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AMENABLE HYPERGROUPS

BY

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A THESIS

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To my Parents
and to all my Brothers, Sisters,
Nephews and Nieces

Abstract

A hypergroup is essentially a locally compact Hausdorff space in which the product of two elements is a probability measure. Such spaces have been studied by Dunkl, Jewett and Spector. Hypergroups naturally arise as double coset spaces of locally compact groups by compact subgroups. The main purpose of this thesis is to initiate a study of amenable hypergroups analogous to that of amenable locally compact groups.

Let K be a hypergroup (convo in the sense of Jewett) admitting a left Haar measure. A left invariant mean on $L_\infty(K)$ is a positive linear functional of norm one, which is invariant under left translations by elements in K. We say that K is amenable if there is a left invariant mean on $L_\infty(K)$. Amenable hypergroups include commutative hypergroups and central hypergroups.

In this thesis, it is shown that several, but not all, characterizations of amenable groups extend to hypergroups. But, in contrast to the group case, a class of commutative hypergroups K for which every invariant mean on $L_\infty(K)$ is a topological invariant mean is exhibited. The cardinality of the set of all topological invariant means on $L_\infty(K)$ is also given.

Let WAP(K) [UC(K)] be the space of all weakly almost periodic [uniformly continuous] functions on K. It is proved that WAP(K) $\subseteq$ UC(K) and that the two spaces coincide if and only if K is compact. It is also shown that there is a class of hypergroups K including amenable hypergroups for which WAP(K)

admits a unique invariant mean.

The Banach algebra $L_1(K)$ is shown to be Arens regular if and only if K is finite. It is further proved that if K is nondiscrete or infinite, discrete and amenable, then the radical of $L_1(K)^{**}$ is not norm separable.

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CHAPTER 1

INTRODUCTION

C. Dunkl [25], I. Jewett [52], and R. Spector [87] independently initiated the theory of hypergroups in the early 1970's . Hypergroups are sufficiently general to cover a variety of important examples including double coset spaces, but yet have enough structure to allow a substantial theory to develop. The definitions of hypergroups given by the three authors are not identical but the ideas are essentially the same and all interesting examples are hypergroups by all the definitions. . Jewett calls hypergroups "Convos" in his paper [52]. In [77], J. S. Pym obtained generalizations of known results about convolution algebras over semigroups which are close to hypergroups. A fairly complete history is given in K. A. Ross's survey article [82]. In recent years, spectral synthesis properties of commutative hypergroups [11], [54], and representation theory on compact and central hypergroups [90], [46], [6], have been extensively studied by harmonic analysts. In [56] and [95], almost periodic functions and weakly almost periodic functions were studied. Some work on probability theory and harmonic analysis for hypergroups can be found in [6], [7], [32] and [33]. The above papers of course do not include all the work done on hypergroups since the theory was born.

Let G be a locally compact group, and let $C(G)$ be the Banach space of bounded continuous functions on G with the supremum norm. A linear functional m on $C(G)$ is called a *mean* if $m(1) = \|m\| = 1$.

For $f \in C(G)$, $a \in G$, define $_a f(g) = f(ag)$, $g \in G$.

A mean m on $C(G)$ is said to be *left invariant* if $m(_a f) = m(f)$ for all $f \in C(G)$, $a \in G$. G is called *amenable* if there is a left invariant mean on $C(G)$. In [72], J. von Neumann made a systematic study of amenable discrete groups. He showed that any solvable group is amenable and that the free group on two generators is not amenable. J. Dixmier obtained some fundamental characterizations of amenable discrete groups in [21] . The term "amenable" was introduced by M. M. Day [17]. Since the appearance of his paper, the subject of amenability has continued to grow, and has been found useful in many areas of mathematics and related fields. Amenability of locally compact groups was initially studied, among others, by E. Granirer, F.P. Greenleaf, A. Hulanicki and H. Reiter. F. P. Greenleaf [42] is a classic reference, and at present the most recent monograph of A. L. T. Paterson [73] and a relatively new book of J. P. Pier [74] contain more up to date material on amenable locally compact groups.

The main purpose of this thesis is to initiate a systematic study of amenable hypergroups, which are analogues to amenable locally compact groups. Many of our results are on hypergroups with left Haar measures. It is still unknown if an arbitrary hypergroup admits a left Haar measure, but all the known examples of hypergroups do. Let K be a hypergroup with a left Haar measure λ, and let $L_\infty(K)$ be the Banach space of all essentially bounded Borel measurable functions on K with the essential supremum norm. Unless otherwise stated, we use the definitions and notations of I. Jewett [52].

Again, a linear functional m on $L_\infty(K)$ is a mean if $m(1) = \|m\| = 1$.

We say that K is *amenable* if there is a mean m on $L_\infty(K)$ such that $m({}_xf) = m(f)$ for all $x \in K$, $f \in L_\infty(K)$, where ${}_xf(y) = f(x * y)$, $y \in K$.

A mean m on $L_\infty(K)$ is called *topologically left [right] invariant* if $m(\phi * f) = m(f)$ $[m(f * \check{\phi}) = m(f)]$ for all $\phi \in L_1(K)$ with $\phi \geq 0$, $\|\phi\|_1 = 1$, and $f \in L_\infty(K)$, where $\check{\phi}(x) = \phi(\check{x})$ and $x \longrightarrow \check{x}$ denotes the involution on K. The set of all two sided [left] topological invariant means on $L_\infty(K)$ is denoted by $TIM(L_\infty(K))$ $[TLIM(L_\infty(K))]$.

A mean m on $L_\infty(K)$ is called *inversion invariant* if $m(\check{f}) = m(f)$ for all $f \in L_\infty(K)$, and the set of all topological and inversion invariant means on $L_\infty(K)$ is denoted by $TIIM(L_\infty(K))$.

This thesis consists of six chapters. Chapter 2 contains a summary of definitions and notations used throughout the thesis.

In Chapter 3 , we define and study the properties of amenable hypergroups. Examples of amenable hypergroups are given in section 3.2 . Compact and commutative hypergroups are amenable, and the maximal subgroup of an amenable hypergroup is amenable. It follows that K is amenable if and only if $TLIM(L_\infty(K)) \neq \phi[TIIM(L_\infty(K)) \neq \phi]$. It is a well known result of E. Granirer [40] and W. Rudin [84] that if G is a non-discrete locally compact group which is amenable as a discrete group, then there is a left invariant mean on $L_\infty(G)$ which is not topologically left invariant. We show in this section that this is not the case in general for hypergroups.

In Section 3.3, we extend some of the important and well known characterizations of amenable locally compact groups to hypergroups. In particular, we prove an analogue of the Day-Rickert fixed point theorem for hypergroups. It is shown that the Reiter-Glicksberg property (RG) characterizes amenability of hypergroups. We also show that K satisfies Reiter's condition (P_1) if and only if K is amenable. Inspired by the work of L. Gallardo and O. Gebuhrer [32, Chapitre IV], we study hypergroups which have property (P_2). A commutative hypergroup satisfies (P_2) if and only if the support of the Plancherel measure on the dual contains the trivial character. We show that every hypergroup which satisfies (P_2) is amenable, or equivalently has property (P_1). The converse is not true in general. However, if K has a compact supernormal subhypergroup [7], then K satisfies (P_2) if (and only if) K is amenable.

In [59], A. T. Lau introduced and studied a class of Banach algebras which he called F-algebras. They include the Banach algebra $L_1(K)$. He obtained several characterizations of F-algebras which admit topological left invariant means. It follows from the definitions that the F-algebra $L_1(K)$ has a topological left invariant mean if and only if K is amenable. In [75, p. 82], J. P. Pier calls F-algebras Lau algebras. Properties related to amenability are also considered in [32, Chapitre IV] for commutative hypergroups.

In [15, § V], C. Chou gave a different proof of his earlier result [12] that if G is a σ-compact non-compact amenable locally compact group, then the cardinality $|TIM(L_\infty(G)| \geq 2^c$. E. Granirer, assuming the continuum hypothesis, proved this result in [39, § III]. Recently, A. T. Lau and A. L. T. Paterson proved in [62, Theorem 1] that if G is an arbitrary non-compact amenable locally compact group, then $|TLIM(L_\infty(G))| = 2^{2^d}$, where d is the smallest cardinality of a cover of G by compact subsets of G . Subsequently, Z. Yang proved in [98, Corollary 3.4], by extending the ideas of C. Chou [15, § V], that if G is non-compact and amenable, then the cardinality of $TIIM(L_\infty(G))$ is at least (and hence equal to) 2^{2^d} . The interested reader is referred to [73, Chapter 7], [74, § 22] and [42] for earlier references on the subject.

Motivated by the work of the last three authors [62], [98], we prove in Section 3.4 that if K is non-compact and amenable, then $|TIIM(L_\infty(K))| = |TIM(L_\infty(K))| = 2^{2^d}$, where as before, d is the smallest cardinality of a cover of K by compact sets. We also prove that if in addition the maximal subgroup of K is open, then $|TLIM(L_\infty(K))| = 2^{2^d}$. It is 1 if K is compact. We give some applications of these results.

Chapter 4 is devoted to a study of weakly almost-periodic functions and almost-periodic functions on hypergroups. This is a continuation of the work of S. Wolfenstetter [95] on weakly almost-periodic functions for hypergroups. Let WAP(K) [AP(K)] be the space of all weakly almost [almost] periodic functions on K, and let UC(K) be the space of bounded uniformly continuous functions. In Section 4.2, we prove that $WAP(K) \subseteq UC(K)$ and that they are equal if and only if K is compact. This result is due to E. Granirer [39, p. 62-64] for the case when K is a group.

It is well known that if G is any locally compact group, then WAP(G) admits a unique invariant mean [42, § 3.1]. We show in Section 4.3 that there is a class of hypergroups K including amenable hypergroups for which WAP(K) admits a unique invariant mean.

Chapter 5 contains two main theorems on the second dual of the Banach algebra $L_1(K)$. The first generalizes a result of N. J. Young [99, p. 59-62]. We prove in Section 5.2 that $L_1(K)$ is

Arens regular if and only if K is finite. E. Granirer proved in [41, p. 321-324] that if G is a non-discrete locally compact group or an infinite discrete amenable group, then the radical of the Banach algebra $L_\infty(G)^*$ is not norm-separable. We show that this result remains valid for hypergroups.

We should mention that several but not all of our proofs are similar to the group case. In places where the proofs are almost identical, we omit the details or only give a sketch. To make it easier for the reader, we give a fairly complete set of references. The reader is referred to the above mentioned books ([42], [73, [74]) on amenability for more information.

In Chapter 6, we state some problems in hypergroups and/or groups which remain open for further research.

CHAPTER 2

PRELIMINARIES

In this chapter, we include some definitions and notations used throughout the thesis. We also obtain a few basic properties of bounded uniformly continuous functions on hypergroups.

§ 2.1 Notations and Definitions.

Let X be a locally compact Hausdorff space. The following notations are used throughout the thesis:

$\mathbb{C}$	The complex numbers
$C(X)$	The bounded complex-valued continuous functions on X
$C_0(X)$, $C_c(X)$	The members of $C(X)$ which are zero at infinity, with compact support respectively
$C_c^+(X)$	The members of $C_c(X)$ which are non-negative

clA or $\bar{A}$	The closure of the set $A \subseteq X$
1_A	The characteristic function of the nonempty set $A \subseteq X$
Borel set	A member of the smallest σ-algebra which contains the open sets
$M(X) = C_0(X)^*$	The regular Borel complex measures on X
$M^+(X)$, $M_c(X)$, and $M_c^+(X)$	Those which are non-negative, with compact support, both
$M'(X)$	Subset of $M^+(X)$ consisting of probability measures
$M'_c(X)$	Those with compact support
δ_x	The point mass at $x \in X$
$\int f(x)d\mu(x)$	$\int f\, d\mu$
$f\mu$	The measure, if it exists, such that $\int g\, d(f\mu) = \int gf\, d\mu$, for all $g \in C_c(X)$

$\bar{f}$	The complex conjugate of a function f on X
$\|f\|_\infty$	$\text{Sup}\|f(x)\|$ for a bounded function f on X
$\|\mu\|$	The total variation norm of $\mu \in M(X)$
spt f	The support of the function f
spt μ	The support of the measure μ

Always, an unspecified topology on $M^+(X)$ is the cone topology. This is the weak topology induced on $M^+(X)$ by the family $C_c^+(X) \cup \{1\}$, and equal to the weak* topology if and only if X is compact, where $1 = 1_X$.

Definition 2.1.1

A non-void locally compact Hausdorff space K will be called a *hypergroup* if the following conditions are satisfied:

(1) $M(K)$ admits a binary operation $*$ under which it is a complex algebra.

(2) The binary mapping $* : M(K) \times M(K) \longrightarrow M(K)$ given by $(\mu,\nu) \longrightarrow \mu * \nu$ is non-negative

($\mu * \nu \geq 0$ whenever $\mu, \nu \geq 0$) and continuous on $M^+(K) \times M^+(K)$.

(3) If $x, y \in K$, then $\delta_x * \delta_y$ is a probability measure with compact support.

(4) The mapping $(x,y) \longrightarrow spt\ \delta_x * \delta_y$ of $K \times K$ into the space $\mathcal{C}(K)$ of compact subsets of K is continuous, where $\mathcal{C}(K)$ is given a topology on subsets studied by Michael in [66]. A subbasis for the topology of $\mathcal{C}(K)$ is given by all $\mathcal{C}_{U,V} = \{ A \in \mathcal{C}(K) : A \cap U \neq \emptyset , A \subseteq V \}$, where U, V are open subsets of K .

(5) There exists a (necessarily unique) element e in K such that $\delta_x * \delta_e = \delta_e * \delta_x = \delta_x$ for all $x \in K$.

(6) There exists a (necessarily unique) involution $x \longrightarrow \mathring{x}$ (a homeomorphism $x \longrightarrow \mathring{x}$ of K onto itself such that $\mathring{\mathring{x}} = x$ for all $x \in K$) such that for $x, y \in K$, $e \in spt\ \delta_x * \delta_y$ if and only if $x = \mathring{y}$, and $(\mu * \nu)^\circ = \mathring{\nu} * \mathring{\mu}$ for all $\mu, \nu \in M(K)$, where $\mathring{\mu} \in M(K)$ is defined by $\mathring{\mu}(A) = \mu(\mathring{A})$ for Borel subsets A of K and $\mathring{A} = \{ \mathring{x} : x \in A \}$.

The element e will be called the identity of K.

Hypergroups naturally arise as double coset spaces of locally compact groups by compact subgroups [52, § 8]. In particular, locally compact groups are hypergroups. The definition of hypergroups above is the one given by I. Jewett [52] who called them convos. A survey of the subject appears in K. A. Ross [82].

If f is a Borel function on K and $x, y \in K$, the left translation ${}_xf$ or L_xf and the right translation f_y or R_yf are given by

$$L_xf(y) = {}_xf(y) = f(x * y) = \int f \, d\, \delta_x * \delta_y = f_y(x) = R_yf(x) ,$$

if this integral exists. The function $\mathring{f}$ is given by $\mathring{f}(x) = f(\mathring{x})$.

Many of our results require the existence of a left Haar measure. Throughout, K will denote a hypergroup with a left Haar measure λ. Hence, by definition, λ is a non-negative regular Borel (not necessarily bounded) measure on K such that $\delta_x * \lambda = \lambda$ for every $x \in K$. The modular function Δ is defined on K by the identity $\lambda * \delta_{\mathring{x}} = \Delta(x)\lambda$. The mapping $x \longrightarrow \Delta(x)$ is a homomorphism from the hypergroup K to the multiplicative group of positive real numbers. Note that Δ is constant on the sets $\{x\} * \{y\}$ $(x, y \in K)$ with the value $\Delta(x)\Delta(y)$ [52, 5.3]. If K is compact or discrete, then it admits a left Haar measure. Compact hypergroups are unimodular [52, § 7]. A hypergroup K is called commutative if $\delta_x * \delta_y = \delta_y * \delta_x$ for all $x, y \in K$. All commutative hypergroups admit Haar measures, as shown by R. Spector [88]. More hypergroups admitting left Haar measures will be mentioned along the way.

The integral $\int$---$d\lambda(x)$ is often denoted by $\int$---dx. If the presence of a left Haar measure is not needed, we use other symbols (eg. J) to denote the corresponding hypergroups. Notations and facts which are used without explicit reference can be found in [52].

Let $(L_p(K), \|\cdot\|_p)$, $1 \le p \le \infty$, denote the usual Banach spaces of Borel functions. For $f \in L_1(K)$, we write $f^{\sim}(x) = \overline{f(\mathring{x})}\Delta(\mathring{x})$ $(x \in K)$. Then $f^{\sim} \in L_1(K)$ with $\|f^{\sim}\|_1 = \|f\|_1$. If $f \in L_p(K)$, $x \in K$, $1 \le p \le \infty$, then $\|{}_xf\|_p \le \|f\|_p$, and this is not an isometry in general [52, 3.3]. The mapping $x \longrightarrow {}_xf$ is continuous from K to $(L_p(K), \|\cdot\|_p)$, $1 \le p < \infty$ [52, 2.2B and 5.4H].

For $x \in K$, $f \in L_p(K)$, write $f \circ \delta_x = f_{\mathring{x}}\Delta(\mathring{x})$ (Note that $f \circ \delta_x$ is denoted $f * \delta_x$ in [47, § 20]). Then, it is easy to see that, for $x \in K$, $f \in L_p(K)$, $1 \le p \le \infty$,

f_x, $f \circ \delta_x \in L_p(K)$, and $\|f_x\| \le \Delta(\mathring{x})^{\frac{1}{p}}\|f\|_p$, and $\| f \circ \delta_x\|_p \le \|f\|_p$. Also, if $1 \le p < \infty$, then the mappings $x \longrightarrow f_x$ and $x \longrightarrow f \circ \delta_x$ from K into $(L_p(K), \|\cdot\|_p)$ are continuous [47, 20.4] .

Write $P(K) = \{ \phi \in L_1(K) : \phi \geq 0 , \|\phi\|_1 = 1 \}$, and $P_c(K) = P(K) \cap C_c(K)$.

The proof of the next result is similar to the group case (see [52, § 5] and [47, 20.15]) .

<u>Lemma 2.1.2</u> Let $\mathcal{U}$ be the family of all neighbourhoods of e and regard $\mathcal{U}$ as a directed set in the usual way :

$$U \geq V \quad \textit{if} \quad U \subseteq V .$$

For each $U \in \mathcal{U}$, choose a function $\phi \in C_c^+(K)$ such that $\int \phi_U(x)dx = 1$ and ϕ_U vanishes outside U . Then

$$\{ \phi_U \}_{U \in \mathcal{U}} \subseteq P_c(K)$$

is a bounded approximate identity for $L_1(K)$.

For non-empty subsets A, B of K , write

$$A * B = \bigcup_{\substack{x \in A \\ y \in B}} spt\ \delta_x * \delta_y \quad .$$

A non-empty closed subset H of K is called a *subhypergroup* of K if $\mathring{H} = H$ and $H * H \subseteq H$. Let

$$G(K) = \{ x \in K : \delta_x * \delta_{\mathring{x}} = \delta_{\mathring{x}} * \delta_x = \delta_e \} .$$

Then $G(K)$ is closed in K and a locally compact group. It is called the *maximal subgroup* of K. For each $x \in K$, $y \in G(K)$, there exists a unique $z \in K$ such that $\delta_x * \delta_y = \delta_z$ [52, 10.4B]. We write $z = xy$.

Let H be compact subhypergroup of K. The double cosets of H in K are the sets $HxH = H * \{x\} * H$, $x \in K$. The collection of double cosets of H will be denoted by $K//H$. It will be given the quotient topology with respect to the natural projection $\Pi : K \longrightarrow K//H$, defined by $\Pi(x) = HxH$. $K//H$ becomes a hypergroup under the convolution defined by

$$\int f \, d\delta_{HxH} * \delta_{HyH} = \int f \circ \Pi \, (x * t * y) \, dt ,$$

$f \in C_c(K//H)$, $x, y \in K$.

There is a left Haar measure on $K//H$, given by

$$\int f \, d \, HxH = \int f \circ \Pi(x) \, dx ,$$

$f \in C_c(K//H)$ [59, 14.2].

We close this section with the definition of hypergroup joins. Let H be a compact hypergroup and J a discrete hypergroup with $H \cap J = \{e\}$, where e is the identity of both hypergroups. Let $K = H \cup J$ have the unique topology for which both H and J are closed subspaces of K. That is, a set $A \subseteq K$ is open in K if and only if $A \cap H$ is open in H. Let σ be the normalized Haar measure on H and define the operation $\cdot$ on K as follows:

(i) If $s, t \in H$ then $\delta_s \cdot \delta_t = \delta_s * \delta_t$

(ii) If $a, b \in J$ and $a \neq b$ then

$\delta_a \cdot \delta_b = \delta_a * \delta_b$

(iii) If $s \in H$ and $a \in J$ $(a \neq e)$ then

$\delta_s \cdot \delta_a = \delta_a \cdot \delta_s = \delta_a$

(iv) If $a \in J$ and $a \neq e$, and

$\delta_{\mathring{a}} * \delta_a = \sum_{b \in J} c_b \delta_b$, where c_b's are non-negative, only finitely many are non-zero and $\sum_{b \in J} c_b = 1$, then

$$\delta_{\mathring{a}} \cdot \delta_a = c_e \sigma + \sum_{b \in J \setminus \{e\}} c_b \delta_b \ .$$

We call the hypergroup K the *join* of H and J and write $K = H \vee J$. Observe that H is an open subhypergroup of K , but J is not a subhypergroup unless or H is equal to $\{e\}$. R. C. Vrem showed in [92, Proposition 1.1] that $K = H \vee J$ always has a left Haar measure. Indeed, if σ is the normalized Haar measure on H , and J has the discrete Haar measure

$$\sum_{x \in J} [x] \delta_x \ ,$$

where

$$[x] = \frac{1}{\delta_{\check{x}} * \delta_x(\{e\})},$$

then

$$\sigma + \sum_{x \in J \setminus \{e\}} [x]\delta_x$$

is a left Haar measure on K . He also showed in [92, Proposition 1.3] that $K//H \cong J$ as hypergroups.

§ 2.2 Invariant Subspaces OF $L_\infty(K)$.

Recall that $L_\infty(K)$ is the space of bounded Borel measurable complex valued functions on K . We identify functions which differ only on a locally null set with respect to the left Haar measure λ . Then $L_\infty(K)$ is a Banach algebra with the essential supremum norm $\|\cdot\|_\infty$, and $L_1(K)^* = L_\infty(K)$ [52, 6.2] . $C(K)$ is a norm closed subspace of $L_\infty(K)$ in a natural way. Write

$UC_r(K) =$

$= \{ f \in C(K) : x \longrightarrow {}_x f$ is continuous from K to $(C(K) , \|\cdot\|_\infty) \}$,

$UC_\ell(K) =$

$= \{ f \in C(K): x \longrightarrow f_x$ is continuous from K into $(C(K) , \|\cdot\|_\infty) \}$,

and $UC(K) = UC_\ell(K) \cap UC_r(K)$. Functions in $UC_r(K)$ [$UC_\ell(K)$] are called *bounded right [left] uniformly continuous* functions on

K , and functions in $UC(K)$ are called *bounded uniformly continuous*. A subset $X \subseteq L_\infty(K)$ is called *left [right] translation invariant* if ${}_x f \in X$ [$f_x \in X$] for all $x \in K$, $f \in X$. Both $C(K)$ and $L_\infty(K)$ are (two-sided) translation invariant [52, 3.1B and 6.2B].

Lemma 2.2.1 Each of the spaces $UC_r(K)$, $UC_\ell(K)$, $UC(K)$ is a norm closed conjugate closed, translation invariant subspace of $C(K)$ containing the constants and the continuous functions vanishing at infinity. Furthermore,

(i) $UC_r(K) = L_1(K) * UC_r(K) = L_1(K) * L_\infty(K)$;

(ii) $UC_\ell(K) = UC_\ell(K)^\circ * L_1(K) = L_\infty(K) * L_1(K)^\circ$;

(iii) $UC_r(K) * L_1(K)^\circ \subseteq UC_r(K)$ and

$L_1(K) * UC_\ell(K) \subseteq UC_\ell(K)$;

(iv) $UC(K) = L_1(K) * UC(K) = UC(K) * L_1(K)$.

Proof: It is easy to see that $UC_\ell(K)$ is a norm closed, conjugate closed right translation invariant subspace of $C(K)$ containing the constants and the continuous functions vanishing at infinity (see [52, 4.2F] . Let $\phi \in L_1(K)$, $f \in L_\infty(K)$. The

$$| \phi * f(x) - \phi * f(y) | = | \int [{}_x\phi(u) - {}_y\phi(u)] f(\mathring{u})du | \leq \|f\|_\infty \| {}_x\phi - {}_y\phi \|_1 .$$

Since $x \longrightarrow {}_x\phi$ is continuous, $\phi * f$ is continuous. Now,

$$\| \delta_x * (\phi * f) - \delta_y * (\phi * f) \|_\infty \le \|f\|_\infty \| \delta_x * \phi - \delta_y * \phi \|_1 ,$$

and also $\| \phi * f \|_\infty \le \|\phi\|_1 \|f\|_\infty$. Therefore, $\phi * f \in UC_r(K)$.

$UC_r(K)$ becomes a Banach left $L_1(K)$ - module (see [8, p. 49] for the definition of a Banach left A-module). Let $\varepsilon > 0$ and $f \in UC_r(K)$ be given. Choose a neighbourhood V of e such that $\| \delta_x * f - f \|_\infty < \varepsilon$ for all $x \in V$. Let ϕ_V be a non-negative function in $L_1(K)$ such that $\|\phi_V\|_1 = 1$ and ϕ_V vanishes outside V . Then

$$| \phi_V * f(x) - f(x) | = | \int_K \phi_V(y) \ \delta_y * f(x) dy - \int_K f(x) \ \phi_V(y) dy |$$

$$\le \int_V \phi_V(y) \ | \delta_y * f(x) - f(x) | \ dy \le \varepsilon .$$

That is, $\| \phi_V * f - f \|_\infty \le \varepsilon$. Thus $L_1(K) * UC_r(K)$ is norm dense in $UC_r(K)$. Since $L_1(K)$ has a bounded approximate identity, by Cohen's factorization theorem [48, 32.22] , we have $L_1(K) * UC_r(K) = UC_r(K)$. This proves (i) .

