MATH 216—Introduction to Analysis (Fall 2021)

Volker Runde

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Contents

> Anyone who cannot cope with mathematics is not fully human. At best he is a tolerable subhuman who has learned to wear shoes, bathe, and not make messes in the house.
>
> Robert A. Heinlein, *Time Enough for Love* (1973).

Introduction

This is the third—and presumably, at least for the foreseeable future, last—incarnation of my notes for MATH 216. Beyond the inevitable debugging and some nip-and-tuck, the most significant difference relative to the previous version is the addition of exercises.

Volker Runde December 2021, Edmonton.

Mathematics rests on proof—and proof is eternal.

Saunders Mac Lane in: *Responses to "Theoretical Mathematics: Toward a cultural synthesis of mathematics and theoretical physics" by A. Jaffe and F. Quinn* (1994).

Introduction (2020)

This is the second iteration of my notes for MATH 216. Several changes have been made relative to the 2019 version:

- There are now a list of symbols and an index, which should make navigating the notes easier.
- Some graphics have been added.
- A new section—numbered as zero—has been added: it is intended to be an appetizer for the course, pointing out the necessity of proof in mathematics and giving some examples for the method of indirect proof.
- The exponential function is now treated in greater detail and has its own section.
- Some material has been added to the section on Riemann sums.

Finally, and perhaps most importantly, the embarrassingly high number of typos has been substantially reduced.

Volker Runde December 2020, Edmonton.

> Ich behaupte aber, daß in jeder besonderen Naturlehre nur so viel eigentliche Wissenschaft angetroffen werden könne, als darin Mathematik anzutreffen ist.
>
> IMMANUEL KANT, *Metaphysische Anfangsgründe der Naturwissenschaft* (1786).

Introduction (2019)

These are the notes for MATH 216 as tought at the University of Alberta in the Fall Term 2019: it is a second year course aimed at students who have taken a full year of calculus, but not yet been exposed to doing mathematics rigorously. The topics covered are the same as in every first year cookie cutter course in calculus, but the level of treatment is different. The focus is on a thorough development of the theory behind the calculations of calculus. In particular, the aim was to introduce the students to techniques of proof such as induction, indirect proof, and ϵ-δ arguments.

Up to Section 8, these notes follow

> KENNETH A. ROSS, *Elementary Analysis*, Second Edition. Springer Verlag (2013)

fairly closely whereas from Section 14 onward, they are more or less patterned after

> OTTO FORSTER, *Analysis* 1. Vieweg (1984).

Volker Runde December 2019, Edmonton.

List of Symbols

0 Why prove things?

Once upon a time, there was a prince who was educated by private tutors. One day, the math tutor set out to explain the Pythagorean Theorem to his royal student. The prince wouldn't believe it. So, the teacher proved the theorem, but the prince was not convinced. The teacher presented another proof of the theorem, and then yet another, but the prince would still shake his head in disbelief. Desperate, the teacher exclaimed: "Your royal highness, I give you my word of honor that this theorem is true!" The prince's face lit up: "Why didn't you say so right away?!"

Wouldn't that be wonderful? A simple word of honor from the teacher, and the student accepts the theorem as true...

Of course, it would be awful. Who makes sure that the teacher can be trusted? Where did he get his knowledge from? Did he rely on another person's word of honor? Was the person from whom the teacher learned the theorem trustworthy? Where did that person get their knowledge from? Did that person, too, trust someone else's word of honor? The longer the chain of words of honor gets, the shakier the theorem starts to look. It can't go on indefinitely: someone must have established the truth of the theorem some other way—that someone must have *proved* it.

Why are mathematicians so obsessed with proofs? The simple answer is: because they are obsessed with the truth. A proof is a procedure which, by applying certain rules, establishes an assertion as true. Proofs do not only occur in mathematics. In a criminal trial, for instance, the prosecution tries to *prove* that the defendant is guilty. Of course, the rules according to which a proof is carried out depend very much on the context: they are quite different in criminal law, experimental science, and mathematics.

Consider a chessboard consisting of 64 squares:

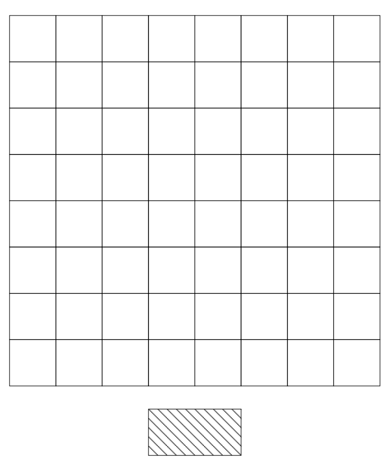

Figure 1: Chessboard and tiles

Then take rectangular tiles as shown below the board: each of them covers precisely two adjacent squares on the chessboard. It's obvious that you can cover the entire board with such tiles without any two of them overlapping. That's straightforward, so why do we need proof here?

To make things slightly more complicated, take a pair of scissors to the chessboard and cut away the squares in the upper left and in the lower right corner. This is how it will look like:

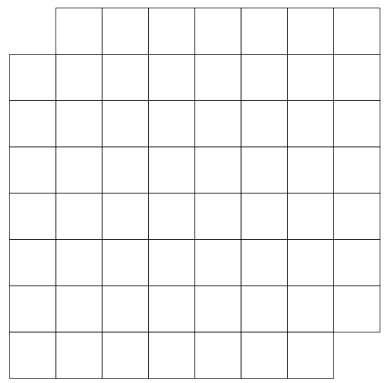

Figure 2: Chessboard with two squares removed

Now, try to cover this altered chessboard with the tiles without any two of them overlapping...

If you really try this (preferably with a chessboard drawn on a piece of paper...), you'll soon find out that—to say the least—it's not easy, and maybe the nagging suspicion will set in that it's not even possible—but why?

There is, of course, the method of brute force to find out. There are only finitely many ways to place the tiles on the chessboard, and if we try them all and see that in no case the area covered by them is precisely the altered chessboard, then we are done. There are two problems with this approach: firstly, we need to determine *every* possible way to arrange the tiles on the chessboard, and secondly, even if we do, the number of possible tile arrangements may be far too large for us to check them all. So, goodbye to brute force...

So, if brute force fails us, what can we do? Remember, we are dealing with a chessboard, and a chessboard not only consists of 64 squares in an eight by eight pattern—the squares alter in color; 32 are white, and 32 are black:

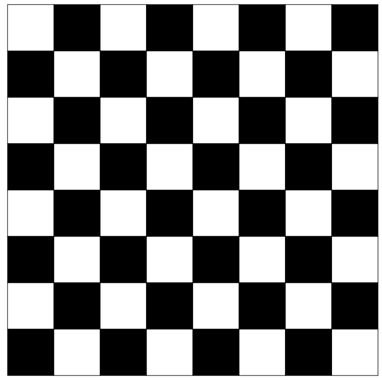

Figure 3: Colored chessboard

Each tile covers precisely two adjacent squares on the board, and two adjacent squares on a chessboard are always different in color; so each tile covers one white square and one black square. Consequently, any arrangement of tiles on the chessboard must cover the same number of white and of black squares. But now, check the altered chessboard:

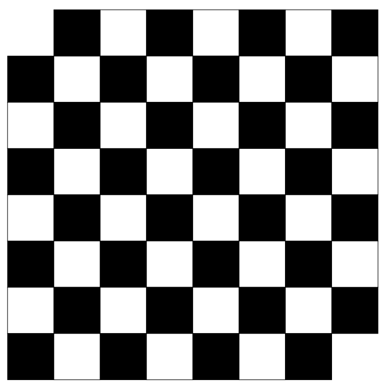

Figure 4: Colored chessboard with two squares removed

We removed two white squares, so the altered board has 30 white squares, but 32 black ones. Therefore, it is impossible to cover it with tiles—we *proved* it.

Does this smell a bit like black magic? Maybe, at the bottom of your heart, you prefer the brute force approach: it's the harder one, but it's still doable, and maybe you just don't want to believe that the tiling problem is unsolvable unless you've tried *every* possibility.

There are situations, however, where a brute force approach to truth is not only inconvenient, but impossible. Have a look at the following mathematical theorem:

Theorem 0.1. *Every integer greater than one is a product of prime numbers.*

Is it true? And if so, how do we *prove* it?

Let's start with checking a few numbers: 2 is prime (and thus a product of prime numbers), so is 3, $4 = 2 \cdot 2$, 5 is prime again, $6 = 2 \cdot 3$, 7 is prime, $8 = 2 \cdot 2 \cdot 2$, $9 = 3 \cdot 3$, and $10 = 2 \cdot 5$. So, the theorem is true for all integers greater than 2 and less than or equal to 10. That's comforting to know, but what about integers greater than 10? Well, 11 is prime, $12 = 2 \cdot 2 \cdot 3$, 13 is prime, $14 = 2 \cdot 7$, $15 = 3 \cdot 5$, $16 = 2 \cdot 2 \cdot 2 \cdot 2$, ... I stop here because it's useless to continue like this. There are infinitely many positive integers, and no matter how many of them we can write as a product of prime numbers, there will always remain infinitely many left for which we haven't shown it yet. Is $10^{10^{10^{10}}} + 1$ a product of prime numbers? That number is awfully large. Even with the help of powerful computers, it might literally take an eternity to find the prime numbers whose product it is (if they exist...). And if we have shown that the theorem holds true for every integer up to $10^{10^{10^{10}}} + 1$, we still don't know about $10^{10^{10^{10^{10}}}} + 1$.

Brute force leads nowhere here. Checking the theorem for certain examples might give you a feeling for it—but it doesn't help to establish its truth for *all* integers greater than one.

Is the theorem possibly wrong? What would that mean? If not every integer greater than one is a product of prime numbers, then there must be at least one integer a_0 which is *not* a product of prime numbers. Maybe, there is another integer a_1 with $1 < a_1 < a_0$ which is also not a product of prime numbers; if so replace a_0 by a_1. If there is an integer a_2 with $1 < a_2 < a_1$ which is not a product of prime numbers, replace a_1 by a_2. And so on... There are only finitely many numbers between 2 and a_0, and so, after a finite number of steps, we hit rock bottom and wind up with an integer $a > 1$ with the following properties: (a) a is not a product of prime numbers, and (b) it is the smallest integer with that property, i.e., every integer greater than one and less than a is a product of prime numbers.

Let's think about this (hypothetical) number a. It exists if the theorem is false. What can we say about it? It can't be prime because then it would be a product (with just one factor) of prime numbers. So, a isn't prime, i.e., $a = bc$ with neither b nor c being a or 1. This, in turn, means that $1 < b, c < a$. By property (b) of a, the numbers b and c are thus products of prime numbers, i.e., there are prime numbers $p_1, \ldots, p_n, q_1, \ldots, q_m$ such that $b = p_1 \cdots p_n$ and $c = q_1 \cdots q_m$. But then

$$a = bc = p_1 \cdots p_n q_1 \cdots q_m$$

holds, and a is product of prime numbers, which contradicts (a).

We assumed that the theorem was wrong, and—based on that assumption—obtained an integer a that is not a product of prime numbers only to see later that this was not possible. The only way out of this dilemma is that our assumption was wrong: the theorem is true! (And we now know that $10^{10^{10^{10}}} + 1$ is a product of prime numbers without having to find them...)

The strategy we used to prove Theorem 0.1 is called *indirect proof*. We can't show something directly, so we assume it's wrong and (hopefully) arrive at a contradiction.

Let's try another (indirect) proof:

Theorem 0.2. *There are infinitely many prime numbers.*

Is this believable? There is no easy formula to calculate the n-th prime number, and after putting down the first few prime numbers, it gets harder and harder to come up with the next prime. So, is the theorem wrong and do we simple run out of prime numbers after a while?

Assume this is so: there are only finitely many prime numbers, say $p_1, \dots, p_n$. Set $a := p_1 \cdots p_n + 1$. By Theorem 0.1, a is a product of prime numbers. In particular, there are a prime number q and a non-negative integer b with $a = qb$. Since $p_1, \dots, p_n$ are all the prime numbers there are, q must be one of them. Let c be the product of all those p_j that aren't q, so that $a = qc + 1$. We then obtain

$$0 = a - a = qc + 1 - qb = q(c - b) + 1,$$

and thus $q(c - b) = -1$. This, however, is impossible because $c - b$ is a non-zero integer and $q \geq 2$.

We have thus again reached a contradiction, and Theorem 0.2 is proven.

1 The Set $\mathbb{N}$ and Induction

We denote the set $\{1, 2, 3, 4, \ldots\}$ of *natural numbers* by $\mathbb{N}$.

Consider the sum of the first n odd natural numbers:

$n = 1$: $1 = 1$;

$n = 2$: $1 + 3 = 4 = 2^2$;

$n = 3$: $1 + 3 + 5 = 9 = 3^2$;

$n = 4$: $1 + 3 + 5 + 7 = 16 = 4^2$.

This suggests:

Guess. *For all $n \in \mathbb{N}$, the sum of the first n odd integers is*

$$1 + 3 + \cdots + (2n - 1) = \sum_{k=1}^{n} (2k - 1) = n^2.$$

How do we prove that this is true for all $n \in \mathbb{N}$?

Properties of $\mathbb{N}$.

(N 1) $1 \in \mathbb{N}$*;*

(N 2) *if $n \in \mathbb{N}$, then it has a* successor $n + 1 \in \mathbb{N}$*;*

(N 3) 1 *is not a successor of any element in* $\mathbb{N}$*;*

(N 4) *if $n, m \in \mathbb{N}$ have the same successor, then $n = m$;*

(N 5) *every $\varnothing \neq S \subset \mathbb{N}$ has a smallest element.*

Claim. *Let $S \subset \mathbb{N}$ be such that:*

(a) $1 \in S$*;*

(b) *if $n \in S$, then $n + 1 \in S$.*

Then $S = \mathbb{N}$.

Proof. Assume that $S \neq \mathbb{N}$. Then

$$\mathbb{N} \setminus S = \{n \in \mathbb{N} : n \notin S\} \neq \varnothing.$$

By (N 5), $\mathbb{N} \setminus S$ then must have a smallest element, say n_0. As $1 \in S$, $n_0 \neq 1$ must hold. Therefore $n_0 - 1 \in \mathbb{N}$ exists and $n_0 - 1 \notin \mathbb{N} \setminus S$, i.e., $n_0 - 1 \in S$. But this means that

$$n_0 = (n_0 - 1) + 1 \in S,$$

which is a contradiction.

Therefore, $S = \mathbb{N}$ must hold. $\square$

Proof of our Guess. Set

$$S := \left\{ n \in \mathbb{N} : \sum_{k=1}^{n} (2k-1) = n^2 \right\}.$$

It is clear that $1 \in S$. Suppose that $n \in S$. We will show that $n+1 \in S$ as well. Consider

$$\begin{aligned}\sum_{k=1}^{n+1} (2k-1) &= \sum_{k=1}^{n} (2k-1) + \underbrace{(2(n+1)-1)}_{=2n+1} \\ &= n^2 + 2n + 1 \\ &= (n+1)^2.\end{aligned}$$

This completes the proof. $\square$

We can distill something far more general out of this:

Principle of Mathematical Induction. *For $n \in \mathbb{N}$, let $P(n)$ be a claim about $n \in \mathbb{N}$ that may be true or not. Suppose that:*

(a) *$P(1)$ is true;*

(b) *whenever $P(n)$ is true, then $P(n+1)$ also true.*

Then $P(n)$ is true for all $n \in \mathbb{N}$.

Proof. Set

$$S := \{n \in \mathbb{N} : P(n) \text{ is true}\}.$$

By (a), $1 \in S$ holds and by (b), $n \in S$ implies $n+1 \in S$. By the previous claim, this means that $S = \mathbb{N}$. $\square$

Example. We claim that

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

for all $n \in \mathbb{N}$. According to the Principle of Induction, we proceed as follows.

$n = 1$ (*induction anchor*): Obviously,

$$\sum_{k=1}^{1} k = 1 = \frac{1 \cdot 2}{2}$$

holds.

$n \rightsquigarrow n+1$ (*induction step*): Suppose that $n \in \mathbb{N}$ is such that $\sum_{k=1}^{n} = \frac{n(n+1)}{2}$. (This is called the *induction hypothesis*). It follows that

$$\begin{aligned}\sum_{k=1}^{n+1} k &= \sum_{k=1}^{n} k + (n+1) \\ &= \frac{n(n+1)}{2} + (n+1), \qquad \text{by the induction hypothesis,} \\ &= \frac{n(n+1)}{2} + \frac{2(n+1)}{2} \\ &= \frac{(n+1)(n+2)}{2},\end{aligned}$$

which proves that the claim is also true for n replaced by $n+1$. By the Principle of Induction, this proves the claim for all $n \in \mathbb{N}$.

Example. We claim that $7^n - 6n - 1$ is divisible by 36 for all $n \in \mathbb{N}$.

$n = 1$ (*induction anchor*): Obviously,

$$7^1 - 6 - 1 = 0$$

is divisible by 36.

$n \rightsquigarrow n+1$ (*induction step*): Note that

$$\begin{aligned}7^{n+1} - 6(n+1) - 1 &= 7^{n+1} - 6n - 6 - 1 \\ &= 7^{n+1} - 6n - 7 \\ &= 7^{n+1} - 42n - 7 + 36n \\ &= 7(7^n - 6n - 1) + 36n.\end{aligned}$$

As $7^n - 6n - 1$ is divisible by 36 by the induction hypothesis, it follows that $7^{n+1} - 6(n+1) - 1$ is divisible by 36 as well. Induction thus proves that the claim is true for all $n \in \mathbb{N}$.

Bernoulli's Inequality. Let $x \geq -1$. We claim that

$$(1+x)^n \geq 1 + nx$$

for all $n \in \mathbb{N}$. We use induction to prove it.

$n = 1$: Clearly,

$$(1+x)^1 = 1 + x = 1 + 1 \cdot x$$

holds.

$n \rightsquigarrow n+1$: We have

$$\begin{aligned}(1+x)^{n+1} &= (1+x)^n \underbrace{(1+x)}_{\geq 0} \\ &\geq (1+nx)(1+x), \quad \text{by the induction hypothesis,} \\ &= 1 + nx + x + \underbrace{nx^2}_{\geq 0} \\ &\geq 1 + (n+1)x.\end{aligned}$$

Induction thus proves Bernoulli's Inequality.

For any set S, its *power set* is defined as

$$\mathfrak{P}(S) := \{A : A \subset S\},$$

i.e., it is the set of all subsets of S.

Examples. 1. If $S = \varnothing$, then $\mathfrak{P}(S) = \{\varnothing\}$;

2. If $S = \{1\}$, then $\mathfrak{P}(S) = \{\varnothing, \{1\}\}$;

3. If $S = \{1, 2\}$, then $\mathfrak{P}(S) = \{\varnothing, \{1\}, \{2\}, \{1, 2\}\}$;

4. If $S = \{1, 2, 3\}$, then $\mathfrak{P}(S) = \{\varnothing, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{1, 3\}, \{1, 2, 3\}\}$.

For the following $n \in \mathbb{N}$, the power set $\mathfrak{P}(\{1, \dots, n\})$ has therefore the following number of elements:

$n = 1$: $2 = 2^1$ elements;

$n = 2$: $4 = 2^2$ elements;

$n = 3$: $8 = 2^3$ elements.

This suggests that the following is true:

Claim. *For all $n \in \mathbb{N}$, the power set $\mathfrak{P}(\{1, \dots, n\})$ has 2^n elements.*

Proof. This induction anchor, i.e., the case $n = 1$ is clear.

Suppose that the claim is true for $n \in \mathbb{N}$. Let $A \subset \{1, \dots, n, n + 1\}$. Then there are two—mutually exclusive—possibilities:

- $n + 1 \notin A$;
- $n + 1 \in A$.

It follows that

$$\mathfrak{P}(\{1, \dots, n, n + 1\}) = \{A \subset \{1, \dots, n, n + 1\} : n + 1 \notin A\} \\ \dot\cup \{A \subset \{1, \dots, n, n + 1\} : n + 1 \in A\}$$

Here, $\dot\cup$ stands for a *disjoint union*, i.e., there is no $A \subset \{1, \dots, n, n + 1\}$ that lies is both subsets of $\mathfrak{P}(\{1, \dots, n, n + 1\})$. It follows that

$$\#\mathfrak{P}(\{1, \dots, n, n + 1\}) = \#\{A \subset \{1, \dots, n, n + 1\} : n + 1 \notin A\} \\ + \#\{A \subset \{1, \dots, n, n + 1\} : n + 1 \in A\},$$

where $\#$ stands for the number of elements in the set concerned.

If $A \subset \{1, \ldots, n, n+1\}$ is such that $n+1 \notin A$, then it is obvious that $A \subset \{1, \ldots, n\}$. It follows that

$$\#\{A \subset \{1, \ldots, n, n+1\} : n+1 \notin A\} = \#\mathfrak{P}(\{1, \ldots, n\}) = 2^n$$

by the induction hypothesis.

For $A \subset \{1, \ldots, n, n+1\}$ such that $n+1 \in A$, define

$$\tilde{A} := A \setminus \{n+1\} \subset \{1, \ldots, n\},$$

and for $B \subset \{1, \ldots, n\}$, set

$$B' := B \cup \{n+1\}.$$

It follows that

$$\left(\tilde{A}\right)' = A \qquad \text{and} \qquad \widetilde{B'} = B.$$

Hence, the subsets of $\{1, \ldots, n, n+1\}$ containing $n+1$ are in a one-to-one correspondence with the subsets of $\{1, \ldots, n\}$; in particular, there are 2^n of them. It follows that

$$\begin{aligned}
&\#\mathfrak{P}(\{1, \ldots, n, n+1\}) \\
&= \#\{A \subset \{1, \ldots, n, n+1\} : n+1 \notin A\} + \#\{A \subset \{1, \ldots, n, n+1\} : n+1 \in A\} \\
&= 2^n + 2^n \\
&= 2^{n+1},
\end{aligned}$$

which proves the claim. □

The Principle of Induction allows for a straightforward and very useful generalization:

Generalization of the Principle of Induction. *Let $n_0 \in \mathbb{Z}$, and let $P(n)$ be a claim about $n \in \mathbb{Z}$ with $n \geq n_0$. Suppose that:*

(a) *$P(n_0)$ is true;*

(b) *whenever $P(n)$ is true, then $P(n+1)$ also true.*

Then $P(n)$ is true for all $n \in \mathbb{Z}$ such that $n \geq n_0$.

Example. Consider n^2 and $n+1$. We obtain for the first four values of n:

$n = 1$: $1^2 = 1 < 2 = 1 + 1$;

$n = 2$: $2^2 = 4 > 3 = 2 + 1$;

$n = 3$: $3^2 = 9 > 4 = 3 + 1$;

$n = 4$: $4^2 = 16 > 5 = 4 + 1$.

This suggests that $n^2 > n + 1$ for $n \geq 2$. We prove this by induction.

The induction anchor with $n = 2$ is clear.

$n \rightsquigarrow n + 1$: Note that

$$\begin{aligned}(n+1)^2 &= n^2 + 2n + 1 \\ &> n + 1 + 2n + 1, \qquad \text{by the induction hypothesis,} \\ &= 3n + 2 \\ &> n + 2,\end{aligned}$$

which proves the claim.

We denote the set of non-negative integers $\{0, 1, 2, 3, \ldots\} = \mathbb{N} \cup \{0\}$ by $\mathbb{N}_0$.

Example. For $n \in \mathbb{N}_0$, its *factorial* $n!$ is defined as follows:

$$n! = \begin{cases} 1, & n = 0, \\ 1 \cdot 2 \cdot \cdots \cdot (n-1) \cdot n, & n \in \mathbb{N}. \end{cases}$$

We observe that

$n = 1$: $1! = 1 = 1^2$;

$n = 2$: $2! = 2 < 4 = 2^2$;

$n = 3$: $3! = 6 < 9 = 3^2$;

$n = 4$: $4! = 24 > 16 = 4^2$;

$n = 5$: $5! = 120 > 25 = 5^2$;

$n = 6$: $6! = 720 > 36 = 6^2$.

This suggests that

$$n! > n^2$$

for $n \geq 4$.

The induction anchor is clear.

$n \rightsquigarrow n + 1$: Observe that

$$\begin{aligned}(n+1)! &= (n+1)n! \\ &> (n+1)n^2, \qquad \text{by the induction hypothesis,} \\ &> (n+1)^2, \qquad \text{by the previous example,}\end{aligned}$$

which proves the claim.

Definition 1.1. Let $n, k \in \mathbb{N}_0$. We define the *binomial coefficient* $\binom{n}{k}$—pronounced: n choose k—as 0 if $k > n$ and

$$\binom{n}{k} := \frac{n!}{k!(n-k)!}$$

if $k \in \{0, \ldots, n\}$.

Remark. It is obvious that

$$\binom{n}{k} = \binom{n}{n-k}$$

for $n \in \mathbb{N}_0$ and $k \in \{0, \ldots, n\}$.

Lemma 1.2 (Pascal's Triangle). *Let $n \in \mathbb{N}$, and let $k \in \{1, \ldots, n\}$. Then*

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

holds.

Remark. The reason why Lemma 1.2 is referred to as "Pascal's Triangle" becomes apparent in the following sketch:

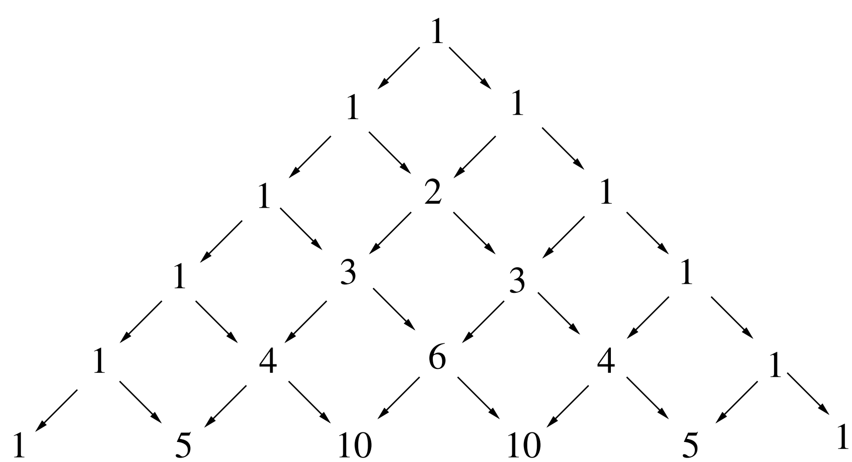

Figure 5: Pascal's Triangle

The first row contains $1 = \binom{0}{0}$, and the second row contains $1 = \binom{1}{0}$ and $1 = \binom{1}{1}$. For the third row, we put $1 = \binom{2}{0} = \binom{2}{2}$ at the endpoints and obtain the midpoint by adding 1 and 1 from the row above to get the midpoint $2 = \binom{2}{1}$. We proceed in this fashion.

Proof. We treat the case where $k = n$ separately. It is then clear that

$$\underbrace{\binom{n}{n}}_{=1} = \underbrace{\binom{n-1}{n-1}}_{=1} + \underbrace{\binom{n-1}{n}}_{=0}.$$

We can thus suppose without loss of generality that $1 \leq k \leq n-1$. It follows that

$$\begin{aligned}\binom{n-1}{k-1} + \binom{n-1}{k} &= \frac{(n-1)!}{(k-1)!(n-k)!} + \frac{(n-1)!}{k!(n-1-k)!} \\ &= \frac{k(n-1)! + (n-k)(n-1)!}{k!(n-k)!} \\ &= \frac{n!}{k!(n-k)!}.\end{aligned}$$

This proves the lemma. ☐

The next lemma reveals why $\binom{n}{k}$ is called n choose k.

Lemma 1.3. *Let $n \in \mathbb{N}$, and let $k \in \mathbb{N}_0$. Then the number of subsets of $\{1, \ldots, n\}$ with exactly k elements is $\binom{n}{k}$.*

Proof. Let c_k^n be the number of subsets of $\{1, \ldots, n\}$ with exactly k elements. It is clear that

$$c_k^n = 0 = \binom{n}{k}$$

if $k > n$ and that

$$c_0^n = 1 = \binom{n}{0}.$$

We can therefore limit ourselves to the case where $1 \leq k \leq n$.

$n = 1$: Then

$$c_1^1 = 1 = \frac{1!}{1!\,0!}$$

holds, which establishes the induction anchor.

$n \rightsquigarrow n+1$: Let $n \in \mathbb{N}$, and suppose that $c_k^n = \binom{n}{k}$ for all $k = 1, \ldots, n$ (and, consequently, for $k = 0, 1, \ldots, n$). We shall deduce that

$$c_k^{n+1} = \binom{n+1}{k}$$

for all $k = 1, \ldots, n, n+1$. As

$$\binom{n+1}{n+1} = 1 = \binom{n+1}{0} = 1 = c_0^{n+1} = c_{n+1}^{n+1},$$

we can limit ourselves to the case where $k = 1, \ldots, n$. Let $A \subset \{1, \ldots, n, n+1\}$ have k elements. Then one and only one of the following cases can occur:

- $n+1 \notin A$;
- $n+1 \in A$.

It follows that

$$\#\{A \subset \{1, \ldots, n, n+1\} : \#A = k,\ n+1 \notin A\} = c_k^n = \binom{n}{k}$$

and

$$\#\{A \subset \{1, \ldots, n, n+1\} : \#A = k,\ n+1 \in A\} = c_{k-1}^n = \binom{n}{k-1},$$

by the induction hypothesis, and therefore

$$c_k^{n+1} = \#\{A \subset \{1, \ldots, n, n+1\} : \#A = k\} = c_k^n + c_{k-1}^n = \binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$$

by Pascal's Triangle. $\square$

Binomial Theorem. *Let $n \in \mathbb{N}_0$. Then*

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

holds for all $x, y \in \mathbb{R}$.

Examples. 1. Let $n = 2$. It follows that

$$(x+y)^2 = y^2 + 2xy + x^2 = x^2 + 2xy + y^2,$$

i.e., the usual (first) binomial formula.

2. Let $x = y = 1$. It follows that

$$2^n = (1+1)^n = \sum_{k=0}^{n} \binom{n}{k} = \sum_{k=0}^{n} c_k^n = \#\mathfrak{P}(\{1, \ldots, n\}).$$

3. Let $n \in \mathbb{N}$, and let $x = -1$ and $y = 1$. It follows that

$$0 = (-1+1)^n = \sum_{k=0}^{n} (-1)^k \binom{n}{k}.$$

Proof. We use induction. If $n = 0$, then both sides of the equality are 1: this establishes the induction anchor.

$n \rightsquigarrow n+1$: Suppose that

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

holds for all $x, y \in \mathbb{R}$. It follows that

$$
\begin{aligned}
(x+y)^{n+1} &= (x+y)^n (x+y) \\
&= \left(\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \right) (x+y), \qquad \text{by the induction hypothesis,} \\
&= \sum_{k=0}^{n} \binom{n}{k} x^{k+1} y^{n-k} + \sum_{k=0}^{n} \binom{n}{k} x^k y^{n+1-k} \\
&= \sum_{k=0}^{n-1} \binom{n}{k} x^{k+1} y^{n-k} + x^{n+1} + y^{n+1} + \sum_{k=1}^{n} \binom{n}{k} x^k y^{n+1-k} \\
&= y^{n+1} + \sum_{k=1}^{n} \binom{n}{k-1} x^k y^{n+1-k} + \sum_{k=1}^{n} \binom{n}{k} x^k y^{n+1-k} + x^{n+1} \\
&= y^{n+1} + \sum_{k=1}^{n} \left(\binom{n}{k-1} + \binom{n}{k} \right) x^k y^{n+1-k} + x^{n+1} \\
&= y^{n+1} + \sum_{k=1}^{n} \binom{n+1}{k} x^k y^{n+1-k} + x^{n+1}, \qquad \text{by Pascal's Triangle,} \\
&= \sum_{k=0}^{n+1} \binom{n+1}{k} x^k y^{n+1-k}.
\end{aligned}
$$

This completes the proof. □

Exercises

1. Show that
$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$
for all $n \in \mathbb{N}$.

2. Show that
$$\sum_{k=1}^{n} (-1)^{k-1} k^2 = (-1)^{n-1} \frac{n(n+1)}{2}$$
for all $n \in \mathbb{N}$.

3. Show that
$$\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$$
for all $n \in \mathbb{N}$.

4. Let $a, b \in \mathbb{R}$. Show that
$$a^{n+1} - b^{n+1} = (a-b) \sum_{k=0}^{n} a^k b^{n-k}$$

for all $n \in \mathbb{N}_0$.

5. Show that
$$\sum_{k=1}^{n} k(k!) = (n+1)! - 1$$
for all $n \in \mathbb{N}$.

6. We use the symbol $\sum$ for finite sums. There is a similar notation for finite products. Let $m, n \in \mathbb{N}$ be such that $m \leq n$, and let $a_m, a_{m+1}, \ldots, a_n \in \mathbb{R}$. Then we define
$$\prod_{k=m}^{n} a_k := a_m a_{m+1} \cdots a_n.$$
Show that
$$\prod_{k=2}^{n} \left(1 - \frac{1}{k^2}\right) = \frac{n+1}{2n}$$
for all $n \geq 2$.

7. Show that
$$\prod_{k=1}^{n-1} \left(1 + \frac{1}{k}\right)^k = \frac{n^n}{n!}$$
for all $n \in \mathbb{N}$ such that $n \geq 2$.

8. For each $n \in \mathbb{N}$, let $P(n)$ be the statement:

   $n^2 + 5n + 1$ *is even.*

   (a) Show that $P(n+1)$ is true whenever $P(n)$ is true.

   (b) For which $n \in \mathbb{N}$ is $P(n)$ actually true?

   What's the moral of this problem?

9. Guess for which $n \in \mathbb{N}$ the inequality $2^n > n^2$ is true, and prove your guess using induction.

2 The Set $\mathbb{Q}$ of Rational Numbers

We denote the set $\{0, 1, -1, 2, -2, \ldots\}$ of *integers* by $\mathbb{Z}$ (presumably because the German word for "number" is "Zahl"), and the set of *rational numbers*

$$\left\{\frac{m}{n} : m, n \in \mathbb{Z},\, n \neq 0\right\}$$

by $\mathbb{Q}$ (presumably because the word "quotient" starts with the letter "q").

The following is (or at least ought to be) well known:

Example. $\sqrt{2}$ is not a rational number.

Definition 2.1. A real number x is called *algebraic* if there are $c_n, \ldots, c_1, c_0 \in \mathbb{Z}$ with $c_n \neq 0$ such that

$$c_n x^n + c_{n-1} x^{n-1} + \cdots + c_1 x + c_0 = 0.$$

Examples.

1. Let $q \in \mathbb{Q}$, i.e., there are $m, n \in \mathbb{Z}$ with $n \neq 0$ such that $q = \frac{m}{n}$. It follows that q solves the equation

$$nx - m = 0$$

and therefore is algebraic.

2. $\sqrt{2}$ solves the equation

$$x^2 - 2 = 0,$$

therefore is algebraic, but fails to be rational.

The following generalizes the statement that $\sqrt{2}$ is not rational:

Rational Zeroes Theorem. *Let $c_n, \ldots, c_1, c_0 \in \mathbb{Z}$ with $n \geq 1$, $c_n \neq 0$, and let $\frac{c}{d}$ with $c, d \in \mathbb{Z}$ and $d \neq 0$ having no prime factor in common solve the equation*

$$c_n x^n + c_{n-1} x^{n-1} + \cdots + c_1 x + c_0 = 0.$$

Then c divides c_0 and d divides c_n.

Proof. We have

$$c_n \left(\frac{c}{d}\right)^n + c_{n-1} \left(\frac{c}{d}\right)^{n-1} + \cdots + c_1 \frac{c}{d} + c_0 = 0.$$

Multiplying by d^n yields

$$c_n c^n + c_{n-1} c^{n-1} d + \cdots + c_1 c d^{n-1} + c_0 d^n = 0. \tag{1}$$

Solving for $c_0 d^n$, we obtain

$$c_0 d^n = -c \left(c_n c^{n-1} + c_{n-1} c^{n-2} d + \cdots + c_1 d^{n-1}\right).$$

This means that c divides $c_0 d^n$. As c and d have no prime factor in common, this is possible only if c divides c_0.

Solving (1) for $c_n c^n$ yields

$$c_n c^n = -d\left(c_{n-1}c^{n-1} + \cdots + c_1 d^{n-2} + c_0 d^{n-1}\right),$$

so that d divides $c_n c^n$. Again, due to the fact that c and d have no prime factor in common, this means that d divides c_n. □

Corollary 2.2. *Consider the equation*

$$x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0 = 0$$

with $c_{n-1}, \ldots, c_1, c_0 \in \mathbb{Z}$ such that $c_0 \neq 0$. Then any rational solution to this equation is an integer dividing c_0.

Proof. Let $c, d \in \mathbb{Z}$ with $d \neq 0$ and having no prime factor in common be such that $\frac{c}{d}$ solves the equation in question. Then, by the Rational Zeroes Theorem, d divides the coefficient of x^n, i.e., 1. It follows that $d = \pm 1$, so that $\frac{c}{d} = \pm c$. Again by the Rational Zeroes Theorem, this means that c divides c_0. □

Examples.

1. Let $m \in \mathbb{N}$ be such that $\sqrt{m} \notin \mathbb{Z}$. Assume that $\sqrt{m}$ is rational. Obviously, $\sqrt{m}$ solves the equation
$$x^2 - m = 0.$$
By Corollary 2.2, this means that $\sqrt{m}$ must be an integer, which is a contradiction. (It follows that the square root of any natural number that is not a perfect square has to be irrational.)

2. Set
$$x = \sqrt{2 + \sqrt[3]{5}}.$$
It follows that $x^2 = 2 + \sqrt[3]{5}$ and thus
$$x^2 - 2 = \sqrt[3]{5}.$$
Cubing this yields
$$5 = (x^2 - 2)^3 = x^6 - 6x^4 + 12x^2 - 8,$$
i.e.,
$$x^6 - 6x^4 + 12x^2 - 13 = 0.$$
If x were rational, then Corollary 2.2 would imply that $x \in \{\pm 1, \pm 13\}$, which is obviously nonsense. Therefore, x must be irrational.

Exercises

1. Is $\sqrt[3]{5 - \sqrt{3}}$ a rational number?

3 The Set $\mathbb{R}$ of Real Numbers

For $x, y \in \mathbb{R}$, the sum $x + y$ and the product xy are defined in $\mathbb{R}$; if $x, y \in \mathbb{Q}$, then $x + y, xy \in \mathbb{Q}$ as well. The following *field axioms* hold for $x, y, z \in \mathbb{R}$:

(F 1) $x + (y + z) = (x + y) + z$ (*associativity of addition*);

(F 2) $x + y = y + x$ (*commutativity of addition*);

(F 3) $x + 0 = x$;

(F 4) there is a unique $-x$ such that $x + (-x) = 0$ (*existence and uniqueness of the additive inverse*);

(F 5) $x(yz) = (xy)z$ (*associativity of multiplication*);

(F 6) $xy = yx$ (*commutativity of multiplication*);

(F 7) $x1 = x$;

(F 8) if $x \neq 0$, the there is a unique x^{-1} such that $xx^{-1} = 1$ (*existence and uniqueness of the multiplicative inverse*);

(F 9) $x(y + z) = xy + xz$ (*distributivity*).

These axioms are satisfied by $\mathbb{R}$ and $\mathbb{Q}$, but much more "exotic" examples are possible:

Example. Let $\mathbb{Z}_2 := \{0, 1\}$ be equipped with an addition $+$ and a multiplication $\cdot$ by the following tables:

$+$	0	1
0	0	1
1	1	0

and

$\cdot$	0	1
0	0	0
1	0	1

It is routinely, albeit tediously verified that $\mathbb{Z}_2$ equipped with these operations satisfies (F 1) to (F 9).

Any set satisfying (F 1) to (F 9) is called a *field*.

Consequences. *For $x, y, z \in \mathbb{R}$, the following hold:*

(i) $x + z = y + z$ *implies* $x = y$;

(ii) $x0 = 0$;

(iii) $(-x)y = -xy$;

(iv) $(-x)(-y) = xy$;

(v) *if* $xz = yz$ *with* $z \neq 0$*, then* $x = y$*;*

(vi) *if* $xy = 0$*, then* $x = 0$ *or* $y = 0$*.*

Proof. (i) Suppose that $x + z = y + z$. This implies that

$$(x + z) + (-z) = (y + z) + (-z)$$

and so, by (F 1),

$$x + (z + (-z)) = y + (z + (-z)),$$

which yields $x = y$ by (F 4) and (F 3).

(ii) By (F 3) and (F 9), we have

$$x0 = x(0 + 0) = x0 + x0,$$

so that

$$0 = x0 + (-x0) = (x0 + x0) + (-x0) = x0 + (x0 + (-x0)) = x0.$$

(iii) As $x + (-x) = 0$, (ii) yields that

$$0 = (x + (-x))y = xy + (-x)y,$$

so that $(-x)y = -xy$.

(iv) Note that, by (iii) and (ii),

$$(-x)(-y) + (-xy) = (-x)(-y) + (-x)y = (-x)((-y) + y) = 0.$$

(v) This is proven in the same way as (i).

(vi) Let $xy = 0$, and suppose that $x \neq 0$. It follows that

$$y = (x^{-1}x)y = x^{-1}(xy) = 0.$$

This completes the proof. □

Remark. All these claims about $\mathbb{R}$ only require the field axioms (F 1) to (F 9). It is clear that any field—such as $\mathbb{Z}_2$ with the addition and multiplication provided—also satisfies the claims above.

