

University of Alberta

**ALGORITHMIC TRADING:
IMPLEMENTING PVOL IN DISCRETE TIME**

by

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Abstract

This thesis considers PVol (Percentage of Volume) strategies, which are an often used type of algorithmic trading strategies. In a PVol strategy, the broker aims to bring the order execution speed in line with a percentage of the market volume. This target percentage and the total order size are typically given by the client. In a discrete-time setting, we analyze the optimization problem of minimizing the expected deviations between the realized portion of trading and the target percentage. Under different assumptions, we either solve the problem explicitly or implement a numerical solution in MATLAB.

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Chapter 1

Problem Formulation

1.1 Introduction

Nowadays, algorithmic trading is very popular for equity trading among investment banks and hedge funds. Algorithmic trading is about computer-based algorithms to execute trading orders. It is used for large orders, which affect stock prices. The main goal of algorithmic trading is to control execution costs and market risk. From a broker's perspective, the task is to execute an order in a way that it is close to a benchmark. The order size (number of shares) and the benchmark are typically chosen by the client. The most popular benchmarks are VWAP (Volume weighted average price), PVol (Percentage of volume; also abbreviated as POV), IS (Implementation shortfall) and TWAP (Time weighted average price).

An IS strategy uses the arrival price as benchmark, which means that one aims to find a strategy such that the price paid is close to the opening price of the stock. While there is an extensive literature on IS strategies, notably the seminal paper by Almgren and Chriss [1], IS strategies use a single price

as benchmark, but it is often more adequate to compare with an average price over a certain period.

Such an average price is TWAP. One splits the time interval, say one trading day, into a number of small time buckets with the same length. A TWAP strategy trades an equal number of shares in each period so that the average cost of the strategy is close to the average intraday price over the trading day. This leads to a strategy which is easy to execute. However, such a strategy may not minimize the market impact because the market volume, which is a key factor for the market impact of a trading order, is not evenly distributed over a trading day. Indeed, intraday market volume is often U-shaped: market volume is high in the morning and evening while the volume in the middle of the day is lower. To account for this, VWAP and PVol strategies are frequently used.

A VWAP strategy uses the VWAP benchmark, which is calculated at the end of the day as the average intraday price weighted by the corresponding market volume. VWAP strategies are among the oldest and most popular trading algorithms, and there is some considerable literature about them, going back to Berkowitz et al. [2], although much less research has been done than for IS strategies.

In practice, also popular are PVol strategies, whose target is to keep the ratio of our volume to the market volume close to a given participation rate. However, there is very little literature about PVol strategies. Recently, Guéant [3] found a closed-form expression for the optimal participation rate in a continuous-time setting under the assumption of deterministic market volume. In contrast, this thesis considers a discrete-time setting from the perspective of a broker with the participation rate fixed by the client and opti-

mization is over trading strategies by minimizing expected deviations between the ratio of our volume to the market volume and the fixed participation rate.

The remainder of this thesis is organized as follows. Section 1.2 introduces the optimization problem corresponding to PVol. Because this optimization problem is involved, the second chapter focuses on a simplified problem, which removes a main difficulty, the random market volumes appearing in the denominator of the optimization problem. At the end of Chapter 2, we use exemplary trading data to analyze the performance of the strategy optimal to the simplified problem. In the third chapter, we return to the original problem with the market volumes in the denominator under i.i.d. assumption on the market volumes.

1.2 Mathematical Formulation

We now present the problem in mathematical terms. A broker needs to buy (or sell) a fixed volume S during the discrete-time periods $1, \dots, T$. The target of PVol is to bring the order execution speed in line with a certain percentage of the market volume. We fix a target percentage $c \in (0, 1)$, and want to have $\frac{Y_j}{X_j + Y_j}$ close to c , where Y_j is our trading volume in the market and X_j is the trading volume of other participants. The aim is to minimize the value

$$E \left[\sum_{j=1}^T \left(\frac{Y_j}{X_j + Y_j} - c \right)^2 \right] \quad (1.1)$$

under the condition $Y_1 + \dots + Y_T = S$. This condition means that the order is completed after T periods. We assume that the market volumes X_1, \dots, X_T are positive random variables on some probability space (Ω, \mathcal{F}, P) with finite

moments $E[X_j^2] < \infty$ and $E[X_j^{-2}] < \infty$ for all j . We denote by \mathcal{F}_t the σ -algebra generated by X_1, \dots, X_t and set $\mathcal{F}_0 = \{\Omega, \emptyset\}$ equal to the trivial σ -algebra. Because the trader needs to choose Y_t with information available at time $t - 1$, Y_t must be \mathcal{F}_{t-1} -measurable, hence the process Y must be predictable. Because the trader can have completed the order already before T , the problem we consider is minimizing

$$E \left[\sum_{j=1}^{\tau} \left(\frac{Y_j}{X_j + Y_j} - c \right)^2 \right]$$

under the condition $Y_1 + \dots + Y_{\tau} = S$ where τ is a stopping time with respect to the filtration $(\mathcal{F}_{t-1})_{t=1, \dots, T}$, because the decision to stop at time t needs to be based on the information from X_1, \dots, X_{t-1} available at time $t - 1$. In mathematical terms, this means that $\{\tau \leq t\} \in \mathcal{F}_{t-1}$ for all $t = 1, \dots, T$. Alternatively, we can see this as first specifying all $\tilde{Y}_1, \dots, \tilde{Y}_T$ with $\sum_{j=1}^T \tilde{Y}_j \geq S$, then defining

$$\tau = \inf \left\{ t \geq 1 \mid \sum_{j=1}^t \tilde{Y}_j \geq S \right\}$$

and finally setting

$$Y_t = \begin{cases} \tilde{Y}_t & , \quad t < \tau \\ S - \sum_{j=1}^{\tau-1} \tilde{Y}_j & , \quad t = \tau. \end{cases}$$

1.2.1 Simplified Problem

Since this question is quite difficult, we modify it into a similar, but easier one.

In a first step, we aim to minimize

$$\sum_{j=1}^T E [(Y_j - c(X_j + Y_j))^2]$$

under the condition $Y_1 \geq 0, \dots, Y_T \geq 0$ and $\sum_{j=1}^T Y_j = S$. Note that this results from multiplying each term in (1.1) by $X_j + Y_j$, which removes a main difficulty in (1.1) that the market volumes X_j appear in the denominator. More generally, we consider a dynamic formulation which means that it starts at time t and the broker has already executed volume s . The value function equals

$$V_1(t, s) = \min_{Y_j \geq 0} E \left[\sum_{j=t+1}^T (Y_j - c(X_j + Y_j))^2 \right]$$

for constants $t \in \{0, 1, \dots, T\}$, $s \in [0, S]$ under the condition $\sum_{j=t+1}^T Y_j = S - s$. We will analyze this problem in Section 2.1 for identically distributed volume and in Section 2.3 for time dependent volume.

We then include the possibility of stopping early, which means the value function is

$$V_2(t, s) = \min_{Y_j \geq 0} E \left[\sum_{j=t+1}^{\tau} (Y_j - c(X_j + Y_j))^2 \right]$$

for constants $t \in \{0, 1, \dots, T\}$, $s \in [0, S]$ under the condition $\sum_{j=t+1}^{\tau} Y_j = S - s$. We will analyze this case in Section 2.2 for identically distributed volume and in Section 2.4 for time dependent volume.

1.2.2 Original Problem

We then get back in Chapter 3 to the original problem formulation of percentage volume. We give a dynamic formulation which means that it starts at time t and the broker has already traded volume s . Again, we consider first

without stopping early, namely

$$V_3(t, s) = \min_{Y_j \geq 0} E \left[\sum_{j=t+1}^T \left(\frac{Y_j}{X_j + Y_j} - c \right)^2 \right]$$

for constants $t \in \{0, 1, \dots, T\}$, $s \in [0, S]$ under the condition $\sum_{j=t+1}^T Y_j = S - s$.

We will analyze this case in Section 3.1.

Including the possibility to finish the order execution early, the value function equals

$$V_4(t, s) = \min_{Y_j \geq 0} E \left[\sum_{j=t+1}^{\tau} \left(\frac{Y_j}{X_j + Y_j} - c \right)^2 \right]$$

for constants $t \in \{0, 1, \dots, T\}$, $s \in [0, S]$ under the condition $\sum_{j=t+1}^{\tau} Y_j = S - s$.

We will analyze this case in Section 3.2.

Chapter 2

Simplified Problem

In this chapter, we approach the simplified problem presented in Section 1.2.1. We divide this chapter into five parts. In Sections 2.1 and 2.2, we analyze the simplified problem when the market volumes are identically distributed while Sections 2.3 and 2.4 consider time dependent volumes. Section 2.5 compares the performance of different trading strategies using real trading volume data.

2.1 Identically Distributed Volumes without Stopping

In this section, we assume that the market volumes X_j are independent and identically distributed with mean $\mu = E[X_j]$ and variance $\sigma^2 = \text{Var}(X_j)$. Recall that we study here

$$V_1(t, s) = \min_{\substack{Y_j \geq 0 \\ j=t+1, \dots, T}} E \left[\sum_{j=t+1}^T (Y_j - c(X_j + Y_j))^2 \right] \quad (2.1)$$

under the condition $\sum_{j=t+1}^T Y_j = S - s$. We use the following well-known auxiliary result; see for example Theorem 6.4 of Kallenberg [4].

Lemma 2.1: *Let X and Y be two random variables on some probability space (Ω, \mathcal{F}, P) . Let $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ be a measurable function with $E[|f(X, Y)|] < \infty$ and let $\mathcal{A} \subseteq \mathcal{F}$ be a σ -algebra. If X is independent of \mathcal{A} and Y is \mathcal{A} -measurable, then*

$$E[f(X, Y)|\mathcal{A}](\omega) = E[f(X, y)]|_{y=Y(\omega)} \quad \text{for almost all } \omega \in \Omega. \quad (2.2)$$

The right-hand side of (2.2) means that we fix the random variable Y and take expectation over the independent random variable X . Using this result, we show that in (2.1), it is enough to minimize over deterministic Y_j . Indeed, we obtain that

$$\begin{aligned} V_1(t, s) &= \min_{Y_j \geq 0} E \left[\sum_{j=t+1}^T (Y_j - c(X_j + Y_j))^2 \right] \\ &= \min_{Y_j \geq 0} E \left[\sum_{j=t+1}^{T-1} (Y_j - c(X_j + Y_j))^2 + (Y_T - c(X_T + Y_T))^2 \right] \\ &= \min_{Y_j \geq 0} E \left[\sum_{j=t+1}^{T-1} E[(Y_j - c(X_j + Y_j))^2 | \mathcal{F}_{j-1}] \right. \\ &\quad \left. + E \left[\left(S - s - \sum_{j=t+1}^{T-1} Y_j - c \left(X_T + S - s - \sum_{j=t+1}^{T-1} Y_j \right) \right)^2 \middle| \mathcal{F}_{T-1} \right] \right] \\ &= \min_{Y_j \geq 0} E \left[\sum_{j=t+1}^{T-1} E[(y_j - c(X_j + y_j))^2] \middle|_{y_j=Y_j(\omega)} + E \left[\left(S - s - \sum_{j=t+1}^{T-1} y_j \right. \right. \right. \\ &\quad \left. \left. - c \left(X_T + S - s - \sum_{j=t+1}^{T-1} y_j \right) \right)^2 \middle|_{y_{t+1}=Y_{t+1}(\omega), \dots, y_{T-1}=Y_{T-1}(\omega)} \right] \right], \end{aligned}$$

where the last equality follows from Lemma 2.1.

Define a function

$$g(y_{t+1}, \dots, y_{T-1}) := \sum_{j=t+1}^{T-1} E [(y_j - c(X_j + y_j))^2] \\ + E \left[\left(S - s - \sum_{j=t+1}^{T-1} y_j - c \left(X_T + S - s - \sum_{j=t+1}^{T-1} y_j \right) \right)^2 \right].$$

This is a deterministic function $g : \mathbb{R}^{T-t+1} \rightarrow \mathbb{R}$ and we can write

$$V_1(t, s) = \min_{Y_j \geq 0} E [g(Y_{t+1}, \dots, Y_{T-1})]$$

under the condition $\sum_{t+1}^{T-1} Y_j \leq S - s$. Hence, the problem is equivalent to minimizing g and finding a deterministic minimizer $y^* = (y_{t+1}^*, \dots, y_{T-1}^*)$ of g . To do so, we take partial derivatives of g with respect to y_ℓ for $\ell = t + 1, \dots, T - 1$, which results in

$$\frac{\partial g}{\partial y_\ell} = E [2(y_\ell - c(X_\ell + y_\ell))(1 - c)] \\ + E \left[2 \left(S - s - \sum_{j=t+1}^{T-1} y_j - c \left(X_T + S - s - \sum_{j=t+1}^{T-1} y_j \right) \right) (-1 + c) \right] \\ = E \left[(y_\ell - c(X_\ell + y_\ell)) - \left(S - s - \sum_{j=t+1}^{T-1} y_j - c \left(X_T + S - s - \sum_{j=t+1}^{T-1} y_j \right) \right) \right] \\ = (1 - c)y_\ell - c\mu - \left(S - s - \sum_{j=t+1}^{T-1} y_j - c\mu - c \left(S - s - \sum_{j=t+1}^{T-1} y_j \right) \right).$$

We set $\frac{\partial g}{\partial y_\ell}$ equal to 0, thus we have

$$y_\ell^* = S - s - \sum_{j=t+1}^{T-1} y_j^*, \quad \ell = t + 1, \dots, T - 1.$$