If $f \in UC_r(K)$, write $f = \phi * h$, $\phi \in L_1(K)$, $h \in UC_r(K)$. Then, for $x \in K$,

$$\delta_x * f = \delta_x * (\phi * h) = (\delta_x * \phi) * h \in UC_r(K)$$

since $\delta_x * \phi \in L_1(K)$. Hence $UC_r(K)$ is left translation invariant. Similar assertions are true for $UC_\ell(K)$ since $f \longrightarrow \check{f}$ is a linear isometry of $C(K)$ onto itself and

$$\left({}_{\overset{\circ}{x}}(f^{\circ}) \right)^{\circ} = f_x ,$$

for $x \in K$, $f \in C(K)$. (Note that $f \in UC_r(K)$ if and only if $\check{f} \in UC_\ell(K)$) .

To prove (iii), let $\phi \in L_1(K)$, $f \in UC_r(K)$, $x, y \in K$. Then $\| {}_x(f * \check{\phi}) - {}_y(f * \check{\phi}) \|_\infty \le \| {}_xf - {}_yf \|_\infty \|\phi\|$, and hence $f * \check{\phi} \in UC_r(K)$. If $f \in UC_\ell(K)$ and $\phi \in L_1(K)$, then

$$\phi * f = (\check{f} * \check{\phi})^\circ \in UC_\ell(K) .$$

To see (iv), note from (i) and (iii) that $L_1(K) * UC(K) \subseteq UC(K)$. Also it follows as in the case of $UC_r(K)$ that $L_1(K) * UC(K) = UC(K)$. Since $f \longrightarrow \check{f}$ is a linear isometry of $UC(K)$ onto itself, we have $UC(K) = UC(K) * L_1(K)$. □

Remark 2.2.2

(a) Let $f \in C(K)$ be such that $x \longrightarrow {}_xf$ is continuous from K into $(C(K), \|\cdot\|_\infty)$ at the identity $x = e$. Then $f \in UC_r(K)$. Indeed, if $(\phi_V)_{V \in \mathcal{U}}$ is the bounded approximate identity for $L_1(K)$ as in 2.1.2, then $(\phi_V * f)_{V \in \mathcal{U}}$ converges to f in the $\|\cdot\|_\infty$ norm.

(b) If the maximal subgroup $G(K)$ of K is open, then $UC_r(K)$ is an algebra. To see this, let $f, g \in UC_r(K)$, $x \in G(K)$, $y \in K$. Then

$${}_x(fg)(y) - (fg)(y) = {}_xf(y)[{}_xg(y) - g(y)] + g(y)[{}_xf(y) - f(y)] ,$$

since $\delta_x * \delta_y = \delta_{xy}$. Thus the mapping $x \longrightarrow {}_x(fg)$ is continuous from K into $(C(K), \|\cdot\|_\infty)$ at e, and hence by (a), $fg \in UC_r(K)$.

(c) If K is compact or discrete, then $C(K) = UC(K)$ [52, 4.2F] .

In contrast to the group case [73, 2.33] , there is a class of nondiscrete noncompact hypergroups K for which $C(K) = UC(K)$ as we shall show next.

<u>Proposition 2.2.3</u> Let $K = H \vee J$, H compact, J discrete, $H \cap J = \{e\}$. Then $C(K) = UC(K)$.

<u>Proof</u>: Let $f \in C(K)$ and write $g = f|_H$. Then $g \in C(H) = UC(H)$ since H is compact. If $x \in H$, then

$$_xf(y) - f(y) = \begin{cases} _xg(y) - g(y) , & y \in H \\ 0 , & y \in J\backslash\{e\} \end{cases} .$$

Because H is open in K , the mapping $x \longrightarrow {}_xf$ (and similarly $x \longrightarrow f_x$) is continuous at e from K to $(C(K) , \|\cdot\|_\infty)$. By 2.2.2(a) , $f \in UC(K)$. □

It is known that if K is nondiscrete, then $C(K) \neq L_\infty(K)$ [91, Theorem A.6].

<u>Proposition 2.2.4</u> If the maximal subgroup $G = G(K)$ of a hypergroup K is open, nondiscrete and noncompact, then

$$UC_r(K) \neq C(K) .$$

Proof: Let $f \in C(G)$, $f \notin UC_r(G)$ [73, Problem 1-3] . Let $\tilde{f}$ be the function on K given by $\tilde{f} = f$ on G and zero otherwise. Then $\tilde{f} \in C(K)$ but $\tilde{f} \notin UC_r(K)$, (if $\tilde{f} \in UC_r(K)$, then $\tilde{f}_{|G} = f \in UC_r(G)$) . □

Lemma 2.2.5

(i) $\overline{Co} \{ {}_x f : x \in K \} = cl\ P(K) * f = cl\ M'(K) * f$, $f \in UC_r(K)$ or $f \in L_p(K)$, $1 \le p < \infty$, where $\overline{Co}\ A$ is the closed convex hull of the set A ;

(ii) $\overline{Co}^{w^*} \{ {}_x f : x \in K \}$

$= w^*\text{-}cl\ P(K) * f = w^*\text{-}cl\ M'(K) * f$, $f \in L_\infty(K)$;

(iii) The sets $\overline{Co} \{ {}_x f : x \in K \}$ in (i) and $\overline{Co}^{w^*} \{ {}_x f : x \in K \}$ in (ii) are left translation invariant ;

(iv) If X is a closed convex subset of $UC_r(K)$ (or $L_p(K)$, $1 \le p < \infty$) , then the following are equivalent:

(a) X *is translation invariant* ;

(b) $P(K) * X \subseteq X$;

(c) $M'(K) * X \subseteq X$.

(v) For a weak* closed convex subset Y of $L_\infty(K)$, (a), (b), (c) in (iv) are equivalent.

There are similar statements for right translations of functions in $UC_\ell(K)$, $L_\infty(K)$, etc. (see [58, § 4]) .

Proof:

(i) If $\phi \in P(K)$, choose a net (μ_α) of probability measures with finite support such that (μ_α) converges to $(\phi\lambda)$. Then, by 2.2.1(i) , it is easy to see that $\| \mu_\alpha * f - f \|_\infty$ converges to zero for $f \in UC_r(K)$ [52, 5.4H] . If $\mu \in M'(K)$, then choose a net $(\phi_\alpha) \subseteq P(K)$ such that $(\phi_\alpha\lambda)$ converges to μ . Then $\| \mu_\alpha * f - \mu * f \|_\infty$ converges to zero. This proves (i) for $f \in UC_r(K)$, and it is even easier for $f \in L_p(K)$, $1 \le p < \infty$.

(iii) To see that $\overline{Co}^{w^*} \{{}_x f : x \in K\}$ is left translation invariant, let $\mu \in M'(K)$. Then there is a net $(\mu_\alpha) \subseteq M(K)$ with *spt* μ_α finite such that (μ_α) converges to μ . Then $\mu_\alpha * f$ converges to $\mu * f$ in the weak* topology of $L_\infty(K)$.

(iv) This follows from (i) .

The rest of the proof is similar. We safely omit the details (see [58, § 4]). □

CHAPTER 3

AMENABLE HYPERGROUPS

§ 3.1 Introduction.

In this chapter, we initiate a systematic study of amenable hypergroups. Let K be a hypergroup with a left Haar measure. A linear functional m on $L_\infty(K)$ is called a *mean* if $m(1) = \|m\| = 1$.

We say that K is *amenable* if there is a mean m on $L_\infty(K)$ such that $m({}_x f) = m(f)$ for all $x \in K$, $f \in L_\infty(K)$.

In Section 3.2 , we give some important examples and discuss stability properties of amenable hypergroups. We prove that a hypergroup K is amenable if and only if

$$TLIM(L_\infty(K)) \neq \phi \quad [TIIM(L_\infty(K)) \neq \phi] .$$

E. Granirer [40] and W. Rudin [84] proved independently that if G is a nondiscrete locally compact group which is amenable as a discrete group, then there is a left invariant mean on $L_\infty(G)$ which is not topologically left invariant. This type of study goes back to S. Banach [3] . In contrast to the group case, we show that there is a class of commutative hypergroups K for which every (left) invariant mean on $L_\infty(K)$ is a topological (left) invariant mean.

In Section 3.3 , we establish some important characterizations of amenable hypergroups. We prove an analogue of the Day-Rickert fixed point theorem. We show that a hypergroup K satisfies the Reiter-Glicksberg property if and only if K is amenable. Reiter's condition (P_1) is also shown to characterize amenability of hypergroups. Finally, we show that if a hypergroup satisfies (P_2) , then it is amenable or equivalently has property (P_1) , and that the converse is not true in general. This is in contrast to the group case.

In [59] , A. T. Lau introduced and studied a class of Banach algebras which includes $L_1(K)$. He called such algebras F-algebras. Using the theory of von Neumann algebras he extended several important characterizations of amenable locally compact groups to F-algebras which admit topological left invariant means. It follows that the F-algebra $L_1(K)$ has a topological left invariant mean if and only if K is amenable. Following [75] , we refer to F-algebras as Lau algebras.

M. M. Day [17] and E. Granirer [36] initiated the study of the cardinality of the set of invariant means. Recently, A. T. Lau and A. L. T. Paterson proved in [62, Theorem 1] that if G is a noncompact amenable locally compact group, then

$$| \text{TLIM} (L_\infty(G) | = 2^{2^d} ,$$

where d is the smallest cardinality of a cover of G by compact sets. By extending C. Chou's ideas [15, § V] , Z. Yang proved in [98, § 3] that if G is noncompact and amenable, then

$$| \text{TLIM } (L_\infty(G) | = | \text{TIM } (L_\infty(G) | = | \text{TIIM } (L_\infty(G) | = 2^{2^d} .$$

Inspired by these, we prove in Section 3.4 that

$$| \text{TIM } (L_\infty(K) | = | \text{TIIM } (L_\infty(K) | = 2^{2^d} ,$$

for an arbitrary amenable noncompact hypergroup, where d is defined exactly as before. We also show that if in addition the maximal subgroup $G(K)$ of K is open, then

$$| \text{TLIM } (L_\infty(K) | = 2^{2^d} .$$

We give some applications of these theorems.

§ 3.2 Amenable Hypergroups, Examples and Stability Properties.

In this section, we define amenable hypergroups, give a few examples, and discuss some stability properties.

Definition 3.2.1 Let K be a hypergroup with a left Haar measure λ , and let X be one of the spaces $UC(K)$, $UC_r(K)$, $C(K)$ or $L_\infty(K)$. A linear functional m on X is called a *mean* if

(i) $m(\overline{f}) = \overline{m(f)}$ for all $f \in X$;

(ii) $f \geq 0$ implies $m(f) \geq 0$ [$f \geq 0$ loc. λ a.e. implies $m(f) \geq 0$] and $m(1) = 1$.

It is easy to see that a linear functional m on X is a mean if and only if $m(1) = \|m\| = 1$ and thus the set $\Sigma(x)$ of all means on X is a nonempty weak* compact convex set in X^* [74, Proposition 3.2] .

A mean m on X is called a *left invariant mean [LIM]* if

$$m({}_x f) = m(f)$$

for all $f \in X$, $x \in K$.

A hypergroup K is called *amenable* if there is a LIM on $C(K)$.

A *right invariant mean [RIM]* on X is a mean such that

$$m(f_x) = m(f)$$

for all $f \in X$, $x \in K$.

A mean m on X is called a *topological left (right) invariant mean [TLIM] ([TRIM])* if

$$m(\phi * f) = m(f) \quad (m(f * \mathring{\phi}) = m(f))$$

for all $f \in X$,

$$\phi \in P(K) = \{ \phi \in L_1(K) : \phi \geq 0 , \|\phi\|_1 = 1 \} .$$

A mean m on X ($= UC(K)$, $C(K)$ or $L_\infty(K)$) is said to be *inversion invariant* if

$$m(f) = m(\mathring{f})$$

for all $f \in X$. Note that if an inversion invariant mean is one sided invariant, then it is automatically two sided invariant.

We denote the set of left [topological] invariant means on X by $LIM(X)$ [$TLIM(X)$] . The sets $IM(X)$, $TIM(X)$, $IIM(X)$ and $TIIM(X)$ are similarly defined. For example, $TIIM(X)$ is the set of all topological inversion invariant means on X ($= UC(K)$, $C(K)$ or $L_\infty(K)$) .

Lemma 3.2.2

(i) Every TLIM on X is a LIM ;

(ii) If $X = UC_r(K)$ or $UC(K)$ then every LIM on X is also a TLIM .

Proof:

(i) Let m be a TLIM on X . Since the modular function Δ is constant on $\{x\} * \{y\}$ with value $\Delta(x)\Delta(y)$ for all $x, y \in K$, it follows that $\phi * {}_x f = (\phi \circ \delta_{\check{x}}) * f$ for $f \in X$, $\phi \in P(K)$. Also $\phi \circ \delta_{\check{x}} \in P(K)$. Hence,

$$m({}_x f) = m(\phi * {}_x f) = m((\phi \circ \delta_{\check{x}}) * f) = m(f) , \quad f \in X , \quad x \in K .$$

(ii) Let m be a LIM on X and $\phi \in P_c(K)$. Since the mapping $x \longrightarrow \delta_x * f$ $(f \in X)$ is continuous from K into $(C(K) , \|\cdot\|_\infty)$ and the point evaluation functionals in X^* separate points of X , we have

$$\phi * f = \int_K (\delta_x * f)\ \phi(x)dx .$$

Thus

$$\begin{aligned} \langle m, \phi * f\rangle &= \langle m, \int_K (\delta_x * f)\ \phi(x)dx \rangle \\ &= \int_K \langle m, \delta_x * f\rangle\ \phi(x)dx \end{aligned}$$

$$= \langle m, f\rangle .$$

Now, (ii) follows by the density of $P_c(K)$ in $P(K)$. □

<u>Theorem 3.2.3</u> Let X be one of the spaces $UC(K)$, $UC_r(K)$, $C(K)$ or $L_\infty(K)$. Then K is amenable if and only if

$$LIM(X) \neq \emptyset \quad [TLIM(X) \neq \emptyset] .$$

In this case, $IM(X) \neq \emptyset$ and thus $TIM(X) \neq \emptyset$. Also $TIIM(X) \neq \emptyset$ for $X = UC(K)$, $C(K)$, $L_\infty(K)$.

<u>Proof</u>: To prove the first statement, let m be a LIM on $UC(K)$, and E a compact symmetric neighbourhood of e . Using an approximate identity for $L_1(K)$ contained in $P(K)$ one can easily see that $m(\phi_1 * f * \check{\psi}_0) = m(\phi_2 * f * \check{\psi}_0)$ for all $\psi_0, \phi_1, \phi_2 \in P(K)$, $f \in L_\infty(K)$ [74, p. 35] .

Let

$$\phi_0 = \frac{1}{\lambda(E)} 1_E \in P(K) .$$

Then $\phi_0 * f * \phi_0 \in UC(K)$ since $\phi_0 = \check{\phi}_0$.

Write $m'(f) = m(\phi_0 * f * \phi_0)$, $f \in L_\infty(K)$. Then m' is a TLIM on $L_\infty(K)$, since

$$m'(\phi * f) = m(\phi_0 * (\phi * f) * \phi_0) = m(\phi_0 * f * \phi_0) = m'(f) ,$$

for all $\phi \in P(K)$, $f \in L_\infty(K)$.

It is easy to see that if m is an IM or IIM then m' is a TIM or TIIM , respectively [74, p. 35-37]. To see the last statement, let m be a LIM on $UC(K)$ and n a right invariant mean on $C(K)$ (If n is a LIM on $C(K)$, then $\mathring{n}(f) = n(\mathring{f})$, $f \in C(K)$ gives a RIM on $C(K)$).

Define

$$F(x) = \langle m, f_x \rangle , \quad f \in UC(K), x \in K .$$

Then $F \in C(K)$. Next, put $m_1(f) = \langle m, f \rangle$, $f \in UC(K)$. Then m_1 is a two sided invariant mean on $UC(K)$. Indeed, since

$$\langle m, ({}_y f)_x \rangle = \langle m, {}_y(f_x) \rangle = \langle m, f_x \rangle = F(x)$$

and

$$\langle m, (f_y)_x \rangle = \int_K \langle m, f_u \rangle \, d\, \delta_x * \delta_y(u)$$

$$= \int_K F(u) \, d\, \delta_x * \delta_y(u) = F_y(x) ,$$

we have

$$\langle m_1, {}_y f \rangle = \langle n, F \rangle = \langle m_1, f \rangle$$

and

$$\langle m_1, f_y \rangle = \langle n, F_y \rangle = \langle n, F \rangle = \langle m_1, f \rangle .$$

So, m_1 is a TLIM on $UC(K)$. m_1 is also a TRIM because

$$f * \mathring{\phi} = \int_K f_y \, \phi(y) dy ,$$

$\phi \in P_c(K)$, $f \in UC(K)$.

To complete the proof, note that if m is an IM [TIM] on X ($= UC(K)$, $C(K)$, $L_\infty(K)$), then

$$m' = \frac{1}{2} (m + \mathring{m})$$

is an IIM [TIIM] on X.

Let $X = UC(K)$, $UC_n(K)$, $C(K)$ or $L_\infty(K)$. The next result is not hard to prove (cf. [30, Proposition 2.1]) .

Lemma 3.2.4 Let $\Pi(X)$ be a conjugate closed subspace of X . The following are equivalent:

(i) $d(1, \Pi(X)) = \inf_{f\in\Pi(X)} \|1 - f\|_\infty = 1$;

(ii) There is a mean m on X such that $m(\Pi(X)) = 0$;

(iii) ess inf $h \le 0$ [inf $h \le 0$] for all real valued $h \in \Pi(X)$.

The equivalence of (i) and (ii) below is known as Dixmier's criteria.

Corollary 3.2.5 Let $\Pi_0(X)$ $[\Pi_1(X)]$ be the subspace of X spanned by $\{{}_x f - f : x \in K , f \in X \}$ $[\{\phi * f - f : \phi \in P((K) , f \in X \}]$. The following are equivalent:

(i) There is a LIM [TLIM] on X ;

(ii) ess inf $h \le 0$ (inf $h \le 0$) for all real valued $h \in \Pi_0(X)$ $[\Pi_1(X)]$;

(iii) $d(1, \Pi_0(X)) = 1$ $[\, d(1, \Pi_1(X)) = 1 \,]$.

Proposition 3.2.6 Let $X = UC_r(K)$, $C(K)$ or $L_\infty(K)$. Then K is amenable if and only if $\Pi_0(X)$ $[\Pi_1(X)]$ is not uniformly dense in X , and in this case we have

(a) $cl\ \Pi_0(X) = \{ f \in X : m(f) = 0$ for all $m \in LIM(X)\}$;

(b) $cl\ \Pi_1(X) = \{ f \in X : m(f) = 0$ for all $m \in TLIM(X)\}$.

If UC(K) is an algebra (in particular, if G(K) is open) then this is also true for $X = UC(K)$.

Proof: If K is amenable then $\Pi_0(X)$ $[\Pi_1(X)]$ is not uniformly dense in X by 3.2.5 . Conversely, suppose first that $\Pi_1(UC_r(K))$ is not dense in $UC_r(K)$. Then there exist $f_0 \in UC_r(K)$, $\phi \in UC_r(K)^*$ such that

$$\phi(f_0) = 1 , \quad \phi(\Pi_1(UC_r(K))) = 0 .$$

Fix $u_0 \in P(K)$ and put $\Phi(f) = \phi(u_0 * f)$, $f \in L_\infty(K)$, so that $\Phi_{|UC_r(K)} = \phi$. As observed earlier (3.2.3) , $\phi(u_1 * f) = \phi(u_2 * f)$ for all $u_1, u_2 \in P(K)$. Hence

$$\Phi(u * f) = \phi(u_0 * (u * f)) = \phi(u_0 * f) = \Phi(f) ,$$

$f \in L_\infty(K)$, $u \in P(K)$. Following an idea of A. T. Lau (see, for example, [86, p. 17]) , write

$$\Psi(f) = \frac{1}{2} (\Phi + \Phi^*) ,$$

where $\Phi^*(f) = \overline{\Phi(\overline{f})}$, $f \in L_\infty(K)$. Then Ψ is topologically left invariant, $\Psi(f_0) = 1$, and $\Psi(\overline{f}) = \Psi(f)$ for all $f \in L_\infty(K)$. Since $L_\infty(K)$ is a C^*-algebra, we can write

$$\Psi = \Psi^+ - \Psi^- ,$$

$$||\Psi|| = ||\Psi^+|| + ||\Psi^-|| ,$$

uniquely [22, 12.34] .

For $u \in P(K)$, let ℓ_u denote the linear operator on $L_\infty(K)$ given by $\ell_u f = u * f \quad (f \in L_\infty(K))$. Then $\ell_u^* \Psi^+$ and $\ell_u^* \Psi^-$ are both positive linear functionals with the same norm as Ψ^+ , Ψ^- , respectively. In fact,

$$\| \ell_u^* \Psi^+ \| = < \ell_u^* \Psi^+, 1> = < \Psi^+, 1> = \| \Psi^+ \|$$

[22, 2.1.4] .

Since $\ell_u^* \Psi = \Psi$, we have $\ell_u^* \Psi^+ = \Psi^+$ and $\ell_u^* \Psi^- = \Psi^-$. If $\Psi^+(f_0) \neq 0$ (say) , let

$$m = \frac{\Psi^+}{\|\Psi^+\|} .$$

Then m is a TLIM on $L_\infty(K)$ with $m(f_0) \neq 0$. This shows K is amenable and

$$cl\ \Pi_1(UC_\eta(K)) = \{ f \in UC_\eta(K) : m(f) = 0$$

for all $m \in TLIM(UC_\eta(K))\}$.

Since $cl\ \Pi_0(UC_\eta(K)) = cl\ \Pi_1(UC_\eta(K))$ and $TLIM(UC_1(K)) = LIM(UC_\eta(K))$, we have the result for $X = UC_\eta(K)$.

When $X = L_\infty(K)$ or $C(K)$ or $UC(K)$ is an algebra, the proofs are even easier because they are C^*-algebras. □

<u>Remark 3.2.7</u>

(i) Let K be amenable. Then we have always $\Pi_0(X) \subseteq \Pi_1(X)$ with $cl\ \Pi_0(X) = cl\ \Pi_1(X)$ if and only if every LIM on X is a TLIM , and hence $cl\ \Pi_0(X) = cl\ \Pi_1(X)$ when $X = UC(K)$ or

$UC_{r}(K)$.

(ii) There are similar statements of 3.2.6 for two sided (and inversion) invariant means. For example, if $\Gamma_1(X)$ is the subspace of X spanned by

$$\{ \phi * f - f , g * \check{\psi} - g ; f, g \in X , \phi, \psi \in P(K) \} ,$$

then K is amenable if and only if $\Gamma_1(X)$ is not dense in X, where $X = UC(K)$, $UC_{r}(K)$, $C(K)$ or $L_\infty(K)$, and in this case

$$cl\ \Gamma_1(X) = \{ f \in X : m(f) = 0 \text{ for all } m \in TIM(X) \} .$$

(iii) The following mappings are one to one and onto:

(a) $m \longrightarrow m_{|UC(K)}$ of $TIM(X)$ onto $TIM(UC(K))$, $X = UC(K)$, $UC_{r}(K)$, $C(K)$, $L_\infty(K)$.

(b) $m \longrightarrow m_{|UC_{r}(K)}$ of $TLIM(X)$ onto $TLIM(UC_{r}(K))$, $X = L_\infty(K)$ or $C(K)$.

(c) $m \longrightarrow m_{|UC(K)}$ from $TIIM(X)$ onto $TIIM(UC(K))$, $X = C(K)$ or $L_\infty(K)$ (see [73, 1.9 Corollary]).

<u>Definition 3.2.8</u> We say that a net $(\phi_\alpha) \subseteq P(K)$ converges *weakly [strongly] to left (right) invariance* if

$$\delta_x * \phi_\alpha - \phi_\alpha \quad (\phi_\alpha \circ \delta_x - \phi_\alpha)$$

converges to zero in the weak [$\|\cdot\|_1$ - norm] topology of $L_1(K)$ for each $x \in K$. It is *weakly [strongly] convergent to topological left (right) invariance* if

$$\phi * \phi_\alpha - \phi_\alpha \quad (\phi_\alpha * \phi - \phi_\alpha)$$

converges to zero in the weak [$\|\cdot\|_1$ - norm] topology for each $\phi \in P(K)$.

The functions in P(K) define means on $L_\infty(K)$ and form a weak* dense convex subset of the weak* compact set $\Sigma(L_\infty(K))$ of all means on $L_\infty(K)$. It is easy to see that there is a net in P(K) weakly convergent to [topological] left (right) invariance if and only if there is a [topological] left (right) invariant mean on $L_\infty(K)$. Furthermore, if $(\phi_\alpha) \subseteq P(K)$ converges weakly to [topological] left (right) invariance then any weak* limit point of (ϕ_α) in $L_\infty(K)^*$ is a [topological] left (right) invariant mean on $L_\infty(K)$ (see [42, 2.4.3]) . If there is a net $(\phi_\alpha) \subseteq P(K)$ converging strongly to left (and right) [topological] invariance, then there is clearly a net $(\psi_\beta) \subseteq P(K)$ weakly converging to left (and right) [topological] invariance. The converse is also true and a consequence of [39, p. 17-18] :

<u>Lemma 3.2.9</u> Let $(\phi_\alpha) \subseteq P(K)$ converge weakly to [topological] left (and right) invariance and m_0 any weak*

cluster point of (ϕ_α) in $L_\infty(K)^*$. Then there is a net (ψ_β) in the convex hull $\mathrm{Co}\,(\phi_\alpha)$ such that (ψ_β) converges strongly to [topological] left (and right) invariance and $w^* - \lim_\beta \psi_\beta = m_0$.

We now give examples and discuss stability properties of amenable hypergroups.

Examples 3.2.10

(a) Commutative hypergroups are amenable: If K is commutative then $L_1(K)$ is a commutative Lau algebra and hence there is a TIM on $L_\infty(K)$. This is a consequence of the Markov-Kakutani fixed point theorem and shown in [59, p. 168] (see also 3.3.1 below) .

