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Geometric Tomography Via Conic Sections

by

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Abstract

A common goal of Geometric Tomography is to find criteria for uniquely determining a convex body. The "solid" and "hollow" n-dimensional cones with central angle $\alpha \in (0, \frac{\pi}{2})$, central axis given by the unit vector ξ, and vertex at the origin, are defined, respectively, as follows:

$$C_{\alpha,\xi} := \{x \in \mathbb{R}^n : \langle \frac{x}{|x|_2}, \xi \rangle \geq \cos(\alpha)\},$$

$$\tilde{C}_{\alpha,\xi} := \{x \in \mathbb{R}^n : \langle \frac{x}{|x|_2}, \xi \rangle = \cos(\alpha)\}.$$

I will examine two problems:

1. Given two convex bodies K and L, and an angle, $\alpha \in (0, \frac{\pi}{2})$, suppose that

$$Vol_n(K \cap C_{\alpha,\xi}) = Vol_n(L \cap C_{\alpha,\xi}) \qquad \forall \xi \in S^{n-1}.$$

   Does this imply $K = L$?

2. Given two convex bodies K and L, and an angle, $\alpha \in (0, \frac{\pi}{2})$, suppose that

$$Vol_{n-1}(K \cap \tilde{C}_{\alpha,\xi}) = Vol_{n-1}(L \cap \tilde{C}_{\alpha,\xi}) \qquad \forall \xi \in S^{n-1}.$$

   Does this imply $K = L$?

For each problem, I will use methods involving spherical harmonics and orthogonal polynomials to provide sufficient conditions for the affirmative answer and a partial answer to the negative answer.

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List of Symbols

n	Dimension ($n \geq 2$)
ϑ	The number $\frac{n-3}{2}$
$\mathbb{N}_0$	The non-negative integers $\mathbb{N} \cup \{0\}$
S	The Schwartz space of rapidly decreasing infinitely differentiable functions
$\langle x, y \rangle$	The inner product of $x, y \in \mathbb{R}^n$
S^{n-1}	The unit sphere in $\mathbb{R}^n$
$\lvert \cdot \rvert_2$	The Euclidean norm on $\mathbb{R}^n$
$\lVert \cdot \rVert_K$	The Minkowski functional of K
$\rho_K(\cdot)$	The Radial function of K
$Vol_n(S)$	The Lebesgue measure of the set $S \subset \mathbb{R}^n$
H_m^n	The space of spherical harmonics of dimension n and degree m
H^n	The space of all finite sums of spherical harmonics of dimension n
$\lvert S^{n-1} \rvert$	The surface area of S^{n-1}
$\xi^\perp$	The hyperplane $\{\langle x, \xi \rangle = 0\} := \{x \in \mathbb{R}^n : \langle x, \xi \rangle = 0\}$
$K \vert \xi^\perp$	The orthogonal projection of the body K onto $\xi^\perp$

Chapter 1

Convex Bodies

In this preliminary section, we introduce necessary concepts from convex geometry and Fourier Analysis. These topics are specific to the problem of calculating volume for n-dimensional bodies and their sections, which is largely what this thesis involves. Most if not all of the results in this section can be found in the book by Koldobsky [5] so they will be stated without proofs. I will sketch the proofs of some results just to illustrate the use of certain Fourier analytic techniques.

1.A Star Bodies

A compact subset of $\mathbb{R}^n$ with non-empty interior, K, is called a *star body* if for every $x \in K$ each point of the interval $[0,x)$ is an interior point of K and the boundary of K is continuous in the sense that the *Minkowski functional* of K defined by

$$\|x\|_K := \min\{a \geq 0 : x \in aK\}$$

is a continuous function on $\mathbb{R}^n$. A compact set with non-empty interior K is called a *convex body* if for every $\lambda \in [0,1]$ and every $x, y \in K$

$$\lambda x + (1-\lambda)y \in K.$$

In this thesis, it will be assumed that every convex body contains 0 as an interior point. The Minkowski functional is homogeneous of degree 1 on $\mathbb{R}^n$ and clearly $x \in K$ if and only if $\|x\|_K \leq 1$. We define the *radial function* of a star body K as the reciprocal of the Minkowski functional:

$$\rho_K(x) := \|x\|^{-1}.$$

S^{n-1} will denote the unit sphere in $\mathbb{R}^n$ (those points in $\mathbb{R}^n$ of Euclidean distance 1 from the origin). If $x \in S^{n-1}$ then $\rho_K(x)$ is the Euclidean distance from the origin to the boundary of K, in the direction of x. A body K is called *origin-symmetric* if $K = -K$ (or $x \in K$ if and only if $-x \in K$). The set of origin-symmetric convex bodies will endowed with the *radial metric* defined as

$$\rho(K, L) = \max_{x \in S^{n-1}} |\rho_K(x) - \rho_L(x)|.$$

Finally, a body is called *k-smooth* if the restriction of its Minkowski functional to S^{n-1} is k times continuously differentiable. A body is called *infinitely smooth* if it is k-smooth for each $k \in \mathbb{N}$. Often a convex body will be assumed to be infinitely smooth since any convex body can be approximated, in the radial metric, by an infinitely smooth convex body, (see [5], Thm 2.10).

Many results involve integrating powers of the radial function, and thus we require the following.

Lemma 1.1: *(5, p.14). Let $K \subset \mathbb{R}^n$ be an origin-symmetric star body. For $0 < p < n$, the function, $\|\cdot\|_K^{-p}$ is locally integrable on $\mathbb{R}^n$. Also, if f is bounded and integrable on $\mathbb{R}^n$, then $\|\cdot\|_K^{-p} f(\cdot)$ is integrable on $\mathbb{R}^n$.*

This work is largely concerned with the study of volume, both of convex bodies and their sections. $Vol_n(S)$ will denote the Lebesgue measure of an n-dimensional set S. Occasionally we will write "volume" in place of $Vol_n(\cdot)$ when it is clear what is actually meant. For $\xi \in S^{n-1}$ the *parallel section function* will be defined on $\mathbb{R}$ as follows

$$A_{K,\xi}(t) := Vol_{n-1}(K \cap \{\xi^\perp + t\xi\}),$$

where $\{\xi^\perp + t\xi\}$ is the hyperplane $\{x \in \mathbb{R}^n : \langle x, \xi \rangle = t\}$. We are mostly concerned with the central hyperplane sections of an origin-symmetric body K. An expression for the volume of these sections can be found using n-dimensional polar cordinates where $x := r\theta$ and the Jacobian is given by r^{n-1}. Taking χ to be the indicator function of the interval [-1,1], then $\chi(\|\cdot\|_K)$ is the indicator function of K and we have the following.

$$\begin{aligned}
A_{K,\xi}(0) = Vol_{n-1}(K \cap \xi^\perp) &= \int_{\langle x,\xi \rangle = 0} \chi(\|x\|_K) dx \\
&= \int_{S^{n-1} \cap \xi^\perp} \left(\int_0^\infty r^{n-2} \chi(r\|\theta\|_K) dr \right) d\theta = \int_{S^{n-1} \cap \xi^\perp} \left(\int_0^{1/\|\theta\|_K} r^{n-2} dr \right) d\theta \\
&= \frac{1}{n-1} \int_{S^{n-1} \cap \xi^\perp} \|\theta\|_K^{-n+1} d\theta = \frac{1}{n-1} \int_{S^{n-1} \cap \xi^\perp} \rho_K^{n-1}(\theta) d\theta,
\end{aligned}$$

where the term r^{n-2} was used as the Jacobian in the second line since the integral is over the $(n-1)$-dimensional hyperplane, $\xi^\perp$. Note that the map

$R : C(S^{n-1}) \to C(S^{n-1})$ defined by

$$Rf(\xi) := \int_{S^{n-1}\cap\xi^\perp} f(x)dx$$

is called the *spherical Radon transform* of $f \in C(S^{n-1})$ (where $C(K)$ is the standard notation for the space of continuous real valued functions on a compact set K). Thus, the above result can be reformulated as

$$A_{K,\xi}(0) = \frac{1}{n-1} R(\|\cdot\|_K^{-n+1})(\xi)$$

for every $n \geq 2$.

1.B The Gamma Function and Fourier Transform

For $z \in \mathbb{C}$, $\mathrm{Re}(z) > 0$, the Γ function is defined by

$$\Gamma(z) := \int_0^\infty t^{z-1}e^{-t}dt.$$

Since $\Gamma(1) = 1$ and, for all z in the above range, $\Gamma(z+1) = z\Gamma(z)$, we have that

$$\Gamma(n+1) = n!$$

for all $n \in \mathbb{N}$.

Differentiating under the integral shows that the Γ function is analytic in the domain $\{z \in \mathbb{C} : \mathrm{Re}(z) > 0\}$. Using $\Gamma(z+1) = z\Gamma(z)$, the Γ function can be extended to an analytic function on the complex plane without the points

$-\mathbb{N}_0 := \{0, -1, -2, ...\}$.

Fourier analytic tools play a large part in what is to follow. We begin by defining the Fourier transform of a function $\phi \in L_1(\mathbb{R}^n)$

$$\hat{\phi}(\xi) := \int_{\mathbb{R}^n} \phi(x) \exp(-i\langle x, \xi \rangle) dx,$$

where $L_1(K)$ is the space of measurable functions on K with the property $\int_K |f| dx < \infty$. $S := S(\mathbb{R}^n)$ will denote the Schwartz space of rapidly decreasing, infinitely differentiable functions on $\mathbb{R}^n$. Namely, $\phi \in S$ if and only if ϕ is C^∞ and ϕ, as well as all of its partial derivatives, converge to zero at infinity faster than any negative power of the Euclidean norm $|\cdot|_2$. Elements of S will be called *test functions* and if ϕ is a test function, then so is $\hat{\phi}$. By S' we denote the space of linear continuous functionals (or *distributions*) on S. We say that a function f has *power growth at infinity* if there exists a number $\gamma > 0$ such that

$$\lim_{|x| \to \infty} \frac{f(x)}{|x|_2^\gamma} = 0.$$

If $f \in S'$ is a locally integrable function with power growth at infinity, then its action on a test function is defined by

$$\langle f, \phi \rangle := \int_{\mathbb{R}^n} f(x)\phi(x) dx.$$

Note that this integral converges. The Fourier transform of such a function f can be also defined by an action on arbitrary test functions,

$$\langle \hat{f}, \phi \rangle := \langle f, \hat{\phi} \rangle = \int_{\mathbb{R}^n} f(x)\hat{\phi}(x) dx.$$

The Fourier transform is invertible on S and its inverse is given by

$$\Psi\phi(\xi) = (2\pi)^{-n} \int_{\mathbb{R}^n} \phi(x) \exp(i\langle x, \xi\rangle) dx.$$

It follows that for every $\phi \in S$, $\hat{\hat{\phi}}(\xi) = (2\pi)^n \phi(-\xi)$. If ϕ is even then so is $\hat{\phi}$ and

$$\langle \hat{f}, \hat{\phi}\rangle = \langle f, \hat{\hat{\phi}}\rangle = (2\pi)^n \langle f, \phi\rangle.$$

Unless otherwise stated, all test functions from now on will be even and real valued. A distribution f is called *even homogeneous of degree* $p \in \mathbb{R}$ if

$$\langle f(x), \phi(\tfrac{x}{\alpha})\rangle = |\alpha|^{n+p} \langle f, \phi\rangle$$

for every test function ϕ and every nonzero $\alpha \in \mathbb{R}$.