The set $\mathbb{R}$ also has an *order structure*, i.e., it satisfies the following *order axioms*. For $x, y, z \in \mathbb{R}$, the following hold:

(O 1) we have $x \leq y$ or $y \leq x$ (note that this in an inclusive "or", i.e., both can hold simultaneously);

(O 2) if $x \leq y$ and $y \leq x$, the $x = y$;

(O 3) if $x \leq y$ and $y \leq z$, then $x \leq z$;

(O 4) if $x \leq y$, then $x + z \leq y + z$;

(O 5) if $x \leq y$ and $z \geq 0$, then $xz \leq yz$.

Consequences. *For $x, y, z \in \mathbb{R}$, the following hold:*

(i) *if $x \leq y$, then $-y \leq -x$;*

(ii) *if $x \leq y$ and $z \leq 0$, then $xz \geq yz$;*

(iii) *if $0 \leq x, y$, then $0 \leq xy$;*

(iv) $0 \leq x^2$;

(v) $0 < 1$;

(vi) *if $0 < x$, then $0 < x^{-1}$;*

(vii) *if $0 < x < y$, then $0 < y^{-1} < x^{-1}$.*

Proof. (i) Set $z := (-x) + (-y)$. It follows from (O 4) that

$$-y = x + z \leq y + z = -x.$$

(ii) From (i), it follows that $0 \leq -z$, so that

$$-xz = x(-z) \leq y(-z) = -yz$$

and therefore $xz \geq yz$ by (i) again.

(iii) Set $x = 0$ in (O 5). It follows that $0 \leq yz$. Relabeling the variables yields the claim.

(iv) If $x \geq 0$, then $x^2 \geq 0$ holds. If $x \leq 0$, then $0 \leq -x$, so that $x^2 = (-x)^2 \geq 0$.

(v) It is clear that $1 = 1^2 \geq 0$. As $x1 = x \neq 0$ for all $x \in \mathbb{R} \setminus \{0\}$, we have $1 \neq 0$ and thus $1 > 0$.

(vi) Assume that $x > 0$, but that $0 \not< x^{-1}$, i.e., $x^{-1} \leq 0$ and therefore $0 \leq -x^{-1}$. It follows that

$$0 \leq x(-x^{-1}) = -1,$$

which contradicts the fact that $1 > 0$.

(vii) Similar.

$\square$

Definition 3.1. For $x \in \mathbb{R}$, define its *absolute value* or *modulus* $|x|$ as

$$|x| := x$$

if $x \geq 0$ and

$$|x| := -x$$

if $x \leq 0$.

Remark. It is clear that $|x|$ is *well defined*, i.e., if the two cases overlap, which means that $x \geq 0$ and $x \leq 0$ and therefore $x = 0$, we have $|x| = 0$ in any case.

Theorem 3.2. *The following are true for $x, y \in \mathbb{R}$:*

(i) $|x| \geq 0$*;*

(ii) $|xy| = |x||y|$*;*

(iii) $|x + y| \leq |x| + |y|$ (triangle inequality).

Proof. (i) Clear.

(ii) We have four cases:

(a) $x, y \geq 0$: then $xy \geq 0$ as well, so that $|x| = x$, $|y| = y$, and $|xy| = xy$;

(b) $x, y \leq 0$: them $|x| = -x$, $|y| = -y$, so that $|x||y| = (-x)(-y) = xy = |xy|$;

(c) $x \leq 0$ and $y \geq 0$: $|x||y| = (-x)y = -xy = |xy|$;

(d) $x \geq 0$ and $y \leq 0$: similar.

(iii) It is clear that

$$-|x| \leq x \leq |x| \qquad \text{and} \qquad -|y| \leq y \leq |y|.$$

Adding those inequalities yields

$$-(|x| + |y|) \leq x + y \leq |x| + |y|.$$

This means that

$$x + y \leq |x| + |y| \qquad \text{and} \qquad -(x + y) \leq |x| + |y|,$$

which yields the claim.

$\square$

Corollary 3.3 ("Phony" Triangle Inequality)**.** *Let $x, y \in \mathbb{R}$. Then*

$$||x| - |y|| \leq |x - y|$$

holds.

Proof. Note that

$$|x| = |y + (x - y)| \leq |y| + |x - y|.$$

Interchanging the rôles of x and y yields

$$|y| \leq |x| + |x - y|,$$

so that

$$\pm(|x| - |y|) \leq |x - y|$$

and therefore

$$||x| - |y|| \leq |x - y|.$$

This proves the claim. $\square$

Exercises

1. Let
$$\mathbb{Q}\left[\sqrt{2}\right] := \left\{p + q\sqrt{2} : p, q \in \mathbb{Q}\right\}.$$
Show that, if $x \in \mathbb{Q}\left[\sqrt{2}\right] \setminus \{0\}$, then $x^{-1} \in \mathbb{Q}\left[\sqrt{2}\right]$ as well.

2. For $n \in \mathbb{N}$, let $x_1, \ldots, x_n \in \mathbb{R}$. Show that
$$\left|\sum_{k=1}^{n} x_k\right| \leq \sum_{k=1}^{n} |x_k|.$$

4 The Difference between $\mathbb{R}$ and $\mathbb{Q}$

Definition 4.1. Let $\varnothing \neq S \subset \mathbb{R}$. Then:

(a) we call $x_0 \in \mathbb{R}$ a *maximum* for S if $x_0 \in S$ and $x \leq x_0$ for all $x \in S$;

(b) we call $x_0 \in \mathbb{R}$ a *minimum* for S if $x_0 \in S$ and $x \geq x_0$ for all $x \in S$.

We use the notation $\max S$ for the maximum and $\min S$ for the minimum of S.

Examples. 1. All non-empty finite subsets of $\mathbb{R}$ have a minimum and a maximum.

2. $\mathbb{Q}$ and $\mathbb{Z}$ have no maximum and no minimum.

3. $\mathbb{N}$ has no maximum, but $\min \mathbb{N} = 1$.

4. The set
$$S := \left\{q \in \mathbb{Q} : q \geq 0,\, q^2 \leq 2\right\}$$
has 0 as its minimum, but no maximum because $\sqrt{2} \notin \mathbb{Q}$.

Definition 4.2. Let $\varnothing \neq S \subset \mathbb{R}$. Then:

(a) if $M \in \mathbb{R}$ satisfies $x \leq M$ for all $x \in S$, we call S *bounded above* by M and M an *upper bound* for S;

(b) if $m \in \mathbb{R}$ satisfies $x \geq m$ for all $x \in S$, we call S *bounded below* by m and m a *lower bound* for S;

(c) if S bounded both above and below, we call S *bounded*.

Examples. 1. Suppose that $M := \max S$ exists. Then M is an upper bound for S.

2. Suppose that $m := \min S$ exists. Then m is a lower bound for S.

3. Neither of $\mathbb{Q}$, $\mathbb{Z}$, or $\mathbb{N}$ is bounded above, and $\mathbb{Q}$ and $\mathbb{Z}$ aren't bounded below either. However, $\mathbb{N}$ is bounded below by its minimum 1.

4. Let $a, b \in \mathbb{R}$ be such that $a < b$, then we define:
$$\begin{aligned}
(a,b) &:= \{x \in \mathbb{R} : a < x < b\} && \text{(open interval from } a \text{ to } b\text{)},\\
[a,b] &:= \{x \in \mathbb{R} : a \leq x \leq b\} && \text{(closed interval from } a \text{ to } b\text{)},\\
(a,b] &:= \{x \in \mathbb{R} : a < x \leq b\} && \text{(left open interval from } a \text{ to } b\text{)},
\end{aligned}$$
and
$$[a,b) := \{x \in \mathbb{R} : a \leq x < b\} \qquad \text{(right open interval from } a \text{ to } b\text{)}.$$

Then all these sets are bounded below by a and above by b. However, a is a minimum for $[a,b)$ and $[a,b]$, but not for (a,b) and $(a,b]$, and b is a maximum for $(a,b]$ and $[a,b]$, but not for (a,b) and $[a,b)$.

5. If S is bounded with lower bound m and upper bound M, then

$$m \leq x \leq M$$

and, consequently,

$$-M \leq -x \leq -m$$

for all $x \in S$. Setting $C := \max\{M, -m\}$, this means that

$$-C \leq \pm x \leq C$$

and therefore

$$|x| \leq C$$

for $x \in S$.

Definition 4.3. Let $\varnothing \neq S \subset \mathbb{R}$. Then:

(a) if S has a *least upper bound*, i.e., there is an upper bound M for S such that, whenever $\tilde{M} < M$, there is $x \in S$ with $x > \tilde{M}$, we call M the *supremum* of S, denoted by $\sup S$;

(b) if S has a *largest lower bound*, we call it the *infimum* of S, denoted by $\inf S$.

Examples. 1. If $\sup S$ and $\inf S$ exist, they are unique.

2. If $\max S$ exists, then $\sup S = \max S$; similarly, if $\min S$ exists, then $\inf S = \min S$.

3. Let $a, b \in \mathbb{R}$ be such that $a < b$. Then we have

$$\sup(a,b) = \sup(a,b] = \sup[a,b) = \sup[a,b] = b$$

and

$$\inf(a,b) = \inf(a,b] = \inf[a,b) = \inf[a,b] = a.$$

4. Consider

$$S = \left\{q \in \mathbb{Q} : 0 \leq q,\, q^2 \leq 2\right\}.$$

Then we have

$$\inf S = \min S = 0 \qquad \text{and} \qquad \sup S = \sqrt{2} \in \mathbb{R} \setminus \mathbb{Q}.$$

The last example shows that the following is false for $\mathbb{Q}$ instead of $\mathbb{R}$:

Completeness Axiom. Let $\varnothing \neq S \subset \mathbb{R}$ be bounded above. Then S has a supremum.

In fact, together with the field axioms and the order axioms the Completeness Axiom characterizes $\mathbb{R}$.

Corollary 4.4. *Let $\varnothing \neq S \subset \mathbb{R}$ be bounded below. Then S has an infimum.*

Proof. Set

$$-S := \{-x : x \in S\}.$$

Let m be a lower bound for S, i.e., $m \leq x$ for all $x \in S$. It follows that $-x \leq -m$ for all $x \in S$, so that $-m$ is an upper bound for $-S$. From the Completeness Axiom, it follows that $\sup(-S)$ exists. It is clear that $-x \leq \sup(-S)$ and therefore $x \geq -\sup(-S)$ for all $x \in S$; hence, $-\sup(-S)$ is a lower bound for S. Assume that $\tilde{m} > -\sup(-S)$ is a lower bound for S. Then it follows that $-\tilde{m} < \sup(-S)$ and thus cannot be an upper bound for $-S$. This means that there is $x \in S$ with $-\tilde{m} < -x$, so that $\tilde{m} > x$, which is impossible if $\tilde{m}$ is a lower bound for S. $\square$

The following is crucial:

Archimedian Property. *Let $a, b > 0$ be reals numbers. Then there is $n \in \mathbb{N}$ such that $na > b$.*

Proof. Assume that the claim is false, i.e., $na \leq b$ for all $n \in \mathbb{N}$. This means that the set

$$S := \{na : n \in \mathbb{N}\}$$

is bounded above by b and therefore has a supremum. As $a > 0$, we have $\sup S - a < \sup S$, so that $\sup S - a$ cannot be an upper bound for S. It follows that there is $n_0 \in \mathbb{N}$ such that $n_0 a > \sup S - a$. This, however, entails that

$$\underbrace{(n_0 + 1)a}_{\in S} > \sup S,$$

which is a contradiction. $\square$

Corollary 4.5. *Let $a, b > 0$ be real numbers. Then there are $n, m \in \mathbb{N}$ such that*

$$b < n \qquad \text{and} \qquad \frac{1}{m} < a.$$

Proof. Applying the Archimedian Property with $a = 1$ yields $n \in \mathbb{N}$ with $n > b$. Applying the Archimedian Property again, with $b = 1$ this time, shows that there is $m \in \mathbb{N}$ such that $ma > 1$, i.e., $\frac{1}{m} < a$. $\square$

Corollary 4.6 (Density of $\mathbb{Q}$)**.** *Let $a, b \in \mathbb{R}$ be such that $a < b$. Then there is $q \in \mathbb{Q}$ such that $a < q < b$.*

Proof. We need to show that there are $m \in \mathbb{Z}$ and $n \in \mathbb{N}$ such that

$$a < \frac{m}{n} < b.$$

As $b - a > 0$, there is $n \in \mathbb{N}$ such that $b - a > \frac{1}{n}$ by the previous corollary. The subset

$$S := \{\nu \in \mathbb{Z} : \nu > na\}$$

of $\mathbb{N}$ is non-empty by the Archimedian Property and therefore has a minimum m. It follows that

$$m > na \qquad \text{and} \qquad m - 1 \leq na,$$

so that

$$na < m \leq na + 1 < na + n(b - a) = nb$$

and therefore

$$a < \frac{m}{n} < b.$$

This completes the proof. □

Corollary 4.7 (Density of $\mathbb{R} \setminus \mathbb{Q}$)**.** *Let $a, b \in \mathbb{R}$ be such that $a < b$. Then there is $r \in \mathbb{R} \setminus \mathbb{Q}$ such that $a < r < b$.*

Proof. Choose $q \in \mathbb{Q}$ such that $a < q < b$. As $b - q > 0$, there is $n \in \mathbb{N}$ such that $n(b - q) > \sqrt{2}$, i.e., $b - q > \frac{\sqrt{2}}{n}$ and therefore

$$a < \underbrace{q + \frac{\sqrt{2}}{n}}_{\notin \mathbb{Q}} < b,$$

which completes the proof. □

The Symbols $+\infty$ and $-\infty$

We add the symbols $+\infty$ and $-\infty$ to $\mathbb{R}$ as objects that do not belong to $\mathbb{R}$ and equip $\mathbb{R} \cup \{+\infty, -\infty\}$ with and order structure with

$$-\infty < x < +\infty$$

for all $x \in \mathbb{R}$. For notational convenience, we will mostly write ∞ instead of $+\infty$. We also set

$$x + \infty = \infty \qquad \text{and} \qquad x - \infty := x + (-\infty) := -\infty$$

for all $x \in \mathbb{R}$ as well as

$$\infty + \infty := \infty \qquad \text{and} \qquad -\infty - \infty := -\infty + (-\infty) := -\infty.$$

Warning. Expressions like

$$\infty - \infty \qquad \text{or} \qquad -\infty + \infty$$

are undefined and—even worse—cannot be defined in any meaningful way. They must be avoided at all cost.

We use the following notation for $a \in \mathbb{R}$:

$$\begin{aligned}
(a, \infty) &:= \{x \in \mathbb{R} : x > a\}, \\
[a, \infty) &:= \{x \in \mathbb{R} : x \geq a\}, \\
(-\infty, a] &:= \{x \in \mathbb{R} : x \leq a\}, \\
(-\infty, a) &:= \{x \in \mathbb{R} : x < a\},
\end{aligned}$$

and

$$(-\infty, \infty) := \mathbb{R}.$$

Moreover, if $\varnothing \neq S \subset \mathbb{R}$ is not bounded above, we set $\sup S := \infty$, and if S is not bounded below, we set $\inf S := -\infty$.

Example. Let $S, T \subset \mathbb{R}$ be non-empty, and define

$$S + T := \{x + y : x \in S,\ y \in T\}.$$

We claim that

$$\sup(S + T) = \sup S + \sup T.$$

Suppose first that S is not bounded above. Let $M \in \mathbb{R}$, and let $y_0 \in T$. As S is not bounded above, there is $x \in S$ such that $x > M - y_0$, so that $x + y_0 > M$. Since $x + y_0 \in S + T$, this means that $S + T$ is not bounded above. Similarly, if T is not bounded above, then $S + T$ is not bounded above either. It follows that, if S or T fails to be bounded above, then

$$\sup(S + T) = \infty = \sup S + \sup T.$$

We can therefore suppose without loss of generality that both S and T are bounded above. Let $x \in S$ and $y \in T$. Then

$$x + y \leq \sup S + \sup T,$$

so that $\sup S + \sup T$ is an upper bound for $S + T$, which entails

$$\sup(S + T) \leq \sup S + \sup T.$$

Assume towards a contradiction that $\sup(S+T) < \sup S + \sup T$, i.e.,

$$\epsilon := \sup S + \sup T - \sup(S+T) > 0.$$

As $\sup S - \frac{\epsilon}{2} < \sup S$, there is $x_0 \in S$ such that $x_0 > \sup S - \frac{\epsilon}{2}$. Similarly, there is $y_0 \in T$ such that $y_0 > \sup T - \frac{\epsilon}{2}$. It follows that

$$\begin{aligned}
\underbrace{x_0 + y_0}_{\in S+T} &> \sup S - \frac{\epsilon}{2} + \sup S - \frac{\epsilon}{2} \\
&= \sup S + \sup T - \epsilon \\
&= \sup S + \sup T - \sup S - \sup T + \sup(S+T) \\
&= \sup(S+T),
\end{aligned}$$

which is a contradiction.

Exercises

1. Let $\varnothing \neq S \subset \mathbb{R}$ be bounded such that $\inf S = \sup S$. What can you say about S?

2. Let $a, b \in \mathbb{R}$ be such that $a < b$. Show that there are infinitely many irrational numbers $r \in \mathbb{R} \setminus \mathbb{Q}$ such that $a < r < b$.

3. Let $\varnothing \neq S, T \subset \mathbb{R}$ be bounded above. Show that

$$\sup(S \cup T) = \max\{\sup S, \sup T\}.$$

4. Determine $\inf S$ and $\sup S$ for

$$S := \left\{ (-1)^n \left(1 - \frac{1}{n}\right) : n \in \mathbb{N} \right\}.$$

5. Determine $\inf S$ and $\sup S$ for

$$S := \left\{ \frac{1}{p} : p \text{ is a prime number} \right\}$$

(*Hint*: You can—and *should*—use the fact that there are infinitely many prime numbers, so that, for each $n \in \mathbb{N}$, there is a prime number p such that $p \geq n$.)

6. Determine $\inf S$ and $\sup S$ for

$$S := \left\{ n((-1)^n + 1) + \frac{1}{n} : n \in \mathbb{N} \right\}.$$

5 Functions

Definition 5.1. Let S and T be non-empty sets. Then a *function* f or *map* from S to T—in symbols: $f\colon S \to T$—is a rule that assigns to each $s \in S$ a unique element $f(s) \in T$.

We use the notation

$$f\colon S \to T, \quad s \mapsto f(s).$$

Examples.

1. Let

$$p(x) := c_n x^n + \cdots + c_1 x + c_0$$

with $c_n, \ldots, c_1, c_0 \in \mathbb{R}$ be polynomial. Then p defines a function

$$p\colon \mathbb{R} \to \mathbb{R}, \quad x \mapsto p(x).$$

2. Let p and $q \neq 0$ be polynomials, and let $Z := \{x \in \mathbb{R} : q(x) = 0\}$. Then

$$r\colon \mathbb{R} \setminus Z \to \mathbb{R}, \quad x \mapsto \frac{p(x)}{q(x)}$$

defines a function from $\mathbb{R} \setminus Z$ into $\mathbb{R}$.

3. sin, cos, and exp define functions from $\mathbb{R}$ to $\mathbb{R}$.

4. The natural logarithm log defines a function

$$\log\colon (0,\infty) \to \mathbb{R}, \quad x \mapsto \log x$$

with $\log(\exp x) = x$ for $x \in \mathbb{R}$ and $\exp(\log x) = x$ for $x \in (0,\infty)$.

5. Define $f\colon \mathbb{R} \to \mathbb{R}$ as follows:

$$f(x) := \begin{cases} 0, & x \notin \mathbb{Q}, \\ \frac{1}{q}, & x = \frac{p}{q} \neq 0 \text{ with } p \in \mathbb{Z} \text{ and } q \in \mathbb{N} \text{ coprime}, \\ 1, & x = 0. \end{cases}$$

Terminology. Let S and T be non-empty sets, and let $f\colon S \to T$ be a function. Then we call

(a) S the *domain* of f,

(b) T the *codomain* of f, and

(c) $f(S) := \{f(s) : s \in S\}$ the *range* of f.

Definition 5.2. Let S and T be non-empty sets, and let $f\colon S \to T$ be a function. Then we call f:

(a) *injective* if, for $s_1, s_2 \in S$ with $s_1 \neq s_2$, we have $f(s_1) \neq f(s_2)$;

(b) *surjective* if $f(S) = T$;

(c) *bijective* if is both injective and surjective.

Remark. For any function $f\colon S \to T$, the function

$$f\colon S \to f(S), \quad s \mapsto f(s)$$

is surjective.

Examples. 1. Consider

$$f\colon \mathbb{R} \to \mathbb{R}, \quad x \mapsto x^2.$$

Then f is neither injective nor surjective because $f(-1) = 1 = f(1)$ and $-1 \notin f(\mathbb{R})$.

2. Consider

$$f\colon \mathbb{R} \to [0, \infty), \quad x \mapsto x^2.$$

The f is not injective, but surjective.

3. Consider

$$f\colon [0, \infty) \to \mathbb{R}, \quad x \mapsto x^2.$$

Then f is not surjective, but injective.

4. Finally,

$$f\colon [0, \infty) \to [0, \infty), \quad x \mapsto x^2$$

is bijective.

Definition 5.3. Let S and T be non-empty sets, and let $f\colon S \to T$ be a function. Then:

(a) for $A \subset S$, define

$$f(A) := \{f(s) : s \in A\},$$

the *image of A under f*;

(b) for $B \subset T$, define

$$f^{-1}(B) := \{s \in S : f(s) \in B\},$$

the *inverse image of B under f*.

Observation. Suppose that $f\colon S \to T$ is injective, let $t \in T$, and set $B := \{t\}$. Then

$$f^{-1}(B) = \{s \in S : f(s) = t\}$$

is either empty—if $t \notin f(S)$—or consists of one single element of S.

Definition 5.4. Let S and T be non-empty sets, and let $f\colon S \to T$ be injective. Then

$$f^{-1}\colon f(S) \to S, \quad f(s) \mapsto s$$

is called the *inverse function* of f.

Examples.

1. If
$$f\colon [0,\infty) \to [0,\infty), \quad x \mapsto x^2,$$
then
$$f^{-1}\colon [0,\infty) \to [0,\infty), \quad y \mapsto \sqrt{y}.$$

2. If
$$f\colon \mathbb{R} \to (0,\infty), \quad x \mapsto e^x,$$
then
$$f^{-1}\colon (0,\infty) \to \mathbb{R}, \quad y \mapsto \log y.$$

3. If
$$f\colon \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{R}, \quad x \mapsto \frac{\sin x}{\cos x} = \tan x,$$
then
$$f^{-1}\colon \mathbb{R} \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \quad y \mapsto \arctan y.$$

Exercises

1. Let S be a non-empty set. Show that there is *no* surjective map from S to $\mathfrak{P}(S)$. (*Hint*: Assume that there is a surjective map $f : S \to \mathfrak{P}(S)$ and consider the set $\{x \in S : x \notin f(x)\}$.)

2. For $n \in \mathbb{N}$, let
$$\mathfrak{S}_n := \{\sigma\colon \{1,\ldots,n\} \to \{1,\ldots,n\} : \sigma \text{ is bijective}\}.$$
Use induction to show that $\#\mathfrak{S}_n = n!$.

6 Convergence of Sequences

Definition 6.1. Let $n_0 \in \mathbb{Z}$ and let T be a non empty set. A *sequence* in T is a function $s\colon \{n \in \mathbb{Z} : n \geq n_0\} \to T$.

Notation. We write:

- $(s_n)_{n=n_0}^{\infty}$ instead of $s\colon \{n \in \mathbb{Z} : n \geq n_0\} \to T$;
- s_n instead of $s(n)$ for $n \in \mathbb{Z}$ with $n \geq n_0$.

We will mostly be concerned with sequences in $\mathbb{R}$. In what follows, we will deal with the case where $n_0 = 1$ only, but it is clear how the statements extend to more general n_0.

Definition 6.2. Let $(x_n)_{n=1}^{\infty}$ be a sequence in $\mathbb{R}$. Then we say that $(x_n)_{n=1}^{\infty}$ *converges* to $x \in \mathbb{R}$ or is *convergent* to x if, for each $\epsilon > 0$, there is $n_\epsilon \in \mathbb{N}$ such that $|x_n - x| < \epsilon$ for all $n \in \mathbb{N}$ with $n \geq n_\epsilon$. We call x the *limit* of $(x_n)_{n=1}^{\infty}$.

Notation. If x is the limit of $(x_n)_{n=1}^{\infty}$, we write

$$x = \lim_{n\to\infty} x_n \qquad \text{or} \qquad x_n \overset{n\to\infty}{\longrightarrow} x \qquad \text{or} \qquad x_n \to x.$$

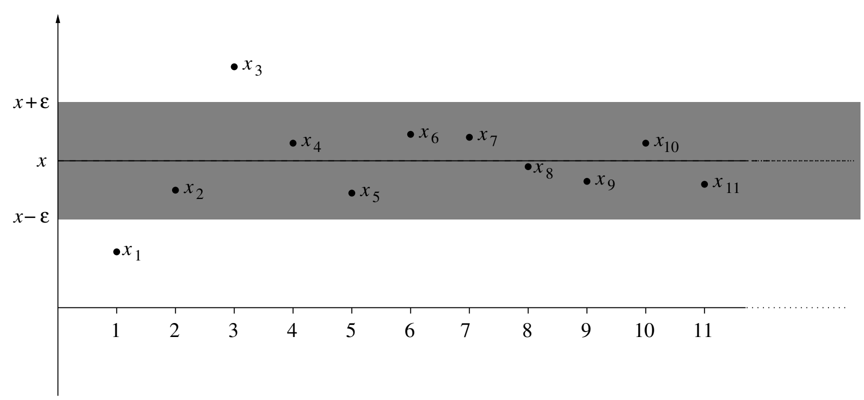

Figure 6: $\lim_{n\to\infty} x_n = x$

Examples.

1. Let $c \in \mathbb{R}$, and let $(x_n)_{n=1}^{\infty}$ be defined as $x_n := c$ for $n \in \mathbb{N}$. Let $\epsilon > 0$, and set $n_\epsilon := 2021$. Then
$$|x_n - c| = 0 < \epsilon$$
holds for all $n \geq n_\epsilon$, so that $c = \lim_{n\to\infty} x_n$.

2. Consider the sequence $\left(\frac{1}{n}\right)_{n=1}^{\infty}$. Let $\epsilon > 0$. By the Archimedian Property of $\mathbb{R}$, there is $n_\epsilon \in \mathbb{N}$ such that $\frac{1}{n_\epsilon} < \epsilon$. Consequently, we have for $n \geq n_\epsilon$ that
$$\left|\frac{1}{n}\right| = \frac{1}{n} \leq \frac{1}{n_\epsilon} < \epsilon.$$

It follows that $\lim_{n\to\infty} \frac{1}{n} = 0$.

3. Let $k \in \mathbb{N}$, and set $x_n := \frac{1}{n^k}$. Let $\epsilon > 0$. By the previous example, there is n_ϵ such that $\frac{1}{n} < \epsilon$ for all $n \geq n_\epsilon$. It follows that

$$\left| \frac{1}{n^k} \right| = \frac{1}{n^k} \leq \frac{1}{n} < \epsilon$$

for $n \geq n_\epsilon$, so that $\frac{1}{n^k} \to 0$.

4. Let $x_n := (-1)^n$ for $n \in \mathbb{N}$. What is the limit of $(x_n)_{n=1}^\infty$?

Claim. $(x_n)_{n=1}^\infty$ *does not converge.*

To see this, assume towards a contradiction that $(x_n)_{n=1}^\infty$ has a limit x. Then there is $n_1 \in \mathbb{N}$ such that $|x_n - x| < 1$ for $n \geq n_1$. It follows that

$$2 = |x_n - x_{n+1}| \leq |x_n - x| + |x - x_{n+1}| < 1 + 1 = 2$$

for $n \geq n_1$, which is impossible.

Proposition 6.3. *Let $(x_n)_{n=1}^\infty$ be a convergent sequence in $\mathbb{R}$. Then the limit of $(x_n)_{n=1}^\infty$ is unique.*

Proof. Assume that there are $x, \tilde{x} \in \mathbb{R}$ with $x \neq \tilde{x}$ that are both limits of $(x_n)_{n=1}^\infty$. Set $\epsilon := |x - \tilde{x}| > 0$. As $x_n \to x$, there is $n_1 \in \mathbb{N}$ such that $|x_n - x| < \frac{\epsilon}{2}$ for $n \geq n_1$. As also $x_n \to \tilde{x}$, there is $n_2 \in \mathbb{N}$ such that $|x_n - \tilde{x}| < \frac{\epsilon}{2}$ for $n \geq n_2$. For $n \geq \max\{n_1, n_2\}$, we therefore have

$$\epsilon = |x - \tilde{x}| = |x - x_n + x_n - \tilde{x}| \leq |x - x_n| + |x_n - \tilde{x}| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

which is a contradiction. □

Theorem 6.4. *Let $(x_n)_{n=1}^\infty$ and $(y_n)_{n=1}^\infty$ be convergent sequences in $\mathbb{R}$ with limits x and y, respectively. Then $(x_n + y_n)_{n=1}^\infty$ is convergent with*

$$\lim_{n\to\infty} (x_n + y_n) = x + y.$$

Proof. Let $\epsilon > 0$. As $x_n \to x$, there is $n_1 \in \mathbb{N}$ such that $|x_n - x| < \frac{\epsilon}{2}$ for $n \geq n_1$, and as $y_n \to y$, there is $n_2 \in \mathbb{N}$ such that $|y_n - y| < \frac{\epsilon}{2}$ for $n \geq n_2$. Set $n_\epsilon := \max\{n_1, n_2\}$. Then it follows for $n \geq n_\epsilon$ that

$$|(x_n + y_n) - (x + y)| = |(x_n - x) + (y_n - y)| \leq |x_n - x| + |y_n - y| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

This means that $x_n + y_n \to x + y$. □

We convene to call a sequence $(x_n)_{n=1}^\infty$ *bounded above*, *bounded below*, or *bounded* if the corresponding statement is true for the set $\{x_n : n \in \mathbb{N}\}$.

Example. Let $\theta > 1$, and set $x_n := \theta^n$ for $n \in \mathbb{N}$. From Bernoulli's Inequality, we obtain that

$$x_n = (1 + \underbrace{(\theta - 1)}_{>0})^n \geq 1 + n(\theta - 1)$$

for $n \in \mathbb{N}$. By the Archimedian Property of $\mathbb{R}$, this means that $(x_n)_{n=1}^\infty$ cannot be bounded.

Proposition 6.5. *Let $(x_n)_{n=1}^\infty$ be a convergent sequence. Then $(x_n)_{n=1}^\infty$ is bounded.*

Proof. Let $x := \lim_{n\to\infty} x_n$. Then there is $n_1 \in \mathbb{N}$ such that $|x_n - x| < 1$ for $n \geq n_1$ and therefore

$$|x_n| = |x_n - x + x| \leq |x_n - x| + |x| < 1 + |x|$$

for those n. Set

$$C := \max\{|x_1|, \ldots, |x_{n_1 - 1}|, 1 + |x|\},$$

so that $|x_n| \leq C$ for $n \in \mathbb{N}$. This proves the claim. $\square$

Theorem 6.6. *Let $(x_n)_{n=1}^\infty$ and $(y_n)_{n=1}^\infty$ be convergent sequences in $\mathbb{R}$ with limits x and y, respectively. Then $(x_n y_n)_{n=1}^\infty$ is convergent with*

$$\lim_{n\to\infty} x_n y_n = xy.$$

Proof. As $(y_n)_{n=1}^\infty$ is convergent, it is bounded, i.e., there is $C \geq 0$ such that $|y_n| \leq C$ for all $n \in \mathbb{N}$.

Let $\epsilon > 0$. Choose $n_1 \in \mathbb{N}$ such that

$$|x_n - x| < \frac{\epsilon}{2(C+1)}$$

for $n \geq n_1$ and $n_2 \in \mathbb{N}$ such that

$$|y_n - y| < \frac{\epsilon}{2(|x| + 1}$$

for $n \geq n_2$. Set $n_\epsilon := \max\{n_1, n_2\}$. For $n \geq n_\epsilon$, we then have

$$\begin{aligned}
|x_n y_n - xy| &= |x_n y_n - x y_n + x y_n - xy| \\
&\leq |x_n y_n - x y_n| + |x y_n - xy| \\
&= |x_n - x||y_n| + |x||y_n - y| \\
&\leq C|x_n - x| + |x||y_n - y| \\
&< \frac{C\epsilon}{2(C+1)} + \frac{|x|\epsilon}{2(|x|+1)} \\
&\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon.
\end{aligned}$$

$\square$

Example.
$$\lim_{n\to\infty}\left(2+\frac{7}{n}-\frac{3}{n^2}\right)\left(2+\frac{13}{n^{42}}\right)=4.$$

Theorem 6.7. *Let $(x_n)_{n=1}^\infty$ be a convergent sequence in $\mathbb{R}$ with limit $x \neq 0$. Then there is $n_0 \in \mathbb{N}$ such that $x_n \neq 0$ for all $n \geq n_0$, and the sequence $\left(\frac{1}{x_n}\right)_{n=n_0}^\infty$ converges to $\frac{1}{x}$.*

Proof. Let $n_0 \in \mathbb{N}$ be such that $|x_n - x| < \frac{|x|}{2}$ for $n \geq n_0$. It follows that
$$|x_n| = |(x_n - x) + x| \geq ||x_n - x| - |x|| > \frac{|x|}{2} > 0$$
and therefore $x_n \neq 0$ for $n \geq n_0$. This also shows that $\frac{1}{|x_n|} < \frac{2}{|x|}$ and therefore $n \geq n_0$. Hence, the sequence $\left(\frac{1}{x_n}\right)_{n=n_0}^\infty$ is bounded, as is the sequence $\left(\frac{1}{x_n x}\right)_{n=n_0}^\infty$. Let $C \geq 0$ be such that $\frac{1}{|x_n x|} \leq C$ for $n \geq n_0$.

Let $\epsilon > 0$. Choose $n_\epsilon \geq n_0$ such that $|x_n - x| < \frac{\epsilon}{C+1}$ for $n \geq n_\epsilon$. For those n, we obtain
$$\left|\frac{1}{x_n}-\frac{1}{x}\right| = \frac{|x_n - x|}{|x_n x|} \leq C|x_n - x| < \frac{C\epsilon}{C+1} < \epsilon.$$
This proves the claim. □

Corollary 6.8. *Let $(x_n)_{n=1}^\infty$ and $(y_n)_{n=1}^\infty$ be convergent sequences in $\mathbb{R}$ with limits x and y, respectively, such that $y \neq 0$. Then there is $n_0 \in \mathbb{N}$ such that $y_n \neq 0$ for all $n \geq n_0$, and $\left(\frac{x_n}{y_n}\right)_{n=n_0}^\infty$ converges such that*
$$\lim_{n\to\infty}\frac{x_n}{y_n} = \frac{x}{y}.$$

Example.
$$\lim_{n\to\infty}\frac{4n^3 - 3n^2 + 17n - 666}{3n^3 + n^2 - n + 2021} = \lim_{n\to\infty}\frac{4 - \frac{3}{n} + \frac{17}{n^2} - \frac{666}{n^3}}{3 + \frac{1}{n} - \frac{1}{n^2} + \frac{2021}{n^3}} = \frac{4}{3}.$$

Together, Theorems 6.4 and 6.6 and Corollary 6.8 are often refereed to as "the Limit Laws".

Divergence of Sequences

Definition 6.9. Let $(x_n)_{n=1}^\infty$ be a sequence in $\mathbb{R}$. We say that $(x_n)_{n=1}^\infty$:

(a) *diverges* or is *divergent* if it does not converge;

(b) *diverges to* ∞ if, for each $R \in \mathbb{R}$, there is $n_R \in \mathbb{N}$ such that $x_n > R$ for $n \geq n_R$;

(c) *diverges to* $-\infty$ if, for each $R \in \mathbb{R}$, there is $n_R \in \mathbb{N}$ such that $x_n < R$ for $n \geq n_R$.

Notation. If $(x_n)_{n=1}^\infty$ diverges to ∞, we write

$$\lim_{n\to\infty} x_n = \infty \qquad \text{or} \qquad x_n \overset{n\to\infty}{\longrightarrow} \infty \qquad \text{or} \qquad x_n \to \infty.$$

Similar conventions apply to $-\infty$.

Examples. 1. The sequence $((-1)^n)_{n=1}^\infty$ is divergent.

2. Every sequence diverging to ∞ or to $-\infty$ is unbounded. However, $(n(-1)^n)_{n=1}^\infty$ is unbounded, but diverges neither to ∞ nor to $-\infty$.

The following is clear:

Proposition 6.10. *Let $(x_n)_{n=1}^\infty$ be a sequence in $\mathbb{R}$. Then $(x_n)_{n=1}^\infty$ diverges to ∞ if and only if $(-x_n)_{n=1}^\infty$ diverges to $-\infty$.*

Theorem 6.11. *Let $(x_n)_{n=1}^\infty$ and $(y_n)_{n=1}^\infty$ be sequences in $\mathbb{R}$. Then:*

(i) *if $\lim_{n\to\infty} x_n = \infty$ and if $y = \lim_{n\to\infty} y_n$ exists in $\mathbb{R} \cup \{\infty\}$, then*

$$\lim_{n\to\infty} (x_n + y_n) = \infty;$$

(ii) *if $\lim_{n\to\infty} x_n = -\infty$ and if $y = \lim_{n\to\infty} y_n$ exists in $\mathbb{R} \cup \{-\infty\}$, then*

$$\lim_{n\to\infty} (x_n + y_n) = -\infty.$$

Proof. We only prove (i). ((ii) can be proven analogously or be deduced from (i) using Proposition 6.10.)

Case 1: $y \in \mathbb{R}$.

In this case $(y_n)_{n=1}^\infty$ is bounded, i.e., there is $C \geq 0$ such that $|y_n| \leq C$ for $n \in \mathbb{N}$. Let $R \in \mathbb{R}$. As $x_n \to \infty$, there is $n_R \in \mathbb{N}$ such that $x_n > R + C$ for all $n \geq n_R$. It follows for those n that

$$x_n + y_n \geq x_n - C > R + C - C = R.$$

Consequently, $\lim_{n\to\infty}(x_n + y_n) = \infty$ holds.

Case 2: $y = \infty$.

Let $R \in \mathbb{R}$. Choose $n_1, n_2 \in \mathbb{N}$ such that $x_n > \frac{R}{2}$ for $n \geq n_1$ and $y_n > \frac{R}{2}$ for $n \geq n_2$. Set $n_R := \max\{n_1, n_2\}$. For $n \geq n_R$, it then follows that

$$x_n + y_n > \frac{R}{2} + \frac{R}{2} = R.$$

This implies $\lim_{n\to\infty}(x_n + y_n) = \infty$. $\square$

What happens with $(x_n + y_n)_{n=1}^\infty$ if $x_n \to \infty$ and $y_n \to -\infty$? In this case, nothing can be said.

Examples. 1. Let $c \in \mathbb{R}$, and define

$$x_n := n + c \qquad \text{and} \qquad y_n := -n$$

for $n \in \mathbb{N}$. Then we have $x_n \to \infty$, $y_n \to -\infty$, and $x_n + y_n = c \to c$.

2. Define

$$x_n := n \qquad \text{and} \qquad y_n := -\sqrt{n}$$

for $n \in \mathbb{N}$, so that $x_n \to \infty$ and $y_n \to -\infty$. Let $R \in \mathbb{R}$. We can suppose without loss of generality that $R \geq 0$. As $\sqrt{n} \to \infty$, there is $n_R \in \mathbb{N}$ such that $\sqrt{n} > \sqrt{R} + 1$ for $n \geq n_R$, so that

$$\sqrt{n} > \sqrt{n} - 1 > \sqrt{R}$$

for those n. We therefore obtain

$$x_n + y_n = n - \sqrt{n} = \sqrt{n}\left(\sqrt{n} - 1\right) > \sqrt{R}\sqrt{R} = R$$

for $n \geq n_R$. It follows that $x_n + y_n \to \infty$.

3. Let $c_k, \ldots, c_1, c_0 \in \mathbb{R}$ with $c_k \neq 0$, and let

$$p(x) := c_k x^k + \cdots + c_1 x + c_0.$$

We then have

$$\lim_{n\to\infty} p(n) = \begin{cases} c_0, & \text{if } k = 0, \\ \infty, & \text{if } k \geq 1 \text{ and } c_k > 0, \\ -\infty, & \text{if } k \geq 1 \text{ and } c_k < 0. \end{cases}$$

Of course, if $k = 0$, the claim is clear.

Suppose that $k \geq 1$ and that $c_k > 0$. Set

$$x_n := c_k + \frac{c_{k-1}}{n} + \cdots + \frac{c_1}{n^{k-1}} + \frac{c_0}{n^k}.$$

The limit laws imply that $x_n \to c_k$; in particular, there is $n_1 \in \mathbb{N}$ such that $|x_n - c_k| < \frac{c_k}{2}$ for all $n \geq n_1$. It follows that

$$x_n = x_n - c_k + c_k \geq c_k - |x_n - c_k| \geq c_k - \frac{c_k}{2} = \frac{c_k}{2}$$

for those n. Let $R > 0$. As $n^k \to \infty$, there is $n_2 \in \mathbb{N}$ such that $n^k > \frac{2R}{c_k}$ for all $n \geq n_2$. Set $n_R := \max\{n_1, n_2\}$. Then we obtain for all $n \geq n_R$ that

$$p(n) = n^k \, x_n > \frac{2R}{c_k}\frac{c_k}{2} = R.$$

It follows that $p(n) \to \infty$.