(b) Compact hypergroups are amenable: the normalized Haar measure is a unique LIM on $C(K)$. It is the unique TLIM on $L_\infty(K)$, and it is also inversion invariant (cf. [52, 7.2A]) .

Example 3.2.11 Let K be a hypergroup such that $(x) * (y)$ is finite for all $x, y \in K$. Write K_d when K is equipped with the discrete topology. In this case, the discrete measures

$$\sum_{i=1}^{\infty} \alpha_i \delta_{x_i} ,$$

$x_i \in K$, (α_i) a sequence of complex numbers such that

$$\sum_{i=1}^{\infty} |\alpha_i| < \infty ,$$

form a closed self adjoint subalgebra of $M(K)$. Hence, the convolution in $M(K)$ naturally induces a hypergroup structure on K_d . Such hypergroups naturally arise as direct products of discrete hypergroups and locally compact groups. They also include the commutative hypergroups of normalized characters on Z-groups [83, 5.5 Theorem] .

We say that K is *amenable as a discrete hypergroup* if K_d is amenable. Clearly, if K_d is amenable then so is K . Also every LIM on $C(K_d)$ is a TLIM .

We next prove that every subgroup of an amenable hypergroup is amenable. We need the following:

Definition 3.2.12 (Bruhat functions)

Let H be a subgroup of the hypergroup K . That is, H is a closed subgroup of the maximal group $G(K)$. Let F be a continuous non-negative function on K such that

(i) For each $x \in K$ there exists $t \in H$ such that $F(xt) > 0$;

(ii) If $W \subseteq K$ is compact, then F coincides on WH with some function $\Psi \in C_c^+(K)$ [46, Lemma 1.2] .

Write

$$F_1(x) = \int_H F(xt)\, dt \quad (x \in K) .$$

Then the integral exists and is positive, and F_1 is continuous.

To see that F_1 is continuous, let W be a compact neighbourhood of $x \in K$ and ψ as in (ii) above. Then for $y \in W$, we have

$$| F_1(x) - F_1(y) | = \left| \int_{H \cap \mathring{W} * spt\, \psi} \psi(xt)dt - \int_{H \cap \mathring{W} * spt\, \psi} \psi(yt)dt \right|$$

If $\varepsilon > 0$ is given, then by [52, 2.2B] and [52, 4.2F] we can find a neighbourhood V of x contained in W such that

$$\| {}_x\psi - {}_y\psi \|_\infty \, \sigma\, (H \cap \mathring{W} * spt\, \psi\,) < \varepsilon\,,$$

for all $y \in V$, where σ is a fixed left Haar measure on H.

So, F_1 is continuous.

Next, put

$$\beta(x) = \frac{F(x)}{F_1(x)}\,, \quad x \in K\,.$$

Then

(i) $\int_H \beta(xt)\, dt = 1$;

(ii) If $W \subseteq K$ is compact, then β coincides on WH with some $\psi \in C_c^+(K)$. β is called a <u>Bruhat function</u> for H [79, Ch. 8, § 1.9] .

N. Rickert initially proved in [81] that a closed subgroup of an amenable locally compact group is amenable. There are many proofs of this fact available in the literature now ([42], [50], [79]) . The one given by H. Reiter [79, Ch. 8, § 5.5] works for hypergroups as we shall see next.

Proposition 3.2.13 Every (closed) subgroup H of an amenable hypergroup K is amenable. In particular, the maximal subgroup $G(K)$ is amenable.

Proof: Let β be a Bruhat function for H. For $\phi \in C(H)$, put

$$f_\phi(x) = \int_H \beta(\overset{\circ}{x}t)\phi(t)\,dt\,, \quad x \in K\,.$$

Then f_ϕ is continuous (this can be proved as above) and

$$\|f_\phi\|_\infty \le \|\phi\|_\infty\,.$$

Let m be a LIM on $C(K)$. Define

$$\langle m', \phi\rangle = \langle m, f_\phi\rangle\,, \quad \phi \in C(H)\,.$$

Then m' is a LIM on $C(H)$ because

$$\begin{aligned}\int_H \beta(\overset{\circ}{x}t)\,\phi(ht)\,dt &= \int_H \beta(\overset{\circ}{x}\overset{\circ}{h}t)\,\phi(t)\,dt \qquad (h \in H)\\ &= \int_H \beta((hx)^{\circ}t)\,\phi(t)\,dt\\ &= f_\phi(hx) = {}_h(f_\phi)(x)\,,\end{aligned}$$

and so

$$\langle m', {}_h\phi\rangle = \langle m, {}_h(f_\phi)\rangle = \langle m, f_\phi\rangle = \langle m', f\rangle\,.$$

Hence, H is amenable. □

A subgroup H of K is called *normal* if $xH = Hx$ for all $x \in K$. Let H be a normal subgroup of K. Let K/H be the set of all cosets xH, $x \in K$, equipped with the quotient topology with respect to the natural projection $p(x) = xH$. Then K/H is a hypergroup under the convolution

$$\int f \, d\delta_{xH} * \delta_{yH} = \int f \circ p \, d\delta_x * \delta_y ,$$

$x,y \in K$, $f \in C_c(K/H)$ [46, p. 84] .

Proposition 3.2.14 K is amenable if and only if both K/H and H are amenable.

Proof: Let m be a LIM on $C(K)$, and write $M(f) = m(f \circ p)$, $f \in C(K/H)$ .This M is a mean on $C(K/H)$. We have

$$_x(f \circ p)(y) = \int f \circ p \, d\delta_x * \delta_y = \int f \, d\delta_{xH} * \delta_{yH}$$
$$= {}_{xH}f \circ p(y)$$

for all $x, y \in K$, $f \in C(K/H)$. (see [52, § 2.3] or [49, 12.46]). Hence, M is a LIM, because

$$M({}_{xH}f) = m({}_{xH}f \circ p) = m({}_x(f \circ p)) = m(f \circ p) = M(f) ,$$

for $f \in C(K/H)$, $x \in K$. So, K/H is amenable, and H is amenable by 3.2.13 .

Conversely, let m_1 be a LIM on $C(H)$ and m_2 a LIM on $C(K/H)$. For $f \in UC_r(K)$ write $f'(x) = \langle m_1, {}_xf|_H \rangle$. Then f' is bounded, continuous and constant on the cosets of H in K , and hence we can write $f' = F \circ p$, $F \in C(K/H)$. Put $\langle m, f\rangle = \langle m_2, F\rangle$. Observe that

$${}_{xH}F(yH) = f'(x * y) = \int \langle m_1, {}_uf|_H \rangle \, d\delta_x * \delta_y(u)$$

$$= \langle m_1, {}_y({}_x f)|_H \rangle ,$$

since $u \longrightarrow {}_u f|_H$ is continuous from K into $(C(H), \|\cdot\|_\infty)$ and the point evaluation functionals in $C(H)^*$ separate points of $C(H)$. That is, ${}_{xH}F \circ p = ({}_x f)'$. This shows that m is a LIM on $UC_r(K)$ and hence K is amenable. □

Let K be a hypergroup, and let $Z(K) = \{x \in K : \delta_y * \delta_x = \delta_x * \delta_y$ for all $y \in K\}$. K is called a *central or a Z-hypergroup* [46] if $K/Z(K) \cap G(K)$ is compact, where $G(K)$ is the maximal subgroup of K. Central hypergroups admit left Haar measures and are unimodular [46, p. 93].

Corollary 3.2.15 Z-hypergroups are amenable.

Proposition 3.2.16 Let H be a compact subhypergroup of K with the normalized Haar measure σ. If K is amenable, so is $K//H$. If $\delta_x * \sigma = \sigma * \delta_x$ for all $x \in K$ (see [7, p. 549] and [54, p. 179]), then the converse is also true.

Proof: Let m be a TLIM on $C(K)$. For $f \in C(K//H)$, write $\langle M, f\rangle = \langle m, f \circ \Pi\rangle$, where Π is the natural projection of K onto $K//H$. Let $\phi \in P_c(K//H)$. Then

$$\phi * f(HxH) = \int_{K//H} \phi(HyH) \, {}_{H\mathring{x}H}(f^{\circ})(HyH) \, d\, HyH$$

$$= \int_K \int_H \phi \circ \Pi(y) \; f \circ \Pi \; (\mathring{y} * t * x) \, dtdy$$

[52, 14.2F]

$$= \int_H (\phi \circ \Pi) * (f \circ \Pi) \; (\check{t} * x) \, dt$$

$$= (\sigma * (\phi \circ \Pi)) * (f \circ \Pi)(x) \; .$$

Hence, $\phi * f \circ \Pi = (\sigma * (\phi \circ \Pi)) * (f \circ \Pi)$. But $\phi \circ \Pi$ and hence $\sigma * (\phi \circ \Pi)$ is in $P(K)$. Now,

$$\langle M, \phi * f\rangle = \langle m, (\phi * f) \circ \Pi\rangle = \langle m, \sigma * (\phi \circ \Pi) * (f \circ \Pi)\rangle$$
$$= \langle m, f \circ \Pi\rangle = \langle M, f\rangle \; .$$

Thus, M is a TLIM on $C(K//H)$ and hence $K//H$ is amenable.

To prove the converse, for $f \in C(K)$ write

$$f'(x) = \int_H f(x * t) \, dt = f * \sigma(x) \; .$$

Then f' is continuous, bounded and constant on cosets (Indeed, if $f \geq 0$, $f'(z_0) = \sup_{z \in \{x\} * H} f'(z)$, $z_0 \in \{x\} * H$, then since $f'(z_0 * t) = f'(z_0)$ for all $t \in H$, we have $f'(u) = f'(z_0)$ for $u \in \{z_0\} * \{t\}$. This shows that f' is constant on $\{x\} * H$. Finally, observe that $\{x\} * H = spt\ \delta_x * \sigma = spt\ \sigma * \delta_x = H * \{x\}$ for all $x \in K$) .

If m is LIM on $C(K//H)$, put $\langle M, f\rangle = \langle m, F\rangle$, where $f' = F \circ \Pi$. Now, note that

$$F(xH * yH) = \int_H f'(x * t * y) \, dt$$

$$= \int_K f' \, d\delta_x * \sigma * \delta_y = \int_K f' \, d\delta_x * \delta_y * \sigma$$
$$= \int_K \int_K f'(u * t) \, d\delta_x * \delta_y(u) \, d\sigma(t)$$
$$= \int_K \int_K f'(u) \, d\delta_x * \delta_y(u) \, d\sigma(t)$$

[$f'(u * t) = f'(u)$ for all $t \in H$] . This implies $({}_x f)'(y) = {}_{xH}F \circ \Pi(y)$, and hence $\langle M, {}_x f\rangle = \langle M, f\rangle$ for all $f \in C(K)$, $x \in K$. Hence, M is a LIM on $C(K)$ and so K is amenable. □

Remark 3.2.17

(i) Let K be an arbitrary hypergroup and H a compact subhypergroup with the normalized Haar measure σ . Then

(a) $\sigma * \delta_x = \delta_x * \sigma$ for all $x \in K$ if and only if $\sigma * f = f * \sigma$ for all $f \in C_c(K)$ $[C(K)]$.

(b) If $\sigma * \delta_x = \delta_x * \sigma$ for all $x \in K$, then it is easy to see that

$$\int f \, d\delta_{xH} * \delta_{yH} = \int f \circ \Pi \, d\delta_x * \delta_y$$

for all $f \in C(K//H)$, $x, y \in K$, and clearly $\{x\} * H = H * \{x\}$ for each $x \in K$. Also, in this case, if there is a left Haar measure on $K//H$ ($= K/H$) then K admits a left Haar measure such that

$$\int_K f(x)\,dx = \int_{K/H}\int_H f(x * t) \, dt \, dHxH ,$$

$f \in C_c(K)$.

(ii) Let $K = H \vee J$, where H is a compact hypergroup and

J a discrete hypergroup with $H \cap J = \{e\}$. If σ is the normalized Haar measure of H , then $\delta_x * \sigma = \sigma * \delta_x$ for all $x \in K$ [92, Proposition 1.2] .

Definition 3.2.18 Let H be a compact subhypergroup of an arbitrary hypergroup K . Following [7] , we say that H is *supernormal* in K if $\{\check{x}\} * H * \{x\} \subseteq H$ for all $x \in K$. If H is supernormal, then $\delta_x * \sigma = \sigma * \delta_x$ for all $x \in K$, where σ is the normalized Haar measure of H [7, 549] .

The converse is not true in general. In fact, $\{e\}$ is supernormal in K if and only if K is a group.

Let H be supernormal in K . Then $K//H$ ($= K/H$) is a group under the convolution

$$\int f \, d\delta_{xH} * \delta_{yH} = \int f \circ \Pi \, d\delta_x * \delta_y = \int_H f \circ \Pi \, (x * t * y) \, dt$$

for all $f \in C_c(K/H)$, $x, y \in K$, where Π is the natural projection $\Pi(x) = xH = HxH$, as shown in [93, Theorem 2.1] (see 3.2.17 (i)(b) above) . Hence, if an arbitrary hypergroup K admits a compact supernormal subhypergroup, then K admits a left Haar measure by 3.2.17(i)(b) .

If H is a compact hypergroup and G any discrete group with $H \cap G = \{e\}$, then H is supernormal in $K = H_\vee G$.

Corollary 3.2.19

(i) If K admits a compact supernormal subhypergroup H

then K is amenable if and only if K/H is amenable.

(ii) If $K = H \vee J$, where H is a compact hypergroup and J a discrete hypergroup with $H \cap J = \{e\}$, then K is amenable if and only if J is amenable.

<u>Proof</u>: Follows from 3.2.16 , 3.2.17 (ii) and 3.2.18 . □

Let G be a locally compact group and let B denote a subgroup of the topological automorphism group Aut G [47, p. 426] . We call G an $[FIA]_B^-$ - group provided the closure B^- of B in Aut G is compact (cf. [69]) . For each $x \in G$, let $[x] = \{\beta(x) : \beta \in B^-\}$, and write $G_B = \{[x] : x \in G\}$, equipped with the quotient topology. Then G_B is a hypergroup with the operation

$$\int f \, d\delta_{[x]} * \delta_{[y]} = \int_{B^-} f \circ \Pi \, (\beta(x)y) \, d\beta = \int_{B^-} f \circ \Pi(x\beta(y)) \, d\beta \, ,$$

$f \in C_c(G_B)$, where Π is the natural projection map $x \longrightarrow [x]$. G_B admits a left Haar measure given by

$$\int_{G_B} f \, d[x] = \int f \circ \Pi(x) \, dx \, ,$$

$f \in C_c(G_B)$. Let $G' = G \times B^-$ and $H' = \{e\} \times B^-$ with the product topology. Define a binary operation on G' by

$$(x,\alpha)(y,\beta) = (\beta(x)y, \alpha\beta) \, .$$

Then G' is a locally compact group, H' a compact subgroup of

G' , and the mapping $H'(x,\alpha)H' \longrightarrow [x]$ is an isomorphism fı the hypergroup $G'//H'$ onto the hypergroup G_B (see [52, § 8.3]) . Note that G' is the semidirect product of G and B^- .

The next result is now immediate from 3.2.16 .

Corollary 3.2.20 Let G be an amenable locally compact $[FIA]_B^-$ - group. Then the hypergroup G_B is amenable.

Example 3.2.21

(a) Let $SL(2,\mathbb{C})$ be the locally compact group (with t usual topology) of all 2×2 complex matrices with determiı 1 , and $SU(2)$ the compact subgroup of unitary matrices in $SL(2,\mathbb{C})$. It is well known that $SL(2,\mathbb{C})$ is nonamenable (s for example [74, § 14]) . However, the hypergroup $SL(2,\mathbb{C})//SU(2)$ is commutative [52, 15.5] and hence amenabl (see also [52, 15.6]) .

(b) Let H be a compact group and G a (discrete) f group on two generators with $H \cap G = \{e\}$. Since G is nonamenable [74, Proposition 14.1] the hypergroup $K = H_\vee G$ nonamenable by 3.2.19 (ii) , but the maximal subgroup of K H which is compact (and hence amenable) .

Let J, L be hypergroups with left Haar measures. Then is easy to see that the hypergroup $J \times L$ has a left Haar measure. The next result is a consequence of 3.2.14 if eit

J or L is a group.

Proposition 3.2.22 $J \times L$ is amenable if and only if both J and L are amenable.

Proof: If m_1 is a LIM on $UC_r(L)$ and m_2 is a LIM on $C(J)$, for $f \in UC_r(J \times L)$, $x \in J$, write $(f:x)(y) = f(x,y)$, $y \in L$. Then $(f:x) \in UC_r(L)$ because

$$\|_y(f:x) - {}_{y_0}(f:x)\|_\infty \le \|_{(e,y)}f - {}_{(e,y_0)}f\|_\infty .$$

Write $F(x) = \langle m_1, (f:x)\rangle$, $x \in J$. Since

$$\|(f:x) - (f:x_0)\|_\infty \le \|_{(x,e)}f - {}_{(x_0,e)}f\|_\infty ,$$

we have $F \in C(J)$. Finally put $\langle m,f\rangle = \langle m_2,F\rangle$. Now,

$$\left({}_{(a,b)}f : x\right)(y) = {}_{(a,b)}f(x,y)$$

$$= \int_J\int_L f(u,v)\, d\delta_a * \delta_x(u)\, d\delta_b * \delta_y(v)$$

$$= \int_J {}_b(f:u)(y)\, d\delta_a * \delta_x(u) .$$

Hence,

$$\left({}_{(a,b)}f : x\right) = \int {}_b(f:u)\, d\delta_a * \delta_x(u) ,$$

because the mapping $u \longrightarrow {}_b(f:u)$ is continuous from J into $(C(L), \|\cdot\|_\infty)$ and the point evaluation functionals in $C(L)^*$ separate points of $C(L)$. Thus

$$<m_1, ({}_{(a,b)}f : x)> = \int <m, {}_b(f:u)> d\delta_a * \delta_x(u) = F_a(x) .$$

So m is a LIM on $UC_r(J \times L)$ and hence $J \times L$ is amenable.

Conversely, if m is a LIM on $C(J \times L)$, write $<M,f> = <m, f \circ \Pi>$ for $f \in C(L)$, where $\Pi(x,y) = x$ for all $(x,y) \in J \times L$. Then M is a LIM on J . Hence J (and similarly L) is amenable. □

Proposition 3.2.23 If the hypergroup K is the directed union of a system of amenable subhypergroups, then K is amenable.

Proof: See [74, Propostion 13.11] . □

E. Granirer [40] and W. Rudin [84] proved independently that if G is a locally compact group which is amenable as a discrete group, then there is a LIM on $L_\infty(G)$ which is not a TLIM . For an earlier work of D. Stafney on this see [86, Chapter V] or [89, § 3] . The next result shows that this is not the case in general for hypergroups.

Example 3.2.24 (C. F. Dunkl and D. E. Ramirez [26])

Let $\mathbb{Z}_+$ be the non-negative integers and $\mathbb{Z}_+ \cup \{\infty\}$ its one point compactification. Let $0 < a \le \frac{1}{2}$. Define:

$$\delta_\infty = \text{the identity element,}$$

$$\delta_m * \delta_n = \delta_{\min(m,n)} , \quad m,n \in \mathbb{Z}^+ , \quad m \neq n ,$$

$$\delta_n * \delta_n(\{t\}) = \begin{cases} 0 & , \ t < n \\ \frac{1-2a}{1-a} & , \ t = n \\ a^k & , \ t = n + k > n , \end{cases}$$

and $\mathring{n} = n$ for all n .

The compact commutative hypergroup obtained this way is denoted by H_a . The normalized Haar measure on H_a is given by

$$\lambda(\{k\}) = \begin{cases} (1-a)a^k & , \ k \neq \infty \\ 0 & , \ k = \infty . \end{cases}$$

It is easy to see that $L_\infty(H_a)$ has a unique LIM , namely the Haar measure. Indeed, let m be a LIM on $L_\infty(H_a)$, $f \in L_\infty(H_a)$, $\phi \in P(H_a)$. Then $\|\phi_n - \phi\|_1$ converges to zero, where $\phi_n = \phi$ on $\{0,1, \ldots ,n\}$, and zero otherwise. Hence

$$m(\phi * f) = \lim_n m(\phi_n * f) = \lim_n \left(\sum_{k=0}^{n} \phi(k)(1-a)\ a^k\ \delta_k * f \right)$$

$$= \lim_n \left(\sum_{k=0}^{n} \phi(k)(1-a)\ a^k \right) m(f) \ = \ m(f) \ .$$

Thus, every LIM on $L_\infty(H_a)$ is a TLIM , and hence the

Haar measure is the unique LIM on $L_\infty(H_a)$ (see 3.2.10 (b)) . □

Consider now the subhypergroup $H = \{1,2, \ldots ,\infty\}$ of the hypergroup H_a in the previous example and the hypergroup $J_0 = \{0,\infty\}$, the convolution on J_0 being given by

$$\delta_0 * \delta_0 = \frac{a}{1-a}\,\delta_\infty + \frac{1-2a}{1-a}\,\delta_0 .$$

Then $H_a = H \vee J_0$ [92, Example 4.5] . Motivated by this and 2.2.3 we have the following:

<u>Theorem 3.2.25</u> Let H be a compact hypergroup, J a discrete hypergroup with $|J| \geq 2$, $H \cap J = \{e\}$, and let $K = H \vee J$. Then every LIM on $L_\infty(K)$ is a TLIM . Furthermore, every LIM M on $L_\infty(K)$ is of the form

$$\langle M,f\rangle = \langle m, f|_{J^*}\rangle + \langle m, 1_{\{e\}}\rangle \int_H f\, d\sigma , \quad f \in L_\infty(K) ,$$

for some LIM m on $\ell_\infty(J)$, where $J^* = J\setminus\{e\}$, and σ is the normalized Haar measure of H .

The correspondence $M \longrightarrow m$ is one to one and onto.

<u>Proof</u>: Let M be a LIM on $L_\infty(K)$, $x \in J^*$. $f \in L_\infty(K)$ (recall that $\cdot$ is the convolution in K , and that the points J^* are isolated points in K) .

$$
{}_x f(y) = f(x\cdot y) = \begin{cases} f(x) , & y \in H \\ f(x \bullet y) , & y \in J^* , \ y \neq \mathring{x} \\ c_e \int_H f\, d\sigma + \sum_{b\in J^*} c_b f(b) , & y = \mathring{x} , \text{ where} \\ \quad \delta_x \bullet \delta_{\mathring{x}} = \sum_{b\in J} c_b \delta_b , \ c_b \geq 0 , \text{ only} & \\ \quad \text{finitely many are non-zero, } \sum_{b\in J} c_b = 1 . & \end{cases}
$$

Now,

$$
{}_x(f|_{J^*})(y) = \begin{cases} f(x) & , \ y \in H \\ f(x \bullet y) & , \ y \in J^* , \ y \neq \mathring{x} \\ \sum_{b\in J^*} c_b f(b) , & y = \mathring{x} \end{cases}
$$

and

$$
{}_x(f|_H)(y) = \begin{cases} 0 , & y \neq \mathring{x} \\ c_e \int_H f\, d\sigma , & y = \mathring{x} , \end{cases}
$$

where $f|_{J^*}$ denotes the function $f\, 1_{J^*}$ on K , etc.. This implies that ${}_x f = {}_x(f|_{J^*}) + {}_x(1_H) \int_H f\, d\sigma$. Hence,

$$
\langle M,f\rangle = \langle M,{}_x f\rangle = \langle M,{}_x(f|_{J^*})\rangle + \langle M,{}_x(1_H)\rangle \int_H f\, d\sigma
$$

$$
= \langle M, f|_{J^*}\rangle + \langle M, 1_H\rangle \int_H f\, d\sigma \tag{1}
$$

Let $\phi \in P(K)$. Since

$$\sum_{y\in J^*} \phi(y)[y] = \int_{J^*} \phi(y)\, dy < \infty ,$$

and

$$\sum_{y\in J^*} \phi(y)[y]\delta_y * f(z) = (\phi|_{J^*}) * f(z) ,$$

we have

$$<M, (\phi|_{J^*}) * f> = \left(\int_{J^*} \phi(y)\, dy\right) <M,f> . \qquad (2)$$

Next,

$$(\phi|_H) * f(z) = \begin{cases} \int_H \phi(y) f(\mathring{y} * z) dy , & z \in H \\ \left(\int_H \phi(y) dy\right) f(z) , & z \in J^* \end{cases} .$$

Hence,

$$<M, (\phi|_H) * f > = <M, (\phi|_H) * f\, |_{J^*} > + <M, 1_H> \int_H (\phi|_H) * f\, d\sigma$$

by (1)

$$= \left(\int_H \phi(y) dy\right) <M, f|_{J^*} > + <M, 1_H> \left(\int_H \phi(y) dy\right) \left(\int_H f\, d\sigma\right) \qquad (3)$$

By (2) and (3) , we have

$<M, \phi * f> =$

$$= <M, 1_H>\left(\int_H \phi(y) dy\right) \left(\int_H f\, d\sigma\right) + <M,f> - \left(\int_H \phi(y) dy\right)<M, f|_H>$$

$= \langle M,f\rangle$, by (1) .

Hence, M is a TLIM, which proves the first statement of the theorem.

If M is a LIM on $L_\infty(K)$ put $\langle m,f\rangle = \langle M, f \circ \Pi\rangle$, where Π is the projection of K onto $K//H$ ($\cong J$) . Then it is easy to see that m is a LIM on $\ell_\infty(J)$ and using (1) that

$$\langle M,f\rangle = \langle m, f|_{J^*}\rangle + \langle m, 1_{\{e\}}\rangle \int_H f\, d\sigma ,$$

$f \in L_\infty(K)$.

Notice the abuse of the notation that $f|_{J^*}$ here means the function on J which is equal to f on J^* and zero at $x = e$.

