ODE To Backstepping...

Oh Backstepping, How do I love thee? Let me count the ways.[1]
You soothe my instabilities,
  Despite my singularities.
In the face of disturbing entities,
  You adapt to my uncertainties.
I love thee for thine flexibility,
  You don't even need controllability.
Your magnitude of feedback is,
  But one of gentle modesty.
You respect my nonlinearities,
  And exploit my structural peculiarities.
As I seek escape (in finite time),
  You bring me back, with Lyapunov tactics so sublime.
Although you're oft chastised for expressions complicated,
  My fears of numerical differentiation you've abated.
I love thee in my states of chaos, rest, or evolution,
  That's right, I love thee in all the meanderings of my solution.
Because of you, invariance my sets will never loose...
Oh Backstepping, to close my loop, there's no one else I'd rather choose!

— Karla Kvaternik

[1] What Elizabeth Barrett Browning would surely have written had she read [1]...

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Globally Stabilizing Output Feedback Methods for Nonlinear Systems

by

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I wholeheartedly dedicate this thesis to my parents, Dragutin and Vera Kvaternik, whose strength and resilience have often surprised me. *Hvala vam za vašu podršku.*

Abstract

The non-local stabilization of nonlinear systems by output feedback is a challenging problem that remains the subject of continuing investigation in control theory. In this thesis we develop two globally asymptotically stabilizing output feedback algorithms for multivariable nonlinear systems. Our first result is an extension of the output feedback method presented in [2] to a class of nonlinear systems whose dynamics can be written as a collection of subsystems that are dynamically coupled through output-dependent nonlinear terms. We show that the method given in [2] must be modified to accommodate this dynamic coupling by introducing additional nonlinear damping terms into each control input. Our second contribution involves the application of observer backstepping to systems in a restricted block-triangular observer form. In this form, the nonlinearities entering each subsystem are allowed to depend on the output associated with the subsystem, and all upper subsystem states, including unmeasured ones. The proposed algorithm is demonstrated on a magnetically levitated ball.

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Nomenclature

List of Acronyms

BTOF	Block Triangular Observer Form, page 42
CL	Closed-Loop, page 2
FT	Filter Transformation, page 84
GAS	Globally Asymptotically Stable, or Stabilizing, page 2
GES	Globally Exponentially Stable, page 5
IOL	Input-Output Linearizable, page 30
ISS	Input to State Stability, page 14
LED	Linear Error Dynamic, page 20
LES	Locally Exponentially Stable, page 6
NF	Normal Form, page 36
NF	Normal Form, page 50
NOF	Nonlinear Observer Form, page 8
OFB	Output Feedback, page 2
OFBF	Output Feedback Form, page 55
RBTOF	Restricted Block-Triangular Observer Form, page 127
ROA	Region of Attraction, page 2
SFBL	State Feedback Linearizable, page 30
SFF	Strict Feedback Form, page 44
SNF	Strict Normal Form, page 50
UCO	Uniform Complete Observability, page 31

VRD	Vector Relative Degree, page 117
ZD	Zero Dynamics, page 51

Chapter 1

Introduction

Aside from its aesthetic pleasures, the study of nonlinear control theory is motivated by the practical need to make control systems more reliable, more precise, and more safe. Most real-world dynamical systems towards which we aim the application of feedback control theory are not inherently linear. Admittedly, one of the simplest approaches to control design is that on the basis of a system's linearization, owing to the well-developed arsenal of tools available for linear designs. For many applications, combining robust techniques [3] with a linear design on the basis of a model's linear approximation may suffice. However, such designs necessarily make use of less information about the dynamics of the plant, generally guaranteeing only local stability[1] of the *closed-loop* (CL) system. In some cases, this design is not only less effective, but also potentially dangerous. Interestingly, it has been shown that for some linear time-varying systems nonlinear control outperforms linear control [4] in some measures of optimality, while for some nonlinear systems the use of linear feedback reduces the size of the closed-loop *region of attraction* (ROA) as the feedback gains are increased [5].

In the study of nonlinear control theory, we seek to develop feedback control laws that guarantee global or semi-global stability of a nonlinear system; that is, the ROA associated with a CL system's equilibrium should at least take a prescribed size, if not cover the entire state-space. A *globally asymptotically stabilizing* (GAS)

[1] In this thesis we are concerned primarily with the problem of stability, although the foregoing discussion applies also to problems such as tracking, disturbance rejection and model matching.

control law ensures the convergence of CL trajectories regardless of “where” the system is initialized within the state-space. In semi-global designs, a control gain is usually increased in order to increase the size of some compact set which is guaranteed to be contained within the ROA. Although global designs are sometimes applicable to a smaller class of systems than semi-global designs, they are generally advantageous in not requiring the use of any high gains that usually accompany semi-global designs.

One of the most practical and fundamental problems in nonlinear control design concerns non-local (i.e. global or semi-global) stabilization using only partial state measurements. A plant’s full state information is usually not available for feedback as the use of additional sensors may not be economically feasible, or the measurement of some states may even be physically impossible. Devising a control strategy when the system’s measured variables – i.e. outputs – can be mapped to only a subset of its state variables, is known as the *output feedback* (OFB) problem. The OFB problem is particularly challenging for nonlinear plants, and has not been fully solved. Although incremental progress is being made, there remain vast classes of systems for which there are no known non-local OFB control strategies.

In this thesis, we investigate non-local OFB stabilization algorithms for multivariable, continuous-time, finite-dimensional, deterministic dynamical systems with no uncertainties or disturbance inputs. In particular, we are interested in the development of *constructive* algorithms – i.e. those that provide explicit control expressions that can be physically implemented. For this reason, we make use of some of the most practical nonlinear design tools, including integrator backstepping, nonlinear damping and differential geometric theory. Different combinations of these tools have found numerous creative applications in global OFB designs such as observer backstepping [6] and what we will refer to as the MT method [2]. Our focus in this thesis is to develop multivariable extensions of these two methods, with the ultimate aim of developing global OFB laws for systems in a block-triangular observer form. In this thesis we document two contributions that we have made in this direction. Chapter 4 presents an extension of the MT method to a class of multivariable nonlinear systems whose subsystems are dynamically coupled through output-dependent nonlinearities. In Chapter 3, we apply observer backstepping to a class of systems in a restricted block-triangular observer form.

Before providing our main results, we present some pertinent background theory and analysis. We begin by giving a precise definition of the OFB problem and the various circumstances under which it can easily be solved. We then try to develop an understanding of the difficulties that present themselves in solving the problem for nonlinear systems by studying the peaking phenomenon and the stability of cascade-connected systems. We use concrete examples wherever possible to fortify our analysis and express our own insights where appropriate. After investigating some of the challenges posed by this difficult problem, we provide a survey of the literature detailing some of the progress that has been made on this problem in various directions.

In the second chapter, we describe the nonlinear design tools and system forms most frequently encountered in the literature on global OFB. We then provide a detailed exposition of observer backstepping and the MT method for the design of GAS OFB algorithms for a class of SISO systems in the output feedback form. These results are then extended in Chapters 3 and 4, and the thesis is concluded in Chapter 5.

1.1 The Output Feedback Problem

In the following, we define the OFB problem, and elaborate on some of the challenges associated with solving it for general nonlinear systems. For simplicity, our discussion will be based on SISO systems of the form:

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{aligned} \tag{1.1}$$

where $x \in \mathbb{R}^n$ is its state, $u \in \mathbb{R}$ its input and $y \in \mathbb{R}$ its output. To ensure the existence of a solution over some time interval, the vector fields f and g are assumed to be at least C^1. Without loss of generality we assume that $f(0) = 0$ so that with

$u = 0$ all motion stops at $x = 0$. If we are interested in regulating the system state to a nonzero equilibrium $x_{eq} \neq 0$, we can re-write the system dynamics in a new set of coordinates $z = x - x_{eq}$ so that the unforced dynamic $\dot{z} = f(z + x_{eq})$ has an equilibrium at $z = 0$.[2]

Definition 1.1.1. Given system (1.1) with $h(0) = 0$, the **non-local OFB Stabilization Problem** is to design a control

$$\begin{aligned} \dot{\xi} &= \Gamma(\xi, y, u), \qquad && \xi \in \mathbb{R}^q \\ u &= \alpha(\xi, y), && u \in \mathbb{R} \end{aligned} \tag{1.2}$$

with $\Gamma(0) = 0$ and $\alpha(0) = 0$, such that all trajectories of the closed-loop system

$$\begin{aligned} \dot{x} &= f(x) + g(x)\alpha(\xi, h(x)) \\ \dot{\xi} &= \Gamma(\xi, h(x), \alpha(\xi, h(x))) \end{aligned} \tag{1.3}$$

initiated inside $\Omega_x \times \Omega_\xi \subseteq \mathbb{R}^n \times \mathbb{R}^q$ (containing the origin) converge asymptotically to $(x, \xi) = (0, 0)$. $\triangleleft$

If the size of the ROA $\Omega_x \times \Omega_\xi$ associated with the origin can explicitly be altered through the control $u = \alpha(\xi, h(x))$, then the stability of the composite system (1.3) is said to be *semi-global*, while global asymptotic stability (i.e. GAS) is achieved if the CL region of attraction is $\Omega_x \times \Omega_\xi = \mathbb{R}^n \times \mathbb{R}^q$. The output feedback control (1.2) is *static*, or memoryless, if $q = 0$; otherwise it is *dynamic*.

Although we are not constrained to relate the dynamic component of (1.2) with the task of reconstructing the state of the plant, a common approach to OFB control involves designing an observer, whose state estimates are then used in lieu of actual plant states in some stabilizing state feedback. We refer to this approach as *estimated state feedback* (ESFB), or the *certainty equivalence* (CE) approach. For linear, time invariant systems with no uncertainties, the global OFB problem is fully solved in this way, owing to the so-called *separation principle*.

[2] It is also possible to express the dynamics (1.1) in a new coordinate set which centers any arbitrary point x_o to its origin; the origin in the new coordinates will be an equilibrium of the unforced system if the vector fields f and g satisfy $f(x_o) = -cg(x_o)$ for some constant c. Then we can define $z = x - x_o$ and $u = v + c$ to obtain $\dot{z} = (f(z + x_o) + cg(z + x_o)) + g(z + x_o)v$ whose drift evaluates to zero at $z = 0$.

1.1.1 The Linear Separation Principle

To demonstrate, we consider the linear case of the system (1.1):

$$\begin{aligned}\dot{x} &= Ax + Bu \\ y &= Cx\end{aligned} \tag{1.4}$$

with (A,B) stabilizable and (A,C) detectable. By these assumptions, there exists a vector $K \in \mathbb{R}^{1\times n}$ and a vector $L \in \mathbb{R}^{n\times 1}$ such that the spectra of $(A-BK)$ and $(A-LC)$ lie in the open left-half of the complex plane. We construct a classical Luenberger observer [7] for (1.4) as

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x}) \tag{1.5}$$

and define the estimation error as $\xi \triangleq x - \hat{x}$. The error dynamics are *globally exponentially stable* (GES) when L is chosen so that $(A-LC)$ is Hurwitz:

$$\dot{\xi} = \dot{x} - \dot{\hat{x}} = (A-LC)\xi. \tag{1.6}$$

If the full state is available for feedback, choosing the control $u = -Kx$ would result in the plant dynamics $\dot{x} = (A-BK)x$ that are likewise GES. However, since the state is unknown, we replace it with its estimate and instead implement the control $u = -K\hat{x}$ to obtain the composite CL system

$$\begin{aligned}\dot{x} &= Ax - BK(x-\xi) \\ \dot{\xi} &= (A-LC)\xi\end{aligned} \tag{1.7}$$

which can be written as

$$\begin{bmatrix}\dot{x}\\ \dot{\xi}\end{bmatrix} = \begin{bmatrix}(A-BK) & BK\\ 0 & (A-LC)\end{bmatrix}\begin{bmatrix}x\\ \xi\end{bmatrix} \triangleq \overline{A}\begin{bmatrix}x\\ \xi\end{bmatrix}. \tag{1.8}$$

Since $\overline{A}$ is block triangular, its eigenvalues are the union of the set of eigenvalues of the diagonal blocks, which can be made Hurwitz by the proper choice of gains K and L. In other words, the combination of *any* K so that $(A-BK)$ is Hurwitz, chosen independently of *any* L such that $(A-LC)$ is Hurwitz yields a Hurwitz composite system matrix. Given that the origin is the only equilibrium for (1.8), we conclude global exponential stability of the composite system (1.8).

1.1.2 Local Separation Principle for Nonlinear Systems

It would be convenient if this separation property also held true for nonlinear systems, since a significant effort has been dedicated to the development of many elegant state feedback and observer design techniques[3] for nonlinear systems. It turns out that locally a separation principle does hold, provided that $f(x)+g(x)u$ in (1.1) is C^1 in both arguments, observable, and stabilizable by a C^1 control $u=\alpha(x)$. Under these conditions, if the observer error $\xi(t)$ is locally asymptotically stable and the exact SFB control locally asymptotically stabilizes the plant about its origin, then it is always possible to initialize the CL system using ESFB sufficiently close to $(x,\xi)=(0,0)$, so that the ensuing trajectories of the composite system asymptotically converge to the origin [11]. In addition to [11], there are several variations of the proof of this fact, also usually relying on converse Lyapunov theory, comparison arguments, and the "theorem of total stability". For example, one may consult the proof of Theorem 10.3.1 in [12], or Proposition 4.1 in [13], or Theorem 3.1 in [14]. A simple Lyapunov-based proof is also given in Lemma 13.1 in [15].

Here, we offer a simpler justification of this fact under stronger assumptions, using Lyapunov's indirect method. Assume that there exists a state feedback law $u=\alpha(x),\ \alpha(0)=0$ such that the origin of

$$\dot{x}=f(x)+g(x)\alpha(x) \tag{1.9}$$

is rendered *locally exponentially stable* (LES). Such would be the case if (1.1) is state-feedback linearizable. Then there exists a scalar function $\phi(x)$ such that the nonlinear coordinate change

$$[z_1,z_2,\ldots,z_n]^T=[\phi(x),L_f\phi(x),\ldots,L_f^{n-1}\phi(x)]^T \triangleq T(x),\ T(0)=0 \tag{1.10}$$

[3] A major novelty in nonlinear control theory resulted from the application of differential geometry to the identification of coordinate and feedback transformations of nonlinear systems into several important canonical forms. For instance, state feedback linearization [8] allows a linear eigenstructure assignment design to be carried out after a preliminary change of coordinates and linearizing state feedback. Likewise, the nonlinear observer form was identified in [9], and allows the design of an observer with a linear error dynamic in a special set of coordinates. The internal structure of a nonlinear system has been characterized in [10], where the geometric existence conditions and diffeomorphisms leading to the so-called *normal form* are developed. Subsequently, there have been hundreds of papers published extending and applying the aforementioned work.

is a diffeomorphism that locally transforms (1.1) into

$$\begin{aligned}
\dot{z}_1 &= z_2 \\
\dot{z}_2 &= z_3 \\
&\vdots \\
\dot{z}_{n-1} &= z_n \\
\dot{z}_n &= L_f^n \phi(x) + u L_g L_f^{n-1} \phi(x)
\end{aligned} \tag{1.11}$$

where $L_g L_f^{n-1}\phi(x)$ is bounded away from zero [16], [17]. In (1.10), we have used $L_f \phi(x) \triangleq \langle d\phi(x), f(x)\rangle$ to denote the Lie derivative of a scalar function $\phi(x)$ in the direction of the vector field $f(x)$, where $d\phi(x)$ denotes $\frac{\partial \phi(x)}{\partial x}$. Multiple Lie derivatives are defined as $L_f^i \phi(x) = \langle dL_f^{i-1}\phi(x), f(x)\rangle$. Next, choosing

$$u = \alpha(x) = \frac{-L_f^n \phi(x)[-KT(x)]}{L_g L_f^{n-1}\phi(x)} \tag{1.12}$$

with $K \in \mathbb{R}^{1\times n}$ renders (1.11):

$$\dot{z} = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ -k_1 & -k_2 & \cdots & -k_n \end{bmatrix} z \triangleq Az. \tag{1.13}$$

Provided the gain K is chosen so that A's characteristic polynomial $s^n + k_n s^{n-1} + \cdots + k_2 s + k_1$, has roots with negative real parts, (1.13) is locally exponentially stable. If $L_g L_f^{n-1}\phi(x) \neq 0$, $\forall x \in \mathbb{R}^n$, then in the z-coordinates (1.13) is GES. Clearly, the linearization of (1.13) about $z = 0$ is Hurwitz. Our objective now is to relate the linearization of (1.13) about $z = 0$ to that of (1.9) about $x = T^{-1}(0) = 0$. To that end, we note that

$$\dot{z} = Az = AT(x).$$

and also

$$\dot{z} = \frac{\partial T(x)}{\partial x}(f(x) + g(x)\alpha(x)).$$

Equating the two expressions and taking the gradient of both sides, we obtain that

$$A\frac{\partial T(x)}{\partial x} = \frac{\partial^2 T(x)}{\partial x^2}(f(x) + g(x)\alpha(x)) + \frac{\partial T(x)}{\partial x}\frac{\partial}{\partial x}(f(x) + g(x)\alpha(x)). \tag{1.14}$$

We denote the Jacobian of $(f(x)+g(x)\alpha(x))$ as $J(x) \triangleq \frac{\partial}{\partial x}(f(x)+g(x)\alpha(x))$ and evaluate (1.14) about $x=0$, noting that $f(0)=0$ and $\alpha(0)=0$:

$$J(0) = \left(\frac{\partial T(x)}{\partial x}\right)^{-1} A \frac{\partial T(x)}{\partial x}\bigg|_{x=0}, \tag{1.15}$$

From (1.15), we see that although the system matrix of the linearization of the CL system in the original coordinates, $J(0)$, is not necessarily the same as that of the CL linearization A in the transformed coordinates, their spectra are identical[4]. Since an equilibrium of a nonlinear system is LES if and only if its linearization about that point is Hurwitz (Theorem 4.15, [15]), we conclude that the state feedback (1.12) renders the origin of (1.1) LES. This fact will become useful in the following discussion.

Suppose that in addition to being state-feedback linearizable, (1.1) also admits a transformation into the *nonlinear observer form* (NOF) [9], and that we are able to find a diffeomorphism $Q : \mathbb{R}^n \to \mathbb{R}^n$, $x \mapsto w$ so that in the new coordinates the dynamic (1.1) can be written as:

$$\begin{aligned} \dot{w} &= A_c w + \gamma(y,u) \\ y &= h(Q^{-1}(w)) = C_c w \end{aligned} \tag{1.16}$$

where

$$A_c = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \qquad C_c = [1, 0, \ldots, 0]. \tag{1.17}$$

Since (A_c, C_c) is observable and the nonlinearity in (1.16) depends on known signals only, we can design an observer for (1.16) as

$$\dot{\hat{w}} = A_c \hat{w} + \gamma(y,u) + L(y - \hat{w}_1), \tag{1.18}$$

which implies that the dynamic of the error $\xi \triangleq w - \hat{w}$

$$\dot{\xi} = (A_c - LC_c)\xi \tag{1.19}$$

[4]For simplicity, assume that the eigenvalues of A are distinct. Then suppose λ_i is an eigenvalue of A, and v_i its associated eigenvector. Given any nonsingular matrix T of appropriate dimension, define a vector $\mu_i = T^{-1}v_i$. Then $AT\mu_i = \lambda_i T\mu_i$, or $(T^{-1}AT)\mu_i = \lambda_i\mu_i$ showing that λ_i is still an eigenvalue of the transformed matrix $(T^{-1}AT)$, with eigenvector μ_i. Since λ_i is arbitrary, this argument holds for the entire spectrum of A, and shows that it is invariant under coordinate transformations

is GES at the origin when the output injection gain L is chosen so that $(A_c - LC_c)$ is Hurwitz. In the original coordinates, the state estimates generated by (1.18) are expressed as $\hat{x} = Q^{-1}(\hat{w}) = Q^{-1}(w - \xi)$. Therefore, implementing the control (1.12) with these state estimates results in the composite system dynamic

$$\dot{x} = f(x) + g(x)\alpha(Q^{-1}(Q(x) - \xi)) \tag{1.20a}$$

$$\dot{\xi} = (A_c - LC_c)\xi. \tag{1.20b}$$

Since (1.20b) is independent of x, to show that the linearization of (1.20) about $(0,0)$ is Hurwitz, it suffices to show that $\frac{\partial \dot{x}}{\partial x}|_{(x,\xi)=(0,0)}$ is Hurwitz. To that end, we apply some basic calculus:

$$\begin{aligned}\frac{\partial \dot{x}}{\partial x}|_{(x,\xi)=(0,0)} = \frac{\partial}{\partial x} f(x)|_{(x,\xi)=(0,0)} + \left(\frac{\partial}{\partial x} g(x)\right) \alpha(Q^{-1}(Q(x) - \xi))\Big|_{(x,\xi)=(0,0)} \\ + g(x) \frac{\partial \alpha(\cdot)}{\partial x}\Big|_{(x,\xi)=(0,0)}\end{aligned} \tag{1.21}$$

Since

$$\alpha(Q^{-1}(Q(x) - \xi))\Big|_{(x,\xi)=(0,0)} = \alpha(Q^{-1}(Q(x))\Big|_{x=0} = \alpha(x)|_{x=0}, \tag{1.22}$$

and by assumption $\alpha(0) = 0$, the second term on the right hand side of (1.21) disappears. The third term is:

$$g(x)\frac{\partial \alpha(\cdot)}{\partial x} = g(x)\left[\frac{\partial \alpha(s)}{\partial s}\Big|_{s=(Q^{-1}(Q(x)-\xi)} \frac{\partial Q^{-1}(p)}{\partial p}\Big|_{p=Q(x)-\xi} \frac{\partial (Q(x) - \xi)}{\partial x}\right]. \tag{1.23}$$

If we evaluate (1.23) at $\xi = 0$ we obtain

$$\begin{aligned} g(x)\frac{\partial \alpha(Q^{-1}(Q(x) - \xi))}{\partial x} &= g(x)\left[\frac{\partial \alpha(x)}{\partial x}\frac{\partial Q^{-1}(w)}{\partial w}\frac{\partial Q(x)}{\partial x}\right] \\ &= g(x)\frac{\partial \alpha(x)}{\partial x}\end{aligned} \tag{1.24}$$

since $\frac{\partial Q^{-1}(w)}{\partial w}\frac{\partial Q(x)}{\partial x} = I$. To see this, we note that since $x = Q^{-1}w$, $\dot{x} = \frac{\partial Q^{-1}(w)}{\partial w}\dot{w}$. But since $w = Q(x)$, $\dot{w} = \frac{\partial Q(x)}{\partial x}\dot{x}$. Therefore, $\dot{x} = \frac{\partial Q^{-1}(w)}{\partial w}\frac{\partial Q(x)}{\partial x}\dot{x} = I\dot{x}$. In consideration of (1.22) and (1.24), we evaluate (1.21) at the equilibrium:

$$\frac{\partial \dot{x}}{\partial x}|_{(x,\xi)=(0,0)} = \frac{\partial f(x)}{\partial x}|_{x=0} + g(x)\frac{\partial \alpha(x)}{\partial x}\Big|_{x=0} = J(0) \tag{1.25}$$

which is shown to be Hurwitz in (1.15). The linearization of the composite CL system (1.20)

$$\begin{bmatrix} \frac{\partial \dot{x}}{\partial x} & \frac{\partial \dot{x}}{\partial \xi} \\ \frac{\partial \dot{\xi}}{\partial x} & \frac{\partial \dot{\xi}}{\partial \xi} \end{bmatrix} \Bigg|_{(x,\xi)=(0,0)} = \begin{bmatrix} J(0) & * \\ 0 & (A_c - LC_c) \end{bmatrix} \tag{1.26}$$

shows that (1.20) is LES at the origin, and that using state-feedback linearization techniques to design a stabilizing SFB control, in combination with a separately designed linear error dynamic observer always results in the local exponential stability of the origin.

1.1.3 Failure of Certainty Equivalence for Nonlinear Systems

One may wonder whether, and to what extent this certainty equivalence approach would hold non-locally for general nonlinear systems. Unfortunately, there is no nonlinear analog of the linear separation principle. We illustrate this fact using a counterexample provided in [18], expanding it here.

Example 1.1.1 (Failure of CE)**.** Consider the system

$$\begin{aligned} \dot{x}_1 &= -x_1 + x_2^2 \\ \dot{x}_2 &= -x_2 + x_1 x_2^2 + u \\ y &= x_2. \end{aligned} \tag{1.27}$$

Given access to the full state, we note that cancelling the nonlinearity in the $\dot{x}_2$-equation using the control $u = -x_1 x_2^2$ renders $(x_1, x_2) = (0,0)$ GAS. This can be seen using the Lyapunov function candidate $V(x) = \frac{1}{2}(x_1^2 + x_2^2) + \frac{1}{4}x_2^4$, whose derivative along the CL solutions of (1.27) is

$$\begin{aligned} \dot{V}(x) &= x_1(-x_1 + x_2^2) + x_2(-x_2) + x_2^3(-x_2) \\ &= -(1 - \tfrac{1}{4})x_1^2 - x_2^2 - \tfrac{1}{4}x_1^2 + x_1 x_2^2 - (x_2^2)^2 \\ &= -\tfrac{3}{4}x_1^2 - x_2^2 - (\tfrac{1}{2}x_1 - x_2^2)^2 \\ &\leq -\tfrac{3}{4}x_1^2 - x_2^2. \end{aligned} \tag{1.28}$$

Global asymptotic stability of $(x_1, x_2) = (0,0)$ then follows from Theorem A.0.1.

We notice also that since x_2 is measured and the first subsystem is stable when $x_2 = 0$, the following observer

$$\dot{\hat{x}}_1 = -\hat{x}_1 + x_2^2, \tag{1.29}$$

with the estimation error defined as $\xi = x_1 - \hat{x}_1$, gives the error dynamic

$$\dot{\xi} = -\xi \tag{1.30}$$

which is GES at $\xi = 0$. If instead of x_1 we use its estimate $\hat{x}_1$ in the previous control law, we obtain the two coupled dynamic equations

$$\begin{aligned} \dot{x}_2 &= -x_2 + x_1 x_2^2 - (x_1 - \xi)x_2^2 \\ &= -x_2 + \xi x_2^2 && \text{(1.31a)} \\ \dot{\xi} &= -\xi && \text{(1.31b)} \end{aligned}$$

which evolve independently of the $\dot{x}_1$ subsystem. To examine the behaviour of (1.31), we proceed to solve it. First, the solution to the error dynamic equation is $\xi(t) = e^{-t}\xi(0)$. We substitute this solution into (1.31b) to obtain

$$\dot{x}_2 = -x_2 + e^{-t}\xi(0)x_2^2. \tag{1.32}$$

Since the "disturbance" term $e^{-t}\xi(0)x_2^2$ decays exponentially, it may seem that the linear stable component of this equation would eventually dominate the motion, bringing x_2 back to the origin. We introduce the change of coordinates $z = T(x_2) = \frac{1}{x_2}$. Then

$$\begin{aligned} \dot{z} &= \frac{dT(x_2)}{dx_2}\dot{x}_2 \circ T^{-1}(z) \\ &= \tfrac{-1}{x_2^2}(-x_2 + e^{-t}\xi(0)x_2^2)\big|_{x_2=\frac{1}{z}} \\ &= z - e^{-t}\xi(0). && \text{(1.33)} \end{aligned}$$

Taking z to the left hand side and multiplying (1.33) by the integrating factor e^{-t} we obtain:

$$\underbrace{e^{-t}\dot{z} - e^{-t}z}_{=\frac{d}{dt}(e^{-t}z)} = -e^{-2t}\xi(0) \tag{1.34}$$

which can be integrated from t to 0 on both sides:

$$e^{-t}z(t) - z(0) = -\xi(0)\int_0^t e^{-2\tau}d\tau$$

$$\begin{aligned}
\Longrightarrow \quad z(t) &= e^t\left(z(0) + \tfrac{\xi(0)}{2}(e^{-2t} - 1)\right) \\
\Longrightarrow \quad x_2(t) &= \frac{2}{2\frac{1}{x_2(0)} + \xi(0)(e^{-t} - e^t)} \\
&= \frac{2x_2(0)}{e^t[2 - x_2(0)\xi(0)] + e^{-t}x_2(0)\xi(0)} \qquad (1.35)
\end{aligned}$$

From the denominator of (1.35), we see that for all "local" initial conditions $\{x_2(0) \in \mathbb{R},\ \xi(0) \in \mathbb{R} : -\infty < x_2(0)\xi(0) < 2\}$, the states of the system (1.31) converge to zero, but all trajectories initiated outside this set escape to infinity at time

$$t_{esc} = \frac{1}{2}\ln\left(\frac{x_2(0)\xi(0)}{x_2(0)\xi(0) - 2}\right). \qquad (1.36)$$

Even though the observer estimate in this example converges exponentially fast to the actual state x_1, this exponentially decaying estimation error causes a disturbance significant enough to destabilize the CL system for some initial conditions. $\triangleleft$

The above example demonstrates that we cannot expect to find a separation principle for nonlinear systems which is as powerful and generic as that for linear systems. Specifically, global OFB designs for nonlinear systems cannot be as simple as combining any convergent observer with any GAS SFB controller under the CE assumption.

One may object to this example, since the ad-hoc construction of this particular observer gives us no opportunity to control its convergence rate, which is dependent on the stability of the linear component in the $\dot{x}_1$ equation. However, assuming that by whatever means a reduced order observer is available for (1.27) so that its error decays at a rate $\xi(t) = e^{-kt}\xi(0)$, where the parameter k can be adjusted, we observe from equation (1.32) that for any such gain $k > 0$ its solution

$$x_2(t) = \frac{(k+1)x_2(0)}{e^t[k+1 - x_2(0)\xi(0)] + e^{-kt}x_2(0)\xi(0)} \qquad (1.37)$$

still has a finite escape time for some initial conditions. In fact, even observers that converge in finite time (such as [19]) may not avoid this problem since the estimation error may reach magnitudes sufficiently large to destabilize the CL system during the error convergence interval [20].

The lack of a generic separation principle for nonlinear systems is a specific instance of a more general problem: our inability to predict the input-output behaviour of a nonlinear system on the basis of its internal stability. Regarding the observer error as an input to the plant under ESFB, we see that when this input is zeroed, the internal response of the CL plant (i.e. response to initial conditions only) is GAS. However, the presence of the multiplicative nonlinearity x_2^2 amplifies even a bounded, decaying input in a way that causes instability. For linear systems, internal stability (i.e. stability of the zero input response) implies, and is implied by input-output stability (i.e. stability of zero-state response), but for nonlinear systems this is not the case.

Even though Example 1.1.1 shows that in general ESFB cannot guarantee non-local stability, there are conditions under which non-local stability is achievable using ESFB for nonlinear systems.

1.2 Stability of Cascade-Connected Systems

From the previous examples, we note that the interconnection of the plant, controller and observer can equivalently be studied within the more general framework of cascade connected systems such as:

$$\dot{x} = f(x, \xi) \tag{1.38a}$$

$$\dot{\xi} = g(\xi), \tag{1.38b}$$

shown in Figure 1.1. We assume that (1.38b), analogous to the observer error, is GAS at $\xi = 0$, and that (1.38a) with $\xi = 0$, analogous to the closed-loop plant under exact state feedback $\dot{x} = f(x, 0)$, is GAS at $x = 0$. We are interested in whether there are conditions under which this cascade connection is globally stable. Such

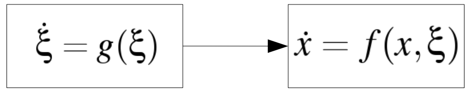

Figure 1.1: Cascade connected subsystems

conditions may offer insight into the limitations of using ESFB to formulate OFB control laws.

The stability of cascaded nonlinear systems has been extensively studied in the literature – for example, in [14], [21], [22] among others. The most prominent result concerning the global stability of (1.38) involves the notion of *input to state stability* (ISS) [23] . If subsystem (1.38a) possesses this property with respect to the signal ξ, then the basic implication is that the magnitude of the state $x(t)$ will be bounded for all bounded inputs $\xi(t)$ and any initial conditions $x(0)$. Clearly, system (1.27) in Example 1.1.1 does not posses this property.

We briefly summarize the meaning and some of the implications of ISS by presenting a few key results related specifically to (1.38), and our previous discussions. Our discussion here is based on the presentation given in Chapter 4 in [15], and Section 10.4 in [12].

Definition 1.2.1 (ISS)**.** System (1.38a) is ISS with respect to its input $\xi(t)$ if there exists a class $\mathcal{KL}$ function $\beta(\cdot,\cdot)$ and a class $\mathcal{K}$ function $\gamma(\cdot)$ such that for all bounded $\xi(\cdot)$ and all $x(0) \in \mathbb{R}^n$, the state satisfies the bound

$$\|x(t)\| \leq \beta(\|x(0)\|, t) + \gamma(\|\xi(t)\|_{\mathcal{L}_\infty}), \quad \forall\, t \geq 0 \tag{1.39}$$

$\triangleleft$

In this definition, a $\mathcal{K}$-class function $\gamma(|r|)$ is a C^1 function that strictly increases with increasing r, with $\gamma(0) = 0$. γ belongs to the class $\mathcal{K}_\infty$ if it is also radially unbounded. A $\mathcal{KL}$-class function $\beta(|r|, t)$ is a function which, for any fixed t is a class $\mathcal{K}$ function, and for any fixed $r = r_o$ is decreasing – i.e. $\lim_{t\to\infty} \beta(|r_o|, t) = 0$. The $\mathcal{L}_\infty$ norm of a signal $\xi(t)$ is defined as $\|\xi(t)\|_{\mathcal{L}_\infty} = \sup_{t\geq 0} \|\xi(t)\|$ and exists for all bounded signals.

The important result involving ISS is the following:

Theorem 1.2.1 (Corollary 10.5.3, [12])**.** *If system* (1.38a) *is ISS with respect to* ξ *as its input, and the system* (1.38b) *is GAS, then the equilibrium* $(x,\xi) = (0,0)$ *of the composite system* (1.38) *is GAS.*

The way to establish that a system is ISS is to produce a so-called "ISS-Lyapunov function", which fully characterizes input-to-state stability in the sense that it is both necessary and sufficient for it. A function $V : \mathbb{R}^n \to \mathbb{R}^+$ is an ISS-Lyapunov function for (1.38a) if it is bounded by two class $\mathcal{K}_\infty$ functions

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|), \quad \forall x \in \mathbb{R}^n \tag{1.40}$$

and its time derivative satisfies

$$\frac{\partial V(x)}{\partial x} f(x,\xi) \leq -\alpha_3(\|x\|), \quad \forall x \in \mathbb{R}^n, \text{ when } \|x\| \geq \alpha_4(\|\xi\|), \tag{1.41}$$

where $\alpha_3(\cdot) \in \mathcal{K}_\infty$ and $\alpha_4(\cdot) \in \mathcal{K}$. A system (1.38a) is ISS if and only if there exists an ISS-Lyapunov function for it (Theorem 10.4.1, [12]).

Condition (1.41) may be difficult to check. Alternatively, that same function V is an ISS-Lyapunov function for (1.38) if and only if (1.40) holds, and there exists a class $\mathcal{K}_\infty$ function σ such that the time derivative of V along the solutions of (1.38a) satisfies (Lemma 10.4.2, [12]):

$$\frac{\partial V(x)}{\partial x} f(x,\xi) \leq -\alpha_3(\|x\|) + \sigma(\|\xi\|) \tag{1.42}$$

To concretely see how a Lyapunov function satisfying (1.40) and (1.42) guarantees (1.39), suppose

$$\dot{x} = -x^3 + x^2\xi. \tag{1.43}$$

Just as in Example 1.1.1, the "disturbance" input ξ is amplified by a quadratic nonlinearity. However in this case, no bounded ξ can destabilize (1.43) owing to the presence of the stronger $-x^3$ term. To show this, we first demonstrate that the function $V = \frac{1}{2}x^2$ is an ISS-Lyapunov function. Its time derivative is:

$$\begin{aligned}
\dot{V} &= -x^4 + x^3\xi(t) \\
&= -\tfrac{3}{4}x^4 + x^2\left[-\tfrac{1}{4}x^2 + x\xi(t) - \xi(t)^2 + \xi(t)^2\right] \\
&= -\tfrac{3}{4}x^4 - x^2\left(\tfrac{1}{2}x - \xi(t)\right)^2 + x^2\xi(t)^2 \\
&\leq -\tfrac{3}{4}x^4 + x^2\xi(t)^2 \qquad (1.44) \\
&= -\tfrac{1}{2}x^4 - \left(\tfrac{1}{2}x^2 - \xi(t)^2\right)^2 + \xi(t)^4 \\
&\leq -\tfrac{1}{2}x^4 + \xi(t)^4. \qquad (1.45)
\end{aligned}$$

Since (1.45) matches the form (1.42), we conclude that $\dot{x} = -x^3 + x^2\xi$ is ISS with respect to ξ. To see why an ISS-Lyapunov function is sufficient to obtain a bound such as (1.39), we will examine the solution to the differential inequality (1.44), which can be written:

$$\dot{V} \leq -3V^2 + 2V\xi(t)^2. \qquad (1.46)$$

Instead of solving (1.46), we will solve the equality

$$\dot{w} = -3w^2 + 2w\xi(t)^2 \qquad (1.47)$$

and then apply the comparison principle (Lemma 3.4 [15]) to obtain a bound on $V(t)$, and hence $x(t)$, in terms of $\|x(0)\|$ and $\|\xi(t)\|_{\mathcal{L}_\infty}$. To solve (1.47), we first employ a similar change of variable as before, in order to transform this nonlinear DE into a linear one. Let $z = \frac{1}{w}$. Then

$$\dot{z} = 3 - 2z\xi(t)^2. \qquad (1.48)$$

This DE can be solved using the integrating factor $e^{2\int \xi(t)^2 dt}$, resulting in:

$$z(t) = e^{-2\int \xi(t)^2 dt}\left[e^{2(\int \xi(t)^2 dt)|_{t=0}} z(0) + 3\int_0^t e^{2(\int \xi(q)^2 dq)|_{q=\tau}} d\tau\right]. \qquad (1.49)$$

Recalling that $w(t) = \frac{1}{z(t)}$ and that $V(t) = \frac{1}{2}x(t)^2 \leq w(t)$ by the comparison principle, we obtain:

$$\tfrac{1}{2}x(t)^2 \leq \frac{x(0)^2 e^{2\int \xi(t)^2 dt}}{e^{2(\int \xi(t)^2 dt)|_{t=0}} + 3x(0)^2 \int_0^t e^{2(\int \xi(q)^2 dq)|_{q=\tau}} d\tau} \qquad (1.50)$$

The first thing to notice in (1.50), is that the denominator can never vanish and that all terms in both numerator and denominator are non-negative for all $\xi(t)$ and all initial conditions $x(0)$ and $\xi(0)$. Therefore the solution $x(t)$ exists for all $t \geq 0$ – i.e. has no finite escape time. To establish that $x(t)$ is bounded and converges according to (1.39), we examine the behaviour of the bound (1.50) when: 1) $\xi(t) \equiv 0$, and 2) $\xi(t) \equiv M \triangleq \|\xi(t)\|_{\mathcal{L}_\infty}$.

1. When $\xi(t) \equiv 0$, the indefinite integral $\int \xi(t)^2 dt$ evaluates to some constant c_1 and e^{2c_1} cancels from numerator and denominator in (1.50), leaving
$$\tfrac{1}{2}x(t)^2 \leq \frac{x(0)^2}{1+3x(0)^2 t} \tag{1.51}$$
which converges to zero as t tends to infinity. We therefore associate $\sqrt{\frac{2x(0)^2}{1+3x(0)^2 t}}$ with the $\mathcal{KL}$-class function $\beta(\|x(0)\|, t)$ in (1.39), since $\gamma(0) = 0$.

2. When $\xi(t)$ is replaced with its supremum M in (1.50), we obtain
$$\begin{aligned}\tfrac{1}{2}x(t)^2 &\leq \frac{x(0)^2 e^{2M^2 t}}{1+3x(0)^2\int_0^t e^{2M^2\tau}d\tau} \\ &= \frac{2M^2 x(0)^2 e^{2M^2 t}}{2M^2 + 3x(0)^2 e^{2M^2 t} - 3x(0)^2}\end{aligned} \tag{1.52}$$
which, as t tends to infinity becomes:
$$\lim_{t\to\infty} \tfrac{1}{2}x(t)^2 \leq \lim_{t\to\infty} \frac{2M^2 x(0)^2 e^{2M^2 t}}{2M^2 + 3x(0)^2 e^{2M^2 t} - 3x(0)^2} = \frac{2}{3}M^2. \tag{1.53}$$
We therefore associate $\sqrt{2M^2/3}$ with the class $\mathcal{K}$ gain function $\gamma(M)$ in (1.39), since $\lim_{t\to\infty}\beta(\|x(0)\|, t) = 0$.

Although the characterization of the ISS property represents a significant advancement in relating the I/O behaviour of a nonlinear system with its internal behaviour, there are two difficulties with it. First, it is only an analysis tool and may not help us actually design stable cascades in the observer-controller context. Second, there is no systematic way of finding an ISS-Lyapunov function for a general nonlinear system; it may be impossible to find such a function even for an ISS system. We quote a result intended to help establish ISS without resorting to a search for an ISS-Lyapunov function:

Theorem 1.2.2 (Lemma 4.6, [15])**.** *If the C^1 vector field $f(x,\xi)$ in* (1.38a) *is globally Lipschitz in x and ξ, and the unforced system $\dot{x} = f(x,0)$ is GES at $x = 0$, then* (1.38a) *is ISS with respect to ξ.*

Certainly, checking whether $f(x,\xi)$ is globally Lipschitz is easier than searching for an ISS-Lyapunov function. However, both requirements of this theorem are extremely restrictive. To require that the growth of a nonlinear system's velocity vector be linearly bounded severely reduces the variety of admissible nonlinearities. Designing a SFB controller for a general nonlinear system so that its origin is GES may also not be an easy task; for instance, if the plant admits a coordinate change in which a SFB control can be designed such that in those coordinates the system is GES (i.e. as the control (1.12) did for system (1.11)), a nonlinear change of coordinates generally does not preserve the states' convergence rates. In other words, in the original coordinates we can expect GAS, but not necessarily GES. Although the linear CL system under ESFB (1.8) certainly qualifies as ISS under the above theorem, possibly not many nonlinear systems do.

From (1.51), it is clear that ISS implies internal stability, and from Example 1.1.1, it is evident that internal stability does not imply ISS. Therefore, ISS is a much stronger property than internal stability and is sufficient to guarantee I/O stability. However, ISS is not necessary for I/O stability, possibly leaving room to weaken conditions such as those of Theorem 1.2.2.

There are in fact several results that attempt to identify a more "minimal" set of conditions under which the cascade interconnection (1.38) is globally stable. For example, let us re-write (1.38) as

$$\dot{x} = f(x,0) + [f(x,\xi) - f(x,0)] \triangleq F(x) + \psi(x,\xi) \tag{1.54a}$$

$$\dot{\xi} = g(\xi), \tag{1.54b}$$

Then, by Proposition 4.11 in [13] the composite system is GAS at $(x,\xi) = (0,0)$

if (1.54b) is GAS at $\xi = 0$, $\dot{x} = F(x)$ is GES at $x = 0$ and the interconnection term $\psi(x,\xi)$ is linearly bounded as

$$\|\psi(x,\xi)\| \leq \gamma_1(\|\xi\|)\|x\| + \gamma_2(\|\xi\|). \tag{1.55}$$

for some class $\mathcal{K}$ functions γ_1 and γ_2. In this case the growth restriction is imposed only on the interconnecting term $\psi(x,\xi)$ as opposed to the entire $f(x,\xi)$ in Theorem 1.2.2. It is also possible to achieve GAS of the composite system by requiring (1.54b) to be GES at the origin (which is plausible, as for example in (1.20b)) and $\dot{x} = F(x)$ to be only GAS, while the interconnecting nonlinearity is still required to satisfy (1.55). In addition, the Lyapunov function associated with $\dot{x} = F(x)$ is then required to satisfy a polynomial growth bound (Theorem 4.7 and Proposition 4.8, [13]). These conditions are still not much weaker than those of Theorem 1.2.2.