Then we sum all the y_ℓ^* for $\ell = t + 1, \dots, T - 1$ so that

$$\sum_{j=t+1}^{T-1} y_j^* = (T-t-1) \left(S-s - \sum_{j=t+1}^{T-1} y_j^* \right).$$

This implies

$$\sum_{j=t+1}^{T-1} y_j^* = \frac{(T-t-1)(S-s)}{T-t} < S-s.$$

Hence, we get

$$y_\ell^* = S-s - \sum_{j=t+1}^{T-1} y_j^* = \frac{S-s}{T-t}, \quad \ell = t+1, \dots, T-1,$$

$$y_T^* = S-s - \sum_{j=t+1}^{T-1} y_j^* = \frac{S-s}{T-t},$$

which means $y_\ell^* = \frac{S-s}{T-t}$ for all ℓ . This shows that the minimizer of $V_1(t, s)$ is deterministic with $y_\ell^* = \frac{S-s}{T-t}$ because g has only one critical point and needs to have a global minimum as continuous function with $g(y) \rightarrow \infty$ for $|y| \rightarrow \infty$. This means that we distribute our trades equally, which is what we expect because the market volume is i.i.d. and we cannot finish our trading early as opposed to the setting of the next section. The value function equals

$$\begin{aligned} V_1(t, s) &= \min_{Y_j \geq 0} E \left[\sum_{j=t+1}^T (Y_j - c(X_j + Y_j))^2 \right] \\ &= \min_{Y_j \geq 0} (T-t) E \left[(1-c)^2 Y_j^2 - 2c(1-c) X_j Y_j + c^2 X_j^2 \right] \\ &= (T-t) \left((1-c)^2 \left(\frac{S-s}{T-t} \right)^2 - 2c(1-c) \frac{S-s}{T-t} E[X_j] + c^2 E[X_j^2] \right) \\ &= (T-t) \left(\left(\frac{(1-c)(S-s)}{T-t} \right)^2 - \frac{2c\mu(1-c)(S-s)}{T-t} + c^2(\sigma^2 + \mu^2) \right). \end{aligned}$$

Alternatively, we can use another method to determine the minimizers for Y_ℓ ,

$\ell = t + 1, \dots, T - 1$ and $V_1(t, s)$. We do this backward in time by dynamic programming. When $t = T - 1$, we need to trade all the remaining volume, hence $Y_T = S - s$ and

$$\begin{aligned}
V_1(T - 1, s) &= \min_{Y_T \geq 0} E [(Y_T - c(X_T + Y_T))^2] \\
&= E [(S - s - c(X_T + S - s))^2] \\
&= ((1 - c)(S - s))^2 - 2c(1 - c)(S - s)E[X_T] + c^2 E[X_T^2] \\
&= ((1 - c)(S - s))^2 - 2c\mu(1 - c)(S - s) + c^2(\mu^2 + \sigma^2).
\end{aligned}$$

When $t = T - 2$, we have

$$\begin{aligned}
Y_T &= S - s - Y_{T-1}, \\
V_1(T - 2, s) &= \min_{Y_T \geq 0, Y_{T-1} \geq 0} E [(Y_{T-1} - c(X_{T-1} + Y_{T-1}))^2 + (Y_T - c(X_T + Y_T))^2] \\
&= \min_{Y_{T-1} \in [0, S-s]} E [(Y_{T-1} - c(X_{T-1} + Y_{T-1}))^2 \\
&\quad + (S - s - Y_{T-1} - c(X_T + S - s - Y_{T-1}))^2].
\end{aligned}$$

Define a function

$$\begin{aligned}
g(y_{T-1}) &:= E [(y_{T-1} - c(X_{T-1} + y_{T-1}))^2] \\
&\quad + E [(S - s - y_{T-1} - c(X_T + S - s - y_{T-1}))^2].
\end{aligned}$$

This is a deterministic function $g : \mathbb{R} \rightarrow \mathbb{R}$. Hence, the problem is equivalent to minimizing g and finding the minimizer y_{T-1}^* of g . We then choose $Y_{T-1} = y_{T-1}^*$

to minimize V_1 . We take the derivative of g , namely,

$$\begin{aligned} g'(y_{T-1}) &= E [2(y_{T-1} - c(X_{T-1} + y_{T-1}))(1 - c) \\ &\quad + 2(S - s - y_{T-1} - c(X_T + S - s - y_{T-1}))(-1 + c)] \\ &= (1 - c)y_{T-1} - c\mu - (S - s - y_{T-1} - c\mu - c(S - s - y_{T-1})). \end{aligned}$$

Setting $g'(y_{T-1}^*)$ equal to 0, we have

$$y_{T-1}^* = \frac{S - s}{2}, \quad y_T^* = S - s - y_{T-1}^* = \frac{S - s}{2}$$

and then

$$\begin{aligned} V_1(T - 2, s) &= \min_{Y_{T-1} \in [0, S-s]} E [(Y_{T-1} - c(X_{T-1} + Y_{T-1}))^2 \\ &\quad + (S - s - Y_{T-1} - c(X_T + S - s - Y_{T-1}))^2] \\ &= 2 \left(\left(\frac{(1 - c)(S - s)}{2} \right)^2 - \frac{2c\mu(1 - c)(S - s)}{2} + c^2(\sigma^2 + \mu^2) \right). \end{aligned}$$

This structure leads us to the induction hypothesis that

$$V_1(T - j, s) = j \left(\left(\frac{(1 - c)(S - s)}{j} \right)^2 - \frac{2c\mu(1 - c)(S - s)}{j} + c^2(\sigma^2 + \mu^2) \right) \quad (2.3)$$

with optimizers $y_\ell^* = \frac{S-s}{j}$, $\ell = T - j, \dots, T$. To confirm this hypothesis, we

deduce

$$\begin{aligned}
& V_1(T - j - 1, s) \\
&= \min_{Y_{T-j} \in [0, S-s]} E \left[(Y_{T-j} - c(X_{T-j} + Y_{T-j}))^2 + V(T - j, s + Y_{T-j}) \right] \\
&= \min_{Y_{T-j} \in [0, S-s]} E \left[(Y_{T-j} - c(X_{T-j} + Y_{T-j}))^2 + j \left(\left(\frac{(1-c)(S-s-Y_{T-j})}{j} \right)^2 \right. \right. \\
&\quad \left. \left. - \frac{2c\mu(1-c)(S-s-Y_{T-j})}{j} + c^2(\sigma^2 + \mu^2) \right) \right].
\end{aligned}$$

Define a function

$$\begin{aligned}
g(y_{T-j}) := E \left[(y_{T-j} - c(X_{T-j} + y_{T-j}))^2 + j \left(\left(\frac{(1-c)(S-s-y_{T-j})}{j} \right)^2 \right. \right. \\
\left. \left. - \frac{2c\mu(1-c)(S-s-y_{T-j})}{j} + c^2(\sigma^2 + \mu^2) \right) \right].
\end{aligned}$$

This is a deterministic function $g : \mathbb{R} \rightarrow \mathbb{R}$. Hence, the problem is equivalent to minimizing g and finding the minimizer y_{T-j}^* of g . We then choose $Y_{T-j} = y_{T-j}^*$ to minimize V_1 . We take the derivative of g , namely,

$$\begin{aligned}
g'(y_{T-j}) &= E \left[2(y_{T-j} - c(X_{T-j} + y_{T-j})) (1-c) - \frac{2}{j}(1-c)^2(S-s) \right. \\
&\quad \left. + \frac{2}{j}(1-c)^2 y_{T-j} + 2c\mu(1-c) \right] \\
&= \frac{2(j+1)}{j}(1-c)^2 y_{T-j} - \frac{2}{j}(1-c)^2(S-s).
\end{aligned}$$

Setting $g'(y_{T-j})$ equal to 0, we have

$$y_{T-j}^* = \frac{S-s}{j+1}$$

and then

$$\begin{aligned} & V_1(T - j - 1, s) \\ &= (j + 1) \left(\left(\frac{(1 - c)(S - s)}{j + 1} \right)^2 - \frac{2c\mu(1 - c)(S - s)}{j + 1} + c^2(\sigma^2 + \mu^2) \right), \end{aligned}$$

which is consistent with our induction hypothesis (2.3). By backward induction, we conclude that this holds for all $j = 0, \dots, T - 1$.

2.2 Identically Distributed Volumes with Stopping

We first show that it is enough to minimize over deterministic stopping times.

Recall that the value function is now given by

$$\begin{aligned} V_2(t, s) &= \min_{Y_j \geq 0} E \left[\sum_{j=t+1}^{\tau} (Y_j - c(X_j + Y_j))^2 \right] \\ &= \min_{Y_j \geq 0} E \left[\sum_{j=t+1}^T (Y_j - c(X_j + Y_j))^2 \mathbb{1}_{\{\tau \geq j\}} \right] \\ &= \min_{Y_j \geq 0} E \left[\sum_{j=t+1}^T (Y_j - c(X_j + Y_j))^2 \mathbb{1}_{\{\tau \leq j-1\}^c} \right] \\ &= \min_{Y_j \geq 0} E \left[\sum_{j=t+1}^T E [(Y_j - c(X_j + Y_j))^2 | \mathcal{F}_{j-1}] \mathbb{1}_{\{\tau \leq j-1\}^c} \right] \\ &= \min_{Y_j \geq 0} E \left[\sum_{j=t+1}^T E [(y_j - c(X_j + y_j))^2] \Big|_{y_j=Y_j(\omega)} \mathbb{1}_{\{\tau \leq j-1\}^c} \right], \end{aligned}$$

using Lemma 2.1 for the last equality and that $\{\tau \leq j - 1\}$ is \mathcal{F}_{j-1} -measurable for the second last equality because τ is a stopping time. Note that the last equality uses that X_j is independent of \mathcal{F}_{j-1} since $(X_j)_{j=1, \dots, T}$ are i.i.d.

Define $g(y) = E[(y - c(X_j + y))^2]$, then

$$\begin{aligned} V_2(t, s) &= \min_{y_j \geq 0} E \left[\sum_{j=t+1}^T g(y_j) \mathbb{1}_{\{\tau \leq j-1\}^c} \right] \\ &= \min_{y_j \geq 0} E \left[\sum_{j=t+1}^{\tau} g(y_j) \right] \end{aligned}$$

subject to $\sum_{j=t+1}^{\tau} y_j = S - s$. Note that this last formulation does not involve any given random variables. We can minimize over deterministic y_j and τ .

We can solve for the value of τ explicitly. From the conclusion of the previous section, we get

$$V_1(t, s) = (T - t) \left(\left(\frac{(1-c)(S-s)}{T-t} \right)^2 - \frac{2c\mu(1-c)(S-s)}{T-t} + c^2(\sigma^2 + \mu^2) \right)$$

and we set

$$\tilde{V}_1(\tau) = (\tau - t) \left(\left(\frac{(1-c)(S-s)}{\tau-t} \right)^2 - \frac{2c\mu(1-c)(S-s)}{\tau-t} + c^2(\sigma^2 + \mu^2) \right)$$

for the value function between time t and τ . The question becomes

$$V_2(t, s) = \min_{\tau=t+1, \dots, T} \tilde{V}_1(\tau).$$

Define a deterministic function $g : \mathbb{R} \rightarrow \mathbb{R}$ by

$$g(p) := (p - t) \left(\left(\frac{(1-c)(S-s)}{p-t} \right)^2 - \frac{2c\mu(1-c)(S-s)}{p-t} + c^2(\sigma^2 + \mu^2) \right).$$

Hence, the problem is equivalent to minimizing g over integers between t and T and finding the integer-valued minimizer p^* of g . We first find the minimizer

over the real numbers in $[t, +\infty)$. We take the derivative of g , resulting in

$$g'(p) = -\frac{((1-c)(S-s))^2}{(p-t)^2} + c^2(\sigma^2 + \mu^2).$$

Then we set $g'(p^*) = 0$ so that

$$p^* = t + \frac{(1-c)(S-s)}{c\sqrt{\sigma^2 + \mu^2}}.$$

Taking the derivative of $g'(p)$, we have

$$g''(p) = \frac{2((1-c)(S-s))^2}{(p-t)^3}.$$

We find that $g''(p) > 0$ for all $p > t$, so we get that if $p \in \left(t, t + \frac{(1-c)(S-s)}{c\sqrt{\sigma^2 + \mu^2}}\right]$, the function g is decreasing, and if $p \in \left[t + \frac{(1-c)(S-s)}{c\sqrt{\sigma^2 + \mu^2}}, +\infty\right)$, the function g is increasing. Hence, we need to consider three situations.

1.case: When $T < t + \frac{(1-c)(S-s)}{c\sqrt{\sigma^2 + \mu^2}}$, we take $\tau^* = T$ to minimize $V_2(t, s)$, which then equals

$$V_2(t, s) = (T-t) \left(\left(\frac{(1-c)(S-s)}{T-t} \right)^2 - \frac{2c\mu(1-c)(S-s)}{T-t} + c^2(\sigma^2 + \mu^2) \right).$$

2.case: When $t+1 > t + \frac{(1-c)(S-s)}{c\sqrt{\sigma^2 + \mu^2}}$, we take $\tau^* = t+1$ to minimize $V_2(t, s)$, which then equals

$$V_2(t, s) = ((1-c)(S-s))^2 - 2c\mu(1-c)(S-s) + c^2(\sigma^2 + \mu^2).$$

3.case: When $t + 1 \leq t + \frac{(1-c)(S-s)}{c\sqrt{\sigma^2+\mu^2}} \leq T$, we set $p_1 = \left\lceil t + \frac{(1-c)(S-s)}{c\sqrt{\sigma^2+\mu^2}} \right\rceil$ and $p_2 = \left\lfloor t + \frac{(1-c)(S-s)}{c\sqrt{\sigma^2+\mu^2}} \right\rfloor$.