The case with $k \geq 1$ and $c_k < 0$ is treated analogously.

4. In the homework, it was shown that $n^2 \le 2^n$ and therefore $n \le \frac{2^n}{n}$ for all $n \ge 5$. It follows that $\frac{2^n}{n} \to \infty$. Let $R \ge 0$. Then there is $n_R \in \mathbb{N}$ such that $\frac{2^n}{n} > R + 1$ for all $n \ge n_R$ and therefore $\frac{2^n}{n} - 1 > R$ for $n \ge n_R$. For those n, we obtain

$$2^n - n = n\left(\frac{2^n}{n} - 1\right) > nR \ge R.$$

All in all, $2^n - n \to \infty$ holds.

Proposition 6.12. *Let $(x_n)_{n=1}^\infty$ be a sequence in $(0, \infty)$. Then the following are equivalent:*

(i) $\lim_{n\to\infty} x_n = \infty$;

(ii) $\lim_{n\to\infty} \frac{1}{x_n} = 0$.

Proof. (i) $\Longrightarrow$ (ii): Let $\epsilon > 0$. Then there is $n_\epsilon \in \mathbb{N}$ such that $x_n > \frac{1}{\epsilon}$ for all $n \ge n_\epsilon$, so that

$$\left|\frac{1}{x_n}\right| = \frac{1}{x_n} < \epsilon$$

for those n. It follows that $\lim_{n\to\infty} \frac{1}{x_n} = 0$.

(ii) $\Longrightarrow$ (i): Let $R \in \mathbb{R}$, and suppose without loss of generality that $R > 0$. As $\frac{1}{x_n} \to 0$, there is $n_R \in \mathbb{N}$ such that $\frac{1}{x_n} < \frac{1}{R}$ for all $n \ge n_R$, so that $x_n > R$ for all $n \ge n_R$. This means that $x_n \to \infty$. $\square$

Example. Let $\theta \in (0, \infty) \setminus \{1\}$.

If $\theta > 1$, Bernoulli's Inequality yields that

$$\theta^n = (1 + (\theta - 1))^n \ge 1 + n(\theta - 1)$$

for $n \in \mathbb{N}$. By the Archimedian Property of $\mathbb{R}$, we have $\lim_{n\to\infty} n(\theta - 1) = \infty$ and, consequently, $\lim_{n\to\infty} 1 + n(\theta - 1) = \infty$. It follows that $\lim_{n\to\infty} \theta^n = \infty$.

If $\theta \in (0, 1)$, then $\frac{1}{\theta} > 1$. It follows that

$$\lim_{n\to\infty} \left(\frac{1}{\theta}\right)^n = \lim_{n\to\infty} \frac{1}{\theta^n} = \infty,$$

so that $\lim_{n\to\infty} \theta^n = 0$.

Monotonic Sequences

Definition 6.13. Let $(x_n)_{n=1}^\infty$ be a sequence in $\mathbb{R}$. Then we call $(x_n)_{n=1}^\infty$:

(a) *increasing* if $x_{n+1} \ge x_n$ for $n \in \mathbb{N}$;

(b) *decreasing* if $x_{n+1} \le x_n$ for $n \in \mathbb{N}$;

(c) *monotonic* if it is increasing or decreasing.

Remark. If we have "$>$" and "$<$" in (a) and (b) instead of "$\geq$" and "$\leq$", we call $(x_n)_{n=1}^\infty$ *strictly* increasing or decreasing, respectively.

Theorem 6.14. *Let $(x_n)_{n=1}^\infty$ be a bounded, monotonic sequence. Then $(x_n)_{n=1}^\infty$ converges.*

Proof. Without loss of generality, suppose that $(x_n)_{n=1}^\infty$ is increasing. Set $x := \sup\{x_n : n \in \mathbb{N}\}$. We claim that $x = \lim_{n\to\infty} x_n$.

Let $\epsilon > 0$. Then there is $n_\epsilon \in \mathbb{N}$, such that $x_{n_\epsilon} > x - \epsilon$. For $n \geq n_\epsilon$, this means that

$$x - \epsilon < x_{n_\epsilon} \leq x_{n_\epsilon+1} \leq \cdots \leq x_{n-1} \leq x_n \leq x < x + \epsilon,$$

i.e., $|x_n - x| < \epsilon$. This proves the claim. $\square$

Example. Define $(x_n)_{n=1}^\infty$ *recursively* by letting

$$x_1 := 5 \qquad \text{and} \qquad x_n := \frac{x_{n-1}^2 + 5}{2x_{n-1}} \quad \text{for } n \geq 2.$$

We claim that

$$\sqrt{5} < x_{n+1} < x_n \leq 5$$

for $n \in \mathbb{N}$ and use induction to prove it.

$n = 1$: Clearly,

$$\sqrt{5} < \underbrace{3}_{=x_2} < \underbrace{5}_{=x_1} = 5$$

holds.

$n \rightsquigarrow n+1$: As $\sqrt{5} < x_{n+1}$, we have $5 < x_{n+1}^2$ and therefore $x_{n+1}^2 + 5 < 2x_{n+1}^2$. Division by $2x_{n+1}$ yields

$$x_{n+2} = \frac{x_{n+1}^2 + 5}{2x_{n+1}} < x_{n+1} \leq 5.$$

As

$$x_{n+1}^2 - 2\sqrt{5}x_{n+1} + 5 = \left(x_{n+1} - \sqrt{5}\right)^2 > 0,$$

we have

$$x_{n+1}^2 + 5 > 2\sqrt{5}x_{n+1}.$$

Division by $2x_{n+1}$ again yields

$$\sqrt{5} < \frac{x_{n+1}^2 + 5}{2x_{n+1}}.$$

Consequently, $(x_n)_{n=1}^\infty$ is bounded and decreasing, so that $x = \lim_{n\to\infty} x_n$ exists in $\mathbb{R}$. Clearly, $x \geq \sqrt{5}$ most hold. On the other hand, we have

$$x = \lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{x_{n-1}^2 + 5}{2x_{n-1}} = \lim_{n\to\infty} \frac{x_n^2 + 5}{2x_n} = \frac{x^2 + 5}{2x},$$

so that $x^2 = 5$. It follows that $x = \sqrt{5}$.

Theorem 6.15. *Let $(x_n)_{n=1}^{\infty}$ be an unbounded sequence in $\mathbb{R}$. Then:*

(i) *if $(x_n)_{n=1}^{\infty}$ is increasing, $\lim_{n\to\infty} x_n = \infty$ holds;*

(ii) *if $(x_n)_{n=1}^{\infty}$ is decreasing, $\lim_{n\to\infty} x_n = -\infty$ holds.*

Proof. We only prove (i).

Let $R \in \mathbb{R}$. As $\{x_n : n \in \mathbb{N}\}$ is bounded below (by x_1), it cannot be bounded above. Therefore, there is $n_R \in \mathbb{N}$ with $x_{n_R} > R$. As $(x_n)_{n=1}^{\infty}$ is increasing, it follows that $x_n \geq x_{n_R} > R$ for all $n \geq n_R$. This means that $\lim_{n\to\infty} x_n = \infty$. $\square$

Exercises

1. Let $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ be convergent sequences in $\mathbb{R}$ with limits x and y, respectively, such that $x_n \leq y_n$ for $n \in \mathbb{N}$. Show that $x \leq y$. (*Hint*: Assume towards a contradiction that $x > y$, and set $\epsilon := \frac{x-y}{2}$. What happens for large enough n?)

2. Let $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ be convergent sequences in $\mathbb{R}$ such that $x_n < y_n$ for $n \in \mathbb{N}$. Does this entail that $\lim_{n\to\infty} x_n < \lim_{n\to\infty} y_n$?

3. Let $(x_n)_{n=1}^{\infty}$ be a sequence in $\mathbb{R}$ converging to zero, and let $(y_n)_{n=1}^{\infty}$ be a bounded sequence. Show that $(x_n y_n)_{n=1}^{\infty}$ is convergent with $\lim_{n\to\infty} x_n y_n = 0$.

4. Is the sequence $(x_n)_{n=1}^{\infty}$ with
$$x_n := \frac{4n^2 + \sin\left(\frac{1}{n^{13}}\right) - 17n}{2n^2 + \cos(n)}$$
for $n \in \mathbb{N}$ convergent? If so, determine its limit.

5. Let $(x_n)_{n=1}^{\infty}$ be a a convergent sequence in $\mathbb{Z}$. Show that there is $n_0 \in \mathbb{N}$ such that $x_n = x_{n_0}$ for all $n \geq n_0$. (*Hint*: Use the definition of convergence with $\epsilon := \frac{1}{2}$.)

6. Let $(x_n)_{n=1}^{\infty}$ be a convergent sequence in $\mathbb{R}$ with limit x. Show that $(|x_n|)_{n=1}^{\infty}$ is also convergent with
$$\lim_{n\to\infty} |x_n| = |x|.$$

7. Let $(x_n)_{n=1}^{\infty}$ be a convergent sequence of non-negative reals with limit x. Show that $\left(\sqrt{x_n}\right)_{n=1}^{\infty}$ is convergent with
$$\lim_{n\to\infty} \sqrt{x_n} = \sqrt{x}.$$
(*Hint*: $\left|\sqrt{x} - \sqrt{y}\right|^2 \leq \left|\sqrt{x} - \sqrt{y}\right| \left|\sqrt{x} + \sqrt{y}\right| = |x - y|$ for all $x, y \geq 0$.)

8. Does the sequence $\left(\sqrt{n^2 + n} - n\right)_{n=1}^{\infty}$ converge? If so, determine its limit.

9. Let p be a non-zero polynomial. Show that $\lim_{n\to\infty} \frac{p(n+1)}{p(n)} = 1$

10. Let $\theta \in \mathbb{R} \setminus \{1\}$, and define $s_n := \sum_{k=0}^{n} \theta^k$ for $n \in \mathbb{N}_0$. Show that
$$s_n = \frac{1-\theta^{n+1}}{1-\theta}$$
for all $n \in \mathbb{N}_0$ and conclude that
$$\lim_{n\to\infty} s_n = \frac{1}{1-\theta}$$
if $\theta \in (-1,1)$.

11. Let $(x_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ be sequences in $(0,\infty)$ such that $\lim_{n\to\infty} x_n = \infty$ and that $\lim_{n\to\infty} y_n$ exists in $(0,\infty) \cup \{\infty\}$. Show that
$$\lim_{n\to\infty} x_n y_n = \infty.$$
What can you say if $\lim_{n\to\infty} y_n = 0$?

12. (a) Show that
$$\sum_{k=1}^{n} \frac{1}{k(k+1)} = 1 - \frac{1}{n+1}$$
for all $n \in \mathbb{N}$.

    (b) For $n \in \mathbb{N}$, let $s_n := \sum_{k=1}^{n} \frac{1}{k^2}$. Show that $(s_n)_{n=1}^{\infty}$ converges. (*Hint*: Show that $(s_n)_{n=1}^{\infty}$ is bounded using (a).)

13. Define the sequence $(x_n)_{n=1}^{\infty}$ recursively by letting
$$x_1 := 2 \qquad \text{and} \qquad x_n := \frac{1}{2}\left(x_{n-1} + \frac{2}{x_{n-1}}\right) \quad \text{for } n \geq 2.$$
Show that
$$(*) \qquad \sqrt{2} \leq x_{n+1} \leq x_n \leq 2$$
for all $n \in \mathbb{N}$. Conclude that $(x_n)_{n=1}^{\infty}$ converges and compute its limit. (*Hint*: Note that $(*)$ means that $x_n - x_{n+1} \geq 0$ and $x_n^2 - 2 \geq 0$ for all $n \in \mathbb{N}$.)

14. Let the sequence $(x_n)_{n=1}^{\infty}$ be defined recursively through
$$x_1 := \frac{1}{2} \qquad \text{and} \qquad x_n := \frac{1}{3}(x_{n-1}^2 + 1) \quad \text{for } n \geq 2.$$
Show that $(x_n)_{n=1}^{\infty}$ converges and compute its limit.

15. Let $\theta > 0$, and define the sequence $(x_n)_{n=1}^{\infty}$ inductively through
$$x_1 := \sqrt{\theta} \qquad \text{and} \qquad x_{n+1} = \sqrt{\theta + x_n} \quad \text{for } n \in \mathbb{N}.$$
Show that $(x_n)_{n=1}^{\infty}$ increases and is bounded above by $1 + \sqrt{\theta}$ (and therefore converges), and compute its limit.

7 $\limsup$, $\liminf$, and Cauchy Sequences

Let $(x_n)_{n=1}^{\infty}$ be a sequence in $\mathbb{R}$ that is bounded above. For $n \in \mathbb{N}$, define

$$v_n := \sup\{x_k : k \geq n\}.$$

As

$$\{x_k : k \geq n+1\} \subset \{x_k : k \geq n\},$$

we have $v_{n+1} \leq v_n$, i.e., the sequence $(v_n)_{n=1}^{\infty}$ is decreasing, so that $\lim_{n\to\infty} v_n$ exists in $\mathbb{R} \cup \{-\infty\}$.

Definition 7.1. Let $(x_n)_{n=1}^{\infty}$ be a sequence in $\mathbb{R}$. Then:

(a) if $(x_n)_{n=1}^{\infty}$ is bounded above, define the *limit superior* of $(x_n)_{n=1}^{\infty}$ as

$$\limsup_{n\to\infty} x_n := \lim_{n\to\infty} \sup\{x_k : k \geq n\};$$

otherwise, define $\limsup_{n\to\infty} x_n = \infty$;

(b) if $(x_n)_{n=1}^{\infty}$ is bounded below, define the *limit inferior* of $(x_n)_{n=1}^{\infty}$ as

$$\liminf_{n\to\infty} x_n := \lim_{n\to\infty} \inf\{x_k : k \geq n\};$$

otherwise, define $\liminf_{n\to\infty} x_n = -\infty$.

Examples. 1. Clearly, we have

$$\limsup_{n\to\infty} (-1)^n = 1 \qquad \text{and} \qquad \liminf_{n\to\infty} (-1)^n = -1.$$

2. Equally clearly,

$$\liminf_{n\to\infty} n = \lim_{n\to\infty} \inf\{k : k \geq n\} = \lim_{n\to\infty} n = \infty = \limsup_{n\to\infty} n$$

holds.

Remark. Note that

$$\inf\{x_n : n \in \mathbb{N}\} \leq \liminf_{n\to\infty} x_n \leq \limsup_{n\to\infty} x_n \leq \sup\{x_n : n \in \mathbb{N}\}$$

and

$$\limsup_{n\to\infty} (-x_n) = -\liminf_{n\to\infty} x_n.$$

Theorem 7.2. *Let $(x_n)_{n=1}^{\infty}$ be a sequence in $\mathbb{R}$. Then:*

(i) *if $\lim_{n\to\infty} x_n$ exists in $\mathbb{R} \cup \{-\infty, \infty\}$, then*

$$\liminf_{n\to\infty} x_n = \lim_{n\to\infty} x_n = \limsup_{n\to\infty} x_n;$$

(ii) *if* $\liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n$, *then* $\lim_{n\to\infty} x_n$ *exists in* $\mathbb{R} \cup \{-\infty, \infty\}$.

Proof. For $n \in \mathbb{N}$, set

$$u_n := \inf\{x_k : k \geq n\} \quad \text{and} \qquad v_n := \sup\{x_k : k \geq n\},$$

so that

$$\lim_{n\to\infty} u_n = \liminf_{n\to\infty} x_n \qquad \text{and} \qquad \lim_{n\to\infty} v_n = \limsup_{n\to\infty} x_n.$$

(i): We first treat the case where $\lim_{n\to\infty} x_n = \infty$. Let $R \in \mathbb{R}$. Then there is $n_R \in \mathbb{N}$ such that $x_n > R$ for all $n \geq n_R$. It follows that

$$u_n = \inf\{x_k : k \geq n\} \geq \inf\{x_k : k \geq n_R\} \geq R$$

for such n. As R was arbitrary, this means that

$$\infty = \lim_{n\to\infty} u_n = \liminf_{n\to\infty} x_n \leq \limsup_{n\to\infty} x_n \leq \infty.$$

The case where where $\lim_{n\to\infty} x_n = -\infty$ is dealt with similarly.

Suppose that $x = \lim_{n\to\infty} x_n$ exists in $\mathbb{R}$. Let $\epsilon > 0$. Then there is $n_\epsilon \in \mathbb{N}$ such that $|x_n - x| < \epsilon$ for $n \geq n_\epsilon$ and, in particular, $x_n < x + \epsilon$ for those n. It follows that

$$v_n = \sup\{x_k : k \geq n\} \leq x + \epsilon$$

for $n \geq n_\epsilon$, so that

$$\limsup_{n\to\infty} x_n = \lim_{n\to\infty} v_n \leq x + \epsilon.$$

As $\epsilon > 0$ was arbitrary, this means that

$$\limsup_{n\to\infty} x_n \leq x = \lim_{n\to\infty} x_n.$$

Similarly, one sees that $\lim_{n\to\infty} x_n \leq \liminf_{n\to\infty} x_n$. As $\liminf_{n\to\infty} x_n \leq \limsup_{n\to\infty}$, this proves the claim.

(ii): Suppose first that $\liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n = \infty$. As $x_n \geq u_n$ for all $n \in \mathbb{N}$, it follows that $\lim_{n\to\infty} x_n = \infty$. The case where $\liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n = -\infty$ is dealt with similarly.

Consider the case where

$$\liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n = x \in \mathbb{R}.$$

Let $\epsilon > 0$. As $x = \lim_{n\to\infty} u_n$, there is $n_1 \in \mathbb{N}$ such that $|u_n - x| < \epsilon$ for all $n \geq n_1$; in particular, we have

$$x - \epsilon < u_n \leq x_n$$

for those n. Similarly, we find $n_2 \in \mathbb{N}$ such that $|v_n - x| < \epsilon$ and therefore

$$x_n \leq v_n < x + \epsilon$$

for $n \geq n_2$. Letting $n_\epsilon := \max\{n_1, n_2\}$, we obtain

$$x - \epsilon < x_n < x + \epsilon$$

for $n \geq n_\epsilon$, so that $x = \lim_{n\to\infty} x_n$. $\square$

Cauchy Sequences

Question. For $n \in \mathbb{N}$, define

$$s_n := \sum_{k=1}^{n} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}.$$

Then $(s_n)_{n=1}^\infty$ is an increasing sequence, so that $\lim_{n\to\infty} s_n$ exists in $\mathbb{R} \cup \{\infty\}$. Is it finite or ∞, i.e., is $(s_n)_{n=1}^\infty$ bounded or not?

Definition 7.3. A sequence $(x_n)_{n=1}^\infty$ in $\mathbb{R}$ is called a *Cauchy sequence* if, for each $\epsilon > 0$, there is $n_\epsilon \in \mathbb{N}$ such that $|x_n - x_m| < \epsilon$ for all $n, m \geq n_\epsilon$.

Proposition 7.4. *Let $(x_n)_{n=1}^\infty$ be a convergent sequence in $\mathbb{R}$. Then $(x_n)_{n=1}^\infty$ is a Cauchy sequence.*

Proof. Set $x := \lim_{n\to\infty} x_n$. Let $\epsilon > 0$. Then there is $n_\epsilon \in \mathbb{N}$ such that $|x_n - x| < \frac{\epsilon}{2}$ for all $n \geq n_\epsilon$. It follows that

$$|x_n - x_m| \leq |x_n - x| + |x_m - x| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

for all $n, m \geq n_\epsilon$. Therefore, $(x_n)_{n=1}^\infty$ is a Cauchy sequence. $\square$

Lemma 7.5. *Let $(x_n)_{n=1}^\infty$ be a Cauchy sequence in $\mathbb{R}$. Then $(x_n)_{n=1}^\infty$ is bounded.*

Proof. Choose $n_0 \in \mathbb{N}$ such that $|x_n - x_m| < 1$ for all $n, m \geq n_0$, so that, in particular, $|x_n - x_{n_0}| < 1$ for $n \geq n_0$. It follows that

$$|x_n| \leq |x_n - x_{n_0}| + |x_{n_0}| < 1 + |x_{n_0}|$$

for $n \geq n_0$. Set

$$C := \max\{|x_1|, \ldots, |x_{n_0 - 1}|, 1 + |x_{n_0}|\},$$

so that $|x_n| \leq C$ for all $n \in \mathbb{N}$. This means that $(x_n)_{n=1}^\infty$ is bounded. $\square$

Theorem 7.6. *The following are equivalent for a sequence $(x_n)_{n=1}^\infty$ in $\mathbb{R}$:*

(i) $(x_n)_{n=1}^\infty$ *converges in* $\mathbb{R}$;

(ii) $(x_n)_{n=1}^\infty$ *is a Cauchy sequence.*

Proof. (i) $\Longrightarrow$ (ii) is Proposition 7.4.

(ii) $\Longrightarrow$ (i): As $(x_n)_{n=1}^\infty$ is bounded by Lemma 7.5, $\liminf_{n\to\infty} x_n$ and $\limsup_{n\to\infty} x_n$ exist in $\mathbb{R}$. We shall invoke Theorem 7.2(ii) and show that

$$\liminf_{n\to\infty} x_n = \limsup_{n\to\infty} x_n.$$

Let $\epsilon > 0$, and choose $n_\epsilon \in \mathbb{N}$ such that $|x_n - x_m| < \epsilon$ and, in particular, $x_n < x_m + \epsilon$ for $n, m \geq n_\epsilon$. It follows that, for all $n, m \geq n_\epsilon$, $x_m + \epsilon$ is an upper bound for $\{x_k : k \geq n\}$, so that

$$v_n = \sup\{x_k : k \geq n\} \leq x_m + \epsilon.$$

Consequently,

$$\limsup_{n\to\infty} x_n - \epsilon = \lim_{n\to\infty} v_n - \epsilon \leq x_m$$

holds for all $m \geq n_\epsilon$. It follows that $\limsup_{n\to\infty} x_n - \epsilon$ is a lower bound for $\{x_k : k \geq m\}$ for all $m \geq n_\epsilon$ so that

$$\limsup_{n\to\infty} x_n - \epsilon \leq u_m = \inf\{x_k : k \geq m\}$$

for those m. This means that

$$\limsup_{n\to\infty} x_n - \epsilon \leq \lim_{n\to\infty} u_n = \liminf_{n\to\infty} x_n.$$

As $\epsilon > 0$ was arbitrary, this means that $\limsup_{n\to\infty} x_n \leq \liminf_{n\to\infty} x_n$ and therefore $\limsup_{n\to\infty} x_n = \liminf_{n\to\infty} x_n$. $\square$

Example. For $n \in \mathbb{N}$, consider $s_n := \sum_{k=1}^n \frac{1}{k}$. Observe that

$$s_{2n} - s_n = \sum_{k=1}^{2n} \frac{1}{k} - \sum_{k=1}^{n} \frac{1}{k} = \sum_{k=n+1}^{2n} \frac{1}{k} \geq \sum_{k=n+1}^{2n} \frac{1}{2n} = n\frac{1}{2n} = \frac{1}{2}$$

for all $n \in \mathbb{N}$. This means that $(s_n)_{n=1}^\infty$ cannot be a Cauchy sequence. It follows that $(s_n)_{n=1}^\infty$ does not converge and therefore has to be unbounded.

Exercises

1. Let $\varnothing \neq S$ be bounded above, let $t \geq 0$, and set

$$tS := \{tx : x \in S\}.$$

Show that

$$\sup(tS) = t \sup S.$$

Use this to conclude that, if $(x_n)_{n=1}^\infty$ is any sequence that is bounded above in $\mathbb{R}$ and $t \geq 0$, then

$$\limsup_{n\to\infty} tx_n = t \limsup_{n\to\infty} x_n.$$

2. Let $(x_n)_{n=1}^\infty$ and $(y_n)_{n=1}^\infty$ be such that $x_n \leq y_n$ for all $n \in \mathbb{N}$. Show that

$$\liminf_{n\to\infty} x_n \leq \liminf_{n\to\infty} y_n \qquad \text{and} \qquad \limsup_{n\to\infty} x_n \leq \limsup_{n\to\infty} y_n.$$

Use this to derive the *Squeeze Theorem*: If $(x_n)_{n=1}^\infty$, $(y_n)_{n=1}^\infty$, and $(z_n)_{n=1}^\infty$ are sequences in $\mathbb{R}$ such that $x_n \leq y_n \leq z_n$ for all $n \in \mathbb{N}$ with $(x_n)_{n=1}^\infty$ and $(z_n)_{n=1}^\infty$ convergent and $\lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n$, then $(y_n)_{n=1}^\infty$ is convergent with $\lim_{n\to\infty} y_n = \lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n$.

3. Let $(x_n)_{n=1}^\infty$ be a sequence in $\mathbb{R}$ such that there is $\theta \in [0,1)$ with

$$|x_{n+1} - x_n| \leq \theta^n$$

for all $n \in \mathbb{N}$. Show that $(x_n)_{n=1}^\infty$ is a Cauchy sequence and therefore converges. (*Hint*: Problem 10 in Section 6.)

8 Subsequences

Definition 8.1. Let $(x_n)_{n=1}^\infty$ be a sequence. Then we call $(y_k)_{k=1}^\infty$ a *subsequence* of $(x_n)_{n=1}^\infty$ if there are $n_1 < n_2 < \cdots < n_k < n_{k+1} < \cdots$ in $\mathbb{N}$ such that $y_k = x_{n_k}$.

Examples. 1. $\left(\frac{1}{n^\nu}\right)_{n=1}^\infty$ is a subsequence of $\left(\frac{1}{n}\right)_{n=1}^\infty$ for every $\nu \in \mathbb{N}$.

2. For $n \in \mathbb{N}$, set $x_n := (-1)^n$. Then $(x_{2n})_{n=1}^\infty$ and $(x_{2n-1})_{n=1}^\infty$ are subsequences of $(x_n)_{n=1}^\infty$. Note that
$$x_{2n} = 1 \qquad \text{and} \qquad x_{2n-1} = -1$$
for $n \in \mathbb{N}$.

3. For $x \in \mathbb{R}$, the *floor* of x is defined as
$$\lfloor x \rfloor := \max\{m \in \mathbb{Z} : m \leq x\}.$$
For $n \in \mathbb{N}$, set
$$x_n := (-1)^n \left\lfloor \frac{n}{2} \right\rfloor.$$
Then $(x_n)_{n=1}^\infty$ enumerates $\mathbb{Z}$: $x_1 = 0$, $x_2 = 1$, $x_3 = -1$, $x_4 = 2$, $x_5 - 2$, etc. Consider the subsequences $(x_{2n})_{n=1}^\infty$ and $(x_{2n-1})_{n=1}^\infty$. We have
$$x_{2n} = n \qquad \text{and} \qquad x_{2n-1} = -(n-1)$$
for $n \in \mathbb{N}$.

Proposition 8.2. *Let $(x_n)_{n=1}^\infty$ be a sequence in $\mathbb{R}$ such that $\lim_{n\to\infty} x_n$ exists in $\mathbb{R} \cup \{-\infty, \infty\}$. Then $\lim_{k\to\infty} x_{n_k} = \lim_{n\to\infty} x_n$ for every subsequence $(x_{n_k})_{k=1}^\infty$ of $(x_n)_{n=1}^\infty$.*

Proof. Let $(x_{n_k})_{k=1}^\infty$ be a subsequence of $(x_n)_{n=1}^\infty$.

We first deal with the case where $\lim_{n\to\infty} x_n = \infty$.

Let $R \in \mathbb{R}$. Then there is $n_R \in \mathbb{N}$ such that $x_n > R$ for all $n \geq n_R$. As $n_1 < n_2 < n_3 < \cdots$, there is $k_R \in \mathbb{N}$ such that $n_{k_R} \geq n_R$. It follows that $n_k \geq n_R$ for all $k \geq k_R$, so that $x_{n_k} > R$ for all $k \geq k_R$. This means that $\lim_{n\to\infty} x_{n_k} = \infty$.

The case where $\lim_{n\to\infty} x_n = -\infty$ is dealt with similarly.

We now turn to the case where $x = \lim_{n\to\infty} x_n \in \mathbb{R}$.

Let $\epsilon > 0$. Then there is $n_\epsilon \in \mathbb{N}$ such that $|x_n - x| < \epsilon$ for $n \geq n_\epsilon$. As $n_1 < n_2 < n_3 < \cdots$, there is $k_\epsilon \in \mathbb{N}$ such that $n_{k_\epsilon} \geq n_\epsilon$. It follows that $n_k \geq n_\epsilon$ for all $k \geq k_\epsilon$, so that $|x_{n_k} - x| < \epsilon$ for all $k \geq k_\epsilon$. This means that $\lim_{k\to\infty} x_{n_k} = x$. □

On the other hand, a sequence need not converge in order to have convergent (in $\mathbb{R} \cup \{-\infty, \infty\}$) subsequences:

- $((-1)^n)_{n=1}^\infty$ diverges whereas the subsequences $((-1)^{2n})_{n=1}^\infty$ and $((-1)^{2n-1})_{n=1}^\infty$ converge to 1 and -1, respectively.

- $\left((-1)^n \left\lfloor \frac{n}{2} \right\rfloor\right)_{n=1}^\infty$ does not converge in $\mathbb{R} \cup \{-\infty, \infty\}$, but

$$\lim_{n\to\infty} (-1)^{2n} \left\lfloor \frac{2n}{2} \right\rfloor = \lim_{n\to\infty} n = \infty$$

$$\text{and} \qquad \lim_{n\to\infty} (-1)^{2n-1} \left\lfloor \frac{2n-1}{2} \right\rfloor = \lim_{n\to\infty} -(n-1) = -\infty.$$

Proposition 8.3. *Let $(x_n)_{n=1}^\infty$ be a sequence in $\mathbb{R}$. Then:*

(i) *if $(x_n)_{n=1}^\infty$ is not bounded above, it has an increasing subsequence $(x_{n_k})_{k=1}^\infty$ such that $\lim_{k\to\infty} x_{n_k} = \infty$;*

(ii) *if $(x_n)_{n=1}^\infty$ is not bounded below, it has a decreasing subsequence $(x_{n_k})_{k=1}^\infty$ such that $\lim_{k\to\infty} x_{n_k} = -\infty$.*

Proof. We only prove (i).

We will inductively construct a sequence $n_1 < n_2 < n_3 < \cdots < n_k < n_{k+1} < \cdots$ in $\mathbb{N}$ such that

$$x_{n_{k+1}} \geq \max\{x_{n_k}, k\}$$

for $k \in \mathbb{N}$. It is then clear that the corresponding subsequence $(x_{n_k})_{k=1}^\infty$ is increasing and satisfies $\lim_{k\to\infty} x_{n_k} = \infty$.

Fix $n_1 \in \mathbb{N}$ arbitrarily.

As $(x_n)_{n=n_1+1}^\infty$ is not bounded above there is $n_2 > n_1$ such that $x_{n_2} \geq \max\{x_{n_1}, 1\}$.

As $(x_n)_{n=n_2+1}^\infty$ is not bounded above, there is $n_3 > n_2$ such that $x_{n_3} \geq \max\{x_{n_2}, 2\}$.

Continue in this fashion. Suppose that $n_1 < n_2 < \cdots < n_k$ have already been constructed such that

$$x_{n_{j+1}} \geq \max\{x_{n_j}, j\}$$

for $j = 1, \ldots, k-1$. As $(x_n)_{n=n_k+1}^\infty$ is not bounded above, there is $n_{k+1} > n_k$ such that $x_{n_{k+1}} \geq \max\{x_{n_k}, k\}$. $\square$

Definition 8.4. Let $(x_n)_{n=1}^\infty$ be a sequence in $\mathbb{R}$. We call $x \in \mathbb{R}$ an *accumulation point* of $(x_n)_{n=1}^\infty$ if there is a subsequence $(x_{n_k})_{k=1}^\infty$ of $(x_n)_{n=1}^\infty$ with $\lim_{k\to\infty} x_{n_k} = x$.

Examples.

1. If $(x_n)_{n=1}^\infty$ is convergent, then $\lim_{n\to\infty} x_n$ is its only accumulation point.

2. For $n \in \mathbb{N}$, let $x_n := (-1)^n$. Then -1 and 1 are accumulation points of $(x_n)_{n=1}^\infty$, and it is easy to see that these are the only ones.

3. For $n \in \mathbb{N}$, set $x_n := n((-1)^n + 1) + \frac{1}{n}$. Then

$$\lim_{n\to\infty} x_{2n} = \lim_{n\to\infty} 4n + \frac{1}{2n} = \infty \qquad \text{and} \qquad \lim_{n\to\infty} x_{2n-1} = \lim_{n\to\infty} \frac{1}{2n-1} = 0.$$

   Therefore, 0 is an accumulation point of $(x_n)_{n=1}^\infty$. Let $x \in \mathbb{R}$ be *any* accumulation point of $(x_n)_{n=1}^\infty$, and let $(x_{n_k})_{k=1}^\infty$ be a subsequence of $(x_n)_{n=1}^\infty$ with $\lim_{k\to\infty} x_{n_k} = x$. Assume that there are infinitely many even numbers among the n_k's. The $x_{n_k} \geq 2n_k$ must hold for infinitely many k's, so that $(x_{n_k})_{k=1}^\infty$ cannot be bounded and therefore not be convergent. Hence, there must be $k_0 \in \mathbb{N}$ such that n_k is odd for all $k \geq k_0$. This, however, means that—except for finitely many terms—$(x_{n_k})_{k=1}^\infty$ is a subsequence of $(x_{2n-1})_{n=1}^\infty$, so that $x = \lim_{n\to\infty} x_{n_k} = \lim_{n\to\infty} x_{2n-1} = 0$. It follows that 0 is the only accumulation point of $(x_n)_{n=1}^\infty$.

4. Every subsequence of $(n(-1)^n)_{n=1}^\infty$ is unbounded and therefore divergent. Consequently, the sequence has no accumulation points.

Theorem 8.5. *Let $(x_n)_{n=1}^\infty$ be a bounded sequence in $\mathbb{R}$, and let S be the set of its accumulation points. Then:*

(i) *$S \neq \varnothing$;*

(ii) *$\max S$ exists and equals $\limsup_{n\to\infty} x_n$;*

(iii) *$\min S$ exists and equals $\liminf_{n\to\infty} x_n$.*

Proof. Set $x := \limsup_{n\to\infty} x_n$. We claim that x belongs to S and is an upper bound for S. For $n \in \mathbb{N}$, set again

$$v_n := \sup\{x_\nu : \nu \geq n\}.$$

We will construct $n_1 < n_2 < n_3 < \cdots$ in $\mathbb{N}$ such that

$$\lim_{k\to\infty} x_{n_k} = \lim_{n\to\infty} v_n = \limsup_{n\to\infty} x_n$$

by defining $n_1 < n_2 < n_3 < \cdots$ inductively such that

$$x - \frac{1}{k} < x_{n_k} < x + \frac{1}{k}$$

for $k \in \mathbb{N}$.

By the definition of $\limsup_{n\to\infty} x_n$, there is $\tilde{n}_1 \in \mathbb{N}$ such that

$$x - 1 < v_n < x + 1$$

for all $n \geq \tilde{n}_1$. As $x - 1$ cannot be an upper bound for $\{x_\nu : \nu \geq \tilde{n}_1\}$ by the definition of $\limsup_{n\to\infty} x_n$, there is must be $n_1 \geq \tilde{n}_1$ such that

$$x - 1 < x_{n_1} \leq v_{n_1} < x + 1.$$

By the definition of $\limsup_{n\to\infty} x_n$ again, there is $\tilde{n}_2 \in \mathbb{N}$ such that

$$x - \frac{1}{2} < v_n < x + \frac{1}{2}$$

for all $n \geq \tilde{n}_2$. As $x - \frac{1}{2}$ cannot be an upper bound for $\{x_\nu : \nu \geq \tilde{n}_2\}$, there is $n_2 \geq \max\{n_1 + 1, \tilde{n}_2\}$ with

$$x - \frac{1}{2} < x_{n_2} < x + \frac{1}{2}.$$

Suppose now that $n_1 < n_2 < \cdots < n_k$ have already been constructed with

$$x - \frac{1}{j} < x_{n_j} < x + \frac{1}{j}$$

for $j = 1, \ldots, k$. Choose $\tilde{n}_{k+1} \in \mathbb{N}$ such that

$$x - \frac{1}{k+1} < v_n < x + \frac{1}{k+1}$$

for all $n \geq \tilde{n}_{k+1}$. Then choose $n_{k+1} \geq \max\{n_k + 1, \tilde{n}_{k+1}\}$ such that

$$x - \frac{1}{k+1} < x_{n_{k+1}} < x + \frac{1}{k+1}.$$

Let $\epsilon > 0$. Choose $k_\epsilon \in \mathbb{N}$ such that $\frac{1}{k} < \epsilon$ for all $k \geq k_\epsilon$; it follows that

$$x - \epsilon < x_{n_k} < x + \epsilon$$

for those k. All in all, $x = \lim_{k\to\infty} x_{n_k}$. This means that $x \in S$, so that, in particular, $S \neq \varnothing$, which proves (i).

Let $\tilde{x} \in S$ be arbitrary. Then there is a subsequence $(x_{n_\nu})_{k=1}^\infty$ of $(x_n)_{n=1}^\infty$ such that $x_{\nu_k} \to \tilde{x}$. As

$$x_{\nu_k} \leq v_{\nu_k}$$

for $k \in \mathbb{N}$ it follows that

$$\tilde{x} = \lim_{k\to\infty} x_{\nu_k} \leq \lim_{k\to\infty} v_{\nu_k} = x.$$

Hence, x is an upper bound for S and therefore its maximum. This proves (ii).

(iii) is proven analogously. $\square$

Corollary 8.6 (Bolzano–Weierstraß Theorem)**.** *Every bounded sequence in $\mathbb{R}$ has a convergent subsequence.*

Corollary 8.7. *The following are equivalent for a bounded sequence $(x_n)_{n=1}^\infty$ in $\mathbb{R}$:*

(i) *$(x_n)_{n=1}^\infty$ is convergent;*

(ii) *$(x_n)_{n=1}^\infty$ has exactly one accumulation point.*

Proof. (i) $\Longrightarrow$ (ii) is clear by Proposition 8.2.

(ii) $\Longrightarrow$ (i): Let S be set of accumulation points of $(x_n)_{n=1}^\infty$ and suppose that S is a singleton set. By Theorem 8.5(i) and (ii), this means that

$$\limsup_{n\to\infty} x_n = \max S = \min S = \liminf_{n\to\infty} x_n.$$

By Theorem 7.2, this means that $(x_n)_{n=1}^\infty$ converges. $\square$

Exercises

1. Let $(x_n)_{n=1}^\infty$ be a sequence in $\mathbb{R}$. Show that $x \in \mathbb{R}$ is an accumulation point of $(x_n)_{n=1}^\infty$ if and only if, for each $\epsilon > 0$, there are infinitely many $n \in \mathbb{N}$ such that $|x_n - x| < \epsilon$. (*Hint*: For the "*only if*" part, inductively construct a subsequence $(x_{n_k})_{k=1}^\infty$ of $(x_n)_{n=1}^\infty$ with $|x_{n_k} - x| < \frac{1}{k}$ for $k \in \mathbb{N}$.)

2. Let $\varnothing \neq S \subset \mathbb{R}$ be bounded above. Show that there is an increasing sequence $(x_n)_{n=1}^\infty$ in S with $\lim_{n\to\infty} x_n = \sup S$.

3. Let $(x_n)_{n=1}^\infty$ be a sequence in $\mathbb{R}$, let x be an accumulation point of $(x_n)_{n=1}^\infty$, and let $(\epsilon_k)_{k=1}^\infty$ be a sequence of strictly positive reals such that $\lim_{k\to\infty} \epsilon_k = 0$. Show that there is a subsequence $(x_{n_k})_{k=1}^\infty$ of $(x_n)_{n=1}^\infty$ such that $|x_{n_k} - x| < \epsilon_k$ for all $k \in \mathbb{N}$.

9 Infinite Series

Let $a_1, a_2, a_3, \ldots \in \mathbb{R}$. What is

$$\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + \cdots$$

supposed to mean? A naive approach leads easily to nonsensical "results". For instance, let $a_k := (-1)^k$ for $k \in \mathbb{N}$. Then—depending on how we bracket the summands—we obtain

$$\begin{aligned}\sum_{k=1}^{\infty} a_k &= -1+1-1+1-1+1-1+1-\ldots \\ &= \begin{cases} (-1+1)+(-1+1)+(-1+1)+(-1+1)-\cdots &= \quad 0, \\ -1+(1-1)+(1-1)+(1-1)+(1-1)+\cdots &= \ -1, \end{cases}\end{aligned}$$

so that $0 = -1$ and thus $1 = 0$...

A more rigorous approach is therefore required.

Definition 9.1. Let $(a_k)_{k=1}^{\infty}$ be a sequence in $\mathbb{R}$. Then the *infinite series* $\sum_{k=1}^{\infty} a_k$ is the sequence $(s_n)_{n=1}^{\infty}$ where

$$s_n := \sum_{k=1}^{n} a_k$$

for $n \in \mathbb{N}$. The terms s_n are called the *partial sums* of $\sum_{k=1}^{\infty} a_k$. We say that $\sum_{k=1}^{\infty} a_k$ *converges* if there is $s \in \mathbb{R}$ such that $s = \lim_{n\to\infty} s_n$; otherwise, we say that $\sum_{k=1}^{\infty} a_k$ diverges.