It appears that most attempts to weaken the ISS requirement do so at the expense of limiting the nonlinear growth rate of the upper subsystem. However this restriction can be considerably weakened if we instead consider semi-global stabilization of the cascade (1.54). For example, recalling equation (1.37):

$$x_2(t) = \frac{(k+1)x_2(0)}{e^t[k+1-x_2(0)\xi(0)] + e^{-kt}x_2(0)\xi(0)} \tag{1.56}$$

which is the solution of $\dot{x}_2 = -x_2 + x_2^2\xi(0)e^{-kt}$, we notice that the the set $\{(x(0),\xi(0)) : -\infty < x(0)\xi(0) < (k+1)\}$, contained within the ROA, can be arbitrarily expanded by increasing the gain k, showing that non-local stabilization is possible in the absence of the ISS property, and in the absence of any growth restrictions on the interconnecting nonlinearity (i.e. x_2^2 is not globally Lipschitz). From this example, it would seem that even though global stabilization may not be possible with ESFB, we may achieve an arbitrarily large ROA for a composite CL system, simply by increasing the convergence rate of the observer. Evidently in some circumstances this strategy may work; however, in [24] and [22] this idea is shown to fail in general due to the so-called “peaking phenomenon”. In the sequel, we will examine more closely the essential reason for the failure of this idea. For stable linear systems, peaking refers to a transitory increase in the magnitude of some states before they decay.

1.2.1 The Peaking Phenomenon

In order to further understand the mechanisms that encumber the nonlinear separation principle, we now examine some aspects of the work in [22]. In this paper, the authors study a "partially linear" cascade such as

$$\dot{x} = F(x) + \psi(x, \xi) \tag{1.57a}$$

$$\dot{\xi} = A\xi + Bu, \tag{1.57b}$$

and investigate various conditions on both subsystems under which the cascade can be globally or semi-globally stabilized by partial state feedback. Such system structures may arise as special cases of systems that have been partially state-feedback linearized [25], or I/O state-feedback linearized [26]. It is found in [22] that if peaking states enter the interconnection term ψ, then the cascade (1.57) cannot be semi-globally stabilized by a feedback of the form $u = \alpha(\xi)$.

In terms of the nonlinear separation principle, we are more interested in systems of the form

$$\dot{x} = F(x) + \psi(x, \xi) \tag{1.58a}$$

$$\dot{\xi} = A\xi. \tag{1.58b}$$

where the input has already been assigned, and ξ represents the state estimation error, resulting from a *linear error dynamic* (LED) observer. Therefore in the sequel, we translate some of the pertinent ideas in [22] to the present context by investigating the possibility of achieving a semi-global nonlinear separation principle that relies on any globally asymptotically stabilizing SFB control using state estimates generated by a LED observer.

For the sake of concreteness, we choose to demonstrate our analysis by means of an example. In the literature, the effects of peaking are often illustrated by examples involving some sort of high-gain design which assigns repeated stable eigenvalues to the linear component (1.57b) or (1.58b) of a cascade – see for instance Example 1.1 in [22], or Example 4.29 in [13], or the example on page 614 in [15]. The frequency with which one encounters this kind of example almost gives the impression that the difficulties caused by peaking result only from designs involving repeated eigenvalues, which is false. For this reason, we choose to analyze the following cascade system:

$$\dot{x} = -kx + \xi_i x^2 \tag{1.59}$$

$$\begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \\ \dot{\xi}_3 \end{bmatrix} = \begin{bmatrix} -L_1 & 1 & 0 \\ -L_2 & 0 & 1 \\ -L_3 & 0 & 0 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} \triangleq A\xi \tag{1.60}$$

where (1.60) could be the error dynamic of a LED observer, and (1.59) could be one of the states in the CL plant under ESFB, with k an adjustable control parameter.

Our objective is to analyze how and when peaking occurs in linear systems, and why it destroys the possibility of even semi-global stabilization. To that end, we will examine the response of (1.59) to each $\xi_i(t)$ as the gains L are increased. Throughout the discussion, we hope to clarify how the so-called "peaking exponents" [22] quantify peaking behaviour in the various states of a linear system, and can be used to predict whether a cascade such as (1.59)-(1.60) can be semi-globally stabilized under minimal assumptions on the individual subsystems.

We assume that the gains L_i are chosen so that the spectrum of A is

$$\sigma(A) = \{\lambda \in \mathbb{C} \,:\, |A - \lambda I| = 0\} = \{-\lambda_1, -\lambda_2, -\lambda_3\},$$

where the λ_i are distinct and strictly positive. We begin by decoupling the subsystem (1.60) usin the change of coordinates $z = P^{-1}\xi$, where we take

$$P = \begin{bmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1\lambda_2 + \lambda_3\lambda_1 & \lambda_1\lambda_2 + \lambda_2\lambda_3 & \lambda_2\lambda_3 + \lambda_3\lambda_1 \\ \lambda_1\lambda_2\lambda_3 & \lambda_1\lambda_2\lambda_3 & \lambda_1\lambda_2\lambda_3 \end{bmatrix}. \tag{1.61}$$

Then

$$\dot{z} = (P^{-1}AP)z = \begin{bmatrix} -\lambda_1 & 0 & 0 \\ 0 & -\lambda_2 & 0 \\ 0 & 0 & -\lambda_3 \end{bmatrix} z \triangleq Dz, \tag{1.62}$$

whose solution is

$$z(t) = \begin{bmatrix} e^{-\lambda_1 t} & 0 & 0 \\ 0 & e^{-\lambda_2 t} & 0 \\ 0 & 0 & e^{-\lambda_3 t} \end{bmatrix} z(0) \triangleq \phi(t) z(0). \tag{1.63}$$

In the original coordinates the solution to (1.60) is

$$\xi(t) = P\phi(t)P^{-1}\xi(0). \tag{1.64}$$

the expressions for the elements of $P\phi(t)P^{-1}$ could be rather complicated, and we omit to write them here. In order to see the effects of peaking, it actually suffices to examine the response of (1.64) to only one of the initial conditions. For simplicity, we arbitrarily choose $(\xi_1(0), \xi_2(0), \xi_3(0)) = (1, 0, 0)$ and instead write the response in terms of the three exponentials governing the motion:

$$\begin{aligned}
\begin{bmatrix}\xi_1(t)\\ \xi_2(t)\\ \xi_3(t)\end{bmatrix} &= \tfrac{1}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)} \begin{bmatrix} \lambda_1^2 & -\lambda_2^2 & \lambda_3^2 \\ \lambda_1^2(\lambda_2+\lambda_3) & -\lambda_2^2(\lambda_1+\lambda_3) & \lambda_3^2(\lambda_2+\lambda_1) \\ \lambda_1^2\lambda_2\lambda_3 & -\lambda_2^2\lambda_1\lambda_3 & \lambda_3^2\lambda_1\lambda_2 \end{bmatrix} \begin{bmatrix} e^{-\lambda_1 t}\\ e^{-\lambda_2 t}\\ e^{-\lambda_3 t}\end{bmatrix} \\
&\triangleq \tfrac{1}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)} \begin{bmatrix} c_{1,1} & c_{1,2} & c_{1,3} \\ c_{2,1} & c_{2,2} & c_{2,3} \\ c_{3,1} & c_{3,2} & c_{3,3} \end{bmatrix} \begin{bmatrix} e^{-\lambda_1 t}\\ e^{-\lambda_2 t}\\ e^{-\lambda_3 t}\end{bmatrix} \qquad (1.65)
\end{aligned}$$

We are now ready to analyze the behaviour of (1.59) when i is 1, 2 and 3. The dynamic (1.59) is familiar by now, and from Example 1.1.1 we know that its solution looks like:

$$x(t) = \frac{x(0)e^{-kt}}{1 + x(0)R_i(t)}, \tag{1.66}$$

where we define

$$R_i(t) \triangleq -\int_0^t e^{-k\tau}\xi_i(\tau)d\tau. \tag{1.67}$$

The expression $R_i(t)$ is important because it relates directly to the ROA of the cascade. From (1.65), we can express each $\xi_i(t)$ as

$$\xi_i(t) = \tfrac{1}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)} \sum_{j=1}^{3} c_{i,j}e^{-\lambda_j t} \tag{1.68}$$

and therefore

$$\begin{aligned}
R_i(t) &= \sum_{j=1}^{3} \frac{-c_{i,j}}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)} \int_0^t e^{-k\tau}e^{-\lambda_j\tau}d\tau \\
&= \sum_{j=1}^{3} \frac{c_{i,j}(e^{-(k+\lambda_j)t}-1)}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(k+\lambda_j)} \qquad (1.69)
\end{aligned}$$

Before discussing the significance of the expression (1.69), we first provide a graphical analysis of the behaviour of both $R_i(t)$ and $\xi_i(t)$, $i \in \{1,2,3\}$, when the convergence rate of the ξ-subsystem is increased. We introduce a gain factor γ by which we amplify all eigenvalues of A simultaneously. In the figures that follow, we take

$$(\lambda_1, \lambda_2, \lambda_3) = (1, 0.5, 1.5). \tag{1.70}$$

and amplify them by $\gamma = 1$, $\gamma = 5$ and $\gamma = 10$.

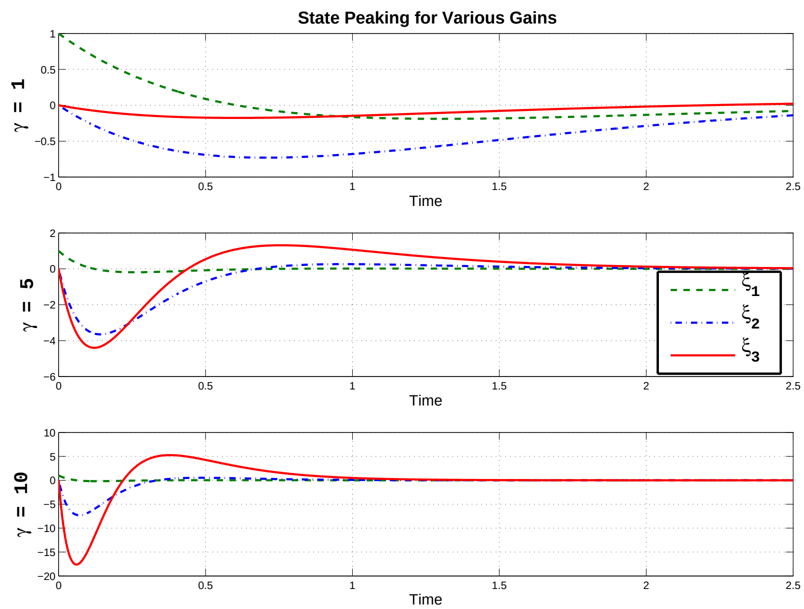

Figure 1.2: State Peaking with Increasing γ. As γ is increased, the third state $\xi_3(t)$ exhibits the greatest peaking effect, while the state $\xi_1(t)$ experiences no peaking at all.

Figures 1.2 and 1.3 show how each of the ξ_i respond when the convergence rate is increased. While all states converge faster as γ is increased, the peaking behaviour of some states becomes more pronounced. From Figure 1.3 it is apparent that the peak in state ξ_3 experiences the greatest increase with γ while the state ξ_1 experiences virtually no transitory increase in magnitude with an increase in γ.

In Figure 1.4 we look for the maximum magnitude of $R_i(t)$ over time, as this quantity determines the set of initial conditions $x(0)$ for which (1.66) does not es-

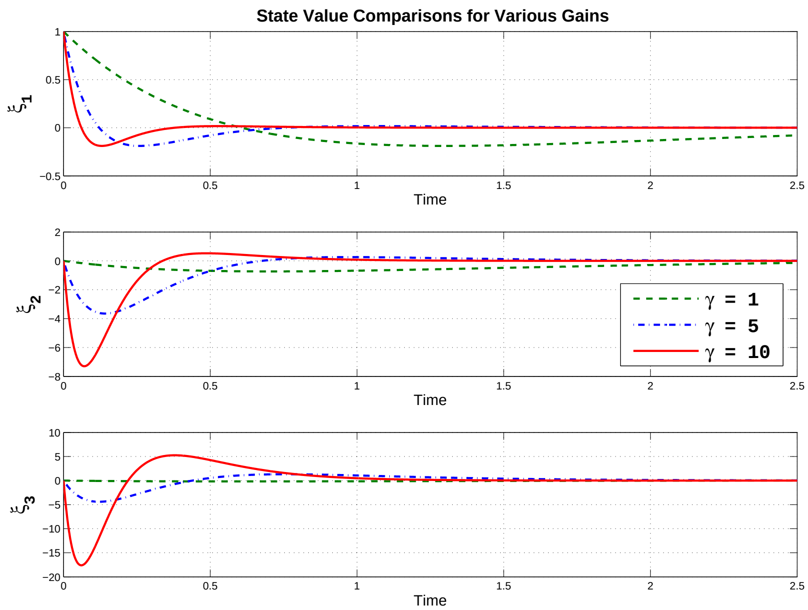

Figure 1.3: A Comparison of State Behaviours for Increasing γ.

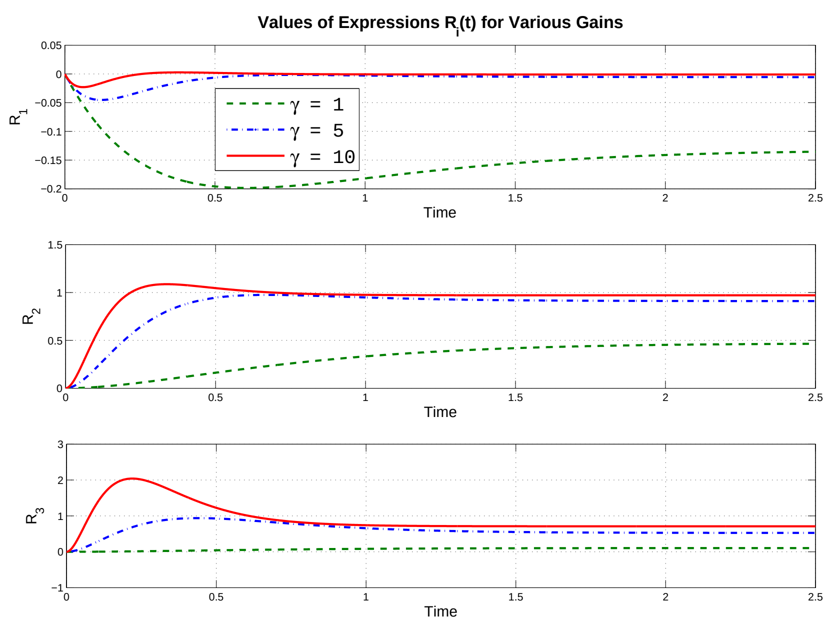

Figure 1.4: The Behaviour of the term $R_i(t)$ as γ is varied.

cape to infinity in finite time. For notational convenience, we define

$$R_i^* \triangleq \max_t |R_i(t)|. \tag{1.71}$$

If $R_i(t)$ is a positive quantity for all time, then the set of initial conditions $x(0) \in (-1/R_i^*, \infty)$ guarantee convergence of $(x(t), \xi(t))$ to the origin. Likewise, if $R_i(t)$ is a negative quantity for all time, then the same set is $x(0) \in (-\infty, 1/R_i^*)$. Therefore, the set of admissible initial condition can concisely be expressed as:

$$\Omega = \left\{ x(0) : -\operatorname{sgn}(R_i(t > 0))x(0) < \frac{1}{R_i^*} \right\} \tag{1.72}$$

If ξ_1 is driving the nonlinear subsystem (1.59), then we note from the corresponding plot of $R_1(t)$ that increasing the gain γ increases the size of Ω since R_1^* decreases. If ξ_2 is drives the nonlinear subsystem, we notice that R_2^* increases with γ to $R_2^* \approx 1.1$ when $\gamma = 10$. This value does not change significantly when γ is further increased – i.e. for $\gamma = 30$, we still observe $R_2^* \approx 1.1$. On the other hand, R_3^* shows an increasing trend with increasing γ. In fact, when $\gamma = 30$, we observe $R_3^* \approx 6.5$. This fact indicates that if the state ξ_3 drives the nonlinear subsystem (1.59), then increasing the convergence rate of the ξ-subsystem actually *decreases* the size of the set Ω, which is related to the size of the composite ROA. We note that "peaking" in the terms $R_i(t)$ correlates to that of the associated state $\xi_i(t)$, which illustrates how the peaking phenomenon can obstruct the possibility of non-local stabilization of a partially linear cascade. These observations are consistent with the following theorem, which we rephrase here in the context of cascades such as (1.58):

Theorem 1.2.3 (Theorem 4.41, [13])**.** *Assume A is Hurwitz, and that its spectrum can be arbitrarily assigned. Also assume that $\dot{x} = F(x)$ is GAS at $x = 0$. If only non-peaking components of the state ξ enter the interconnection term $\psi(x, \xi)$, then semi-global asymptotic stability of the cascade* (1.58) *is achievable via the eigenstructure assignment for A.*

To complete our analysis, we make some interesting observations linking our graphical arguments and the structure of equation (1.69) to the notion of *peaking exponents* discussed in [22]. We consider all Hurwitz matrices in the observer canonical form

$$A = \begin{bmatrix} -L_1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -L_{q-1} & 0 & \cdots & 1 \\ -L_q & 0 & \cdots & 0 \end{bmatrix}$$

and define $a \triangleq \min_i\{\mathrm{Re}(\lambda_i) : -\lambda_i \in \sigma(A)\}$. Then, the solution to $\dot{\xi} = A\xi$ is bounded by $\|\xi(t)\| \leq \kappa\|\xi(0)\|e^{-at}$. It is noted in [22] that in linear systems peaking occurs because in general it is not possible to choose the gains L_i to make a larger without also making κ larger – i.e. $\kappa = \kappa(a)$. According to [22], any given linear combination of states $y = C\xi$ will have associated with it a peaking exponent π, so that it is bounded by $\|y(t)\| \leq a^{\pi}\|\xi(0)\|e^{-at}$. It is interesting to associate this fact with the following observation. If all eigenvalues of A are multiplied simultaneously by a gain γ in our previous example, then in consideration of (1.68) and (1.65), the solution of (1.60) can be re-written as:

$$\xi_i(t) = \frac{\gamma^{i+1}}{\gamma^2(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)} \sum_{j=1}^{3} c_{i,j} e^{-\gamma\lambda_j t}, \tag{1.73}$$

which offers clear insight into the peaking behaviour of the three states. For $i = 1$, the state $\xi_1(t)$ does not get amplified as γ is increased, since γ cancels from numerator and denominator in (1.73). In effect, we may say that $\xi_1(t)$ responds with a peaking exponent of zero – i.e. although the convergence rate of ξ_1 increases with an increased γ, $\xi_1(t)$ itself is amplified by $\gamma^{\pi} = \gamma^0$. On the other hand, by this interpretation the peaking exponents associated with ξ_2 and ξ_3 are 1 and 2, respectively.[5] For peaking exponents $\pi > 0$, the associated signals exhibit peaking, and prevent semi-global stability of cascades such as (1.58) if they are present in the interconnection term $\psi(x, \xi)$ [13].

The example cascade (1.59)-(1.60) is intended to draw an analogy with a CL plant under ESFB, using state estimates generated by a LED observer. In general, a SFB control law relies on all state estmates, implying that all states of the error

[5] Actually, this can also be intuited by observing that the canonical form A is "almost" a chain of integrators. Therefore, if the state ξ_1 converges at a rate of $\xi_1(t) \approx e^{-at}$, then the state ξ_2 is roughly its time derivative, and will converge at $\xi_2(t) \approx -ae^{-at}$, and so on as we progress through the chain. This observation implies that higher-indexed states in this chain will have higher associated peaking exponents, exhibiting greater peaking behaviour [27].

dynamic are present in the coupling term $\psi(x,\xi)$. Then from the preceding analysis, it is clear that we cannot even expect to have a "pseudo-separation principle" for nonlinear systems involving a LED observer and any non-locally asymptotically stabilizing SFB control.

1.3 Nonlinear Separation

The preceding discussion makes clear that non-local OFB design for nonlinear systems must take into consideration the I/O behaviour of the plant if it is to consist of a dynamic component. Any such design cannot ignore the interconnection properties of the plant to dynamic components such as observers, as is done in the classical separation design. Therefore, we must broaden our notion of "separation", and hence our perspective on the possible approaches in addressing the non-local OFB problem for nonlinear systems.

Since the observer-plus-SFB-control paradigm seems natural to OFB design, we must clarify exactly what we mean by the word "separation" in the nonlinear case. We broadly categorize all separation approaches to the OFB problem as those that loosely associate the role of the dynamic component of the OFB with the estimation of the plant's state, with the intention of using these estimates in a (possibly modified) SFB control. Then, the word *separation* refers to some degree of anticipated design freedom in terms of choosing $u = \alpha(\xi, y)$ and $\dot{\xi} = \Gamma(\xi, y, u)$ in (1.2) independently of one another. With this clarification, we offer the following categorization of possible approaches to the non-local OFB problem:[6]

1. *Separation*: Design $u = \alpha(x)$ so that $\dot{x} = f(x) + g(x)\alpha(x)$ is non-locally asymptotically stable. Then, independently design a system $\dot{\xi} = \Gamma(\xi, y)$, whose function is associated with generating an estimate $\hat{x}$ of the true state x. Implement $u = \alpha(\hat{x})$.

[6] in Remark 10 of [20], a similar categorization is given, but the context is different. Freeman interprets the dynamic component strictly as an observer, and the controller is static and belonging to a specific class of controllers. Furthermore the interest in [20] is whether one can make a general statement on the possibility of global asymptotic stabilization within any of the four nonlinear separation categories, whereas our interest is in classifying the various available solutions and possible approaches to the problem.

2. *Controller Separation*: Design any $u = \alpha(x)$ so that $\dot{x} = f(x) + g(x)\alpha(x)$ is non-locally asymptotically stable. Then, design a system $\dot{\xi} = \Gamma(\xi, y)$ so that the coupled system

$$\dot{x} = f(x) + g(x)\alpha(\xi)$$
$$\dot{\xi} = \Gamma(\xi, h(x))$$

is non-locally asymptotically stable.

3. *Observer Separation*: Design $\dot{\xi} = \Gamma(\xi, y)$ whose function is associated with generating an estimate $\hat{x}$ of the true state x. Then design $u = \alpha(\hat{x}, y)$ in such a way that the coupled system is non-locally asymptotically stable.

4. *Non-Separation Approach*: Design $u = \alpha(\xi, y)$ and $\dot{\xi} = \Gamma(\xi, y)$ interdependently so that the composite system (1.3) is non-locally asymptotically stable.

Observer and Controller separation suggest the possibility of modifying the control design or observer design (respectively) in order to compensate for the dynamic coupling between the ξ subsystem and the plant, while leaving freedom in the design of the observer or controller (respectively). We will see in the literature survey that follows, that many existing solutions to the OFB problem fall within one of these four categories.

1.4 Literature Review

We divide the following literature survey according to whether the design is semi-global or global. This is a natural categorization which recognizes the different set of tools and system restrictions (with some overlap) usually applied in either design. At the end, we give an overview of literature dealing specifically with multivariable OFB results.

1.4.1 Semi-Global OFB

Research into semi-global OFB designs became popular after the publication of [28], where a restrictive class of "fully linearizable" systems is considered – i.e. systems that are simultaneously linearizable by state feedback from input to output (IOL) and from input to state (SFBL)[7] and can therefore be expressed as

$$\begin{aligned}\dot{x} &= A_c x + B_c(\phi_1(x) + \phi_2(x)u)\\ y &= C_c x\end{aligned}$$

where A_c has 1's on the superdiagonal and zeros elsewhere, $B_c = [0, \ldots, 0, 1]^T$, $C_c = [1, 0, \ldots, 0]$, and $\phi_1(x)$ and $\phi_2(x)$ are some nonlinearities, with $\phi_2(x)$ nonsingular for all x. First, the authors design a globally asymptotically stabilizing state feedback controller $u(x) = (\phi_{20}(x))^{-1}(-\phi_{10}(x) + \alpha(x))$, where $\alpha(x)$ is any control capable of compensating for the inexact cancellation of the nonlinearities $\phi_1(x)$ and $\phi_2(x)$ by their nominal values $\phi_{10}(x)$ and $\phi_{20}(x)$[8]. Then, they design a *high-gain* (HG) observer much like the one originally introduced in [29], and analyze the effects of implementing the ESFB $u(\hat{x})$. Aside from the modeling uncertainty (which they assume is compensated for in the control $u(x)$), they identify a second disturbance term resulting from inexact cancellation of the nonlinearities due to the use of the state estimate $\hat{x}$ instead of the true state x in $u(x)$. Although the HG observer has excellent disturbance rejection properties with respect to this second disturbance as its gain is increased, the inevitable presence of peaking in the observer states is shown to have a destabilizing effect on the composite system. The authors solve this problem simply by saturating the control $u(\hat{x})$ and show that this technique enables semi-global stability, with the observer gain affecting the size of the CL ROA. This design is an example of controller separation, as freedom is given in the design of $u(x)$.

An important result that followed was given in [30], and elaborated further in [31] and [32]. As the title suggests, in [30] the authors demonstrate that if a given

[7] Please see Theorem 4.2.2 and Theorem 2.2.1 (respectively) in [16] for sufficient conditions and associated transformations.

[8] They give a concrete example of a Lyapunov redesign SFB that accomplishes this.

smooth nonlinear system is globally asymptotically stabilizable by exact SFB and is *uniformly completely observable* (UCO),[9] then it can be semi-globally stabilized by OFB. In this paper Teel and Praly append a number of integrators to the input side of the system to avoid using input derivatives, and work with this augmented plant:

$$
\begin{aligned}
\dot{x} &= f(x, u_0) \\
\dot{u}_0 &= u_1 \\
&\vdots \\
\dot{u}_{n_u} &= v
\end{aligned}
$$

Since $\dot{x} = f(x,u)$ is assumed stabilizable by u, there exists a state feedback control law $v = \alpha(x, u_0, \ldots, u_{n_u})$ that globally asymptotically stabilizes the augmented system (this can be done by backstepping, for instance). Then, similarly to equation

[9]The notion of UCO (also referred to as "uniform observability") was first introduced in [33] for nonlinear systems. The essential implication of complete observability is that theoretically it is possible to extract full state information at an instant of time given exact information about the input $u(t)$, the output $y(t)$, and a sufficient number of their derivatives at that time. If the preceding statement is true for any time instant, then the observability is said to be uniform. For the system (1.1), there are two equivalent characterizations of UCO systems:

1. the system is diffeomorphic to [29]:

$$
\begin{aligned}
\dot{x}_i &= x_{i+1} + g_i(x_1, \ldots, x_i)u, \qquad 1 \le i \le (n-1) \\
\dot{x}_n &= \phi(x) + g_n(x)u,
\end{aligned}
\tag{1.74}
$$

2. there exists a unique smooth function $\psi(\cdot)$ such that (Section 9.6, [17]):

$$
x(t) = \psi(y, \dot{y}, \ldots, y^{(n-1)}, u, \dot{u}, \ldots, u^{(n-2)}). \tag{1.75}
$$

The second characterization can be seen as follows. Define $\overline{y} \triangleq [y, \dot{y}, \ldots, y^{(n-1)}]^T$ and $\overline{u} \triangleq [u, \dot{u}, \ldots, u^{(n-2)}]^T$. Then, taking successive derivatives of the output results in $\overline{y} = \Phi(x, \overline{u})$, with $\Phi(0,0) = 0$ when $y = h(x)$ is zero at $x = 0$. By definition, a UCO system has

$$
\operatorname{rank}\left[\frac{\partial \Phi(x, \overline{u})}{\partial x}\right] = n, \qquad \forall\, x \in \mathbb{R}^n,\ \forall\, \overline{u} \in \mathbb{R}^{n-1} \tag{1.76}
$$

which, by the Implicit Function Theorem implies (1.75). Effectively, (1.76) means that there is no input u that can destroy "observability" – i.e. falsify the equality (1.76). We note two facts: first, it can easily be shown that the usual observability condition requiring $\operatorname{span}\{dh(x), dL_f h(x), \ldots, dL_f^{n-1} h(x)\} = \mathbb{R}^n$ is necessary for (1.76) to hold when $u \equiv 0$. Second, (1.76) allways holds for linear systems, for any input – i.e. plain "observability" is equivalent to UCO for linear systems.

(1.75), they express the state as $x(t) = \phi(y, \dot{y}, \ldots, y^{(n_y)}, u_0, u_1, \ldots, u_{n_u})$ where u_i is the state of the $(i+1)$th integrator after the plant input, and design a HG observer to estimate the n_y derivatives of the output needed for the knowledge of $x(t)$. Finally, they implement $v = \alpha(\phi(\hat{y}, \dot{\hat{y}}, \ldots, \hat{y}^{(n_y)}, u_0, u_1, \ldots, u_{n_u}), u_0, u_1, \ldots, u_{n_u})$, applying the same saturation technique used in [28] in order to prevent the destabilizing effects of peaking. The dynamic component of this OFB law consists of the n_u appended integrator dynamics, and the n_yth order HG observer. If for a given nonlinear system the function $\phi(\cdot)$ can easily be found, then this method is constructive. The existence of such a function is guaranteed by the Implicit Function Theorem, provided the system is UCO; however finding an explicit expression for $\phi(\cdot)$ may be difficult. Therefore, this result is generally regarded as an existence result, rather than a practical result.

In [34] Khalil improves his previous result [28] by broadening the class of systems to which the techniques in [28] are applicable, and incorporating some of the techniques introduced in [31] into his design and analysis. He extends the analysis in [31] to demonstrate that if a semi-globally asymptotically stabilizing exact SFB exists for a nonlinear system belonging to a certain class, then ESFB "recovers" the performance of the exact SFB controller, in the sense that as the HG observer gain is increased, the trajectories and the ROA of the CL system under ESFB approach those of the CL system under exact SFB. Although the set of systems he considers is not as broad as the set of all stabilizable UCO systems, his result is fully constructive and easily implemented.

The results in [34] and [31] are furthered in [35] and [36], which likely represent the current state of the art in semi-global OFB stabilization of nonlinear systems.

Although there are many extensions available, the aforementioned papers cover the main ideas, generally used in most other papers on semi-global OFB. For example, in [37], the authors consider a multi-input, single output system that is not affine in the control u, and whose model ($\dot{x} = f(x) + g(x,u)$, $y = h(x)$, $x \in \mathbb{R}^n$) is not necessarily valid over all of $\mathbb{R}^n$. They identify a set of conditions that guarantee that such a system is transformable to the form (1.74) so that a HG observer can be designed. The conditions they provide are very similar to those guaranteeing the existence of a NOF, and they point out that for systems not affine in the control, transformability into (1.74) is stronger than UCO. Then, assuming there exits a semi-globally stabilizing exact SFB $u = \alpha(x)$, they design an nth order HG observer on the basis of the model's Lipschitz extension and explicitly provide a dynamic OFB in terms of $\alpha(\hat{x})$.

One may argue that given the unrestrictive set of conditions sufficient for the existence of semi-global OFB relative to the ISS or growth-rate restrictions that allow global nonlinear OFB designs, it appears that semi-global designs are more practical and more worthy of research attention. However, in all of the above references the key ingredient is a HG observer whose use in physical applications has serious practical limitations. The use of high gains is usually associated with "differentiation", and hence the amplification of noise. Global designs generally dispense of this difficulty, and in that sense are more elegant. Furthermore, the designer does not have to worry about sizes of compact sets of initial conditions for which the system states are guaranteed to converge. The development of global OFB designs is deemed important, since a designer should have such tradeoffs at her disposal.

1.4.2 Global OFB

The global OFB problem is significantly more difficult than the semi-global OFB problem, especially if we wish to relax the growth restrictions usually imposed on the system's nonlinearities. From Example 1.1.1 and our discussion in Section 1.2, it is natural to expect that global OFB stabilization of nonlinear systems requires a more stringent set of restrictions on the dynamic structure of (1.1) than semi-global OFB stabilization. Indeed, Mazenc et al have shown in [38] by several counterexamples that even if a system is globally asymptotically stabilizable by exact SFB and satisfies a strong observability condition such as UCO, it may not be globally stabilizable by any OFB. They have shown that for an entire class of nonlinear systems diffeomorphic to the following form[10]

$$\begin{aligned}
\dot{\eta} &= \Gamma(\eta, z_1, \ldots, z^{r-1}) \\
\dot{z}_i &= z_{i+1}, \qquad 1 \le i \le (r-1) \\
\dot{z}_r &= z_r^n + f_1(\eta, z_1, \ldots, z_{r-1}) + g_1(\eta, z_1, \ldots, z_{r-1})u \\
y &= h(z) = z_1
\end{aligned} \tag{1.77}$$

there is no globally stabilizing OFB when

$$n \ge \frac{r}{r-1},$$

regardless of the stability properties of the η-subsystem. The obstacle identified in [38] is a so-called "unboundedness unobservability" (UU) property, which implies that even though (1.77) may be observable and stabilizable, it may have some states that escape to infinity in finite time without being "noticed" through the output y. In other words, even if those states are observable through y, the fact that they are escaping to infinity in finite time may not be observable.

Though there appear to be some theoretical limitations on what is possible for global OFB designs, the complete set of systems for which global OFB is possible, and conversely the set of systems for which global OFB is not possible, have not been fully characterized. The set of systems for which we know how to design globally stabilizing OFB laws is likely not the largest such set. We proceed to give an overview of the key contributions to the global OFB problem.

One of the earlier investigations of a global separation principle is provided by Tsinias in [39] and [40]. In [40], Sontag's ISS condition for the global stability of cascades is applied and extended to ESFB for a class of generalized bilinear systems of the form

$$\begin{aligned}
\dot{x} &= f(x, u) + uBx \\
y &= Cx.
\end{aligned}$$

[10] This is a special case of the normal form introduced in [10]. The normal form plays an important role in global OFB designs for nonlinear systems and will be discussed in the next chapter. Some variant of the normal form is usually a starting point for global OFB designs.

where the norm of the rate of change of f with respect to x is assumed to be bounded for all x and u. For this class of systems, Tsinias translates the ISS condition into a concise set of algebraic conditions, which, if satisfied, guarantee the global asymptotic stability of the composite system consisting of an observer and the CL plant under ESFB.

A more general result also relying on the concept of ISS is given in [41] for a class of SISO systems of the form

$$\begin{aligned} \dot{\eta} &= \Gamma(\eta, z_1) \\ \dot{z}_i &= z_{i+1} + f_i(\eta, z_1) + g_i(z_1), \qquad 1 \leq i \leq (r-1) \\ \dot{z}_r &= u + f_r(\eta, z_1) + g_r(z_1) \\ y &= z_1, \end{aligned} \tag{1.78}$$

where in particular it is important to note that η is allowed to enter the chain of integrators nonlinearly. Under the relatively strong assumptions that the inverse system $\dot{\eta} = \Gamma(\eta, y)$ (i.e. the zero dynamics) are ISS with respect to y and that the nonlinearity $\phi(\eta, y) \triangleq [f_1(\eta, y), \ldots, f_r(\eta, y)]^T$ satisfies a small-gain condition, the system (1.78) can be globally asymptotically stabilized by an rth order dynamic OFB. The construction of the OFB law is similar to that in [6], relying on a SFB design *for* an observer for (1.78). However, the final expressions depend on a gain function similar to the one in (1.39) whose existence is posited by the assumption that the η-subsystem is ISS. Since such a function is generally difficult to find, this paper presents more of an existence result than a practical result.

Another generalization is presented in [42] to systems similar to (1.78):

$$\begin{aligned} \dot{\eta} &= \Gamma(\eta, z_1) \\ \dot{z}_i &= z_{i+1} + g_i(\eta, z_1, \ldots z_i), \qquad 1 \leq i \leq (r-1) \\ \dot{z}_r &= u + g_r(\eta, z_1, \ldots, z_r) \\ y &= z_1, \end{aligned} \tag{1.79}$$

where the η-subsystem is assumed to be locally exponentially stable and ISS with respect to y. However, the nonlinearities g_i are required to satisfy the following rather restrictive assumption:

$$\left|\frac{\partial g_i}{\partial \eta}\right| + \sum_{j=1}^{i} \left|\frac{\partial g_i}{\partial z_j}\right| \leq M_i(x,y) \tag{1.80}$$

for some continuous function M_i which is strictly positive everywhere. Under this assumption, Tsinias shows that there exists a dynamic OFB that globally asymptotically stabilizes the origin of (1.79).

Under slightly stronger structural conditions, it is possible to develop fully constructive global OFB laws for minimum phase systems whose nonlinearities depend on the output only, without imposing any growth restrictions on such nonlinearities. This has been demonstrated primarily in [6], [43] and [2] which will be discussed in detail in the next chapter. The system structures considered in these papers are all variants of the *normal form* (NF), or a subclass of the normal form known as the *output feedback form* (OFBF). We postpone a detailed discussion of these system forms until Section 2.2. For the systems studied in these papers it is not necessary to assume any sort of “small gain” property for the nonlinearities or any ISS condition on the zero dynamics, although as pointed out in [41], such assumptions are often made implicit by the system structure.

In the category of controller separation, Arcak, Praly and Kotkotović have contributed [44], [45] and [46]. In [44] an OFB design is given on the basis of a “circle-criterion observer”, designed for the following multivariable system structure:

$$\begin{aligned} \dot{x} &= Ax + G\gamma(Hx) + \rho(y,u) \\ y &= Cx \end{aligned} \tag{1.81}$$

where each element of the nonlinearity $\gamma(\cdot)$ is assumed to be a nondecreasing function of a linear combination of the state. Owing to this special plant structure, a linear matrix inequality can be set up to solve for a set of two gain vectors used in the observer design. The OFB design they propose treats the observer error as a disturbance, and introduces terms that allow the plant-observer interconnection to satisfy a small gain criterion similar to the one introduced in [47]. An interesting property of circle-criterion observers is that their error convergence rate is dependent on the magnitude of the plant state. This fact is exploited in [45] where Praly and Arcak develop a set of sufficient conditions guaranteeing GAS of a CE design. Their assumptions include the standard stabilizability assumption, and a specialized detectability assumption in the sense that the observer's convergence rate should be dependent on the plant state. A circle-criterion observer design is further refined in [46] for the purpose of OFB design.

An interesting result is given Tan et al in [48] in which a global stabilization and tracking OFB is proposed for a system of the form

$$
\begin{aligned}
\dot{x}_1 &= x_2 + \phi_1(y) \\
\dot{x}_i &= x_{i+1} + \phi_i(y) + \gamma_i(y)x_2, \quad 1 \le i \le (r-1) \\
\dot{x}_{r+j} &= x_{r+j+1} + \phi_j(y) + \gamma_j(y)x_2 + g_{r+j}\sigma(y)u, \quad 0 \le j \le (n-1-r) \\
\dot{x}_n &= \phi_n(y) + \gamma_n(y)x_2 + g_n\sigma(y)u \\
y &= x_1
\end{aligned}
\tag{1.82}
$$

where the $\phi(\cdot)$ and $\gamma(\cdot)$ are smooth nonlinearities, $\sigma(\cdot)$ is bounded away from zero, and the coefficients g_i, $r \le i \le n$ form a *Hurwitz vector* (please see Section 2.2). This system form slightly generalizes the OFBF, in that it allows the unmeasured x_2 to appear affinely in all equations after the first. The appearance of x_2 implies that the form (1.82) no longer directly admits a LED observer design as the OFBF does. Nevertheless, their design is based on an observer backstepping technique [6]. The clever trick used here is a nonlinear transformation of the form $\xi_i = x_2 - w_i(x_1)$ which results in $\dot{\xi}$-dynamic that is linear and stable in ξ, driven by nonlinearities which are exclusively functions of the output y, thus admitting an observer design which cancels the nonlinearities by an appropriate output injection term; in this way, the technique is strongly reminiscent of Marino and Tomei's *Filtered Transformations* (FTs)[11], and suggests that FTs may find other creative applications in the future.

A series of papers influenced in part by the result in Tan et al is [49], [50]

[11] Please see section 2.4.1, and [2] for more details.

and [51]. In [49] the following system is studied:

$$\begin{aligned} \dot{\eta} &= A(y,u)\eta + B(u,y) \\ \dot{y} &= \psi_0(y,u) + \psi_1(y,u)\eta \\ \rho &= h(y,\eta) \end{aligned} \tag{1.83}$$

where $\eta \in \mathbb{R}^n$, the output $y \in \mathbb{R}^P$, the input $u \in \mathbb{R}^m$, and $\rho(y,\eta)$ is a "performance variable" which is to be regulated to zero by dynamic OFB while all other state variables remain bounded. System (1.83) is more general than the NF or the OFBF studied in [43], [6] or [2] in that it does not necessarily have stable zero dynamics, nor is it necessarily affine in the control u. However, the system must satisfy two main assumptions. One, there must exist SFB $u = \alpha(\eta,y)$ which regulates $\rho(y,\eta)$ to zero, keeps all other variables bounded, and renders the CL system globally "bounded input bounded state" (BIBS) with respect to any input d, when $u = \alpha(y,\eta + d(t))$. Second, there must exist a function $\beta(y)$ so that the system

$$\dot{z} = \left(A(y,u) - \frac{\partial \beta(y)}{\partial y} \psi_1(y,u) \right) z \tag{1.84}$$

is stable for any y and u. Furthermore, $z(t)$ is required to be such that $\lim_{t\to\infty} \alpha(y,\eta + z(t)) = \alpha(y,\eta)$ (which does not necessitate $\lim_{t\to\infty} z(t) = 0$). Analogously to Tan et al [48], the $\dot{z}$-subsystem is not an observer, but a dynamic equation governing the behaviour of an auxiliary variable defined as $z \triangleq M\hat{\eta} - \eta + \beta(y)$ where M is some invertible matrix and $\hat{\eta}$ is the state of the dynamical component of the OFB:

$$\begin{aligned} \dot{\hat{\eta}} &= w \\ u &= \alpha(y, M\hat{\eta} + \beta(y)) \end{aligned} \tag{1.85}$$

with w a new control signal whose expression is chosen to make $\dot{z}$ take the form (1.84). Here, $\beta(y)$ is analogous to the $w_i(x_1)$ in Tan et al.

It is also interesting to note that the dynamic component $\dot{\hat{\eta}}$ is not designed for the purpose of reconstructing the state η so its state is not required to converge to the latter. In effect, such a design does not really belong to a separation category; the objective is to estimate asymptotically the required *control* $u = \alpha(y,\eta)$, not to estimate the state η and then use it in a CE design.

In [50] and [51] these results are extended to systems of the form (1.83) with the unmeasured η variables entering the equations nonlinearly. Although these results are quite novel and interesting, the method is demonstrated ad-hoc on a few physical and mathematical examples; no systematic method or even guidelines for constructing a function $\beta(y)$ are provided in terms of the system functions $A(y,u)$ and $\psi_1(y,u)$.

A recent paper that provides a higher-level analysis of existing observer separation methods is [52]. Therein the authors examine some of the above-mentioned global OFB methods which apply specifically to systems in the *strict normal form* (SNF)[12] and try to classify them according to certain properties of the observer.

Other recent directions in the OFB problem include the removal of the restrictive minimum-phase condition required in many of the methods noted above, as well as considering the problem in a stochastic setting, or in the presence of uncertain parameters. Some work in this direction includes [53], [54], [55], [56]. It is worth noting that most global OFB methods rely on the technique of backstepping, which is applicable to systems in the *strict feedback form* (SFF). Historically, this technique has augmented, and in some cases generalized global nonlinear designs based on passivity methods. Passivity methods rely on a very specific set of structural restrictions; namely, the system must have relative degree one, and be minimum phase [13]. Backstepping has removed the "relative degree one" requirement. There is an analogous method known as "forwarding" for systems in *feedforward form* and it is found to be useful in global OFB methods seeking to remove the minimum-phase requirement incumbent in passivity methods. For example, forwarding is used in [57] to remove the minimum-phase requirement often seen global OFB methods.