Lemma 1.2: *(5, p.35). The Fourier transform of an even homogeneous distribution of degree p is an even homogeneous distribution of degree $-n-p$.*

Finally, we will say that a distribution f is *positive definite* if its Fourier transform is a positive distribution. That is, $\langle \hat{f}, \phi\rangle \geq 0$ for every non-negative test function ϕ. Positive definite distributions are of great importance in the study of intersection bodies, a topic related to the study of volume but not covered in this thesis.

1.C A First Look at Volume

At the end of Section 1.1, we saw an explicit formula for the volume of $(n-1)$-dimensional central hyperplane sections of a body K. Repeating this derivation with "$\mathbb{R}^n$" in place of "$\langle x, \xi\rangle = 0$" gives us a formula for the volume of any star body:

$$Vol_n(K) = \frac{1}{n}\int_{S^{n-1}} \rho_K^n(\theta)d\theta.$$

Using some properties of the Γ function as well as the equation for $A_{K,\xi}$ mentioned in the first section, we can establish the following result for the unit ball in the l^p norm.

Lemma 1.3: *(5, p.32).*

$$Vol_n(B_p^n) = \frac{2^n\big(\Gamma(1+1/p)\big)^n}{\Gamma(1+n/p)},$$

where $B_p^n \subset \mathbb{R}^n$ *is the unit ball wi1th respect to the norm* $\|x\|_p := (\sum_{i=1}^n |x_i|^p)^{1/p}$.

Everything that follows involves the study of volume using the Fourier transform. The next result is the most important in this section.

Theorem 1.1: *(5, p.54). Let* K *be an origin-symmetric star body in* $\mathbb{R}^n$. *The Fourier transform of the function* $\|\cdot\|_K^{n-1}$ *is a homogeneous of degree* -1 *function on* $\mathbb{R}^n$, *continuous on* $\mathbb{R}^n\backslash\{0\}$ *and such that*

$$A_{K,\xi}(0) = Vol_{n-1}(K\cap\xi^\perp) = \frac{1}{\pi(n-1)}(\|\cdot\|_K^{-n+1})^\wedge(\xi) \qquad \forall\xi\in S^{n-1}.$$

In particular this gives a relationship between the Fourier transform and the spherical Radon transform. If f is an even function that is continuous and homogeneous of degree $-n+1$ then

$$Rf(\xi) = \frac{1}{\pi}\hat{f}(\xi).$$

Since $\left(\|\cdot\|_K^{-n+1}\right)^\wedge(\xi)$ is homogeneous, its values on all of $\mathbb{R}^n$ are determined by those on S^{n-1}. Consider origin-symmetric star bodies K and L with the property that $A_{K,\xi}(0) = A_{L,\xi}(0)$ for all $\xi \in S^{n-1}$. By the above result, we have that $\left(\|\cdot\|_K^{-n+1}\right)^\wedge(\xi) = \left(\|\cdot\|_L^{-n+1}\right)^\wedge(\xi)$ for all $\xi \in S^{n-1}$. By homogeneity, we can extend this equality to $\mathbb{R}^n$ and applying the inverse Fourier transform yields $\|x\|_K = \|x\|_L$ for all $x \in \mathbb{R}^n$.

Corollary 1.1.1: An origin-symmetric star body in $\mathbb{R}^n$ is uniquely determined by the $(n-1)$-dimensional volume of its central hyperplane sections.

In a similar way, one can prove a more general result.

Corollary 1.1.2: (5, p.55). Let $1 \leq m < n$ and let $f, g \in C(S^{n-1})$ be two even functions so that for any m-dimensional subspace H of $\mathbb{R}^n$

$$\int_{S^{n-1}\cap H} f(\theta)d\theta = \int_{S^{n-1}\cap H} g(\theta)d\theta.$$

Then $f = g$.

This last result can also be used to uniquely determine origin-symmetric bodies by their m-dimensional central sections since in this case, $\|\cdot\|_K$ and $\|\cdot\|_L$ are even.

Determining a body K uniquely usually involves finding a quantity that is proportional to its radial function. If two bodies share this quantity, they have the same radial function and are thus equal. If K is origin-symmetric then we can also study the Fourier transform of powers of $\|\cdot\|_K$ and then proceed as in the argument before Corollary 1.1.1. Later, we will use a different Fourier analytic tool, spherical harmonics, to uniquely determine a convex body by its volume of intersection with a fixed cone. For now, we will discuss the derivatives of $A_{K,\xi}$.

Let $m \in \mathbb{N}_0$ and h be a continuous integrable function on $\mathbb{R}$, m times continuously differentiable in some neighborhood of zero. Let $q \in \mathbb{C}$, $-1 < \mathrm{Re}(q) < m$, $q \neq 0, 1, ..., m-1$. The *fractional derivative* of the order q of the function h at zero is defined as follows:

$$h^{(q)}(0) := \frac{1}{\Gamma(-q)} \int_0^1 t^{-1-q} \Big(h(t) - h(0) - h'(0)\frac{t}{1!} - ... - h^{(m-1)}(0)\frac{t^{m-1}}{(m-1)!} \Big) dt$$
$$+ \frac{1}{\Gamma(-q)} \int_1^\infty t^{-1-q} h(t) dt + \frac{1}{\Gamma(-q)} \sum_{k=0}^{m-1} \frac{h^{(k)}(0)}{k!(k-q)}.$$

The next result gives us the fractional derivatives of $A_{K,\xi}(t)$ at $t = 0$.

Theorem 1.2: *(5, p.60). Let D be an infinitely smooth origin-symmetric convex body in $\mathbb{R}^n$, and $\xi \in S^{n-1}$. Then for every $q \in (-1, \infty)$, $q \neq n-1$, the fractional derivative of the order q of the parallel section funtion at zero is given by*

$$A_{D,\xi}^{(q)}(0) = \frac{\cos(\pi q/2)}{\pi(n-q-1)} (\| \cdot \|_D^{-n+q+1})^\wedge(\xi).$$

Chapter 2

Spherical Harmonics

Functions of the form $\cos(\alpha x)$ and $\sin(\beta x)$, α, $\beta \in \mathbb{N}$, form a basis for the Hilbert space $C[-\pi, \pi]$ (the space of real valued continuous functions on $[-\pi, \pi]$ equiped with the inner product $\langle f, g \rangle := \int\limits_{-\pi}^{\pi} f(x)g(x)dx$). If $f \in C[-\pi, \pi]$, then f has a *Fourier series* expansion with respect to this basis. In a similar way we can define a Fourier series for a square integrable function on S^{n-1}. The elements of the corresponding basis for $L_2(S^{n-1})$ are referred to as Spherical Harmonics.

2.A Definition and Properties

Recall that the Laplacian of a twice differentiable function $f(x_1, x_2, \ldots, x_k)$ is defined as

$$\Delta f := \sum_{i=1}^{k} \frac{\partial^2 f}{\partial x_i^2}.$$

f is *harmonic* if $\Delta f = 0$. A *spherical harmonic* of dimension n and degree m is the restriction of a homogeneous of degree m harmonic polynomial to S^{n-1}. The motivation for such a definition is clear if we consider the two-dimensional case. A function $f = f(x_1, x_2)$ restricted to S^1 can be written in the form $a\sin(m\theta)+b\cos(m\theta)$ if and only if f is harmonic and homogeneous of degree m (see [4], Prop 3.1.1). If f is a series of harmonic homogeneous polynomials, then its restriction to S^1 is a series of trigonometric terms like those above. Hence, this definition generalizes the classical Fourier series to multivariable functions defined on the sphere. Define the following vector spaces over $\mathbb{R}$.

H_m^n: The space of spherical harmonics that are obtained by restricting a harmonic, homogenous of degree m, polynomial in n variables to S^{n-1}.

H^n: The space of all finite sums of spherical harmonics of dimension n.

Both of the above spaces will be equiped with the inner product

$$\langle H, G\rangle := \int_{S^{n-1}} G(\xi)H(\xi)d\xi.$$

The dimension of H_m^n, denoted $N(n, m)$, is finite and equals

$$N(n, m) = \frac{2m+n-2}{m+n-2}\binom{m+n-2}{n-2}$$

(see [4], Thm 3.1.4).

Examples

1. Constant functions on S^{n-1} are spherical harmonics of degree 0.

2. $f(x_1, x_2, ..., x_n) = c_1x_1 + c_2x_2 + ... + c_nx_n$ is a harmonic of degree 1 once

restricted to S^{n-1}.

3. As mentioned above, all spherical harmonics of dimension 2 are of the form $a\sin(m\theta) + b\cos(m\theta)$.

4. Let $f(x, y, z)$ be a harmonic function that is homogeneous of degree m. Using these well known parametric equations to restrict f to S^2

$$x = \sin(\phi)\cos(\theta), \qquad y = \sin(\phi)sin(\theta), \qquad z = \cos(\phi)$$

produces a spherical harmonic of dimension 3 which is now written as a trigonometric function of the variable $\xi = (\theta, \phi) \in S^2$.

Theorem 2.1: *(4, p.68). If* $G \in H_j^n$, $H \in H_k^n$ *with* $j \neq k$ *then* G *and* H *are orthogonal. In particular, since* $G = 1$ *is a spherical harmonic of degree 0, for every* $H \in H_k^n$, $k \neq 0$, *we have*

$$\int_{S^{n-1}} H(\xi)d\xi = 0.$$

A finite set of linearly independent harmonics of the same dimension, need not be mutually orthogonal. However, the Gram-Schmidt orthogonalization procedure can be used to produce a mutually orthogonal sequence with the same number of elements. Such a sequence is called a *standard sequence* of spherical harmonics. While spherical harmonics form a rather specific class of functions on the sphere, they can be used to approximate any continuous function on S^{n-1}.