To establish the converse, let m be a LIM on $\ell_\infty(J)$. Write $f'(x) = \int_H f(x * t)dt$, $f' = F \circ \Pi$, $f \in L_\infty(K)$. Put $\langle M,f\rangle = \langle m,F\rangle$. Then M is a LIM on $L_\infty(K)$ since $({}_xf)'(y) = {}_x(f')(y) = {}_{xH}F(yH)$ for all $x,y \in K$. Now,

$$\langle M,f\rangle = \langle M, f|_{J^*}\rangle + \langle M, 1_H\rangle \int_H f\, d\sigma$$

by (1)

$$= \langle m, f|_{J^*}\rangle + \langle m, 1_{\{e\}}\rangle \int_H f\, d\sigma$$

since $(f|_{J^*})' = f|_{J^*}$ and $(1_H)' = 1_H$.

Finally, it is easy to see that the mapping $M \longrightarrow m$ is one to one. □

Proposition 3.2.26 Let K be a hypergroup with a left Haar measure such that $(x) * (y)$ is finite for all $x,y \in K$. Suppose that K contains a normal nondiscrete subgroup H of finite index. If H is amenable as a discrete group, then there is a LIM on $L_\infty(K)$ which is not a TLIM.

Proof: Let m be a LIM on $L_\infty(H)$ which is not a TLIM, and let ν be the normalized Haar measure on K/H. We take the restriction of λ to H to be the left Haar measure on H (H is open in K). For $x \in K$, $f \in L_\infty(K)$, put $f'(x) = \langle m, {}_x f|_H \rangle$. Then f' is bounded, continuous and constant on the cosets of H in K (see [40, p. 619-620]). Next, put $\langle M,f\rangle = \langle \nu,F\rangle$, where $f' = F \circ \Pi$. Then,

$$_{xH}F \circ \Pi = F(xH * yH) = f'(x * y) = \langle m, {}_y({}_x f)\rangle = ({}_x f)'(y) ,$$

for all $x,y \in K$, since $(x) * (y)$ is finite. Hence M is a LIM on $L_\infty(K)$.

For $f \in L_\infty(H)$, let $\tilde{f}$ be the function in $L_\infty(K)$ given by $\tilde{f} = f$ on H and zero otherwise. Then

$$(\tilde{f})'(x) = \begin{cases} \langle m,f\rangle , & x \in H \\ 0 & \text{otherwise} . \end{cases}$$

Then, $\langle M, \tilde{f}\rangle = \nu((H))\ \langle m,f\rangle$.

Let $u \in P(H)$, $f \in L_\infty(H)$ be such that $m(u * f) \neq m(f)$. Then $\tilde{u} * \tilde{f} = \widetilde{u*f}$, and $\tilde{u} \in P(K)$. Therefore,

$$M(\tilde{u} * \tilde{f}) = \nu((H))\ m(u * f) \neq \nu((H))m(f) = \langle M, \tilde{f}\rangle .$$

Thus, M is not topologically left invariant. □

Corollary 3.2.27 Let G be any locally compact nondiscrete group which is amenable as a discrete group, J a finite hypergroup, and $K = G \times J$. Then there is a LIM on $L_\infty(K)$ which is not a TLIM .

§ 3.3 Characterizations of Amenable Hypergroups.

In this section, we generalize some of the very important characterizations of amenable locally compact groups to hypergroups.

Throughout let K be a hypergroup with a left Haar measure λ . We begin with an analogue of the Day-Rickert fixed point theorem. Let E be a locally convex Hausdorff linear topological space (l.c.s.) and S a nonempty compact convex set in E . Suppose that there is a [separately] continuous mapping $(x,s) \longrightarrow x\cdot s$ of $K \times S$ into S . Then the weak vector valued integral $\int_K u\cdot s\, d\delta_x * \delta_y(u)$ $(s \in S\ ,\ \ x,y \in K)$ exists, is unique and belongs to S [85, 3.27 Theorem] . We denote this integral by $(\delta_x * \delta_y) \cdot s$, so that

$$\langle \phi, (\delta_x * \delta_y) \cdot s \rangle = \int_K \langle \phi, u \cdot s \rangle \, d\delta_x * \delta_y(u) ,$$

for all $\phi \in E^*$.

The next result is due to M. M. Day [18, Theorem 1] and N. W. Rickert [81, Theorem 4.2] for the case when K is a group. (See also P. Eymard [31, p. 6-15]) .

Theorem 3.3.1 K is amenable if and only if K satisfies the following fixed point property F_1 $[F_2]$:

Whenever there is a jointly [separately] continuous mapping $(x,s) \longrightarrow x \cdot s$ from $K \times S$ into S , where S is a nonempty compact convex set in a l.c.s. E such that

(i) $e \cdot s = s$ for each $s \in S$;

(ii) $s \longrightarrow x \cdot s$ is affine for each $x \in K$;

(iii) $x \cdot (y \cdot s) = (\delta_x * \delta_y) \cdot s$ for all $x, y \in K$, $s \in S$,

then there is a point $s_0 \in S$ such that $x \cdot s_0 = s_0$ for all $x \in K$.

Proof: Suppose K is amenable. Let $(x,s) \longrightarrow x \cdot s$ be a separately continuous mapping of $K \times S$ into S satisfying the hypothesis (i), (ii), (iii), and let m be a LIM on C(K) . Let A(S) be the Banach space of all continuous complex valued affine functions on S . Define $f_\phi(x) = \langle \phi, x \cdot s \rangle$, for $x \in K$,

$\phi \in A(S)$, where $s \in S$ is fixed. Then f_ϕ is continuous and $\|f_\phi\|_\infty \le \|\phi\|_\infty$.

Let $T : A(S) \longrightarrow C(K)$ be the bounded linear operator given by $T(\phi) = f_\phi$, $\phi \in A(S)$. Since T^*m is a mean on $A(S)$, there is a point $s_0 \in S$ such that $\langle T^*m,\phi\rangle = \langle\phi,s_0\rangle$, $\phi \in A(S)$ [10, Lemma 1.23] . For $\phi \in E^*$, $x \in K$, define $\langle\phi\Box x,s'\rangle = \langle\phi,x\cdot s'\rangle$, $s' \in S$. Then $\phi\Box x \in A(S)$ and

$$f_{\phi\Box x}(y) = \langle\phi\Box x,\ y\cdot s\rangle = \langle\phi,\ x\cdot(y\cdot s)\rangle = \langle\phi,\ (\delta_x * \delta_y)\cdot s\rangle \qquad (y \in K)$$

$$= \int_K \langle\phi,u\cdot s\rangle\, d\delta_x * \delta_y(u) = \int_K f_\phi(u)\, d\delta_x * \delta_y(u) \qquad (E^*|_S \subseteq A(S)) .$$

$$= {}_x(f_\phi)(y) .$$

Since m is left invariant,

$$\langle\phi,\ x\cdot s_0\rangle = \langle\phi\Box x,\ s_0\rangle = \langle T^*m,\ \phi\Box x\rangle = \langle m,\ T(\phi\Box x)\rangle = \langle m,\ f_{\phi\Box x}\rangle$$

$$= \langle m,{}_x(f_\phi)\rangle = \langle m,f_\phi\rangle = \langle m,T\phi\rangle = \langle T^*m,\phi\rangle = \langle\phi,s_0\rangle$$

for all $\phi \in E^*$. Hence $x\cdot s_0 = s_0$ for each $x \in K$, because E^* separates the points of E . Thus K satisfies (F_2) .

If K satisfies the fixed point property (F_1) , let Σ be the set of all means on $UC_r(K)$. Then Σ is a nonempty compact convex set in $(UC_r(K)^*$, weak$^*)$. Define $(x,m) \longrightarrow x\cdot m$ from $K \times \Sigma$ into Σ by $\langle x\cdot m,f\rangle = \langle m,{}_xf\rangle$, $x \in K$, $f \in UC_r(K)$, $m \in \Sigma$. Then it is easy to see that the mapping $(x,m) \longrightarrow x\cdot m$ is continuous. We have

$$\delta_x * \delta_y * f = \int (\delta_u * f)\, d\delta_x * \delta_y(u) \qquad (x,y \in K\ ,\ f \in UC_r(K))$$

and hence

$$\langle x\cdot(y\cdot m),\ f\rangle = \langle m,\ \delta_{\mathring{y}} * \delta_{\mathring{x}} * f\rangle = \int_K \langle m,\ \delta_u * f\rangle\, d\delta_{\mathring{y}} * \delta_{\mathring{x}}(u)$$

$$= \int_K \langle m,\ {}_u f\rangle\, d\delta_x * \delta_y(u) = \int \langle u\cdot m, f\rangle\, d\delta_x * \delta_y(u) .$$

Thus $x\cdot(y\cdot m) = (\delta_x * \delta_y)\cdot m$ for all $x,y \in K$, $m \in \Sigma$, because the weak* topology is the $\sigma(UC_r(K)^*, UC_r(K))$ topology. Hence there is a mean $m \in \Sigma$ such that $x\cdot m = m$ for all $x \in K$. That is, m is a LIM on $UC_r(K)$ and hence K is amenable. □

<u>Definition 3.3.2</u> By a *representation* T of K on a Banach space E , we mean a mapping $T : x \longrightarrow T_x$ of K into $\mathcal{B}(E)$, the bounded linear operators on E , such that

(i) $T_e = I$, the identity operator ;

(ii) $\|T_x\| \le 1$ for each $x \in K$;

(iii) $x \longrightarrow T_x s$ $(s \in E)$ is continuous and

$$\langle\phi,\ T_x\cdot(T_y s)\rangle = \int_K \langle\phi,\ T_u s\rangle\, d\delta_x * \delta_y(u) ,$$

$x, y \in K$, $s \in E$, $\phi \in E^*$. (That is, $T_x T_y = T_{\delta_x * \delta_y}$, $x, y \in K$; see also 4.2.3(1)) .

For $s \in E$, let $C_s = co\{T_x s : x \in K\}$ and J the subspace of E spanned by $\{T_x s - s : x \in K, s \in E\}$. Observe that J and $cl\ C_s$ are invariant under T (cf. 2.2.5(iii)). We say that K satisfies the *Reiter-Glicksberg property (RG)* if whenever T is a representation of K on a Banach space E, then $d(0,C_s) = d(s,J)$ for each $s \in E$, where $d(s',A) = \inf_{t \in A} \|s' - t\|$ for a subset $A \subseteq E$ and $s' \in E$.

<u>Theorem 3.3.3</u> K is amenable if and only if K satisfies the Reiter-Glicksberg property (RG).

<u>Proof</u>: Let K be amenable. Let $s_0 \in E$ be such that $d(0,C_{s_0}) = \delta > 0$, and write

$$\Sigma = \{ \phi \in E^* : \|\phi\| \le \frac{1}{\delta} \text{ and } \operatorname{Re} \langle \phi, T_x s_0 \rangle \ge 1 \text{ for all } x \in K \}.$$

Then Σ is a convex weak* compact set in E^*. It is non-empty by the Hahn-Banach theorem [79, Ch. 8, 6.2].

Next consider the map $(x,\phi) \longrightarrow x\cdot\phi$ from $K \times \Sigma$ to Σ given by $\langle x\cdot\phi, s\rangle = \langle\phi, T_x s\rangle$, $x \in K$, $\phi \in \Sigma$, $s \in E$ (note that

$$\begin{aligned} \operatorname{Re}\langle x\cdot\phi, T_y s_0\rangle &= \operatorname{Re}\langle\phi, T_x(T_y s_0)\rangle \\ &= \operatorname{Re} \int \langle\phi, T_u s_0\rangle \, d\delta_x * \delta_y(u) \\ &= \int \operatorname{Re} \langle\phi, T_u s_0\rangle \, d\delta_x * \delta_y(u) \ge 1). \end{aligned}$$

It is easy to see that $(x,\phi) \longrightarrow x\cdot\phi$ is continuous and that the hypotheses (i), (ii) and (iii) of the fixed point property are satisfied. Thus there is a $\phi \in \Sigma$ such that $x \cdot \phi = \phi$ for each $x \in K$. Hence, it follows that $\langle \phi, s_0 \rangle = 1$, $\langle \phi m\ T_x s \rangle = \langle \phi, s \rangle$, for $x \in K$, $s \in E$, and $\|\phi\| = \frac{1}{\delta}$. This implies $d(0, C_s) = d(s, J)$ for each $s \in E$ (see [79, p. 183]). That is, K satisfies the property (RG).

Conversely, if K satisfies the property (RG), consider the representation $x \longrightarrow \delta_x * f$ of K on $UC_\eta(K)$. Then $1 = d(0, C_1) = d(1, \Pi_0(UC_\eta(K)))$, and hence there is a LIM on $UC_\eta(K)$ by 3.2.5 (cf. [38, Theorem 5]). So, K is amenable. □

Let $D = co\ \{\delta_x : x \in K\}$, and $\Pi_D(X) = \{ f \in X : 0 \in$ norm closure of $D * f \}$, where $X = UC(K)$, $UC_\eta(K)$, $C(K)$, or $L_\infty(K)$. The next result is due to W. R. Emerson [30, Theorem 1] and J. C. S. Wong and A. Riazi [97, Theorem 4.1] for the case when K is a group.

Theorem 3.3.4

(i) If $X = UC_\eta(K)$ or $UC(K)$, then $\Pi_D(X)$ is closed under addition if and only if K is amenable, and in this case $\Pi_D(X) = cl\ \Pi_0(X)$.

(ii) Suppose that $spt\ \delta_x * \delta_y$ is finite for all $x, y \in K$.

(a) If K_d is amenable, then

$$\Pi_D(X) = cl\ \Pi_0(X) \quad \text{for} \quad X = C(K),\ L_\infty(K).$$

(b) If $\Pi_D(X)$ is closed under addition, then K is amenable and $\Pi_D(X) = cl\ \Pi_0(X)$, $X = C(K)$ or $L_\infty(K)$.

<u>Proof</u>:

(i) If there is a LIM on $UC_\eta(K)$, consider the representation $x \longrightarrow \delta_x \bullet f$ of K on $UC_\eta(K)$. Then $d(0, D\bullet f) = d(f, \Pi_0(UC_\eta(K))$ by 3.3.3 , and hence $\Pi_D(UC_\eta(K)) = cl\ \Pi_0(UC_\eta(K))$.

Conversely, if $\Pi_D(UC_\eta(K))$ is closed under addition, then as in the group case [30, Proposition 1.3] , $\Pi_D(UC_\eta(K))$ is a conjugate closed subspace of $UC_\eta(K)$ with $\inf h \le 0$ for all real valued $h \in \Pi_D(UC_\eta(K))$. Indeed, if $h \in \Pi_D(UC_\eta(K))$ is real valued and $\inf h = \varepsilon > 0$, then

$$\left(\sum_{i=1}^{n} c_i \delta_{x_i} \right) \bullet h(y) = \sum_{i=1}^{n} c_i \int_K h(u)\ d\delta_{\mathring{x}_i} \bullet \delta_y(u) \ge \varepsilon ,$$

for $x_i \in K$, $c_i \ge 0$, $\sum_{i=1}^{n} c_i = 1$.

Next, for $x \in K$ and $f \in UC_\eta(K)$, write

$$\nu_n = \frac{1}{n} (\delta_x + \delta_x \bullet \delta_x + \dots + \underbrace{\delta_x \bullet \dots \bullet \delta_x}_{n \text{ times}}) ,$$

and

$$f_n = \nu_n \bullet (\delta_x \bullet f - f) .$$

Then $\|f_n\|_\infty \le \frac{2}{n}\|f\|_\infty$ which converges to zero. Thus,

$\delta_x * f - f \in \Pi_D(UC_r(K))$ by 2.2.5(i) . That is, $\Pi_D(UC_r(K))$ contains $\Pi_0(UC_r(K))$, and hence there is a LIM on $UC_r(K)$ by 3.2.4 .

(ii) (a) Follows as in (i) since $x \longrightarrow \delta_x * f$ is a representation of K_d on X (= $C(K)$, $L_\infty(K)$) and K_d is amenable. To prove (b) , first observe that if $h \in L_\infty(K)$ is real valued and ess inf $h = c > 0$, then ess inf $\mu * h \ge c$ for $\mu \in D$. Since the ν_n's are contained in D by the hypothesis, if follows that Π_D $(L_\infty(K))$ is a conjugate closed subspace of $L_\infty(K)$ containing $\Pi_0(L_\infty(K))$ and hence amenable by 2.2.5 . Now, if $f \in \Pi_D(L_\infty(K))$ and m is a LIM on $L_\infty(K)$, then $m(f) = 0$. Hence, $cl\ \Pi_0(L_\infty(K)) = \Pi_D(L_\infty(K))$ by 3.2.6 (a) . □

<u>Remark 3.3.5</u> T. Miao has recently solved Emerson's original problem [30, p. 187] . Indeed, he has proved in [65] that if G is any locally compact group then G is amenable as a discrete group if (and only if) Π_D $(L_\infty(G))$ is closed under addition.

<u>Corollary 3.3.6</u> Let K be amenable. If α is a complex number and $f \in X$ (= $UC(K)$, $UC_r(K)$) then $\alpha \cdot 1 \in cl\ D*f$ if and only if $m(f) = \alpha$ for all LIMS m on X .

Proof: Follows from 3.3.4(i) (See also [79, 7.3(c)]) . □

The interested reader is referred to [59] for more characterizations of amenable hypergroups which are consequences of the Reiter-Glicksberg property (RG) .

Definition 3.3.7 We say that K satisfies *Reiter's condition* (P_1) $[P_1^*]$ if whenever $\varepsilon > 0$ and a compact [finite] set $E \subseteq K$ are given, then there exists $\phi \in P(K)$ such that $\|\delta_x * \phi - \phi\| < \varepsilon$ for every $x \in E$.

The proof of the next result is adapted from A. Hulanicki [50, § 4] .

Theorem 3.3.8 K is amenable if and only if K satisfies Reiter's condition (P_1) $[(P_1^*)]$.

Proof: If K is amenable, let $\varepsilon > 0$ and $E \subseteq K$ compact be given. Fix $\beta \in P(K)$. Choose $x_1, x_2, \ldots, x_n \in K$ and open neighbourhoods V_{x_i} of x_i , $1 \le i \le n$ such that $E \subseteq \bigcup_{i=1}^{n} V_{x_i}$, and $\|\delta_y * \beta - \delta_{x_i} * \beta\|_1 < \varepsilon$ for all $y \in V_{x_i}$, $1 \le i \le n$.

Next, using an approximate identity for $L_1(K)$ contained in P(K) , we can find $\psi \in P(K)$ such that $\|\psi * \beta - \beta\| < \varepsilon$. By

3.2.8 and 3.2.9 , there exists $\phi_0 = \phi_{\alpha_0} \in P(K)$ such that $\|\beta * \phi_0 - \phi_0\|_1 < \varepsilon$, and $\|(\delta_{x_k} * \psi) * \phi_0 - \phi_0\|_1 < \varepsilon$, $1 \le k \le n$, $\left(\delta_{x_k} * \psi \in P(K)\right)$. Put $\phi = \beta * \phi_0 \in P(K)$. We then have $\|\psi * \phi - \phi\|_1 = \|(\psi * \beta) * \phi_0 - (\beta * \phi_0)\|_1 \le \|\psi * \beta - \beta\|_1 < \varepsilon$. This implies $\|\delta_{x_k} * (\psi * \phi) - \delta_{x_k} * \phi\|_1 < \varepsilon$, $1 \le k \le n$. Let $z \in V_{x_k}$ for some $1 \le k \le n$. Then

$$
\begin{aligned}
\|\delta_z * \phi - \phi\| \quad & \|\delta_z * \phi - \delta_{x_k} * \phi\|_1 + \|\delta_{x_k} * \phi - \phi\|_1 \\
&= \|(\delta_z * \beta) * \phi_0 - (\delta_{x_k} * \beta) * \phi_0\|_1 + \|\delta_{x_k} * \phi - \phi\|_1 \\
&\le \|\delta_z * \beta - \delta_{x_k} * \beta\|_1 + \|\delta_{x_k} * \phi - \phi\|_1 \\
&< \varepsilon + \|\delta_{x_k} * \phi - \delta_{x_k} * (\psi * \phi)\|_1 + \|\delta_{x_k} * (\psi * \phi) - \phi\|_1 \\
&< 2\varepsilon + \|\delta_{x_k} * (\psi * \phi) - \phi\|_1 \\
&= 2\varepsilon + \|\delta_{x_k} * \psi) * \beta * \phi_0 - \beta * \phi_0\|_1 \\
&\le 2\varepsilon + \|(\delta_{x_k} * \psi) * \beta * \phi_0 - (\delta_{x_k} * \psi) * \phi_0\|_1 + \\
&\qquad\qquad + \|(\delta_{x_k} * \psi) * \phi_0 - \phi_0\|_1 + \|\beta * \phi_0 - \phi_0\|_1 \\
&< 2\varepsilon + 2\|\beta * \phi_0 - \phi_0\|_1 + \|(\delta_{x_k} * \psi) * \phi_0 - \phi_0\|_1 < 5\varepsilon .
\end{aligned}
$$

If K satisfies (P_1^*) then it is easy to see that there is a net $(\phi_\alpha) \subseteq P(K)$ converging strongly to left invariance. □

<u>Corollary 3.3.9</u> K is amenable if and only if there is a net $(\phi_\alpha) \subseteq P(K)$ such that $\|\mu * \phi_\alpha - \phi_\alpha\|_1$ converges to zero for all $\mu \in M'(K)$.

Proof: See [73, 44 Theorem] . □

Let E be a two sided Banach $L_1(K)$-module [8, p. 49] . Then E^* is also a two sided Banach $L_1(K)$-module. By a *derivation* D of $L_1(K)$ into E^* we mean a linear map

$$D : L_1(K) \longrightarrow E^*$$

such that $D(\phi * \psi) = D(\phi)\cdot\psi + \phi\cdot D(\psi)$ for all $\phi,\psi \in L_1(K)$. If $f \in E^*$, then the map $\delta_f : L_1(K) \longrightarrow E^*$ given by $\delta_f(\phi) = \phi\cdot f - f\cdot\phi$ is a bounded derivation. A derivation of this form is called an *inner derivation.* Following B. E. Johnson [53, p. 60] , we say that the *Banach algebra* $L_1(K)$ *is amenable* if every bounded derivation of $L_1(K)$ into E^* is an inner derivation. The next result follows from [59, Theorem 4.1] .

Proposition 3.3.10 If the Banach algebra $L_1(K)$ is amenable then K is amenable.

The converse is not true in general. The author is very thankful to Dr. Brian Forrest for suggesting the following:

Example 3.3.11 This is the same as [11, Example 4.5] . Let $G = \mathbb{R}^n$ and let B be the group of rotations in G . Consider the hypergroup $K = G_B$ (see the discussion preceeding 3.2.20) .

As a set, K is identified with $\mathbb{R}^+ = [0,\infty)$. The hypergroup $\hat{K}$ is isomorphic with K and so $L_1(K)$ and $A(K)$ are isometrically isomorphic, where $A(K)$ is the pointwise algebra of Fourier transforms on K . Fix ρ in $(0,\infty)$. It is known that all of the functions in $A(K)$ are continuously differentiable in $(0,\infty)$ [79, Ch. 2, 6.3(4)] . Let δ be the first derivative evaluated at $\rho : \delta(f) = f'(\rho)$ for $f \in A(K)$. For $n \geq 3$, δ is continuous in the topology of $A(K)$ and can be defined on all of $A(K)$. Accordingly, δ is a point derivation at ρ (see [11, p. 326]) . This shows that $L_1(K)$ is not weakly amenable (We say $L_1(K)$ is weakly amenable if every bounded derivation of $L_1(K)$ into a commutative Banach module E $[\phi \cdot s = s \cdot \phi$ for all $\phi \in L_1(K)$, $s \in E]$ is necessarily zero [2]) . Finally, note that if $L_1(K)$ is amenable, then it is weakly amenable [2] , and that K is amenable since it is commutative. Since $L_1(K)$ is isometrically isomorphic to the closed subalgebra of $L_1(\mathbb{R}^n)$ consisting of the radial functions, we have a closed subalgebra of an amenable algebra which is not weakly amenable.

The remaining results in this section are motivated by the work of L. Gallardo and O. Gebuhrer [32, Chapitre V] and J. Dieudonné [20] . Let $P_2(K) = \{ \phi \in L_2(K) ; \phi \geq 0 , \|\phi\|_2 = 1\}$. We say that K satisfies (P_2) , if whenever $\varepsilon > 0$ and a compact $E \subseteq K$ are given there exists $\phi \in P_2(K)$ such that $\|\delta_x * \phi - \phi\|_2 < \varepsilon$, for all $x \in E$. The proof of the next result is slightly more delicate than the group case because the relation ${}_x(fg) = {}_xf\,{}_xg$ does not hold in general for hypergroups.

<u>Theorem 3.3.12</u> If K satisfies (P_2) then it satisfies Reiter's condition (P_1). Conversely, if ϕ in Reiter's condition (P_1) can be chosen of the form $\frac{1}{\lambda(A)} 1_A$, where A is a Borel set in K with $0 < \lambda(A) < \infty$, then K has property (P_2).

<u>Proof</u>: If K satisfies (P_2), let $\varepsilon > 0$ and compact $E \subseteq K$ be given. Let $\phi \in P_2(K)$ be such that $\|\delta_x * \phi - \phi\|_2 < \varepsilon$ for all $x \in E$, and put $\psi = \phi^2 \in P(K)$. Following [11, p. 319], write

$$\begin{aligned}\delta_x * \psi(y) - \psi(y) &= \int_K [\psi(z) - \psi(y)]\, d\delta_{\check{x}} * \delta_y(z) \\ &= \int_K [\phi(z) - \phi(y)]^2\, d\delta_{\check{x}} * \delta_y(z) + 2[\phi(y)\delta_x * \phi(y) - \phi^2(y)] \\ &= G_1(y) + G_2(y) .\end{aligned}$$

$$\int_K |G_2(y)|\, dy \le 2\|\phi\|_2\, \|\delta_x * \phi - \phi\|_2 < 2\varepsilon , \quad \text{all} \quad x \in E ,$$

$$\begin{aligned}G_1(y) &= \int_K [\phi^2(z) - 2\phi(z)\phi(y) + \phi^2(y)]\, d\delta_{\check{x}} * \delta_y(z) \\ &= \delta_x * \phi^2(y) - 2\delta_x * \phi(y)\phi(y) + \phi^2(y) \\ &= [\delta_x * \phi(y) - \phi(y)]^2 + \delta_x * \phi^2(y) - (\delta_x * \phi)^2(y) .\end{aligned}$$

So $$\begin{aligned}\int_K G_1(y)\,dy &= \|\delta_x * \phi - \phi\|_2^2 + \|\phi\|_2^2 - \|\delta_x * \phi)\|_2^2 \\ &\le 2\, \|\delta_x * \phi - \phi\|_2 + 2[\|\phi\|_2 - \|\delta_x * \phi\|_2]\end{aligned}$$

$\leq 4 \, \|\delta_x * \phi - \phi\|_2 < 4\varepsilon$ for all $x \in E$.