If $g(p_1) \leq g(p_2)$, then we take $\tau^* = p_1$ so that

$$V_2(t, s) = (p_1 - t) \left(\left(\frac{(1-c)(S-s)}{p_1 - t} \right)^2 - \frac{2c\mu(1-c)(S-s)}{p_1 - t} + c^2(\sigma^2 + \mu^2) \right).$$

If $g(p_2) < g(p_1)$, then we take $\tau^* = p_2$ so that

$$V_2(t, s) = (p_2 - t) \left(\left(\frac{(1-c)(S-s)}{p_2 - t} \right)^2 - \frac{2c\mu(1-c)(S-s)}{p_2 - t} + c^2(\sigma^2 + \mu^2) \right).$$

Following this idea, we program the MATLAB code presented in Appendix A.1. This function returns the values of τ and V . For example, choosing $T = 50, t = 5, s = 500, c = 0.5, S = 600, \mu = 5, \sigma = 5$, we get $\tau = 19$ and $V = 103.57$. Similarly to the exercise boundary for American options, we

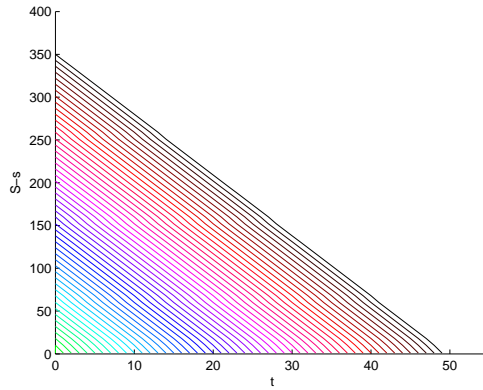


Figure 2.1: Optimal stopping regions depending on t and $S - s$.

can plot the optimal stopping regions, which are displayed in Figure 2.1. The parameters chosen are $S = 600, T = 50, c = 0.5, \mu = 5$ and $\sigma = 5$. The main idea of the corresponding MATLAB code presented in Appendix A.1 is to

compare $V_1(t, s)$ given different end times $t + 1, \dots, T$ and to find the stopping time τ which is the end time where $V_1(t, s)$ is minimal. In Figure 2.1, we see, for example, that a curve starts at $(t, S - s) = (0, 210)$ goes via $(t, S - s) = (20, 70)$ to $(t, S - s) = (30, 0)$. This means that with remaining volume 210 at time zero it is optimal to trade over 30 periods while with remaining volume 70 at time 20, it is optimal to trade over 10 periods. Note that the curves are linear because the optimal strategies are the equal distributions of volume until the stopping time thanks to the i.i.d assumption of market volume.

2.3 Time Dependent Volume without Stopping

In this section, we consider the case where X_j are still independent, but not necessarily identically distributed. Analogously to Section 2.1, we can minimize over deterministic Y_j . However, here we need to make sure that the optimizers Y_j^* are nonnegative, because this does not automatically follow from setting the partial derivatives equal to zero; see the example at the end of this section. We use a set $A \subseteq \{t + 1, \dots, T\}$ which contains all j such that $Y_j > 0$. We can rewrite the optimization problem as

$$\begin{aligned} V_1(t, y) &= \min_{Y_j \geq 0} E \left[\sum_{j=t+1}^T (Y_j - c(X_j + Y_j))^2 \right] \\ &= \min_{A, Y_j \geq 0} E \left[g \left(A; (Y_j)_{j \in A \setminus \{\tilde{T}\}} \right) \right], \end{aligned}$$

where $\tilde{T} = \max A$ and g is defined by

$$g\left(A; (y_j)_{j \in A \setminus \{\tilde{T}\}}\right) := c^2 \sum_{j \notin A} (\sigma_j^2 + \mu_j^2) + \sum_{j \in A} E \left[(y_j - c(X_j + y_j))^2 \right] \\ + E \left[\left(S - s - \sum_{j \in A \setminus \{\tilde{T}\}} y_j - c \left(X_{\tilde{T}} + S - s - \sum_{j \in A \setminus \{\tilde{T}\}} y_j \right) \right)^2 \right].$$

Hence, the problem is equivalent to minimizing g and finding a deterministic minimizer. To do so, we take partial derivatives of g with respect to $y_\ell, \ell \in A \setminus \{\tilde{T}\}$, which yields

$$\frac{\partial g}{\partial y_\ell} = E \left[2(y_\ell - c(X_\ell + y_\ell))(1 - c) \right] \\ + E \left[2 \left(S - s - \sum_{j \in A \setminus \{\tilde{T}\}} y_j - c \left(X_{\tilde{T}} + S - s - \sum_{j \in A \setminus \{\tilde{T}\}} y_j \right) \right) (-1 + c) \right] \\ = E \left[y_\ell - c(X_\ell + y_\ell) - \left(S - s - \sum_{j \in A \setminus \{\tilde{T}\}} y_j - c \left(X_{\tilde{T}} + S - s - \sum_{j \in A \setminus \{\tilde{T}\}} y_j \right) \right) \right] \\ = (1 - c)y_\ell - c\mu_\ell - \left(S - s - \sum_{j \in A \setminus \{\tilde{T}\}} y_j - c\mu_T - c \left(S - s - \sum_{j \in A \setminus \{\tilde{T}\}} y_j \right) \right).$$

We set $\frac{\partial g}{\partial y_\ell}$ equal to 0 to find y_ℓ^* , which results in

$$y_\ell^* = S - s - \sum_{j \in A \setminus \{\tilde{T}\}} y_j^* + \frac{c(\mu_\ell - \mu_{\tilde{T}})}{1 - c}, \quad \ell = t + 1, \dots, T - 1.$$

Then we sum all the y_ℓ^* for $\ell \in A \setminus \{\tilde{T}\}$ so that

$$\sum_{j \in A \setminus \{\tilde{T}\}} y_j^* = (|A| - 1) \left(S - s - \sum_{j \in A \setminus \{\tilde{T}\}} y_j^* \right) + \sum_{j \in A \setminus \{\tilde{T}\}} \frac{c(\mu_j - \mu_{\tilde{T}})}{1 - c}.$$

This implies

$$\sum_{j \in A \setminus \{\tilde{T}\}} y_j^* = \frac{(|A|-1)(S-s)}{|A|} + \sum_{j \in A \setminus \{\tilde{T}\}} \frac{c(\mu_j - \mu_{\tilde{T}})}{|A|(1-c)}.$$

Hence, we get

$$\begin{aligned} y_\ell^* &= S - s - \sum_{j \in A \setminus \{\tilde{T}\}} y_j^* + \frac{c(\mu_\ell - \mu_{\tilde{T}})}{1-c} \\ &= \frac{S-s}{|A|} - \sum_{j \in A \setminus \{\tilde{T}\}} \frac{c(\mu_j - \mu_{\tilde{T}})}{|A|(1-c)} + \frac{c(\mu_\ell - \mu_{\tilde{T}})}{1-c}, \\ y_T^* &= S - s - \sum_{j \in A \setminus \{\tilde{T}\}} y_j = \frac{S-s}{|A|} - \sum_{j \in A \setminus \{\tilde{T}\}} \frac{c(\mu_j - \mu_{\tilde{T}})}{|A|(1-c)}. \end{aligned}$$

So, we have

$$\begin{aligned} y_\ell^* &= \frac{S-s}{|A|} - \sum_{j \in A \setminus \{\tilde{T}\}} \frac{c(\mu_j - \mu_{\tilde{T}})}{|A|(1-c)} + \frac{c(\mu_\ell - \mu_{\tilde{T}})}{1-c} \\ &= \frac{S-s}{|A|} + \frac{c}{1-c} \left(\mu_\ell - \frac{1}{|A|} \sum_{j \in A} \mu_j \right) \quad \text{for all } \ell = t+1, \dots, T. \end{aligned} \quad (2.4)$$

This is the minimizer of $g(A; \cdot)$ because it has only one critical point and needs to have a global minimum as continuous function with $g(A; y) \rightarrow \infty$ for $|y| \rightarrow \infty$. We then choose $Y_\ell = y_\ell^*$ to minimize $V_1(t, s)$, which gives

$$\begin{aligned} V_1(t, s) &= \min_A \sum_{\ell \in A} E [(y_\ell^* - c(X_j + y_\ell^*))^2] + c^2 \sum_{\ell \notin A} (\sigma_\ell^2 + \mu_\ell^2) \\ &= \min_A \sum_{\ell \in A} E \left[\left(\frac{(S-s)(1-c)}{|A|} - \frac{c}{|A|} \sum_{j \in A} \mu_j + c\mu_\ell - cX_\ell \right)^2 \right] \\ &\quad + c^2 \sum_{\ell \notin A} (\sigma_\ell^2 + \mu_\ell^2). \end{aligned} \quad (2.5)$$

We set

$$\alpha_A := \frac{(S-s)(1-c)}{|A|} - \frac{c}{|A|} \sum_{j \in A} \mu_j, \quad (2.6)$$

then

$$\begin{aligned} V_1(t, s) &= \min_A \sum_{\ell \in A} E [(\alpha_A + c\mu_\ell - cX_\ell)^2] + c^2 \sum_{\ell \notin A} (\sigma_\ell^2 + \mu_\ell^2) \\ &= \min_A \sum_{\ell \in A} E [\alpha_A^2 + 2\alpha_A(c\mu_\ell - cX_\ell) + (c\mu_\ell - cX_\ell)^2] + c^2 \sum_{\ell \notin A} (\sigma_\ell^2 + \mu_\ell^2) \\ &= \min_A \sum_{\ell \in A} (\alpha_A^2 + 2\alpha_A c E[\mu_\ell - X_\ell] + c^2 E[(\mu_\ell - X_\ell)^2]) + c^2 \sum_{\ell \notin A} (\sigma_\ell^2 + \mu_\ell^2) \\ &= \min_A \sum_{\ell \in A} (\alpha_A^2 + c^2 \sigma_\ell^2) + c^2 \sum_{\ell \notin A} (\sigma_\ell^2 + \mu_\ell^2). \end{aligned}$$

The minimization is over A such that all y_j^* , $j \in A$, given in (2.4), are nonnegative. This can be done by ordering the μ_j , starting with the j corresponding to the biggest μ_j and then continuing until a y_j becomes negative. We denote the resulting minimizing set by A^* , so that

$$V_1(t, s) = \sum_{\ell \in A^*} (\alpha_{A^*}^2 + c^2 \sigma_\ell^2) + c^2 \sum_{\ell \notin A^*} (\sigma_\ell^2 + \mu_\ell^2).$$

To see that one really needs to introduce the set A , we consider a simple situation with $t = T - 2$, $\mu_T = \frac{S}{c}$, $\mu_{T-1} = 0$ and $s = 0$. In this extreme example, we would get from (2.4) for $A = \{T - 1, T\}$ that

$$\begin{aligned} Y_{T-1} &= \frac{S}{2} - \frac{S}{2(1-c)} < 0, \\ Y_T &= \frac{S}{2} + \frac{S}{2(1-c)} > S. \end{aligned}$$

This makes no sense as Y_j denotes our volume, which cannot be negative. With the choice $A = \{T\}$, we have $Y_{T-1} = 0$ and $Y_T = S$, which is the correct optimizer. The introduction of the set A allows us to perform the optimization over the nonnegative Y_j .

2.4 Time Dependent Volume with Stopping

In this section, we consider the same situation as in Section 2.3, but allow for stopping. Recall that the value function is now given by

$$V_2(t, s) = \min_{Y_j \geq 0} E \left[\sum_{j=t+1}^{\tau} (Y_j - c(X_j + Y_j))^2 \right].$$

We can rewrite this as a deterministic minimization problem. From the conclusion of the previous section, we get

$$V_1(t, s) = \sum_{\ell \in A_{t,T}^*} \left(\alpha_{A_{t,T}^*}^2 + c^2 \sigma_\ell^2 \right) + c^2 \sum_{\ell \notin A^*} (\sigma_\ell^2 + \mu_\ell^2).$$

where α is given in (2.6) and $A_{t,T}^* \subseteq \{t+1, \dots, T\}$ is the optimal set for (2.5).

Set

$$\tilde{V}_1(\tau) = c^2 \sum_{\ell=t+1}^{\tau} \sigma_\ell^2 + |A_{t,\tau}^*| \alpha_{A_{t,\tau}^*}^2 + c^2 \sum_{\ell \notin A_{t,\tau}^*} \mu_\ell^2$$

for the value function between time t and τ .

The question becomes

$$V_2(t, s) = \min_{\tau=t+1, \dots, T} \tilde{V}_1(\tau).$$

Define a deterministic function $g : \mathbb{R} \rightarrow \mathbb{R}$ by

$$g(p) = c^2 \sum_{\ell=t+1}^T \sigma_\ell^2 + |A_{t,p}^*| \alpha_{A_{t,p}^*}^2 + c^2 \sum_{\ell \notin A_{t,p}^*} \mu_\ell^2.$$

Hence, the problem is equivalent to minimizing g over integers and finding the integer-valued minimizer p^* of g . As illustration, we next consider different examples of the time dependence of σ_t and μ_t .

1.case: Assume $\sigma_t = \sigma$ is constant and μ_t depends linearly on time while the total volume $\sum_{t=1}^T \mu_t = \mu_{\text{total}}$ is fixed. When we start trading at time 1, this implies $\mu_t = \frac{bt+1}{\frac{bT(T+1)}{2} + T} \mu_{\text{total}}$, $b \in (\frac{-1}{T}, +\infty)$. The coefficient b gives how fast the volume increases (or decreases) over the different trading periods while the total volume over all trading periods is fixed.