1.4.3 Multivariable OFB

Although some of the aforementioned work has natural extensions to the more realistic multivariable case, for several methods such extensions are not obvious. For

[12] Please see Section 2.2

this reason, we list a few references dealing specifically with MIMO OFB designs.

A direct extension of Teel and Praly's semi-global existence result [30] is given in [58], where a slightly simpler proof is also provided. Khalil's semi-global result [34] is developed within a multivariable setting. According to the authors, the global result in [43] is easy to extend to the MIMO case (although the extension is not given in that paper). An extension to [6] is possible on the basis of the MIMO strict normal form identified in [59].

Other MIMO OFB methods are obtained by neural networks [60], or by sliding mode control [61]. In [62], the authors consider a MIMO system whose state-dependent nonlinearity is globally Lipschitz bounded, and they develop a very simple linear HG observer and a linear HG controller, showing that the combination yields global asymptotic stability if the gains are selected sufficiently high.

The more challenging multivariable adaptive output feedback problem is addressed in papers such as [63], [64] and [65]. Because the authors consider the adaptive problem, the structural restrictions on their systems are often more severe; for example, in [65], in addition to requiring the number of inputs to be equal to the number of outputs, the MIMO system these authors consider also must consist of subsystems with identical dimensions and relative degree. Another similar result is given in [66] where a HG observer is used.

1.5 Thesis Objectives

From the preceding literature review, it is evident that the study of non-local OFB for nonlinear systems is not complete. Many of the available results are applicable to very specific, and often restrictive system structures. Furthermore, many seminal results referenced here are still only at their "theoretical infancy" stage – that is, they are theoretical existence results which are not constructive or cannot be easily implemented.

In terms of globally stabilizing OFB, we know that the well-known ISS and Lipschitz growth restrictions are stronger than necessary. The search for a more "minimal" set of sufficient conditions guaranteeing the existence of global OFB (akin to Teel and Praly's existence result [30]) is still open. Once we get closer to a full characterization of such conditions, there will still be the task of "activating"[13] such existence theories or analytical tools into explicit, constructive engineering control designs. Therefore, a well-motivated effort would be in one of two directions:

1. the development of new tools and constructive nonlinear design techniques (such as backstepping) or,

2. the expansion of the classes of systems to which existing tools can be applied to design OFB laws,

or some combination thereof.

Another observation that can be made from this literature survey is that many global results rely in some sense on a globally convergent observer, or a system structure that admits the design of an observer. Such is the case with the OFBF and results such as (but not exclusively) [6], [43] and [2]. This fact suggests that different observer forms might well serve as platforms for the development of new OFB algorithms using existing tools.

Multivariable systems sometimes exhibit certain structural flexibilities with no analogue in their SISO counterparts. For instance, such is the case in nonlinear observer design, where several different generalizations of the SISO LED observer are possible to the MIMO case [68], [69]; in particular, some less obvious extensions allow for the presence of unmeasured states in some error subsystems while still guaranteeing asymptotic convergence of the whole [70]. In some cases it may be possible to exploit the additional freedoms offered by the multivariable nature of a system for the development of new OFB laws.

The objective of the work we present in this thesis is to develop and analyze a preliminary set of MIMO global OFB designs on the basis of the *block triangular observer form* (BTOF) introduced in [70]. To that end, we explore the possibility of applying some of the techniques presented by Marino and Tomei in [2]. As an initial step in this direction, we develop MIMO extension of Marino and Tomei's

[13] Kokotović used this terminology in the interesting article [67].

global SISO OFB method[14], which is published in [71] and recapitulated in Chapter 3. In Chapter 4 we present an application of observer backstepping to a subset of systems equivalent to the BTOF, where the output dependency is triangular [72]. Finally, we present an OFB algorithm on the basis the BTOF, using tools from [2] and [1]. Along the way, we hope to provide a clear exposition of the most relevant tools currently used in most constructive OFB designs.

In the next chapter we provide the preliminary background used to develop our results.

[14]From now on, we refer to Marino and Tomei's global SISO OFB method as the MT method.

Chapter 2

Preliminary Background

This chapter will focus primarily on global output feedback results based on the work of Kanellakopoulos, Kokotović and Morse in [6], and Marino and Tomei in [2], as these results will later be extended to a broader class of systems. Prior to detailing the work in these two papers, we introduce what is probably the most important tool in constructive nonlinear control: integrator backstepping. This simple design technique has been widely used in many creative ways and is worth reviewing. We then present and analyze some of the basic system forms that enabled the designs in [6] and [2], and finally give a thorough exposition of the relevant techniques developed in the two papers.

2.1 A Basic Tool: Integrator Backstepping

The historical origin of the technique known as *backstepping* is not clear; reportedly it has been used implicitly by researchers as early as 1966 [73]. However, its most interesting applications have been collected and formalized in [1] (and [13]), which are most frequently cited as sources for background on backstepping. Backstepping is most well known for being an iterative method of constructing explicit expressions for smooth, globally-stabilizing control laws for a relatively unrestrictive class of systems whose nonlinearities are not required to satisfy any growth rate bounds. The most general class of systems to which backstepping applies is the class of systems in *pure feedback form* [1], however a more practical form to consider is the subset class of systems in *strict feedback form* (SFF):

$$\Sigma_{SFF}:\begin{cases}\dot{\eta}=\Gamma(\eta,x_1)\\ \dot{x}_1= f_1(\eta,x_1)+g_1(\eta,x_1)x_2\\ \dot{x}_2= f_2(\eta,x_1,x_2)+g_2(\eta,x_1,x_2)x_3\\ \quad\vdots\\ \dot{x}_{r-1}= f_{r-1}(\eta,x_1,\dots,x_{r-1})+g_{r-1}(\eta,x_1,\dots,x_{r-1})x_r\\ \dot{x}_r= f_r(\eta,x_1,\dots,x_r)+g_r(\eta,x_1,\dots,x_r)u\end{cases}\tag{2.1}$$

where $\eta\in\mathbb{R}^m$, $x_i\in\mathbb{R}$, $i\in\{1,\dots,r\}$, and the functions Γ, f_i and g_i are assumed to be smooth with $f_i(0)=0$. The functions $g_i(\cdot)$ are assumed to be bounded away from zero for all valuations of their arguments and it is further assumed that there exists a smooth function $\alpha(\eta)$, with $\alpha(0)=0$, so that

$$\dot{\eta}=\Gamma(\eta,\alpha(\eta)))\tag{2.2}$$

is GAS at $\eta=0$. With those assumptions we have:

Lemma 2.1.1. *For system Σ_{SFF}, there exists a smooth, globally asymptotically stabilizing state feedback $u=\vartheta(\eta,x)$.*

The complete formal proof is inductive, and unnecessary here. To demonstrate all of the "mechanics" of backstepping, it suffices to consider the special case where $r=2$[1]:

$$\begin{aligned}\dot{\eta}&=\Gamma(\eta,x_1)\\ \dot{x}_1&= f_1(\eta,x_1)+g_1(\eta,x_1)x_2\\ \dot{x}_2&= f_2(\eta,x_1,x_2)+g_2(\eta,x_1,x_2)u.\end{aligned}\tag{2.3}$$

This system is shown in Figure 2.1 which shows why this form is referred to as the "strict feedback" form.

Ideally we would like $x_1=\alpha(\eta)$ since by assumption, the η-subsystem would then be GAS. To that end, we introduce an error variable

$$w_1\triangleq x_1-\alpha(\eta)$$

[1]Here we will show a combination of concepts presented in Lemmas 9.2.1 and 9.2.2 in [17], and Section 2.3.1 in [1]. A slightly different presentation is also given in the proof of Theorem 2.5.1 in [16]. We provide our own interpretations where appropriate.

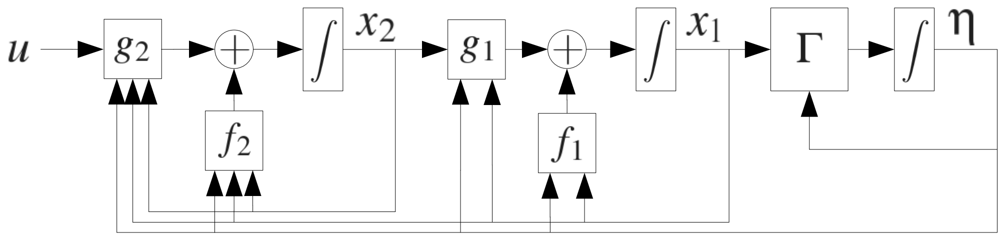

Figure 2.1: A system in strict-feedback form. All states are "fed back" towards the input; each state is driven by only lower-indexed states and the input enters only the last state.

which we wish to regulate to zero. The dynamic behaviour of this new variable is described by

$$\dot{w}_1 = f_1(\eta, x_1) + g_1(\eta, x_1)x_2 - \frac{\partial \alpha(\eta)}{\partial \eta}\Gamma(\eta, w_1 + \alpha(\eta)). \tag{2.4}$$

Therefore in the new coordinates, the system (2.3) can be written as:

$$\begin{aligned} \dot{\eta} &= \overline{\Gamma}(\eta, w_1) + \Gamma(\eta, \alpha(\eta)) \\ \dot{w}_1 &= f_1(\eta, w_1 + \alpha(\eta)) + g_1(\eta, w_1 + \alpha(\eta))x_2 - \frac{\partial \alpha(\eta)}{\partial \eta}\Gamma(\eta, w_1 + \alpha(\eta)) \\ \dot{x}_2 &= f_2(\eta, w_1 + \alpha(\eta), x_2) + g_2(\eta, w_1 + \alpha(\eta), x_2)u, \end{aligned} \tag{2.5}$$

where

$$\overline{\Gamma}(\eta, w_1) \triangleq \Gamma(\eta, w_1 + \alpha(\eta)) - \Gamma(\eta, \alpha(\eta)). \tag{2.6}$$

Before constructing the feedback $u = \vartheta(\eta, x_1)$ for this system, it is helpful to decompose the function $\overline{\Gamma}(\eta, w_1)$ as

$$\overline{\Gamma}(\eta, w_1) = \gamma(\eta, w_1)w_1. \tag{2.7}$$

This decomposition is always possible since $\overline{\Gamma}(\cdot)$ is at least C^1 (because $\Gamma(\cdot)$ is assumed to be smooth) and by construction, $\overline{\Gamma}(\eta, 0) \equiv 0$. As a consequence of those two facts, we can apply the following trick to construct $\gamma(\eta, x_1)$; express

$$\begin{aligned} \overline{\Gamma}(\eta, w_1) &= \int_0^1 \frac{\partial \overline{\Gamma}(\eta, s w_1)}{\partial s} ds && \text{(tautology)} \\ &= \int_0^1 \left(\frac{\partial \overline{\Gamma}(\eta, z)}{(1/w_1)\partial z} \right) \Bigg|_{z = s w_1} ds && \text{(change of differentiation variable)} \\ &\triangleq w_1 \gamma(\eta, w_1) \end{aligned} \tag{2.8}$$

The first iteration of backstepping begins by introducing another coordinate shift

$$w_2 \triangleq x_2 - \alpha_1(\eta, w_1). \tag{2.9}$$

Our intention is to design the smooth function $\alpha_1(\eta, w_1)$, with $\alpha_1(0,0) = 0$, so that the subsystem consisting of the (η, w_1) dynamics

$$\begin{aligned} \dot{\eta} &= \Gamma(\eta, \alpha(\eta)) + w_1\gamma(\eta, w_1) \\ \dot{w}_1 &= f_1(\eta, w_1 + \alpha(\eta)) + g_1(\eta, w_1 + \alpha(\eta))[w_2 + \alpha_1] \\ &\qquad - \frac{\partial\alpha(\eta)}{\partial\eta}\Gamma(\eta, w_1 + \alpha(\eta)). \end{aligned} \tag{2.10}$$

is GAS at $(\eta, w_1) = (0,0)$ when $w_2 \equiv 0$. To simplify our expressions, we first assign

$$\alpha_1 = \frac{1}{g_1(\eta, x_1)}\left(-f_1(\eta, x_1) + \frac{\partial\alpha(\eta)}{\partial\eta}\Gamma(\eta, w_1 + \alpha(\eta)) + v\right) \tag{2.11}$$

where v is to be determined, and re-write (2.10) as

$$\begin{aligned} \dot{\eta} &= \Gamma(\eta, \alpha(\eta)) + \gamma(\eta, w_1)w_1 \\ \dot{w}_1 &= v + g_1(\eta, w_1 + \alpha(\eta))w_2. \end{aligned} \tag{2.12}$$

In this form it is easy to see that with $w_2 \equiv 0$, (2.12) can be globally asymptotically stabilized by v. Since the system $\dot{\eta} = \Gamma(\eta, \alpha(\eta))$ is assumed to be GAS at $\eta = 0$, the Converse Lyapunov Theorem A.0.2 guarantees the existence of a smooth, proper, positive definite function $V(\eta)$, and a continuous positive definite function $W(\eta)$ such that

$$\frac{\partial V(\eta)}{\partial\eta}\Gamma(\eta, \alpha(\eta)) \leq -W(\eta). \tag{2.13}$$

We then consider the candidate Lyapunov function $V_1(\eta, w_1) = V(\eta) + \frac{1}{2}w_1^2$ and find its gradient along the solution of (2.12):

$$\dot{V}_1 \leq -W(\eta) + \frac{\partial V(\eta)}{\partial\eta}\gamma(\eta, w_1)w_1 + w_1 v + w_1 g_1(\eta, w_1 + \alpha(\eta))w_2. \tag{2.14}$$

Because of the common factor w_1 in the middle two terms, $\dot{V}_1$ can be rendered negative definite (when $w_2 \equiv 0$) by choosing

$$v = -c_1 w_1 - \frac{\partial V(\eta)}{\partial\eta}\gamma(\eta, w_1). \tag{2.15}$$

We emphasize the importance of the second term in this expression; it is the means by which backstepping is able to ensure the internal stability of the *interconnection* of the η and w_1 subsystems.

The variable x_2 in equation (2.10) is known as a *virtual control* and

$$\alpha_1(\eta, w_1) = \frac{1}{g_1(\eta, w_1 + \alpha(\eta))}\Bigg(- f_1(\eta, w_1 + \alpha(\eta)) + \frac{\partial \alpha(\eta)}{\partial \eta}\Gamma(\eta, w_1 + \alpha(\eta)) - c_1 w_1 - \frac{\partial V(\eta)}{\partial \eta}\gamma(\eta, w_1)\Bigg) \tag{2.16}$$

is its associated *stabilizing function*. We note that this expression for the stabilizing function is not unique; our only objective in its design is to render $\dot{V}_1$ negative definite when the error variable $w_2 = x_2 - \alpha_1(\eta, w_1) = 0$. This observation may sometimes help us avoid the cancellation of nonlinearities that are actually helpful to the stability of subsystem (2.10).

In the second iteration of the backstepping algorithm, we wish to regulate w_2 to zero using the actual control u. Noting that $\dot{w}_2 = \dot{x}_2 - \dot{\alpha}_1(\eta, w_1)$ we re-write the original system dynamic (2.3) as:

$$\begin{aligned} \dot{\eta} &= \Gamma(\eta, \alpha(\eta)) + w_1\gamma(\eta, w_1) \\ \dot{w}_1 &= -c_1 w_1 - \frac{\partial V(\eta)}{\partial \eta}\gamma(\eta, w_1) + g_1(\eta, w_1 + \alpha(\eta))w_2 \\ \dot{w}_2 &= \left[f_2(\eta, x_1, x_2) + g_2(\eta, x_1, x_2)u - \frac{\partial \alpha_1}{\partial \eta}\dot{\eta} - \frac{\partial \alpha_1}{\partial w_1}\dot{w}_1 \right]_{\substack{x_1 = w_1 + \alpha(\eta) \\ x_2 = w_2 + \alpha_1(\eta, w_1)}} . \end{aligned} \tag{2.17}$$

From here, we repeat the procedure by forcing the new function $V_2(\eta, w_1, w_2) = V_1(\eta, w_1) + \frac{1}{2}w_2^2$ to be a Lyapunov function for the system (2.17). Taking into account (2.14), its derivative along the trajectories of (2.17) is:

$$\dot{V}_2 \leq -W(\eta) - c_1 w_1^2 + w_2\left[w_1 g_1 + f_2 + g_2 u - \frac{\partial \alpha_1}{\partial \eta}\dot{\eta} - \frac{\partial \alpha_1}{\partial w_1}\dot{w}_1 \right]. \tag{2.18}$$

where we drop all function arguments from notation for simplicity. Letting

$$u = \frac{1}{g_2}\Bigg(- w_1 g_1 - f_2 + \frac{\partial \alpha_1}{\partial \eta}\dot{\eta} + \frac{\partial \alpha_1}{\partial w_1}\dot{w}_1 - c_2 w_2 \Bigg), \tag{2.19}$$

we obtain

$$\dot{V}_2 \leq -W(\eta) - c_1 w_1^2 - c_2 w_2^2, \tag{2.20}$$

and the CL system

$$\begin{aligned}\dot{\eta} &= \Gamma(\eta,\alpha(\eta)) + w_1\gamma(\eta,w_1)\\ \dot{w}_1 &= -c_1w_1 - \frac{\partial V(\eta)}{\partial\eta}\gamma(\eta,w_1) + g_1(\eta,w_1+\alpha(\eta))w_2 \\ \dot{w}_2 &= -c_2w_2 - w_1g_1(\eta,w_1+\alpha(\eta)).\end{aligned} \tag{2.21}$$

By Theorem A.0.1 this CL system is GAS at $(\eta,w_1,w_2) = (\eta,x_1-\alpha(\eta),x_2-\alpha_1(\eta,w_1)) = (0,0,0)$, provided that the constants c_1 and c_2 are chosen greater than zero. From (2.16) we see that $\alpha_1(0,0)=0$, and by assumption $\alpha(0)=0$. Therefore,

$$\lim_{t\to\infty}(\eta,w_1,w_2) = (0,0,0) \implies \lim_{t\to\infty}(\eta,x_1,x_2) = (0,0,0) \tag{2.22}$$

and we conclude that the CL system in the original coordinates is GAS at the origin under the SFB law (2.16) and (2.19).

Several remarks are in order:

Remark 2.1.1. We note that the transformation $(\eta,x_1,x_2)\mapsto(\eta,w_1,w_2)$ is a globally defined change of coordinates since the Jacobian

$$\begin{bmatrix} d\eta \\ dw_1 \\ dw_2 \end{bmatrix} = \begin{bmatrix} I & 0 & 0 \\ * & 1 & 0 \\ * & * & 1 \end{bmatrix} \tag{2.23}$$

is nonsingular for all $\eta\in\mathbb{R}^m$ and $[x_1,x_2]^T\in\mathbb{R}^2$. Therefore (2.17) represents the same motion as equation (2.3) – i.e. we do not have to worry about "lost dynamics" in this new representation.

In fact, the backstepping procedure may be interpreted as the iterative search for a new set of (globally defined) coordinates in which the system dynamic admits a quadratic Lyapunov function. In essence, at every iteration we define a coordinate shift $w_i = x_i - \alpha_{i-1}(\eta,w_1,\ldots,w_{i-1})$, designing all the stabilizing functions and finally the control u such that the transformed, CL system acquires a "skew-plus-diagonal" symmetry. To see this, consider the CL system (2.21), except assume that $\eta\in\mathbb{R}^0$ so that we have:

$$\begin{aligned}\dot{w}_1 &= -c_1w_1 + g_1(w_1)w_2\\ \dot{w}_2 &= -c_2w_2 - w_1g_1(w_1),\end{aligned}$$

or,

$$\begin{bmatrix} \dot{w}_1 \\ \dot{w}_2 \end{bmatrix} = \begin{bmatrix} -c_1 & g_1(w_1) \\ -g_1(w_1) & -c_2 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}. \tag{2.24}$$

An interesting converse of this fact is that any system whose structure exhibits this kind of symmetry can be shown to be asymptotically stable at the origin by means of a quadratic Lyapunov function. ◁

Remark 2.1.2. From our study of cascade-connected systems, we know that the interconnection term (i.e. $\psi(x,\xi)$ in equation (1.54)) plays an important role in determining what is possible in terms of achieving the internal stability of two connected subsystems. At every iteration, backstepping introduces coupling terms that specifically ensure the internal stability of the interconnection of two subsystems – the one-dimensional x_i-subsystem receiving the interconnection term, and an $(m+i-1)$ dimensional subsystem for which the design is already complete. In the above example, the two coupling terms that accomplished the internal stability of the whole are $-\frac{\gamma(\eta,w_1)}{g_1}\frac{\partial V(\eta)}{\partial \eta}$ in the first iteration, and $-\frac{g_1}{g_2}w_1$ in the second iteration. ◁

Remark 2.1.3 (The importance of triangularity)**.** One may wonder why a system must have a triangular structure like Σ_{SFF} in order to admit a backstepping design. To investigate, suppose we have

$$\begin{aligned} \dot{x}_1 &= \phi_1(x_1) + x_2 + u \\ \dot{x}_2 &= u \end{aligned} \tag{2.25}$$

which is not in a strict feedback form since u appears in the $\dot{x}_1$ equation. If we attempt to proceed with the usual backstepping design, we find that the stabilizing function associated with the virtual control x_2 must cancel u in the first equation. Therefore, the error variable $w \triangleq x_2 - \alpha(x_1,u)$ has the dynamics

$$\dot{w} = u - \frac{\partial \alpha}{\partial x_1}(\phi_1(x_1) + w + \alpha(x_1,u) + u) - \frac{\partial \alpha}{\partial u}\dot{u}. \tag{2.26}$$

The presence of $\dot{u}$ in the $\dot{w}$ equation implies that in order to design a globally asymptotically stabilizing SFB control law by backstepping, we would have to solve a differential equation in u which may not be possible in general. ◁

Remark 2.1.4. Even though we have advertised backstepping as a very practical, constructive method, equation (2.16) is dependent on the knowledge of some function $V(\eta)$ which is simply assumed into existence by virtue of the fact that $\dot{\eta} = \Gamma(\eta, \alpha(\eta))$ is GAS at $\eta = 0$. Depending on the nature of $\Gamma(\eta, x_1)$, searching for a Lyapunov function $V(\eta)$ may be futile, even if an $\alpha(\eta)$ is known.

However, if in (2.3) $\dot{\eta} = \Gamma\eta + x_1\gamma(\eta, x_1)$ with $\Gamma \in \mathbb{R}^{m\times m}$ Hurwitz, then the expression (2.16) (and hence (2.19)) can be easily calculated with $\alpha(\eta) \equiv 0$ and $V(\eta) = \eta^T P\eta$ where P is the unique solution of the Lyapunov equation $\Gamma^T P + P\Gamma = -I$. The positive definite, symmetric matrix P is guaranteed to exist since Γ is Hurwitz. This fact will become relevant in the discussion on Marino and Tomei's OFB method. $\triangleleft$

With this understanding of backstepping, we are ready to examine several system forms that are very frequently encountered in the nonlinear OFB literature. These system forms also play an important role in our own results.

2.2 Important System Forms

The most ubiquitous system forms in the nonlinear OFB literature are the *normal form* (NF) and the *strict normal form* (SNF), both originally identified by coordinate-free differential geometric conditions in [10].

2.2.1 The Normal Form

Nonliner SISO systems of the form

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{aligned} \tag{2.27}$$

with $x \in \mathbb{R}^n$ that have a well-defined relative degree[2] r in some neighbourhood U_o of the origin, are locally diffeomorphic to the form

$$\Sigma_{NF} : \begin{cases} \dot{\eta} = \Gamma(\eta, \xi) \\ \dot{\xi}_1 = \xi_2 \\ \dot{\xi}_2 = \xi_3 \\ \quad \vdots \\ \dot{\xi}_{r-1} = \xi_r \\ \dot{\xi}_r = (L_f^r h(x) + u L_g L_f^{r-1} h(x)) \circ T^{-1}(\eta, \xi) \\ y = \xi_1 \end{cases} \tag{2.29}$$

by a change of coordinates of the form

$$\begin{bmatrix} \eta \\ \xi \end{bmatrix} = \begin{bmatrix} \phi_1(x) \\ \vdots \\ \phi_m(x) \\ h(x) \\ L_f h(x) \\ \vdots \\ L_f^{r-1} h(x) \end{bmatrix} \triangleq T(x). \tag{2.30}$$

The functions $\phi_i(x)$ can be chosen arbitrarily, as long as the Jacobian $\frac{\partial T(x)}{\partial x}$ is non-singular for all x in U_o. Such functions are guaranteed to exist, and can further be chosen so that the η subsystem is independent of u – i.e. choose ϕ_i so that their exact one-forms annihilate the input vector g:

$$\langle d\phi_i(x), g(x) \rangle = 0, \quad i \in \{1, \ldots, m\}$$

(Proposition 4.1.3 [17]). The system Σ_{NF} is globally *minimum phase* if the *zero dynamics* (ZD) $\dot{\eta} = \Gamma(\eta, 0)$ are GAS at $\eta = 0$. The zero dynamics describe the

[2]Relative degree for SISO systems is defined as an integer r such that

$$\begin{aligned} L_g L_f^i h(x) &= 0, \qquad \forall x \in U_o \subseteq \mathbb{R}^n, \ 0 \leq i \leq (r-2) \\ L_g L_f^{r-1} h(x) &\neq 0, \qquad \forall x \in U_o \subseteq \mathbb{R}^n. \end{aligned} \tag{2.28}$$

If $U_o = \mathbb{R}^n$ and the vector fields $f(x) - \frac{L_f^r h(x)}{L_g L_f^{r-1} h(x)} g(x)$ and $\frac{1}{L_g L_f^{r-1} h(x)} g(x)$ are complete, then there exists a globally valid change of coordinates into the normal form. The relative degree is the number of times the signal y must be differentiated before the signal u appears.

motion of the system when its output and all its r derivatives are identically zero. Often the subsystem $\dot{\eta} = \Gamma(\eta, \xi)$ is also referred to as the "zero dynamics", although this is technically incorrect.

2.2.2 The Strict Normal Form

A special case of the NF is the strict normal form (SNF), in which the zero dynamics are driven by only the output $y = \xi_1$, and not its derivatives. A sufficient geometric condition for the existence of the SNF (in addition to requiring a well-defined relative degree) is that each of the distributions

$$G_i \triangleq \mathrm{span}\{g(x), ad_f g(x), \ldots, ad_f^i g(x)\} = \mathbb{R}^{i+1}, \ \forall\, x \in U_o, \quad 0 \leq i \leq (r-1) \tag{2.31}$$

and are involutive. To check the involutivity of a distribution it suffices to check that the Lie bracket of any two of its spanning vector fields belongs to the distribution – i.e. G_i is involutive if

$$[ad_f^k g(x), ad_f^j g(x)] \triangleq \chi_{k,j}(x) \in G_i, \quad \forall\, k, j \in \{0, 1, \ldots, i\} \tag{2.32}$$

or, if there exist $i+1$ functions $c_j(x)$ such that

$$\chi_{k,j}(x) = \sum_{j=0}^{i} c_j(x) ad_f^j g(x), \quad \forall\, k, j \in \{0, 1, \ldots, i\} \tag{2.33}$$

In that case, we choose the functions $\eta_i = \phi_i(x),\ i \in \{1, \ldots, m\}$ such that not only do their exact one-forms annihilate the input vector g, but also all the other vector fields in the distribution G_{r-1}. This choice will guarantee that the change of coordinates $[\eta^T, \xi^T]^T = T(x)$, defined as in (2.30), transforms (2.27) into the SNF:

$$\Sigma_{SNF} : \begin{cases} \dot{\eta} = \Gamma(\eta, \xi_1) \\ \dot{\xi}_1 = \xi_2 \\ \dot{\xi}_2 = \xi_3 \\ \quad \vdots \\ \dot{\xi}_{r-1} = \xi_r \\ \dot{\xi}_r = (L_f^r h(x) + u L_g L_f^{r-1} h(x)) \circ T^{-1}(\eta, \xi) \\ y = \xi_1 \end{cases} \tag{2.34}$$

We based this assertion on Theorem 2.4.3 in [16], which actually concerns partial linearization of systems into triangular form by SFB – i.e. no output is considered. We now justify this assertion. In the first part of the proof of this theorem, the existence of some smooth function $\psi(x)$, with $\psi(0) = 0$ which solves the linear partial differential equations

$$\begin{aligned} \langle d\psi(x), G_{r-2} \rangle &= 0 \\ \langle d\psi(x), ad_f^{r-1} g(x) \rangle &= \varphi(x) \neq 0 \end{aligned} \tag{2.35}$$

is posited owing to Frobenius's Theorem[3] and the assumed involutivity and constant rank of the distribution G_{r-2} in U_o. We denote

$$Q(x) = [\psi(x), L_f\psi(x), \ldots, L_f^{r-1}\psi(x)]^T.$$

Then, the linear independence in U_o of the one-forms

$$\{d\psi(x), dL_f\psi(x), \ldots, dL_f^{r-1}\psi(x)\} \tag{2.36}$$

is ascertained via the matrix

$$\begin{aligned} N(x) &\triangleq \frac{\partial Q(x)}{\partial x}[g(x), ad_f g(x), \ldots, ad_f^{r-1} g(x)] \\ &= \begin{bmatrix} \langle d\psi, g \rangle & \cdots & \langle d\psi, ad_f^{r-1} g \rangle \\ \langle dL_f\psi, g \rangle & \cdots & \langle dL_f\psi, ad_f^{r-1} g \rangle \\ \vdots & \ddots & \vdots \\ \langle dL_f^{r-1}\psi, g \rangle & \cdots & \langle dL_f^{r-1}\psi, ad_f^{r-1} g \rangle \end{bmatrix} \\ &= \begin{bmatrix} 0 & \cdots & 0 & \varphi \\ 0 & \cdots & \varphi & * \\ \vdots & \iddots & \vdots & \vdots \\ \varphi & \cdots & * & * \end{bmatrix} \end{aligned} \tag{2.37}$$

which is clearly nonsingular since $\varphi \neq 0$. Since the distribution G_{r-1} is assumed to have constant rank r, for $N(x)$ to remain full rank the matrix $\frac{\partial Q(x)}{\partial x}$ must also be full rank in U_o, and therefore the one-forms (2.36) must be linearly independent in U_o. The third equality in (2.37) is demonstrated in Theorem A.3.1 in

[3]Please see Theorem A.4.3 in [16].

[16], and involves the use of (2.35) and the repeated application of Leibniz's Rule: $L_{[f,g]}h(x) = L_f L_g h(x) - L_g L_f h(x)$.

In the second part of the proof, the assumed involutivity and constant rank of the distribution G_{r-1} is used to demonstrate that the functions $\eta_i = \phi_i(x),\ i \in \{1,\dots,m\}$ (whose existence is likewise guaranteed by Frobenius's Theorem), if chosen such that

$$\langle d\phi(x), G_{r-1}\rangle = 0, \tag{2.38}$$

guarantee that the transformed η subsystem is independent of $\xi_j,\ j \in \{2,\dots,r\}$.

In the case of system (2.27) where the output $y = h(x)$ is defined, and we have further assumed that its associated relative degree is well-defined, we see that the index r in their theorem is identified with the relative degree in our discussion and therefore the first part of their proof is superfluous for our purposes. That is, if we identify the function $\psi(x)$ with $h(x)$, then by the assumption that the relative degree is well defined we can independently assert that

$$\operatorname{rank}(\operatorname{span}\{dh(x),\dots,dL_f^{r-1}h(x)\}) = r$$

for all $x \in U_o$ (as in the proof of Lemma 4.1.1 in [16], for example). However, the second part of their proof provides an explicit means of calculating the transformation $\eta_i = \phi_i(x),\ i \in \{1,\dots,m\}$ so that the ZD subsystem is driven only by $y = \xi_1 = h(x)$ – i.e. we can solve (2.38). In summary, for the system (2.27), the set of conditions:

- The relative degree r is well defined:

$$\begin{aligned} L_g L_f^i h(x) &= 0, \qquad \forall x \in U_o \subseteq \mathbb{R}^n,\ 0 \le i \le (r-2) \\ L_g L_f^{r-1} h(x) &\ne 0, \qquad \forall x \in U_o \subseteq \mathbb{R}^n. \end{aligned} \tag{2.39}$$

- Each of the distributions

$$G_i \triangleq \operatorname{span}\{g(x), ad_f g(x), \dots, ad_f^i g(x)\}, \quad 0 \le i \le (r-1) \tag{2.40}$$

  has constant rank $(i+1)$ and is involutive for all x in U_o, is more than what is required to guarantee the existence of the SNF. To transform system (2.27) into the SNF, we apply the change of coordinates (2.30), with the functions $\phi_i(x)$ $i \in \{1,\ldots,m\}$ chosen so that $\frac{\partial T(x)}{\partial x}$ is full rank for all x in U_o and so that $\langle d\phi_i(x), G_{r-1}\rangle = 0$.

Proposition 9.1.1 in [17] gives the more minimal condition which is both sufficient and necessary for a system to be equivalent to the SNF by a coordinate and feedback transformation. This condition requires that the vector fields

$$\tau_i \triangleq (-1)^{i-1} ad_{\tilde{f}}\tilde{g}(x), \quad 1 \leq i \leq r$$

where

$$\tilde{f} = f(x) - L_f^r h(x)\tilde{g}(x) \qquad \text{and} \qquad \tilde{g}(x) = \frac{1}{L_g L_f^{r-1} h(x)},$$

must commute; that is, $[\tau_j, \tau_i] = 0$, $\forall\ i, j \in \{1,\ldots,r\}$. The proof of this theorem relies on a more abstract set of mathematical concepts, and the explicit means of calculating the transformation $\eta_i = \phi_i(x)$ is not stated. However, we note that when $r = n$, either set of conditions identify the class of systems which are equivalent by state and feedback transformation to a controllable and observable linear system. Indeed then, both the controllability condition

$$\text{span}\{g(x), ad_f g(x), \ldots, ad_f^{n-1} g(x)\} = \mathbb{R}^n, \quad \forall x \in U_o \tag{2.41}$$

and the observability condition

$$\text{span}\{dh(x), dL_f h(x), \ldots, dL_f^{n-1} h(x)\} = \mathbb{R}^n, \quad \forall x \in U_o \tag{2.42}$$

are implied by either set of conditions.

2.2.3 The Output Feedback Form

Another important special case of the NF is the *output feedback form* (OFBF), which is actually a subset of the class of systems in SNF. Systems of the form (2.27) that satisfy the following differential geometric conditions (Theorem 6.3.1 in [16]):

1. The one-forms
$$\{dh(x), dL_f h(x), \ldots, dL_f^{n-1} h(x)\} \tag{2.43}$$
are linearly independent,

2. The vector fields $ad^i_{-f}\rho(x),\ i \in \{0, 1, \ldots, (n-1)\}$ commute with each other – i.e. $[ad^i_{-f}\rho(x), ad^j_{-f}\rho(x)] = 0,\ \forall i, j \in \{0, 1, \ldots, (n-1)\}$, where the *starting vector* $\rho(x)$ is the vector that uniquely solves the equations
$$\begin{bmatrix} \langle dh(x), \rho(x)\rangle \\ \langle dL_f h(x), \rho(x)\rangle \\ \vdots \\ \langle dL_f^{n-1} h(x), \rho(x)\rangle \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \tag{2.44}$$

3. The input vector $g(x)$ satisfies:
$$[g(x), ad_f^k \rho(x)] = 0, \qquad 0 \leq k \leq (n-2) \tag{2.45}$$

4. There exists a smooth function $\sigma : \mathbb{R} \to \mathbb{R}$, and $n - r + 1$ real numbers $(d_r, d_{r+1}, \ldots, d_n)$ such that the input vector takes the form
$$g = \sigma \circ h \cdot \sum_{j=1}^{n-r+1} d_{n-j+1} ad_{-f}^{j-1} \rho \tag{2.46}$$
and the real numbers $(d_r, d_{r+1}, \ldots, d_n)$ are required to be such that the polynomial
$$d_r s^{n-r} + d_{r+1} s^{n-r-1} + \cdots + d_{n-1} s + d_n \tag{2.47}$$
has roots with strictly negative real parts.

are state equivalent to the OFBF:
$$\Sigma_{OFBF} : \begin{cases} \dot{\zeta} &= A_c \zeta + \psi(y) + \sigma(y) d u \\ y &= C_c \zeta = \zeta_1 \end{cases} \tag{2.48}$$
where the $\sigma(y) \neq 0,\ \forall y$, the matrix
$$A_c = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \end{bmatrix} \tag{2.49}$$

and d is the *Hurwitz vector*:

$$d = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ d_r \\ d_{r+1} \\ \vdots \\ d_n \end{bmatrix} \tag{2.50}$$

in the sense of (2.47). The nonlinearity $\psi(y)$ is smooth with $\psi(0) = 0$, if $f(x)$ and $g(x)$ are smooth and $f(0) = 0$. The system (2.27) is globally diffeomorphic to Σ_{OFBF} if these four conditions hold globally, and in addition the vector fields $ad^i_f \rho(x)$, $i \in \{1, \ldots, (n-1)\}$ are complete.

The first three conditions are identical to those required for a system to be diffeomorphic to the nonlinear observer form. In fact, we note that Σ_{OFBF} is a special case of the NOF, admitting the design of a LED observer. The first condition ensures that the drift component of the system is observable. Conditions 1 and 2 together are shown[4] to be equivalent to the existence of a local diffeomorphism $\zeta = T(x)$, $T(0) = 0$ that transforms the drift component of (2.27)

$$\begin{aligned} \dot{x} &= f(x) \\ y &= h(x) \end{aligned} \tag{2.52}$$

into

$$\begin{aligned} \dot{\zeta} &= A_c\zeta + \psi(y) \\ y &= C_c\zeta = \zeta_1. \end{aligned} \tag{2.53}$$

[4]Please see Theorem 5.2.1 in [16]. Actually, in the proof of this theorem, the authors transform the drift component of (2.27) into the following form, which is not quite what we want:

$$\begin{aligned} \dot{\zeta} &= A_o\zeta + \psi(y) \\ y &= C_o\zeta = \zeta_n \end{aligned} \tag{2.51}$$

where A_o has ones on the sub-diagonal and zeros elsewhere. In the sequel, we show how the same conditions 1 and 2 can transform the unforced component of (2.43) into (2.53). Please note that we have changed equation (2.54) accordingly.

in which coordinates the vector fields $ad_{-f}^{i-1}\rho(x)$ are "rectified" into the unit vectors[5]

$$\frac{\partial T(x)}{\partial x} ad_{-f}^{i}\rho(x) \circ T^{-1}(\zeta) = \frac{\partial}{\partial \zeta_{n-i}}, \qquad 0 \le i \le n-1, \tag{2.54}$$

The unit vector representation of the vector fields $ad_{-f}^{i}\rho(x)$ in new coordinates is a consequence of our assumption that they commute, the fact that they are linearly independent[6] for all x, and the application of the Simultaneous Rectification Theorem (for example, see Theorem A.4.5 in [16]). The third condition ensures that the input vector in the transformed coordinates depends only on $y = \zeta_1$; that is, $\frac{\partial T(x)}{\partial x} g(x) \circ T^{-1}(\zeta) = \beta(y)$, for some smooth function $\beta(y)$. Finally, due to (2.54), the fourth condition implies that this transformed input vector $\beta(y)$ takes the special structure $\sigma(y)d$, with d as in (2.50).

Albeit being rather restrictive, these conditions are satisfied by several physical systems, including[7] a single-link flexible robot and a third order model of a synchronous generator.

Since we will be working extensively with this system form, we are interested in how exactly to construct a transformation into Σ_{OFBF}. The key to finding such a transformation lies in the statement of the Simultaneous Rectification Theorem itself. From equation (2.54), we can write

$$\frac{\partial T(x)}{\partial x}[ad_{-f}^{n-1}\rho(x), ad_{-f}^{n-2}\rho(x), \ldots, ad_{-f}\rho(x), \rho(x)] \circ T^{-1}(\zeta) = I. \tag{2.56}$$

Since the right-hand side of this equation is a constant, we do not care whether we work in the x or ζ coordinates. Therefore, to find the transformation $\zeta = T(x)$, we

[5]The notation $\frac{\partial}{\partial \zeta_i}$ signifies a unit vector in the ζ_i direction. For example, a vector field such as $F(x) = [f_1(x), f_2(x)]^T$ can be written as $F(x) = f_1(x)\frac{\partial}{\partial x_1} + f_2(x)\frac{\partial}{\partial x_2}$.

[6]The linear independence of the vector fields $ad_{-f}^{j}\rho$ can be shown using an argument similar to that in equation (2.37), except that

$$N(x) = \begin{bmatrix} dh \\ \vdots \\ dL_f^{n-1}h \end{bmatrix} [\rho, \ldots, ad_{-f}^{n-1}\rho] \tag{2.55}$$

is shown to be nonsingular for all x by virtue of the definition of the starting vector ρ in (2.44) and Theorem A.3.1 in [16]. Then, the linear independence of the vector fields $ad_{-f}^{j}\rho$ follows from the assumption that the one-forms $\{dh, \ldots, dL_f^{n-1}h\}$ are linearly independent, as per condition 1.

[7]Please see Section 6.5 in [16] for more details and other examples.

must solve the following n PDEs for the n unknown components of $T(x)$:

$$\frac{\partial T(x)}{\partial x}[ad_{-f}^{n-1}\rho(x), ad_{-f}^{n-2}\rho(x), \ldots, ad_{-f}\rho(x), \rho(x)] = I. \tag{2.57}$$

To see why the transformation $T(x)$, which rectifies the vector fields $[ad_f^{n-1}\rho(x), \ldots, \rho(x)]$, would also bring the original system (2.27) into the OFBF, we first note that

$$\frac{\partial T(x)}{\partial x} ad_f^i \rho(x) \circ T^{-1}(\zeta) \equiv ad_{f^*}^i \rho^*(\zeta) \tag{2.58}$$

where

$$\rho^*(\zeta) = \frac{\partial T(x)}{\partial x}\rho(x) \circ T^{-1}(\zeta)$$

and $f^* = f^*(\zeta)$ is similarly obtained. For $i = 0$, this fact is obvious from the preceding equation. For $i = 1$, we write:

$$\begin{aligned}
ad_{f^*}\rho^*(\zeta) &= \left(\frac{\partial}{\partial \zeta}\rho^*(\zeta)\right) f^*(\zeta) - \left(\frac{\partial}{\partial \zeta} f^*(\zeta)\right)\rho^*(\zeta) \\
&= \left(\frac{\partial}{\partial \zeta}\frac{\partial T(x)}{\partial x}\rho(T^{-1}(\zeta))\right)\frac{\partial T(x)}{\partial x} f(T^{-1}(\zeta)) \\
&\quad - \left(\frac{\partial}{\partial \zeta}\frac{\partial T(x)}{\partial x} f(T^{-1}(\zeta))\right)\frac{\partial T(x)}{\partial x}\rho(T^{-1}(\zeta)) \\
&= \left(\frac{\partial T(x)}{\partial x}\frac{\partial \rho(x)}{\partial x}\frac{\partial T^{-1}(\zeta)}{\partial \zeta}\right)\frac{\partial T(x)}{\partial x} f(T^{-1}(\zeta)) \\
&\quad - \left(\frac{\partial T(x)}{\partial x}\frac{\partial f(x)}{\partial x}\frac{\partial T^{-1}(\zeta)}{\partial \zeta}\right)\frac{\partial T(x)}{\partial x}\rho(T^{-1}(\zeta))
\end{aligned}$$

Then, as in Subsection 1.1.2, we note that $\frac{\partial T^{-1}(\zeta)}{\partial \zeta}\frac{\partial T(x)}{\partial x} = I$ and therefore write

$$ad_{f^*}\rho^*(\zeta) = \frac{\partial T(x)}{\partial x}\left(\frac{\partial \rho(x)}{\partial x} f(x) - \frac{\partial f(x)}{\partial x}\rho(x)\right) \circ T^{-1}(\zeta),$$

which is the same as the left-hand side of equation (2.58) for $i = 1$. The same argument can be applied for all $i > 1$ owing to the recursive definition of repeated Lie brackets. With (2.58), we can now re-write (2.56) as

$$[ad_{-f^*}^{n-1}\rho^*(\zeta), ad_{-f^*}^{n-2}\rho^*(\zeta), \ldots, ad_{-f^*}\rho^*(\zeta), \rho^*(\zeta)] = [e_1, e_2, \ldots, e_n] \tag{2.59}$$

where e_i is a column vector with all zero elements except the ith, which is 1. We then examine the ith column. By the definition of a Lie bracket,

$$ad_{-f^*}^{n-i}\rho^*(\zeta) = [-f^*(\zeta), ad_{-f^*}^{n-i-1}\rho^*(\zeta)], \qquad 1 \le i \le (n-1) \tag{2.60}$$

and by (2.59),

$$\begin{aligned} ad^{n-i}_{-f^*}\rho^*(\zeta) &= [-f^*(\zeta), e_{i+1}] \\ &= \frac{\partial f^*(\zeta)}{\partial \zeta} e_{i+1} - \underbrace{\frac{\partial e_{i+1}}{\partial \zeta} f^*(\zeta)}_{=0}. \end{aligned} \tag{2.61}$$

Again by (2.59), we obtain:

$$\frac{\partial f^*(\zeta)}{\partial \zeta} e_{i+1} = e_i, \qquad 1 \le i \le (n-1) \tag{2.62}$$

which indicates that the drift component in transformed coordinates must have the following state dependencies:

$$\frac{\partial f_k^*(\zeta)}{\partial \zeta_j} = \begin{cases} 1, & j = k+1 \\ 0, & j \in \{2,3,\ldots,n\} - \{(k+1)\}, \end{cases} \tag{2.63}$$

meanwhile nothing can be said in general about f^*'s dependency on $y = \zeta_1$ from (2.62). Therefore, we conclude that in the new coordinates, the drift component of (2.27) takes the form $\dot{\zeta} = f^*(\zeta)$, with:

$$\begin{aligned} \dot{\zeta}_1 &= \zeta_2 + \psi_1(y) \\ \dot{\zeta}_2 &= \zeta_3 + \psi_2(y) \\ &\vdots \\ \dot{\zeta}_n &= \psi_n(y), \end{aligned} \tag{2.64}$$

which is what we wanted to show. To see how the input vector $g(x)$ and output function $h(x)$ transform, one may follow the remainder of the proof of Theorem 5.2.1 in [16].