Notation. If $\sum_{k=1}^{\infty} a_k$ converges and $s = \lim_{n\to\infty} s_n$, we write

$$\sum_{k=1}^{\infty} a_k = s.$$

The symbol $\sum_{k=1}^{\infty} a_k$ can therefore mean two objects: the sequence $(s_n)_{n=1}^{\infty}$ of partial sums and—in the case of convergence—its limit. The reasons for this are historical. If $(s_n)_{n=1}^{\infty}$ diverges to $-\infty$ or ∞, we write

$$\sum_{k=1}^{\infty} a_k = -\infty \qquad \text{or} \qquad \sum_{k=1}^{\infty} a_k = \infty,$$

respectively.

Examples. 1. Consider $\sum_{k=1}^{\infty}(-1)^k$. As

$$s_n = \sum_{k=1}^{n}(-1)^k = \begin{cases} 0, & \text{for even } n \in \mathbb{N}, \\ -1, & \text{for odd } n \in \mathbb{N}, \end{cases}$$

The series $\sum_{k=1}^{\infty}(-1)^k$ diverges.

2. *Geometric Series*: As shown in the exercises,

$$\sum_{k=0}^{\infty} \theta^k = \frac{1}{1-\theta}$$

holds for $\theta \in (-1, 1)$.

3. *Harmonic Series*: As seen before,

$$\sum_{k=1}^{\infty} \frac{1}{k} = \infty.$$

4. On the other hand, $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges as shown in the exercises; in fact, $\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$.

The following is immediate:

Lemma 9.2. *Let $a_1, a_2, a_3, \ldots \geq 0$. Then $\sum_{k=1}^{\infty} a_k$ converges if and only if $(s_n)_{n=1}^{\infty}$ is bounded.*

In this situation, we write

$$\sum_{k=1}^{\infty} a_k < \infty.$$

Example. Let $p > 0$. We have

$$\sum_{k=1}^{\infty} \frac{1}{k^p} \begin{cases} = & \infty, \quad \text{for } p = 1, \\ < & \infty, \quad \text{for } p = 2. \end{cases}$$

Which precisely are the $p > 0$ for which $\sum_{k=1}^{\infty} \frac{1}{k^p} < \infty$? As

$$\sum_{k=1}^{n} \frac{1}{k^p} \geq \sum_{k=1}^{n} \frac{1}{k}$$

for $p \leq 1$ and

$$\sum_{k=1}^{n} \frac{1}{k^p} \leq \sum_{k=1}^{n} \frac{1}{k^2}$$

for $p \geq 2$, it is clear that $\sum_{k=1}^{\infty} \frac{1}{k^p} = \infty$ for $p \in (0, 1]$ and $\sum_{k=1}^{\infty} \frac{1}{k^p} < \infty$ for $p \in [2, \infty)$. But what if $p \in (1, 2)$?

Theorem 9.3 (Cauchy's Compression Theorem)**.** *Let $(a_k)_{k=1}^{\infty}$ be a decreasing sequence of non-negative reals. Then we have*

$$\sum_{k=1}^{\infty} a_k < \infty \quad \Longleftrightarrow \quad \sum_{k=1}^{\infty} 2^k a_{2^k} < \infty.$$

Proof. For $n \in \mathbb{N}$, set

$$s_n := \sum_{k=1}^{n} a_k \qquad \text{and} \qquad S_n := \sum_{k=1}^{n} 2^k a_{2^k}.$$

"$\Longrightarrow$": Set $s := \sum_{k=1}^{\infty} a_k$, and note that

$$\begin{aligned}
s &\geq s_{2^n} \\
&= a_1 + a_2 + a_3 + \cdots + a_{2^n} \\
&= a_1 + a_2 + \underbrace{(a_3 + a_4)}_{\geq 2a_4} + \underbrace{(a_5 + \cdots + a_8)}_{\geq 4a_8} + \cdots + \underbrace{(a_{2^{n-1}+1} + \cdots + a_{2^n})}_{\geq 2^{n-1}a_{2^n}} \\
&\geq a_1 + a_2 + 2a_4 + 4a_8 + \cdots + 2^{n-1}a_{2^n} \\
&= a_1 + \frac{1}{2}S_n
\end{aligned}$$

and therefore

$$S_n \leq 2(s - a_1).$$

for $n \in \mathbb{N}$. It follows that $(S_n)_{n=1}^{\infty}$ is bounded, so that $\sum_{k=1}^{\infty} 2^k a_{2^k} < \infty$.

"$\Longleftarrow$": As $(s_n)_{n=1}^{\infty}$ is increasing, it is enough to show that $(s_{2^n})_{n=1}^{\infty}$ is bounded. Set $S := \sum_{k=1}^{\infty} 2^k a_{2^k}$. For $n \in \mathbb{N}$, we have

$$\begin{aligned}
s_{2^n} &= a_1 + a_2 + a_3 + \cdots + a_{2^n} \\
&= a_1 + \underbrace{(a_2 + a_3)}_{\leq 2a_2} + \underbrace{(a_4 + \cdots + a_7)}_{\leq 4a_4} + \cdots + \underbrace{(a_{2^{n-1}} + \cdots + a_{2^n - 1})}_{\leq 2^{n-1}a_{2^{n-1}}} + a_{2^n} \\
&\leq a_1 + S_{n-1} + a_{2^n} \\
&\leq 2a_1 + S,
\end{aligned}$$

which proves the boundedness of $(s_{2^n})_{n=1}^{\infty}$. $\square$

Example. Let $p > 0$. As $\left(\frac{1}{k^p}\right)_{k=1}^{\infty}$ is decreasing, Cauchy's Compression Theorem applies, so that

$$\begin{aligned}
\sum_{k=1}^{\infty} \frac{1}{k^p} < \infty \quad &\iff \quad \sum_{k=1}^{\infty} \frac{2^k}{(2^k)^p} = \sum_{k=1}^{\infty} (2^{1-p})^k < \infty \\
&\iff \quad 2^{1-p} < 1 \\
&\iff \quad p > 1.
\end{aligned}$$

Hence, for instance, $\sum_{k=1}^{\infty} \frac{1}{k^{\frac{3}{2}}} < \infty$ holds.

Theorem 9.4 (Cauchy Criterion for Infinite Series)**.** *Let $(a_k)_{k=1}^{\infty}$ be a sequence in $\mathbb{R}$. Then $\sum_{k=1}^{\infty} a_k$ converges if and only if, for each $\epsilon > 0$, there is $n_\epsilon \in \mathbb{N}$ such that*

$$\left| \sum_{k=m+1}^{n} a_k \right| < \epsilon$$

for all $n > m \geq n_\epsilon$.

Proof. Let $(s_n)_{n=1}^\infty$ be the sequence of partial sums of $\sum_{k=1}^\infty a_k$ and note that

$$\begin{aligned}
\textstyle\sum_{k=1}^\infty a_k \text{ converges} \quad &\iff \quad (s_n)_{n=1}^\infty \text{ converges} \\
&\iff \quad (s_n)_{n=1}^\infty \text{ is a Cauchy sequence} \\
&\iff \quad \text{for every } \epsilon > 0, \text{ there is } n_\epsilon \in \mathbb{N} \text{ such that} \\
&\qquad\quad |s_n - s_m| < \epsilon \text{ for all } n, m \geq n_\epsilon \\
&\iff \quad \text{for every } \epsilon > 0, \text{ there is } n_\epsilon \in \mathbb{N} \text{ such that} \\
&\qquad\quad |s_n - s_m| < \epsilon \text{ for all } n > m \geq n_\epsilon.
\end{aligned}$$

As

$$s_n - s_m = \sum_{k=1}^n a_k - \sum_{k=1}^m a_k = \sum_{k=m+1}^n a_k$$

for $n > m$, this yields the claim. □

Corollary 9.5. *Let $(a_k)_{k=1}^\infty$ be a sequence in $\mathbb{R}$ such that $\sum_{k=1}^\infty a_k$ converges. Then $\lim_{k\to\infty} a_k = 0$.*

Proof. Let $\epsilon > 0$, and let $n_\epsilon \in \mathbb{N}$ be as in the Cauchy Criterion. Then

$$|a_n| = \left| \sum_{k=(n-1)+1}^n a_k \right| < \epsilon$$

for all $n \geq n_\epsilon + 1$. □

Absolute Convergence

Definition 9.6. Let $(a_k)_{k=1}^\infty a_k$ be a sequence in $\mathbb{R}$. We call $\sum_{k=1}^\infty a_k$ *absolutely convergent* if $\sum_{k=1}^\infty |a_k| < \infty$.

Example. Let $\theta \in (-1, 1)$. Then the geometric series $\sum_{k=0}^\infty \theta^k$ is absolutely convergent.

Proposition 9.7. *Let $(a_k)_{k=1}^\infty$ be a sequence in $\mathbb{R}$ such that $\sum_{k=1}^\infty a_k$ converges absolutely. Then $\sum_{k=1}^\infty a_k$ converges.*

Proof. Let $\epsilon > 0$. Applying the Cauchy Criterion to $\sum_{k=1}^\infty |a_k|$, we obtain $n_\epsilon \in \mathbb{N}$ such that

$$\sum_{k=m+1}^n |a_k| < \epsilon$$

for all $n > m \geq n_\epsilon$. It follows that

$$\left| \sum_{k=m+1}^n a_k \right| \leq \sum_{k=m+1}^n |a_k| < \epsilon$$

for $n > m \geq n_\epsilon$. Applying the Cauchy Criterion again—this time to $\sum_{k=1}^\infty a_k$—we obtain that $\sum_{k=1}^\infty a_k$ converges. □

Does the converse hold?

Theorem 9.8 (Alternating Series Test)**.** *Let $(a_k)_{k=1}^\infty$ be a decreasing sequence of non-negative reals such that $\lim_{k\to\infty} a_k = 0$. Then $\sum_{k=1}^\infty (-1)^{k-1} a_k$ converges.*

Proof. For $n \in \mathbb{N}$, set

$$s_n := \sum_{k=1}^n (-1)^{k-1} a_k.$$

As

$$s_{2n+2} - s_{2n} = -a_{2n+2} + a_{2n+1} \geq 0$$

for $n \in \mathbb{N}$, the subsequence $(s_{2n})_{n=1}^\infty$ of $(s_n)_{n=1}^\infty$ is increasing. Similarly, we have

$$s_{2n+1} - s_{2n-1} = a_{2n+1} - a_{2n} \leq 0$$

for $n \in \mathbb{N}$, so that $(s_{2n-1})_{n=1}^\infty$ is decreasing. Moreover,

$$s_2 \leq s_{2n} = s_{2n-1} - a_{2n} \leq s_{2n-1} \leq s_1$$

holds for $n \in \mathbb{N}$, so that both $(s_{2n})_{n=1}^\infty$ and $(s_{2n-1})_{n=1}^\infty$ are bounded and therefore convergent.

Set $s := \lim_{n\to\infty} s_{2n-1}$. We claim that $s = \sum_{k=1}^\infty (-1)^{k-1} a_k$.

Let $\epsilon > 0$. Then there is $n_1 \in \mathbb{N}$ such that $|s_{2n-1} - s| < \frac{\epsilon}{2}$ for all $n \geq n_1$. As $\lim_{k\to\infty} a_k = 0$, there is $n_2 \in \mathbb{N}$ such that $|a_n| < \frac{\epsilon}{2}$ for all $n \geq n_2$. Set $n_\epsilon := \max\{2n_1, n_2\}$, and let $n \geq n_\epsilon$.

Case 1: n is odd, i.e., $n = 2m-1$ for some $m \in \mathbb{N}$. As $n \geq 2n_1$, we have $2m \geq 2n_1 + 1$, so that $m \geq n_1$. It follows that

$$|s_n - s| = |s_{2m-1} - s| < \frac{\epsilon}{2} < \epsilon.$$

Case 2: n is even, i.e., $n = 2m$ for some $m \in \mathbb{N}$. It follows that

$$|s_n - s| = |s_{2m} - s| = |s_{2m-1} - a_n - s| \leq |s_{2m-1} - s| + |a_n| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

This completes the proof. □

Example. As $\left(\frac{1}{k}\right)_{k=1}^\infty$ is decreasing and converges to zero, the Alternating Series Test yields the convergence of the *alternating harmonic series* $\sum_{k=1}^\infty \frac{(-1)^{k-1}}{k}$; in fact, $\sum_{k=1}^\infty \frac{(-1)^{k-1}}{k} = \log 2$. But as $\sum_{k=1}^\infty \frac{1}{k} = \infty$, it does not converge absolutely.

Proposition 9.9. *Let $(a_k)_{k=1}^\infty$ and $(b_k)_{k=1}^\infty$ be sequences in $\mathbb{R}$, and let $\alpha, \beta \in \mathbb{R}$. Then:*

(i) *if $\sum_{k=1}^\infty a_k$ and $\sum_{k=1}^\infty b_k$ converge, $\sum_{k=1}^\infty (\alpha a_k + \beta b_k)$ converges as well, and*

$$\sum_{k=1}^\infty (\alpha a_k + \beta b_k) = \alpha \sum_{k=1}^\infty a_k + \beta \sum_{k=1}^\infty b_k$$

holds;

(ii) *if* $\sum_{k=1}^\infty a_k$ *and* $\sum_{k=1}^\infty b_k$ *converge absolutely,* $\sum_{k=1}^\infty (\alpha a_k + \beta b_k)$ *converges absolutely as well.*

Proof. (i) follows immediately from the limit laws.

For (ii), let $n \in \mathbb{N}$, and note that

$$\begin{aligned}\sum_{k=1}^n |\alpha a_k + \beta b_k| &\leq \sum_{k=1}^n (|\alpha a_k| + |\beta b_k|) \\ &= |\alpha| \sum_{k=1}^n |a_k| + |\beta| \sum_{k=1}^n |b_k| \\ &\leq |\alpha| \sum_{k=1}^\infty |a_k| + |\beta| \sum_{k=1}^\infty |b_k|.\end{aligned}$$

It follows that $\sum_{k=1}^\infty |\alpha a_k + \beta b_k| < \infty$. □

Exercises

1. For $k \in \mathbb{N}$, let
$$a_k := \left| \frac{1}{k} - \frac{(-1)^k}{\sqrt{k}} \right|,$$
so that $\lim_{k \to \infty} a_k = 0$. Show that $\sum_{k=1}^\infty (-1)^{k-1} a_k$ diverges. Why doesn't this contradict the Alternating Series Test?

2. Let $(a_k)_{k=1}^\infty$ and $(b_k)_{k=1}^\infty$ be sequences of non-negative reals such that $\sum_{k=1}^\infty a_k < \infty$ and $\sum_{k=1}^\infty b_k < \infty$. Show that $\sum_{k=1}^\infty \sqrt{a_k b_k} < \infty$. (*Hint*: First, prove the *inequality between the arithmetic and the geometric mean*: $\sqrt{ab} \leq \frac{a+b}{2}$ for all $a, b \geq 0$.)

3. Let $(\epsilon_n)_{n=1}^\infty$ be a sequence of non-negative reals such that $\sum_{k=1}^\infty \epsilon_k < \infty$, and let $(x_n)_{n=1}^\infty$ be a sequence in $\mathbb{R}$ such that
$$|x_{n+1} - x_n| \leq \epsilon_n$$
for all $n \in \mathbb{N}$. Show that $(x_n)_{n=1}^\infty$ converges. Does this conclusion remain valid if we only require that $\lim_{n \to \infty} \epsilon_n = 0$?

10 Convergence Criteria

Theorem 10.1 (Comparison Test)**.** *Let $(a_k)_{k=1}^\infty$ and $(b_k)_{k=1}^\infty$ with $b_k \geq 0$ for $k \in \mathbb{N}$.*

(i) *Suppose that $\sum_{k=1}^\infty b_k < \infty$ and that there is $n_0 \in \mathbb{N}$ such that $|a_k| \leq b_k$ for $k \geq n_0$. Then $\sum_{k=1}^\infty a_k$ converges absolutely.*

(ii) *Suppose that $\sum_{k=1}^\infty b_k = \infty$ and that there is $n_0 \in \mathbb{N}$ such that $a_k \geq b_k$ for $k \geq n_0$. Then $\sum_{k=1}^\infty a_k$ diverges.*

Proof. (i): For $n \geq n_0$, we have

$$\sum_{k=1}^{n} |a_k| = \sum_{k=1}^{n_0-1} |a_k| + \sum_{k=n_0}^{n} |a_k| \leq \sum_{k=1}^{n_0-1} |a_k| + \sum_{k=n_0}^{n} b_k \leq \sum_{k=1}^{n_0-1} |a_k| + \sum_{k=1}^{\infty} b_k.$$

Hence, the sequence of partial sums of $\sum_{k=1}^\infty |a_k|$ is bounded. This means that $\sum_{k=1}^\infty a_k$ converges absolutely.

(ii): For $n \geq n_0$, we have

$$\sum_{k=1}^{n} a_k = \sum_{k=1}^{n_0-1} a_k + \sum_{k=n_0}^{n} a_k \geq \sum_{k=1}^{n_0-1} a_k + \sum_{k=n_0}^{n} b_k,$$

so that the sequence of partial sums of $\sum_{k=1}^\infty a_k$ is unbounded. Therefore, $\sum_{k=1}^\infty a_k$ diverges. □

Example. For $k \in \mathbb{N}$, let

$$a_k := \frac{\sin(43k^7 - 4k + 13)}{4k^2 + \cos(e^{k^4} - 7k)}.$$

As

$$|a_k| \leq \frac{1}{3k^2}$$

for $k \in \mathbb{N}$ and $\sum_{k=1}^\infty \frac{1}{3k^2} < \infty$, the series $\sum_{k=1}^\infty a_k$ converges absolutely.

Corollary 10.2 (Limit Comparison Test)**.** *Let $(a_k)_{k=1}^\infty$ and $(b_k)_{k=1}^\infty$ with $b_k \geq 0$ for $k \in \mathbb{N}$.*

(i) *Suppose that $\sum_{k=1}^\infty b_k < \infty$ and that $\lim_{k\to\infty} \frac{|a_k|}{b_k}$ exists in $\mathbb{R}$. Then $\sum_{k=1}^\infty a_k$ converges absolutely.*

(ii) *Suppose that $\sum_{k=1}^\infty b_k = \infty$ and that $\lim_{k\to\infty} \frac{a_k}{b_k}$ exists in $(0,\infty) \cup \{\infty\}$. Then $\sum_{k=1}^\infty a_k$ diverges.*

Proof. (i): Let $n_0 \in \mathbb{N}$ be such that $b_k \neq 0$ for all $k \geq n_0$. As $\left(\frac{|a_k|}{b_k}\right)_{k=n_0}^\infty$ converges, it is bounded, which means that there is $C \geq 0$ with $\frac{|a_k|}{b_k} \leq C$, i.e., $|a_k| \leq C b_k$. Apply the Comparison Test.

(ii): Let $n_0 \in \mathbb{N}$ and $\delta > 0$ be such that $b_k \neq 0$ and $\frac{a_k}{b_k} > \delta$, i.e., $a_k > \delta b_k$ for $k \geq n_0$. Again, apply the Comparison Test. □

Example. For $k \in \mathbb{N}$, let

$$a_k := \frac{4k^2 - 3k + 3}{3k^3 + 2k - 4}$$

and

$$b_k := \frac{1}{k}.$$

It follows that

$$\frac{a_k}{b_k} = \frac{4k^3 - 3k^2 + 3k}{3k^3 + 2k - 4} = \frac{4 - \frac{3}{k} + \frac{3}{k^2}}{3 + \frac{2}{k^2} - \frac{4}{k^3}} \to \frac{4}{3} > 0.$$

Therefore, $\sum_{k=1}^{\infty} a_k$ diverges.

Theorem 10.3 (Ratio Test)**.** *Let $(a_k)_{k=1}^{\infty}$ be a sequence in $\mathbb{R}$.*

(i) *Suppose that there are $n_0 \in \mathbb{N}$ and $\theta \in (0,1)$ such that $a_k \neq 0$ and $\frac{|a_{k+1}|}{|a_k|} \leq \theta$ for all $k \geq n_0$. Then $\sum_{k=1}^{\infty} a_k$ converges absolutely.*

(ii) *Suppose that there are $n_0 \in \mathbb{N}$ and $\theta \geq 1$ such that $a_k \neq 0$ and $\frac{|a_{k+1}|}{|a_k|} \geq \theta$ for all $k \geq n_0$. Then $\sum_{k=1}^{\infty} a_k$ diverges.*

Proof. (i): Clearly, $|a_{k+1}| \leq \theta |a_k|$ holds for all $k \geq n_0$.

We claim that $|a_k| \leq \theta^{k-n_0} |a_{n_0}|$ for all $k \geq n_0$.

This is clear for $k = n_0$. Suppose the claim is true for $k \geq n_0$. Then we obtain

$$|a_{k+1}| \leq \theta |a_k| \leq \theta\, \theta^{k-n_0} |a_{n_0}| = \theta^{(k+1)-n_0} |a_{n_0}|.$$

Induction yields the claim.

As $\sum_{k=1}^{\infty} \theta^{k-n_0} |a_{n_0}| = \theta^{-n_0} |a_{n_0}| \sum_{k=1}^{\infty} \theta^k < \infty$, it follows from the Comparison Test that $\sum_{k=1}^{\infty} a_k$ converges absolutely.

(ii): Clearly, $|a_{k+1}| \geq \theta |a_k|$ holds for all $k \geq n_0$. Inductively, we obtain that

$$|a_k| \geq \theta^{k-n_0} |a_{n_0}| \geq |a_{n_0}| > 0$$

for $k \geq n_0$. This means that $(a_k)_{k=1}^{\infty}$ cannot converge to zero, so that $\sum_{k=1}^{\infty} a_k$ must diverge. $\square$

Corollary 10.4 (Limit Ratio Test)**.** *Let $(a_k)_{k=1}^{\infty}$ be a sequence in $\mathbb{R}$ such that there is $n_0 \in \mathbb{N}$ with $a_k \neq 0$ for all $k \geq n_0$. Then:*

(i) $\sum_{k=1}^{\infty} a_k$ *converges absolutely if* $\lim_{k \to \infty} \frac{|a_{k+1}|}{|a_k|} < 1$*;*

(ii) $\sum_{k=1}^{\infty} a_k$ *diverges if* $\lim_{k \to \infty} \frac{|a_{k+1}|}{|a_k|} > 1$ *in* $\mathbb{R} \cup \{\infty\}$*.*

Remark. Nothing can be said if $\lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|} = 1$. For instance, let

$$a_k := \frac{1}{k^2} \qquad \text{and} \qquad b_k := \frac{1}{k}$$

for $k \in \mathbb{N}$. Then $\lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|} = \lim_{k\to\infty} \frac{|b_{k+1}|}{|b_k|} = 1$ holds whereas

$$\sum_{k=1}^{\infty} a_k < \infty \qquad \text{and} \qquad \sum_{k=1}^{\infty} b_k = \infty.$$

Example. Fix $x \in \mathbb{R}$. For $k \in \mathbb{N}$, let $a_k := \frac{x^k}{k!}$. Clearly, $\sum_{k=0}^{\infty} a_k$ converges if $x = 0$. If $x \neq 0$, we have $a_k \neq 0$ for all $k \in \mathbb{N}_0$ and

$$\frac{|a_{k+1}|}{|a_k|} = \frac{|x|^{k+1}}{(k+1)!} \frac{k!}{|x|^k} = \frac{|x|}{k} \to 0.$$

Therefore, $\sum_{k=0}^{\infty} a_k$ converges absolutely.

Theorem 10.5 (Root Test)**.** *Let $(a_k)_{k=1}^{\infty}$ be a sequence in $\mathbb{R}$.*

(i) *Suppose that there are $n_0 \in \mathbb{N}$ and $\theta \in (0,1)$ such that $\sqrt[k]{|a_k|} \leq \theta$ for all $k \geq n_0$. Then $\sum_{k=1}^{\infty} a_k$ converges absolutely.*

(ii) *Suppose that there is $\theta \geq 1$ such that $\sqrt[k]{|a_k|} \geq \theta$ for infinitely many $k \in \mathbb{N}$. Then $\sum_{k=1}^{\infty} a_k$ diverges.*

Proof. (i): We have $|a_k| \leq \theta^k$ for $k \geq n_0$. Applying the Comparison Test yields the claim.

(ii): We have $|a_k| \geq \theta^k \geq 1$ for infinitely many $k \in \mathbb{N}$. This rules out that $(a_k)_{k=1}^{\infty}$ converges to zero, so that $\sum_{k=1}^{\infty} a_k$ must diverge. $\square$

Example. For $k \in \mathbb{N}$, let

$$a_k := \frac{2 + (-1)^k}{2^{k-1}},$$

so that

$$\frac{a_{k+1}}{a_k} = \frac{2 + (-1)^{k+1}}{2^k} \frac{2^{k-1}}{2 + (-1)^k} = \frac{1}{2} \frac{2 + (-1)^{k+1}}{2 + (-1)^k} = \begin{cases} \frac{1}{6}, & \text{for even } k, \\ \frac{3}{2}, & \text{for odd } k. \end{cases}$$

Hence, we cannot apply the Ratio Text to the series $\sum_{k=1}^{\infty} a_k$. However, we have

$$\sqrt[k]{a_k} = \sqrt[k]{\frac{2(2 + (-1)^k)}{2^k}} \leq \frac{\sqrt[k]{6}}{2} \to \frac{1}{2}$$

Therefore, there is $n_0 \in \mathbb{N}$ such that $\sqrt[k]{a_k} < \frac{2}{3}$ for $k \geq n_0$, so that $\sum_{k=1}^{\infty} a_k$ converges absolutely.

Corollary 10.6 (Limit Root Test)**.** *Let $(a_k)_{k=1}^{\infty}$ be a sequence in $\mathbb{R}$. Then:*

(i) $\sum_{k=1}^\infty a_k$ *converges absolutely if* $\limsup_{k\to\infty} \sqrt[k]{|a_k|} < 1$;

(ii) $\sum_{k=1}^\infty a_k$ *diverges if* $\limsup_{k\to\infty} \sqrt[k]{|a_k|} > 1$ *in* $\mathbb{R} \cup \{\infty\}$.

Proof. (i): Let $\theta \in \mathbb{R}$ be such that $\limsup_{k\to\infty} \sqrt[k]{|a_k|} < \theta < 1$. Then there is $n_0 \in \mathbb{N}$ such that $\sqrt[k]{|a_k|} < \theta$. The claim then follows from the Root Test.

(ii): Let $k_1 < k_2 < k_3 < \cdots$ be such that $\lim_{j\to\infty} \sqrt[k_j]{|a_{k_j}|} = \limsup_{k\to\infty} \sqrt[k]{|a_k|} > 1$. Then there must be $j_0 \in \mathbb{N}$ such that $\sqrt[k_j]{|a_{k_j}|} \geq 1$ for all $j \geq j_0$, i.e., $\sqrt[k]{|a_k|} \geq 1$ for infinitely many $k \in \mathbb{N}$. The Root Test then yields the divergence of $\sum_{k=1}^\infty a_k$. $\square$

As for the Limit Ratio Test, no conclusion is possible if $\lim_{k\to\infty} \sqrt[k]{|a_k|} = 1$.

Exercises

1. Determine whether or not the following series converge or converge absolutely:

$$\text{(a)} \quad \sum_{\nu=0}^\infty \frac{\cos(\nu\pi)}{\nu+1}; \qquad \text{(b)} \quad \sum_{k=1}^\infty \frac{k!}{k^k}; \qquad \text{(c)} \quad \sum_{m=2}^\infty \frac{1}{m\log m}.$$

   (*Hint for* (c): Cauchy's Comppression Theorem.)

2. Determine whether or not the following series diverge, converge, or converge absolutely:

   (a) $\displaystyle\sum_{k=1}^\infty (-1)^{k^3+3k^2-7k+13} \frac{\sin k}{\cos k + k^2}$;

   (b) $\displaystyle\sum_{\nu=1}^\infty \frac{1}{\nu^\nu}\binom{2\nu}{\nu}$;

   (c) $\displaystyle\sum_{n=1}^\infty \frac{n^n}{n!}$;

   (d) $\displaystyle\sum_{n=1}^\infty \binom{2n}{n} \Big/ \binom{3n}{n}$;

   (e) $\displaystyle\sum_{\nu=1}^\infty \frac{(-1)^\nu + 2}{(-1)^{\nu-1}\nu}$;

   (f) $\sum_{m=1}^\infty \frac{\sin\left(\frac{\pi}{2}+m\pi\right)}{\sqrt{m}}$.

3. Let p be a polynomial, and let $\theta \in (-1,1)$. Show that the series $\sum_{k=1}^\infty p(k)\theta^k$ converges absolutely.

4. (a) Show that $\limsup_{n\to\infty} \sqrt[n]{n} \leq 1$.

(b) Conclude that that $\lim_{n\to\infty} \sqrt[n]{n} = 1$ and $\lim_{n\to\infty} \sqrt[n]{r} = 1$ for all $r > 0$.

(*Hint for* (a): Assume that $\limsup_{n\to\infty} \sqrt[n]{n} > 1$ and arrive a contradiction to the fact that $\sum_{n=1}^{\infty} n\,\theta^n < \infty$ for all $\theta \in (0,1)$.)

5. Let p and q be polynomials, let ν be the degree of p, and let μ be the degree of q. Suppose that n_0 is such that $q(k) \neq 0$ for all $k \geq n_0$. Show that the series $\sum_{k=n_0}^{\infty} \frac{p(k)}{q(k)}$ converges if and only if $\mu - \nu \geq 2$. (*Hint*: Limit Comparison Test.)

6. Let $(a_n)_{n=1}^{\infty}$ be a sequence in $\mathbb{R}$ such that $\liminf_{n\to\infty} |a_n| = 0$, and let $(R_k)_{k=1}^{\infty}$ be a sequence of non-zero reals. Show that $(a_n)_{n=1}^{\infty}$ has a subsequence $(a_{n_k})_{k=1}^{\infty}$ such that $\sum_{k=1}^{\infty} R_k a_{n_k}$ converges absolutely.

7. Show that that
$$\sum_{k=1}^{n} kx^k = x\frac{nx^{n+1} - (n+1)x^n + 1}{(1-x)^2}$$
for all $n \in \mathbb{N}$, and conclude that
$$\sum_{k=1}^{\infty} kx^k = \frac{x}{(1-x)^2}$$
if $|x| < 1$.

11 More on Absolute Convergence

Theorem 11.1. *Let $(a_k)_{k=1}^\infty$ be a sequence in $\mathbb{R}$ such that $\sum_{k=1}^\infty a_k$ converges absolutely, and let $\sigma : \mathbb{N} \to \mathbb{N}$ be bijective. Then $\sum_{k=1}^\infty a_{\sigma(k)}$ is also absolutely convergent with $\sum_{k=1}^\infty a_{\sigma(k)} = \sum_{k=1}^\infty a_k$.*

Proof. Set $x := \sum_{k=1}^\infty a_k$. Let $\epsilon > 0$. By the Cauchy Criterion for infinite series, there is $n_0 \in \mathbb{N}$ such that $\sum_{k=n_0}^n |a_k| < \frac{\epsilon}{3}$ for all $n \geq n_0$, so that

$$\sum_{k=n_0}^\infty |a_k| \leq \frac{\epsilon}{3} < \frac{\epsilon}{2}.$$

It follows that

$$\left| x - \sum_{k=1}^{n_0-1} a_k \right| = \left| \sum_{k=n_0}^\infty a_k \right| \leq \sum_{k=n_0}^\infty |a_k| < \frac{\epsilon}{2}.$$

Choose $n_\epsilon \in \mathbb{N}$ so large that

$$\{\sigma^{-1}(1), \ldots, \sigma^{-1}(n_0 - 1)\} \subset \{1, \ldots, n_\epsilon\},$$

i.e.,

$$\{1, \ldots, n_0 - 1\} \subset \{\sigma(1), \ldots, \sigma(n_\epsilon)\}.$$

Let $n \geq n_\epsilon$. We obtain

$$\begin{aligned}
\left| x - \sum_{k=1}^n a_{\sigma(k)} \right| &\leq \left| x - \sum_{k=1}^{n_0-1} a_k \right| + \left| \sum_{k=1}^{n_0-1} a_k - \sum_{k=1}^n a_{\sigma(k)} \right| \\
&< \frac{\epsilon}{2} + \sum_{k \in \{\sigma(1), \ldots, \sigma(n)\} \setminus \{1, \ldots, n_0-1\}} |a_k| \\
&\leq \frac{\epsilon}{2} + \sum_{k=n_0}^\infty |a_k| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon.
\end{aligned}$$

This proves that $\sum_{k=1}^\infty a_{\sigma(k)} = x$.

The same argument applied to the series $\sum_{k=1}^\infty |a_k|$ shows that $\sum_{k=1}^\infty |a_{\sigma(k)}|$ converges, so that $\sum_{k=1}^\infty a_k$ is absolutely convergent. □

Theorem 11.2 (Riemann's Rearrangement Theorem)**.** *Let $(a_k)_{k=1}^\infty$ be a sequence in $\mathbb{R}$ such that $\sum_{k=1}^\infty a_k$ converges, but is not absolutely convergent, and let $x \in \mathbb{R} \cup \{-\infty, \infty\}$. Then there is a bijection $\sigma : \mathbb{N} \to \mathbb{N}$ such that $\sum_{k=1}^\infty a_{\sigma(k)} = x$.*

Proof. We need to show to find a rearrangement of $a_1, a_2, a_3, \ldots$ such that the rearrange series converges to x.

Let $b_1, b_2, b_3, \ldots$ denote the non-negative terms of $(a_k)_{k=1}^\infty$, and let $c_1, c_2, c_3, \ldots$ the strictly negative ones. We claim that $\sum_{k=1}^\infty b_k = \infty$. Otherwise,

$$\sum_{k=1}^\infty c_k = \sum_{k=1}^\infty a_k - \sum_{k=1}^\infty b_k$$

would converge as would consequently

$$\sum_{k=1}^\infty |a_k| = \sum_{k=1}^\infty b_k - \sum_{k=1}^\infty c_k.$$

This would contradict the hypothesis that $\sum_{k=1}^\infty a_k$ does not converge absolutely. Similarly, one sees that $\sum_{k=1}^\infty c_k = -\infty$.

Case 1: $x = \infty$.

As $\sum_{k=1}^\infty b_k = \infty$, there is $n_1 \in \mathbb{N}$ such that

$$\sum_{k=1}^{n_1} b_k > 1 - c_1.$$

As $\sum_{k=n_1+1}^\infty b_k = \infty$, there is $n_2 \in \mathbb{N}$, $n_2 > n_1$ such that

$$\sum_{k=1}^{n+1} b_k + c_1 + \sum_{k=n_1+1}^{n_2} b_k > 2 - c_2.$$

As $\sum_{k=n_2+1}^\infty b_k = \infty$, there is $n_3 \in \mathbb{N}$, $n_3 > n_2$ such that

$$\sum_{k=1}^{n+1} b_k + c_1 + \sum_{k=n_1+1}^{n_2} b_k + c_2 + \sum_{k=n_2+1}^{n_3} > 3 - c_3.$$

Continuing in this fashion, we obtain a rearrangement

$$b_1, \ldots, b_{n_1}, c_1, b_{n_1+1}, \ldots b_{n_2}, c_2, b_{n_2+1}, \ldots, b_{n_3}, c_3, b_{n+3+1}, \ldots$$

of $a_1, a_2, a_3, \ldots$.

For $N \in \mathbb{N}$, let s_N denote the N^{th} partial sum of the rearranged series. If $\nu \in \mathbb{N}$ is such that $N \geq n_\nu$, then $s_N \geq \nu$. It follows that the rearranged series diverges to ∞.

Case 2: $x = -\infty$.

This is dealt with similarly.

Case 3: $x \in \mathbb{R}$.

Choose $n_1 \in \mathbb{N}$ *minimal* such that

$$\sum_{k=1}^{n_1} b_k > x.$$

Then choose $m_1 \in \mathbb{N}$ *minimal* such that

$$\sum_{k=1}^{n_1} b_k + \sum_{k=1}^{m_1} c_k < x.$$

Now choose $n_2 \in \mathbb{N}$, $n_2 > n_1$ minimal such that

$$\sum_{k=1}^{n_1} b_k + \sum_{k=1}^{m_1} c_k + \sum_{k=n_1+1}^{n_2} b_k > x.$$

Then choose $m_2 \in \mathbb{N}$, $m_2 > m_1$ such that

$$\sum_{k=1}^{n_1} b_k + \sum_{k=1}^{m_1} c_k + \sum_{k=n_1+1}^{n_2} b_k + \sum_{k=m_1+1}^{m_2} c_k < x.$$

Continuing in this fashion, we obtain a rearrangement

$$b_1, \ldots, b_{n_1}, c_1, \ldots, c_{m_1}, b_{n_1+1}, \ldots b_{n_2}, c_{m_1+1}, \ldots c_{m_2}, b_{n_2+1}, \ldots, b_{n_3}, c_{m_2+1}, \ldots$$

of $a_1, a_2, a_3, \ldots$.

For $N \in \mathbb{N}$, let s_N denote the N^{th} partial sum of the rearranged series. Then s_N is of one of the following forms for some $\nu \in \mathbb{N}$:

$$\sum_{k=1}^{n_1} b_k + \sum_{k=1}^{m_1} c_k + \cdots + \sum_{k=n_\nu+1}^{n} b_k \tag{2}$$

with $n \le n_{\nu+1}$ or

$$\sum_{k=1}^{n_1} b_k + \sum_{k=1}^{m_1} c_k + \cdots + \sum_{k=n_\nu+1}^{n} b_k + \sum_{k=m_\nu+1}^{m} c_k \tag{3}$$

with $m \le m_{\nu+1}$.

Suppose that S_N is of the form (2). Then the minimality of $n_{\nu+1}$ yields that

$$|x - S_N| \le b_{n_{\nu+1}},$$

whereas if $n = n_{n+1}$ and

$$|x - S_N| \le -c_{m_\nu}$$

if $n < n_{\nu+1}$. Similar estimates hold if S_N is of the form (3). All in all

$$|x - S_N| \le \max\{b_{n_{\nu+1}}, c_{m_\nu}, -c_{m_{\nu+1}}, b_{n_\nu}\}$$

holds.

As $\sum_{k=1}^{\infty} a_k$ converges, $\lim_{k\to\infty} a_k = 0$ holds and therefore $\lim_{k\to\infty} b_k = \lim_{k\to\infty} c_k = 0$ as well. It follows that $x = \lim_{N\to\infty} s_N$. $\square$

Theorem 11.3 (Cauchy Product Formula). *Let $(a_k)_{k=0}^\infty$ and $(b_k)_{k=0}^\infty$ be sequences in $\mathbb{R}$ such that $\sum_{k=0}^\infty a_k$ and $\sum_{k=0}^\infty b_k$ converge absolutely. For $k \in \mathbb{N}_0$, set*

$$c_k := \sum_{j=0}^{k} a_j b_{k-j}.$$

Then the series $\sum_{k=0}^\infty c_k$ converges absolutely such that

$$\sum_{k=0}^\infty c_k = \left(\sum_{k=0}^\infty a_k\right)\left(\sum_{k=0}^\infty b_k\right).$$

Proof. For $n \in \mathbb{N}$, set

$$A_n := \sum_{k=0}^{n} a_k, \qquad B_n := \sum_{k=1}^{n} b_k, \qquad C_n := \sum_{k=1}^{n} c_k$$

and

$$P_n := \left(\sum_{k=0}^{n} |a_k|\right)\left(\sum_{k=0}^{n} |b_k|\right).$$

We claim that

$$\lim_{n\to\infty}(A_n B_n - C_n) = 0.$$

Note first that, for $n \in \mathbb{N}$, we have

$$C_n = \sum_{k=0}^{n}\sum_{j=0}^{k} a_j b_{k-j} = \sum_{\substack{0\le j,l \\ j+l\le n}} a_j b_l \qquad \text{and} \qquad A_n B_n = \sum_{0\le j,l\le n} a_j b_l,$$

so that

$$A_n B_n - C_n = \sum_{\substack{0\le j,l\le n \\ j+l>n}} a_j b_l.$$

Let $\epsilon > 0$. By the absolute convergence of $\sum_{k=0}^\infty a_k$ and $\sum_{k=0}^\infty b_k$, the sequence $(P_n)_{n=1}^\infty$ converges and therefore is a Cauchy sequence. This means that there is $n_\epsilon \in \mathbb{N}$ such that

$$P_n - P_{n_\epsilon} = |P_n - P_{n_\epsilon}| < \epsilon$$

for all $n \geq n_\epsilon$. For $n \geq 2n_\epsilon$, we obtain

$$\begin{aligned}
|A_n B_n - C_n| &\leq \sum_{\substack{0 \leq j,l \leq n \\ j+l > n}} |a_j b_l| \\
&\leq \sum_{\substack{0 \leq j,l \leq n \\ j+l > 2n_\epsilon}} |a_j b_l| \\
&\leq \sum_{\substack{0 \leq j,l \leq n \\ j > n_\epsilon \text{ or } l > n_\epsilon}} |a_j b_l| \\
&= \sum_{0 \leq j,l \leq n} |a_j||b_l| - \sum_{0 \leq j,l \leq n_\epsilon} |a_j||b_l| \\
&= P_n - P_{n_\epsilon} \\
&< \epsilon.
\end{aligned}$$

This proves our claim, so that

$$\sum_{k=0}^{\infty} c_k = \left(\sum_{k=0}^{\infty} a_k \right) \left(\sum_{k=0}^{\infty} b_k \right).$$

holds.