Hence, $\|\delta_x * \psi - \psi\|_1 < 6\varepsilon$ for all $x \in E$. Conversely, if $\phi = \frac{1}{\lambda(A)} 1_A \in P(K)$, where A is a Borel set in K with $0 < \lambda(A) < \infty$, satisfies $\|\delta_x * \phi - \phi\|_1 < \varepsilon^2$ for all $x \in E$, let

$$\psi = \phi^{\frac{1}{2}} = \frac{1}{\lambda(A)^{\frac{1}{2}}} 1_A \in P_2(K)$$

(cf. [50, p. 100]) . Then

$$\|\delta_x * \psi - \psi\|_2^2 = \frac{1}{\lambda(A)} \int_K |\delta_x * 1_A(y) - 1_A(y)|^2 dy$$

$$\leq \frac{2}{\lambda(A)} \int_K |\delta_x * 1_A(y) - 1_A(y)| \, dy = 2 \, \|\delta_x * \phi - \phi\|_1 .$$

Hence, $\|\delta_x * \psi - \psi\|_2 < \varepsilon$ for all $x \in E$. □

Let $\mu \longrightarrow T_\mu$ be the left regular representation of K on $L_2(K)$, given by $T_\mu f = \mu * f$, $f \in L_2(K)$, $\mu \in M(K)$.

Proposition 3.3.13 The following two statements are equivalent:

(i) K satisfies (P_2) ;

(ii) K satisfies (F) : There is a net $(f_\alpha) \subseteq L_2(K)$, $\|f_\alpha\|_2 = 1$, such that $f_\alpha * f_\alpha^{\#}$ converges to 1 uniformly on compact subsets of K .

In this case, we have

(iii) (G) : $|\int_K d\mu| \leq \|T_\mu\|$ for all $\mu \in M(K)$;

(iv) (D_2) : $\|T_\mu\| = \|\mu\|$ for all $\mu \in M^+(K)$.

<u>Proof</u>:

(ii) $\longrightarrow$ (i) If $\varepsilon > 0$ and a compact set $E \subseteq K$ are given choose $f \in L_2(K)$, $\|f\|_2 = 1$ such that $|1 - f{*}f^{\#}(x)| < \varepsilon$ for all $x \in E$, and let $\phi = |f|$. Then, $0 \le |f{*}f^{\#}(x)| \le \phi{*}\phi^{\sim}(x)$, and $0 \le 1 - \phi{*}\phi^{\#}(x) \le 1 - |f{*}f^{\#}(x)| \le |1 - f{*}f^{\#}(x)| < \varepsilon$, $x \in E$. Now,

$$\|{}_x\phi - \phi\|_2^2 = \langle {}_x\phi - \phi\ {}_x\phi - \phi\rangle = \|{}_x\phi\|_2^2 + \|\phi\|_2^2 - 2\langle \phi, {}_x\phi\rangle$$
$$\le 2\,(1 - \langle \phi, {}_x\phi\rangle) < 2\varepsilon \quad \text{for all } x \in E .$$

(i) $\longrightarrow$ (ii) is easy.

(ii) $\longrightarrow$ (iii) If $f_\alpha{*}f_\alpha{}^{\#}$ converges to 1 uniformly on compact subsets of K, then

$$\lim_\alpha \langle T_\mu f_\alpha, f_\alpha\rangle = \lim_\alpha \int_K f_\alpha{*}f_\alpha^{\#}\, d\mu = \int_K d\mu \qquad (\mu \in M(K))$$

This implies that $\left|\int_K d\mu\right| \le \|T_\mu\|$ for all $\mu \in M(K)$.

(iii) $\longrightarrow$ (iv) is clear. □

The next result is in [32] for a second countable commutative hypergroup. For the sake of completeness, we give a proof.

<u>Lemma 3.3.14</u> Let K be commutative hypergroup with the Plancherel measure π on the dual $\hat{K}$. Then the trivial character 1 is in $spt\ \pi$ if and only if K satisfies (F) or equivalently (P_2).

<u>Proof</u>: If $1 \in spt\ \pi$ then by considering the inverse Fourier

transform one can easily find a net $(f_\alpha) \subseteq L_2(K)$, $\|f_\alpha\|_2 = 1$ such that $f_\alpha * f_\alpha^{\#}$ converges to 1 uniformly on compact subsets of K (see [52, 13.7B] and [11, Lemmas 3.1 & 3.2]) . The converse follows from the implication (ii) $\longrightarrow$ (iii) of 3.3.13 and [52, p. 41] . □

Example 3.3.15 Let K be the hypergroup given in [52, 9.5] (known as Naimark's example) . Then $1 \notin spt\ \pi$ and hence K does not satisfy (P_2) . But K is amenable (or satisfies (P_1)) because it is commutative. An example of a commutative discrete hypergroup which does not satisfy (P_2) (That is, the trivial character $1 \notin spt\ \pi$) can be found in [33, 6.2.3] .

Theorem 3.3.16

(i) If H is a compact sub-hypergroup of K such that $\delta_x * \sigma = \sigma * \delta_x$ for all $x \in K$, where σ is the normalized Haar measure of H , and K//H satisfies (F) , so does K ;

(ii) If H is a compact normal subgroup of K such that K/H has property (F) then K satisfies (F) ;

(iii) If K_1 and K_2 are hypergroups having property (P_2) , then $K_1 \times K_2$ has property (P_2) ;

(iv) If $K = H \vee J$, H compact, J discrete, $H \cap J = \{e\}$, then K has property (F) provided J satisfies (F) . In particular, if J is an amenable (discrete) group then $H \vee J$ satisfies (F) ;

(v) If a hypergroup K has a compact supernormal sub-hypergroup H then K has property (P_2) if (and only if) K is amenable.

__Proof__:

(i) If $f,g \in C_c(K//H)$, then it is easy to see that $(f*g)\circ\Pi = (f\circ\Pi)*(g\circ\Pi)$. This shows that if $(f_\alpha) \subseteq L_2(K//H)$, $\|f_\alpha\|_2 = 1$ is such that $f_\alpha * f_\alpha^{\#}$ converges to 1 uniformly on compact subsets of $K//H$, then $(f_\alpha\circ\Pi)*(f_\alpha\circ\Pi)^{\#}$ converges to 1 uniformly on compact subsets of K, $f_\alpha\circ\Pi \in L_2(K)$, and $\|f_\alpha\circ\Pi\|_2 = 1$.

(ii) Same as (i).

(iii) Let $\varepsilon > 0$ and a compact set $E \subseteq K_1 \times K_2$ be given. We assume that $E = E_1 \times E_2$, $E_i \subseteq K_i$ compact, $i = 1,2$. Let $\phi_i \in P_2(K_i)$ $(i = 1,2)$ satisfy $\|\delta_x * \phi_i - \phi_i\|_2 < \varepsilon$, all $x \in E_i$, and write $\phi(x,y) = \phi_1(x)\phi_2(y)$, $(x,y) \in K_1 \times K_2$. Then $\phi \in L_2(K_1 \times K_2)$, $\phi \geq 0$, $\|\phi\|_2 = 1$, and it is easy to see that $\|\delta_{(x,y)} * \phi - \phi\|_2 < 2\varepsilon$ for all $(x,y) \in E_1 \times E_2$.

(iv) If J has property (F) then so does K by (i) and [92, Proposition 1.2].

(v) If K is amenable then $K//H$ is amenable by 3.2.16. Since $K//H$ is a group [93, Theorem 2.1] it has property (F). Thus K has property (F) by (i) and [7, 2.2.1 Lemma]. □

§ 3.4 On the Size of the Set of Topological Invariant Means on $L_\infty(K)$.

In this section, we obtain the exact cardinality of the set of topological invariant means on $L_\infty(K)$. Some applications are also given. The main results and their proofs are inspired by recent papers of A. T. Lau and A. L. T. Paterson [62] and Z. Yang [98] .

Throughout this section, K will denote a noncompact amenable hypergroup with left Haar measure λ . Let d be the smallest cardinality of a cover of K by compact sets.

<u>Lemma 3.4.1</u> Let A be a closed set in K that can be written as the union of less than d compact subsets of K . Then $m(A) = 0$ for all LIMs m on $L_\infty(K)$, where $m(A) = m(1_A)$.

<u>Proof</u>: For $x \in K$, we have $\{x\} * A \cap A \neq \emptyset$ if and only if $x \in A * \mathring{A}$ [52, 4.1A] . Since $A * \mathring{A}$ is the union of less than d compact subsets of K [52, 3.2B] , there exists $x \in K$ such that $\{x\} * A \cap A = \emptyset$. By induction, we can find a sequence $\{x_n\}_{n=1}^{\infty} \subseteq K$ such that $\{x_i\} * A \cap \{x_j\} * A = \emptyset$ $(i \neq j)$. For $x, y \in K$, we have $\{\mathring{x}\} * \{y\} \cap A \neq \emptyset$ if and only if $y \in \{x\} * A$. Also $\delta_{\mathring{x}} * \delta_y$ is a probability measure. Hence, $\delta_x * 1_A$ vanishes outside $\{x\} * A$, and less than or equal to one on $\{x\} * A$. That is,

$\delta_x \cdot 1_A \le 1_{\{x\}\cdot A}$ for $x \in K$. This implies that $m(A) \le \frac{1}{n}$ for each positive integer n, and hence $m(A) = 0$. □

Remark 3.4.2

(a) If $f \in C_0(K)$ and m a LIM on $UC(K)$, then $m(f) = 0$. For, if $f \in C_c(K)$, $x \in K$, then $spt\ \delta_x \cdot f \subseteq \{x\} \cdot spt\ f$. The rest follows as in the proof of the lemma (see [74, Proposition 2.1.2]).

(b) If $f \in L_\infty(K) \cap L_1(K)$ then $m(\mathring{f}) = 0$ for every TLIM on $L_\infty(K)$. In particular, if A is a Borel set in K with $\lambda(A) < \infty$, then $m(\mathring{A}) = 0$ (See [40, Proposition 1]).

Let V be a cover of K by compact sets with $|V| = d$, where $|V|$ is the cardinality of V. Let $\Omega = \Omega(V)$ be the set of all finite subsets of V and consider Ω as a directed set in the usual way : $\lambda \ge \lambda'$ if $\lambda \subseteq \lambda'$. Fix a TLIM m_0 on $L_\infty(K)$. Let U be a compact symmetric neighbourhood of e and $(f_k)_{k=1}^{\infty}$ a countable set in $L_\infty(K)$. The next result is due to Z. Yang [98, Theorem 3.3] for a locally compact noncompact amenable group. Similar formats appear in [15, § V] and [39, Theorem 5] for the case when G is σ-compact.

Lemma 3.4.3 There exists a net $(\psi_\lambda) \subseteq P_c(K) = P(K) \cap C_c(K)$ of means on $L_\infty(K)$ with the directed set Ω such that

(i) If $\lambda \ne \lambda'$, then $U \cdot spt\ \psi_\lambda \cap U \cdot spt\ \psi_{\lambda'} = \emptyset$;

(ii) (ψ_λ) converges strongly to topological left invariance (and right invariance if m is a TIM) ;

(iii) If m_0 is inversion invariant so is each ψ_λ ;

(iv) If m is any weak* cluster point of (ψ_λ) in $L_\infty(K)^*$, then $m(f_n) = m_0(f_n)$, $n = 1, 2, \ldots$.

If K is σ-compact then we can find a sequence $(\psi_n) \subseteq P_c(K)$ satisfying (i), (ii), (iii), and (iv) .

Proof: We assume that m_0 is inversion invariant (the other cases are even easier) and for convenience that $\|f_n\|_\infty \le 1$ for all n . Let (ϕ_α) be a net in $P_c(K)$ converging to m_0 in the weak* topology with $\phi_\alpha^{\#} = \phi_\alpha$ for all α . Since (ϕ_α) converges weakly to topological invariance, by 3.2.9 , we can assume that (ϕ_α) converges strongly to topological invariance and that $w^*\text{-}\lim\limits_\alpha \phi_\alpha = m_0$. Now, well order the set Ω by $\{\lambda_\alpha\}_{1\le\alpha<d}$ and let $\alpha < d$ be an ordinal. Suppose that for each $\beta < \alpha$ we have constructed a mean $\psi_{\lambda_\beta} \in P_c(K)$ satisfying

(a) If $\beta < \gamma < \alpha$ then $U * spt\,\psi_{\lambda_\beta} \cap U * spt\,\psi_{\lambda_\gamma} = \emptyset$;

(b) If $\beta < \alpha$, then $\|\delta_s * \psi_{\lambda_\beta} - \psi_{\lambda_\beta}\|_1 < \frac{1}{|\lambda_\beta|}$, and

$\|\psi_{\lambda_\beta} \circ \delta_s - \psi_{\lambda_\beta}\|_1 < \frac{1}{|\lambda_\beta|}$, for all $s \in \cup\, \lambda_\beta$;

(c) If $\beta < \alpha$, then ψ_{λ_β} is inversion invariant ;

(d) If $\beta < \alpha$, then $|\psi_{\lambda_\beta}(f_i) - m_0(f_i)| < \frac{1}{|\lambda_\beta|}$, for

$1 \leq i \leq |\lambda_\beta|$.

Write $A_\alpha = \bigcup_{\beta<\alpha} spt\ \psi_{\lambda_\beta}$, $A_1 = \emptyset$. For $s \in K$, $U \cdot s \cap spt\ \psi_\beta = \emptyset$ $(\beta < \alpha)$ if and only if $s \in U \cdot spt\ \psi_\beta$. That is, the neighbourhood $U \cdot s$ of s meets at most one element of the family $(spt\ \psi_\beta)_{1\leq\beta<\alpha}$. Thus A_α is closed and therefore $U^3 \cdot A_\alpha \cdot U^3$ is closed [52, 4.1E] .

Since the latter set is the union of less than d compact sets, by 3.4.1 , $m_0(U^3 \cdot A_\alpha \cdot U^3) = 0$. Fix $\phi \in P_c(K)$ with $spt\ \phi \subseteq U$, $\phi = \phi^\#$. Let $0 < \varepsilon < 1$ be given. Choose $s_1, \ldots, s_n$ in $U\lambda_\alpha$ and neighbourhoods V_i of s_i , $1 \leq i \leq n$ such that $U\lambda_\alpha \subseteq \bigcup_{i=1}^{n} V_i$, $\|\delta_s \cdot \phi - \delta_{s_i} \cdot \phi\|_1 < \epsilon$ and $\|\phi\circ\delta_s - \phi\circ\delta_{s_i}\|_1 < \varepsilon$, for $s \in V_i$, $1 \leq i \leq n$. Next find $\psi \in P_c(K)$ such that $\|\phi\cdot\psi - \phi\|_1 < \varepsilon$, $\|\psi\cdot\phi - \phi\|_1 < \varepsilon$. Finally, since (ϕ_α) converges strongly to topological invariance and $w^*\text{-}\lim_\alpha \phi_\alpha = m_0$, we can find $\phi_{\alpha_0} = \phi_0$ such that

$\|\phi\cdot\phi_0 - \phi_0\|_1 < \varepsilon$, $\|\phi_0\cdot\phi - \phi_0\| < \varepsilon$, $\|(\delta_{s_i}\cdot\psi)\cdot\phi_0 - \phi_0\|_1 < \epsilon$,

$\|\phi_0\cdot(\psi\circ\delta_{s_i}) - \phi_0\|_1 < \varepsilon$, for $1 \leq i \leq n$,

$|\phi_0(f_i) - m_0(f_i)| < \varepsilon$, $1 \leq i \leq |\lambda_\alpha|$, and $\phi_0(U^3\cdot A_\alpha\cdot U^3) < \varepsilon$.

Next, define $\phi_0' \in P(K)$ by $\langle\phi_0' , f\rangle = \langle\phi_0 , \frac{f}{\phi_0(B_\alpha)} 1_{B_\alpha}\rangle$, $f \in L_\infty(K)$, where $B_\alpha = K \setminus U^3 \cdot A_\alpha \cdot U^3$. Then

$\|\phi_0' - \phi_0\|_1 < \varepsilon'$, $\varepsilon < \varepsilon' < \frac{2\varepsilon}{1-\varepsilon}$. Let $\psi_{\lambda_\alpha} = \phi * \phi_0' * \phi$. Then $\psi_{\lambda_\alpha} \in P_c(K)$, and it is easy to see that $\|\delta_s * \psi_{\lambda_\alpha} - \psi_{\lambda_\alpha}\|_1 < 7\varepsilon'$, $\|\psi_{\lambda_\alpha} \circ \delta_s - \psi_{\lambda_\alpha}\|_1 < 7\varepsilon'$, for $s \in \bigcup_{i=1}^{n} V_i$, and $|\psi_\alpha(f_i) - m_0(f_i)| < 4\varepsilon'$, for $1 \le i \le |\lambda_\alpha|$ (see the proof of 3.3.8) . Now, $spt\ \psi_\alpha \subseteq U * spt\ \phi_0' * U$, and $U * spt\ \psi_{\lambda_\alpha} \cap U * A_\alpha = \emptyset$ if and only if $U^2 * A_\alpha \cap spt\ \psi_\alpha = \emptyset$ [32, 4.1B] . Also, $U * spt\ \phi_0' * U \cap U^2 * A_\alpha = \emptyset$ if and only if $spt\ \phi_0' * U \cap U^3 * A_\alpha = \emptyset$ if and only if $spt\ \phi_0' \cap U^3 * A_\alpha * U = \emptyset$. Since $spt\ \phi_0' \subseteq B_\alpha$, we have $U * A_\alpha \cap U * spt\ \psi_{\lambda_\alpha} = \emptyset$. Each $spt\ \psi_{\lambda_\beta}$, $\beta < \alpha$, is symmetric, and hence A_α is symmetric. This shows that ϕ_0' and hence ψ_{λ_α} is inversion invariant. So, ψ_{λ_α} satisfies (a), (b), (c), and (d) . Thus by transfinite induction we have a net $(\psi_\lambda)_{\lambda\in\Omega} \subseteq P_c(K)$ of means on $L_\infty(K)$ such that each ψ_λ satisfies (a), (b), (c), and (d) . It is not hard to show now that the net $(\psi_\lambda)_{\lambda\in\Omega}$ satisfies all the properties of the lemma. By easy modifications of the above arguments, we have the last statement (see [15, p. 225-226]) . □

Let Ω be a directed set, and let $\ell_\infty(\Omega)$ be the Banach space of bounded real valued functions on Ω , with the supremum norm. Write

$$\Phi = \{ \phi \in \ell_\infty(\Omega)^* : \phi(x) \le \limsup_{\lambda\in\Omega} x(\lambda) \quad \text{for all} \quad x \in \ell_\infty(\Omega) \} .$$

Then Φ is the set of all $\phi \in \ell_\infty(\Omega)^*$ such that $\phi \ge 0$, $\|\phi\| = 1$, and $\phi(x) = \lim_{\lambda} x(\lambda)$ whenever the limit exists. Let A be an infinite set and $\Omega = \Omega(A)$ is the set of all finite subsets of A directed by inclusion.

The next result is in [98, Lemma 2.1].

<u>Lemma 3.4.4</u> If Φ is as above for the directed set $\Omega(A)$, then $|\Phi| = 2^{2^{|A|}}$.

Let $L^r_\infty(K)$ be the Banach space of real valued essentially bounded Borel measurable functions on K (with respect to the left Haar measure λ). Let $(\phi_\lambda)_{\lambda\in\Omega}$ be a net of means on $L^r_\infty(K)$ given by functions in $P(K)$. Suppose that for each $s \in K$ there is a neighbourhood U of s which meets at most one element of the family $(spt\ \phi_\lambda)_{\lambda\in\Omega}$. Let Ψ be the weak* closed convex hull of the set of all weak* cluster points of $(\phi_\lambda)_{\lambda\in\Omega}$ in $L^r_\infty(K)^*$. Then Ψ is a nonempty weak* compact convex subset of the set $\Sigma(L^r_\infty(K))$ of all means on $L^r_\infty(K)$. Let Φ be defined for the directed set Ω as before. Define $\Pi : L^r_\infty(K) \longrightarrow \ell_\infty(\Omega)$ by $\Pi(f)(\lambda) = \phi_\lambda(f)$, $f \in L^r_\infty(K)$, $\lambda \in \Omega$. The proof of the next result is exactly as in [98, Lemma 3.1]. Note that the group structure and the topological invariance of the net $(\mu_\lambda)_{\lambda\in\Lambda}$ are not used there.

Lemma 3.4.5 There exists a linear isometry of $\ell_\infty(\Omega)^*$ into $L^r_\infty(K)^*$ which maps Φ weak* homeomorphically onto Ψ.

The next theorem is due to C. Chou [12, Theorem 5.3] for a σ-compact and noncompact amenable locally compact group. E. Granirer, assuming the continuum hypothesis, gives a different proof of this result in [40, p. 61]. It is due to A. T. Lau and A. L. T. Paterson [62, Theorem 1] and Z. Yang [98, Theorem 3.3] for an arbitrary locally compact noncompact amenable group.

Theorem 3.4.6 Let K be a noncompact amenable hypergroup. Then $|TIIM\ (L_\infty(K))| \geq 2^{2^d}$.

Proof: The net $(\psi_\lambda)_{\lambda\in\Omega}$ constructed in 3.4.3 has at least 2^{2^d} weak* cluster points in $L^r_\infty(K)^*$ by 3.4.4 and 3.4.5. We next observe that each TIIM on $L^r_\infty(K)$ extends to a TIIM on $L_\infty(K)$: If m is a TIIM on $L^r_\infty(K)$, write
$M(f + ig) = m(f) + im(g)$, $f,g \in L^r_\infty(K)$.

The next result is due to A. T. Lau and A. L. T. Paterson [62, Theorem 1] for the case when K is a group (see also [98, Corollary 3.4]).

__Theorem 3.4.7__ Let K be a noncompact amenable hypergroup. Then $|TIIM(L_\infty(K))| = |TIM(L_\infty(K))| = 2^{2^d}$. If the maximal subgroup $G(K)$ is open, then $|TLIM(L_\infty(K))| = 2^{2^d}$.

__Proof__: To prove the first statement, by 3.4.6, we only need to prove $|TIM(L_\infty(K))| \le 2^{2^d}$.

Let H be a compact subhypergroup of K such that $K//H$ is metrizable [91, Theorem A.4]. Let H_0 be an open noncompact σ-compact subhypergroup of K containing H (see [52, 10.1B] and [95, p. 71B]). The smallest cardinality of the cover of $K//H$ by compact sets is d. Let E be a compact subset of $K//H$. For $x \in K$, the set $(H_0//H) * HxH * (H_0//H)$ is open and σ-compact in $K//H$. Since E can be covered by a finite number of such sets, it is separable by [91, Lemma A.2]. Hence, there is a dense subset T of $K//H$ with cardinality d. Let σ be the normalized Haar measure of H, and let $\phi \in P(K)$. For $f \in L_\infty(K)$, the function $(\sigma*\phi) * f * (\sigma*\phi)^\circ = \sigma * (\phi * f * \mathring{\phi}) * \sigma$ is continuous and constant on the double cosets of H in K. Indeed, if g is a nonnegative bounded continuous function and $h = \sigma * g * \sigma$, then there is a $z_0 \in HxH$ such that $h(z_0) = \sup_{z \in HxH} h(z)$. If $s,t \in H$, then

$\delta_{\mathring{s}} * h * \delta_{\mathring{t}}(z_0) = h(z_0) = \int h \, d\delta_s * \delta_{z_0} * \delta_t$, and h is constant on $\{s\} * \{z_0\} * \{t\}$. This shows that h is constant on $Hz_0H = HxH$. Consider $A = \{(\sigma*\phi) * f * (\sigma*\phi)^\circ : f \in L_\infty(K)\}$ as a subspace of $C(K//H)$. Since every function in $C(K//H)$ is determined by its values on T , we have $|C(K//H)| \le c^d = 2^d$.

If m is a TIM on $L_\infty(K)$, then $m((\sigma*\phi) * f * (\sigma*\phi)^\circ) = m(f)$ for all $f \in L_\infty(K)$, and hence m can be considered as a continuous linear functional on A . Therefore, $|\mathrm{TIM}\ (L_\infty(K))| \le |A^*| \le |C(K//H)| = c^{2^d} = 2^{2^d}$.

To see the second statement, we first assume that $G = G(K)$ is open and noncompact. Let L be a σ-compact noncompact open subgroup of $G(K)$ and H a compact normal subgroup of L such that L/H is separable [62, p. 79] . Write $(K/H)_r = \{Hx : x \in K\}$. Using [52, 10.3B and 10.4B] , one can show that for each $x \in K$, the mapping $Hg \longrightarrow Hgx$ of L/H into $(K/H)_r$ is continuous $(g \in L)$. Also, the set $(L/H)x = \{Hgx : g \in L\} = \{Hy : y \in Lx\}$ is open in $(K/H)_r$. Therefore every compact set $E \subseteq (K/H)_r$ is separable, and hence there is a dense set in $(K/H)_r$ with cardinality d . If G is compact and open then $(K/G)_r$ is discrete and $|(K/G)_r| = d$. The rest of the proof now follows as in the group case [62, Theorem 1] . □

The next result is due to C. Chou for the case when K is a group [13] .

Corollary 3.4.8 If K is an infinite discrete amenable hypergroup, then

$$|IM(\ell_\infty(K))| = |IIM(\ell_\infty(K))| = |LIM(\ell_\infty(K))| = 2^{2^{|K|}} .$$

Corollary 3.4.9 Let $K = H \vee J$, H compact, J discrete and infinite with $J \cap H = \{e\}$. If K is amenable then

$$|TLIM(L_\infty(K))| = |TIM(L_\infty(K))| = |TIIM(L_\infty(K))| = 2^{2^{|J|}} .$$

Proof: This follows from 3.4.7 and 3.2.25 . □

Corollary 3.4.10 Let K be a noncompact amenable hypergroup. If any of the sets $TLIM(L_\infty(K))\backslash TIM(L_\infty(K))$, $TIM(L_\infty(K))\backslash TIIM(L_\infty(K))$ or $TLIM(L_\infty(K))\backslash TIIM(L_\infty(K))$ is nonempty, then its cardinality is at least 2^{2^d} . It is equal to 2^{2^d} in the second case (and in all cases if the maximal subgroup of K is open) .