Now that we have discussed the geometric conditions that identify the class of systems diffeomorphic to Σ_{OFBF}, we provide an analysis of some of its important properties.

2.2.3.1 The Internal Structure of Σ_{OFBF}

For convenience, we re-write (2.48) here:

$$\Sigma_{OFBF}: \begin{cases} \dot{\zeta}_1 = \zeta_2 + \psi_1(y) \\ \dot{\zeta}_2 = \zeta_3 + \psi_2(y) \\ \quad\vdots \\ \dot{\zeta}_{r-1} = \zeta_r + \psi_{r-1}(y) \\ \dot{\zeta}_r = \zeta_{r+1} + \psi_r(y) + d_r\sigma(y)u \\ \quad\vdots \\ \dot{\zeta}_{n-1} = \zeta_n + \psi_{n-1}(y) + d_{n-1}\sigma(y)u \\ \dot{\zeta}_n = \psi_n(y) + d_n\sigma(y)u \\ y = \zeta_1 \end{cases} \tag{2.65}$$

or

$$\begin{aligned} \dot{\zeta} &= A_c\zeta + \psi(y) + du \triangleq F(\zeta) + du \\ y &= \zeta_1, \end{aligned} \tag{2.66}$$

where for notational convenience we have absorbed the scalar function $\sigma(y)$ into the control u – i.e. let $u = \frac{1}{\sigma(y)}v$, then re-notate v back to u.

The first thing to notice about this system form is that it has a globally well-defined relative degree which can be discerned from the input vector d. The number r in this case corresponds to the relative degree of Σ_{OFBF} since $y = \zeta_1$, $\psi(\cdot)$ is a function of ζ_1 only and the linear component constitutes a chain of integrators. In that case the number of times y has to be time-differentiated before u appears is equal to one plus the number of leading zeros in the input vector. Alternatively, this fact can be verified using the definition of relative degree, as in (2.39). Since a well-defined relative degree is the only pre-requisite for a system to be diffeomorphic to the NF, we conclude that Σ_{OFBF} can be transformed into Σ_{NF} using a change of coordinates of the form (2.30). We also notice that Σ_{OFBF} satisfies the second condition, equation (2.40) in Subsection 2.2.2, which together with a well-defined relative degree is sufficient to guarantee that it can also be transformed to the SNF.

To justify this claim, we consider the Lie brackets $ad_F^i d$, $0 \leq i \leq (r-1)$:

$$ad_F d = [F,d] = \frac{\partial d}{\partial \zeta}F - \frac{\partial F(\zeta)}{\partial \zeta}d = -A_c d - \underbrace{\begin{bmatrix} \psi_1'(y) & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \psi_n'(y) & 0 & \cdots & 0 \end{bmatrix}}_{\triangleq \mu(\zeta_1)} d \tag{2.67}$$

$$= -A_c d + 0 \tag{2.68}$$

since the first $r-1$ elements of d are zero. Then

$$\begin{aligned} ad_F^2 d = [F, ad_F d] &= \frac{\partial ad_F d}{\partial \zeta}F - \frac{\partial F(\zeta)}{\partial \zeta} ad_F d \\ &= -\left(A_c + \mu(\zeta_1)\right)(-A_c d) \\ &= A_c^2 d \end{aligned} \tag{2.69}$$

Clearly, this pattern continues for all subsequent Lie brackets:

$$ad_F^j d = A_c^j d, \qquad 0 \leq j \leq (r-1) \tag{2.70}$$

since even when $j = r-1$, the term

$$\mu(\zeta_1) A_c^{r-2} d = \begin{bmatrix} \psi_1'(y) & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \psi_n'(y) & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} 0 \\ d_r \\ d_{r+1} \\ \vdots \\ d_n \\ 0 \\ \vdots \\ 0 \end{bmatrix} = 0. \tag{2.71}$$

Noting that the matrix

$$G \triangleq [d, A_c d, \ldots, A_c^{r-1} d] = \begin{bmatrix} 0 & 0 & \cdots & 0 & d_r \\ 0 & 0 & \cdots & d_r & d_{r+1} \\ \vdots & \vdots & \iddots & \vdots & \vdots \\ 0 & d_r & \iddots & * & * \\ d_r & d_{r+1} & \iddots & * & * \\ \vdots & \vdots & \iddots & \vdots & \vdots \\ d_{n-1} & d_n & \cdots & 0 & 0 \\ d_n & 0 & \cdots & 0 & 0 \end{bmatrix} \tag{2.72}$$

has rank r, it is clear that the distributions

$$G_i = \text{span}\{d, ad_F d, \ldots, ad_F^i d\}, \qquad i \in \{1, 2, \ldots, (r-1)\} \tag{2.73}$$

each have constant rank $i+1$. These distributions are also all involutive, since the Lie bracket of any two consant vectors is identically zero, and zero belongs to any distribution.

Therefore, according to our discussion in Subsection 2.2.2, every system which is diffeomorphic to Σ_{OFBF} is also transformable to the SNF, which shows that the class of systems that are state equivalent to the OFBF is a subset of the class of systems that are state equivalent to the SNF.

In order to examine the stability of the zero dynamics of Σ_{OFBF} we need to transform it into either the NF or the SNF. To that end, define the change of coordinates

$$\begin{aligned}
\xi_1 &= y = \zeta_1 \\
\xi_2 &= \dot{y} = L_F \zeta_1 = \zeta_2 + \psi_1(\zeta_1) \triangleq \zeta_2 + \theta_1(\zeta_1) \\
\xi_3 &= \ddot{y} = L_F^2 \zeta_1 = \langle [\psi_1'(\zeta_1), 1, 0, \ldots, 0], F \rangle = \underbrace{\psi_1'(\zeta_1)(\zeta_2 + \psi_1(\zeta_1)) + \zeta_3 + \psi_2(\zeta_1)}_{\triangleq \zeta_3 + \theta_2(\zeta_1, \zeta_2)} \\
\xi_4 &= \dddot{y} = L_F^3 \zeta_1 \triangleq \zeta_4 + \theta_3(\zeta_1, \zeta_2, \zeta_3) \\
&\vdots \\
\xi_{r-1} &= y^{(r-2)} = L_F^{r-2} \zeta_1 \triangleq \zeta_{r-1} + \theta_{r-2}(\zeta_1, \ldots, \zeta_{r-2}) \\
\xi_r &= y^{(r-1)} = L_F^{r-1} \zeta_1 \triangleq \zeta_r + \theta_{r-1}(\zeta_1, \ldots, \zeta_{r-1})
\end{aligned} \tag{2.74}$$

naming it

$$\xi = T_1(\zeta_1, \ldots, \zeta_r). \tag{2.75}$$

Because of this definition, the ξ-dynamic has the following structure:

$$\begin{aligned}
\dot{\xi}_i &= \xi_{i+1}, \qquad 1 \leq i \leq r-1 \\
\dot{\xi}_r &= L_F^r \zeta_1 + u L_d L_F^{r-1} \zeta_1
\end{aligned} \tag{2.76}$$

Furthermore, by Theorem 2.4.3 in [16], it is guaranteed that we can find another $m = n - r$ functions $\phi_j(\zeta)$ such that $[\phi_1(\zeta), \ldots, \phi_m(\zeta), T_1(\zeta)^T]^T$ forms a global diffeomorphism, and all the inner products $\langle d\phi_j, G_{r-1} \rangle$ are equal to zero. In fact, these remaining m functions can be defined as linear combinations of the state ζ, so that

$$\eta = T_2\zeta, \qquad T_2 \in \mathbb{R}^{m\times n}$$

and

$$T_2 G = \mathbf{0}_{m\times r}. \tag{2.77}$$

Instead of finding a general expression for T_2 in terms of d so that (2.77) is satisfied, we opt for the simpler task of finding a T_2 so that only the input vector d is annihilated instead of the entire distribution G_{r-1}. In that case we will obtain the normal form instead of the SNF; however, we wish to examine the stability of the zero dynamics of Σ_{OFBF} – a property that is invariant under state transformation, and hence will be identical whether we transform Σ_{OFBF} into NF or SNF.

In order to satisfy the requirement that

$$\begin{bmatrix} \eta \\ \xi \end{bmatrix} = T(\zeta) \triangleq \begin{bmatrix} T_2\zeta \\ T_1(\zeta) \end{bmatrix} \tag{2.78}$$

have an inverse mapping defined everywhere, we will choose T_2 so that

$$\frac{\partial T(\zeta)}{\partial \zeta} = \left[\begin{array}{ccccc:cccc} 1 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ * & 1 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ * & * & \cdots & * & 1 & 0 & 0 & \cdots & 0 \\ \hdashline 0 & 0 & \cdots & 0 & a_1 & 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & a_2 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & a_m & 0 & 0 & \cdots & 1 \end{array}\right].$$

Then, for $T_2 d = \mathbf{0}$, we choose

$$a_i = \frac{-d_{i+r}}{d_r}$$

so that

$$T_2: \quad \eta_i = \frac{-d_{i+r}}{d_r}\zeta_r + \zeta_{r+i}, \qquad 1 \le i \le m. \tag{2.79}$$

Considering the structure of Σ_{OFBF} in (2.65), with this transformation we obtain:

$$\begin{aligned}\dot{\eta}_i &= \frac{-d_{i+r}}{d_r}\left(\zeta_{r+1}+\psi_r(\zeta_1)+d_r u\right)+\left(\zeta_{r+i+1}+\psi_{r+i}(\zeta_1)+d_{r+i}u\right)\\ &= \frac{-d_{i+r}}{d_r}\left(\eta_1+\frac{d_{r+1}}{d_r}\zeta_r\right)+\left(\eta_{i+1}+\frac{-d_{i+r+1}}{d_r}\zeta_r\right)+\psi_{r+i}(\zeta_1)-\frac{d_{i+r}}{d_r}\psi_r(\zeta_1)\\ &= \frac{-d_{i+r}}{d_r}\eta_1+\eta_{i+1}+b_i\zeta_r+\gamma_i(\zeta_1)\end{aligned} \tag{2.80}$$

where we have substituted for ζ_{r+1} and ζ_{r+i+1} from (2.79), and introduced the constant b_i and function $\gamma_i(y)$ for notational brevity. As desired, the $\dot{\eta}$ equation is independent of u; from (2.80) it is evident that u is cancelled exactly because of our choice of a_i in T_2. However, since T_2 was not designed to annihilate any of the other vectors $ad_F^j d,\ j\in\{1,\ldots,(r-1)\}$, the η-dynamic is driven by all the ξ-states through ζ_r. This can be seen from our definition of the ξ-coordinates in (2.74); an inverse transformation can easily be established owing to the triangular structure of $\xi = T_1(\zeta)$. In fact, we can obtain the inverse transformation

$$\begin{aligned}\zeta_1 &= \xi_1\\ \zeta_i &= \xi_i-\overline{\theta}_{i-1}(\xi_1,\ldots,\xi_{i-1}), \qquad 2\le i\le r,\end{aligned} \tag{2.81}$$

and therefore

$$\dot{\eta}_i = \frac{-d_{i+r}}{d_r}\eta_1+\eta_{i+1}+b_i\xi_r-b_i\overline{\theta}_{i-1}(\xi_1,\ldots,\xi_{i-1})+\gamma_i(y). \tag{2.82}$$

In consideration of (2.76) and (2.80), the complete system dynamic in transformed coordinates takes the form:

$$\begin{aligned}\dot{\eta} &= \Gamma\eta+\Gamma_\xi(\xi_1,\ldots,\xi_r)\\ \dot{\xi} &= A_c\xi+B_c(L_F^r\zeta_1+uL_dL_F^{r-1}\zeta_1)\circ T^{-1}(\eta,\xi)\\ y &= \xi_1\end{aligned} \tag{2.83}$$

where

$$\Gamma = \begin{bmatrix} \frac{-d_{r+1}}{d_r} & 1 & 0 & \cdots & 0\\ \frac{-d_{r+2}}{d_r} & 0 & 1 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ \frac{-d_{n-1}}{d_r} & 0 & 0 & \cdots & 1\\ \frac{-d_n}{d_r} & 0 & 0 & \cdots & 0\end{bmatrix}, \tag{2.84}$$

$B_c = [0,\ldots,0,1]^T$, A_c is an $r \times r$ matrix with ones on the superdiagonal and zeros elsewhere, and $\Gamma_\xi(\xi)$ is a smooth function with $\Gamma_\xi(0) = 0$ (this fact is a consequence of our definitions of the functions θ_i in (2.74), which are smooth and zero at zero since the original nonlinearity $\psi(y)$ is assumed to be smooth and zero at zero). With y constrained to zero, all of its derivatives are also zero, and we are left with the zero dynamics:

$$\dot{\eta} = \Gamma\eta$$

which are GAS at $\eta = 0$ owing to the canonical structure of Γ, and the fact that the vector d was assumed to be Hurwitz in the sense of (2.47). Therefore, systems that are diffeomorphic to Σ_{OFBF} are globally minimum phase, and are state equivalent to systems in the SNF. Figure 2.2 shows a Venn diagram depicting the class inclusions of the aforementioned system forms.

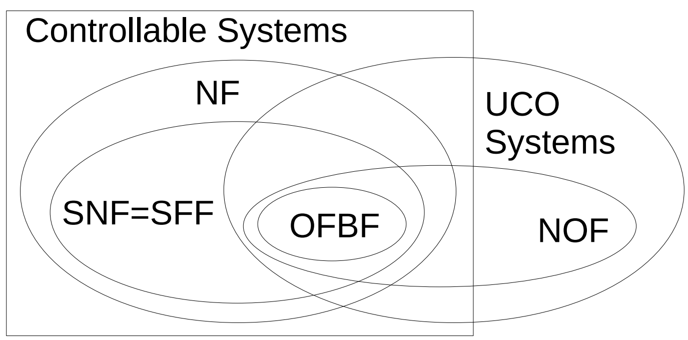

Figure 2.2: Set inclusion relationships between the various system forms important in nonlinear OFB design. The set inclusions are based on state equivalence.

It is interesting to note that all systems that belong to the same equivalence class as Σ_{SFF} discussed in Section 2.1, are also diffeomorphic to systems in the SNF (equation (2.1)) by the following change of coordinates:

$$\begin{bmatrix} \eta^* \\ \xi \end{bmatrix} = T(\eta,x) = \begin{bmatrix} T_2(\eta,x) \\ T_1(\eta,x) \end{bmatrix} \tag{2.85}$$

where $T_2(\eta,x) = \eta$ and $T_1(\eta,x) = T_1(x) = [x_1, \dot{x}_1, \ldots, x_1^{(r-1)}]^T$. Therefore, the class of systems in SFF (with an output defined as $y = x_1$) is included in the class of systems in the SNF. On the other hand, the SNF is a special case of the SFF and therefore the set inclusion works the other way as well. We therefore conclude that the class of systems that are diffeomorphic to the SNF is identical to those that are diffeomorphic to the SFF.

The two global OFB designs that are of interest in this thesis both rely on the properties of the OFBF. Having discussed these properties, we next describe the OFB methods presented in [6] and [2]. Extensions of these methods are provided in subsequent chapters.

2.3 Observer Backstepping

In this section we describe the globally stabilizing OFB design method of *observer backstepping* presented by Kanellakopoulos et al in [6]. In addition to making use of our understanding of backstepping and the properties of the OFBF, we introduce the concept of *nonlinear damping*, which is an important design technique that accompanies backstepping in global partial SFB designs. The basic idea in this method is to design a globally exponentially convergent observer for a system in OFBF, and then design a backstepping-based control law for the observer instead of the plant. Then, nonlinear damping is used to strengthen this controller, directly accounting for the observer error which is treated as a disturbance.

We note that our presentation and analysis of the method given in [6] is different and significantly elaborated relative to what what is provided in either [6] or [1]. Also, in [6] this result is stated in terms of tracking, which includes stabilization as a special case. For simplicity we choose to discuss stabilization here, and remark that the extension to tracking is not difficult; in extending this method to systems in a restricted block triangular observer form in Chapter 4, we demonstrate a tracking design on a MAGLEV system.

This OFB design method is most concretely illustrated by means of a lowdimensional example; all of its key features are demonstrable on the OFBF system

$$\Sigma : \begin{cases} \dot{x}_1 = x_2 + \psi_1(y) \\ \dot{x}_2 = x_3 + \psi_2(y) + d_2 u \\ \dot{x}_3 = x_4 + \psi_3(y) + d_3 u \\ \dot{x}_4 = \psi_4(y) + d_4 u \\ y = x_1 \end{cases} \tag{2.86}$$

from which it will be clear that the same strategies apply in general to any system in the OFBF. Here we assume that the nonlinearity $\Psi(y) = [\psi_1(y), \ldots, \psi_4(y)]^T$ is smooth with $\Psi(0) = 0$ and $d = [0, d_2, d_3, d_4]^T$ is a Hurwitz vector; that is, the polynomial $d_2 s^2 + d_3 s + d_4$ has roots with negative real parts. This system has a globally well-defined relative degree $r = 2$ with respect to $y = x_1$ and its $(n - r = 2)$-dimensional zero dynamics are GES according to our discussion in Subsection 2.2.3.1. The design begins with the construction of an observer for Σ:

$$\hat{\Sigma} : \begin{cases} \dot{\hat{x}}_1 = \hat{x}_2 + \psi_1(y) + L_1(y - \hat{x}_1) \\ \dot{\hat{x}}_2 = \hat{x}_3 + \psi_2(y) + d_2 u + L_2(y - \hat{x}_1) \\ \dot{\hat{x}}_3 = \hat{x}_4 + \psi_3(y) + d_3 u + L_3(y - \hat{x}_1) \\ \dot{\hat{x}}_4 = \psi_4(y) + d_4 u + L_4(y - \hat{x}_1) \end{cases} , \tag{2.87}$$

whose error dynamic is GES:

$$\tilde{\Sigma} : \begin{cases} \dot{\tilde{x}}_1 = -L_1 \tilde{x}_1 + \tilde{x}_2 \\ \dot{\tilde{x}}_2 = -L_2 \tilde{x}_1 + \tilde{x}_3 \\ \dot{\tilde{x}}_3 = -L_3 \tilde{x}_1 + \tilde{x}_4 \\ \dot{\tilde{x}}_4 = -L_4 \tilde{x}_1 \end{cases} , \tag{2.88}$$

when $\tilde{x}_i \triangleq x_i - \hat{x}_i$ and the output injection gain $L = [L_1, \ldots, L_4]^T$ is chosen so that the polynomial $s^3 + L_1 s^2 + L_2 s + L_3$ is Hurwitz. Expressing $\tilde{\Sigma}$ as:

$$\tilde{\Sigma} : \quad \dot{\tilde{x}} = (A_c - LC_c)\tilde{x}, \tag{2.89}$$

we re-write the composite dynamic consisting of Σ and $\hat{\Sigma}$ as

$$\Sigma_c : \begin{cases} \dot{x}_1 = \hat{x}_2 + \tilde{x}_2 + \psi_1(y) \\ \dot{\hat{x}}_2 = \hat{x}_3 + \psi_2(y) + d_2 u + L_2(y - \hat{x}_1) \\ \dot{\hat{x}}_3 = \hat{x}_4 + \psi_3(y) + d_3 u + L_3(y - \hat{x}_1) \\ \dot{\hat{x}}_4 = \psi_4(y) + d_4 u + L_4(y - \hat{x}_1) \\ \dot{\tilde{x}} = (A_c - LC_c)\tilde{x} \\ y = x_1 \end{cases} \tag{2.90}$$

where we have replaced the dynamics of all unmeasured plant states with their estimates. To be sure that Σ_c fully represents the Σ-$\hat{\Sigma}$ dynamics, we note that the transformation

$$\begin{bmatrix} x_1 \\ \hat{x}_2 \\ \hat{x}_3 \\ \hat{x}_4 \\ \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \\ \tilde{x}_4 \end{bmatrix} = \left[\begin{array}{cccc:cccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \hdashline 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \end{array}\right] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \\ \hat{x}_4 \end{bmatrix} \tag{2.91}$$

is nonsingular. Therefore, any controller that stabilizes Σ_c will also stabilize our original plant-observer interconnection.

Instead of following exactly the proof given in Section 3 of [6], here we first introduce one more transformation for Σ_c which will allow us to show more directly how the stability of the overall CL system relies on the properties of the OFBF. Our transformation involves

$$\begin{bmatrix} x_1 \\ \hat{x}_2 \\ \hat{x}_3 \\ \hat{x}_4 \end{bmatrix} \mapsto \begin{bmatrix} x_1 \\ \hat{x}_2 \\ \eta_1 \\ \eta_2 \end{bmatrix} \tag{2.92}$$

with

$$\eta_1 = \frac{-d_3}{d_2}\hat{x}_2 + \hat{x}_3$$
$$\eta_2 = \frac{-d_4}{d_2}\hat{x}_2 + \hat{x}_4$$

which is very similar to the transformation T_2 in (2.79). With this (nonsingular) change of coordinates, Σ_c takes the form

$$\Sigma^* : \begin{cases} \dot{\eta} = \Gamma\eta + \gamma(y) + B_1\tilde{x}_1 + B_2\hat{x}_2 \\ \dot{x}_1 = \hat{x}_2 + \tilde{x}_2 + \psi_1(y) \\ \dot{\hat{x}}_2 = \eta_1 + \frac{d_3}{d_2}\hat{x}_2 + \psi_2(y) + d_2 u + L_2\tilde{x}_1 \\ \dot{\tilde{x}} = (A_c - LC_c)\tilde{x} \\ y = x_1 \end{cases} \tag{2.93}$$

where $B_1 = [b_{1,1}, b_{1,2}]^T$, and $B_2 = [b_{2,1}, 0]^T$, the constants

$$b_{1,1} = L_3 - \tfrac{d_3L_2}{d_2}, \qquad b_{1,2} = L_4 - \tfrac{d_4L_2}{d_2}, \qquad b_{2,1} = \tfrac{d_4}{d_2} - \tfrac{d_3^2}{d_2^2},$$

the function $\gamma(y) = [\gamma_1(y), \gamma_2(y)]^T$, where

$$\gamma_1(y) = \psi_3(y) - \tfrac{d_3}{d_2}\psi_2(y), \qquad \gamma_2(y) = \psi_4(y) - \tfrac{d_4}{d_2}\psi_2(y),$$

and

$$\Gamma = \begin{bmatrix} \frac{-d_3}{d_2} & 1 \\ \frac{-d_4}{d_2} & 0 \end{bmatrix}.$$

The system Σ^* is the starting form for the backstepping design, and it possesses several important features. First, the linear system $\dot{\eta} = \Gamma\eta$ is GAS at $\eta = 0$. Therefore, when the "disturbance" inputs $\gamma(y)$, $\tilde{x}_1$ and $\hat{x}_2$ are bounded, the state η is also bounded. Furthermore, from its solution we can conclude that if each of these disturbance inputs is decaying, then η itself converges to zero. We will formalize this statement later. The disturbance $\tilde{x}_1$ decays to zero because of our choice of observer gains. Since $\gamma(y)$ is a smooth function with $\gamma(0) = 0$, it also decays as $y \to 0$. It therefore remains to design the input $u = \alpha(x_1, \hat{x}_2, \eta)$ so that the $(x_1, \hat{x}_2)$-subsystem is also asymptotically stabilized at zero.

To that end, we proceed with the backstepping procedure as detailed in Section 2.1, considering the following as a candidate Lyapunov function:

$$V_1 = \tfrac{1}{2}x_1^2 + V_e$$

where

$$V_e = \tilde{x}^T P \tilde{x}$$

is the Lyapunov function associated with the observer error subsystem, with P being the unique, symmetric, positive definite matrix solution of the Lyapunov equation

$$(A_c - LC_c)^T P + P(A_c - LC_c) = -\kappa I$$

for some $\kappa > 0$. The derivative of V_e along $\tilde{x}(t)$ is

$$\begin{aligned} \dot{V}_e &= \dot{\tilde{x}}^T P \tilde{x} + \tilde{x}^T P \dot{\tilde{x}} \\ &= [(A_c - LC_c)\tilde{x}]^T P \tilde{x} + \tilde{x}^T P[(A_c - LC_c)\tilde{x}] \\ &= \tilde{x}^T [(A_c - LC_c)^T P + P(A_c - LC_c)]\tilde{x} \\ &= -\kappa \tilde{x}^T \tilde{x} \\ &= -\kappa \|\tilde{x}\|^2. \end{aligned} \tag{2.94}$$

Then, the time derivative of V_1 is

$$\dot{V}_1 = -\kappa\|\tilde{x}\|^2 + x_1(\hat{x}_2 + \tilde{x}_2 + \psi_1(y)). \tag{2.95}$$

We must choose a virtual control and design its associated stabilizing function so that $\dot{V}_1$ is rendered negative definite. The natural choice for a virtual control is the known state $\hat{x}_2$. However, the difficulty in choosing its stabilizing function is that the signal $\tilde{x}_2$ is unknown, and therefore cannot be cancelled. The way around this difficulty is to apply the very simple, and beautiful nonlinear design technique known as *nonlinear damping*. To demonstrate, we define $w_2 \triangleq \hat{x}_2 - \alpha_1$ and let

$$\alpha_1(x_1) = -\psi_1(y) - c_1 x_1 - \delta_1 x_1. \tag{2.96}$$

where the constant δ_1 is known as a *damping coefficient*. Then $\dot{V}_1$ becomes

$$\dot{V}_1 = -(\kappa - \tfrac{1}{4\delta_1})\|\tilde{x}\|^2 - \tfrac{1}{4\delta_1}\|\tilde{x}\|^2 + x_1\tilde{x}_2 - \delta_1 x_1^2 - c_1 x_1^2 + x_1 w_2.$$

Since $x_1\tilde{x}_2 \leq |x_1||\tilde{x}_2| \leq |x_1|\|\tilde{x}\|$, we have that

$$\begin{aligned}\dot{V}_1 &\leq -(\kappa - \tfrac{1}{4\delta_1})\|\tilde{x}\|^2 \underbrace{- \tfrac{1}{4\delta_1}\|\tilde{x}\|^2 + |x_1|\|\tilde{x}\| - \delta_1|x_1|^2}_{\text{complete the square}} - c_1 x_1^2 + x_1 w_2 \\ &= -(\kappa - \tfrac{1}{4\delta_1})\|\tilde{x}\|^2 - \delta_1\left(\frac{\|\tilde{x}\|}{2\delta_1} - |x_1|\right)^2 - c_1 x_1^2 + x_1 w_2. \end{aligned} \tag{2.97}$$

Thus we see that when $w_2 \equiv 0$, $\dot{V}_1$ is is made negative definite by the last term in (2.96), without cancelling the unknown signal $\tilde{x}_2$.

In the next iteration we proceed as usual, with the intention of regulating w_2 to zero. We therefore consider the time derivative of the new Lyapunov function candidate $V_2 = V_1 + \frac{1}{2}w_2$:

$$\begin{aligned}\dot{V}_2 &= -(\kappa - \tfrac{1}{4\delta_1})\|\tilde{x}\|^2 + \tau_1 + w_2[x_1 + \dot{\hat{x}}_2 - \dot{\alpha}_1(x_1)] \\ &= -(\kappa - \tfrac{1}{4\delta_1})\|\tilde{x}\|^2 + \tau_1 + w_2\Bigg[x_1 + \eta_1 + \tfrac{d_3}{d_2}\hat{x}_2 + \psi_2(y) + d_2 u \\ &\qquad + L_2\tilde{x}_1 - \frac{\partial\alpha_1}{\partial x_1}(\hat{x}_2 + \psi_1(y)) - \frac{\partial\alpha_1}{\partial x_1}\tilde{x}_2\Bigg] \end{aligned} \tag{2.98}$$

where we have defined the negative definite term

$$\tau_1 \triangleq -\delta_1 \left(\frac{\|\tilde{x}\|}{2\delta_1} - |x_1| \right)^2 - c_1 x_1^2 \tag{2.99}$$

to simplify the notation. We note that both $\tilde{x}_1$ and η_1 are known to us, and we therefore assign

$$\begin{aligned} u = \frac{1}{d_2} \Bigg(& -c_2 w_2 - x_1 - \eta_1 - \tfrac{d_3}{d_2} \hat{x}_2 - \psi_2(y) - L_2 \tilde{x}_1 \\ & + \frac{\partial \alpha_1}{\partial x_1} (\hat{x}_2 + \psi_1(y)) - \delta_2 \left(\frac{\partial \alpha_1}{\partial x_1} \right)^2 w_2 \Bigg) \end{aligned} \tag{2.100}$$

to obtain

$$\begin{aligned} \dot{V}_2 = & -(\kappa - \tfrac{1}{4\delta_1} - \tfrac{1}{4\delta_2}) \|\tilde{x}\|^2 + \tau_1 - c_2 w_2^2 \\ & \underbrace{- \tfrac{1}{4\delta_2} \|\tilde{x}\|^2 - \left(\frac{\partial \alpha_1}{\partial x_1} w_2 \right) \tilde{x}_2 - \delta_2 \left(\frac{\partial \alpha_1}{\partial x_1} w_2 \right)^2}_{\text{complete the square}} \\ \leq & -(\kappa - \tfrac{1}{4\delta_1} - \tfrac{1}{4\delta_2}) \|\tilde{x}\|^2 + \tau_1 + \tau_2 \end{aligned} \tag{2.101}$$

where

$$\tau_2 \triangleq -c_2 w_2^2 - \delta_2 \left(\frac{\|\tilde{x}\|}{2\delta_2} + \left| \frac{\partial \alpha_1}{\partial x_1} w_2 \right| \right)^2. \tag{2.102}$$

Then, in consideration of our definition of the terms τ_1 and τ_2, we conclude by Theorem A.0.1 that the closed-loop system

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{w}_2 \end{bmatrix} &= \begin{bmatrix} -c_1 & 1 \\ -1 & -c_2 \end{bmatrix} \begin{bmatrix} x_1 \\ w_2 \end{bmatrix} + \begin{bmatrix} 1 \\ \frac{\partial \alpha_1(x_1)}{\partial x_1} \end{bmatrix} \tilde{x}_2 - \begin{bmatrix} \delta_1 & 0 \\ 0 & \delta_2 \left(\frac{\partial \alpha_1(x_1)}{\partial x_1} \right)^2 \end{bmatrix} \begin{bmatrix} x_1 \\ w_2 \end{bmatrix} \\ \dot{\tilde{x}} &= (A_c - L C_c) \tilde{x} \end{aligned} \tag{2.103}$$

is rendered GAS at its origin by the control law (2.100) and (2.96) provided the control coefficients c_1, c_2 and the damping coefficients δ_1, δ_2 are chosen positive, and

$$\kappa > \tfrac{1}{4\delta_1} + \tfrac{1}{4\delta_2}.$$

A direct consequence of the fact that the system (2.103) is GAS at its origin and that it evolves independently of the η-subsystem in Σ^*, is that its states are bounded by some $\mathcal{KL}_\infty$ function $\beta(\cdot,\cdot)$ as:

$$\|\chi(t)\| \leq \beta(\|\chi(0)\|, t) \tag{2.104}$$

where we have defined

$$\chi \triangleq \begin{bmatrix} x_1 \\ w_2 \\ \tilde{x} \end{bmatrix} \in \mathbb{R}^{r+n=6}$$

Recalling the expression for the η-dynamics in Σ^* (equation (2.93)), we write:

$$\begin{aligned} \dot{\eta} &= \Gamma\eta + \gamma(x_1) + B_1\tilde{x}_1 + B_2(w_2 + \alpha_1(x_1)) \\ &= \Gamma\eta + \underbrace{\gamma(x_1) + B_2\alpha_1(x_1)}_{\text{let}=\varphi(x_1)} + [0, B_1, B_2] \begin{bmatrix} x_1 \\ w_2 \\ \tilde{x}_1 \end{bmatrix}. \end{aligned} \tag{2.105}$$

We note that $\varphi(x_1)$ is smooth and zero at zero, since both $\gamma(x_1)$ and $\alpha_1(x_1)$ are smooth and zero at zero. This observation has two consequences: first, for any bounded x_1, $\|\varphi(x_1)\|$ is bounded. Second,

$$\lim_{x_1 \to 0} \varphi(x_1) = \varphi(0) = 0.$$

Then, by the global asymptotic stability of (2.103), we have that x_1 is bounded and $x_1(t) \to 0$ as $t \to \infty$, and therefore there exists another $\mathcal{KL}_\infty$-class function $\beta_1(\cdot,\cdot)$ such that

$$\|\varphi(x_1(t))\| \le \beta_1(r,t)$$

where

$$r \triangleq \|\varphi(\|x_1(t)\|_{\mathcal{L}_\infty})\|.$$

Also, we note that

$$\left\| [0, B_1, B_2] \begin{bmatrix} x_1 \\ w_2 \\ \tilde{x}_1 \end{bmatrix} \right\| \le \|[0, B_1, B_2]\| \left\| \begin{bmatrix} x_1 \\ w_2 \\ \tilde{x}_1 \end{bmatrix} \right\| \le b\|\chi\| \le b\beta(\|\chi(0)\|, t)$$

for some $b > 0$. Then, to show that $\eta \to 0$ as $t \to \infty$, we solve the differential equation (2.105) using the usual matrix exponential integrating factor:

$$\eta(t) = e^{\Gamma t}\eta(0) + \int_0^t e^{\Gamma(t-\tau)}\big(\gamma(x_1) + B_1\tilde{x}_1 + B_2(w_2 + \alpha_1(x_1))\big)d\tau. \tag{2.106}$$

Taking the norm of both sides and appealing to the triangle inequality, we obtain

$$\|\eta(t)\| \le \|e^{\Gamma t}\|\|\eta(0)\| + \int_0^t \|e^{\Gamma(t-\tau)}\|\|\big(\gamma(x_1) + B_1\tilde{x}_1 + B_2(w_2 + \alpha_1(x_1))\big)\|d\tau. \tag{2.107}$$

Then, noting that $\|e^{\Gamma t}\| \leq ke^{-at}$ for some constants $k > 0$ and $a > 0$, we can write:

$$\|\eta(t)\| \leq ke^{-at}\|\eta(0)\| + \int_0^t ke^{-a(t-\tau)}\Big(\beta_1(r,\tau) + \beta(\|\chi(0)\|,\tau)\Big)d\tau. \qquad (2.108)$$

The first term clearly converges to zero exponentially. To show that the entire second term also decays to zero as time advances, we examine the following (notationally simpler) quantity

$$I(t) \triangleq \int_0^t e^{-a(t-\tau)}\beta_2(s,\tau)d\tau$$

where $\beta_2(s,t)$ is any class $\mathcal{KL}_\infty$ function. If we can show that $I(t) \to 0$ as $t \to \infty$, then it becomes clear that the second term in (2.108) also decays as time advances. By splitting the integral, we obtain

$$\begin{aligned} I(t) &= e^{-at}\left[\int_{t/2}^t e^{a\tau}\beta_2(s,\tau)d\tau + \int_0^{t/2} e^{a\tau}\beta_2(s,\tau)d\tau\right] \\ &\leq e^{-at}\left[\int_{t/2}^t e^{a\tau}\underbrace{\left(\sup_{t/2\leq\tau\leq t}\beta_2(s,\tau)\right)}_{=\beta_2(s,\frac{t}{2})}d\tau + \int_0^{t/2} e^{a\tau}\underbrace{\left(\sup_{0\leq\tau\leq t/2}\beta_2(s,\tau)\right)}_{=\beta_2(s,0)}d\tau\right], \end{aligned} \qquad (2.109)$$

since, by definition, $\beta_2(\cdot,\cdot) > 0$ for all its arguments and it decreases as $t \to \infty$ for any fixed s. Then,

$$\begin{aligned} I(t) &\leq \left[e^{-at}\beta_2(s,\tfrac{t}{2})\left(\frac{e^{at} - e^{at/2}}{a}\right) + e^{-at}\beta_2(s,0)\left(\frac{e^{at/2} - 1}{a}\right)\right] \\ &= \underbrace{\frac{\beta_2(s,\frac{t}{2})}{a}}_{\text{decaying}}\left(1 - e^{-\frac{at}{2}}\right) + \frac{\beta_2(s,0)}{a}\underbrace{\left(e^{-\frac{at}{2}} - e^{-at}\right)}_{\text{decaying}} \end{aligned} \qquad (2.110)$$

and we conclude that

$$\lim_{t\to\infty} I(t) = 0.$$

It is not difficult to see that similar arguments would lead to the conclusion that the second term in (2.108) also converges to zero as t tends to infinity.

We have therefore shown by (2.108) that for all bounded, converging x_1, $w_2 = \hat{x}_2 - \alpha_1(x_1)$ and $\tilde{x}$, the state of the η-subsystem also globally asymptotically converges to $\eta = 0$. We can thus conclude that the composite CL system

$$\Sigma_{CL}^{*}\begin{cases}\dot{\eta} = \Gamma\eta + \gamma(y) + B_1\tilde{x}_1 + B_2(w_2 + \alpha_1(x_1)) \\ \dot{x}_1 = -c_1x_1 + w_2 + \tilde{x}_2 - \delta_1 x_1 \\ \dot{w}_2 = -x_1 - c_2w_2 + \frac{\partial\alpha_1(x_1)}{\partial x_1}\tilde{x}_2 - \delta_2\left(\frac{\partial\alpha_1(x_1)}{\partial x_1}\right)^2 w_2 \\ \dot{\tilde{x}} = (A_c - LC_c)\tilde{x} \\ \alpha_1(x_1) = -\psi_1(y) - c_1x_1 - \delta_1x_1 \\ y = x_1\end{cases} \tag{2.111}$$

is GAS at the origin.

Remark 2.3.1. For any system diffeomorphic to Σ_{OFBF} with a relative degree r and dimension n, the controller is designed in the same way; a full order observer is implemented, and then r iterations of backstepping are performed on the dynamics of the $(x_1, \hat{x}_2, \ldots, \hat{x}_r)$ states, while nonlinear damping is used to account for the observation error. In the preceding example, if $n > r > 2$, then the expression (2.100) would be assigned instead to the next stabilizing function α_2, and the error variable $w_3 \triangleq \hat{x}_3 - \alpha_2$ would be defined. It is easy to see that since each of the subsequent virtual controls – i.e. $\hat{x}_2$, $\hat{x}_3$,..., $\hat{x}_{r-1}$ – appear linearly in the form Σ_c, each iteration follows in a manner identical to what we have illustrated here, until finally the actual control u appears in the rth iteration. The closed-loop system consisting of those r "backstepped" states and the remaining $n-r$ states is guaranteed to be GAS at the origin. ◃

Remark 2.3.2. Throughout the entire procedure, regardless of the dimension n or relative degree r, the only unknown signal appearing in the starting form Σ^* is $\tilde{x}_2$, which it furthermore enters affinely. At each iteration the stabilizing functions pick up the following signal dependencies:

$$\begin{aligned}\alpha_1 &= \alpha_1(x_1) \\ \alpha_2 &= \alpha_2(x_1, w_2, \tilde{x}_1) \\ &\vdots \\ \alpha_i &= \alpha_i(x_1, w_2, \ldots, w_i, \tilde{x}_1) \\ &\vdots\end{aligned}$$

and therefore at every iteration $(i > 1)$, the quadratic Lyapunov function derivative contains terms of the form

$$\begin{aligned}\dot{V}_{i+1} &= \ldots + w_{i+1}\dot{w}_{i+1} \\ &= \ldots + w_{i+1}\big(\dot{\hat{x}}_{i+1} - \tfrac{\partial\alpha_i}{\partial x_1}\dot{x}_1 - \sum_{j=2}^{i}\tfrac{\partial\alpha_i}{\partial w_j}\dot{w}_j - \tfrac{\partial\alpha_i}{\partial\tilde{x}_1}\dot{\tilde{x}}_1\big).\end{aligned}$$

It is important to note that the latter $(i+1)$ terms all contain an *affine* dependency on $\tilde{x}_2$, which implies that a coefficient $C_i(x_1, w_2, \ldots, w_i)$ can be extracted so that

$$\dot{V}_{i+1} = \ldots + w_{i+1}\big(\dot{\hat{x}}_{i+1} - K_i(x_1, w_2, \ldots, w_i) - C_i(x_1, w_2, \ldots, w_i)\tilde{x}_2\big).$$

Therefore it is possible to once again apply nonlinear damping via the $(i+1)$th stabilizing function by introducing a term of the form $\delta_i C(\cdot)^2 w_{i+1}$, while cancelling the fully known term $K_i(x_1, w_2, \ldots, w_i)$. This observation may be useful in implementing symbolic math algorithms that calculate the final control expressions for higher order systems.

We also remark that in general, nonlinear damping can only be applied to disturbances that enter the system equations affinely, which is why many global OFB stabilization solutions still require the system to be either affine or “linear in the unmeasured states” [43]. $\triangleleft$

Remark 2.3.3. Our preceding discussions may give the impression that backstepping procedures are fixed and always yield the same outcome in terms of controller structure, for a given set of system equations. On the contrary, backstepping is more of a technique than a pre-defined design, contrasting with control designs by feedback linearization for instance. As an example of this flexibility, we note that at each iteration it is possible to separate the coefficient $C_i(x_1, w_2, \ldots, w_i)$ that multiplies $\tilde{x}_2$ (cf. Remark 2.3.2) into several smaller terms in different ways. In other words, for any given set of system equations several solutions are possible. One may find that implementing two damping terms like $\delta_{i,1}C_{i,1}(\cdot)^2 w_{i+1} + \delta_{i,2}C_{i,2}(\cdot)^2 w_{i+1}$ instead of $\delta_i C_i(\cdot)^2 w_{i+1} = \delta_i(C_{i,1}(\cdot) + C_{i,2}(\cdot))^2 w_{i+1}$ results in a reduced control effort, or a simpler control expression. $\triangleleft$

It is important to notice that even though the OFBF appears to be a rather restrictive system form, the nonlinearities $\Psi(y)$ are not growth-restricted in any way. In this way, the method of observer backstepping represents a powerful advancement in nonlinear control.

The observer backstepping method applied to the OFBF is constructive, but it requires a full order observer in order to be implemented. From a practical point of view, the order of the dynamic component of any controller should be kept minimal to reduce the computational burden. In the next section, we present an alternative global OFB method for systems in OFBF due to [2], which requires a dynamic component of order $(r-1)$.

