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Gauge fields in homogeneous and inhomogeneous cosmologies

By
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to

my Mother

and

my Father.

Abstract

Despite its formidable appearance, the study of classical Yang-Mills (YM) fields on homogeneous cosmologies is amenable to a formal treatment. This dissertation is a report on a systematic approach to the general construction of invariant YM fields on homogeneous cosmologies undertaken for the first time in this context. This construction is subsequently followed by the investigation of the behavior of YM field variables for the most simple of self-gravitating YM fields.

Particularly interesting was a dynamical system analysis and the discovery of chaotic signature in the axially symmetric Bianchi I-YM cosmology. Homogeneous YM fields are well studied and are known to have chaotic properties. The chaotic behavior of YM field variables in homogeneous cosmologies might eventually lead to an invariant definition of chaos in (general) relativistic cosmological models.

By choosing the gauge fields to be Abelian, the construction and the field equations presented so far reduce to that of electromagnetic field in homogeneous cosmologies. A perturbative analysis of gravitationally interacting electromagnetic and scalar fields in inhomogeneous cosmologies is performed via the Hamilton-Jacobi formulation of general relativity. An essential feature of this analysis is the spatial gradient expansion of the generating functional (Hamilton principal function) to solve the Hamiltonian constraint. Perturbations of a spatially flat Friedman-Robertson-Walker cosmology with an exponential potential for the scalar field are presented.

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Σ or M .... the space-like hypersurfaces of simultaneity,
$\overset{(4)}{g}$ .... the four-metric of the space-time,
$\overset{(3)}{g}$ or g .... the (pullback) induced-metric on Σ,
$\otimes$ .... product of two vectors,
$\wedge$ .... exterior product,
$e_\perp, e_i$ .... the left-invariant frame field,
$\theta^\perp, \theta^i$ .... the left-invariant form field,
$c^i{}_{jk}$ .... the structure constants of the isometry Lie algebra,
$\xi_i, \bar{e}_a, \bar{e}_\Gamma$ .... Killing vector fields,
A, ... the pullback of the Yang-Mills connection to the base manfold,
ad .... the adjoint representation of a Lie group in its Lie algebra,
$\ltimes$, ... semi-direct product of Lie groups,
K_0 .... the isotropy Lie group,
K .... the isometry Lie group,
λ .... homomorphsims from the isotropy group into the gauge group,
$\mathfrak{k}_0$ .... the isotropy Lie algebra,
$\mathfrak{k}$ .... the isometry Lie algebra,
G .... the gauge group,
$\mathfrak{g}$ .... the gauge Lie algebra,
Λ_i^A .... linear maps from $\mathfrak{k}/\mathfrak{k}_0$ into the gauge Lie algebra,
$\mathcal{L}_{\xi_i}$ .... the Lie derivative with respect to the vector field ξ_i,
d .... exterior derivative,
D_μ .... gauge-covariant derivative,
$;$ or $|$ or ∇ .... covariant derivative on Σ,
K_{ij} .... the extrinsic curvature on Σ,

$\overset{(4)}{R}$ or R .... the curvature of the space-time,

$\overset{(3)}{R}$ .... the curvature of Σ,

P .... a principal fiber bundle,

$\tilde{\psi}_a$ .... an automorphism of P induced by the action of $a \in K$,

$:=$ .... a definition,

$\in$ .... a member of a group,

$\bar{\psi}_a$ .... the left action of $a \in K$ on M,

$\forall$ .... for all members of a set ,

$\exists$ .... there exists at least one member of the set,

: or | .... such that,

$\cong$ .... diffeomorphic,

R_g .... the right action of the gauge group G on the P,

π .... the projection map from P to Σ or M,

$\tilde{\omega}$ .... the invariant Yang-Mills connection on P,

$\tilde{\Omega}$ .... the invariant Yang-Mills curvature on P,

$\mathbf{e}$ .... the generators of $\mathfrak{k}$,

F^A_{ab} .... the field strength tensor for Yang-Mills potential A^A_a,

E^A_a .... the electric field of the Yang-Mills potential,

$B^A{}_{ab}$ or $F^A{}_{ab}$ .... the magnetic field of the Yang-Mills potential,

ϵ_{ijk} .... the completely antisymmetric tensor,

Λ^K_i .... a basis of the solution space of Wang's equation,

Φ_K .... a solution of Wang's equation,

$SO(n)$ .... the n-dimensional special orthogonal group,

$SU(n)$ .... the n-dimensioanl special unitary group,

$\mathfrak{so}(n)$ .... the Lie algebra of $SO(n)$,

$\mathfrak{su}(n)$ .... the Lie algebra of $SU(n)$,

$GL(n, R)$ .... the general real n-dimensional linear group,

$\mathfrak{gl}(n, R)$ .... the Lie algebra of $GL(n, R)$,

$\mathfrak{m}$ .... the tangent space to Σ,

π .... the momentum conjugate to g,

S .... a solution of the Hamilton-Jacobi equation,

$\mathcal{H}$ .... the Hamiltonian constraint,

$\mathcal{H}_\mu$, ... momentum constraints,

δ, ... functional derivative,

π^{ij}, ... the momentum conjugate to g_{ij},
π^{ϕ}, ... the momentum conjugate to ϕ,
$<,>$, ... invariant product on $\mathfrak{g}$,
$(,)$, ... invariant product on K.

Chapter 1

Prelude

One of the most remarkable achievements of modern theoretical physics has been the discovery and a precise formulation of the classical and quantum theory of gauge fields. Modern theoretical physics is an amalgamation of the various theories of gauge fields. These theories can be divided into two major groups. In one group there is a spin-two field, gravity, which is the characteristic interaction field of large scale structures in the universe, and in the other group there are spin-one Yang-Mills (YM) fields like electromagnetism or weak and strong interactions which are basically microscopic phenomena best described by a quantum theory of fields. A. Einstein presented general relativity in the language of Riemannian geometry. However, a geometric interpretation of the other gauge fields was put forward by C.N. Yang and R.L. Mills after their development [1]. The object of chief importance in YM theories is the YM connection (potential), whereas, in general relativity, all the relevant information regarding the structure of the space-time is encoded in the metric[1]. Many other dissimilarities have surfaced in trying to formulate a quantum theory of gravity.

Many attempts to reach a unified theory of gravity and YM fields have so far been unsuccessful. One such attempt was the formulation of Kaluza-Klein theories in which gravity, electromagnetism and the other gauge and scalar fields are the different manifestations of a unified theory in higher dimensions that undergoes the so-called process of dimensional reduction [2]. To achieve charge quantization,

[1]Ashtekar's formalism based on Palatini's formulation of gravity is an attempt to resurrect the role of connection to its perhaps rightful place.

the dimensional reduction is followed by the spatial compactification of these extra dimensions. In addition to the absence of any good reason as to why these extra dimensions have to compactify, a Kaluza-Klein approach to a unification of YM theories and gravity yields inconsistent results [3].

Despite the apparent lack of success, Kaluza-Klein theories resulted in many new ideas and concepts. Superstring theory is a progeny of these theories and many results from differential geometry used in this dissertation were obtained by the method of dimensional reduction.

Another viewpoint is that, rather than trying to cast either general relativity or YM theory in a form that resembles the other, one might try to investigate the properties of the systems that interact via both the YM and gravitational fields. The simplest of such models are self-gravitating source free homogeneous YM fields. Given that the gravitational interaction is many orders of magnitude weaker than the YM interactions, gravitationally interacting YM fields could only be physically relevant near the Planck regime when the universe was in its infancy. One might argue that because there is a fundamental length-scale associated with quark confinement, non-Abelian YM fields (weak and strong interactions) have no classical manifestation. Hence, a classical treatment of the YM fields can have no physical significance. However, the Hubble length is also a fundamental length scale present in the early universe that could feasibly distort the quark confinement.

Because of the symmetries present in the action

$$\mathcal{I} = \int \sqrt{\overset{(4)}{g}} \left(\overset{(4)}{R} - \tfrac{1}{4} F^{A}_{\mu\nu} F^{\mu\nu}_{A} \right) d^4x \tag{1.1}$$

associated with the gauge degrees of freedom, such fields are characterized by conserved charges (conserved flux). Therefore, one might argue that during the De Sitter expansion phase of the inflation such fields are diluted away so rapidly that they can never play any significant role in the subsequent evolution of the universe. Nevertheless, in the past few years, there have been some efforts to formulate a vector field driven inflation by adding gauge-breaking potentials to the action (1.1)[4]. Moreover, modified minimally coupled gravitational and YM fields could become important after inflation during the reheating and particle production. Of course, one might have to add source terms to (1.1) and treat the space-time metric as a background field.

This dissertation is mainly a presentation of the construction and study of the YM fields in homogeneous cosmologies (the so-called mini-superspace cosmologies). There is a Lie group of isometries associated with every homogeneous cosmology. The study of YM fields in homogeneous cosmologies involves devising a procedure to find the YM connections that are invariant under the action of the group of isometries. This procedure utilizes the theory of invariant connections in homogeneous spaces (the so-called Wang's theorem [5]) in the same spirit as that of Kaluza-Klein theories. The last part of the dissertation is a discussion of the gauge fields in inhomogeneous cosmologies for the simplest gauge field (electromagnetism), which employs the Einstein-Hamilton-Jacobi formulation of general relativity.

1.1. Overview of the dissertation

We now proceed with an overview of the dissertation. Throughout the dissertation, the signature of the four-metric $\overset{(4)}{g}$ and the three-metric $\overset{(3)}{g}$ are $+2$ and $+3$ respectively. The problem at hand is diverse and has many aspects. Consequently, the sign conventions can be very confusing. The sign conventions are consistent throughout. However, to serve clarity some of the more important ones are repeated in each chapter.

Chapter 2

This chapter is a modified version of Ref. [6]. In this chapter we make use of Wang's theorem and Wang's equations to construct the invariant YM connections in the space-time of Bianchi cosmologies ($\Sigma \times \mathbb{R}$ where Σ is a space-like three-manifold and $\mathbb{R}$ is the time) with the isometry Lie algebra $\mathfrak{k}$ and isotropy Lie subalgebra $\mathfrak{k}_0$. EYM equations are written in an invariant basis. In doing so we implicitly make the assumption that in a reductive decomposition of the isometry Lie algebra: $\mathfrak{k} = \mathfrak{k}_0 + \mathfrak{m}$ such that $\mathfrak{k}_0 \cap \mathfrak{m} = 0$ and $[\mathfrak{k}_0, \mathfrak{m}] \subset \mathfrak{m}$ where $\mathfrak{m}$ is the tangent space of Σ, Σ is a group manifold. In other words, the space-like hypersurfaces of simultaneity admit a global frame field. Therefore, the derived equations of motion are valid for all spatially homogeneous cosmologies except Kantowski-Sachs and locally rotationally symmetric (LRS) Bianchi III models. However, this is not a severe restriction because the construction of the equations of motion is completely local and can be generalized to the equations of motion for non-reductive and non-parallelizable models . Indeed, the equations of motion given in chapter 4 include the LRS Bianchi III and Kantowski-Sachs cosmologies. If the bundle of linear frames on Σ is non-trivial, Σ is non-parallelizable. The triviality of a fiber bundle on Σ is sensitive to the topology of Σ. It is known that the non-trivial topology of Σ affects the number of degrees of freedom of a mini-superspace cosmology. In this dissertation, we implicitly make the assumption that Σ is simply connected. However, for a discussion of the role of topology on the number of degrees of freedom and equations of motion see Sect. 2.2.1.

We derive the Einstein-YM (EYM) equations for general Bianchi and axially symmetric Bianchi-I YM cosmology by the ADM reduction of the EYM equations. The system of ordinary differential equations describes the motion of a point particle in a certain potential well whose exact form depends on the type of the Bianchi cosmology. Derivation of the EYM equations is followed by a dynamical system analysis of the axially symmetric Bianchi I-YM cosmology. The potential well for this system has moving walls with two open channels (see Fig. 2.4) that are reminiscent of the open channels of mixmaster cosmologies. Mixmaster cosmologies are known to have chaotic properties [7].

An attractive property of Bianchi I cosmologies is that the space-like hypersurfaces of simultaneity are flat. Spatial flatness greatly simplifies the equations of motion. This means that the true dynamical variable is the time derivative of the metric (extrinsic curvature) not the metric itself. Therefore, many essential features of the axially symmetric Bianchi I YM cosmology, we suspected, must resemble the behavior of axially symmetric homogeneous YM fields in flat space. Indeed the dynamics of homogeneous YM fields, dubbed YM classical mechanics, have similar features and exhibit chaotic properties.

We numerically computed the Liapunov exponent for axially symmetric homogeneous YM fields in flat space, axially symmetric Bianchi I cosmology in synchronous time, and axially symmetric Bianchi I cosmology in conformal time. The numerical computations indicated that the Liapunov exponent is non-vanishing in the flat space model and in axially symmetric Bianchi I cosmology in conformal time and is vanishing in the latter model in the synchronous time (see Fig. 2.5). The apparent discrepancy between the two different time parameterizations is not alarming since the Liapunov exponent is known to be sensitive to time reparametrizations. However, after observing the apparent similarities between the behavior of the YM field variables in the flat space and Bianchi I cosmology, we concluded that the Bianchi I model has chaotic properties and further conjectured that the chaotic properties are milder than the flat space model. Indeed very recently, J.D. Barrow and J. Levin have used our equations of motion for axially symmetric Bianchi I-YM cosmology and the method of fractal basins to show that this system does have chaotic properties[8].

Chapter 3

This chapter is a modified version of [9]. In this chapter we again employ Wang's theorem to obtain the EYM equations for all four-dimensionally homogeneous space-times (manifolds) (e.g. de Sitter cosmologies) and space-times that are only spatially homogeneous (e.g. Bianchi cosmologies) and are either spherically symmetric or have local rotational symmetry. One important difference between the construction of EYM equations in this chapter and the previous chapter is that the EYM equations are constructed not on the (left)-invariant basis but on the basis of Killing vector fields. Consequently, the (left)-invariant quantities no longer have constant components in this basis and the field equations are constructed at one point only and are carried over all the homogeneous manifold by the action of the group.

The use of Wang's theorem in this context in not completely new[10], [11]. Wang's theorem has been used in the past to formulate the EYM systems in Friedmann-Robertson-Walker (FRW) cosmologies. Nevertheless, our approach has several advantages and is new in that:

1) We convert Wang's equation into a very simple linear algebraic equation

$$\Lambda[\bar{e}_i, \bar{e}_\Gamma] = [\Lambda(\bar{e}_i), \lambda(\bar{e}_\Gamma)], \tag{1.2}$$

in which $\bar{e}_i$ is a Killing vector field, $\bar{e}_\Gamma$ is a Killing vector field that generates the isotropy subalgebra, and λ is a homomorphism from the isotropy group into the gauge group.

2) We fully utilize the theory of principal fiber bundles on homogeneous manifolds to derive the invariant connections for the YM fields and the invariant

curvature of the geometry.

3) Of course, many invariant connections on the homogeneous space are related to each other by gauge transformations. Wang's theorem states that the invariant connections on a principal fiber bundle are in one-to-one correspondence with the conjugacy classes of homomorphisms. For the first time we relate the problem of finding such conjugacy classes of homomorphisms in this context to the representation theory of compact Lie groups (e.g. $SU(2)$ and $U(1)$). We present a systematic method to obtain the invariant $SU(n)$ and $SO(n)$ YM connections in FRW and LRS Bianchi and Kantowski-Sachs cosmologies. The field equations for FRW-$SU(n)$-YM for simplest of representations are solved and the field equations for locally rotationally symmetric (LRS) models are given.

Chapter 4

Chapter 4 is mainly based on [12]. This is an effort to use the Hamilton-Jacobi formulation of general relativity to derive a perturbative solution to the Hamiltonian constraint for inhomogeneous gravitationally interacting electromagnetic and scalar fields. The author starts with the action for minimally coupled electromagnetic and scalar fields. The ADM reduction of the action is followed by a Legendre transformation for fully constrained systems that leads to

$$\text{Hamiltonian} = \int (N^\mu \mathcal{H}_\mu + A_0 \mathcal{G})\, d^3x \tag{1.3}$$

where N, N^i, A_0 (the lapse, shift, and the temporal component of the electromagnetic field, respectively) act as Lagrange multipliers and are the "gauge-evolvers". $\mathcal{H} = 0$, $\mathcal{H}^i = 0$ and $\mathcal{G} = 0$ are the Hamiltonian, the momentum and the Gauss law constraints. The Hamilton-Jacobi formulation of general relativity proceeds similarly to that of the Hamilton-Jacobi theory in classical mechanics where the idea is finding S, the Hamilton principal function. S is the generating function of a canonical transformation in which the old Hamiltonian H and the new Hamiltonian H' are related by

$$H' = H + \frac{\partial S}{\partial t} \tag{1.4}$$

such that $H' = 0$. This canonical transformation is the solution of the equations of motion. The (Einstein-)Hamilton-Jacobi theory for general relativity proceeds similarly to that of the Hamilton-Jacobi theory in classical mechanics. However, there are two major differences between these two theories. First, since we are dealing with fields rather than particles, partial derivatives and partial differential equations are replaced by functional derivatives and functional differential equations. This means that there is a partial differential equation for every spatial point and therefore, one has to deal with integrability conditions. Secondly and more importantly, since the Hamiltonian $\equiv 0$ on the constraint surface, the canonical transformation generated by S is a "gauge" transformation[2]. After a particular choice of the space-like hypersurfaces of simultaneity, this "gauge" transformation manifests itself in the time evolution of the three-geometry.

Diffeomorphism and gauge invariance of S guarantees that S satisfies the momentum and gauss law constraints, respectively. To solve the Hamiltonian constraint, the author uses the spatial gradient expansion of the generating functional recently developed by Parry, Salopek, and Stewart. The spatial gradient expansion gives rise to an order-by-order solution to the Hamiltonian constraint [13]. A conformal transformation and functional integral are used to derive the generating functional up to the terms of fourth order in spatial gradients. The integrability at each order is guaranteed by the lower order generating functionals satisfying the momentum constraints. The perturbations of a spatially flat FRW cosmology with a scalar field are given up to second order in spatial gradients. The application of this formalism is demonstrated in the specific example of the exponential potential $V = V_0 \exp\{-\sqrt{\frac{2}{p}}\phi\}$.

Chapter 5

Finally, this chapter is a summary of our work and possible future directions.

[2]In this context, gauge loosely refers to the diffeomorphisms of the four-geometry. Throughout this dissertation, we preserve the word gauge only for gauge transformation associated with either electromagnetism for YM fields.

Chapter 2

Axially Symmetric Bianchi I Yang-Mills Cosmology as a Dynamical System

2.1. Introduction

The effects of anisotropy on the dynamics of the early universe have been a point of interest to cosmologists from time to time. This interest stems from the fact that by adding more degrees of freedom to any isotropic minisuperspace model one might hope to gain a better understanding of the behavior of the model generalized to the full superspace. Bianchi cosmologies with fluid sources are such models. The matter in these models is either a perfect fluid [14] or consists of massive or massless vector fields [15],[4].

There has also been interest in the study of homogeneous source-free Yang-Mills fields as a dynamical system in the hope that a non-perturbative treatment might yield a better understanding of the vacuum state in YM theories, despite the fact that strong and weak interactions have no classical counterpart. The theory of these finite dimensional dynamical systems is dubbed Yang-Mills classical mechanics [16]. Similarly a non-perturbative mini-superspace Einstein-Yang-Mills (EYM) theory might eventually result in a better understanding of the vacuum state of YM fields in the Planck regime. EYM cosmology is not new. There has been extensive work on various Friedmann-Robertson-Walker (FRW) cosmologies with a YM field source that has a stress-energy tensor of the form of a tracefree perfect fluid [17],[18],[19],[20],[10].

In this chapter our aim is to relax the requirement of full isotropy. After adopting and refining a general scheme developed to construct YM fields on homogeneous spaces, we examine, as a specific model, the dynamical properties of the EYM equations in *axially symmetric* Bianchi I cosmologies with an $SU(2)$-YM field. The organization is as follows: In section 2.2 after introducing the basic notation, we give a brief account of how invariant YM fields in Bianchi cosmologies with a given isometry group are constructed. This involves gauge fixing for both the space-time metric and the YM connection. The general field equations for invariant YM fields in Bianchi cosmologies are given in section 2.3. Then we use these equations to derive the evolution equations for axially symmetric YM fields in a Bianchi I cosmology followed by a brief review of how these equations are related to the known exact solution of axially symmetric electromagnetic fields in Bianchi I cosmologies and $SU(2)$-YM fields in FRW cosmologies. Section 2.5 contains a numerical analysis of the obtained EYM equations as a dynamical system, computation of the Liapunov exponent and a comparison with the flat space behavior. It is shown that, surprisingly, in synchronous time the EYM system obtained has substantially milder stochastic properties than the corresponding flat space system. In conformal time, the Liapunov exponent is non-vanishing and the dynamical system is numerically less stable.

2.2. Invariant fields in Bianchi cosmologies

2.2.1. Invariant metrics in Bianchi cosmologies

We consider Bianchi cosmologies where the space-time manifold is of the form $\mathbb{R} \times \Sigma$ with a metric that admits an isometry group whose orbits are the space sections $\Sigma_t = \{t\} \times \Sigma$ where Σ is a three-dimensional group manifold with a (t-dependent) invariant metric g. (This excludes the so-called Kantowski-Sachs solutions where Σ is not a group but only a homogeneous Riemannian manifold.) The space-time metric can then always be written in the synchronous form [21]

$$\overset{(4)}{g} = -\theta^{\perp} \otimes \theta^{\perp} + g = -\theta^{\perp} \otimes \theta^{\perp} + \overset{(3)}{g}_{ij}(t)\theta^i \otimes \theta^j, \tag{2.1}$$

where the lapse and shift satisfy the relations $N = 1$, $N^i = 0$ respectively, the θ^i ($i = 1,2,3$) are the components of the (left-invariant) Maurer-Cartan form on the group Σ and $\theta^\perp = dt$. If $\{e_\perp, e_i\}$ is the (left-invariant) frame field dual to $\{\theta^\perp, \theta^i\}$ then the right-invariant vector fields ξ_i are Killing vector fields of g (and $\overset{(4)}{g}$), and they commute with the e_j since $\mathcal{L}_{\xi_i} e_j = [\xi_i, e_j]$.

The question of the most general form of the tensor fields on Σ invariant under certain group actions is extensively addressed in [22]. Here we only briefly discuss a special case. We assume that the space-time admits a four-dimensional isometry group K whose orbits are the Σ_t so that there is a one-dimensional isotropy subgroup K_0 at each point. These so-called locally rotationally symmetric (LRS) cosmologies have been all been classified (see, for example, [23]). The isotropy group K_0 is then necessarily isomorphic to $U(1)$ (as a one-dimensional subgroup of $SO(3)$) and the metric $\overset{(3)}{g}(t)$ can in all cases be chosen diagonal with two equal entries,

$$(g_{ab}(t)) = \mathrm{diag}(b_1, b_1, b_3), \tag{2.2}$$

say. In all cases but one (Bianchi III) it turns out that the action of $U(1)$ on Σ is an automorphism (a homomorphism of a Lie group into itself) of the group Σ that leaves the metric at the identity invariant from which it follows that K is a semi-direct product of $U(1)$ with Σ (i.e. $K = \Sigma \ltimes U(1)$). The generator ξ_ϕ of the isotropy group at the identity then acts as an infinitesimal orthogonal transformation, and it follows that the commutation relations are

$$[\xi_i, \xi_j] = -c^r{}_{ij}\xi_r, \quad [\xi_i, \xi_\phi] = c^r{}_{i\phi}\xi_r \tag{2.3}$$

where explicitly

$$[\xi_1, \xi_\phi] = -\xi_2, \quad [\xi_2, \xi_\phi] = \xi_1, \quad [\xi_3, \xi_\phi] = 0. \tag{2.4}$$

Let us denote by $g_{ij}(t_0)$ the three-metric in (2.1). It is known that not all different metrics $g_{ij}(t_0)$ lead to different three-geometries. All the different invariant metrics corresponding to the same geometry are related to each other by those diffeomorphisms of Σ that leave the group structure and the identity of Σ intact. By definition, such diffeomorphisms are the automorphisms ($\mathrm{Aut}(\Sigma)$) of Σ. The induced action of these automorphisms on the Lie algebra of Σ, leaves the structure constants of the Lie algebra invariant. One can use these automorphisms to cast $g_{ij}(t_0)$ in a canonical form. In the canonical form, $g_{ij}(t_0)$ has the fewest number of variables; usually $g_{ij}(t_0)$ is in the diagonal form. These variables are the true dynamical degrees of freedom. However, depending on the type of Bianchi cosmology and the nature of the matter field present, one might have to introduce time-dependent automorphisms (inner automorphisms) to keep $g_{ij}(t)$ diagonal in the time evolution of Σ. Such time-dependent automorphisms are generated by the shift vectors. For example, in a Bianchi IX cosmology where $\Sigma = \text{Aut}(\Sigma) = SU(2)$, on purely kinematical grounds (i.e. without using the evolution equations) one can show that N^i generate spatial rotations that can keep $g_{ij}(t)$ diagonal regardless of the nature of the matter field present.

The construction of the EYM equations as given in this chapter and chapter 3 is completely local. However, $\text{Aut}(\Sigma)$ depends on the topology of Σ, simply because under the identifications of $\tilde{\Sigma}$, the universal covering group of Σ, $\text{Aut}(\Sigma)$ might no longer leave the identity of Σ invariant. Therefore, $\text{Aut}(\Sigma)$ becomes smaller. Consequently, the number of the dynamical degrees of freedom is altered by the change in the topology. Heuristically, upon spatial identifications, some degrees of freedom associated with $\text{Aut}(\tilde{\Sigma})$ become dynamical. More precisely, these extra degrees of freedom are Teichmüller parameters of Σ[24]. Teichmüller parameters parametrize the space of equivalence classes of the isomorphic imbeddings of $\pi_1(\Sigma)$ (the fundamental group of Σ into the group of isometries of Σ.