Proof: See [98, p. 323] . □

Corollary 3.4.11

(i) $|TIM(X)| = 2^{2^d}$ for $X = L_\infty(K)$, $C(K)$, $UC_n(K)$ or $UC(K)$;

(ii) $|TIIM(X)| = 2^{2^d}$ for $X = L_\infty(K)$, $C(K)$ or $UC(K)$;

(iii) $|TLIM(X)| = 2^{2^d}$ if the maximal subgroup is open, where $X = L_\infty(K)$, $C(K)$, or $UC_\eta(K)$.

Proposition 3.4.12 Let K be a noncompact amenable hypergroup. Then the convex sets $TLIM(L_\infty(K))$, $TIM(L_\infty(K))$ and $TIIM(L_\infty(K))$ do not have any weak* exposed points or weak* G_δ points (see [39, p. 11-13]) .

Proof: If m_0 is a weak* G_δ point then it is easy to see that there exists a sequence $(f_n) \subseteq L_\infty(K)$ such that $m_0(f_n) = m(f_n)$ for all $n = 1,2, \ldots$, and for all $m \in TLIM(L_\infty(K))$. This is impossible by 4.3.3 , 4.3.4 and 4.3.5 . □

The following is due to E. Granirer [40, Proposition 5] for the case when K is a σ-compact, noncompact, amenable, locally compact group.

Proposition 3.4.13 Let K be a noncompact amenable hypergroup. Let $\Pi_1(X)$ $[\Gamma_1(X)]$ be the subspace of X , spanned by $\{\phi * f - f : \phi \in P(K) , f \in X\}$ $[\{\phi * f - f , h * \check{\psi} - h, f,h \in X , \phi,\psi \in P(K)\}]$, where $X = UC(K)$, $UC_\eta(K)$, $C(K)$ or $L_\infty(K)$. Then $X/ cl\, \Pi_1(X) \oplus \mathbb{C}1$ and $X/ cl\, \Gamma_1(X) \oplus \mathbb{C}1$ are not norm separable.

<u>**Proof**</u>: Suppose that $X/\, cl\, \Gamma_1(X) \oplus \mathbb{C}1$ is norm separable. Choose a sequence $(f_n) \subseteq X$ such that $B + cl\, \Gamma_1(X) \oplus \mathbb{C}1$ is dense in X, where B is the linear span of (f_n). Let m_0 be a TIM on $L_\infty(K)$ and consider the set

$$\mathcal{M} = \{m \in \text{TIM}\,(L_\infty(K)) : m(f_n) = m_0(f_n)\ ,\ n = 1,2, \ldots\}\ .$$

If $m \in \mathcal{M}$, then since $m(\, cl\, \Gamma_1(X)) = 0$ and $m(1) = 1$, so we have $m(f) = m_0(f)$, all $f \in X$. Hence $m = m_0$ by 3.2.7. This is a contradiction since the set

$\{ m \in \text{TIM} : m(f_n) = m_0(f_n)\ ,\ n = 1,2, \ldots \}$ has cardinality 2^{2^d}. Thus $X/\, cl\, \Gamma_1(X) \oplus \mathbb{C}1$ and hence $X/\, cl\, \Pi_1(X) \oplus \mathbb{C}1$ is not norm separable. □

<u>Remark 3.4.14</u> Let K be a hypergroup. Then

$$w^*\text{-}cl\ \Pi_0(L_\infty(K)) = w^*\text{-}cl\ \Pi_1(L_\infty(K))$$

$$= \begin{cases} L_\infty(K) & \text{if } K \text{ is noncompact} \\ cl\ \Pi_1(L\ (K)) = \{ f \in L\ (K) : \int f\, d\lambda = 0 \} & \\ \qquad\qquad \text{if } K \text{ is compact}\ . \end{cases}$$

(See [40, p. 621-622]).

CHAPTER 4

WEAKLY ALMOST PERIODIC FUNCTIONS ON HYPERGROUPS

§ 4. Introduction.

This chapter deals with weakly almost periodic functions on hypergroups. Let K be a hypergroup with a left Haar measure, and WAP(K) the space of all weakly almost periodic functions on K . In [10, Theorem 4.10] , R. Burckel proved that if G is an arbitrary locally compact group, then WAP(G) = C(G) if and only if G is compact. E. Granirer improved this in [39, p. 62-64] . In fact, he showed that UC(G) = WAP(G) if and only if G is compact, and that if G is noncompact and amenable, then UC(G)/WAP(G) is not norm separable. Both proofs use the well known result that WAP(G) admits a unique invariant mean for any locally compact group. Recently, S. Wolfenstetter proved in [95, 2.6 Theorem] that an arbitrary hypergroup J is compact if and only if WAP(J) = C(J) . His proof is different from the group case, and based on this result, he showed that neither almost periodic functions nor weakly almost periodic functions are algebras in general. In section 4.2 , we prove that WAP(K) $\subseteq$ UC(K) , and that they are equal if and only if K is

compact. Our methods of proof are inspired by those of B. E. Johnson [53, p. 23-27] and H. A. M. Dzinotyiweyi [27, Theorem 3.2] . In this section, we also obtain some results on almost periodic functions AP(K) .

In section 4.3 , by using some of the methods in section 3.2, we show that there is a class of hypergroups K , including amenable hypergroups, for which both AP(K) and WAP(K) admit unique invariant means.

§ 4.2 Weakly Almost Periodic Functions.

This section is devoted to a study of basic properties of almost periodic functions and weakly almost periodic functions on hypergroups.

Let J be an arbitrary hypergroup. For $f \in C(J)$, write $\mathcal{O}_L(f) = \{{}_xf : x \in J\}$ and $\mathcal{O}_R(f) = \{f_x : x \in J\}$. A function $f \in C(J)$ is called *[weakly] almost periodic* if the orbit $\mathcal{O}_L(f)$ is relatively [weakly] norm compact in C(J) . We denote the set of all [weakly] almost periodic functions on J by [WAP(J)] AP(J) . Parts of the next result are in [95] but notice that the functions in $UC(J) \cap WAP(J)$ are called weakly almost periodic there.

<u>Lemma 4.2.1</u> WAP(J) and AP(J) are norm closed conjugate closed translation invariant subspaces of C(J) containing the

constants. Furthermore,

(i) $AP(J)$, $C_0(J) \subseteq WAP(J)$;

(ii) If $f \in C(J)$, then $f \in WAP(J)$ $[AP(J)]$ if and only if $\mathcal{O}_R(f)$ is relatively weakly [norm] compact in $C(J)$;

(iii) Both $WAP(J)$ and $AP(J)$ are closed under the operation $f \longrightarrow \check{f}$.

<u>Proof</u>: It is easy to see that $WAP(J)$ $[AP(J)]$ is a norm closed conjugate closed right translation invariant subspace of $C(J)$ containing the constants. To see that $WAP(J)$ is left translation invariant, let $f \in WAP(J)$, $x \in J$. The weak topology of the weak closure of $\mathcal{O}_L(f)$ is a compact topology, stronger that the pointwise topology, and the latter is Hausdorff. Hence, the two topologies coincide, and since $y \longrightarrow \delta_y * f$ is continuous in the pointwise topology [52, 31A] , it is indeed weakly continuous. By the Krein-Smulian theorem [24, p. 434-435] , the closed convex hull $C_L(f)$ of [illegible]) is relatively weakly compact. Hence, the weak vector valued integral

$\delta_y*(\delta_x*f) = \int_K(\delta_u*f)\ d\delta_y*\delta_x(u)$,

$y \in J$, exists and is contained in $C_L(f)$ ([9, Chapter III, § 3] or [85, 3.27 Theorem]) . Since $\mathcal{O}_L(\delta_x*f)$ is contained in the weak compact set $C_L(f)$, it is relatively weakly compact. That is $WAP(J)$ [and similarly $AP(J)$] is left translation invariant.

(i) is proved in [95, 2.6 Theorem] .

To prove (ii), we follow the lines of [95, 2.4 Theorem]. Embed J into the topological semigroup $M_c'(J)$ by the homeomorphism $x \longrightarrow \delta_x$ [52, 2.2B]. Each $f \in C(J)$ extends to an $\tilde{f} \in C(M_c'(J))$, given by $\tilde{f}(\mu) = \int_J f\,d\mu$, $\mu \in M_c'(J)$ [52, 2.2D], Then $f \longrightarrow \tilde{f}$ is a linear isometry and hence continuous in the norm (and weak) topologies of $C(J)$ and $C(M_c'(J))$. Now, if $f \in WAP(J)$, $\mu \in M_c'(J)$, then $\tilde{O}_L(f) = \{\mu * f : \mu \in M_c'(J)\}$ is contained in the closed convex hull of $O_L(f)$ by the above arguments, and hence $\tilde{O}_L(f)$ is relatively weakly compact. It is easy to verify that $(\mu * f)^{\sim} = \delta_\mu * \tilde{f}$ for $\mu \in M_c'(J)$, where $\delta_\mu * \tilde{f}(\nu) = \tilde{f}(\mathring{\mu} * \nu)$, $\nu \in M_c'(J)$, and therefore $O_L(\tilde{f}) = (\tilde{O}_L(f))^{\sim}$. The righthand side being the continuous image of $\tilde{O}_L(f)$, is relatively compact in the weak topology of $C(M_c'(J))$. Since $M_c'(J)$ is a topological semigroup, $O_R(\tilde{f})$ is relatively weakly compact in $C(M_c'(J))$. It follows easily that $O_R(f)$ is relatively weakly compact in $C(J)$. By symmetry, we have (ii), and (iii) follows from (ii) easily. □

It is shown in [95, p. 68-69] that $AP(J) \subseteq UC(J)$. In what follows, K will denote a hypergroup with a left Haar measure λ. The next result is due to T. Mitchell [67, Theorem 7] for a locally compact group. The interested reader is referred to [73, 2.33] for more information. Our methods of proof follow ideas of B. E. Johnson [53, p. 23-27].

Theorem 4.2.2 $UC_r(K) = \{f \in C(K) : x \longrightarrow {}_x f$ is continuous from K to $(C(K), \text{weak})\}$, and similarly
$UC_\ell(K) = \{f \in C(K) : x \longrightarrow f_x$ is weakly continuous from K to $C(K)\}$.

Proof: Let $f \in C(K)$ be such that $x \longrightarrow \delta_x \bullet f$ is weakly continuous. If $u \in C_c(K)$, then the weak vector valued integral $\int_K (\delta_x \bullet f)\, u(x)dx$ exists in $C(K)$. Because the point evaluations in $C(K)^*$ separate points in $C(K)$, we have

$$u \bullet f = \int_K (\delta_x \bullet f)\, u(x)\, dx .$$

Let $(\phi_U)_{U\in\mathcal{U}} \subseteq P_c(K)$ be the bounded approximate identity for $L_1(K)$, as in 2.1.2. Let $\varepsilon > 0$ and $\psi \in C(K)^*$ be given. Choose a neighbourhood W of e such that
$|\langle\psi, \delta_x \bullet f\rangle - \langle\psi, f\rangle| < \varepsilon$ for all $x \in W$. If V is a neighbourhood of e contained in W, then
$|\langle\psi, \phi_V \bullet f\rangle - \langle\psi, f\rangle| = \left|\int_K \langle\psi, \delta_x \bullet f\rangle\, \phi_V(x)\, dx - \int_K \langle\psi, f\rangle\, \phi_V(x)\, dx\right|$
$\le \int_K |\langle\psi, \delta_x \bullet f\rangle - \langle\psi, f\rangle|\, \phi_V(x)\, dx \le \varepsilon$. Thus, the net $(\phi_U \bullet f)_{U\in\mathcal{U}}$ converges to f in the weak topology of $C(K)$. Thus $f \in UC_r(K)$ since each $\phi_U \bullet f \in UC_r(K)$. If $f \in C(K)$ is such that $x \longrightarrow f_x$ is continuous from K to $(C(K), \text{weak})$, then by similar arguments the net $(f \bullet \check{\phi}_U)_{U\in\mathcal{U}}$ converges weakly to f, and hence $f \in UC_\ell(K)$ since each $f \bullet \check{\phi}_U \in UC_\ell(K)$. □

Remarks 4.2.3

(i) Let $x \longrightarrow T_x$ be a mapping of K into $B(E)$, the bounded linear operators on a Banach space E, such that

(i) $T_e = I$, the identity operator;

(ii) $\|T_x\| \le M < \infty$ for all $x \in K$;

(iii) $x \longrightarrow T_x s$ $(s \in E)$ is weakly continuous and $\langle \phi, T_x(T_y s)\rangle = \int_K \langle \phi, T_u s\rangle \, d\delta_x * \delta_y(u)$ for all $x, y \in K$, $s \in E$, $\phi \in E$. That is, $T_x T_y = T_{\delta_x * \delta_y}$, $x, y \in K$.

Then the mapping $x \longrightarrow T_x s$ $(s \in E)$ is norm continuous (see 3.3.2).

To prove this, put $\mu \cdot s = \int_K T_u s \, d\mu(u)$, $s \in E$, $\mu \in M_c(K)$. If $\mu \in M(K)$, choose a sequence $(\mu_n) \subseteq M_c(K)$ such that $\|\mu_n - \mu\|$ converges to zero, and write $\mu \cdot s = \lim_n \mu_n \cdot s$. Then E becomes a unital left Banach $M(K)$-module. Consider E as a left Banach $L_1(K)$-module given by

$$\phi \cdot s = (\lambda\phi) \cdot s \qquad (\phi \in L_1(K), \ s \in E).$$

Let $(\phi_U)_{U \in \mathcal{U}} \subseteq P_c(K)$ be the bounded approximate identity for $L_1(K)$ as in 2.1.2. Then it is easy to see that $\phi_U \cdot s$ converges to s weakly as in the previous result (see also [53, p. 27]). Hence, by Cohen's factorization theorem, $L_1(K) \cdot E = E$. Now, if $s \in E$, write $s = \phi \cdot s'$, $\phi \in L_1(K)$, $s' \in E$. If (x_α) converges to x in K, then

$\|T_{x_\alpha} s - T_x s\| = \|x_\alpha \cdot s - x \cdot s\| \le M \|\delta_{x_\alpha} * \phi - \delta_x * \phi\| \|s'\|$ which converges to zero. Hence, the mapping $x \longrightarrow T_x s$ $(s \in E)$ is norm continuous.

(ii) If T is as in (i) (That is, T is a representation of K on E), then E can be given a unital left Banach $M(K)$-module structure so that $\langle \phi, \mu \cdot s\rangle = \int_K \langle \phi, u \cdot s\rangle \, d\mu(u)$, $\mu \in M(K)$, $s \in E$, $\phi \in E^*$. In particular, the mapping $\mu \longrightarrow \mu \cdot s$ is weakly continuous on $M^+(K)$ [52, 2.2D] . Conversely, if E is a unital Banach $M(K)$-module such that $\mu \longrightarrow \mu \cdot s$ $(s \in E)$ is weakly continuous on $M^+(K)$, then $T_x s = \delta_x \cdot s$ $(x \in K, s \in E)$ will define a representation of K on E . This correspondence is a bijection (for a fixed Banach space E).

(iii) If E is a unital left Banach $M(K)$-module, put

$$\langle \phi \cdot \mu, s\rangle = \langle \phi, \mu \cdot s\rangle , \quad \phi \in E^* , s \in E , \mu \in M(K) .$$

Then E^* is a unital right Banach $M(K)$-module. If $\mu \longrightarrow \mu \cdot s$ is weakly continuous on $M^+(K)$, then $\mu \longrightarrow \phi \cdot \mu$ is weak* continuous on $M^+(K)$ and this is also weakly continuous if E is reflexive or K is discrete. □

<u>Propostion 4.2.4</u> $WAP(K) \subseteq UC(K)$.

Proof: Let $f \in WAP(K)$. Then $O_L(f)$ $[O_R(f)]$ is relatively weakly compact in $C(K)$, and hence $x \longrightarrow {}_x f$ $[x \longrightarrow f_x]$ is weakly continuous as before. Thus $f \in UC(K)$ by 4.2.2. □

Let G be a locally compact group. Then WAP(G) admits a unique invariant mean [42, § 3.1]. Using this fact, E. Granirer proved in [39, p. 62-64] that WAP(G) = UC(G) if and only if G is compact and that if G is noncompact and amenable then UC(G)/WAP(G) is not norm separable. We do not know whether for an arbitrary hypergroup WAP(K) admits an invariant mean, and hence Granirer's method cannot be used to prove that if K is noncompact, then $UC(K) \neq WAP(K)$. However, if WAP(K) admits an invariant mean then it is unique. Thus, it follows as in the group case that if K is noncompact and amenable, then UC(K)/WAP(K) is not norm separable (see 4.3.6). H. A. M. Dzinotyiweyi showed in [27, Theorem 3.2] that WAP(S) = UC(S) if and only if S is compact for a large class of semigroups S which include all locally compact groups. His proof is elementary and does not use the theory of invariant means. We shall prove next, by adapting his methods, that if K is noncompact then $UC(K) \neq WAP(K)$. We need the following:

Lemma 4.2.5 Let K be any noncompact hypergroup, and E a compact set in K. Then there exist infinite sequences (x_n),

(y_m) of distinct points of K such that

$$\bigcup_{m=1}^{\infty} \Big(\bigcup_{n>m} E\cdot(x_n)\cdot(y_m)\cdot E\Big) \cap \bigcup_{n=1}^{\infty} \Big(\bigcup_{n<m} E\cdot(x_n)\cdot(y_m)\cdot E\Big) = \emptyset .$$

<u>Proof</u>: The proof proceeds by induction. Suppose, for some positive integer p we have finite sequences $(x_1, \dots, x_p)$ and $(y_1, \dots, y_p)$ such that

$$\bigcup_{m=1}^{p-1} \Big(\bigcup_{p\ge n>m} E\cdot(x_n)\cdot(y_m)\cdot E\Big) \cap \bigcup_{n=1}^{p-1} \Big(\bigcup_{n<m\le p} E\cdot(x_n)\cdot(y_m)\cdot E\Big) = \emptyset .$$

Write

$$X_p = (x_1, \dots, x_p) , \quad Y_p = (y_1, \dots, y_p) ,$$

$$L_p = \bigcup_{m=1}^{p-1} \Big(\bigcup_{p\ge n>m} E\cdot(x_n)\cdot(y_m)\cdot E\Big) ,$$

and

$$U_p = \bigcup_{n=1}^{p-1} \Big(\bigcup_{n<m\le p} \Gamma\cdot(x_n)\cdot(y_m)\cdot E\Big) .$$

Then

$$L_p \cap U_p = \emptyset . \tag{1}$$

The set $(\mathring{\hat{E}} \cdot U_p) \cdot (Y_p \cdot E)^{\circ}$ is compact [52, 3.2B] , and since K is noncompact there exists $x_{p+1} \in K \backslash X_p$ such that $x_{p+1} \notin (\mathring{\hat{E}} \cdot U_p) \cdot (Y_p \cdot E)^{\circ}$. Then

$$U_p \cap E \cdot x_{p+1} \cdot Y_p \cdot E = \emptyset \tag{2}$$

by [52, 41B] . Similarly, since
$(E * X_p)^\circ * (L_p \cup E * \{x_{p+1}\} * Y_p * E) * \hat{E}$ is compact, we can find $y_{p+1} \in K \backslash Y_p$ such that
$y_{p+1} \notin (E * X_p)^\circ * (L_p \cup E * \{x_{p+1}\} * Y_p * E) * \hat{E}$. Hence,

$$E * X_p * \{y_{p+1}\} * E \cap (L_p \cup E * \{x_{p+1}\} * Y_p * E) = \emptyset . \qquad (3)$$

Now, $U_{p+1} \cap L_{p+1} =$

$$= (U_p \cup E * X_p * \{y_{p+1}\} * E) \cap (L_p \cup E * \{x_{p+1}\} * Y_p * E)$$

$$= (U_p \cap L_p) \cup (U_p \cap (E * \{x_{p+1}\} * Y_p * E))$$

$$\cup ((E * X_p * \{y_{p+1}\} * E) \cap (L_p \cup E * \{x_{p+1}\} * Y_p * E)$$

$$= \emptyset$$

by (1), (2) and (3) . □

The next result is a consequence of a well known theorem of A. Grothendieck [44, Theorem 6] .

Lemma 4.2.6 Let K be a hypergroup and $f \in C(K)$. Then $f \in WAP(K)$ if and only if
$\lim_m \lim_n f(x_n * y_m) = \lim_n \lim_m f(x_n * y_m)$, whenever (x_n) , (y_m) are sequences in K such that all relevant limits exist.

<u>Theorem 4.2.7</u> Let K be a hypergroup with a left Haar measure λ. Then K is compact if and only if $UC(K) = WAP(K)$.

<u>Proof</u>: If K is compact, then it is easy to see that $UC(K) = C(K) = WAP(K)$ [52, 2.2D and 4.2F]. If K is noncompact, let $\phi \in C_c^+(K)$, $\int_K \phi \, d\lambda = 1$, $E = spt\ \phi$. Let $\{x_n\}$ and $\{y_m\}$ be sequences of distinct elements of K such that if $H_1 = \bigcup_{m=1}^{\infty} (\bigcup_{n>m} E*\{x_n\}*\{y_m\}*E)$ and $H_2 = \bigcup_{n=1}^{\infty} (\bigcup_{n<m} E*\{x_n\}*\{y_m\}*E)$, then $H_1 \cap H_2 = \emptyset$. Let $h \in L_\infty(K)$ be defined by $h(x) = 1$ if $x \in H_1$ and zero otherwise (Note: H_1 is σ-compact). Let
$f(x) = \int_K h \, d(\phi\lambda) * \delta_x * (\phi\lambda) = (\phi\Delta)^\circ * h * \mathring{\phi}(x)$, $x \in K$. Then $f \in UC(K)$. To see that $f \notin WAP(K)$, note that

$$f(x_n * y_m) = \int_K f(u) \, d\delta_{x_n} * \delta_{y_m}(u)$$

$$= \int_K \int_K h \, d(\phi\lambda) * \delta_u * (\phi\lambda) \, d\delta_{x_n} * \delta_{y_m}(u)$$

$$= \int_K h \, d(\phi\lambda) * d\delta_{x_n} * \delta_{y_m} * (\phi\lambda) ,$$

and

$$spt\ (\phi\lambda) * \delta_{x_n} * \delta_{y_m} * (\phi\lambda) = E * \{x_n\} * \{y_m\} * E$$

$\subseteq H_1$ if $n > m$ and $\subseteq H_2$ if $m > n$. Thus, $f(x_n * y_m) = 1$ if $n > m$ and zero if $n < m$. Hence, $\lim_m \lim_n f(x_n * y_m) = 1$, while $\lim_n \lim_m f(x_n * y_m) = 0$. So, $f \notin WAP(K)$ by 4.2.6 . □

Proposition 4.2.8 WAP(K) [AP(K)] contains all the bounded positive definite [multiplicative bounded continuous] functions on K .

Proof: The proof is standard (see [10, Corollary 3.3] and [52, § 11]) . □

We now obtain some stability properties of WAP(J) and AP(J) for arbitrary hypergroups J . We only give the proofs for WAP(J) ; the proofs for AP(J) are similar.

Definition 4.2.9 Let J , L be hypergroups. A continuous mapping ϕ of J onto L is called a *homomorphism* if $\int_L f \, d\delta_{\phi(x)} * \delta_{\phi(y)} = \int_J f \circ \phi \, d\delta_x * \delta_y$ for all $f \in C_c(L)$, $x, y \in J$. In this case, L is called a *homomorphic image* of J . If ϕ is a homomorphism of J onto L , then there is a positive continuous map $\phi_* : M(J) \longrightarrow M(L)$, which preserves convolution and involution (see [55, Propostion 2.3] and the proof of the next lemma) .

If H is a closed normal subgroup of J , then the natural projection Π is a homomorphism of J onto J/H . Also, if H is a compact subhypergroup of J such that $\delta_x * \sigma = \sigma * \delta_x$ for every $x \in J$, then J//H (= J/H) is a homomorphic image of

J , where σ is the normalized Haar measure of H (3.2.17) .

The next result is of independent interest.

Lemma 4.2.10 Let J, L be hypergroups, and ϕ a homomorphism of J onto L . Then

(i) $\ker \phi = \{x \in J : \phi(x) = e_L\}$ is a normal subhypergroup of J ;

(ii) $\phi_{|G(J)}$ maps $G(J)$ into $G(L)$;

(iii) If ϕ is open and $\ker \phi \subseteq G(J)$, then $J/\ker \phi \cong L$.

Proof:

(i) If $z \in \{\phi(x)\} * \{\phi(e_J)\}$, $x \in J$, $z \neq \phi(x)$, then there exists $f \in C_L^+(L)$ such that $f(z) > 0$, $f(\phi(x)) = 0$, and hence

$$0 < \int_L f \, d\delta_{\phi(x)} * \delta_{\phi(e_J)} = \int_J f\circ\phi \, d\delta_x * \delta_{e_J} = f(\phi(x)) = 0 .$$

This shows $e_J \in \ker \phi$. Let $x,y \in \ker \phi$. Then $f(e_L) = \int f\circ\phi \; \delta_x * \delta_y$, $f \in C_c^+(L)$, and hence $\phi(z) = e_L$ for all $z \in \{x\}*\{y\}$. To see $\phi(\mathring{x}) = \phi(x)$, $x \in J$, we note that $\int_J f \, d\delta_{\phi(x)} * \delta_{\phi(\mathring{x})} = \int f\circ\phi \, d\delta_x * \delta_{\mathring{x}}$, $f \in C_c^+(L)$. If $f(e_L) = f(\phi(e_J)) > 0$, $f \in C_c^+(L)$, then $\int f \, d\delta_{\phi(x)} * \delta_{\phi(\mathring{x})} > 0$. Hence, $\ker \phi$ is a subhypergroup of J . To see that $\ker \phi$ is normal, we verify that $\{x\} * \ker \phi = \{z \in J : \phi(x) = \phi(z)\}$. If

$z \in \{x\}*\{y\}$, $y \in \ker\phi$, $\phi(z) \neq \phi(x)$, choose $f \in C_c^+(L)$ such that $f(\phi(z)) > 0$, $f(\phi(x)) = 0$. Then
$0 < \int f\circ\phi \, d\delta_x * \delta_y = f(\phi(x)) = 0$, a contradiction.