2.4 The MT Method

Observer backstepping is clearly an observer separation method, in which the controller is designed to compensate for the effects of the observer error. The method given by Marino and Tomei in [2] is interesting because it is not a separation method; that is, the role of the dynamic component of their control law is not related to the task of state estimation. Instead, its main purpose is to help transform the system into a set of coordinates in which it "appears" to have relative degree one (with respect to a new input) with stable, linear zero dynamics. This method is also fully constructive and relies on the properties of the OFBF. The key tools used are backstepping, and *filtered transformations*.

The original idea for the method was given in [2], where it was described within an adaptive control context, and a cleaner version is provided in Chapter 6 of [16]. An alternative, and beautifully systematic analysis of the MT method is also provided in Section 11.3 of [12]. Here, we give a much more detailed exposition than provided in either [16], [12] or in the original papers [2]. We regard as part of our contribution this more thourough, but simplified presentation that attempts to clarify the method and its motivations, and more closely adheres to the standard backstepping framework that one would recognize from [1].

We start by examining the OFBF, with input v and relative degree $r = 1$:

$$\begin{aligned} \dot{x}_i &= x_{i+1} + \psi_i(y) + \overline{d}_i\sigma(y)v, \qquad 1 \le i \le (n-1) \\ \dot{x}_n &= \psi_n(y) + \overline{d}_n\sigma(y)v \qquad\qquad\qquad\qquad , \\ y &= x_1 \end{aligned} \tag{2.112}$$

where the nonlinearities $\psi_j(y)$, $1 \le j \le n$ are assumed to be smooth everywhere and zero at zero, the input vector $\overline{d} = [\overline{d}_1, \ldots, \overline{d}_n]^T$ is assumed to be Hurwitz as usual, and the scalar function $\sigma(y)$ is nonsingular everywhere. We assign $v = \frac{1}{d_1\sigma(y)}u$ and define

$$d_i \triangleq \frac{\overline{d}_i}{d_1}$$

so that d_1 is normalized to 1 and we can write more simply:

$$\Sigma : \begin{cases} \dot{x}_i = x_{i+1} + \psi_i(y) + d_i u, \qquad 1 \le i \le (n-1) \\ \dot{x}_n = \psi_n(y) + d_n u \\ y = x_1 \end{cases} , \tag{2.113}$$

or

$$\Sigma : \begin{cases} \dot{x} = A_c x + \Psi(y) + du \\ y = x_1 \end{cases} \tag{2.114}$$

where $d = [1, d_2, \ldots, d_n]^T$ is Hurwitz since $\overline{d}$ is assumed Hurwitz. For this system, we can design a static, globally asymptotically stabilizing OFB law. To see this, we first apply the following linear change of coordinates:

$$\begin{bmatrix} \eta \\ y \end{bmatrix} = \begin{bmatrix} -d_2 & 1 & 0 & \cdots & 0 \\ -d_3 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -d_n & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{bmatrix} x \triangleq Tx \tag{2.115}$$

where $\eta \in \mathbb{R}^{n-1}$ and the output y is as before. In consideration of our definition of d_i, we note that this transformation is identical to the one discussed in Subsection 2.2.3.1 (cf. equation (2.79)). In fact, the purpose of (2.115) is to obtain the normal form for Σ, with the η-subsystem corresponding to the zero-dynamics. However, since Σ's relative degree is $r = 1$, the chain of integrators in its normal form consists of only one state – y itself. Therefore, transforming Σ into the NF is by default the same as transforming it to the SNF; this fact is taken advantage of throughout MT's method.

Proceeding with the transformation, we obtain:

$$\begin{bmatrix} \dot{\eta} \\ \dot{y} \end{bmatrix} = (TA_cT^{-1}) \begin{bmatrix} \eta \\ y \end{bmatrix} + T\Psi(y) + Tdu \tag{2.116}$$

where

$$(TA_cT^{-1}) = \begin{bmatrix} -d_2 & 1 & 0 & \cdots & 0 & d_3 - d_2d_2 \\ -d_3 & 0 & 1 & \cdots & 0 & d_4 - d_2d_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -d_{n-1} & 0 & 0 & \cdots & 1 & d_n - d_2d_{n-1} \\ -d_n & 0 & 0 & \cdots & 0 & -d_2d_n \\ 1 & 0 & 0 & \cdots & 0 & d_2 \end{bmatrix} \tag{2.117}$$

and

$$Td = \begin{bmatrix} d_2 - d_2d_1 \\ d_3 - d_3d_1 \\ \vdots \\ d_n - d_nd_1 \\ d_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \tag{2.118}$$

since we have made $d_1 \equiv 1$. The product (2.118) is to be expected, since T was designed to annihilate the presence of u in the zero-dynamics subsystem. Also of interest is the transformed nonlinearity, which can be expressed as:

$$T\Psi(y) = \begin{bmatrix} \psi_2(y) - d_2\psi_1(y) \\ \psi_3(y) - d_3\psi_1(y) \\ \vdots \\ \psi_n(y) - d_n\psi_1(y) \end{bmatrix} = \begin{bmatrix} \phi_2(y) - d_2\phi_1(y) \\ \phi_3(y) - d_3\phi_1(y) \\ \vdots \\ \phi_n(y) - d_n\phi_1(y) \end{bmatrix} y \triangleq \Phi(y)y \tag{2.119}$$

where we have factored each $\psi_i(y)$ as $\psi_i(y) = y\phi_i(y),\ 1 \leq i \leq n$. The functions $\psi_i(y)$ can always be factored this way owing to our assumptions that $\psi_i(y) \in C^\infty$ everywhere, and that $\psi(0) = 0$. These assumptions guarantee[8] that the functions

[8]This construction is similar to the one discussed in Section 2.1 (cf. equation (2.8)). For interest's sake we mention that in the case of single variable scalar function $\psi(y)$ there is an alternative proof of the fact that the factorization $\psi(y) = y\phi(y)$ (with $\phi(y) \in C^0$) is always possible when $\phi(y) \in C^1$, and $\psi(0) = 0$. The proof is again by construction. First we note that if $\psi(y)$ is C^1 everywhere, then $\psi'(y)$ exists and is continuous everywhere. Intuitively, we would like to let $\phi(y) = \frac{\psi(y)}{y}$; unfortunately at $y = 0$ this may lead to an indefinite form. However, by L'Hospital's rule, we have that

$$\lim_{y\to 0} \frac{\psi(y)}{y} = \lim_{y\to 0} \frac{\psi'(y)}{1} = \psi'(0) \tag{2.120}$$

$\phi_i(y)$ can always be constructed as:

$$\phi_i(y) = \int_0^1 \frac{d\psi_i(\zeta)}{d\zeta}\bigg|_{\zeta=sy} ds. \tag{2.121}$$

The η-dynamic is then written as:

$$\dot{\eta} = \Gamma\eta + By + \Phi(y)y \tag{2.122}$$

where Γ is the Hurwitz matrix

$$\Gamma = \begin{bmatrix} -d_2 & 1 & 0 & \cdots & 0 \\ -d_3 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -d_{n-1} & 0 & 0 & \cdots & 1 \\ -d_n & 0 & 0 & \cdots & 0 \end{bmatrix}, \tag{2.123}$$

which is identical to (2.84), and the constant vector B is

$$B = \begin{bmatrix} d_3 - d_2 d_2 \\ d_4 - d_2 d_3 \\ \vdots \\ d_n - d_2 d_{n-1} \\ -d_2 d_n \end{bmatrix}. \tag{2.124}$$

Letting

$$\gamma(y) \triangleq B + \Phi(y)$$

we write the transformed system as:

$$\Sigma^* : \begin{cases} \dot{\eta} = \Gamma\eta + \gamma(y)y \\ \dot{y} = \eta_1 + d_2 y + \psi_1(y) + u. \end{cases} \tag{2.125}$$

We now make some observations concerning the form (2.125). Unlike in observer backstepping, no observer has been constructed here, and therefore the states of the

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and therefore by defining

$$\phi(y) = \begin{cases} \frac{\psi(y)}{y}, & y \neq 0 \\ \psi'(0), & y = 0 \end{cases}$$

we ensure that it is continuous, since $lim_{y \to y^*} \phi(y) = y^*$, for all $y^* \in \mathbb{R}$.

Alternatively, we may impose a stronger requirement on $\psi(y)$. If $\psi(y)$ is analytic everywhere and is zero at zero, then there exists a Taylor series that converges to $\psi(y)$ everywhere. Since $\psi(0) = 0$, the constant term in this series is zero. We can therefore factor a y out of each term in the series, and collect all terms in the new factored series into another function $\phi(y)$.

η-subsystem are unknown and cannot be cancelled by u. However, we notice that if $\gamma(y)y \equiv 0$, then the only coupling between the η and y subsystems is through the state η_1. Further, since the unmeasured η_1 enters $\dot{y}$ linearly and the η subsystem is exponentially convergent, we could apply nonlinear damping to compensate for its effect on the y-subsystem. In that case we would only have to worry about stabilizing the y-subsystem, taking into account the effect of the "disturbance" η_1. However, since y drives the η-subsystem through $\gamma(y)y$, this dynamic coupling must also be accounted for. Fortunately, it is possible to do so owing to our factorization of $T\Psi(y)$ in (2.119). To demonstrate, we propose the Lyapunov function candidate

$$V = \eta^T Q \eta + \tfrac{1}{2} y^2$$

where $Q = Q^T > 0$ is the unique solution of the Lyapunov equation

$$\Gamma^T Q + Q\Gamma = -\kappa I$$

for some real number $\kappa > 0$. We then find the derivative of V along the solutions of Σ^*:

$$\begin{aligned}
\dot{V} &= \dot{\eta}^T Q \eta + \eta^T Q \dot{\eta} + y\dot{y} \\
&= \big(\Gamma\eta + \gamma(y)y\big)^T Q\eta + \eta^T Q\big(\Gamma\eta + \gamma(y)y\big) + y\dot{y} \\
&= \eta^T \left(\Gamma^T Q + Q\Gamma\right)\eta + 2\eta^T Q\gamma(y)y + y\dot{y}
\end{aligned} \tag{2.126}$$

where we have used the fact that

$$y\gamma(y)^T Q\eta \equiv \eta^T Q\gamma(y)y$$

due to the symmetry of Q. Continuing with (2.126), we obtain

$$\dot{V} = -\kappa\eta^T\eta + y\left[2\eta^T Q\gamma(y) + \eta_1 + d_2 y + \psi_1(y) + u\right], \tag{2.127}$$

which can be rendered negative definite by choosing

$$u = -\psi_1(y) - d_2 y - \delta y - \beta(Q\gamma(y))^T(Q\gamma(y))y - cy \tag{2.128}$$

where δ and β are damping coefficients to be chosen positive, and c is a control coefficient, also to be chosen positive. In that case $\dot{V}$ becomes

$$\begin{aligned}\dot{V} \le & -(\kappa - \tfrac{1}{4\delta} - \tfrac{1}{\beta})\eta^T\eta \underbrace{-\tfrac{1}{\beta}\eta^T\eta + 2\eta^T(Q\gamma(y)y) - \beta(Q\gamma(y)y)^T(Q\gamma(y)y)}_{\text{complete the square}} \\ & -\tfrac{1}{4\delta}\|\eta\|^2 + |y|\|\eta\| - \delta y^2 - cy^2 \qquad (2.129)\\ = & -(\kappa - \tfrac{1}{4\delta} - \tfrac{1}{\beta})\eta^T\eta - \beta\left(\tfrac{1}{\beta}\eta - Q\gamma(y)y\right)^T\left(\tfrac{1}{\beta}\eta - Q\gamma(y)y\right) \\ & -\delta\left(\frac{\|\eta\|}{2\delta} - |y|\right)^2 - cy^2 \\ \le & -(\kappa - \tfrac{1}{4\delta} - \tfrac{1}{\beta})\eta^T\eta - cy^2 \qquad (2.130)\end{aligned}$$

which shows that the CL system (2.125) and (2.128) is GAS at $(\eta, y) = (\mathbf{0}, 0)$ by Theorem A.0.1.

The true ingenuity in MT's method reveals itself for the case $r > 1$, where a dynamic component is used to obtain a system structure that is similar to Σ for the $r = 1$ case.

2.4.1 The Relative Degree $r > 1$ Case

We once again consider the OFBF system

$$\Sigma: \begin{cases} \dot{x} = A_c x + \Psi(y) + du \\ y = x_1 \end{cases} \qquad (2.131)$$

where without loss of generality we assume that the Hurwitz input vector is normalized with respect to d_r – i.e. $d = [0, \ldots, 0, 1, d_{r+1}, \ldots, d_n]^T$. We now introduce the linear system

$$\begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \\ \vdots \\ \dot{\xi}_{r-2} \\ \dot{\xi}_{r-1} \end{bmatrix} = \begin{bmatrix} -\lambda_1 & 1 & \cdots & 0 \\ 0 & -\lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & -\lambda_{r-1} \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_{r-2} \\ \xi_{r-1} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u, \qquad (2.132)$$

or

$$\dot{\xi} = \Lambda\xi + Bu, \qquad (2.133)$$

where we choose the numbers $\lambda_i > 0,\ 1 \le i \le (r-1)$. We claim that the following dynamic change of coordinates:

$$z = x - D\xi, \tag{2.134}$$

where $D \in \mathbb{R}^{n\times(r-1)}$ is to be determined, can transform Σ into the coupled system

$$\begin{aligned} \dot{z} &= A_c z + \Psi(y) + g\xi_1 \\ \dot{\xi} &= \Lambda\xi + Bu \qquad\qquad , \\ y &= z_1 \end{aligned} \tag{2.135}$$

where the vector $g = [1, g_2, \ldots, g_n]^T$ is Hurwitz. We will prove this claim in Subsection 2.4.1.1. For the time being, assume that we indeed have (2.135). In that case, if we regard ξ_1 as an input, then the z-subsystem resembles exactly Σ with relative degree $r = 1$, a case we know how to solve as per our foregoing discussion. Continuing in exactly the same way as for the $r = 1$ case, we transform the z-subsystem using the linear transformation (2.115) (with d_i's replaced by g_i's, for $2 \le i \le n$) to obtain the starting form:

$$\Sigma^* : \begin{cases} \dot{\eta} = \Gamma\eta + \gamma(y)y \\ \dot{y} = \eta_1 + g_2 y + \psi_1(y) + \xi_1, \\ \dot{\xi} = \Lambda\xi + Bu \end{cases} \tag{2.136}$$

where Γ is a Hurwitz matrix as before, and the state ξ_1 takes the place of u. The solution to the OFB problem is now all but complete. It remains only to point out that the ξ-subsystem is in a SFF and hence admits a backstepping design. To proceed, we would consider the Lyapunov function candidate $V_1 = \eta^T Q\eta + \frac{1}{2}y^2$ as before. This time, we would use ξ_1 as the virtual control and define the error variable $w_2 \triangleq \xi_1 + \alpha_1(y)$, where the expression in equation (2.128) is now assigned to $\alpha_1(y)$. In the next iteration we would take the derivative of $V_2 = V_1 + \frac{1}{2}w_2^2$:

$$\dot{V}_2 \le -(\kappa - \tfrac{1}{4\delta})\eta^T\eta - cy^2 + w_2 y + w_2[-\lambda_1\xi_1 + \xi_2 - \tfrac{\partial\alpha_1(y)}{\partial y}\dot{y}]$$

and choose ξ_2 as the next virtual control. Its associated stabilizing function must cancel the third term in $\dot{V}_2$, which is the usual “leftover” term from the previous coordinate shift, as well as apply nonlinear damping to the affine η term that appears through $\dot{y}$. One can see that the remainder of the iterations proceed in a similar manner, according to our previous discussions of backstepping and nonlinear damping. In the rth iteration, the actual control u appears and is selected similarly to yield a GAS CL system. Figure 2.3 shows the CL system under this OFB law.

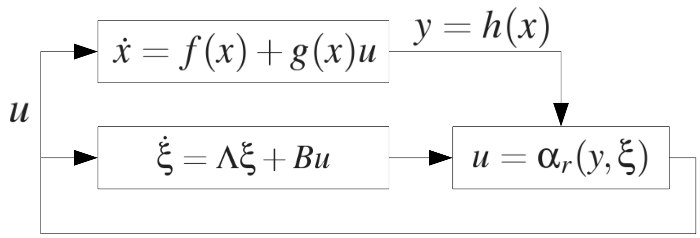

Figure 2.3: The closed-loop system implementing Marino and Tomei's globally stabilizing OFB law.

For a greater level of computational detail the reader is referred to Chapter 3, where the extension of this method to MIMO systems is provided along with a more formal inductive proof. A special case of the proof provided therein includes this SISO formulation. Here, we are more eager to explain filtered transformations, which lie at the heart of this clever method.

In the sequel, we give a significantly more extensive and detailed analysis of the filtered transformation than is provided in any of [16], [2] or [12]. We hope that our insights are helpful in making the MT method easier to understand and possibly adapt in new applications.

2.4.1.1 The Filtered Transformation

The linear system (2.132) is referred to as the filter, and ultimately becomes incorporated as the dynamic component of the OFB law. The associated transformation (2.134) is known as a *filter transformation* (FT), a vehicle that takes a system in OFBF of any relative degree, and makes it look like a system with relative degree one. The FT

$$z = x - D\xi$$

can be viewed as a time-varying coordinate shift, and its design is motivated as follows.

$$\begin{aligned}\dot{z} &= A_c[z+D\xi]+\Psi(y)+du-D[\Lambda\xi+Bu] \\ &= A_c z+\Psi(y)+[A_cD-D\Lambda]\xi+[d-DB]u. \end{aligned} \tag{2.137}$$

Comparing (2.137) with our desired form

$$\dot{z} = A_c z+\Psi(y)+g\xi_1$$

in (2.135), we see that the matrix D must satisfy the following conditions

$$A_cD-D\Lambda = gC \tag{2.138}$$

$$d-DB=0 \tag{2.139}$$

where we define $C=[1,0,\ldots,0]_{1\times(r-1)}$ so that $C\xi=\xi_1$. Since $B=[0,\ldots,0,1]^T_{1\times(r-1)}$, (2.139) implies that the $(r-1)$th column of D must equal the original input vector d. We first notice that (2.138) looks very much like a Sylvester equation, with A_c and Λ being two square matrices with no two eigenvalues in common, and D the matrix to be solved for. It differs from a standard Sylvester equation in that D is not square, and that (2.139) imposes an additional restriction on its solution. Nevertheless, a unique solution does exist for D, and can be obtained recursively by analyzing (2.138) on a column-by-column basis. To that end, we denote the columns of D as

$$D=[d[2],d[3],\ldots,d[r-1],d[r]]. \tag{2.140}$$

By this notation and by condition (2.139), we have that

$$d[r]\equiv d,$$

the original input vector in (2.131). It is also helpful to express Λ as

$$\Lambda = \begin{bmatrix} -\lambda_1 & 0 & \cdots & 0 & 0 \\ 0 & -\lambda_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -\lambda_{r-2} & 0 \\ 0 & 0 & \cdots & 0 & -\lambda_{r-1} \end{bmatrix} + \begin{bmatrix} 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \triangleq \overline{\Lambda}+A_{c_\xi}$$

and to note that post-multiplying D by A_{c_ξ} shifts all of D's columns to the right. Therefore (2.138) can be re-written as

$$\begin{aligned}\big[A_c d[2], A_c d[3], \ldots, A_c d[r]\big] - \big[-\lambda_1 d[2], -\lambda_2 d[3], \ldots, -\lambda_{r-1} d[r]\big] \\ - \big[0, d[2], \ldots, d[r-1]\big] = \big[g, 0, \ldots, 0\big],\end{aligned} \tag{2.141}$$

or, column-by-column we have

$$(A_c + \lambda_1 I)d[2] = g \tag{2.142a}$$

$$(A_c + \lambda_i I)d[i+1] = d[i], \quad 2 \le i \le (r-1). \tag{2.142b}$$

Recalling that $d[r] = d$ is known, we can use (2.142) to solve recursively for the columns of the matrix D. Denoting

$$d[1] \triangleq g$$

we express this recursive solution as:

$$\begin{aligned} d[r] &= d \\ d[i-1] &= (A_c + \lambda_{i-1} I)d[i], \quad r \ge i \ge 2 \end{aligned} \tag{2.143}$$

which is clearly a unique solution.

To show that $g = d[1]$ is a Hurwitz vector, we re-arrange equation (2.142) to obtain

$$(A_c + \lambda_1 I)(A_c + \lambda_2 I)\cdots(A_c + \lambda_{r-2} I)(A_c + \lambda_{r-1} I)d[r] = g, \tag{2.144}$$

and note that the matrix

$$\begin{aligned} M &\triangleq (A_c + \lambda_1 I)(A_c + \lambda_2 I)\cdots(A_c + \lambda_{r-2} I)(A_c + \lambda_{r-1} I) \\ &= A_c^{r-1} + a_1 A_c^{r-2} + a_2 A_c^{r-3} + \ldots + a_{r-2} A_c + a_{r-1} I \end{aligned} \tag{2.145}$$

forms the following pattern

$$M = \begin{bmatrix} a_{r-1} & a_{r-2} & \cdots & a_1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & a_{r-1} & \cdots & a_1 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & a_{r-1} & a_{r-2} \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & a_{r-1} \end{bmatrix} \tag{2.146}$$

where the a_i are clearly the coefficients of the polynomial

$$\prod_{i=1}^{r-1}(s+\lambda_i) = s^{r-1} + a_1 s^{r-2} + \ldots + a_{r-2}s + a_{r-1}s$$

which is Hurwitz if and only if the filter is stable. Then, since M is a triangular Toeplitz matrix with polynomial coefficients on its diagonals, the equation

$$Md = g \tag{2.147}$$

(i.e. equation (2.144)) can be interpreted as the numerical computation of the following polynomial product:

$$g_1 s^n + \ldots + g_{n-1}s + g_n = \prod_{i=1}^{r-1}(s+\lambda_i)\left[s^{n-r} + d_{r+1}s^{n-r-1} + \ldots + d_{n-1}s + d_n\right]. \tag{2.148}$$

To justify this statement, we remark that polynomial multiplication is equivalent to the convolution of their two respective coefficient vectors, and the convolution of two vectors can be expressed as a multiplication between a Toeplitz matrix such as M above, and a vector. To illustrate, suppose we have two polynomials

$$a(s) = s^2 + a_1 s + a_2$$

and

$$b(s) = s^3 + b_1 s^2 + b_2 s + b_3.$$

Then,

$$\begin{aligned} a(s)b(s) = s^5 &+ (a_1+b_1)s^4 + (a_2+a_1b_1+b_2)s^3 + (a_2b_1+a_1b_2+b_3)s^2 \\ &+ (a_2b_2+a_1b_3)s + (a_3b_3). \end{aligned}$$

The coefficients of this product can be computed from

$$\begin{bmatrix} a_2 & a_1 & 1 & 0 & 0 \\ 0 & a_2 & a_1 & 1 & 0 \\ 0 & 0 & a_2 & a_1 & 1 \\ 0 & 0 & 0 & a_2 & a_1 \\ 0 & 0 & 0 & 0 & a_2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} b_3 & b_2 & b_1 & 1 & 0 \\ 0 & b_3 & b_2 & b_1 & 1 \\ 0 & 0 & b_3 & b_2 & b_1 \\ 0 & 0 & 0 & b_3 & b_2 \\ 0 & 0 & 0 & 0 & b_3 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} a_1+b_1 \\ a_2+a_1b_1+b_2 \\ a_2b_1+a_1b_2+b_3 \\ a_2b_2+a_1b_3 \\ a_2b_3 \end{bmatrix}$$

We can now see why the stability of the zero dynamics is important to this design. To recapitulate, once the FT $z = x - D\xi$ has been applied, we obtain the system dynamic (2.135) which now resembles a relative degree one system when ξ_1 is regarded as its input. Then, applying the linear transformation

$$T = \begin{bmatrix} -g_2 & 1 & 0 & \cdots & 0 \\ -g_3 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -g_n & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{bmatrix} \tag{2.149}$$

to the z-subsystem results in the starting form

$$\Sigma^* : \begin{cases} \dot{\eta} = \Gamma\eta + \gamma(y)y \\ \dot{y} = \eta_1 + g_2 y + \psi_1(y) + \xi_1, \\ \dot{\xi} = \Lambda\xi + Bu \end{cases}$$

where the matrix

$$\Gamma = \begin{bmatrix} -g_2 & 1 & 0 & \cdots & 0 \\ -g_3 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -g_{n-1} & 0 & 0 & \cdots & 1 \\ -g_n & 0 & 0 & \cdots & 0 \end{bmatrix} \tag{2.150}$$

is Hurwitz if, and only if its characteristic polynomial

$$|\Gamma - sI| = g_1 s^n + \ldots + g_{n-1}s + g_n \tag{2.151}$$

has roots with negative real parts. From Σ^*, the method proceeds as described in Section 2.4 and hence relies on the stability of the $\dot{\eta} = \Gamma\eta$ system. On the other hand, we observe from (2.148) that the polynomial $g_1 s^n + \ldots + g_{n-1}s + g_n$ has roots with negative real parts if and only if the filter matrix Λ is Hurwitz, and the original input vector d is Hurwitz. From the properties of the OFBF described in Section 2.2.3.1, we know that the zero dynamics of Σ_{OFBF} are stable if, and only if d is a Hurwitz vector. Therefore, the minimum phase assumption is just as critical for this method as it is for observer backstepping.

Remark 2.4.1. It is well known that the relative degree or the stability of the zero dynamics cannot be changed by SFB or static changes of coordinates. The FT

(2.134) also does not change a system's relative degree, a fact which is clear from the structure of Σ^*; the signal y and u are still separated by exactly r integrators. From (2.148) it is also clear that the stability of the zero dynamics cannot be altered by the FT. However, the dynamics of the η subsystem no longer represent the zero dynamics of the original system Σ_{OFBF}; they are now the zero dynamics of Σ_{OFBF} augmented by the filter. $\triangleleft$

2.5 Summary

The two global OFB design methods we described in this chapter are applicable to a rather restrictive class of nonlinear systems – those diffeomorphic to Σ_{OFBF}. However, these methods posses two important virtues. First, they are able to deal with systems whose nonlinearities are not restricted in any way in terms of growth rate or structure. Even more importantly, both methods are fully constructive, giving the designer a means of systematically obtaining an explicit expression for a globally asymptotically stabilizing dynamic controller. Much of the currently available nonlinear control theory offers only analytical tools or theoretical existence results that, although bearing significant importance in the advancement of the field, are often difficult to translate into constructive design methodologies. Appart from being early examples of constructive nonlinear control, these two methods are also valuable because they demonstrate the use of several general *techniques* – not just fixed designs – such as integrator backstepping, nonlinear damping and filtered transformations. The application of such techniques may be extended to other scenarios. In the next chapter we explore one such extension to a class of multivariable nonlinear systems.

Chapter 3

Extension of the MT Method to Multivariable Systems

This chapter is based largely on [71].

3.1 Introduction

The design of globally stabilizing output feedback (OFB) for nonlinear systems is an important control problem that is actively being studied. The objective is to develop a systematic design of a static or dynamic control expression dependent on known (measured or generated) signals only, such that in closed-loop, a dynamic system of interest is globally stabilized. It has been an equally important objective to broaden the class of systems to which such methods are applicable.

Important work in papers such as [22], [74], [38], [11], [39] [75] and the references therein has elaborated the central issues involved in designing output feedback controllers for nonlinear systems. Unlike for linear systems, fundamental obstacles such as the peaking phenomenon and the absence of a generic separation principle make nonlinear output feedback design a challenging problem.

Although our understanding has significantly progressed in the last few decades, there are still very few systematic, constructive solutions to the problem. Of the constructive solutions that do exist, the main approaches include a local result in [76], the semi-global approach originated in [28] and further developed in [31], [37], [34], and global approaches including observer backstepping [1], [6], and the adaptive OFB method presented in [2].

Even though from a practical point of view dynamic models generally do not emulate the behaviour of a physical system globally, global OFB stabilization is still of interest. Semi-global formulations, albeit generally less restrictive than global formulations, usually require that certain gains be “sufficiently high” in correspondence to a desired size of the region of attraction associated with a closed-loop equilibrium. Global formulations have no such requirements, but usually impose restrictive conditions identifying the class of systems to which they apply. For this reason, our interest is in extending the class of systems that are globally stabilizable by OFB using the method presented in [2].

The results in both [76] and [34] are directly applicable to MIMO systems; however, a MIMO generalization of the OFB scheme presented in [2] cannot be found in the literature. Our objective in this chapter is to provide such an extension. Our approach can be regarded as an alternative to the ones taken in [64], [61] and [60]. Using [2] as our starting point, our method inherits the key benefits of MT’s method – namely, we impose no growth restrictions on the model’s nonlinearities, and the dynamic order of our OFB is strictly less than that of the plant. The class of systems we consider is similar to the SISO output feedback form, but we allow dynamic coupling between the subsystems through the output-dependent nonlinearity. Our main contribution is the modification of the SISO algorithm presented in [2], so that it can accommodate the dynamic coupling between subsystems in the MIMO case. Specifically, the problem can be solved within the backstepping framework if additional nonlinear damping terms are included at every iteration to account for this coupling.

This chapter is organized as follows: in Section 3.2 we formulate the problem to be solved and provide the solution in Section 3.3. We illustrate the presented theory on a mathematical example in Section 3.4 and provide a summary in Section 3.6.

3.1.1 Notation

For most quantities, the first subscript identifies the subsystem number while the second identifies the index within that subsystem. If only one subscript is used then the reference is to the subsystem. For all virtual controls $\alpha_{i,j}$ the first subscript refers to the subsystem number and the second to the iteration number. Subscripts on Lyapunov functions indicate iteration number.

3.2 Problem Statement

In this chapter we solve the problem of global asymptotic stabilization by output feedback for the follwing class of MIMO nonlinear systems:

$$\begin{aligned} \dot{x} &= \mathrm{blkdiag}(A_{c_1}, A_{c_2}, \ldots, A_{c_m})x + \Psi(y) + \sum_{i=1}^{m} D_i u_i, \\ y &= \mathrm{blkdiag}(C_{c_1}, C_{c_2}, \ldots, C_{c_m})x \end{aligned} \tag{3.1}$$

where $x = (x_1^T, \ldots, x_m^T)^T \in \mathbb{R}^n$, $x_i \in \mathbb{R}^{k_i}$, $y = (y_1, \ldots, y_m)^T \in \mathbb{R}^m$, $u = (u_1, \ldots, u_m)^T \in \mathbb{R}^m$, and the matrices

$$A_{c_i} = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \end{bmatrix}_{(k_i \times k_i)}, \qquad C_{c_i} = [1, 0, \ldots, 0]_{(1 \times k_i)},$$

$$D_i^T = [\mathbf{0}_{1\times k_1} \vdots \cdots \vdots \mathbf{0}_{1\times k_{i-1}} \vdots d_i \vdots \mathbf{0}_{1\times k_{i+1}} \vdots \cdots \vdots \mathbf{0}_{1\times k_m}]$$

with

$$d_i = [0, 0, \ldots, 0, d_{i,\rho_i}, d_{i,\rho_i+1}, d_{i,\rho_i+2}, \ldots, d_{i,k_i}], \quad \text{for } 1 \le i \le m.$$

The nonlinearity $\Psi(y) = [\psi_1(y)^T, \ldots, \psi_m(y)^T]^T$, where $\psi_i(y) = [\psi_{i,1}(y), \ldots, \psi_{i,k_i}(y)]^T$. We assume that (3.1) has an equilibrium at the origin when $u_i \equiv 0$, $1 \le i \le m$ and that the functions $\psi_{i,j}(y)$ are C^∞. An important consequence of these two usual assumptions is that each $\psi_{i,j}(y)$ can be expressed as

$$\psi_{i,j}(y) = \sum_{k=1}^{m} \bar{\phi}_{i,j,k}(y) y_k, \tag{3.2}$$

with $\overline{\phi}_{i,j,k}(y)$ also C^{∞}. To see this, suppose $m = 2$ and we have a scalar function $\psi(y_1, y_2)$ which is smooth everywhere in both arguments and is such that $\psi(0,0) = 0$. We use a constructive argument similar to the one given for single-variable functions in Section 2.1 to demonstrate that (3.2) is always attainable. First, we define

$$\overline{\psi}_2(y_1, y_2) \triangleq \psi(y_1, y_2) - \psi(y_1, 0)$$

which implies that $\overline{\psi}_2(y_1, 0) = 0$ and can therefore be expressed as

$$\overline{\psi}_2(y_1, y_2) = \int_0^1 \frac{\partial \overline{\psi}_2(y_1, sy_2)}{\partial s} ds.$$

Then, by changing the variable of differentiation to $\zeta = sy_2$, we can write

$$\int_0^1 \frac{\partial \overline{\psi}_2(y_1, sy_2)}{\partial s} ds = y_2 \left(\int_0^1 \left(\frac{\partial \overline{\psi}_2(y_1, \zeta)}{\partial \zeta} \right) \Bigg|_{\zeta = sy_2} ds \right) \triangleq y_2 \phi_2(y_1, y_2).$$

Similarly, define

$$\overline{\psi}_1(y_1, y_2) = \psi(y_1, 0) - \psi(0, 0).$$

Since we assumed that $\psi(0,0) = 0$, we have that $\overline{\psi}_1(0, y_2) = 0$ and therefore by a similar argument, we can define

$$\phi_1(y_1, y_2) \triangleq \int_0^1 \left(\frac{\partial \overline{\psi}_1(\zeta, y_2)}{\partial \zeta} \right) \Bigg|_{\zeta = sy_1} ds$$

so that finally

$$\psi(y_1, y_2) = y_1 \phi_1(y_1, y_2) + y_2 \phi_2(y_1, y_2).$$

From this simplified argument, it is clear that the factorization (3.2) is possible in general, provided the $\psi_{i,j}(y)$ are at least continuously differentiable, and zero at zero.

We define the number ρ_i as the number of integrations separating y_i from its associated input u_i. This number does not necessarily correspond to the traditional definition of "relative degree" associated with the ith output, since differentiating the signal y_i fewer than ρ_i times may in fact result in the appearance of some $u_j,\ j \neq i$.

The k_i's can be regarded as the observability indices uniquely associated with the system. In that case, the differential geometric conditions characterizing the drift portion of (3.1) are the same as those characterizing the multi-output Nonlinear Observer Form (NOF) [68]. In the general NOF the input vectors are free to be functions of all the outputs whereas we restrict their structure to take the special form of the D_i. Each d_i is assumed to be a known *Hurwitz vector* in the sense that the roots of the polynomial

$$d_{i,\rho_i} s^{n-\rho_i} + \ldots + d_{i,k_i-1} s + d_{i,k_i}$$

have negative real parts. When $m = 1$ the system (3.1) is in the well-known OFBF, and the restriction on the structure of the input vectors implies that there exists state coordinates in which the zero, or tracking dynamics of (3.1) are linear and stable, as shown in Section 2.2.3.1.

The dynamic coupling between the subsystems is entirely due to the dependence of each nonlinearity $\psi_{i,j}$ on outputs associated with all other subsystems. This coupling makes the OFB design more difficult in the MIMO case. In the sequel, we demonstrate that the MIMO design is still possible, owing to the special structure of the input vectors D_i.

3.3 Main Result

Theorem 3.3.1. For any multivariate nonlinear system diffeomorphic to the form (3.1), there exists a globally asymptotically stabilizing control law dependent on known signals only, whose dynamic order does not exceed $\sum_{i=1}^{m}(\rho_i - 1)$.

For clarity, we provide some preliminary discussion before presenting the proof. Prior to constructing the control law it is necessary to carry out two transformations on each subsystem in (3.1).

STEP 1: Filter Transformation For each subsystem in (3.1), we apply the following Filter Transformation according to its number ρ_i. For the ith subsystem, let

$$z_i = x_i - \sum_{j=2}^{\rho_i} d_i[j]\xi_{i,j-1}, \tag{3.3}$$

where the constant vectors $d_i[j]$ are iteratively defined as:

$$\begin{aligned} d_i[\rho_i] &\triangleq d_i \\ d_i[j-1] &= A_{c_i} d_i[j] + \lambda_{i,j-1} d_i[j], \quad \rho_i \geq j \geq 2, \end{aligned} \tag{3.4}$$

and the $\xi_{i,j}$ are the states of the m filters with dynamics:

$$\dot{\xi}_i \triangleq \Lambda_i \xi_i + b_i u_i, \quad 1 \leq i \leq m, \tag{3.5}$$

where

$$\Lambda_i = \begin{bmatrix} -\lambda_{i,1} & 1 & \cdots & 0 \\ 0 & -\lambda_{i,2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & -\lambda_{i,\rho_i-1} \end{bmatrix} \tag{3.6}$$

and $b_i = [0, \ldots, 0, 1]^T$. We remark that if $\rho_i = 1$ for any $i \in \{1, \ldots, m\}$, then no FT is applied to this subsystem. For stability, the $\lambda_{i,j}$ must be chosen positive, and can be tuned to adjust the closed-loop system performance. In the z coordinates the extended system is:

$$\Sigma_{z_i} : \begin{cases} \dot{z}_i = A_{c_i} z_i + \psi_i(y) + d_i[1]\xi_{i,1} \\ \dot{\xi}_i = \Lambda_i \xi_i + b_i u_i \\ y_i = C_{c_i} z_i \end{cases} \tag{3.7}$$

for $1 \leq i \leq m$. It is important to note that the final vector $d_i[1]$ defined by (3.4) is Hurwitz, and takes the form

$$d_i[1] = [1, d_{i,2}[1], \ldots, d_{i,k_i}[1]]^T,$$

as discussed in Section 2.4.1.1.

STEP 2: Linear Transformation To each subsystem we apply the linear transformation

$$[\eta_i^T, y_i]^T = T_i z_i$$

where

$$T_i = \begin{bmatrix} -d_{i,2}[1] & 1 & 0 & \dots & 0 \\ -d_{i,3}[1] & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -d_{i,k_i}[1] & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \end{bmatrix}, \tag{3.8}$$

which results in the following dynamics:

$$\Sigma_{\eta_i,y_i} : \begin{cases} \dot{\eta}_i = \Gamma_i \eta_i + \gamma_i(y) \\ \dot{y}_i = \eta_{i,1} + d_{i,2}[1] y_i + \psi_{i,1}(y) + \xi_{i,1}, \\ \dot{\xi}_i = \Lambda_i \xi_i + b_i u_i \end{cases} \tag{3.9}$$

for all $1 \le i \le m$. Here, $\eta_i \in \mathbb{R}^{k_i - 1}$, $\xi_i \in \mathbb{R}^{\rho_i - 1}$, the matrix

$$\Gamma_i = \begin{bmatrix} -d_{i,2}[1] & 1 & 0 & \dots & 0 \\ -d_{i,3}[1] & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -d_{i,k_i-1}[1] & 0 & 0 & \dots & 1 \\ -d_{i,k_i}[1] & 0 & 0 & \dots & 0 \end{bmatrix},$$

and

$$\begin{bmatrix} \gamma_i(y) \\ \psi_{i,1}(y) \end{bmatrix} = T_i \psi_i(y) = T_i \begin{bmatrix} \sum_{k=1}^m y_k \bar{\phi}_{i,1,k} \\ \vdots \\ \sum_{k=1}^m y_k \bar{\phi}_{i,k_i,k} \end{bmatrix} \triangleq \sum_{k=1}^m \begin{bmatrix} \Phi_{i,k}(y) \\ \phi_{i,k_i,k}(y) \end{bmatrix} y_k \tag{3.10}$$

where we have used (3.2) to express $\psi_i(y)$, we have denoted

$$\Phi_{i,k}(y) \triangleq \begin{bmatrix} \phi_{i,1,k}(y) \\ \vdots \\ \phi_{i,k_i-1,k}(y) \end{bmatrix},$$

and the $\phi_{i,j,k}$ represent appropriate linear combinations of the $\bar{\phi}_{i,j,k}$.

The system in (3.9) has three structural features that enable the OFB design. Taking into consideration the fact that the $d_i[1]$ are Hurwitz vectors, we see that the linear component of the η_i dynamic is exponentially stable. The second important feature is that the filter state $\xi_{i,1}$ appears only in the $\dot{y}_i$ equation, which is due to the fact that $T_i d_i[1] = [0, \dots, 0, 1]^T$. Subsystems whose $\rho_i = 1$ do not require a filter transformation; in that case the linear transformation is applied directly and in (3.9) u_i appears instead of $\xi_{i,1}$ in the $\dot{y}_i$ equation. We observe that since the filter dynamic is in strict feedback form, the (y_i, ξ_i) subsystem of (3.9) is amenable to backstepping provided that a function α_1 (of known signals only) can be found, such that if $\xi_{i1} = \alpha_1$ the $\dot{y}_i$ subsystem is stabilized. The third structural feature of importance is the $\psi_i(y)$'s dependence on the outputs y alone; this dependence enables us to influence the η_i dynamic ensuring the stability of the overall interconnection of the η_i, y_i, and ξ_i subsystems.

It should be noted that even though the η_i dynamic is ISS with respect to $\psi_i(y)$, a backstepping design on the (y_i, ξ_i) subsystem alone cannot guarantee the stability of the composite system due to the presence of the unknown $\eta_{i,1}$ term in each $\dot{y}_i$ equation. This inconvenience can be overcome by applying (3.2) and the idea of *nonlinear damping*. In the following, we demonstrate the design.

STEP 3: Backstepping Algorithm Up to this point our development follows closely that in [2]. From here, the design demands several new considerations to be made in order to accommodate the coupled MIMO structure in (3.1).

Iteration 1: For the MIMO formulation, the presentation is clearest if we apply the backstepping procedure to each system in "parallel" – i.e. complete each iteration for every subsystem before the next iteration. We begin by proposing the positive definite, proper Lyapunov function candidate

$$V_1 = \sum_{i=1}^{m} (\eta_i^T Q_i \eta_i + \tfrac{1}{2} y_i^2) \tag{3.11}$$

where $Q_i = Q_i^T > 0$ is guaranteed to uniquely solve the Lyapunov equation

$$\Gamma_i^T Q_i + Q_i \Gamma_i = -\kappa_i I, \qquad 1 \leq i \leq m,$$

for some $\kappa_i > 0$, since each Γ_i is Hurwitz. The derivative of V_1 along the solutions of all (η_i, y_i) subsystems (3.9) is:

$$\begin{aligned} \dot{V}_1 = \sum_{i=1}^{m} \big[& -\kappa_i \eta_i^T \eta_i + 2\eta_i^T Q_i \gamma_i(y) \\ & + y_i(\eta_{i,1} + d_{i,2}[1] y_i + \psi_{i,1}(y) + \chi_{i,1}) \big] \end{aligned} \tag{3.12}$$

Here we must account for the possibility that one or more of the m subsystems has $\rho = 1$, in which case their inputs appear in the associated $\dot{y}$ equations. To that end, we have introduced the variable $\chi_{i,1}$ in (3.12), which we define in general for the rth iteration as:

$$\chi_{i,r} \triangleq \begin{cases} u_i, & i \in P_r \\ \xi_{i,r}, & i \in (M - P_r^*) \end{cases}, \tag{3.13}$$

where $P_r^* \triangleq (P_r \cup P_{r-1} \cup \cdots \cup P_1)$ and P_r is defined as the index subset at the rth iteration identifying precisely those subsystems with number ρ equal to that iteration number:

$$\begin{aligned} & P_r \subseteq M : (i \in P_r) \Leftrightarrow (\rho_i = r) \\ & M = \{1, 2, \ldots, m\} \end{aligned} \tag{3.14}$$

As the design iterations progress, some subsystems' designs will reach completion before others; the foregoing notation will help us most easily and succinctly express this "parallel" design process. For an example clarifying the usage of this notation, please see the forthcoming Remark 3.3.4.