2.2.2. Invariant YM fields in Bianchi cosmologies

The question of invariance of the YM connection (or potential) in homogeneous spaces was addressed by Harnad et al. [25]. We give a short account of how such invariant connections are constructed in the case of Bianchi cosmologies. The complication is that an invariant YM connection is not necessarily constant in a left-invariant frame (just as a Riemannian connection depending on the coordinate system does not necessarily vanish in Euclidean flat space), but any change in the field variables is merely due to a gauge transformation. However, it can be shown that for the YM potential

$$\overset{(4)}{A} = A_\perp(x,t)\theta^\perp + A \tag{2.5}$$

in which $A = A_i(x,t)\theta^i$ is the connection form on the homogeneous 3-space and $A_\perp(x,t)$ is a Lie algebra-valued scalar, there is always a gauge such that the A_i are only functions of time in a left invariant frame $\{\theta^i\}$ provided the 3-space is a group manifold (otherwise the θ_i are the pull back of Maurer-Cartan form components from the manifold of the isometry group to the homogeneous space). This requires that all the Lie-algebra-valued fields that transform according to the adjoint representation (e.g. $A_\perp$) have constant components in a left-invariant frame. Therefore the YM connection (2.5) reduces to

$$\overset{(4)}{A} = A_\perp(t)\theta^\perp + A_i(t)\theta^i. \tag{2.6}$$

Several important facts should be mentioned regarding the YM connection constructed so far.

(1) There is no local gauge freedom left in A_i and the remaining gauge freedom is global, i.e. only transformations of the form

$$(A_i^B) \mapsto \gamma(t)(A_i^B)\gamma^{-1}(t), \quad (\gamma \in G) \tag{2.7}$$

are allowed. (Here we have written $A_i = A_i^B \mathbf{E}_B$ where $\{\mathbf{E}_A\}$ is a basis of the Lie algebra $\mathfrak{g}$ of the gauge group G.

(2) With $A_\perp(t) \neq 0$, one can use the global gauge freedom above to make A_i^A upper-triangular. For $SU(2)$ gauge group, the remaining six variables represent the dynamical degrees of freedom of the $SU(2)$-YM field.

(3) If an additional Killing vector field ξ_ϕ generates an isotropy group $U(1)$, it has a non-trivial action on the tangent space in view of the commutation relations (2.3)/(2.4). The invariance of the YM connection requires the induced action of ξ_ϕ on the cotangent space and on the A_i to be equivalent to a gauge transformation.

(4)The gauge transformations mentioned in (2) are automorphisms of the gauge group G which in the light of (3), are sensitive to the toplogy of Σ. Therefore, similar to g_{ij}, the dynamical degrees of freedom in A_i can also be altered by the non-trivial topology of Σ.

To classify the possible K-invariant gauge fields systematically the following approach is needed [25],[5]. The equivalence classes of K-principal bundles P over Σ (where K is a Lie group that acts on P and acts via its projection by isometries on Σ) are in one-to-one correspondence with conjugacy classes of homomorphisms of the isotropy group K_0 ($= U(1)$ in our case) into the gauge group G (see [25]). These equivalence classes are well known from the investigations of spherically symmetric EYM-fields ([26],[27],[28]) and are for $K = U(1) \ltimes \Sigma$ and $G = SU(2)$ classified by (nonnegative) integers n such that the n-th equivalence class is represented, for example, by

$$\lambda : U(1) \to SU(2),\ e^{i\phi} \mapsto e^{n\phi\tau_3} \tag{2.8}$$

where $\tau_B = -i\sigma_B/2 \in \mathfrak{su}(2)$ for $B = 1,2,3$ are used as a basis of $\mathfrak{su}(2)$. On the other hand Wang's theorem (cf.[5]) states that there is a one-to-one correspondence between the K-invariant G-connections on P and linear maps $\Lambda : \mathfrak{k} \to \mathfrak{g}$ such that

$$\Lambda \circ \mathrm{ad}_z = \mathrm{ad}_{\lambda(z)} \circ \Lambda \qquad \forall z \in K_0. \tag{2.9}$$

and the connection components at the group identity can be chosen such that $A_i = \Lambda(\xi_i)$.

Infinitesimally (2.9) means in our case ($K = U(1) \ltimes \Sigma$, $K_0 = U(1)$, $G = SU(2)$) that $\lambda_*(\xi_\phi) = n\tau_3$ and

$$\Lambda([\xi_i, \xi_\phi]) = n[\Lambda_i, \tau_3] \tag{2.10}$$

where we have put $\Lambda_i = \Lambda(\xi_i) = \Lambda^B{}_i\tau_B$. Solving (2.10), which becomes more explicitly

$$\Lambda^A{}_r c^r_{i\phi} = n\epsilon^A{}_{B3}\Lambda^B{}_i \tag{2.11}$$

with $c^r_{i\phi}$ as in (2.3) and (2.4), gives for $n = 0$

$$\Lambda = \begin{pmatrix} 0 & 0 & \delta \\ 0 & 0 & \varepsilon \\ 0 & 0 & \gamma \end{pmatrix}, \tag{2.12}$$

for $n = 1$

$$\Lambda = \begin{pmatrix} \alpha & \beta & 0 \\ -\beta & \alpha & 0 \\ 0 & 0 & \gamma \end{pmatrix}, \tag{2.13}$$

and for $n > 1$

$$\Lambda = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \gamma \end{pmatrix}. \tag{2.14}$$

When these parameters α, β, γ, δ, ε are given as functions of t the YM connection is determined uniquely.

2.3. Field equations for axially symmetric YM fields in Bianchi cosmology

We shall use units where 8π(Newton's constant) = (speed of light) = (YM coupling constant) = 1. The Yang-Mills field determined by $\Lambda(t)$ in the gauge $A_\perp = 0$ is then

$$F^A = E^A_a \theta^\perp \wedge \theta^a + \tfrac{1}{2} B^A_{ab} \theta^a \wedge \theta^b, \tag{2.15}$$
$$E^A_a = \dot{\Lambda}^A_a, \tag{2.16}$$
$$B^A_{ab} = \epsilon^A{}_{BC} \Lambda^B_a \Lambda^C_b - \Lambda^A_r c^r_{ab}. \tag{2.17}$$

where $\dot{} = \mathcal{L}_{e_\perp}$ and the Lie algebra indices are raised and lowered with $(\delta_{ab}) = \mathrm{diag}(1,1,1) = (\delta_{AB})$ (the latter representing the invariant metric on $su(2)$). If $(e_\perp)^\mu (e_\perp)_\mu = -1$, $[e_\perp, e_i] = 0$, the ADM reduction of The Einstein-Yang-Mills equations is achieved by noting that

$$\overset{(4)}{\Gamma}{}^\perp{}_{\perp\perp} = \overset{(4)}{\Gamma}{}^b{}_{\perp\perp} = \overset{(4)}{\Gamma}{}^\perp{}_{\perp b} = \overset{(4)}{\Gamma}{}^\perp{}_{b\perp} = 0, \; \overset{(4)}{\Gamma}{}^a{}_{\perp b} = \overset{(4)}{\Gamma}{}^a{}_{b\perp} = K^a_b, \tag{2.18}$$
$$\overset{(4)}{\Gamma}{}^\perp{}_{ab} = K_{ab}, \; \overset{(4)}{\Gamma}{}^a{}_{bc} = \overset{(3)}{\Gamma}{}^a{}_{bc} = \tfrac{1}{2}(c^a{}_{bc} + c_b{}^a{}_c + c_c{}^a{}_b), \tag{2.19}$$
$$\nabla_i e_j = \overset{(3)}{\Gamma}{}^k{}_{ij} e_k \tag{2.20}$$

in which $K_{ab} = \frac{1}{2}\dot{g}_{ab}$. The YM equations are

$$D^\mu F^A_{\perp\mu} = \epsilon^A{}_{CD} \Lambda^{Ci} E^D_i + c^r{}_{rs} E^{As} = 0, \qquad \text{(YM constraint)} \tag{2.21}$$
$$D^\mu F^A_{i\mu} = \dot{E}^A_i + K^r_r E^A_i - 2K^j_i E^A_j - c^s{}_{rs} B^{Ar}_i - \tfrac{1}{2} g_{ir} c^r{}_{pq} B^{Apq} + \epsilon^A{}_{BC} \Lambda^{Br} B^C{}_{ir} = 0, \tag{2.22}$$

where D_μ is the four-dimensional-gauge-covariant derivative $D := \nabla \;\; + [A\wedge \;\;]$ and ∇ is the covariant derivative on Σ. Since

$$T_{\mu\nu} = F^A{}_{\mu\alpha} F_{A\nu}{}^\alpha - \frac{1}{4} g_{\mu\nu} F^A_{\rho\sigma} F^{\rho\sigma}_A \tag{2.23}$$

the Einstein equations become

$$(K^r_r)^2 - K^{rs}K_{rs} + \overset{(3)}{R} = E^2 + B^2 \quad \text{(scalar constraint)}, \tag{2.24}$$
$$K^r_i c^s_{rs} + K^r_s c^s_{ri} = B^A_{ir} E^r_A, \quad \text{(momentum constraint)} \tag{2.25}$$
$$\dot{K}_{ij} - 2K^r_i K_{rj} + K^r_r K_{ij} + \overset{(3)}{R}_{ij} = B^A{}_{ir} B_{Aj}{}^r - E^A_i E_{Aj} + \tfrac{1}{2} g_{ij}(E^2 - B^2), \tag{2.26}$$

where $E^2 = E^A_r E^r_A$ and $B^2 = \frac{1}{2} B^A_{rs} B^{rs}_A$ and

$$\overset{(3)}{R}_{ij} = c_{ikl} c_j^{\ kl}/4 - c^k_{\ kl}\left(c_{ij}^{\ \ l} + c_{ji}^{\ \ l}\right)/2 - c^l_{\ kj} c_l^{\ k}{}_i/2 - c^l_{\ kj} c^k_{\ li}/2. \tag{2.27}$$

For the metric (2.2) the YM constraint is trivially satisfied if Λ has the form (2.14). For the form (2.12) it yields

$$\dot{\delta}\varepsilon - \delta\dot{\varepsilon} = \dot{\varepsilon}\gamma - \varepsilon\dot{\gamma} = \dot{\delta}\gamma - \delta\dot{\gamma} = 0 \Rightarrow (\text{const.})\varepsilon = (\text{const.})\delta = \gamma. \tag{2.28}$$

The YM constraint for (2.13) yields

$$\dot{\alpha}\beta - \alpha\dot{\beta} = 0 \Rightarrow \alpha = (\text{const.})\beta. \tag{2.29}$$

The above equations show that after a time independent gauge transformation, (2.12) and (2.13) can be written in the following respective forms

$$\Lambda = \text{diag}(0, 0, \gamma), \tag{2.30}$$

$$\Lambda = \text{diag}(\alpha, \alpha, \gamma). \tag{2.31}$$

Therefore modulo a gauge transformation, (2.31) is the most general form of an invariant $SU(2)$-YM connection in Bianchi cosmologies with a fourth Killing vector field obeying the commutation relations (2.4). With this choice of the connection and the metric (2.2) inserting $c^k_{\ ij} = 0$ implies $\overset{(3)}{R}_{ij} = 0$, i.e. the 3-space for Bianchi I models is flat.

The evolution equations for axially symmetric YM fields in a Bianchi I cosmology (Bianchi I-EYM) are now

$$\ddot{\alpha} + \frac{\dot{\alpha}\dot{b}_3}{2b_3} + \alpha\left(\frac{\gamma^2}{b_3} + \frac{\alpha^2}{b_1}\right) = 0, \tag{2.32}$$

$$\ddot{\gamma} + \dot{\gamma}\left(\frac{\dot{b}_1}{b_1} - \frac{\dot{b}_3}{2b_3}\right) + \frac{2\alpha^2\gamma}{b_1} = 0, \tag{2.33}$$

$$\frac{\alpha^2}{b_1}\left(\frac{\alpha^2}{2b_1} + \frac{\gamma^2}{b_3}\right) + \frac{\dot{\alpha}^2}{b_1} + \frac{\dot{\gamma}^2}{2b_3} = \frac{\dot{b}_1\dot{b}_3}{2b_1b_3} + \frac{\dot{b}_1^2}{4b_1^2}, \tag{2.34}$$

$$\frac{\alpha^4}{b_1^2} + \frac{\dot{\gamma}^2}{b_3} = \frac{\ddot{b}_1}{b_1} + \frac{\dot{b}_1\dot{b}_3}{2b_1b_3}, \tag{2.35}$$

$$-\frac{\alpha^4}{b_1^2} - \frac{\dot{\gamma}^2}{b_3} = \frac{\ddot{b}_3}{2b_3} - \frac{1}{4}\left(\frac{\dot{b}_3^2}{b_3^2} + \frac{\dot{b}_1^2}{b_1^2}\right), \tag{2.36}$$

in which (2.32),(2.33),(2.35) and (2.36) are the dynamical equations, (2.34) is the scalar constraint and $\dot{} = \frac{d}{dt}$ (where is t is the synchronous time).

We consider first two special cases.

Electromagnetism: With $\alpha = 0$ the case (2.13) reduces to (2.14). With $(d/dt) = (\sqrt{b_3}/b_1)(d/d\tau)$, the general solution to the YM equations is $\gamma = c_1\tau$. Subtracting (2.34) from (2.35) and adding (2.34) to (2.36), respectively, with a time reparametrization $(d/dt) = \sqrt{b_3}(d/d\tau')$ gives the solution

$$b_1 = (c_0\tau' + \sqrt{B_0})^2, \tag{2.37}$$

$$b_3 = \frac{2A_0}{c_0\tau' + \sqrt{B_0}} - \frac{A_0^2}{(c_0\tau' + \sqrt{B_0})^2} \tag{2.38}$$

in which c_0, c_1, B_0 and A_0 are the integration constants. This solution is equivalent to the known solution of the Einstein-Maxwell equations for an electromagnetic field in an axially symmetric Bianchi I universe [29]. The energy-momentum tensor in an orthonormal frame is $(T^\mu_\nu) = \mathrm{diag}(-\rho, \rho, \rho, -\rho)$ in which ρ is the matter energy density. Heuristically, the positive principal pressures in directions 1-2 and negative pressure in direction 3 explain why such a universe evolves as equations (2.37)and (2.38) indicate. During any expansion in direction 3 energy is transferred from the gravitational field to the EM field whereas in any expansion in the 1-2 directions, energy is transferred from the EM field to the gravitational field. However there is no potential energy associated with the gravitational field. Therefore there is an expansion in 1-2 directions and any expansion in direction 3 can not be sustained for a long time. In this model the Ricci tensor uniquely determines the EM field tensor up to a constant duality transformation.

Isotropic case: Imposing spherical symmetry such that $K_0 = SU(2)$ requires $\alpha = \gamma$ and $b_1 = b_3$ in which case the EYM equations reduce to those for a $SU(2)$-YM field in a FRW cosmology. In conformal time the EYM equations are given in [19]. The solution for the YM field variables is given by elliptic integrals. The energy-momentum tensor is that of a radiation perfect fluid with energy-momentum tensor $(T^\mu_\nu) = \mathrm{diag}(-\rho, \rho/3, \rho/3, \rho/3)$ and the geometry is that of a Tolman universe in which the space-like hypersurfaces of homogeneity are flat. In synchronous time $b_1 = b_3 = c_1 t + c_2$ where c_1 and c_2 are integration constants. In this particular example, one can easily show that any axially symmetric YM connection must necessarily be spherically symmetric. A comprehensive treatment of Einstein-$SU(n)$-YM system in

FRW cosmologies is given in the next chapter.

2.4. The qualitative analysis of the dynamical system

To further facilitate the analysis of the Hamiltonian dynamical system (2.32)-(2.36) one can use the Hamiltonian

$$H = [b_3(2P_3P_1 - \frac{b_3}{b_1}P_3^2) - \frac{b_3}{2b_1}(P_\gamma^2 + \alpha^4) - (\frac{P_\alpha^2}{4} + \alpha^2\gamma^2)]/\sqrt{b_3} \tag{2.39}$$

in which P_1, P_3, P_α and P_γ are the momenta conjugate to b_1, b_3, α and γ respectively. The Hamiltonian above is basically $G_\perp^\perp - T_\perp^\perp$. By using Hamilton's equations, the system (2.32)-(2.36) can be written in the equivalent form

$$\begin{aligned}
\dot{b_1} &= 2\sqrt{b_3}P_3, & \dot{P_1} &= -\frac{\sqrt{b_3}}{b_1^2}(\frac{P_\gamma^2}{2} + b_3P_3^2 + \frac{\alpha^4}{2}), \\
\dot{b_3} &= 2\sqrt{b_3}(P_1 - \frac{b_3}{b_1}P_3), & \dot{P_3} &= \frac{1}{\sqrt{b_3}}(-\frac{P_\alpha^2}{4b_3} + \frac{b_3}{b_1}P_3^2 - \frac{\alpha^2\gamma^2}{b_3}), \\
\dot{\alpha} &= -\frac{P_\alpha}{2\sqrt{b_3}}, & \dot{P_\alpha} &= \frac{2\alpha}{\sqrt{b_3}}(\gamma^2 + \frac{\alpha^2 b_3}{b_1}), \\
\dot{\gamma} &= -\frac{\sqrt{b_3}}{b_1}P_\gamma, & \dot{P_\gamma} &= \frac{2\alpha^2\gamma}{\sqrt{b_3}},
\end{aligned} \tag{2.40}$$

and the constraint $H = 0$. To ease the dynamical system analysis of (2.40), one can convert the system above of equations into polynomial form by the time reparametrization $d\tau = dt/(b_1\sqrt{b_3})$ and a transformation $s_i = P_ib_i$, $i = 1, 3$. After defining $' := \frac{d}{d\tau}$, the above sytem is transformed into

$$\begin{aligned}
b_1' &= 2b_1s_3, & s_1' &= b_1(\frac{P_\alpha^2}{4} + \alpha^2\gamma^2), \\
b_3' &= 2b_3(s_1 - s_3), & s_3' &= \frac{b_3}{2}(P_\gamma^2 + \alpha^4), \\
\alpha' &= -\frac{b_1P_\alpha}{2}, & P_\alpha' &= 2\alpha(b_1\gamma^2 + b_3\alpha^2), \\
\gamma' &= -b_3P_\gamma, & P_\gamma' &= 2\alpha^2\gamma b_1,
\end{aligned} \tag{2.41}$$

and the constraint $2s_3s_1 - s_3^2 = \frac{b_1P_\alpha^2}{4} + \frac{b_3P_\gamma^2}{2} + \frac{\alpha^4 b_3}{2} + b_1\alpha^2\gamma^2$. The above system consists of first order equations which are of the form $\dot{\mathbf{x}} = V(\mathbf{x})$ where $\mathbf{x}$ is an eight-dimensional vector in the eight-dimensional phase space of the dynamical system. A qualitative analysis of the dynamical system (2.41) is achieved by the study of the behavior of the trajectories in the vicinity of the critical (equilibrium) points defined as the solutions of the algebraic equation $V(\mathbf{x}) = 0$. The set of the critical points of (2.41) in the physical region $b_1 > 0, b_3 > 0$ is the three-dimensional manifold $s_1 = s_3 = \alpha = P_\alpha = P_\gamma = 0$ that corresponds to vanishing YM fields and the extrinsic curvature (the flat space-time). By definition, M, an invariant submanifold of a dynamical system is a submanifold of the phase space such that every trajectory in M stays entirely in M for $-\infty < \tau < +\infty$. Obviously, the critical points of the dynamical system are a subset of M. The set of the critical points corresponding to flat space is an invariant submanifold. There is also a two-dimensional invariant submanifold associated with (2.37) and (2.38). Another invariant manifold is the set of trajectories that correspond to Kasner solution $P_\alpha = P_\gamma = \alpha = \gamma = 0$. A comprehensive analysis of Kasner solutions is given in Ref. [30].

It is well known that if the real parts of the eigenvalues of $\frac{\partial V^i}{\partial \mathbf{x}^j}$ do not vanish, the behavior of the trajectories of any system $\dot{\mathbf{x}} = V(\mathbf{x})$ in some neighborhood of the critical points $\mathbf{x}_0$ is qualitatively equivalent to the behavior of the trajectories of its linear part $\dot{\mathbf{x}}^i = \frac{\partial V^i}{\partial \mathbf{x}^j}(\mathbf{x}^j - \mathbf{x}_0^j)$. Unfortunately, all the eigenvalues of the Jacobian matrix $\frac{\partial V^i}{\partial \mathbf{x}^j}$ of (2.41) have vanishing real parts at the critical points. In other words, the qualitative behavior of this system is given by the higher order parts of $\dot{\mathbf{x}} = V(\mathbf{x})$ and is highly non-trivial. Elaborate means to analyze the behavior of a dynamical system in the vicinity of degenerate critical points are available (see Ref. [30] for example). However, such a treatment of the degenerate critical points of (2.41) is beyond the scope of the present thesis and would be distracting to the study of the stochastic properties of axially symmetric Bianchi I Yang-Mills cosmology. Correspondingly, the dynamical equations of motion in the conformal time $d\eta := (b_1^2 b_3)^{-1/6} dt$ are derived from the Hamiltonian $H_{\text{conformal}} = (b_1/b_3)^{(1/3)}\sqrt{b_3}H$.

The system above is invariant under the group of scale transformations $\alpha \to c\alpha, \gamma \to c\gamma, P_\alpha \to c^2 P_\alpha, P_\gamma \to c^2 P_\gamma, P_1 \to cP_1, P_3 \to cP_3, b_1 \to c^2 b_1, b_3 \to c^2 b_3$. One can use this symmetry to reduce the order of the above system from eight to six by the transformations $B_1 = \frac{\dot{b}_1}{b_1}$, $B_3 = \frac{\dot{b}_3}{b_3}$, $A_1 = \frac{\alpha^2}{b_1}$, $A_2 = \frac{\dot{\alpha}}{\alpha}$, $G_1 = \frac{\gamma^2}{b_3}$, and $G_2 = \frac{\dot{\gamma}}{\gamma}$. However, due to the singular nature of this transformation, the resulting system is not suitable for numerical analysis.

To gain a better understanding of the long-time behavior of the system, we note that the energy-momentum tensor in an orthonormal frame is $(T^\mu_\nu) = \text{diag}(-A-$

$B, B, B, A - B)$ where

$$A = \frac{(\alpha^2\gamma^2 + P_\alpha^2/4)}{b_1 b_3}, \tag{2.42}$$

$$B = \frac{(P_\gamma^2 + \alpha^4)}{2b_1^2}. \tag{2.43}$$

Contrary to the electromagnetic and fully isotropic cases, the principal pressure in direction 3 does not have a definite sign. As will be seen from numerical investigations, the numerators of both expressions (2.42)and (2.43) have the same order of magnitude. However, any decrease in b_3 will cause the positive term in T_3^3 to dominate and (note the discussion after (2.38)) b_3 starts to increase. Hence, generally speaking, one would expect both b_1 and b_3 to be increasing functions of time. In fact, we used a

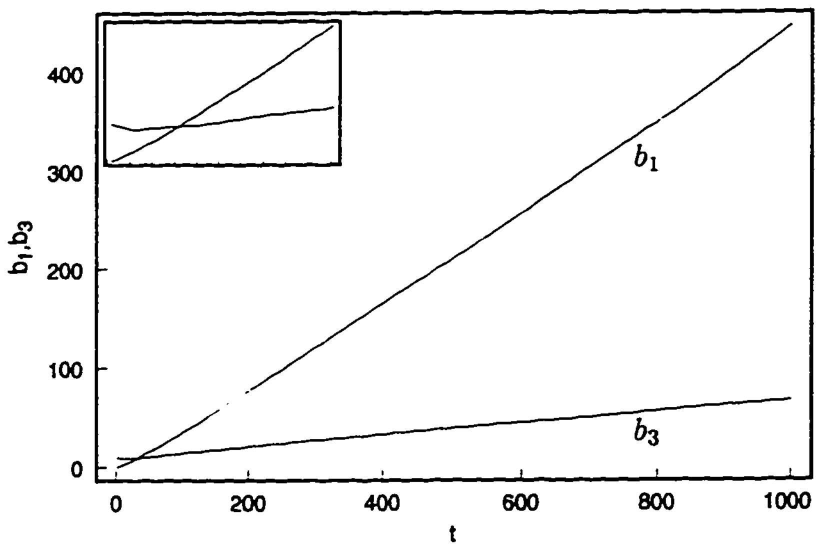

Figure 2.1: The behavior of metric variables for initial conditions $b_1 = 0.50, b_3 = 11.12, P_3 = 0.03, \alpha = \gamma = 0.1, P_\alpha = -0.67, P_\gamma = 0.03$. The inserted figure covers $0 \leq t \leq 100$ and indicates an initial decrease in b_3 which quickly reverses direction.

fifth order Runge-Kutta integrator to integrate the system (2.40) and computed the Hamiltonian constraint to check the accuracy of the numerical integration.

As figure 2 and the YM equations indicate, the general behavior of the YM fields is that of two coupled anharmonic oscillators with time-dependent frequencies.

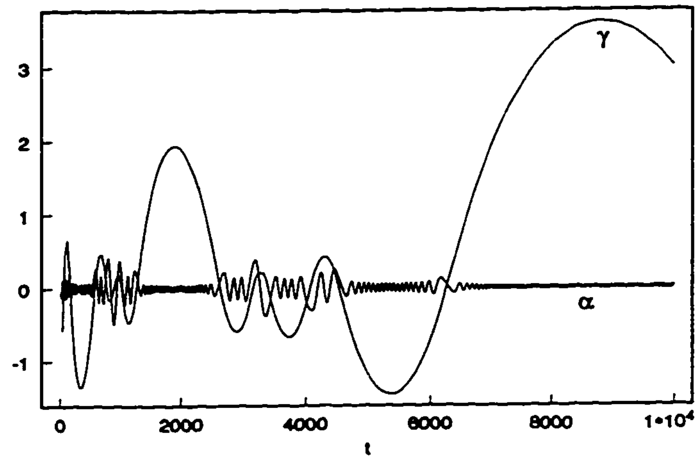

Figure 2.2: The behavior of YM field variables in axially symmetric Bianchi I-EYM cosmology for the initial conditions $b_1 = 2, b_3 = 1, P_3 = \alpha = \gamma = P_\alpha = 0.1, P_\gamma = 0.2$. The oscillations of γ are characterized by higher amplitudes and lower frequencies.