To see that $\sum_{k=0}^{\infty} c_k$ actually converges absolutely, let $n \in \mathbb{N}$, and observe that

$$\sum_{k=0}^{n} |c_k| \leq \sum_{k=0}^{n} \sum_{j=0}^{k} |a_j||b_{k-j}| \leq \left(\sum_{k=0}^{n} |a_k| \right) \left(\sum_{k=0}^{n} |b_k| \right) \leq \left(\sum_{k=0}^{\infty} |a_k| \right) \left(\sum_{k=0}^{\infty} |b_k| \right),$$

which completes the proof. □

Exercises

1. (*Failure of the Cauchy Product Formula without absolute convergence.*) For $k \in \mathbb{N}_0$, set

$$a_k := b_k := \frac{(-1)^k}{\sqrt{k+1}} \qquad \text{and} \qquad c_k := \sum_{j=0}^{k} a_j b_{k-j}.$$

Show that $\sum_{k=0}^{\infty} a_k$ and $\sum_{k=0}^{\infty} b_k$ converge, but that $(c_k)_{k=0}^{\infty}$ does not converge to zero (so that, in particular, $\sum_{k=0}^{\infty} c_k$ does not converge).

12 The Exponential Function

The *exponential function* $\exp\colon \mathbb{R} \to \mathbb{R}$ is defined for $x \in \mathbb{R}$ as

$$\exp(x) := \sum_{k=0}^{\infty} \frac{x^k}{k!}.$$

As we saw, the series converges absolutely for each $x \in \mathbb{R}$. For $x, y \in \mathbb{R}$, we obtain from the Cauchy Product Formula

$$\begin{aligned}
\exp(x)\exp(y) &= \left(\sum_{k=0}^{\infty} \frac{x^k}{k!}\right)\left(\sum_{k=0}^{\infty} \frac{y^k}{k!}\right) \\
&= \sum_{k=0}^{\infty}\sum_{j=0}^{k} \frac{x^j}{j!}\frac{y^{k-j}}{(k-j)!}, \qquad \text{by Theorem 11.3,} \\
&= \sum_{k=0}^{\infty} \frac{1}{k!}\sum_{j=0}^{k} \underbrace{\frac{k!}{j!(k-j)!}}_{=\binom{k}{j}} x^j y^{k-j} \\
&= \sum_{k=0}^{\infty} \frac{(x+y)^k}{k!}, \qquad \text{by the Binomial Theorem,} \\
&= \exp(x+y).
\end{aligned}$$

By induction, we see that

$$\exp(nx) = \exp(\underbrace{x + \cdots + x}_{n \text{ times}}) = \exp(x)^n$$

for any $x \in \mathbb{R}$ and $n \in \mathbb{N}$. We define *Euler's constant* as $e := \exp(1)$, so that $\exp(n) = e^n$ for all $n \in \mathbb{N}$. Clearly, $\exp(0) = 1 = e^0$ holds. For any $x \in \mathbb{R}$, we have

$$1 = \exp(0) = \exp(x - x) = \exp(x)\exp(-x),$$

so that $\exp(x) \neq 0$ and $\exp(-x) = \frac{1}{\exp(x)}$. In particular,

$$\exp(-n) = \frac{1}{\exp(n)} = \frac{1}{e^n} = e^{-n}$$

holds for all $n \in \mathbb{N}$. All in all, we have $\exp(m) = e^m$ for all $m \in \mathbb{Z}$. Next, note that

$$\exp(x) = \exp\left(\frac{x}{2} + \frac{x}{2}\right) = \exp\left(\frac{x}{2}\right)^2 > 0$$

for any $x \in \mathbb{R}$. Let $q \in \mathbb{Q}$, and choose $m \in \mathbb{Z}$ and $n \in \mathbb{N}$ such that $q = \frac{m}{n}$. We have

$$\exp(q)^n = \exp(nq) = \exp(m) = e^m,$$

so that

$$\exp(q) = \sqrt[n]{e^m} = e^q.$$

In view of this, we *define*

$$e^x := \exp(x)$$

for all $x \in \mathbb{R}$.

We *defined* Euler's constant e to be $\exp(1)$. We *claim* that

$$e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$$

We proceed by proving several inequalities.

Claim 1.

$$2 \leq \left(1 + \frac{1}{n}\right)^n \leq \sum_{k=0}^{n} \frac{1}{k!}.$$

for all $n \in \mathbb{N}$*.*

Proof. By Bernoulli's Inequality, we have

$$\left(1 + \frac{1}{n}\right)^n \geq 1 + \frac{n}{n} = 2$$

for all $n \in \mathbb{N}$, which proves the first inequality.

For $n \in \mathbb{N}$, note that

$$\begin{aligned}
\left(1 + \frac{1}{n}\right)^n &= \sum_{k=0}^{n} \binom{n}{k} \frac{1}{n^k} \\
&= 1 + \sum_{k=1}^{n} \left(\prod_{j=1}^{k} \frac{n-k+j}{j}\right) \frac{1}{n^k} \\
&= 1 + \sum_{k=1}^{n} \frac{1}{k!} \prod_{j=1}^{k} \underbrace{\frac{n-k+j}{n}}_{\leq 1} \\
&\leq 1 + \sum_{k=1}^{n} \frac{1}{k!} \\
&= \sum_{k=0}^{n} \frac{1}{k!},
\end{aligned}$$

which proves the second inequality. $\square$

Claim 2.

$$e = \sum_{k=0}^{\infty} \frac{1}{k!} \leq 3.$$

Proof. First note that, by induction, it is clear that

$$k! \geq 2^{k-1}$$

for all $k \in \mathbb{N}$.

Let $n \in \mathbb{N}$, and note that

$$\sum_{k=0}^{n} \frac{1}{k!} = 1 + \sum_{k=1}^{n} \frac{1}{k!} \leq 1 + \sum_{k=1}^{n} \frac{1}{2^{k-1}} = 1 + \sum_{k=0}^{n-1} \frac{1}{2^k} = 1 + 2\left(1 - \frac{1}{2^n}\right).$$

Letting $n \to \infty$ yields the equality. $\square$

In particular, Claims 1 and 2 guarantee that the sequence $\left(\left(1+\frac{1}{n}\right)^n\right)_{n=1}^{\infty}$ is bounded.

Claim 3.

$$\left(1 + \frac{1}{n}\right)^n > \left(1 + \frac{1}{m}\right)^m$$

for all $n, m \in \mathbb{N}$ with $n > m$.

Proof. Let $n, m \in \mathbb{N}$ with $n > m$. Then the Binomial Theorem yields

$$\begin{aligned}
\left(1 + \frac{1}{n}\right)^n &= \sum_{k=0}^{n} \binom{n}{k} \frac{1}{n^k} \\
&> \sum_{k=0}^{m} \binom{n}{k} \frac{1}{n^k} \\
&= 1 + \sum_{k=1}^{m} \frac{1}{k!} \prod_{j=1}^{k} \frac{n-j+1}{n} \\
&= 1 + \sum_{k=1}^{m} \frac{1}{k!} \prod_{j=1}^{k} \left(1 - \frac{j-1}{n}\right) \\
&\geq 1 + \sum_{k=1}^{m} \frac{1}{k!} \prod_{j=1}^{k} \left(1 - \frac{j-1}{m}\right) \\
&= 1 + \sum_{k=1}^{m} \frac{1}{k!} \prod_{j=1}^{k} \frac{m-j+1}{m} \\
&= \sum_{k=0}^{m} \binom{m}{k} \frac{1}{m^k} \\
&= \left(1 + \frac{1}{m}\right)^m,
\end{aligned}$$

which proves the claim. $\square$

By Claims 1 and 2, the sequence $\left(\left(1+\frac{1}{n}\right)^n\right)_{n=1}^{\infty}$ is bounded below by 2 and above by 3. Claim 3 asserts that it is also increasing. Therefore, we now know that $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$ exists. It remains to be shown that it equals e.

Claim 4.

$$\left(1+\frac{1}{n}\right)^n \geq 1+1+\left(1-\frac{1}{n}\right)\frac{1}{2!}+\cdots+\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{m-1}{n}\right)\frac{1}{m!}$$

for all $n, m \in \mathbb{N}$ with $n > m$.

Proof. For $n > m$, the Binomial Theorem yields

$$\begin{aligned}
\left(1+\frac{1}{n}\right)^n &= \sum_{k=0}^{n}\binom{n}{k}\frac{1}{n^k} \\
&\geq \sum_{k=0}^{m}\binom{n}{k}\frac{1}{n^k} \\
&= 1+\sum_{k=1}^{m}\frac{1}{k!}\prod_{j=1}^{k}\frac{n-j+1}{n} \qquad , \\
&= 1+\sum_{k=1}^{m}\frac{1}{k!}\prod_{j=1}^{k}\left(1-\frac{j-1}{n}\right) \\
&= 1+1+\left(1-\frac{1}{n}\right)\frac{1}{2!}+\cdots+\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{m-1}{n}\right)\frac{1}{m!}.
\end{aligned}$$

which proves the claim. □

We can now put it all together.

Fix $m \in \mathbb{N}$. Letting $n \to \infty$, we obtain that

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \geq 1+1+\frac{1}{2!}+\cdots+\frac{1}{m!}=\sum_{k=0}^{m}\frac{1}{k!}.$$

Letting $m \to \infty$ yields $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \geq e$. The reversed inequality follows from Claim 1.

More generally, one can show that

$$e^x = \exp(x) = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n$$

for all $x \in \mathbb{R}$ (see Exercise 2 below).

Exercises

1. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a non-zero, continuous function such that $f(x+y) = f(x)f(y)$ for all $x, y \in \mathbb{R}$. Show that there is $C \in \mathbb{R}$ such that $f(x) = \exp(Cx)$ for $x \in \mathbb{R}$.

2. Show that show that
$$\exp(x) = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n$$
for all $x \in \mathbb{R}$. Proceed as follows:

- for $x \geq 0$, modify the argument in the case where $x = 1$;
- for $x < 0$, note that $\left(1 + \frac{x}{n}\right)^n \left(1 - \frac{x}{n}\right)^n = \left(1 - \frac{x^2}{n^2}\right)^n$, and observe what happens as $n \to \infty$.

13 Closed and Compact Subsets of $\mathbb{R}$

Definition 13.1. Let $F \subset \mathbb{R}$. We call F *closed* if, for every convergent sequence $(x_n)_{n=1}^\infty$ in F, i.e., $x_n \in F$ for all $n \in \mathbb{N}$, its limit also lies in F.

Examples. 1. $\varnothing$ and $\mathbb{R}$ are closed for trivial reasons.

2. All *singleton subsets* of $\mathbb{R}$, i.e., sets of the form $\{x\}$ with $x \in \mathbb{R}$ are closed.

3. Let $a < b$. Then $(-\infty, a]$, $[a, b]$, and $[b, \infty)$ are closed.

4. The set $(0, 1]$ is *not* closed. To see this, let $x_n := \frac{1}{n} \in (0, 1]$ for $n \in \mathbb{N}$. Then $0 = \lim_{n\to\infty} x_n$ does not lie in $(0, 1]$.

5. Let $x \in \mathbb{R}$. For each $n \in \mathbb{N}$, there is $q_n \in \mathbb{Q}$ with $x < q_n < x + \frac{1}{n}$, so that $x = \lim_{n\to\infty} q_n$. As x may be irrational, this means that $\mathbb{Q}$ is not closed.

6. Let $(x_n)_{n=1}^\infty$ be a sequence in $\mathbb{R}$, and let

$$S := \{x \in \mathbb{R} : x \text{ is an accumulation point of } (x_n)_{n=1}^\infty\}.$$

We claim that S is closed. To see this, let $(s_\nu)_{\nu=1}^\infty$ be a convergent sequence in S, and let $s := \lim_{\nu\to\infty} s_\nu$. We must find a subsequence $(x_{n_k})_{k=1}^\infty$ of $(x_n)_{n=1}^\infty$ such that $s = \lim_{k\to\infty} x_{n_k}$. We can find $\nu_1 < \nu_2 < \nu_3 < \cdots$ such that

$$|s_{\nu_k} - s| < \frac{1}{k}$$

for all $k \in \mathbb{N}$. For any $k \in \mathbb{N}$, set

$$\mathbb{N}_k := \left\{ n \in \mathbb{N} : |s_{\nu_k} - x_n| < \frac{1}{k} \right\}.$$

As s_{ν_k} is an accumulation point for $(x_n)_{n=1}^\infty$ for each $k \in \mathbb{N}$, the set $\mathbb{N}_k$ is infinite for each $k \in \mathbb{N}$. Pick $n_1 \in \mathbb{N}_1$. As $\mathbb{N}_2$ is infinite, there is $n_2 \in \mathbb{N}_2$ with $n_2 > n_1$. Continue in this fashion. Suppose that $n_1 < n_2 < \cdots < n_k$ have already been chosen such that $n_j \in \mathbb{N}_j$ for $j = 1, \ldots, k$. As $\mathbb{N}_{k+1}$ is infinite, there is $n_{k+1} \in \mathbb{N}_{k+1}$ with $n_{k+1} > n_k$. For $n_1 < n_2 < n_3 < \cdots$ chosen this way, we have

$$|s - x_{n_k}| \le |s - s_{\nu_k}| + |s_{\nu_k} - x_{n_k}| < \frac{1}{k} + \frac{1}{k} = \frac{2}{k}$$

for all $k \in \mathbb{N}$. This means that $s = \lim_{k\to\infty} x_{n_k}$, so that $s \in S$.

Definition 13.2. Let $S \subset \mathbb{R}$. The *closure* of S is defined as

$$\overline{S} := \left\{ x \in \mathbb{R} : \text{there is a sequence } (x_n)_{n=1}^\infty \text{ in } S \text{ with } x = \lim_{n\to\infty} x_n \right\}.$$

Remarks. 1. $S \subset \overline{S}$ for every $S \subset \mathbb{R}$.

2. $\overline{S} = S$ if and only if S is closed.

3. $\overline{\mathbb{Q}} = \mathbb{R}$.

Proposition 13.3. *Let $S \subset \mathbb{R}$. Then $\overline{S}$ is closed. Moreover, $\overline{S} \subset F$ for any closed $F \subset \mathbb{R}$ with $S \subset F$.*

Proof. We first prove the "moreover" part.

Let $F \subset \mathbb{R}$ be closed such that $S \subset F$, and let $x \in \overline{S}$. Then there is a sequence $(x_n)_{n=1}^\infty$ in S such that $x = \lim_{n\to\infty} x_n$. As $S \subset F$, the sequence $(x_n)_{n=1}^\infty$ also lies in F, and as F is closed, $x = \lim_{n\to\infty} x_n \in F$ holds.

To see that $\overline{S}$ is indeed closed, let $(s_n)_{n=1}^\infty$ be a convergent sequence in $\overline{S}$, and set $s := \lim_{n\to\infty} s_n$. We need to find a sequence $(x_k)_{k=1}^\infty$ in S such that $s = \lim_{k\to\infty} x_k$. Choose $n_1 < n_2 < n_3 < \cdots$ such that

$$|s_{n_k} - s| < \frac{1}{k}$$

for $k \in \mathbb{N}$. For each $k \in \mathbb{N}$, the term s_{n_k} is the limit of a convergent sequence in S; in particular, there is $x_k \in S$ such that $|s_{n_k} - x_k| < \frac{1}{k}$. It follows that

$$|x_k - s| \le |x_k - s_{n_k}| + |s_{n_k} - s| < \frac{1}{k} + \frac{1}{k} = \frac{2}{k}$$

for $k \in \mathbb{N}$, so that $s = \lim_{k\to\infty} x_k$. □

Definition 13.4. A set $K \subset \mathbb{R}$ is called *compact* if it is both closed and bounded.

Proposition 13.5. *The following are equivalent for $K \subset \mathbb{R}$:*

(i) *K is compact;*

(ii) *every sequence in K has a convergent subsequence with limit in K.*

Proof. (i) $\Longrightarrow$ (ii): Let $(x_n)_{n=1}^\infty$ be a sequence in K. As K is bounded, so is $(x_n)_{n=1}^\infty$. The Bolzano–Weierstraß Theorem then yields a convergent subsequence $(x_{n_k})_{k=1}^\infty$ of $(x_n)_{n=1}^\infty$. As K is closed, $\lim_{k\to\infty} x_{n_k} \in K$ holds.

(ii) $\Longrightarrow$ (i): Assume that (ii) holds, but that K is not compact. That leaves two (possibly overlapping) possibilities: K is not closed or K is not bounded.

Case 1: K is not closed. Then there is a convergent sequence $(x_n)_{n=1}^\infty$ in K such that $\lim_{n\to\infty} x_n \notin K$. If $(x_{n_k})_{k=1}^\infty$ is any subsequence of $(x_n)_{n=1}^\infty$, then $\lim_{k\to\infty} x_{n_k} = \lim_{n\to\infty} x_n \notin K$, so that (ii) is violated.

Case 2: K is not bounded. In this case there is, for each $n \in \mathbb{N}$, some $x_n \in K$ with $|x_n| \ge n$. The sequence $(x_n)_{n=1}^\infty$ has no bounded and therefore no convergent subsequence. This contradicts (ii). □

Exercises

1. Let $S \subset \mathbb{R}$. Show that

$$\overline{S} = \{x_0 \in \mathbb{R} : \text{for each } \epsilon > 0, \text{ there is } x \in S \text{ with } |x - x_0| < \epsilon\}.$$

2. Show the following:

   (a) if $F_1, \ldots, F_m \subset \mathbb{R}$ are closed, then $F_1 \cup \cdots \cup F_m$ is closed;

   (b) if $\mathcal{F} \subset \mathfrak{P}(\mathbb{R})$ is such that each $F \in \mathcal{F}$ is closed, then $\bigcap\{F : F \in \mathcal{F}\}$ is closed.

3. A set $U \subset \mathbb{R}$ is called *open* if its complement $\mathbb{R} \setminus U$ is closed. Show that $U \subset \mathbb{R}$ is open if and only if, for each $x \in U$, there is $\epsilon > 0$ such that $(x - \epsilon, x + \epsilon) \subset U$. Conclude that (a, b) is open for $a < b$.

14 Limits of Functions and Continuity

Definition 14.1. Let $\varnothing \neq D \subset \mathbb{R}$, let $f\colon D \to \mathbb{R}$, and let $x_0 \in \overline{D}$.

(a) Suppose that $y_0 \in \mathbb{R}$ is such that $\lim_{n\to\infty} f(x_n) = y_0$ for *every* sequence $(x_n)_{n=1}^\infty$ in D with $\lim_{n\to\infty} x_n = x_0$. Then we say that $f(x)$ *converges to* y_0 as x tends to x_0 and call y_0 the *limit* of $f(x)$ as x tends to x_0; we write

$$\lim_{x\to x_0} f(x) = y_0 \qquad \text{or} \qquad f(x) \overset{x\to x_0}{\longrightarrow} y_0 \qquad \text{or} \qquad f(x) \to y_0.$$

(b) Suppose that $\lim_{n\to\infty} f(x_n) = \infty$ for every sequence $(x_n)_{n=1}^\infty$ in D with $\lim_{n\to\infty} x_n = x_0$. Then we say that $f(x)$ *diverges to* ∞ as x tends to x_0; we write

$$\lim_{x\to x_0} f(x) = \infty \qquad \text{or} \qquad f(x) \overset{x\to x_0}{\longrightarrow} \infty \qquad \text{or} \qquad f(x) \to \infty.$$

(c) Suppose that $\lim_{n\to\infty} f(x_n) = -\infty$ for every sequence $(x_n)_{n=1}^\infty$ in D with $\lim_{n\to\infty} x_n = x_0$. Then we say that $f(x)$ *diverges to* $-\infty$ as x tends to x_0; we write

$$\lim_{x\to x_0} f(x) = -\infty \qquad \text{or} \qquad f(x) \overset{x\to x_0}{\longrightarrow} -\infty \qquad \text{or} \qquad f(x) \to -\infty.$$

Example. Define

$$f\colon \mathbb{R} \to \mathbb{R}, \quad x \mapsto \begin{cases} 0, & x \in \mathbb{Q}, \\ 1, & x \in \mathbb{R} \setminus \mathbb{Q}, \end{cases}$$

and let $x_0 \in \mathbb{R}$ be arbitrary. For every $n \in \mathbb{N}$, there are $q_n \in \mathbb{Q}$ and $r_n \in \mathbb{R} \setminus \mathbb{Q}$ such that

$$x_0 < q_n, r_n < x_0 + \frac{1}{n},$$

so that $x_0 = \lim_{n\to\infty} q_n = \lim_{n\to\infty} r_n$. As

$$\lim_{n\to\infty} f(q_n) = 0 \neq 1 = \lim_{n\to\infty} f(r_n),$$

the limit $\lim_{x\to x_0} f(x)$ does not exist.

Theorem 14.2. *Let $\varnothing \neq D \subset \mathbb{R}$, let $f\colon D \to \mathbb{R}$, and let $x_0 \in \overline{D}$. Then the following are equivalent for $y_0 \in \mathbb{R}$:*

(i) $\lim_{x\to x_0} f(x) = y_0$;

(ii) *for every $\epsilon > 0$, there is $\delta > 0$ such that $|f(x) - y_0| < \epsilon$ for every $x \in D$ with $|x - x_0| < \delta$.*

Proof. (i) $\Longrightarrow$ (ii): We proceed indirectly. Assume that (ii) is false. This means that there is $\epsilon_0 > 0$ such that for every $\delta > 0$, there is $x_\delta \in D$ with $|x_\delta - x_0| < \delta$, but $|f(x_\delta) - y_0| \geq \epsilon_0$. In particular, for each $n \in \mathbb{N}$, there is $x_n \in D$ with $|x_n - x_0| < \frac{1}{n}$ and $|f(x_n) - y_0| \geq \epsilon_0$. It follows that $x_n \to x_0$, but that $f(x_n) \not\to y_0$. This is a contradiction.

(ii) $\Longrightarrow$ (i): Let $(x_n)_{n=1}^\infty$ be a sequence in D such that $x_n \to x_0$. Let $\epsilon > 0$, and let $\delta > 0$ be as specified by (ii). Then there is $n_\epsilon \in \mathbb{N}$ such that $|x_n - x_0| < \delta$ for all $n \geq n_\epsilon$ and consequently $|f(x_n) - y_0| < \epsilon$ for $n \geq n_\epsilon$. This means that $\lim_{n\to\infty} f(x_n) = y_0$. $\square$

Proposition 14.3. *Let* $\varnothing \neq D \subset \mathbb{R}$, *let* $f, g : D \to \mathbb{R}$, *let* $x_0 \in \overline{D}$, *and suppose that* $\lim_{x\to x_0} f(x)$ *and* $\lim_{x\to x_0} g(x)$ *exist in* $\mathbb{R}$. *Then the limits* $\lim_{x\to x_0}(f(x) + g(x))$ *and* $\lim_{x\to x_0} f(x)g(x)$ *exist in* $\mathbb{R}$, *namely*

$$\lim_{x\to x_0}(f(x) + g(x)) = \lim_{x\to x_0} f(x) + \lim_{x\to x_0} g(x)$$

and

$$\lim_{x\to x_0} f(x)g(x) = \left(\lim_{x\to x_0} f(x)\right)\left(\lim_{x\to x_0} g(x)\right).$$

Moreover, if $\lim_{x\to x_0} g(x) \neq 0$, *then there is* $\delta > 0$ *such that* $g(x) \neq 0$ *for all* $x \in D$ *with* $|x - x_0| < \delta$, *and we have*

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = \frac{\lim_{x\to x_0} f(x)}{\lim_{x\to x_0} g(x)}.$$

Proof. The first part is clear by the Limit Laws for sequences.

For the "moreover" part, set $y_0 := \lim_{x\to x_0} g(x)$, so that $|y_0| > 0$. Choose $\delta > 0$ such that $|g(x) - y_0| < \frac{|y_0|}{2}$ for all $x \in D$ with $|x - x_0| < \delta$. It then follows that

$$|g(x)| = |g(x) - y_0 + y_0| \geq ||g(x) - y_0| - |y_0|| > \frac{|y_0|}{2} > 0$$

for all $x \in D$ with $|x - x_0| < \delta$. The formula then follows again from the limit laws. $\square$

Just for the record, we define:

Definition 14.4. Let $D \subset \mathbb{R}$ not be bounded above, and let $f : D \to \mathbb{R}$.

(a) Suppose that $y_0 \in \mathbb{R}$ is such that $\lim_{n\to\infty} f(x_n) = y_0$ for *every* sequence $(x_n)_{n=1}^\infty$ in D with $\lim_{n\to\infty} x_n = \infty$. Then we say that $f(x)$ *converges to* y_0 as x tends to ∞ and call y_0 the *limit* of $f(x)$ as x tends to ∞; we write

$$\lim_{x\to\infty} f(x) = y_0 \qquad \text{or} \qquad f(x) \overset{x\to\infty}{\longrightarrow} y_0 \qquad \text{or} \qquad f(x) \to y_0.$$

(b) Suppose that $\lim_{n\to\infty} f(x_n) = \infty$ for every sequence $(x_n)_{n=1}^\infty$ in D with $\lim_{n\to\infty} x_n = \infty$. Then we say that $f(x)$ *diverges to* ∞ as x tends to ∞; we write

$$\lim_{x\to\infty} f(x) = \infty \qquad \text{or} \qquad f(x) \overset{x\to\infty}{\longrightarrow} \infty \qquad \text{or} \qquad f(x) \to \infty.$$

(c) Suppose that $\lim_{n\to\infty} f(x_n) = -\infty$ for every sequence $(x_n)_{n=1}^\infty$ in D with $\lim_{n\to\infty} x_n = \infty$. Then we say that $f(x)$ *diverges to* $-\infty$ as x tends to ∞; we write

$$\lim_{x\to\infty} f(x) = -\infty \qquad \text{or} \qquad f(x) \stackrel{x\to\infty}{\longrightarrow} -\infty \qquad \text{or} \qquad f(x) \to -\infty.$$

It is obvious how the analogous definition with D not bounded below and $x \to -\infty$ will look like.

Example. Let $c_0, c_1, \ldots, c_n \in \mathbb{R}$ with $c_n \neq 0$, and let

$$p(x) = c_n x^n + \cdots + c_1 x + c_0.$$

We the have

$$\lim_{x\to\infty} p(x) = \begin{cases} c_0, & \text{if } n = 0, \\ \infty, & \text{if } n \geq 1 \text{ and } c_n > 0, \\ -\infty, & \text{if } n \geq 1 \text{ and } c_n < 0, \end{cases}$$

and

$$\lim_{x\to-\infty} p(x) = \begin{cases} c_0, & \text{if } n = 0, \\ \infty, & \text{if } n \geq 1 \text{ is even and } c_n > 0, \\ -\infty, & \text{if } n \geq 1 \text{ is even and } c_n < 0, \\ \infty, & \text{if } n \geq 1 \text{ is odd and } c_n < 0, \\ -\infty, & \text{if } n \geq 1 \text{ is odd and } c_n > 0. \end{cases}$$

Definition 14.5. Let $\varnothing \neq D \subset \mathbb{R}$, let $f : D \to \mathbb{R}$, and let $x_0 \in D$. We say that f is *continuous* at x_0 if $\lim_{x\to x_0} f(x) = f(x_0)$.

The following are immediate from Theorem 14.2 and Proposition 14.3:

Corollary 14.6. *Let $\varnothing \neq D \subset \mathbb{R}$, let $f\colon D \to \mathbb{R}$, and let $x_0 \in D$. Then the following are equivalent:*

(i) *f is continuous at x_0;*

(ii) *for every $\epsilon > 0$, there is $\delta > 0$ such that $|f(x) - f(x_0)| < \epsilon$ for every $x \in D$ with $|x - x_0| < \delta$.*

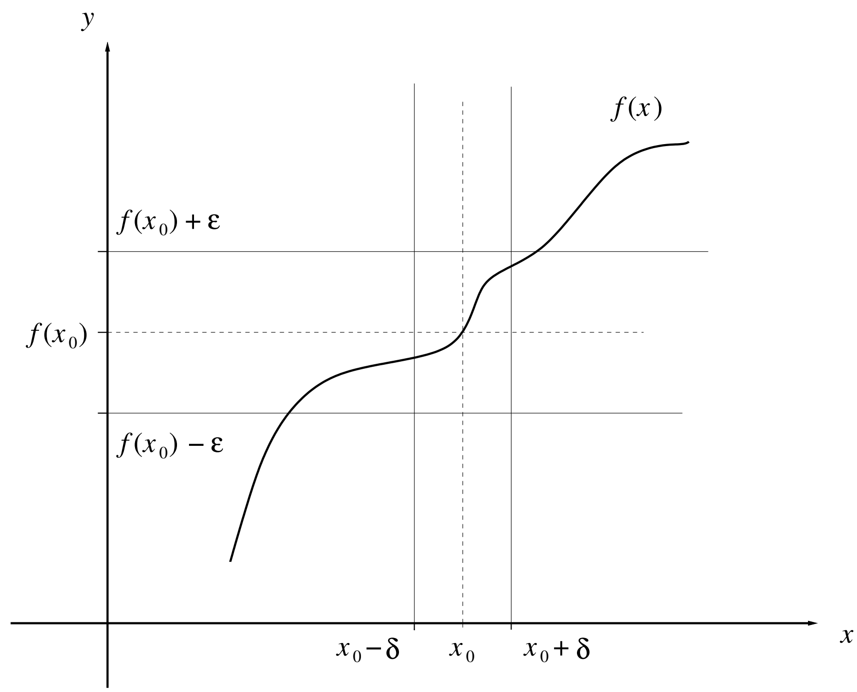

Figure 7: Continuity of f at x_0 in terms of ϵ and δ

Corollary 14.7. *Let $\varnothing \neq D \subset \mathbb{R}$, let $f, g\colon D \to \mathbb{R}$, let $x_0 \in D$, and suppose f and g are continuous at x_0. Then $f+g$ and fg are continuous at x_0. Moreover, if $g(x_0) \neq 0$, then there is $\delta > 0$ such that $g(x) \neq 0$ for all $x \in D$ with $|x - x_0| < \delta$, and the restriction of $\frac{f}{g}$ to $(x_0 - \delta, x_0 + \delta) \cap D$ is continuous at x_0.*

Definition 14.8. Let $\varnothing \neq D \subset \mathbb{R}$, and let $f\colon D \to \mathbb{R}$. We say that f is *continuous* if it is continuous at each $x_0 \in D$.

Examples. 1. Polynomials are continuous on all of $\mathbb{R}$.

2. Rational functions are continuous on their natural domain, i.e., there where the denominator is non-zero.

Continuity of the Exponential Function

Recall that the exponential function $\exp\colon \mathbb{R} \to \mathbb{R}$ is defined as

$$\exp(x) := \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

for $x \in \mathbb{R}$. Is it continuous?

Let $x_0 \in \mathbb{R}$, and let $(x_n)_{n=1}^{\infty}$ be such that $x_n \to x_0$. It follows from the Limit Laws that

$$\lim_{n\to\infty} \exp(x_n) = \lim_{n\to\infty} \sum_{k=0}^{\infty} \frac{x_n^k}{k!} = \sum_{k=0}^{\infty} \lim_{n\to\infty} \frac{x_n^k}{k!} = \sum_{k=0}^{\infty} \frac{x_0^k}{k!} = \exp(x_0),$$

so that exp is continuous at x_0. What is problematic with this argument?

The problem is that the summation symbol in the definition of exp is not a sum, but a limit of sums, so that the above chain of equalities becomes, in fact,

$$\begin{aligned}\lim_{n\to\infty} \exp(x_n) &= \lim_{n\to\infty} \lim_{m\to\infty} \sum_{k=0}^{m} \frac{x_n^k}{k!} \\ &= \lim_{m\to\infty} \lim_{n\to\infty} \sum_{k=0}^{m} \frac{x_n^k}{k!} = \lim_{m\to\infty} \sum_{k=0}^{\infty} \lim_{n\to\infty} \frac{x_n^k}{k!} = \lim_{m\to\infty} \sum_{k=0}^{m} \frac{x_0^k}{k!} = \exp(x_0).\end{aligned}$$

The problem lies with the second equality: we are interchanging the order of $\lim_{n\to\infty}$ and $\lim_{m\to\infty}$. Interchanging the order of limits, however, can be treacherous:

$$\lim_{n\to\infty} \lim_{m\to\infty} \left(1 - \frac{1}{n}\right)^m = 0 \neq 1 = \lim_{m\to\infty} \lim_{n\to\infty} \left(1 - \frac{1}{n}\right)^m.$$

To prove that exp is indeed continuous, we therefore need to do more work.

Example. We will show that exp is continuous. First, we will see that exp is continuous at 0. Let $(x_n)_{n=1}^{\infty}$ be a sequence in $\mathbb{R}$ such that $\lim_{n\to\infty} x_n = 0$. We need do show that $\lim_{n\to\infty} \exp(x_n) = 1$. Let $\epsilon > 0$, and let $C \geq 0$ be such that $|x_n| \leq C$ for $n \in \mathbb{N}$. As $\sum_{k=0}^{\infty} \frac{C^k}{k!}$ converges, there is $m_\epsilon \in \mathbb{N}$ such that $\sum_{k=m_\epsilon+1}^{m} \frac{C^k}{k!} < \frac{\epsilon}{2}$ for all $m > m_\epsilon$ and, consequently, $\sum_{k=m_\epsilon+1}^{\infty} \frac{C^k}{k!} \leq \frac{\epsilon}{2}$. Define

$$p(x) := \sum_{k=0}^{m_\epsilon} \frac{x^k}{k!}$$

for $x \in \mathbb{R}$. Then p is a polynomial—and therefore continuous—with $p(0) = 1$. It follows that there is $n_\epsilon \in \mathbb{N}$ such that $|p(x_n) - 1| < \frac{\epsilon}{2}$ for all $n \geq n_\epsilon$. We therefore obtain for $n \geq n_\epsilon$ that

$$\begin{aligned}|\exp(x_n) - 1| &= \left| \sum_{k=0}^{\infty} \frac{x_n^k}{k!} - 1 \right| \\ &\leq |p(x_n) - 1| + \left| \sum_{k=m_\epsilon+1}^{\infty} \frac{x_n^k}{k!} \right| \\ &< \frac{\epsilon}{2} + \sum_{k=m_\epsilon+1}^{\infty} \frac{C^k}{k!} \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon.\end{aligned}$$

This proves the continuity of exp at 0. Let $x_0 \in \mathbb{R}$ be arbitrary, and let $(x_n)_{n=1}^\infty$ be a sequence in $\mathbb{R}$ such that $x_n \to x_0$. It follows that $\lim_{n\to\infty}(x_n - x_0) = 0$, so that

$$\lim_{n\to\infty} \exp(x_n) = \lim_{n\to\infty} \exp((x_n - x_0) + x_0) = \lim_{n\to\infty} \exp(x_n - x_0) \exp(x_0) = \exp(x_0).$$

Hence, exp is continuous.

Exercises

1. Let $\varnothing \neq D \subset \mathbb{R}$, let $f : D \to \mathbb{R}$, and let $x_0 \in \overline{D}$. Show that the following are equivalent:

   (i) $\lim_{x\to x_0} f(x) = \infty$;

   (ii) for every $R \in \mathbb{R}$, there is $\delta > 0$ such that $f(x) > R$ for all $x \in D$ with $|x - x_0| < \delta$.

2. Let $\varnothing \neq D \subset \mathbb{R}$ be not bounded above, and let $f : D \to \mathbb{R}$. Show that the following are equivalent for $y_0 \in \mathbb{R}$:

   (i) $\lim_{x\to\infty} f(x) = y_0$;

   (ii) for every $\epsilon > 0$, there is $R > 0$ such that $|f(x) - y_0| < \epsilon$ for all $x \in D$ with $x > R$.

3. Define $f : \mathbb{R} \to \mathbb{R}$ by letting

$$f(x) := \begin{cases} 0, & x \notin \mathbb{Q}, \\ \frac{1}{q}, & x = \frac{p}{q} \neq 0 \text{ with } p \in \mathbb{Z} \text{ and } q \in \mathbb{N} \text{ coprime}, \\ 1, & x = 0. \end{cases}$$

   Show that f is discontinuous at every $x \in \mathbb{Q}$, but continuous at every $x \in \mathbb{R} \setminus \mathbb{Q}$.

15 Properties of Continuous Functions and Uniform Continuity

Let $\varnothing \neq S$ be any set, and let $f : S \to \mathbb{R}$ be any function. We call f *bounded* if its range $f(S)$ is bounded.

Theorem 15.1. *Let $\varnothing \neq K \subset \mathbb{R}$ be a compact set, and let $f : K \to \mathbb{R}$ be continuous. Then f is bounded, and there are $x_{\min}, x_{\max} \in K$ such that*

$$f(x_{\min}) = \inf f(K) \qquad \textit{and} \qquad f(x_{\max}) = \sup f(K).$$

Proof. Assume towards a contradiction that f is not bounded, i.e., for each $n \in \mathbb{N}$, there is $x_n \in K$ such that $|f(x_n)| \geq n$. As K is compact, there is a subsequence $(x_{n_k})_{k=1}^{\infty}$ of $(x_n)_{n=1}^{\infty}$ converging to some $x_0 \in K$. As f is continuous, this means that $\lim_{k\to\infty} f(x_{n_k}) = f(x_0)$, which means that $(f(x_{n_k}))_{k=1}^{\infty}$ is bounded. This contradicts the fact that

$$|f(x_{n_k})| \geq n_k \geq k$$

for all $k \in \mathbb{N}$.

For any $n \in \mathbb{N}$, there is $x_n \in K$ such that

$$f(x_n) > \sup f(K) - \frac{1}{n}.$$

The sequence $(x_n)_{n=1}^{\infty}$ has a convergent subsequence $(x_{n_k})_{k=1}^{\infty}$ with $\lim_{k\to\infty} x_{n_k} =: x_{\max} \in K$. The continuity of f yields that

$$f(x_{\max}) = \lim_{k\to\infty} f(x_{n_k}) \geq \lim_{k\to\infty} \left(\sup f(K) - \frac{1}{n_k} \right) = \sup f(K) \geq f(x_{\max}).$$

In a similar vein, the existence of $x_{\min} \in K$ is proven. $\square$

Theorem 15.2 (Intermediate Value Theorem)**.** *Let $a < b$, and let $f : [a,b] \to \mathbb{R}$ be continuous. Then, for any c between $f(a)$ and $f(b)$, there is $x_c \in [a,b]$ such that $f(x_c) = c$.*

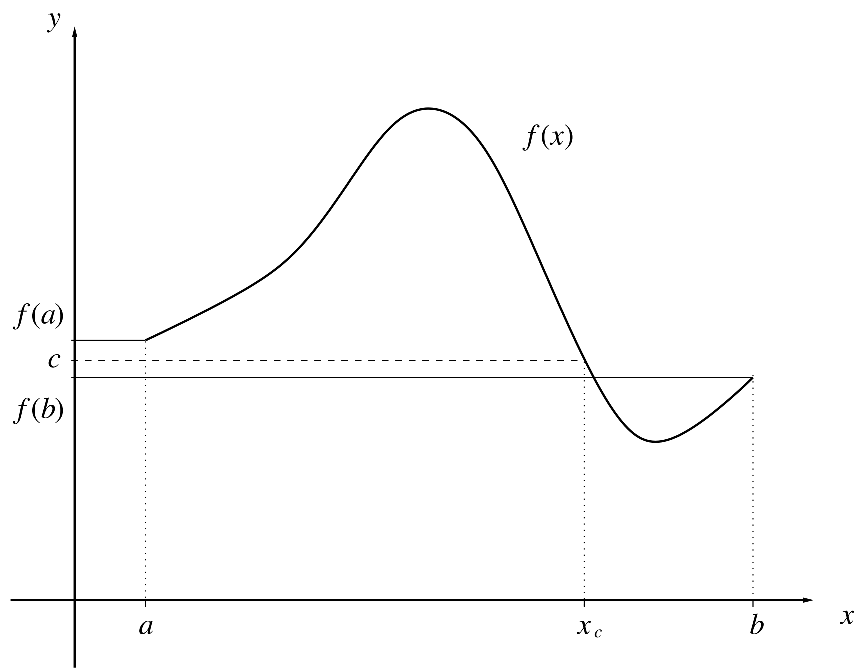

Figure 8: Intermediate Value Theorem

Proof. Without loss of generality, we can focus on the case where $f(a) < c < f(b)$. (If $f(a) = c$ or if $f(b) = c$ the case is clear, and if $f(a) > c > f(b)$, we just replace f by $-f$ and c by $-c$.)

Let

$$S := \{x \in [a,b] : f(x) \leq c\}.$$

Then S is a non-empty, bounded subset of $\mathbb{R}$ and therefore has a supremum x_c. We claim that $f(x_c) = c$. Let $(x_n)_{n=1}^{\infty}$ be a sequence in S such that $\lim_{n\to\infty} x_n = x_c$. The continuity of f yields that $f(x_c) = \lim_{n\to\infty} f(x_n) \leq c$. As $f(b) > c$, it is clear that $x_c < b$. Therefore there is $n_0 \in \mathbb{N}$ such that $x_c + \frac{1}{n} < b$ for $n \geq n_0$. By the definition of x_c, it is clear that

$$f\left(x_c + \frac{1}{n}\right) > c$$

for all $n \geq n_0$. As $x_c + \frac{1}{n} \to x_c$, the continuity of f again yields that

$$f(x_c) = \lim_{n\to\infty} f\left(x_c + \frac{1}{n}\right) \geq c.$$

This completes the proof. $\square$

Example. Let p be a polynomial of odd degree, i.e., there are $c_0, c_1, \ldots, c_\nu$ with $c_\nu \neq 0$ and ν odd such that

$$p(x) = c_\nu x^\nu + \cdots + c_1 x + c_0.$$

If $c_\nu > 0$, then

$$\lim_{x\to\infty} p(x) = \infty \qquad \text{and} \qquad \lim_{x\to-\infty} p(x) = -\infty.$$

Hence, there are $a < 0$ such that $p(a) < 0$ and $b > 0$ such that $p(b) > 0$. By the Intermediate Value Theorem, there is $x \in (a,b)$ such that $p(x) = 0$. If $c_\nu < 0$, replace p by $-p$. In any case, a polynomial of odd degree has a zero in $\mathbb{R}$.