Proposition 2.1: *(4, p.70). Let* f *be a continuous function on* S^{n-1}. *For every* $\epsilon > 0$ *there exist spherical harmonics* $H_0, H_1, \ldots, H_k$ *such that* $H_i \in$

H_i^n *and for every* $\xi \in S^{n-1}$

$$|f(\xi) - \sum_{i=0}^{k} H_i(\xi)| < \epsilon.$$

For instance, if L is a star body, $\|\cdot\|_L^{-n}$ is continuous on S^{n-1} and can be expanded as a series of spherical harmonics (much like a square integrable function on $[-\pi, \pi]$ can be written as a Fourier series).

$$\|\xi\|_L^{-n} \sim \sum_{m=0}^{\infty} \lambda_m H_m(\xi),$$

where $\{H_0, H_1, ...\}$ is a standard sequence of spherical harmonics with $H_m \in H_m^n$ $\forall m \in \mathbb{N}$. As is always the case with a Fourier series on a Hilbert space, if $f = \sum_{m \geq 0} \lambda_m H_m$ then these coefficients are given by

$$\lambda_m = \frac{\langle f, H_m \rangle}{\|H_m\|},$$

where $\|\cdot\|$ denotes the norm $\|f\| := \langle f, f \rangle^{1/2}$. Multiplying a spherical harmonic by a non-zero scalar does not change its degree or dimension. Given the above expression for f we will often denote $\lambda_m H_m$ by H_m as this still has degree m. We may also collect terms of the same order in the aforementioned series to produce the *condensed expansion of* f:

$$f \sim \sum_{m=0}^{\infty} Q_m.$$

Unless otherwise stated, the *harmonic expansion of* f will refer to the condensed harmonic expansion of f. We will conclude this section with one more familiar result about Fourier series which also holds in our current setting.

Proposition 2.2: *(4, p.73). Let* $F, G \in L_2(S^{n-1})$ *with harmonic expansions*

$$F \sim \sum_{m=0}^{\infty} Q_m, \qquad G \sim \sum_{m=0}^{\infty} R_m.$$

Then

$$\langle F, G \rangle = \sum_{m=0}^{\infty} \langle Q_m, R_m \rangle .$$

2.B Orthogonal Polynomials

Let $\zeta(x)$ be a real, non-decreasing, bounded function taking infinitely many values. Assume also that ζ is differentiable almost everywhere. We call ζ a *distribution function with momenta* or an *m-distribution* if the improper integral

$$\int_{-\infty}^{\infty} x^n \zeta'(x) dx$$

exists for all $n \in \mathbb{N}_0$.

In most cases, ζ will have compact support $[a, b]$ and will be differentiable on (a, b). The interval $[a, b]$ is referred to as the *interval of orthogonality* and $\int_{-\infty}^{\infty}$ in the definition above can be replaced with $\int_a^b$.

The term $\zeta'(x)$ is referred to as a *weight factor* and we denote

$$d\zeta(x) := \zeta'(x) dx$$

and for a polynomial $p(x)$

$$\int_{-\infty}^{\infty} p(x) d\zeta(x) := \int_{-\infty}^{\infty} p(x) \zeta'(x) dx.$$

The term $d\zeta$ will also be referred to as an m-distribution.

Theorem 2.2: *(2, p.13). For any m-distribution* $d\zeta$ *there exists a sequence of polynomials* $\{p_n(x)\}_{n\geq 0}$ *with the following properties.*

a) $p_n(x)$ *has degree* n.

b) The leading coefficient of p_n *is positive.*

c) $\forall n, k \in \mathbb{N}$ *we have* $\int\limits_{-\infty}^{\infty} p_n(x)p_k(x)d\zeta(x) = 0$ *if and only if* $k \neq n$.

This sequence of polynomials is unique up to scalar multiplication. That is, if $\{p_n(x)\}_{n\geq 0}$ and $\{q_n(x)\}_{n\geq 0}$ satisfy the above conditions, then there exists a non-zero sequence of scalars $\{\lambda_n\}_{n\geq 0}$, such that $p_n(x) = \lambda_n q_n(x)$ for all $n \in \mathbb{N}_0$. For an appropriate choice of scalars, condition c) can be replaced with a more specific condition.

c*) $\forall n, k \in \mathbb{N}$ one has

$$\int\limits_{-\infty}^{\infty} p_n(x)p_k(x)d\zeta(x) = \begin{cases} 1 & \text{for } n = k \\ 0 & \text{for } n \neq k. \end{cases}$$

There exists a unique family of polynomials satisfying a), b), and c*) which will be denoted $p_n(d\zeta, x)$, *"the family of orthogonal polynomials induced by* $d\zeta$*"*. We will make use of some important results involving the roots of $p_n(d\zeta, x)$ (see Theorem 2.3 and 2.4). Since the roots of a polynomial are not effected by scalling, these results hold for any family of polynomials satisfying a), b), and c), but not necessarily c*).

Examples

1. The Chebyshev polynomials (of the first kind) are given explicitly by the formula $p_n(x) := \cos(n \arccos(x))$ for all $n \in \mathbb{N}_0$

$$\{p_0(x), p_1(x), p_2(x), ...\} = \{1, x, 2x^2 - 1, 4x^3 - 3x, 8x^4 - 8x^2 + 1, ...\}.$$

This family is orthogonal on the interval [-1,1] with weight factor $\frac{1}{\sqrt{1-x^2}}$. Hence, $\zeta(x) = \arcsin(x)$ is the m-distribution associated to this family of polynomials (see [2], p.34).

2. The classical *Legendre Polynomials* can be defined by the relation

$$P_n(x) := \frac{1}{2^n n!} \frac{d^n}{dx^n} [(x^2 - 1)^n]$$

These polynomials are also orthogonal on the interval [-1,1] with a weight factor of 1 (see [2], p.35,36).

To be consistant with the definition at the beginning of this section, the m-distribution associated to each of the examples above is extended to $\mathbb{R}$ by defining them to be zero outside of the interval [-1,1]. Thus, these m-distributions are differentiable on $\mathbb{R}$ except ± 1. I will conclude this section with two deep results that will play an important role in the next chapter.

Theorem 2.3: *(2, p.17). The roots of $p_n(d\zeta, x)$ are simple and are contained in the interval of orthogonality.*

Theorem 2.4: *(2, p.130). Suppose the roots of $p_n(d\zeta, x)$ lie in the interval $[-1, 1]$ and that $\zeta(x)$ is differentiable on $(-1, 1)$. Let $N(\theta_1, \theta_2, n)$ denote the number of roots of $p_n(x)$ in the interval $[\cos(\theta_2), \cos(\theta_1)]$. Then*

$$\lim_{n \to \infty} \frac{1}{n} N(\theta_1, \theta_2, n) = \frac{\theta_2 - \theta_1}{\pi}.$$

In particular, this result implies that if a family of orthogonal polynomials is supported in [-1,1] its roots form a dense subset of [-1,1].

2.C Legendre Polynomials

In this section, we generalize the definition of the Legendre Polynomials given above and state some necessary properties of these polynomials as well as the well known Funk-Hecke Theorem.

Theorem 2.5: *(4, p.80). Let $n \geq 2$ and $m \geq 0$. There exists a unique polynomial P_m^n with the following property. If $\{H_1, ...H_N\}$ is any orthonormal basis of H_m^n then*

$$\sum_{i=1}^{N} H_i(u)H_i(v) = \frac{N}{|S^{n-1}|} P_m^n(u \cdot v)$$

for all $u, v \in S^{n-1}$, where $|S^{n-1}|$ refers to the $(n-1)$-dimensional surface area of S^{n-1}. Furthermore, for every fixed $v \in S^{n-1}$, $P_m^n(u \cdot v)$ is an n-dimensional spherical harmonic of degree m.

P_m^n is called the *Legendre polynomial of dimension n and degree m* and has a much more explicit definition.

Proposition 2.3: *(4, p.84). The Legendre polynomials have the property that*

$$P_m^n(t) = \frac{(-1)^m}{2^m(\vartheta+1)(\vartheta+2)\cdots(\vartheta+m)}(1-t^2)^{-\vartheta}\frac{d^m}{dt^m}\big((1-t^2)^{\vartheta+m}\big),$$

where $\vartheta := \frac{n-3}{2}$ and the coefficient of $(1-t^2)^{-\vartheta}$ is assumed to be 1 if $m = 0$.

The Chebyshev and classical Legendre polynomials discussed in the last section are special cases of the above formula corresponding to $n = 2$ and $n = 3$ respectively. Also note that the above formula is defined at $t = \pm 1$, since the term $(1-t^2)^{-\vartheta}$ is multiplied by the mth derivative of $(1-t^2)^{\vartheta+m}$; an expression containing the factor $(1-t^2)^\vartheta$. In fact, the next lemma completely characterizes these polynomials at 0 and 1.

Lemma 2.1: *(4, p.82,85). For all $n \in \mathbb{N}$, the Legendre polynomials have the following properties:*

a)$P_m^n(1) = 1$ where $0 \leq m \leq N(n,m)$.

b) If m is odd, then $P_m^n(0) = 0$, and if $m \neq 0$ is even, then

$$P_m^n(0) = (-1)^{m/2} \frac{1 \cdot 3 \cdots (m-1)}{(n-1)(n+1) \cdots (n+m-3)},$$

where $P_0^n = 1$.

The most important property of the Legendre polynomials is that they are orthogonal on [-1,1] with a weight factor of $(1-t^2)^\vartheta$. Specifically:

$$[P_j^n, P_k^n] := \int_{-1}^{1} P_j^n(t) P_k^n(t)(1-t^2)^\vartheta dt = \delta_{j,k} \frac{|S^{n-1}|}{|S^{n-2}|N(n,k)}$$

(see [4], Prop 3.3.6).

Once again, we don't require $[P_j^n, P_j^n] = 1$ only that $[P_j^n, P_j^n] \neq 0$ for all $j \in \mathbb{N}_0$, $n \in \mathbb{N}$. By the remarks following Theorem 2.2, this latter condition implies that $P_k^n(t)$ is, up to a non-zero constant, "the" family of orthogonal polynomials corresponding to the weight factor $(1-t^2)^\vartheta$.

As we have seen, the Legendre polynomials of dimension 2 and 3 are well known and are usually simplier to work with. Hence, the following result is of use to us as it shows that any Legendre polynomial can be reduced to one of dimension 2 or 3 by passing to derivatives.

Lemma 2.2: *(4, p.87). If $j \geq 0$ and $m \geq -1$, then*

$$\frac{d^j}{dt^j} P^n_{m+j}(t) = c_{n,m,j} P^{n+2j}_m(t),$$

where

$$c_{n,m,j} = \tfrac{N(n+2j,m)}{N(n,m+j)} n(n+2) \cdots (n+2j-2) \qquad (c_{n,-1,j} = 0, c_{n,m,0} = 1).$$

Now to state the most important result in this chapter.