The behavior of the YM field variables in the above system resembles the dynamical properties of homogeneous YM fields in flat space known as Yang-Mills Classical Mechanics (YMCM) [31].

2.5. Axially symmetric YM fields in flat space and regularizing effects of gravitational self-interaction

A full analysis of the dynamical system (2.40) is an insurmountable task. Therefore we decided to start our analysis from the simpler system of axially symmetric YM fields in flat space. Fortunately, the procedure described in section 2.2 encompasses the gauge fixing for homogeneous YM fields in flat space. It is well known that YMCM has stochastic properties [31],[16],[32],[33]. In these models the reduction from the full space of dynamical variables to lower dimensions to make the dynamical evolution tractable is via some ansatz. In our model, the reduction is an inevitable consequence of the space-time symmetry.

The two dimensional flat system $\Lambda_1^1 \neq 0, \Lambda_2^2 \neq 0$ and the three dimensional flat system $\Lambda_1^1 \neq 0, \Lambda_2^2 \neq 0, \Lambda_3^3 \neq 0$, all other components vanishing, have been extensively covered in [16] and [32]. The stochastic character of these systems is demonstrated by numerically computing the Liapunov exponent. Such numerical computations are achieved by the simultaneous integration of the first order system $\dot{\mathbf{x}} = \mathbf{V}(\mathbf{x})$ and the linearized first order system

$$\dot{\mathbf{w}} = M(\mathbf{x}).\mathbf{w} \tag{2.44}$$

in which $\mathbf{w}$ is the perturbation vector connecting two nearby trajectories and $M(\mathbf{x})$ is the Jacobian matrix of $\mathbf{V}(x)$ [34]. The Liapunov exponent is defined as

$$\sigma = \lim_{t\to\infty} \frac{1}{t} \ln \frac{|\mathbf{w}(t)|}{|\mathbf{w}(0)|}. \tag{2.45}$$

Fig. 2.3 is a schematic diagram of two nearby trajectories connected by the vector

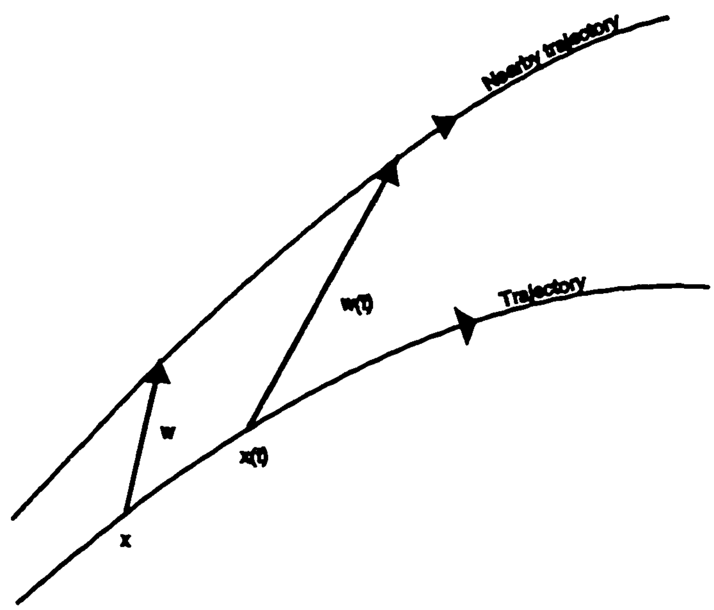

Figure 2.3: Two nearby initial conditions that are separated by $\mathbf{w}(0)$ initially are separated by $\mathbf{w}(t)$ as time evolves. For a stochastic system, $\mathbf{w}(t)$ grows exponentially.

$\mathbf{w}(t)$. A positive Liapunov exponent corresponds to the exponential divergence of two nearby trajectories. More specifically, the Liapunov exponent is the mean exponential rate of the divergence of two nearby trajectories. Heuristically, a positive Liapunov exponent is an indication of extreme sensitivity of the time evolution to the initial conditions. This means that due to the finite randomness in the choice of initial conditions in any stochastic (chaotic) physical system, predictability is lost in a time scale that is inversely proportional to the Liapunov exponent.

It can be shown that, for completely integrable Hamiltonian systems (*i.e.* systems for which there are as many integrals as the degrees of freedom) $\sigma = 0$. In principle, there can be more than one positive Liapunov exponent corresponding to positive eigenvalues $0 < \lambda_1 < \cdots < \lambda_m$ of $M(x)$. But, due to the exponential growth in $\mathbf{w}(t)$, unless $\mathbf{w}(t)$ points exactly in the direction of the eigenvectors corresponding to $\lambda_{i<m}$, (2.45) yields the maximal Liapunov exponent. In other words, growth in the direction of the eigenvector corresponding to λ_m dominates in the subsequent evolution of $\mathbf{w}(t)$.

In practice, numerical computation of (2.45) is marred by the overflow as a consequence of the exponential growth in the ratio $\frac{|\mathbf{w}(t)|}{|\mathbf{w}(0)|}$. We follow the procedure explained in appendix A to compute the Liapunov exponent for axially symmetric YM fields first in flat space and later in a Bianchi I cosmology.

The dynamics of axially symmetric YM fields in flat space is governed by the system

$$\ddot{\alpha} + \alpha(\gamma^2 + \alpha^2) = 0, \tag{2.46}$$

$$\ddot{\gamma} + 2\alpha^2\gamma = 0, \tag{2.47}$$

which correspond to a set of two strongly coupled oscillators with varying frequencies and amplitudes. These equations are derived from the Hamiltonian

$$H = \frac{1}{2}(\dot{\gamma}^2 + 2\dot{\alpha}^2 + 2\alpha^2\gamma^2 + \alpha^4). \tag{2.48}$$

The system describes the motion of a point particle moving in a potential well $U = \alpha^2\gamma^2 + \alpha^4/2$ with two open channels in the directions of positive and negative γ (figure 2.4)[1]. In these channels the term quartic in α in U is much smaller than the term quadratic in α. Therefore the behavior of the point particle in each channel is basically the same as that of the point particle in the two dimensional Hamiltonian system

$$H = \frac{1}{2}(\dot{\alpha}^2 + \dot{\gamma}^2 + \alpha^2\gamma^2) \tag{2.49}$$

[1]These channels resemble the open channels in mixmaster cosmologies.

treated in [31]. The potential barrier in this system has open channels in both the α and γ directions and such systems have been extensively studied because of their relation to the plasma confinement problem. Unless $\alpha \equiv 0$ or $\gamma \equiv 0$, as the particle moves deeper and deeper into the γ channel, say, the frequency of oscillations in α increases while the amplitude decreases. However, at a finite value of γ, $\dot{\gamma} = 0$ at which point the particle returns to the $\alpha \sim \gamma$ region. In this system, the stochastic regions occupy a significant portion of the phase space and the part of the regular region found so far is limited to a very small region of the phase space [35].

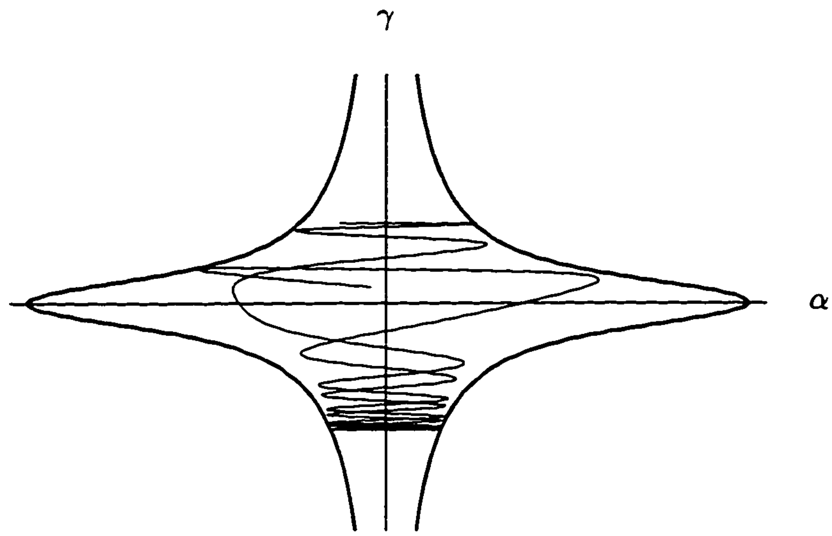

Figure 2.4: The behavior of a particle in potential well $U = \alpha^2\gamma^2 + \alpha^4/2$.

The behavior of a particle in a system with potential barrier U is basically the same. However, because of the lack of the existing channels in directions α, oscillations of γ are characterized by larger amplitudes and smaller frequencies.

Following [36] we numerically computed the Liapunov exponent for the system (2.46) and (2.47) for randomly selected initial conditions satisfying (2.48) with $H = 1$. It turns out that the Liapunov exponent for this system is positive and is of the same order of magnitude as the one for the system (2.49)(see figure 2.5). Numerical investigations for randomly selected initial conditions indicate that in this system, the stochastic regions occupy a large portion of the phase space also. The author is not aware of any systematic search to find regular regions in the phase space associated with this potential.

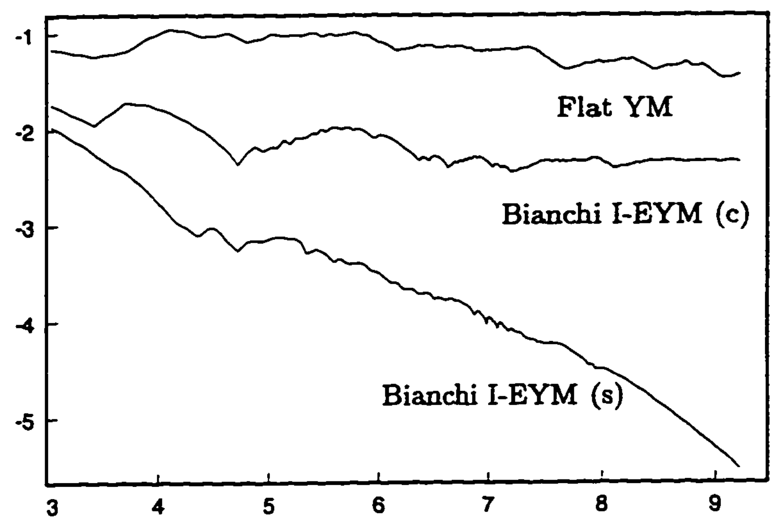

Figure 2.5: ln of the Liapunov exponent vs. ln of the evolution parameter for axially symmetric YM fields in flat space and in Bianchi I-EYM cosmology for initial conditions as in Fig. 2.2. This was a typical behavior for randomly selected initial conditions. (s) and (c) refer to synchronous and conformal time respectively.

As the potential term in (2.39) reveals, in a Bianchi I Yang-Mills system, the walls of the potential well in Fig. 2.4 move inwards for the initial conditions chosen in Fig. 2.1. There are at least two problems associated with generalizing the study of the stochastic properties of the flat space model to axially symmetric Bianchi I-EYM model represented by the system (2.40). One is related to the strongly coupled nature of the ODE system and the higher number of degrees of freedom which are known to cause sophisticated stochastic phase space properties like Arnold diffusion [34]. Numerical investigations (see figure 2.1) point to a non-compact phase space. Thus we can say that axially symmetric Bianchi I-EYM systems are not globally ergodic. However, we do not rule out the existence of ergodic components.

The other problem is related to the inherent gauge dependence in the definition of the Liapunov exponent and the non-existence of a satisfactory gauge-covariant definition of chaos in general relativity. It is known that in mixmaster models the positivity of Liapunov exponents depends on the choice of time reparametrization [7]. However, in a mixmaster cosmology, the stochasticity is associated with the behavior of the metric variables in the vicinity of the cosmological singularity where cosmological time is not well defined. In Bianchi I-EYM, ergodicity, if there is any, is mainly in the YM field variables, far away from the cosmological singularity.

Similar to the flat space scenario, we calculated the Liapunov exponents in both synchronous and conformal time of Bianchi I-EYM system for randomly selected initial values (see figure 2.5). The vanishing of Liapunov exponents in synchronous time points to a dynamical system in which the stochastic regions, if there are any, occupy a much smaller portion of the phase space. However , it also underlines the known fact that the Liapunov exponent is sensitive to time reparametrization. The Liapunov exponent is non-vanishing in conformal time with a value smaller than the corresponding flat space model. At this point we would like to add that it is more difficult to preserve the constraint in the conformal time and the numerical stability of the dynamical system is substantially enhanced in the synchronous time.

We also tried to use the notion of correlation between various YM field variables at different times during the evolution. The correlation coefficient for two random variables X and Y is defined as:

$$\rho = \frac{<(X - <X>)(Y - <Y>)>}{[(<X^2> - <X>^2)(<Y^2> - <Y>^2)]^{(1/2)}}, \tag{2.50}$$

where $<X>$ refers to the expectation value of X. Following [37] we computed the correlation between the initial and final values of YM field variables as an indication of a particular statistical independence in the dynamical evolution of the field variables. As figure 2.6 demonstrates, after a large time evolution, there is a loss of correlation between γ_i and γ_f (respectively α_i and α_f). Therefore $\gamma_i(\alpha_i)$ and $\gamma_f(\alpha_f)$ can be regarded as two stochastically independent random variables. One should note that the loss of correlation between these two variables is an indication of only a very specific kind of statistical independence. It is possible to find two variables X and Y with a very simple functional dependence $Y = Y(X)$ such that $\rho(X,Y) = 0$ in a particular range[2].

[2] For example, $Y = X^2$, for $-1 \leq X \geq 1$ is such a function.

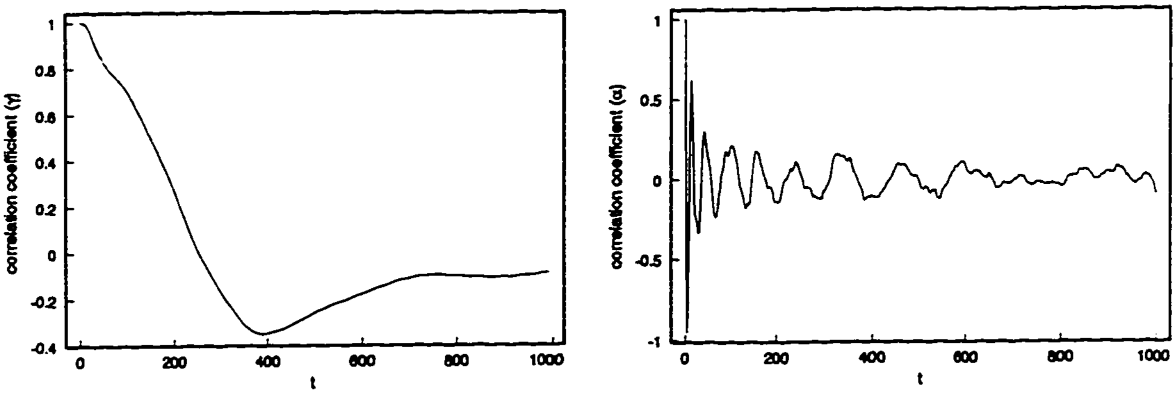

Figure 2.6: The correlation between γ_f and γ_i in terms of time for the initial conditions $b_1 = b_3 = 1, P_1 = 2, P_3 = 1, P_\alpha = -.28$. In the left figure $\alpha = 0.1, 0.15 \leq \gamma \leq 17.25$ and in the right figure $\gamma = 0.1, 0.15 \leq \alpha \leq 1.56$. The large time behavior indicates that there is no clear correlation between $\gamma_f(\alpha_f)$ and $\gamma_i(\alpha_i)$.

2.6. Conclusion

We systematically derived the most general form of the YM connection and the EYM field equations in Bianchi cosmologies with a four-dimensional isometry group in which the Killing vector fields obey the commutation relations (2.4). For the simplest of Bianchi cosmologies, namely Bianchi I, we investigated the resulting dynamical system. In doing so, one realizes that there is little hope of finding a non-trivial exact solution. Numerical integration suggests a non-compact phase space and oscillatory behavior for the YM field variables. However, one can easily use the scheme mentioned to construct invariant YM connections in flat space. There has been extensive work on the dynamical properties of homogeneous YM fields in flat space (YMCM) which are known to have stochastic properties. We used some methods to investigate chaos in YMCM (*i.e* numerical computation of Liapunov exponent) to see how gravitational self-interaction can affect the stochastic behavior. It turned out that the system with gravitational self-interaction has milder stochastic properties. We hope to extend this work to other Bianchi cosmologies.

Chapter 3

Cosmological Einstein-Yang-Mills equations

3.1. Introduction

There has been extensive work on Einstein-Yang-Mills (EYM) cosmological models in the last decade. This work was partly motivated by the successes of inflationary models driven by scalar fields in solving flatness and (to a large extent) horizon problems in cosmology. The interest in inflationary models driven by fields other than scalar fields is a consequence of less attractive features of the latter among which one could mention the lack of any concrete experimental evidence on the support of the existence of fundamental scalar fields [4]. Mini-superspace EYM cosmology is a natural extension of a non-perturbative treatment of self-gravitating scalar fields. It has been realized that despite a large phase space associated with seemingly redundant extra gauge degrees of freedom, there already exists a systematic mathematical method (based on Wang's theorem [38]) for the construction of invariant connections over homogeneous spaces in the same spirit as that of Kaluza-Klein theories. As will be seen in section 3.2, such invariant connections are related to the representation theory of compact Lie algebras. For some of the most easily constructed cases of $SO(n)$-YM fields the solutions were obtained on closed Friedmann-Robertson-Walker (FRW) cosmologies [10],[39],[40]. $SU(2)$-YM fields on open FRW cosmologies have also been of some interest [19],[17]. In this particular representation there is one degree of freedom associated with the YM fields.

Conformal invariance of the YM field equations (due to the fact that they are zero-rest-mass fields) results for the homogeneous and isotropic case in a decoupling of the gravitational and YM degrees of freedom in the conformal time. The energy momentum tensor is that of a radiation perfect fluid and the geometry is that of a Tolman universe.

Despite the fact that it is known that the construction of invariant YM connections could be generalized — at least in principle — to other compact gauge groups and cosmological models with compact and non-compact spatial sections, a systematic attempt to study models based on more complicated representations in FRW and anisotropic homogeneous cosmologies has not been conducted.

In the present chapter we derive the EYM equations for $SU(n)$-FRW and $SU(n)$ locally rotationally symmetric (LRS) cosmologies.

Section 3.2 is an exposition of a general but rather explicit construction of the Riemann and YM curvatures based on the theory of connections invariant under symmetry groups that act transitively on the base manifold. It turns out that the resulting purely algebraic Yang-Mills equations do not require any explicit choice of gauge. Such space-time homogeneous models are not considered to be realistic physically and we make no attempt in this paper to find any exact solutions.

In section 3.3 we derive the EYM equations for spatially homogeneous cosmological models. The result is a system of ordinary differential equations where again the YM gauge needs to be fixed only mildly, for example, by setting the temporal component of the potential to zero. The spatially homogeneous and isotropic models are discussed in section 3.4. Although the space-time geometry is completely determined independently of the YM fields, the latter satisfy in general some complicated coupled system of evolution equations. We derive here a few general facts for arbitrary gauge groups and some more explicit equations corresponding to different possible YM fields for the gauge groups $SU(n)$ and $SO(n)$.

Finally, in section 3.5 we consider, in a unified way, all LRS cosmological models with a $SU(n)$ Yang-Mills source. In such models, after solving for the constraints, there are $2(n-1)$ degrees of freedom associated with the YM fields. Here we just concentrate on what we consider the simplest YM-connections that contain a 'magnetic' part and derive the full evolution equations of the EYM-system. An analysis of the solutions of this quite complicated system is beyond the scope of this paper.

Even the system for homogeneous YM fields in two-dimensional flat space is known to be non-integrable. A dynamical system analysis of LRS Bianchi I models with $SU(2)$-YM fields was given in [6].

3.2. Einstein-Yang-Mills equations on homogeneous space-time

Following the conventions of [5] we let (M,g) be a connected pseudo-Riemannian manifold with its Levi-Civita connection, and K its isometry group. Its (left) action,

$$\bar{\psi} : K \times M \to M : (a,x) \mapsto \bar{\psi}_a x \tag{3.1}$$

has the following properties:

1.

$$\bar{\psi}_e(x) = x \tag{3.2}$$

where e is the identity element of K.

2.

$$\bar{\psi}_a(\bar{\psi}_{a'}(x)) = \bar{\psi}_{aa'}(x) \tag{3.3}$$

therefore, the transformations $\bar{\psi}_a$ form a group isomorphic with K.

3. The transformation is transitive on M (i.e.

$$\forall x, x' \in M,\ \exists a \in K \ :\ x' = \bar{\psi}_a(x)). \tag{3.4}$$

Fixing a point $x_0 \in M$ (to be called the *origin*) the isotropy subgroup K_0 of K is defined by

$$K_0 := \{a \in K \,|\, \bar{\psi}_a x_0 = x_0\}. \tag{3.5}$$

If the isotropy group at $x_1 = \bar{\psi}_{a_1} x_0$ is denoted by K_1, and $a \in K_0$, then

$$\bar{\psi}_a x_0 = x_0 \Rightarrow \bar{\psi}_a \bar{\psi}_{a_1^{-1}}(x_1) = \bar{\psi}_{a_1^{-1}}(x_1) \Rightarrow \bar{\psi}_{a_1} \bar{\psi}_a \bar{\psi}_{a_1^{-1}}(x_1) = \bar{\psi}_{a_1 a a_1^{-1}}(x_1) = x_1 \Rightarrow a_1 a a_1^{-1} \in K_1. \tag{3.6}$$

In other words, the isotropy groups at different points of M are conjugate.

An equivalence relation in K is defined with respect to the action of K_0 on K from the right. Therefore, two elements $k_1, k_2 \in K$ are in the same equivalence class (i.e. $[k_1] = k_2]$) if $k_1 = k_2 k_0$ for some $k_0 \in K_0$. The manifold M is diffeomorphic to the set of equivalence classes (left cosets) of K with respect to K_0, i.e. $M \cong K/K_0$.

3.2.1. Invariant metrics on homogeneous space-times

The classification of all geometries on M invariant under the left action of K (K-invariant pseudo-Riemannian metrics) started with the pioneering work of L.P. Eisenhart [41] and was later completed by A.Z. Petrov [42]. The action of K on M is generated by a set of vector fields $\{\bar{\mathrm{e}} \mid \quad = 1 \ldots m\}$, ($m \leq$ dim K) on M. Since dim M $\leq$dim K, at each point on M, say x_0, one can choose a set of vector fields $\{\bar{\mathrm{e}}_a | a = 1 \ldots n\}$ ($n =$ dim M) such that the $\bar{\mathrm{e}}_a$ are linear combinations of $\bar{\mathrm{e}}$ and span the tangent space $T_{x_0}M$ at x_0. One can also choose a set of vector fields $\{\bar{\mathrm{e}}_\Gamma | \Gamma = n+1 \ldots m\}$ such that the $\bar{\mathrm{e}}_\Gamma$ are linear combinations of $\bar{\mathrm{e}}$ and vanish (in general only) at x_0 (this is always possible if K_0 is not a normal subgroup of K). The $\bar{\mathrm{e}}_\Gamma$ are the generators of the action of K_0 on M. A subtlety here requires further attention: $\mathfrak{k}$ is the Lie algebra of left-invariant vector fields on K. These vector fields do not project unambiguously under the projection map $\pi : K \to K/K_0 \sim M$, $k_1 \in K \mapsto [k_1]$ on which the equivalence is defined by the action of the elements of K_0 from the right [1]. To find the commutation relations among $\bar{\mathrm{e}}_\alpha$ which are the generators of the left action of K on M, one notes that the left action of K on itself is generated by the right-invariant vector fields on K with commutation relations opposite to those of the basis of $\mathfrak{k}$ and are related to the left-invariant vector fields by the adjoint action. The right-invariant vector fields on $\mathfrak{k}$ project unambiguously under π. Therefore, $[\bar{\mathrm{e}}_\alpha, \bar{\mathrm{e}}_\beta] = -[\mathrm{e}_\alpha, \mathrm{e}_\beta]$ where e_α are a basis of $\mathfrak{k}$. In other words, the Lie algebra of the generators $\bar{\mathrm{e}}_\alpha$ is anti-isomorphic to $\mathfrak{k}$.

The invariance of any metric g is given by the Killing equations $\mathcal{L}_{\bar{\mathrm{e}}_\alpha} g = 0$ where $\mathcal{L}$ is the Lie derivative. If the action of K on M is simply transitive (i.e. $\bar{\mathrm{e}}_\Gamma \equiv 0$, K_0 is a normal subgroup of K), an invariant metric at any point $x \in M$ is obtained simply by Lie dragging any symmetric bilinear form at x_0 to x. However, since e_Γ

[1] This is simply because the left-invariant vector fields do not stay invariant under the right action of K_0 on K.

vanishes *only* at x_0, it has a non-trivial action on $T_{x_0}M$. This action translates into

$$\mathcal{L}_{\bar{\mathrm{e}}_\Gamma} g = \mathcal{L}_{\bar{\mathrm{e}}_\Gamma}(g^{ab}\bar{\mathrm{e}}_a \otimes \bar{\mathrm{e}}_b) = \bar{\mathrm{e}}_\Gamma(g^{ab})\bar{\mathrm{e}}_a \otimes \bar{\mathrm{e}}_b + g^{ab}[\bar{\mathrm{e}}_\Gamma, \bar{\mathrm{e}}_a] \otimes \bar{\mathrm{e}}_b + g^{ab}\bar{\mathrm{e}}_a \otimes [\bar{\mathrm{e}}_\Gamma, \bar{\mathrm{e}}_b] \quad (3.7)$$

which is equivalent to ad_{K_0}-invariance of g. Therefore there is a one-to-one correspondence between K-invariant pseudo-Riemannian metrics on M and ad_{K_0}-invariant non-degenerate symmetric bilinear forms $\mathring{g}$ on the quotient space $\mathfrak{k}/\mathfrak{k}_0$ of the corresponding Lie algebras.

Not all different symmetric bilinear forms $\mathring{g}$ give rise to non-equivalent metrics. It is known that if $K_0 = e$, all the equivalent metrics are related by the automorphism group of K, Aut(K) where by definition Aut(K) $\subset$ Diff(K) is a subgroup of the diffeomorphism group of M that leaves the group structure of M invariant [43].