We can run the program shown in Appendix A.2 to minimize g . Choosing the parameters $S = 600, \mu_{\text{total}} = 10,000, T = 50, c = 0.1, t = 0, s = 0$ and $\sigma = 5$, we obtain the optimal stopping time displayed in Figure 2.2.

With the parameter given, we get $b \in (-0.02, +\infty)$. We can find that the shape of τ is increasing with b which means when the other participants' volumes change faster over the different trading periods, then we stop our trading later. From Figure 2.2, we see that the stopping time is close to 27 for $b = 0$. Actually, this is the same special case as the third case in Section 2.2, because $\mu_t = \frac{1}{T} \mu_{\text{total}}$ is then constant over time. Recall that when $t+1 \leq t + \frac{(1-c)(S-s)}{c\sqrt{\sigma^2 + \mu^2}} \leq T$, we get $\tau = t + \frac{(1-c)(S-s)}{c\sqrt{\sigma^2 + \mu^2}}$. When $b = 0$, we get $\mu = \frac{\mu_{\text{total}}}{T} = 200$, so $\tau = 27$, which is the same as shown in the graph.

2.case: Assume $\sigma_\ell = \sigma$ is constant and μ_t is quadratic function of time, while the total volume $\sum_{t=1}^T \mu_t = \mu_{\text{total}}$ is fixed. This implies that μ_t is of the form $\mu_t = \frac{at^2 + bt + 1}{\frac{aT(T+1)(2T+1)}{6} + \frac{bT(T+1)}{2} + T} \mu_{\text{total}}$ for some constants a and b . The coefficients

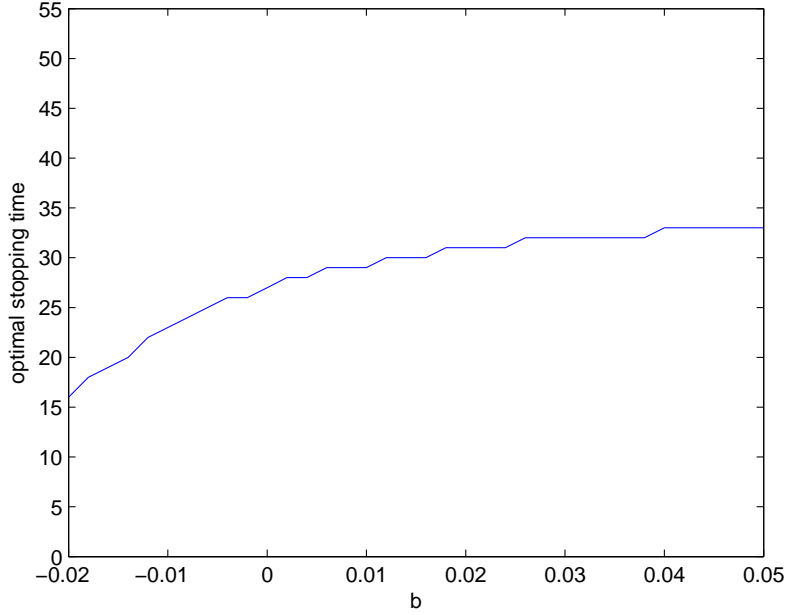


Figure 2.2: The optimal stopping time depending on the parameter b of the expected volume $\mu_t = \frac{bt+1}{\frac{bT^2}{2}+T}\mu_{\text{total}}$.

a and b control how fast the volume increases (or decreases) over the different trading periods. The parameter a has more influence on the speed than b , because of the quadratic structure.

We can run the program shown in Appendix A.2.2 to minimize g . This results in the optimal stopping time of Figure 2.3 for the parameters $S = 600$, $\mu_{\text{total}} = 10000$, $T = 50$, $c = 0.1$, $t = 0$, $s = 0$ and $\sigma = 5$.

We can find that τ is increasing with a and b which means when other participants' volumes change faster over the different trading periods, then we stop our trading later. When $a = 0$ and $b = 0$, we get $\mu = \frac{\mu_{\text{total}}}{T} = 200$ and can find $\tau = t + \frac{(1-c)(S-s)}{c\sqrt{\sigma^2+\mu^2}} = 27$ from the conclusion in Section 2.2, which is in line with the value in Figure 2.3.

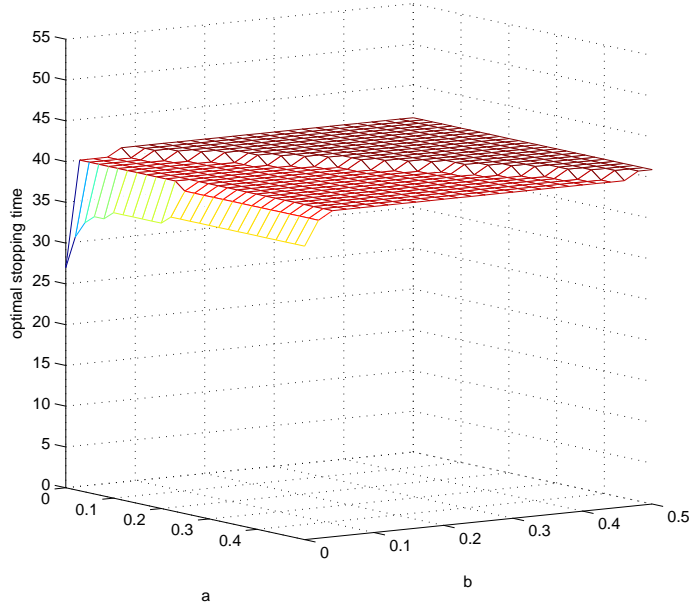


Figure 2.3: The optimal stopping time depending on the parameters a and b of the expected volume $\mu_t = \frac{at^2+bt+1}{\frac{aT(T+1)(2T+1)}{6} + \frac{bT(T+1)}{2} + T} \mu_{\text{total}}$.

3. case: Assume $\mu_\ell = \mu$ is constant and $\sigma_t^2 = k + nt$.

We can get

$$\alpha_{A_{t,p}^*} = \frac{(S-s)(1-c)}{|A_{t,p}^*|} - c\mu$$

If $|\mu_\ell - \mu_j| < \frac{(1-c)(S-s)}{c(T-t)}$ for all ℓ and j , then $y_\ell^* > 0$ for all ℓ , which means $A_{t,p}^* = \{t+1, \dots, p\}$. Indeed, we want for any A the positivity of

$$y_\ell^* = \frac{S-s}{|A|} + \frac{c}{1-c} \left(\mu_\ell - \frac{1}{|A|} \sum_{j \in A} \mu_j \right) > 0,$$

which means

$$\mu_\ell - \frac{1}{|A|} \sum_{j \in A} \mu_j > -\frac{(1-c)(S-s)}{c|A|}.$$

We know that

$$\begin{aligned} \left| \mu_\ell - \frac{1}{|A|} \sum_{j \in A} \mu_j \right| &= \left| \frac{\sum_{j \in A} (\mu_\ell - \mu_j)}{|A|} \right| \\ &\leq \frac{\sum_{j \in A} |\mu_\ell - \mu_j|}{|A|}, \end{aligned}$$

which means that if $|\mu_\ell - \mu_j| < \frac{(1-c)(S-s)}{c(T-t)}$ for all ℓ and j , then $y_\ell^* > 0$ for all ℓ .

Hence, in this case, we have

$$\alpha_{A_{t,p}^*} = \frac{(S-s)(1-c)}{p-t} - c\mu,$$

$$g(p) = c^2(p-t) \left(k + \mu^2 + \frac{n(p+t+1)}{2} \right) + \frac{(S-s)^2(1-c)^2}{p-t} - 2c\mu(S-s)(1-c).$$

Let $q = p - t$, $C_1 = c^2 \left(k + \mu^2 + \frac{n(2t+1)}{2} \right)$, $C_2 = \frac{c^2 n}{2}$, $C_3 = (S-s)^2(1-c)^2$ and $C_4 = -2c\mu(S-s)(1-c)$. We then obtain

$$\tilde{g}(q) = g(q+t) = C_2 q^2 + C_1 q + \frac{C_3}{q} + C_4.$$

Hence, the problem is equivalent to minimizing \tilde{g} over integers and finding the integer-valued minimizer q of \tilde{g} . We first find the minimizer over the real numbers in $[1, +\infty)$. We take the derivative of \tilde{g} , resulting in

$$\tilde{g}'(q) = 2C_2 q - \frac{C_3}{q^2} + C_1.$$

Then we set $\tilde{g}'(q^*) = 0$, in other words, we need to find the solution q^* to

$$2C_2 q^3 + C_1 q^2 - C_3 = 0.$$

The discriminant of this cubic function equals

$$\Delta = 4C_1^3C_3 - 108C_2^2C_3^2.$$

The following cases need to be considered:

If $\Delta > 0$, then the equation has three distinct real roots.

If $\Delta = 0$, then the equation has a multiple root and all its roots are real.

If $\Delta < 0$, then the equation has one real root and two nonreal complex conjugate roots.

The general formula of the roots is given by

$$\begin{aligned} x_1 &= -\frac{C_1}{6C_2} - \frac{C}{6C_2} - \frac{C_1^2}{6C_2C}, \\ x_2 &= -\frac{C_1}{6C_2} + \frac{C(1+i\sqrt{3})}{12C_2} + \frac{C_1^2(1-i\sqrt{3})}{12C_2C}, \\ x_3 &= -\frac{C_1}{6C_2} + \frac{C(1-i\sqrt{3})}{12C_2} + \frac{C_1^2(1+i\sqrt{3})}{12C_2C}, \end{aligned}$$

where $Q = \sqrt{-108C_2^2\Delta}$ and $C = \sqrt[3]{\frac{Q + 2C_1^3 - 108C_2^2C_3}{2}}$.

Case A: $\Delta < 0$

Set $v_1 = \lfloor x_1 \rfloor$, $v_2 = \lceil x_1 \rceil$,

1.case: $T - t \leq x_1$ or $x_1 \leq 1$

If $\tilde{g}(1) \leq \tilde{g}(T - t)$, then we take $\tau^* = t + 1$ so that

$$V_2(t, s) = C_2 + C_1 + C_3 + C_4. \tag{2.7}$$

If $\tilde{g}(T-t) \leq \tilde{g}(1)$, then we take $\tau^* = T$ so that

$$V_2(t, s) = C_2(T-t)^2 + C_1(T-t) + \frac{C_3}{T-t} + C_4. \quad (2.8)$$

2.case: $1 < x_1 < T-t$

If $\tilde{g}(v_1)$ is smallest, then we take $\tau^* = v_1$ so that

$$V_2(t, s) = C_2v_1^2 + C_1v_1 + \frac{C_3}{v_1} + C_4.$$

If $\tilde{g}(v_2)$ is smallest, then we take $\tau^* = v_2$ so that

$$V_2(t, s) = C_2v_2^2 + C_1v_2 + \frac{C_3}{v_2} + C_4.$$

If $\tilde{g}(1)$ is smallest, then we take $\tau^* = t+1$ so that (2.7) holds.

If $\tilde{g}(T-t)$ is smallest, then we take $\tau^* = T$ so that (2.8) holds.

Case B: $\Delta \geq 0$

Let m_1, m_2 and m_3 be the largest, medium and smallest numbers, respectively, among x_1, x_2, x_3 ; We set $p_1 = \lfloor m_3 \rfloor$, $p_2 = \lceil m_3 \rceil$, $p_3 = \lfloor m_2 \rfloor$ and $p_4 = \lceil m_2 \rceil$, $p_5 = \lfloor m_1 \rfloor$ and $p_6 = \lceil m_1 \rceil$ and consider the following situations:

1.case: $T-t < m_3$ or $m_3 \leq 1 < T-t \leq m_2$ or $m_2 \leq 1 < T-t \leq m_1$ or $m_1 \leq 1$

If $\tilde{g}(1) \leq \tilde{g}(T-t)$, then we take $\tau^* = t+1$ so that (2.7) holds.

If $\tilde{g}(T-t) \leq \tilde{g}(1)$, then we take $\tau^* = T$ so that (2.8) holds.

2.case: $1 \leq m_3 \leq T-t \leq m_2$

Compare $\tilde{g}(p_1), \tilde{g}(p_2), \tilde{g}(1), \tilde{g}(T-t)$. If $\tilde{g}(p_j)$ is smallest for $j = 1$ or 2 , then we

take $\tau^* = t + p_j$ so that

$$V_2(t, s) = C_2 p_j^2 + C_1 p_j + \frac{C_3}{p_j} + C_4. \quad (2.9)$$

If $\tilde{g}(1) \leq \tilde{g}(T - t)$, then we take $\tau^* = t + 1$ so that (2.7) holds.

If $\tilde{g}(T - t) \leq \tilde{g}(1)$, then we take $\tau^* = T$ so that (2.8) holds.

3.case: $1 \leq m_3$ and $m_2 \leq T - t \leq m_1$

Compare $\tilde{g}(p_1), \tilde{g}(p_2), \tilde{g}(p_3), \tilde{g}(p_4), \tilde{g}(1), \tilde{g}(T - t)$

If $\tilde{g}(p_j)$ is smallest for some $j = 1, \dots, 4$, then we take $\tau^* = t + p_j$ so that (2.9) holds.

If $\tilde{g}(1) \leq \tilde{g}(T - t)$, then we take $\tau^* = t + 1$ so that (2.7) holds.

If $\tilde{g}(T - t) \leq \tilde{g}(1)$, then we take $\tau^* = T$ so that (2.8) holds.