In consideration of (3.10), we re-write (3.12) as

$$\begin{aligned} \dot{V}_1 = & \sum_{i=1}^{m} -\kappa_i \eta_i^T \eta_i + \sum_{i=1}^{m} \left(\sum_{k=1}^{m} 2\eta_i^T Q_i \Phi_{i,k}(y) y_k \right) \\ & + \sum_{i=1}^{m} y_i (\eta_{i,1} + d_{i,2}[1] y_i + \psi_{i,1}(y) + \chi_{i,1}), \end{aligned} \tag{3.15}$$

Re-naming the summation indices from i to k in the first and last terms and collecting terms in y_k, we then re-write (3.15) as

$$\begin{aligned} \dot{V}_1 = \sum_{k=1}^{m} \Bigg[& -\kappa_k \eta_k^T \eta_k + y_k \Bigg(\sum_{i=1}^{m} 2\eta_i^T Q_i \Phi_{i,k}(y) + \eta_{k,1} \\ & + d_{k,2}[1] y_k + \psi_{k,1}(y) + \chi_{k,1} \Bigg) \Bigg]. \end{aligned} \tag{3.16}$$

We now define a stabilizing function for the kth subsystem as $\alpha_{k,1} : \mathbb{R}^m \to \mathbb{R}$:

$$\begin{aligned} \alpha_{k,1}(y) = & -\psi_{k,1}(y) - d_{k,2}[1] y_k - c_{k,1} y_k - \delta_{k,1} y_k \\ & - \sum_{i=1}^{m} \beta_{k,i} (Q_i \Phi_{i,k})^T (Q_i \Phi_{i,k}) y_k, \end{aligned} \tag{3.17}$$

and remark that $\alpha(0) = 0$. To the subsystems with $\rho_k = 1,\ k \in P_1$, we assign $u_k = \alpha_{k,1}(y)$, while for the remaining subsystems we introduce the coordinate change $w_{k,1} = \xi_{k,1} - \alpha_{k,1}(y),\ k \in (M - P_1)$. Then $\dot{V}_1$ becomes:

$$\begin{aligned}\dot{V}_1 \leq \sum_{k=1}^{m} \Bigg[& -\Big(\kappa_k - \frac{1}{4\delta_{k,1}} - \sum_{i=1}^{m} \frac{1}{\beta_{i,k}}\Big)\eta_k^T\eta_k - c_{k,1}y_k^2 \\ & - \left(\frac{1}{4\delta_{k,1}}\|\eta_k\|^2 - |y_k|\|\eta_k\| + \delta_{k,1}y_k^2\right) - \sum_{i=1}^{m}\Big(\frac{1}{\beta_{k,i}}\eta_i^T\eta_i \\ & - 2\eta_i^T Q_i\Phi_{i,k}y_k + \big(\beta_{k,i}(Q_i\Phi_{i,k})^T(Q_i\Phi_{i,k})y_k^2\big)\Big)\Bigg] \\ & + \sum_{k\in(M-P_1)} y_k w_{k,1},\end{aligned} \tag{3.18}$$

where we have split off the negative definite term $\eta_k^T\eta_k$ and used the fact that

$$\eta_{k,1}^2 \leq \eta_k^T\eta_k = \|\eta_k\|^2, \qquad \forall \eta_k \in \mathbb{R}^{k_k-1},$$

and the vacuously true fact that

$$\sum_{k=1}^{m}\sum_{i=1}^{m}\frac{1}{\beta_{i,k}}\eta_k^T\eta_k = \sum_{k=1}^{m}\sum_{i=1}^{m}\frac{1}{\beta_{k,i}}\eta_i^T\eta_i.$$

In introducing the last two terms in (3.17) we applied the notion of nonlinear damping [1]. These terms now allow us to dominate the affine cross terms involving the unknown η_k in (3.18) by “completing the square”, and generating a negative definite quadratic term, instead of cancelling. The damping coefficients $\beta_{k,i} > 0$ and $\delta_{k,1} > 0$ can be adjusted to affect the performance of the closed-loop system.

One difference between the MIMO and the SISO formulation is the form of $\alpha_{k,1}$ in (3.17). Here we must include m damping terms $\beta_{k,i}(Q_i\Phi_{i,k})^T(Q_i\Phi_{i,k})y_k$ to account for the nonlinearities' dependence on all outputs. In effect, the control input to every subsystem must provide terms that help stabilize all other subsystems, due to the general dynamic coupling between them.

We proceed by re-writing (3.18) to obtain

$$\begin{aligned}\dot{V}_1 \leq \sum_{k=1}^{m}\Bigg[&-\big(\kappa_k - \frac{1}{4\delta_{k,1}} - \sum_{i=1}^{m}\frac{1}{\beta_{i,k}}\big)\eta_k^T\eta_k - c_{k,1}y_k^2 \\ &- \sum_{i=1}^{m}\beta_{k,i}\Big(\frac{1}{\beta_{k,i}}\eta_k - Q_i\gamma_i\Big)^T\Big(\frac{1}{\beta_{k,i}}\eta_k - Q_i\gamma_i\Big) \\ &- \delta_{k,1}\Big(\frac{\|\eta_k\|}{2\delta_{k,1}} - |y_k|\Big)^2\Bigg] + \sum_{k\in(M-P_1)} y_k w_{k,1}. \end{aligned} \tag{3.19}$$

For convenience, we define the negative definite term

$$\begin{aligned}\tau_1 \triangleq \sum_{k=1}^{m}\Bigg[&-\sum_{i=1}^{m}\beta_{k,i}\Big(\frac{1}{\beta_{k,i}}\eta_k - Q_i\gamma_i\Big)^T\Big(\frac{1}{\beta_{k,i}}\eta_k - Q_i\gamma_i\Big) \\ &- c_{k,1}y_k^2 - \delta_{k,1}\Big(\frac{\|\eta_k\|}{2\delta_{k,1}} - y_k\Big)^2\Bigg], \end{aligned} \tag{3.20}$$

so that

$$\begin{aligned}\dot{V}_1 = \tau_1 + \sum_{k=1}^{m}\Bigg[-\big(\kappa_k - \frac{1}{4\delta_{k,1}} - \sum_{i=1}^{m}\frac{1}{4\beta_{k,i}}\big)\eta_k^T\eta_k\Bigg] \\ + \sum_{k\in(M-P_1)} y_k w_{k,1}. \end{aligned} \tag{3.21}$$

Since (3.21) is rendered negative definite when $w_{k,1} \equiv 0$, our next concern is the regulation of $w_{k,1}$.

Iteration 2: We consider the new Lyapunov function candidate

$$V_2 = V_1 + \tfrac{1}{2}\sum_{k\in(M-P_1)} w_{k,1}^2,$$

whose time derivative along the solutions of the $(\eta_i, y_i, w_{q,1})$, $(1 \leq i \leq m)$, $q \in$

$(M-P_1)$ subsystem is

$$\dot{V}_2 = \tau_1 + \sum_{k=1}^{m}\left[-\left(\kappa_k - \frac{1}{4\delta_{k,1}} - \sum_{i=1}^{m}\frac{1}{\beta_{i,k}}\right)\eta_k^T\eta_k\right] + \sum_{k\in(M-P_1)} w_{k,1}[y_k + \dot{w}_{k,1}]$$
$$= \tau_1 + \sum_{k=1}^{m} -\left(\kappa_k - \frac{1}{4\delta_{k,1}} - \sum_{i=1}^{m}\frac{1}{\beta_{i,k}}\right)\eta_k^T\eta_k + \sum_{k\in(M-P_1)} w_{k,1}\Big[y_k + (-\lambda_{k,1}\xi_{k,1} + \chi_{k,2}) - \sum_{l=1}^{m}\frac{\partial\alpha_{k,1}}{\partial y_l}(\eta_{l,1} + d_{l,2}[1]y_l + \psi_{l,1}(y) + \chi_{l,1})\Big], \tag{3.22}$$

where $\chi_{k,2}$ is defined according to (3.13) with $r=2$. This time we design a stabilizing function $\alpha_{k,2}$ for $\chi_{k,2}$ as:

$$\alpha_{k,2}(y,\xi_{q,1}) = \lambda_{k,1}\xi_{k,1} - c_{k,2}w_{k,1} - \sum_{l=1}^{m}\left(\delta_{l,2}\left(\frac{\partial\alpha_{k,1}}{\partial y_l}\right)^2 w_{k,1}\right) + \sum_{l=1}^{m}\frac{\partial\alpha_{k,1}}{\partial y_l}(d_{l,2}[1]y_l + \psi_{l,1}(y) + \chi_{l,1}) - y_k, \tag{3.23}$$

for $q\in(M-P_1)$, and remark that it depends on known signals only. We assign this expression to all u_k, $k\in P_2$ and the new coordinate shift $w_{k,2} = \xi_{k,2} - \alpha_{k,2}$, $k\in(M-P_2^*)$ to obtain

$$\dot{V}_2 \le \tau_1 + \sum_{k=1}^{m} -\left(\kappa_k - \frac{1}{4\delta_{k,1}} - \sum_{i=1}^{m}\frac{1}{\beta_{k,i}} - \frac{|M-P_1|}{4\delta_{k,2}}\right)\eta_k^T\eta_k + \sum_{k\in(M-P_1)}\Big[-c_{k,2}w_{k,1}^2 - \sum_{l=1}^{m}\frac{\|\eta_l\|^2}{4\delta_{l,2}} - \left|w_{k,1}\sum_{l=1}^{m}\frac{\partial\alpha_{k,1}}{\partial y_l}\right|\|\eta_l\| - \sum_{l=1}^{m}\delta_{l,2}\left(\frac{\partial\alpha_{k,1}}{\partial y_l}\right)^2 w_{k,1}^2\Big] + \sum_{k\in(M-P_2^*)} w_{k,1}w_{k,2}, \tag{3.24}$$

where $|M-P_1|$ denotes the cardinality of the set $M-P_1$ (i.e. $|M-P_1|$ is the number of subsystems that have $\rho>1$, and have therefore not yet had an expression assigned to their associated input u). To obtain the term

$$\sum_{k\in(M-P_1)}\sum_{l=1}^{m}\frac{\|\eta_l\|^2}{4\delta_{l,2}}$$

in (4.21) (second line), we have added and subtracted $|M-P_1|\eta_k^T\eta_k/4\delta_{k2}$, and used the trivial fact that

$$\sum_{k=1}^{m}\frac{|M-P_1|}{4\delta_{k,2}}\eta_k^T\eta_k = \sum_{k=1}^{m}\sum_{l\in(M-P_1)}\frac{1}{4\delta_{k2}}\eta_k^T\eta_k$$
$$= \sum_{k\in(M-P_1)}\sum_{l=1}^{m}\frac{1}{4\delta_{l,2}}\eta_l^T\eta_l. \tag{3.25}$$

Completing the square we re-write (4.21):

$$\begin{aligned}\dot{V}_2 \leq \tau_1 &+ \sum_{k=1}^{m} -\big(\kappa_k - \frac{1}{4\delta_{k,1}} - \sum_{i=1}^{m}\frac{1}{\beta_{i,k}} - \frac{|M-P_1|}{4\delta_{k,2}}\big)\eta_k^T\eta_k \\ &+ \sum_{k\in(M-P_1)}\Big[-\sum_{l=1}^{m}\delta_{l,2}\Big(\frac{\|\eta_l\|}{2\delta_{l,2}} + \big|\frac{\partial\alpha_{k,1}}{\partial y_l}w_{k,1}\big|\Big)^2 \\ &- c_{k,2}w_{k,1}^2\Big] + \sum_{k\in(M-P_2^*)} w_{k,1}w_{k,2}.\end{aligned} \tag{3.26}$$

For convenience, we define the negative definite term τ_r, pertaining to the rth iteration, for all $r \geq 2$:

$$\begin{aligned}\tau_r \triangleq -\sum_{k\in(M-P_{r-1})}\Bigg(&c_{k,r}w_{k,r-1}^2 \\ &+ \sum_{l=1}^{m}\delta_{l,r}\Big(\frac{1}{2\delta_{l,r}}\eta_{l,1} + \frac{\partial\alpha_{k,r-1}}{\partial y_l}w_{k,r-1}\Big)^2\Bigg),\end{aligned} \tag{3.27}$$

and observe that

$$\begin{aligned}\dot{V}_2 \leq &\sum_{k=1}^{m} -\big(\kappa_k - \frac{1}{4\delta_{k,1}} - \sum_{i=1}^{m}\frac{1}{\beta_{i,k}} - \frac{|M-P_1|}{4\delta_{k,2}}\big)\eta_k^T\eta_k \\ &+ \tau_1 + \tau_2 + \sum_{k\in(M-P_2^*)} w_{k,1}w_{k,2}\end{aligned} \tag{3.28}$$

is negative definite when $w_{k,2} \equiv 0$. Again, the expression for $\alpha_{k,2}$ is different from the SISO case where $m = 1$. Additional damping terms were included in order to compensate for $\alpha_{k,1}$'s dependence on all output components.

From now on the algorithm proceeds in exactly the same way as it did starting at the second iteration; we introduce a new Lyapunov function candidate $V_3 = V_2 + \frac{1}{2}\sum_{k\in(M-P_2^*)} w_{k,2}^2$. Owing to the strict-feedback structure of the filter, the derivative of V_3 produces an affine $\xi_{k,3}$ for $k \in (M-P_3^*)$ and u_k for $k \in P_3$. We then force $\dot{V}_3$

to be negative definite by choosing $\alpha_{k,3}$ appropriately and assigning its expression to all $u_k,\ k \in P_3$ or $\xi_{k,3} = w_{k,3} + \alpha_{k,3},\ k \in (M - P_3^*)$.

It is important to note that in the second iteration the stabilizing function $\alpha_{k,2}(y, \xi_{q,1})$ has a dependence only on the outputs and the first filter states belonging to all subsystems $q \in (M - P_1)$ because of the second last term in (3.23). To clarify, we note that the appearance of the variable $\chi_{l,1}$ in that term implies that $\alpha_{k,2}$ also has dependence on the $u_k,\ k \in P_1$ which in turn depend only on y. Owing to $\alpha_{k,2}$'s filter state dependence, $\dot{V}_3$ necessarily contains terms of the form $\sum_{q\in(M-P_1)} \frac{\partial \alpha_{k,2}}{\partial \xi_{q,1}}(\lambda_{q,1}\xi_{q,1} + \chi_{q,2})$. Since these terms are comprised of known signals only, they can be cancelled directly by $\alpha_{k,3}$ which thus picks up a dependence on the second filter states of all subsystems $q \in (M - P_2^*)$, as well as $u_k,\ k \in P_2$ which have the same functional dependence as $\alpha_{k,2}$. We therefore observe that in the rth iteration $(r > 1)$, the stabilizing function associated with the kth subsystem in general has the following functional dependence:

$$\begin{aligned} \alpha_{k,r} = \alpha_{k,r}(y_i, \xi_{q,j}), \quad & k \in (M - P_{r-1}^*) \\ & 1 \le i \le m, \\ & q \in M = \{1, \ldots, m\}, \\ & 1 \le j \le \min\{(\rho_q - 1), (r-1)\}. \end{aligned} \tag{3.29}$$

We point out that (3.29) correctly indicates that for subsystems with $\rho_q = 1$, there are no filter states and therefore $\alpha_{k,r}$ is independent of any such $\xi_{q,j}$. Since $\alpha_{k,r}$ is assigned to $u_k,\ k \in P_r$, (3.29) implies that the control $u_k,\ k \in P_r$ must effectively cancel all previously assigned $u_k,\ k \in P_{r-1}^*$. This cancellation presents no difficulty, as these previously assigned control expressions depend exclusively on known signals, and never on one another or yet-unassigned control variables. Therefore, in the MIMO case with output-dependent coupling it is still possible to find explicit expressions for the $u_k,\ k \in M$ without having to solve a system of equations[1] in the u_k. This fact is a consequence of our choice of input vectors in (3.1) as well as the strict-feedback structure of the filters.

[1] See for example the ad hoc MIMO OFB design for an induction motor presented in Section 7.2 in [1].

The Backstepping Algorithm continues for

$$N = \max_{k \in M}(\rho_k)$$

iterations, after which the set $(M - P_N^*)$ is empty since the kth subsystem's filter has exactly $\rho_k - 1$ states, and exactly ρ_k iterations are required to complete its design. We now show that it is possible to choose expressions for inputs and stabilizing functions at every such iteration, regardless of the system dimension n or the number ρ_k associated with each subsystem.

Lemma 3.3.1. Define the set of functions $V_r = V_{r-1} + \frac{1}{2}\sum_{k \in (M-P_{r-1}^*)} w_{k,r-1}^2$ for $2 \le r \le N$, with V_1 given in (3.11). Then, for every iteration r, $2 \le r \le N$ of the Backstepping Algorithm there exist stabilizing functions $\alpha_{k,r},\ k \in (M - P_{r-1}^*)$ rendering

$$\begin{aligned}\dot{V}_r \le \sum_{q=1}^{r} \tau_q + \sum_{k=1}^{m} -\Big(\kappa_k - \frac{1}{4\delta_{k1}} - \sum_{i=1}^{m} \frac{1}{\beta_{i,k}} - \sum_{j=1}^{r-1} \frac{|M - P_j^*|}{4\delta_{k,j+1}}\Big)\eta_k^T \eta_k \\ + \sum_{k \in (M-P_r^*)} w_{k,r-1} w_{k,r},\end{aligned} \tag{3.30}$$

once the coordinate change $w_{k,r} = \xi_{k,r} - \alpha_{k,r},\ k \in (M - P_r^*)$ is defined or $u_k = \alpha_{k,r},\ k \in P_r$ is assigned.

Proof. We begin by considering the rth iteration.

Iteration r: Suppose there exists a stabilizing function $\alpha_{k,r-1}$ and an associated coordinate change $w_{k,r-1} = \xi_{k,r-1} - \alpha_{k,r-1}$ such that the expression for $\dot{V}_{r-1}$ has the following final form, identical to (3.28) when $r = 3$:

$$\begin{aligned}\dot{V}_{r-1} \le \sum_{k=1}^{m} -\Big(\kappa_k - \frac{1}{4\delta_{k,1}} - \sum_{i=1}^{m} \frac{1}{\beta_{i,k}} - \sum_{j=1}^{r-2} \frac{|M - P_j^*|}{4\delta_{k,j+1}}\Big)\eta_k^T \eta_k \\ + \sum_{q=1}^{r-1} \tau_q + \sum_{k \in (M-P_{r-1}^*)} w_{k,r-2} w_{k,r-1}\end{aligned} \tag{3.31}$$

where τ_1 is defined in (3.20) and τ_q as in (3.27). We proceed as usual and propose the subsequent Lyapunov function candidate

$$V_r = V_{r-1} + \tfrac{1}{2} \sum_{k \in (M-P_{r-1}^*)} w_{k,r-1}^2$$

whose time derivative is

$$\begin{aligned}
\dot{V}_r &= \sum_{q=1}^{r-1}\tau_q + \sum_{k=1}^{m} -\Big(\kappa_k - \frac{1}{4\delta_{k,1}} - \sum_{i=1}^{m}\frac{1}{\beta_{i,k}} - \sum_{j=1}^{r-2}\frac{|M-P_j^*|}{4\delta_{k,j+1}}\Big)\eta_k^T\eta_k \\
&\quad + \sum_{k\in(M-P_{r-1}^*)} w_{k,r-1}[w_{k,r-2}+\dot{w}_{k,r-1}] \\
&= \sum_{q=1}^{r-1}\tau_q + \sum_{k=1}^{m} -\Big(\kappa_k - \frac{1}{4\delta_{k1}} - \sum_{i=1}^{m}\frac{1}{\beta_{i,k}} - \sum_{j=1}^{r-2}\frac{|M-P_j^*|}{4\delta_{k,j+1}}\Big)\eta_k^T\eta_k \\
&\quad + \sum_{k\in(M-P_{r-1}^*)} \Bigg(w_{k,r-1}\Big[w_{k,r-2} + \Big(-\lambda_{k,r-1}\xi_{k,r-1} + \chi_{k,r}\Big) \\
&\quad - \Big(\sum_{l=1}^{m}\frac{\partial\alpha_{k,r-1}}{\partial y_l}(\eta_{l1} + d_{l2}[1]y_l + \psi_{l1}(y) + \chi_{l1}) + S_r\Big)\Big]\Bigg), \qquad (3.32)
\end{aligned}$$

where $\chi_{k,r}$ is defined as in (3.13), and we have shortened the notation further by introducing the term S_r which we define as

$$S_r \triangleq \sum_{j=1}^{min\{(\rho_q-1),(r-2)\}}\Big(\sum_{q=1}^{m}\frac{\partial\alpha_{k,r-1}}{\partial\xi_{q,j}}\dot{\xi}_{q,j}\Big). \qquad (3.33)$$

S_r represents the time derivative of $\alpha_{k,r-1}$ along the trajectories of all relevant filter states, given in (3.29). This term is much simpler in the SISO case. Here, (3.33) shows how the stabilizing functions themselves couple the dynamics of the closed-loop system in the MIMO case.

We observe that just as before, we are in a position to choose a stabilizing function $\alpha_{k,r}$ for $\chi_{k,r}$ in (3.32) as

$$\begin{aligned}
\alpha_{k,r} &= \lambda_{k,r-1}\xi_{k,r-1} - w_{k,r-2} - c_{k,r}w_{k,r-1} \\
&\quad + \sum_{l=1}^{m}\frac{\partial\alpha_{k,r-1}}{\partial y_l}(d_{l2}[1]y_l + \psi_{l,1}(y) + \chi_{l,1}) \\
&\quad - \sum_{l=1}^{m}\delta_{l,r}\Big(\frac{\partial\alpha_{k,r-1}}{\partial y_l}\Big)^2 w_{k,r} + S_r, \qquad (3.34)
\end{aligned}$$

so that it depends on known signals only. Again, we assign $u_k = \alpha_{k,r}$ for $k \in P_r$, while for $k \in (M-P_r^*)$ we define a new coordinate shift $w_{k,r} = \xi_{k,r} - \alpha_{k,r}$. We therefore obtain:

$$
\begin{aligned}
\dot{V}_r \leq & \sum_{q=1}^{r-1}\tau_q+\sum_{k=1}^{m}-\big(\kappa_k-\frac{1}{4\delta_{k,1}}-\sum_{i=1}^{m}\frac{1}{\beta_{i,k}}-\sum_{j=1}^{r-1}\frac{|M-P_j^*|}{4\delta_{k,j+1}}\big)\eta_k^T\eta_k \\
& -\sum_{k\in(M-P_{r-1}^*)}\sum_{l=1}^{m}\frac{\|\eta_l\|^2}{4\delta_{l,r}}+\sum_{k\in(M-P_{r-1}^*)}\Bigg(-c_{k,r}w_{k,r-1}^2 \\
& -\sum_{l=1}^{m}\Big|w_{k,r-1}\frac{\partial\alpha_{k,r-1}}{\partial y_l}\Big|\|\eta_l\|-\sum_{l=1}^{m}\delta_{l,r}\Big(\frac{\partial\alpha_{k,r-1}}{\partial y_l}\Big)^2w_{k,r}^2\Bigg) \\
& \sum_{k\in(M-P_r^*)}w_{k,r-1}w_{k,r}.
\end{aligned}
\tag{3.35}
$$

Applying completion of squares we write out the expressions in more detail and obtain:

$$
\begin{aligned}
\dot{V}_r \leq & \sum_{q=1}^{r-1}\tau_q+\sum_{k=1}^{m}-\big(\kappa_k-\frac{1}{4\delta_{k1}}-\sum_{i=1}^{m}\frac{1}{\beta_{i,k}}-\sum_{j=1}^{r-1}\frac{|M-P_j^*|}{4\delta_{k,j+1}}\big)\eta_k^T\eta_k \\
& +\sum_{k\in(M-P_{r-1}^*)}\Bigg(-c_{k,r}w_{k,r-1}^2-\sum_{l=1}^{m}\delta_{lr}\Big[\frac{\|\eta_l\|^2}{4\delta_{lr}^2} \\
& -\Big|w_{k,r-1}\frac{\partial\alpha_{k,r-1}}{\partial y_l}\Big|\frac{\|\eta_l\|}{\delta_{lr}}-\Big(\frac{\partial\alpha_{k,r-1}}{\partial y_l}w_{k,r}\Big)^2\Big]\Bigg) \\
\leq & \sum_{q=1}^{r-1}\tau_q+\sum_{k=1}^{m}-\big(\kappa_k-\frac{1}{4\delta_{k1}}-\sum_{i=1}^{m}\frac{1}{\beta_{i,k}}-\sum_{j=1}^{r-1}\frac{|M-P_j^*|}{4\delta_{k,j+1}}\big)\eta_k^T\eta_k \\
& +\sum_{k\in(M-P_r^*)}w_{k,r-1}w_{k,r}+\sum_{k\in(M-P_{r-1}^*)}\Bigg(-c_{k,r}w_{k,r-1}^2 \\
& -\sum_{l=1}^{m}\delta_{lr}\Big(\frac{\|\eta_l\|}{2\delta_{lr}}+\Big|\frac{\partial\alpha_{k,r-1}}{\partial y_l}w_{k,r}\Big|\Big)^2\Bigg).
\end{aligned}
\tag{3.36}
$$

Defining τ_r as in (3.27), we can write

$$
\begin{aligned}
\dot{V}_r \leq & \sum_{k=1}^{m}-\big(\kappa_k-\frac{1}{4\delta_{k1}}-\sum_{i=1}^{m}\frac{1}{\beta_{i,k}}-\sum_{j=1}^{r-1}\frac{|M-P_j^*|}{4\delta_{k,j+1}}\big)\eta_k^T\eta_k \\
& +\sum_{q=1}^{r}\tau_q+\sum_{k\in(M-P_r^*)}w_{k,r-1}w_{k,r},
\end{aligned}
\tag{3.37}
$$

which has the desired form (3.30).

Taking (3.28) as the base step and (3.31) as our hypothesis, the Lemma is proven by (3.37) and the principle of induction. ■

The Backstepping Algorithm terminates when we reach the Nth iteration at which the index set $(M - P_N^*)$ is empty. In that case the last term in (3.37) is zero, and we are left with a quadratic negative definite Lyapunov function derivative along the trajectories of all closed-loop system states. We now ready to state the proof for Theorem 1:

Proof. (of Theorem 1) From *Lemma 1* we conclude the existence of stabilizing functions $\alpha_{k,r}$ as in (3.29). Therefore within N iterations of the Backstepping Algorithm it is possible to assign expressions to all $u_k,\ 1 \leq k \leq m$ as:

$$u_k = \alpha_{k,\rho_k}(y, \xi_{q,1}, \ldots, \xi_{q,\min\{\rho_k - 1, \rho_q - 1\}}) \tag{3.38}$$

for all $1 \leq k \leq m$ and $1 \leq q \leq m$. A byproduct of the design is the iterative construction of a quadratic Lyapunov function

$$V = \sum_{k=1}^{m} (\eta_k^T P_k \eta_k + \tfrac{1}{2} y_k^2) + \tfrac{1}{2} \sum_{r=1}^{N} \left(\sum_{k \in (M - P_r^*)} w_{k,r}^2 \right)$$

whose time derivative is

$$\begin{aligned} \dot{V}_N \leq & \sum_{k=1}^{m} -\big(\kappa_k - \frac{1}{4\delta_{k1}} - \sum_{i=1}^{m} \frac{1}{\beta_{i,k}} - \sum_{j=1}^{N-1} \frac{|M - P_j^*|}{4\delta_{k,j+1}}\big) \eta_k^T \eta_k \\ & + \sum_{q=1}^{N} \tau_q. \end{aligned} \tag{3.39}$$

Taking into consideration our definition for τ_r in (3.27) and τ_1 in (3.20), we neglect the negative definite "cross terms" in these definitions and therefore express (3.39) as:

$$\begin{aligned} \dot{V}_N \leq & \sum_{k=1}^{m} -\Bigg[\big(\kappa_k - \frac{1}{4\delta_{k,1}} - \sum_{i=1}^{m} \frac{1}{\beta_{i,k}} - \sum_{j=1}^{N-1} \frac{|M - P_j^*|}{4\delta_{k,j+1}}\big) \eta_k^T \eta_k \\ & - c_{k,1} y_k^2 + \sum_{q=2}^{\rho_k} -c_{k,q} w_{k,q-1}^2 \Bigg] \\ \leq & \sum_{k=1}^{m} \Big(-\underline{\kappa}_k \eta_k^T \eta_k - c_{k,1} y_k^2 + \sum_{q=2}^{\rho_k} -c_{k,q} w_{k,q-1}^2 \Big), \end{aligned} \tag{3.40}$$

where

$$\underline{\kappa}_k \triangleq \kappa_k - \frac{1}{4\delta_{k,1}} - \sum_{i=1}^{m} \frac{1}{\beta_{i,k}} - \sum_{j=1}^{N-1} \frac{|M - P_j^*|}{4\delta_{k,j+1}}. \tag{3.41}$$

We stipulate that the coefficients $c_{k,j},\ 1 \leq j \leq \rho_k$ be chosen positive, and the coefficients λ, β, δ be chosen such that $\underline{\kappa}_k > 0,\ 1 \leq k \leq m$. In that case the origin of the closed-loop system in (η, y, w) coordinates is globally exponentially stable. In the original coordinates (3.1), the system is at least globally asymptotically stable. ■

Remark 3.3.1. It is always possible to choose the coefficients δ and β so that $\underline{\kappa}_k > 0$ since they are independent of λ. The magnitude of κ_k can be influenced by tuning the filter's eigenvalues λ, which affect Q_i in (3.11). ◃

Remark 3.3.2. Unlike in [2], here we have included the damping coefficients δ and β in order to point out the possible performance tradeoffs in this design. For pedagogical purposes, we have attempted to cast our presentation in a form that more closely adheres to the familiar backstepping framework presented in [1]. ◃

Remark 3.3.3. We note that the dynamic order of the OFB law is equal to the number of filter states associated with all the subsystems – i.e. $\sum_{k=1}^{m}(\rho_k - 1)$. It is interesting to note that in the best case it is possible to have a globally asymptotically stabilizing OFB law which is static, when $\rho_k = 1,\ 1 \leq k \leq m$. ◃

Remark 3.3.4. The notation and indexing in this chapter are admittedly difficult. In an attempt to clarify the usage and efficacy of this notation, we consider an example. Suppose a MIMO nonlinear system (3.1), with $m = 4$ inputs and outputs, has been transformed into subsystems of the form

$$\begin{array}{lll}
\dot{\eta}_1 = \Gamma_1\eta_1 + \gamma_1(y), & \dot{\eta}_2 = \Gamma_2\eta_2 + \gamma_2(y), & \dot{\eta}_3 = \Gamma_3\eta_3 + \gamma_3(y) \\
\dot{y}_1 = \eta_{11} + d_{12}[1]y_1 & \dot{y}_2 = \eta_{21} + d_{22}[1]y_2 & \dot{y}_3 = \eta_{31} + d_{32}[1]y_3 \\
\qquad +\psi_{11}(y) + \xi_{11} & \qquad +\psi_{21}(y) + \xi_{21} & \qquad +\psi_{31}(y) + \xi_{31} \\
\dot{\xi}_{11} = -\lambda_{11}\xi_{11} + \xi_{12}, & \dot{\xi}_{21} = -\lambda_{21}\xi_{21} + \xi_{22}, & \dot{\xi}_{31} = -\lambda_{31}\xi_{31} + \xi_{32} \\
\dot{\xi}_{12} = -\lambda_{12}\xi_{12} + u_1, & \dot{\xi}_{22} = -\lambda_{22}\xi_{22} + \xi_{23}, & \dot{\xi}_{32} = -\lambda_{32}\xi_{32} + u_3 \\
 & \dot{\xi}_{23} = -\lambda_{23}\xi_{23} + u_2 &
\end{array}$$

and

$$\dot{\eta}_4 = \Gamma_4\eta_4 + \gamma_4(y)$$

$$\dot{y}_4 = \eta_{41} + d_{42}[1]y_4 + \psi_{41}(y) + u_4$$

so that we have

$$\rho_1 = 3, \qquad \rho_2 = 4, \qquad \rho_3 = 3, \qquad \rho_4 = 1$$

The index subsets are

$$\begin{array}{ll} P_1 = \{4\}, & P_1^* = \{4\} \\ P_2 = \{\emptyset\}, & P_2^* = \{4\} \\ P_3 = \{1,3\}, & P_3^* = \{1,3,4\} \\ P_4 = \{2\}, & P_4^* = \{1,2,3,4\} \equiv M \end{array}.$$

For this system we require four iterations, and the stabilizing function associated with the kth subsystem has the following variable dependencies:

$$\begin{array}{ll} \alpha_{k,1} = \alpha_{k,1}(y_1,\ldots,y_4), & \forall k \in M \\ \alpha_{k,2} = \alpha_{k,2}(y_1,\ldots,y_4,\xi_{11},\xi_{21},\xi_{31}), & \forall k \in (M - P_1^*) \\ \alpha_{k,3} = \alpha_{k,3}(y_1,\ldots,y_4,\xi_{11},\xi_{21},\xi_{31},\xi_{12},\xi_{22},\xi_{32}), & \forall k \in (M - P_2^*) \\ \alpha_{k,4} = \alpha_{k,4}(y_1,\ldots,y_4,\xi_{11},\xi_{21},\xi_{31},\xi_{12},\xi_{22},\xi_{32},\xi_{23}), & \forall k \in (M - P_3^*) \end{array},$$

and we assign

$$u_1 = \alpha_{13}(\cdot)$$

$$u_2 = \alpha_{24}(\cdot)$$

$$u_3 = \alpha_{33}(\cdot)$$

$$u_4 = \alpha_{41}(\cdot)$$

◁

3.4 Mathematical Example

We illustrate the application of the theory presented in this chapter using the following mathematical example.

Example 3.4.1 (Multivariable OFB)**.** Consider the following system in the form (3.1), with $m = 2$ subsystems, the numbers $(\rho_1, \rho_2) = (3,1)$, and $(k_1, k_2) = (3,1)$:

$$\begin{bmatrix} \dot{x}_{11} \\ \dot{x}_{12} \\ \dot{x}_{13} \\ \dot{x}_{21} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{12} \\ x_{13} \\ x_{21} \end{bmatrix} + \underbrace{\begin{bmatrix} \phi_{1,1,1}(y) \\ \phi_{1,2,1}(y) \\ \phi_{1,3,1}(y) \\ \phi_{2,1,1}(y) \end{bmatrix} y_1 + \begin{bmatrix} \phi_{1,1,2}(y) \\ \phi_{1,2,2}(y) \\ \phi_{1,3,2}(y) \\ \phi_{2,1,2}(y) \end{bmatrix} y_2}_{=\Psi(y)=[\psi_{11}(y),\psi_{12}(y),\psi_{13}(y),\psi_{21}(y)]^T} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_{11} \\ x_{21} \end{bmatrix}. \tag{3.42}$$

Since the second subsystem has $\rho_2 = 1$, we do not need to construct a filter for it. However, the first subsystem requires a second-order filter

$$\begin{bmatrix}\dot{\xi}_{11}\\ \dot{\xi}_{12}\end{bmatrix} = \begin{bmatrix}-\lambda_{11} & 1\\ 0 & -\lambda_{12}\end{bmatrix}\begin{bmatrix}\xi_{11}\\ \xi_{12}\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}u_1 \tag{3.43}$$

and the filtered transformation

$$z_1 = x_1 - d_1[2]\xi_{11} - d_1[3]\xi_{12} \tag{3.44}$$

with

$$d_1[3] = \begin{bmatrix}0\\0\\1\end{bmatrix}, \qquad d_1[2] = \begin{bmatrix}0\\1\\ \lambda_{12}\end{bmatrix}, \qquad d_1[1] = \begin{bmatrix}1\\ \lambda_{11}+\lambda_{12}\\ \lambda_{11}\lambda_{12}\end{bmatrix},$$

designed according to (3.3) and (3.4). It is apparent that $d_1[1]$ is a Hurwitz vector, since

$$s^2 + (\lambda_{11}+\lambda_{12})s + (\lambda_{11}\lambda_{12}) = (s+\lambda_{11})(s+\lambda_{12})$$

and $\lambda_{1,j}$ $j = 1,2$ are assumed to be chosen positive. With some algebra, we can work out that the (x_1, ξ_1) dynamic in the (z_1, ξ_1) coordinates is represented by

$$\begin{aligned}\dot{z}_1 &= A_{c_1}z_1 + \psi_1(y) + d_1[1]\xi_{11}\\ \dot{\xi}_{11} &= -\lambda_{11}\xi_{11} + \xi_{12}\\ \dot{\xi}_{12} &= -\lambda_{12}\xi_{12} + u_1,\end{aligned}$$

and that applying the linear transformation

$$T_1 = \begin{bmatrix}-d_{12}[1] & 1 & 0\\ -d_{13}[1] & 0 & 1\\ 1 & 0 & 0\end{bmatrix} = \begin{bmatrix}-(\lambda_{11}+\lambda_{12}) & 1 & 0\\ -(\lambda_{11}\lambda_{12}) & 0 & 1\\ 1 & 0 & 0\end{bmatrix} \tag{3.45}$$

to the z_1 subsystem allows us to write the entire system dynamic as

$$\begin{aligned}\dot{\eta}_1 &= \Gamma\eta_1 + \gamma_1(y)y_1 + \gamma_1(y)y_2\\ \dot{y}_1 &= \eta_{11} + d_{12}[1]y_1 + \psi_{11}(y) + \xi_{11}\end{aligned} \tag{3.46a}$$

$$\begin{aligned}\dot{\xi}_{11} &= -\lambda_{11}\xi_{11} + \xi_{12}\\ \dot{\xi}_{12} &= -\lambda_{12}\xi_{12} + u_1\end{aligned} \tag{3.46b}$$

$$\dot{y}_2 = \psi_{21}(y) + u_2 \tag{3.46c}$$

where we denote

$$\eta_1 = \begin{bmatrix} \eta_{11} \\ \eta_{12} \end{bmatrix}, \qquad \Gamma = \begin{bmatrix} -(\lambda_{11}+\lambda_{12}) & 1 \\ -(\lambda_{11}\lambda_{12}) & 0 \end{bmatrix},$$

and

$$\gamma_1(y) = \begin{bmatrix} \phi_{1,2,1}(y) - d_{12}[1]\phi_{1,1,1}(y) + d_{13}[1] - d_{12}[1]^2 \\ \phi_{1,3,1}(y) - d_{13}[1]\phi_{1,1,1}(y) - d_{13}[1]d_{12}[1] \end{bmatrix},$$
$$\gamma_2(y) = \begin{bmatrix} \phi_{1,2,2}(y) - d_{12}[1]\phi_{1,1,2}(y) \\ \phi_{1,3,2}(y) - d_{13}[1]\phi_{1,1,2}(y) \end{bmatrix}.$$

From (3.46), it is tempting to examine (3.46c) and immediately select

$$u_2 = -\psi_{21}(y) - cy_2 \tag{3.47}$$

for some $c > 0$. However, such a choice cannot guarantee that the subsystem (3.46a)-(3.46b) can be stabilized using u_1. We demonstrate by carrying out our “parallel design” for the first two of the $N = 3$ iterations of backstepping that are required to complete the design.

Iteration 1: As usual, let

$$V_1 = \eta_1^T Q\eta_1 + \tfrac{1}{2}y_1^2 + \tfrac{1}{2}y_2^2$$

where $Q = Q^T > 0$ uniquely solves[2] the Lyapunov equation

$$\Gamma^T Q + Q\Gamma = -\kappa I, \qquad \kappa \in \mathbb{R}^+.$$

Then,

$$\begin{aligned} \dot{V}_1 = -\kappa\|\eta_1\|^2 &+ 2\eta_1^T Q\gamma_1(y)y_1 + 2\eta_1^T Q\gamma_2(y)y_2 \\ &+ y_1\left[\eta_{11} + d_{12}[1]y_1 + \psi_{11}(y) + \xi_{11}\right] + y_2\left[\psi_{21}(y) + u_2\right] \end{aligned} \tag{3.48}$$

From this expression it becomes clear how and why the design for the SISO case must be altered if general dynamic coupling between the subsystems is allowed via

[2] In fact, we have

$$Q = \begin{bmatrix} q_1 & q_2 \\ q_2 & q_3 \end{bmatrix},$$

with $\quad q_1 = \frac{\kappa(1+d_{13}[1])}{2d_{12}[1]}, \qquad q_2 = \frac{-\kappa}{2} \qquad$ and $\qquad q_3 = \frac{\kappa}{2}\left(\frac{1+d_{13}[1]}{d_{12}[1]d_{13}[1]} + \frac{d_{12}[1]}{d_{13}[1]}\right).$

the outputs. If it were possible to factor the nonlinearity $\psi_1(y)$ as

$$\psi_1(y) = \begin{bmatrix} \psi_{11}(y) \\ \psi_{12}(y) \\ \psi_{13}(y) \end{bmatrix} = \begin{bmatrix} \phi_{11}(y) \\ \phi_{12}(y) \\ \phi_{13}(y) \end{bmatrix} y_1$$

for some smooth functions $\phi_{1,j}(y),\ j = 1,2,3$, then we could indeed choose u_2 as in (3.47). However, in general we cannot expect such a factorization to be possible, as we have assumed that each $\psi_{i,j}(y),\ 1 \leq i \leq m,\ 1 \leq j \leq k_i$ is allowed to have a general dependency on all system outputs $y_1, \ldots, y_m$. One situation in which such a factorization is impossible is when $\psi_1 = \psi_1(y_2)$. In that case, we can still solve the problem according to our method by introducing a damping term inside u_2, designed to dominate the cross-term between y_2 and the unknown η_1. Specifically, if we choose

$$\begin{aligned} \alpha_{11}(y) &= -\psi_{11}(y) - d_{12}[1]y_1 - \delta_{11}y_1 - c_{11}y_1 - \beta_{11}\big(Q\gamma_1(y)\big)^T\big(Q\gamma_1(y)\big)y_1 \\ \alpha_{21}(y) &= -\psi_{21}(y) - c_{21}y_2 - \beta_{21}\big(Q\gamma_2\big)^T\big(Q\gamma_2\big)y_2, \end{aligned} \tag{3.49}$$

define the error variable $w_{11} \triangleq \xi_{11} - \alpha_{11}(y)$, and assign $u_2 = \alpha_{21}(y)$, then $\dot{V}_1$ becomes:

$$\begin{aligned} \dot{V}_1 &\leq -\big(\kappa - \tfrac{1}{4\delta_{11}} - \tfrac{1}{\beta_{11}} - \tfrac{1}{\beta_{21}}\big)\|\eta_1\|^2 - c_{11}y_1^2 - c_{21}y_2^2 - \delta_{11}\left(\frac{\|\eta_1\|}{2\delta_{11}} - |y_1|\right)^2 \\ &\quad - \beta_{11}\big(\tfrac{1}{\beta_{11}}\eta_1 - Q\gamma_1(y)y_1\big)^T\big(\tfrac{1}{\beta_{11}}\eta_1 - Q\gamma_1(y)y_1\big) + y_1 w_{11} \\ &\triangleq -\big(\kappa - \tfrac{1}{4\delta_{11}} - \tfrac{1}{\beta_{11}} - \tfrac{1}{\beta_{21}}\big)\|\eta_1\|^2 + \tau_1 + y_1 w_{11} \end{aligned}$$

where τ_1 is clearly negative definite.