3.2.2. Invariant Yang-Mills fields on homogeneous space-times

We wish to describe Yang-Mills connections that have as many symmetries as the metric of space-time and therefore assume that the full isometry group also acts by principal bundle automorphisms

$$\tilde{\psi} : K \times P \to P \quad (3.8)$$

on the principal bundle P that projects onto isometries on M thus satisfying

$$\pi \circ \tilde{\psi} = \tilde{\psi} \circ \pi \quad \text{and} \quad \tilde{\psi}_a \circ R_g = R_g \circ \tilde{\psi}_a \quad \forall\, a \in K\ \forall\, g \in G \quad (3.9)$$

where π is the projection, G the structure (gauge) group, and R the right action of G on P (see Fig. 3.1). A principal fiber bundle automorphism is a fiber preserving map of a principal fiber bundle into itself that, when restricted to M, is an isometry on M [2]. If the gauge potential is invariant under this action, i.e. if the connection form $\tilde{\omega}$ on P is invariant, $\tilde{\psi}_a^*\tilde{\omega} = \tilde{\omega}$ for all $a \in K$, then so is the curvature form $\tilde{\Omega}$, $\tilde{\psi}_a^*\tilde{\Omega} = \tilde{\Omega}$. This is because the exterior derivative and the action of $\tilde{\psi}$ commute. It follows that

$$\mathcal{L}_{\tilde{X}}\tilde{\omega} = 0 \quad \text{and} \quad \mathcal{L}_{\tilde{X}}\tilde{\Omega} = 0 \quad \forall\, X \in \mathfrak{k}. \quad (3.10)$$

[2]The definition of a principal fiber bundle automorphism given here is a restricted version of a more general definition in which $\tilde{\psi}$ is a fiber preserving map of a principal fiber bundle that 1) is a homomorphism of the structure group when restricted to the fibers and 2) is a diffeomorphism of the base manifold M when restricted to M.

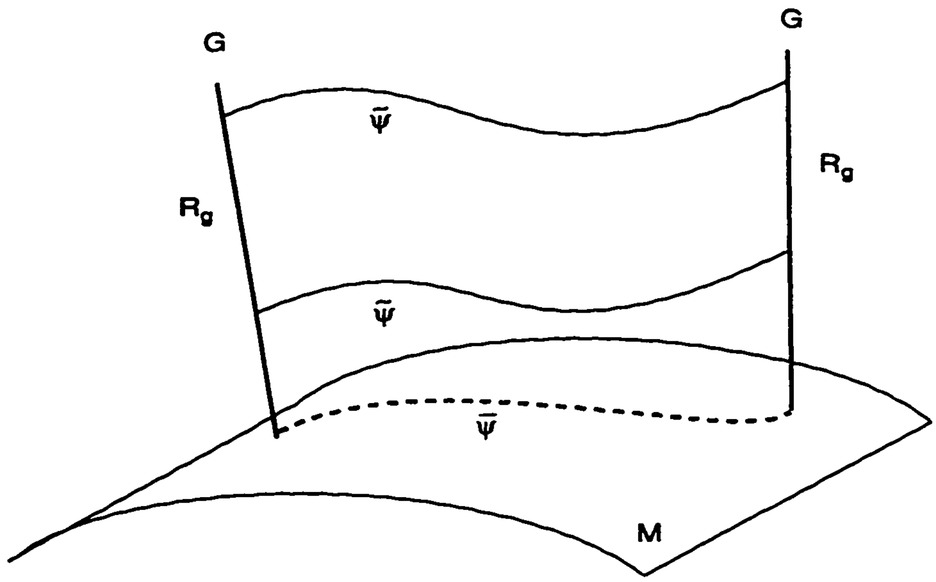

Figure 3.1: An automorphism of a principal fiber bundle commutes with R_g and projects to the isometry $\tilde{\psi}$ when restricted to M.

where $\tilde{X}$ is the infinitesimal generator of the action $\tilde{\psi}$ on P corresponding to $X \in \mathfrak{k}$.

If $\phi : P \to P'$ is a principal fiber bundle isomorphism, one can define an equivalence relation among different $\tilde{\psi}$-invariant principal fiber bundles. Two $\tilde{\psi}$-invariant and $\tilde{\psi}'$-invariant principal fiber bundles are equivalent if the commutative diagram

$$\begin{array}{ccc} P & \xrightarrow{\phi} & P' \\ \downarrow \tilde{\psi} & & \downarrow \tilde{\psi}' \\ P & \xrightarrow{\phi} & P' \end{array}$$

is valid. Now it is known (see, for example, [25]) that the equivalence classes of such $\tilde{\psi}$-invariant principal fiber bundles P over M are in one-to-one correspondence with the conjugacy classes of homomorphisms $\lambda : K_0 \to G$. Here λ and $\tilde{\psi}$ are related by

$$\tilde{\psi}_a(u_0) = R_{\lambda(a)} u_0 \quad \forall\, a \in K_0 \tag{3.11}$$

where u_0 is any fixed element of $\pi^{-1}(x_0)$.

Moreover, Wang's theorem ([38], see also [5]) states that (for fixed λ) the set of $\tilde{\psi}$-invariant connections on P is in one-to-one correspondence with the set of linear maps $\Lambda : \mathfrak{k} \to \mathfrak{g}$ that satisfy

$$\begin{aligned} \Lambda(X) &= \lambda(X) && (X \in \mathfrak{k}_0), \\ \Lambda \circ ad_k &= ad_{\lambda(k)} \circ \Lambda && (k \in K_0) \end{aligned} \tag{3.12}$$

where λ now also denotes the induced Lie algebra homomorphism. The second condition in (3.12) is similar to (3.7). The difference in appearance is due to the gauge degrees of freedom associated with the Yang-Mills connection. This equation states that to have K_0-invariant physical quantities, any action of K_0 on tensors that are valued in the gauge Lie algebra $\mathfrak{g}$ (the adjoint action ad_{K_0}) must be compensated by a gauge transformation (the action of $\mathrm{ad}_{\lambda(K_0)}$ on the fibers). The invariant connection and curvature on P are then given by

$$<\tilde{X}, \tilde{\omega}> \overset{o}{=} \Lambda(X) \qquad (X \in \mathfrak{k}), \tag{3.13}$$

$$<\tilde{X} \wedge \tilde{Y} / \tilde{\Omega}> \overset{o}{=} [\Lambda(X), \Lambda(Y)] - \Lambda([X,Y]) \qquad (X, Y \in \mathfrak{k}). \tag{3.14}$$

The symbol $\overset{o}{=}$ indicates that these equations only hold at the fixed point $u_0 \in P$. The second equation of (3.12) becomes infinitesimally

$$\Lambda([X,Y]) = [\lambda(X), \Lambda(Y)] \quad \forall\, X \in \mathfrak{k}_0, \forall\, Y \in \mathfrak{k}. \tag{3.15}$$

Let us, as before, choose $\{\mathrm{e}\ |\quad = 1 \dots m\}$ to be a basis of the Lie algebra $\mathfrak{k}$ such that the corresponding generators $\{\mathrm{e}_a | a = 1 \dots n\}$ span the tangent space $T_{x_0}M$ at x_0 while $\{\mathrm{e}_\Gamma | \Gamma = n+1 \dots m\}$ span the Lie subalgebra $\mathfrak{k}_0$. Note that the vector fields introduced in Sec. 3.2.1 are the projections of these vector fields on M. If the structure constants c^λ are introduced by

$$[\mathrm{e}\ , \mathrm{e}\ ] = c^\lambda\ \mathrm{e}_\lambda, \tag{3.16}$$

then

$$c^a_{\Gamma\Delta} = 0, \tag{3.17}$$

which simply means that K_0 is a (closed) subalgebra. The infinitesimal generators $\bar{\mathrm{e}}_a$ on M corresponding to e_a form a frame field in a neighborhood of x_0, but this will, in general, only be global on M if M admits a simply transitive isometry subgroup and is thus a group manifold. Let $\{\bar{\theta}^a\}$ be the local 1-form field dual to $\{\bar{\mathrm{e}}_a\}$.

A pseudo-Riemannian metric g on M can now be written in the form

$$g = g_{ab}\bar{\theta}^a \otimes \bar{\theta}^b. \tag{3.18}$$

From here on the symbol $\overset{o}{=}$ denotes equality at x_0 only. The components $\overset{o}{g}_{ab} := g_{ab}(x_0)$ satisfy

$$\mathcal{L}_{\bar{e}_\Gamma} \overset{o}{g}_{ab} = -2 \overset{o}{g}{}^{c(b} c^{a)}{}_{\Gamma c} = 0 \tag{3.19}$$

The coefficients of the Levi-Civita connection and the curvature tensor with respect to this frame at x_0 are then given by

$$\Gamma^a_{bc} \overset{o}{=} -\tfrac{1}{2}c^a_{bc} + g^{ar}c^s_{r(b}g_{c)s} \tag{3.20}$$
$$R^a{}_{bcd} \overset{o}{=} \Gamma^a_{rc}\Gamma^r_{bd} - \Gamma^a_{rd}\Gamma^r_{bc} - c^r_{cd}\Gamma^a_{br} + c^a_{b\Sigma}c^\Sigma_{cd}. \tag{3.21}$$

Note that none of the components of g, Γ, nor R are constant on M, in general.

Equation (3.20) is easily derived from the commutation relations

$$2\Gamma^d_{[bc]}\bar{e}_d = \nabla_{\bar{e}_b}\bar{e}_c - \nabla_{\bar{e}_c}\bar{e}_b = [\bar{e}_b, \bar{e}_c] = -c^d{}_{bc}\bar{e}_d \tag{3.22}$$

for the right-invariant vector fields for a torsion-free metric and $\nabla_{\bar{e}_a} g_{bc} = 0$ for a metric connection. Equation (3.21) is obtained most conveniently from Wang's theorem applied to the bundle of pseudo-orthogonal frames over M. Here, however, the principal bundle and the connection on it are already fixed as well as the action of K on the bundle, which is the natural lift of the action on M. $\tilde{\omega}$ in (3.13) and the the connection components in (3.21) are related by

$$\tilde{\omega}^a_b(\tilde{e}_c) = \Gamma^a_{bc} \tag{3.23}$$

in which $\tilde{e}_c$ is the natural lift of $\bar{e}_c$ to the bundle of pseudo-orthonormal frames. Thus (3.13) fixes the Wang map together with the requirement of zero torsion (i.e.. (3.22)) and (3.14) then leads to (3.21) (cf. [5], Ch. X). In a systematic study of EYM-systems from a Kaluza-Klein perspective in [44], the Riemann tensor for metrics on homogeneous spaces is also calculated in a very explicit form in terms of the structure constants of the symmetry group by calculating $\left(\nabla_{[\bar{e}_a}\nabla_{\bar{e}_b]} - \nabla_{[\bar{e}_a,\bar{e}_b]}\right)\bar{e}_c = R^d{}_{cab}\bar{e}_d$ which leads to an equivalent expression.

The gauge fields being invariant under a transitive symmetry group are also determined by their values at just one point of M which we take to be the origin x_0.

Their derivatives which occur in the Yang-Mills equations can be computed using again Wang's theorem so that the field equations are reduced to a purely algebraic form. Let σ be a local section of P, thus satisfying $\pi \circ \sigma = \mathrm{id}_M$, and introduce the local gauge potential A and the gauge field F by the pull back of the invariant connection and curvature on the principal fiber bundle P and the base manifold M respectively. If the contraction of $\hat{X}$ and A is denoted by $\iota_{\hat{X}} A \equiv \langle \hat{X}, A \rangle$ and

$$A = \sigma^* \tilde{\omega}, \qquad F = \sigma^* \tilde{\Omega}, \tag{3.24}$$

then we have

Lemma 1 *Under the assumptions above, the Lie derivative of the gauge curvature F at $x_0 \in M$ can be written in the form*

$$\mathcal{L}_{\bar{X}} F \overset{\circ}{=} [\Lambda(X), F] - [\langle \hat{X}, A \rangle, F]. \tag{3.25}$$

Proof:

Since $\bar{X} = \pi_* \circ \sigma_* \bar{X} = \pi_* \tilde{X}$ the vector field $\hat{X} = \sigma_* \bar{X} - \tilde{X}$ is vertical on P (a vector field is vertical if it is tangent to the fibers). Now $\mathcal{L}_{\bar{X}} F = \mathcal{L}_{\bar{X}} \sigma^* \tilde{\Omega} = \sigma^* \mathcal{L}_{\sigma_* \bar{X}} \tilde{\Omega} = \sigma^* (\mathcal{L}_{\tilde{X} + \hat{X}} \tilde{\Omega}) = \sigma^* \mathcal{L}_{\hat{X}} \tilde{\Omega}$ in view of (3.10). But

$$\mathcal{L}_{\hat{X}} \tilde{\Omega} = \iota_{\hat{X}} d\tilde{\Omega} + d\iota_{\hat{X}} \tilde{\Omega} = -\iota_{\hat{X}} [\tilde{\omega} \wedge \tilde{\Omega}] = -[\langle \hat{X}, \tilde{\omega} \rangle, \tilde{\Omega}] + [\tilde{\omega} \wedge \iota_{\hat{X}} \tilde{\Omega}] = -[\langle \hat{X}, \tilde{\omega} \rangle, \tilde{\Omega}]. \tag{3.26}$$

The first identity $\mathcal{L}_{\hat{X}} = [\iota_{\hat{X}}, d]$ gives the Lie derivative with respect to $\hat{X}$ in terms of the contraction with $\hat{X}$ and the exterior differentiation. The second equality holds in view of the Bianchi identities, $d\tilde{\Omega} + [\tilde{\omega} \wedge \tilde{\Omega}] = 0$, and the fact that $\iota_Z \tilde{\Omega} = 0$ for any vertical vector field Z. In the third equality the contraction has propagated through the wedge product according to the Leibniz rule. In the fourth equality, we have used $\iota_{\hat{X}} \tilde{\Omega} = 0$.

Pulling back (3.26) to M by σ, $\mathcal{L}_{\bar{X}} F = \sigma^* \mathcal{L}_{\hat{X}} \tilde{\Omega} = -\sigma^* [\langle \hat{X}, \tilde{\omega} \rangle, \tilde{\Omega}] = -[\sigma^* \langle \hat{X}, \tilde{\omega} \rangle . F]$. But $\sigma^* \langle \hat{X}, \tilde{\omega} \rangle = \sigma^* \iota_{\sigma_* \bar{X}} \tilde{\omega} - \sigma^* \langle \tilde{X}, \tilde{\omega} \rangle = \iota_{\bar{X}} \sigma^* \tilde{\omega} - \sigma^* \langle \tilde{X}, \tilde{\omega} \rangle \overset{\circ}{=} \iota_{\bar{X}} A - \Lambda(X)$ by (3.13) and $\sigma^* \langle \tilde{X}, \tilde{\omega} \rangle = \sigma^* \Lambda(X) = \Lambda(X)$. □

We choose now for the vector field $\bar{X}$ the local space-time frame vectors $\bar{\mathrm{e}}_a$ and frame forms θ^a and let

$$A = A_b \bar{\theta}^b, \qquad F = \tfrac{1}{2} F_{ab} \bar{\theta}^a \wedge \bar{\theta}^b. \tag{3.27}$$

Then, introducing the (space-time) covariant derivatives $F_{ab/c} = (\mathcal{L}_{\bar{e}_c}F)_{ab} + 2F_{r[a}\Gamma^r_{b]c}$, together with (3.25) we have

$$F_{ab/c} \overset{o}{=} [\Lambda_c - A_c, F_{ab}] + 2F_{r[a}\Gamma^r_{b]c}. \tag{3.28}$$

where $\Lambda_c := \Lambda(e_c)$.

Since the gauge-covariant derivative of F is defined by

$$D_\alpha F_{\beta\gamma} = F_{\beta\gamma/\alpha} + [A_\alpha, F_{\beta\gamma}] \tag{3.29}$$

we now find, interestingly, that the Yang-Mills equations, $D^\lambda F_{\lambda\alpha} = 0$ can be written in these frame components without involving the gauge potentials,

$$[\Lambda^r, F_{ra}] + \Gamma^\ell_{ar}F_\ell{}^r + F_{a\ell}\Gamma^\ell_{rs}g^{rs} \overset{o}{=} 0. \tag{3.30}$$

In view of (3.14), the frame components F_{ab} of the Yang-Mills field are given by

$$F_{ab} \overset{o}{=} [\Lambda_a, \Lambda_b] - c^r_{ab}\Lambda_r - c^\Sigma_{ab}\lambda_\Sigma. \tag{3.31}$$

Einstein's equations are also easily formulated in these frame components,

$$R_{ab} = \kappa T_{ab} \tag{3.32}$$

where $\kappa = 8\pi$(Newton's constant), the velocity of light is set to unity,

$$T_{ab} = X_{ab} - \tfrac{1}{4}X^r_r g_{ab}, \quad X_{ab} := \langle F_{ar}, F_b{}^r \rangle \tag{3.33}$$

and $\langle , \rangle$ represents a bi-invariant scalar product on the gauge group Lie algebra $\mathfrak{g}$. The stress energy tensor T_{ab} has zero trace, and the Ricci tensor components are obtained from (3.21).

All these equations hold only at the origin $x_0 \in M$ and they form a complicated algebraic system. For a given isometry group K of space-time and a chosen basis of $\mathfrak{k}$ the structure constants can be considered fixed. The homomorphism λ can be chosen arbitrarily and then fixed. Possible choices are found by considering the subgroups of the gauge group G onto which there are homomorphisms from the isotropy group K_0, in particular, imbeddings of K_0 in G. This classification is discussed (for semisimple K_0 and semisimple G) in [45],[46]. After the choice of a particular homomorphism, equations (3.12) or, infinitesimally, (3.15), i.e.

$$[\Lambda_a, \lambda_\Gamma] + c^r_{a\Gamma}\Lambda_r = -c^\Sigma_{a\Gamma}\lambda_\Sigma \tag{3.34}$$

must be solved for Λ which is then substituted into (3.30), (3.31) and into Einstein's equations (3.32).

In the (most important) case of a reductive homogeneous space $c_{a\Gamma}^{\Sigma} = 0$ and (3.34) is a homogeneous linear system. Then Λ can also be regarded as an intertwining operator between two linear representations of the isotropy group K_0 in the following way. We have $\mathfrak{k} = \mathfrak{k}_0 \oplus \mathfrak{m}$ as a vector space and the map Λ in (3.12) is fully determined by the linear map $\bar{\Lambda} : \mathfrak{m} \to \mathfrak{g}$ that satisfies

$$\bar{\Lambda} \circ \phi = \psi \circ \bar{\Lambda} \tag{3.35}$$

where $\phi : K_0 \times \mathfrak{m} \to \mathfrak{m} : (a, X) \mapsto \mathrm{ad}_a X$ and $\psi : K_0 \times \mathfrak{g} \to \mathfrak{g} : (a, Z) \mapsto \mathrm{ad}_{\lambda(a)} Z$. Then $\bar{\Lambda}$ is an intertwining operator between these representations of K_0, namely the adjoint representation ϕ on $\mathfrak{m}$ and the representation ψ on $\mathfrak{g}$

Also the g_{ab} are arbitrary, subject to (3.19). But not all choices need lead to nonisometric space-times. One can reduce the number of free parameters by bringing g_{ab} into a canonical form using basis transformations by automorphisms of K that leave the subgroup K_0 invariant.

3.3. EYM equations in spatially homogeneous cosmological models

Let (M, g) now be an $n+1$-dimensional space-time manifold with an isometry group K whose orbits are n-dimensional space-like hypersurfaces so that $M = \Sigma \times \mathbb{R}$ with K acting transitively on Σ and K_0 the isotropy subgroup at $x_0 \in \Sigma$. We choose to describe the metric by a coordinate time t and a frame field $\{\bar{e}_a\}$ of Killing vector fields on Σ,

$$g = -dt \otimes dt + g_{ab} \bar{\theta}^a \otimes \bar{\theta}^b. \tag{3.36}$$

Assume also that the $\bar{e}_\Gamma$ ($\Gamma = n+1 \ldots m$) vanish at a fixed point $x_0 \in \Sigma$. It then follows that the Σ_t-coordinate components of the frame vectors $\bar{e}_a$ do not depend on the time t so that

$$[\partial_t, \bar{e}\ \ ] = 0 \quad \forall \quad\ \ = 1 \ldots m. \tag{3.37}$$

The connection and curvature components with respect to the local space-time frame field $\{\bar{e}_0 = \partial_t, \bar{e}_a\}$ can then be calculated in the standard fashion. If

$$K_{ab} = \tfrac{1}{2}\dot{g}_{ab} \tag{3.38}$$

is the extrinsic curvature of the hypersurfaces and a dot denotes the time derivative, we have for the Ricci tensor components

$$R_{00} \overset{o}{=} -g^{rs}\dot{K}_{rs} + K^r_s K^s_r, \tag{3.39}$$

$$R_{0b} \overset{o}{=} K^r_b c^s_{rs} + K^r_s c^s_{rb}, \tag{3.40}$$

$$R_{ab} \overset{o}{=} \dot{K}_{ab} + K^r_r K_{ab} - 2K_{ar}K^r_b + \overset{\Sigma}{R}_{ab}. \tag{3.41}$$

Here $\overset{\Sigma}{R}_{ab} = \overset{\Sigma}{R}{}^s{}_{asb}$ is the Ricci tensor on Σ and is given, according to (3.21), by

$$\overset{\Sigma}{R}_{ab} \overset{o}{=} \Gamma^r_{rs}\Gamma^r_{ab} - \Gamma^s_{rb}\Gamma^r_{as} - \Gamma^s_{ar}c^r_{sb} + c^r_{a\Sigma}c^{\Sigma}_{rb}. \tag{3.42}$$

The g_{ab} and K_{ab} depend on t, the c are constant (on $\{x_0\} \times \mathbb{R}$) and the Γ^a_{bc} are still given by (3.20).

The calculation of the Yang-Mills equations for a gauge connection invariant under a symmetry group with orbits on surfaces of constant t is analogous to the one on spherically symmetric static space-times and is done as first outlined in [25] (see also [27]). Locally one can introduce a gauge potential $\mathcal{A} = A_0 dt + A$ where A is the potential of a (t-dependent) invariant connection on Σ and A_0 is a $\mathfrak{g}$-valued scalar, invariant under $Ad_{\lambda(K_0)}$. In practice (unless there are incompatible boundary conditions in the time evolution) A_0 can be gauged away. This is because a time-dependent gauge transformation to achieve such a result needs to satisfy an ordinary differential equation on the gauge group that can always be solved, at least locally in t.

In terms of the space-time co-frame $\{\bar{\theta}^0 = dt, \bar{\theta}^a\}$ we now write for the Yang-Mills field

$$F = E_a dt \wedge \bar{\theta}^a + \tfrac{1}{2}B_{ab}\bar{\theta}^a \wedge \bar{\theta}^b. \tag{3.43}$$

Then the Lie derivative of F in the time direction is

$$\mathcal{L}_{\partial_t} F = \dot{E}_a dt \wedge \bar{\theta}^a + \tfrac{1}{2}\dot{B}_{ab}\bar{\theta}^a \wedge \bar{\theta}^b \tag{3.44}$$

and those along Σ are still given by (3.25). Just as in section 3.2 we can then compute the frame components of the covariant derivatives and find

$$F_{ab/c} \overset{o}{=} [\Lambda_c - A_c, F_{ab}] + 2B_{r[a}\Gamma^r_{b]c} + 2E_{[a}K_{b]c}, \tag{3.45}$$

$$F_{0b/c} \overset{o}{=} [\Lambda_c - A_c, E_b] - E_r\Gamma^r_{bc} + B_{br}K^r_c, \tag{3.46}$$

$$F_{ab/0} \overset{o}{=} \dot{B}_{ab} + 2B_{r[a}K^r_{b]}, \tag{3.47}$$

$$F_{0b/0} \overset{o}{=} \dot{E}_b - E_rK^r_b. \tag{3.48}$$

The Yang-Mills equations thus become

$$[E^r, \Lambda_r] - c^r_{rs}E^s \overset{o}{=} 0, \tag{3.49}$$

$$\dot{E}_a + [A_0, E_a] + K^r_rE_a - 2K^r_aE_r - [B_{ar}, \Lambda^r] + B_a{}^sc^r_{rs} - \tfrac{1}{2}g_{ar}c^r_{pq}B^{pq} \overset{o}{=} 0, \tag{3.50}$$

where

$$B_{ab} \overset{o}{=} [\Lambda_a, \Lambda_b] - c^r_{ab}\Lambda_r - c^{\Sigma}_{ab}\lambda_{\Sigma}, \tag{3.51}$$

$$E_a \overset{o}{=} \partial_t\Lambda_a + [A_0, \Lambda_a] \tag{3.52}$$

and we may choose the gauge such that $A_0 = 0$.

For the stress-energy tensor components we find (if we now restrict to $n = 3$)

$$T_{00} = \tfrac{1}{2}(E^2 + B^2), \tag{3.53}$$

$$T_{0a} = \epsilon_a{}^{rs}\langle E_r, B_s\rangle, \tag{3.54}$$

$$T_{ab} = -\langle E_a, E_b\rangle - \langle B_a, B_b\rangle + \tfrac{1}{2}(E^2 + B^2)g_{ab} \tag{3.55}$$

where $B_a := \frac{1}{2}\epsilon_a{}^{rs}B_{rs}$, $E^2 := \langle E_r, E^r\rangle$ and $B^2 := \langle B_r, B^r\rangle$. Here, $\langle\ ,\ \rangle$ is a biinvariant inner product on $\mathfrak{g}$ (see the discussion before (3.92)). Einstein's equations (3.32) can now be brought into the form

$$\overset{\Sigma}{R} + (K^r_r)^2 - K^{rs}K_{rs} = \kappa(E^2 + B^2), \text{ Hamiltonian constraint} \tag{3.56}$$

$$K^r_ac^s_{rs} + K^r_sc^s_{ra} = \kappa\epsilon_a{}^{rs}\langle E_r, B_s\rangle, \text{ momentum constraint} \tag{3.57}$$

$$\dot{K}_{ab} - 2K_{ar}K^r_b + K^r_rK_{ab} + \overset{\Sigma}{R}_{ab} = \kappa T_{ab}\,. \tag{3.58}$$

If we choose the gauge such that $A_0 = 0$ then, after a basis of the symmetry Lie algebra $\mathfrak{k}$ and the homomorphism $\lambda : K_0 \to G$ are chosen and a point $x_0 \in \Sigma$

is fixed, we have as dynamical variables the functions $g_{ab}(t)$, subject to (3.19), and the $\mathfrak{g}$-valued functions $\Lambda_a(t)$, subject to (3.34). Equations (3.56) and (3.57) can be considered the Hamiltonian and the momentum constraints, respectively. They restrict somewhat the choice of initial values for an initial time but will afterwards be preserved by the time evolution. This follows as a special case from the general analysis of the Cauchy problem in EYM theory.