4.case: $1 \leq m_3$ and $m_1 \leq T - t$

Compare $\tilde{g}(p_1), \tilde{g}(p_2), \tilde{g}(p_3), \tilde{g}(p_4), \tilde{g}(p_5), \tilde{g}(p_6), \tilde{g}(1), \tilde{g}(T - t)$.

If $\tilde{g}(p_j)$ is smallest for some $j = 1, \dots, 6$, then we take $\tau^* = t + p_j$ so that (2.9) holds.

If $\tilde{g}(1) \leq \tilde{g}(T - t)$, then we take $\tau^* = t + 1$ so that (2.7) holds.

If $\tilde{g}(T - t) \leq \tilde{g}(1)$, then we take $\tau^* = T$ so that (2.8) holds.

5.case: $m_3 \leq 1 \leq m_2$ and $m_2 \leq T - t \leq m_1$

If $\tilde{g}(p_j)$ is smallest for some $j = 3$ or 4 , then we take $\tau^* = t + p_j$ so that (2.9) holds.

If $\tilde{g}(1) \leq \tilde{g}(T - t)$, then we take $\tau^* = t + 1$ so that (2.7) holds.

If $\tilde{g}(T - t) \leq \tilde{g}(1)$, then we take $\tau^* = T$ so that (2.8) holds.

6.case: $m_3 \leq 1 \leq m_2$ and $m_1 \leq T - t$

If $g(p_3)$ is smallest, then we take $\tau^* = t + p_3$ so that If $\tilde{g}(p_j)$ is smallest for $j = 3, \dots, 6$, then we take $\tau^* = t + p_j$ so that (2.9) holds.

If $\tilde{g}(1) \leq \tilde{g}(T - t)$, then we take $\tau^* = t + 1$ so that (2.7) holds.

If $\tilde{g}(T - t) \leq \tilde{g}(1)$, then we take $\tau^* = T$ so that (2.8) holds.

7.case When $m_2 \leq 1 \leq m_1$ and $m_1 \leq T - t$,

If $\tilde{g}(p_j)$ is smallest for $j = 5$ or 6 , then we take $\tau^* = t + p_j$ so that (2.9) holds.

If $\tilde{g}(1) \leq \tilde{g}(T - t)$, then we take $\tau^* = t + 1$ so that (2.7) holds.

If $\tilde{g}(T - t) \leq \tilde{g}(1)$, then we take $\tau^* = T$ so that (2.8) holds.

Following this idea, we program the MATLAB code presented in Appendix A.2.3, which returns τ and V_2 . For example, choosing $T = 50, t = 30, s = 500, c = 0.5, S = 600, \mu = 5, k = 5, n = 10$, we get $\tau = 35$ and $V_2 = 700$.

2.5 Performance Analysis using Trading Data

In this section, we analyze the performance of the strategy from Section 2.4 using real trading data.¹ As examples, we consider trading volumes from four large US companies: General Electric Company (GE), Microsoft Corporation (MSFT), Wal-Mart Stores Inc. (WMT) and Exxon Mobil Corporation (XOM). We use intraday volume observed at 5 minute frequencies over the trading days from January 2010 to December 2012. We compare the average error of our method and two alternative strategies $\tilde{Y}_{t,j}$ and $\bar{Y}_{t,j}$: as long as the volume is below S , they are given by $\tilde{Y}_{t,j} = \frac{c}{40} \sum_{\ell=1}^{40} X_{t-41+\ell,j}$ and $\bar{Y}_{t,j} = \frac{1}{1-c} \tilde{Y}_{t,j}$, where t is the trading day and j is the period. These are two reasonable alternatives:

¹Wharton Research Data Service (WRDS) was used for this study. This service and the data available thereon constitute valuable intellectual property and trade secrets of WRDS and/or its third-party suppliers.

the first is just c times the historical mean of market volume over the last 40 trading days while the second is chosen such that $\frac{\bar{Y}_{t,j}}{\bar{Y}_{t,j} + \mu_{t,j}} = c$, where $\mu_{t,j}$ is the historical mean.

We can run the program shown in Appendix A.3 to compare the performance of the different strategies. We choose as parameters a trading volume of $S = 100,000$ shares and a participation rate $c = 0.1$. Tables 2.1 and 2.2 show the results for the different strategies: Strategy 1 is based on our method from Section 2.4, Strategy 2 uses $\tilde{Y}_{t,j}$ and Strategy 3 uses $\bar{Y}_{t,j}$. Error 1 means the average error of the simplified problem $\sum_{j=1}^{\tau} (Y_j - c(X_j + Y_j))^2$ and Error 2 means the average error of the original problem $\sum_{j=1}^{\tau} \left(\frac{Y_j}{X_j + Y_j} - c \right)^2$. We see in all the examples of Tables 2.1 and 2.2 that Strategy 1 has the lowest Errors 1 and 2.

Error 1	Strategy 1	Strategy 2		Strategy 3	
GE	8.3897×10^9	1.115×10^{10}	+32.9%	9.9214×10^9	+18.3%
MSFT	6.1815×10^9	7.9237×10^9	+28.2%	6.8805×10^9	+11.3%
WMT	9.0539×10^8	1.1743×10^9	+29.7%	1.0826×10^9	+19.6%
XOM	8.9490×10^8	1.3960×10^9	+55.9%	1.4021×10^9	+56.7%

Table 2.1: Comparison of Error 1 for different strategies

Error 2	Strategy 1	Strategy 2		Strategy 3	
GE	0.0022870	0.0041416	+81.1%	0.0037910	+65.8%
MSFT	0.0020464	0.0035300	+72.5%	0.0026895	+31.4%
WMT	0.0158720	0.0162800	+2.6%	0.0180120	+13.5%
XOM	0.0060703	0.0073188	+20.6%	0.0083281	+37.2%

Table 2.2: Comparison of Error 2 for different strategies

Chapter 3

Original Problem

3.1 Without Stopping

In this section, we assume that the market volumes X_j are independent and identically distributed with mean $\mu = E[X_j]$ and variance $\sigma^2 = \text{Var}(X_j)$. We now turn our attention to the problem

$$V_3(t, s) = \min_{\substack{Y_j \geq 0 \\ j=t+1, \dots, T}} E \left[\sum_{j=t+1}^T \left(\frac{Y_j}{X_j + Y_j} - c \right)^2 \right]$$

where the minimization is under the constraint that $\sum_{j=t+1}^T Y_j = S - s$. Similarly to Section 2.1, we obtain that

$$\begin{aligned} V_3(t, s) &= \min_{Y_j \geq 0} E \left[\sum_{j=t+1}^T \left(\frac{Y_j}{X_j + Y_j} - c \right)^2 \right] \\ &= \min_{Y_j \geq 0} E \left[\sum_{j=t+1}^{T-1} \left(\frac{Y_j}{X_j + Y_j} - c \right)^2 + \left(\frac{Y_T}{X_T + Y_T} - c \right)^2 \right], \end{aligned}$$

which equals

$$\begin{aligned}
V_3(t, s) &= \min_{Y_j \geq 0} E \left[\sum_{j=t+1}^{T-1} E \left[\left(\frac{Y_j}{X_j + Y_j} - c \right)^2 \middle| \mathcal{F}_{j-1} \right] \right. \\
&\quad \left. + E \left[\left(\frac{S - s - \sum_{j=t+1}^{T-1} Y_j}{X_T + S - s - \sum_{j=t+1}^{T-1} Y_j} - c \right)^2 \middle| \mathcal{F}_{T-1} \right] \right] \\
&= \min_{Y_j \geq 0} E \left[\sum_{j=t+1}^{T-1} E \left[\left(\frac{y_j}{X_j + y_j} - c \right)^2 \right] \middle|_{y_j = Y_j(\omega)} \right. \\
&\quad \left. + E \left[\left(\frac{S - s - \sum_{j=t+1}^{T-1} y_j}{X_T + S - s - \sum_{j=t+1}^{T-1} y_j} - c \right)^2 \right] \middle|_{y_{t+1} = Y_{t+1}(\omega), \dots, y_{T-1} = Y_{T-1}(\omega)} \right]
\end{aligned}$$

where the last equality follows from Lemma 2.1.

Define a function

$$\begin{aligned}
g(y_{t+1}, \dots, y_{T-1}) &:= \sum_{j=t+1}^{T-1} E \left[\left(\frac{y_j}{X_j + y_j} - c \right)^2 \right] \\
&\quad + E \left[\left(\frac{S - s - \sum_{j=t+1}^{T-1} y_j}{X_T + S - s - \sum_{j=t+1}^{T-1} y_j} - c \right)^2 \right].
\end{aligned}$$

This is a deterministic function $g : \mathbb{R}^{T-t+1} \rightarrow \mathbb{R}$ and we can write

$$V_3(t, s) = \min_{Y_j \geq 0} E [g(Y_{t+1}, \dots, Y_{T-1})].$$

Hence, the problem is equivalent to minimizing g and finding a deterministic minimizer $y^* = (y_{t+1}^*, \dots, y_{T-1}^*)$ of g . To do so, we take partial derivatives of

g with respect to y_ℓ for $\ell = t + 1, \dots, T - 1$, which results in

$$\begin{aligned} \frac{\partial g}{\partial y_\ell} &= E \left[2 \left(\frac{y_\ell}{X_\ell + y_\ell} - c \right) \frac{X_\ell}{(X_\ell + y_\ell)^2} \right] \\ &\quad - E \left[2 \left(\frac{S - s - \sum_{j=t+1}^{T-1} y_j}{X_T + S - s - \sum_{j=t+1}^{T-1} y_j} - c \right) \frac{X_T}{(X_T + S - s - \sum_{j=t+1}^{T-1} y_j)^2} \right] \\ &= E \left[2 \left(\frac{y_\ell}{X_\ell + y_\ell} - c \right) \frac{X_\ell}{(X_\ell + y_\ell)^2} \right] - E \left[2 \left(\frac{y_T}{X_T + y_T} - c \right) \frac{X_T}{(X_T + y_T)^2} \right]. \end{aligned}$$

We set $\frac{\partial g}{\partial y_\ell}$ equal to 0. While we cannot solve this explicitly in general, we give three examples.

1. Example: Consider the simple situation with $T = 2$ and $X_1 = X_2 = \mu$ deterministic, which results in

$$\left(\frac{y_1}{\mu + y_1} - c \right) \frac{\mu}{(\mu + y_1)^2} = \left(\frac{y_2}{\mu + y_2} - c \right) \frac{\mu}{(\mu + y_2)^2}.$$

We can change this equation to

$$\frac{c\mu - (1 - c)y_1}{c\mu - (1 - c)y_2} = \frac{(\mu + y_1)^3}{(\mu + y_2)^3}.$$

Set $k_1 = c\mu$, $k_2 = 1 - c$ and $k_3 = S - s$, then simplify the equation to

$$(y_2 - y_1) \left((3\mu^2 k_1 + 3\mu k_1 k_3 + k_1 k_3^2 + \mu^3 k_2) - (k_1 + 3\mu k_2 + k_2 k_3) y_1 y_2 \right) = 0.$$

Let $u = 3\mu^2 k_1 + 3\mu k_1 k_3 + k_1 k_3^2 + \mu^3 k_2$ and $v = k_1 + 3\mu k_2 + k_2 k_3$, then

$$(y_2 - y_1)(u - v y_1 y_2) = 0.$$

Thus, $y_1 = y_2$ or $y_1 y_2 = \frac{u}{v}$.

If $y_1 = y_2$, with condition $y_1 + y_2 = S - s$, we get

$$y_1 = y_2 = \frac{S - s}{2}. \quad (3.1)$$

If $y_1 y_2 = \frac{u}{v}$, with condition $y_1 + y_2 = S - s$, we distinguish three cases.

1. *Case:* If $k_3^2 - \frac{4u}{v} < 0$, then there is no solution for y_1 and y_2 in addition to (3.1).

2. *Case:* If $k_3^2 - \frac{4u}{v} = 0$, then

$$y_1 = y_2 = \frac{S - s}{2},$$

which corresponds to (3.1).

3. *Case:* If $k_3^2 - \frac{4u}{v} > 0$, then there is a solution given by

$$y_{3,4} = \frac{k_3 \pm \sqrt{k_3^2 - \frac{4u}{v}}}{2}.$$

In conclusion, if $k_3^2 - \frac{4u}{v} \leq 0$, there exists a unique solution given by (3.1) while in the case $k_3^2 - \frac{4u}{v} > 0$, there exist two solutions. We next examine which of the two leads to the minimizer.

