 5

Conclusion

In this thesis, we extended the concept of piecewise affine functions with finite components in $C(\mathbb{R}^m)$ by introducing the set of locally piecewise affine functions which could possibly have infinite components. We discussed the relationship between the two sets under the umbrella of order theory.

Our first original contribution was the definition of piecewise affine functions on arbitrary subsets of $\mathbb{R}^m$ and the extension lemma (**Lemma** 3.1.4). In that lemma we proved that a piecewise affine function on a closed and convex set in $\mathbb{R}^m$ can be naturally extended to a piecewise affine function on the whole space. We also introduced a conjecture that such an extension can also be achieved for any open, connected, bounded set whose closure is equal to the union of its its boundary and its interior.

Our second original contribution was the definition of locally piecewise affine functions and showing that this set of functions as a subset of $C(\mathbb{R}^m)$ is a vector lattice (**Lemma** 3.2.3). We also showed that the properties of the set S of piecewise affine functions can be naturally extended to the set S_{lp} of locally piecewise affine functions.

Our third original contribution was the comparison of the various definitions of order convergence in $C(\mathbb{R}^m)$. We first showed that $C(\mathbb{R}^m)$ possesses the countable sup property which enabled us to show that for bounded nets, the two main defintions of order convergence in the literature coincide (**Theorem** 4.1.12 and the following corollaries and remark).

Our fourth original contribution was the study of S and S_{lp} as sublattices of $C(\mathbb{R}^m)$. We first introduced three different definitions for order dense subsets of vector lattices (**Definitions** 4.2.1, 4.2.2, and 4.2.3). We showed that in Archimedean vector lattices an order dense minorizing sublattice is a stronger condition than a topologically order dense sublattice (**Lemma** 4.2.6). We introduced the conjecture that in $C(\mathbb{R}^m)$, order dense minorizing sublattices are equivalent to topologically order dense sublattices. We then showed that both S and subsequently S_{lp} are order dense minorizing sublattices in $C(\mathbb{R}^m)$.

Our fifth contribution was the study of the relationship betwen S and S_{lp} by introducing the definition of locally finite sets of functions (**Definition** 4.2.17). Then we showed that any locally finite set of functions in S has a supremum and an infimum both of which are in S_{lp} (**Lemma** 4.2.18). In addition, we showed that any function in S_{lp} can be expressed as the difference of the supremums of two locally finite sequence of functions in S.

Our final result was to show that S_{lp} uniformly approximates functions in $C(\mathbb{R}^m)$. While the result was straight forward when $m = 1$ (**Lemma** 4.2.15), we utilized the definition of locally finite sets of functions to show the result for the general case (**Theorem** 4.2.25).

Bibliography

[1] Charalambos D. Aliprantis and Rabee Tourky. *Cones and Duality*. Number 84 in Graduate Studies in Mathematics. American Mathematical Society.

[2] R.F. Anderson and J.C. Mathews. A comparison of two modes of order convergence. 18:100–104.

[3] Charalambos D. Aliprantis, David Harris, and Rabee Tourkey. Riesz estimators. (136):431–456.

[4] C.D. Aliprantis and O. Burkinshaw. *Principles of Real Analysis*. Academic Press, 3rd edition edition.

[5] Y. Abramovich and G. Sirotkin. On order convergence of nets. 9:287–292.

[6] P. Meyer-Nieberg. *Banach Lattices*. Springer-Verlag.

[7] Y. Abramovich and C.D. Aliprantis. *An Invitation to Operator Theory*. Number 50 in Graduate Studies in Mathematics. American Mathematical Society.

[8] C.D. Aliprantis and O. Burkinshaw. *Locally solid Riesz Spaces with Applications to Economics*. Number 105 in Mathematica Surveys and Monographs. Second edition edition.

[9] C.D. Aliprantis and K.C. Border. *Infinite Dimensional Analysis. A Hitchiker's Guide*. Springer-Verlag.