Only a time-independent basis transformation in $\mathfrak{k}$ by automorphisms leaving $\mathfrak{k}_0$ invariant can now be used to possibly eliminate some variables. The algebraic problem of finding the possible homomorphisms λ and solving for Λ is similar to the one mentioned in section 3.2 but a little simpler. The isotropy group K_0 is now a subgroup of $SO(3)$ and thus compact so that the homogeneous space is reductive (see Sec. 3.5 for a full discussion). Moreover, on the three-dimensional space-like space sections the isotropy group can only be either $SO(3)$ or $U(1)$ (or trivial). We will consider in the following sections some of these cases that can be handled without recourse to the more advanced techniques of the theory of Lie algebra representations.

3.4. Isotropic cosmological models

The isotropy subgroup K_0 of a space-time transitive isometry group must be a subgroup of the Lorentz group and a classification of all homomorphisms of such a subgroup into any compact gauge group G is a nontrivial algebraic problem. For a cosmological model with three-dimensional homogeneous spatial sections the situation is much simpler, since K_0 must be a subgroup of $SO(3)$ which leaves only $SO(3)$, $U(1)$ or the trivial subgroup. In this section we consider the "physically isotropic" models where K_0 is $SO(3)$. There are still many possible conjugacy classes of homomorphisms λ and a complete classification for arbitrary compact groups G may not be known. We will here mainly consider the case when G is either $SU(n)$ or a real orthogonal group.

When $SO(3)$ is the isotropy group of an isometric action on the three-dimensional maximally symmetric manifold Σ the $(\Sigma, \overset{(3)}{g})$ must be of constant curvature k and its isometry group K is $SO(4)$, $E(3)$ or $SO(3,1)$, respectively, depending on whether k is positive, zero or negative. The Lie algebra has a basis $\{e_i, f_i\}$ $(i = 1 \ldots 3)$ with commutators

$$[e_i, e_j] \;=\; k\epsilon_{ij}{}^r f_r, \tag{3.59}$$
$$[e_i, f_j] \;=\; \epsilon_{ij}{}^r e_r, \tag{3.60}$$
$$[f_i, f_j] \;=\; \epsilon_{ij}{}^r f_r. \tag{3.61}$$

where the f_i span the Lie algebra of the isotropy group. We can choose k to be ± 1 or 0 and the $\epsilon_{ij}{}^r$ in this section now refers to the Euclidean metric in $\mathbb{R}^3$.

The geometry of these isotropic models is then already determined, namely the one of the well known Friedmann-Robertson-Walker space-times. We have in the terminology of section 3.3

$$g_{ab} = a(t)\delta_{ab}, \quad K_{ab} = \tfrac{1}{2}\dot{a}\delta_{ab}, \quad \overset{\Sigma}{R}_{ab} = 2k\delta_{ab} \tag{3.62}$$

where the bar was dropped and $\{\theta^i\}$ is the co-frame dual to $\{e_i\}$. In terms of the *conformal time* τ the metric is

$$g = R(\tau)^2(-d\tau^2 + \delta_{ab}\theta^a \otimes \theta^b) \tag{3.63}$$

so that $a = R^2$ and $\dot{\phi} = d\phi/dt = R^{-1}d\phi/d\tau = R^{-1}\phi'$ for any function ϕ. The stress tensor, being isotropic, is of the form

$$T_{ab} = pg_{ab} \tag{3.64}$$

where p is the pressure and, since the source will be a zero-rest-mass Yang-Mills field, the mass-energy density is $\mu = 3p$. Einstein's equations are now equivalent to

$$\ddot{a} = -2k \quad \text{and} \quad \kappa p = \tfrac{1}{4}a^{-2}\dot{a}^2 + ka \tag{3.65}$$

or, in terms of the conformal time,

$$R'' + kR = 0 \quad \text{and} \quad \kappa p = R^{-4}R'^2 + kR^{-2} = (\text{ const.})R^{-4}. \tag{3.66}$$

The complete time evolution of the geometry and thus the stress-energy tensor is therefore easily obtained explicitly. It remains to formulate the equations for the Yang-Mills field.

If we use again the notation $\Lambda_i = \Lambda(e_i)$ and now $\lambda_i = \lambda(f_i)$ then equations (3.34) become

$$[\lambda_i, \Lambda_j] = \epsilon_{ij}{}^r \Lambda_r. \tag{3.67}$$

They represent a system of linear equations for the Λ_i once the λ_i, i.e. the homomorphism is chosen. We have from (3.51) and (3.52)

$$E_i = \dot{\Lambda}_i = R^{-1}\Lambda_i' \quad \text{and} \quad B_i = R^{-1}\left(\tfrac{1}{2}\epsilon_i{}^{rs}[\Lambda_r, \Lambda_s] - k\lambda_i\right) \tag{3.68}$$

for the Yang-Mills fields (where the indices on Λ and λ are raised and lowered with respect to δ_{ij}) so that

$$E^2 = R^{-4}\delta^{rs}\langle\Lambda_r', \Lambda_s'\rangle \tag{3.69}$$

$$B^2 = \tfrac{1}{2}R^{-4}\left(\langle[\Lambda_r, \Lambda_s], [\Lambda^r, \Lambda^s]\rangle - 4k\langle\Lambda_r, \Lambda^r\rangle + 2k^2\langle\lambda_r, \lambda^r\rangle\right) \tag{3.70}$$

The YM field equations become

$$\Lambda_i'' - 2k\Lambda_i - [[\Lambda_i, \Lambda_r], \Lambda^r] = 0 \tag{3.71}$$

$$[\Lambda_r', \Lambda^r] = 0. \tag{3.72}$$

From (3.56), (3.57) and (3.64) we have, moreover,

$$\epsilon_i{}^{rs}\langle\lambda_r, \Lambda_s'\rangle = 0 \tag{3.73}$$

$$\langle E_i, E_j\rangle + \langle B_i, B_j\rangle = 2pg_{ij}. \tag{3.74}$$

To derive these expressions we have used, whenever convenient, (3.67) as well as the invariance of the inner product $\langle , \rangle$ on $\mathfrak{g}$.

We can go a little further before we need to specify the gauge group G, but the specific structure of the isotropy group and its action on Σ incorporated in equations (3.67) are essential. Equations (3.67) are a system of linear equations for the ($\mathfrak{g}$-valued) Λ_i. Let $\Lambda_i = \Lambda_i^K\phi_K(t)$ and $\{\Lambda_i^K, K = 0, \ldots, r-1\}$ be a basis of the solution space where $\Lambda_i^0 = \lambda_i$ since λ_i is always a solution and is nonzero except if λ is the trivial homomorphism. The following lemma is needed to write the YM equations in a more simple form.

Lemma 2 *The basis vectors* $\{\Lambda_i^K, K = 0, \ldots, r-1\}$ *of the solution space of Wang's* *conditions* (3.67) *satisfy the following relations*

$$\epsilon_i{}^{rs}[\Lambda_r^K, \Lambda_s^L] = \gamma_S^{KL}\Lambda_i^S \tag{3.75}$$

$$[\Lambda_i^{(K}, \Lambda_j^{L)}] = \tfrac{1}{2}\epsilon_{ij}{}^r\gamma_S^{KL}\Lambda_r^S \tag{3.76}$$

$$\gamma_M^{KL} = \gamma_M^{LK} \tag{3.77}$$

$$L^{KL} := \delta^{rs}[\Lambda_r^K, \Lambda_s^L] = -L^{LK} \tag{3.78}$$

$$<\Lambda_i^K, \Lambda_j^L> = \alpha^{KL}\delta_{ij} \quad \textit{with} \quad \alpha^{KL} = \alpha^{LK} \tag{3.79}$$

$$\gamma_S^{KL}\alpha^{SM} = \alpha^{KS}\gamma_S^{LM} \tag{3.80}$$

$$\gamma_L^{0K} = 2\delta_L^K \quad \textit{and} \quad L^{0K} = 0 \tag{3.81}$$

Proof: To prove (3.75) let L_i and M_j be solutions of (3.67) and $N_i = \epsilon_i{}^{rs}[L_r, M_s]$. Then we can show that $[\lambda_i, N_j] = \epsilon_{ij}{}^r N_r$ by a simple calculation using the Jacobi identity and the identities satisfied by the Levi-Civita symbol ϵ_{ijk}. Thus N_i is also a solution of (3.67). (However, the full solution space need not be a Lie subalgebra of $\mathfrak{g}$, in general.)
Equations (3.76) and (3.78) follow immediately from the antisymmetry of the Lie bracket and (3.77) is a consequence of either (3.75) or (3.76).
To prove (3.79) we let $\alpha_{ij}^{KL} := <\Lambda_i^K, \Lambda_j^L>$ and use (3.67) and the invariance of the scalar product $<,>$,

$$\begin{aligned} \epsilon_{ij}{}^r\alpha_{rk}^{KL} &= <\epsilon_{ij}{}^r\Lambda_r^K, \Lambda_k^L> = <[\lambda_i, \Lambda_j^K], \Lambda_k^L> = -<\Lambda_j^K, [\lambda_i, \Lambda_k^L]> \\ &= -<\Lambda_j^K, \epsilon_{ik}{}^r\Lambda_r^L> = -\epsilon_{ik}{}^r\alpha_{jr}^{KL}, \end{aligned}$$

from which the result easily follows.
Finally, (3.80) follows directly from the invariance of the scalar product and (3.81) is an immediate consequence of (3.67) since $\Lambda_i^0 = \lambda_i$. □

The only time dependent quantities are now the amplitudes $\Phi_K(\tau)$ which satisfy the Yang-Mills equations in the form

$$L^{KL}\Phi'_K\Phi_L = 0, \tag{3.82}$$

$$\Phi''_K - 2k\Phi_K + \tfrac{1}{2}\gamma_K^{LM}\gamma_M^{PQ}\Phi_L\Phi_P\Phi_Q = 0. \tag{3.83}$$

Here (L^{KL}), defined in (3.78), is an array of skewsymmetric matrices one for each dimension of the Lie algebra $\mathfrak{g}$. From (3.68) we have

$$E_i = R^{-1}\Phi'_K\Lambda_i^K \quad \text{and} \quad B_i = R^{-1}\left(\tfrac{1}{2}\gamma_M^{KL}\Phi_K\Phi_L - k\delta_M^0\right)\Lambda_i^M \tag{3.84}$$

and, in view of (3.79), Einstein's equations (3.73) and (3.74) reduce to (3.66) and the following expression for the mass-energy density

$$\mu = \tfrac{1}{2}(E^2 + B^2) \tag{3.85}$$

where now

$$E^2 = 3R^{-4}\alpha^{KL}\Phi'_K\Phi'_L, \tag{3.86}$$

$$B^2 = 3R^{-4}\left(k^2\alpha^{00} - k\alpha^{0M}\gamma_M^{KL}\Phi_K\Phi_L + \tfrac{1}{4}\gamma_R^{KL}\alpha^{RS}\gamma_S^{PQ}\Phi_K\Phi_L\Phi_P\Phi_Q\right). \tag{3.87}$$

Using the relations of Lemma 2 it can be verified that μR^4 is constant as it should be.

The quantities α^{KL}, γ_M^{KL} and L^{KL} depend only on the Lie algebra $\mathfrak{g}$ and the homomorphism $\lambda : \mathfrak{su}(2) \to \mathfrak{g}$. Hence, to find all possible isotropic EYM equations one has to find all $\mathfrak{su}(2)$ subalgebras of $\mathfrak{g}$ (up to inner isomorphism), thus choosing the homomorphism λ (see Ref. [46]) and then solve the equation (3.35) for the intertwining operator $\bar{\Lambda} = (\Lambda_i^K)$. This can be done in a systematic way using a Cartan-Weyl basis of $\mathfrak{g}$ by the methods given in [45]. Here we will only consider those examples which can be dealt with in a more elementary way, without involving the theory of Lie algebra root systems.

We know that all (connected) compact gauge groups can be imbedded as subgroups of $GL(n,\mathbb{R})$ or $GL(n,\mathbb{C})$ (in fact in $SO(n)$) for some n. Moreover, all finite dimensional complex (real) representations of $SU(2)$ are equivalent to unitary (real orthogonal) ones and decompose orthogonally into irreducible parts. Thus at least for the unitary and the real orthogonal groups we can determine the possible homomorphisms directly from the well known representation theory. If, for example, $\tilde{\lambda}$ is a $n \times n$-unitary representation of K_0, i.e. $\tilde{\lambda} : a \mapsto U_a \; \forall\, a \in K_0$ where U_a is a unitary matrix, then $\lambda : a \mapsto (\det U_a)^{-1/n} U_a$ is a homomorphism into $SU(n)$. Moreover, it is easily seen that equivalent representations define conjugate homomorphisms and that, in fact, *conjugacy classes of homomorphisms of K_0 into $SU(n)$ are in one-to-one correspondence with equivalence classes of n-dimensional unitary representations of K_0*. Similarly, any real n-dimensional orthogonal representation of K_0 immediately defines a homomorphism into $SO(n)$.

If now $K_0 = SU(2)$[3] then any n-dimensional unitary (or real orthogonal) representation is a direct sum of irreducible unitary (real orthogonal) representations,

[3]Perhaps one should take $SO(3)$ rather than $SU(2)$ as the isotropy group. But it is clear that

i.e. any homomorphism $\lambda : SU(2) \to SU(n)$ is conjugate to one that maps into block matrices

$$\lambda(a) = \begin{pmatrix} D_{k_1}(a) & & \\ & \ddots & \\ & & D_{k_r}(a) \end{pmatrix} \tag{3.88}$$

where each D_{k_i} is an irreducible k_i-dimensional representation and where $k_1 + \cdots + k_r = n$. As is well known, the Lie algebra representation corresponding to an n-dimensional irreducible representation can be written as follows. If $\{\tau_1, \tau_2, \tau_3\}$ is the standard basis of $su(2)$ in terms of anti-Hermitian matrices and $\lambda_k = \lambda(\tau_k)$ are the images in $su(n)$ then the latter can be represented by the matrices

$$(\lambda_+)_{\ell m} = \sqrt{m(n-m)}\delta_{\ell,m+1}, \quad \lambda_- = \lambda_+^H \tag{3.89}$$

$$\lambda_1 = -\tfrac{i}{2}(\lambda_+ + \lambda_-), \quad \lambda_2 = -\tfrac{1}{2}(\lambda_+ - \lambda_-), \quad (\lambda_3)_{\ell m} = -i(\frac{n+1}{2} - m)\delta_{\ell m} \tag{3.90}$$

where $m, l \leq n$. Consider first a homomorphism class from $SU(2)$ to $SU(n)$, that arises from an irreducible unitary representation in $\mathbb{C}^n$. Then the λ_i in (3.67) can be chosen as the matrices (3.90) and the system (3.67) can be explicitly solved (this also follows the from more general results of representation theory) for the Λ_j which can now be taken to be $(n \times n)$ skew-Hermitian matrices. It follows that

$$\Lambda_i = \Phi\lambda_i \tag{3.91}$$

i.e. the solution space is one-dimensional. In this case the YM-potential is thus determined by a single function $\Phi(\tau)$. For a simple Lie algebra like $su(n)$ the invariant product $<,>$ must be a multiple of the Killing form,

$$<X,Y> = -c_n\kappa(X,Y) \propto \text{ trace } (XY) \tag{3.92}$$

for some constants $c_n > 0$ which we will choose to be 1. It follows from (3.75) and (3.76) that $\gamma_0^{00} = 2$ and $\alpha^{00} = n^2(n^2-1)/6$ so that the Yang-Mills equations become

$$\Phi'' - 2(k - \Phi^2)\Phi = 0 \tag{3.93}$$

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a covering group of an isometry group of a Riemannian manifold M will also act in a well defined way via projections at least if M is also simply connected. We will not consider these topological questions here.

whence

$$\frac{d\Phi}{\sqrt{c^2 - (\Phi^2 - k)^2}} = d\tau \tag{3.94}$$

where the constant $c^2 = 4\mu R^4/(n^2(n^2-1))$. Thus $\Phi(\tau)$ is periodic in the cosmological time τ and can be expressed in terms of an inverse elliptic integral. It is easily seen that the "electric" and "magnetic" contributions to the energy density μ oscillate in the time τ. These equations (for $G = SU(2)$) have previously been derived and analyzed by Gal'tsov and Volkov [19].

If the homomorphism class is not induced by an irreducible representation, the gauge field may be more complicated. However, since the evolution of the geometry of space-time is already determined, only the evolution of the gauge fields can be affected. Table 3.1 shows the dimensions d of the solution space of (3.67) for some homomorphisms $\lambda : su(2) \to su(n)$. Here $1 \oplus 2$, for example, means that λ is obtained from a representation in $\mathbb{C}^3$ that decomposes into a (trivial) one-dimensional one D_{k_1} and an irreducible two-dimensional one D_{k_2}. In these cases, according to (3.83), the YM field depends on d independent amplitudes $\Phi_K(\tau)$ which each satisfy a second order equation. However, at least for $n \leq 6$, the c constraint conditions (3.82)(which are not linearly independent in general) simply imply that many of the Φ_K are proportional to each other so that the remaining number n_{eq} of second order equations that must be solved is much smaller.

To give one example, for an $SU(5)$-theory with the homomorphism λ corresponding to a representation of the type $1 \oplus 1 \oplus 3$ we find the Yang-Mills equations

$$\Phi'' + 2\Phi(\Phi^2 + 3\Psi^2 - k) = 0 \tag{3.95}$$

$$\Psi'' + 2\Psi(3\Phi^2 + \Psi^2 - k) = 0 \tag{3.96}$$

and

$$E^2 = 20R^{-4}\left(\Phi'^2 + \Psi'^2\right) \quad \text{and} \quad B^2 = 20R^{-4}\left[\left(\Phi^2 + \Psi^2 - k\right)^2 + 4\Phi^2\Psi^2\right]. \tag{3.97}$$

The contribution of the electric and the magnetic part to the mass-energy density changes in time similarly as in the 'irreducible' case, but the gauge fields now 'rotate' in the Lie algebra in more dimensions.

If the gauge group is $SO(n)$ we can similarly classify the λ by considering all n-dimensional real orthogonal representations of $\mathfrak{su}(2)$. These decompose into irreducible blocks of dimensions $2k+1$ or $4k$ for integer k, but not $2k+2$ (see, e.g. [47]). It does not seem to be simple to write down formulae for these representations for arbitrary n as in (3.89) and (3.90). But there exists an algorithm to construct them explicitly. First note that an irreducible complex representation of $\mathfrak{su}(2)$ leaves invariant a bilinear form β on $\mathbb{C}^n$. For the choice of λ in (3.89) and (3.90) we find that $\beta_{k\ell} = (-1)^k \delta_{\ell,n+1-k}$ which is symmetric for odd n and skew for even n.

Thus if n is odd then λ is of real type, i.e. the representation is unitarily equivalent to one by real orthogonal matrices. In fact,

$$\tilde{\lambda}_k = U^H \lambda_k U \quad \text{where} \quad U^H U = \mathrm{id} \quad \text{and} \quad U^T \beta U = \mathrm{id} \tag{3.98}$$

are the generators of the orthogonal representation. The matrices U can be easily computed by diagonalizing β by congruence. For a $\lambda : \mathfrak{su}(2) \to \mathfrak{so}(2k+1)$ that corresponds to an irreducible representation it now follows easily from the complex case that the solutions of (3.67) are again of the form (3.91) and the single time dependent amplitude Φ satisfies (3.93).

For $n = 4k$ the explicit irreducible representations are obtained via the Lie algebra homomorphism

$$\rho : \mathfrak{gl}(\ell, \mathbb{C}) \to \mathfrak{gl}(2\ell, \mathbb{R}) : A = A_1 + iA_2 \mapsto \tilde{A} = \begin{pmatrix} A_1 & -A_2 \\ A_2 & A_1 \end{pmatrix} \tag{3.99}$$

which maps $\mathfrak{su}(\ell)$ into $\mathfrak{so}(2\ell)$. For $\ell = 2k$ the image of the matrices λ_k generate an irreducible $4k$-dimensional real orthogonal representation of $\mathfrak{su}(2)$. Thus, one finds a $4k$-dimensional representation of $\mathfrak{su}(2)$ via its representation on $\mathfrak{su}(2k)$. Again, it can be verified explicitly that (3.67) has only the solutions (3.91) and that the only amplitude satisfies (3.93). For $l = 2k+1$, one obtains a $2k+2$-dimensional representation which is always reducible.

The remaining equivalence classes of homomorphisms λ into $\mathfrak{so}(n)$ can now be obtained from reducible orthogonal representations in the same way as those for $\mathfrak{su}(n)$. Some examples are tabulated in Table 3.2. The corresponding equations and expressions for E^2 and B^2 are very similar to (3.4) and (3.97).

3.5. Locally rotationally symmetric cosmological models

Spatially homogeneous cosmological models with $K_0 = U(1)$ have been extensively studied and are known as locally rotationally symmetric (LRS) models. Our construction of four-dimensional isometry groups of LRS models is along the lines with [23]. If K_0 is compact, one can use an arbitrary inner product $(\ ,\ )'$ on K to define a new $\mathrm{ad}_{(K_0)}$-invariant inner product given by

$$(X, Y) = \int_{K_0} (\mathrm{ad}_{K_0}(X), \mathrm{ad}_{K_0}(Y))'\, dk \tag{3.100}$$

where dk is the Haar measure on K_0. If $\mathcal{M}$ is the complement of K_0 in K with respect to this inner product, namely, $\mathcal{M} = \{m \in K_0\ : (m, k_0) = 0,\ \forall k_0 \in K_0\}$ then there exists a reductive decomposition of $\mathfrak{k}$. At the Lie algebra level this means that there is a subspace $\mathfrak{m}$ such that $\mathfrak{k} = \mathfrak{k}_0 + \mathfrak{m}$ and $[\mathfrak{k}_o, \mathfrak{m}] \subset \mathfrak{m}$ and $\mathfrak{k}_0 \cap \mathfrak{m} = 0$. The choice of such a reductive decomposition is not unique and depends on the choice of the inner product. As it will be seen shortly, a judicious choice of a reductive decomposition, greatly simplifies the EYM equations. It is interesting to note that for all Bianchi cosmologies except Bianchi III, there is a reductive decomposition in which $\mathfrak{m}$ is a Lie subalgebra [4]. In a suitable basis $e_1, \cdots, e_4$ such that e_1, e_2, e_3 span $\mathfrak{m}$ and e_4 span $\mathfrak{k}_0$,

$$-c_{14}^2 = c_{24}^1 = 1,\ c_{34}^a = 0 \qquad (a = 1, 2, 3). \tag{3.101}$$

The $\mathrm{ad}(K_0)$-invariance of the metric expressed via (3.19) then restricts the space-metric to the form $\mathrm{diag}(f^2, f^2, f^2\sigma^2)$ where f and σ are functions of t. Given an invariant basis on a homogeneous space, one can start from this metric and, after integrating the Killing equations, find out which spatially homogeneous space-times admit the action of a four-dimensional isotropy group (cf. Table 3.3 and [48]). Kramer *et al.* [23] have classified all such space-times with two integers ℓ and k (Bianchi V (BV) does not fall into this category and is treated separately). All homogeneous spaces which have the same four-dimensional isometry group, belong to the group manifolds (Bianchi cosmologies). Such group manifolds correspond to different three-dimensional subgroups of the isometry group which act simply transitively on the

[4] Such a decomposition for BIII would require $SU(1,1)$ to be solvable which contradicts the simplicity of $SU(1,1)$. A Lie group is solvable if the commutator subalgebra (the algebra of all the commutations of the Lie algebra elements) of the corresponding Lie algebra at each step leads to smaller and smaller subalgebras until only the identity element of the algebra survives.

hypersurfaces of homogeneity. Kramer *et al.'s* classification (cf. [23] section 11.1) is based on the metric

$$
\begin{aligned}
g &= f^2[2C^{-2}(dx^2+dy^2)+\tfrac{1}{4}\sigma^2dz^2-\ell\sigma^2C^{-1}(ydx-xdy)dz \\
&\quad +\ell^2\sigma^2C^{-2}(ydx-xdy)^2], \\
&\quad \text{where } C := 1+1/2k(x^2+y^2) \qquad (3.102)
\end{aligned}
$$

or, for Bianchi V,

$$
g = f^2[e^{2z}(dy^2+dx^2)+\sigma^2dz^2].
$$

These metrics all have (generically) four-dimensional isometry groups. We must now select a frame field of Killing vectors in such a way as to let e_4 generate the isotropy group and the structure constants to satisfy (3.101). The following choice achieves this.

$$
\begin{aligned}
e_1 &= -\tfrac{k}{\sqrt{2}}xy\partial_y-\tfrac{1}{\sqrt{2}}(1+K)\partial_x+\sqrt{2}\ell y\partial_z, && (\partial_x),\\
e_2 &= \tfrac{k}{\sqrt{2}}xy\partial_x+\tfrac{1}{\sqrt{2}}(1-K)\partial_y+\sqrt{2}\ell x\partial_z, && (\partial_y),\\
e_3 &= -2\partial_z, && (-x\partial_x-y\partial_y+\partial_z),\\
e_4 &= x\partial_y-y\partial_x, && (y\partial_x-x\partial_y),
\end{aligned} \qquad (3.103)
$$

where $K := (k/2)(x^2-y^2)$ and the entries of the right column are the Killing vector fields of BV. The above Killing vector fields and non-vanishing structure constants (3.101) and

$$
c^3_{12}=\ell, \quad c^4_{12}=k \quad \text{or} \quad c^1_{13}=c^2_{23}=-1 \text{ for BV}, \qquad (3.104)
$$

determine the isometry group, embeddings of the isotropy group in the isometry group up to conjugacy class, and identify the three-dimensional homogeneous spaces which admit an action of a four-dimensional isometry group. Here Σ is simply connected. It is known that the number of degrees of freedom in mini-superspace models depends on the choice of topology [49].