Recall that the objective function is $g = f(\tilde{y}) + f(S - s - \tilde{y})$, where $f(y) = \left(\frac{y}{\mu + y} - c\right)^2$. To find out more about the minimizer of g , we study the function f . We take the first derivative of $f(y)$ and get

$$f'(y) = 2 \left(\frac{y}{\mu + y} - c \right) \frac{\mu}{(\mu + y)^2}.$$

For $f'(y) = 0$, we get $y^* = \frac{c\mu}{1-c}$. Taking the second derivative of $f(y)$, we obtain

$$\begin{aligned} f''(y) &= \frac{2\mu^2}{(\mu+y)^4} + 2 \left(\frac{y}{\mu+y} - c \right) \frac{-2\mu}{(\mu+y)^3} \\ &= \frac{2\mu}{(\mu+y)^3} \left(\frac{\mu}{\mu+y} - \frac{2y}{\mu+y} + 2c \right) \\ &= \frac{2\mu}{(\mu+y)^3} \left(\frac{\mu-2y}{\mu+y} + 2c \right). \end{aligned}$$

For $f''(y) = 0$, we get $y^* = \frac{(1+2c)\mu}{2(1-c)}$. Figure 3.1 illustrates the function f for the parameters $\mu = 300$ and $c = 0.5$, based on the MATLAB code presented in Appendix A.4. Choosing $y = \frac{c\mu}{1-c}$ (300 in the plot), we can get the minimal

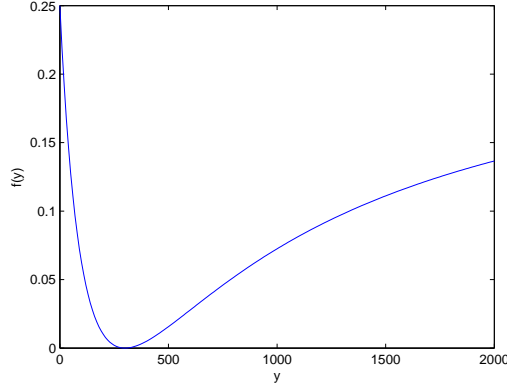


Figure 3.1: Illustration of the penalty function f as function of our trading volume y .

value which is equal to 0; when choosing $y^* = \frac{(1+2c)\mu}{2(1-c)}$, we can get the point $(y^*, f(y^*))$, which is the inflection point.

Note that $f''(y) \leq 0$ for all $y \in \left[0, \frac{(1+2c)\mu}{2(1-c)}\right]$, hence f is convex on $\left[0, \frac{(1+2c)\mu}{2(1-c)}\right]$.

Thus, we can get the conclusion that when $0 \leq S - s \leq \frac{(1+2c)\mu}{2(1-c)}$, then the minimizer is $y_1 = y_2 = \frac{S-s}{2}$, since $f(y_3) + f(y_4) \geq 2f\left(\frac{y_3+y_4}{2}\right) = f(y_1) + f(y_2)$.

Realistically, we should satisfy the condition $\frac{c\mu}{1-c}(T-t) > S - s$, where $\frac{c\mu}{1-c}$

means the optimal volume per period without any constraint. This condition says that the client has chosen c such that following the expected market volume optimally will give at least the required amount of shares. In particular, when considering two periods, we should satisfy $\frac{2c\mu}{1-c} > S - s$. We have

$$\frac{(1+2c)\mu}{2(1-c)} - \frac{2c\mu}{1-c} = \frac{1-2c}{2(1-c)}.$$

Therefore, if we make $c < 0.5$, then $\frac{2c\mu}{1-c} > S - s$ implies $\frac{(1+2c)\mu}{2(1-c)} > S - s$ and thus the minimizer is $y_1 = y_2 = \frac{S-s}{2}$. Actually, in the second example presented next, we will find the result that the minimizer is the equal distribution of volume holds for all c .

2. Example: We now consider the multiperiod generalization of Example 1. The objective function is to minimize $\sum_{\ell=t+1}^T (\frac{y_\ell}{y_\ell+\mu} - c)^2$ under the constraint $\sum_{\ell=t+1}^T y_\ell = S - s$.

Lemma 3.1: *Under the reasonable assumption $\frac{S-s}{T-t} < \frac{c\mu}{1-c}$, the optimal strategy is $y_{t+1}^* = \dots = y_T^* = \frac{S-s}{T-t}$.*

Proof. Similarly to the first example, we can write $\sum_{\ell=t+1}^T (\frac{y_\ell}{y_\ell+\mu} - c)^2 = \sum_{\ell=t+1}^T f(y_\ell)$ for $f(y) = (\frac{y}{y+\mu} - c)^2$. Recall that f is convex on $[0, \frac{(1+2c)\mu}{2(1-c)}]$. Let y_{t+1}, \dots, y_T be nonnegative numbers such that $\sum_{\ell=t+1}^T y_\ell = S - s$. We set $\bar{y}_\ell = \min\{y_\ell, \frac{c\mu}{1-c}\}$. Since f is increasing on $[\frac{c\mu}{1-c}, +\infty)$, we have $f(y_\ell) \geq f(\bar{y}_\ell)$ for all ℓ . Therefore, we obtain

$$\sum_{\ell=t+1}^T f(y_\ell) \geq \sum_{\ell=t+1}^T f(\bar{y}_\ell) \geq (T-t)f\left(\frac{\sum_{\ell=t+1}^T \bar{y}_\ell}{T-t}\right)$$

by Jensen's inequality using that f is convex on $[0, \frac{c\mu}{1-c}]$.

Because $\bar{y}_\ell \leq y_\ell$ and $\sum_{\ell=t+1}^T y_\ell = S - s$, we have $\frac{\sum_{\ell=t+1}^T \bar{y}_\ell}{T-t} \leq \frac{S-s}{T-t}$, and hence

$(T - t)f\left(\frac{\sum_{\ell=t+1}^T \bar{y}_\ell}{T-t}\right) \geq (T - t)f\left(\frac{S-s}{T-t}\right)$, since f is decreasing on $[0, \frac{c\mu}{1-c}]$ and $\frac{S-s}{T-t} \leq \frac{c\mu}{1-c}$ by assumption.

In conclusion, this shows

$$\sum_{\ell=t+1}^T f(y_\ell) \geq (T - t)f\left(\frac{S - s}{T - t}\right),$$

which proves that the minimizer is $y_{t+1}^* = \dots = y_T^* = \frac{S-s}{T-t}$. \square

3. *Example:* We now consider uniformly distributed volumes. Assume that

$$o(\tilde{y}) = E \left[2 \left(\frac{\tilde{y}}{X_\ell + \tilde{y}} - c \right) \frac{X_\ell}{(X_\ell + \tilde{y})^2} \right]$$

and $X_\ell \sim U[a, b]$, $a, b > 0$. The optimizers y_ℓ satisfy $o(y_\ell) = o(y_T)$. If o is strictly monotone, then we can say that the solution is unique with

$$y_\ell^* = y_T^*, \quad \ell = t + 1, \dots, T - 1.$$

Then we sum all the y_ℓ^* for $\ell = t + 1, \dots, T$ so that

$$\sum_{j=t+1}^T y_j^* = (T - t)y_\ell^* = S - s.$$

Hence, we get

$$y_\ell^* = \frac{S - s}{T - t}, \quad \ell = t + 1, \dots, T.$$

Instead, if for example o is strictly increasing and there exists $y_\ell < \frac{S-s}{T-t}$, then there must exist $y_j > \frac{S-s}{T-t}$ because $\sum_{j=t+1}^T y_j = S - s$. But then $o(y_j) > o(\frac{S-s}{T-t}) > o(y_\ell)$, which contradicts to $o(y_j) = o(y_\ell)$.

Take the derivative of $o(\tilde{y})$, which results in

$$\begin{aligned}
o'(\tilde{y}) &= E \left[\frac{2X_\ell((1+2c)X_\ell - (2-2c)\tilde{y})}{(X_\ell + \tilde{y})^4} \right] \\
&= \frac{1}{b-a} \int_a^b \left[\frac{2+4c}{(x+\tilde{y})^2} + \frac{(-4c-8)\tilde{y}}{(x+\tilde{y})^3} + \frac{6\tilde{y}^2}{(x+\tilde{y})^4} \right] dx \\
&= \frac{1}{b-a} \left(-\frac{2+4c}{x+\tilde{y}} - \frac{(-4c-8)\tilde{y}}{2(x+\tilde{y})^2} - \frac{6\tilde{y}^2}{3(x+\tilde{y})^3} \right) \Big|_a^b \\
&= \frac{1}{(a+\tilde{y})(b+\tilde{y})} \left[2+4c + \frac{(-4c-8)\tilde{y}(a+b+2\tilde{y})}{2(a+\tilde{y})(b+\tilde{y})} \right. \\
&\quad \left. + \frac{6\tilde{y}^2(a^2+ab+b^2+3a\tilde{y}+3b\tilde{y}+3\tilde{y}^2)}{3(a+\tilde{y})^2(b+\tilde{y})^2} \right] \\
&= \frac{1}{(a+\tilde{y})(b+\tilde{y})} \left[\frac{6(1+2c)(a+\tilde{y})^2(b+\tilde{y})^2 + 6\tilde{y}^2(4\mu^2 - ab + 6\mu\tilde{y} + 3\tilde{y}^2)}{3(a+\tilde{y})^2(b+\tilde{y})^2} \right. \\
&\quad \left. - \frac{12(c+2)\tilde{y}(\mu+\tilde{y})(a+\tilde{y})(b+\tilde{y})}{3(a+\tilde{y})^2(b+\tilde{y})^2} \right] \\
&= \frac{1}{(a+\tilde{y})(b+\tilde{y})} \left[\frac{6(1+2c)(a+\tilde{y})^2(b+\tilde{y})^2 + 18\tilde{y}^2(\mu+\tilde{y})^2 + \frac{3\tilde{y}^2(a-b)^2}{2}}{3(a+\tilde{y})^2(b+\tilde{y})^2} \right. \\
&\quad \left. - \frac{12(c+2)\tilde{y}(\mu+\tilde{y})(a+\tilde{y})(b+\tilde{y})}{3(a+\tilde{y})^2(b+\tilde{y})^2} \right] \\
&= \frac{2}{(a+\tilde{y})^3(b+\tilde{y})^3} \left[\left((2\tilde{y}^2 + 2(a+b)\tilde{y})c - 2\tilde{y}^2 - \frac{(a+b)\tilde{y}}{2} + (1+2c)ab \right) \right. \\
&\quad \left. \times \left((a+\tilde{y})(b+\tilde{y}) - \tilde{y}(\tilde{y}+\mu) \right) + \frac{3\tilde{y}^2(a-b)^2}{2} \right].
\end{aligned}$$

Set $k = \frac{a}{b} \in (0, 1)$, and consider

$$\begin{aligned}
f(c) &= \left((2\tilde{y}^2 + 2(a+b)\tilde{y})c - 2\tilde{y}^2 - \frac{(a+b)\tilde{y}}{2} + (1+2c)ab \right) \\
&\quad \times \left((a+\tilde{y})(b+\tilde{y}) - \tilde{y}(\tilde{y}+\mu) \right) + \frac{3\tilde{y}^2(a-b)^2}{2}.
\end{aligned}$$

For $o'(\tilde{y}) > 0$, which means $f(c) > 0$, it is enough to have $f(0) \geq 0$ and $f(1) \geq 0$, because f depends linearly on c .

When $c = 1$, we get

$$f(1) = \left[\frac{3(a+b)\tilde{y}}{2} + (1+2c)ab \right] \left[\frac{(k+1)b\tilde{y}}{2} + kb^2 \right] + \frac{3\tilde{y}^2(k-1)^2b^2}{2} > 0.$$

When $c = 0$, we get

$$\begin{aligned} f(0) &= \left[-2\tilde{y}^2 - \frac{(k+1)b\tilde{y}}{2} + kb^2 \right] \left[\frac{(k+1)b\tilde{y}}{2} + kb^2 \right] + \frac{3\tilde{y}^2(k-1)^2b^2}{2} \\ &= k^2b^3 + \frac{5}{4}(k^2 - 4k + 1)\tilde{y}^2b - (k+1)\tilde{y}^3 \\ &> k^2\tilde{y}^3 + \frac{5}{4}(k^2 - 4k + 1)\tilde{y}^2\tilde{y} - (k+1)\tilde{y}^3 \\ &= \frac{1}{4}(9k^2 - 24k + 1)\tilde{y}^3. \end{aligned}$$

Thus, if we make $9k^2 - 24k + 1 \geq 0$, then $f(0) > 0$, which means we need to have $0 < k < \frac{4-\sqrt{15}}{3}$. In this case, the minimizer of $V_3(t, s)$ is deterministic with $y_\ell^* = \frac{s-s}{T-t}$. This means that we distribute our trades equally, which is what we expect because the market volume is i.i.d. and we cannot finish our trading early as opposed to the setting of the next section. The corresponding value function equals

$$\begin{aligned} V_3(t, s) &= \min_{Y_j \geq 0} E \left[\sum_{j=t+1}^T \left(\frac{Y_j}{X_j + Y_j} - c \right)^2 \right] \\ &= (T-t)E \left[\left(\frac{Y_\ell^*}{X_\ell + Y_\ell^*} \right)^2 - \frac{2cY_\ell^*}{X_\ell + Y_\ell^*} + c^2 \right] \\ &= \frac{T-t}{b-a} \int_a^b \left[\left(\frac{y_\ell^*}{x + y_\ell^*} \right)^2 - \frac{2cy_\ell^*}{x + y_\ell^*} + c^2 \right] dx, \end{aligned}$$

which can be simplified to

$$\begin{aligned}
V_3(t, s) &= \frac{T-t}{b-a} \left[-\frac{y_\ell^{*2}}{x+y_\ell^*} - 2cy_\ell^* \ln|x+y_\ell^*| + c^2x \right] \Big|_a^b \\
&= \frac{T-t}{b-a} \left[-\frac{(S-s)^2}{b(T-t)^2 + (S-s)(T-t)} - 2c\frac{S-s}{T-t} \ln \left| b + \frac{S-s}{T-t} \right| + c^2b \right] \\
&\quad - \frac{T-t}{b-a} \left[-\frac{(S-s)^2}{a(T-t)^2 + (S-s)(T-t)} - 2c\frac{S-s}{T-t} \ln \left| a + \frac{S-s}{T-t} \right| + c^2a \right] \\
&= \frac{(T-t)(S-s)^2}{ab(T-t)^2 + (a+b)(T-t)(S-s) + (S-s)^2} \\
&\quad + \frac{2c(S-s)}{(b-a)} \ln \left(\frac{a(T-t) + S-s}{b(T-t) + S-s} \right) + (T-t)c^2.
\end{aligned}$$

3.2 With Stopping

We first show that it is enough to minimize over deterministic stopping times.