Definition 15.3. Let $\varnothing \neq D \subset \mathbb{R}$. Then $f\colon D \to \mathbb{R}$ is called *uniformly continuous* if, for each $\epsilon > 0$, there is $\delta > 0$ such that $|f(x) - f(y)| < \epsilon$ for all $x, y \in D$ such that $|x - y| < \delta$.

By Corollary 14.6, it is clear that every uniformly continuous function is continuous. But does the converse hold?

Examples. 1. The function

$$f\colon (0,\infty) \to \mathbb{R}, \quad x \mapsto \frac{1}{x}$$

is clearly continuous. Let $\epsilon := 1$, and assume that f is uniformly continuous. Let $\delta > 0$ be as in the definition of uniform continuity. As $\lim_{n\to\infty} \frac{1}{n} = 0$, there is $n_\delta \in \mathbb{N}$ such that $\left|\frac{1}{n+1} - \frac{1}{n}\right| < \delta$ for $n \geq n_\delta$. However, we have

$$\left| f\left(\frac{1}{n+1}\right) - f\left(\frac{1}{n}\right)\right| = |(n+1) - n| = 1 \geq \epsilon$$

for all $n \geq n_\delta$. Therefore, f is not uniformly continuous.

2. The function

$$f\colon [0,\infty) \to \mathbb{R}, \quad x \mapsto x^2$$

is continuous. Assume that f is uniformly continuous, and let $\epsilon := 1$. Then there is $\delta > 0$ such that $|f(x) - f(y)| < 1$ for all $x, y \geq 0$ with $|x - y| < \delta$. Choose, $x := \frac{2}{\delta}$ and $y := \frac{2}{\delta} + \frac{\delta}{2}$. It follows that $|x - y| = \frac{\delta}{2} < \delta$. However, we have

$$\begin{aligned} |f(x) - f(y)| &= |x - y|(x + y) \\ &= \frac{\delta}{2}\left(\frac{2}{\delta} + \frac{2}{\delta} + \frac{\delta}{2}\right) \\ &\geq \frac{\delta}{2}\frac{4}{\delta} \\ &= 2. \end{aligned}$$

Therefore, f is not uniformly continuous.

3. The function

$$f\colon [0,1] \to \mathbb{R}, \quad x \mapsto x^2$$

is uniformly continuous continuous. Let $\epsilon > 0$, and note that

$$|f(x) - f(y)| = |x - y|(x + y) \leq 2|x - y|$$

for $x, y \in [0,1]$. Set $\delta := \frac{\epsilon}{2}$.

That last example is no coincidence:

Theorem 15.4. *Let $\varnothing \neq K \subset \mathbb{R}$ be compact, and let $f\colon K \to \mathbb{R}$ be continuous. Then f is uniformly continuous.*

Proof. Assume that f is *not* uniformly continuous. This means that there is $\epsilon_0 > 0$ such that, for each $\delta > 0$, there are $x_\delta, y_\delta \in K$ with $|x_\delta - y_\delta| < \delta$ such that $|f(x_\delta) - f(y_\delta)| \geq \epsilon_0$. In particular, for each $n \in \mathbb{N}$, there are $x_n, y_n \in K$ such that $|x_n - y_n| < \frac{1}{n}$ and $|f(x_n) - f(y_n)| \geq \epsilon_0$. As K is compact, $(x_n)_{n=1}^\infty$ has a subsequence $(x_{n_k})_{k=1}^\infty$ converging to some $x_0 \in K$. As clearly $\lim_{n\to\infty}(x_n - y_n) = 0$, it follows that $x_0 = \lim_{k\to\infty} y_{n_k}$ as well. The continuity of f yields

$$\lim_{k\to\infty} f(x_{n_k}) = f(x_0) = \lim_{k\to\infty} f(y_{n_k}),$$

which contradicts that $|f(x_n) - f(y_n)| \geq \epsilon_0$ for all $n \in \mathbb{N}$. □

Exercises

1. Let $\varnothing \neq K \subset \mathbb{R}$ be compact, and let $f\colon K \to \mathbb{R}$ be continuous. Show that $f(K)$ is compact.

2. Show that the following are equivalent for a set $\varnothing \neq K \subset \mathbb{R}$:

   (i) K is compact;

   (ii) every continuous function $f\colon K \to \mathbb{R}$ is bounded.

3. Let $f\colon [0,1] \to \mathbb{R}$ be continuous such that $f([0,1]) \subset [0,1]$. Show that f has a fixed point, i.e., there is $x_0 \in [0,1]$ such that $f(x_0) = x_0$. (*Hint*: Apply the Intermediate Value Theorem to the function $[0,1] \ni x \mapsto x - f(x)$.)

4. Let $a < b$, and let $f, g : [a,b] \to \mathbb{R}$ be continuous such that $f(a) \leq g(a)$ and $f(b) \geq g(b)$. Show that there is $x_0 \in [a,b]$ such that $f(x_0) = g(x_0)$.

5. Let $f : [0,2] \to \mathbb{R}$ be continuous such that $f(0) = f(2)$. Show that there are $x, y \in [0,2]$ with $|x - y| = 1$ and $f(x) = f(y)$. (*Hint*: Apply the Intermediate Value Theorem to the auxiliary function $[0,1] \ni x \mapsto f(x+1) - f(x)$.)

6. Show that there is *no* continuous function $f : [0,1] \to \mathbb{R}$ that attains each of its values exactly twice. (*Hint*: Assume that there is such a function, and play around with the Intermediate Value Theorem.)

7. Let $\varnothing \neq D \subset \mathbb{R}$, let $f : D \to \mathbb{R}$ be uniformly continuous, and let $(x_n)_{n=1}^\infty$ be a Cauchy sequence in D. Show that $(f(x_n))_{n=1}^\infty$ is a Cauchy sequence.

16 Continuity of Inverse Functions

Definition 16.1. Let $\varnothing \neq D \subset \mathbb{R}$, and let $f\colon D \to \mathbb{R}$. We call f:

(a) *increasing* if $f(x_1) \leq f(x_2)$ for all $x_1, x_2 \in D$ with $x_1 < x_2$;

(b) *strictly increasing* if $f(x_1) < f(x_2)$ for all $x_1, x_2 \in D$ with $x_1 < x_2$;

(c) *decreasing* if $f(x_1) \geq f(x_2)$ for all $x_1, x_2 \in D$ with $x_1 < x_2$;

(d) *strictly decreasing* if $f(x_1) > f(x_2)$ for all $x_1, x_2 \in D$ with $x_1 < x_2$.

If f is increasing or decreasing, we call it *monotonic*; if it is strictly increasing or strictly decreasing, we call it *strictly monotonic*.

Clearly, every strictly monotonic function is injective.

Proposition 16.2. *Let $a < b$, and let $f\colon [a,b] \to \mathbb{R}$ be continuous and injective. Then f is strictly monotonic.*

Proof. As f is injective, $f(a) \neq f(b)$ must hold, i.e., $f(a) < f(b)$ or $f(a) > f(b)$. We can suppose without loss of generality that $f(a) < f(b)$.

We first claim that

$$f(a) = \min f([a,b]) \qquad \text{and} \qquad f(b) = \max f([a,b]).$$

To see that $f(a) = \min f([a,b])$, assume that there is $x_0 \in (a,b]$ such that $f(x_0) < f(a)$. As $f(x_0) < f(a) < f(b)$, the Intermediate Value Theorem yields $c \in (x_0, b)$ such that $f(c) = f(a)$. This violates the injectivity of f. Similarly, the claim for $f(b)$ is proven.

Assume towards a contradiction that f is *not* increasing, i.e., there are $x_1, x_2 \in [a,b]$ with $x_1 < x_2$ such that $f(x_1) \geq f(x_2)$. By the injectivity of f, the case $f(x_1) = f(x_2)$ cannot occur, so that $f(x_1) > f(x_2)$. It is clear that both $x_1 = a$ and $x_2 = b$ cannot happen. This means that $x_1 \neq a$ or $x_2 \neq b$.

Suppose that $x_1 \neq a$, i.e., $a < x_1$. Choose c such that $f(x_2) < c < f(x_1)$. By the Intermediate Value Theorem, there is $x_3 \in (x_1, x_2)$ such that $f(x_3) = c$. As

$$f(a) < f(x_2) < c < f(x_1),$$

the Intermediate Value Theorem also yields $x_4 \in (a, x_1)$ with $f(x_4) = c$. This contradicts the injectivity of f.

Similarly, we deal with the case where $x_2 \neq b$. $\square$

Theorem 16.3. *Let $a < b$, and let $f : [a,b] \to \mathbb{R}$ be continuous and injective. Then f maps $[a,b]$ onto $[\alpha, \beta]$—with $\alpha = f(a)$ and $\beta = f(b)$ if f is strictly increasing and $\alpha = f(b)$ and $\beta = f(a)$ if f is strictly decreasing—, and the inverse function $f^{-1}\colon [\alpha,\beta] \to [a,b]$ is also continuous.*

Proof. The part about f mapping $[a,b]$ onto $[\alpha,\beta]$ with α and β as stated is clear by Proposition 16.2. What remains to be shown is the continuity of f^{-1}.

Let $y_0 \in [\alpha,\beta]$, and let $(y_n)_{n=1}^{\infty}$ be a sequence in $[\alpha,\beta]$ with $y_0 = \lim_{n\to\infty} y_n$. We need to show that $\lim_{n\to\infty} f^{-1}(y_n) = f^{-1}(y_0)$. We assume that this is not the case. This means that there are $\epsilon_0 > 0$ and $n_1 < n_2 < n_3 < \cdots$ such that $|f^{-1}(y_{n_k}) - f^{-1}(y_0)| \geq \epsilon_0$ for all $k \in \mathbb{N}$. As the sequence $(f^{-1}(y_{n_k}))_{k=1}^{\infty}$ is bounded, it has a convergent subsequence by the Bolzano–Weierstraß Theorem. We may replace $(f^{-1}(y_{n_k}))_{k=1}^{\infty}$ by this subsequence and suppose that $(f^{-1}(y_{n_k}))_{k=1}^{\infty}$ already converges. (This spares us atrocious notation like $y_{n_{k_l}}$.) Set $x_0 := \lim_{k\to\infty} f^{-1}(y_{n_k})$. As f is continuous, we have

$$y_0 = \lim_{k\to\infty} y_{n_k} = \lim_{k\to\infty} f(f^{-1}(y_{n_k})) = f(x_0)$$

and therefore $f^{-1}(y_0) = x_0$, so that $|f^{-1}(y_{n_k}) - x_0| \geq \epsilon_0$ for all $k \in \mathbb{N}$. This is a contradiction. $\square$

Examples. 1. Let $n \in \mathbb{N}$. The function

$$f\colon [0,\infty) \to [0,\infty), \quad x \mapsto x^n$$

is strictly increasing and maps $[0,\infty)$ onto $[0,\infty)$. For any $0 \leq a < b$, the restriction of f to $[a,b]$ has a continuous inverse by Theorem 16.3, namely

$$[a^n, b^n] \to [a,b], \quad x \mapsto \sqrt[n]{x}.$$

This means that the restriction of

$$f^{-1}\colon [0,\infty) \to [0,\infty), \quad x \mapsto \sqrt[n]{x}$$

onto $[\alpha,\beta]$ is continuous for all $0 \leq \alpha < \beta$. As any convergent sequence in $[0,\infty)$ is contained in an interval $[0,R]$ with $R > 0$, this means that f^{-1} is continuous.

2. We claim that the exponential function $\exp\colon \mathbb{R} \to \mathbb{R}$ is strictly increasing. To see this, let $x > 0$, and note that

$$\exp(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + \underbrace{\sum_{k=1}^{\infty} \frac{x^k}{k!}}_{>0} > 1.$$

For $x_1 < x_2$, we therefore obtain

$$\exp(x_2) = \exp(x_1 + x_2 - x_1) = \exp(x_1) \underbrace{\exp(\underbrace{x_2 - x_1}_{>0})}_{>1} > \exp(x_1).$$

Next, we claim that

$$\lim_{x\to\infty}\exp(x)=\infty \qquad \text{and} \qquad \lim_{x\to-\infty}\exp(x)=0.$$

As

$$\exp(-x)=\frac{1}{\exp(x)}$$

for all $x\in\mathbb{R}$, it is enough to prove that $\lim_{x\to\infty}\exp(x)=\infty$. Recall that

$$\exp(n)=e^n$$

for $n\in\mathbb{N}$ where

$$e=\exp(1)=\sum_{k=0}^{\infty}\frac{1}{k!}=2+\sum_{k=2}^{\infty}\frac{1}{k!}>2.$$

Let $(x_n)_{n=1}^{\infty}$ be a sequence in $\mathbb{R}$ with $\lim_{n\to\infty}x_n=\infty$, and note that $\lim_{n\to\infty}\lfloor x_n\rfloor=\infty$ as well. We obtain

$$\exp(x_n)\geq\exp(\lfloor x_n\rfloor)=e^{\lfloor x_n\rfloor}>2^{\lfloor x_n\rfloor}$$

for $n\in\mathbb{N}$, so that $\lim_{n\to\infty}\exp(x_n)=\infty$.

Let $c\in(0,\infty)$. As $\lim_{x\to-\infty}\exp(x)=0$, there is $a<0$ with $\exp(a)<c$, and since $\lim_{x\to\infty}\exp(x)=\infty$, there is $b>0$ with $\exp(b)>c$. The continuity of exp and the Intermediate Value Theorem yield $x_c\in(a,b)$ with $\exp(x_c)=c$. This means that $\exp(\mathbb{R})=(0,\infty)$.

By Theorem 16.3, the restriction of exp to $[a,b]$ has a continuous inverse for every $a<b$, namely

$$[\exp(a),\exp(b)]\to[a,b],\quad x\mapsto\log x.$$

As in the previous example, the continuity of

$$\log\colon(0,\infty)\to\mathbb{R},\quad x\mapsto\log x$$

follows.

Just for the record, we define now a^x for $a>0$ and *any* $x\in\mathbb{R}$.

Definition 16.4. Let $a>0$. Define

$$\exp_a\colon\mathbb{R}\to\mathbb{R},\quad x\mapsto\exp(x\log a).$$

Properties. For any $a>0$:

1. $\exp_a$ is continuous;

2. $\exp_a(x+y)=\exp_a(x)\exp_a(y)$ for all $x,y\in\mathbb{R}$;

3. $\exp_a(q) = a^q$ for all $q \in \mathbb{Q}$.

The third property motivates to define

$$a^x = \exp_a(x)$$

for all $a > 0$ and $x \in \mathbb{R}$.

17 Differentiation

Definition 17.1. Let $\varnothing \neq D \subset \mathbb{R}$, and let $f\colon D \to \mathbb{R}$. Then f is called *differentiable* at $x_0 \in D$ if

$$\lim_{\substack{x \to x_0 \\ x \in D \setminus \{x_0\}}} \frac{f(x) - f(x_0)}{x - x_0}$$

exists. This limit is called the *(first) derivative* of f at x_0. If f is differentiable at each point of D, we call it differentiable on D.

Notation. Symbols for the first derivative of f at x_0 are

$$f'(x_0) \qquad \text{or} \qquad \frac{df}{dx}(x_0) \qquad \text{or} \qquad \left.\frac{df}{dx}\right|_{x=x_0}$$

Remarks. 1. It is tacitly understood throughout that, whenever we speak of differentiability at a point $x_0 \in D$ that $x_0 \in \overline{D \setminus \{x_0\}}$, i.e., there is at least one sequence in $D \setminus \{x_0\}$ converging to x_0.

2. Clearly, f is differentiable at x_0 if and only if

$$\lim_{\substack{h \to 0 \\ h \neq 0 \\ x_0 + h \in D}} \frac{f(x_0 + h) - f(x_0)}{h}$$

exists.

Examples. 1. Let $n \in \mathbb{N}$, and let

$$f\colon \mathbb{R} \to \mathbb{R}, \quad x \mapsto x^n.$$

Let $x_0, h \in \mathbb{R}$ be such that $h \neq 0$. The Binomial Theorem yields

$$f(x_0 + h) = (x_0 + h)^n = \sum_{k=0}^{n} \binom{n}{k} x_0^k h^{n-k},$$

so that

$$f(x_0 + h) - f(x_0) = \sum_{k=0}^{n-1} \binom{n}{k} x_0^k h^{n-k}$$

and therefore

$$\frac{f(x_0 + h) - f(x_0)}{h} = \sum_{k=0}^{n-1} \binom{n}{k} x_0^k h^{n-1-k} = \sum_{k=0}^{n-2} \binom{n}{k} x_0^k h^{n-1-k} + n x_0^{n-1}.$$

As $\lim_{h \to 0} \sum_{k=0}^{n-2} \binom{n}{k} x_0^k h^{n-1-k} = 0$, this means that f is differentiable at x_0 with $f'(x_0) = n x_0^{n-1}$.

2. We claim that the exponential function is differentiable at 0 and that

$$\left.\frac{d\exp}{dx}\right|_{x=0} = 1.$$

For $h \neq 0$, note that

$$\frac{\exp(h)-1}{h} = \frac{1}{h}\left(\sum_{k=0}^{\infty}\frac{h^k}{k!} - 1\right) = \sum_{k=1}^{\infty}\frac{h^{k-1}}{k!}.$$

Let $(h_n)_{n=1}^{\infty}$ be a sequence in $\mathbb{R}\setminus\{0\}$ converging to zero. Let $C \geq 0$ be such that $|h_n| \leq C$ for all $n \in \mathbb{N}$. Let $\epsilon > 0$. It follows from the Limit Ratio Test that $\sum_{k=1}^{\infty}\frac{C^{k-1}}{k!} < \infty$; by the Cauchy Criterion, there is therefore $m_\epsilon \in \mathbb{N}$ such that $\sum_{k=m_\epsilon+1}^{\infty}\frac{C^{k-1}}{k!} \leq \frac{\epsilon}{2}$. Set

$$p(x) := \sum_{k=1}^{m_\epsilon}\frac{x^{k-1}}{k!}.$$

Then p is a polynomial—and thus continuous—and $p(0) = 1$. Choose $n_\epsilon \in \mathbb{N}$ such that $|p(h_n) - 1| < \frac{\epsilon}{2}$ for all $n \geq n_\epsilon$. For those n, we have

$$\begin{aligned}
\left|\frac{\exp(h_n)-1}{h_n} - 1\right| &= \left|\sum_{k=1}^{\infty}\frac{h_n^{k-1}}{k!} - 1\right| \\
&\leq \left|\sum_{k=1}^{m_\epsilon}\frac{h_n^{k-1}}{k!} - 1\right| + \sum_{k=m_\epsilon+1}^{\infty}\frac{|h_n|^{k-1}}{k!} \\
&\leq |p(h_n) - 1| + \sum_{k=m_\epsilon+1}^{\infty}\frac{C^{k-1}}{k!} \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon.
\end{aligned}$$

This proves that $\lim_{\substack{h\to 0\\ h\neq 0}}\frac{\exp(h)-1}{h} = 1$, i.e., $\left.\frac{d\exp}{dx}\right|_{x=0} = 1$ as claimed. Let $x_0, h \in \mathbb{R}$ be such that $h \neq 0$. As

$$\frac{\exp(x_0+h)-\exp(x_0)}{h} = \frac{\exp(x_0)\exp(h)-\exp(x_0)}{h} = \exp(x_0)\frac{\exp(h)-1}{h},$$

it follows that exp is differentiable at x_0 such that

$$\left.\frac{d\exp}{dx}\right|_{x=x_0} = \exp(x_0).$$

Proposition 17.2. *Let $\varnothing \neq D \subset \mathbb{R}$, let $x_0 \in D$, and let $f\colon D \to \mathbb{R}$ be differentiable at x_0. Then f is continuous at x_0.*

Proof. Let $(x_n)_{n=1}^\infty$ be a sequence in D with $x_0 = \lim_{n\to\infty} x_n$. We distinguish two cases.

Case 1: There is $n_0 \in \mathbb{N}$ such that $x_n = x_0$ for all $n \geq n_0$. Then it is obvious that $\lim_{n\to\infty} f(x_n) = f(x_0)$.

Case 2: There are $n_1 < n_2 < n_3 < \cdots$ with $x_{n_k} \neq x_0$ for all $k \in \mathbb{N}$. We can suppose that

$$\{n_1, n_2, n_3, \ldots\} = \{n \in \mathbb{N} : x_n \neq x_0\}.$$

We have

$$\begin{aligned}\lim_{k\to\infty}(f(x_{n_k}) - f(x_0)) &= \lim_{k\to\infty} \frac{f(x_{n_k}) - f(x_0)}{x_{n_k} - x_0}(x_{n_k} - x_0)\\ &= \lim_{k\to\infty} \frac{f(x_{n_k}) - f(x_0)}{x_{n_k} - x_0} \lim_{k\to\infty}(x_{n_k} - x_0) = f'(x_0)0 = 0.\end{aligned}$$

Let $\epsilon > 0$, and choose $k_\epsilon \in \mathbb{N}$ such that $|f(x_{n_k}) - f(x_0)| < \epsilon$ for all $k \geq k_\epsilon$. Set $n_\epsilon := n_{k_\epsilon}$, and let $n \geq n_\epsilon$. Then there are two possibilities: either $x_n = x_0$ or $x_n \neq x_0$, i.e., $n \in \{n_1, n_2, n_3, \ldots\}$, so that n is of the form n_k, necessarily with $k \geq k_\epsilon$. In either case, $|f(x_n) - f(x_0) < \epsilon$ holds. All in all, we have $\lim_{x\to x_0} f(x) = f(x_0)$. □

Proposition 17.3 (Differentiation Laws)**.** *Let $\varnothing \neq D \subset \mathbb{R}$, let $x_0 \in D$, and let $f, g\colon D \to \mathbb{R}$ be differentiable at x_0. Then:*

(i) *$f + g$ is differentiable at x_0 with*

$$(f+g)'(x_0) = f'(x_0) + g'(x_0);$$

(ii) *fg is differentiable at x_0 with*

$$(fg)'(x_0) = f(x_0)g'(x_0) + f'(x_0)g(x_0);$$

(iii) *if $g(x_0) \neq 0$ there is $\delta > 0$ such that $g(x) \neq 0$ for all $x \in D$ with $|x - x_0| < \delta$ and $\frac{f}{g}$ is differentiable at x_0 with*

$$\left(\frac{f}{g}\right)'(x_0) = \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{g(x_0)^2}.$$

Proof. (i) is immediate from the Limit Laws.

(ii): Let $h \neq 0$ be such that $x_0 + h \in D$, and note that

$$\begin{aligned}&\frac{f(x_0+h)g(x_0+h) - f(x_0)g(x_0)}{h}\\ &\quad= \frac{f(x_0+h)g(x_0+h) - f(x_0+h)g(x_0) + f(x_0+h)g(x_0) - f(x_0)g(x_0)}{h}\\ &\quad= \underbrace{f(x_0+h)}_{\overset{h\to 0}{\to} f(x_0)} \underbrace{\frac{g(x_0+h) - g(x_0)}{h}}_{\overset{h\to 0}{\to} g'(x_0)} + \underbrace{\frac{f(x_0+h) - f(x_0)}{h}}_{\overset{h\to 0}{\to} f'(x_0)} g(x_0)\end{aligned}$$

(iii): The existence of δ follows from the continuity of g at x_0. We first treat the case where $f \equiv 1$. Let $h \neq 0$ be such that $|h| < \delta$ and $x_0 + h \in D$. We then have

$$\begin{aligned}
\frac{\frac{1}{g(x_0+h)} - \frac{1}{g(x_0)}}{h} &= \frac{1}{h}\left(\frac{1}{g(x_0+h)} - \frac{1}{g(x_0)}\right) \\
&= \frac{1}{h}\left(\frac{g(x_0) - g(x_0+h)}{g(x_0+h)g(x_0)}\right) \\
&= -\underbrace{\frac{g(x_0+h) - g(x_0)}{h}}_{\stackrel{h\to 0}{\to} g'(x_0)} \underbrace{\frac{1}{g(x_0+h)g(x_0)}}_{\stackrel{h\to 0}{\to} g(x_0)^2} \\
&\stackrel{h\to 0}{\to} -\frac{g'(x_0)}{g(x_0)^2},
\end{aligned}$$

which proves the claim in this particular case. For general f, apply (ii) and the special case:

$$\begin{aligned}
\left(\frac{f}{g}\right)'(x_0) &= f'(x_0)\left(\frac{1}{g}\right)(x_0) + f(x_0)\left(\frac{1}{g}\right)'(x_0) \\
&= \frac{f'(x_0)}{g(x_0)} - \frac{f(x_0)g'(x_0)}{g(x_0)^2} \\
&= \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{g(x_0)^2}.
\end{aligned}$$

This completes the proof. □

Theorem 17.4 (Chain Rule). *Let $\varnothing \neq D_g \subset \mathbb{R}$ and $\varnothing \neq D_f \subset \mathbb{R}$, let $g\colon D_g \to \mathbb{R}$ and $f\colon D_f \to \mathbb{R}$ be such that $g(D_g) \subset D_f$, and let $x_0 \in D_g$ be such that g is differentiable at x_0 and f is differentiable at $g(x_0)$. Then $f \circ g$ is differentiable at x_0 with*

$$(f \circ g)'(x_0) = f'(g(x_0))g'(x_0).$$

Proof. Define $\tilde{f}\colon D_f \to \mathbb{R}$ by letting

$$\tilde{f}(y) := \begin{cases} \frac{f(y) - f(g(x_0))}{y - g(x_0)}, & \text{if } y \neq g(x_0), \\ f'(g(x_0)), & \text{otherwise.} \end{cases}$$

As f is differentiable in $g(x_0)$, we have

$$\lim_{y \to g(x_0)} \tilde{f}(y) = f'(g(x_0)) = \tilde{f}(g(x_0));$$

moreover,

$$f(y) - f(g(x_0)) = \tilde{f}(y)(y - g(x_0))$$

holds for all $y \in D_f$. It follows that

$$\begin{aligned}
\lim_{\substack{x \to x_0 \\ x \in D_g \setminus \{x_0\}}} \frac{f(g(x)) - f(g(x_0))}{x - x_0} &= \lim_{\substack{x \to x_0 \\ x \in D_g \setminus \{x_0\}}} \frac{\tilde{f}(g(x))(g(x) - g(x_0))}{x - x_0} \\
&= \lim_{\substack{x \to x_0 \\ x \in D_g \setminus \{x_0\}}} \tilde{f}(g(x)) \lim_{\substack{x \to x_0 \\ x \in D_g \setminus \{x_0\}}} \frac{g(x) - g(x_0)}{x - x_0} \\
&= f'(g(x_0))g'(x_0),
\end{aligned}$$

which proves the claim. □

Theorem 17.5 (Differentiability of the Inverse Function)**.** *Let $a < b$, and let $f \colon [a, b] \to \mathbb{R}$ be continuous and strictly monotonic, and let $[\alpha, \beta] := f([a, b])$. Then, if f is differentiable at $x_0 \in [a, b]$ with $f'(x_0) \neq 0$, the inverse function $f^{-1} \colon [\alpha, \beta] \to [a, b]$ is differentiable at $f(x_0)$ with*

$$\left. \frac{df^{-1}}{dy} \right|_{y = f(x_0)} = \frac{1}{f'(x_0)}$$

Proof. Set $y_0 := f(x_0)$, and let $y \in [\alpha, \beta] \setminus \{y_0\}$; set $x := f^{-1}(y)$. It follows that

$$\frac{f^{-1}(y) - f^{-1}(y_0)}{y - y_0} = \frac{x - x_0}{f(x) - f(x_0)}.$$

As f^{-1} is continuous by Theorem 16.3, x tends to x_0 as y tends to y_0. This means that

$$\lim_{\substack{y \to y_0 \\ y \neq y_0}} \frac{f^{-1}(y) - f^{-1}(y_0)}{y - y_0} = \lim_{\substack{x \to x_0 \\ x \neq x_0}} \frac{x - x_0}{f(x) - f(x_0)} = \lim_{\substack{x \to x_0 \\ x \neq x_0}} \frac{1}{\frac{f(x) - f(x_0)}{x - x_0}} = \frac{1}{f'(x_0)},$$

which proves the theorem. □

Remark. With $y = f(x)$, so that $x = f^{-1}(y)$, Theorem 17.5 takes on the catchy form

$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}},$$

i.e., the derivative of the inverse is the inverse of the derivative.

Examples. 1. As $\exp : \mathbb{R} \to (0, \infty)$ is differentiable $\exp^{-1} = \log : (0, \infty) \to \mathbb{R}$ is differentiable at every $y_0 \in (0, \infty)$ and

$$\left. \frac{d \log}{dy} \right|_{y = y_0} = \frac{1}{\left. \frac{d \exp}{dx} \right|_{x = \log y_0}} = \frac{1}{\exp(\log y_0)} = \frac{1}{y_0}.$$

2. For $r \in \mathbb{R}$, define

$$f_r \colon (0, \infty) \to \mathbb{R}, \quad x \mapsto x^r := \exp_x(r) := \exp(r \log x).$$

Then f_r is differentiable, and the rules of differentiation and the previous example yield

$$\begin{aligned} f_r'(x) &= \frac{r}{x}\exp(r\log x) = \frac{r}{\exp(\log x)}\exp(r\log x) \\ &= r\exp(-\log x)\exp(r\log x) = r\exp((r-1)\log x) = rf_{r-1}(x) = rx^{r-1} \end{aligned}$$

for $x > 0$.

Higher Derivatives

If $\varnothing \neq D \subset \mathbb{R}$, and let $f : D \to \mathbb{R}$ be differentiable on D. Suppose that the derivative $f' \colon D \to \mathbb{R}$ is again differentiable. Then the derivative of f' is called the *second derivative* of f and denoted by f'' or $\frac{d^2 f}{dx^2}$. Inductively, one can go on an define, for each n, the n-th derivative $f^{(n)}$ or $\frac{d^n f}{dx^n}$. It is customary, to identify the zeroth derivative $f^{(0)}$ with f. If f', f'', ..., and $f^{(n)}$ exist, we call f *n-times differentiable*; if, furthermore, $f^{(n)}$ is continuous, we call f *n-times continuously differentiable* (in the case where $n = 1$, we simply call f continuously differentiable).

Exercises

1. Let $\varnothing \neq D \subset \mathbb{R}$, let $n \in \mathbb{N}_0$, and let $f, g \colon D \to \mathbb{R}$ be n-times differentiable. Show that fg is n-times differentiable with

$$(fg)^{(n)}(x) = \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(x) g^{(n-k)}(x).$$

for all $x \in D$. (*Hint*: Review the proof of the Binomial Theorem.)

2. Show that the function

$$f \colon [0,1] \to \mathbb{R}, \quad x \mapsto \begin{cases} x\sin\left(\frac{1}{x}\right), & x \neq 0, \\ 0, & x = 0, \end{cases}$$

is continuous on $[0, 1]$, differentiable on $(0, 1]$, but not differentiable at 0.

3. Show that the function

$$f \colon [0,1] \to \mathbb{R}, \quad x \mapsto \begin{cases} x^2\sin\left(\frac{1}{x}\right), & x \neq 0, \\ 0, & x = 0, \end{cases}$$

is differentiable on $[0, 1]$ whereas $f' \colon [0, 1] \to \mathbb{R}$ is not continuous at 0.

4. Let $\varnothing \neq D \subset \mathbb{R}$. Then $f : D \to \mathbb{R}$ is called *Lipschitz continuous* if there is $C \geq 0$ such that

$$|f(x) - f(y)| \leq C|x - y|$$

for all $x, y \in D$.

(a) Show that every Lipschitz continuous function is uniformly continuous.

(b) Suppose that $D = [a, b]$ with $a < b$ and that f is continuously differentiable on $[a, b]$. Show that f is Lipschitz continuous. (*Hint*: Mean Value Theorem)

(c) Show that

$$f\colon [0,1] \to \mathbb{R}, \quad x \mapsto \sqrt{x}$$

is uniformly continuous, but not Lipschitz continuous. (*Hint*: What do you observe if you assume that f is Lipschitz continuous and choose $x = \frac{1}{(C+1)^2}$ and $y = 0$?)

18 Local Extrema, the Mean Value Theorem, and Taylor's Theorem

Definition 18.1. Let $\varnothing \neq D \subset \mathbb{R}$, let $f\colon D \to \mathbb{R}$, and let $x_0 \in D$. Then we say that f has:

(a) a *local maximum* at x_0 if there is $\epsilon > 0$ with $(x_0 - \epsilon, x_0 + \epsilon) \subset D$ and $f(x) \leq f(x_0)$ for all $x \in (x_0 - \epsilon, x_0 + \epsilon)$;

(b) a *local minimum* at x_0 if there is $\epsilon > 0$ with $(x_0 - \epsilon, x_0 + \epsilon) \subset D$ and $f(x) \geq f(x_0)$ for all $x \in (x_0 - \epsilon, x_0 + \epsilon)$.

If f has a local maximum or a local minimum at x_0, we say that it has a *local extremum* at x_0.

Theorem 18.2 (First Derivative Test)**.** *Let $\varnothing \neq D \subset \mathbb{R}$, let $f\colon D \to \mathbb{R}$, and let $x_0 \in D$ be such that f has a local extremum at x_0 and is differentiable at x_0. Then $f'(x_0) = 0$.*

Proof. We can suppose without loss of generality that f has a local maximum at x_0.

Let $\epsilon > 0$ be as in the definition of a local maximum. Let $h \in (-\epsilon, 0)$. Then it follows that $f(x_0 + h) \leq f(x_0)$ and $h < 0$ and therefore

$$\frac{f(x_0+h) - f(x_0)}{h} \geq 0,$$

so that

$$f'(x_0) = \lim_{\substack{h \to 0 \\ h \neq 0 \\ x_0 + h \in D}} \frac{f(x_0+h) - f(x_0)}{h} = \lim_{\substack{h \to 0 \\ h \in (-\epsilon, 0)}} \frac{f(x_0+h) - f(x_0)}{h} \geq 0.$$

On the other hand, we have

$$\frac{f(x_0+h) - f(x_0)}{h} \leq 0$$

for $h \in (0, \epsilon)$ and therefore

$$f'(x_0) = \lim_{\substack{h \to 0 \\ h \neq 0 \\ x_0 + h \in D}} \frac{f(x_0+h) - f(x_0)}{h} = \lim_{\substack{h \to 0 \\ h \in (0, \epsilon)}} \frac{f(x_0+h) - f(x_0)}{h} \leq 0.$$

This means that $f'(x_0) = 0$ as claimed. $\square$

Remarks. 1. The existence of $\epsilon > 0$ with $(x_0 - \epsilon, x_0 + \epsilon) \subset D$ in Definition 18.1 is crucial for Theorem 18.2 to hold: the function

$$f\colon [0, 1] \to \mathbb{R}, \quad x \mapsto x$$

attains its minimum at 0 and its maximum at 1, but $f'(0) = f'(1) = 1$.

2. If $f : D \to \mathbb{R}$ is differentiable at x_0 such that $f'(x_0) = 0$, then f need not have a local extremum there: consider

$$f : \mathbb{R} \to \mathbb{R}, \quad x \mapsto x^3$$

at $x_0 = 0$.

Lemma 18.3 (Rolle's Theorem)**.** *Let $a < b$, let $f : [a, b] \to \mathbb{R}$ be continuous and differentiable on (a, b), and suppose that $f(a) = f(b)$. Then there is $\xi \in (a, b)$ such that $f'(\xi) = 0$.*

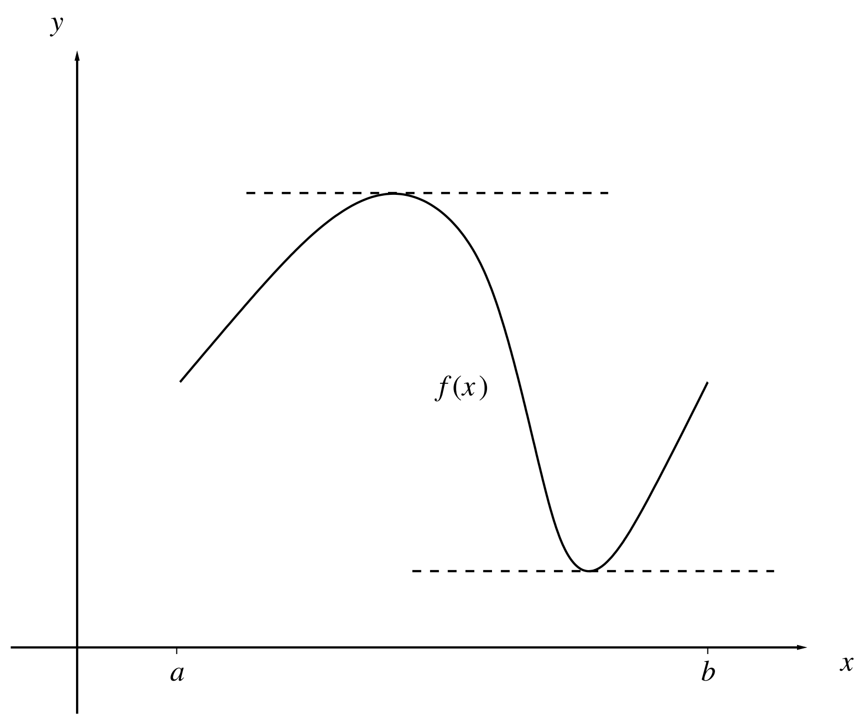

Figure 9: Rolle's Theorem

Proof. If f is constant, then the claim is trivial.

We can therefore suppose that f is *not* constant, i.e., there is $x_0 \in (a, b)$ with $f(x_0) > f(a) = f(b)$ or $f(x_0) < f(a) = f(b)$. In any case, f attains its maximum or minimum on $[a, b]$—which it must attain because f is continuous and $[a, b]$ is compact—at point $\xi \in (a, b)$. Then f has a local extremum at ξ, so that $f'(\xi) = 0$ by Theorem 18.2. $\square$

Theorem 18.4 (Mean Value Theorem)**.** *Let $a < b$, and let $f : [a, b] \to \mathbb{R}$ be continuous and differentiable on (a, b). Then there is $\xi \in (a, b)$ such that*

$$f'(\xi) = \frac{f(b) - f(a)}{b - a}.$$

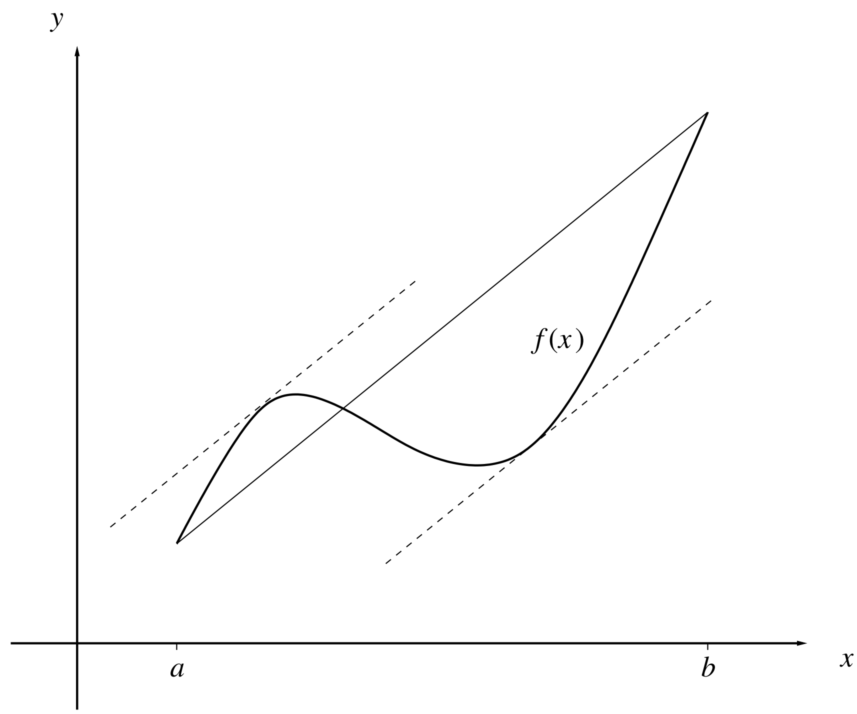

Figure 10: Mean Value Theorem

Proof. Define

$$g\colon [a,b] \to \mathbb{R}, \quad x \mapsto f(x) - \frac{f(b)-f(a)}{b-a}(x-a).$$

Then g is continuous, differentiable on (a,b), and satisfies $g(a) = f(a) = g(b)$. By Rolle's Theorem, there is $\xi \in (a,b)$ with

$$0 = g'(\xi) = f'(\xi) - \frac{f(b)-f(a)}{b-a},$$

which proves the claim. □

Corollary 18.5. *Let $a < b$, and let $f : [a,b] \to \mathbb{R}$ be continuous and differentiable on (a,b) such that $f'(x) = 0$ for all $x \in (a,b)$. Then f is constant.*

Proof. Assume otherwise, i.e., there are $x, y \in [a,b]$ with $f(x) \neq f(y)$, i.e., $\frac{f(x)-f(y)}{x-y} \neq 0$. By the Mean Value Theorem, however, there is $\xi \in (\min\{x,y\}, \max\{x,y\})$ with

$$0 = f'(\xi) = \frac{f(x)-f(y)}{x-y}.$$

This is a contradiction. □

Theorem 18.6 (Taylor's Theorem)**.** *Let $a < b$, let $n \in \mathbb{N}_0$, and let $f : [a,b] \to \mathbb{R}$ be $(n+1)$-times differentiable on $[a,b]$. Then, for any $x, x_0 \in [a,b]$ with $x \neq x_0$, there is*

$\xi \in (\min\{x, x_0\}, \max\{x, x_0\})$ *such that*

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_0)^{n+1}.$$

Proof. Let $x, x_0 \in [a, b]$ be such that $x \neq x_0$; without loss of generality, suppose that $x < x_0$. Set

$$y := \frac{(n+1)!}{(x - x_0)^{n+1}} \left(f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k \right),$$

so that

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k + \frac{y}{(n+1)!}(x - x_0)^{n+1}.$$

We will show that there is $\xi \in (x, x_0)$ such that $f^{(n+1)}(\xi) = y$.