Our aim is to construct the invariant $SU(n)$-YM connections for homogeneous spaces listed in the table 3.3. In doing so, we have to find all the conjugacy classes of homomorphisms $\lambda : U(1) \to SU(n)$. Such conjugacy classes of homomorphisms are well understood for spherically symmetric solutions of the EYM equations (cf. [27]). These classes of homomorphisms are basically of the same form as (3.88). However, since the irreducible representations of $U(1)$ are one-dimensional, D_k have only one entry. Therefore if $U(1) = \{z \in \mathbb{C} : |z| = 1\}$, then

$$\lambda : z \mapsto \mathrm{diag}(z^{j_1}, \cdots, z^{j_n}) \quad (\sum_{i=1}^{n} j_i = 0,\ j_i = \text{an integer}) \tag{3.105}$$

is clearly a homomorphism of $U(1)$ into $SU(n)$. The set of integers j_p $(p = 1, \cdots, n)$ such that $j_p \geq j_q$ for $p < q$, yields all conjugacy classes of homomorphisms $\lambda : U(1) \to SU(n)$.Denoting $\mathcal{D} := (i/2)\mathrm{diag}(j_1, \cdots, j_n)$ we have

$$\Lambda[e_4, e_i] = [\lambda(e_4), \Lambda_i] = [\mathcal{D}, \Lambda_i] \quad \Longrightarrow \quad c^r_{4i}\Lambda_r = [\mathcal{D}, \Lambda_i] \tag{3.106}$$

in which Λ_i are traceless antihermitian matrices as in section 3.4. These equations and (3.101) give

$$\Lambda_2 = -[\mathcal{D}, \Lambda_1], \quad \Lambda_1 = [\mathcal{D}, \Lambda_2], \quad [\mathcal{D}, \Lambda_3] = 0, \tag{3.107}$$

which in turn yield

$$(\Lambda_l)_{pq}[4 - (j_p - j_q)^2] = 0, \quad l = (1, 2). \tag{3.108}$$

The solution to the equations above is

$$\Lambda_1 = i/2(\Lambda_+ - \Lambda_-), \quad \Lambda_2 = -1/2(\Lambda_+ + \Lambda_-), \quad \Lambda_+ = -(\Lambda_-)^H \tag{3.109}$$

where $j_p \geq j_q$ for $p < q$ and therefore $\Lambda_+(\Lambda_-)$ is a strictly upper (lower) triangular matrix. Moreover, $(\Lambda_+)_{pq} \neq 0$ only if $j_p = j_q + 2$. The general solution of the above equations is in the root space corresponding to $\mathcal{D} \subset$ (Cartan subalgebra of $su(n)$) and in principle could be obtained for any compact group. However, such a general treatment is out of the scope of the present paper (cf. [28]). Some interesting special cases to consider are

(a) $j_p = 0, \forall\, p \in \{1, \cdots, n\}$, (trivial homomorphism) requires $\Lambda_1 = \Lambda_2 = 0$ and Λ_3 is completely undetermined.

(b) If $|j_p - j_q| \neq 2\ \forall\ p, q \in \{1, \ldots, n\}$ then $\Lambda_1 = \Lambda_2 = 0$ and Λ_3 is a diagonal traceless anti-hermitian matrix. In this case the gauge group reduces to its maximal torus (i.e. $U(1) \otimes \cdots \otimes U(1) \subset SU(n)$).

(c) If $j_p = j_{p+1}+2, \forall\, p \in \{1,\cdots,n-1\} \Rightarrow \mathcal{D} = (i/2)\mathrm{diag}(n-1, n-3, \cdots, -n+1)$. Then (3.107) and (3.108) respectively imply that Λ_3 is an anti-hermitian traceless diagonal matrix and $(\Lambda_+)_{p,p+1} = -(\Lambda_-^H)_{p+1,p}$ are the only non-vanishing entries of $\Lambda_\pm$.

In (b) the EYM equations for $SU(2)$-YM fields reduce to that of axially symmetric electromagnetic fields and one can show that (a) and (b) are gauge equivalent [6]. We consider (c) the simplest non-Abelian YM field in which the entries of $\mathcal{D}$ correspond to the magnetic quantum numbers in the n-dimensional unitary representation of $SU(2)$. Up to a gauge transformation, this representation yields the only possible non-Abelian connection for $SU(2)$-YM fields. Therefore we derive the EYM equations for this particular example starting with

$$\begin{aligned}(\Lambda_+)_{p,p+1} &= \omega_p e^{i\gamma_p}, \quad p \in \{1,\cdots,n-1\}\\ \Lambda_3 &= i \;\mathrm{diag}\,(\alpha_1,\ldots,\alpha_p-\alpha_{p-1},\ldots,-\alpha_{n-1}).\end{aligned} \tag{3.110}$$

The YM constraints (3.49) in terms of these variables are as follows

$$\omega_p^2\dot{\gamma}_p + 2\dot{\alpha}_p\sigma^{-2}\left\{\begin{array}{c}0\\1\end{array}\right\} = 0. \tag{3.111}$$

Terms in the upper (lower) part of the braces refer to 'general' (BV) case. The YM dynamical equations (3.50) consist of

$$\begin{aligned}&\ddot{\omega}_p + (f^{-1}\dot{f}+\sigma^{-1}\dot{\sigma})\dot{\omega}_p + f^{-2}\omega_p\left(\sigma^{-2}\tilde{\alpha}_p^2 + \tfrac{1}{2}\tilde{W}_p - f^2\dot{\gamma}_p^2 - \left\{\begin{array}{c}k\\ \sigma^{-2}\end{array}\right\}\right) = 0,\\ &\ddot{\gamma}_p + (2\dot{\omega}_p\omega_p^{-1} + f^{-1}\dot{f}+\sigma^{-1}\dot{\sigma})\dot{\gamma}_p + 2(f\sigma)^{-2}\tilde{\alpha}_p\left\{\begin{array}{c}0\\1\end{array}\right\} = 0,\\ &\ddot{\alpha}_p + (f^{-1}\dot{f}-\sigma^{-1}\dot{\sigma})\dot{\alpha}_p + f^{-2}\tilde{\alpha}_p\omega_p^2 - \tfrac{1}{2}\ell\sigma^2 f^{-2}\left[W_p + p(n-p)k\right]\left\{\begin{array}{c}1\\0\end{array}\right\} = 0\end{aligned} \tag{3.112}$$

and Einstein equations (3.56-3.58) are, respectively,

$$\begin{aligned}3f^{-2}\dot{f}^2 + 2f^{-1}\dot{f}\sigma^{-1}\dot{\sigma} + f^{-2}\left\{\begin{array}{c}k-\frac{1}{4}\ell^2\sigma^2\\ -3\sigma^{-2}\end{array}\right\} &= \kappa f^{-2}(T_1+T_2),\\ \ddot{f} + 2f^{-1}\dot{f}^2 + \dot{f}\sigma^{-1}\dot{\sigma} + f^{-1}\left\{\begin{array}{c}k-\frac{1}{2}\ell^2\sigma^2\\ -2\sigma^{-2}\end{array}\right\} &= \kappa f^{-1}T_1,\\ \ddot{\sigma} + 3f^{-1}\dot{f}\dot{\sigma} - f^{-2}(k\sigma-\ell^2\sigma^3)\left\{\begin{array}{c}1\\0\end{array}\right\} &= \kappa\sigma f^{-2}(T_2-2T_1).\end{aligned} \tag{3.113}$$

with the only non-trivial momentum constraint given by

$$\sigma^{-1}\dot{\sigma}\left\{\begin{array}{c}0\\1\end{array}\right\} = \kappa n f^{-2}\left(\sum_p \tilde{\alpha}_p\dot{\gamma}_p\omega_p^2 - \sum_p \omega_p\dot{\omega}_p\left\{\begin{array}{c}0\\1\end{array}\right\}\right). \tag{3.114}$$

Here we have used the abbreviations

$$\begin{aligned}
\tilde{\alpha}_p &:= 2\alpha_p - \alpha_{p-1} - \alpha_{p+1},\\
W_p &:= \omega_p^2 - \left\{\begin{array}{c}2\ell\alpha_p\\0\end{array}\right\},\\
\tilde{W}_p &:= 2W_p - W_{p-1} - W_{p+1} + \left\{\begin{array}{c}4k\\0\end{array}\right\}.
\end{aligned} \tag{3.115}$$

and

$$\begin{aligned}
T_1 &:= n\left[\sigma^{-2}\sum_p \dot{\tilde{\alpha}}_p\dot{\alpha}_p + \tfrac{1}{4}f^{-2}\left(\sum_p \tilde{W}_pW_p + (1/3)n(n^2-1)k^2\left\{\begin{array}{c}1\\0\end{array}\right\}\right)\right],\\
T_2 &:= n\sum_p\left[\dot{\omega}_p^2 + \omega_p^2\dot{\gamma}_p^2 + \omega_p^2(f\sigma)^{-2}\left(\tilde{\alpha}_p^2 + \left\{\begin{array}{c}0\\1\end{array}\right\}\right)\right],
\end{aligned} \tag{3.116}$$

and it is understood that all subscripted quantities are zero when the index is outside the range $\{1,\dots,n-1\}$.

At this point, we do not intend to give a complete analysis of the above system of differential equations. However, a few points are in order. For the general case, if $\omega_p \neq 0\ \forall p$, $\dot{\gamma}_p = 0$ and the first equation in (3.113), the Hamiltonian constraint, is the only constraint of the system. The dynamical evolution is expected to preserve the constraint $\dot{H} = 0$. Indeed, as a check on the consistency of the above equations, one can show, for example for $G = SU(2)$, that $\dot{H} = -(6\dot{f}/f + 2\dot{\sigma}/\sigma)H$. One observes that there are $2(n-1)$ degrees of freedom associated with YM fields. Such an explicit integration is very complicated for the Bianchi V case, but as mentioned at the end of section 3.3 we would expect the constraints to be conserved in view of the general consistency of the Cauchy problem.

The system above is the set of $SU(n)$-EYM equations for the particular homomorphism from $U(1)$ to $SU(n)$ chosen above for all spatially homogeneous cosmologies with isotropy group $U(1)$. These equations are mildly gauge dependent (A_0 was set to 0). Nevertheless, the gauge-invariant quantities, like the various components of the energy-momentum tensor, are easily expressible in terms of α_p, γ_p, and ω_p.

Table 3.1: This table gives for different homomorphisms $\lambda : \mathbf{su}(2) \to \mathbf{su}(n)$ the number d of dimensions of the solution space of (3.67), the number c of nonzero constraint conditions (3.82) and the number n_{eq} of independent amplitudes that satify second order equations in time. (Trivial homomorphisms and those arising from irreducible representations are not included.)

n	λ	d	c	n_{eq}
3	$1 \oplus 2$	1	0	1
4	$1 \oplus 1 \oplus 2$	1	0	1
	$1 \oplus 3$	3	1	2
	$2 \oplus 2$	4	3	2
5	$1 \oplus 1 \oplus 1 \oplus 2$	1	0	1
	$1 \oplus 1 \oplus 3$	5	6	2
	$1 \oplus 4$	1	0	1
	$1 \oplus 2 \oplus 2$	4	4	2
	$2 \oplus 3$	2	0	2
6	$1 \oplus 1 \oplus 1 \oplus 1 \oplus 2$	1	0	1
	$1 \oplus 1 \oplus 1 \oplus 3$	7	11	2
	$1 \oplus 1 \oplus 4$	1	0	1
	$1 \oplus 1 \oplus 2 \oplus 2$	4	4	2
	$1 \oplus 2 \oplus 3$	4	3	3
	$1 \oplus 5$	1	0	1
	$2 \oplus 2 \oplus 2$	9	11	2
	$2 \oplus 4$	4	4	3
	$3 \oplus 3$	4	5	2

Table 3.2: Values d, c and n_{eq} for the equivalence classes of homomorphisms $\lambda : su(2) \to so(n)$ for small n. Trivial homomorphisms and those arising from irreducible representations are not included. The question marks indicate cases where the constraint equations do not simply imply that some amplitudes are proportional to others.

n	λ	d	c	n_{eq}
4	$1 \oplus 3$	2	0	2
5	$1 \oplus 1 \oplus 3$	3	1	2
	$1 \oplus 4$	1	0	1
6	$1 \oplus 1 \oplus 1 \oplus 3$	4	3	2
	$1 \oplus 1 \oplus 4$	1	0	1
	$1 \oplus 5$	1	0	1
	$3 \oplus 3$	3	1	2
7	$1 \oplus 1 \oplus 1 \oplus 1 \oplus 3$	5	6	2
	$1 \oplus 1 \oplus 1 \oplus 4$	1	0	1
	$1 \oplus 1 \oplus 5$	1	0	1
	$1 \oplus 3 \oplus 3$	5	1	?
	$3 \oplus 4$	2	0	2
8	$1 \oplus 1 \oplus 1 \oplus 1 \oplus 1 \oplus 3$	6	10	2
	$1 \oplus 1 \oplus 1 \oplus 1 \oplus 4$	1	0	1
	$1 \oplus 1 \oplus 1 \oplus 5$	1	0	1
	$1 \oplus 1 \oplus 3 \oplus 3$	7	2	?
	$1 \oplus 7$	1	0	1
	$3 \oplus 5$	3	0	3
	$4 \oplus 4$	6	17	2

Table 3.3: The three-homogeneous cosmologies with a four-dimensional isometry group. WH refers to Weyl-Heisenberg group.

Class	Homogeneous cosmology	Isometry group	l	k
A	BI	$E(2) \otimes U(1)$	0	0
A	BVII$_0$			
B	BV	$BVII_h \otimes U(1)$	-	-
B	BVII$_h$			
B	BIII	$SU(1,1) \otimes U(1)$	0	-1
A	BVIII		1	-1
A	BII	$WH \otimes U(1)$	1	0
A	BIX	$SU(2) \otimes U(1)$	1	1
-	Kantowski-Sachs	$SU(2) \otimes R$	0	1

Chapter 4

Solving the Hamilton-Jacobi equation for gravitationally interacting electromagnetic and scalar fields

4.1. Introduction

Hamilton-Jacobi (HJ) theory has many applications in the perturbative and non-perturbative analysis of dynamical systems in classical mechanics. Peres developed an Einstein-Hamilton-Jacobi (EHJ) formulation of general relativity in which a generating functional has to satisfy the momentum and Hamiltonian constraints of general relativity [50]. In the framework of quantum cosmology, it was known that the momentum constraints require any wave functional to be diffeomorphism invariant [51]. In a WKB approximation, such a requirement translates into the diffeomorphism invariance of the generating functional. The Hamiltonian constraint is a non-linear functional partial differential equation that governs the time evolution of the generating functional.

Based on the formalism developed by Peres, Parry, Salopek and Stewart used a series expansion of the generating functional in spatial gradients of the fields to derive an order-by-order solution of the Hamiltonian constraint for general relativity with matter fields (see Ref. [13] from now on referred to as PSS). Such a generating functional is diffeomorphism invariant in each order of the expansion. Salopek and Bond used this formalism to show how non-linear effects of the metric and scalar fields may be included in stochastic inflationary models. The main advantage of this analysis is that the lapse function and shift vectors do not appear in the EHJ equations. Therefore, one obtains a coordinate free approach to cosmological perturbations. In the above models, matter fields consist of self-interacting scalar and dust fields.

In this chapter the formalism above is extended to minimally coupled gravitationally interacting scalar and electromagnetic fields. Such minimally coupled electromagnetic fields give rise to conformally invariant field equations. Hence, the electromagnetic field energy density is proportional to $1/a^4$ where a is the scale factor. Consequently, the electromagnetic field is diluted away during the de Sitter expansion phase of the inflationary cosmologies. To break the conformal invariance, a direct coupling of gravity to electromagnetism [52] or corrections due to the quantum conformal anomaly have been considered [53]. A coordinate free approach to the perturbative analysis of cosmological models with electromagnetic fields could eventually lead to a better understanding of the primordial magnetic fields. The generating functional up to the third order in the spatial gradient expansion is given in section 4.2. Following PSS, section 4.3 is a demonstration of how a recursion relation and a functional integral in superspace can be used to derive the higher order terms in the spatial gradient expansion from the previous terms. Section 4.4 is an exhibition of the gauge fixing and the solution of the field equations. The perturbations of a flat Friedmann-Robertson-Walker cosmology with a scalar field, up to second order in spatial gradients are given. The application of this formalism is demonstrated in the specific example of the exponential potential $V = V_0 \exp\{-\sqrt{\frac{2}{p}}\phi\}$.

4.2. ADM reduction and the EHJ equations

The action for minimally coupled gravitationally interacting neutral scalar and electromagnetic fields can be written as

$$\mathcal{I} = \int \sqrt{\overset{(4)}{g}}[\tfrac{1}{2}\overset{(4)}{R} - \tfrac{1}{2}\phi_{,\mu}\phi^{,\mu} - V(\phi) - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}]d^4x,\ \mu = (0,1,2,3). \qquad (4.1)$$

where $F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}$ is the electromagnetic field strength and $V(\phi)$ is the scalar field potential. ADM reduction of the above action is achieved by defining the 3-

metric $\gamma_{\mu\nu} := g_{\mu\nu} + n_\mu n_\nu$ and the vector potential $\mathbf{A} = \mathbf{A}_{||} + \mathbf{A}_\perp$, such that $(\mathbf{A}_{||})_\mu = \gamma^\nu_\mu A_\nu$, $(\mathbf{A}_\perp)_\mu = -(n^\nu A_\nu)n_\mu$, where n_μ is the unit vector field normal to space-like hypersurfaces of simultaneity parametrized by t. In the basis $(\partial_t, \partial_i), i = 1, 2, 3$, such that $n^\mu(\partial_i)_\mu = 0$ and $n^\mu = (\partial_t, -N^i\partial_i)/N$ (no sum), the following relations hold: $(\mathbf{A}_{||})_i = \gamma^\nu_i A_\nu$, $A_0 = NA_\perp + N^i A_i$, and

$$ds^2 = -N^2dt^2 + \gamma_{ij}(dx^i + N^i dt)(dx^j + N^j dt). \tag{4.2}$$

Then one proceeds with the procedure outlined in Ref. [54] to derive the Lagrangian L for gravitationally interacting electromagnetic and scalar fields. With the momenta $\pi^{ij} = \dfrac{\delta L}{\delta \dot{\gamma}_{ij}}$, $\pi^\phi = \dfrac{\delta L}{\delta \dot{\phi}}$, $E^i = \dfrac{\delta L}{\delta \dot{A}_i}$ where $\dot{} := d/dt$, after a Legendre transformation, the Hamiltonian is

$$\text{Hamiltonian} = \int (N^\mu \mathcal{H}_\mu + A_0 \mathcal{G}) d^3x, \tag{4.3}$$

where

$$\begin{aligned}
\mathcal{H}_0 &= \gamma^{-1/2}\pi^{ij}\pi^{kl}(2\gamma_{il}\gamma_{jk} - \gamma_{ij}\gamma_{kl}) + \gamma^{1/2}[V(\phi) - R/2 + F^{il}F_{il}/4 + \phi^{,i}\phi_{,i}/2] \\
&+ \gamma^{-1/2}[E^iE_i + (\pi^\phi)^2]/2 = 0, \text{ Hamiltonian constraint,} \\
\mathcal{H}_i &= -2\pi^j_{i\,|j} + F_{il}E^l + \pi^\phi\phi_{,i} = 0, \text{ momentum constraint,} \\
\mathcal{G} &= -E^i{}_{|i} = 0, \text{ Gauss law constraint.}
\end{aligned} \tag{4.4}$$

$_{|i}$ is the 3-covariant derivative (for covariant derivatives of tensor densities and sign conventions see [55]). Utilizing Hamilton's equations, the evolution equations for the fields are as follows:

$$\dot{\phi} = N\gamma^{-1/2}\pi^\phi + N^i\phi_{,i}, \tag{4.5}$$

$$\dot{\gamma}_{ij} = N2\gamma^{-1/2}\pi^{kl}(2\gamma_{il}\gamma_{jk} - \gamma_{ij}\gamma_{kl}) + 2N_{(i|j)}, \tag{4.6}$$

$$\dot{A}_i = N\gamma^{-1/2}E_i + N^jF_{ji} + A_{0,i}. \tag{4.7}$$

The evolution equations for the momenta are considerably more complicated. They are given by

$$\begin{aligned}
\dot{\pi}^{\phi} &= -N\gamma^{1/2}\frac{dV}{d\phi} - \tfrac{1}{2}\gamma^{1/2}\left(N_{,m}\phi^{,m} + N\phi_{|l}{}^{|l}\right) - \left(N^m\pi^{\phi}\right)_{,m}, && (4.8)\\
\dot{\pi}^{ij} &= N\gamma^{-1/2}\left\{\gamma^{ij}\left[\pi^{mn}\pi_{mn} - \tfrac{1}{2}(\pi)^2\right] - 4\pi^{im}\pi^{j}_{m} + 2\pi^{ij}\pi\right.\\
&+ \left.\tfrac{1}{4}\gamma^{ij}E^lE_l - \tfrac{1}{2}E^iE^j + \tfrac{1}{2}\gamma^{ij}(\pi^{\phi})^2\right\} - \tfrac{1}{2}N\gamma^{1/2}\left\{R^{ij} - \tfrac{1}{2}\gamma^{ij}R\right.\\
&- \left.F^i{}_kF^{jk} + \tfrac{1}{4}\gamma^{ij}F_{lm}F^{lm} + \tfrac{1}{2}\gamma^{ij}\phi_{,l}\phi^{,l} - \phi^{,i}\phi^{,j} + \gamma^{ij}V(\phi)\right\}\\
&+ \tfrac{1}{2}\gamma^{1/2}\left(N^{|ij} - N_{|l}{}^{|l}\gamma^{ij}\right) + \left(N^n\pi^{ij}\right)_{,n} - 2\pi^{n(i}N^{j)}{}_{,n}, && (4.9)\\
\dot{E}^i &= \gamma^{1/2}\left(N_{,m}F^{mi} + NF^{mi}{}_{|m}\right) + \left(N^mE^i - N^iE^m\right)_{,m}. && (4.10)
\end{aligned}$$

Instead of solving the evolution equations for the fields and momenta, one can try to solve the EHJ equations. The EHJ equations are derived by the substitutions

$$\pi^{ij} = \frac{\delta S}{\delta\gamma_{ij}},\ \pi^{\phi} = \frac{\delta S}{\delta\phi},\ E^i = \frac{\delta S}{\delta A_i}, \qquad (4.11)$$

in $\mathcal{H}_{\mu}$ and $\mathcal{G}$. $S = S[\gamma_{ij}, \phi, A_i]$ is the generating functional (Hamilton's principal function) [56]. The Hamiltonian constraint is a hyperbolic functional partial differential equation for S. After solving the EHJ equations, (4.5-4.7) and (4.11) yield the full set of the evolution equations.

4.3. The spatial gradient expansion and the order-by-order solution of the EHJ equations

The momentum constraint implies that the generating functional is diffeomorphism invariant [50]-[51]. One such diffeomorphism invariant quantity is $S = \int f[\phi, \gamma_{ij}, A_i]\gamma^{1/2}d^3x$. More generally, a diffeomorphism invariant S can be a multiple integral of some multi-point functions. The contribution of such highly non-local terms could be important, for example, if the spatial inhomogeneities are correlated. However, at least in the lowest orders, the contribution of such terms to the generating functional are expected to be insignificant. Likewise, the Gauss law constraint implies that S is gauge-invariant, e.g. $S = S[F_{ij}]$. Other gauge-invariant quantities like $\oint A_l dx^l$ could also be included in S. However, if the space-like surfaces of simultaneity are simply connected, one can write all such quantities in terms of F_{ij} using Stokes theorem. Non-simply connected three-manifolds are not considered here. The

Hamiltonian constraint determines the time evolution of the fields. Following PSS, an order-by-order solution of the Hamiltonian constraint is achieved by the expansion of the generating functional in spatial gradients:

$$S = \sum_{n=0}^{\infty} \lambda^n S^{(n)}, \tag{4.12}$$

where n denotes the number of spatial gradients in $S^{(n)}$. It turns out that in the limit $R \to 0$, in the absence of electromagnetism, the zeroth order solution is exact. This suggests that the expansion parameter should be $\lambda := \frac{\bar{R}}{V(\phi)}$ where $\bar{R}$ is an appropriate combination of the curvature invariants of dimension L^{-2} such that in the flat space limit where the three-curvature is vanishing, $\bar{R} = 0$. $V(\phi)$ represents the energy density of the scalar field. Therefore, the scalar field is indispensable in this model. As it will be seen, the scalar field dominates the dynamics of the space-time. The convergence of the above series is an unsolved problem [57]. An order-by-order solution of the EHJ equation is achieved by substituting (4.12) in the first equation in (4.4) and the subsequent expansion of $\mathcal{H}_0$ in spatial gradients:

$$\mathcal{H}_0 = \sum_{n=0}^{\infty} \lambda^n \mathcal{H}^{(n)} \tag{4.13}$$

and requiring the EHJ equation to vanish at each order. In the equation above

$$\mathcal{H}^{(0)} = \gamma^{-1/2} \frac{\delta S^{(0)}}{\delta \gamma_{ij}} \frac{\delta S^{(0)}}{\delta \gamma_{kl}} (2\gamma_{il}\gamma_{jk} - \gamma_{ij}\gamma_{kl}) + \gamma^{1/2} V(\phi) + \tfrac{1}{2}\gamma^{-1/2} (\frac{\delta S^{(0)}}{\delta \phi})^2 = 0 \tag{4.14}$$

One can easily obtain the first few terms in (4.12) by an ansatz. For the zeroth order term

$$S^{(0)} = -2 \int \gamma^{1/2} H(\phi) d^3x, \tag{4.15}$$

called the long-wavelength approximation (LWA) which is the same as in PSS for some function $H(\phi)$. Inserting $S^{(0)}$ in (4.14) yields

$$-3H^2 + V(\phi) + 2(\frac{dH}{d\phi})^2 = 0 \tag{4.16}$$

which determines $H(\phi)$ up to some arbitrary integration constant. Electromagnetism has no dynamical degrees of freedom at this order. The LWA is very important in structure formation after inflation. Therefore, it is unlikely that electromagnetic fields play a significant role in structure formation in this model. For $n > 0$

$$\begin{aligned}
\mathcal{H}^{(n)} &= \gamma^{-1/2}(2\gamma_{il}\gamma_{jk} - \gamma_{ij}\gamma_{kl}) \left(2\frac{\delta S^{(0)}}{\delta \gamma_{ij}} \frac{\delta S^{(n)}}{\delta \gamma_{kl}} + \sum_{p=1}^{n-1} \frac{\delta S^{(p)}}{\delta \gamma_{ij}} \frac{\delta S^{(n-p)}}{\delta \gamma_{kl}} \right) \\
&+ \tfrac{1}{2}\gamma^{-1/2}\gamma_{ij} \sum_{p=1}^{n-1} \frac{\delta S^{(p)}}{\delta A_i} \frac{\delta S^{(n-p)}}{\delta A_j} + \gamma^{-1/2} \frac{\delta S^{(0)}}{\delta \phi} \frac{\delta S^{(n)}}{\delta \phi} + \tfrac{1}{2}\gamma^{-1/2} \sum_{p=1}^{n-1} \frac{\delta S^{(p)}}{\delta \phi} \frac{\delta S^{(n-p)}}{\delta \phi} \\
&+ \nu^{(n)}, \\
\nu^{(2)} &= \gamma^{1/2}(-R + \phi^{,i}\phi_{,i} + F_{il}F^{il}/2)/2, \\
\nu^{(n)} &= 0, \text{ for } n \neq 2.
\end{aligned}$$

At each order, $\mathcal{H}^{(n)} = 0$ is a linear hyperbolic functional differential equation in the unknown functional $S^{(n)}$.