Recall that the value function is now given by

$$\begin{aligned}
V_4(t, s) &= \min_{Y_j \geq 0} E \left[\sum_{j=t+1}^{\tau} \left(\frac{Y_j}{X_j + Y_j} - c \right)^2 \right] \\
&= \min_{Y_j \geq 0} E \left[\sum_{j=t+1}^T \left(\frac{Y_j}{X_j + Y_j} - c \right)^2 \mathbb{1}_{\{\tau \geq j\}} \right] \\
&= \min_{Y_j \geq 0} E \left[\sum_{j=t+1}^T \left(\frac{Y_j}{X_j + Y_j} - c \right)^2 \mathbb{1}_{\{\tau \leq j-1\}^c} \right] \\
&= \min_{Y_j \geq 0} E \left[\sum_{j=t+1}^T E \left[\left(\frac{Y_j}{X_j + Y_j} - c \right)^2 \middle| \mathcal{F}_{j-1} \right] \mathbb{1}_{\{\tau \leq j-1\}^c} \right] \\
&= \min_{Y_j \geq 0} E \left[\sum_{j=t+1}^T E \left[\left(\frac{y_j}{X_j + y_j} - c \right)^2 \right] \middle|_{y_j = Y_j(w)} \mathbb{1}_{\{\tau \leq j-1\}^c} \right],
\end{aligned}$$

using Lemma 2.1 for the last equality and that $\{\tau \leq j-1\}$ is \mathcal{F}_{j-1} -measurable for the second last equality because τ is a stopping time. Note that the last

equality uses that X_j is independent of \mathcal{F}_{j-1} since $(X_j)_{j=1,\dots,T}$ are i.i.d.

Define $g(y) = E \left[\left(\frac{y}{X_j+y} - c \right)^2 \right]$, then

$$\begin{aligned} V_4(t, s) &= \min_{y_j \geq 0} E \left[\sum_{j=t+1}^T g(y_j) \mathbb{1}_{\{\tau \leq j-1\}^c} \right] \\ &= \min_{y_j \geq 0} E \left[\sum_{j=t+1}^{\tau} g(y_j) \right] \end{aligned}$$

subject to $\sum_{j=t+1}^{\tau} y_j = S - s$. Note that this last formulation does not involve any given random variables. We can minimize over deterministic Y_j and τ .

1. Example: Consider the multiperiod situation with deterministic volumes $X_\ell = \mu, \ell \in [t+1, T]$. By using Example 2 of Section 3.1, we obtain

$$V_3(t, s) = (T - t) \left(\frac{\frac{S-s}{T-t}}{\mu + \frac{S-s}{T-t}} - c \right)^2$$

and we set

$$\tilde{V}_3(\tau) = (\tau - t) \left(\frac{\frac{S-s}{\tau-t}}{\mu + \frac{S-s}{\tau-t}} - c \right)^2$$

for the value function between time t and τ . The question becomes

$$V_4(t, s) = \min_{\tau=t+1,\dots,T} \tilde{V}_3(\tau).$$

Define a deterministic function $g : \mathbb{R} \rightarrow \mathbb{R}$ by

$$g(p) := (p - t) \left(\frac{\frac{S-s}{p-t}}{\mu + \frac{S-s}{p-t}} - c \right)^2.$$

Hence, the problem is equivalent to minimizing g over integers and finding the integer-valued minimizer p^* of g . We first find the minimizer over the real

numbers in $[t, +\infty)$. We take the derivative of g , namely,

$$g'(p) = \left(\frac{S-s}{(p-t)\mu + S-s} - c \right) \times \left(\frac{(1-c)(S-s)^2 - (1+2c)\mu(S-s)(p-t) - c\mu^2(p-t)^2}{((p-t)\mu + S-s)^2} \right).$$

Then we set $g'(p^*) = 0$ so that

$$p_1^* = t + \frac{(1-c)(S-s)}{c\mu},$$

$$p_2^* = t + \frac{(\sqrt{8c+1} - 2c - 1)(S-s)}{2c\mu},$$

$$p_3^* = t + \frac{(-\sqrt{8c+1} - 2c - 1)(S-s)}{2c\mu} < t,$$

so that p_3^* is outside the relevant interval and $p_2^* - p_1^* = \frac{(\sqrt{8c+1}-3)(S-s)}{2c\mu} < 0$.

From $\frac{c\mu}{1-c}(T-t) > S-s$, which means $\frac{(1-c)(S-s)}{c\mu} < T-t$, we get $p_1^* < T$.

We find that g is increasing on $[t, p_2^*)$ and $(p_1^*, T]$, and it is decreasing on $[p_2^*, p_1^*]$.

We set $q_1 = t+1$, $q_2 = \left\lceil t + \frac{(1-c)(S-s)}{c\mu} \right\rceil$ and $q_3 = \left\lceil t + \frac{(1-c)(S-s)}{c\mu} \right\rceil$.

If $g(q_j) = \min[g(q_1), g(q_2), g(q_3)]$ for $j = 1, 2$ or 3 , then we take $\tau^* = q_j$ so that

$$V_4(t, s) = (q_j - t) \left(\frac{\frac{S-s}{q_j-t}}{\mu + \frac{S-s}{q_j-t}} - c \right)^2.$$

2. Example: Consider the same situation as in Example 3 of Section 3.1 with uniformly distributed volume $X_\ell \sim U[a, b]$, $\ell = t+1, \dots, T$ for $a, b > 0$. In this example, it is hard to find the solution explicitly, but we can solve it

numerically.

$$\begin{aligned}
V_4(t, s) &= \min_{y_j \geq 0} E \left[\sum_{j=t+1}^{\tau} E \left[\left(\frac{y_j}{X + y_j} - c \right)^2 \right] \right] \\
&= \min_{y_j \geq 0} \sum_{j=t+1}^{\tau} \int_a^b \left[\left(\frac{y_j}{x + y_j} - c \right)^2 \right] dx \\
&= \min_{y_j \geq 0} \sum_{j=t+1}^{\tau} \frac{1}{b-a} \left[-\frac{y_j^2}{x + y_j} - 2cy_j \ln(x + y_j) + c^2 x \right] \Big|_a^b \\
&= \min_{y_j \geq 0} \sum_{j=t+1}^{\tau} \left[\frac{y_j^2}{(a + y_j)(b + y_j)} + \frac{2cy_j \ln \left(\frac{a+y_j}{b+y_j} \right)}{b-a} + c^2 \right].
\end{aligned}$$

We can run the program shown in Appendix A.5 to find the stopping time and optimal trading volumes.

Choosing $a = 30$, $b = 50$, $S = 0.5$, $s = 0$, $T = 10$, $t = 0$, $c = 0.1$, we will get the stopping time $\tau = 6$, the minimizer $y_\ell^* = 0.0833$, $\ell = 1, \dots, 6$ and $V_4 = 0.12635$.

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- [3] O. Guéant (2013): Execution and Block Trade Pricing with Optimal Constant Rate of Participation. Preprint available at arXiv:1210.7608.
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Appendix A

MATLAB Code

A.1 MATLAB Code to Section 2.2

In the code, we write p_1 instead of p_1 , p_2 instead of p_2 , p instead of \tilde{p} , g_1 instead of $g(p_1)$ and g_2 instead of $g(p_2)$.

```
function [tau,V]=compare4(T,t,s,c,S,mu,sigma)
    if T<t+(1-c)*(S-s)/(c*sqrt(mu^2+sigma^2))
        tau=T;
        V=(T-t)*(((1-c)*(S-s)/(T-t))^2-2*c*mu*(1-c)*(S-s)/(T-t)
            +c^2*(sigma^2+mu^2));
    elseif t+1>t+(1-c)*(S-s)/(c*sqrt(mu^2+sigma^2))
        tau=t+1;
        V=((1-c)*(S-s))^2-2*c*mu*(1-c)*(S-s)
            +c^2*(sigma^2+mu^2);
    else
        p1=floor(t+(1-c)*(S-s)/(c*sqrt(mu^2+sigma^2)));
        p2=ceil(t+(1-c)*(S-s)/(c*sqrt(mu^2+sigma^2)));
        g1=(p1-t)*(((1-c)*(S-s)/(p1-t))^2
```

```

        -2*c*mu*(1-c)*(S-s)/(p1-t)+c^2*(sigma^2+mu^2));
g2=(p2-t)*(((1-c)*(S-s)/(p2-t))^2
        -2*c*mu*(1-c)*(S-s)/(p2-t)+c^2*(sigma^2+mu^2));
if g1<=g2
    tau=p1;
    V=g1;
else
    tau=p2;
    V=g2;
end
end

```

```

function tau=compare(T,t,s,c,S,mu,sigma)
V=zeros(1,T-t);
a=t;
tau=t+1;
for ii=1:T-t
    a=a+1;
    V(ii)=(a-t)*(((1-c)*(S-s)/(a-t))^2
        -2*c*mu*(1-c)*(S-s)/(a-t)+c^2*(sigma^2+mu^2));
    % assign V matrix
end
for ii=1:T-t-1
    if V(ii+1)<=V(ii)
        % compare which element of V
        tau=ii+t+1;
        % find tau when the element of V is smallest
    else
        V(ii+1)=V(ii);
    end
end

```

```

        end
    end
end

```

The following code generates Figure 2.1.

```

S=600;
T=50;
s=0:S-1;
sigma=zeros(50)+5;
c=.5;
mu=zeros(50)+5;
t=0:T-1;
O=zeros(T,S);
    for j=0:T-1
        mu=mu(1:T-j);
        sigma=sigma(1:T-j);
        for k=0:S-1
O(j+1,k+1)=compare(T,j,k,c,S,mu,sigma);
%0 matrix stores tau, the values of its rows
%correspond to t-1 and the values of its columns
%correspond to s-1 since j and k can be zero, but the
%subscript of O matrix cannot be
%zero, we need to add one.
        end
    end
    z=ones(1,T);
%predefine z, z(j) is used to account the
%amounts of the points when given tau=j, j=1,...,T
    for j=1:T

```

```

        eval(['a', num2str(j), '= [0;0];']);
%initialize the matrix aj, j=1,...,T, aj is used to
%store the points when given tau=j, j=1,...,T
    end
    for i=1:T
        for g=1:S
            for j=1:T
                if O(i,g)==j
                    eval(['a', num2str(j), '(:,z(j))=[i;g];']);
%find the points, if the t is same, we
%just store the point which s is the maximum.
%In other words,
%each t maps to one point at most
%and assign the matrix aj, j=1,...,T,
                    end
                end
            end
            for j=i+1:T
                if eval(['a', num2str(j), '(:,z(j))~= [0;0]']);
%since when given tau, it is not possible to
%find the points spreading all the t, so
%we need to account of the existing points
                    z(j)=z(j)+1;
                    end
                end
            end
        end
    for g=1:S
        if O(1,g)==1
            eval(['a', num2str(1), '(:,z(1))=[1;g];']);
%record the point when given tau=1
            z(1)=z(1)+1;
        end
    end
end
end

```



```

        end
    end
    h = figure;
    ColorSet = varycolor(T); set(gca, 'ColorOrder', ColorSet);
    hold all;
    for i=1:T
        eval(['k','=a', num2str(i), '(1,:)']);
        % k presents the vector storing the t+1 when given tau=i
        eval(['sa','=a', num2str(i), '(2,:)']);
        % sa presents the vector storing the s+1 when given tau=i
        plot(k-1,S-sa+1);
        % plot the points(t,S-s) when given each tau
    end
    xlabel('t');ylabel('S-sa');
    axis(gca,[0 T+5 0 400]);
    set(h, 'Units', 'Inches');
    pos = get(h, 'Position');
    set(h, 'PaperPositionMode', 'Auto', 'PaperUnits',
        'Inches', 'PaperSize', [pos(3), pos(4)])
    print(h, 'filename', '-dpdf', '-r0')

```

A.2 MATLAB Code to Section 2.4

A.2.1 Case 1

```

function [n,v,u]=compare22(T,g,t,s,c,e,S,b)
l=[];%keep the y-j
k=[];%keep the \mu_j^2, which j is not in optimal set A

```

```

w=[];%keep the \mu_j, which j is not in optimal set A
x=0;
for ii=1:g-t
    x=x+(b*ii+1)/(b*T*(T+1)/2+T)*e;
end
for i=1:g-t
    q=(S-s)/(g-t)+c/(1-c)*((b*i+1)/(b*T*(T+1)/2+T)*e-x/(g-t));
    z=(b*i+1)/(b*T*(T+1)/2+T)*e;
    if q<0
        q=0;
    else
        z=0;
    end
    l=[1;q];
    k=[k;z^2];
    w=[w;z];
end
m=find(l(:)~=0);%remove y_j=0
n=length(m);%|A|
u=x-sum(w);%\sum\mu_j, which j is in optimal set A
v=sum(k);%\sum\mu_j^2, which j is not in optimal set A
end

```

```

function tau=compare7(T,t,s,c,e,S,b,sigma)
    tau=t+1;
    V=zeros(1,T-t);
    for i=1:T-t
        [n,v,u]=compare22(T,i+t,t,s,c,e,S,b);
        al=(S-s)*(1-c)/n-c*u/n;
    end
end

```

```

        V(i)=c^2*sigma^2*i+n*al^2+c^2*v;% assign V matrix
    end
    for ii=1:T-t-1
        if V(ii+1)<=V(ii)% compare which element of V
            tau=ii+t+1;
        % find tau when the element of V is smallest
        else
            V(ii+1)=V(ii);
        end
    end
end
end
end