Iteration 2: Again, proceeding as usual we introduce the new candidate Lyapunov function $V_2 = V_1 + \frac{1}{2}w_{11}^2$ whose time derivative is

$$\dot{V}_2 \leq -\big(\kappa - \tfrac{1}{4\delta_{11}} - \tfrac{1}{\beta_{11}} - \tfrac{1}{\beta_{21}}\big)\|\eta_1\|^2 + \tau_1 + w_{11}\big[y_1 + \dot{\xi}_{11} - \tfrac{\partial\alpha_{11}(y)}{\partial y_1}\dot{y}_1 - \tfrac{\partial\alpha_{11}(y)}{\partial y_2}\dot{y}_2\big] \tag{3.50}$$

$$\begin{aligned} &= -\big(\kappa - \tfrac{1}{4\delta_{11}} - \tfrac{1}{\beta_{11}} - \tfrac{1}{\beta_{21}}\big)\|\eta_1\|^2 + \tau_1 + w_{11}\big[y_1 - \lambda_{11}\xi_{11} + \xi_{12} \\ &\quad - \tfrac{\partial\alpha_{11}(y)}{\partial y_1}\big(\eta_{11} + d_{12}[1]y_1 + \psi_{11}(y) + \xi_{11}\big) - \tfrac{\partial\alpha_{11}(y)}{\partial y_2}\big(\psi_{21}(y) + \alpha_{21}(y)\big)\big] \end{aligned} \tag{3.51}$$

from which it is clear that we select ξ_{12} as the next virtual control. The design for the second subsystem is complete within one iteration, and the stabilizing function $\alpha_{12}(y,\xi_{11})$ must cancel all known terms including $\frac{\partial\alpha_{11}(y)}{\partial y_2}(\psi_{21}(y)+\alpha_{21}(y))$ and damp the unknown term $\frac{\partial\alpha_{11}(y)}{\partial y_1}\eta_{11}$. We mention this because we mean to illustrate the fact that controls associated with subsystems having a higher number ρ ultimately must cancel (i.e. pick up a dependency) on control expressions designed in previous iterations. Such cancellation presents no difficulty as these previously designed controls in turn depend on known signals only. Thus, our sequential design is made possible because of our restriction on the input vectors D_i in (3.1); specifically, in this example suppose we allowed the input matrix to take the form

$$[D_1,D_2] = \left[\begin{array}{c:c} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ \hdashline a & 1 \end{array}\right]$$

for some $a \neq 0$. Then, within the first iteration the control u_1 appears prematurely in (3.48) and must be cancelled by u_2. However, u_1 has not yet been assigned at this iteration, and hence $u_2 = \alpha_{21}(y,u_1)$. In the next iteration the stabilizing function $\alpha_{12} = \alpha_{12}(y,\xi_{11},u_1)$ would cancel the term $\frac{\partial\alpha_{11}(y)}{\partial y_2}(\psi_{21}(y)+\alpha_{21}(y,u_1))$ so that eventually the control u_1 must be selected to depend on terms containing $\dot{u}_1$ and $\ddot{u}_1$. In other words we would have to solve an ODE in u_1 in order to find the complete OFB law. For this reason, we restrict all off-diagonal blocks of the input matrix $[D_1,\ldots,D_m]$ to be zero[3].

From (3.50) it is easy to discern the form of α_{12} and the final iteration proceeds in a similar fashion. The complete control law for this system is:

$$u_1 = \alpha_{13}(y,\xi_{11},\xi_{12}) \tag{3.52a}$$

$$u_2 = \alpha_{21}(y) \tag{3.52b}$$

[3] This restriction can be loosened to allow all lower-triangular blocks to depend on functions of y, as long as a block-strict feedback structure is imposed on the entire system, including the input matrix.

where

$$\alpha_{11}(y) = -\psi_{11}(y) - d_{12}[1]y_1 - \delta_{11}y_1 - c_{11}y_1 \qquad (3.53a)$$
$$-\beta_{11}\left(Q\gamma_1(y)\right)^T\left(Q\gamma_1(y)\right)y_1$$
$$\alpha_{21}(y) = -\psi_{21}(y) - c_{21}y_2 - \beta_{21}\left(Q\gamma_2\right)^T\left(Q\gamma_2\right)y_2 \qquad (3.53b)$$
$$\alpha_{12}(y,\xi_{11}) = \lambda_{11}\xi_{11} - y_1 - c_{12}w_{11} + \frac{\partial\alpha_{11}(y)}{\partial y_2}(\psi_{21}(y)+\alpha_{21}(y)) \qquad (3.53c)$$
$$-\delta_{12}\left(\frac{\partial\alpha_{11}(y)}{\partial y_1}\right)^2 w_{11} + \frac{\partial\alpha_{11}(y)}{\partial y_1}(d_{12}[1]y_1 + \psi_{11}(y) + \xi_{11})$$
$$\alpha_{13}(y,\xi_{11},\xi_{12}) = \lambda_{12}\xi_{12} - w_{11} - c_{13}w_{12} - \delta_{13}\left(\frac{\partial\alpha_{12}(y,\xi_{11})}{\partial y_1}\right)^2 w_{12} \qquad (3.53d)$$
$$+\frac{\partial\alpha_{12}(y,\xi_{11})}{\partial y_1}(d_{12}[1]y_1 + \psi_{11}(y) + \xi_{11})$$
$$+\frac{\partial\alpha_{12}(y,\xi_{11})}{\partial y_2}(\psi_{21}(y)+\alpha_{21}(y)) + \frac{\partial\alpha_{12}(y,\xi_{11})}{\partial\xi_{11}}(-\lambda_{11}\xi_{11}+\xi_{12})$$

To demonstrate the efficacy of our method, we calculated the expressions (3.53) using symbolic math software when the system's nonlinearities are chosen as

$$\Psi(y) = \begin{bmatrix}\psi_{11}(y)\\ \psi_{12}(y)\\ \psi_{13}(y)\\ \psi_{21}(y)\end{bmatrix} = \begin{bmatrix} y_2^2 \\ y_2^2 + y_1y_2 + y_1^2 \\ y_1^2 + y_2^2 \\ y_1^3 \end{bmatrix}. \qquad (3.54)$$

None of these nonlinearities are Lipschitz and some do not allow the output variable of their associated subsystem to be factored out – for example, the second subsystem is driven by only y_1. We chose to factor $\Psi(y)$ as

$$\Psi(y) = \begin{bmatrix}\phi_{1,1,1}(y)\\ \phi_{1,2,1}(y)\\ \phi_{1,3,1}(y)\\ \phi_{2,1,1}(y)\end{bmatrix} y_1 + \begin{bmatrix}\phi_{1,1,2}(y)\\ \phi_{1,2,2}(y)\\ \phi_{1,3,2}(y)\\ \phi_{2,1,2}(y)\end{bmatrix} y_2 = \begin{bmatrix}0\\ y_1+y_2\\ y_1\\ y_1^2\end{bmatrix} y_1 + \begin{bmatrix}y_2\\ y_2\\ y_2\\ 0\end{bmatrix} y_2 \qquad (3.55)$$

and emphasize that this is not a unique choice. We thus highlight another difference between the SISO and MIMO case – in the SISO case, this factorization is usually unique, as there is only one variable to factor. However, in the MIMO case, the designer has more flexibility in her choice of factorizations, some of which may lead to simpler control expressions or reduced control effort.

The calculated control expressions are lengthy, and will not be given here. Instead we show the results of simulating equations (3.42)-(3.52) using ode45 in MATLAB with the following parameters:

$$\begin{bmatrix} x_{11}(0) \\ x_{12}(0) \\ x_{13}(0) \\ x_{21}(0) \\ \xi_{11}(0) \\ \xi_{12}(0) \end{bmatrix} = \begin{bmatrix} 10 \\ -16 \\ 4 \\ 6 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} \lambda_{11} \\ \lambda_{12} \\ \kappa \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 6 \end{bmatrix}, \quad \begin{bmatrix} \delta_{11} \\ \delta_{12} \\ \delta_{13} \\ \beta_{11} \\ \beta_{21} \end{bmatrix} = \begin{bmatrix} 1 \\ 0.01 \\ 0.001 \\ 0.01 \\ 0.01 \end{bmatrix}, \quad \begin{bmatrix} c_{11} \\ c_{12} \\ c_{13} \\ c_{21} \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 1 \end{bmatrix}$$

In Figure 3.1 we show the behaviour of the CL system states, while Figures 3.2 and 3.3 show how the filter states evolve and the magnitudes of the control inputs.

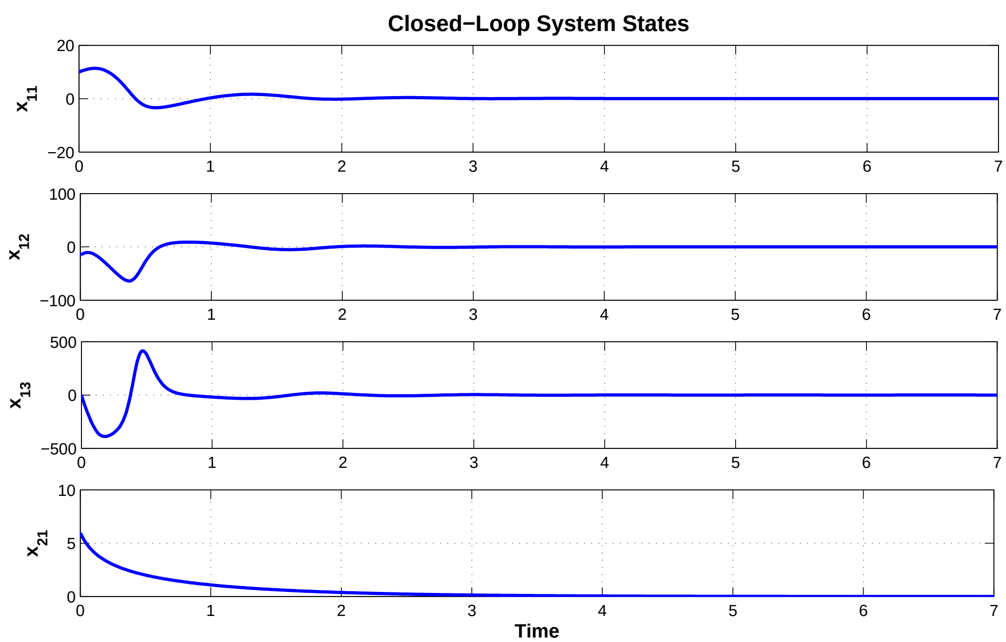

Figure 3.1: Behaviour of the closed-loop system.

The choice of simulation parameters was arbitrary, and the plots are not meant to show efficient control; rather, we are only concerned with the functionality of our algorithm. However, our design allows many degrees of freedom in altering the CL system performance by tuning the control and damping coefficients. $\triangleleft$

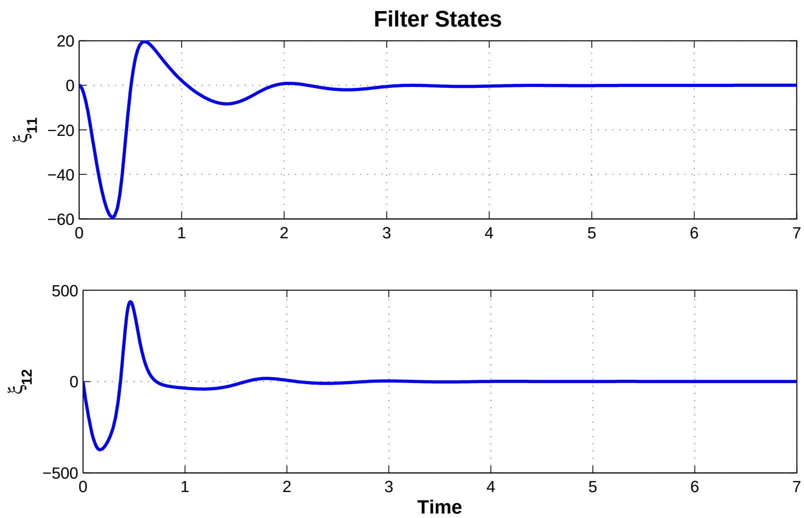

Figure 3.2: Behaviour of the filter states.

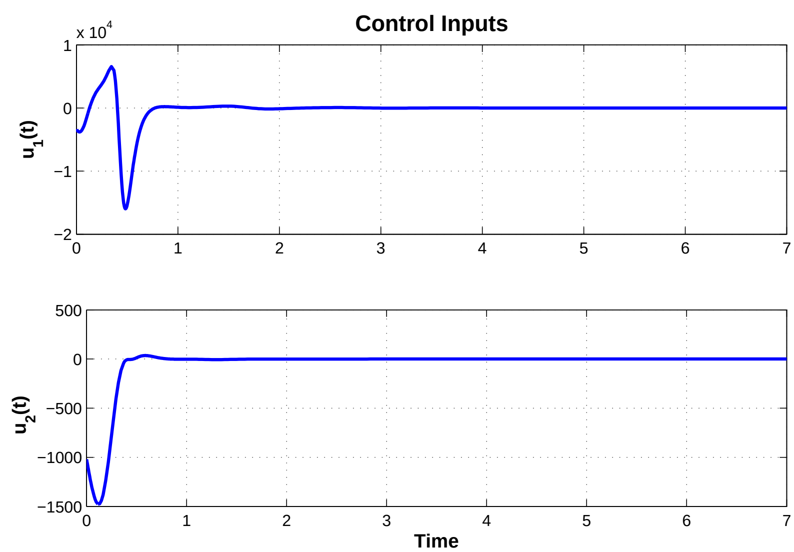

Figure 3.3: The control inputs.

3.5 Internal System Structure

In this section we investigate the internal structure of the system form (3.1), which has some curious features.

3.5.1 Vector Relative Degree

In Section 3.2, we mentioned that the number ρ_i does not necessarily correspond to the relative degree r_i associated with the ith output. To illustrate, we once again consider the fourth order system (3.42) in Example 3.4.1. First we recall that for a MIMO nonlinear system of the form

$$\begin{aligned} \dot{x} &= f(x) + G(x)u \\ y &= H(x) \end{aligned} \tag{3.56}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^m$ and

$$G(x) = [g_1(x), \ldots, g_m(x)], \qquad H(x) = \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix} = \begin{bmatrix} h_1(x) \\ \vdots \\ h_m(x) \end{bmatrix}, \tag{3.57}$$

the *vector relative degree* (VRD)

$$r = \{r_1, \ldots, r_m\}$$

is said to exist and be well-defined[4] at a point x_o if

$$L_{g_j} L_f^k h_i(x) = 0, \qquad \begin{array}{l} 1 \le j, i \le m \\ 0 \le k < r_i - 1 \end{array}, \tag{3.58}$$

for all x in a neighbourhood of x_o, and the *decoupling matrix*:

$$A(x) = \begin{bmatrix} L_{g_1} L_f^{r_1 - 1} h_1(x) & L_{g_2} L_f^{r_1 - 1} h_1(x) & \cdots & L_{g_m} L_f^{r_1 - 1} h_1(x) \\ L_{g_1} L_f^{r_2 - 1} h_2(x) & L_{g_2} L_f^{r_2 - 1} h_2(x) & \cdots & L_{g_m} L_f^{r_2 - 1} h_2(x) \\ \vdots & \vdots & \ddots & \vdots \\ L_{g_1} L_f^{r_m - 1} h_m(x) & L_{g_2} L_f^{r_m - 1} h_m(x) & \cdots & L_{g_m} L_f^{r_m - 1} h_m(x) \end{bmatrix} \tag{3.59}$$

is nonsingular at x_o. The number r_i associated with the ith output represents the number of times y_i must be time-differentiated before *any* $u_j,\ j \in \{1, \ldots, m\}$ appears. To apply this definition we examine one output at a time. Considering the output y_i, it is necessary for the ith row of $A(x)$

$$[L_{g_1} L_f^{k-1} h_i(x), L_{g_2} L_f^{k-1} h_i(x), \cdots, L_{g_m} L_f^{k-1} h_i(x)]$$

[4] Please see Section 5.1 in [17].

to be nonzero in order to have $A(x_o)$ nonsingular. However by (3.58), we expect each of the entries in the above row-vector to be zero for all $k = 0, 1 \ldots, (r_i - 2)$. When $k = r_i - 1$, then at least one of the entries in this row vector becomes nonzero. The most computationally efficient way to apply this definition in order to discern a system's VRD is to consider one output at a time, and compute each of the entries of this row vector for increasing k until a nonzero entry appears. At least one entry must be nonzero for all x in a neighbourhood of x_o otherwise the VRD is not well defined.

We proceed in this way to identify the numbers r for the system (3.42) in Example 3.4.1. For this system we obtain $(r_1, r_2) = (2, 1)$, which is not the same as the pair of numbers $(\rho_1, \rho_2) = (3, 1)$ that dictate the dynamic order of our OFB design. Also interesting is that the decoupling matrix

$$A(x) = \begin{bmatrix} L_{g_1} L_f h_1(x) & L_{g_2} L_f h_1(x) \\ L_{g_1} h_2(x) & L_{g_2} h_2(x) \end{bmatrix} = \begin{bmatrix} 0 & \frac{\partial \psi_{11}(x_{11}, x_{21})}{\partial x_{21}} \\ 0 & 1 \end{bmatrix} \tag{3.60}$$

is singular for all x which implies that the VRD is not defined for this system.

The fact that $\rho \neq r$ necessarily is interesting for several reasons. First, in the SISO version of (3.1) (which is the OFBF), the number ρ is identical to the relative degree r which is inherently well-defined. Therefore, a well-defined relative degree is *necessary* in order for a SISO system to admit an OFB design by the MT method. For MIMO systems, this is clearly not the case as we have demonstrated in Example 3.4.1. Furthermore, many OFB methods for SISO systems implicitly rely on the existence of a well-defined relative degree, as they are based on some variant of the normal form. For MIMO systems, a well-defined relative degree has many similar implications as that for SISO systems; for instance, if r exists and is well defined for (3.56), then there exists a diffeomorphism that transforms (3.56) to a MIMO normal form [59] from where the system can be I/O linearized by SFB, and the problem of *noninteracting control*[5] can then be solved. One would therefore

[5]Please see Section 5.3 in [17]. The problem of noninteracting control is to design a state feedback that decouples the I/O behaviour of the individual subsystems and is analogous to I/O SFB linearization for SISO systems. If such a feedback exists, then it is possible to regulate the behaviour of each output y_i individually through the m inputs.

expect that the existence of a well-defined VRD would also play an important role in OFB designs for MIMO systems. Contrary to this expectation, our example shows how the multivariable structure of a system may in itself provide design flexibilities not enjoyed by SISO systems. In particular, it is interesting to note that the global MIMO OFB problem is solvable for the form (3.1) even when the problem of noninteracting control is not solvable by static state feedback[6].

3.5.2 Stability of the Zero Dynamics

Fortunately, the existence of a well-defined relative degree is not necessary for the stability analysis of a system's zero dynamics. We carry out such an analysis for the sake of completeness and to help further characterize the system (3.1).

It is useful to revisit the definition of "zero dynamics" (ZD). These are the dynamics of an unforced system $\dot{x} = F(x)$, comprising the largest subset of the set of all its trajectories, with the property that $y \equiv 0$. Since we generally assume that all the solutions of $\dot{x} = F(x)$ depend continuously on the initial conditions, searching for such a set is equivalent to identifying the largest set of initial conditions from which the system evolves with $y = 0$. It is well known that the stability properties of the (ZD) are invariant under static or dynamic state feedback. Therefore, the task of analyzing the ZD stability of a forced system such as (3.1) can be accomplished by first finding an invertible change of state coordinates and a (full) state feedback $u = \theta(x)$ such that in the new coordinates the unforced system clearly reveals the largest set of initial conditions for which the system's output is identically zero. As discussed in greater detail in Section 2.2, for SISO systems like

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ y &= h(x) \end{aligned} \tag{3.61}$$

with a well-defined relative degree r, a transformation of the form

$$\begin{aligned} z_j &= L_f^{j-1} h(x), \qquad 1 \leq j \leq r \\ \eta &= \phi(x) \end{aligned}$$

[6] A well-defined vector relative degree is sufficient and necessary for the problem of noninteracting control to be solvable by static state-feedback (Proposition 5.3.1 [17]).

brings the system dynamic into the normal form

$$\begin{aligned}
\dot{\eta} &= \Gamma(\eta, z) \\
\dot{z}_j &= z_{j+1} \\
\dot{z}_r &= L_f^r h(x) + u L_g L_f^{r-1} h(x) \\
y &= z_1.
\end{aligned} \tag{3.62}$$

Then, with the SFB

$$u = (L_g L_f^{r-1} h(x))^{-1} (-L_f^r h(x)) \tag{3.63}$$

(3.62) shows that for any initial conditions in the set

$$\{(\eta(0), z(0)) \in \mathbb{R}^{n-r} \times \mathbb{R}^r \mid z(0) \equiv 0\}, \tag{3.64}$$

the trajectories of (3.62) evolve in such a way that $y \equiv 0$, $\forall\, t \geq 0$, with $\eta = \Gamma(\eta, 0)$ being the ZD of (3.61).

Since the VRD is not necessarily defined for systems of the form (3.1), we cannot apply the analogous change of coordinates (cf. Lemma 4.6.1 in [16]):

$$\begin{aligned}
z_{i,j} &= L_f^{j-1} h_i(x), \qquad 1 \leq i \leq m, \qquad 1 \leq j \leq r_i \\
\eta_j &= \phi_j(x), \qquad 1 \leq j \leq n - \sum_{i=1}^{m} r_i
\end{aligned} \tag{3.65}$$

to obtain the multivariable NF that reveals the ZD. To ascertain the form of the ZD, we could apply the so-called "zero dynamics algorithm" described in Chapter 6 in [17]. However, this algorithm is rather involved and we opt for a simpler approach that takes advantage of the structure of system (3.1). We proceed as follows. Without loss of generality, assume that

$$\rho_1 = \max\{\rho_1, \ldots, \rho_m\}.$$

Define the number

$$s_i \triangleq \rho_1 - \rho_i.$$

Then, append s_i integrators to the input associated with the ith subsystem $(1 \leq i \leq$

m) to obtain the extended system:

$$\begin{aligned}
\dot{x}_{i,j} &= x_{i,j+1} + \psi_{i,j}(y), \qquad 1 \le j \le (\rho_i - 1) \\
\dot{x}_{i,j} &= x_{i,j+1} + \psi_{i,j}(y) + d_{i,j}\zeta_{i,1}, \qquad \rho_i \le j \le (n-1) \\
\dot{x}_{i,k_i} &= \psi_{i,n}(y) + d_{i,k_i}\zeta_{i,1} \\
\dot{\zeta}_{i,j} &= \zeta_{i,j+1}, \qquad 1 \le j \le (s_i - 1) \\
\dot{\zeta}_{i,s_i} &= v_i \\
y_i &= x_{i,1}
\end{aligned} \tag{3.66}$$

Our technique is a variant on the *dynamic extension* algorithm given in Section 5.4 of [17], in that we "delay" the appearance of certain inputs in all output derivatives. The variables $\zeta_{i,j}$ are the states of the appended integrators, with $\zeta_{i,1} = u_i$ and v_i the input to the last integrator appended to the ith subsystem. Augmenting the system in this way allows us to define a diffeomorphism and state feedback (applied to the new inputs) that transforms this extended system into a form where its zero dynamics become apparent. We note that no (static or dynamic) state feedback can affect the stability of a system's ZD, and therefore if the extended system's ZD are stable, then so are those of the original system.

We then claim that the change of coordinates (for $1 \le i \le m$)

$$\begin{aligned}
z_{i,j} &= L_f^{j-1} h_i(x), \qquad 1 \le j \le \rho_1 \\
\eta_{i,j} &= \tfrac{-d_{\rho_i + j}}{d_{\rho_i}} x_{i,\rho_i} + x_{i,\rho_i + j}
\end{aligned} \tag{3.67}$$

where the vector field $f(x) = [f_1^T, \ldots, f_m^T]^T$ is defined so that the system dynamic (3.66) can be written as

$$\begin{bmatrix} \dot{x}_{i,1} \\ \vdots \\ \dot{x}_{i,k_i} \\ \dot{\zeta}_{i,1} \\ \vdots \\ \dot{\zeta}_{i,s_i} \end{bmatrix} = f_i(x) + g_i v_i \tag{3.68}$$

with

$$g_i = [0, \ldots, 0, 1]^T_{1 \times \rho_1},$$

forms a diffeomorphism that transforms (3.66) into a MIMO normal form where a state feedback and a set of initial conditions can be found so that the zero dynamics are shown to take the form

$$\dot{\eta} = \mathrm{blkdiag}(\Gamma_1, \ldots, \Gamma_m)\eta \tag{3.69}$$

where the Hurwitz $\Gamma_i \in \mathbb{R}^{(k_i-\rho_i)\times(k_i-\rho_i)}$.

We justify this claim by illustrating the procedure on a system similar to (3.42) studied in Example 3.4.1. We add an extra state to the first subsystem to ensure that the zero dynamics are not trivial as they are for system (3.42). The system we consider is

$$\begin{aligned}
\dot{x}_{11} &= x_{12} + \psi_{11}(y)\\
\dot{x}_{12} &= x_{13} + \psi_{12}(y)\\
\dot{x}_{13} &= x_{14} + \psi_{13}(y) + u_1\\
\dot{x}_{14} &= \psi_{14}(y) + au_1, \qquad a > 0\\
\dot{x}_{21} &= \psi_{21}(y) + u_2\\
y &= [x_{11}, x_{21}]^T,
\end{aligned} \tag{3.70}$$

which can be shown to have a singular decoupling matrix, identical to the one for system (3.42). Since $s_2 = \rho_1 - \rho_2 = 2$, we re-write (3.70) as

$$\begin{aligned}
\dot{x}_{11} &= x_{12} + \psi_{11}(y)\\
\dot{x}_{12} &= x_{13} + \psi_{12}(y)\\
\dot{x}_{13} &= x_{14} + \psi_{13}(y) + u_1\\
\dot{x}_{14} &= \psi_{14}(y) + au_1, \qquad a > 0\\
\dot{x}_{21} &= \psi_{21}(y) + \zeta_{21}\\
\dot{\zeta}_{21} &= \zeta_{22}\\
\dot{\zeta}_{22} &= v_2
\end{aligned} \tag{3.71}$$

appending two integrators to the second input. We note that y_2 now must be differentiated ρ_1 times before the new input v_2 appears. We define the change of coordinates

$$
\begin{aligned}
z_{11} &= y_1 = x_{11} \\
z_{12} &= \dot{y}_1 = x_{12} + \psi_{11} \\
z_{13} &= \ddot{y}_1 = (x_{13} + \psi_{12}) + \tfrac{\partial \psi_{11}}{\partial y_1}(x_{12} + \psi_{11}) + \tfrac{\partial \psi_{11}}{\partial y_2}(\psi_{21} + \zeta_{21}) \\
&\triangleq x_{13} + \theta_{13}(y_1, y_2, x_{12}, \zeta_{21}) \\
\eta &= -a x_{13} + x_{14} \\
z_{21} &= y_2 = x_{21} \\
z_{22} &= \dot{y}_2 = \psi_{21} + \zeta_{21} \\
z_{23} &= \ddot{y}_2 = \tfrac{\partial \psi_{21}}{\partial y_1}(x_{12} + \psi_{11}) + \tfrac{\partial \psi_{11}}{\partial y_2}(\psi_{21} + \zeta_{21}) + \zeta_{22} \\
&\triangleq \zeta_{22} + \theta_{23}(y_1, y_2, x_{12}, \zeta_{21})
\end{aligned} \tag{3.72}
$$

whose Jacobian is clearly nonsingular for all x. In the new coordinates, we have

$$
\begin{aligned}
\dot{\eta} &= -a\eta - a^2\big[z_{13} + \theta_{13}(z_{11}, z_{21}, (z_{12} - \psi_{11}), (z_{22} - \psi_{21}))\big] \\
&\qquad + \big(\psi_{14} - a\psi_{13}\big) \\
\dot{z}_{11} &= z_{12} \\
\dot{z}_{12} &= z_{13} \\
\dot{z}_{13} &= x_{14} + \psi_{13} + u_1 + \dot{\theta}_{13}(y_1, y_2, x_{12}, \zeta_{21}) \\
\dot{z}_{21} &= z_{22} \\
\dot{z}_{22} &= z_{23} \\
\dot{z}_{23} &= v_2 + \dot{\theta}_{23}(y_1, y_2, x_{12}, \zeta_{21})
\end{aligned} \tag{3.73}
$$

We notice here that the expressions for $\dot{\theta}_{i,3}(\cdot),\ i = 1,2$ do not produce a dependency on either u_1 or v_2. Therefore, the z_1 and z_2 subsystems can be decoupled by choosing

$$
\begin{aligned}
u_1 &= -\psi_{13} - x_{14} - \dot{\theta}_{13}(y_1, y_2, x_{12}, \zeta_{21}) \\
v_2 &= -\dot{\theta}_{23}(y_1, y_2, x_{12}, \zeta_{21}).
\end{aligned} \tag{3.74}
$$

Secondly, by their definitions, the $\theta_{i,3}(\cdot),\ i = 1,2$ vanish when their arguments are zero, and if we initialize (3.73)-(3.74) on the set

$$
\{(\eta(0), z(0)) \in \mathbb{R}^1 \times \mathbb{R}^6 \mid z_(0) \equiv 0\}, \tag{3.75}
$$

we see that the output y remains zero for all time, and the zero dynamics

$$\dot{\eta} = -a\eta$$

are GES.

3.6 Summary

In this chapter we provided an extension to the globally asymptotically stabilizing output feedback design method presented in [2], to MIMO nonlinear systems. The efficacy of our method is demonstrated via simulation on a mathematical example. We have shown that if the structure of the input vectors in the starting form is restricted as in (3.1), the nonlinear output-coupling between the subsystems poses no difficulty in formulating a globally stabilizing OFB. In the MIMO case, it is necessary to include additional nonlinear damping terms to compensate for this dynamic coupling.

Chapter 4

Application of Observer Backstepping to Systems in a Restricted BTOF

This chapter is based largely on [72].

4.1 Introduction

In this chapter we investigate an application of observer backstepping to a class of nonlinear, multivariable systems in a restricted BTOF (block-triangular observer form). The BTOF was introduced in [70] where the differential geometric conditions fully characterizing this class of systems are derived. The BTOF observer is a generalization of the linear error dynamic observer designed on the basis of the NOF (nonlinear observer form) [68]. We note that the basic result in [6], and its extension to MIMO systems is obtained using an observer based on the OFBF, which is a restricted subset of systems in NOF. The contribution in this chapter is to demonstrate a similar design using an observer that does not have a linear error dynamic.

We illustrate our method on a model of a physical system, the MAGLEV, consisting of a magnetically levitated ball, and show how the design can easily be made robust with respect to variations in the electromagnet's coil resistance. Two additional benefits of the design are that it does not rely on the use of high-gains in order to guarantee boundedness or convergence of solutions on any compact set, and it does not impose growth restrictions on the system's nonlinearities.

We first present the BTOF, designing an exponentially convergent observer for this class of systems. In the next section we consider a subset of systems in BTOF and present our main result. We then carry out the design on the MAGLEV. In the conclusion, we note several observations on the practicability of recursive methods such as the one presented here.

4.2 The BTOF Observer

In this chapter we consider the following class of systems, where the ith subsystem can be written as:

$$\Sigma_{BTOF_i}: \begin{cases} \dot{x}_{i,j} = x_{i,j+1} + \gamma_{i,j}(x_{\langle i-1\rangle}, y_{\langle m\rangle}) \\ \dot{x}_{i,\lambda_i} = \gamma_{i,\lambda_i}(x_{\langle i-1\rangle}, y_{\langle m\rangle}) + b_i\sigma(y)u \\ y_i = C_i x_i = x_{i,1} \end{cases} \tag{4.1}$$

for $1 \leq i \leq m$ and $1 \leq j \leq \lambda_i - 1$. In (4.1), $x \in \mathbb{R}^n$, $n = \sum_{i=1}^{m} \lambda_i$, $x_i \in \mathbb{R}^{\lambda_i}$, $y \in \mathbb{R}^m$, $u \in \mathbb{R}$ and $b_i = 0$ for $1 \leq i \leq m-1$, while $b_m \neq 0$, and $\sigma(y)$ is a smooth function bounded away from zero. We denote

$$x_{\langle i-1\rangle} = (x_{1,2}, \ldots, x_{1,\lambda_1}, \ldots, x_{i-1,2}, \ldots, x_{i-1,\lambda_{i-1}})$$

and

$$y_{\langle i\rangle} = (y_1, \ldots, y_i).$$

Each number λ_i could be chosen as the observability index associated with the ith output but in general it is not necessarily uniquely defined. We require the functions γ_i to be smooth, to satisfy $\gamma_i(0) = 0$, $1 \leq i \leq m$, and to be globally Lipschitz in $x_{\langle i-1\rangle}$, uniformly in y. Alternatively, we could weaken this assumption and require the γ_i to be Lipschitz in $x_{\langle i-1\rangle}$ on some compact and connected set Ω, uniformly in y. The necessary and sufficient conditions characterizing the structure of the drift component of (4.1) are derived in [70]. Whereas in [70] each nonlinearity γ_i can depend on all subsystem outputs, here we consider a subset of the class of systems equivalent to the BTOF by requiring the following assumption:

A 4.2.1.

$$\gamma_{i,j} = \begin{cases} \gamma_{i,j}(x_{\langle i-1 \rangle}, y_{\langle i \rangle}) & \text{for } j < \lambda_i \\ B_i(y_{\langle i \rangle}) + C_i(y_{\langle i \rangle})\phi_i(y_{i+1}) & \text{for } j = \lambda_i \end{cases}$$

for $1 \le i \le m-1$, *where* C_i, B_i *and* ϕ_i *are smooth functions, with* C_i *and the derivative* $\phi_i'(\cdot)$ *being bounded away from zero on some operating region* $\Omega \subseteq \mathbb{R}^n$.

Assumption A4.2.1 enforces a strict feedback form on the BTOF, linking the subsystems in a way that allows us to carry out a backstepping design. We refer to the BTOF satisfying A4.2.1 as the RBTOF (restricted BTOF). According to [70], an observer for systems in RBTOF can be constructed as

$$\hat{\Sigma}_{RBTOF_i} : \begin{cases} \dot{\hat{x}}_{i,j} = \hat{x}_{i,j+1} + \gamma_{i,j}(\hat{x}_{\langle i-1 \rangle}, y_{\langle i \rangle}) + L_{i,j}\tilde{x}_{i,1} \\ \dot{\hat{x}}_{i,\lambda_i} = \gamma_{i,\lambda_i}(\hat{x}_{\langle i-1 \rangle}, y_{\langle i+1 \rangle}) + b_i \sigma(y) u + L_{i,\lambda_1}\tilde{x}_{i,1} \end{cases} \tag{4.2}$$

for $1 \le j \le \lambda_i$, $1 \le i \le m-1$, with $\tilde{x}_{i,1} = x_{i,1} - \hat{x}_{i,1}$. The mth subsystem is written similarly, except that all $\gamma_{m,j}$'s are allowed to depend on all system outputs $y_{\langle m \rangle}$. If the observer gains $L_{i,j}$ are chosen such that the polynomials $s^{\lambda_i} + L_{i,1}s^{\lambda_i - 1} + \ldots + L_{i,\lambda_i - 1}s + L_{i,\lambda_i}$ are Hurwitz, then the dynamics of the error $\tilde{x} = x - \hat{x}$ are

$$\tilde{\Sigma}_{RBTOF_i} : \begin{cases} \dot{\tilde{x}}_{i,j} = \tilde{x}_{i,j+1} + \triangle\gamma_{i,j} - L_{i,j}\tilde{x}_{i,1} \\ \dot{\tilde{x}}_{i,\lambda_i} = \triangle\gamma_{i,\lambda_i} - L_{i,\lambda_1}\tilde{x}_{i,1} \end{cases} \tag{4.3}$$

where $\triangle\gamma_{i,j} \triangleq \gamma_{i,j}(x_{\langle i-1 \rangle}, y_{\langle i \rangle}) - \gamma_{i,j}(\hat{x}_{\langle i-1 \rangle}, y_{\langle i \rangle})$, $1 \le j \le m-1$ (with $\triangle\gamma_{i,\lambda_i}$ defined similarly), which is shown in [70] to converge exponentially to its origin. This fact can be intuited by observing that for the first subsystem, $\triangle\gamma_1 \equiv 0$ and the $\tilde{x}_1$-dynamic is linear and stable. In the second subsystem the linear stable error dynamic is perturbed by the disturbance $\triangle\gamma_2$ whose norm, by assumption, is less than or equal to a linear term in $\|\tilde{x}_1\|$. From the convergence of the first subsystem's error, the magnitude of this disturbance decays exponentially to zero, and the entire $\tilde{\Sigma}_{RBTOF}$ can be shown to converge by induction.

In our main result, we make use of the observer (4.2) to construct a globally stabilizing output feedback law.

4.3 Main Result

Theorem 4.3.1. *For any system globally diffeomorphic to the form* (4.1)*, satisfying assumption A4.2.1, there exists a stabilizing control law based on known signals only. If the* $\gamma_{i,j}$ *satisfy*

$$\|\gamma_{i,j}(x_{\langle i-1\rangle}, y_{\langle i\rangle}) - \gamma_{i,j}(0, y_{\langle i\rangle})\| \leq K_{i,j}\|x_{\langle i-1\rangle}\|$$

on $\Omega = \mathbb{R}^{\nu_{i-1}}$, $\nu_{i-1} = \sum_{l=1}^{i-1}\lambda_l$*, then the control law globally stabilizes the system.*

Proof. We first re-write the composite system consisting of Σ_{RBTOF} and its observer $\hat{\Sigma}_{RBTOF}$ dividing each into its subsystem components for clarity:

$$\begin{cases} \Sigma_1: & \begin{cases} \dot{x}_{1,1} = \hat{x}_{1,2} + \tilde{x}_{1,2} + \gamma_{1,1}(y_1) \\ \dot{\hat{x}}_{1,j} = \hat{x}_{1,j+1} + \gamma_{1,j}(y_1) + L_{1,j}\tilde{x}_{1,1} \\ \dot{\hat{x}}_{1,\lambda_1} = \gamma_{1,\lambda_1}(y_1, y_2) + L_{1,\lambda_1}\tilde{x}_{1,1} \end{cases} \\ \Sigma_i: & \begin{cases} \dot{x}_{i,1} = \hat{x}_{i,2} + \tilde{x}_{i,2} + \gamma_{i,1}(\hat{x}_{\langle i-1\rangle} + \tilde{x}_{\langle i-1\rangle}, y_{\langle i\rangle}) \\ \dot{\hat{x}}_{i,j} = \hat{x}_{i,j+1} + \gamma_{i,j}(\hat{x}_{\langle i-1\rangle}, y_{\langle i\rangle}) + L_{i,j}\tilde{x}_{i,1} \\ \dot{\hat{x}}_{i,\lambda_i} = B_i(y_{\langle i\rangle}) + C_i(y_{\langle i\rangle})\phi_i(y_{i+1}) + L_{i,\lambda_i}\tilde{x}_{i,1} \end{cases} \\ \Sigma_m: & \begin{cases} \dot{x}_{m,1} = \hat{x}_{m,2} + \tilde{x}_{m,2} + \gamma_{m,1}(\hat{x}_{\langle m-1\rangle} + \tilde{x}_{\langle m-1\rangle}, y_{\langle m\rangle}) \\ \dot{\hat{x}}_{m,j} = \hat{x}_{m,j+1} + \gamma_{m,j}(\hat{x}_{\langle m-1\rangle}, y_{\langle m\rangle}) + L_{m,j}\tilde{x}_{m,1} \\ \dot{\hat{x}}_{m,\lambda_m} = \gamma_{m,\lambda_m}(\hat{x}_{\langle m-1\rangle}, y_{\langle m\rangle}) + b_m\sigma(y)u + L_{m,\lambda_m}\tilde{x}_{m,1} \end{cases} \\ \tilde{\Sigma}: & \begin{cases} \dot{\tilde{x}}_{1,k} = \tilde{x}_{1,k+1} - L_{1,k}\tilde{x}_{1,1} \\ \dot{\tilde{x}}_{1,\lambda_1} = -L_{1,\lambda_1}\tilde{x}_{1,1} \\ \dot{\tilde{x}}_{l,k} = \tilde{x}_{i,k+1} + \triangle\gamma_{i,k} - L_{i,k}\tilde{x}_{i,1} \\ \dot{\tilde{x}}_{l,\lambda_l} = \triangle\gamma_{l,\lambda_l} - L_{l,\lambda_l}\tilde{x}_{l,1} \end{cases} \end{cases} \tag{4.4}$$

for $2 \leq i \leq m-1$, $2 \leq j \leq \lambda_i - 1$, $1 \leq k \leq \lambda_i - 1$ and $2 \leq l \leq m$. Replacing all unmeasured states with their estimates in this way is equivalent to applying the following non-singular linear transformation to the ith subsystem in $(x, \hat{x})$ coordinates:

$$[x_{i,1}, \hat{x}_{i,2}, \ldots, \hat{x}_{i,\lambda_i}, \tilde{x}_i^T]^T = \begin{bmatrix} T_1 & T_2 \\ I & -I \end{bmatrix} \begin{bmatrix} x_i \\ \hat{x}_i \end{bmatrix} \tag{4.5}$$

where

$$[T_1]_{i,j} = \begin{cases} 1 & \text{if } i = j = 1 \\ 0 & \text{otherwise} \end{cases} \text{ and } [T_2]_{i,j} = \begin{cases} 1 & \text{if } i = j > 1 \\ 0 & \text{otherwise.} \end{cases}$$

First, we notice that the nonlinearities in the first subsystem Σ_1 consist of known signals only, and can be directly cancelled. Thus, the backstepping procedure for Σ_1 is straightforward for the first $\lambda_1 - 1$ iterations.

Second, since the error subsystem in (4.4) is exponentially convergent, through the Converse Lyapunov Theorem A.0.2 we infer the existence of a positive definite, proper function $V_e : \mathbb{R}^n \to \mathbb{R}^+$, with a time derivative

$$\frac{\partial V_e(\tilde{x})}{\partial \tilde{x}} \dot{\tilde{x}} \leq -\kappa \|\tilde{x}\|^2, \;\; \kappa > 0 \tag{4.6}$$

where κ is related to the convergence rate of the observer error, and can be increased by increasing the observer gains $L_{i,j}$.

Consider the following sequence of definitions for the stabilizing functions and associated coordinate shifts:

$$\alpha_{1,1} = -c_{1,1}x_{1,1} - \gamma_{1,1}(y_1) - d_{1,1}x_{1,1} \tag{4.7}$$

$$w_{1,j} \triangleq \hat{x}_{1,j} - \alpha_{1,j-1}, \quad 2 \leq j \leq \lambda_1 \tag{4.8}$$

$$\begin{aligned} \alpha_{1,2} = & -x_{1,1} - c_{1,2}w_{1,2} - \gamma_{1,2}(y_1) - L_{1,2}\tilde{x}_{1,1} \\ & - d_{1,2}\left(\frac{\partial \alpha_{1,1}}{\partial x_{1,1}}\right)^2 w_{1,2} + \frac{\partial \alpha_{1,1}}{\partial x_{1,1}}(\hat{x}_{1,2} + \gamma_{1,1}(y_1)) \end{aligned} \tag{4.9}$$

$$\begin{aligned} \alpha_{1,k} = & -w_{1,k-1} - c_{1,k}w_{1,k} - \gamma_{1,k}(y_1) - L_{1,k}\tilde{x}_{1,1} \\ & - d_{1,k}\left(\frac{\partial \alpha_{1,k-1}}{\partial x_{1,1}} + \frac{\partial \alpha_{1,k-1}}{\partial \tilde{x}_{1,1}}\right)^2 w_{1,j} + \sum_{q=2}^{k-1} \frac{\partial \alpha_{1,k-1}}{\partial \hat{x}_{1,q}} \dot{\hat{x}}_{1,q} \\ & + \frac{\partial \alpha_{1,k-1}}{\partial x_{1,1}}(\hat{x}_{1,2} + \gamma_{1,1}(y_1)) - \frac{\partial \alpha_{1,k-1}}{\partial \tilde{x}_{1,1}} L_{1,1}\tilde{x}_{1,1} \end{aligned} \tag{4.10}$$

for $3 \leq k \leq \lambda_1 - 1$. The $c_{1,j} > 0$ (and $c_{1,1}$) are control coefficients and the $d_{1,j} > 0$ (and $d_{1,1}$) are the damping coefficients as in [1].