Theorem 2.6: *(Funk-Hecke)[4, p.98]. Let Φ be an integrable function on [-1,1] and $H \in H^n_m$. For every fixed $u \in S^{n-1}$, $\Phi(\langle u, v \rangle)$ is an integrable function of v on S^{n-1} and*

$$\int_{S^{n-1}} \Phi(\langle u, v \rangle) H(v) dv = \beta_{n,m}(\Phi) H(u)$$

with

$$\beta_{n,m}(\Phi) := |S^{n-2}| \int_{-1}^{1} \Phi(t) P^n_m(t) (1-t^2)^{\vartheta} dt.$$

The proof given in [4] also assumes that Φ is bounded, a condition that can be relaxed in certain cases. For instance, if Φ is the pointwise limit of an increasing sequence of bounded, non-negative functions, then Theorem 2.6 follows from the Monotone Convergence Theorem.

Notice that if we set $\Phi(t) = 1$, then $\beta_{n,m}(\Phi) = 0$ for all $m \geq 1$ since $\{P^n_m\}_{m \geq 0}$ are mutually orthogonal on [-1,1] with a weight factor of $(1-t^2)^{\vartheta}$. This is consistent with the second part of Theorem 2.1. Suppose now that we only wish to integrate a harmonic H over a portion of S^{n-1}. We could choose Φ as follows. For some $c \in (-1, 1)$

$$\Phi(t) := \begin{cases} 1 & \text{for } t \geq c, \\ 0 & \text{for } t < c. \end{cases}$$

Then $\int_{S^{n-1}} \Phi(\langle u, v \rangle) H(v) dv$ is the integral of H over the spherical cap $\{v \in S^{n-1} : \langle u, v \rangle \geq c\}$. By Funk-Hecke, this otherwise difficult surface integral is nothing more than $H(u)$ multiplied by the much simpler integral

$$|S^{n-2}| \int_{-1}^{1} \Phi(t) P_m^n(t)(1-t^2)^{\vartheta} dt = |S^{n-2}| \int_{c}^{1} P_m^n(t)(1-t^2)^{\vartheta} dt.$$

This formula will be used in the next chapter.

Chapter 3

The Volume Problem

A common goal of Geometric Tomography is to find conditions which uniquely determine a convex body. That is, if K and L are convex bodies satisfying a certain condition, then $K = L$. Clearly, a body is uniquely determined by its radial function, and if the body is origin-symmetric, it is uniquely determined by the Fourier transform of its radial function. By Corollary 1.1, an origin-symmetric convex body is also uniquely determined by the $(n-1)$-dimensional volume of its central hyperplane sections. In the next two chapters, we will provide sufficient conditions, involving a cone, for uniquely determining a convex body. For similar results see [8] and [9].

3.A Statement of the Problem

Let $\alpha \in (0, \pi/2)$. The "solid" n-dimensional cone with central angle α, central axis given by the unit vector ξ, and vertex at the origin is defined as follows:

$$C_{\alpha,\xi} = \{x \in \mathbb{R}^n : \left\langle \tfrac{x}{|x|_2}, \xi \right\rangle \geq \cos(\alpha)\}.$$

Notice that this is not a truncated cone: it is unbounded and has infinite "height" and volume. However, we are interested in the volume of intersection of $C_{\alpha,\xi}$ and a convex body K (recall that K is assumed to contain the origin as an interior point). Clearly this volume is finite. The answer to the following question, which we will simply refer to as the *Volume Problem*, is the goal of this chapter.

The Volume Problem.

Given two convex bodies K and L, and an angle $\alpha \in (0, \frac{\pi}{2})$, suppose that

$$Vol_n(K \cap C_{\alpha,\xi}) = Vol_n(L \cap C_{\alpha,\xi}) \qquad \forall \xi \in S^{n-1}.$$

Does this imply $K = L$?

In the above problem, ξ varies but α is fixed. Also note that we are comparing the volume of n-dimensional subsets of K and L as opposed to $(n-1)$-dimensional hyperplane sections which are mentioned throughout Chapter 1. To produce a counterexample to the volume problem, we will use a standard argument (see [5], p.96 and Lemma 5.16). Suppose K is a 2-smooth convex body whose boundary has strictly positive curvature, and f is a twice continuously differentiable function on S^{n-1}. Let $p \in \mathbb{R}\backslash\{0\}$. Given $\epsilon > 0$ we define a new body L by its Minkowski functional

$$\|x\|_L^p := \|x\|_K^p + \epsilon f(x/|x|_2)|x|_2^p.$$

Choosing ϵ so that L also has strictly positive curvature will imply that L is convex.

If $K \subset \mathbb{R}^2$ is a body whose boundary is the parametric curve $r = r(t)$, $a \leq t \leq b$, then its curvature is given by the well known formula

$$\kappa(t) = \frac{|r'(t) \times r''(t)|}{|r'(t)|^3}.$$

It is more convenient to work with polar curves so proceed as follows. If the boundary curve mentioned above is instead defined by the polar equation $r = \rho(\theta)$ then it has parametric equations

$$r(\theta) := \langle x(\theta), y(\theta), z(\theta) \rangle = \langle \rho(\theta)\cos(\theta), \rho(\theta)\sin(\theta), 0 \rangle.$$

Using these equations in the curvature formula produces the polar equation for curvature of a plane curve

$$\kappa(\theta) = \frac{\rho^2 + 2(\rho')^2 - \rho\rho''}{[(\rho')^2 + \rho^2]^{3/2}}$$

(see [3], p.25).

Lemmas 3.1 and 4.1 deal with bodies in $\mathbb{R}^2$. Abusing notation slightly, $\rho_K(\theta)$ will refer to the polar equation of the boundary of K. Also, the direction ξ will refer to an angle in $[0, 2\pi)$ for these two lemmas only.

Lemma 3.1: *The answer to the volume problem in $\mathbb{R}^2$ is negative if α is a rational multiple of π and affirmative if α is an irrational multiple of π.1*

Proof. If α is a rational divisor of π, that is, $\alpha = \frac{j}{k}\pi$ for some integers j and k ($k \neq 0$), we may produce a counterexample as follows. Let K be the unit disk, $\rho_K(\theta) = 1$ and define the body L by

$$\rho_L(\theta) := \sqrt{1 + \epsilon \cos(k\theta)}.$$

Using the equation for the area inside a polar curve ($A = \frac{1}{2}\int_{\theta_1}^{\theta_2} r^2(\theta)d\theta$), we have

$$\begin{aligned} Vol_n(L \cap C_{\alpha,\xi}) &= \frac{1}{2}\int_{\xi-\alpha}^{\xi+\alpha} \left(\sqrt{1+\epsilon\cos(k\theta)}\right)^2 d\theta \\ &= \frac{1}{2}\left(\int_{\xi-\alpha}^{\xi+\alpha} d\theta + \int_{\xi-\alpha}^{\xi+\alpha} \epsilon\cos(k\theta)d\theta\right) \\ &= \frac{1}{2}\int_{\xi-\alpha}^{\xi+\alpha} d\theta + 0 = Vol_n(K \cap C_{\alpha,\xi}) \end{aligned}$$

for all $\xi \in [0, 2\pi)$. Choosing $\epsilon < \frac{1}{2k^2}$, the curvature formula proceeding this lemma shows that the boundaries of L and K have strictly positive curvature. Thus, L and K are convex.

Now suppose that α is not a rational divisor of π. The conditions of the Volume Problem imply the existence of $\theta_0 \in [0, 2\pi)$ such that $\rho_K(\theta_0) = \rho_L(\theta_0)$. Indeed, by the continuity of ρ_K and ρ_L, if such an angle did not exist then either $\rho_K > \rho_L$ or $\rho_L > \rho_K$. Either of these conditions contradict $Vol_n(K \cap C_{\alpha,\xi}) = Vol_n(L \cap C_{\alpha,\xi})$. Differentiating $\int_a^{a+2\alpha} \rho_K^2(\theta)d\theta = \int_a^{a+2\alpha} \rho_L^2(\theta)d\theta$ with respect to a gives

$$\rho_L^2(\theta + 2\alpha) - \rho_L^2(\theta) = \rho_K^2(\theta + 2\alpha) - \rho_K^2(\theta) \qquad \forall \theta \in [0, 2\pi)$$

and, in particular, $\rho_K(\theta_0 + 2\alpha) = \rho_L(\theta_0 + 2\alpha)$. Since α is not a rational divisor of π,

$$\{2m\alpha + \theta_0 : m \in \mathbb{Z}\}$$

is an infinite (and dense) subset of S^1 on which $\rho_K = \rho_L$. By continuity, $\rho_k = \rho_L$ on all of S^{n-1} and we have that $K = L$. $\square$

3.B The Affirmative Answer

Now we pass to the n-dimensional case. We begin by deriving an expression for $Vol_n(C_{\alpha,\xi} \cap K)$. Fix $\xi \in S^{n-1}$. For $x \in \mathbb{R}^n \backslash \xi^{\perp}$, $x \in C_{\alpha,\xi}$ if and only if $\langle x, \xi \rangle \geq |x|_2 \cos(\alpha) \geq 0$ which is equivalent to $1 \geq \frac{|x|_2 \cos(\alpha)}{\langle x,\xi \rangle} \geq 0$. Let ψ be the indicator function of the interval [0,1]. Then

$$
\begin{aligned}
Vol_n(C_{\alpha,\xi} \cap K) &= \int_K \psi\left(\frac{|x|_2 \cos(\alpha)}{\langle x, \xi \rangle}\right) dx \\
&= \int_{\mathbb{R}^n} \psi\left(\frac{|x|_2 \cos(\alpha)}{\langle x, \xi \rangle}\right) \chi(\|x\|_K) dx \\
&= \int_{S^{n-1}} \int_0^{\infty} r^{n-1} \psi\left(\frac{r \cos(\alpha)}{r \langle \theta, \xi \rangle}\right) \chi(r\|\theta\|_K) dr d\theta \\
&= \int_{S^{n-1}} \psi\left(\frac{\cos(\alpha)}{\langle \theta, \xi \rangle}\right) \int_0^{\frac{1}{\|\theta\|_K}} r^{n-1} dr d\theta \\
&= \frac{1}{n} \int_{S^{n-1}} \psi\left(\frac{\cos(\alpha)}{\langle \theta, \xi \rangle}\right) \|\theta\|_K^{-n} d\theta \\
&= \frac{1}{n} \int_{S^{n-1}} \Phi(\langle \theta, \xi \rangle) \rho_K^n(\theta) d\theta,
\end{aligned}
$$

where $\Phi(t) = \psi\left(\frac{cos(\alpha)}{t}\right) = \begin{cases} 1 & \text{for } t \geq \cos(\alpha) \\ 0 & \text{for } t < \cos(\alpha) \end{cases}$.