```

```

S=600;
e=10000;
T=50;
sigma=5;
c=.1;
k=[];
for i=-0.05:0.002:0.05
    f=compare7(T,0,0,c,e,S,i,sigma);
    k=[k;f];
end
b2=-0.05:0.002:0.05;
plot(b2,k);
ylim([0 55]);
xlim([-0.02 0.05]);
xlabel('b');ylabel('optimal stopping time');

```

A.2.2 Case 2

```
function [n,v,u]=compare23(T,g,t,s,c,e,S,a,b)
l=[];%keep the y-j
k=[];%keep the \mu-j^2, which j is not in optimal set A
w=[];%keep the \mu-j,which j is not in optimal set A
x=0;
for ii=1:g-t
    x=x+(a*ii^2+b*ii+1)/(a*T*(T+1)*(2*T+1)/6+b*T*(T+1)/2+T)*e;
end
for i=1:g-t
    q=(S-s)/(g-t)+c/(1-c)*((a*i^2+b*i+1)/
    (a*T*(T+1)*(2*T+1)/6+b*T*(T+1)/2+T)*e-x/(g-t));
    z=(a*i^2+b*i+1)/(a*T*(T+1)*(2*T+1)/6+b*T*(T+1)/2+T)*e;
    if q<0
        q=0;
    else
        z=0;
    end
    l=[l;q];
    k=[k;z^2];
    w=[w;z];
end
m=find(l(:)~=0);%remove y-j=0
n=length(m);%|A|
u=x-sum(w);%\sum\mu-j,which j is in optimal set A
v=sum(k);%\sum\mu-j^2,which j is not in optimal set A
end
```

```

function tau=compare9(T,t,s,c,e,S,a,b,sigma)
    V=zeros(1,T-t);
    tau=t+1;
    for i=1:T-t
        [n,v,u]=compare23(T,i+t,t,s,c,e,S,a,b);
        alpha=(S-s)*(1-c)/n-c*u/n;
        V(i)=c^2*sigma^2*i+n*alpha^2+c^2*v;% assign V matrix
    end
    for ii=1:T-t-1
        if V(ii+1)<=V(ii)
            % compare which element of V
            tau=ii+t+1;
            % find tau when the element of V is smallest
        else
            V(ii+1)=V(ii);
        end
    end
end
end
end

```

```

S=600;
T=50;
e=10000;
sigma=5;
c=.1;
l=[];
for i=0:0.02:0.5
    temp=[];
    for j=0:0.02:0.5

```

```

f=compare9(T,0,0,c,e,S,i,j,sigma);
temp=[temp,f];
end
l=[1;temp];
end
a=0:0.02:0.5;
b=0:0.02:0.5;
mesh(a,b,l);
axis ([0 0.5 0 0.5 0 55]);
xlabel('a');ylabel('b');zlabel('optimal stopping time');

```

A.2.3 Case 3

```

function [tau,V]=compare11(T,t,s,c,S,mu,k,n)
C1=c^2*(k+mu^2+n*(2*t+1)/2);
C2=c^2*n/2;
C3=(S-s)^2*(1-c)^2;
C4=-2*c*mu*(S-s)*(1-c);
delta=4*C1^3*C3-108*C2^2*C3^2;
Q=(-108*C2^2*delta)^.5;
C=((Q+2*C1^3-108*C2^2*C3)/2)^(1/3);
x1=-C1/(6*C2)-C/(6*C2)-C1^2/(6*C2*C);
x2=-C1/(6*C2)+C*(1+sqrt(3)*1i)/12/C2
+C1^2*(1-sqrt(3)*1i)/12/C2/C;
x=-C1/(3*C2)+C/(6*C2)+C1^2/(6*C2*C);
x3=x-x2;
v=[floor(x1),ceil(x1)];
m=[max([x1 x2 x3]),median([x1 x2 x3]),min([x1 x2 x3])];
p=[floor(m(3)),ceil(m(3)),floor(m(2))]

```

```

    ,ceil(m(2)),floor(m(1)),ceil(m(1))];
g1=C1+C2+C3+C4;
gtT=C2*(T-t)^2+C1*(T-t)+C3/(T-t)+C4;
gv=C2*v.^2+C1*v+C3./v+C4;
gp=C2*p.^2+C1*p+C3./p+C4;
r1=[g1 gtT gv];
m1=[t+1,T,t+v];
r2=[gp(1:2) g1 gtT];
m2=[t+p(1:2),t+1,T];
r3=[gp(1:4) g1 gtT];
m3=[t+p(1:4),t+1,T];
r4=[gp g1 gtT];
m4=[t+p,t+1,T];
r5=[gp(3:4) g1 gtT];
m5=[t+p(3:4),t+1,T];
r6=[gp(3:6) g1 gtT];
m6=[t+p(3:6),t+1,T];
r7=[gp(5:6) g1 gtT];
m7=[t+p(5:6),t+1,T];
if delta<0
if T-t<=x1 || x1<=1
    if g1<=gtT
        tau=t+1;
        V=g1;
    else
        tau=T;
        V=gtT;
    end
end
if 1<x1<T-t
    tau=m1(min(r1)==r1);

```

```

    V=min(r1);
end
end
if delta>=0
    if T-t<=m(3) || m(3)<=1<T-t<=m(2) || m(2)<=1<T-t<=m(1)
        if g1<=gtT
            tau=t+1;
            V=g1;
        else
            tau=T;
            V=gtT;
        end
    end
    if 1<=m(3)<=T-t<=m(2)
        tau=m2(min(r2)==r2);
        V=min(r2);
    end
    if 1<=m(3) && m(2)<=T-t<=m(1)
        tau=m3(min(r3)==r3);
        V=min(r3);
    end
    if 1<=m(3) && m(1)<=T-t
        tau=m4(min(r4)==r4);
        V=min(r4);
    end
    if m(3)<=1<=m(2) && m(2)<=T-t<=m(1)
        tau=m5(min(r5)==r5);
        V=min(r5);
    end
    if m(3)<=1<=m(2) && m(1)<=T-t
        tau=m6(min(r6)==r6);
    end
end
end

```



```

        V=min(r6);
    end
    if m(2)<=1<=m(1) && m(1)<=T-t
        tau=m7(min(r7)==r7);
        V=min(r7);
    end
end

```

A.3 MATLAB Code to Section 2.5

```

function [op,tau,l]=compare24(t,s,c,S,mu,ss)
    tau=t+1;
    g=size(mu(:,1),1);%keep the size of mu
    V=zeros(1,g-t);
    for i=t+1:g
        [n,v,u,l]=compare25(i+t,t,s,c,S,mu);
        al=(S-s)*(1-c)/n-c*u/n;
        V(i)=c^2*sum(ss(1:g-t,1),1)+n*al^2+c^2*v;
        % assign V matrix
    end
    for ii=1:g-1
        if V(ii+1)<=V(ii)
            % compare which element of V
            tau=ii+t+1;
            % find tau when the element of V is smallest
            op=V(ii+1);
            % find the optimal value of V
        else

```

```

        V(ii+1)=V(ii);
        op=V(ii+1);    % find the optimal value of V
    end
end
[n,v,u,l]=compare25(tau,t,s,c,S,mu);
for i=t+1:tau
if l(i)~=0
    l(i)=(S-s)/n+c/(1-c)*(mu(i)-u/n);
    %keep the optimal strategy
end
end
end
end

```

```

function [n,v,u,l]=compare25(g,t,s,c,S,mu)
l=[];%keep the y-j
w=[];
%keep the \mu-j, which j is not in optimal set A
x=sum(mu(1:g-t),1);
for i=1:g-t
    q=(S-s)/(g-t)+c/(1-c)*(mu(i)-x/(g-t));
    z=mu(i);
    if q<0
        q=0;
    else
        z=0;
    end
    l=[l;q];
    w=[w;z];
end
end

```

```

m=find(1(:)~=0);%remove y-j=0
n=length(m);%|A|
u=x-sum(w);
%sum\mu-j,which j is in optimal set A
v=sum(w.^2);
%sum\mu-j^2,which j is not in optimal set A
end

```

```

S=100000;
Aa =xlsread('WMT.xlsx','vlm');
Aa = Aa(3:78,844:end-2);
Aa(isnan(Aa))=0;
%replace undefined values by zero
err1=0;
err2=0;
err3=0;
err4=0;
err5=0;
err6=0;
c=.1;
l2=zeros(n,754);
%initialize one alternative strategy with \tilde{Y}
l3=zeros(n,754);
%initialize another alternative strategy with \overline{Y}
for i=0:753
mu=mean(Aa(:,1+i:40+i),2);
%using the previous 40 columns data to estimate mu
ss=var(Aa(:,1+i:40+i),0,2);
%using the previous 40 columns data to estimate the variance

```

```

[op,tau,l1]=compare24(0,0,c,S,mu,ss);
err=sum((l1(1:tau)-c*(Aa(1:tau,40+i)+l1(1:tau))).^2);
err0=sum((l1(1:tau)./(l1(1:tau)+Aa(1:tau,40+i))-c).^2);
err1=err1+err;
err2=err2+err0;
l2(:,i+1)=mu*c;
muc=cumsum(mu*c);
[~,tau0]=max(muc>S);
l2(tau0,i+1)=S-sum(l2(1:tau0-1,i+1));
err=sum((l2(1:tau0,i+1)-c*(Aa(1:tau0,40+i)+l2(1:tau0,i+1))).^2);
err0=sum((l2(1:tau0,i+1)./(l2(1:tau0,i+1)
+Aa(1:tau0,40+i+1))-c).^2);
err3=err3+err;
err4=err4+err0;
l3=l2/(1-c);
muc1=cumsum(mu*c/(1-c));
[~,tau1]=max(muc1>S);
l3(tau1,i+1)=S-sum(l3(1:tau1-1,i+1));
err=sum((l3(1:tau1,i+1)-c*(Aa(1:tau1,40+i)+l3(1:tau1,i+1))).^2);
err0=sum((l3(1:tau1,i+1)./(l3(1:tau1,i+1)
+Aa(1:tau1,40+i+1))-c).^2);
err5=err5+err;
err6=err6+err0;
end
err1=err1/754;
%average error of 754 days in simplified form with our method
err2=err2/754;
%average error of 754 days in original form with our method
err3=err3/754;
%average error of 754 days in simplified form in strategy ...
with \tilde{Y}

```

```

err4=err4/754;
%average error of 754 days in original form in strategy with ...
  \tilde{Y}
err5=err5/754;
%average error of 754 days in simplified form in strategy ...
  with \overline{Y}
err6=err6/754;
%average error of 754 days in original form in strategy with ...
  \overline{Y}

```

A.4 MATLAB Code to Section 3.1

```

x=300;
c=.5;
l=[];
for i=0:10:2000
f=(i/(x+i)-c)^2;
l=[1;f];
end
y=0:10:2000;
plot(y,l);
xlabel('y');ylabel('f(y)');

```

```

function [g1,g2]=compare12(x,c,k3)
k1=c*x;
k2=1-c;
u=3*x^2*k1+3*x*k1*k3+k1*k3^2+x^3*k2;

```

```

v=k1+3*x*k2+k2*k3;
g1=2*(k3/(2*x+k3)-c)^2;
if k3^2-4*u/v==0
    y1=k3/2;
    y2=k3/2;
    g2=(y1/(x+y1)-c)^2+(y2/(x+y2)-c)^2;
elseif k3^2-4*u/v>0
    y1=(k3+sqrt(k3^2-4*u/v))/2;
    y2=(k3-sqrt(k3^2-4*u/v))/2;
    g2=(y1/(x+y1)-c)^2+(y2/(x+y2)-c)^2;
elseif k3^2-4*u/v<0
    g2=NaN;
end

```

```

x=300;
c=.5;
l=[];
for i=0:5:200
    [a,b]=compare12(x,c,i);
    f=[a,b];
    l=[l;f];
end
b1=0:5:200;
plot(b1,l(:,1));
hold all;
plot(b1,l(:,2));
xlabel('S-s');ylabel('minimal value');
legend('g1','g2')

```

```

x=50;
k3=300;
l=[];
for i=0.005:.05:0.995
[a,b]=compare12(x,i,k3);
f=[a,b];
l=[l;f];
end
b1=0.005:.05:0.995;
plot(b1,l(:,1));
hold all;
plot(b1,l(:,2));
xlabel('c');ylabel('minimal value');
legend('g1','g2')

```

A.5 MATLAB Code to Section 3.2

```

function y=fun(x)%find the minimal value of V3 and minimizer y^*
y=0;
a=30;
b=50;
c=0.1;
for i=1:size(x,1)
    y=sum(x.^2./(a+sqrt(x)).*(b+x)
        +2*c*x.*log((a+x)/(b+x))/(b-a)+c^2);
end
end

```

```

T=10;
f=[];
k=zeros(T);
S=0.5;
s=0;
t=0;
for i=1:T
e=eye(i-t)*(-1);%e*x<=m, which means y-j>=0
m=zeros(i-t,1);
n=ones(1,i-t);%n*x=S-s; which means \sum y-j=S-s
x0=ones(i-t,1)*(S-s)/(i-t);
V=zeros(1,i-t);
[X,V] = fmincon(@fun,x0,e,m,n,S-s);
k(1:i,i)=X;
f=[f;V];
end
for ii=1:T-t-1
    if f(ii+1)<=f(ii)% compare which element of V
        temp=ii+t+1;
        % find tau when the element of V is smallest
        q=k(:,ii+t+1);
        s=f(ii+t+1);
    else
        f(ii+1)=f(ii);
    end
end
q(q==0)=[];
tau=temp% the stopping time
y=q% the minimizer y-j
V4=s% V_4

```