Conversely, if $\phi(z) = \phi(x)$, then

$$\int_J f\circ\phi \, d\delta_{\mathring{x}} * \delta_z = \int_L f \, d\delta_{\phi(\mathring{x})} * \delta_{\phi(z)} = \int_L f \, d\delta_{\phi(\mathring{x})} * \delta_{\phi(x)} .$$

Now, $z \in \{x\} * \ker\phi$ if and only if $\ker\phi \cap \{\mathring{x}\} * \{z\} = \emptyset$. If $\{\mathring{x}\} * \{z\} \cap \ker\phi = \emptyset$, then $e_L \notin \phi(\{\mathring{x}\} * \{z\})$, a compact set. Choose $f \in C_c^+(L)$ such that $f(e_L) > 0$ and $f(\phi(\{\mathring{x}\} * \{z\})) = 0$ .This implies

$$0 = \int_J f\circ\phi \, d\delta_{\mathring{x}} * \delta_z = \int_L f \, \delta_{\phi(\mathring{x})} * \delta_{\phi(z)} = \int_L f \, d\delta_{\phi(\mathring{x})} * \delta_{\phi(x)} > 0 ,$$

and thus $\{\mathring{x}\} * \{z\} \cap \ker\phi \neq \emptyset$.

(ii) If $x \in G(J)$, then

$f(e_L) = \int f\circ\phi \, d\delta_{\mathring{x}} * \delta_x = \int f \, d\delta_{\phi(\mathring{x})} * \delta_{\phi(x)}$ for all $f \in C_c(L)$, and hence $\phi(x) \in G(J)$.

(iii) is easy. □

<u>Propostion 4.2.11</u> Let J, L be hypergroups, and ϕ a homomorphism of J onto L . Let $\tilde{\phi}$ be the induced map given by $\tilde{\phi} : C(L) \longrightarrow C(J)$, $\tilde{\phi}(f) = f\circ\phi$, $f \in C(L)$. Then $\tilde{\phi}(WAP(L)) = WAP(J) \cap \tilde{\phi}(C(L))$, and $\tilde{\phi}(AP(L)) = AP(J) \cap \tilde{\phi}(C(L))$.

__Proof__: We have ${}_x(f\circ\phi) = {}_{\phi(x)}f\circ\phi$ for all $f \in C(L)$, $x \in J$, and hence $\mathcal{O}_L(f\circ\phi) = \tilde{\phi}(\mathcal{O}_L(f))$ since ϕ is onto. $\tilde{\phi}$ is a linear isometry of $C(L)$ into $C(J)$, and thus $\tilde{\phi}$ (and $\tilde{\phi}^{-1}$) is norm-norm and weak-weak continuous. It follows easily that $\mathcal{O}_L(f\circ\phi)$ is relatively weakly [norm] compact if and only if $\mathcal{O}_L(f)$ is relatively weakly [norm] compact (see [64, Lemma 5.2]) . □

Let J be a hypergroup. Let H be a normal subgroup of J or a compact subhypergroup of J such that $\delta_x * \sigma = \sigma * \delta_x$ for all $x \in J$, where σ is the normalized Haar measure of H. We denote the projection of J onto J/H by Π (in both cases) . We have the following:

__Corollary 4.2.12__ $\tilde{\Pi}(WAP(J/H)) = WAP(J) \cap \tilde{\Pi}(C(J/H))$, and $\tilde{\Pi}(AP(J/H)) = AP(J) \cap \tilde{\Pi}(C(J/H))$.

__Corollary 4.2.13__

(i) If a hypergroup K contains a compact supernormal subhypergroup H, then $WAP(K) \cap \tilde{\Pi}(C(K/H)) = \tilde{\Pi}(WAP(K/H))$, and $AP(K) \cap \tilde{\Pi}(C(K/H)) = \tilde{\Pi}(AP(K/H))$.

(ii) If $K = H \vee J$, H compact, J discrete, $H \cap J = \{e\}$, then every $f \in WAP(J)$ $[AP(J)]$ is of the form $f = g_{|J}$ for some $g \in WAP(K)$ $[AP(K)]$.

__Proof__: In both cases, we have $\delta_x * \sigma = \sigma * \delta_x$ for every $x \in K$.

□

__Corollary 4.2.14__ Let J be a hypergroup such that $(x)*(y)$ is finite for all $x, y \in J$. Then $f \in WAP(J)$ $[AP(J)]$ if and only if $f \in WAP(J_d)$ $[AP(J_d)]$ and $f \in C(J)$.

__Propostion 4.2.15__ Let H be a comapct subhypergroup of a hypergroup J with the normalized Haar measure σ. Then
$\sigma*(AP(J))*\sigma = AP(J) \cap \tilde{\Pi}(C(J//H)) \subseteq \tilde{\Pi}(AP(J//H))$,
$\sigma*(WAP(J))*\sigma = WAP(J) \cap \tilde{\Pi}(C(J//H)) \subseteq \tilde{\Pi}(WAP(J//H))$.

__Proof__: Let $f \in WAP(J) \cap \tilde{\Pi}(C(J//H))$. Then, $f = F \circ \Pi$, where $\Pi : J \longrightarrow J//H$ is the projection, and $F \in C(J//H)$. We have

$$F(HxH*HyH) = \int_H f(x*t*y)\, dt \qquad [52,\ 14.2F]$$

$$= (\sigma*\delta_{\overset{\circ}{x}}) * f(y).$$

That is, ${}_{HxH}F\circ\Pi = (\sigma*\delta_{\overset{\circ}{x}}) * f$, $x \in J$. Hence,
$\tilde{\Pi}(O_L(F)) = \{F_{HxH} \circ \Pi : x \in J\} = \{(\sigma*\delta_{\overset{\circ}{x}}) * f : x \in J\} \subseteq C_L(f)$,
the closed convex hull of $O_L(f)$ (This fact was observed earlier). Since $\tilde{\Pi}$ is a linear isometry and $C_L(f)$ is weakly compact, it follows that $O_L(F)$ is relatively weakly compact. Thus, $F \in WAP(J//H)$. This proves that the right hand side

inclusion holds for the weakly almost periodic functions.

If $f \in WAP(J)$, then $\sigma \bullet f \in WAP(J)$ since $\mathcal{O}_L(\sigma \bullet f) \subseteq C_L(f)$, and hence $\sigma \bullet f \bullet \sigma = \sigma \bullet ((\sigma \bullet \check{f})^{\circ}) \in WAP(J)$. We observed in the proof of 3.4.7 that $\sigma \bullet f \bullet \sigma$ is constant on the double cosets of H in J (for any $f \in C(J)$). Hence, $\sigma \bullet (WAP(J)) \bullet \sigma \subseteq WAP(J) \cap \tilde{\Pi}(C(J//H))$, and the converse inclusion follows easily because $\sigma \bullet f \bullet \sigma = f$ for all $f \in \tilde{\Pi}(C(J//H))$. □

Remark 4.2.16

(i) If follows easily from [95, § 3] that the right hand side inclusions in 4.2.15 may be proper. In fact, let $G = \mathbb{R}^2 \otimes_s SO(2)$ be the semidirect product of $\mathbb{R}^2$ and $SO(2)$, and $H = \{(0,0) \times SO(2)\}$ the compact subgroup of G. Then $G//H \cong (\mathbb{R}^2)_{SO(2)}$ as hypergroups. If $A(G)$ $[A(\mathbb{R}^2)]$ is any class of functions on G $[\mathbb{R}^2]$, we denote by $A_H(G)$ $[A_{SO(2)}(\mathbb{R}^2)]$ the set of those functions in $A(G)$ $[A(\mathbb{R}^2)]$, which are constant on the double cosets of H in G [SO(2)-orbits in $\mathbb{R}^2$]. Now, by considering the homomorphism $j : x \longrightarrow (x,e)$ of $\mathbb{R}^2$ into $\mathbb{R}^2 \otimes_s SO(2) = G$, one can easily see that $WAP_H(G) \subseteq WAP_{SO(2)}(\mathbb{R}^2)$, $AP_H(G) \subseteq AP_{SO(2)}(\mathbb{R}^2)$. Finally, use [95, 3.9 Remark] (see also [14]).

(ii) In [95, p. 71-72], S. Wolfenstetter has given an example of a hypergroup K for which both AP(K) and WAP(K) are not algebras (see also [56, Remark 1]). This is a hypergroup on $\mathbb{N}_0$, the nonnegative integers, arising from Jacobi polynomials $P^{(\frac{1}{2},\frac{1}{2})}(x)$. The interested reader is referred to R. Lasser [57] for a class of hypergroups on $\mathbb{N}_0$ arising from certain polynomials. Using this example K we can easily produce many hypergroups J for which WAP(J) and AP(J) are not algebras. To see this, let J, L be hypergroups, and ϕ a homomorphism of J onto L . For f, g $\in$ C(L) , we have $\tilde{\phi}(fg) = fg\circ\phi = (f\circ\phi)(g\circ\phi) = \tilde{\phi}(f)\tilde{\phi}(g)$. If WAP(J) [AP(J)] is an algebra, then so is WAP(L) [AP(L)] by 4.2.11 . Now, let K be the (discrete) hypergroup as in [95, p. 71-72] . If G is any locally compact group then AP(K×G) and WAP(K×G) are not algebras. Also, if G is a compact group such that $K\cap G = \{e\}$, then AP($G_\vee K$) and WAP($G_\vee K$) are not algebras.

§ 4.3 Invariant means on WAP(K) .

In this section, we shall show that there is a class of hypergroups J including amenable hypergroups for which AP(J) and WAP(J) admit unique (left) invariant means.

Let J be a hypergroup. For f $\in$ WAP(J) , $\phi \in \mathrm{WAP}(J)^*$, $f_\phi(x) = \langle\phi, {}_xf\rangle$ $(x \in J)$. Since $x \longrightarrow {}_xf$ is weakly continuous $f_\phi \in C(J)$, and

${}_x(f_\phi)(y) = \int_J \langle\phi, {}_uf\rangle \, d\delta_x * \delta_y(u) = \langle\phi, {}_y({}_xf)\rangle = ({}_xf)_\phi(y)$. Thus $O_L(f_\phi) = (O_L(f))_\phi$. Also, the mapping $f \longrightarrow f_\phi$ of WAP(J) into

$C(J)$ is bounded and linear. Hence, $f_\phi \in WAP(J)$. Similarly, if $f \in AP(J)$, then $f_\phi \in AP(J)$.

<u>Lemma 4.3.1</u> Let m be a mean on $WAP(J)$ $[AP(J)]$. Then, for each $f \in WAP(J)$ $[AP(J)]$ there exists a unique $P(f) \in C_L(f)$, the closed convex hull of $\mathcal{O}_L(f)$, such that $\langle m, f_\phi\rangle = \langle \phi, P(f)\rangle$ for $\phi \in WAP(J)^*$ $[AP(J)^*]$, and $P : f \longrightarrow P(f)$ is a bounded linear operator on $WAP(J)$ $[AP(J)]$ with $P(1) = \|P\| = 1$. Furthermore, (i) if m is a LIM , then $P({}_xf) = P(f)$ for $x \in J$, $f \in WAP(J)$ $[AP(J)]$; (ii) if m is a RIM , then $P(f)$ is a constant for $f \in WAP(J)$ $[AP(J)]$.

<u>Proof</u>: Let $f \in WAP(J)$. Then there exists $F \in WAP(J)^{**}$ such that $\langle F, \phi\rangle = \langle m, f_\phi\rangle$ for $\phi \in WAP(J)^*$ $\left(F = \int_J {}_uf \, dm(u)\right)$. Let (μ_α) be a net of finite means on $WAP(J)$ converging to m in the weak* topolcgy. Since the closed convex hull $C_L(f)$ of $\mathcal{O}_L(f)$ is weakly compact we can (and will) assume that $\int {}_uf \, d\mu_\alpha(u)$ converges to some $P(f) \in C_L(f)$ in the weak topology. It is now easy to see that $P(f) = F$ (see [42, p. 84-85]) . To see (i) , if m is a LIM, then

$$\langle P({}_xf), \phi\rangle = \langle m, ({}_xf)_\phi\rangle = \langle m, {}_x(f_\phi)\rangle = \langle m, f_\phi\rangle = \langle \phi, P(f)\rangle ,$$

and hence $P({}_xf) = P(f)$, $x \in J$. If m is a RIM , since

$$(f_\phi)_x(y) = \int_J f_\phi(u) \, d\delta_y \bullet \delta_x(u) = \int \langle \phi, {}_uf\rangle \, d\delta_y \bullet \delta_x(u)$$

$$= \langle \phi, {}_x({}_yf)\rangle = \langle \phi\square x, {}_yf\rangle ,$$

where $\langle \phi\square x, f\rangle = \langle \phi, {}_xf\rangle$, we have

$$\langle \phi, P(f)\rangle = \langle m, f_\phi\rangle = \langle m, (f_\phi)_x\rangle = \langle m, f_{\phi\square x}\rangle = \langle \phi\square x, P(f)\rangle = \langle \phi, {}_x(P(f))\rangle .$$

Hence $P(f) = {}_x(P(f))$, for each $x \in J$, and so $P(f)$ is a constant. We have thus proved (ii) . □

A version of the next result for topological semigroups appears in [42, Theorem 3.8.4] . J. Dixmier also gives a result of this sort for discrete amenable semigroups in [21, Theorem 7] .

<u>Propostion 4.3.2</u> Let m be a LIM on WAP(J) , and for $f \in$ WAP(J) , let $C_L(f)$ denote the closed convex hull of $\mathcal{O}_L(f)$. Then

(i) $C_L(f)$ has a unique constant function, namely $m(f)1$;

(ii) $\mathrm{WAP}(J) = \mathbb{C}1 \oplus cl \langle \{{}_xf - f : x \in J, f \in \mathrm{WAP}(J)\}\rangle$.

Similar statements are true for AP(J) .

<u>Proof</u>: Let n be a RIM on WAP(J) , and write

$P(f) = \int_J {}_uf \, dn(u)$, $Q(f) = \int_J {}_uf \, dm(u)$ for $f \in$ WAP(J) . Then

$P(f), Q(f) \in C_L(f)$. Since

$$Q\left(\sum_{i=1}^{n} \lambda_i\, {}_{x_i}f\right) = \sum_{i=1}^{n} \lambda_i\, Q\left({}_{x_i}f\right) = Q(f) \quad \text{for } x_i \in J\,, \quad \lambda_i \geq 0\,,$$

$\sum_{i=1}^{n} \lambda_i = 1$ (by 4.3.1 (i)), we have $Q(g) = Q(f)$ for all $g \in C_L(f)$. But $P(f) \in C_L(f)$, and thus $Q(f) = Q(P(f)) = \int_J u(P(f)\, dm(u) = P(f)$, because $P(f)$ is a constant by 4.3.1(ii). Also, ${}_x(P(f)) = {}_x(Q(f)) = Q({}_xf) = P({}_xf)$ for all $x \in J$. Now, if $f \in \ker P = \{f \in WAP(J) : P(f) = 0\}$, then $P({}_xf) = 0$ for all $x \in J$. So, $P(g) = 0$ for all $g \in C_L(f)$. Let $f \in WAP(J)$ be arbitrary. Then $P(f - P(f)) = 0$, and hence $C_L(f - P(f)) = 0$. But $C_L(f) = C_L((f - P(f)) + P(f)) \subseteq C_L(f - P(f)) + P(f)$. Then $C_L(f) \subseteq P(f) + \ker P$. Also, $P(f) \in C_L(f)$. If there were two constant functions $\alpha 1, \beta 1 \in C_L(f)$, then write $\alpha 1 = P(f) + g$, $\beta 1 = \beta(f) + g'$, for some $g, g' \in \ker P$. Then $\alpha 1 - \beta 1 = g - g' \in \ker P$, and this implies $\alpha = \beta$. Hence, $C_L(f)$ has a unique constant, namely $P(f)$. If $f \in WAP(J)$, then $P(f - {}_xf) = P(f) - P({}_xf) = P(f) - P(f) = 0$, for $x \in J$. Hence, $cl \langle \{{}_xf - f : x \in J\,,\ f \in WAP(J)\}\rangle \subseteq \ker P$. Conversely, if $f \in \ker P$, then $0 = P(f) \in C_L(f)$, and therefore, if $\varepsilon > 0$ is given, there exist $x_i \in J$, $\lambda_i \geq 0$ with $\sum_{i=1}^{n} \lambda_i = 1$ such that $\left\| \sum_{i=1}^{n} \lambda_i\, {}_{x_i}f \right\|_\infty < \varepsilon$. This implies

$$f = \left(f - \sum_{i=1}^{n} \lambda_i \, {}_{x_i}f \right) + \sum_{i=1}^{n} \lambda_i \, {}_{x_i}f = \sum_{i=1}^{n} \lambda_i \left(f - {}_{x_i}f \right) + g ,$$

with $\|g\|_\infty < \varepsilon$. Hence, $f \in cl \langle \{ {}_x f - f : x \in J, f \in WAP(J) \} \rangle$. Thus

$$WAP(J) = \mathbb{C}1 \oplus \ker P = \mathbb{C}1 \oplus cl \langle \{ {}_x f - f : x \in J, f \in WAP(J) \} \rangle ,$$

and $P(f) = m(f)$. The proof is similar for $AP(J)$. □

<u>Corollary 4.3.3</u> Let m be a LIM and n a RIM on $WAP(J)$ [$AP(J)$] . Then $m = n$. In particular, if K is an amenable hypergroup, then $WAP(K)$ has a unique invariant mean.

<u>Proof</u>: It follows as in the group case [42, § 3.1] that $C_L(f) \cap \mathbb{C}1 = C_R(f) \cap \mathbb{C}1 = \{m(f)1\} = \{n(f)1\}$ for $f \in WAP(J)$, where $C_R(f)$ is the closed convex hull of $\mathcal{O}_R(f)$. □

Let G be a locally compact group. Then it follows from a well known fixed point theorem of C. Ryll-Nardzewski that $WAP(G)$ admits a unique IM [42, § 3.1] . But, for hypergroups, the operators $L_x : f \longrightarrow {}_x f$ $(f \in WAP(J))$ are far from being non-contractive [52, 3.3] , and hence his theorem cannot be applied here in general. However, by making use of some methods of §3.2 , we shall show that there is a class of hypergroups K which properly contains amenable hypergroups and for which $AP(K)$ and $WAP(K)$ admit unique invariant means (see also [64, § 5]) .

Propostion 4.3.4

(i) Let J, L be hypergroups, and ϕ a homomorphism of J onto L. If there is a LIM m on WAP(J), then there is a LIM M on WAP(L) ;

(ii) If H is a normal subgroup of J, then there is a LIM on WAP(J) if and only if there is a LIM on WAP(J/H) ;

(iii) There is a LIM on WAP(J×L) if and only if both WAP(J) and WAP(L) admit LIM's ;

(iv) If H is compact subhypergroup of J such that $\delta_x * \sigma = \sigma * \delta_x$ for all $x \in J$, then there is a LIM on WAP(J) if and only if there is one on WAP(J/H) ;

(v) If $K = H \vee J$, J discrete and H compact with $H \cap J = \{e\}$, then WAP(K) has a LIM if and only if WAP(J) admits a LIM. Similar results are true for almost periodic functions.

Proof:

(i) By 4.2.11, we can put $\langle M, f\rangle = \langle m, f\circ\phi\rangle$, $f \in WAP(L)$. Then M is a LIM on WAP(L) because $_{\phi(x)}f\circ\phi = {}_x(f\circ\phi)$ for all $x \in J$, $f \in WAP(L)$, and ϕ is onto .

(ii) If there is a LIM on WAP(J), then there is one on WAP(J/H) by (i) . Conversely, let m be the unique IM on WAP(H) . For $f \in WAP(J)$, write $f'(x) = \langle m, {}_xf_{|H}\rangle$. Then

$f \longrightarrow f'$ is a bounded linear mapping of WAP(J) into C(J) and $O_L(f)' = O_L(f')$. Hence, $f' \in$ WAP(J) . Also f' is constant on the cosets, and hence we can write $f' = F \circ \Pi$, $F \in$ WAP(L) by 4.2.11 . If m_1 is a LIM on WAP(J/H) , put $\langle M, f\rangle = \langle m_1, F\rangle$. Then M is a LIM on WAP(J) .

(iii) (cf. 3.2.22) Let m_1 be a LIM on WAP(J) and m_2 a LIM on WAP(L) , respectively. For $f \in$ WAP(J×L) , $x \in J$, write $(f:x)(y) = f(x,y)$ $(y \in L)$. Then

$$_y(f:x)(z) = \int_L (f:x)(u)\ d\delta_y * \delta_z(u) = \int_L f(x,u)\ d\delta_y * \delta_z(u)$$
$$= {}_{(e,y)}f(x,z) = \left({}_{(e,y)}f : x\right)(z) .$$

This implies $O_L(f:x) \subseteq (O_L(f):x)$. Since $f \longrightarrow (f:x)$ is a bounded linear mapping of WAP(J×L) into C(L) , $(f:x) \in$ WAP(L) . Next write $f'(x) = \langle m_2, (f:x)\rangle$, $x \in J$. Then $f' \in C(J)$ [To see this, consider the map $j : L \longrightarrow J \times L$ given by $j(y) = (e,y)$, $y \in L$, and define the map $\tilde{j} : C(J\times L) \longrightarrow C(L)$ by $\tilde{j}(f) = f \circ j$. It is clear $(f:x) = {}_{(x,e)}f \circ j$, for $f \in C(J\times L)$, $x \in J$. Since $\tilde{j}$ is weak-weak continuous, the mapping $x \longrightarrow (f:x)$ is weakly continuous from J to C(J)] . The mapping $f \longrightarrow f'$ is bounded and linear from WAP(J×L) into C(J) . Also, it is easy to see that

$$\left({}_{(a,b)}f : x\right) = \int_J {}_b(f:u)\ d\delta_a * \delta_x(u)$$

$(u \longrightarrow (f:u)$ is weakly continuous) . This implies that $({}_a(f')(x)) = \left({}_{(a,e)}f\right)'(x)$, and hence $O_L(f') \subseteq O_L(f)'$. Thus

$f' \in WAP(J)$. Finally, put $\langle m,f\rangle = \langle m,f'\rangle$, $f \in WAP(J\times L)$. Then it is easy to see that m is a LIM on $WAP(J\times L)$, as in 3.2.22 . The converse follows from (i) since the projection of $J\times L$ onto L (J) is a homomorphism.

(iv) For $f \in WAP(J)$, write $f'(x) = \int_H f(x*t)\,dt$. Then $f' \in WAP(J)$. The rest is as in (ii) (see 3.2.16) .

(v) This follows from (iv) since $\delta_x*\sigma = \sigma*\delta_x$ for every $x \in K$. □

We now state the main result of this section which follows almost immediately from 4.3.3 and 4.3.4 .

Theorem 4.3.5 Let K be a hypergroup. In each of the following cases $WAP(K)$ $[AP(K)]$ admits a unique invariant mean:

(i) $K = J\times G$, where J is an amenable hypergroup and G is any locally compact group;

(ii) K contains a compact supernormal subhypergroup;

In particular, this is the case if $K = H_\vee G$ where H is any compact hypergroup and G a discrete group with $H\cap G = \{e\}$.

Proof: If G is a locally compact group, then $WAP(G)$ admits a unique invariant mean [42, § 3.1] . Hence (i) follows from 4.3.4(ii) or (iii) . To prove (ii) , recall that $K//H$ is a locally compact group, and $\delta_x*\sigma = \sigma*\delta_x$ for every $x \in K$, and use 4.3.4(iv) . □

Remark 4.3.6 Let K be an amenable noncompact hypergroup. Then UC(K)/WAP(K) is not norm separable. This follows from the fact that if m_0 is a TIM on $L_\infty(K)$, and $(f_n)_{n=1}^{\infty} \subseteq L_\infty(K)$ then the cardinality of the set

$\{m \in TIM(L_\infty(K)) : m(f_n) = m_0(f_n),\ n = 1,2, \ldots\}$ is 2^{2^d},

where d is the smallest cardinality of a cover of K by compact sets (§3.4) (see [39, p. 62-64] for details) . It is now known that if G is any locally compact, noncompact group, then UC(G)/WAP(G) contains a linear isometric copy of ℓ_∞ and hence not norm separable. The interested reader is referred to [28, Chapter 4] for more information.

CHAPTER 5

ON THE SECOND CONJUGATE OF THE BANACH ALGEBRA $L_1(K)$

§ 5.1 Introduction.

In this rather short chapter, we extend two known results on the second dual of the group algebra $L_1(G)$ to hypergroups. N. J. Young proved in [99, p. 59-62] that if G is a locally compact group then $L_1(G)$ is Arens regular if and only if G is finite. In [41, p. 321-324], E. Granirer, by developing ideas of S. L. Gulick [45, Lemma 5.2], proved that if G is a nondiscrete locally compact group, then the radical of $L_\infty(G)^*$ is not norm separable. We show that both of these results remain valid for hypergroups.

§ 5.2 On the Banach Algebra $L_\infty(K)^*$.