Since $\rho_K^n(\theta)$ is continuous on S^{n-1} we can expand it as a series of spherical harmonics

$$\rho_K^n(\theta) \sim \sum_{m \geq 0} H_m,$$

where m is the degree of the harmonic. Applying this to the integral above yields

$$\int_{S^{n-1}} \Phi(\langle\theta,\xi\rangle)\rho_K^n(\theta)d\theta \sim \sum_{m\geq 0}\int_{S^{n-1}} \Phi(\langle\theta,\xi\rangle)H_m(\theta)d\theta.$$

Applying the Funk-Hecke Theorem to each integral on the right hand side, we obtain

$$\int_{S^{n-1}} \Phi(\langle\theta,\xi\rangle)\rho_K^n(\theta)d\theta \sim \sum_{m\geq 0}\beta_{n,m}(\Phi)H_m(\xi),$$

where

$$\beta_{n,m}(\Phi) = |S^{n-2}|\int_{-1}^{1} \Phi(x)P_m^n(x)(1-x^2)^{\vartheta}ds = |S^{n-2}|\int_{\cos(\alpha)}^{1} P_m^n(x)(1-x^2)^{\vartheta}dx$$

and as before, $P_m^n(x)$ is the Legendre Polynomial of degree m and dimension n. This provides the following,

$$nVol_n(C_{\alpha,\xi}\cap K) \sim \sum_{m\geq 0}\beta_{n,m}(\Phi)H_m(\xi).$$

Thus, if L is another convex body satisfying the condition

$$Vol_n(C_{\xi,\alpha}\cap K) = Vol_n(C_{\xi,\alpha}\cap L) \qquad \forall\xi\in S^{n-1}$$

and if we write $\rho_L^n(\theta)$ as its harmonic expansion, $\rho_L^n(\theta)\sim\sum_{m\geq 0}G_m$, then

$$\sum_{m\geq 0}\beta_{n,m}(\Phi)H_m(\xi) = \sum_{m\geq 0}\beta_{n,m}(\Phi)G_m(\xi) \qquad \forall\xi\in S^{n-1}.$$

By the independence of spherical harmonics of different degrees, we have that $\beta_{n,m}(\Phi)H_m(\xi) = \beta_{n,m}(\Phi)G_m(\xi)$ for all $\xi\in S^{n-1}$ and $m\in\mathbb{N}_0$. If $\beta_{n,m}\neq 0$

$\forall m \in \mathbb{N}_0$ then

$$\rho_K^n(\theta) \sim \sum_{m\geq 0} H_m \sim \sum_{m\geq 0} G_m \sim \rho_L^n(\theta) \quad \Rightarrow K = L.$$

On the other hand, if $\beta_{n,k} = 0$ for some $k \in \mathbb{N}_0$, then the condition $\beta_{n,k}(\Phi)H_k(\xi) = \beta_{n,k}(\Phi)G_k(\xi)$ does not imply $H_k(\xi) = G_k(\xi)$. In fact, if $H_k(\xi) \neq G_k(\xi)$ we have

$$\sum_{m\geq 0} \beta_{n,m}(\Phi)H_m(\xi) \neq \sum_{m\geq 0} \beta_{n,m}(\Phi)G_m(\xi) \quad \Rightarrow \rho_K^n(\theta) \neq \rho_L^n(\theta) \Rightarrow K \neq L.$$

In particular, we may take K to be the unit sphere (so that $\rho_K = 1$) and define $\rho_L := (1 + \epsilon H_k^n)^{1/n}$. By the argument prior to Lemma 3.1, L is convex if ϵ is sufficiently small. $\rho_K^n(\theta)$ is constant so its spherical harmonic expansion only involves the constant term $H_0^n = 1$ while $\rho_L^n(\theta) = 1 + \epsilon H_k^n = H_0^n + \epsilon H_k^n$. Using the expression for $Vol_n(C_{\alpha,\xi} \cap L)$ above, we observe

$$\begin{aligned} nVol_n(C_{\alpha,\xi} \cap L) &\sim \sum_{m\geq 0} \beta_{n,m}(\Phi)H_m(\xi) \\ &= \beta_{n,0}(\Phi)H_0^n(\xi) + \beta_{n,k}(\Phi)H_k^n(\xi) \\ &= \beta_{n,0}(\Phi)H_0^n(\xi) + 0 \\ &\sim nVol_n(C_{\alpha,\xi} \cap K). \end{aligned}$$

This gives us our main result.

Theorem 3.1: *The answer to the volume problem in* $\mathbb{R}^n$ *is affirmative if and only if*

$$\beta_{n,m}(\Phi) = \int_{\cos(\alpha)}^{1} P_m^n(x)(1-x^2)^{\vartheta} dx$$

is non-zero for all $m \in \mathbb{N}_0$.

Since $P_0^n(x) = 1$, it is clear from the integral above that $\beta_{n,0} > 0$ for all $n \in \mathbb{N}$. Thus, "$\mathbb{N}_0$" in the final line of the theorem may be replaced with "$\mathbb{N}$". Also notice that this theorem is consistent with the two dimensional result we had before (Lemma 3.1).

Corollary 3.1.1: The answer to the volume problem in $\mathbb{R}^2$ is affirmative if and only if α is an irrational multiple of π.

Proof. $P_m^2(x)$ is the class of Chebyshev Polynomials defined by $P_m^2(x) := \cos(m \arccos(x))$. Using the substitution $t := \arccos(x)$ to compute the terms $\beta_{2,m}(\Phi)$ with $m \in \mathbb{N}$, we observe

$$\beta_{2,m}(\Phi) = \int_{\cos(\alpha)}^{1} P_m^2(x)(1-x^2)^{\vartheta} dx = \int_{\cos(\alpha)}^{1} \cos(m \arccos(x)) \frac{1}{\sqrt{1-x^2}} dx$$
$$= \frac{1}{m} \sin(m\alpha).$$

If α is an irrational multiple of π, this is clearly non-zero for all $m \in \mathbb{N}$. If $\alpha = \frac{p}{q}\pi$ for integers p and q, $q \neq 0$, then $\sin(m\alpha) = 0$ when $m = q$. □

Theorem 3.1 provides us with some insight behind the proof of Lemma 3.1. As mentioned in the previous chapter, every spherical harmonic of degree m in $\mathbb{R}^2$ is of the form $a\cos(m\theta) + b\sin(m\theta)$. In Lemma 3.1, the term $\cos(k\theta)$ is a spherical harmonic such that $\beta_{2,k}(\Phi) = 0$ and the proof of the negative part of Lemma 3.1 is essentially the same as the argument preceding Theorem 3.2.

By Proposition 2.3, the Legendre polynomials can be written as

$$P_m^n(x) = \frac{(-1)^m}{2^m(\vartheta+1)(\vartheta+2)\cdots(\vartheta+m)}(1-x^2)^{-\vartheta} \frac{d^m}{dx^m}\left((1-x^2)^{\vartheta+m}\right),$$

where $\vartheta = \frac{n-3}{2}$. Using this in our expression for $\beta_{n,m}$

$$\begin{aligned}
\beta_{n,m}(\Phi) &= |S^{n-2}| \int\limits_{\cos(\alpha)}^{1} P_m^n(x)(1-x^2)^{\vartheta} dx \\
&= \frac{(-1)^m |S^{n-2}|}{2^n(\vartheta+1)(\vartheta+2)\cdot\dots\cdot(\vartheta+m)} \int\limits_{\cos(\alpha)}^{1} \frac{d^m}{ds^m}(1-x^2)^{\vartheta+m} dx \\
&= \frac{(-1)^m |S^{n-2}|}{2^m(\vartheta+1)(\vartheta+2)\cdot\dots\cdot(\vartheta+m)} \frac{d^{m-1}}{dx^{n-1}}(1-x^2)^{\vartheta+m} \Big|_{x=\cos(\alpha).}^{x=1}
\end{aligned}$$

With the following lemma, it is only necessary to evaluate this function at the left endpoint.

Lemma 3.2:

$$\frac{d^{m-1}}{dx^{m-1}}(1-x^2)^{\vartheta+m} \Big|_{x=\pm 1} = 0.$$

Proof. Using the binomial derivative formula $(fg)^{(m)} = \sum_{l=0}^{m} \binom{m}{l} f^{(m-l)} g^{(l)}$ with $f(x) = (1-x)^{m+\vartheta}$, $g(x) = (1+x)^{m+\vartheta}$

$$\begin{aligned}
[(1-x^2)^{m+\vartheta}]^{(m-1)} &= [(1-x)^{m+\vartheta}(1+x)^{m+\vartheta}]^{(m-1)} \\
&= \sum_{l=0}^{m-1} \binom{m-1}{l} [(1-x)^{m+\vartheta}]^{(m-l-1)} [(1+x)^{m+\vartheta}]^{(l)}.
\end{aligned}$$

Since both l and $m-l-1$ are less than $\vartheta + m$ each term in this expansion contains the factor $(1+x)(1-x)$. Thus each term in this expansion vanishes at $x = \pm 1$. $\square$

Hence the answer to the Volume Problem is affirmative if and only if

$$\frac{d^{m-1}}{dx^{m-1}}(1-x^2)^{\vartheta+m}\Big|_{x=\cos(\alpha)} \neq 0$$

$\forall m \in \mathbb{N}$. Recall that a complex number is called *algebraic* if it is the root of a polynomial having rational coefficients. A number which is not algebraic is called *transcendental*. It is known from analysis that if α is a nonzero algebraic number then $\cos(\alpha)$ is transcendental (see, e.g. [1], Thm 1.4).

Theorem 3.2: *If α is a nonzero algebraic number, than the answer to the volume problem is affirmative.*

Proof. If n is odd then $\vartheta = \frac{n-3}{2} \in \mathbb{N}_0$ and $\frac{d^{m-1}}{dx^{m-1}}(1-x^2)^{\vartheta+m}$ is a polynomial with integer coefficients so the result follows from the fact mentioned above. Suppose n is even. Refering to the binomial derivative formula in Lemma 3.2, we get

$$[(1-x^2)^{m+\vartheta}]^{(m-1)} = \sum_{l=0}^{m-1} \binom{m-1}{l} [(1-x)^{m+\vartheta}]^{(m-l-1)}[(1+x)^{m+\vartheta}]^{(l)}.$$

Every term in this expansion contains the factor $(1-x)^{\frac{j}{2}}(1+x)^{\frac{k}{2}}$ for some odd numbers j and k. We can thus remove the factor $(1-x)^{\frac{1}{2}}(1+x)^{\frac{1}{2}}$ producing

$$[(1-x^2)^{m+\vartheta}]^{(m-1)} = (1-x)^{\frac{1}{2}}(1+x)^{\frac{1}{2}}Q(x),$$

where $Q(x)$ is some polynomial with integer coefficients. Since $\cos\alpha \neq \pm 1$, and $\cos(\alpha)$ is not a root of $Q(x)$, $[(1-x^2)^{n+\vartheta}]^{(n-1)}|_{x=\cos\alpha} \neq 0$. □

The condition in Theorem 3.2 is sufficient but not necessary. If a particular value of α provides a negative answer to the Volume Problem, then $\cos(\alpha)$

must be an algebraic number. This means there are only countably many possibilities for the value of α. The set of $\alpha \in (0, \frac{\pi}{2})$ providing an affirmative answer to the volume problem is thus an uncountable set, containing the algebraic numbers as a proper subset.