$S^{(2)}$ is the next non-vanishing term given by

$$S^{(2)} = \int \gamma^{1/2}(J(\phi)R + K(\phi)\phi_{,i}\phi^{,i} + L(\phi)F_{ij}F^{ij})d^3x. \tag{4.17}$$

The terms above are not the only terms quadratic in spatial gradients in $S^{(2)}$. However, the remaining terms are either equal to the above terms modulo surface integrals or vanish identically. For example, $\int \gamma^{1/2}F^{ij}\epsilon_{ijk}\phi^{,k}d^3x = -\int \gamma^{1/2}(F^{ij|k}\epsilon_{ijk}\phi)d^3x = 0$ due to the Maxwell equations $F_{[ij|k]} = 0$. It is easy to verify that $S^{(2)}$ satisfies the momentum and Gauss law constraints. One notices that $\frac{\delta S^{(2)}}{\delta A_i}$ is already quadratic in spatial derivatives and does not appear in the second order EHJ equation $\mathcal{H}^{(2)} = 0$. After inserting $S^{(2)}$ in the second order EHJ equation and grouping together the coefficients of R, $\phi_{|l}{}^{|l}$, $\phi^{,l}\phi_{,l}$, and $F_{ik}F^{ik}$ one respectively has

$$\begin{aligned}
s_1 &:= HJ - 2\frac{dH}{d\phi}\frac{dJ}{d\phi} - \frac{1}{2} = 0, \quad & s_3 &:= HK - 4J\frac{d^2J}{d\phi^2} + 2\frac{dH}{d\phi}\frac{dk}{d\phi} + \frac{1}{2} = 0, \\
s_2 &:= -H\frac{dJ}{d\phi} + K\frac{dH}{d\phi} = 0, & s_4 &:= HL + 2\frac{dH}{d\phi}\frac{dL}{d\phi} - \frac{1}{4} = 0.
\end{aligned} \tag{4.18}$$

At first sight, (4.18) seems to be an over-determined system for three unknown functions, J, K, and L. However, by solving $s_2 = 0$ for K, one can show that s_3 is not independent and obeys the relation $s_3 = 2\frac{ds_2}{d\phi} - s_1 + \frac{ds_1}{d\phi}H\left(\frac{dH}{d\phi}\right)^{-1}$.

In the spatial gradient expansion of the generating functional S for the scalar fields in the absence of electromagnetism, there is no contribution from odd order terms. Electromagnetism makes non-trivial contributions to odd orders. The only non-vanishing term in $S^{(3)}$ is

$$S^{(3)} = \int \gamma^{1/2} M(\phi) F^{ij}{}_{|j} \epsilon_{ikl} F^{kl} d^3x. \tag{4.19}$$

All other third order terms like $F^{ij}{}_{|j}\phi_{,i}$, $F^{mn}F^{lp}\epsilon_{mnl}\phi_{,p}$, $F^{ij}{}_{|ij}$ either vanish or are total divergences. Because of ϵ_{ikl}, provided that $M(\phi)$ is a scalar, (4.19) is not invariant under parity transformations. Therefore it is not invariant under the large group of diffeomorphisms and should vanish if invariance under such transformations is required.

To solve the third-order EHJ equation $\mathcal{H}^{(3)} = 0$ one has to use the relation $\Gamma^i_{ji} = \gamma^{-1/2}(\gamma^{1/2})_{,i}$. The solution yields

$$MH + \frac{dM}{d\phi}\frac{dH}{d\phi} = 0. \tag{4.20}$$

$S^{(3)}$ non-trivially satisfies the momentum and Gauss law constraints. To show this, one frequently uses the identity $V_s\epsilon_{ijk} = V_i\epsilon_{sjk} + V_j\epsilon_{isk} + V_k\epsilon_{ijs}$ for any vector field V_i, achieved from $\frac{\delta}{\delta\gamma_{rs}}\left(\epsilon^{ijk}\right) = \frac{\delta}{\delta\gamma_{rs}}\left(\gamma^{il}\gamma^{jm}\gamma^{kn}\epsilon_{lmn}\right)$ and multiplication of both sides by V_r. Equations (4.16), (4.18) and (4.20) form a set of differential equations, easily solvable for most relevant potentials. Nevertheless, the full set of differential equations becomes increasingly complicated at higher orders. As in PSS one can use the expression for $S^{(0)}$ and the conformal transformation

$$f_{ij} = \gamma_{ij}\Omega^{-2}(u),\ u := \int (-2\frac{dH}{d\phi})^{-1} d\phi,\ \frac{d\Omega}{du} = H\Omega, \tag{4.21}$$

to solve the EHJ equations $\mathcal{H}^{(n)}$. The EHJ equations transforms into

$$\frac{\delta S^{(n)}}{\delta u(x)}\Big|_{f_{ij},A_i} = -\tilde{\mathcal{R}}^{(n)} \tag{4.22}$$

where

$$
\begin{aligned}
\tilde{\mathcal{R}}^{(n)} &= f^{-1/2}\Omega^{-3}(2f_{il}f_{jk} - f_{ij}f_{kl})\sum_{p=1}^{n-1}\frac{\delta S^{(p)}}{\delta f_{ij}}\frac{\delta S^{(n-p)}}{\delta f_{kl}} \\
&+ \frac{f^{-1/2}}{8\Omega^3}\left(\frac{dH}{d\phi}\right)^{-2}\sum_{p=1}^{n-1}(\frac{\delta S^{(p)}}{\delta u} - 2\frac{\delta S^{(p)}}{\delta f_{lm}}f_{lm}H)(\frac{\delta S^{(n-p)}}{\delta u} \\
&- 2\frac{\delta S^{(n-p)}}{\delta f_{pq}}f_{pq}H) + \frac{f^{-1/2}}{2\Omega}f_{ij}\sum_{p=1}^{n-1}\frac{\delta S^{(p)}}{\delta A_i}\frac{\delta S^{(n-1)}}{\delta A_j} + \tilde{\nu}^{(n)} \\
\tilde{\nu}^{(2)} &= f^{1/2}\{\frac{\Omega}{2}\tilde{R} + \frac{d}{du}(H\Omega)u_{,i}u_{,j}f^{ij} - \frac{1}{4\Omega}F_{ik}F_{lm}f^{il}f^{km}\},\ \tilde{\nu}^{(n)} = 0, \text{for } n \neq 2.
\end{aligned}
\tag{4.23}
$$

$\tilde{R}$ is the conformal curvature and all indices are raised and lowered with the conformal metric f_{ij}.

The proof of the integrability of (4.22) for gravitationally interacting electromagnetic and scalar fields proceeds similarly to what was outlined in [58]. By using the expression for $\tilde{\nu}^{(2)}$ from (4.23), a functional integration of (4.22) gives rise to the following expression for $S^{(2)}$:

$$S^{(2)} = \int f^{1/2}\left(j(u)\tilde{R} + k(u)u_{,i}u^{,i} + l(u)F_{ik}F^{ik}\right)d^3x, \tag{4.24}$$

where

$$j(u) := \int_0^u \frac{\Omega(u')}{2}du' + D,\ l(u) := -\int_0^u \frac{du'}{4\Omega(u')} + D',\ k(u) := H\Omega. \tag{4.25}$$

The complementary functionals D and D' are constants of integration. In the next order $\frac{\delta S^{(3)}}{\delta u} = 0$, therefore $S^{(3)} = \int f^{1/2}F^{ij}{}_{;j}\varepsilon_{ikl}F^{kl}d^3x$ is the most general form of $S^{(3)}$ in which ; and ε_{ijk} are the covariant derivative and Levi-Civita tensor associated with f_{ij}, respectively. Conformal transformation of this expression gives rise to

$$S^{(3)} = \int \gamma^{1/2}\Omega^2 F^{ij}{}_{|j}\epsilon_{ikl}F^{kl}d^3x \tag{4.26}$$

Functional integration of (4.22) in the next order gives rise to

$$
\begin{aligned}
S^{(4)} &= \int d^3x f^{1/2}\{-\ell(u)\tilde{R}^{ij}\tilde{R}_{ij} - (3\ell(u)/8 + m(u))\tilde{R}^2 - n(u)(\tilde{R}^{ij} - f^{ij}\tilde{R}/2)u_{,i}u_{,j} \\
&+ r(u)u_{;l}^{;l}u^{;j}u_{;j} + s(u)(F_{lm}F^{lm})^2 + t(u)u_{,p}u^{,p}F_{lm}F^{lm} + v(u)u_{,m}u^{,n}F^{mi}F_{ni} \\
&+ w(u)u_{,m}F^{mi}F^{n}{}_{i;n} + x(u)F^{mi}{}_{;mn}F^{n}{}_{i} + y(u)F_{km}F_{l}{}^{m}F^{kn}F^{l}{}_{n} + z(u)\tilde{R}F_{ln}F^{ln} \\
&+ a(u)\tilde{R}_{kl}F^{km}F^{l}{}_{m}\}.
\end{aligned}
\tag{4.27}
$$

In the above expression $\ell(u), \cdots, a(u)$ are defined as

$$
\begin{aligned}
\ell'(u) &= \frac{-2j^2(u)}{\Omega^3(u)}, \\
n(u) &= -\frac{j}{\Omega^2}, \\
m'(u) &= -\frac{\left(\frac{\Omega}{2} - jH\right)^2}{8\left[\frac{dH}{d\phi}(u)\right]^2 \Omega^3(u)}, \\
r(u) &= -\frac{1}{4\Omega(u)}, \\
s'(u) &= \Omega^{-3}[-11l^2/4 + (\frac{dH}{d\phi})^{-2}(\Omega^{-1}/4 + Hl)^2/8], \\
a'(u) &= 8jl\Omega^{-3}, \\
z'(u) &= \Omega^{-3}\left(-\frac{5}{2}jl + \tfrac{1}{4}(\frac{dH}{d\phi})^{-2}(1/8 + Hl\Omega/2 - Hj\Omega^{-1}/4 - H^2jl)\right), \\
x'(u) &= -8l^2\Omega^{-1}, \\
v'(u) &= \Omega^{-2}(-4lH + 3\Omega^{-1}/2), \\
y'(u) &= 8l^2\Omega^{-3}, \\
w'(u) &= 4l\Omega^{-1}(2lH + \Omega^{-1}), \\
t'(u) &= \Omega^{-2}(Hl - \frac{\Omega^{-3}}{3}),
\end{aligned}
\tag{4.28}
$$

in which $' := d/du$.

4.4. The evolution equations of the fields

Once the EHJ equations are solved, the evolution equations for the fields are obtained from (4.5-4.7) and (4.11). A judicious choice of gauge greatly simplifies the field equations. In the almost synchronous gauge ($N^i = 0$) if u is the time parameter, from (4.5) it follows that in the LWA (i.e $S = S^{(0)}$) the lapse obeys $N^{(1)} = 1$. The superscript (n) means that the right hand side of the equation contains terms up to $(n-1)$th order in spatial gradients.

The choice of u as the time parameter is valid as long as the geometry is sufficiently close to that of the homogeneous models (for a relevant discussion see Ref. [59]). Then it is useful to replace (4.6) with the equivalent evolution equation

$$
\dot{f}_{ij} = 2N\Omega^{-3}f^{-1/2}\frac{\delta S}{\delta f_{kl}}(2f_{jk}f_{il} - f_{ij}f_{kl}) - 2Hf_{ij} \tag{4.29}
$$

for the conformal metric f_{ij} which is related to γ_{ij} via

$$\gamma_{ij} = \exp\left\{-\int (\frac{dH}{d\phi})^{-1} H d\phi\right\} f_{ij}. \tag{4.30}$$

In LWA $\dot{f}_{ij}^{(1)} = 0$. $f_{ij}^{(1)}$ is the seed metric that contains no dynamical degrees of freedom. The first non-trivial evolution equation for $\dot{A}_i$ in the temporal gauge $A_0 = 0$, is obtained from the conformal transformation of (4.7):

$$\dot{A}_i^{(3)} = N f^{-1/2} \Omega^{-1} f_{li} \frac{\delta S^{(2)}}{\delta A_l} = -4\Omega^{-1} F^m{}_{i;m}. \tag{4.31}$$

Since d/du and ; commute, the evolution equation for F_{ij} is easily derived from the equations above to be

$$\dot{F}_{ij}^{(3)} = 8\Omega^{-1} l(u) F^m{}_{[j;i]m}, \tag{4.32}$$

with the solution

$$F_{ij}^{(3)} = \exp\left\{-\left(8\int du\, \Omega^{-1} l(u)\right) \delta^k_{[i}\delta^l_{j]} \nabla_k \nabla^m\right\} \mathcal{F}_{ml}. \tag{4.33}$$

In the equation above and what follows, $\mathcal{F}_{ij}$ is an arbitrary antisymmetric tensor field, ∇ refers to the covariant derivative with respect to the seed metric and the indices are raised with the seed metric. The exponential of the matrix differential operator is defined as:

$$\exp\{\cdots\} := \left\{\delta_i^m \delta_j^l - \left(8\int \Omega^{-1} l(u)\, du\right) \delta^k_{[i}\delta^l_{j]} \nabla_k \nabla^m \right. \tag{4.34}$$

$$\left. + \frac{1}{2!}\left(8\int \Omega^{-1} l(u)\, du\right)^2 \delta^p_{[i}\delta^q_{j]} \nabla_p \nabla^r \delta^k_{[q}\delta^l_{r]} \nabla_k \nabla^m + \cdots \right\} \mathcal{F}_{ml}. \tag{4.35}$$

Once the evolution equations for the fields are solved, the evolution equations for the momenta are easily derived from (4.11). In particular, the electric field obeys the equation

$$E^{(3)i} = \frac{\delta S^{(2)}}{\delta A_i} = -4 f^{1/2} l(u) F^{mi}{}_{;m} \tag{4.36}$$

In higher orders (4.5) shows that $N^{(n)} \neq 1$. For example,

$$N^{(3)} = 1 - \Omega^{-3}\left[\tilde{R}^{(1)}\left(\frac{dj}{du} - jH\right) + \mathcal{F}_{kl}\mathcal{F}_{mn} f^{(1)km} f^{(1)ln}\left(lH + \frac{dl}{du}\right)\right]\left(-2\frac{dH}{d\phi}\right)^{-2} \tag{4.37}$$

where $\tilde{R}^{(1)}$ is the three-curvature associated with the seed metric $f_{ij}^{(1)}$. Obviously, the higher order evolution equations are more complicated. The third order evolution of the conformal metric is derived from (4.29) and (4.37) to obey

$$\begin{aligned} \dot{f}_{ij}^{(3)} &= \Omega^{-3}\left\{\tilde{R}^{(1)} f_{ij}^{(1)}\left[\frac{H}{2}(\frac{dH}{d\phi})^{-2}(jH - \frac{dj}{du}) + j\right] - 4j\tilde{R}_{ij}^{(1)}\right. \\ &+ \mathcal{F}_{kl}\mathcal{F}_{mn} f^{(1)km} f^{(1)ln} f_{ij}^{(1)}\left[-\frac{H}{2}(\frac{dH}{d\phi})^{-2}(lH + \frac{dl}{du}) + 3l\right] \\ &- \left. 8l\mathcal{F}_{in}\mathcal{F}_{jm} f^{(1)mn}\right\}. \end{aligned} \tag{4.38}$$

As a demonstration of an application of the formalism developed so far, one could compute $\dot{f}_{ij}^{(3)}$ for an arbitrary magnetic field $\mathcal{F}_{kl}$ and seed metric $\dot{f}_{ij}^{(1)}$ with a scalar potential $V = V_0\exp\left\{-\sqrt{\frac{2}{p}}\phi\right\}$. The general solution of (4.16) for $p \neq 1/3$ is given in [60]. The general parametric solution of (4.16) for $p = 1/3$ with H and ϕ as functions of an independent variable v is,

$$H = \left[\frac{V_0}{3}\right]^{\frac{1}{2}} \exp\{-\phi\sqrt{\frac{3}{2}}\} \cosh v, \tag{4.39}$$

$$\phi = \phi_m + \sqrt{\frac{3}{2}}\left(\pm\frac{v}{2} + \frac{e^{\pm 2v}}{4}\right), \tag{4.40}$$

where ϕ_m is the integration constant. A special solution of (4.16) for $p \neq 1/3$ is

$$H = \left[\frac{V_0}{3 - 1/p}\right]^{\frac{1}{2}} \exp\{\frac{\phi}{\sqrt{2p}}\}, \tag{4.41}$$

that corresponds to the Halliwell attractor [61]. By using (4.21), (4.38), (4.41) and with a choice of time parameter such that $\lim_{u\to 0}\phi = -\infty$, the contribution of spatial gradients to the evolution of the conformal metric at this order is

$$\begin{aligned} \dot{f}_{ij}^{(3)} &= \frac{-2c^{-2}}{(p+1)}\left[\frac{V_0}{p(3p-1)}\right]^{-p} u^{-2p+1}\tilde{R}_{ij}^{(1)} \\ &+ \frac{c^{-4}}{(1-p)}\left[\frac{V_0}{p(3p-1)}\right]^{-2p} u^{-4p+1}\left(-\tfrac{1}{2}\mathcal{F}_{mn}\mathcal{F}^{mn} f_{ij}^{(1)} + 2\mathcal{F}_i{}^n\mathcal{F}_{jn}\right), \; p \neq 1, \end{aligned} \tag{4.42}$$

where c is the integration constant and the seed metric is absorbed to raise indices. Integration of the equation above yields

$$\begin{aligned} f_{ij}^{(3)} &= \frac{c^{-4}}{(1-p)(1-2p)}\left[\frac{V_0}{p(3p-1)}\right]^{-2p} u^{-4p+2}\left(-\tfrac{1}{4}\mathcal{F}_{mn}\mathcal{F}^{mn} f_{ij}^{(1)} + \mathcal{F}_i{}^n \mathcal{F}_{jn}\right) \\ &+ \frac{c^{-2}}{p^2-1}\left[\frac{V_0}{p(3p-1)}\right]^{-p} u^{-2p+2}\tilde{R}_{ij} + f_{ij}^{(1)}. \end{aligned} \tag{4.43}$$

After taking (4.30) and (4.42) into account, the three-metric is given by

$$\begin{aligned} \gamma_{ij}^{(3)} &= \frac{c^{-2}}{(1-p)(1-2p)}\left[\frac{V_0}{p(3p-1)}\right]^{-p} u^{-2p+2}\left(-\tfrac{1}{4}\mathcal{F}_{mn}\mathcal{F}^{mn} f_{ij}^{(1)} + \mathcal{F}_i{}^n \mathcal{F}_{jn}\right) \\ &+ \frac{1}{p^2-1}u^2\tilde{R}_{ij}^{(1)} + c^2\left[\frac{V_0}{p(3p-1)}\right]^{p} u^{2p} f_{ij}^{(1)}. \end{aligned} \tag{4.44}$$

Two points regarding the equation above are in order. First, as a perturbation of a spatially flat Friedmann-Robertson-Walker (FRW) model, the growth of inhomogeneities from terms second order in spatial gradients is encoded in $\gamma_{ij}^{(3)}$. One notices that with the choice of u as the time parameter, such a growth resulting from the curvature is insensitive to the exact form of the exponential potential. In other words, deviations from a flat FRW cosmology resulting from a magnetic field evolve quite differently from that of the inhomogeneities due to space-time geometry. In particular, one notices that if $p > 1$, the contribution of a primordial magnetic field to spatial inhomogeneities decays rapidly.

Secondly, the reader is cautioned against a mini-superspace perturbation of a flat FRW model, namely, a perturbation within a mini-superspace cosmology. For example, for a vanishing magnetic field in a Bianchi I cosmology in which $\tilde{R}_{ij} = 0$, one would get the result $\gamma_{ij}^{(n)} = \gamma_{ij}^{(1)}$ for $n \neq 1$. However, this result can not be correct. This is because in a Bianchi I cosmology with a homogeneous scalar field, the time evolution of the three-metric is not a conformal transformation of an initial three-metric. This problem also exists in the dust model treated in PSS which has been quite successful in deriving the contribution of short wavelength fields to structure formation and the anisotropies of microwave background radiation.

4.5. Conclusion

The spatial gradient expansion of the generating functional was developed by PSS to solve the Hamiltonian constraint in EHJ formulation of general relativity for gravitationally interacting dust and scalar fields. The spatial gradient expansion could be consistently applied to solve the Hamiltonian constraint for gravitationally interacting electromagnetic and scalar fields. At each order, the EHJ equation is a linear functional partial differential equation in the unknown functional $S^{(n)}$ which after a conformal transformation could be integrated to yield $S^{(n)}$. $S^{(2)}$ and $S^{(3)}$ were calculated in detail and $S^{(4)}$ was given. Such an order-by-order solution of the EHJ equation gives rise to order-by-order corrections to the fields evolving in a flat FRW model. The corrections are due the presence of spatial inhomogeneities and magnetic field. Not surprisingly, such corrections start with terms second order in spatial gradients. The formalism was applied to the specific example of a scalar field with potential $V = V_0 \exp\{-\sqrt{\frac{2}{p}}\phi\}$. Contributions of all the terms second order in spatial gradients to the metric were derived.

Chapter 5

Epilogue

We have presented a rather formal approach to the study of YM fields in homogeneous cosmologies based on the theory of invariant connections on principal fiber bundles that admit the transitive action of a Lie group. When projected on the base manifold, this action is by isometries of the space-time. Basically, there is a way of finding the YM potentials that give rise to the gauge invariant quantities like $F^A_{\mu\nu}F^{\mu\nu}_A$ or the energy momentum tensor that respect the symmetries of the space-time geometry. Whether such connections exhaust the set of all the YM connections that give rise to gauge invariant quantities, is an open question that awaits further research.

In general, if there is no isotropy, in the temporal gauge, the $SO(n)$-YM and $SU(n)$-YM connections have $\frac{3}{2}n(n-1)$ and $3(n^2-1)$ degrees of freedom, respectively. However, $\frac{n(n-1)}{2}$ components of the $SO(n)$-YM potential and n^2-1 components of the $SU(n)$-YM potential are gauge degrees of freedom which are eliminated if the Gauss law constraints are not satisfied trivially. We showed that if the space-time is axially or spherically symmetric, one can utilize a theorem regarding the invariant connections on homogeneous spaces to restrict the YM connection by solving an algebraic equation. The solution to this algebraic equation depends, first on how the isotropy group is embedded in the isometry group of the space-time and, secondly, on how the isotropy group can be homomorphically embedded in the gauge group. The first problem was tackled in various papers in the literature, although not in a systematic fashion. We showed that the answer to the latter part is related to the representation theory of compact Lie groups. For $SU(n)$ gauge groups and for isotropic ($K_0 = SU(2)$) spatially homogeneous cosmologies, we gave the solution to the field equations for the simplest of such representations, namely, irreducible representations. For $K_0 = SU(2)$ and for reducible representations, the field equations reduce to that of the coupled irreducible ones that have to satisfy the YM constraints. The question of finding all non-equivalent spherically symmetric YM potentials for an arbitrary gauge group $\mathfrak{g}$ is equivalent to finding all embeddings of $SU(2)$ in $\mathfrak{g}$ that are not related by gauge transformations (the adjoint action). This is done by using a Cartan-Weyl basis of $\mathfrak{g}$ and for the $\mathfrak{g}$-YM potential. To the best of our knowledge, this is an open problem that deserves further investigation.

The question of finding all the equivalence classes of $SU(n)$-YM connections for the LRS cosmologies (when the isotropy group is $U(1)$) was partly answered during the investigations of $SU(n)$-YM potentials for static spherically symmetric solutions of the EYM equations where the isotropy group is also $U(1)$. We derived the field equations for all the LRS cosmologies and proved the consistency of the Cauchy problem (except for Bianchi V).

We analyzed the LRS Bianchi I cosmology with $SU(2)$-YM matter field, as a dynamical system. Some of the invariant submanifolds were determined. Not surprisingly, one of the invariant submanifolds has turned out to be the flat space. The investigation of the stability of the flat space as a solution of the field equations was marred by the vanishing of the real parts of the eigenvalues of the corresponding linearized system. A stability analysis of the other solutions that correspond to the LRS electromagnetic fields in Bianchi I cosmology and spatially flat FRW cosmologies was not carried out.