Let K be a hypergroup with a left Haar measure λ. We recall the definitions of two Arens products in $L_\infty(K)^* = L_1(K)^{**}$ [1]. The first Arens product is defined as follows: Given $\phi, \psi \in L_1(K)$, $f \in L_\infty(K)$, $F, G \in L_\infty(K)^*$, define $f\phi$, $Ff \in L_\infty(K)$, $FG \in L_\infty(K)^*$ by

$$\langle f\phi, \psi\rangle = \langle f, \phi * \psi\rangle ,$$

$$\langle Ff,\phi\rangle = \langle F,f\phi\rangle , \quad \text{and}$$

$$\langle FG,f\rangle = \langle F,Gf\rangle .$$

The second Arens product is defined in a similar way: We define ϕf, $fF \in L_\infty(K)$, $F\cdot G \in L_\infty(K)^*$ by

$$\langle \phi f,\psi\rangle = \langle f,\psi * \phi\rangle ,$$

$$\langle fF,\phi\rangle = \langle F,\phi f\rangle , \quad \text{and}$$

$$\langle F\cdot G,f\rangle = \langle G,fF\rangle .$$

Then $L_\infty(K)^*$ is a Banach algebra under either Arens product [1]. It is easy to see that $f\phi = \phi^{\sim} * f$ and $\phi f = f * \phi^{\sim}$, $\phi \in L_1(K)$, $f \in L_\infty(K)$ [96, Lemma 4.1] . We say that $L_1(K)$ is *Arens regular* if $F\cdot G = FG$ for all $F,G \in L_\infty(K)$.

If $L_1(K)$ is commutative (that is, K is commutative), then $L_1(K)$ is Arens regular if and only if $L_\infty(K)^*$ is commutative under the first (or second) Arens product [23, Proposition 1] .

In [99, p. 59-62] , N. J. Young proved that if G is a locally compact group then $L_1(G)$ is Arens regular if and only if G is finite. H. A. M. Dzinotyiwei [27, § 4] and J. S. Pym [77] obtained this result for a class of semigroups which include all locally compact groups. We show below that Young's theorem remains valid for hypergroups.

<u>Lemma 5.2.1</u> The following are equivalent:

(i) $L_1(K)$ is Arens regular;

(ii) Given bounded sequences (ϕ_n) , (ψ_m) in $L_1(K)$ and

$f \in L_\infty(K)$, the iterated limits $\lim_n \lim_m \langle f, \phi_n * \psi_m \rangle$, $\lim_m \lim_n \langle f, \phi_n * \psi_m \rangle$ are equal when they both exist.

Proof: This is a special case of a theorem of J. S. Pym ([23, Theorem 1] and [76, § 4]) . □

Lemma 5.2.2 Let J be a nondiscrete hypergroup, and W an open neighbourhood of e in J . Then there exist infinite sequences $\{C_n\}$, $\{D_m\}$ of compact sets with nonempty interior and contained in W such that

$$\bigcup_{m=1}^{\infty} \left(\bigcup_{n>m} C_n * D_m \right) \cap \bigcup_{n=1}^{\infty} \left(\bigcup_{n<m} C_n * D_m \right) = \emptyset .$$

Proof: The proof proceeds by induction. Suppose we have finite sequences $\{C_1, \dots, C_p\}$ and $\{D_1, \dots, D_p\}$ of compact sets with nonempty interior and contained in W such that if

$$X_p = \bigcup_{k=1}^{p} C_k , \quad Y_p = \bigcup_{k=1}^{p} D_k . \quad L_p = \bigcup_{m=1}^{p-1} \left(\bigcup_{p \geq n > m} C_n * D_m \right) , \text{ and}$$

$$U_p = \bigcup_{n=1}^{p-1} \left(\bigcup_{n<m\leq p} C_n * D_m \right) , \text{ then}$$

(a) $e \notin X_p \cup Y_p \cup L_p \cup U_p$;

(b) $X_p \cap Y_p = \emptyset$;

(c) $X_p \cap L_p = \emptyset$;

(d) $Y_p \cap U_p = \emptyset$; and

(e) $L_p \cap U_p = \emptyset$ (See [27, Lemma 4.2]) .

Since the compact sets $\{e\}$ and Y_p are contained in the open set $W\setminus(X_p \cup U_p)$ there is an open neighbourhood V of e such that $V \subseteq W\setminus(X_p \cup Y_p \cup L_p \cup U_p)$ and $V{*}Y_p \subseteq W\setminus(X_p \cup U_p)$ (see [52, 3.2D]). Next, choose a symmetric neighbourhood U of e such that $U{*}U \subseteq V$. Let C_{p+1} be any compact set in $U\setminus\{e\}$ with nonempty interior. Then

(1) $e \notin C_{p+1} \cup C_{p+1}{*}Y_p$ by [52, 4.1B] ;

(2) $C_{p+1} \cap (Y_p \cup L_p) = \emptyset$;

(3) $C_{p+1}{*}Y_p \cap (U_p \cup X_p \cup C_{p+1}) = \emptyset$.

Similarly, by making use of (a), (b), (c), (1) and (3), we can find a compact set $D_{p+1} \subseteq W$ with nonempty interior such that

(4) $e \notin D_{p+1} \cup X_p * D_{p+1}$;

(5) $D_{p+1} \cap (X_p * D_{p+1} \cup U_p \cup X_{p+1}) = \emptyset$;

(6) $X_p * D_{p+1} \cap (C_{p+1} * Y_p \cup L_p \cup Y_p \cup D_{p+1}) = \emptyset$.

It is now easy to see that the sets X_{p+1}, Y_{p+1}, $U_{p+1} = U_p \cup X_p * D_{p+1}$ and $L_{p+1} = L_p \cup C_{p+1} * Y_p$ satisfy (a), (b), (c), (d) and (e), and this completes the proof of the lemma by induction. □

We now state and prove the first main result of this chapter.

Theorem 5.2.3 Let K be a hypergroup with a left Haar measure λ. Then $L_1(K)$ is Arens regular if and only if K is finite.

Proof: If K is infinite, then by the previous result and 4.2.5, there exist sequences of compact sets (C_n), (D_m) with nonempty interior such that if $H_1 = \bigcup_{m=1}^{\infty}\left(\bigcup_{n>m} C_n * D_m\right)$ and $H_2 = \bigcup_{n=1}^{\infty}\left(\bigcup_{m>n} C_n * D_m\right)$ then $H_1 \cap H_2 = \emptyset$. Let (f_n), (g_m) be sequences in $L_1(K)$ given by $f_n = \frac{1}{\lambda(C_n)} 1_{C_n}$ and $g_m = \frac{1}{\lambda(D_m)} 1_{D_m}$ and let $h \in L_\infty(K)$ be defined by $h = 1$ on the σ-compact set H_1 and zero otherwise. Then $f_n * g_m$ vanishes outside $C_n * D_m$, and hence

$$\langle h, f_n * g_m\rangle = \int_K h(x)\; f_n * g_m(x)\; dx = 1 \text{ if } n>m \text{ and zero if } n<m .$$

Hence, the two iterated limits of $\langle h, f_n * g_m\rangle$ clearly exist and are unequal. So, $L_1(K)$ is not Arens regular by 5.2.1.

If K is finite, then $L_1(K)$ is reflexive and hence Arens regular. □

There is a natural multiplication on $UC_\ell(K)^*$ under which it is a Banach algebra: For $f \in UC_\ell(K)$, $\phi \in UC_\ell(K)^*$, define $\phi f(x) = \phi({}_x f)$ $(x \in K)$. Then $\phi f \in UC_\ell(K)$. Indeed,
${}_x(\phi f)(y) = \int_K (\phi f)(u)\, d\delta_x * \delta_y(u) = \int_K \langle \phi, {}_u f\rangle\, d\delta_x * \delta_y(u) = \langle \phi, {}_y({}_x f)\rangle$.
Hence, if (x_α) converges to x in K, then

$\|{}_{x_\alpha}(\phi f) - {}_x(\phi f)\|_\infty \le \|\phi\| \, \|{}_{x_\alpha}f - {}_x f\|_\infty$, which converges to zero. Next, define $\phi\psi \in UC_r(K)^*$ by $\langle \phi\psi, f\rangle = \langle \phi, \psi f\rangle$ $(\psi \in UC_r(K)^*)$. Then $UC_r(K)^*$ becomes a Banach algebra with a unit (see [37, p. 130] or [71, § 4]) . The next result shows that $UC_r(K)^*$ can be identified with a closed subalgebra of $L_\infty(K)^*$ with respect to the first Arens product. The ideas involved in the proof are essentially in [51, § 3], [34, Theorem] and [89, 2.3, 5.2, 5.3] .

Propostion 5.2.4 Let K be nondiscrete and let Π be the adjoint of the inclusion map of $UC_r(K)$ into $L_\infty(K)$. Then

(i) The following are equivalent:

(a) E is a right annihilator in $L_\infty(K)^*$;

(b) $E\, L_\infty(K) = \{0\}$;

(c) $E \in \ker \Pi = \{\phi \in L_\infty(K)^* : \phi(UC_r(K)) = 0\}$

$$= UC_r(K)^\perp .$$

(ii) E is a right identity of $L_\infty(K)^*$ if and only if $E = \Pi^{-1}(\delta_e)$, where $\delta_e \in UC_r(K)^*$ is given by $\delta_e(f) = f(e)$ for $f \in UC_r(K)$;

(iii) There are no left annihilators and no left identities in $L_\infty(K)^*$;

(iv) $F \in UC_r(K)^\perp$ if and only if $F \cdot L_1(K) = \{0\}$;

(v) Let E be a right identity in $L_\infty(K)^*$. Then $E\, L_\infty(K)^*$ is a closed right ideal in $L_\infty(K)^*$, and

the decomposition $L_\infty(K)^* = UC_\eta(K)^\perp \oplus E\,L_\infty(K)^*$ is a topological and an algebraic direct sum. If in addition, $\|E\| = 1$, then
$E\,L_\infty(K)^* \cong UC_\eta(K)^* \cong L_\infty(K)^*/UC_\eta(K)^\perp$ (as Banach algebras).

<u>Proof</u>:

(i) (b) $\longleftrightarrow$ (c) : If $E \in UC_\eta(K)^\perp$, $f \in L_\infty(K)$, then $\langle Ef,\phi\rangle = \langle E,\phi^{\sim}\bullet f\rangle = 0$ for $\phi \in L_1(K)$. Conversely, if $Ef = 0$, $f \in L_\infty(K)$, $E \in L_\infty(K)^*$, then $\langle E,\phi\bullet f\rangle = \langle E,(\Delta\phi^{\sim})^{\sim}\bullet f\rangle = \langle Ef,(\Delta\phi)^{\sim}\rangle = 0$. Hence, $E \in UC_\eta(K)^\perp$ since $UC_\eta(K) = L_1(K)\bullet L_\infty(K)$.

(a) $\longleftrightarrow$ (c) : If $L_\infty(K)\,E = \{0\}$, then $\langle GE,f\rangle = \langle G,Ef\rangle = 0$ for all $f \in L_\infty(K)$, $G \in L_\infty(K)^*$. Thus $Ef = 0$ for $f \in L_\infty(K)$ by the Hahn-Banach theorem. Conversely, if $Ef = 0$ for all $f \in L_\infty(K)$, clearly $L_\infty(K)^*\,E = 0$.

(ii) If E is a right identity for $L_\infty(K)^*$, then $\langle E,\phi\bullet f\rangle = \langle Ef,\tilde{\phi}\rangle = \langle\hat{\tilde{\phi}},Ef\rangle = \langle\hat{\tilde{\phi}}E,f\rangle = \langle f,\tilde{\phi}\rangle = \phi\bullet f(e) = \langle\delta_e,\phi\bullet f\rangle$.

Conversely, if $\Pi(e) = \delta_e$, then $\langle FE,f\rangle = \lim_\alpha \langle\hat{\phi}_\alpha E,f\rangle$, (where $(\phi_\alpha) \subseteq L_1(K)$ converges to F in the weak* topology (see [96, Lemma 4.1E])), $= \lim_\alpha \langle\hat{\phi}_\alpha,Ef\rangle = \lim_\alpha \langle Ef,\phi_\alpha\rangle = \lim_\alpha \langle E,\tilde{\phi}_\alpha\bullet f\rangle$ $= \lim_\alpha \tilde{\phi}_\alpha\bullet f(e) = \lim_\alpha \langle\phi_\alpha,f\rangle = \langle F,f\rangle$, as required.

(iii) This follows from (i) and (ii).

(iv) It is easy to see that $\hat{\phi}f = f\cdot\tilde{\phi}$, $\phi \in L_1(K)$, $f \in L_\infty(K)$ (see [96, Lemma 4.1]) . Hence, if $F \in UC_\ell(K)$ then $\langle F\hat{\phi}, f\rangle = \langle F,\hat{\phi}f\rangle = 0$ and since every element in $UC_\ell(K)$ is of the form $\hat{g}f$ the converse is also true.

(v) $E\, L_\infty(K)^*$ is closed since E is idempotent, and clearly it is a right ideal in $L_\infty(K)^*$. Now, if $\phi \in L_1(K)$, $f \in L_\infty(K)$, $G \in L_\infty(K)^*$, then

$\langle EG,\phi\cdot f\rangle = \langle (EG)f,\tilde{\phi}\rangle = \langle G,\phi^{\sim\sim}\cdot f\rangle = \langle G,\phi\cdot f\rangle$. Hence $EG = G$ on $UC_r(K)$. For $G \in L_\infty(K)^*$ write $G = (G - EG) + EG$. Suppose $EG = 0$ on $UC_r(K)$. Let $(\phi_\alpha) \subseteq L_1(K)$ converge to E in the weak* topology. Then $\langle EG,f\rangle = \lim_\alpha \langle\hat{\phi}_\alpha G,f\rangle = \lim_\alpha \langle\hat{\phi}_\alpha,Gf\rangle$ $= \lim_\alpha \langle Gf,\phi_\alpha\rangle = \lim_\alpha \langle G,\tilde{\phi}_\alpha\cdot f\rangle = \lim_\alpha \langle EG,\tilde{\phi}_\alpha\cdot f\rangle = 0 \quad (f \in L_\infty(K))$. That is, if $EG = 0$ on $UC_r(K)$, then it is zero. Hence, the sum is an algebraic and a topological direct sum. To see that Π is a homomorphism, note that

$\langle Gf,\phi\rangle = \langle G,\tilde{\phi}\cdot f\rangle = \langle \Pi(G),\tilde{\phi}\cdot f\rangle = \int_K \overline{\langle \Pi(G),f\rangle}\, \phi(x)\, dx = \langle \Pi(G)f,\phi\rangle$, $G \in L_\infty(K)^*$, $\phi \in C_c(K)$, $f \in UC_r(K)$. Hence, $\Pi(G)f = Gf$ for $G \in L_\infty(K)^*$, $f \in UC_r(K)$. Next,

$\langle \Pi(GF),f\rangle = \langle GF,f\rangle = \langle G,Ff\rangle = \langle G,\Pi(F)f\rangle = \langle \Pi(G),\Pi(F)f\rangle = \langle \Pi(G)\Pi(F),f\rangle$. That is, $\Pi(GF) = \Pi(G)\Pi(F)$ for all $G,F \in L_\infty(K)^*$. To complete the proof of (v) , we have to show that the restriction map $\Pi|_{EL_\infty(K)^*} : EL_\infty(K)^* \longrightarrow UC_r(K)^*$ is an onto linear isometry when $\|E\| = 1$. If $\Phi \in UC_r(K)^*$, by the Hahn-Banach theorem there exists a norm preserving extension $\Psi \in L_\infty(K)^*$. Then $\Phi = E\Psi|_{UC_r(K)}$ and hence $\|\phi\| \le \|E\Psi\|$. But $\|E\psi\| \le \|E\| \, \|\Psi\| = \|\Phi\|$, and therefore $\|E\Psi\| = \|\Phi\|$. Finally, note that $E\Psi_1 = E\Psi_2$ on $UC_r(K)$ implies $\Psi_1 = \Psi_2$. □

E. Granirer proved in [41, p. 321-324], by developing ideas of S. L. Gulick [45, Lemma 5.2], that if G is a nondiscrete locally compact group (or an infinite amenable discrete group), then the radical of $L_\infty(G)^*$ is not norm separable. We shall next extend this result to hypergroups. K. A. Ross proved in [91, Theorem A6] that if K is nondiscrete then $L_\infty(K) \ne C(K)$. We need the following lemma which is due to S. L. Gulick [45, § 5] for the case when K is a group (see also [41, p. 322]).

<u>Lemma 5.2.5</u> If K is a nondiscrete hypergroup then $L_\infty(K)/C(K)$ is not norm separable.

<u>Proof</u>: We first claim that there is a sequence $(U_n)_{n=1}^\infty$ of open sets in K such that $U_n \cap U_m = \emptyset$ $(n \ne m)$ and $e \in \overline{U_n}$ for each n (see [91, Lemma A5]). Suppose first that K is metrizable. Choose a sequence of points (x_n) converging to e and a sequence of open neighbourhoods $(W_n)_{n=1}^\infty$ of these points with the property that the sets $\overline{W_n}$ are pairwise disjoint. Let

$(k_i)_{i=1}^{\infty}$, $k = 1,2, \ldots$ be pairwise disjoint subsequences of positive integers. If $U_k = \bigcup_{i=1}^{\infty} W_{k_i}$, then U_k is open, $U_k \cap U_\ell = \emptyset$ $(k \neq \ell)$ and $e \in \overline{U_k}$ for each k. If K is nonmetrizable, then (since K is nondiscrete) there is a compact subhypergroup H of K such that $K//H$ is metrizable and nondiscrete (see the proof of [91, Lemma A5]). Hence, there exists a sequence $(V_n)_{n=1}^{\infty}$ of pairwise disjoint open sets in $K//H$ such that $H \in \bigcup_{n=1}^{\infty} \overline{V_n}$. Finally, if $U_n = \Pi^{-1}(V_n)$, $n = 1,2, \ldots$, where Π is the projection of K onto $K//H$, then $(U_n)_{n=1}^{\infty}$ is the desired sequence of open sets. This proves the claim. Next, for each subsequence $\alpha = (n_i)$ of positive integers write $h_\alpha = 1_{\bigcup_{i=1}^{\infty} U_{n_i}}$. Then $\|h_\alpha - f\|_\infty \geq \frac{1}{4}$ for each $f \in C(K)$ Indeed, if $|f(e)| < \frac{1}{2}$ there is an open neighbourhood W of e such that $|f(e)| < \frac{1}{2}$ for $x \in W$. Then

$|h_\alpha(x) - f(x)|_\infty \geq \frac{1}{2}$ for $x \in \bigcup_{i=1}^{\infty} U_{n_i} \cap W$, and each $U_{n_i} \cap W$ is open and nonempty since $e \in \bigcap_{i=1}^{\infty} \overline{U_{n_i}}$. If $|f(e)| \geq \frac{1}{2}$, then we can find an open neighbourhood W of e such that $|f(x)| \geq \frac{1}{4}$ for all $x \in W$. Pick any U_{n_0} different from U_{n_i}, $i = 1,2, \ldots$. Then $|h_\alpha(x) - f(x)|_\infty \geq \frac{1}{4}$ for $x \in W \cap U_{n_0}$, a nonempty open set. Also, if α and β are two different subsequences of positive integers then $\|h_\alpha - h_\beta\|_\infty = 1$. Hence, $L_\infty(K)/C(K)$ is not norm separable. □

<u>Corollary 5.2.6</u> If K is nondiscrete, then $L_\infty(K)/UC_r(K)$, $L_\infty(K)/UC_\ell(K)$ and $L_\infty(K)/UC(K)$ are not norm separable.

<u>Proof</u>: This follows easily from the remark in [39, p. 62-63] . □

<u>Corollary 5.2.7</u> The space $UC_r(K)^\perp$ of right annihilators in $L_\infty(K)^*$ is not norm separable, provided K is nondiscrete.

<u>Theorem 5.2.8</u> Let K be a nondiscrete hypergroup with a left Haar measure. Then the radical $\mathcal{R}(L_\infty(K)^*)$ of $L_\infty(K)^*$ is not norm separable.

<u>Proof</u>: $\mathcal{R}(L_\infty(K)^*)$ contains the nonseparable space $UC_r(K)^\perp$ by 5.2.4(i) [80, p. 56] . □

<u>Remark 5.2.9</u> Let K be amenable. Then the radical $\mathcal{R}(UC_r(K)^*) = \{0\}$ if and only if K is compact. If K is noncompact, then

(i) $\mathcal{R}(UC_r(K)^*)$ is not norm separable, and

(ii) $\mathcal{R}(L_\infty(K)^*)$ properly contains $UC_r(K)^\perp$. To see this, consider the set

$J = \{m \in L_\infty(K)^* : m$ is topologically left invariant and $m(1) = 0\}$.
Then J is a left ideal and $J^2 = \{0\}$. Furthermore, if E is a right identity in $L_\infty(K)^*$ then $J \subseteq E\,L_\infty(K)^*$: Let $m,n \in J$, $F \in L_\infty(K)^*$, $f \in L_\infty(K)$. Then $\langle Ff,g\rangle = \langle F,fg\rangle = \langle F,(\mathring{g}\mathring{\Delta})\cdot f\rangle$ $(g \in L_1(K))$, and hence $\langle Ff,g\rangle = \left(\int_K g\,d\lambda\right)\langle F,f\rangle$ for all $f \in L_\infty(K)$, $g \in L_1(K)$, if and only if F is topologically left invariant. Thus F is topologically left invariant if and only if $Ff = \langle F,f\rangle 1$ for all $f \in L_\infty(K)$, and this is equivalent to $EF = F$ since $\langle E,1\rangle = 1$. That is, $J \subseteq E\,L_\infty(K)^*$. The rest of the claim is easy to see [37, p. 130-133]. Now, let m_0 be a TLIM on $L_\infty(K)$ and $A = \{m - m_0 : m$ is a TLIM on $L_\infty(K)\} \subseteq J$. If K is noncompact, then $|A| \geq 2^{2^d}$, where d is the smallest cardinality of a cover of K by compact sets (by 3.4.6). Hence, we have (i) and (ii) by 5.2.4(v). If K is compact, then $UC_\eta(K) = C(K)$, and $C(K)^* = M(K)$ is semisimple since the left regular representation is faithful ([52, 6.2I] and [80, Theorem 4.6.7]).

CHAPTER 6

QUESTIONS AND REMARKS

In this chapter, we state some problems in hypergroups and/or groups which remain open for further research. Throughout, let K denote a hypergroup with a left Haar measure.

1. We observed in 2.2.2 that if the maximal subgroup G(K) of K is open then $UC_r(K)$ is an algebra.

Is $UC_r(K)$ always an algebra?

It is well known that if G is a locally compact group, then $UC_r(G) = C(G)$ if and only if G is compact or discrete [73, 2.33] . This is not the case in general for hypergroups (2.2.3 and 2.2.4) . Characterize all hypergroups K for which $C(K) = UC_r(K)$.

2. Let K be amenable. We proved in 3.2.13 that every (closed) subgroup of K is amenable. Is every subhypergroup of K necessarily amenable?

3. The Banach algebra $L_1(K)$ is left amenable if and only if K is amenable [59, Theorem 4.1] . This shows that if $L_1(K)$ is amenable, then K is amenable. There is a commutative hypergroup

K (3.3.11) for which $L_1(K)$ is not weakly amenable and hence not amenable. Find all amenable (commutative) hypergroups K for which the Banach algebra $L_1(K)$ is amenable (weakly amenable) (see [2], [53]) . It would be worthwhile to study these problems at least for the following classes:

(1) Hypergroups arising from certain polynomials introduced and studied by R. Lasser [57] ;

(2) Hypergroups G_B induced by $[FIA]_B$-groups [83] ;

(3) Z-hypergroups ([46], [83]) .

4. Amenability of K is characterized by Reiter's condition (P_1) (3.3.8) . (P_2) always implies (P_1) , but the converse need not be true (3.3.12 and 3.3.15) . A commutative hypergroup K satisfies (P_2) if and only if the support of the Plancherel measure on the dual has the trivial character. Such hypergroups seem to be useful in probability theory ([32],[33]). Hence, it is not unreasonable to pursue further studies on hypergroups which satisfy (P_2) .

5. Let H be a compact hypergroup, J a discrete hypergroup with $J \cap H = \{e\}$, $|J| \geq 2$, and $K = H \vee J$. Then every LIM on $L_\infty(K)$ is a TLIM (3.2.25) , which is in contrast to the group case ([40,[84]) . On the other hand, if G is a locally compact nondiscrete group which is amenable as a discrete group, and J a finite hypergroup, then $LIM(L_\infty(K)) \setminus TLIM(L_\infty(K)) \neq \emptyset$, where

$K = G \times J$. It is interesting to study this problem in more detail. In particular, the following question seems to be reasonable: Let K be a commutative nondiscrete hypergroup. Assume that $(x)*(y)$ is finite for all $x,y \in K$. Is $LIM(L_\infty(K)) \setminus TLIM(L_\infty(K)) \neq \emptyset$?

6. We established in § 3.4 that if K is noncompact and amenable then $|TIM(L_\infty(K))| = |TIIM(L_\infty(K))| = 2^{2^d}$, where d is the smallest cardinality of a cover of K by compact sets. We also proved that if the maximal subgroup $G(K)$ of K is open then $|TLIM(L_\infty(K))| = 2^{2^d}$, and it is conceivable that this is true for all hypergroups. Does there exist an explicit one to one correspondence between $TIM(L_\infty(K))$ and $TLIM(L_\infty(K))$? Note that this result (if established) may be new even in the group case.

7. We showed in § 4.3 that there is a class of hypergroups K including amenable hypergroups for which the [weakly] almost periodic functions [WAP(K)] AP(K) admit a unique invariant mean. Does WAP(K) or AP(K) admit a (unique) invariant mean for every hypergroup ? There are also other questions available on [weakly] almost periodic functions for hypergroups. For example, one can study the structure of WAP(K) (see [64, § 5] , [95, § 3] and [14]) and the sizes of the spaces WAP(K)/AP(K) , $WAP(K)/C_0(K)$, UC(K)/WAP(K) etc. (see [28, Chapter 4] and [14]) .

8. **The Banach algebra $L_1(K)$ is Arens regular if and only if K is finite (5.2.3). A stronger result is known for the case when K is a group. In fact, if G is any locally compact group, then the topological center of $L_1(G)^{**}$ is $L_1(G)$ ([61],[63]). For a compact group, this is in [51]. The center of $UC_r(G)^*$ is M(G) [61]. Study these problems for hypergroups. It is interesting to note that $UC_r(K)^*$ is isometrically isomorphic to a closed subalgebra of $L_\infty(K)^*$. The ideas in [51] may be useful here.**

9. It is still not known if the radical of $\ell_\infty(G)^*$ is not norm separable for the case when G is an infinite nonamenable discrete group. Let G be a noncompact locally compact group. Is the radical $\mathcal{R}(UC_r(G)^*)$ nonseparable ? ([41])

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