3.C The Negative Answer

By the comments following Lemma 3.2, the answer to the volume problem is negative if and only if there exists some $m \in \mathbb{N}$ such that

$$\frac{d^{m-1}}{dx^{m-1}}(1-x^2)^{\vartheta+m}\Big|_{x=\cos(\alpha)} = 0.$$

In this case, K and L could be different bodies whose radial functions differ by a spherical harmonic of degree m and these bodies would still satisfy

$$Vol_n(K \cap C_{\alpha,\xi}) = Vol_n(L \cap C_{\alpha,\xi}) \qquad \forall \xi \in S^{n-1}.$$

To fully answer the negative part of the volume problem would involve finding all roots of the family of polynomials listed above. Since the degree of these polynomials becomes arbitrarily large as m increases, this is not possible in general. Even the classic Legendre polynomials (which are a special case of P_m^n with $n = 3$) do not have easily determined roots. The Chebyshev polynomials (P_m^n with $n = 2$) have roots given by the expression

$$x_{k,n} = \cos\left(\frac{2k-1}{2n}\pi\right) \qquad k = 1, 2, ..., n$$

(see [7], Section 2.2). Thus, it was possible to completely solve the volume problem in $\mathbb{R}^2$. If a particular value of $\alpha \in (0, \frac{\pi}{2})$ produces a negative answer to the volume problem, it must be a root of one of the countably many polynomials given above. Thus, there are at most a countable number of α for which this is the case. It will be shown in this section that the set of α corresponding to a negative answer of the volume problem is a countable dense subset of the interval $(0, \frac{\pi}{2})$.

Proposition 3.1: *If* $q_m(x) := \frac{d^{m-1}}{dx^{m-1}}(1-x^2)^{\vartheta+m}$, *then* $\{q_m(x)\}_{m\geq 0}$ *is orthogonal with a weight factor of* $(1-x^2)^{-(\vartheta+1)}$.

Proof. Recall that $\{P_m^n(x)\}_{m\geq 0}$ is orthogonal on $[-1, 1]$ with weight factor $(1-x^2)^{\vartheta}$ for all $n \in \mathbb{N}$ (see the comment after Lemma 2.1). So if $j \neq k$

$$\begin{aligned} 0 &= \int_{-1}^{1} P_j^n(x) P_k^n(x)(1-x^2)^{\vartheta} dx \\ &= c(j)c(k) \int_{-1}^{1} (1-x^2)^{-\vartheta} \left(\frac{d^j}{dx^j}(1-x^2)^{\vartheta+j} \right) \left(\frac{d^k}{dx^k}(1-x^2)^{\vartheta+k} \right) dx, \end{aligned}$$

where $c(k) := \frac{(-1)^k}{2^k(\vartheta+1)(\vartheta+2)\cdots(\vartheta+k)}$.

Since this holds $\forall n \in \mathbb{N}$, we may replace n with $n+2$ (i.e. ϑ with $\vartheta+1$) getting

$$\begin{aligned} 0 &= \int_{-1}^{1} (1-x^2)^{-(\vartheta+1)} \left(\frac{d^j}{dx^j}(1-x^2)^{\vartheta+j+1} \right) \left(\frac{d^k}{dx^k}(1-x^2)^{\vartheta+k+1} \right) dx \\ &= \int_{-1}^{1} (1-x^2)^{-(\vartheta+1)} q_{j+1}(x) q_{k+1}(x) dx. \end{aligned}$$

This proves that $q_l(x)$ and $q_m(x)$ are orthogonal if $l \neq m$ and $l, m \geq 2$. It still remains to show that $q_1(x)$ (which equals $(1-x^2)^{\vartheta+1}$) is orthogonal to q_m when $m \neq 1$. Using the same reasoning as in Lemma 3.2 (since $m-2 < \vartheta+m$)

$$\begin{aligned}\int_{-1}^{1} q_1(x)q_m(x)(1-x^2)^{-(\vartheta+1)}dx &= \int_{-1}^{1} q_m(x)dx \\ &= \int_{-1}^{1} \frac{d^{m-1}}{dx^{m-1}}(1-x^2)^{\vartheta+m}dx \\ &= \frac{d^{m-2}}{dx^{m-2}}(1-x^2)^{\vartheta+m}|_{x=-1}^{x=1} = 0.\end{aligned}$$

Finally $\int_{-1}^{1}(1-x^2)^{-(\vartheta+1)}q_k(x)q_k(x)dx \neq 0 \; \forall n \in \mathbb{N}$, since the integrand is non-negative. This completes the proof. □

To be consistent with the given definition of orthogonal polynomials, $(1-x^2)^{-(\vartheta+1)}$ must be the derivative of an m-distribution. For every $n \in \mathbb{N}$, let $\zeta_n(x)$ be an antiderivative of $(1-x^2)^{-(\vartheta+1)}$. ζ_n is non-decreasing and has infinite range, since its derivative, $(1-x^2)^{-(\vartheta+1)}$, is strictly positive on the interval $(-1,1)$. So $\{q_m(x)\}_{m\geq 0}$ satisfy all the criteria needed to be called an orthogonal family of polynomials and we now apply the main result of Section 2.2.

Theorem 3.3: *The set of all $\alpha \in (0, \frac{\pi}{2})$ for which the answer to the volume problem is negative is dense in $(0, \frac{\pi}{2})$.*

Proof. We have established that $\{q_m(x)\}_{m\geq 0}$ is orthogonal on $[-1,1]$ which, by Theorem 2.3, implies that the roots of these polynomials lie in the same interval. We thus have the required conditions to use Theorem 2.4. For an arbitrary subinterval $[\cos(\theta_2), \cos(\theta_1)]$, there exists $m \in \mathbb{N}$ sufficiently large so that $N(\theta_1, \theta_2, m) \geq 1$. Hence, the roots of $\{q_m(x)\}_{m\geq 0}$ are dense in $[-1, 1]$ and the result follows from the continuity of the arccos function. $\square$

Chapter 4

The Surface Area Problem

In this chapter we consider a problem similar to the one in the previous chapter, but involving surface area instead of volume. The strategy to solving this problem will be the same. Derive a formula for the quantity that two bodies, K and L, are assumed to share, express this quantity in terms of the integral over S^{n-1}, then use the Funk-Hecke Theorem and proceed as before.

4.A Statement of the Problem

The "hollow" cone is the $(n-1)$ dimensional surface of the solid cone described before. Specifically, if it has its vertex at the origin, a central axis in the direction $\xi \in S^{n-1}$, and a central angle of α it is defined as follows:

$$\tilde{C}_{\alpha,\xi} = \{x \in \mathbb{R}^n : \left\langle \frac{x}{|x|_2}, \xi \right\rangle = \cos(\alpha)\}.$$

We may ask an analogous question to the one stated at the begining of Chapter 3.

The Surface Area Problem.

Given two convex bodies K and L, and an angle $\alpha \in (0, \frac{\pi}{2})$, suppose that

$$Vol_{n-1}(K \cap \tilde{C}_{\alpha,\xi}) = Vol_{n-1}(L \cap \tilde{C}_{\alpha,\xi}) \qquad \forall \xi \in S^{n-1}.$$

Does this imply $K = L$?

As is the case with most of the results in Chapter 1, the problem aims to uniquely determine a body by subsets of dimension $n-1$. However, $\tilde{C}_{\alpha,\xi}$ is not a "flat" hyperplane section like those we dealt with before. A proof very similar to that of Lemma 3.1 shows that the Volume and Surface Area Problems have the same answer in $\mathbb{R}^2$.

Lemma 4.1: *The answer to the surface area problem in $\mathbb{R}^2$ is affirmative if and only if α is an irrational multiple of π.*

Proof. Let $\rho(\theta)$ denote the polar equation for the boundary of a body in $\mathbb{R}^2$. The conditions of the Surface Area Problem imply that

$$\rho_K(\theta) + \rho_K(\theta + 2\alpha) = \rho_L(\theta) + \rho_L(\theta + 2\alpha)$$

for all $\theta \in [0, 2\pi)$. This implies that there exists $\theta_0 \in [0, 2\pi)$ such that $\rho_K(\theta_0) = \rho_L(\theta_0)$. Indeed, if such a θ_0 did not exist, then by the continuity of the radial functions, either $\rho_L > \rho_K$ or $\rho_L < \rho_K$. Either of these relations contradict the assumed equation above. The above equation then also establishes $\rho_K(\theta_0 + 2n\alpha) = \rho_L(\theta_0 + 2n\alpha)$ for all $n \in \mathbb{N}$. If α is an irrational multiple of π then, as was the case in the proof of Lemma 3.1, the previous relation produces an infinite set of angles, dense in the interval $[0, 2\pi)$, for which ρ_K and ρ_L are equal. By continuity of these radial functions, we have that $\rho_K = \rho_L$

on $[0, 2\pi)$ and $K = L$.

If $\alpha = \frac{m}{n}\pi$ for some $n, m \in \mathbb{Z}$, $n \neq 0$ then we can produce a counterexample as follows. Take K to be the unit disk and L to be the body whose boundary is the polar curve

$$\rho_L(\theta) := 1 + \epsilon \cos\left(\frac{n}{2m}\theta\right).$$

Then for every $\theta \in [0, 2\pi)$

$$\begin{aligned} Vol_{n-1}(L \cap \tilde{C}_{\alpha,\theta}) &= \rho_L(\theta - \alpha) + \rho_L(\theta + \alpha) \\ &= 1 + \epsilon \cos\left(\frac{n}{2m}\theta\right) + 1 + \epsilon \cos\left(\frac{n}{2m}\left(\theta + \frac{2m}{n}\pi\right)\right) \\ &= 1 + \epsilon \cos\left(\frac{n}{2m}\theta\right) + 1 - \epsilon \cos\left(\frac{n}{2m}\theta\right) \\ &= 1 + 1 \\ &= \rho_K(\theta) + \rho_K(\theta + 2\alpha) \\ &= Vol_{n-1}(K \cap \tilde{C}_{\alpha,\theta}). \end{aligned}$$

K is convex, and by the argument preceding Lemma 3.1, L is convex for sufficiently small ϵ. □

4.B The Affirmative Answer

In this section, we consider the higher dimensional case $n \geq 3$ (so $\vartheta \geq 0$). For a convex body K deriving an expression for $Vol_{n-1}(K \cap \tilde{C}_{\alpha,\xi})$ is more difficult than it was for $Vol_n(K \cap C_{\alpha,\xi})$. By projecting this surface onto the central hyperplane $\xi^{\perp}$, we are able to work with quantities involving the central hyperplane sections of K. Also, the following result shows that this quantity differs from $Vol_{n-1}(K \cap \tilde{C}_{\alpha,\xi})$ by a factor of $\sin(\alpha)$.