Define

$$g\colon [a, b] \to \mathbb{R}, \qquad t \mapsto f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(t)}{k!}(x - t)^k - \frac{y}{(n+1)!}(x - t)^{n+1},$$

so that $g(x_0) = g(x) = 0$. By Rolle's Theorem, there is $\xi \in (x, x_0)$ such that $g'(\xi) = 0$. Note that

$$\begin{aligned} g'(t) = -f'(t) - \sum_{k=1}^{n} \left(\frac{f^{(k+1)}(t)}{k!}(x - t)^k - \frac{f^{(k)}(t)}{(k-1)!}(x - t)^{k-1} \right) + \frac{y}{n!}(x - t)^n \\ = -\frac{f^{(n+1)}(t)}{n!}(x - t)^n + \frac{y}{n!}(x - t)^n, \end{aligned}$$

so that

$$0 = -\frac{f^{(n+1)}(\xi)}{n!}(x - \xi)^n + \frac{y}{n!}(x - \xi)^n$$

and thus $y = f^{(n+1)}(\xi)$. □

Remark. For $n = 0$, Taylor's Theorem is just the Mean Value Theorem (up to a slightly stronger differentiability hypothesis).

Taylor's Theorem can be used to derive the Second Derivative Test for local extrema:

Corollary 18.7 (Second Derivative Test)**.** *Let $a < b$, let $f\colon (a, b) \to \mathbb{R}$ be twice continuously differentiable, and let $x_0 \in (a, b)$ be such that $f'(x_0) = 0$. Then:*

(i) *if $f''(x_0) < 0$, then f has a local maximum at x_0;*

(ii) *if $f''(x_0) > 0$, then f has a local minimum at x_0.*

Proof. We only deal with the case where $f''(x_0) < 0$.

Since f'' is continuous and (a,b) is an open interval, there is $\epsilon > 0$ with $(x_0-\epsilon, x_0+\epsilon) \subset (a,b)$ such that $f''(x) < 0$ for all $x \in (x_0-\epsilon, x_0+\epsilon)$. Fix $x \in (x_0-\epsilon, x_0+\epsilon) \setminus \{x_0\}$. By Taylor's Theorem, there is ξ between x and x_0 such that

$$f(x) = f(x_0) + \underbrace{f'(x_0)(x-x_0)}_{=0} + \underbrace{\frac{\overbrace{f''(\xi)}^{<0}\,\overbrace{(x-x_0)^2}^{\geq 0}}{2!}}_{\leq 0} \leq f(x_0),$$

which proves the claim. $\square$

Exercises

1. Let $f\colon \mathbb{R} \to \mathbb{R}$ be a differentiable function such that $f' = f$ and $f(0) = 1$. Show that $f = \exp$. (*Hint*: Differentiate $\frac{f}{\exp}$.)

2. Let $r \in \mathbb{R}$, and let $f\colon (0,\infty) \to \mathbb{R}$ be a differentiable function such that $f(1) = 1$ and
$$x\,f'(x) = r\,f(x)$$
all $x > 0$. Show that
$$f(x) = x^r$$
for $x > 0$. (*Hint*: Consider the function $(0,\infty) \ni x \mapsto \frac{f(x)}{x^r}$ and differentiate it.)

3. Let $f, g\colon \mathbb{R} \to \mathbb{R}$ be differentiable functions such that
$$f(0) = 0, \qquad g(0) = 1, \qquad f' = g, \qquad \text{and} \qquad g' = -f.$$
Show that
$$f(x) = \sin x \qquad \text{and} \qquad g(x) = \cos x$$
for $x \in \mathbb{R}$. (*Hint*: Consider the function
$$\mathbb{R} \to \mathbb{R}, \quad x \mapsto (f(x) - \sin x)^2 + (g(x) - \cos x)^2,$$
and differentiate it.)

4. Let $a < b$, and let $f, g\colon [a,b] \to \mathbb{R}$ be continuous and differentiable on (a,b). Show that there is $\xi \in (a,b)$ such that
$$f'(\xi)(g(b) - g(a)) = g'(\xi)(f(b) - f(a)).$$
(*Hint*: Apply Rolle's Theorem to a suitable auxiliary function.)

5. Let $f : \mathbb{R} \to \mathbb{R}$ be such that there is $C \geq 0$ with

$$|f(x) - f(y)| \leq C(x-y)^2$$

for all $x, y \in \mathbb{R}$. Show that f is constant.

6. Let $a < b$, and let $f : [a,b] \to \mathbb{R}$ be continuous and differentiable on (a,b). Show that:

   (a) if $f'(x) \geq 0$ for all $x \in (a,b)$, then f is increasing, and if $f'(x) > 0$ for all $x \in (a,b)$, then f is strictly increasing;

   (b) if f is increasing, then $f'(x) \geq 0$ for all $x \in (a,b)$.

   Give an example of a strictly increasing function f that is differentiable on (a,b) such that there is $\xi \in (a,b)$ with $f'(\xi) = 0$. (*Hint for* (a): Mean Value Theorem.)

7. Let $n \in \mathbb{N}$, and let $f : \mathbb{R} \to \mathbb{R}$ be n-times differentiable such that $f^{(n)} \equiv 0$. Show that f is a polynomial of degree at most $n-1$. (*Hint*: Taylor's Theorem.)

8. Let $f : [a,b] \to \mathbb{R}$ be differentiable, and let $c \in \mathbb{R}$ be such that $f'(a) \leq c \leq f'(b)$. Show that there is $\xi \in [a,b]$ such that $f'(\xi) = c$.

   Hint: First, consider the case where $c = 0$.

   Warning: You must not suppose that f' is continuous!

19 The Riemann Integral

Intuitively, the Riemann integral of a (non-negative) function over an interval can be thought of as the area between the graph of the function and the x-axis. We want to make this notion precise.

For constant functions, it is straightforward how one would define a Riemann integral:

Definition 19.1. Let $a < b$, and let $f : [a, b] \to \mathbb{R}$ be constant, i.e., $f(x) = c$ for all $x \in [a, b]$ and fixed $c \in \mathbb{R}$. Then the *Riemann Integral* of f from a to b is defined as

$$\int_a^b f(x)\,dx := c(b-a).$$

Of course, we want to integrate more functions.

Given $a < b$, we call a finite number of points with

$$a = x_0 < x_1 < \cdots < x_n = b$$

a *partition* of $[a, b]$, and we call a function $f\colon [a, b] \to \mathbb{R}$ a *step function* if there is a partition $a = x_0 < x_1 < \cdots < x_n = b$ such that f is constant on (x_{j-1}, x_j) for $j = 1, \ldots, n$.

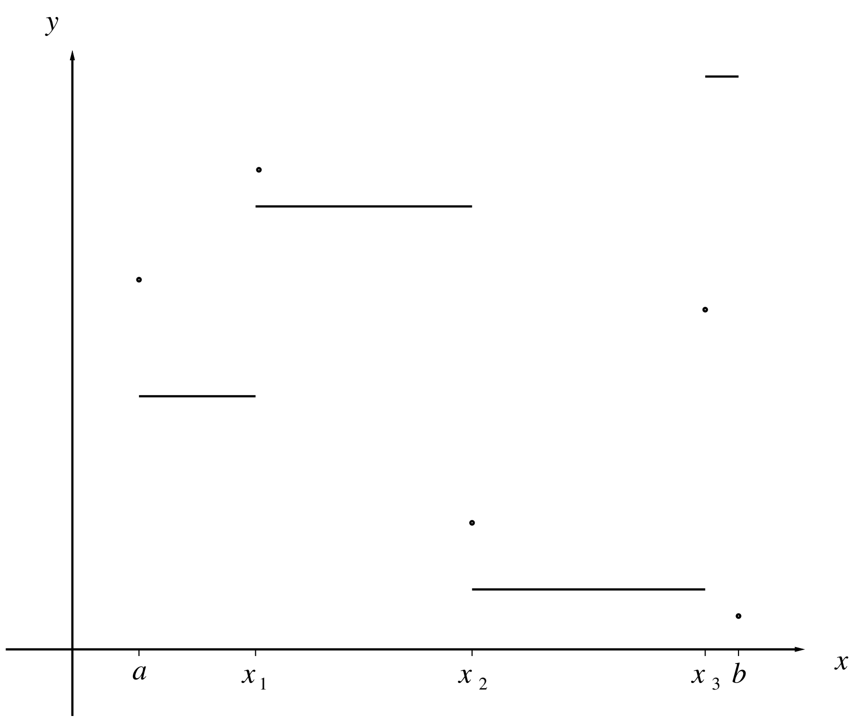

Figure 11: Graph of a step function

Definition 19.2 (Riemann Integral of Step Functions)**.** Let $a < b$, and let $f : [a, b] \to \mathbb{R}$ be a step function defined with respect to the partition $a = x_0 < x_1 < \cdots < x_n = b$. Then the *Riemann Integral* of f from a to b is defined as

$$\int_a^b f(x)\, dx := \sum_{j=1}^n f(\xi_j)(x_j - x_{j-1})$$

with $\xi_j \in (x_{j-1}, x_j)$ for $j = 1, \ldots, n$.

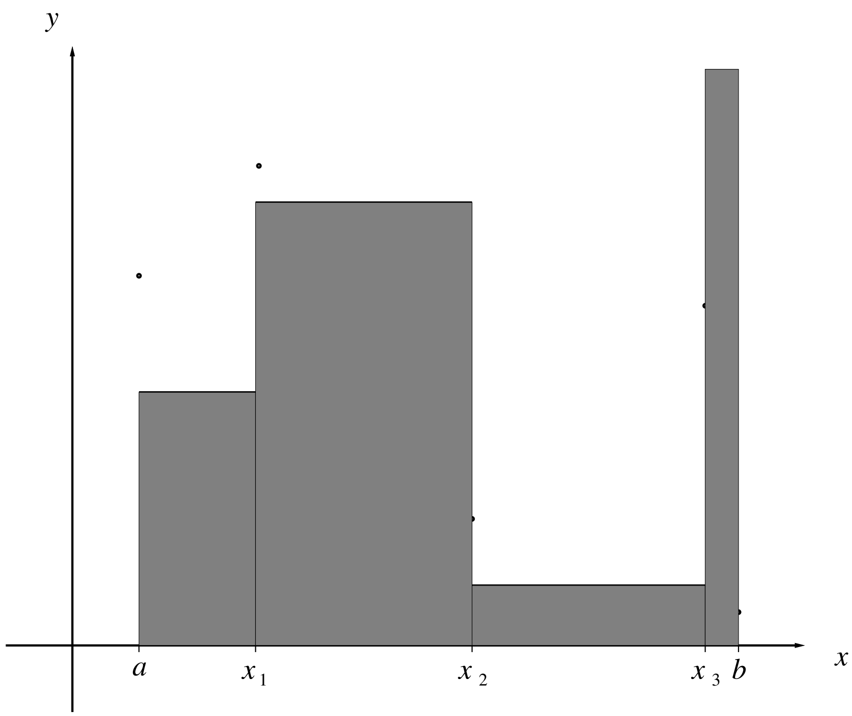

Figure 12: Integral of a step function

Question. Let $a = y_0 < y_1 < \cdots < y_m = b$ be another partition such that f is constant on (y_{j-1}, y_j) for $j = 1, \ldots, m$. This would then yield

$$\int_a^b f(x)\, dx = \sum_{j=1}^m f(\eta_j)(y_j - y_{j-1})$$

with $\eta_j \in (y_{j-1}, y_j)$ for $j = 1, \ldots, n$. Does this give the same value?

The answer is "yes" (fortunately...): Let $a = z_0 < z_1 < \cdots < z_p = b$ be a *common refinement* of $a = x_0 < x_1 < \cdots < x_n = b$ and $a = y_0 < y_1 < \cdots < y_m = b$, i.e.,

$$\{z_0, z_1, \ldots, z_p\} \supset \{x_0, x_1, \cdots, x_n\} \cup \{y_0, y_1, \ldots, y_m\}.$$

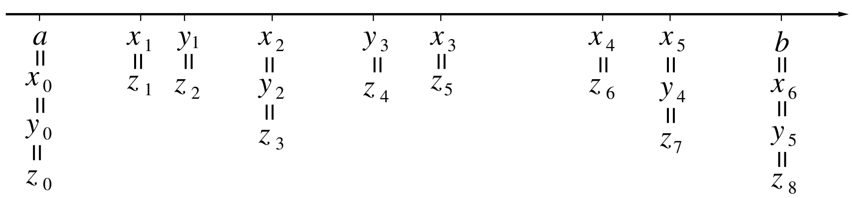

Figure 13: Common refinement $\{z_0, \ldots, z_8\}$ of $\{x_0, \ldots, x_6\}$ and $\{y_0, \ldots, y_5\}$

It follows that f is constant on (z_{j-1}, z_j) for $j = 1, \ldots, p$. For each $j = 1, \ldots, p$, choose $\zeta_j \in (z_{j-1}, z_j)$. We then obtain

$$\begin{aligned}
\sum_{j=1}^{n} f(\xi_j)(x_j - x_{j-1}) &= \sum_{j=1}^{n} f(\xi_j) \sum_{(z_{\nu-1}, z_\nu) \subset (x_{j-1}, x_j)} (z_\nu - z_{\nu-1}) \\
&= \sum_{j=1}^{n} \sum_{(z_{\nu-1}, z_\nu) \subset (x_{j-1}, x_j)} f(\zeta_\nu)(z_\nu - z_{\nu-1}) \\
&= \sum_{\nu=1}^{p} f(\zeta_\nu)(z_\nu - z_{\nu-1}) \\
&= \sum_{j=1}^{m} \sum_{(z_{\nu-1}, z_\nu) \subset (y_{j-1}, y_j)} f(\zeta_\nu)(z_\nu - z_{\nu-1}) \\
&= \sum_{j=1}^{m} f(\eta_j) \sum_{(z_{\nu-1}, z_\nu) \subset (y_{j-1}, y_j)} (z_\nu - z_{\nu-1}) \\
&= \sum_{j=1}^{m} f(\eta_j)(y_j - y_{j-1}).
\end{aligned}$$

Remark. If we define a step function $f\colon [a,b] \to \mathbb{R}$ with respect to a partition $a = x_0 < x_1 < \cdots < x_n = b$, we do not make any requirements for the values of f at the partition points $x_0, x_1, \ldots, x_n$. This means, in particular, that if we change a step function's values at only finitely many points, the value of its Riemann integral will not be affected.

We collect the basic properties of the Riemann integral of step functions:

Theorem 19.3. *Let $a < b$, let $f, g\colon [a,b] \to \mathbb{R}$ be step functions, and let $\alpha, \beta \in \mathbb{R}$. Then:*

(i) *$\alpha f + \beta g$ is a step function such that*

$$\int_a^b \alpha f(x) + \beta g(x)\, dx = \alpha \int_a^b f(x)\, dx + \beta \int_a^b g(x)\, dx;$$

(ii) *if $f \leq g$, i.e., $f(x) \leq g(x)$ for all $x \in [a,b]$, then*

$$\int_a^b f(x)\, dx \leq \int_a^b g(x)\, dx$$

holds.

Proof. (i): Let $a = x_0 < x_1 < \cdots < x_n = b$ be a partition of $[a,b]$ such that f is a step function with respect to it, and let $a = y_0 < y_1 < \cdots < y_m = b$ be a partition of $[a,b]$ such that g is a step function with respect to it. Replacing both $a = x_0 < x_1 < \cdots < x_n = b$ and $a = y_0 < y_1 < \cdots < y_m = b$ with a common refinement if necessary, we can suppose without loss of generality that f and g are both step functions with respect to $a = x_0 < x_1 < \cdots < x_n = b$, i.e., both functions are constant on (x_{j-1}, x_j) as is, consequently, $\alpha f + \beta g$ for $j = 1, \ldots, n$, so that $\alpha f + \beta g$ is a step function as well. Choose $\xi_j \in (x_{j-1}, x_j)$ for $j = 1, \ldots, n$, and note that

$$\begin{aligned}\int_a^b \alpha f(x) + \beta g(x)\,dx &= \sum_{j=1}^n (\alpha f(\xi_j) + \beta g(\xi_j))(x_j - x_{j-1})\\ &= \alpha \sum_{j=1}^n f(\xi_j)(x_j - x_{j-1}) + \beta \sum_{j=1}^n g(\xi_j)(x_j - x_{j-1})\\ &= \alpha \int_a^b f(x)\,dx + \beta \int_a^b g(x)\,dx.\end{aligned}$$

(ii) is straightforward. □

Definition 19.4. Let $a < b$, and let $f\colon [a,b] \to \mathbb{R}$ be bounded. Then:

(a) the *lower integral* of f from a to b is defined as

$$\underset{*}{\int_a^b} f(x)\,dx := \sup\left\{\int_a^b \phi(x)\,dx : \phi\colon [a,b] \to \mathbb{R} \text{ is a step function with } \phi \le f\right\};$$

(b) the *upper integral* of f from a to b is defined as

$$\int_a^{*b} f(x)\,dx := \inf\left\{\int_a^b \psi(x)\,dx : \psi\colon [a,b] \to \mathbb{R} \text{ is a step function with } \psi \ge f\right\}.$$

Examples. 1. Let $a < b$, and let $f : [a,b] \to \mathbb{R}$ be a step function. It follows that the supremum and the infimum the the definitions of the lower and the upper integral of f are in fact attained—at $\phi = \psi = f$—, so that

$$\underset{*}{\int_a^b} f(x)\,dx = \int_a^b f(x)\,dx = \int_a^{*b} f(x)\,dx.$$

2. Consider

$$f\colon [0,1] \to \mathbb{R}, \quad x \mapsto \begin{cases} 1, & x \notin \mathbb{Q}, \\ 0, & x \in \mathbb{Q}. \end{cases}$$

Let $\phi, \psi \colon [0,1] \to \infty$ be step functions with $\phi \leq f \leq \psi$, and let $a = x_0 < x_1 < \cdots < x_n = b$ be such that ϕ and ψ are constant on (x_{j-1}, x_j) for $j = 1, \ldots, n$. Choose $\xi_j \in (x_{j-1}, x_j) \cap \mathbb{Q}$ and $\eta_j \in (x_{j-1}, x_j) \setminus \mathbb{Q}$ for $j = 1, \ldots, n$. It follows that

$$\int_a^b \phi(x)\,dx = \sum_{j=1}^n \phi(\xi_j)(x_j - x_{j-1}) \leq \sum_{j=1}^n f(\xi_j)(x_j - x_{j-1}) = 0$$

and

$$\int_a^b \psi(x)\,dx = \sum_{j=1}^n \psi(\eta_j)(x_j - x_{j-1}) \geq \sum_{j=1}^n f(\eta_j)(x_j - x_{j-1}) = 1$$

and therefore

$$\int_{*\,a}^{b} f(x)\,dx = 0 < 1 = \int_a^{*\,b} f(x)\,dx.$$

(This function is known as the *Dirichlet Function*.)

Proposition 19.5. *Let $a < b$, let $f, g : [a,b] \to \mathbb{R}$ be bounded, and let $t \in \mathbb{R}$ be non-negative. Then we have*

$$\int_a^{*\,b} f(x) + g(x)\,dx \leq \int_a^{*\,b} f(x)\,dx + \int_a^{*\,b} g(x)\,dx \qquad \textit{and} \qquad \int_a^{*\,b} tf(x)\,dx = t\int_a^{*\,b} f(x)\,dx.$$

Proof. For the first claim, let $\epsilon > 0$. By the definition of the upper integral, there are step functions $\psi_1, \psi_2 \colon [a,b] \to \mathbb{R}$ with $\psi_1 \geq f$ and $\psi_2 \geq g$ such that

$$\int_a^b \psi_1(x)\,dx < \int_a^{*\,b} f(x)\,dx + \frac{\epsilon}{2} \qquad \text{and} \qquad \int_a^b \psi_2(x)\,dx < \int_a^{*\,b} g(x)\,dx + \frac{\epsilon}{2}.$$

It follows that

$$\int_a^b \psi_1(x) + \psi_2(x)\,dx = \int_a^b \psi_1(x)\,dx + \int_a^b \psi_2(x)\,dx < \int_a^{*\,b} f(x)\,dx + \int_a^{*\,b} g(x)\,dx + \epsilon.$$

As $\psi_1 + \psi_2$ is a step function with $\psi_1 + \psi_2 \geq f + g$, it follows that

$$\int_a^{*\,b} f(x) + g(x)\,dx \leq \int_a^b \psi_1(x) + \psi_2(x)\,dx < \int_a^{*\,b} f(x)\,dx + \int_a^{*\,b} g(x)\,dx + \epsilon.$$

Since $\epsilon > 0$ is arbitrary, this proves the claim.

For the second claim, we can suppose without loss of generality that $t > 0$. Let $\psi \colon [a,b] \to \mathbb{R}$ be a step function with $\psi \geq f$. Then $t\psi$ is a step function with $t\psi \geq tf$. It follows that

$$\int_a^{*\,b} tf(x)\,dx \leq \int_a^b t\psi(x)\,dx = t\int_a^b \psi(x)\,dx$$

and thus

$$\overline{\int_a^b} t f(x)\,dx \leq t \overline{\int_a^b} f(x)\,dx.$$

On the other hand, we have

$$\overline{\int_a^b} f(x)\,dx = \overline{\int_a^b} t\frac{1}{t} f(x)\,dx \leq \frac{1}{t}\overline{\int_a^b} t f(x)\,dx,$$

so that multiplication with t yields the reversed inequality. □

Corollary 19.6. *Let $a < b$, let $f, g\colon [a,b] \to \mathbb{R}$ be bounded, and let $t \in \mathbb{R}$ be non-negative. Then we have*

$$\underline{\int_a^b} f(x) + g(x)\,dx \geq \underline{\int_a^b} f(x)\,dx + \underline{\int_a^b} g(x)\,dx \qquad \text{and} \qquad \underline{\int_a^b} t f(x)\,dx = t\underline{\int_a^b} f(x)\,dx.$$

Proof. Observe that

$$\underline{\int_a^b} f(x)\,dx = -\overline{\int_a^b} -f(x)\,dx$$

(and same for g), and apply the previous proposition. □

Definition 19.7 (Riemann Integral)**.** Let $a < b$. We call $f\colon [a,b] \to \mathbb{R}$ *Riemann integrable* if it is bounded and satisfies

$$\overline{\int_a^b} f(x)\,dx = \underline{\int_a^b} f(x)\,dx.$$

In this case, we define

$$\int_a^b f(x)\,dx := \overline{\int_a^b} f(x)\,dx$$

as the *Riemann integral* of f from a to b.

Examples.

1. Step functions are Riemann integrable.
2. The Dirichlet Function is not Riemann integrable.

Proposition 19.8. *Let $a < b$. Then $f\colon [a,b] \to \mathbb{R}$ is Riemann integrable if and only if, for each $\epsilon > 0$, there are step functions $\phi, \psi\colon [a,b] \to \mathbb{R}$ with $\phi \leq f \leq \psi$ such that*

$$\int_a^b \psi(x)\,dx - \int_a^b \phi(x)\,dx < \epsilon.$$

Proof. Suppose that f is Riemann integrable, and let $\epsilon > 0$. Then there are $\phi, \psi \colon [a,b] \to \mathbb{R}$ with $\phi \leq f \leq \psi$ such that

$$\int_a^b \phi(x)\,dx > \int_a^b f(x)\,dx - \frac{\epsilon}{2} \qquad \text{and} \qquad \int_a^b \psi(x)\,dx < \int_a^b f(x)\,dx + \frac{\epsilon}{2}.$$

Multiplying the first inequality with -1 and adding it to the second yields the claim.

Conversely, let $\epsilon > 0$, and let $\phi, \psi \colon [a,b] \to \mathbb{R}$ be step functions with $\phi \leq f \leq \psi$ and

$$\int_a^b \psi(x)\,dx - \int_a^b \phi(x)\,dx < \epsilon.$$

As ϕ and ψ are bounded, so is f, and we have

$$\begin{aligned}\int_{*\,a}^{b} f(x)\,dx &\geq \int_a^b \phi(x)\,dx \\ &= \int_a^b \psi(x)\,dx - \int_a^b \psi(x) - \phi(x)\,dx \geq \int_a^b \psi(x)\,dx - \epsilon \geq \int_a^{*\,b} f(x)\,dx - \epsilon.\end{aligned}$$

As $\epsilon > 0$ is arbitrary, this means that

$$\int_{*\,a}^{b} f(x)\,dx \geq \int_a^{*\,b} f(x)\,dx.$$

The reversed inequality holds trivially. This completes the proof. $\square$

Examples. 1. Let $a < b$, and let $f : [a,b] \to \mathbb{R}$ be continuous. Then f is Riemann integrable. To see this, let $\epsilon > 0$. As f is uniformly continuous, there is $\delta > 0$ such that $|f(x) - f(y)| < \frac{\epsilon}{b-a}$ for all $x, y \in [a,b]$ with $|x - y| < \delta$. Choose a partition $a = x_0 < x_1 < \cdots < x_n = b$ such that $x_j - x_{j-1} < \delta$ for $j = 1, \ldots, n$. Set

$$m_j := \min f([x_{j-1}, x_j]) \qquad \text{and} \qquad M_j := \max f([x_{j-1}, x_j])$$

for $j = 1, \ldots, n$, and define

$$\phi \colon [a,b] \to \mathbb{R}, \quad x \mapsto \begin{cases} f(x_j), & \text{if } x = x_j \text{ with } j \in \{0, 1, \ldots, n\}, \\ m_j, & \text{if } x \in (x_{j-1}, x_j) \text{ with } j \in \{1, \ldots, n\}, \end{cases}$$

and

$$\psi \colon [a,b] \to \mathbb{R}, \quad x \mapsto \begin{cases} f(x_j), & \text{if } x = x_j \text{ with } j \in \{0, 1, \ldots, n\}, \\ M_j, & \text{if } x \in (x_{j-1}, x_j) \text{ with } j \in \{1, \ldots, n\}. \end{cases}$$

It is then clear that $\phi, \psi : [a,b] \to \mathbb{R}$ are step functions with $\phi \leq f \leq \psi$. For $j = 1, \ldots, n$, let $\xi_j, \eta_j \in [x_{j-1}, x_j]$ be such that $f(\xi_j) = m_j$ and $f(\eta_j) = M_j$. As $x_j - x_{j-1} < \delta$, it follows that $|\xi_j - \eta_j| < \delta$ and, consequently,

$$M_j - m_j = |f(\xi_j) - f(\eta_j)| < \frac{\epsilon}{b-a}.$$

We conclude that

$$\int_a^b \psi(x)\,dx - \int_a^b \phi(x)\,dx = \sum_{j=1}^n (M_j - m_j)(x_j - x_{j-1}) < \sum_{j=1}^n \frac{\epsilon}{b-a}(x_j - x_{j-1}) = \epsilon.$$

2. Let $a < b$, and let $f\colon [a,b] \to \mathbb{R}$ be monotonic. Then f is Riemann integrable. To see this, suppose first without loss of generality that f is increasing. Let $\epsilon > 0$, and let $n \in \mathbb{N}$. For $j = 0, 1, \ldots, n$, set

$$x_j := a + \frac{b-a}{n} j.$$

Define

$$\phi\colon [a,b] \to \mathbb{R}, \quad x \mapsto \begin{cases} f(x_{j-1}), & \text{if } x \in [x_{j-1}, x_j) \text{ with } j \in \{1, \ldots, n\}, \\ f(b), & x = b, \end{cases}$$

and

$$\psi\colon [a,b] \to \mathbb{R}, \quad x \mapsto \begin{cases} f(x_j), & \text{if } x \in [x_{j-1}, x_j) \text{ with } j \in \{1, \ldots, n\}, \\ f(b), & x = b. \end{cases}$$

Then $\phi, \psi\colon [a,b] \to \mathbb{R}$ are step functions with $\phi \leq f \leq \psi$ such that

$$\begin{aligned} \int_a^b \psi(x)\,dx - \int_a^b \phi(x)\,dx &= \sum_{j=1}^n f(x_j)(x_j - x_{j-1}) - \sum_{j=1}^n f(x_{j-1})(x_j - x_{j-1}) \\ &= \frac{b-a}{n} \sum_{j=1}^n (f(x_j) - f(x_{j-1})) \\ &= \frac{b-a}{n}(f(b) - f(a)). \end{aligned}$$

For sufficiently large n, we therefore obtain $\int_a^b \psi(x)\,dx - \int_a^b \phi(x)\,dx < \epsilon$.

Theorem 19.9 (Properties of the Riemann Integral)**.** *Let $a < b$, let $f, g\colon [a,b] \to \mathbb{R}$ be Riemann integrable, and let $\alpha \in \mathbb{R}$. Then:*

(i) *$f + g$ is Riemann integrable with*

$$\int_a^b f(x) + g(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx;$$

(ii) *αf is Riemann integrable with*

$$\int_a^b \alpha f(x)\,dx = \alpha \int_a^b f(x)\,dx;$$

(iii) *if $f \leq g$, then*

$$\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx;$$

(iv) $|f|$ *is Riemann integrable with*

$$\left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx;$$

(v) fg *is Riemann integrable.*

Proof. (i): From Proposition 19.5 and Corollary 19.6, we conclude that

$$\begin{aligned}\int_a^{*b} f(x)+g(x)\,dx &\leq \int_a^{*b} f(x)\,dx + \int_a^{*b} g(x)\,dx \\ &= \int_{*a}^{b} f(x)\,dx + \int_{*a}^{b} g(x)\,dx \leq \int_{*a}^{b} f(x)+g(x)\,dx,\end{aligned}$$

which yields the claim.

(ii): Suppose first that $\alpha \geq 0$. By Proposition 19.5 and Corollary 19.6, we then have

$$\int_a^{*b} \alpha f(x)\,dx = \alpha \int_a^b f(x)\,dx = \alpha \int_{*a}^{b} f(x)\,dx,$$

which proves the claim for non-negative α. Suppose now that $\alpha < 0$, and observe that

$$\int_{*a}^{b} \alpha f(x)\,dx = -\int_a^{*b} -\alpha f(x)\,dx = -(-\alpha)\int_a^{*b} f(x)\,dx = \alpha \int_a^b f(x)\,dx.$$

and, similarly,

$$\int_a^{*b} \alpha f(x)\,dx = \alpha \int_a^b f(x)\,dx.$$

This proves the claim for negative α as well.

(iii) is obvious.

(iv): Define $f^+, f^-\colon [a,b] \to \mathbb{R}$ as follows:

$$f^+(x) := \begin{cases} f(x), & f(x) \geq 0, \\ 0, & \text{otherwise,}\end{cases} \qquad \text{and} \qquad f^-(x) := \begin{cases} -f(x), & f(x) \leq 0, \\ 0, & \text{otherwise.}\end{cases}$$

It is clear that

$$f = f^+ - f^- \qquad \text{and} \qquad |f| = f^+ + f^-.$$

We claim that both f^+ and f^- are Riemann integrable. Let $\epsilon > 0$, and let $\phi, \psi\colon [a,b] \to \mathbb{R}$ be step functions such that $\phi \leq f \leq \psi$ and $\int_a^b \psi(x)\,dx - \int_a^b \phi(x)\,dx < \epsilon$. It is then clear that ϕ^+ and ψ^+ are step functions with $\phi^+ \leq f^+ \leq \psi^+$ such that

$$\begin{aligned}\int_a^b \psi^+(x)\,dx - \int_a^b \phi^+(x)\,dx &= \int_a^b \psi^+(x) - \phi^+(x)\,dx \\ &\leq \int_a^b \psi^+(x) - \phi^+(x) - \underbrace{(\psi^-(x) - \phi^-(x))}_{\leq 0}\,dx \int_a^b \psi(x) - \phi(x)\,dx < \epsilon.\end{aligned}$$

It follows that f^+ is Riemann integrable. Similarly, one sees that f^- is Riemann integrable. It follows that $|f| = f^+ + f^-$ is Riemann integrable and that

$$\begin{aligned}&\left|\int_a^b f(x)\,dx\right|\\ &\quad = \left|\int_a^b f^+(x)\,dx - \int_a^b f^-(x)\,dx\right| \leq \int_a^b f^+(x)\,dx + \int_a^b f^-(x)\,dx = \int_a^b |f(x)|\,dx.\end{aligned}$$

(v): We first treat the case where $g = f$, i.e., we show that f^2 is integrable. As $f^2 = |f|^2$, we my replace f by $|f|$ and therefore suppose that $f \geq 0$. As f is bounded, we can multiply it with a suitable scalar and also suppose that $f \leq 1$. Let $\epsilon > 0$, and let $\phi, \psi \colon [a,b] \to \mathbb{R}$ be step functions with $\phi \leq f \leq \psi$ and $\int_a^b \psi(x)\,dx - \int_a^b \phi(x)\,dx < \frac{\epsilon}{2}$. Replacing ϕ by ϕ^+ and ψ by

$$[a,b] \to \mathbb{R}, \quad x \mapsto \begin{cases} \psi(x), & \psi(x) \leq 1, \\ 1, & \psi(x) > 1, \end{cases}$$

we can suppose without loss of generality that $0 \leq \phi \leq f \leq \psi \leq 1$. Then ϕ^2 and ψ^2 are also step functions with $\phi^2 \leq f^2 \leq \psi^2$. It follows that

$$\begin{aligned}\int_a^b \psi(x)^2\,dx - \int_a^b \phi(x)^2\,dx &= \int_a^b \psi(x)^2 - \phi(x)^2\,dx \\ &= \int_a^b (\psi(x) - \phi(x))(\psi(x) + \phi(x))\,dx \\ &\leq 2\int_a^b \psi(x) - \phi(x)\,dx \\ &< 2\frac{\epsilon}{2} \\ &= \epsilon,\end{aligned}$$

so that f^2 is Riemann integrable.

For the general case, just note that

$$fg = \frac{1}{4}((f+g)^2 - (f-g)^2),$$

which completes the proof. $\square$

Corollary 19.10 (Mean Value Theorem of Integration)**.** *Let $a < b$, let $f \colon [a,b] \to \mathbb{R}$ be continuous, and let $g \colon [a,b] \to [0,\infty)$ be Riemann integrable. Then there is $\xi \in [a,b]$ such that*

$$\int_a^b f(x)g(x)\,dx = f(\xi)\int_a^b g(x)\,dx.$$

Proof. Let

$$m := \inf f([a,b]) \qquad \text{and} \qquad M := \sup f([a,b]),$$

so that $mg \le fg \le Mg$ and, consequently,

$$m \int_a^b g(x)\,dx \le \int_a^b f(x)g(x)\,dx \le M \int_a^b g(x)\,dx.$$

Hence, there is $c \in [m, M]$ with

$$\int_a^b f(x)g(x)\,dx = c \int_a^b g(x)\,dx.$$

By the Intermediate Value Theorem, there is $\xi \in [a,b]$ with $f(\xi) = c$. □

Exercises

1. Is the function

$$f\colon [0,1] \to \mathbb{R}, \quad x \mapsto \begin{cases} x, & x \in \mathbb{Q}, \\ 0, & x \notin \mathbb{Q}, \end{cases}$$

   Riemann integrable? If so, evaluate its integral.

2. For $n \in \mathbb{N}$, define step functions $\phi_n, \psi_n\colon [0,1] \to \mathbb{R}$ by letting

$$\phi_n(x) = \begin{cases} \frac{k-1}{n}, & \text{if } x \in \left[\frac{k-1}{n}, \frac{k}{n}\right) \text{ for some } k \in \{1, \ldots, n\}, \\ 1, & x = 1, \end{cases}$$

   and

$$\psi_n(x) = \begin{cases} \frac{k}{n}, & \text{if } x \in \left[\frac{k-1}{n}, \frac{k}{n}\right) \text{ for some } k \in \{1, \ldots, n\}, \\ 1, & x = 1. \end{cases}$$

   Compute $\int_0^1 \phi_n(x)\,dx$ and $\int_0^1 \psi_n(x)\,dx$ and use this to evaluate $\int_0^1 x\,dx$.

3. Let

$$f\colon [0,1] \to \mathbb{R}, \quad x \mapsto \begin{cases} 1, & x \in \left\{\frac{1}{n} : n \in \mathbb{N}\right\}, \\ 0, & \text{otherwise.} \end{cases}$$

   and, for $n \in \mathbb{N}$, let $\phi_n, \psi_n\colon [0,1] \to \mathbb{R}$ be defined through

$$\phi_n(x) = \begin{cases} 0, & 0 \le x < \frac{1}{n}, \\ f(x), & \frac{1}{n} \le x \le 1, \end{cases}$$

   and

$$\psi_n(x) = \begin{cases} 1, & 0 \le x < \frac{1}{n}, \\ f(x), & \frac{1}{n} \le x \le 1. \end{cases}$$

   Show that f is Riemann integrable and that $\int_0^1 f(x)\,dx = 0$.

4. Let $a < b$, and let f be the function defined Exercise 3 in Section 14. Show that f is Riemann integrable on $[a, b]$ with $\int_a^b f(x)\,dx = 0$.

5. Let $a < b$, and let $f\colon [a, b] \to \mathbb{R}$ be a function such that $|f|\colon [a, b] \to \mathbb{R}$ is Riemann integrable. Does this necessarily mean that f is Riemann integrable?

6. Let $a < b$, and let $f\colon [a, b] \to [0, \infty)$ be continuous such that $\int_a^b f(x)\,dx = 0$. Show that $f(x) = 0$ for all $x \in [a, b]$. (*Hint*: A sketch might help.)

7. Let $a < b$, let $f\colon [a, b] \to \mathbb{R}$ be Riemann integrable, and let $g\colon [a, b] \to \mathbb{R}$ be such that $\{x \in [a, b] : f(x) \neq g(x)\}$ is finite. Show that g is Riemann integrable and $\int_a^b g(x)\,dx = \int_a^b f(x)\,dx$. Proceed as follows:

   - first suppose, that f is a step function and show that g is also a step function and that $\int_a^b g(x)\,dx = \int_a^b f(x)\,dx$;
   - then use the definition of the Riemann integral to prove the general case.

20 Integration and Differentiation

Lemma 20.1. *Let $a < b < c$. Then $f\colon [a,c] \to \mathbb{R}$ is Riemann integrable if and only if the restrictions of f to both $[a,b]$ and $[b,c]$ are Riemann integrable, in which case*

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$$

holds.

Proof. Boring. □

For convenience, we agree to define

$$\int_a^a f(x)\,dx = 0 \qquad \text{and} \qquad \int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

if $a > b$. With these conventions, the formula of Lemma 20.1 remains valid for any choice of $a, b, c \in [\min\{a,b,c\}, \max\{a,b,c\}]$.

In what follows, we use the term *interval* for any set of one of the following forms:

- (a,b), $[a,b]$, $(a,b]$, or $[a,b)$ with $a < b$;
- (a,∞), $[a,\infty)$, $(-\infty,a)$, or $(-\infty,a]$ with $a \in \mathbb{R}$;
- $\mathbb{R}$.

Theorem 20.2. *Let $I \subset \mathbb{R}$ be an interval, let $f\colon I \to \mathbb{R}$ be continuous, and let $x_0 \in I$. Then*

$$F\colon I \to \mathbb{R}, \quad x \mapsto \int_{x_0}^x f(t)\,dt$$

is an antiderivative *of f, i.e., F is differentiable such that $F' = f$. Moreover, if $G\colon I \to \mathbb{R}$ is any antiderivative of f, then $F - G$ is constant.*

Proof. Let $x \in I$, and let $(h_n)_{n=1}^\infty$ be a sequence in $\mathbb{R} \setminus \{0\}$ such that $h_n \to 0$ and $x + h_n \in I$ for all $n \in \mathbb{N}$. We obtain

$$\begin{aligned}
\frac{F(x+h_n) - F(x)}{h_n} &= \frac{1}{h_n}\left(\int_{x_0}^{x+h_n} f(t)\,dt - \int_{x_0}^{x} f(t)\,dt\right) \\
&= \frac{1}{h_n}\left(\int_{x_0}^{x+h_n} f(t)\,dt + \int_{x}^{x_0} f(t)\,dt\right) \\
&= \frac{1}{h_n}\int_{x}^{x+h_n} f(t)\,dt
\end{aligned}$$

for all $n \in \mathbb{N}$ by Lemma 20.1. By the Mean Value Theorem of Integration (with $g \equiv 1$), there is, for each $n \in \mathbb{N}$, a number ξ_n between x and $x + h_n$ such that

$$\int_x^{x+h_n} f(t)\,dt = f(\xi_n)\int_x^{x+h_n} 1\,dt = f(\xi_n)h_n,$$

so that

$$\frac{F(x+h_n)-F(x)}{h_n} = f(\xi_n).$$

As $h_n \to 0$, it follows that $\xi_n \to x$, and since f is continuous, we conclude that

$$\lim_{n\to\infty} \frac{F(x+h_n)-F(x)}{h_n} = \lim_{n\to\infty} f(\xi_n) = f(x).$$

Hence, F is an antiderivative of f.