If we define

$$V_{1,\lambda_1 - 1} = V_e + \tfrac{1}{2}x_{1,1}^2 + \tfrac{1}{2}\sum_{k=2}^{\lambda_1 - 1} w_{1,k}^2$$

(where V_e is as in (4.6)), it can be verified that (4.7) to (4.10) lead to

$$\dot{V}_{1,\lambda_1 - 1} \leq -\left(\kappa - \sum_{k=1}^{\lambda_1 - 1} \frac{1}{4d_{1,k}}\right)\|\tilde{x}\|^2 + \sum_{k=1}^{\lambda_1 - 1} \tau_{1,k} + w_{1,\lambda_1 - 1}w_{1,\lambda_1}$$

with

$$\tau_{1,1} \triangleq -c_{1,1}x_{1,1}^2 - d_{1,1}\big(\frac{\|\tilde{x}\|}{2d_{1,1}} - \|x_{1,1}\|\big)^2$$

and for $k > 1$,

$$\tau_{1,k} \triangleq -c_{1,k}w_{1,k}^2 - d_{1,k}\big(\frac{\|\tilde{x}\|}{2d_{1,k}} - \|\big(\frac{\partial\alpha_{1,k-1}}{\partial x_{1,1}} + \frac{\partial\alpha_{1,k-1}}{\partial\tilde{x}_{1,1}}\big)w_{1,k}\|\big)^2.$$

The expressions for the stabilizing functions (4.7) to (4.10) were derived using the techniques presented in Sections 2.1 and 2.3. Our design method for the RBTOF deviates from these techniques starting with the next iteration – i.e. the transition to the second subsystem – which is what we now show. Assumption A4.2.1 allows us to express $\gamma_{1,\lambda_1}(y_1,y_2)$ as

$$\gamma_{1,\lambda_1}(y_1,y_2) = B_1(y_1) + C_1(y_1)\phi_1(y_2),$$

and we define

$$w_{2,1} \triangleq \phi_1(y_2) - \alpha_{1,\lambda_1}.$$

We propose the next Lyapunov function candidate $V_{1,\lambda_1} = V_{1,\lambda_1-1} + \frac{1}{2}w_{1,\lambda_1}^2$, and choose

$$\begin{aligned}\alpha_{1,\lambda_1} = \frac{1}{C_1(y_1)}\big[&-B_1(y_1) - w_{1,\lambda_1-1} - c_{1,\lambda_1}w_{1,\lambda_1}\\ &- d_{1,\lambda_1}\Big(\frac{\partial\alpha_{1,\lambda_1-1}}{\partial x_{1,1}} + \frac{\partial\alpha_{1,\lambda_1-1}}{\partial\tilde{x}_{1,1}}\Big)^2 w_{1,\lambda_1} - \frac{\partial\alpha_{1,\lambda_1-1}}{\partial\tilde{x}_{1,1}}L_{1,1}\tilde{x}_{1,1}\\ &+ \sum_{k=2}^{\lambda_1-1}\frac{\partial\alpha_{1,\lambda_1-1}}{\partial\hat{x}_{1,k}}\dot{\hat{x}}_{1,k} + \frac{\partial\alpha_{1,\lambda_1-1}}{\partial x_{1,1}}(\hat{x}_{1,2} + \gamma_{1,1}(y_1))\big]\end{aligned}$$

to obtain

$$\dot{V}_{1,\lambda_1} \leq -\Big(\kappa - \sum_{k=1}^{\lambda_1}\frac{1}{4d_{1,k}}\Big)\|\tilde{x}\|^2 + \sum_{k=1}^{\lambda_1}\tau_{1,k} + C_1(y_1)w_{2,1}w_{1,\lambda_1}$$

which is also different from the previous iterations because of the appearance of $C_1(y_1)$.

The design for the first subsystem is thus complete. The design for subsystems $\Sigma_i,\ 2 \leq i \leq m-1$ is different because we must account for the unknown observer errors in the $\gamma_{i,j}$, which cannot be cancelled as they were in the design for the first subsystem. This fact can be seen from the time derivative of

$$V_{2,1} = V_{1,\lambda_1} + \tfrac{1}{2} w_{2,1}^2,$$

which is

$$\begin{aligned}\dot{V}_{2,1} \leq &-\Big(\kappa - \sum_{k=1}^{\lambda_1} \frac{1}{4d_{1,k}}\Big)\|\tilde{x}\|^2 + \sum_{k=1}^{\lambda_1} \tau_{1,k} + w_{2,1}\Big[C_1(y_1)w_{1,\lambda_1} \\ &+ \phi_1'(y_2)\big(\hat{x}_{2,2} + \tilde{x}_{2,2} + \gamma_{2,1}(y_1, y_2, x_{\langle 1\rangle})\big) - \frac{\partial\alpha_{1,\lambda_1}}{\partial x_{1,1}}\tilde{x}_{1,2} \\ &- \frac{\partial\alpha_{1,\lambda_1}}{\partial x_{1,1}}(\hat{x}_{1,2} + \gamma_{1,2}(y_1)) - \sum_{k=2}^{\lambda_1} \frac{\partial\alpha_{1,\lambda_1}}{\partial \hat{x}_{1,k}}\dot{\hat{x}}_{1,k} - \frac{\partial\alpha_{1,\lambda_1}}{\partial \tilde{x}_{1,1}}\dot{\tilde{x}}_{1,1}\Big].\end{aligned}$$

This time we define $w_{2,2} \triangleq \hat{x}_{2,2} - \alpha_{2,1}$ and since we cannot cancel the nonlinearity $\gamma_{2,1}$, we choose for the stabilizing function the expression

$$\begin{aligned}\alpha_{2,1} = &-\hat{\gamma}_{2,1} - \delta_{2,1}\phi_1'(K_{2,1}+1)^2 w_{2,1} + \frac{1}{\phi_1'}\Big[-C_1 w_{1,\lambda_1} \\ &- d_{2,1}\Big(\frac{\partial\alpha_{1,\lambda_1}}{\partial x_{1,1}} + \frac{\partial\alpha_{1,\lambda_1}}{\partial \tilde{x}_{1,1}}\Big)^2 w_{2,1} + \frac{\partial\alpha_{1,\lambda_1}}{\partial x_{1,1}}(\hat{x}_{1,2} + \gamma_{1,2}) \\ &+ \sum_{k=2}^{\lambda_1} \frac{\partial\alpha_{1,\lambda_1}}{\partial \hat{x}_{1,k}}\dot{\hat{x}}_{1,k} - \frac{\partial\alpha_{1,\lambda_1}}{\partial \tilde{x}_{1,1}} L_{1,1}\tilde{x}_{1,1} - c_{2,1}w_{2,1}\Big]\end{aligned} \tag{4.11}$$

where $\hat{\gamma}_{2,1}$ denotes $\gamma_{2,1}(y_1, y_2, \hat{x}_{\langle 1\rangle})$ and we have dropped all function arguments from notation. In (4.11), the number $K_{2,1}$ represents the Lipschitz constant bounding the growth of the nonlinearity $\gamma_{2,1}$. In the following, we use the assumption that

$$\|\gamma_{2,1}(y_1, y_2, \hat{x}_{\langle 1\rangle}) - \gamma_{2,1}(y_1, y_2, \hat{x}_{\langle 1\rangle} + \tilde{x}_{\langle 1\rangle})\| \leq K_{2,1}\|\tilde{x}\|, \tag{4.12}$$

for all $\hat{x} \in \Omega$. With (4.11) we obtain

$$\begin{aligned}\dot{V}_{2,1} \leq &-\Big(\kappa - \sum_{k=1}^{\lambda_1} \frac{1}{4d_{1,k}} - \frac{1}{4d_{2,1}}\Big)\|\tilde{x}\|^2 + \sum_{k=1}^{\lambda_1} \tau_{1,k} - c_{2,1}w_{2,1}^2 \\ &- d_{2,1}\Big(\frac{\|\tilde{x}\|}{2d_{2,1}} - \Big\|\Big(\frac{\partial\alpha_{1,\lambda_1}}{\partial x_{1,1}} + \frac{\partial\alpha_{1,\lambda_1}}{\partial \tilde{x}_{1,1}}\Big)w_{2,1}\Big\|\Big)^2 + w_{2,1}\phi_1' w_{2,2} \\ &+ w_{2,1}\phi_1'\tilde{x}_{2,2} + w_{2,1}\phi_1'\big(\gamma_{2,1} - \hat{\gamma}_{2,1}\big) - \delta_{2,1}(\phi_1' w_{2,1})^2(K_{2,1}+1)^2.\end{aligned}$$

Using (4.12), we note that

$$w_{2,1}\phi_1'\big(\gamma_{2,1}-\hat{\gamma}_{2,1}\big) \leq \|w_{2,1}\phi_1'\|\cdot K_{2,1}\cdot\|\tilde{x}\|$$

and re-write the last three terms in $\dot{V}_{2,1}$:

$$\begin{aligned}\dot{V}_{2,1} \leq &-\Big(\kappa-\sum_{k=1}^{\lambda_1}\frac{1}{4d_{1,k}}-\frac{1}{4d_{2,1}}-\frac{1}{4\delta_{2,1}}\Big)\|\tilde{x}\|^2+\sum_{k=1}^{\lambda_1}\tau_{1,k}\\ &-d_{2,1}\Big(\frac{\|\tilde{x}\|}{2d_{2,1}}-\Big\|\Big(\frac{\partial\alpha_{1,\lambda_1}}{\partial x_{1,1}}+\frac{\partial\alpha_{1,\lambda_1}}{\partial\tilde{x}_{1,1}}\Big)w_{2,1}\Big\|\Big)^2\\ &+w_{2,1}\phi_1' w_{2,2}-c_{2,1}w_{2,1}^2\\ &-\frac{\|\tilde{x}\|^2}{4\delta_{2,1}}+\|w_{2,1}\phi_1'\|\|\tilde{x}\|(K_{2,1}+1)-\delta_{2,1}\|w_{2,1}\phi_1'\|^2(K_{2,1}+1)^2\end{aligned}$$

By completion of squares, the last line can be re-written as:

$$-\delta_{2,1}\big(\frac{\|\tilde{x}\|^2}{2\delta_{2,1}}-(K_{2,1}+1)\|w_{2,1}\phi_1'\|\big)^2$$

and we see that with $w_{2,2}\equiv 0$, $\dot{V}_{2,1}$ becomes negative definite. Owing to notational complexity, we forgo presenting the remainder of the formal inductive proof, which proceeds without significant variation from this point onwards. The only deviation from the observer backstepping algorithm described in Section 2.3 exists in the last two iterations just shown. □

Remark 4.3.1. For simplicity, we have chosen to present the stabilization result here. Re-formulating the problem in terms of tracking the output $y_1 = x_{1,1}$ can easily be done as in [1] by introducing the tracking error variable $w_{1,1}\triangleq x_{1,1}-y_r$ for some smooth, known reference signal y_r. This will be demonstrated in the example that follows. ◁

Remark 4.3.2. If we had allowed for multiple inputs – i.e. $b_i\neq 0$, for all $1\leq i\leq m$, then the proof actually becomes simpler. We would no longer need assumption $A4.2.1$ and could replace it with the requirement that each $\gamma_{i,j}=\gamma_{i,j}(x_{\langle i-1\rangle},y_{\langle i\rangle})$, $1\leq j\leq\lambda_i$. ◁

4.4 A Physical Example

The MAGLEV is an example of a system in RBTOF, for which an output feedback control can be designed using Theorem 4.3.1.

4.4.1 The MAGLEV Model

We briefly derive a model for the magnetically levitated ball shown in Figure 1. The

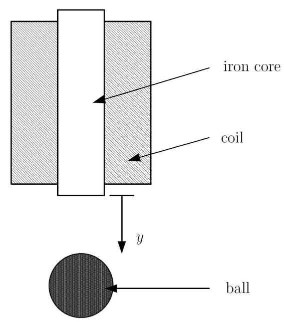

Figure 4.1: Magnetically levitated ball (MAGLEV)

coil creates a magnetic force F_m upward on the steel ball of mass m. Gravitational force mg acts in the opposite direction to F_m. The force F_m is the spatial rate of change of the energy W in the magnetic field of coil: $F_m = \frac{\partial W}{\partial y}$, where y denotes the ball's position. Assuming the electromagnet's core does not saturate during operation, the energy $W = L(y)i^2/2$ where i is the current in the coil and the coil inductance $L(y)$ is a function of the ball's position. Hence, we have

$$F_m(y,i) = \frac{\partial W(i,y)}{\partial y} = \frac{i^2}{2}L'(y) \tag{4.13}$$

where $L'(y) = \frac{dL}{dy}(y)$. The dynamics governing the mechanical subsystem are

$$\ddot{y} = g + \frac{i^2}{2m}\frac{dL(y)}{dy}. \tag{4.14}$$

Following [77] we take the approximation

$$L(y) = \alpha + \frac{\beta}{y+\kappa} \tag{4.15}$$

where α, β, κ are positive constants. The electrical subsystem is modeled using Kirchhoff's voltage law. We obtain

$$u(t) = i(t)R + \frac{d}{dt}L(y(t))i(t),$$

where R is the coil resistance and u is the control input voltage applied to the coil. Hence, we obtain

$$\frac{di}{dt} = -i\frac{R}{L(y)} - \frac{i}{L(y)}\frac{dL(y)}{dy}\dot{y} + \frac{1}{L(y)}u. \tag{4.16}$$

Choosing the state variables $x_1 = y + \kappa$, $x_2 = \dot{y}$ and $x_3 = i$, the dynamics for the MAGLEV are

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= g + L'(y)\frac{x_3^2}{2m} \\
\dot{x}_3 &= \frac{1}{L(y)}\big(-Rx_3 - L'(y)x_2x_3 + u\big).
\end{aligned} \tag{4.17}$$

Prior to applying our result in Section 4.3, we make some observations concerning (4.17). We first note that if the ball's position x_1 and the current x_3 are measured, then the MAGLEV is in BTOF with (x_1, x_2) forming the first subsystem state and x_3 the second (i.e. $(\lambda_1, \lambda_2) = (2, 1)$). We see that (4.17) furthermore satisfies A4.2.1 with $B_1 = g$, $C_1 = L'(y)/2m$ and $\phi_1 = x_3^2$. The derivative $\phi'(x_3) > 0$ for $x_3 > 0$. We also note that the model itself has a singularity at $x_1 = 0$ and so we assume the system state evolves on $\Omega = \{(x_1, x_2, x_3) : x_1 > 0, x_3 > 0\}$. Since (4.17) is in RBTOF, Theorem 4.3.1 can be applied.

4.4.2 Control Design

First we require an observer to estimate the ball's velocity x_2. Following (4.2), we design the observer as

$$\begin{aligned}
\dot{\hat{x}}_1 &= \hat{x}_2 + L_{1,1}(x_1 - \hat{x}_1) \\
\dot{\hat{x}}_2 &= g + L'(y)\frac{x_3^2}{2m} + L_{1,2}(x_1 - \hat{x}_1) \\
\dot{\hat{x}}_3 &= \frac{1}{L(y)}\big(-Rx_3 - L'(y)\hat{x}_2x_3 + u\big) + L_{2,1}(x_3 - \hat{x}_3).
\end{aligned} \tag{4.18}$$

In consideration of Remark 4.3.1, we formulate a tracking controller for the MAGLEV, a special case of which includes set-point regulation. We define the tracking error variable $w_1 \triangleq x_1 - y_r$ and re-write the composite system (omitting the error dynamics, which are known to converge exponentially) as in (4.4):

$$\begin{aligned}
\dot{w}_1 &= \hat{x}_2 + \tilde{x}_2 - \dot{y}_r \\
\dot{\hat{x}}_2 &= g + L'(y)\frac{x_3^2}{2m} + L_{1,2}\tilde{x}_1 \\
\dot{x}_3 &= \frac{1}{L(y)}\big(-(R+\triangle)x_3 - L'(y)(\hat{x}_2+\tilde{x}_2)x_3 + u\big).
\end{aligned} \tag{4.19}$$

In (4.19) we have included an uncertainty term $\triangle$ to account for unknown changes in the coil's resistance, possibly due to a temperature increase which is to be expected as the system runs for an extended period of time. We note that the signals $\tilde{x}_2$ and $\triangle$ are unknown and cannot be directly cancelled; we apply nonlinear damping to account for their effect.

The design proceeds by choosing the known signal $\hat{x}_2$ as the first virtual control and defining the coordinate shift $w_2 \triangleq \hat{x}_2 - \alpha_1$. Choosing

$$\alpha_1(x_1, y_r) = -c_1 w_1 - d_1 w_1 + \dot{y}_r,$$

the Lyapunov function candidate $V_1 = V_e + \frac{1}{2}w_1^2$ becomes:

$$\begin{aligned}
\dot{V}_1 &= -\kappa\|\tilde{x}\|^2 - c_1 w_1^2 - d_1 w_1^2 + w_1\tilde{x}_2 + w_1 w_2 \\
&\leq -\left(\kappa - \frac{1}{4d_1}\right)\|\tilde{x}\|^2 - c_1 w_1^2 - d_1\left(\frac{\|\tilde{x}\|}{2d_1} + w_1\right)^2 + w_1 w_2 \\
&\triangleq -\left(\kappa - \frac{1}{4d_1}\right)\|\tilde{x}\|^2 + \tau_1 + w_1 w_2
\end{aligned}$$

where V_e is as in (4.6) and we have defined the negative definite term τ_1 for notational convenience. The second iteration becomes more interesting since x_3 does not enter the $\dot{\hat{x}}_2$ equation affinely. If the next coordinate shift is chosen as $w_3 = x_3 - \alpha_2$, then upon substituting for $(x_3)^2 = (w_3 + \alpha_2)^2$, we would find it difficult to design α_2 so that the second subsystem is stabilized. We therefore employ assumption A4.2.1 which, for systems in RBTOF, makes possible the use of a function of x_3 as a virtual control. We note that the requirement $\phi'(\cdot) \neq 0$ ensures that the complete coordinate shift (defined iteratively) $w = x - \alpha$ is a diffeomorphism on Ω. Further, if $w_3 \triangleq \phi(x_3) - \alpha_2$, then a term involving $1/\phi'(\cdot)$ necessarily becomes incorporated into the control law.

Since A4.2.1 is satisfied for the MAGLEV, we proceed by defining $w_3 \triangleq x_3^2 - \alpha_2$ and examine the time derivative of the Lyapunov function candidate $V_2 = V_1 + \frac{1}{2}w_2^2$:

$$\begin{aligned} \dot{V}_2 \leq & -\Big(\kappa - \frac{1}{4d_1}\Big)\|\tilde{x}\|^2 + \tau_1 + w_2\Big[w_1 + \Big(g + \frac{L'(y)}{2m}(w_3 + \alpha_2) \\ & + L_{1,2}\tilde{x}_1\Big) - \frac{\partial \alpha_1}{\partial x_1}(\hat{x}_2 + \tilde{x}_2 - \dot{y}_r) - \frac{\partial \alpha_1}{\partial y_r}\dot{y}_r\Big] \end{aligned} \tag{4.20}$$

where $\tilde{x}_2$ again appears through the $\dot{x}_1$ equation. Noting that the error term $\tilde{x}_1$ is a known signal, we assign

$$\alpha_2 = (2m/L'(y))\big(-g - w_1 - c_2 w_2 - L_{1,2}\tilde{x}_1 + \frac{\partial \alpha_1}{\partial x_1}(\hat{x}_2 - \dot{y}_r) - d_2\big(\frac{\partial \alpha_1}{\partial x_1}\big)^2 w_2 + \frac{\partial \alpha_1}{\partial x_1}\dot{y}_r\big)$$

to obtain

$$\begin{aligned} \dot{V}_2 \leq & -\Big(\kappa - \frac{1}{4d_1} - \frac{1}{4d_2}\Big)\|\tilde{x}\|^2 + \tau_1 - c_2 w_2^2 + \frac{L'(y)}{2}w_2 w_3 \\ & - \frac{1}{4d_2}\|\tilde{x}\|^2 - \tilde{x}_2\frac{\partial \alpha_1}{\partial x_1}w_2 - d_2\big(\frac{\partial \alpha_1}{\partial x_1}w_2\big)^2. \end{aligned} \tag{4.21}$$

Replacing $\tilde{x}_2\frac{\partial \alpha_1}{\partial x_1}w_2$ with the larger $\|\tilde{x}\|\|\frac{\partial \alpha_1}{\partial x_1}w_2\|$ term and completing the square with the last three terms in (4.21) we have

$$\begin{aligned} \dot{V}_2 \leq & -\Big(\kappa - \frac{1}{4d_1} - \frac{1}{4d_2}\Big)\|\tilde{x}\|^2 + \tau_1 + \frac{L'(y)}{2}w_2 w_3 \\ & - d_2\Big(\frac{\|\tilde{x}\|}{2d_2} - \Big\|\frac{\partial \alpha_1}{\partial x_1}w_2\Big\|\Big)^2 - c_2 w_2^2 \\ \triangleq & -\Big(\kappa - \frac{1}{4d_1} - \frac{1}{4d_2}\Big)\|\tilde{x}\|^2 + \tau_1 + \frac{L'(y)}{2}w_2 w_3 + \tau_2, \end{aligned} \tag{4.22}$$

where the negative definite term τ_2 is introduced to simplify notation.

In the third and final iteration we design the control u. The time derivative of $V_3 = V_2 + \frac{1}{2}w_3^2$ along the trajectories of the $(x_1, \hat{x}_2, x_3, \tilde{x})$ system is

$$\begin{aligned} \dot{V}_3 \leq & -\Big(\kappa - \frac{1}{4d_1} - \frac{1}{4d_2}\Big)\|\tilde{x}\|^2 + \tau_1 + \tau_2 + w_3\Big[\frac{L'(y)}{2}w_2 \\ & + 2x_3\Big(\frac{1}{L(y)}\big(-(R + \triangle)x_3 - L'(y)(\hat{x}_2 + \tilde{x}_2)x_3 + u\big)\Big) \\ & - \frac{\partial \alpha_2}{\partial x_1}\dot{x}_1 - \frac{\partial \alpha_2}{\partial y_r}\dot{y}_r - \frac{\partial \alpha_2}{\partial \dot{y}_r}\ddot{y}_r - \frac{\partial \alpha_2}{\partial \ddot{y}_r}\dddot{y}_r - \frac{\partial \alpha_2}{\partial \tilde{x}_1}\dot{\tilde{x}}_1 - \frac{\partial \alpha_2}{\partial \hat{x}_2}\dot{\hat{x}}_2\Big]. \end{aligned} \tag{4.23}$$

We now make a number of observations about (4.23).

- Considering (4.19), the unknown signal $\tilde{x}_2$ enters (4.23) through $\dot{x}_1$, $\dot{\hat{x}}_2$ and the shown time derivative of x_3^2. It also appears in the $\dot{\tilde{x}}_1$ equation since $\dot{\tilde{x}}_1 = -L_{1,1}\tilde{x}_1 + \tilde{x}_2$, as in (4.3).

- Unlike in the proof of Theorem 4.3.1, here we are fortunate to have $\gamma_{1,2}(y_1, y_2, x_2)$ be affine in $\tilde{x}_2$, with its coefficient
$$C(y_1, y_2) \triangleq \frac{2L'(y)x_3^2}{L(y)}$$
consisting of known signals only. In this case, we apply nonlinear damping by introducing a term such as
$$\frac{L(y)}{2x_3}\big(-d_4 C(y_1, y_2)^2 w_3\big)$$
into the control u (which can be simplified).

- Since $w_3 \triangleq x_3^2 - \alpha_2$, its derivative $\dot{w}_3 = 2x_3\dot{x}_3 - \dot{\alpha}_2$, where the control u appears in $\dot{x}_3$. We now see why, in general, we expect that u would contain a $1/\phi'$ term.

- It is important to keep track of the signal dependencies that the stabilizing functions α_i pick up at every iteration, so that they can be accounted for in subsequent stabilizing functions.

Without expanding the complicated expression for $\dot{V}_3$, we give the expression for u:

$$\begin{aligned} u = {} & Rx_3 - d_3\frac{2x_3^3}{L(y)}w_3 + L'(y)\hat{x}_2 x_3 - d_4(L'(y)x_3)^2\frac{2x_3}{L(y)}w_3 \\ & + \frac{L(y)}{2x_3}\Big[\frac{-L'(y)}{2}w_2 + \frac{\partial\alpha_2}{\partial x_1}\hat{x}_2 - d_5\Big(\frac{\partial\alpha_2}{\partial x_1}\Big)^2 w_3 + \frac{\partial\alpha_2}{\partial y_r}\dot{y}_r \\ & + \frac{\partial\alpha_2}{\partial \dot{y}_r}\ddot{y}_r + \frac{\partial\alpha_2}{\partial \ddot{y}_r}\dddot{y}_r - \frac{\partial\alpha_2}{\partial \tilde{x}_1}L_{1,1}\tilde{x}_1 - d_6\Big(\frac{\partial\alpha_2}{\partial \tilde{x}_1}\Big)^2 w_3 \\ & + \frac{\partial\alpha_2}{\partial \hat{x}_2}\big(g + \tfrac{1}{2}L'(y)x_3^2 + L_{1,2}\tilde{x}_1\big) - c_3 w_3\Big] \end{aligned} \tag{4.24}$$

where each of the terms involving a damping coefficient d_i, $1 \le i \le 6$, applies non-linear damping to the unknown signals. Together with the expressions chosen for the virtual controls α_1 and α_2, (4.24) can be shown to yield the time derivative of the Lyapunov function $V_3 = V_e + \frac{1}{2}(w_1^2 + w_2^2 + w_3^2)$ negative definite and proper:

$$\dot{V}_3 \le -\big(\kappa - \frac{1}{4}\sum_{i=1}^{6}\frac{1}{d_i}\big)\|\tilde{x}\|^2 + \tau_1 + \tau_2 - c_3 w_3^2 + \tau_3,$$

where

$$\begin{aligned}\tau_3 \triangleq &-d_3\Big(\frac{\|\tilde{x}\|}{2d_3} - \Big\|\frac{2x_3}{L(y)}w_3\Big\|\Big)^2 - d_5\Big(\frac{\|\tilde{x}\|}{2d_5} - \Big\|\frac{\partial\alpha_2}{\partial x_1}w_3\Big\|\Big)^2 \\ &- d_4\Big(\frac{\|\tilde{x}\|}{2d_4} - \Big\|\frac{2L'(y)x_3^2}{L(y)}w_3\Big\|\Big)^2 - d_6\Big(\frac{\|\tilde{x}\|}{2d_6} - \Big\|\frac{\partial\alpha_2}{\partial\tilde{x}_1}w_3\Big\|\Big)^2.\end{aligned}$$

Remark 4.4.1. In this example, the dynamic order of the feedback law could have been reduced by using a second-order linear error dynamic observer to reconstruct the ball's velocity using the first subsystem alone. However, in this chapter our objective was to demonstrate an output feedback design on the basis of the RBTOF, which is more general than the NOF or its multi-variable version. $\triangleleft$

4.4.2.1 A Semi-Global Formulation

We continue with the control design for the MAGLEV. For simplicity, in the following let us consider the non-robust case $\triangle \equiv 0$, re-notate

$$\begin{aligned}\dot{x}_3 &= \frac{1}{L(y_1)}(-Ry_2 - L'(y_1)x_{1,2}y_2) + \frac{1}{L(y_1)}u \\ &= \gamma_{2,1}(y_1, y_2, x_{1,2}) + \frac{1}{L(y_1)}u.\end{aligned}$$

and neglect for the moment the surrounding terms in (4.23). According to the proof of Theorem 4.3.1, we could have chosen

$$u = L(y)(-\gamma_{2,1}(y_1, y_2, \hat{x}_2) + u^*)$$

(where u^* is some auxiliary control variable that can be used to cancel or damp the surrounding terms in (4.23)) instead of applying nonlinear damping to the $\tilde{x}_2$ term which happens to appear affinely in $\gamma_{2,1}$. In that case,

$$\begin{aligned}
\dot{x}_3 &= \gamma(y_1, y_2, \hat{x}_2 + \tilde{x}_2) - \gamma(y_1, y_2, \hat{x}_2) + u^* \\
&\leq \|\gamma(y_1, y_2, \hat{x}_2 + \tilde{x}_2) - \gamma(y_1, y_2, \hat{x}_2)\| + u^* \\
&= \left\| \frac{-L'(y)x_3}{L(y)} \tilde{x}_2 \right\| + u^* \\
&\leq K\|\tilde{x}_2\| + u^*
\end{aligned}$$

with K being an estimate of the upper bound of the magnitude of the nonlinearity $\frac{-L'(y)x_3}{L(y)}$ on some compact set Ω^* – i.e.

$$K = \max_{x \in \Omega^*} \left\| \frac{-L'(y)x_3}{L(y)} \right\| = \max_{x \in \Omega^*} \left\| \frac{\beta x_3}{\alpha x_1^2 + \beta x_1} \right\|.$$

The norm is maximized when the current x_3 is largest and the ball position x_1 is smallest within some set Ω^*. Thus, since $\gamma_{2,1}$ is locally Lipschitz in x_2, we could implement a semi-global control by replacing the term associated with d_4 in (4.24) with the term

$$-d_4 K^2 \frac{2x_3}{L(y)} w_3.$$

The number $d_4 K^2$ then becomes a control parameter that can be adjusted to increase the size of the region Ω^* in closed-loop.

It is important to note that if the nonlinearities $\gamma_{i,j}(\hat{x}_{\langle i-1 \rangle}, y)$ are only *locally* Lipschitz in $\hat{x}_{\langle i-1 \rangle}$, the design of the BTOF observer must be modified by saturating these states beyond a compact set Ω. Then, if the system's state does not leave Ω, the observer error is guaranteed to be GES at its origin (Lemma 1, [70]). Specifically, the observer must be redesigned as

$$\hat{\Sigma}_{RBTOF_i} : \begin{cases} \dot{\hat{x}}_{i,j} = \hat{x}_{i,j+1} + \gamma_{i,j}(\mathrm{sat}(\hat{x}_{\langle i-1 \rangle}), y_{\langle i \rangle}) + L_{i,j}\tilde{x}_{i,1} \\ \dot{\hat{x}}_{i,\lambda_i} = \gamma_{i,\lambda_i}(\mathrm{sat}(\hat{x}_{\langle i-1 \rangle}), y_{\langle i+1 \rangle}) + b_i \sigma(y) u + L_{i,\lambda_1}\tilde{x}_{i,1} \end{cases} \tag{4.25}$$

for $1 \leq j \leq \lambda_i$, $1 \leq i \leq m-1$ (the mth subsystem being written similarly), where $\mathrm{sat}(\cdot)$ is an element-wise saturation function:

$$\mathrm{sat}(x_{\langle j \rangle}) \triangleq \begin{cases} x_{\langle j \rangle}, & x_{\langle j \rangle} \in \Omega \\ \overline{x}_{\langle j \rangle}, & \text{otherwise} \end{cases}$$

where Ω denotes a compact subset of $\mathbb{R}^j$, $\partial\Omega$ its boundary, and

$$\overline{x}_{\langle j\rangle} = \{x_{\langle j\rangle} \in \partial\Omega\}.$$

Saturating the estimated states in this way allows us to employ a *Lipschitz extension* technique which guarantees that

$$|\gamma_{i,j}(\mathrm{sat}(x_{\langle i-1\rangle}), y_{\langle i\rangle}) - \gamma_{i,j}(\mathrm{sat}(\hat{x}_{\langle i-1\rangle}), y_{\langle i\rangle})| \leq M(y_{\langle i\rangle})\|x - \hat{x}\|$$

for some bounded function $M(y_{\langle i\rangle})$, and therefore makes it possible to prove the exponential convergence of the observer's error in BTOF coordinates.

Although this semi-global formulation may be the only alternative for some systems, it should be avoided if possible; K may be a very conservative estimate of the size of this nonlinearity under normal operating conditions, while under some other conditions the state may leave Ω^*, after which time the damping term involving K would not be strong enough to guarantee the boundedness of solutions.

4.4.3 Simulation

We simulated the MAGLEV system (4.17) using the tracking controller (4.24) as the feedback. In the following simulation, we used the initial conditions

$$(x_1(0), x_2(0), x_3(0)) = (0.004 \text{ m}, 0 \text{ m/s}, 1 \text{ A}),$$

with

$$(\hat{x}_1(0), \hat{x}_2(0), \hat{x}_3(0)) = (0.014 \text{ m}, 0.04 \text{ m/s}, 0.2 \text{ A})$$

to introduce a realistic observer error. We chose the observer gains so that

$$(s^2 + L_{1,1}s + L_{1,2})(s + L_{2,1}) = (s + 1 + j)(s + 1 - j)(s + 1.5).$$

The nominal values of the parameters for the system are:

$$\alpha = 0.4 \text{ H},$$
$$\beta = 90 \times 10^{-6} \text{ m}^2/\text{A}^2,$$
$$m = 0.068 \text{ kg},$$
$$R = 11\ \Omega$$
$$\kappa = 3.4 \text{ mm}.$$

In all simulations we used the control coefficients $c_1 = c_2 = c_3 = 4$. We introduced an uncertainty of $2\ \Omega$ into the resistance in the model, and applied two versions of the controller: one implementing nonlinear damping to compensate for both observer error and the resistance uncertainty with $d_i = 1,\ 1 \leq i \leq 6$, and the other with damping coefficients set to zero.

Figures 4.2 and 4.3 show the behaviour of the plant and observer states in closed-loop, as well as tracking error when the damping coefficients are $d_i = 1,\ 1 \leq i \leq 6$.

In Figure 4.2 shows that the current in the coil is nonzero for all $0 \leq t \leq 10$ s, even though its estimate evaluates to zero at approximately .5 s (and also in the brief transitory period at the immediate start of the simulation). This behaviour is acceptable, since the boundedness of the feedback (4.24) requires only $|x_3| > 0$.

When the damping coefficients are all set to zero, the system is unstable for the set of aforementioned initial conditions, and no simulation can be obtained. For this reason, we include Figure 4.4 which was obtained by initializing the observer's estimate of the ball's velocity at $\hat{x}_2 = 0.02$ m/s as opposed to $\hat{x}_2 = 0.04$ m/s used previously. In that case, the closed-loop behaviour of the ball's position x_1 remains bounded, but the tracking behaviour is clearly deteriorated by comparison to that of x_1 when the damping coefficients are non-zero. We include Figure 4.4 as an interesting demonstration of the performance benefits gained by taking direct account of the observer error in the control design, which is a central idea in observer backstepping.

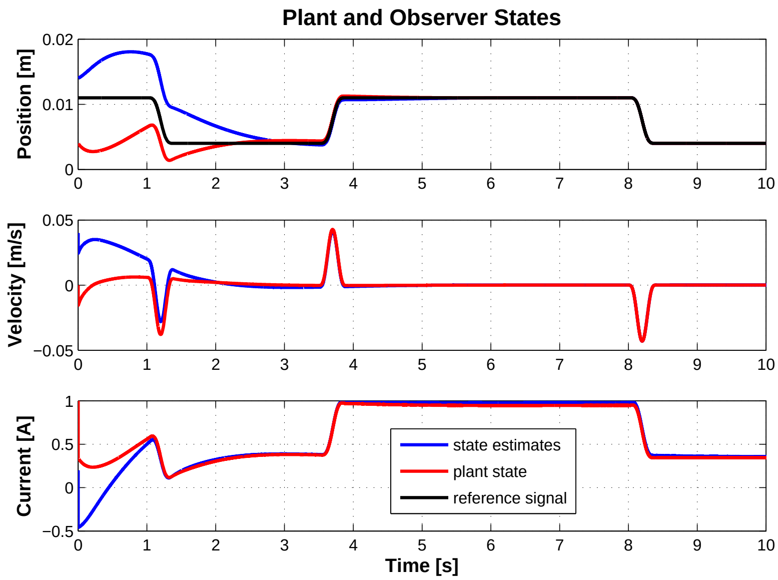

Figure 4.2: Simulation of the closed-loop system (4.17), (4.18) and (4.24), showing the behaviour of all three plant and observer states when nonzero damping coefficients are employed.

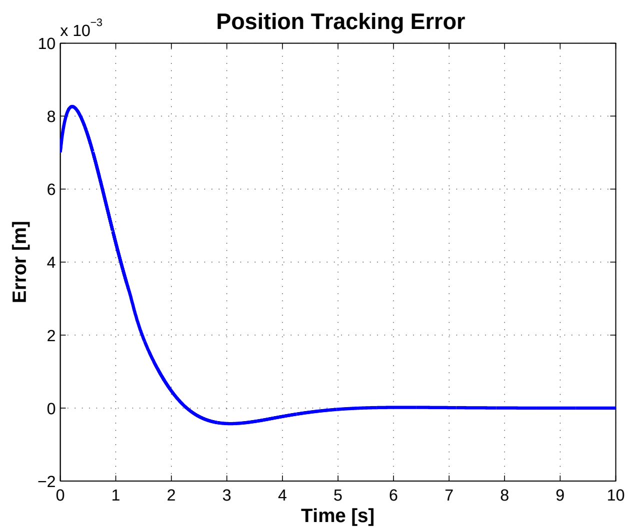

Figure 4.3: The closed-loop tracking error $y_r - x_1$.

Primarily, the simulations shown in this section are intended to demonstrate the functionality of our proposed algorithm, and to reinforce the design procedure. For that reason, no effort was made to optimize our choice of control or damping

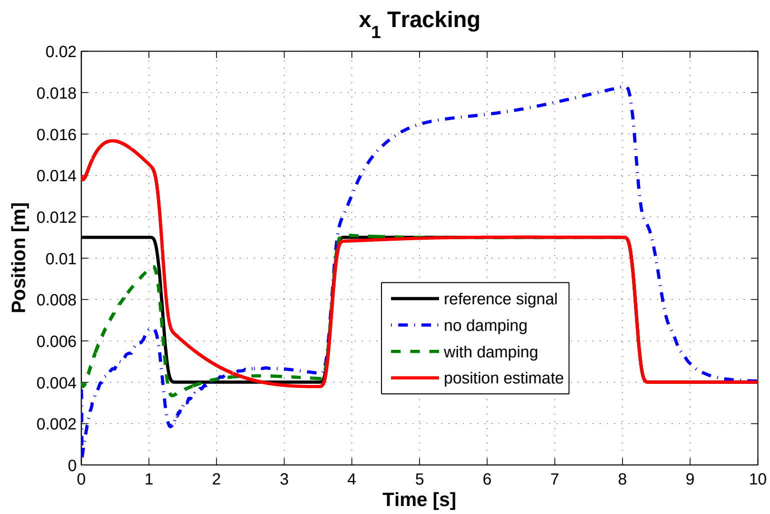

Figure 4.4: Simulation of the closed-loop system (4.17), (4.18) and (4.24), showing the position state x_1 with and without damping, and its estimate with damping.

coefficients to improve the system's closed-loop performance.

The MAGLEV model (4.17) itself may not be the best physical example on which to test the performance of our method since it possesses several features that make control design difficult. First, the model has a singularity at $x_1 = 0$, and the "interconnecting term" x_3^2, which we have used as a virtual control in the second design iteration is nonlinear with a derivative that approaches zero as $x_3 \to 0$. We are therefore forced to impose the cumbersome *assumption* that the system state evolves on the set $\Omega = \{(x_1, x_2, x_3) : x_1 > 0, |x_3| > 0\}$. In fact, we acknowledge that there may be some initial conditions and time $t^* > 0$ for which $x_3(t^*) = 0$ in which case we expect the control signal to become unbounded.

4.5 Summary

In this chapter we have developed an output feedback algorithm for systems in RBTOF which is a strict-feedback form of the BTOF introduced in [70]. The method uses the idea of observer backstepping. Since it is based on a BTOF, the approach can allow for a more general class of admissible systems relative to existing work. The design was implemented in simulation on the model of a magnetically levitated ball.

Recursive control design methods such as backstepping have created many possibilities for theoretical advances in the problem of output feedback for nonlinear systems. However, several difficulties remain, including complex and difficult to implement expressions for the control and a lack of systematic methods for choosing values for the damping coefficients. These two aspects are areas of potential future work.

Chapter 5

Concluding Remarks

In this thesis we studied the problem of non-local stabilization of nonlinear systems by output feedback. Nonlinear systems exhibit a rich variety of behaviours and the set of analysis and design tools available is much more limited than that for linear systems. For this reason, the nonlinear non-local output feedback problem is challenging, and has no general solution. The most significant difficulty in the control design for nonlinear systems stems from the fact that their internal stability is generally not indicative of their input-output behaviour. In many existing approaches to the output feedback problem, the lack of full state measurement necessitates the use of an additional dynamic component – usually an observer – whose states become incorporated into the feedback. If non-local asymptotic (i.e. internal) stability of the interconnection of such dynamic components with the plant is to be achieved, the output feedback law must be designed to account for the nature of the interconnection; it must take into consideration the input-output behaviour of the plant.

The set of systems for which constructive, globally stabilizing output feedback designs are known is likely not the largest set of systems for which such designs are possible. Integrator backstepping, nonlinear damping, and differential geometric techniques are among the few crucial mathematical tools employed in constructive designs. Therefore broadly speaking, two research directions that remain important in nonlinear control theory involve the development of new constructive design techniques, and the identification of new system forms that admit output feedback designs using existing techniques.

We have presented two contributions in this thesis. The first is an extension of Marino and Tomei's globally stabilizing output feedback law for single-input, single-output systems in the output feedback form, to multivariable nonlinear systems whose dynamics can be written in a subsystem form, with dynamic coupling between the subsystems via an output-dependent nonlinearity. We have shown that the global output feedback problem for this class of systems is solvable, provided additional nonlinear damping terms are included in each input to account for the dynamic coupling between the subsystems. Our second result involves the application of observer backstepping to multivariable nonlinear systems in a restricted block-triangular observer form. This result is applied to a dynamic model of a magnetically levitated ball.

Future work in the direction of this thesis may involve:

- Casting the multivariable output feedback result presented in Chapter 3 into an adaptive framework, considering the case where the elements of the input vector are not necessarily known. It is also possible to incorporate robust design techniques into this method.

- Finding a systematic means of choosing an optimal value for the damping coefficients in observer backstepping designs.

- Extending the result in Chapter 3 to systems in the block-triangular observer form.

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Appendix A

Some Useful Theorems

Theorem A.0.1 (Lyapunov's Direct Method). *([15] Chapter 4, or [17] Appendix B)*

Given a system $\dot{x} = f(x)$ with $x \in \mathbb{R}^n$, let $V : \mathbb{R}^n \to \mathbb{R}^+$ be a C^1 function with $V(0) = 0$, and $W : \mathbb{R}^n \to \mathbb{R}^+$ be a C^0 function with $W(0) = 0$. If

$$\frac{\partial V(x)}{\partial x} f(x) < -W(x),$$

then $x = 0$ is an asymptotically stable equilibrium of $\dot{x} = f(x)$. If additionally $V(x)$ is radially unbounded[1]*and $w(x)$ is positive definite, then $x = 0$ is globally asymptotically stable.*

Theorem A.0.2 (Converse Lyapunov Theorem). *([15] Theorem 4.17)*

Let $x = 0$ be an asymptotically stable equilibrium point of the system $\dot{x} = f(x)$, where $f(x) \in C^1$, $x \in \mathbb{R}^n$, with R_A its associated region of attraction. Then there exists a positive definite function $V(x) \in C^\infty$ and a positive definite function $W(x) \in C^0$, defined for all $x \in R_A$, and satisfying:

$$\frac{\partial V(x)}{\partial x} f(x) \leq -W(x), \quad \forall x \in R_A \tag{A.1}$$

and

$$x \to \partial R_A \implies V(x) \to \infty, \tag{A.2}$$

where ∂R_A denotes the boundary of the region R_A. If $R_A = \mathbb{R}^n$, then $V(x)$ is radially unbounded.

If $x = 0$ is an exponentially stable equilibrium point of this system, then the

[1] V is radially unbounded if $V(x) \to \infty$ when $\|x\| \to \infty$. We could instead stipulate that $V(x)$ be *proper*, in which case for any $a \in \mathbb{R}^+$, the set $\{x \in \mathbb{R}^n : 0 \leq V(x) \leq a\}$ is compact.

functions $V(\cdot)$ *and* $W(\cdot)$ *satisfy ([15], Theorem 4.14):*

$$\begin{aligned} & c_1\|x\|^2 \leq V(x) \leq c_2\|x\|^2 \\ & \frac{\partial V(x)}{\partial x} f(x) \leq -c_3\|x\|^2 \\ & \left\| \frac{\partial V(x)}{\partial x} \right\| \leq c_4\|x\| \end{aligned} \tag{A.3}$$

for some positive constants $c_i,\ i \in \{1,2,3,4\}$.