Define the *surface area measure*, $S(K, \cdot)$, of the convex body K as follows. For every Borel set $E \subset S^{n-1}$, $S(K, E)$ is the Lebesgue measure of the part of the boundary of K, where normal vectors belong to E.

Theorem 4.1: *(Cauchy Projection Formula)[3, p.361]. For $\xi \in S^{n-1}$, let $K|\xi^{\perp}$ denote the orthogonal projection of the n-dimensional convex body K onto the hyperplane $\xi^{\perp}$. Then for every $\xi \in S^{n-1}$,*

$$Vol_{n-1}\big(K|\xi^{\perp}\big) = \frac{1}{2}\int_{S^{n-1}} |\langle \xi, \theta \rangle| dS(K, \theta).$$

The factor of $\frac{1}{2}$ is in place because each point of the projection $K|\xi^{\perp}$ is covered twice by K. By definition, if $\theta \in S^{n-1} \cap \tilde{C}_{\alpha,\xi}$, then $\langle \theta, \xi \rangle = \cos(\alpha)$. So, if $\bar{x}$ is a unit normal vector to $\tilde{C}_{\alpha,\xi}$ then

$$|\langle \bar{x}, \xi \rangle| = \sin(\alpha).$$

Since $\tilde{C}_{\alpha,\xi}$ is an "uncapped" cone, the factor of $\frac{1}{2}$ in Theorem 4.1 is unnecessary. This gives us the following

$$Vol_{n-1}\big((K \cap \tilde{C}_{\alpha,\xi})|\xi^{\perp}\big) = \sin(\alpha)\int_{S^{n-1}} dS(\tilde{C}_{\alpha,\xi}, \theta) = \sin(\alpha)Vol_{n-1}\big(K \cap \tilde{C}_{\alpha,\xi}\big).$$

That is, $Vol_{n-1}(K \cap \tilde{C}_{\alpha,\xi})$ equals the area of the projection of this surface onto $\xi^{\perp}$, divided by the constant $\sin(\alpha)$. If $\theta \in S^{n-1} \cap \tilde{C}_{\alpha,\xi}$, then $\rho_K(\theta)$ is the radius of K in the direction of $\theta \in \tilde{C}_{\alpha,\xi}$. That is, the radius of K along the boundary of the cone. Projecting $\tilde{C}_{\alpha,\xi}$ onto the central hyperplane $\xi^{\perp}$, the projection has a radius of $\sin(\alpha)\rho_K(\theta)$ (once again, $\theta \in S^{n-1} \cap \tilde{C}_{\alpha,\xi}$). Since $\theta \in S^{n-1} \cap \tilde{C}_{\alpha,\xi}$, it can be written as

$$\theta = \xi \cos(\alpha) + \eta \sin(\alpha)$$

for some $\eta \in S^{n-1} \cap \xi^{\perp}$. The formula for surface area can now be written as an integral over a central hyperplane section:

$$\sin(\alpha) Vol_{n-1}\big(K \cap \tilde{C}_{\alpha,\xi}\big) = \frac{(\sin(\alpha))^{n-1}}{n-1} \int_{S^{n-1}\cap\xi^{\perp}} \rho_K^{n-1}(\xi \cos(\alpha) + \eta \sin(\alpha)) d\eta.$$

Finally, since this is the area of the projection, we can recover the area of the cone by dividing this expression by $\sin(\alpha)$.

$$Vol_{n-1}(K \cap \tilde{C}_{\alpha,\xi}) = \frac{(\sin(\alpha))^{n-2}}{n-1} \int_{S^{n-1}\cap\xi^{\perp}} \rho_K^{n-1}(\xi \cos(\alpha) + \eta \sin(\alpha)) d\eta.$$

Instead of finding the spherical harmonic expansion for the above function of ξ, we consider related functions, and then use a limiting argument. We will first need two lemmas. The proof of this first lemma is similar to that of [4], Lemma 1.3.1. Also see [7], p.1.

Lemma 4.2: *Let $f(\theta)$ be continuous on S^{n-1} and $\Phi(x)$ integrable on [-1,1]. Then*

$$\int_{S^{n-1}} \Phi(\langle \theta, \xi \rangle) f(\theta) d\theta = \int_{-1}^{1} \Phi(x) F_\xi(x) dx,$$

where

$$F_\xi(x) := (1 - x^2)^{\vartheta} \int_{S^{n-1}\cap\xi^{\perp}} f(x\xi + \sqrt{1 - x^2}\theta) d\theta.$$

Lemma 4.3: *Let f be continuous on $\mathbb{R}$ with compact support. Then*

$$\lim_{q\to 0^-} \frac{1}{\Gamma(-q)} \int_0^\infty t^{-1-q} f(t)dt = f(0).$$

Proof. Let $c > 0$ be such that $f(x) = 0$ on (c, ∞) and let $\epsilon > 0$. Since f is continuous, there exists $\delta > 0$ such that

$$\sup_{x\in(0,\delta)} \mid f(x) - f(0) \mid < \epsilon.$$

The integral can be broken up as follows:

$$\lim_{q\to 0^-} \frac{1}{\Gamma(-q)} \Big(\int_0^\delta t^{-1-q} f(t)dt + \int_\delta^c t^{-1-q} f(t)dt \Big).$$

The limit of the second integral is zero because the integrand is bounded on the interval (δ, c) and $\frac{1}{\Gamma(-q)}$ approaches zero as $q \to 0^-$. Using $(-q)\Gamma(-q) = \Gamma(1-q)$, it's easy to check that

$$\lim_{q\to 0^-} \frac{1}{\Gamma(-q)} \int_0^\delta t^{-1-q} dt = 1.$$

Observe that

$$\begin{aligned} \lim_{q\to 0^-} \left| \frac{1}{\Gamma(-q)} \int_0^\delta t^{-1-q} f(t)dt - f(0) \right| &= \lim_{q\to 0^-} \left| \frac{1}{\Gamma(-q)} \int_0^\delta t^{-1-q} \big(f(t) - f(0)\big) dt \right| \\ &\le \lim_{q\to 0^-} \frac{1}{\Gamma(-q)} \int_0^\delta t^{-1-q} |f(t) - f(0)| dt \end{aligned}$$

$$
\begin{aligned}
&\leq \epsilon \lim_{q\to 0^-} \frac{1}{\Gamma(-q)} \int\limits_0^\delta t^{-1-q} dt \\
&= \epsilon.
\end{aligned}
$$

Since ϵ was arbitrary, this completes the proof. $\square$

This lemma can be generalized in several ways. Clearly f need not have a compact support if the integral was over the interval $[0, c]$ for some $c > 0$. A similar proof would be used if the integral was over $[-c, 0]$. Also

$$
\lim_{q\to 0^-} \frac{1}{\Gamma(-q)} \int\limits_{-1}^{1} t^{-1-q} f(t) dt = 2f(0)
$$

because the integral can be divided into $\int\limits_{-1}^{0} \cdots + \int\limits_{0}^{1} \cdots$ and the above argument applies to each integral. A simple substitution also yields

$$
\lim_{q\to 0^-} \frac{1}{\Gamma(-q)} \int\limits_{-1}^{1} (t-c)^{-1-q} f(t) dt = 2f(c)
$$

for all $c \in (0, 1)$.

Our goal is to write $Vol_{n-1}(K \cap \tilde{C}_{\alpha,\xi})$ as a series of spherical harmonics. We may then proceed in a way similar to the solution of the Volume Problem. Let $\{G_{mj}\}_{\substack{m\geq 0 \\ 1\leq j\leq N(n,m)}}$ be an orthonormal basis of spherical harmonics for $L_2(S^{n-1})$. Denote

$$
f(\xi) := \int_{S^{n-1}\cap\xi^\perp} \rho_K^{n-1}(\xi\cos(\alpha) + \theta\sin(\alpha)) d\theta,
$$

so that $f(\xi)$ only differs from $Vol_{n-1}(K \cap \tilde{C}_{\alpha,\xi})$ by a constant. Expressing $f(\xi)$

as a harmonic expansion with respect to the basis above, we have

$$f(\xi) \sim \sum_{m\geq 0} \sum_{j=1}^{N(n,m)} \gamma_{mj} G_{mj},$$

where γ_{mj} are the Fourier coefficients of this expansion, given explicitly as

$$\gamma_{mj} := \langle f, G_{mj} \rangle = \int_{S^{n-1}} f(\xi) G_{mj}(\xi) d\xi.$$

As was the case in the solution to the Volume Problem, we are interested only in which harmonic coefficients γ_{mj} equal zero. Once again, $f(\xi)$ and $Vol_{n-1}(K \cap \tilde{C}_{\alpha,\xi})$ differ by a non-zero constant, so it will suffice to consider the harmonic expansion of $f(\xi)$. Fix m and j, and simplify the expression for γ_{mj} as follows.

$$\begin{aligned}
\gamma_{mj} &= \int_{S^{n-1}} f(\xi) G_{mj}(\xi) d\xi \\
&= \int_{S^{n-1}} G_{mj} \left(\int_{S^{n-1} \cap \xi^{\perp}} \rho_K^{n-1}(\xi \cos(\alpha) + \theta \sin(\alpha)) d\theta \right) d\xi \\
&= \frac{1}{(\sin(\alpha))^{n-3}} \int_{S^{n-1}} G_{mj}(\xi) F_{\xi}(\cos(\alpha)) d\xi,
\end{aligned}$$

where F_{ξ} is defined in Lemma 4.2 as

$$F_{\xi}(x) := (1 - x^2)^{\vartheta} \int_{S^{n-1} \cap \xi^{\perp}} \rho_K^{n-1}(x\xi + \sqrt{1 - x^2}\theta) d\theta.$$

Define $\Phi_q(x) := \frac{1}{|x - \cos(\alpha)|^{q+1}}$ (which is integrable on $[-1, 1]$ for all $q < 0$). Using

Lemmas 4.3, 4.2, and Fubini's theorem,

$$\begin{aligned}
\gamma_{mj} &= \frac{1}{2\left(\sin(\alpha)\right)^{n-3}} \lim_{q\to 0^-} \frac{1}{\Gamma(-q)} \int_{S^{n-1}} G_{mj}(\xi) \left(\int_{-1}^{1} \Phi_q(x) F_\xi(x) dx \right) d\xi \\
&= \frac{1}{2\left(\sin(\alpha)\right)^{n-3}} \lim_{q\to 0^-} \frac{1}{\Gamma(-q)} \int_{S^{n-1}} G_{mj}(\xi) \left(\int_{S^{n-1}} \Phi_q\left(\langle \theta, \xi \rangle\right) \rho_K^{n-1}(\theta) d\theta \right) d\xi \\
&= \frac{1}{2\left(\sin(\alpha)\right)^{n-3}} \lim_{q\to 0^-} \frac{1}{\Gamma(-q)} \int_{S^{n-1}} \rho_K^{n-1}(\theta) \left(\int_{S^{n-1}} \Phi_q\left(\langle \theta, \xi \rangle\right) G_{mj}(\xi) d\xi \right) d\theta.
\end{aligned}$$