Let $G\colon I \to \mathbb{R}$ be any antiderivative of f. As

$$(F-G)' = F' - G' = f - f = 0,$$

it is clear that $F - G$ must be constant. □

Corollary 20.3 (Fundamental Theorem of Calculus)**.** *Let $I \subset \mathbb{R}$ be an interval, let $f\colon I \to \mathbb{R}$ be continuous, and let $F\colon I \to \mathbb{R}$ be an antiderivative of f. Then*

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

for all $a, b \in I$.

Proof. Let $a, b \in I$. Define

$$\tilde{F}\colon I \to \mathbb{R}, \quad x \mapsto \int_a^x f(t)\,dt.$$

By Theorem 20.2, $\tilde{F}$ is an antiderivative of f with

$$\tilde{F}(a) = 0 \qquad \text{and} \qquad \tilde{F}(b) = \int_a^b f(x)\,dx,$$

so that

$$\int_a^b f(x)\,dx = \tilde{F}(b) - \tilde{F}(a).$$

As $F - \tilde{F}$ is constant, this yields the claim. □

Notation. For the right hand side of the formula in Corollary 20.3, we often write

$$F(x)\Big|_a^b \qquad \text{or} \qquad F(x)\Big|_{x=a}^{x=b}.$$

If F is an antiderivative of f, we use the symbols

$$F(x) = \int f(x)\,dx \qquad \text{or} \qquad F(x) = \int f(x)\,dx + C,$$

the *indefinite integral* of f, where the last expression emphasizes that an antiderivative is unique only up to an additive constant.

Examples. 1.
$$\int x^r \, dx = \frac{x^{r+1}}{r+1}$$
for $r \in \mathbb{R} \setminus \{-1\}$.

2.
$$\int \frac{1}{x} \, dx = \log x.$$

3.
$$\int \exp(x) \, dx = \exp(x).$$

4.
$$\int \sin x \, dx = -\cos x \qquad \text{and} \qquad \int \cos x \, dx = \sin x.$$

Corollary 20.4 (Change of Variables)**.** *Let $I \subset \mathbb{R}$ be an interval, let $f : I \to \mathbb{R}$ be continuous, let $a < b$, and let $\phi : [a, b] \to \mathbb{R}$ be continuously differentiable such that $\phi([a, b]) \subset I$. Then*
$$\int_a^b f(\phi(t))\phi'(t) \, dt = \int_{\phi(a)}^{\phi(b)} f(x) \, dx$$
holds.

Proof. Let $F \colon I \to \mathbb{R}$ be an antiderivative of f. The Chain Rule then yields
$$(F \circ \phi)' = (F' \circ \phi)\phi' = (f \circ \phi)\phi',$$
i.e., $F \circ \phi$ is an antiderivative of $(f \circ \phi)\phi'$. It follows from Corollary 20.3 that
$$\int_a^b f(\phi(t))\phi'(t) \, dt = F(\phi(b)) - F(\phi(a)) = \int_{\phi(a)}^{\phi(b)} f(x) \, dx$$
as claimed. □

Corollary 20.5 (Integration by Parts)**.** *Let $a < b$, and let $f, g \colon [a, b] \to \mathbb{R}$ be continuously differentiable. Then*
$$\int_a^b f(x)g'(x) \, dx = f(x)g(x)\Big|_a^b - \int_a^b f'(x)g(x) \, dx$$
holds.

Proof. Set $F := fg$, so that $F' = f'g + fg'$ and therefore
$$\int_a^b f'(x)g(x) \, dx + \int_a^b f(x)g'(x) \, dx = \int_a^b F'(x) \, dx = F(b) - F(a) = f(x)g(x)\Big|_a^b$$
as claimed. □

Exercises

1. Prove Lemma 20.1.

2. Let $a < b$, and let $f, g\colon [a,b] \to \mathbb{R}$ be continuously differentiable such that $f(a) \leq g(a)$ and $f' \leq g'$. Show that $f \leq g$.

3. Let $a < b$, let $f\colon [a,b] \to \mathbb{R}$ be continuously differentiable, and let

$$F\colon \mathbb{R} \to \mathbb{R}, \quad x \mapsto \int_a^b f(t) \sin(xt)\, dt.$$

   Show that $\lim_{x\to\infty} F(x) = \lim_{x\to-\infty} F(x) = 0$. (*Hint*: Integration by Parts.)

4. Let $a < b$, and let $f, g\colon [a,b] \to \mathbb{R}$ be Riemann integrable. By Theorem 19.9(v), fg is also Riemann integrable. Does then

$$\int_a^b f(x)g(x)\, dx = \left(\int_a^b f(x)\, dx\right)\left(\int_a^b g(x)\, dx\right)$$

   necessarily hold?

21 Riemann Sums

Our approach to Riemann integration—via order theoretic arguments—is not the original one. It has the advantage that it is fast and almost effortlessly yields the Riemann integrability of monotonic functions. Usually, Riemann integrability—and the Riemann integral—are defined in terms of Riemann sums.

Definition 21.1. Let $a < b$, let $f : [a, b] \to \mathbb{R}$, let $a = x_0 < x_1 < \cdots < x_n = b$, and let $\xi_k \in [x_{k-1}, x_k]$ for $k = 1, \ldots, n$. Then the expression

$$\sum_{k=1}^{n} f(\xi_k)(x_k - x_{k-1})$$

is called the *Riemann sum* of f with respect to the partition $a = x_0 < x_1 < \cdots < x_n = b$ with support points $\xi_1, \ldots, \xi_n$.

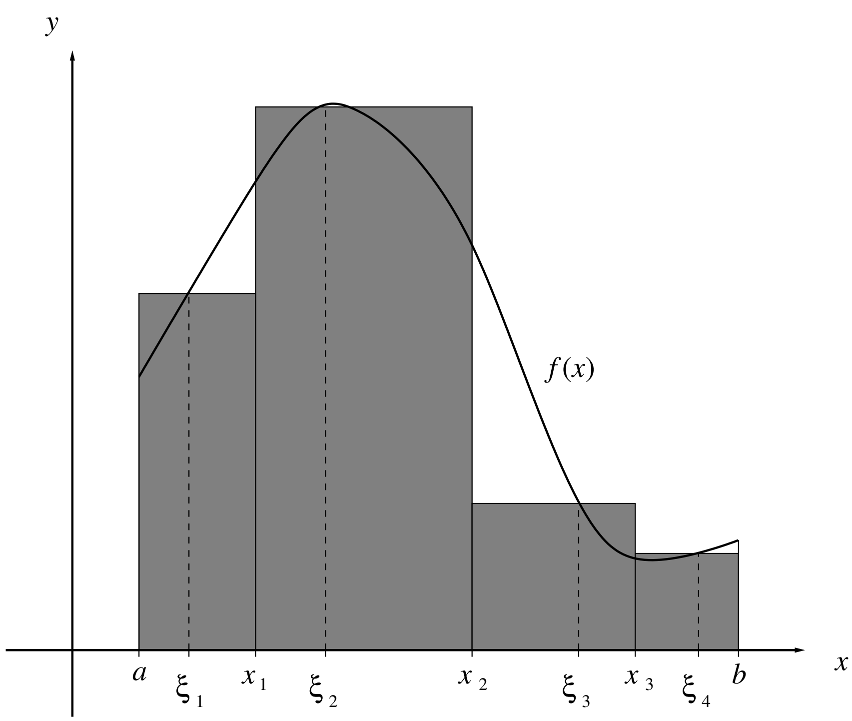

Figure 14: A Riemann sum

If a function is Riemann integrable, then its Riemann integral can be arbitrarily well approximated by Riemann sums:

Theorem 21.2. *Let $a < b$, and let $f : [a, b] \to \mathbb{R}$ be Riemann integrable. Then, for each $\epsilon > 0$, there is $\delta > 0$ such that, for every partition $a = x_0 < x_1 < \cdots < x_n = b$ with*

$\max_{k=1,\ldots,n} x_k - x_{k-1} < \delta$ *and any choice of support points* $\xi_1, \ldots, \xi_n$ *with* $\xi_k \in [x_{k-1}, x_k]$ *for* $k = 1, \ldots, n$*, we have*

$$\left| \int_a^b f(x)\,dx - \sum_{k=1}^n f(\xi_k)(x_k - x_{k-1}) \right| < \epsilon.$$

Proof. Let $\epsilon > 0$. Then there are step functions $\phi, \psi \colon [a,b] \to \mathbb{R}$ with $\phi \leq f \leq \psi$ such that $\int_a^b \psi(x) - \phi(x)\,dx < \frac{\epsilon}{2}$. Choose $a = t_0 < t_1 < \cdots < t_m = b$ such that ϕ and ψ restricted to (t_{j-1}, t_j) are constant for $j = 1, \ldots, m$. Choose $C > 0$ such that $C \geq |f(x)|$ for all $x \in [a,b]$, and set $\delta := \frac{\epsilon}{8Cm}$.

Let $a = x_0 < x_1 < \cdots < x_n = b$ be a partition of $[a,b]$ with $\max_{k=1,\ldots,n} x_k - x_{k-1} < \delta$ and pick $\xi_k \in [x_{k-1}, x_k]$ for $k = 1, \ldots, n$. Define

$$\tilde{f} \colon [a,b] \to \mathbb{R}, \quad x \mapsto \begin{cases} 0, & \text{if } x \in \{x_0, x_1, \ldots, x_n\}, \\ f(\xi_k), & \text{if } x \in (x_{k-1}, x_k) \text{ with } k \in \{1, \ldots, n\}. \end{cases}$$

Clearly, $\tilde{f}$ is a step function and its Riemann integral from a to b is

$$\int_a^b \tilde{f}(x)\,dx = \sum_{k=1}^n f(\xi_k)(x_k - x_{k-1}),$$

i.e., the Riemann sum of f with respect to the partition $a = x_0 < x_1 < \cdots < x_n = b$ with support points $\xi_1, \ldots, \xi_n$.

The following are clear:

- $\phi - 2C \leq \tilde{f} \leq \psi + 2C$;
- if $k \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, m\}$ are such that $[x_{k-1}, x_k] \subset (t_{j-1}, t_j)$ then

  $$\phi(x) \leq \tilde{f}(x) \leq \psi(x)$$

  holds for all $x \in (x_{k-1}, x_k)$.

If $k \in \{1, \ldots, n\}$ is such that there is *no* $j \in \{1, \ldots, m\}$ with $[x_{k-1}, x_k] \subset (t_{j-1}, t_j)$, then there is $j_k \in \{0, 1, \ldots, n\}$ such that $t_{j_k} \in [x_{k-1}, x_k]$. As any two of the intervals $[x_0, x_1], \ldots, [x_{n-1}, x_n]$ have at most one point in common, this means that the number of all such k is at most $2m$. Set

$$\mathbb{I} = \bigcup_{\substack{k=1 \\ \text{there is } j \in \{1,\ldots,m\} \text{ such that } [x_{k-1},x_k] \subset (t_{j-1},t_j)}}^n (x_{k-1}, x_k).$$

Define

$$\Phi \colon [a,b] \to \mathbb{R}, \quad x \mapsto \begin{cases} 0, & x \in \mathbb{I}, \\ 2C, & \text{otherwise.} \end{cases}$$

Obviously, Φ is a step function and satisfies

$$\phi - \Phi \le \tilde{f} \le \psi + \Phi.$$

There are at most $2m$ intervals (x_{k-1}, x_k) on which Φ is non-zero. This means that

$$\int_a^b \Phi(x)\,dx < 2C(2m\delta) = \frac{\epsilon}{2}.$$

This implies

$$\int_a^b \phi(x)\,dx - \frac{\epsilon}{2} \le \int_a^b \tilde{f}(x)\,dx \le \int_a^b \psi(x)\,dx + \frac{\epsilon}{2}.$$

As

$$\int_a^b f(x)\,dx - \frac{\epsilon}{2} \le \int_a^b \phi(x)\,dx \qquad \text{and} \qquad \int_a^b \psi(x)\,dx \le \int_a^b f(x)\,dx + \frac{\epsilon}{2},$$

it follows that

$$\left|\int_a^b f(x)\,dx - \int_a^b \tilde{f}(x)\,dx\right| < \epsilon.$$

This completes the proof. $\square$

In fact, more is true:

Theorem 21.3. *Let $a < b$. Then the following are equivalent for $f\colon [a,b] \to \mathbb{R}$:*

(i) *f is Riemann integrable;*

(ii) *there is $I \in \mathbb{R}$ such that, for each $\epsilon > 0$, there is $\delta > 0$ such that, for every partition $a = x_0 < x_1 < \cdots < x_n = b$ with $\max_{k=1,\ldots,n} x_k - x_{k-1} < \delta$ and any choice of support points $\xi_1, \ldots, \xi_n$ with $\xi_k \in [x_{k-1}, x_k]$ for $k = 1, \ldots, n$, we have*

$$\left| I - \sum_{k=1}^{n} f(\xi_k)(x_k - x_{k-1}) \right| < \epsilon.$$

In this case, we have

$$I = \int_a^b f(x)\,dx.$$

Proof. (i) $\Longrightarrow$ (ii) is the content of Theorem 21.2

(ii) $\Longrightarrow$ (ii): We first claim that f must be bounded. Assume towards a contradiction that f is unbounded. Choose $\delta > 0$ and a partition $a = x_0 < x_1 < \cdots < x_n = b$ with $\max_{k=1,\ldots,n} x_k - x_{k-1} < \delta$ such that

$$\left| I - \sum_{k=1}^{n} f(\xi_k)(x_k - x_{k-1}) \right| < 1.$$

for all $\xi_k \in [x_{k-1}, x_k]$ for $k = 1, \ldots, n$. As f is unbounded, there must be $k_0 \in \{1, \ldots, n\}$ such that f is unbounded on $[x_{k_0-1}, x_{k_0}]$. For $k \in \{1, \ldots, n\} \setminus \{k_0\}$, fix $\xi_k \in [x_{k-1}, x_k]$. Choose $\xi_{k_0} \in [x_{k_0-1}, x_{k_0}]$ such that

$$|f(\xi_{k_0})| \geq \frac{1}{x_{k_0} - x_{k_0-1}} \left| I - \sum_{\substack{k=1 \\ k \neq k_0}}^{n} f(\xi_k)(x_k - x_{k-1}) \right| + \frac{1}{x_{k_0} - x_{k_0-1}}.$$

It follows that

$$\left| I - \sum_{k=1}^{n} f(\xi_k)(x_k - x_{k-1}) \right| \geq |f(\xi_{k_0})|(x_{k_0} - x_{k_0-1}) - \left| I - \sum_{\substack{k=1 \\ k \neq k_0}}^{n} f(\xi_k)(x_k - x_{k-1}) \right| \geq 1,$$

which is a contradiction. Therefore, f is bounded.

Let $\epsilon > 0$. Choose $\delta > 0$ such that such that, for every partition $a = x_0 < x_1 < \cdots < x_n = b$ with $\max_{k=1,\ldots,n} x_k - x_{k-1} < \delta$ and any choice of support points $\xi_1, \ldots, \xi_n$ with $\xi_k \in [x_{k-1}, x_k]$ for $k = 1, \ldots, n$, we have

$$\left| I - \sum_{k=1}^{n} f(\xi_k)(x_k - x_{k-1}) \right| < \frac{\epsilon}{4}.$$

Let $a = x_0 < x_1 < \cdots < x_n = b$ be such a partition. For $k = 1, \ldots, n$, set

$$m_k := \inf f([x_{k-1}, x_k]) \qquad \text{and} \qquad M_k := \sup f([x_{k-1}, x_k]).$$

Define

$$\phi \colon [a,b] \to \mathbb{R}, \quad x \mapsto \begin{cases} f(x), & \text{if } x \in \{x_0, x_1, \ldots, x_n\}, \\ m_k, & \text{if } x \in (x_{k-1}, x_k) \text{ with } k \in \{1, \ldots, n\}, \end{cases}$$

and

$$\psi \colon [a,b] \to \mathbb{R}, \quad x \mapsto \begin{cases} f(x), & \text{if } x \in \{x_0, x_1, \ldots, x_n\}, \\ M_k, & \text{if } x \in (x_{k-1}, x_k) \text{ with } k \in \{1, \ldots, n\}. \end{cases}$$

Then ϕ and ψ are step functions such that $\phi \leq f \leq \psi$. For $k = 1, \ldots, n$ choose $\xi_k, \eta_k \in [x_{k-1}, x_k]$ such that

$$f(\xi_k) - m_k < \frac{\epsilon}{4(b-a)} \qquad \text{and} \qquad M_k - f(\eta_k) < \frac{\epsilon}{4(b-a)}.$$

It follows that

$$\begin{aligned}
\left| \int_a^b \phi(x)\,dx - \sum_{k=1}^n f(\xi_k)(x_k - x_{k-1}) \right| &= \left| \sum_{k=1}^n m_k(x_k - x_{k-1}) - \sum_{k=1}^n f(\xi_k)(x_k - x_{k-1}) \right| \\
&= \sum_{k=1}^n (f(\xi_k) - m_k)(x_k - x_{k-1}) \\
&< \frac{\epsilon}{4(b-a)} \sum_{k=1}^n x_k - x_{k-1} \\
&< \frac{\epsilon}{4}
\end{aligned}$$

and, analogously,

$$\left| \int_a^b \psi(x)\,dx - \sum_{k=1}^n f(\eta_k)(x_k - x_{k-1}) \right| < \frac{\epsilon}{4}.$$

We conclude that

$$\begin{aligned}
&\left| I - \int_a^b \phi(x)\,dx \right| \\
&\quad \le \left| I - \sum_{k=1}^n f(\xi_k)(x_k - x_{k-1}) \right| + \left| \sum_{k=1}^n f(\xi_k)(x_k - x_{k-1}) - \int_a^b \phi(x)\,dx \right| < \frac{\epsilon}{4} + \frac{\epsilon}{4} = \frac{\epsilon}{2}
\end{aligned}$$

as well as

$$\left| I - \int_a^b \psi(x)\,dx \right| < \frac{\epsilon}{2},$$

so that

$$\int_a^b \psi(x)\,dx - \int_a^b \phi(x)\,dx \le \left| \int_a^b \psi(x)\,dx - I \right| + \left| I - \int_a^b \phi(x)\,dx \right| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

In view of Proposition 19.8, this means that f is Riemann integrable. Finally, Theorem 21.2 yields that $I = \int_a^b f(x)\,dx$ because I has to be unique (see Exercise 1 below). $\square$

In fact, the original definition of the Riemann integral uses Theorem 21.3(ii) and *defines* the Riemann integral to be I.

Exercises

1. Show that I in Theorem 21.3(ii) is unique.

22 The Complex Numbers $\mathbb{C}$, and the Truth about $\exp$, $\cos$, and $\sin$

Definition 22.1. The *complex numbers* $\mathbb{C}$ are the set

$$\mathbb{R}^2 := \{(x, y) : x, y \in \mathbb{R}\}$$

equipped with addition $+$ and multiplication $\cdot$ defined by

$$(x_1, y_1) + (x_2, y_2) := (x_1 + x_2, y_1 + y_2)$$

and

$$(x_1, y_1) \cdot (x_2, y_2) := (x_1 x_2 - y_1 y_2, x_1 y_2 + y_1 x_2)$$

for $(x_1, y_1), (x_2, y_2) \in \mathbb{C}$.

Proposition 22.2. *$\mathbb{C}$ is a field (with zero element $(0,0)$ and identity element $(1,0)$).*

Proof. Except for the existence of the multiplicative inverse, verifying the field axioms is routine.

Let $(x, y) \in \mathbb{C} \setminus \{(0,0)\}$, so that $x^2 + y^2 > 0$. It follows that

$$\begin{aligned}
(x, y) \cdot \left(\frac{x}{x^2+y^2}, \frac{-y}{x^2+y^2} \right) &= \left(x \frac{x}{x^2+y^2} - y \frac{-y}{x^2+y^2}, x \frac{-y}{x^2+y^2} + y \frac{x}{x^2+y^2} \right) \\
&= \left(\frac{x^2+y^2}{x^2+y^2}, \frac{-xy+yx}{x^2+y^2} \right) \\
&= (1, 0),
\end{aligned}$$

i.e., $\left(\frac{x}{x^2+y^2}, \frac{-y}{x^2+y^2} \right)$ is the multiplicative inverse of (x, y). □

We identify $\mathbb{R}$ with the set

$$\{(x, 0) : x \in \mathbb{R}\} \subset \mathbb{C}.$$

Addition and multiplication in $\mathbb{C}$ restricted to this set just "are" addition and multiplication of $\mathbb{R}$. For $x \in \mathbb{R}$, we therefore simply write x to denote the complex number $(x, 0)$.

Set $i := (0, 1)$, the *imaginary unit*. We have

$$i^2 = (0, 1) \cdot (0, 1) = (-1, 0) = -1,$$

i.e., the equation $z^2 = -1$ has the solution i in $\mathbb{C}$ ($-i$ also solves the equation).

Warning. One often sees the expression $\sqrt{-1}$ for i, but this is problematic because one can easily arrive at nonsense this way:

$$1 = \sqrt{1} = \sqrt{(-1)(-1)} = \sqrt{-1}\sqrt{-1} = -1.$$

The notation $\sqrt{-1}$ should therefore be avoided.

The reason for the apparent paradox is the following:

Proposition 22.3. $\mathbb{C}$ *cannot be ordered.*

Proof. Assume that we can define an order relation "$\leq$" on $\mathbb{C}$ such that the order axioms are satisfied.

Consider 0 and i.

By (O 1), $0 \leq i$ or $i \leq 0$ must hold.

If $0 \leq i$, then $0 \leq -1$ by (O 5) and therefore $1 \leq 0$, which cannot be.

If $i \leq 0$, then $0 \leq -i$, and—again by (O 5)—

$$0 \leq (-i)^2 = i^2 = -1.$$

This proves the claim. $\square$

If $y > 0$, the equation $x^2 = y$ has two solutions, of which $\sqrt{y}$ is defined to be the positive one. If $y < 0$, there is no such way to tell the two solutions of $x^2 = y$ apart.

Given $z = (x, y) \in \mathbb{C}$, we have

$$z = (x, 0) + (0, y) = (x, 0)(1, 0) + (y, 0)(0, 1) = x + iy,$$

which is the usual way to write complex numbers.

Definition 22.4. Let $z = x + iy \in \mathbb{C}$. Then:

(a) $\operatorname{Re} z := x$ is the *real part* of z;

(b) $\operatorname{Im} z := y$ is the *imaginary part* of z;

(c) $\bar{z} := x - iy$ is the *complex conjugate* of z;

(d) $|z| := \sqrt{x^2 + y^2}$ is the *absolute value* or *modulus* of z.

In terms of these definitions, we have for $z, w \in \mathbb{C}$:

$$\begin{aligned}
z &= \operatorname{Re} z + i \operatorname{Im} z, \\
\bar{z} &= \operatorname{Re} z - i \operatorname{Im} z, \\
\bar{\bar{z}} &= z, \\
\operatorname{Re} z &= \frac{1}{2}(z + \bar{z}), \\
\operatorname{Im} z &= \frac{1}{2i}(z - \bar{z}), \\
\overline{z + w} &= \bar{z} + \bar{w}, \\
\overline{zw} &= \bar{z}\bar{w}, \\
|z| &= \sqrt{z\bar{z}},
\end{aligned}$$

and

$$z^{-1} = \frac{\bar{z}}{|z|^2}$$

if $z \neq 0$.

As for the absolute value on $\mathbb{R}$, we have:

Theorem 22.5. *The following are true for $z, w \in \mathbb{C}$:*

(i) $|z| \geq 0$ *with* $|z| = 0$ *if and only if* $z = 0$*;*

(ii) $|zw| = |z||w|$*;*

(iii) $|z + w| \leq |z| + |w|$.

Proof. (i) is clear.

For (ii), note that

$$|zw|^2 = zw\overline{zw} = (z\bar{z})(w\bar{w}) = |z|^2|w|^2,$$

and take roots.

(iii): First note that

$$\operatorname{Re} z\bar{w} \leq |z\bar{w}| = |z\bar{w}| = |z||\bar{w}| = |z||w|,$$

so that

$$\begin{aligned}
|z + w|^2 &= (z + w)\left(\overline{z + w}\right) \\
&= z\bar{z} + z\bar{w} + w\bar{z} + w\bar{w} \\
&= |z|^2 + z\bar{w} + \overline{z\bar{w}} + |w|^2 \\
&= |z|^2 + 2 \operatorname{Re} z\bar{w} + |w|^2 \\
&\leq |z|^2 + 2|z||w| + |w|^2 \\
&= (|z| + |w|)^2.
\end{aligned}$$

Taking roots again yields the claim. $\square$

Given the absolute value on $\mathbb{C}$, we can define convergence of sequences in $\mathbb{C}$:

Definition 22.6. Let $(z_n)_{n=1}^\infty$ be a sequence in $\mathbb{C}$. Then we say that $(z_n)_{n=1}^\infty$ *converges* to $z \in \mathbb{C}$ or is *convergent* to z if, for each $\epsilon > 0$, there is $n_\epsilon \in \mathbb{N}$ such that $|z_n - z| < \epsilon$ for all $n \in \mathbb{N}$ with $n \geq n_\epsilon$. We call z the *limit* of $(z_n)_{n=1}^\infty$.

Remarks. 1. We use the same notation for limits of complex sequences as for real ones.

2. The Limit Laws hold for complex sequences as for real ones.

Theorem 22.7. *Let $(z_n)_{n=1}^\infty$ be a sequence in $\mathbb{C}$. Then the following are equivalent:*

(i) *$(z_n)_{n=1}^\infty$ converges in $\mathbb{C}$;*

(ii) *$(\operatorname{Re} z_n)_{n=1}^\infty$ and $(\operatorname{Im} z_n)_{n=1}^\infty$ converge in $\mathbb{R}$.*

In this case, we have

$$\lim_{n\to\infty} z_n = \lim_{n\to\infty} \operatorname{Re} z_n + i \lim_{n\to\infty} \operatorname{Im} z_n.$$

Proof. (i) $\Longrightarrow$ (ii): Let $z \in \mathbb{C}$ be the limit of $(z_n)_{n=1}^\infty$. As

$$|\overline{z_n} - \bar{z}| = |\overline{z_n - z}| = |z_n - z|$$

for $n \in \mathbb{N}$, we see that $\lim_{n\to\infty} \overline{z_n} = \bar{z}$. The Limit Laws yield

$$\lim_{n\to\infty} \operatorname{Re} z_n = \lim_{n\to\infty} \frac{1}{2}\left(z_n + \overline{z_n}\right) = \frac{1}{2}(z + \bar{z}) = \operatorname{Re} z.$$

A similar argument yields $\lim_{n\to\infty} \operatorname{Im} z_n = \operatorname{Im} z$.

(ii) $\Longrightarrow$ (i): Let $x := \lim_{n\to\infty} \operatorname{Re} z_n$ and $y := \lim_{n\to\infty} \operatorname{Im} z_n$. The Limit Laws then yield that

$$\lim_{n\to\infty} z_n = \lim_{n\to\infty} \operatorname{Re} z_n + i \operatorname{Im} z_n = x + iy.$$

This completes the proof. $\square$

Definition 22.8. A sequence $(z_n)_{n=1}^\infty$ in $\mathbb{C}$ is called *bounded* if $(|z_n|)_{n=1}^\infty$ is bounded in $\mathbb{R}$.

Remark. It is easy to see that $(z_n)_{n=1}^\infty$ is bounded if and only if both $(\operatorname{Re} z_n)_{n=1}^\infty$ and $(\operatorname{Im} z_n)_{n=1}^\infty$ are bounded.

Corollary 22.9. *Every bounded sequence in $\mathbb{C}$ has a convergent subsequence.*

Proof. We cannot just copy the proof of the real case because it involves order arguments. Theorem 22.7, however, allows us to steer around this difficulty.

Let $(z_n)_{n=1}^\infty$ be a bounded sequence in $\mathbb{C}$. Then $(\operatorname{Re} z_n)_{n=1}^\infty$ is a bounded sequence in $\mathbb{R}$ and therefore has a convergent subsequence $(\operatorname{Re} z_{n_k})_{k=1}^\infty$. As $(\operatorname{Im} z_n)_{n=1}^\infty$ is bounded in $\mathbb{R}$, so is its subsequence $(\operatorname{Im} z_{n_k})_{k=1}^\infty$, which therefore has a convergent subsequence $(\operatorname{Im} z_{n_{k_\nu}})_{\nu=1}^\infty$. Then $(\operatorname{Re} z_{n_{k_\nu}})_{\nu=1}^\infty$ and $(\operatorname{Im} z_{n_{k_\nu}})_{\nu=1}^\infty$ are convergent sequences in $\mathbb{R}$, so that $(z_{n_{k_\nu}})_{\nu=1}^\infty$ converges in $\mathbb{C}$. $\square$

Of course, we can also define complex Cauchy sequences:

Definition 22.10. A sequence $(z_n)_{n=1}^\infty$ in $\mathbb{C}$ is called a *Cauchy sequence* if, for each $\epsilon > 0$, there is $n_\epsilon \in \mathbb{N}$ such that $|z_n - z_m| < \epsilon$ for all $n, m \geq n_\epsilon$.

As for real sequences, we have:

Theorem 22.11. *The following are equivalent for a sequence $(z_n)_{n=1}^\infty$ in $\mathbb{C}$:*

(i) $(z_n)_{n=1}^\infty$ *converges in $\mathbb{C}$;*

(ii) $(z_n)_{n=1}^\infty$ *is a Cauchy sequence.*

Proof. (i) $\Longrightarrow$ (ii): The proof for real sequences carries over verbatim.

(ii) $\Longrightarrow$ (i): As in the proof of Theorem 22.7, we see that $(\overline{z_n})_{n=1}^\infty$ is also a Cauchy sequence. It is easy to see that the sum of two Cauchy sequences and the product of a Cauchy sequence with a constant are again Cauchy sequences. As in the proof of Theorem 22.7, it then follows that $(\operatorname{Re} z_n)_{n=1}^\infty$ and $(\operatorname{Im} z_n)_{n=1}^\infty$ are Cauchy sequences and therefore convergent in $\mathbb{R}$. By Theorem 22.7, this means that $(z_n)_{n=1}^\infty$ converges in $\mathbb{C}$. $\square$

One can now go on and define:

- convergence and absolute convergence of infinite series in $\mathbb{C}$;
- limits of functions from subsets of $\mathbb{C}$ into $\mathbb{C}$;
- continuity of functions from subsets of $\mathbb{C}$ into $\mathbb{C}$.

In most cases, it is straightforward how the definitions from the real case can be adapted to the complex situation. Most of the results from Sections 6 to 16—as long as they do not involve the fact that $\mathbb{R}$ can be ordered—carry over. In particular, this is true for everything we proved for absolutely convergent series.

$\exp(z)$ for $z \in \mathbb{C}$

We defined

$$\exp(x) = \sum_{k=0}^\infty \frac{x^k}{k!}$$

for $x \in \mathbb{R}$: this was possible because the series converges absolutely by the Limit Ratio Test. The same argument shows that the series also converges if $x \in \mathbb{R}$ is replaced by some $z \in \mathbb{C}$.

Definition 22.12. The *exponential function* $\exp\colon \mathbb{C} \to \mathbb{C}$ is defined by letting

$$\exp(z) := \sum_{k=0}^\infty \frac{z^k}{k!}$$

for $z \in \mathbb{C}$.

It is straightforward that

$$\exp(\bar{z}) = \sum_{k=0}^{\infty} \frac{\bar{z}^k}{k!} = \overline{\sum_{k=0}^{\infty} \frac{z^k}{k!}} = \overline{\exp(z)}$$

for all $z \in \mathbb{N}$. The Cauchy Product Formula holds for absolutely convergent complex series as it holds over $\mathbb{R}$: the proof carries over verbatim. As a consequence, we obtain—as over $\mathbb{R}$—that

$$\exp(z + w) = \exp(z)\exp(w)$$

for $z, w \in \mathbb{C}$. As over $\mathbb{R}$, it follows that $\exp(\mathbb{C}) \subset \mathbb{C} \setminus \{0\}$. Furthermore, the proof of the continuity of exp carries over from $\mathbb{R}$ to $\mathbb{C}$. As over $\mathbb{R}$, we also write e^z instead of $\exp(z)$ for $z \in \mathbb{C}$.

Definition 22.13. *Define* $\cos, \sin \colon \mathbb{R} \to \mathbb{R}$ by letting

$$\cos x := \operatorname{Re} e^{ix} \qquad \text{and} \qquad \sin x := \operatorname{Im} e^{ix}$$

for $x \in \mathbb{R}$.

For the remainder of this section, we shall only work with *this* definition of cos and sin. It is immediate that

$$e^{ix} = \cos x + i \sin x$$

for $x \in \mathbb{R}$.

Properties.

1. cos and sin are continuous, even differentiable.

   This follows from Theorem 22.7.

2. For $x \in \mathbb{R}$, we obtain

$$(\cos x)^2 + (\sin x)^2 = |e^{ix}|^2 = e^{ix}\overline{e^{ix}} = e^{ix}e^{\overline{ix}} = e^{ix}e^{-ix} = e^0 = 1.$$

3. Let $x, y \in \mathbb{R}$. By definition,

$$e^{i(x+y)} = \cos(x + y) + i \sin(x + y)$$

   holds. On the other hand, the exponential law implies

$$\begin{aligned} e^{i(x+y)} &= e^{ix}e^{iy} \\ &= (\cos x + i \sin x)(\cos y + i \sin y) \\ &= (\cos x)(\cos y) - (\sin x)(\sin y) + i((\sin x)(\cos y) + (\cos x)(\sin y)). \end{aligned}$$

   Comparing the two expressions, we obtain

$$\cos(x + y) = (\cos x)(\cos y) - (\sin x)(\sin y) \qquad \text{and} \qquad \sin(x + y) = (\sin x)(\cos y) + (\cos x)(\sin y),$$

   i.e., the *addition formulae* for cos and sin.

4. It is obvious that
$$i^k = \begin{cases} 1, & k \equiv 0 \mod 4, \\ i, & k \equiv 1 \mod 4, \\ -1, & k \equiv 2 \mod 4, \\ -i, & k \equiv 3 \mod 4, \end{cases}$$
for $k \in \mathbb{N}_0$. It follows that
$$\begin{aligned} e^{ix} &= \sum_{k=0}^{\infty} \frac{(ix)^k}{k!} \\ &= \sum_{\substack{k=0 \\ k \text{ even}}}^{\infty} \frac{(ix)^k}{k!} + \sum_{\substack{k=0 \\ k \text{ odd}}}^{\infty} \frac{(ix)^k}{k!} \\ &= \sum_{k=0}^{\infty} \frac{(ix)^{2k}}{(2k)!} + \sum_{k=0}^{\infty} \frac{(ix)^{2k+1}}{(2k+1)!} \\ &= \underbrace{\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}}_{=\cos x} + i \underbrace{\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}}_{=\sin x} \end{aligned}$$
for $x \in \mathbb{R}$, i.e.,
$$\cos x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} \qquad \text{and} \qquad \sin x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}.$$

5. We saw that $\frac{d}{dx} e^x = e^x$ for $x \in \mathbb{R}$; in a similar way, one sees that
$$\frac{d}{dx} e^{ix} = i e^{ix}.$$
It follows that
$$\frac{d}{dx} \cos x + i \frac{d}{dx} \sin x = \frac{d}{dx} e^{ix} = i e^{ix} = -\sin x + i \cos x,$$
so that
$$\cos' x = -\sin x \qquad \text{and} \qquad \sin' x = \cos x$$
for $x \in \mathbb{R}$.

What is π?

It is obvious that $\cos 0 = 1$.

Claim.
$$\cos 2 \leq -\frac{1}{3}.$$

Proof. Apply Taylor's Theorem with $f = \cos$, $x_0 = 0$, $x = 2$, and $n = 3$: there is $\xi \in (0,2)$ such that

$$\cos 2 = 1 - 2 + \frac{\cos \xi}{24} 16.$$

It follows that

$$\cos 2 \leq 1 - 2 + \frac{16}{24} = -1 + \frac{2}{3} = -\frac{1}{3},$$

which proves the claim. $\square$

The Intermediate Value Theorem immediately yields that cos has a zero in $(0, 2)$.

Definition 22.14. Define $\pi \in \mathbb{R}$ as

$$\pi := 2 \inf\{x \geq 0 : \cos x = 0\}.$$

Claim. *π is well defined, strictly positive and less than 4, and* $\cos\left(\frac{\pi}{2}\right) = 0$. *In particular, $\frac{\pi}{2}$ is the least non-negative zero of* cos.

Proof. Let

$$S := \{x \geq 0 : \cos x = 0\}.$$

Clearly, S is bounded below and not empty. So, π is well defined. Let $(x_n)_{n=1}^{\infty}$ be a sequence in S such that $x_n \to \inf S$. As cos is continuous, it follows that

$$\cos\left(\frac{\pi}{2}\right) = \cos(\inf S) = \lim_{n\to\infty} \cos x_n = 0.$$

As $\cos 0 = 1$, and since cos has a zero in $(0, 2)$, it follows that $\frac{\pi}{2} \in (0, 2)$, i.e., $\pi \in (0, 4)$. $\square$

Claim (Values of e^{ix})**.** *The following are true:*

$$e^{i\frac{\pi}{2}} = i, \quad e^{i\pi} = -1, \quad e^{i\frac{3\pi}{2}} = -i, \qquad \textit{and} \qquad e^{2\pi i} = 1.$$

Proof. As $\cos\left(\frac{\pi}{2}\right) = 0$, we have

$$\left(\sin\left(\frac{\pi}{2}\right)\right)^2 = 1 - \left(\cos\left(\frac{\pi}{2}\right)\right)^2 = 1,$$

i.e., $\sin\left(\frac{\pi}{2}\right) = \pm 1$. Assume towards a contradiction that $\sin\left(\frac{\pi}{2}\right) = -1$. Then the Mean Value Theorem yields $\xi \in \left(0, \frac{\pi}{2}\right)$ such that

$$\cos \xi = \frac{\sin\left(\frac{\pi}{2}\right) - \sin 0}{\frac{\pi}{2}} = -\frac{2}{\pi} < 0,$$

so that cos has a zero in $\left(0, \frac{\pi}{2}\right)$ by the Intermediate Value Theorem. This contradicts the definition of π. It follows that $\sin\left(\frac{\pi}{2}\right) = 1$ and therefore

$$e^{i\frac{\pi}{2}} = \cos\left(\frac{\pi}{2}\right) + i \sin\left(\frac{\pi}{2}\right) = i.$$

As $e^{i\frac{n\pi}{2}} = i^n$ for $n \in \mathbb{N}$, the other claims follow. $\square$

In particular, *Euler's Identity* holds:

$$e^{i\pi} + 1 = 0.$$

We obtain the following table of values for cos and sin:

x	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$\cos x$	1	0	-1	0	1
$\sin x$	0	1	0	-1	0

Together with the addition formulae for cos and sin, this yields:

Claim. *The following are true for all $x \in \mathbb{R}$:*

$$\cos(x + 2\pi) = \cos x \quad \textit{and} \quad \sin(x + 2\pi) = \sin x,$$
$$\cos(x + \pi) = -\cos x \quad \textit{and} \quad \sin(x + \pi) = -\sin x,$$

and

$$\cos x = \sin\left(\frac{\pi}{2} - x\right) \quad \textit{and} \quad \sin x = \cos\left(\frac{\pi}{2} - x\right)$$

Claim.

$$\{x \in \mathbb{R} : \sin x = 0\} = \{n\pi : n \in \mathbb{Z}\}.$$

Proof. Clearly, $\cos(-x) = \cos x$ for $x \in \mathbb{R}$, so that $\cos x > 0$ for $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. As $\sin x = \cos\left(\frac{\pi}{2} - x\right)$, it follows that $\sin x > 0$ for $x \in (0, \pi)$, and since $\sin(x + \pi) = -\sin x$, we conclude that $\sin x < 0$ for all $x \in (\pi, 2\pi)$. Consequently, sin only has the zeroes 0 and π in $[0, 2\pi)$.

Let $x \in \mathbb{R}$ be such that $\sin x = 0$, and set $m := \left\lfloor \frac{x}{2\pi} \right\rfloor$, so that $x = 2m\pi + \xi$ with $\xi \in [0, 2\pi)$. It follows that

$$\sin \xi = \sin(x - 2m\pi) = \sin x = 0,$$

so that $\xi \in \{0, \pi\}$, i.e., $x = 2m\pi$ or $x = (2m + 1)\pi$. $\square$

Claim.

$$\{x \in \mathbb{R} : \cos x = 0\} = \left\{n\pi + \frac{\pi}{2} : n \in \mathbb{Z}\right\}.$$

Proof. This follows from the previous claim because $\cos x = -\sin\left(x - \frac{\pi}{2}\right)$ for all $x \in \mathbb{R}$. $\square$

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