We computed the Liapunov exponent for the LRS $SU(2)$-YM field equations in Bianchi I cosmological models in both synchronous time (the coordinate system of non-accelerating observers) and the conformal time. The Liapunov exponent is vanishing in synchronous time and non-vanishing in conformal time. This apparent discrepancy was not surprising since the Liapunov exponent is known to depend on the time reparametrizations and a satisfactory invariant definition of chaos in general relativity has yet to be found. By the study of the behavior of the YM field variables, we made the statement that $SU(2)$-YM fields in the LRS Bianchi cosmology have milder stochastic properties than that of the flat space homogeneous YM fields that are known to have chaotic properties. We computed the (two point) correlation function between the initial and final values of the YM field variables and showed that it is a decreasing function of time. Although this is a very weak indication of any existing statistical independence as a result of the time evolution, it could be developed into a more powerful indicator of chaos in cosmological models, perhaps by the use of multi-point correlation functions.

Recently, by using the method of fractal basin boundaries, J.D. Barrow and J. Levin showed that the LRS Bianchi I-YM system is indeed chaotic. The method of fractal basins is basically used for dynamical systems that have repellors. A more thorough investigation of this approach to a definition of chaotic behavior in relativistic cosmological models is needed.

Up to this point in the dissertation, the study of gravitationally interacting YM fields has been restricted to mini-superspace (spatially homogeneous) models. However, mini-superspace cosmologies are the singular points of the superspace of the field configurations. In other words, the mini-superspace models are exactly those points in the superspace where the moduli space of gauge connections fails to be a manifold[1]. Therefore, the behavior of the YM fields in mini-superspace models might not be an indicative of the behavior of the YM fields in the whole mini-superspace (non-homogeneous models).

One can use the (Einstein-)Hamilton-Jacobi formulation of general relativity to formulate a perturbative approach to the effects of inhomogeneities in cosmological models. In this approach the recently developed spatial gradient expansion of the generating functional is used to derive an order-by-order solution of the Hamiltonian constraint. In doing so, one has to derive all the diffeomorphism and gauge invariant terms that can contribute to the generating functional at a given order either by ansatz or by a functional integration in superspace. Construction of all such gauge and diffeomorphism invariant terms for non-Abelian YM fields turned out to be a formidable task. Therefore, the study of YM fields in inhomogeneous cosmologies was restricted to electromagnetism.

In this dissertation, the EHJ formulation of general relativity was used to solve the dynamical constraints for gravitationally interacting electromagnetic and scalar fields. The generating functional up to terms fourth order in spatial gradients was

[1]The moduli space of gauge connections is the set that is obtained by factoring out the space of diffeomorphically equivalent field configurations from the superspace. Here gauge loosely refers not only to the gauge transformation of the gauge fields, but also to the diffeomorphisms of the space-like hypersurfaces of homogeneity.

given. The perturbations of the metric and electromagnetic field tensor for a spatially flat FRW cosmology were given. The equations were solved for the specific example of an exponential potential. It turns out that one can not use these equations to investigate minisuperspace perturbations of a flat FRW model. For example, using these equations to add anisotropies to the FRW model, yields contradictory results.

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Appendix A

The numerical computation of Liapunov exponent

To avoid the numerical overflow in the computation of Liapunov exponents, instead of computing ln for a very large number, one divides the evolution time t into n steps such that $t = n\tau$ and defines

$$\sigma_n = \frac{1}{n\tau} \ln \frac{w(n\tau)}{w(0)} = \frac{1}{n\tau} \ln \prod_{m=1}^{n} \frac{w(m\tau)}{w[(m-1)\tau]} = \frac{1}{n\tau} \sum_{m=1}^{n} \ln \frac{w(m\tau)}{w[(m-1)\tau]}, \tag{A.1}$$

where $w := |\mathbf{w}|$ and $\mathbf{w} := w^\mu \partial_\mu$ is the perturbation vector connecting two nearby trajectories. The right hand side of (A.1) still involves the ratios of large numbers and therefore the problem with the overflow is not resolved yet. Let us set $\mathbf{w}_0(0) := \mathbf{w}(0)$, $\mathbf{w}_1(0) := w(0)\dfrac{\mathbf{w}_0(\tau)}{w_0(\tau)}$, $\mathbf{w}_2(0) := w(0)\dfrac{\mathbf{w}_1(\tau)}{w_1(\tau)}$, and in general by $\mathbf{w}_m(0)$ the vector obtained by normalizing the magnitude of $\mathbf{w}_{m-1}(\tau)$ to be $w(0)$. Here, $\mathbf{w}_m(\tau)$ is the vector that is obtained by integrating

$$\dot{\mathbf{w}} = M(\mathbf{x}).\mathbf{w} \tag{A.2}$$

from $\mathbf{x}(m\tau)$ to $\mathbf{x}[(m+1)\tau]$ with the initial condition $\mathbf{w}(0) := \mathbf{w}_m(0) = w(0)\dfrac{\mathbf{w}_{m-1}(\tau)}{w_{m-1}(\tau)}$ (see (2.44) and Fig. A.1). Note that according to these definitions $w_m(0) \equiv w(0)$. One can prove that the following relation holds:

$$\frac{w(m\tau)}{w[(m-1)\tau]} = \frac{w_m(\tau)}{w(0)}. \tag{A.3}$$

Proof: The above relation is proved by induction: $w(\tau) = w_0(\tau)$ holds by definition. If the dragging of the vector $w^\mu(0)$ from $\mathbf{x}(0)$ to $\mathbf{x}(\tau)$ is denoted by $w^\mu(0)\partial_\mu x^\nu(\tau)$ where $\partial_\mu x^\nu(\tau) = \dfrac{\partial x^\nu(\tau)}{\partial x^\mu(0)}$, then the following string of identities hold:

$$\frac{w_1^\nu(\tau)}{w(0)} = \frac{w_1^\mu(0)\partial_\mu x^\nu(2\tau)}{w(0)} = \frac{1}{w(0)}\left(w(0)\frac{w_0^\mu(\tau)}{w_0(\tau)}\right)\partial_\mu x^\nu(2\tau) = \frac{w^\mu(\tau)}{w_0(\tau)}\partial_\mu x^\nu(2\tau) = \frac{w^\nu(2\tau)}{w(\tau)} \tag{A.4}$$

□

Now the right hand side of (A.3) involves the ratios of small quantities. Then one can use (A.1) and (A.3) to obtain the following relation for the Lyapunov exponent.

$$\sigma = \lim_{n\to\infty} \sigma_n. \tag{A.5}$$

It is known that the Lyapunov exponent is independent of $\mathbf{x}(0)$ and $\mathbf{w}(0)$ as long as the integration is performed in the stochastic region.

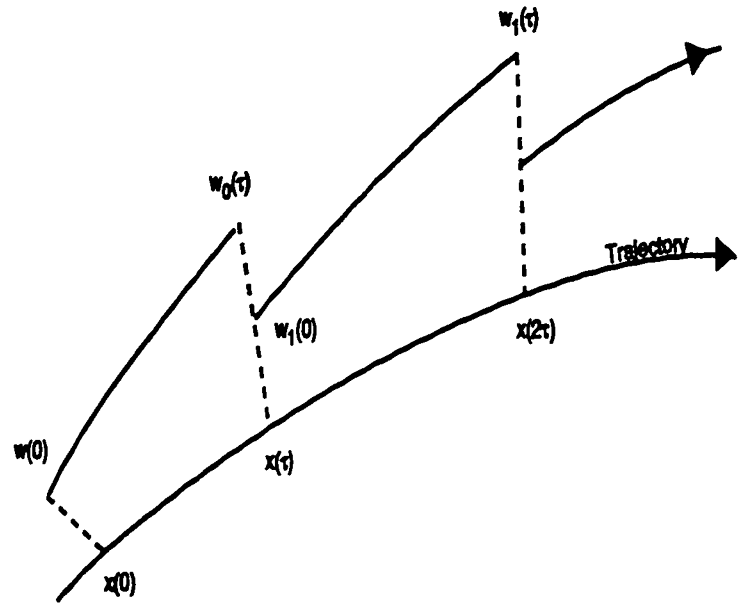

Figure A.1: Successive renormalizations of the perturbation vector prevents the numerical overflow.

Appendix B

The ADM reduction of the action and Hamiltonian formalism

B.1. ADM reduction

Canonical gravity starts with the assumption that the space-time admits a congruence of non-singular vector field T (which we assume to be time-like for convenience) parametrized by t. One can always choose t such that for at least some open interval $t \in (a, b)$, surfaces of $t =$ const., Σ are space-like. Choosing t to be the time parameter is equivalent to choosing Σ to be the surfaces of simultaneity. Given a metric $\overset{(4)}{g}_{\mu\nu}$, the foliation of the space-time with the surfaces Σ defines a time-like vector field n^ν such that $n^\nu n_\nu = -1$ and n^ν is orthogonal to Σ [1]. One can use n^ν to define a projection operator $\gamma^\mu_\nu = \delta^\mu_\nu + n^\mu n_\nu$. Any form field $\mathbf{A}$ or vector field $\mathbf{B}$ can be written as $\mathbf{A} = \mathbf{A}_{||} + \mathbf{A}_\perp$ and $\mathbf{B} = \mathbf{B}_{||} + \mathbf{B}_\perp$ such that $(\mathbf{A}_{||})_\mu = \gamma^\nu_\mu A_\nu$, $(\mathbf{A}_\perp)_\mu = -(n^\nu A_\nu) n_\mu$, $(\mathbf{B}_{||})^\mu = \gamma^\mu_\nu B^\nu$, and $(B_\perp)^\mu = -(n_\nu B^\nu) n^\mu$. One can show that $\gamma_{\mu\nu} = \gamma^\alpha_\mu \gamma^\beta_\nu \overset{(4)}{g}_{\alpha\beta}$. In this sense, $\gamma_{\mu\nu}$ is the three-metric of Σ. In the basis $(\partial_t, \hat{e}_i)$, $i = 1, 2, 3$ such that $n^\mu (\hat{e}_i)\mu = 0$, $\partial_t = T$, and

$$n^\mu = (\partial_t - N^i \partial_i)/N, \tag{B.1}$$

[1]Here I have used the conventions of Ref. [62] where n^ν and n_ν represent a vector field and its associated form field. Latin indices refer to the components of the spatial coordinates.

the following relations hold:

$$(\mathbf{A}_{||})_i = \gamma_i^\nu A_\nu, \; A_0 = NA_\perp + N^i A_i, \tag{B.2}$$

$$ds^2 = -N^2 dt^2 + \gamma_{ij}(dx^i + N^i dt)(dx^j + N^j dt), \tag{B.3}$$

where γ_{ij} is the pull back of $g_{\mu\nu}$ to Σ. One can easily show that in the above basis, $\gamma_j^i = \delta_j^i$, $\gamma_0^i = N^i$, and $\gamma_i^0 = 0$. In the present context, we are only concerned with projection of forms to Σ. This means that $(\mathbf{A}_{||})_i = \gamma_i^\mu A_\mu = A_i$. The relation between these basis is shown in the following figure:

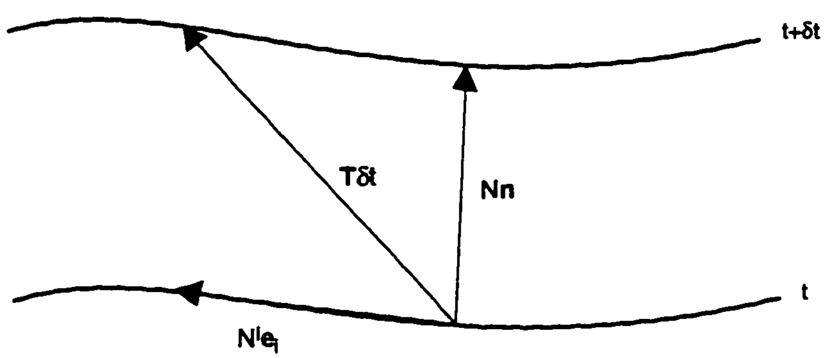

Figure B.1: The lapse and shift as related to the evolution of space-like hypersurfaces.

$\hat{e}_i$ is referred to as a triad of vector basis field. If $\hat{n} := n^\mu$, the choice of $(\hat{n}, \hat{e}_i)$ as a basis is referred to the synchronous gauge. If $d\hat{n} := -n_\mu = Ndt$, the basis dual to $(\partial_t, \hat{e}_k)$ and $(\hat{n}, \hat{e}_i)$ are (dt, θ^i) and $(d\hat{n}, \omega^i)$ respectively. Utilizing (B.1), one can show $\omega^i = N^i dt + \theta^i$. If $\hat{e}_i$ satisfy the integrability conditions $[\hat{e}_i, \hat{e}_j] = 0$, then $e_i = \partial_i$.

B.2. Hamiltonian formulation of gravitationally interacting electromagnetic and scalar fields

P.A.M. Dirac presented a systematic study of the Hamiltonian formulation of constrained systems in 1950 [63]. It was not until 1960 that ADM developed a consistent Hamiltonian formulation of general relativity [64]. One difficulty is that Einstein's theory of general relativity is a fully covariant theory and the splitting of the four-geometry into a dynamical three-geometry and time is not unique. In other words, the solution is a four-geometry. There is some arbitrariness in the choice of the time and introducing dynamics into the system at the beginning is done via ADM reduction explained in Sec. B.1 and is somewhat artificial. A more elaborate notion of time like *many-fingered-time* requires a more detailed investigation of the properties of superspace which is outside of the scope of the present thesis. For a good review of this notion see [65]. Intertwined with the freedom in the choice of a time parameter is the freedom associated with the choice of a coordinate system on Σ. Time reparametrizations and coordinate transformations are the gauge freedoms of general relativity. Such gauge transformations are generated by the constraints of the theory. In a consistent Hamiltonian formulation of a dynamical system, the algebra of constraints is necessarily closed under the Poisson brackets.

To obtain a Hamiltonian formulation for gravitationally interacting scalar and electromagnetic fields, I followed the following steps [54]: One starts with the action

$$\begin{aligned} \mathcal{I} &= \int \sqrt{\overset{(4)}{g}} \left[\tfrac{1}{2} \overset{(4)}{R} - \tfrac{1}{2}\phi_{,\mu}\phi^{,\mu} - V(\phi) - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} \right] d^4x \\ &= \mathcal{I}_{\text{Einstein-Hilbert}} + \mathcal{I}_{\text{scalar field}} + \mathcal{I}_{\text{Maxwell}}, \end{aligned} \tag{B.4}$$

where $F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}$ is the electromagnetic field tensor and $V(\phi)$ is the potential of the scalar field. The reduction of the above action is achieved by the embedding relations of Gauss-Godazzi that relate the 4-curvature $\overset{(4)}{R}$ to the intrinsic curvature R and the extrinsic curvature $k_{ab} = \gamma^{\mu}_{a}\gamma^{\nu}_{b} n_{\mu;\nu}$ of Σ. A semi-colon denotes the 4-covariant derivative. Utilizing Gauss-Godazzi relations, one has

$$\overset{(4)}{R} = R + K^2 - K_{ij}K^{ij} - 2\,\overset{(4)}{R}{}^{\alpha}{}_{\perp\alpha\perp} = R - K^2 + K_{ij}K^{ij} - V^{\alpha}{}_{;\alpha} \tag{B.5}$$

in which $K := \gamma^{ij}K_{ij}$ and $V^{\alpha} = 2(n^{\beta}n^{\alpha}{}_{;\beta} - n^{\alpha}n^{\beta}{}_{;\beta})$. The last term is a total divergence and does not affect the equations of motions. The ADM reduction of the action can be written as $\mathcal{I} = \int L dt = \int \mathcal{L} d^3x\, dt$ where

$$\begin{aligned} \mathcal{L} &= \gamma^{1/2} \left[\tfrac{1}{2}\left(R - K^2 + K^{ij}K_{ij}\right) + \frac{1}{2N}\dot{\phi}^2 - \frac{N^i}{N}\dot{\phi}\phi_{,i} \right. \\ &+ \tfrac{1}{2}\phi_{,i}\phi^{,i} - \frac{N^iN^j}{2N}\phi_{,i}\phi_{,j} - NV(\phi) + \frac{\gamma^{ij}}{2N}\left(\dot{A}_i\dot{A}_j + A_{0,i}A_{0,j} - 2\dot{A}_iA_{0,j}\right) \\ &- \left. \frac{N^i}{2N}\left(\gamma^{ij} - \frac{N^jN^k}{N^2}\right)\left(\dot{A}_j - A_{0,j}\right)F_{ik} - \frac{N}{4}F_{ik}F^{ik} \right], \end{aligned}$$

$\dot{} := \frac{d}{dt}$, and all indices are raised and lowered with γ_{ij}. One can show that

$$K_{ij} = \gamma_i^\mu \gamma_j^\nu n_{\mu;\nu} = -\frac{1}{2}\mathcal{L}_{\hat{n}}\gamma_{ij} = -\frac{1}{2N}\left(\dot{\gamma}_{ij} - 2N_{(i|j)}\right) \tag{B.6}$$

where $_{|j}$ is the three-covariant derivative and $\mathcal{L}_{\hat{n}}$ is the Lie derivative with respect to $\hat{n}$. To perform a Legendre transformation, one defines the momenta

$$\pi^{ij} = \frac{\delta L}{\delta \dot{\gamma}_{ij}} = \gamma^{\frac{1}{2}}\left(\gamma^{ij}K - K^{ij}\right), \tag{B.7}$$

$$E^i = \frac{\delta L}{\delta \dot{A}_i} = \frac{\gamma^{\frac{1}{2}}}{N}\left[\gamma^{ij}\left(\dot{A}_j - A_{0,j}\right) - N^j\gamma^{ij}F_{jk}\right], \tag{B.8}$$

$$\pi^{\phi} = \frac{\delta L}{\delta \dot{\phi}} = \frac{\gamma^{\frac{1}{2}}}{N}\left(\dot{\phi} - N^i\phi_{,i}\right), \tag{B.9}$$

$$\pi^{(N)} = \frac{\delta L}{\delta \dot{N}} = 0, \tag{B.10}$$

$$\pi^{(N^i)} = \frac{\delta L}{\delta \dot{N}^i} = 0, \tag{B.11}$$

$$\pi^{(A_0)} = \frac{\delta L}{\delta \dot{A}_0} = 0. \tag{B.12}$$

The canonical Hamiltonian is given by the Legendre transformation

$$H_c = \int \left(\pi^{ij}\dot{\gamma}_{ij} + E^i\dot{A}_i + \pi^{\phi}\dot{\phi} + \pi^{(N)}\dot{N} + \pi^{(N^i)}\dot{N}_i + \pi^{(A_0)}\dot{A}_0 - \mathcal{L}\right) d^3x. \tag{B.13}$$

(B.10-B.12) are the primary first class constraints of the system [66] and one has to modify the Hamiltonian by

$$\text{Hamiltonian} = H_c + v_N\pi^{(N)} + v_{N^i}\pi^{(N^i)} + v_{A_0}\pi^{(A_0)}. \tag{B.14}$$

The closure of the algebra of constraints under the Poisson brackets $\left\{\text{Hamiltonian}, \pi^{(N,N^i,A_0)}\right\}$ gives rise to secondary first class constraints. However, from the beginning, before performing a Legendre transformation, one can impose the primary first class constraints and treat N, N^i, and A_0 as Lagrange multipliers. After subtracting the surface term $2\oint \pi^{ab}N_b d_a^s$ one arrives at the Hamiltonian

$$\text{Hamiltonian} = \int \left(N^\mu \mathcal{H}_\mu + A_0\mathcal{G}\right) d^3x \tag{B.15}$$

in a fully constrained form in which

$$\begin{aligned}
\mathcal{H}_0 &= \gamma^{-1/2}\pi^{ij}\pi^{kl}(2\gamma_{il}\gamma_{jk} - \gamma_{ij}\gamma_{kl}) + \gamma^{1/2}[V(\phi) - R/2 + F^{il}F_{il}/4 + \phi^{,i}\phi_{,i}/2] \\
&+ \gamma^{-1/2}[E^iE_i + (\pi^\phi)^2]/2 = 0, \text{ Hamiltonian constraint,} \quad \text{(B.16)} \\
\mathcal{H}_i &= -2\pi_{i\,|j}^{j} + F_{il}E^l + \pi^\phi\phi_{,i} = 0, \text{ momentum constraint,} \quad \text{(B.17)} \\
\mathcal{G} &= -E^i{}_{|i} = 0, \text{ Gauss law constraint.} \quad \text{(B.18)}
\end{aligned}$$

For any coordinate q^i and its conjugate momentum p_i, $\dot{q} = \frac{\delta}{\delta p_i}$(Hamiltonian) and $\dot{p}_i = -\frac{\delta}{\delta q_i}$(Hamiltonian) subject to (B.16-B.18) yield the full set of evolution equations of motion. In doing so, one frequently uses the relation

$$\delta(\gamma^{1/2}R) = \frac{\gamma^{1/2}}{2}\left[-R^{lm}\delta\gamma_{lm} + \gamma^{lm}\gamma^{np}(\delta\gamma_{pl|mn} - \delta\gamma_{pn|lm})\right]. \quad \text{(B.19)}$$

The closure of the algebra of constraints under the Poisson brackets is guaranteed for all Hamiltonians of the form (B.14) which are in a fully constrained form [54].

B.3. Hamilton-Jacobi theory for general relativity

The object of chief importance in the Hamilton-Jacobi theory for a dynamical system with the Hamiltonian H in classical mechanics is the Hamilton principal function S that satisfies the differential equation

$$H(p_i, q^i) + \frac{\partial S}{\partial t} = 0, \; p_i = \frac{\partial S}{\partial q^i}. \quad \text{(B.20)}$$

$S = S(q^i, p_i)$ could be regarded as the generator of a canonical transformation $(q^i, p_i) \to (Q^i, P_i)$ such that $H_{\text{new}} = 0$. Therefore, $\dot{P}_i = -\frac{\partial H_{\text{new}}}{\partial Q^i} = 0$ and the new momenta are the constants of integration. In other words, all new coordinates are cyclic [56]. Further insight into the nature of S is gained from the simple consideration that

$$\frac{dS}{dt} = \frac{\partial S}{\partial q^i}\dot{q}^i + \frac{\partial S}{\partial t} = p_i\dot{q}^i - H = L, \quad \text{(B.21)}$$

where L is the Lagrangian of the dynamical system. Therefore, $p_i(t), q^i(t)$, the classical trajectories, are the characteristics of the Hamilton-Jacobi equation. Modulo an additive constant, S is the extremized action of the dynamical system and, hence, $S = S(q^i_{initial}, p_{i\,initial}; q^i_{final}, p_{i\,final})$.

In the Einstein-Hamilton-Jacobi formulation of general relativity, S is a functional of the fields on $\Sigma_{initial}$ and Σ_{final}. Moreover, as was mentioned in the previous section, general relativity with any matter field that enters the action in a covariant form, is a fully parametrized system that obeys the constraints (B.16- B.18). The classical trajectories lie on the constraint surface of the vanishing Hamiltonian and any functional change in the extremized action S due to a change in the final values of the fields on Σ_{final} for gravitationally interacting electromagnetic and scalar fields is given by

$$\delta S = \int \left(\pi^{ij}\delta\gamma_{ij} + E^j\delta A_j + \pi^\phi \delta\phi\right) d^3x. \tag{B.22}$$

Under any infinitesimal coordinate transformation $x^i \to x^i + \xi^i$ generated by the vector field ξ^i, the following relations hold:

$$\begin{aligned} \delta\gamma_{ij} &= \mathcal{L}_{\xi^l}\gamma_{ij} &= \xi_{i|j} + \xi_{j|i}, \\ \delta A_i &= \mathcal{L}_{\xi^l} A_i &= \xi^j A_{i|j} + A_j \xi^j{}_{|i}, \\ \delta\phi &= \mathcal{L}_{\xi^l}\phi &= \phi_{,i}\xi^i, \end{aligned} \tag{B.23}$$

where $\mathcal{L}_{\xi^i}$ is the Lie derivative with respect to the vector field ξ^i. After integration by parts and substitutions

$$\pi^{ij} = \frac{\delta S}{\delta\gamma_{ij}},\ \pi^\phi = \frac{\delta S}{\delta\phi},\ E^i = \frac{\delta S}{\delta A_i}, \tag{B.24}$$

one obtains

$$\delta S = \int \left[-2\gamma_{ki}\left(\frac{\delta S}{\delta\gamma_{kj}}\right)_{|j} + \frac{\delta S}{\delta A_j}F_{ji} + \frac{\delta S}{\delta\phi}\phi_{,i}\right]\xi^i d^3x = \int \mathcal{H}_i\xi^i d^3x. \tag{B.25}$$

Invariance of S under infinitesimal coordinate transformations would require $\delta S = 0$ which in turn due to the arbitrariness of ξ^i translates into $\mathcal{H}_i = 0$. Hence, if S is invariant under the coordinate transformations generated by a vector field, the diffeomorphism constraint (B.17) is satisfied. Likewise, by a similar procedure, one can easily show that the invariance of S under gauge transformations $A_i \to A_i + \Lambda_{,i} \Rightarrow \delta A_i = \Lambda_{,i}$ would require S to satisfy the Gauss law constraint (B.18).

What is shown so far gives clues on how to construct a Hamilton-Jacobi formulation for any field theory in general and general relativity in particular. Substitutions (B.24) in (B.16-B.18) give rise to a system of functional partial differential equations that the Hamilton principal function has to satisfy. Of course, solving such a system of coupled non-linear equations is a non-trivial task. Fortunately, the diffeomorphism and Gauss law constraints have clear geometrical meanings and are solved by a functional S that is both diffeomorphism and gauge-invariant.

More effort is required to find a solution for the Hamiltonian constraint. Due to the functional nature of the constraint equation, one has to prove some integrability conditions. Borrowing some terminology from differential geometry, one has to prove that $\frac{\delta S}{\delta q^i}$ where q^i stand for generalized field variables, are "exact". Moncrief and Teitelboim proved the "exactness" of $\frac{\delta S}{\delta q^i}$ by showing that they are "closed" provided that the momentum constraints are satisfied [67][2]. The Hamiltonian constraint is the generator of the time evolution. Chapter 4 is basically an exposition of a scheme called the spatial gradient expansion to derive an order-by-order solution for the Hamiltonian constraint [13].

[2]The conditions for a closed form to be exact is given by the Poincare lemma [68].