Now applying the Funk-Hecke Theorem and Lemma 4.3 once more,

$$\begin{aligned}
\gamma_{mj} &= \frac{|S^{n-2}|}{2\left(\sin(\alpha)\right)^{n-3}} \left(\lim_{q\to 0^-} \frac{1}{\Gamma(-q)} \int_{-1}^{1} \Phi_q(x) P_m^n(x) (1-x^2)^{\vartheta} dx \right) \int_{S^{n-1}} \rho_K^{n-1}(\theta) G_{mj}(\theta) d\theta \\
&= \frac{|S^{n-2}|}{2\left(\sin(\alpha)\right)^{n-3}} \left(2 P_m^n(\cos(\alpha))\right) \left(1-\cos^2(\alpha)\right)^{\vartheta} \lambda_{mj}(\rho_K^{n-1}) \\
&= |S^{n-2}| \left(P_m^n(\cos(\alpha))\right) \lambda_{mj}(\rho_K^{n-1}),
\end{aligned}$$

where

$$\lambda_{mj}(\rho_K^{n-1}) := \langle \rho_k^{n-1}, G_{mj} \rangle = \int_{S^{n-1}} \rho_K^{n-1}(x) G_{mj}(x) dx.$$

It follows that $\gamma_{mj} = 0$ whenever $\cos(\alpha)$ is a root of P_m^n. We will denote

$$H_m := \sum_{j=1}^{N(n,m)} \lambda_{mj}\left(\rho_K^{n-1}\right) G_{mj}$$

which is a spherical harmonic of degree m, so the harmonic expansion for ρ_K^{n-1} can be written as

$$\rho_K^{n-1} \sim \sum_{m\geq 0} \sum_{j=1}^{N(n,m)} \langle \rho_k^{n-1}, G_{mj} \rangle G_{mj} \sim \sum_{m\geq 0} H_m.$$

Returning to $Vol_{n-1}(K \cap \tilde{C}_{\alpha,\xi})$, this quantity can now be expressed in terms of the basis $\{H_m\}_{m\geq 0}$:

$$
\begin{aligned}
Vol_{n-1}(K \cap \tilde{C}_{\alpha,\xi}) &= \frac{(\sin(\alpha))^{n-2}}{n-1} f(\xi) \\
&\sim \frac{(\sin(\alpha))^{n-2}}{n-1} \sum_{m \geq 0} \sum_{j=1}^{N(n,m)} \gamma_{mj} G_{mj} \\
&\sim \frac{|S^{n-2}|(\sin(\alpha))^{n-2}}{n-1} \sum_{m \geq 0} \sum_{j=1}^{N(n,m)} \lambda_{mj}(\rho_K^{n-1}) G_{mj} P_m^n(\cos(\alpha)) \\
&\sim \frac{|S^{n-2}| \sin^{n-2}(\alpha)}{(n-1)} \sum_{m \geq 0} H_m(\xi) P_m^n(\cos(\alpha)).
\end{aligned}
$$

The argument preceding Theorem 3.1 applies here as well. If $P_m^n(\cos(\alpha)) \neq 0$ for all $m \in \mathbb{N}$ and if L is another convex body (with harmonic expansion $\rho_L^{n-1} \sim \sum_{m \geq 0} F_m$), satisfying the conditions of the Surface Area Problem, then

$$
\sum_{m \geq 0} H_m(\xi) P_m^n(\cos(\alpha)) = \sum_{m \geq 0} F_m(\xi) P_m^n(\cos(\alpha)) \qquad \forall \xi \in S^{n-1} \qquad (\star)
$$

which, by the independence of spherical harmonics of different degrees, implies that $H_m = F_m$ for all $m \in \mathbb{N}$. Since $\rho_K^{n-1} \sim \sum_{m \geq 0} H_m$ and $\rho_L^{n-1} \sim \sum_{m \geq 0} F_m$ we conclude that $\rho_K^{n-1}(\xi) = \rho_L^{n-1}(\xi)$ for all $\xi \in S^{n-1}$ and therefore $K = L$.

On the other hand, if $P_k^n(\cos(\alpha)) = 0$ for some $k \in \mathbb{N}$ then $(\star)$ does not imply $H_k = F_k$. A counterexample can be constructed as it was in Chapter 3. Let K be the unit ball, $\rho_K(\theta) := 1$, and L be the body given by

$$
\rho_L(\theta) := (1 + \epsilon H_k)^{\frac{1}{n-1}}.
$$

$\rho_K^{n-1}(\theta)$ and $\rho_L^{n-1}(\theta)$ are continuous on S^{n-1} and have finite harmonic expansions 1 and $(1+\epsilon H_k)$ respectively. Hence,

$$\begin{aligned}
Vol(L\cap \tilde{C}_{\alpha,\xi}) &\sim \frac{|S^{n-2}|\sin^{n-2}(\alpha)}{(n-1)}\sum_{m\geq 0} H_m(\xi)P_k^n(\cos(\alpha))\\
&= \frac{|S^{n-2}|\sin^{n-2}(\alpha)}{(n-1)}(H_0(\xi)P_0^n(\alpha)+\epsilon H_k(\xi)P_k^n(\cos(\alpha))\\
&= \frac{|S^{n-2}|\sin^{n-2}(\alpha)}{(n-1)}(1)P_0^n(\cos(\alpha))+\epsilon H_k(\xi)(0)\\
&= \frac{|S^{n-2}|\sin^{n-2}(\alpha)}{(n-1)}P_0^n(\cos(\alpha))\\
&= Vol(K\cap \tilde{C}_{\alpha,\xi}).
\end{aligned}$$

By choosing ϵ sufficiently small, L is convex.

Theorem 4.2: *The answer to the Surface Area Problem in $\mathbb{R}^n$ is affirmative if and only if $P_m^n(\cos(\alpha))\neq 0$ for all $m\in\mathbb{N}$.*

By Proposition 2.3, the family of polynomials $\{P_m^n\}_{m\geq 0}$ has rational coefficients. Once again, if α is a non-zero algebraic number, $\cos(\alpha)$ is a transcendental number meaning that it is not the root of any polynomial with rational coefficients.

Theorem 4.3: *If α is a non-zero algebraic number, then the answer to the Surface Area Problem is affirmative.*

As before, the condition in Theorem 4.3 is sufficient but not necessary. There are only countably many algebraic numbers, so there are only countably many values of α for which $\cos(\alpha)$ is a root of $P_m^n(x)$ for some $m\in\mathbb{N}$. Thus, there are uncountably many values of $\alpha\in(0,\frac{\pi}{2})$ producing an affirmative answer to the Surface Area Problem. On the other hand, only countably many such α are given by Theorem 4.3.

4.C The Negative Answer

Since $\{P_m^n\}_{m\geq 0}$ is orthogonal on $[-1,1]$, the same density argument as was used at the end of Chapter 3 applies here.

Theorem 4.4: *The set of all* $\alpha \in (0, \frac{\pi}{2})$ *for which the answer to the Surface Area Problem is negative is dense in* $(0, \frac{\pi}{2})$.

Proof. By the comments following Lemma 2.1, $\{P_m^n\}_{m\geq 0}$ is orthogonal on $[-1,1]$, which, by Theorem 2.3, implies that the roots of $\{P_m^n\}_{m\geq 0}$ lie in this interval. Now applying Theorem 2.4, if $[\cos(\theta_2), \cos(\theta_1)] \subset [-1,1]$ is an arbitrary subinterval, then for m sufficiently large, $P_m^n(x)$ has a root in this interval. Hence, there exists $\alpha \in [\theta_1, \theta_2]$, such that the answer to the Surface Area Problem is negative. □

Chapter 5

Remarks

We have provided sufficient conditions for the affirmative answer to the Volume Problem but we have not provided a complete solution. To determine exactly when the answer to the Volume Problem is negative, it is necessary to find the roots of the family $\{q_m(x)\}_{m\geq 0}$ defined in Chapter 3. Obviously the roots of high-degree polynomials cannot be found in general. Also, in light of Theorem 3.3, there is no value in estimating values of α for which $\cos(\alpha)$ is a root of some $q_m(x)$. Similar difficulties arise in finding a complete answer to the Surface Area Problem so we cannot expect a complete answer to either of these problems in higher dimensions. However, complete solutions to these problems are possible in some lower dimensional cases. The following result answers the Volume Problem in $\mathbb{R}^4$.

Lemma 5.1: *The answer to the volume problem in $\mathbb{R}^4$ is affirmative if and only if*

$$\frac{\sin(m\alpha)}{\sin\big((m+2)\alpha\big)} \neq \frac{m}{m+2}$$

for all $m \geq 1$.

Note that the solutions to the Volume Problem in $\mathbb{R}^2$ and $\mathbb{R}^4$ are different. In particular, the values $\alpha = \pi/6, \pi/4$ or $\pi/3$ produce an affirmative answer to the Volume Problem in $\mathbb{R}^4$, but a negative answer to the Volume Problem in $\mathbb{R}^2$.

Proof. By Theorem 3.1, the answer is affirmative if and only if

$$\beta_{4,m}(\Phi) = \int_{\cos(\alpha)}^{1} P_m^4(x)(1-x^2)^{\vartheta} dx$$

is non-zero for all $m \in \mathbb{N}$. Using Lemma 2.2 (where $n = 4$ implies $\vartheta = 1/2$),

$$\begin{aligned}\int_{\cos(\alpha)}^{1} P_m^4(x)(1-x^2)^{1/2} dx &= \frac{1}{c_{2,m,1}} \int_{\cos(\alpha)}^{1} \frac{d}{dx}\left(P_{m+1}^2(x)\right)(1-x^2)^{1/2} dx \\ &= \frac{m+1}{c_{2,m,1}} \int_{\cos(\alpha)}^{1} \sin((m+1)\arccos x)) dx,\end{aligned}$$

where we also used $P_m^2(x) := \cos(m \arccos(x))$. Applying the substitution $x := \cos(t)$ and the well known identity for $\sin(jx)\sin(kx)$, the above integral becomes

$$\begin{aligned}&\frac{m+1}{c_{2,m,1}} \int_{0}^{\alpha} \sin\big((m+1)(t)\big)\sin(t) dt \\ &= \frac{m+1}{2c_{2,m,1}} \int_{0}^{\alpha} \big(\cos(mt) - \cos\big((m+2)t\big) dt \\ &= \frac{m+1}{2c_{2,m,1}} \left(\frac{\sin(m\alpha)}{m} - \frac{\sin((m+2)\alpha)}{m+2} \right).\end{aligned}$$

Since the coefficient $c_{2,m,1}$ is non-zero, the previous expression is zero if and only if the two sine terms are equal. $\square$

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